13. Lagrange Formalism 2

Chapters 6 of my lecture notes provides the background to solve the following exercises. Your solution to the problems 13.1–13.3 should uploaded

to your Moodle account

as a PDF-file

by Wednesday, Jan 24, 10:30 (with a grace time till the start of the seminars).

The parts marked by ⋆ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

The self-test problems serve to check you understanding of the analysis of the motion of different systems.

Problems

Problem 1. Non-dimensionalization and phase-space portraits

We consider the EOM

$$m\,\ddot{x}(t) = a\,x(t) - b\,x^3(t) \tag{13.1}$$

where $x(t)$ is the position of a particle of mass m.

a) What are the dimensions of a and b?

b) We consider the case where a and b take positive values. Determine a length scale L and a time scale T that provide dimensionless coordinates

$$\tau = (t - t_0)/T \qquad\qquad \xi(\tau) = x(t)/L$$

such that the EOM takes the dimensionless form

$$\ddot{\xi}(\tau) = \xi(\tau) - \xi^3(\tau)$$

Determine the dimensionless energy $\mathcal{E}$, and sketch the phase-space portrait.

c) We consider the case where a takes a negative and b takes a positive value. Determine a length scale L and a time scale T that provide dimensionless coordinates

$$\tau = (t - t_0)/T \qquad\qquad \xi(\tau) = x(t)/L$$

such that the EOM takes the dimensionless form

$$\ddot{\xi}(\tau) = -\xi(\tau) - \xi^3(\tau)$$

Determine the dimensionless energy $\mathcal{E}$, and sketch the phase-space portrait.

Remark: Beware the different sign of the linear term. How does it come about?

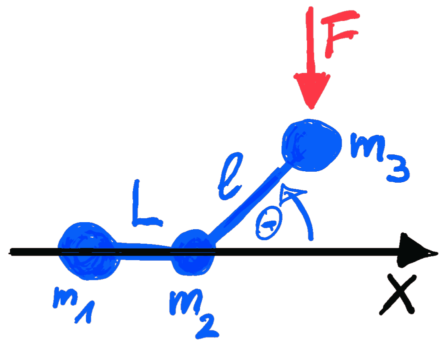
⋆ d) Verify that the minimal dimensionless energy takes the value $-1/2$ and 0 for positive and negative a, respectively. How can it change *discontinuously* when a changes sign?

⋆ e) Linearize the dimensionless EOM for positions $\xi = \xi_c + \epsilon$ close to the stable fixed points ξ_c of the dimensionless EOM, i.e. determine a constant c such that

$$\ddot{\epsilon} \simeq c\,\epsilon \qquad \text{for} \quad |\epsilon| \ll 1$$

How does the value of c influence the shape of the phase-space portrait in the vicinity of the fixed points?

Problem 2. Oscillator on rails



We consider a sled that is composed of two particles of mass m_1 and m_2 that keep a fixed distance L, while they move without friction on a straight trail. The position of particle 2 will be denoted as x. At that particle we attach an arm of fixed length ℓ with a particle of mass m_3 attached at its other end. The motion of particles 1 and 2 is constrained to the trail. The arm can rotate in the plane. Its deflection from the positive x-axis is denoted as θ. However, the rotation is not free: Particle 3 is subjected to a harmonic force $\mathbf{F}$ that is pushing the particle back towards the trail: $\mathbf{F}$ always acts perpendicular to the trail with magnitude amounts to $-k\,y$ proportional to the distance y from the trail.

a) Determine the kinetic energy of the cart, the kinetic energy of particle 3, and the potential energy due to the harmonic force acting on particle 3.
Provide the resulting Lagrange function for the system.

b) Identify the equation of motion for x, and show that it leads to a conserved quantity of the form

$$P = \alpha\,\dot{x} + \beta\,\ell\,\dot{\theta}\,\sin\theta$$

How do α and β depend on the masses and on the spring constant, k?

