Lecture "Experimental Physics I"

(Prof. Dr. R. Seidel)

Lecture 14

Torque & moment of inertia

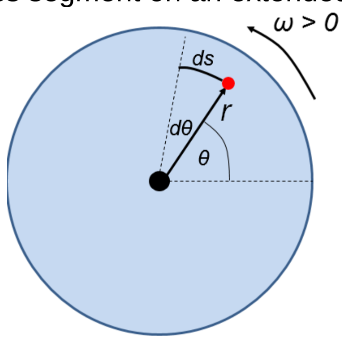
- Angular velocity and acceleration
- Torque
- Moment of inertia
- Parallel axis theorem

1) Angular velocity & acceleration

So far, we looked at mass points and systems of mass points, where the motion of each mass point was a simple translation (even if occurring on a more complicated track). Now we want to describe the dynamics of **extended, rigid objects**, where in addition to the position of the mass point the orientation (i.e. its rotation angle) and changes of its orientation need to be considered. Here we will obtain some fascinating quantities that describe the rotary motion of such objects.

A) Tangential motion

Let us first repeat and extend some of the previous knowledge of circular motion. We first look at the (tangential) velocity of a mass segment on an extended rotating disk:



Its displacement is given by:

$$ds = r\,d\theta$$

such that we get for the tangential velocity $v_{(t)}$:

$$v_{(t)} = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$

With this we have for the angular velocity:

$$\omega = \frac{v_{(t)}}{r} = \frac{d\theta}{dt}$$

Typically, ω is positive for counterclockwise motion and negative for clock-wise rotation. Now we look at the time derivative of the angular velocity to get an **angular acceleration** as the change of the angular velocity over time:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{\frac{1}{r}dv_{(t)}}{dt} = \frac{a_{(t)}}{r}$$

Looking at these equations we see that **angular kinematic quantities are obtained by dividing the translational quantities by $\boldsymbol{r}$**. For the rotation of an extended **rigid body about an axis it is better to use angular position, velocity and acceleration**, since all mass segments are at a fixed angular position with respect to each other and thus experience the same changes in angular position. Therefore, all segments are described by the same angular velocity and acceleration, while their absolute velocities and accelerations are dependent on their radial positions.

We defined **angular velocity and acceleration as the successive time derivatives of the angular position** of the object of interest. This is the same type of definition as for translational velocity and acceleration in kinematics. Therefore, if we have a constant angular acceleration (linear change of angular velocity over time) we can **derive the same equations for the time dependence of $\boldsymbol{\omega(t)}$ and $\boldsymbol{\varphi(t)}$ as in kinematics for translation**:

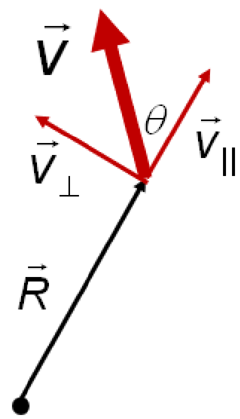
constant angular acceleration

#	Along line	Rotational
1	$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
2	$x = x_0 + v_0 t + at^2/2$	$\theta = \theta_0 + \omega_0 t + \alpha t^2/2$
3	$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
4	$x = x_0 + (v + v_0)t/2$	$\theta = \theta_0 + (\omega + \omega_0)t/2$
5	$x = x_0 + vt - at^2/2$	$\theta = \theta_0 + \omega t - \alpha t^2/2$

Remember: We already introduced angular velocity and acceleration from general circular motion as the quantities describing the tangential motion (**show slide**).

