Mathematics II

Model Question paper - 1

part -A (Answer all questions. Each carries 1 Mark)

1) Evaluate $\begin{vmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{vmatrix}$

A) $\begin{vmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{vmatrix} = \sin x \times \sin x - (\cos x \times -\cos x)$

$= \sin^2 x + \cos^2 x$

$= \underline{\underline{1}}$

2) find $A-B$ if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 \\ -3 & -3 \end{bmatrix}$

A) $A-B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ -3 & -3 \end{bmatrix}$

$= \begin{bmatrix} 1-0 & 2-(-2) \\ 3-(-3) & 4-(-3) \end{bmatrix} = \begin{bmatrix} 1 & 2+2 \\ 3+3 & 4+3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix}$

3) If $\vec{a} = i + j + k$, $\vec{b} = 2i - j + 3k$.
find $\vec{a} \cdot \vec{b}$

A) $\vec{a} \cdot \vec{b} = (i + j + k) \cdot (2i - j + 3k)$

$= 1 \times 2 + 1 \times -1 + 1 \times 3$

$= 2 - 1 + 3 = 1 + 3 = \underline{\underline{4}}$

4) Find unit vector in the direction of
$\vec{a} = 2i + 3j + 4k$.

A) $|\vec{a}| = \sqrt{2^2 + 3^2 + 4^2}$

$= \sqrt{4 + 9 + 16} = \underline{\underline{\sqrt{29}}}$

$\therefore$ the unit vector in the direction of
$\vec{a}$ is $\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{2i + 3j + 4k}{\sqrt{29}}$

5) Evaluate $\int (2x+3)\,dx$.

A) $\int (2x+3)\,dx$

$= \int 2x\,dx + \int 3\,dx$

$= 2\int x\,dx + 3\int 1\,dx$

$= 2\,\frac{x^2}{2} + 3x + C$

$= \underline{\underline{x^2 + 3x + C}}$

6) Evaluate $\int \sec x(\sec x + \tan x)\,dx$.

A) $\int \sec x\,(\sec x + \tan x)\,dx$

$= \int (\sec^2 x + \sec x \tan x)\,dx$

$= \int \sec^2 x\,dx + \int \sec x \tan x\,dx$

$= \underline{\underline{\tan x + \sec x + C}}$

7) Evaluate $\int_0^1 x\,dx$.

A) $\int_0^1 x\,dx = \left(\frac{x^2}{2}\right)_0^1$

$$= \frac{1^2}{2} - \frac{0^2}{2}$$

$$= \frac{1}{2} - 0 = \underline{\underline{1/2}}$$

8) Find order and degree of

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{d^3y}{dx^3} + 5\frac{dy}{dx} = y.$$

A) order = 3, degree = 1

9) Solve $\frac{dy}{dx} = \frac{x}{y}$.

A) $\frac{dy}{dx} = \frac{x}{y}$

$$\Rightarrow y\,dy = x\,dx.$$

Integrating on bothsides,

$$\int y\,dy = \int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

Part-B

(Answer all questions. Each Carries 3 Marks)

1) If $\begin{vmatrix} x & 1 & 3 \\ 4 & 1 & -1 \\ 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix}$. find x

A) $$\begin{vmatrix} x & 1 & 3 \\ 4 & 1 & -1 \\ 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix}$$

$$\Rightarrow x\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ -1 & 0 \end{vmatrix}$$

$\Rightarrow x(3+0) - 1(\cancel{4}12-2) + 3(0-2)$
$= 2(0-0) + 1(6-(-1)) + 1(0-0)$