⋆ c) Determine the x-component Q of the center of mass of the system. Which interpretation does this provide for the result of part (c)?

d) Determine the EOM for θ, and use the conservation law to eliminate x and its derivaties from this equation.

Introducing the mass ratio $\mu = m_3/(m_1 + m_2 + m_3)$ and adopting a suitable time scaling provides

$$\ddot{\theta} = -\sin\theta\,\cos\theta\,\frac{1 - \mu\,\dot{\theta}^2}{1 - \mu\,\sin^2\theta}$$

Which time scaling has been adopted to arrive at this form of the equation?

e) Let us discuss the limit where the sled is much heavier than the mass of particle three. Show that $\alpha = 2\theta$ will then follow the equation of motion

$$\ddot{\alpha} = -\sin\alpha$$

Determine the fixed points of this dynamics, and sketch the evolution in the phase space $(\theta, \dot{\theta})$.

Bonus: Describe the motion of the sled and the arm for motion

1. close to the fixed points, and
2. during the heteroclines.

⋆ f) Consider now the limit where particle 3 is much heavier than the sled. Which value will μ take in this limit? Show that the equation of motion can then be rewritten in the form

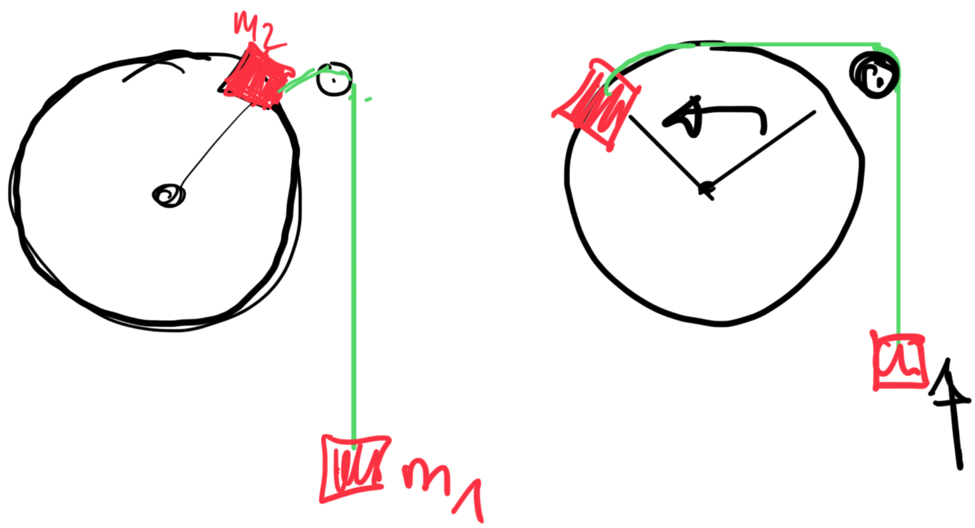
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sin\theta = -\sin\theta$$

Determine the general solution for $\sin\theta$.

Provide a physical description of the small-amplitude and large amplitude solutions.

Problem 3. Particles on a wheel and pulley

We consider two particles of masses m_1 and m_2 that are connected by a rope of fixed length. The rope is led over a pulley such that mass 1 is hanging straight down at a height z. Mass 2 is attached to a wheel such that the chord wraps along the wheel circumference when the wheel is turned. The sketch to the left shows the setting when mass 2 is attached at the closest point on the wheel. When the wheel is then turned (in either direction!) the chord wraps around the wheel, and mass 1 moves up, as shown in the following sketch:



a) Let the wheel have radius R and let it be turned by an angle θ, i.e. $\theta = 0$ in the sketch to the left. Mark these notations in the sketches.

b) For $|\theta| \gtrsim \pi/4$ the function $z(\theta)$ is linear. Determine its slope for positive and negative θ. How are the slopes related?