B) Non-tangential velocity vectors

We also defined in the past lectures an angular velocity about any point in space even for non-tangential velocities by taking the projection of the velocity onto the tangential direction:



$$\omega = \frac{d\varphi}{dt} = \frac{v_\perp}{R} = \frac{1}{R} v \sin\theta = \frac{1}{R^2} R\, v \sin\theta$$

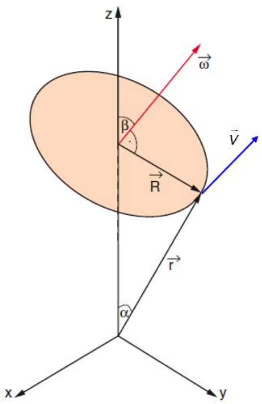
To this end we used a simplified description of ω using the vector product:

$$\vec{\omega} = \frac{1}{R^2} \vec{R} \times \vec{v}$$

which is equivalent to:

$$\vec{v}_\perp = \vec{\omega} \times \vec{R}$$

The **angular velocity** is thus taken as a vector that stands perpendicular on the plane of rotation, i.e. it **points along the rotation axis**. The right-hand rules define hereby the orientation of $\vec{\omega}$ with respect to the rotation direction.



An angular velocity can thus be defined for any particle (even unidirectional motion) passing by a point in space!

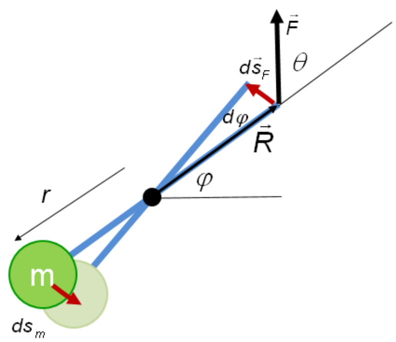
Similarly, we can also define a **vectorial form of the angular acceleration** (see slide):

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$
which is connected to the linear acceleration by:
$$\vec{\alpha} = \frac{1}{R^2}\vec{R} \times \vec{a} \quad \text{and} \quad \vec{a} = \vec{\alpha} \times \vec{R}$$
It points along the rotation axis that is driven by $\vec{a}$. Note, as for linear acceleration there is a non-zero angular acceleration if we change the magnitude of $\vec{\omega}$, i.e. accelerate a given rotation, but also if we change the direction of $\vec{\omega}$, i.e. the rotation axis.

2) Torque and moment of inertia

A) Torque

By defining the angular acceleration, we have a kinematic quantity to derive the trajectory of any point of a rotating rigid body. To go one step further, we want to move towards dynamics and find an "rotation" equivalent to the force. To this end we look at the work, that is required when we accelerate the rotation of a mass point attached on a mass-free rod by applying a force on a distant end:



For the work necessary to accelerate the mass at distance r from the rotation center we can write:
$$dW_m = ma\, ds_m = mr\alpha \underbrace{rd\varphi}_{ds_m} = mr^2\alpha\, d\varphi$$
This work must be done by the external force acting on the point at distance R from the rotation center:
$$dW_F = \vec{F} \cdot d\vec{s}_F = F \sin\theta\, ds_F = RF \sin\theta\, d\varphi$$
Since $dW_F = dW_m$, we get after division by $d\varphi$:
$$RF\, sin\, \theta = mr^2\alpha$$
In case we have multiple masses on the rod, the required work for the acceleration is described by a simple sum of the works to accelerate the individual masses:
$$dW_m = \left(\sum_i m_i {r_i}^2\right)\alpha\, d\varphi$$
Thus, we have:
$$RF\, sin\, \theta = \underbrace{\sum_i m_i {r_i}^2}_{I} \alpha = I\, \alpha$$
We now call the sum: I **… the moment of inertia or rotational inertia**:
$$I = \sum_i \underbrace{m_i {r_i}^2}_{po\, int\, mass}$$
The term inside the sum is the moment of inertia of a point mass and the total moment of inertia is the sum over all individual moment of inertia.