$\Rightarrow 3x - 14 - 6 = 7 + 0$

$\Rightarrow 3x - 20 = 7$

$\Rightarrow 3x = 7 + 20 = 27$

$\therefore x = \frac{27}{3} = \underline{\underline{9}}$

2) Find inverse of $\begin{bmatrix} 4 & 1 \\ 6 & 5 \end{bmatrix}$

A) $\boxed{A^{-1} = \frac{\text{Adj} A}{|A|}}$

Cofactor of $4 = C_{11} = (-1)^{1+1} |5|$

$= (-1)^2 \times 5$

$= 1 \times 5 = \underline{\underline{5}}$

Cofactor of 1 $= C_{12} = (-1)^{1+2} \times |6|$

$= (-1)^3 \times 6 = -1 \times 6 = \underline{\underline{-6}}$

Cofactor of 6 $= C_{21} = (-1)^{2+1} |1|$

$= (-1)^3 \times 1 = -1 \times 1 = \underline{\underline{-1}}$

Cofactor of 5 $= C_{22} = (-1)^{2+2} \times |4|$

$= (-1)^4 \times 4 = 1 \times 4 = \underline{4}$

Cofactor Matrix $= \begin{bmatrix} 5 & -6 \\ -1 & 4 \end{bmatrix}$

$\therefore$ Adj A $= \begin{bmatrix} 5 & -1 \\ -6 & 4 \end{bmatrix}$

$|A| = \begin{vmatrix} 4 & 1 \\ 6 & 5 \end{vmatrix} = 4 \times 5 - 1 \times 6$

$= 20 - 6$

$= \underline{\underline{14}}$

$\therefore \quad A^{-1} = \dfrac{\text{Adj } A}{|A|}$

$= \dfrac{\begin{bmatrix} 5 & -1 \\ -6 & 4 \end{bmatrix}}{14}$

3) Find a vector perpendicular to the vectors $\vec{a} = 2i + 3j + 4k$ and $\vec{b} = i + j + k$.

A) Vector perpendicular to $\vec{a}$ and $\vec{b}$ is $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix}$

$= \hat{i}\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$

$= \hat{i}(3-4) - \hat{j}(2-4) + \hat{k}(2-3)$

$= \hat{i}\times -1 - \hat{j}\times -2 + \hat{k}\times -1$

$= -\hat{i} + 2\hat{j} - \hat{k}$

4) Find the angle between the vectors $6i - 3j + 2K$ and $2i + 2j - k$.

A) angle between two vectors $\vec{a}$ and $\vec{b}$ is given by $\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right)$

Here let $\vec{a} = 6i - 3j + 2K$ and

$\vec{b} = 2i + 2j - k$.

$\vec{a}\cdot\vec{b} = (6i - 3j + 2k)\cdot(2i + 2j - k)$

$= 6\times 2 + -3\times 2 + 2\times -1$

$= 12 - 6 - 2 = 6 - 2 = \underline{\underline{4}}$

$a = |\vec{a}| = \sqrt{6^2 + (-3)^2 + 2^2} = \sqrt{36 + 9 + 4}$

$= \sqrt{49} = \underline{\underline{7}}$

$b = |\vec{b}| = \sqrt{2^2 + 2^2 + (-1)^2}$

$= \sqrt{4+4+1} = \sqrt{9} = \underline{3}$

$\therefore \theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{ab}\right)$

$= \cos^{-1}\left(\dfrac{4}{7\cdot 3}\right) = \cos^{-1}\left(\dfrac{4}{21}\right).$

5) Find the workdone by a force $\vec{F} = i + 2j + K$ acting on a particle which is displaced from a point with position vector $2i + j + k$ to the point with position vector $3i + 2j + 4K$.

A) Given $\vec{F} = i + 2j + K$.

Position vector of A $= 2i + j + K$ and

Position vector of B $= 3i + 2j + 4K$.

$\therefore \vec{AB}$ = position vector of B − position vector of A

$= 3i + 2j + 4k - (2i + j + k)$

$= 3i + 2j + 4k - 2i - j - k$

$= \underline{\underline{i + j + 3k}}$

$\therefore$ work done $W = \vec{F} \cdot \vec{AB}$

$= (i + 2j + k) \cdot (i + j + 3k)$

$= 1 \times 1 + 2 \times 1 + 1 \times 3$

$= 1 + 2 + 3 = \underline{\underline{6}}$ units.

6) Evaluate $\int \frac{\sin^{-1} 2x}{\sqrt{1 - 4x^2}} \, dx$.

A) put $\sin^{-1} 2x = u$

$\frac{1}{\sqrt{1-(2x)^2}} \times \frac{d}{dx}(2x) = \frac{du}{dx}$

ie, $\dfrac{du}{dx} = \dfrac{1}{\sqrt{1-4x^2}} \times 2$

ie, $\dfrac{du}{2} = \dfrac{1}{\sqrt{1-4x^2}}\, dx$.

$\therefore \displaystyle\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}}\, dx$

$= \displaystyle\int u \cdot \frac{du}{2}$

$= \dfrac{1}{2} \displaystyle\int u\, du$

$= \dfrac{1}{2} \dfrac{u^2}{2} + c$

$= \dfrac{1}{4} u^2 + C$

$= \dfrac{1}{4} \left(\sin^{-1}(2x)\right)^2 + C$

7) Evaluate $\int x \sin x \, dx$.

A) $\int x \sin x \, dx = x \int \sin x \, dx - \int \left(\frac{d}{dx}(x) \int \sin x \, dx \right) dx$

$= x \times -\cos x - \int 1 \cdot x - \cos x \, dx$

$= -x \cos x + \int \cos x \, dx$

$= -x \cos x + \sin x + C$

8) Evaluate $\int_0^{\pi/2} \cos 4x \cos x \, dx$.

