Problems for Exam Preparation

Problem 13.1. Is there a sequence of irrational numbers that converges to 1 ? Is there a sequence of irrational numbers that converges to $\sqrt{2}$?

Problem 13.2. Prove that

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

Problem 13.3. In how many ways one can choose 4 chairs from a room with 10 chairs ?

Problem 13.4. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Problem 13.5. Compute the limit

$$\lim_{n\to\infty} \left(\frac{n^2+1}{n+1} - n \right).$$

Problem 13.6. Compute the limit

$$\lim_{n\to\infty} \frac{n2^n + n^{1/n}3^n + n^4}{3^n\left(2 + n^{1/n}\right) + 2n^4 + n^2 2^n}.$$

Problem 13.7. Compute the limit

$$\lim_{n\to\infty} \left(\sqrt{n^4 + n^2} - n^2 \right).$$

Problem 13.8. Compute the limit

$$\lim_{n\to\infty} \left(\frac{4n+1}{4n+8} \right)^{3n+2}.$$

Problem 13.9. Assume that $0 < a_n < 1$ for all $n \in \mathbb{N}$. Does it imply that $\lim_{n\to\infty} a_n^n = 0$?

Problem 13.10. Prove that if $\lim_{n\to\infty} a_n = A$, then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = A.$$

Problem 13.11. Check the following series for convergence

$$\sum_{n=1}^{\infty} \frac{n^{100}}{1.01^n}.$$

Problem 13.12. Compute the infinite series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{2}{3^n} \right)$$

Problem 13.13. Check the following series for convergence

$$\sum_{n=1}^{\infty} \frac{(2n)!}{5^n (n!)^2}.$$

Problem 13.14. Check the following series for convergence

$$\sum_{n=2}^{\infty} \frac{n+4}{n^{5/2}-2}.$$

Problem 13.15. Compute the infinite series

$$\sum_{n=2}^{\infty} \left(\frac{3^n}{n!}\right)$$

Problem 13.16. Does the following limit exist ?

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}.$$

Problem 13.17. Compute the following limit

$$\lim_{x\to 0} \frac{\ln(1+2x)}{x}.$$

Problem 13.18. Compute the following limit

$$\lim_{x\to+\infty} \frac{x^2+5x+7}{-x^3+2x^2-3x+4}.$$

Problem 13.19. Compute the following limit

$$\lim_{x\to 1/2} \left(\frac{x+2}{2x-1}\right)^{4x^2-1}.$$

Problem 13.20. Calculate the derivative of the following function

$$\sin(x)^{\cos(x)}.$$

Problem 13.21. Prove that the function $x^5 - 5x + 2$ has at least 3 real roots.

Problem 13.22. Find all local and global extrema of the function

$$f(x) = xe^{-x}, \qquad f : [-2; 2] \to \mathbb{R}.$$

Problem 13.23. Find all local and global extrema of the function

$$f(x) = \frac{\ln(x)}{x}, \qquad f : [1; \infty] \to \mathbb{R}.$$

Problem 13.24. Compute the following limit

$$\lim_{x\to 0} \frac{\cos(x) - e^{-\frac{x^2}{2}}}{x^4}.$$

Problem 13.25. Compute the following integral

$$\int_1^3 x^2 \ln(x) dx$$

Problem 13.26. Compute the following integral

$$\int_1^2 \frac{2}{\sqrt{x-1}} dx$$

Problem 13.27. Compute the area of a region bounded by curves $y = x^2$, $y = 2^x$, and $x = 0$.

Problem 13.28. Write three nonparallel vectors from $\mathbb{R}^3$ which are perpendicular to the vector $(3, -1, 2)$.

Problem 13.29. Which of the following triples of vectors form a basis of $\mathbb{R}^3$?

a) $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$

b) $(1, 0, 0)$, $(2, 1, 1)$, $(1, 1, 1)$

c) $(3, 1, -2)$, $(1, 1, -1)$, $(1, -1, 0)$

Problem 13.30. Compute the intersection of the following two planes in $\mathbb{R}^3$:

$$x + 2y - 3z + 1 = 0, \qquad -x + y - z - 1 = 0.$$