Above we introduced that $\vec{\alpha}$ is a vector that stands as $\vec{\omega}$ perpendicular to the plane of rotation, i.e. it points out of the plane in the figure above. To obtain a vectorial form of the force-acceleration equation, we express the left side of the equation as vector product:

$$\underbrace{\vec{R} \times \vec{F}}_{\vec{\tau}} = I\,\vec{\alpha}$$

We call the vector product the **torque $\vec{\tau}$** which points out of the rotation plane in the figure abve, i.e. in the direction of $\vec{\alpha}$. The obtained equation is analogous to Newtons 2nd law ($\vec{F} = m\vec{a}$):

$$\vec{\tau} = I\,\vec{\alpha}$$

The scalar moment of inertia I is for rotations of rigid bodies the equivalent for the mass. Torque is thus the equivalent of force for rotational motion.

Going back to the work done by the force at radial distance r (**see slide**), we see that it comprises the torque:

$$dW_F = \vec{F} \cdot d\vec{s}_F = \underbrace{F\sin\theta\, R}_{\tau}\; d\varphi$$

Thus, **torque is the work done per angular displacement**.

Experiment: This expression for the work during an angular displacement can be verified using a beam connected to a spiral-shaped spring. To reach the same angular displacement, i.e. the same performed work, one always needs to apply the same torque, such that one has to apply **different forces depending on the radial distance and pulling angle**.

This allows to write for the work done:

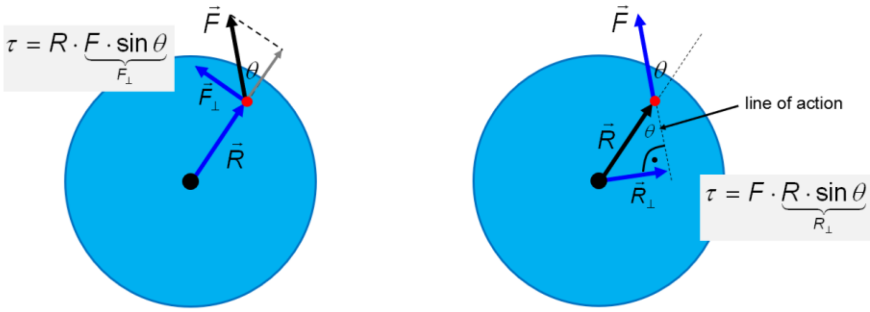
$$dW_F = \vec{\tau} \cdot d\vec{\varphi}$$

with $d\vec{\varphi} = \vec{\omega} \cdot dt$. This is again analogous to the corresponding expression for the work for translational motion $dW = \vec{F} \cdot d\vec{s}$. The **vectorial notation considers only torque in the direction of the angular displacement**. Using the expression of work, the definition of torque, in particular the proportionality to R, makes sense, since a **force acting at a larger distance from the rotation center does more work** due to a larger arc length over which the force is acting.

Torques also add up like forces, since they contribute to work in an additive manner.

$$\vec{\tau}_{tot} = \sum_i \vec{R}_i \times \vec{F}_i$$

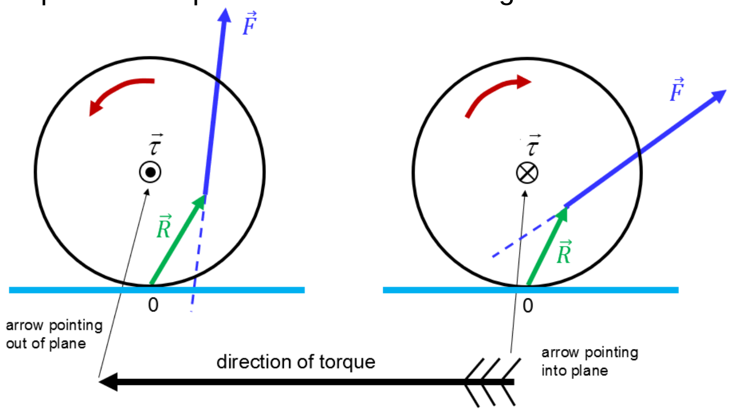
There are two ways how to imagine the magnitude of torque from the cross product:



$F\sin\theta$ is the tangential force component acting perpendicular to the radius vector (see left figure part). **$F\sin\theta$ defines the shorted distance of the "line of action" of the force vector from the rotation axis,** which is thus the effective radius at which the force acts. The latter perspective is useful to understand the following experiments:

Experiment: Line of action. **An acting force can be translated along the force direction from its original point of action to a new point of** action since only the effective distance enters the torque calculation (see slides).

