MODULE 3

CENTROID CENTRE OF GRAVITY AND MOMENT OF INERTIA

Centre of Gravity: Centre of gravity of a body is the point through which the whole weight of the body acts. It is represented by C.G. or simply G

Centroid: The point at which the total area of a plane figure (like rectangle, square, triangle, quadrilateral, circle etc.) is assumed to be concentrated, is known as the centroid of that area

CENTROID OR CENTRE OF GRAVITY OF SIMPLE PLANE FIGURES

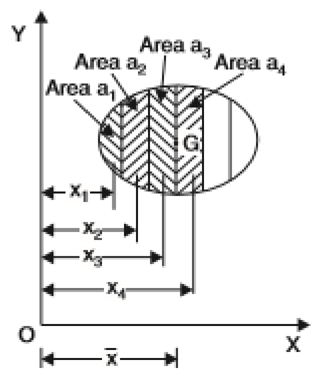
(i) The centre of gravity (C.G.) of a uniform rod lies at its middle point.

(ii) The centre of gravity of a triangle lies at the point where the three medians* of the triangle meet.

(iii) The centre of gravity of a rectangle or of a parallelogram is at the point, where its diagonal meet each other. It is also the point of intersection of the lines joining the middle points of the opposite sides.

(iv) The centre of gravity of a circle is at its centre.

CENTROID (OR CENTRE OF GRAVITY) OF AREAS OF PLANE FIGURES BY THE METHOD OF MOMENTS



Let x1 = The distance of the C.G. of the area a1 from axis OY

x2 = The distance of the C.G. of the area a2 from axis OY

x3 = The distance of the C.G. of the area a3 from axis OY

x4 = The distance of the C.G. of the area a4 from axis OY and so on.

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots}{A}$$

...

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 + \ldots}{A}$$

Problem .1. *Find the centre of gravity of the T-section shown in Fig. 9.2 (a).*

Sol. The given T-section is split up into two rectangles $ABCD$ and $EFGH$ as shown in Fig. 9.2 (b). The given T-section is symmetrical about Y-Y axis. Hence the C.G. of the section will lie on this axis. The lowest line of the figure is line GF. Hence the moments of the areas are taken about this line GF, which is the axis of reference in this case.

Let $\bar{y}$ = The distance of the C.G. of the T-section from the bottom line GF (which is axis of reference)

a_1 = Area of rectangle $ABCD = 12 \times 3 = 36\ \text{cm}^2$

y_1 = Distance of C.G. of area a_1 from bottom line $GF = 10 + \frac{3}{2} = 11.5$ cm

a_2 = Area of rectangle $EFGH = 10 \times 3 = 30\ \text{cm}^2$

y_2 = Distance of C.G. of area a_2 from bottom line $GF = \frac{10}{2} = 5$ cm.

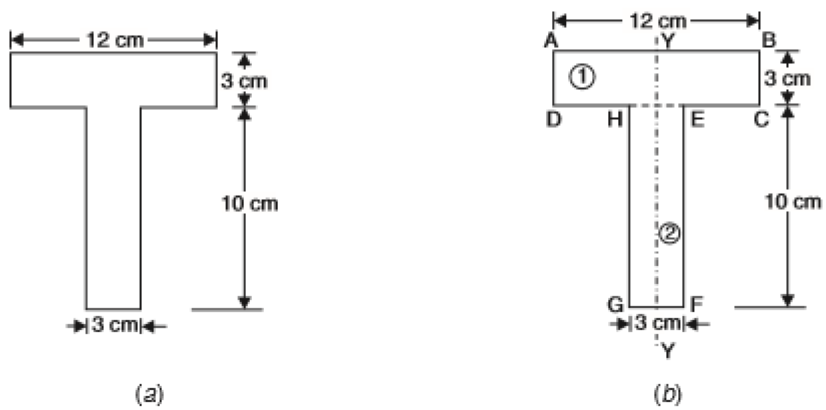


Fig. 9.2

Using equation (9.2), we have

$$\bar{y} = \frac{a_1y_1 + a_2y_2}{A} = \frac{a_1y_1 + a_2y_2}{a_1 + a_2} \qquad (\because \quad A = a_1 + a_2)$$

$$= \frac{36 \times 11.5 + 30 \times 5}{36 + 30} = \frac{414 + 150}{66} = \mathbf{8.545\ cm.} \quad \textbf{Ans.}$$