A) we have,

$\cos A \cos B = \frac{1}{2} \left[\cos(A+B) + \cos(A-B) \right]$

$\therefore \cos 4x \cos x = \frac{1}{2} \left[\cos(4x+x) + \cos(4x-x) \right]$

$= \frac{1}{2} \left[\cos 5x + \cos 3x \right]$

$\therefore \int_0^{\pi/2} \cos 4x \cos x \, dx = \int_0^{\pi/2} \frac{1}{2} \left(\cos 5x + \cos 3x \right) dx$

$$= \frac{1}{2}\left[\int_0^{\pi/2} (\cos 5x + \cos 3x)\, dx\right]$$

$$= \frac{1}{2}\left[\frac{\sin 5x}{5} + \frac{\sin 3x}{3}\right]_0^{\pi/2}$$

$\boxed{\sin 0 = 0}$

$$= \frac{1}{2}\left[\left(\frac{\sin \frac{5\pi}{2}}{5} + \frac{\sin \frac{3\pi}{2}}{3}\right) - \left(\frac{\sin 0}{5} + \frac{\sin 0}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin 450}{5} + \frac{\sin 270}{3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{5} + \frac{-1}{3}\right]$$

$$= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{3}\right)$$

$$= \frac{1}{2}\left(\frac{3-5}{15}\right)$$

$$= \frac{1}{2} \times \frac{-2}{15}$$

$$= -1/15$$

$\sin 450^\circ = \sin(360+90)$

$= \sin 90 = 1$

$\sin 270 = \sin(180+90)$

$= -\sin 90$

$= -1$

Q) obtain the area enclosed between the parabola $y = x^2 - x - 2$ and the x-axis.

A) when the curve meets the x-axis,

$x^2 - x - 2 = 0$.

For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here $a = 1$, $b = -1$, $c = -2$

$$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times -2}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{1+8}}{2}$$

$$= \frac{1 \pm \sqrt{9}}{2}$$

$$= \frac{1 \pm 3}{2}$$

$$= \frac{1+3}{2}, \frac{1-3}{2}$$

$$= \frac{4}{2}, \frac{-2}{2}$$

$$= \underline{\underline{2, -1}}.$$

$$\therefore \text{Area } A = \int_a^b y \, dx$$

$$= \int_{-1}^{2} (x^2 - x - 2) \, dx$$

$$= \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)_{-1}^{2}$$

$$= \left(\frac{2^3}{3} - \frac{2^2}{2} - 2\times 2\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2} - 2\times -1\right)$$

$$= \left(\frac{8}{3} - \frac{4}{2} - 4\right) - \left(\frac{-1}{3} - \frac{1}{2} + 2\right)$$

$$= \frac{8}{3} - 2 - 4 + \frac{1}{3} + \frac{1}{2} - 2$$

$$= \frac{8}{3} + \frac{1}{3} + \frac{1}{2} - 8$$

$$= \frac{9}{3} + \frac{1}{2} - 8$$

$$= 3 + \frac{1}{2} - 8$$

$$= \frac{1}{2} - 5$$

$$= \frac{1-10}{2} = \underline{\underline{-9/2}}$$

10) Solve $\dfrac{dy}{dx} = \dfrac{xy^2+x}{yx^2+y}$.

A) $$\frac{dy}{dx} = \frac{xy^2+x}{yx^2+y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x(y^2+1)}{y(x^2+1)}$$

$$\Rightarrow y(x^2+1)\,dy = x(y^2+1)\,dx$$

$$\Rightarrow \frac{y}{y^2+1}\,dy = \frac{x}{x^2+1}\,dx.$$

Integrating on both sides,

$$\int \frac{y}{y^2+1}\,dy = \int \frac{x}{x^2+1}\,dx \quad \text{—— (1).}$$

$$\int \frac{y}{y^2+1}\,dy \qquad \left\{ \begin{array}{l} \text{put } y^2+1 = u \\ 2y = \dfrac{du}{dy} \\ 2y\,dy = du \\ y\,dy = \dfrac{du}{2} \end{array} \right.$$

$$= \int \frac{1}{u} \cdot \frac{du}{2}$$

$$= \frac{1}{2} \int \frac{1}{u}\,du$$

$$= \frac{1}{2} \log u + c_1 .$$

$$= \frac{1}{2} \log (y^2+1) + c_1$$

$$\int \frac{x}{x^2+1}\,dx \qquad \left\{ \begin{array}{l} \text{put } x^2+1 = u \\ 2x = \dfrac{du}{dx} \\ 2x\,dx = du \\ x\,dx = \dfrac{du}{2} \end{array} \right.$$

$$= \int \frac{1}{u} \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\log u + c_2$$

