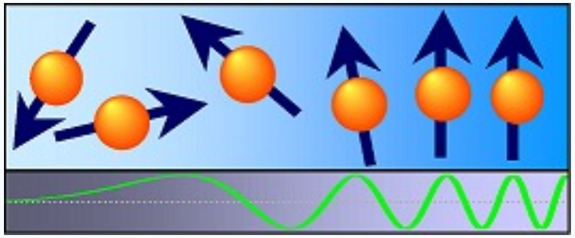
Experimental Physics
EP1 MECHANICS

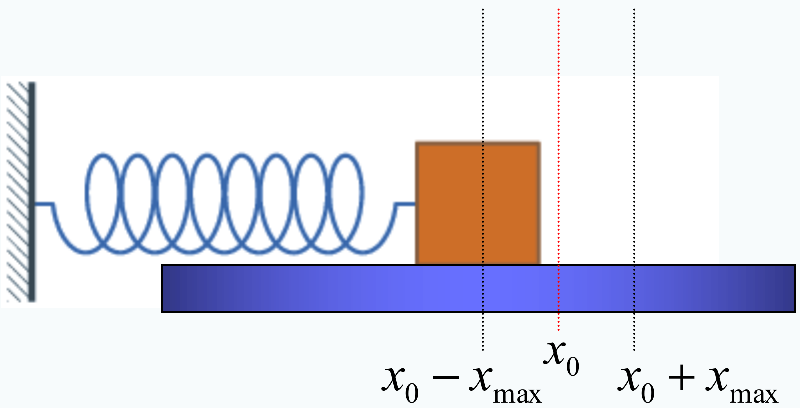
– Oscillations –



Rustem Valiullin

https://www.physgeo.uni-leipzig.de/en/fbi/applied-magnetic-resonance

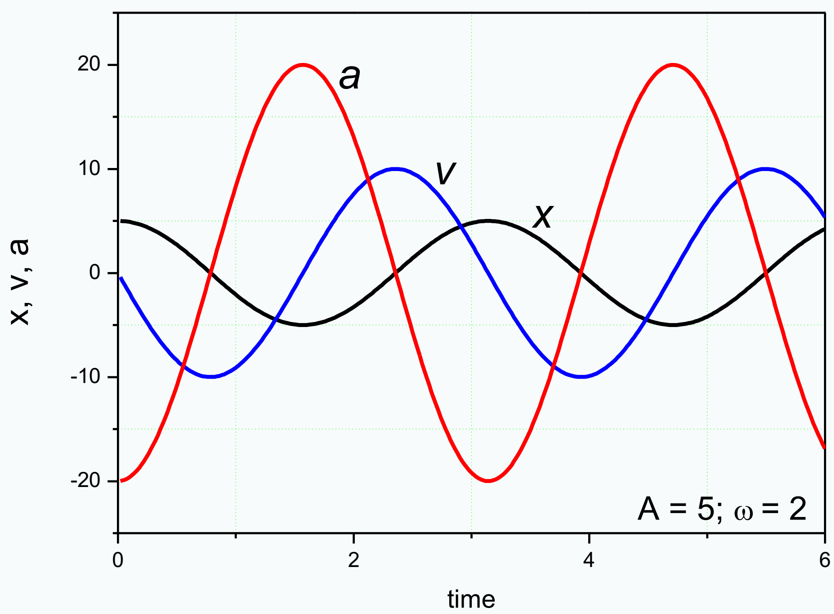
Simple harmonic motion



$$F_x = -kx = ma = m\frac{d^2x}{dt^2}$$

$$\underbrace{a}_{} = \frac{d^2x}{dt^2} = \underbrace{-\left(\frac{k}{m}\right)}_{} \underbrace{x}_{}$$