   Sketch $z(\theta)$.

c) Provide the kinetic energy and the potential energy of particle 1 and 2.

d) Determine the Lagrange function and express it in terms of $z(\theta(t))$, $\theta(t)$ and their time derivatives.

e) Show that the equation of motion for $\theta(t)$ takes the form

$$\ddot{\theta} = K \left(\kappa \, s(\theta) - \sin(\theta + \theta_2) \right) \tag{13.2}$$

where K is a constant that can be absorbed into the time scale and κ is a dimensionless parameter of the system. Moreover, $s(\theta)$ is a rescaled version of $z'(\theta)$ that takes the values ± 1 for large arguments and it crosses over from -1 to $+1$ in a narrow range of angles close to $\theta = 0$.
Determine K and κ.

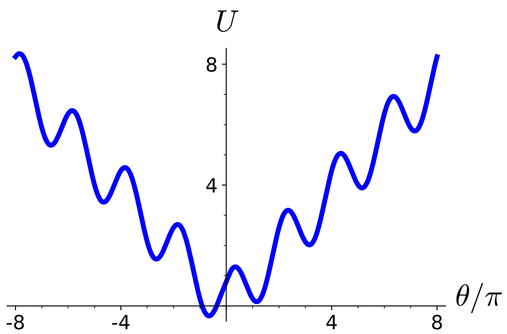
f) Show that the system has a single fixed point when $\kappa \gg 1$.
Hint: The fixed points are the roots of Eq. (13.2).

What does this mean physically?

How does the phase-space plot look like in that case?

$\star$ g) Show that the equation of motion has infinitely many fixed points for $\kappa \ll 1$.

What does this mean physically?



h) The plot to the left shows the effective potential of the dynamics for $\kappa \ll 1$. Provide the corresponding phase-space plot.

$\star$ i) Let the rope have a length L. For which range of angles θ would you trust the predictions of the model?

Self Test

Problem 4. The driven oscillator

We consider a weight of mass M attached to a Hookian spring with spring constant k and rest length ℓ. The weight is moving vertically in a gravitational field, while the free end of the spring is moved vertically such that it resides at $z(t)$ at time t. The length of the spring will be denoted as $L(t)$.

a) Sketch the problem and setup and indicate the relevant parameters and coordinates.

b) Determine the kinetic energy and the potential energy.
Provide the resulting expressions in terms of $z(t)$ and $L(t)$.

c) Determine the equation of motions for $L(t)$.

d) To be concrete we specify now that $z(t) = \Delta z_0 \cos(\Omega t)$.
Provide the nondimensionalized equation of motion for $\lambda(\tau) = \big(L(t) - \ell - mg/k\big)/\ell$ where τ is an appropriate dimensionless time $\tau = \omega t$, and determine its solution.

Problem 5. Two masses hanging at a rubber band

Two weights of the same mass m are attached on opposite ends of a rubber band that is hanging over a roll. The weights are at height h_1 and h_2. They move only vertically, either one up and one down at a fixed length of the band, or stretching the band, or releasing tension on the band. We assume that friction and the mass of the band are negligible.

a) Sketch the problem, and indicate the relevant parameters and coordinates.

b) We describe the problem by adopting the coordinates $H = h_1 + h_2$ and $D = h_1 - h_2$. Verify that the Lagrange function will then take the following form

$$\mathcal{L}(H, D, \dot{H}, \dot{D}) = \frac{\mu}{2} \left(\dot{H}^2 + \dot{D}^2\right) - mg\,H - \frac{k}{2}\,H^2$$

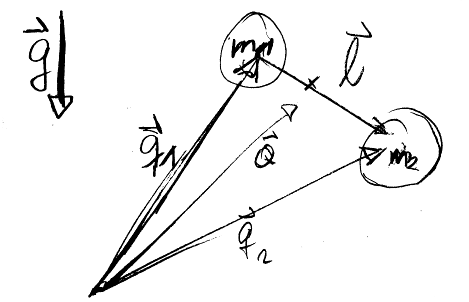
Here k is the elastic module of the rubber band, and μ is an effective mass. How is μ related to m?
The expression for $\mathcal{L}$ adopts a particular choice of the origin used to specify h_1 and h_2. Which choice has been used?

c) Determine the equations of motion for H and D.

d) Solve the equations of motion and interpret the result. For which values of H and D will you trust the result?

