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Theoretical Physics I

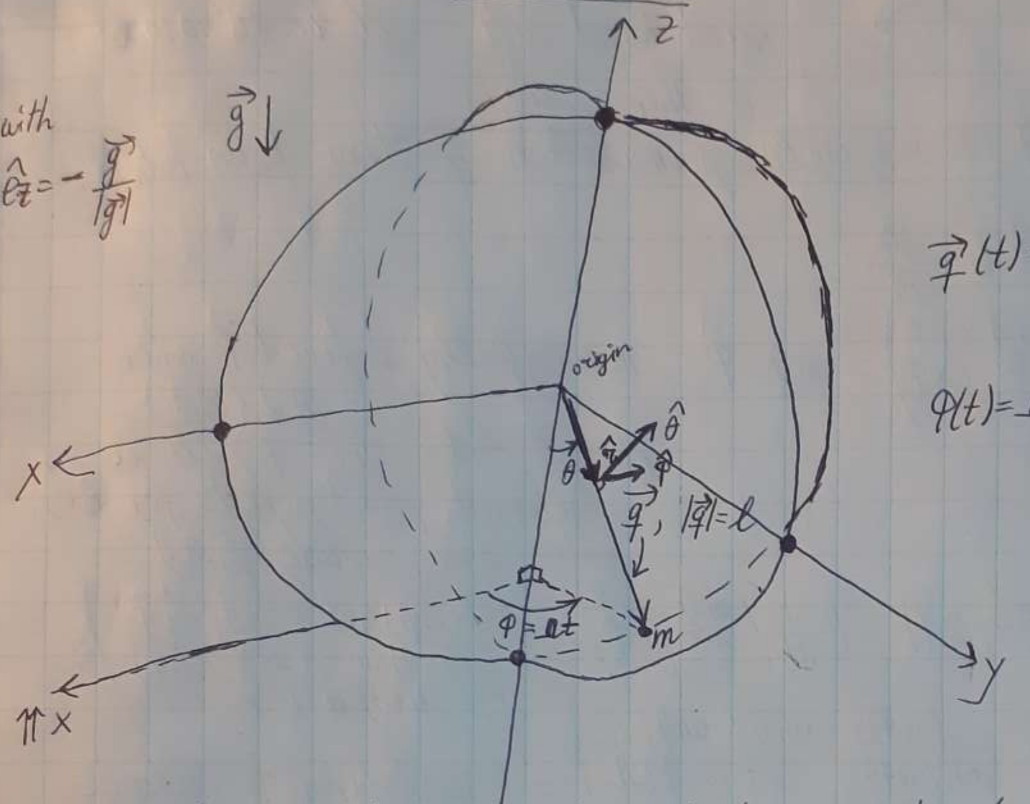
Prof Dr Jürgen Vollmer

HW Exercise Sheet 11

Lagrange Formalism

Problem 11.1 Motion of pearl on rotating ring

with $\hat{e}_z = -\frac{\vec{g}}{|\vec{g}|}$ $\quad \vec{g}\downarrow$



$\vec{q}(t) = \ell\,\hat{r}(\theta(t), \Omega t = \varphi_{(t)})$

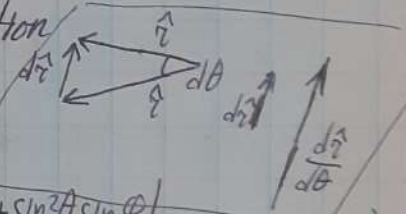
$\varphi(t) = \Omega t$

a) Components of $\hat{r}(\theta, \phi)$ in Cartesian coordinates:

$$\hat{r}(\theta(t), \varphi(t)) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ -\cos\theta \end{pmatrix}$$

$$\hat{\theta} = \frac{\partial\hat{r}}{\partial\theta} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ \sin\theta \end{pmatrix}$$

(see the direction on diagram, check with either geometric intuition or check ortogonality by $\hat{r}\cdot\hat{\theta} = 0$)



$$\hat{\varphi} = \hat{\theta}\times\hat{r} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ \sin\theta \end{pmatrix} \times \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ -\cos\theta \end{pmatrix} = \begin{pmatrix} -(\cos^2\theta\sin\varphi + \sin^2\theta\sin\varphi) \\ +\sin^2\theta\cos\varphi + \cos^2\theta\cos\varphi \\ \sin\theta\cos\theta\sin\varphi\cos\varphi - \sin\theta\cos\theta\sin\varphi\cos\varphi \end{pmatrix} =$$

$$= \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$ – is in horizontal plane, ~~[illegible]~~ matches 2D case.