Experiment: We have a spool on a string that goes in different directions depending on the pulling geometry. To understand in which direction the spool is going one has to consider that the spool is pivoting around the point where it touches the surface. Calculating the vector product of the torque provides the torque on the spool and thus the turning direction.

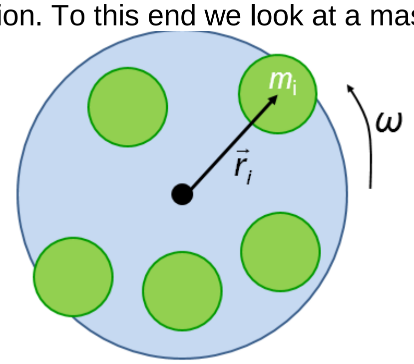


Instead of the cross product one can also look at the effective radius at which the force acts and deduce from this the turning direction of the spool. When **the force is acting along the radius vector there is no torque**. This is the case were we can **drag the spool.**

Movie: Actio = Reactio for torque. Also for torque we formulate an equivalent to Newton's 3rd law.

B) Kinetic energy for rotation of objects

We now have an equivalent to Newton's 2nd law for the rotation of a rigid body. Let us see how general the definition of the moment of inertia is by deriving an expression for the kinetic energy of a rotating rigid mass distribution. To this end we look at a mass-free disk with point masses



We can write for the kinetic energy of the masses in case of rotation:

$$E_k = \sum_i \frac{m_i}{2} v_i^2 = \sum_i \frac{m_i}{2} (r_i \omega)^2$$

Transformation provides again the moment of inertia:

$$E_k = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

$$E_k = \frac{1}{2} I \omega^2$$

i.e. we can take the moment of inertia as mass equivalent. Thus, **for rotational motion the moment of inertia replaces mass when angular quantities (φ, ω, α) are taken!**
The dependence of the moment of inertia on the squared distance from the rotation axis makes sense, since for a given ω, the tangential velocity increases linearly with the radial distance and thus the kinetic energy increases quadratically.

C) Moment of inertia for different geometries

To obtain the moment of inertia for rigid three-dimensional objects, we first need to define a **rotation axis.** For discrete point masses we have then as before:

$$I = \sum_i m_i r_i^2$$

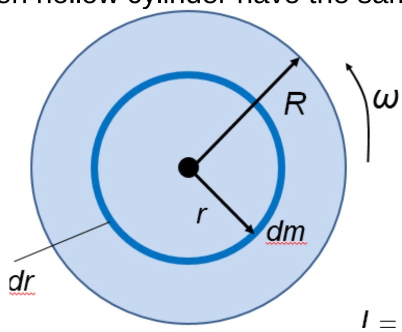
For continuous bodies the sum becomes an integral:

$$I = \int r^2 dm$$

***r* is hereby always the shortest distance to the rotation axis,** i.e. it is measured within the rotation plane**!**

Moment of inertia for a cylinder

Let us practice the calculation of the moment of inertia. We first consider a **cylinder** with mass M, radius R and height h and will calculate the moment of inertia for a rotation about the cylinder axis. For this, we will slice the cylinder into concentric hollow cylinders with wall thickness dr, since all mass elements in a given hollow cylinder have the same distance r.