$$= \frac{1}{2}\log(x^2+1) + c_2 .$$

$\therefore$ ① becomes

$$\frac{1}{2}\log(y^2+1) + c_1 = \frac{1}{2}\log(x^2+1) + c_2$$

$$\Rightarrow \log(y^2+1)^{1/2} + c_1 = \log(x^2+1)^{1/2} + c_2$$

$$\Rightarrow \log(y^2+1)^{1/2} - \log(x^2+1)^{1/2} = c_2 - c_1 = C$$

$$\Rightarrow \log\sqrt{y^2+1} - \log\sqrt{x^2+1} = C$$

$$\Rightarrow \log\frac{\sqrt{y^2+1}}{\sqrt{x^2+1}} = C$$

$$\Rightarrow \log\sqrt{\frac{y^2+1}{x^2+1}} = C$$

Part-C

Answer all questions. Each question Carries Seven Marks.

III) a) Solve for x if,

$$\begin{vmatrix} 2 & 1 & x \\ 3 & -1 & 2 \\ 1 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 4 & x \\ 3 & 2 \end{vmatrix}$$

A) $$\begin{vmatrix} 2 & 1 & x \\ 3 & -1 & 2 \\ 1 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 4 & x \\ 3 & 2 \end{vmatrix}$$

$$\Rightarrow 2\begin{vmatrix} -1 & 2 \\ 1 & 6 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} + x\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 4\times 2 - 3\times x$$

$$\Rightarrow 2(-6-2) - 1(18-2) + x(3+1) = 8-3x$$

$$\Rightarrow 2\times -8 - 1\times 16 + x\times 4 = 8-3x$$

$\Rightarrow -16 - 16 + 4x = 8 - 3x$

$\Rightarrow -32 + 4x = 8 - 3x$

$\Rightarrow 4x + 3x = 8 + 32$

$\Rightarrow 7x = 40$

$\Rightarrow x = \underline{\underline{\frac{40}{7}}}$.

b) Find the inverse of the Matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

A) $|A| = \begin{vmatrix} 1 & 2 \\ 4 & 9 \end{vmatrix} = 1 \times 9 - 4 \times 2 = 9 - 8 = \underline{\underline{1}}$

Cofactor of $1 = C_{11} = (-1)^{1+1} |9| = (-1)^2 \times 9 = 1 \times 9 = \underline{\underline{9}}$

" $2 = C_{12} = (-1)^{1+2} |4| = (-1)^3 \times 4 = -1 \times 4 = \underline{\underline{-4}}$

" $4 = C_{21} = (-1)^{2+1} |2| = (-1)^3 \times 2 = -1 \times 2 = \underline{\underline{-2}}$

" $9 = C_{22} = (-1)^{2+2} |1| = (-1)^4 \times 1 = 1 \times 1 = \underline{\underline{1}}$

$\therefore$ Cofactor Matrix $= \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$

$\text{Adj } A = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$

$$\therefore A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= \frac{\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}}{1} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$$

$\boxed{\text{OR}}$

IV) a) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

Compute AB and BA.

A) $AB = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1\times1+0\times2+2\times-3 & 1\times-2+0\times3+2\times1 & 1\times3+0\times-1+2\times2 \\ 0\times1+1\times2+2\times-3 & 0\times-2+1\times3+2\times1 & 0\times3+1\times-1+2\times2 \\ 1\times1+2\times2+0\times-3 & 1\times-2+2\times3+0\times1 & 1\times3+2\times-1+0\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-6 & -2+0+2 & 3+0+4 \\ 0+2-6 & 0+3+2 & 0-1+4 \\ 1+4+0 & -2+6+0 & 3-2+0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+-2\times0+3\times1 & 1\times0+-2\times1+3\times2 & 1\times2+-2\times2+3\times0 \\ 2\times1+3\times0+-1\times1 & 2\times0+3\times1+-1\times2 & 2\times2+3\times2+-1\times0 \\ -3\times1+1\times0+2\times1 & -3\times0+1\times1+2\times2 & -3\times2+1\times2+2\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+3 & 0-2+6 & 2-4+0 \\ 2+0-1 & 0+3-2 & 4+6+0 \\ -3+0+2 & 0+1+4 & -6+2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

b) Find the values of a, b, c that satisfy the matrix relation,

$$\begin{bmatrix} a+3 & 3a-2b \\ 3a-c & a+b+c \end{bmatrix} = \begin{bmatrix} 2 & -7+2b \\ b+4 & 8a \end{bmatrix}$$

$a+3=2 \Rightarrow a=2-3=\underline{\underline{-1}}$

$3a-2b=-7+2b$

$3\times -1-2b=-7+2b$

$-3-2b=-7+2b$

$-2b-2b=-7+3$

$-4b=-4$

$b=-4/-4=\underline{1}$

$3a-c=b+4$

$3\times -1-c=1+4$

$-3-c=5$

$-c = 5+3 = 8$

$\therefore c = -8$

ii) a) A force $\vec{F} = 4i - 3k$ passes through the point A whose position vector is $2i - 2j + 5k$. Find the moment of the force about the point B whose position vector is $i - 3j + k$.