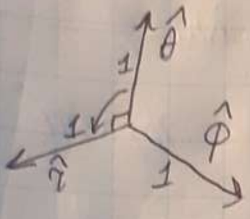
Check they form RHB:

1) $|\hat{r}| = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$

$|\hat{\theta}| = \sqrt{\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$\hat{r} \perp \hat{\theta}$ because $\hat{r}\cdot\hat{\theta} = \underbrace{\sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi}_{\sin\theta\cos\theta} - \sin\theta\cos\theta = 0$

Then:



$\hat\varphi$ is as follows by definition of right hand rule will be orthogonal to them & unit.

So $\hat r, \hat\varphi, \hat\theta$ or any cyclic permutation of them form RHB.

Relation of $\hat\varphi$ to $\frac{\partial \hat r}{\partial\varphi}$:

$$\frac{\partial\hat{r}}{\partial\varphi} = \begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix} = \sin\theta\cdot\hat\varphi,$$ meaningfully defined $\hat\varphi$ where $\theta \notin \{0,\pi\}$.

(though later, since φ will be irrelevant, fixed points for $(\theta,\dot\theta)$ phase space include $\theta\in\{0,\pi\}$)

b) $$\dot{\vec{q}}(t) = \ell\dot{\hat{r}}(\theta,\varphi) = \ell\frac{d}{dt}\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi \\ -\cos\theta\end{pmatrix} = \ell\begin{pmatrix}(\dot{\sin\theta})\cos\varphi + \sin\theta(\dot{\cos\varphi}) \\ (\dot{\sin\theta})\sin\varphi + \sin\theta(\dot{\sin\varphi}) \\ (\sin\theta)\dot\theta + 0\end{pmatrix} =$$

$$= \ell\begin{pmatrix}\cos\theta(\cos\varphi)\dot\theta + \sin\theta(-\sin\varphi)\Omega \\ \cos\theta(\sin\varphi)\dot\theta + \sin\theta(\cos\varphi)\Omega \\ (\sin\theta)\dot\theta + 0\end{pmatrix} = \text{(simplifying to vectors)} =$$

$$= \underbrace{\ell\dot\theta\hat\theta}_{\text{moving on ring}} + \underbrace{\ell(\sin\theta)\Omega\hat\varphi}_{\text{rotation of ring}} = \ell\dot\theta\hat\theta + \ell(\sin\theta)\dot\varphi\hat\varphi.$$

moving on ring + rotation of ring = moving on sphere

c) $$T = \frac{m}{2}\dot{q}^2 = \frac{m}{2}\dot{\vec q}\cdot\dot{\vec q} = \frac{m}{2}\left(\ell^2\dot\theta^2 + \ell^2\sin^2\theta\cdot\dot\varphi^2\right) = \frac{m\ell^2}{2}\left(\dot\theta^2 + \sin^2\theta\cdot\Omega^2\right)$$

$V = -mg\ell\cos\theta$ (definining $V=0$ on plane of $\hat e_x, \hat e_y$), rotating Ω-part does not play role.

d) Lagrange formalism $\to$ EOM,

$$L(\theta,\Omega) = T - V = \frac{m\ell^2}{2}\left(\dot\theta^2 + \sin^2\theta\,\Omega^2\right) + mg\ell\cos\theta,$$ where $\Omega = \dot\varphi$

$$L = \frac{m\ell^2}{2}\left(\dot\theta^2 + \sin^2\theta\cdot(\dot\varphi)^2\right) + mg\ell\cos\theta$$

From Euler-Lagrange equation then:

* for θ-coordinate;