The mass of such a hollow cylinder is given by:

$$dm = \rho\, h\, \underbrace{2\pi r\, dr}_{dA}$$

and its moment of inertia is provided by

$$dI = r^2\, \underbrace{\rho\, h\, 2\pi r\, dr}_{dm}$$

The total moment of inertia of the whole cylinder is then given by integrating the moment of inertia over all hollow cylinder shells

$$I = \int dI = \int_0^R \rho\, 2\pi h\, r^3\, dr = \frac{1}{2}\rho\, h\, \pi R^4$$

Replacing the total mass of the cylinder $M = \rho\, h\, \pi R^2$ finally gives:

$$I = \frac{1}{2} M R^2$$

To obtain the moment of inertia for a hollow cylinder with a thick wall (inner radius R_1 and outer radius R_2) we just have to adapt the integration boundaries in the integral above:

$$I = \int_{R_1}^{R_2} \rho\, 2\pi h\, r^3\, dr = \frac{1}{2}\rho\, h\, \pi(R_2^4 - R_1^4) = \frac{1}{2}\rho\, h\, \underbrace{\pi(R_2^2 - R_1^2)}_{A}(R_2^2 + R_1^2)$$

Replacing again the mass of the hollow cylinder gives:

$$I = \frac{1}{2} M(R_2^2 + R_1^2)$$

Since the mass is distributed at a larger distance from the rotation axis, the moment of inertia of a hollow cylinder is larger than of a filled cylinder with equal outer diameter.
For a hollow cylinder with a very thin wall we have thin wall we have $R_1 \approx R_2$ such that:

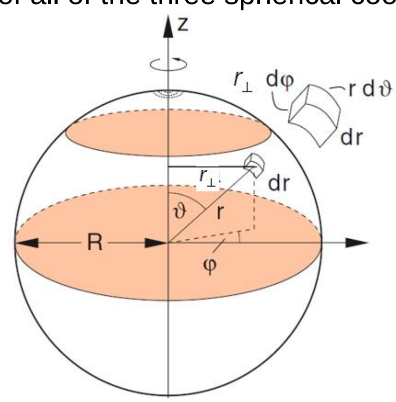
$$I \approx \frac{1}{2}M(R_2^2 + R_2^2) = MR_2^2$$

This equation is consistent with our definition of the moment of inertia since all mass elements are located at about the same distance R_2 from the rotation axis

Moment of inertia of a homogeneous sphere: (not part of lecture only calculation of d*m* and first integral)
To derive the moment of inertia for a homogenous sphere that rotates around an axis through its center (like the earth), it is convenient to use **spherical coordinates**. For these coordinates the position of a little mass or volume element within sphere is described by the **radial distance $\boldsymbol{r}$** from the sphere center, the **polar angle $\boldsymbol{\vartheta}$** (angle of $\vec{r}$ with z-axis) and the azimuthal angle φ, i.e. the angular position of the component of radial vector in the x-y-plane (see figure below).
The moment of inertia for the rotation around an axis through the sphere center (z-axis in the following) is given by:

$$I_z = (I_{CM}) = \int {r_\perp}^2 dm$$

where $r_\perp$ is the distance of dm from the rotation axis (measured in the rotation plane of dm). The mass of a suitable element dm is obtain by considering the volume element dV that would be obtained by very small changes of all of the three spherical coordinates $dr, d\vartheta, d\varphi$:



According to the figure, we can write for the volume/mass of this element:

$$dm = \rho\, dV = \rho \underbrace{r_\perp d\varphi}_{\substack{edge\ length\\ for\ displacement\\ by\ d\varphi}} \underbrace{r\, d\vartheta}_{\substack{edge\ length\\ for\ displacement\\ by\ d\vartheta}} dr$$

The moment of inertia of the sphere is then obtained by integration over all three spherical coordinates (see slides):

$$I_z = \rho \underbrace{\int_0^R \int_0^\pi \underbrace{\int_0^{2\pi} {r_\perp}^3 r\ d\varphi}_{\text{ring with radius } r_\perp} d\vartheta}_{\text{hollow sphere with radius r}}\ dr$$

where the first integral is an integration over a ring with radius $r_\perp$ and the second integral an integration over all rings located at distance r, which forms a hollow sphere with radius r. The final integral sums up all concentric hollow spheres.