Problem 6. Flight of a dumbbell

We explore the flight of a dumbbell under the influence of gravity $\mathbf{g}$ in our three-dimensional space. The dumbbell is idealized as two particles of masses m_1 and m_2. The positions $\mathbf{q_1}(t)$ and $\mathbf{q}_2(t)$ will be kept at an approximately constant distance $\ell_\star$ by a bar of negligible mass. We denote the center of mass of the dumbbell as $\mathbf{Q}$ and the relative coordinate as $\boldsymbol{\ell} = \mathbf{q_2} - \mathbf{q}_1$.



a) We express the relation between $(\mathbf{Q}, \boldsymbol{\ell})$ and the positions $\mathbf{q}_i$, $i \in \{1, 2\}$ as $\mathbf{q}_i = \mathbf{Q} + \alpha_i \, \boldsymbol{\ell}$. Determine the real numbers α_i, $i \in \{1, 2\}$.

b) Show that the kinetic energy and the potential energy of the dumbbell have the form

$$T = \frac{M}{2} \, \dot{\mathbf{Q}}^2 + \frac{\mu}{2} \, \dot{\boldsymbol{\ell}}^2 ,$$
$$V = -M\mathbf{g} \cdot \mathbf{Q} + \Phi(\ell)$$

where $\ell = |\boldsymbol{\ell}|$, and $\Phi(\ell)$ is a potential that generates the force fixing the distance of the masses to the value of about $\ell_\star$.[1] How do M and μ depend on m_1 and m_2?

c) Show that

$$\ddot{\mathbf{Q}} = \mathbf{g}$$

How does the trajectory of the center of mass of the dumbbell look like when the dumbbell is thrown at time t_0 from a position $\mathbf{Q}_0$ with a velocity $\mathbf{V}_0$?

[1] Hence, $\ell_\star$ is the length of the bar, when no forces are acting, and the potential counteracts centrifugal forces such that ℓ always takes a value very close to $\ell_\star$.

d) Show that

$$\mu\,\ddot{\boldsymbol{\ell}} = -\hat{\boldsymbol{\ell}}\,\frac{\mathrm{d}\Phi(\ell)}{\mathrm{d}\ell} \quad \text{with} \quad \hat{\boldsymbol{\ell}} = \frac{\boldsymbol{\ell}}{\ell}\,.$$

e) Show that the energy $E = \mu\dot{\boldsymbol{\ell}}^2/2 + \Phi(\ell)$ and the angular momentum $\mathbf{L} = \mu\boldsymbol{\ell}\times\dot{\boldsymbol{\ell}}$ are constants of the motion of the dumbbell.

f) Discuss the evolution of $\boldsymbol{\ell}$ in terms of spherical coordinates (r, θ, ϕ) that are chosen such that initially $\boldsymbol{\ell}$ and $\dot{\boldsymbol{\ell}}$ lie in the equatorial plane, $\theta = \pi/2$ of the coordinate system:

- Show that ϕ is a cyclic coordinate. How is the associated conservation law $\mu\,\ell^2\,\dot{\phi}$ related the $\mathbf{L}$?
- Show that $\theta = \pi/2$ is a fixed point of the θ dynamics. How is this fixed point related to the conservation of $\mathbf{L}$?

g) Provide the position of the masses $\mathbf{q}_1(t)$ and $\mathbf{q}_2(t)$ for the initial conditions provided in (c), some fixed $\boldsymbol{\Omega}$, and $\boldsymbol{\ell}_0$.