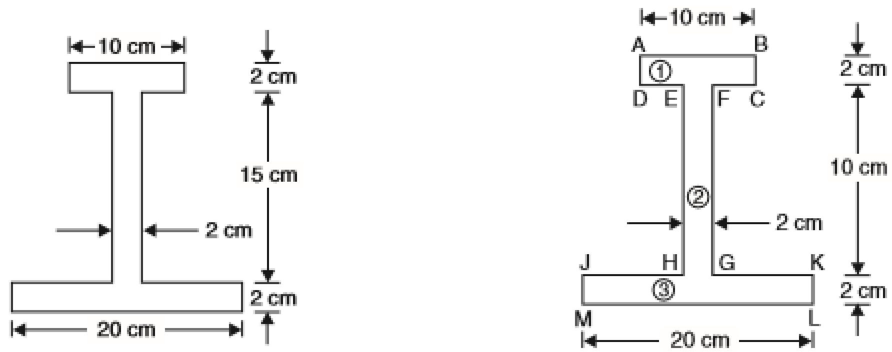
Problem .2. *Find the centre of gravity of the I-section shown in Fig. 9.3 (a).*

Sol. The *I*-section is split up into three rectangles *ABCD, EFGH* and *JKLM* as shown in Fig. 9.3 (*b*). The given *I*-section is symmetrical about *Y-Y* axis. Hence the C.G. of the section will lie on this axis. The lowest line of the figure line is *ML*. Hence the moment of areas are taken about this line, which is the axis of reference.

Let $\bar{y}$ = Distance of the C.G. of the *I*-section from the bottom line *ML*

a_1 = Area of rectangle $ABCD = 10 \times 2 = 20 \text{ cm}^2$

y_1 = Distance of C.G. of rectangle *ABCD* from bottom line $ML = 2 + 15 + \frac{2}{2} = 18$ cm



(*a*) (*b*)

a_2 = Area of rectangle $EFGH = 15 \times 2 = 30 \text{ cm}^2$

y_2 = Distance of C.G. of rectangle *EFGH* from bottom line $ML = 2 + \frac{15}{2} = 2 + 7.5 = 9.5$ cm

a_3 = Area of rectangle $JKLM = 20 \times 2 = 40 \text{ cm}^2$

y_3 = Distance of C.G. of rectangle *JKLM* from bottom line $ML = \frac{2}{2} = 1.0$ cm

Now using equation (9.2), we have

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{A}$$

$$= \frac{a_1y_1 + a_2y_2 + a_3y_3}{a_1 + a_2 + a_3} \qquad (\because \quad A = a_1 + a_2 + a_3)$$

$$= \frac{20 \times 18 + 30 \times 9.5 + 40 \times 1}{20 + 30 + 40}$$

$$= \frac{360 + 285 + 40}{90} = \frac{685}{90}$$

$$= \mathbf{7.611 \text{ cm.}} \quad \textbf{Ans.}$$

Problem 9.3. *Find the centre of gravity of the L-section shown in Fig. 9.4.*

Sol. The given L-section is not symmetrical about any section. Hence in this case, there will be two axis of references. The lowest line of the figure (*i.e.*, line GF) will be taken as axis of reference for calculating $\bar{y}$. And the left line of the L-section (*i.e.*, line AG) will be taken as axis of reference for calculating $\bar{x}$.