Inserting $r_\perp = r\sin\vartheta$ and rewriting the integral yields:

$$I_z = \rho \int_0^R \int_0^\pi \int_0^{2\pi} r^4\, sin^3\,\vartheta\, d\varphi\, d\vartheta\, dr = \rho \int_0^\pi sin^3\,\vartheta \int_0^R r^4 \int_0^{2\pi} d\varphi\ dr\, d\vartheta$$

carrying out the integration over $d\varphi$ (no φ dependent function) gives simply:

$$I_z = 2\pi\rho \int_0^\pi sin^3\,\vartheta \int_0^R r^4\, dr\, d\vartheta$$

and integration over dr gives:

$$I_z = \frac{2\pi}{5}\rho R^5 \int_0^\pi sin^3\,\vartheta\ d\vartheta$$

Final integration over $d\vartheta$ (integral solved in seminar) provides:

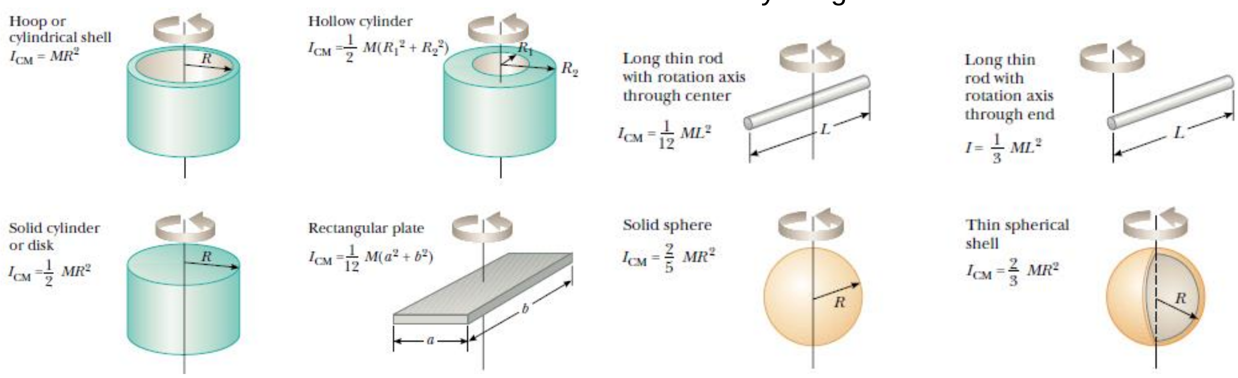
$$I_z = \frac{2\pi}{5}\rho R^5\,\frac{4}{3} = \frac{2}{5}\rho R^2\ \underbrace{\frac{4}{3}\pi R^3}_{V_{sphere}}$$

and we finally get for the moment of inertia of a sphere:

$$I_z = \frac{2}{5} M R^2$$

Moment of inertia for other objects:

The following table provides an overview about the moment of inertia for different objects around the indicated rotation axes that can be calculated by integration:



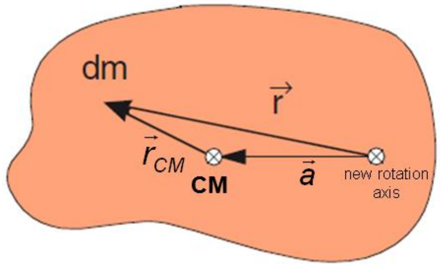
Experiment: We generate a constantly accelerated rotation generated by a constant torque that is produced by attaching a weight to a cord spooled on a rotatable axis:

$$\tau = r\,F = I\,\alpha$$

Onto the axis we mount two different cylinders of equal rotating mass but different mass distribution (hollow and filled cylinder) and see that the hollow cylinder experiences a slower acceleration due to its higher moment of inertia.