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right)$$

* for φ coordinate trivial

$$\varphi = \Omega t$$

$$\frac{ml^2}{2}\cdot 2\sin\theta\cos\theta\,(\dot\varphi)^2 - mgl\sin\theta = \frac{d}{dt}(ml^2\dot\theta)$$

$$\frac{1}{2}ml^2(\dot\varphi)^2\sin 2\theta - mgl\sin\theta = ml^2\ddot\theta \;\Big|\cdot\frac{1}{ml^2}$$

$$\frac{(\dot\varphi)^2}{2}\sin 2\theta - \frac{g\sin\theta}{l} = \ddot\theta \rightarrow \frac{\Omega^2}{2}\sin 2\theta - \frac{g}{l}\sin\theta = \ddot\theta,$$

$$\sin\theta\left(\Omega^2\cos\theta - \frac{g}{l}\right) = \ddot\theta;$$ for small angles

becomes $\theta\left(\Omega^2 - \frac{g}{l}\right) = \ddot\theta$
is easy Linear ODE
of oscillation if no Ω^2
(anyway Ω is const)

Therefore equations are

$$\begin{cases} \varphi = \Omega t \\ \ddot\theta = \sin\theta\left(\Omega^2\cos\theta - \frac{g}{l}\right) \end{cases}$$

e) Determine fixed points for motion, stability as $f(\Omega)$.

In phase space for θ $\vec{V}_\theta = \begin{pmatrix}\dot\theta \\ \ddot\theta\end{pmatrix} = \vec{0}$. In phase space for φ $\vec{V}_\varphi = \begin{pmatrix}\dot\varphi \\ \ddot\varphi\end{pmatrix} = \begin{pmatrix}\Omega \\ 0\end{pmatrix} = \vec{0}$

iff $\Omega = 0$ (no ring rotation $\rightarrow$ nothing interesting)

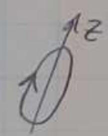
$$\begin{cases} \dot\theta = 0 \quad \text{(horizontal axis in PS)} \\ \ddot\theta = \sin\theta\left(\Omega^2\cos\theta - \frac{g}{l}\right) = 0 \end{cases}$$

$\theta \in \{0, \pi\}$
up & bottom points

$\cos\theta = \frac{g}{l\Omega^2}$

$\theta = \arccos\left(\frac{g}{l\Omega^2}\right)$

frequency $\omega_0 = \sqrt{\frac{g}{l}}$ from

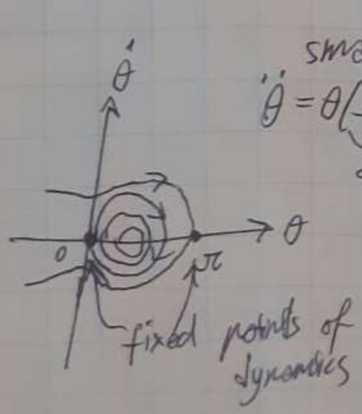


if $\frac{g}{l\Omega^2} = \frac{\omega_0^2}{\Omega^2} > 1$, no solutions, so only $\theta \in \{0, \pi\}$, for small θ

$\ddot\theta = \theta(\underbrace{\Omega - \tfrac{g}{l}}_{<0})$ oscillation in circle in PS (harmonic)

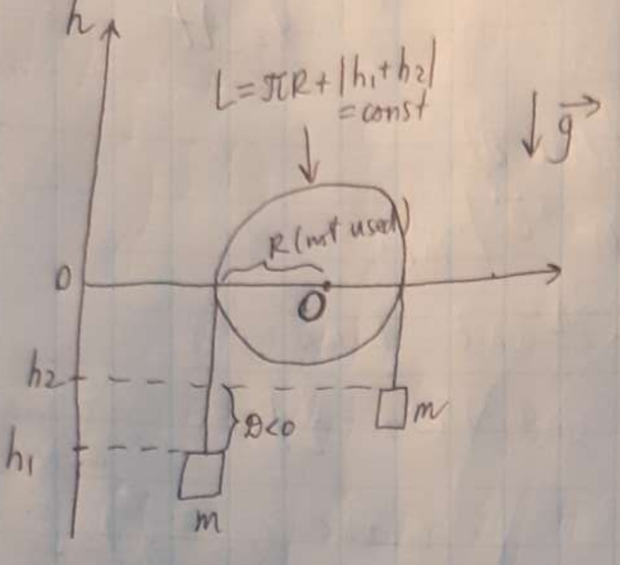
if $\frac{g}{l\Omega^2} = \left(\frac{\omega_0}{\Omega}\right)^2 \leq 1$,
3 fixed points in PS;