conditions for simple harmonic motion



angular frequency

phase

$$x = A\cos(\overbrace{\omega t + \varphi}^{\text{phase}})$$

amplitude

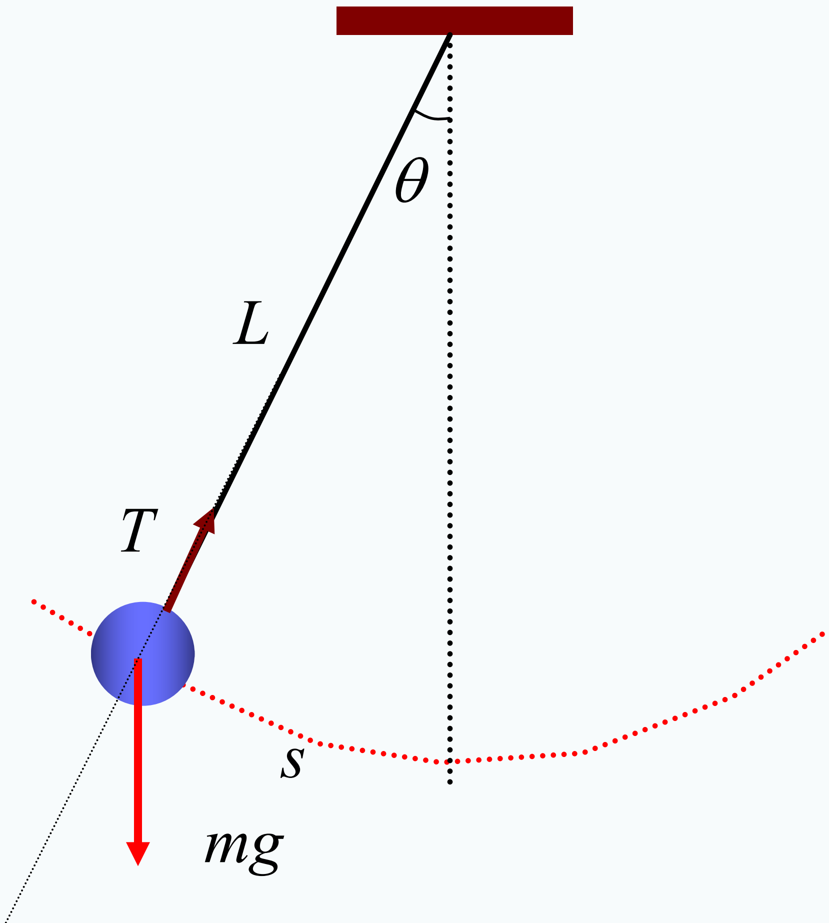
phase constant

$$\omega(t+T)+\varphi = 2\pi + \omega t + \varphi \Rightarrow \omega = 2\pi/T$$

$$v = -A\omega\sin(\omega t + \varphi)$$

$$a = -A\omega^2\cos(\omega t + \varphi) = -\omega^2 x$$

A simple pendulum



$$F_{\mathrm{t}} = -mg\sin\theta = m\frac{d^2 s}{dt^2}$$

$$s = L\theta \Rightarrow \theta = s/L$$

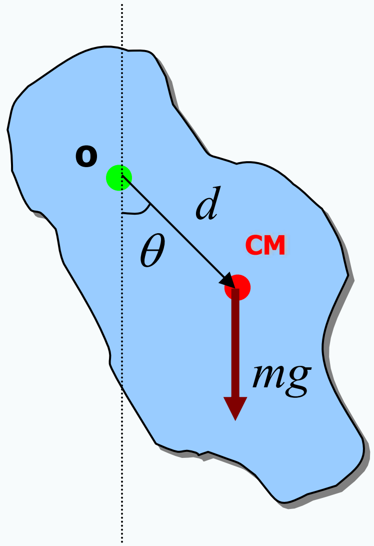
$$\frac{d^2 s}{dt^2} = -\left(\frac{g}{L}\right)s \qquad = -\omega^2 s$$

$$s = s_0\cos(\omega t + \varphi)$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

In the same way (!): $$\theta = \theta_0\cos(\omega t + \varphi)$$

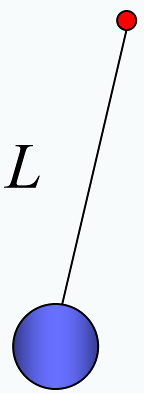
A physical pendulum



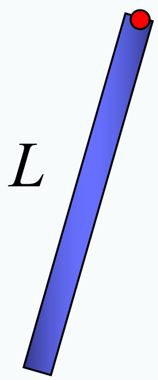
$$F = ma \quad \Rightarrow \quad \tau = I\alpha = I\frac{d^2\theta}{dt^2}$$