A) $\vec{F} = 4i - 3k$.

$\vec{r} = \vec{BA}$

$=$ position vector of A $-$ position vector of B

$= (2i - 2j + 5k) - (i - 3j + k)$

$= 2i - 2j + 5k - i + 3j - k$

$= i + j + 4k$

$\therefore m = \vec{r} \times \vec{F}$

$= (i + j + 4k) \times (4i - 3k)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 1 & 4 \\ 0 & -3 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 4 \\ 4 & -3 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix}$$

$$= \hat{i}(-3-0) - \hat{j}(-3-16) + \hat{k}(0-4)$$

$$= -3 \times \hat{i} - \hat{j} \times -19 + \hat{k} \times -4$$

$$= -3\hat{i} + 19\hat{j} - 4\hat{k}$$

The modules of the above vector gives the moment.

ie, moment $= \sqrt{(-3)^2 + 19^2 + (-4)^2}$

$= \sqrt{386}$ units

1) Find area of the triangle formed by O, A and B when $\vec{OA} = i + 2j + 3k$ and $\vec{OB} = -3i - 2j + k$.

$$\vec{OA} \times \vec{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 3 \\ -3 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ -3 & -2 \end{vmatrix}$$

$$= \hat{i}(2+6) - \hat{j}(1+9) + \hat{k}(-2+6)$$

$$= \hat{i} \times 8 - \hat{j} \times 10 + \hat{k} \times 4$$

$$= 8\hat{i} - 10\hat{j} + 4\hat{k}.$$

$$|\vec{OA} \times \vec{OB}| = \sqrt{8^2 + (-10)^2 + 4^2}$$

$$= \sqrt{180}.$$

$\therefore$ area $= \frac{1}{2}|\vec{OA} \times \vec{OB}| = \frac{\sqrt{180}}{2}$ sq. units.

$\boxed{OR}$

VI) a) The Constant forces $2i-5j+6K$, $-i+2j-K$ and $2i+7j$ act on a particle from the position $4i-3j-2K$ to $6i+j-3K$. find the total workdone.

A) Total force,

$$\vec{F} = 2i-5j+6K-i+2j-K+2i+7j$$

$$= 3i+4j+5K.$$

$$\vec{AB} = 6i+j-3K-(4i-3j-2K)$$

$$= 6i+j-3K-4i+3j+2K$$

$$= 2i+4j-K.$$

$$\text{Workdone} = \vec{F}\cdot\vec{AB}$$

$$= (3i+4j+5K)\cdot(2i+4j-K)$$

$$= 3\times2+4\times4+5\times-1$$

$$= 6+16-5 = 17 \text{ units}$$

b) Find a unit vector perpendicular to the vectors $i - j + k$ and $2i + j - k$.

A) Let $\vec{a} = i - j + k$ and $\vec{b} = 2i + j - k$.

vector perpendicular to $\vec{a}$ and $\vec{b}$ is,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= \hat{i}(1-1) - \hat{j}(-1-2) + \hat{k}(1+2)$$

$$= \hat{i} \times 0 - \hat{j} \times -3 + \hat{k} \times 3$$

$$= \underline{\underline{3\hat{j} + 3\hat{k}}}$$

$$|\vec{a} \times \vec{b}| = \sqrt{3^2 + 3^2} = \sqrt{9+9} = \underline{\underline{\sqrt{18}}}$$

$\therefore$ the unit vector perpendicular to $\vec{a}$ & $\vec{b}$

is $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \underline{\underline{\dfrac{3\hat{j} + 3\hat{k}}{\sqrt{18}}}}$

VII) a) Find angle between $7i-j+11k$ and $i+j+k$.

A) let $\vec{a} = 7i-j+11k$ & $\vec{b} = i+j+k$.

$$\vec{a}\cdot\vec{b} = (7i-j+11k)\cdot(i+j+k)$$

$$= 7\times1 + -1\times1 + 11\times1$$

$$= 7-1+11$$

$$= 6+11 = \underline{\underline{17}}$$

$$a = |\vec{a}| = \sqrt{7^2+(-1)^2+11^2}$$

$$= \sqrt{49+1+121} = \sqrt{171}$$

$$b = |\vec{b}| = \sqrt{1^2+1^2+1^2} = \underline{\underline{\sqrt{3}}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right)$$

$$= \underline{\underline{\cos^{-1}\left(\frac{17}{\sqrt{3}\cdot\sqrt{171}}\right)}}$$

VII) Find the Value of 'p' so that two vectors $2i - 3j - k$ and $4i - pj - 2k$ are perpendicular to each other.