$\theta_1 = 0$
$\theta_2 = \pi$
$\theta_3 = \arccos\left(\frac{\omega_0}{\Omega}\right)^2$
$\dot\theta_{1,2,3} = 0$



Problem 11.2 Two masses at rubber band

a)



b) $H = h_1 + h_2$ – const

$(-[\text{total length}] - \pi R)$

$\mathcal{D} = h_1 - h_2$ – difference with sign between mass positions

$L(H, \mathcal{D}, \dot{H}, \dot{\mathcal{D}}) = T - V$

Now $T = \frac{m(\dot{h}_1)^2}{2} + \frac{m(\dot{h}_2)^2}{2} = \frac{m}{2}\left((\dot{h}_1)^2 + (\dot{h}_2)^2\right) = \frac{\mu}{2}\left((\dot{H})^2 + (\dot{\mathcal{D}})^2\right)$ for some $\mu(m)$.

So this must hold:

$$\frac{m}{2}\left((\dot{h}_1)^2 + (\dot{h}_2)^2\right) = \frac{\mu}{2}\left((\dot{h}_1 + \dot{h}_2)^2 + (\dot{h}_1 - \dot{h}_2)^2\right) = \frac{\mu}{2}\left(2\dot{h}_1^2 + 2\dot{h}_2^2\right) =$$

$$= \underline{\mu(\dot{h}_1^2 + \dot{h}_2^2), \text{ so } \mu = \frac{m}{2}}$$ ("reduced/effective mass" in general $\mu = \frac{m_1 m_2}{m_1 + m_2}$, see Kepler problem also)

Now $V = \underbrace{mgh_1 + mgh_2}_{\text{gravitational PE}} + \underbrace{\frac{k}{2}(-h_1 - h_2)^2}_{\text{elastic PE (total elongation from 0 is stretch, assume this)}} =$

$$= mgH + \frac{k}{2}H^2$$

The choice $V = 0$ when $h = 0$ was used (note that for E-L equation it makes no difference which reference to take – it contains only derivatives)

Then $L(H, \mathcal{D}, \dot{H}, \dot{\mathcal{D}}) = T - V = \frac{\mu}{2}(\dot{H}^2 + \dot{\mathcal{D}}^2) - mgH - \frac{k}{2}H^2$

c) From Euler-Lagrange equation:

c.1) H-coordinate: $\frac{\partial L}{\partial H} = \frac{d}{dt} \cdot \left(\frac{\partial L}{\partial \dot{H}}\right)$

$$-mg\underbrace{-\frac{kH}{}}_{<0} = \frac{d}{dt}(\mu \dot{H}) = \mu \ddot{H}$$

$\mu \ddot{H} + kH + mg = 0$ (same as in Newton approach)

Note

(Question to Dr. Vollmer/Johannes Ewald, which approach works best in general, when simulating system with billion points on computer → Newton or Lagrange? Does it depend on how forces look like? (e.g. for planets – Newton's will work since there are just many forces, but they all are exact & trivial)

c.2) D-coordinate: $\frac{\partial L}{\partial D} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{D}}\right)$

$$0 = \frac{d}{dt}(\mu \dot{D}) = \mu \ddot{D}$$

$$\ddot{D} = 0$$

EOM $\begin{cases} \mu \ddot{H} + kH + mg = 0 \\ \ddot{D} = 0 \end{cases}$

d) Solve equations, interpret, when I will solve in dimensionless form first? for practice/fun:

d.1) $\frac{m}{2}\ddot{H} + kH + mg = 0 \,\Big|\cdot \frac{2}{m}$

$$\ddot{H} + \left(\frac{2k}{m}\right)H + (2g) = 0$$

k, m, g can all be removed

$$\underbrace{\tilde{H} = \frac{H}{(mg/k)} \quad \tilde{\tau} = \frac{t}{\sqrt{m/k}}}_{\text{dimensionless quantities}}$$