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2} \qquad \frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta$$

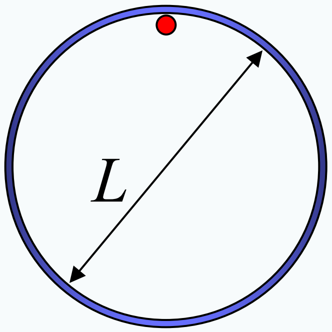
$$T = 2\pi\sqrt{I/mgd}$$



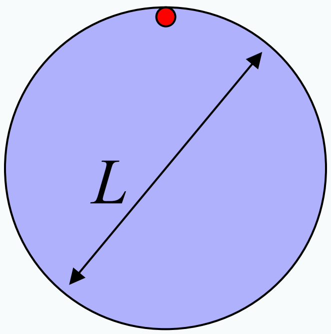
$$T = 2\pi\sqrt{L/g}$$



$$T = 2\pi\sqrt{\tfrac{2}{3}(L/g)}$$

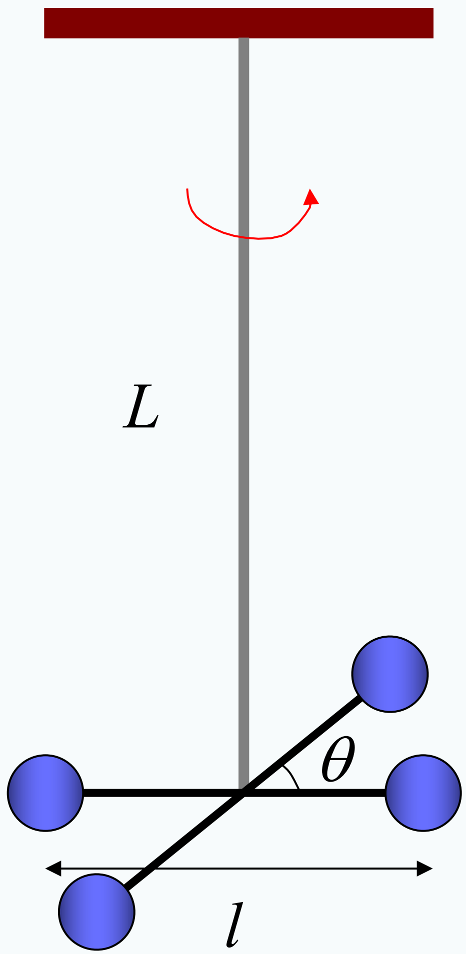


$$T = 2\pi\sqrt{L/g}$$



$$T = 2\pi\sqrt{\tfrac{3}{4}(L/g)}$$

A torsional pendulum



$$\tau = -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

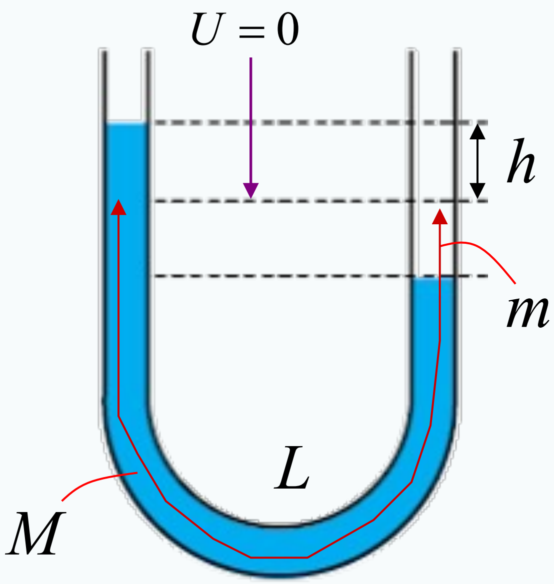
torsional constant (~Y; ~A; ~L^{-1})

$$\theta = \theta_{\max}\cos(\omega t + \varphi)$$

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \qquad I = 2M\frac{l^2}{4} = \frac{1}{2}Ml^2$$

This equation provides a means to determine the torsional constant.

