Problem 1: Surface tension 2+2+3 Points

a) $\Delta W = \sigma(A_1 - A_2) = \sigma(4\pi r_1^2 \cdot 8000 - 4\pi r_2^2)$

It applies: $8000 \cdot V_1 = V_2 \quad \leftrightarrow \quad r_2 = r_1 \sqrt[3]{8000} = 0.1\ \mathrm{mm}\ \cdot 20 = 20\ \mathrm{mm}$

$$\Delta W = \sigma 4\pi \left(8000 r_1^2 - \left(r_1 \sqrt[3]{8000}\right)^2\right) = 4.44 \cdot 10^{-4}\mathrm{nm}$$

b) From script: $p = \frac{2\sigma}{r}$

$$p_1 = \frac{2 \cdot 0.465\,\frac{\mathrm{N}}{\mathrm{m}}}{0.1 \cdot 10^{-3}\mathrm{m}} = 9.3 \cdot 10^3\,\frac{\mathrm{N}}{\mathrm{m}^2}$$

$$p_2 = 465\,\frac{\mathrm{N}}{\mathrm{m}^2}$$

There was a typo in the exercise sheet which gave both the small and the one big bead the same radius (which makes no sense at all of course). Students that used the same radius should have the same value of $9.3\ \cdot 10^3\,\frac{N}{m^2}$ for both p-values and got full points for that!

c) The capillary depression:

$$h = \frac{2\sigma}{\rho g r}$$

so $$\Delta h = \frac{2\sigma}{\rho g}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 18\ \mathrm{cm}$$

Problem 2: Fluids under Gravity **$1 + 2 + 2$ Points**

Solution

(a)

Pressure at the left and right sides of the membrane creates the resulting force on the movable membrane. It stays still only when those resulting forces produce 0 as a sum. Thus, resulting integral of the pressure on the left side over the membrane area should be equal to the integral on the right side.

Since it was given in the problem that d is small comparing to heights of the liquids, one can neglect the difference of the pressure from the top to ground levels of the membrane height. Then, the resulting forces can be written:

$$\begin{cases} F_1 = S_{membrane} * \rho_1 * g * h_1 \\ F_2 = S_{membrane} * \rho_2 * g * h_2 \end{cases}$$

$$And\ F_1 = F_2$$

$$\downarrow$$

$$\frac{h_1}{h_2} = \frac{\rho_2}{\rho_1} = 0.789\ (1P)$$

(b)

After this action, the membrane is going to be displaced since F_1 is going to become bigger than F_2. At the still state forces should become equal again which would result in:

$$\frac{h'_1}{h'_2} = \frac{\rho_2}{\rho_1}$$

But the total amount of liquid is going to change by $\Delta V = (h_2 - h_1) \cdot \frac{\pi \cdot d^2}{4}$.

Note: here we assume, that the tube has a circular cross-section, which was not specified in the task (sorry about that). But the answer should be the same for circular and square tubes, as long as the area of the cross-section scales with square of tube diameters d and D.

This volume is going to result in increase in the levels of both liquids (at the end):

$$\Delta V = \Delta h_1 \cdot \pi \cdot d^2 + \Delta h_2 \cdot \pi \cdot D^2$$

Combining it with the height's ratio formula, will have:

$$\Delta V = h'_1 \cdot \pi \cdot d^2 - h_1 \cdot \pi \cdot d^2 + h'_2 \cdot \pi \cdot D^2 - h_2 \cdot \pi \cdot D^2 \ \rightarrow$$

$$\Delta V = h'_1 \cdot \pi \cdot d^2 - h_1 \cdot \pi \cdot d^2 + h'_1 \cdot \frac{\rho_1}{\rho_2} \cdot \pi \cdot D^2 - h_2 \cdot \pi \cdot D^2 \ \rightarrow$$

$$(h_2 - h_1) \cdot \pi \cdot d^2 = h'_1 \cdot \pi \cdot (d^2 + \frac{\rho_1}{\rho_2} D^2) - h_1 \cdot \pi \cdot d^2 - h_2 \cdot \pi \cdot D^2 \ \rightarrow$$

$$h_2 \cdot (d^2 + D^2) = h'_1 \cdot (d^2 + \frac{\rho_1}{\rho_2} D^2) \rightarrow$$

$$h'_1 = h_2 \frac{(d^2 + D^2)}{(d^2 + \frac{\rho_1}{\rho_2} D^2)} (1\mathrm{P})$$

This will be height of the water in relaxed state, when the forces and potential energies are balanced (we will use it in c).

Thus, the displacement of the top level of water, which is equal to the displacement of the membrane, is going to be

$$\Delta x = h_2 - h_1' = h_2 \frac{D^2 \left(\frac{\rho_1}{\rho_2} - 1\right)}{\left(d^2 + \frac{\rho_1}{\rho_2} D^2\right)} \; or \; h_2 \cdot \frac{\rho_1 - \rho_2}{\rho_2} \cdot \frac{D^2}{\left(d^2 + \frac{\rho_1}{\rho_2} D^2\right)}$$

$$or \;\; h_1 \cdot \rho_1 \cdot \frac{\rho_1 - \rho_2}{\rho_2^2} \cdot \frac{D^2}{\left(d^2 + \frac{\rho_1}{\rho_2} D^2\right)} \; (1\mathrm{P})$$

(c)

So, after the membrane is released, it will start to accelerate until the displacement of the membrane reaches the balance point Δx calculated in b), after this the membrane will be decelerating since the pressure of ethanol will become higher than from the water. Thus, the maximum speed of the membrane will be achieved in the balance point Δx, where all the potential energy of such oscillating system is minimized and transformed into kinetics energy.