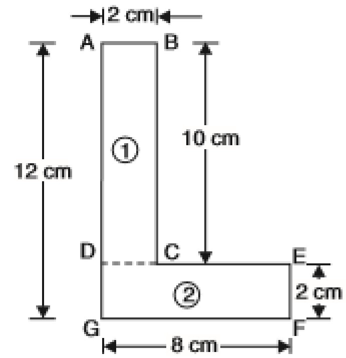


Fig. 9.4

The given L-section is split up into two rectangles $ABCD$ and $DEFG$, as shown in Fig. 9.4.

To Find $\bar{y}$

Let $\bar{y}$ = Distance of the C.G. of the L-section from bottom line GF

a_1 = Area of rectangle $ABCD = 10 \times 2 = 20$ cm^2

y_1 = Distance of C.G. of rectangle $ABCD$ from bottom line GF

$= 2 + \frac{10}{2} = 2 + 5 = 7$ cm

a_2 = Area of rectangle $DEFG = 8 \times 2 = 16$ cm^2

y_2 = Distance of C.G. of rectangle $DEFG$ from bottom line GF

$= \frac{2}{2} = 1.0$ cm.

Using equation (9.2), we have

$$\bar{y} = \frac{a_1y_1 + a_2y_2}{A}, \quad \text{where} \quad A = a_1 + a_2$$

$$= \frac{a_1y_1 + a_2y_2}{a_1 + a_2} = \frac{20 \times 7 + 16 \times 1}{20 + 16} = \frac{140 + 16}{36}$$

$$= \frac{156}{36} = \frac{13}{3} = 4.33 \text{ cm.}$$

To Find $\bar{x}$

Let $\bar{x}$ = Distance of the C.G. of the L-section from left line AG

x_1 = Distance of the rectangle $ABCD$ from left line AG

$= \frac{2}{2} = 1.0$ cm

x_2 = Distance of the rectangle $DEFG$ from left line AG

$= \frac{8}{2} = 4.0$ cm.

Using equation (9.1), we get

$$\bar{x} = \frac{a_1x_1 + a_2x_2}{A}, \quad \text{where} \quad A = a_1 + a_2$$

$$= \frac{a_1x_1 + a_2x_2}{a_1 + a_2} = \frac{20 \times 1 + 16 \times 4}{20 + 16} \qquad (\because \quad a_1 = 20 \text{ and } a_2 = 16)$$

$$= \frac{20 + 64}{36} = \frac{84}{36} = \frac{7}{3} = 2.33 \text{ cm.}$$

Hence the C.G. of the L-section is at a distance of 4.33 cm from the bottom line GF and 2.33 cm from the left line AG. **Ans.**

Problem 9.5. *From a rectangular lamina ABCD 10 cm × 12 cm a rectangular hole of 3 cm × 4 cm is cut as shown in Fig. 9.6.*

Find the c.g. of the remainder lamina.

Sol. The section shown in Fig. 9.6, is having a cut hole. The centre of gravity of a section with a cut hole is determined by considering the main section first as a complete one, and then subtracting the area of the cut-out hole, *i.e.*, by taking the area of the cut-out hole as negative.

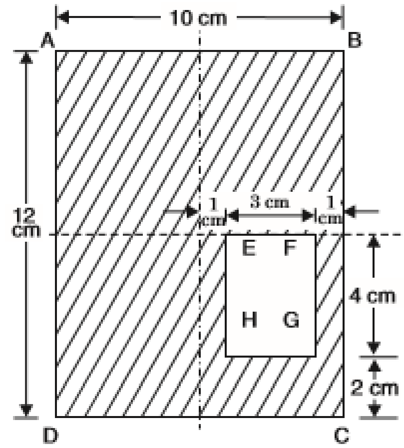


Fig. 9.6

Let $\bar{y}$ is the distance between the C.G. of the section with a cut hole from the bottom line DC.