A water pendulum



$$E = E_k + U = const$$

$$E = \frac{1}{2}Mv^2 + mgh = const$$

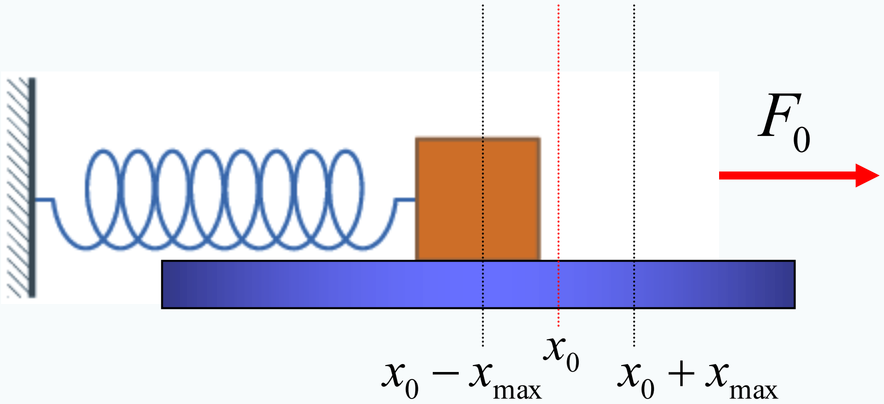
$$E = \frac{1}{2}\rho AL\left(\frac{dh}{dt}\right)^2 + \rho Ahgh = const$$

$$\frac{dE}{dt} = L\frac{dh}{dt}\left(\frac{d^2h}{dt^2}\right) + 2hg\frac{dh}{dt} = 0$$

$$\boxed{\frac{d^2h}{dt^2} = -\frac{2g}{L}h}$$

$$T = 2\pi\sqrt{\frac{L}{2g}}$$

Oscillations under force action

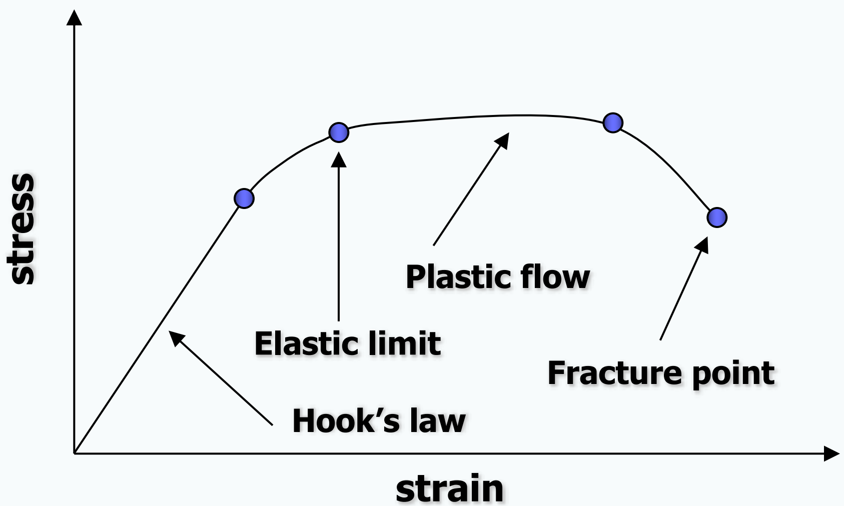


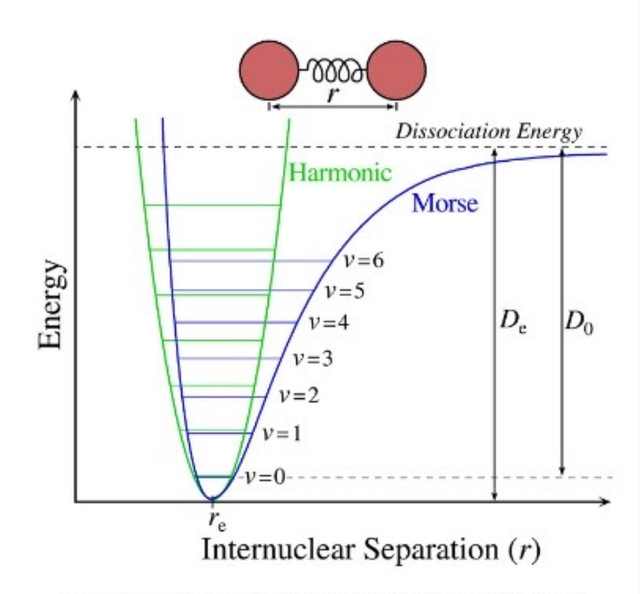
$$m\frac{d^2x}{dt^2} = -kx \qquad \omega_0^2 \equiv \frac{k}{m}$$