D) Parallel axis theorem (Steiner's theorem)

Suppose we know the moment of inertia for the rotation of an object around its center of mass. Now we want to choose a different rotation axis that is parallel to the previous one but displaced by distance vector $\vec{\boldsymbol{a}}$ from the center of mass:



For the moment of inertia around the new axis we can write:

$$I = \int |\vec{r}|^2 dm = \int |\vec{r}_{CM} + \vec{a}|^2 dm = \int (\vec{r}_{CM} + \vec{a}) \cdot (\vec{r}_{CM} + \vec{a}) dm$$

$$I = \int r_{CM}^2 dm + 2\vec{a} \underbrace{\int \vec{r}_{CM} dm}_{0} + \int a^2 dm$$

The integral in the central term just defines the center of mass in the center of mass system (if devided by M) and is thus zero. With this we get:

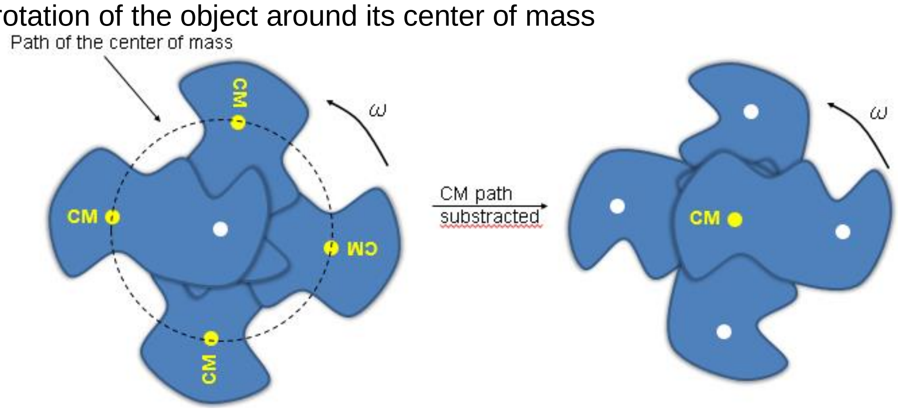
$$I = I_{CM} + Ma^2$$

This is the **parallel axis theorem** also called the **Steiner's theorem**. With this theorem, one only needs to know the moment of inertia for a rotation about a single axis (e.g. through the CM) in order to obtain the moment of inertia about any parallel rotation axis by a simple calculation.

One can intuitively understand parallel axis theorem by dissecting a rotation about an arbitrary axis into:

(1) a circular motion of the center of mass and

(2) a rotation of the object around its center of mass

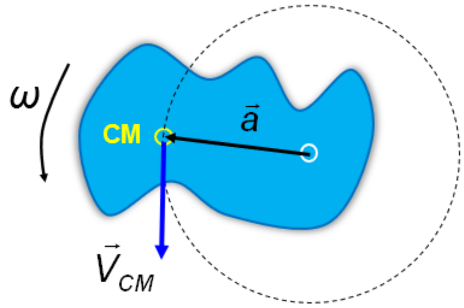


The parallel axis theorem contains consequently the moment of inertia for the rotation about the CM as well as for the movement of the CM on a circular track with radius a:

$$I = \underbrace{I_{CM}}_{\substack{\text{rotation} \\ \text{around CM}}} + \underbrace{Ma^2}_{\substack{\text{circular} \\ \text{path of CM}}}$$

Notably, from this superposition we also see that the angular velocity is independent of the particular rotation axis (i.e. reference frame).