a_1 = Area of rectangle $ABCD = 10 \times 12 = 120\ \text{cm}^2$

y_1 = Distance of C.G. of the rectangle $ABCD$ from bottom line DC

$$= \frac{12}{2} = 6\ \text{cm}$$

a_2 = Area of cut-out hole, *i.e.*, rectangle $EFGH$,

$$= 4 \times 3 = 12\ \text{cm}^2$$

y_2 = Distance of C.G. of cut-out hole from bottom line DC

$$= 2 + \frac{4}{2} = 2 + 2 = 4\ \text{cm.}$$

Now using equation (9.2) and taking the area (a_2) of the cut-out hole as negative, we get

$$\bar{y} = \left(\frac{a_1 y_1 - a_2 y_2}{A}\right)^* \quad \text{where} \quad A = a_1 - a_2$$

$$= \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2} \text{ (–ve sign is taken due to cut-out hole)}$$

$$= \frac{120 \times 6 - 12 \times 4}{120 - 12} = \frac{720 - 48}{108} = 6.22\ \text{cm.}$$

To Find $\bar{x}$

Let $\bar{x}$ = Distance between the C.G. of the section with a cut hole from the left line AD

x_1 = Distance of the C.G. of the rectangle $ABCD$ from the left line AD

$$= \frac{10}{2} = 5\ \text{cm}$$

x_2 = Distance of the C.G. of the cut-out hole from the left line AD

$$= 5 + 1 + \frac{3}{2} = 7.5\ \text{cm.}$$

Using equation (9.1) and taking area (a_2) of the cut hole as negative, we get

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2} \qquad (\because \quad A = a_1 - a_2)$$

$$= \frac{120 \times 5 - 12 \times 7.5}{120 - 12} = \frac{600 - 90}{108} = \frac{510}{108} = 4.72\ \text{cm.}$$

Hence the C.G. of the section with a cut hole will be at a distance of 6.22 cm from bottom line DC and 4.72 cm from the line AD. **Ans.**

DEFINITION OF MOMENT OF INERTIA OF AREA

The product of the area (or mass) and the square of the distance of the centre of gravity of the area (or mass) from an axis is known as moment of inertia of the area (or mass) about that axis. Moment of inertia is represented by I. Hence moment of inertia about the axis OX is represented by Ixx whereas about the axis OY by Iyy.

Polar moment of inertia

The product of the area (or mass) and the square of the distance of the centre of gravity of the area (or mass) from an axis perpendicular to the plane of the area is known as polar moment of inertia and is represented by J.

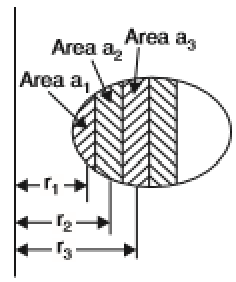
Radius of Gyration. Radius of gyration of a body about an axis is a distance such that its square multiplied by the area gives moment of inertia of the area about the given axis.

$$I = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 + \dots \qquad \dots(i)$$

Let the whole mass (or area) of the body is concentrated at a distance k from the axis of reference, then the moment of inertia of the whole area about the given axis will be equal to Ak^2.

If $Ak^2 = I$, then k is known as radius of gyration about the given axis.

$$\therefore \qquad k = \sqrt{\frac{I}{A}}. \qquad \dots(9.6)$$



PERPENDICULAR AXIS THEOREM AND POLAR MOMENT OF INERTIA

Theorem of the perpendicular axis states that if I_{XX} and I_{YY} be the moment of inertia of a plane section about two mutually perpendicular axis X-X and Y-Y in the plane of the section, then the moment of inertia of the section I_{ZZ} about the axis Z-Z, perpendicular to the plane and passing through the intersection of X-X and Y-Y is given by

$$I_{ZZ} = I_{XX} + I_{YY}.$$

The moment of inertia I_{ZZ} is also known as **polar moment of inertia.**

PARALLEL AXIS THEOREM

It states that if the moment of inertia of a plane area about an axis in the plane of area through the C.G. of the plane area be represented by I_G, then the moment of the inertia of the given plane area about a parallel axis AB in the plane of area at a distance h from the C.G. of the area is given by

$$I_{AB} = I_G + Ah^2.$$

where I_{AB} = Moment of inertia of the given area about AB

I_G = Moment of inertia of the given area about C.G.

