Express $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$

Solution

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

$$\frac{2n}{x} J_n x = J_{n-1}(x) + J_{n+1}(x)$$

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \rightarrow (1)$$

Put $n=1$ in (i)

$$J_2(x) = \frac{2}{x}\left[J_1(x) - J_0(x)\right]$$

Put $n=2$ in (1) we get

$$\Rightarrow J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$\Rightarrow J_3(x) = \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$$

$$\Rightarrow J_3(x) = \frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)$$

$$\Rightarrow J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

Putting $n=3$ in (1) we get

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$$

$$\Rightarrow J_4(x) = \frac{6}{x}\left[\left(\frac{8}{x^3}-1\right)J_1(x) - \frac{4}{x}J_0(x)\right] - \left[\frac{2}{x}J_1(x) - J_0(x)\right]$$

$$\Rightarrow J_4(x) = \left(\frac{48}{x^3} - \frac{6}{x}\right)J_1(x) - \frac{24}{x^2}J_0(x) - \frac{2}{x}J_1(x) + J_0(x)$$

$$\Rightarrow J_4(x) = \left(\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right)J_1(x) + J_0(x) - \frac{24}{x^2}J_0(x)$$

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$$

EX05: Show that $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

Solution:

As from the Recurrence formula

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$$n x^{n-1} J_n(x) + x^n J_n'(x) = x^n J_{n-1}(x)$$

Dividing by x^n, we get

$$\frac{n}{x} J_n(x) + J_n'(x) = J_{n-1}(x) \quad \text{--- (1)}$$

Similarly,

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

$$-n x^{-n-1} J_n(x) + x^{-n} J_n'(x) = -x^{-n} J_{n+1}(x)$$

Dividing by x^{-n}, we get

$$\frac{-n}{x} J_n(x) + J_n'(x) = -J_{n+1}(x) \quad \text{--- (2)}$$

By (1) and (2) we get

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$