A) $(2i - 3j - k) \cdot (4i - pj - 2k) = 0$

$\Rightarrow 2 \times 4 + -3 \times -p + -1 \times -2 = 0$

$\Rightarrow 8 + 3p + 2 = 0$

Two vectors are perpendicular if dot product is zero.

$\Rightarrow 3p + 10 = 0$

$\Rightarrow 3p = -10$

$\Rightarrow p = -10/3$

OR

VIII) a) Find area of a parallelogram whose adjacent sides are determined by the vectors $\vec{a} = i - j + 3k$ and $\vec{b} = 2i - 7j + k$

A) let the adjacent sides be

$\vec{a} = i - j + 3k$ & $\vec{b} = 2i - 7j + k$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} -1 & 3 \\ -7 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -1 \\ 2 & -7 \end{vmatrix}$$

$$= \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2)$$

$$= \hat{i} \times 20 - \hat{j} \times -5 + \hat{k} \times -5$$

$$= 20\hat{i} + 5\hat{j} - 5\hat{k}$$

$\therefore$ Area $= |\vec{a} \times \vec{b}|$

$$= \sqrt{20^2 + 5^2 + (-5)^2} = \sqrt{400+25+25}$$

$$= \sqrt{450}$$

Q) Find the dot product of $2i+3j-k$ & $i-2j+4k$.

A) $(2i+3j-k)\cdot(i-2j+4k)$

$= 2\times 1 + 3\times -2 + -1\times 4$

$= 2-6-4 = 2-10 = \underline{\underline{-8}}$

IX) a) Evaluate $\int_0^{\pi} \frac{1-\sin x}{x+\cos x}\,dx$.

A) put $u = x+\cos x$

$\therefore \frac{du}{dx} = 1-\sin x$

$\Rightarrow (1-\sin x)\,dx = du$

$\therefore \int \frac{1-\sin x}{x+\cos x}\,dx = \int \frac{1}{u}\,du$

$= \log u + C$

$= \log(x+\cos x) + C.$

$\therefore \int_0^{\pi} \frac{1-\sin x}{x+\cos x}\,dx = \Big[\log(x+\cos x)\Big]_0^{\pi}$

$= \log(\pi + \cos\pi) - \log(0 + \cos 0)$

$= \log(\pi - 1) - \log 1$

$= \log\left(\frac{\pi - 1}{1}\right)$ $\log a - \log b = \log\left(\frac{a}{b}\right)$

$= \log(\pi - 1)$

b) Evaluate $\int_0^{\pi/2} \cos^3 x \, dx$.

A) We have $\cos 3x = 4\cos^3 x - 3\cos x$

$\Rightarrow 4\cos^3 x = \cos 3x + 3\cos x$

$\Rightarrow \cos^3 x = \frac{\cos 3x + 3\cos x}{4}$

$\therefore \int_0^{\pi/2} \cos^3 x \, dx = \int_0^{\pi/2} \frac{\cos 3x + 3\cos x}{4} \, dx$

$= \frac{1}{4}\int_0^{\pi/2} (\cos 3x + 3\cos x) \, dx$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(\frac{\sin \frac{3\pi}{2}}{3} + 3\sin\frac{\pi}{2}\right) - \left(\frac{\sin 3\times 0}{3} + 3\sin 0\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{\sin \frac{3\pi}{2}}{3} + 3\sin\frac{\pi}{2}\right) - 0\right]$$

$$= \frac{1}{4}\left[\frac{-1}{3} + 3\times 1\right]$$

$\sin \frac{3\pi}{2} = \sin 270 = \underline{\underline{-1}}$

$\sin \frac{\pi}{2} = \underline{\underline{1}}$

$$= \frac{1}{4}\left(\frac{-1}{3} + 3\right)$$

$$= \frac{1}{4}\left(\frac{-1+9}{3}\right)$$

$$= \frac{1}{4} \times \frac{8}{3}$$

$$= \underline{\underline{2/3}}$$

$\boxed{\text{OR}}$

X) a) Evaluate $\displaystyle\int \frac{\left(\tan^{-1}(5x)\right)^2}{1+25x^2}\,dx$.

A) We know, $\dfrac{d}{dx}\left(\tan^{-1}x\right) = \dfrac{1}{1+x^2}$.