A = Area of the section

h = Distance between the C.G. of the section and the axis AB.

Moment of Inertia of a Rectangular Section

Consider a rectangular elementary strip of thickness dy at a distance y from the X-X axis as shown in Fig. 9.15.

Area of the strip $= b\ .\ dy$.

Moment of inertia of the area of the strip about X-X axis

$$= \text{Area of strip} \times y^2$$
$$= (b\ .\ dy) \times y^2 = by^2dy.$$

Moment of inertia of the whole section will be obtained by integrating the above equation between the limits $-\dfrac{d}{2}$ to $\dfrac{d}{2}$.

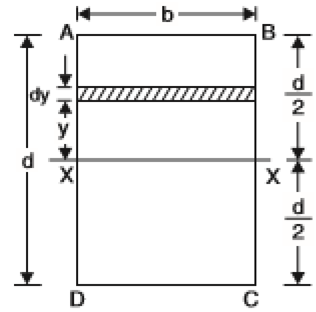


Fig. 9.15

$$\therefore \qquad I_{XX} = \int_{-d/2}^{d/2} by^2dy = b\int_{-d/2}^{d/2} y^2dy$$

($\because$ b is constant and can be taken outside the integral sign)

$$= b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2} = \frac{b}{3}\left[\left(\frac{d}{2}\right)^3 - \left(-\frac{d}{2}\right)^2\right]$$

$$= \frac{b}{3}\left[\frac{d^3}{8} - \left(-\frac{d^3}{8}\right)\right] = \frac{b}{3}\left[\frac{d^3}{8} + \frac{d^3}{8}\right]$$

$$= \frac{b}{3}\cdot\frac{2d^3}{8} = \frac{bd^3}{12}. \qquad \text{...(9.9)}$$

Similarly, the moment of inertia of the rectangular section about Y-Y axis passing through the C.G. of the section is given by

$$I_{YY} = \frac{db^3}{12}. \qquad \text{...(9.10)}$$

Refer to Fig. 9.15 (a)

Area of strip, $dA = d \times dx$

M.O.I. of strip above Y-Y axis $= dA \times x^2$

$$= (d \times dx) \times x^2 \qquad (\because \quad dA = d\ .\ dx)$$
$$= d \times x^2 \times dx$$

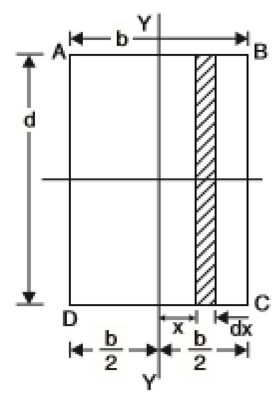


Fig. 9.15 (a)

$$\therefore \qquad I_{YY} = \int_{-b/2}^{b/2} d \times x^2 \times dx = d\left[\frac{x^3}{3}\right]_{-b/2}^{b/2}$$

$$= \frac{d}{3}\left[\left(\frac{b}{2}\right)^3 - \left(-\frac{b}{2}\right)^3\right]$$

$$= \frac{d}{3}\left[\frac{b^3}{8} + \frac{b^3}{8}\right] = \frac{d}{3}\cdot\frac{b^3}{4} = \frac{db^3}{12}.$$

Problem ! *Fig. 9.23 shows a T-section of dimensions 10 × 10 × 2 cm. Determine the moment of inertia of the section about the horizontal and vertical axes, passing through the centre of gravity of the section. Also find the polar moment of inertia of the given T-section.*

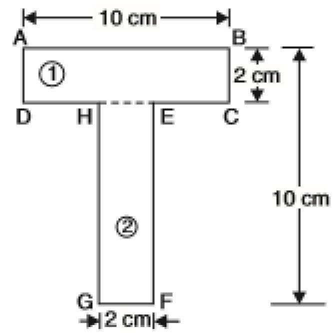


Fig. 9.23

Sol. First of all, find the location of centre of gravity of the given *T*-section. The given section is symmetrical about the axis *Y-Y* and hence the C.G. of the section will lie on *Y-Y* axis. The given section is split up into two rectangles *ABCD* and *EFGH* for calculating the C.G. of the section.