$$\frac{d^2H}{dt^2} + \left(\frac{2k}{m}\right)H + (2g) = 0$$

$$\frac{mg}{k}\cdot\frac{d^2(\tilde{H})}{d\left(\sqrt{\frac{m}{k}}\tilde{\tau}\right)^2} + \left(\frac{2k}{m}\right)\tilde{H}\cdot\frac{mg}{k} + (2g) = 0$$

$$g\frac{d^2\tilde{H}}{d\tilde{\tau}^2} + 2g\tilde{H} + 2g = 0 \;\Big|\cdot\frac{1}{g}, \quad {}^{\cdot} \text{ is } \frac{d}{d\tilde{\tau}}$$

$\ddot{\tilde{H}} + 2\tilde{H} + 2 = 0$ Charac. equation $\lambda^2 + 2\lambda + 2 = 0$

$$D = 4 - 8 = -4$$

$$\lambda_{1,2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\tilde{H}(\tilde{\tau}) = e^{-\tilde{\tau}}(C_1 \cos\tilde{\tau} + C_2 \sin\tilde{\tau})$$

Restoring dimensions:

$H(t) = \frac{mg}{k} e^{-\sqrt{\frac{k}{m}}t}\left(C_1 \cos\left(-\sqrt{\frac{k}{m}}\,t\right) + C_2 \sin\left(-\sqrt{\frac{k}{m}}\,t\right)\right)$, $\sqrt{\frac{k}{m}} = \omega$, then

first root is +, but $\cos(+) = \cos(-)$

$H(t) = \frac{g}{\omega^2} e^{-\omega t}(C_1 \cos\omega t - C_2 \sin\omega t) = A e^{-\omega t}\cos(\omega t + \varphi_0)$,

A includes $\frac{g}{\omega^2}$

$$H(t) = A e^{-\sqrt{\frac{k}{m}} t} \cos\left(\sqrt{\frac{k}{m}} t + \varphi_0\right) \quad (A, \varphi_0 \text{ are initial conditions})$$

d.2) $\ddot{D} = 0$ (or recall formulas)

$$\int_{t_0}^{t} \ddot{D}(t')dt' = \int_{t_0}^{t} 0\, dt'$$

$$\dot{D}(t) - \dot{D}(t_0) = 0$$

$$\dot{D}(t) = \dot{D}(t_0) = \dot{D}$$

$$\int_{t_0}^{t} \dot{D}(t')dt' = \int_{t_0}^{t} \dot{D}\, dt' = \dot{D}\int_{t_0}^{t} dt' = \dot{D}(t - t_0)$$

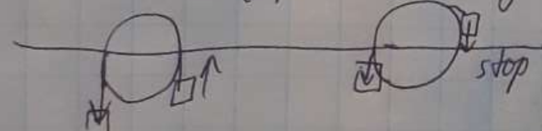
$$D(t) - D(t_0) = \dot{D}(t - t_0)$$

$D(t) = D_0 + \dot{D}(t - t_0)$, if $t_0 = 0$,

$D(t) = D_0 + \dot{D} \cdot t$, linear law

Interpretation: ~~difference~~ distance between weights changes ~~since one weighs more (in general) → accelerates more,~~ move ~~(stupid)~~ since they move into opposite directions, but changes with constant speed.

Sum of distances oscillates ~~befo~~ becoming 0 with large k (rope resists moving of masses) and small m.