Put $u = \tan^{-1}(5x)$.

$$\frac{du}{dx} = \frac{1}{1+(5x)^2} \times \frac{d}{dx}(5x)$$

$$= \frac{1}{1+25x^2} \times 5 \times 1$$

$$\therefore \frac{du}{dx} = \frac{5}{1+25x^2}$$

$$\therefore \frac{1}{1+25x^2}\,dx = \frac{du}{5}$$

$$\therefore \int \frac{\left(\tan^{-1}(5x)\right)^2}{1+25x^2}\, dx$$

$$= \int u^2 \cdot \frac{du}{5}$$

$$= \frac{1}{5} \int u^2\, du$$

$$= \frac{1}{5} \frac{u^{2+1}}{2+1} + C$$

$$= \frac{1}{5} \frac{u^3}{3} + C$$

$$= \frac{u^3}{15} + C$$

$$= \frac{\left(\tan^{-1}(5x)\right)^3}{15} + C$$

b) Evaluate $\int_0^{\pi/2} \sin 2x \cos x \, dx$.

A) We know that,

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right].$$

$$\therefore \sin 2x \cos x = \frac{1}{2}\left[\sin(2x+x) + \sin(2x-x)\right]$$

$$= \frac{1}{2}\left[\sin 3x + \sin x\right]$$

$$\therefore \int_0^{\pi/2} \sin 2x \cos x \, dx$$

$$= \int_0^{\pi/2} \frac{1}{2}\left[\sin 3x + \sin x\right] dx$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\sin 3x + \sin x\right] dx.$$

$$= \frac{1}{2}\left[-\frac{\cos 3x}{3} - \cos x\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(-\frac{\cos\frac{3\pi}{2}}{3} - \cos\frac{\pi}{2}\right) - \left(-\frac{\cos 3\times 0}{3} - \cos 0\right)\right]$$

$$= \frac{1}{2}\left[-\frac{\cos\frac{3\pi}{2}}{3} - \cos\frac{\pi}{2} + \frac{\cos 0}{3} - \cos 0\right]$$

$$= \frac{1}{2}\left[-\frac{0}{3} - 0 + \frac{1}{3} - 1\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} - 1\right]$$

$$\left\{\begin{array}{l} \cos\frac{3\pi}{2} = \cos 270 = 0 \\ \cos\frac{\pi}{2} = 0 \\ \cos 0 = 1 \end{array}\right.$$

$$= \frac{1}{2} \times \left(\frac{1-3}{3}\right)$$

$$= \frac{1}{2} \times \frac{-2}{3}$$

$$= \underline{\underline{-1/3}}$$

XI) a) Evaluate $\int_0^{3\pi/2} x \cdot \cos 3x \, dx$.

A) $\int x \cos 3x \, dx$

(Product Rule)

$= x \int \cos 3x \, dx - \int \left(\frac{d}{dx}(x) \int \cos 3x \, dx \right) dx$

$= x \frac{\sin 3x}{3} - \int 1 \cdot \frac{\sin 3x}{3} \, dx$

$= \frac{x \sin 3x}{3} - \frac{1}{3} \times -\frac{\cos 3x}{3}$

$= \frac{x \sin 3x}{3} + \frac{\cos 3x}{9}$

$\therefore \int_0^{3\pi/2} x \cos 3x \, dx$

$= \left[\frac{x \sin 3x}{3} + \frac{\cos 3x}{9} \right]^{\frac{3\pi}{2}}$

$$= \left(\frac{\frac{3\pi}{2} \times \sin 3 \times \frac{3\pi}{2}}{3} + \frac{\cos 3 \times \frac{3\pi}{2}}{9} \right) - \left(\frac{0 \times \sin 3 \times 0}{3} + \frac{\cos 3 \times 0}{9} \right)$$

$$= \frac{\frac{3\pi}{2} \times \sin \frac{9\pi}{2}}{3} + \frac{\cos \frac{9\pi}{2}}{9} - \frac{0}{3} - \frac{\cos 0}{9}$$

$) dx$

$$= \frac{\frac{3\pi}{2} \times 1}{3} + \frac{0}{9} - \frac{0}{3} - \frac{1}{9}$$

$$= \frac{3\pi}{2} \times \frac{1}{3} - \frac{1}{9} = \frac{\pi}{2} - \frac{1}{9} = \frac{9\pi - 2}{18}$$

$$\sin \frac{9\pi}{2} = \sin 810 = \sin (360 + 450)$$

$$= \sin 450 \quad \{\text{Since } \sin(360+\theta) = \sin\theta\}$$

$$= \sin (360+90) = \sin 90 = \underline{\underline{1}}$$

Similarly $\cos \frac{9\pi}{2} = \cos 810 = \cos (360 + 450)$

$$= \cos 450 \quad \{\text{Since } \cos(360+\theta) = \cos\theta\}$$

$$= \cos (360+90) = \cos 90 = \underline{\underline{0}}$$

b) Evaluate $\int x^2 \log x \, dx$.