Kinetic energy using the parallel axis theorem (on slide):



Using the parallel axis theorem, we can also calculate the kinetic energy for the rotation around any axis. Inserting into the kinetic energy equation from before provides:

$$E_k = \frac{I}{2}\omega^2 = \frac{I_{CM} + Ma^2}{2}\omega^2$$

$$E_k = \frac{I_{CM}}{2}\omega^2 + \frac{M}{2}\underbrace{(a\omega)}_{v_{CM}}{}^2$$

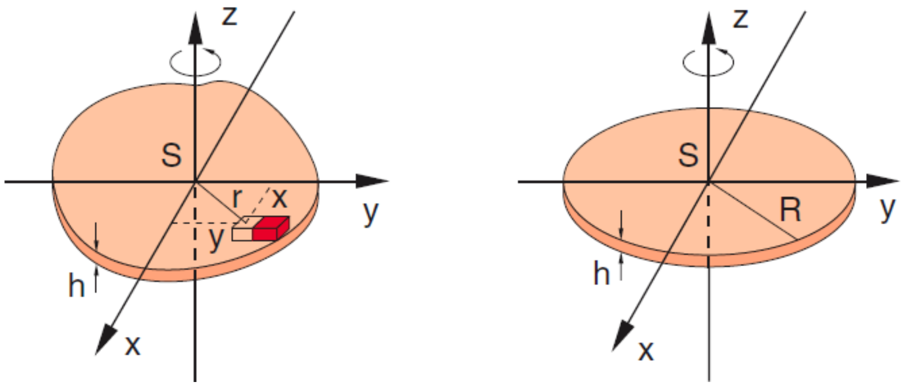
This is nothing else as the sum of the kinetic energies of the rotation about the CM and the circular motion of the CM with tangential velocity $\boldsymbol{a\omega}$ itself:

$$E_k = \underbrace{E_{kCM}}_{\text{rotation around CM}} + \underbrace{\frac{M}{2}V_{CM}^2}_{\text{motion of CM}}$$

Again, we get that **rotating of an object around an arbitrary point can be seen as a rotation of the object about its CM plus a circular motion of the CM around the rotation point**. It is also equivalent to the expression of the kinetic energy which we derived for particle systems with respect to the center of mass frame.

E) Perpendicular axis theorem

For planar object, such as a disk with $h \ll x_{max,}y_{max}$ we can derive another simple relation for the moment of inertia about its main axes. Consider a planar object within the xy plane:



To calculate the moment of inertia about the three perpendicular axes of the coordinate system we can write:

$$I_z = \rho \int_V (x^2 + y^2)dV$$

$$I_x = \rho \int_V (y^2 + z^2)dV \approx \rho \int_V y^2 dV$$

$$I_y = \rho \int_V (x^2 + z^2)dV \approx \rho \int_V x^2 dV$$

Since for the rotation about the x and the y axis we can for a thin object neglect the contribution of z^2 to the distance of the rotation axis, since we have $|z| < h/2 \ll x_{max,} y_{max}$.
Replacing the integrals in the first equation by I_x and I_y provides the useful expression:

$$I_z = I_x + I_y$$

It can, for example, be applied to calculate the moment of inertia when spinning a circular disk around an axis within the disk plane that goes through the CM (see figure on the right, see slide). Due to the disk symmetry, we can write:

$$I_x = I_y$$

$$I_z = I_x + I_y = 2I_x$$

Using the moment of inertia of a cylinder we thus obtain:

$$I_x = \frac{I_z}{2} = \frac{1}{4} M R^2$$

Lecture 14: Experiments

1) Line of action: A force can be translated along its direction without changing the torque on a system
2) „Folgsame Rolle“
3) Expression for the work during an angular displacement is verified using a lever connected to a spiral-shaped spring. To reach the same angular displacement, i.e. the same performed work, one needs half the force if the distance R is doubled. The lowest force is required if the pulling directions occurs along the tangent.
4) A constantly accelerated rotation by a constant force from a weight acting at constant distance r from an axis. On the rotation axis we mount two different cylinders of equal rotating mass but different mass distribution (hollow vs. filled cylinder) and see that the hollow cylinder experiences a slower acceleration due to its higher moment of inertia.