Let $\bar{y}$ = Distance of the C.G. of the section from the bottom line *GF*

a_1 = Area of rectangle *ABCD* = $10 \times 2 = 20 \text{ cm}^2$

y_2 = Distance of C.G. of the area a_1 from the bottom line *GF* = 8 + 1 = 9 cm

a_2 = Area of rectangle *EFGH* = $8 \times 2 = 16 \text{ cm}^2$

y_2 = Distance of C.G. of rectangle *EFGH* from the bottom line $GF = \frac{8}{2} = 4$ cm

Using the relation, $$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{20 \times 9 + 16 \times 4}{20 + 16} = \frac{180 + 64}{36} = \frac{244}{36} = 6.777 \text{ cm.}$$

Hence the C.G. of the given section lies at a distance of 6.777 cm from *GF*. Now find the moment of inertia of the *T*-section.

Now, Let I_{G_1} = Moment of inertia of rectangle (1) about the horizontal axis and passing through its C.G.

I_{G_2} = Moment of inertia of rectangle (2) about the horizontal axis and passing through the C.G. of the rectangle (2)

h_1 = The distance between the C.G. of the given section and the C.G. of the rectangle (1)

$= y_1 - \bar{y} = 9.0 - 6.777 = 2.223$ cm

h_2 = The distance between the C.G. of the given section and the C.G. of the rectangle (2)

$= \bar{y} - y_2 = 6.777 - 4.0 = 2.777$ cm.

Now, $$I_{G_1} = \frac{10 \times 2^3}{2} = 6.667 \text{ cm}^4$$

$$I_{G_2} = \frac{2 \times 8^3}{12} = 85.333 \text{ cm}^4.$$

From the theorem of parallel axes, the moment of inertia of the rectangle (1) about the horizontal axis passing through the C.G. of the given section

$$= I_{G_1} + a_1 h_1^2 = 6.667 + 20 \times (2.223)^2$$
$$= 6.667 + 98.834 = 105.501 \text{ cm}^4.$$

Similarly, the moment of inertia of the rectangle (2) about the horizontal axis passing through the C.G. of the given section

$$= I_{G_2} + a_2 h_2^2 = 85.333 + 16 \times (2.777)^2$$
$$= 85.333 + 123.387 = 208.72 \text{ cm}^4.$$

$\therefore$ The moment of inertia of the given section about the horizontal axis passing through the C.G. of the given section is,

$$I_{xx} = 105.501 + 208.72 = \mathbf{314.221\ cm^4}. \quad \textbf{Ans.}$$

The moment of inertia of the given section about the vertical axis passing through the C.G. of the given section is,

$$I_{yy} = \frac{2 \times 10^3}{12} + \frac{8 \times 2^3}{12}$$
$$= 166.67 + 5.33 = \mathbf{172\ cm^4}. \quad \textbf{Ans.}$$

Now the polar moment of inertia (I_{zz}) is obtained from equation (9.7) as

$$I_{zz} = I_{xx} + I_{yy}$$
$$= 314.221 + 221 = \mathbf{486.221\ cm^4}. \quad \textbf{Ans.}$$

POLAR MOMENT OF INERTIA OF MASSES

Let x = Distance of the centre of gravity of mass M from axis OY

y = Distance of the C.G. of mass M from axis OX

Then moment of the mass about the axis $OY = M \,.\, x$

The above equation is known as *first moment* of mass about the axis OY.