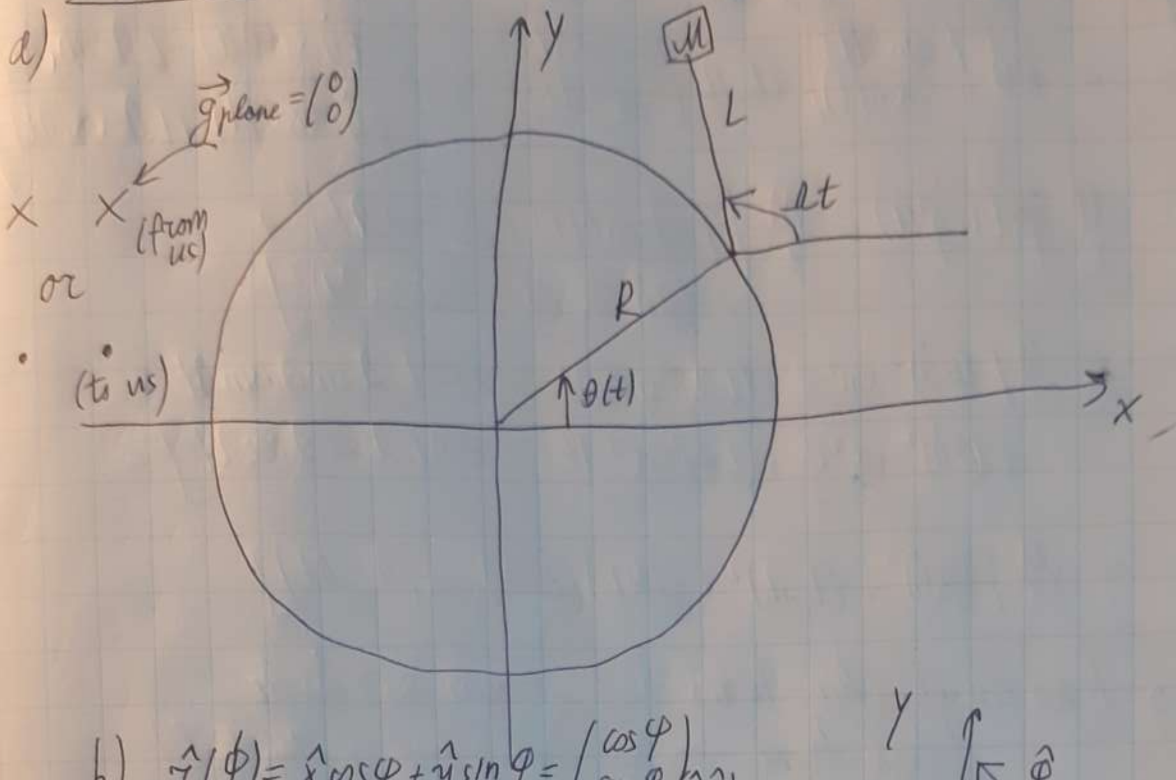


$h_1 + h_2 = 0$
$h_1 < 0$, $h_2 > 0$

But I will trust these results only when Hooke law applies for small $|h_i|$ and thus $|H|$, and the result for H for very small $|H|$ as well, since H is const and since move opposite directions, and depicted on picture above is possible only then, when $|h_i|_{max} \approx R$ of roll.

Problem 11.3 Horizontal driven double pendulum.

a)



b) $\hat{r}(\phi) = \hat{x}\cos\varphi + \hat{y}\sin\varphi = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}_{(\hat{x},\hat{y})}$

$\hat{\varphi}(\phi) = \frac{d}{d\varphi}\hat{r}(\phi) = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}_{(\hat{x},\hat{y})}$

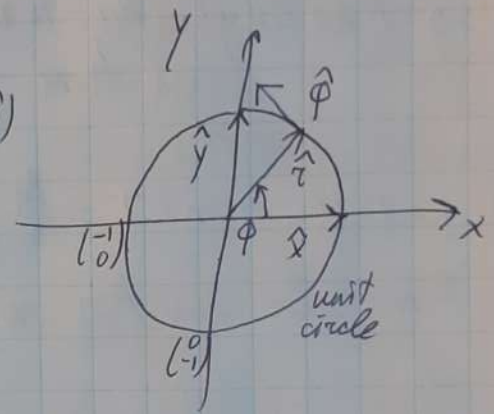
theory to use in c)