A) $\int x^2 \log x \, dx = \int \log x \cdot x^2 \, dx$

$= \log x \int x^2 dx - \int \left(\frac{d}{dx}(\log x) \cdot \int x^2 dx\right) dx$

$= \log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} \, dx$

$= \log x \cdot \frac{x^3}{3} - \frac{1}{3} \int x^2 \, dx$

$= \log x \cdot \frac{x^3}{3} - \frac{1}{3} \frac{x^3}{3} + C$

$= \log x \cdot \frac{x^3}{3} - \frac{x^3}{9} + C$

OR

XII) a) Prove that $\int \sec x \, dx = \log(\sec x + \tan x) + C$

A) $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

{Multiply nr & dr by $\sec x + \tan x$}

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$ ——— (1)

put $u = \sec x + \tan x$

$\therefore \frac{du}{dx} = \sec x \tan x + \sec^2 x$

$\therefore (\sec x \tan x + \sec^2 x) dx = du$

$\therefore \int \sec x \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$ from (1)

x

$$= \int \frac{du}{u}$$

$$= \log u + c$$

$$= \log (\sec x + \tan x) + c$$

b) Evaluate $\int \frac{2x^4}{1+x^{10}} \, dx$.

A) $\int \frac{2x^4}{1+x^{10}} \, dx = \int \frac{2x^4}{1+(x^5)^2} \, dx$

put $u = x^5$

$$\frac{du}{dx} = 5x^4$$

$$\therefore 5x^4 \, dx = du$$

$$\therefore x^4 \, dx = \frac{du}{5}$$

$$\therefore \int \frac{2x^4}{1+(x^5)^2}\,dx = \int 2\cdot\frac{1}{1+u^2}\,\frac{du}{5}$$

$$= \frac{2}{5}\int \frac{1}{1+u^2}\,du$$

$$= \frac{2}{5}\tan^{-1}(u) + c$$

$$= \frac{2}{5}\tan^{-1}(x^5) + c$$

XIII) a) Find area bounded by the curve $x = y^2 - 2y$, the Y-axis and the abscissae at $y=1$ and $y=2$.

A) $$A = \int_a^b x\,dy$$

$$= \int_1^2 (y^2 - 2y)\,dy$$

$$= \left(y^3 - \frac{2y^2}{2} \right)_1^2$$

$$= \left(y^3 - y^2 \right)_1^2$$

$$= \left(2^3 - 2^2\right) - \left(1^3 - 1^2\right)$$

$$= (8-4) - (1-1)$$

$$= 8 - 4 - 0 = \underline{\underline{4}} \text{ Sq. units}$$

b) Solve $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$.

A) $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

$$\Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}} = \frac{-\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\therefore \sqrt{1-x^2}\, dy = -\sqrt{1-y^2}\, dx$$

$$\therefore \frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}}$$

Integrating on both sides,

$$\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$$

ie, $\sin^{-1}(y) = -\sin^{-1}(x) + C$

OR

XIV) a) Find the area under the straight line $y = 2x+3$ bounded by the X-axis and the ordinates $x=1$ and $x=3$.

A) $A = \int_a^b y \, dx$

$= \int_1^3 (2x+3) \, dx$

$$= \left[2\frac{x^2}{2} + 3x\right]_1^3$$

$$= \left[x^2 + 3x\right]_1^3$$

$$= (3^2 + 3\times3) - (1^2 + 3\times1)$$

$$= (9+9) - (1+3)$$

$$= 18 - 4$$

$$= \underline{\underline{14}} \text{ Sq. units.}$$

b) Solve $\frac{dy}{dx} + y\cot x = \operatorname{cosec} x$.

A) It is of the form $\frac{dy}{dx} + Py = Q$.

Here $P = \cot x$, $Q = \operatorname{cosec} x$

Integrating factor (IF) $= e^{\int P\,dx}$

$$= e^{\int \cot x\,dx}$$

$= e^{\log(\sin x)}$

$\boxed{e^{\log x} = x}$

$= \sin x$

$\therefore$ the Solution is given by,

$$\boxed{y \cdot IF = \int IF \cdot Q \, dx}$$

ie,

$$y \cdot \sin x = \int \sin x \cdot \operatorname{cosec} x \, dx$$

$$= \int \sin x \cdot \frac{1}{\sin x} \, dx$$

$$= \int 1 \, dx$$

$$= x + C$$

$\therefore$ the Solution is,

$$y \sin x = x + C$$