If the moment of mass given by the above equation is again multiplied by the perpendicular distance between the C.G. of the mass and axis OY, then the quantity $(M \,.\, x) \,.\, x = M \,.\, x^2$ is known as *second* moment of mass about the axis OY. This second moment of the mass (*i.e.,* quantity $M \,.\, x^2$) is known as mass moment of inertia about the axis OY.

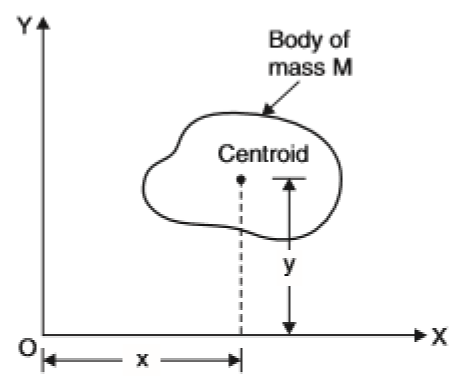


Fig. 9.26

Similarly, the second moment of mass or mass moment of inertia about the axis OX

$$= (M \,.\, y) \,.\, y = M \,.\, y^2$$

Centroid of areas

When a plane area has an axis of symmetry, the centroid will be on that axis. When there are two axes of symmetry, the point of intersection of the axes of symmetry will be the centroid of the area.

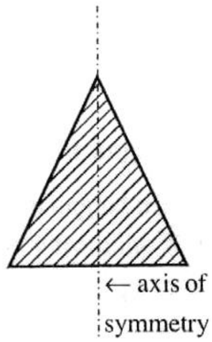


Fig.2.20

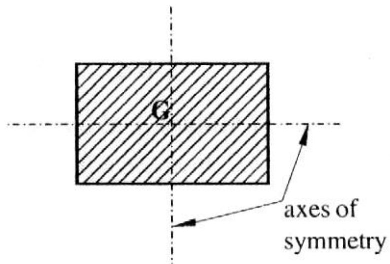


Fig.2.21

In the previous section it was proved that the distance of centroid of an area from Y axis is given by,

$$\bar{x} = \frac{\int x dA}{\int dA}$$

. where dA is an elemental area and x is the distance of this elemental area from Y axis.

Similarly $$\bar{y} = \frac{\int y dA}{\int dA}$$ **. where y is the distance of elemental area dA from X axis.**

Centroid of rectangle

To find $\bar{y}$

Consider a horizontal strip of thickness by dy at a distance y from the X axis as shown in fig. 2.22.

Elemental area dA = b × dy

$$\bar{y} = \frac{\int y dA}{\int dA} = \frac{\int_0^d y dA}{\int_0^d dA} = \frac{\int_0^d y b dy}{\int_0^d b dy}$$

$$= \frac{b\left[\frac{y^2}{2}\right]_0^d}{b[y]_0^d} = \frac{\frac{b}{2}d^2}{bd} = \frac{d}{2} \qquad \bar{y} = \frac{d}{2}$$

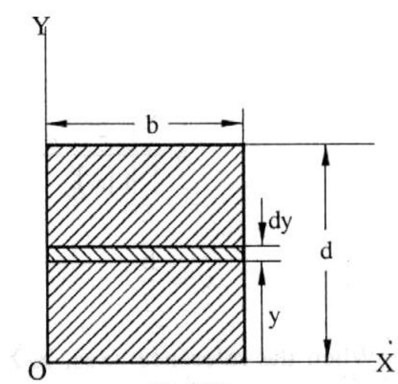
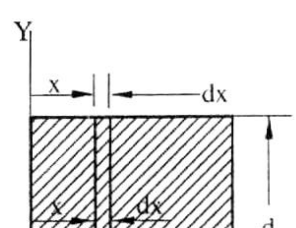


Fig.2.22

To find $\bar{x}$

Consider a vertical strip of thickness dx at a distance *x* from the Y axis as shown in fig. 2.23.

