Mathematics 1. Selected proofs
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Continuity. Intermediate value theorems

THEOREM 1. f is continuous on $[a,b]$, $f(a) \le 0$, $f(b) \ge 0 \quad \Longrightarrow \quad \exists\, c \in [a,b]\colon\ f(c) = 0$

PROOF. Denote $a_0 = a$, $b_0 = b$

1. Construct the sequence of nested intervals:

   Step 1.

   - Split the interval $[a_0, b_0]$ onto $\left[a_0, \frac{a_0+b_0}{2}\right]$ and $\left[\frac{a_0+b_0}{2}, b_0\right]$
   - If $f\left(\frac{a_0+b_0}{2}\right) \ge 0$ denote $[a_1, b_1] := \left[a_0, \frac{a_0+b_0}{2}\right]$
   - If $f\left(\frac{a_0+b_0}{2}\right) < 0$ denote $[a_1, b_1] := \left[\frac{a_0+b_0}{2}, b_0\right]$
   - In any case we obtain the interval $[a_1, b_1] \subset [a_0, b_0]$ such that $f(a_1) \le 0$ and $f(b_1) \ge 0$

   Step 2.

   - Split the interval $[a_1, b_1]$ onto $\left[a_1, \frac{a_1+b_1}{2}\right]$ and $\left[\frac{a_1+b_1}{2}, b_1\right]$
   - If $f\left(\frac{a_1+b_1}{2}\right) \ge 0$ denote $[a_2, b_2] := \left[a_1, \frac{a_1+b_1}{2}\right]$
   - If $f\left(\frac{a_1+b_1}{2}\right) < 0$ denote $[a_2, b_2] := \left[\frac{a_1+b_1}{2}, b_1\right]$
   - In any case we obtain the interval $[a_2, b_2] \subset [a_1, b_1]$ such that $f(a_2) \le 0$ and $f(b_2) \ge 0$

   ...

   Step n.

   - Split the interval $[a_{n-1}, b_{n-1}]$ onto $\left[a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}\right]$ and $\left[\frac{a_{n-1}+b_{n-1}}{2}, b_{n-1}\right]$
   - If $f\left(\frac{a_{n-1}+b_{n-1}}{2}\right) \ge 0$ denote $[a_n, b_n] := \left[a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}\right]$
   - If $f\left(\frac{a_{n-1}+b_{n-1}}{2}\right) < 0$ denote $[a_n, b_n] := \left[\frac{a_{n-1}+b_{n-1}}{2}, b_{n-1}\right]$
   - In any case we obtain $[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$ such that $f(a_n) \le 0$ and $f(b_n) \ge 0$

   ...

2. Use the monotone sequence theorem:

   $[a_n, b_n] \subset [a_{n-1}, b_{n-1}] \quad \Longrightarrow \quad \forall\, n \in \mathbb{N} \quad a_{n-1} \le a_n \le b_n \le b_{n-1}$

   $\{a_n\}_{n=1}^{\infty}$ — $\nearrow$ and bounded from above $(a_n \le b_0) \quad \Longrightarrow \quad \exists\, c_1 \in [a,b]\colon\ c = \lim\limits_{n\to\infty} a_n$

   $\{b_n\}_{n=1}^{\infty}$ — $\searrow$ and bounded from below $(b_n \ge a_0) \quad \Longrightarrow \quad \exists\, c_2 \in [a,b]\colon\ c_2 = \lim\limits_{n\to\infty} b_n$

3. Show that $c_1 = c_2$:

   $c_2 - c_1 \;=\; \lim\limits_{n\to\infty} (b_n - a_n) \;=\; \lim\limits_{n\to\infty} \frac{b_0-a_0}{2^n} \;=\; 0 \quad \Longrightarrow \quad c := c_1 = c_2$

4. Use continuity of f:

   f is continuous on $[a, b]$, $c \in [a, b] \implies f$ is continuous at c

   f is continuous at c, $a_n \to c \implies f(c) = \lim_{n\to\infty} f(a_n)$

   f is continuous at c, $b_n \to c \implies f(c) = \lim_{n\to\infty} f(b_n)$

5. Show that $f(c) = 0$:

   If limits exist, one can pass to the limit in the inequality: $x_n \le y_n \implies \lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$

   $\forall\, n \in \mathbb{N} \quad f(a_n) \le 0 \implies f(c) = \lim_{n\to\infty} f(a_n) \le 0$

   $\forall\, n \in \mathbb{N} \quad f(b_n) \ge 0 \implies f(c) = \lim_{n\to\infty} f(b_n) \ge 0$

   $f(c) \le 0$ and $f(c) \ge 0 \implies f(c) = 0$

□

THEOREM 2. f is continuous on $[a, b]$, $f(a) = y_1$, $f(b) = y_2$, $y_1 \le y_2 \implies [y_1, y_2] \subset R(f)$

PROOF.

6. Use the definition of $R(f)$

   Take any $y_0 \in [y_1, y_2]$. We want to show: $\exists\, x_0 \in [a, b]\ \ f(x_0) = y_0$

7. Construct an auxiliary function $g : [a, b] \to \mathbb{R}$:

   Define $g(x) := f(x) - y_0$, $x \in [a, b] \implies g$ is continuous on $[a, b]$

   As $y_0 \in [y_1, y_2]$ we obtain $y_1 \le y_0 \le y_2$.

   Hence $g(a) \le 0$ and $g(b) \ge 0$.

8. Apply Theorem 1:

   $g : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, $g(a) \le 0$, $g(b) \ge 0 \implies \exists\, x_0 \in [a, b]\colon \quad g(x_0) = 0$

   $g(x_0) = 0 \iff f(x_0) = y_0$

□