$$m\frac{d^2x}{dt^2} = -kx + F_0$$

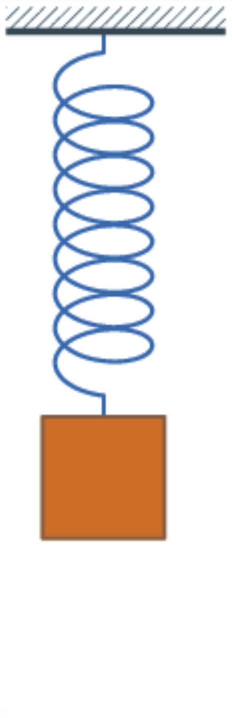
amplitude and frequency are independent $\quad x = B + A\cos(\omega_0 t) \qquad B = F_0 / k$

Anharmonicity





A vertical spring



***y* = 0 – spring equilibrium without mass; *y*_eq – with mass**

$$F_{\mathrm{y}} = -k(-y_{eq}) - mg = 0 \qquad y_{eq} = mg/k$$

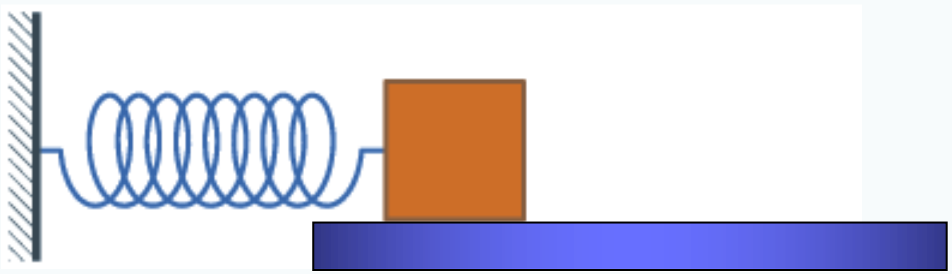
$$F_{\mathrm{y}} = -ky - mg = m\frac{d^2y}{dt^2} \qquad y = y' - y_{eq}$$

$$-ky' + k(mg/k) - mg = m\frac{d^2y'}{dt^2}$$

$$-\left(\frac{k}{m}\right)y' = \frac{d^2y'}{dt^2} \qquad y = A\cos(\omega t + \varphi) - y_{eq}$$

If displacements are measured from the equilibrium position, the effect of gravity is excluded.

Damped oscillations: Amplitude



$$m\frac{d^2x}{dt^2} = -kx - bv$$

$$x = A(t)\cos(\omega t)$$

Substituting this general solution to the equation above one gets:

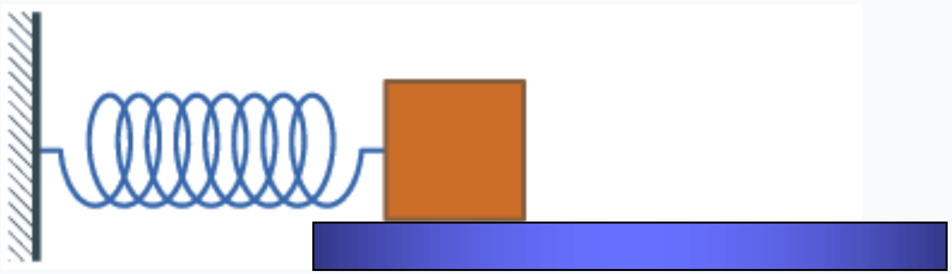
$$\omega_0^2 \equiv \frac{k}{m}$$

$$A\cdot\left(\omega_0^2 - \omega^2\right) - \frac{b}{m}A\omega\tan(\omega t) + \frac{dA}{dt}\cdot\left(\frac{b}{m} - 2\omega\tan(\omega t)\right) + \frac{d^2A}{dt^2} = 0$$