$|\hat{r}| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$

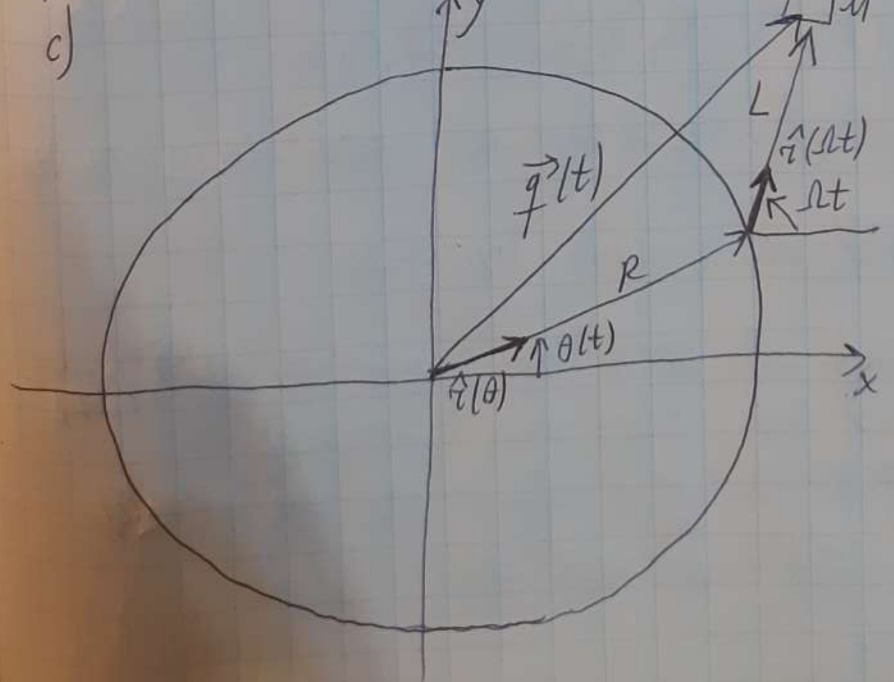
$|\hat{\varphi}| = \sqrt{\sin^2\varphi + \cos^2\varphi} = 1$

$\hat{r}\cdot\hat{\varphi} = \cancel{\sqrt{\quad}} -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0.$

Hense $\hat{r}, \hat{\varphi}$ form ONB in x-y plane.



c)



$\vec{q}(t) = \hat{r}(\theta)R + \hat{r}(\Omega t)\cdot L,$

$\hat{r}(\theta) = \begin{pmatrix}\cos\theta(t)\\ \sin\theta(t)\end{pmatrix}$

$\hat{r}(\Omega t) = \begin{pmatrix}\cos(\Omega t)\\ \sin(\Omega t)\end{pmatrix}$

d) $\dot{\vec{q}}(t) = R\,\dot{\hat{r}}(\theta) + L\,\dot{\hat{r}}(\Omega t) = R\frac{d\hat{r}}{d\theta}\dot{\theta} + L\frac{d\hat{r}}{d(\Omega t)}(\dot{\Omega t}) =$

$$= R\dot{\theta}\begin{pmatrix}-\sin\theta(t)\\ \cos\theta(t)\end{pmatrix} + L\Omega\begin{pmatrix}-\sin(\Omega t)\\ \cos(\Omega t)\end{pmatrix} = \begin{pmatrix}-R\dot{\theta}\sin\theta(t) - L\Omega\sin(\Omega t)\\ R\dot{\theta}\cos\theta(t) + L\Omega\cos(\Omega t)\end{pmatrix}.$$

$$T = \frac{M}{2}\dot{\vec{q}}(t)\cdot\dot{\vec{q}}(t) = \frac{M}{2}\Big[(R\dot{\theta}\sin\theta(t) + L\Omega\sin\Omega t)^2 + (R\dot{\theta}\cos\theta(t) + L\Omega\cos\Omega t)^2\Big] =$$

$$= \frac{M}{2}\Big[\underline{(R\dot{\theta})^2\sin^2\theta} + (L\Omega)^2\sin^2\Omega t + 2R\dot{\theta}L\Omega\sin\theta\cdot\sin\Omega t + (R\dot{\theta})^2\underline{\cos^2\theta} + (L\Omega)^2\cos^2\Omega t + 2R\dot{\theta}L\Omega\cos\theta\cos\Omega t\Big] =$$

$$= \frac{M}{2}\Big[(R\dot{\theta})^2 + (L\Omega)^2 + 2(RL)(\dot{\theta}\Omega)\cos(\theta - \Omega t)\Big]$$

e) I) $V = 0$ (no conservative forces act, assume no elastic deformations $\rightarrow$ no concept of potential)

In ~~fact~~

II) $L = T - V = T - 0 = T = \frac{M}{2}\Big[(R\dot{\theta})^2 + (L\Omega)^2 + 2(RL)(\dot{\theta}\Omega)\cos(\theta - \Omega t)\Big]$