Since mass of the liquids is much less than the mass of the membrane, kinetic energy will be mostly accumulated in the membrane, thus, we neglect the kinetic energy of the liquid.

Then, energy conservation law:

$$\Delta E_{pot} = M_m \frac{v_{membrane\,max}^2}{2} \rightarrow$$

And

$$\Delta E_{pot} = \Delta x \cdot \frac{\pi \cdot d^2}{4} \cdot \rho_1 \cdot g \cdot \left(h_2 - \frac{\Delta x}{2}\right) - \Delta x \cdot \frac{\pi \cdot d^2}{4} \cdot \rho_2 \cdot g \cdot \left(h_2 + \frac{\Delta x}{2} \frac{d^2}{D^2}\right) \; (1\mathrm{P})$$

The first term is from the segment of water which is going down, and the second negative term is going from the segment of ethanol which is lifted up.

Simplifying and leaving Δx as calculated value (otherwise too lengthily expression):

$$v_{membrane\,max} = \sqrt{\frac{\Delta x \cdot \pi \cdot d^2 \cdot g}{M_m} \left(h_2(\rho_1 - \rho_2) - \frac{\Delta x}{2} \left(\rho_1 + \rho_2 \frac{d^2}{D^2}\right)\right)}$$

$$\downarrow$$

$$v_{membrane\,max} = \sqrt{\frac{\Delta x \cdot \pi \cdot d^2 \cdot g \cdot h_2 \cdot (\rho_1 - \rho_2)}{2 \cdot M_m}} = \sqrt{\frac{\pi \cdot d^2 \cdot g \cdot h_2^2 \cdot (\rho_1 - \rho_2)^2}{2 \cdot M_m \cdot \left(\rho_2 \cdot \frac{d^2}{D^2} + \rho_1\right)}} =$$

$$= d \cdot D \cdot h_2 \cdot (\rho_1 - \rho_2) \cdot \sqrt{\frac{\pi \cdot g}{2 \cdot M_m \cdot (\rho_2 \cdot d^2 + \rho_1 \cdot D^2)}} \text{ (1P)}$$

Problem 3. **Balloon with hydrogen gas 2+2+2 Points**

a) The resulting force is given by the difference of gravity force and buoyancy:

$$F = F_B - F_g = (\rho_{0,O} - \rho_{0,H})gV - m_B g$$

$$= \left((1{,}29 - 0{,}09)\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot \frac{4\pi}{3}(1{,}5\mathrm{m})^3 - 2\mathrm{kg}\right) 9{,}81\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

$$= 146{,}8\mathrm{N}$$

b) At maximum height we have equilibrium of forces:

$$0 = F_B - F_G \rightarrow F_B = F_G$$

To calculate density in dependence of height h we use barometric formular

$$p = p_0 e^{-\frac{\rho_{0,O} g h}{p_0}}$$

Since we assume ideal gasses, we have $\rho \sim p$

$$F_B = F_G$$

$$(\rho_{0,O} - \rho_{0,H})gV = m_B g$$

$$V(\rho_{0,O} - \rho_{0,H}) \cdot exp\left(-\frac{\rho_{0,O} g h}{p_0}\right) = m_B$$

$$h = \frac{p_0}{\rho_{0,O} g} log\left(\frac{V(\rho_{0,O} - \rho_{0,H})}{m_B}\right) =$$

$$= 16.9 \text{ km}$$

c) We only have to substitute $\rho_{0,He}$ for $\rho_{0,H}$ in the previous formular and get:

$$h = 16.3 \text{ km}$$

A lot of students took the ideal-gas-hint as a suggestion to express p in terms of the ideal gas law and got a maximum height that was a few kilometres lower than the one given in the solution. If the solution was coherent and expressed comprehensibly, they still got points.

Solution 4: friction on ice

$$a = g \cdot \mu = 9.81\,\frac{\mathrm{N}}{\mathrm{kg}} \cdot 0.0012 = 0.117\,\frac{\mathrm{N}}{\mathrm{kg}}$$

$$t = \frac{v}{a} = \frac{50.76\,\frac{\mathrm{km}}{\mathrm{h}}}{0.117\,\frac{\mathrm{N}}{\mathrm{kg}}} = \frac{14.1\,\frac{\mathrm{m}}{\mathrm{s}}}{0.117\,\frac{\mathrm{N}}{\mathrm{kg}}} = 120.5s$$

$$s = \tfrac{1}{2}at^2 = \tfrac{1}{2}0.117\,\tfrac{\mathrm{N}}{\mathrm{kg}} \cdot (120.5)^2 = 849\ m$$