Elemental area, dA = d × dx



$$\bar{x} = \frac{\int x dA}{\int dA} = \frac{\int_0^b x d \times dx}{\int_0^b d \times dx}$$

$$= \frac{d\left[\frac{x^2}{2}\right]_0^b}{d[x]_0^b} = \frac{d\frac{b^2}{2}}{db} = \frac{b}{2}$$

$$\bar{x} = \frac{b}{2}$$

When the reference axes, X-X and YY axis, pass through the centre of rectangle,

$\bar{x} = 0$ **and** $\bar{y} = 0$ **as shown in fig. 2.24.**

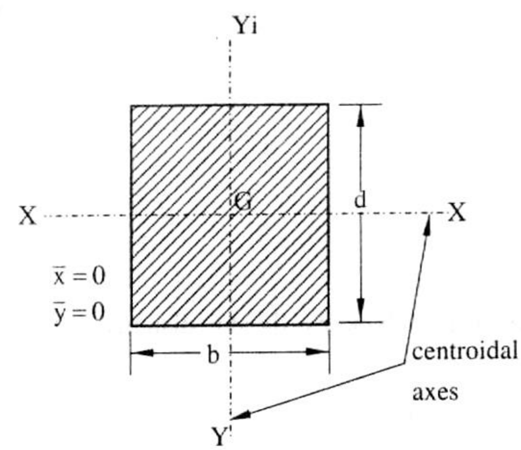


Fig.2.24

Locate the centroid of the triangular lamina as shown in fig. 2.27.

Solution

$$\bar{x} = b - \frac{1}{3}b = \frac{2}{3}b$$

$$\bar{y} = h - \frac{1}{3}h = \frac{2}{3}h$$

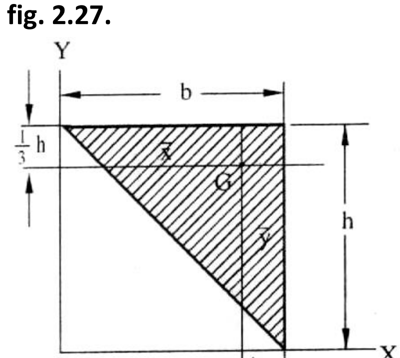


Fig.2.27

Centroid of a Circle

When the reference axes are selected as shown in fig. 2.29,

$$\bar{x} = \bar{y} = \frac{d}{2}$$

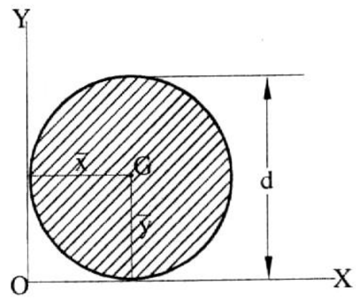


Fig.2.29

Example 1

Locate the centroid of the 'T' section shown in fig. 2.37.

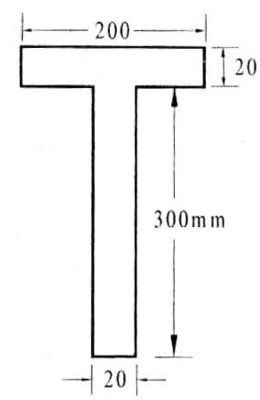


Fig.2.37

Solution

Since the section is symmetrical with respect to the Y axis, $\bar{x} = 0$.

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

a_1 = 300 × 20 = 6000 cm²

a_2 = 200 × 20 = 4000 cm²

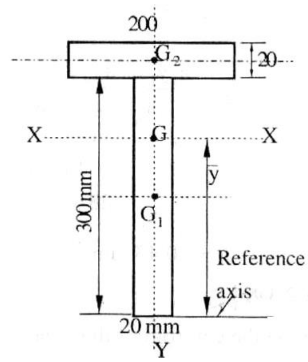


Fig.2.38

$$y_1 = \frac{300}{2} = 150\,\text{cm}$$

$$y_2 = 300 + \frac{20}{2} = 310\,\text{cm}$$

$$\therefore\ \bar{y} = \frac{6000 \times 150 + 4000 \times 310}{6000 + 4000} = 214\,\text{mm}$$