***t* = 0:** $$A\left(\omega_0^2 - \omega^2\right) + \frac{b}{m}\frac{dA}{dt} + \frac{d^2A}{dt^2} = 0 \quad \Rightarrow \quad -\frac{b}{m}A\omega - 2\omega\frac{dA}{dt} = 0$$

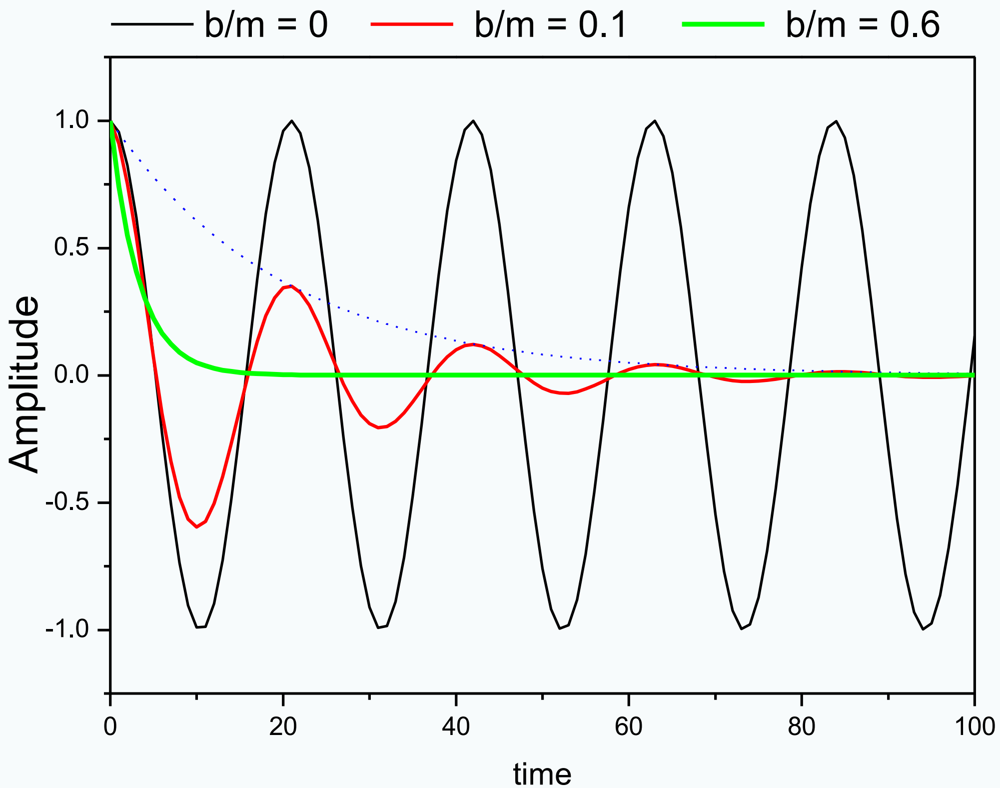
$$\frac{dA}{dt} = -\frac{b}{2m}A \qquad \ln A = -\frac{b}{2m}t + C \qquad t = 0,\ A = A_0 \Rightarrow A = A_0 e^{-\frac{b}{2m}t}$$

Damped oscillations: Frequency



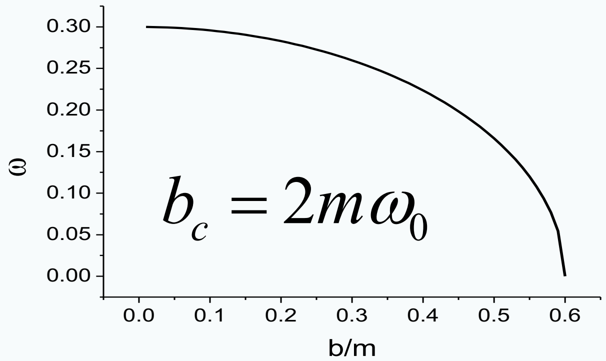
$$A\left(\omega_0^2-\omega^2\right)+\frac{b}{m}\frac{dA}{dt}+\frac{d^2A}{dt^2}=0$$

