

CS 61B Lab 9
March 20-21, 2014

Goal: to practice proving asymptotic (big-Oh) results. The class notes from Lecture 20 may come in handy; however, we will try to be even more rigorous here than in lecture.

Recall the definition of big-Oh:

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| O(f(n)) is the SET of ALL functions T(n) that satisfy:
|
|   There exist positive constants c and N such that, for all n >= N,
|   T(n) <= c f(n)
|=====
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EXAMPLE

Formally prove that $n^2 + n + 1$ is in $O(n^2)$.

Solution:

Let $T(n) = n^2 + n + 1$. Let $f(n) = n^2$.

Choose $c = 3$, and $N = 1$. Then, we know $T(n)$ is in $O(n^2)$ if we can prove

$$\begin{aligned} T(n) &\leq c f(n), \\ \text{or equivalently, } n^2 + n + 1 &\leq 3 n^2, & \text{for all } n \geq 1. \end{aligned}$$

Is this inequality true? Well, for any $n \geq 1$, we know that $1 \leq n \leq n^2$. Hence, all of the following are true:

$$\begin{aligned} 1 &\leq n^2 \\ n &\leq n^2 \\ n^2 &= n^2 \end{aligned}$$

Adding the left and right sides of these inequalities together, we have

$$n^2 + n + 1 \leq 3 n^2, \text{ which completes the proof.}$$

Part I: (1 point)

Formally prove that $2^n + 1$ is in $O(4^n - 16)$.

HINT: You may assert without proof that, for all $n \geq 1$, $2^n \geq 1$. (You may also assert without proof that 4^n and 2^n are monotonically increasing, if you find it useful.)

Part II: (1 point)

Formally prove that if $f(n)$ is in $O(g(n))$, and $g(n)$ is in $O(h(n))$, then $f(n)$ is in $O(h(n))$.

NOTE: The values of c and N used to prove that $f(n)$ is in $O(g(n))$ are not necessarily the same as the values used to prove that $g(n)$ is in $O(h(n))$. Hence, assume that there are positive c' , N' , c'' , and N'' such that

$$\begin{aligned} f(n) &\leq c' g(n) & \text{for all } n \geq N', \text{ and} \\ g(n) &\leq c'' h(n) & \text{for all } n \geq N''. \end{aligned}$$

Part III: (2 points)

Formally prove that $0.01 n^2 - 1$ is NOT in $O(n)$.

We need to show that, no matter how large we choose c and N , we will never obtain the desired inequality. We cannot prove this by picking a specific value of c and N . Instead, we must study how the two functions behave as n approaches infinity.

Let $T(n) = 0.01 n^2 - 1$, and let $f(n) = n$. Prove that

$$\lim_{n \rightarrow \infty} \frac{c f(n)}{T(n)} = 0,$$

no matter how large we choose c to be. You will need to scale both the numerator and the denominator by a well-chosen multiplier to get the result.

Use this result to show that there are no values c , N such that $T(n) \leq c f(n)$ for all $n \geq N$.

Postscript

The functions $|\cos(n)|$ and $|\sin(n)|$ are interesting, because neither is dominated by the other. Can you informally suggest why $|\cos(n)|$ is not in $O(|\sin(n)|)$, and $|\sin(n)|$ is not in $O(|\cos(n)|)$?

How would you prove that, for all $n \geq 1$, $2^n \geq 1$? (Hint: use calculus.)