III) From Euler equation only for $\theta(t)$ (for Ωt it is given)

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

$* \frac{\partial L}{\partial\theta} = \frac{\partial}{\partial\theta}\left(MRL\,\Omega\dot{\theta}\cos(\theta - \Omega t)\right) = MRL\Omega\dot{\theta}\left(-\sin(\theta - \Omega t)\right) = -MRL\Omega\dot{\theta}\cdot\sin(\theta - \Omega t)$

$* \frac{\partial L}{\partial\dot{\theta}} = \frac{\partial}{\partial\dot{\theta}}\left(\frac{M}{2}R^2\dot{\theta}^2 + \frac{M}{\cancel{2}}\cancel{2}RL\Omega\cos(\theta - \Omega t)\dot{\theta}\right) =$

$\downarrow$

$$= \frac{M}{\cancel{2}}R^2\cancel{2}\dot{\theta} + MRL\,\Omega\cos(\theta - \Omega t) = MR^2\dot{\theta} + MRL\,\Omega\cos(\theta - \Omega t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = MR^2\ddot{\theta} + MRL\,\Omega\left[-\sin(\theta - \Omega t)\right](\dot{\theta} - \Omega)$$

So: $-MRL\Omega\dot{\theta}\sin(\theta - \Omega t) = MR^2\ddot{\theta} + MRL\,\Omega\sin(\theta - \Omega t)(\Omega - \dot{\theta}) \Big/ \cdot\frac{1}{MRL}$

$$-\Omega\dot{\theta}\sin(\theta - \Omega t) = \frac{R}{L}\ddot{\theta} + \Omega\sin(\theta - \Omega t)(\Omega - \dot{\theta})$$

$$-\Omega\dot{\theta}\,\cancel{\sin(\theta - \Omega t)} = \frac{R}{L}\ddot{\theta} + \Omega^2\sin(\theta - \Omega t) - \Omega\dot{\theta}\,\cancel{\sin(\theta - \Omega t)}$$

$$\frac{R}{L}\ddot{\theta} = -\Omega^2\sin(\theta - \Omega t)$$

f) $\alpha(t) = \theta(t) - \Omega t$

$\dot{\alpha}(t) = \dot{\theta}(t) - \Omega$

$\ddot{\alpha}(t) = \ddot{\theta}(t)$

So $\frac{R}{L}\ddot{\alpha} = -\Omega^2 \sin\alpha$

$\frac{R}{L}\frac{d^2\alpha}{dt^2} = -\Omega^2 \sin\alpha$, $t = K\tau$, so goal is to find K. (τ — dimensionless time)

$\frac{R}{L} \cdot \frac{1}{K^2}\frac{d^2\alpha}{d\tau^2} = -\Omega^2 \sin\alpha$, $\underbrace{\frac{R}{K^2 L \Omega^2}}_{1} \cdot \frac{d^2\alpha}{d\tau^2} = -\sin\alpha$

$K^2 = \frac{R}{L\Omega^2}$, $K = \sqrt{\frac{R}{L}} \cdot \frac{1}{\Omega}$

Then $t = \sqrt{\frac{R}{L}}\,\frac{1}{\Omega}\tau$, $\tau = \frac{t \cdot \Omega}{\sqrt{\frac{R}{L}}} = \left(\sqrt{\frac{L}{R}}\,\Omega\right)t$ — time scale in units of $\sqrt{\frac{R}{L}} \cdot \frac{1}{\Omega}$.

g) $E = \frac{\dot{\alpha}^2}{2} - \cos\alpha$ is COM

Prove this: $\frac{dE}{dt} = \frac{2\dot{\alpha}\ddot{\alpha}}{2} + \sin\alpha \cdot \dot{\alpha} =$

$= \dot{\alpha}(\ddot{\alpha} + \sin\alpha) =$ [using EOM]

$= \dot{\alpha}(\underbrace{\ddot{\alpha} - \ddot{\alpha}}_{0}) = 0.$

Question to Dr Vollmer / J. Ewald: how to „generate“/find different formulas for constants of motion like this, except guess?