$$A=A_0\exp\left\{-\frac{b}{2m}t\right\}$$



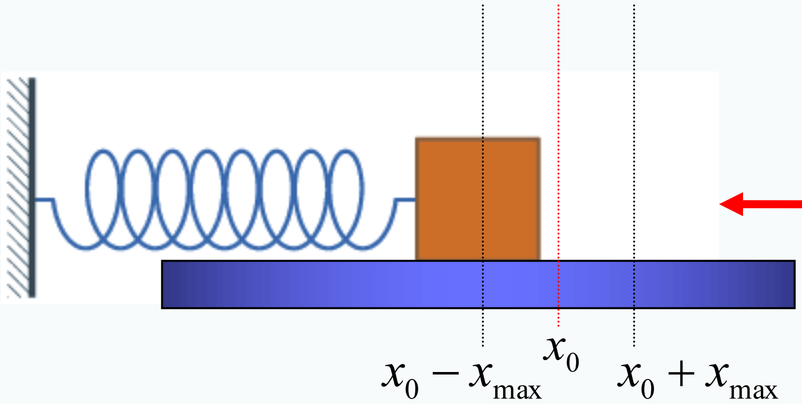
$$\left(\omega_0^2-\omega^2\right)+\frac{b}{m}\left(-\frac{b}{2m}\right)+\left(-\frac{b}{2m}\right)^2=0$$

$$\Rightarrow \omega=\omega_0\sqrt{1-\left(\frac{b}{2m\omega_0}\right)^2}$$



$$b_c=2m\omega_0$$

Driven oscillations



$$F_0 \cos(\omega_d t)$$

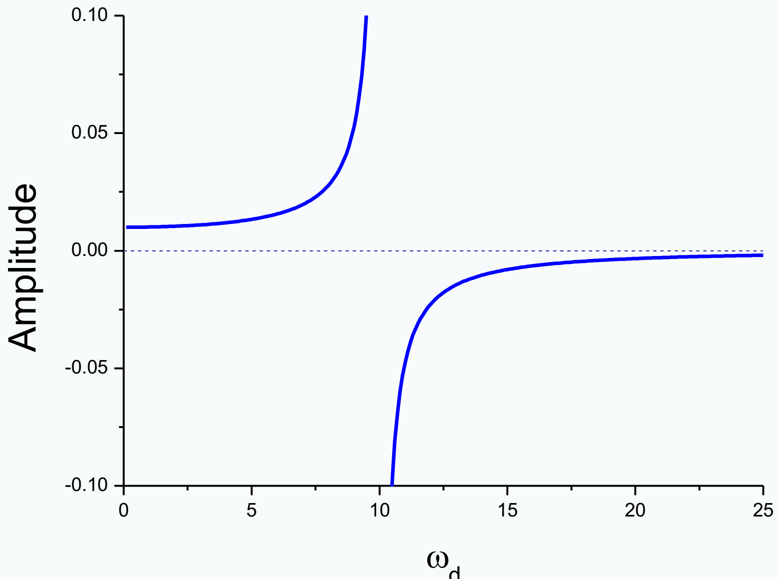
$$\omega_0^2 \equiv \frac{k}{m}$$

$$m\frac{d^2x}{dt^2} = -kx + F_0\cos(\omega_d t)$$

$$x = A\cos(\omega t)$$

$$A\cos(\omega t)\left(\omega_0^2 - \omega^2\right) = \left(F_0/m\right)\cos(\omega_d t)$$

$$\boxed{A = \frac{F_0/m}{\omega_0^2 - \omega^2} \qquad \omega = \omega_d}$$

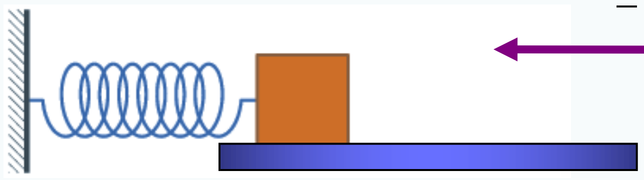


$$\begin{cases} \omega_d << \omega_0;\ A = \dfrac{F_0}{m\omega_0^2} \\ \omega_d = \omega_0;\ A \to \infty \\ \omega_d >> \omega_0;\ A = \dfrac{F_0}{m\omega_d^2} \end{cases}$$

resonance

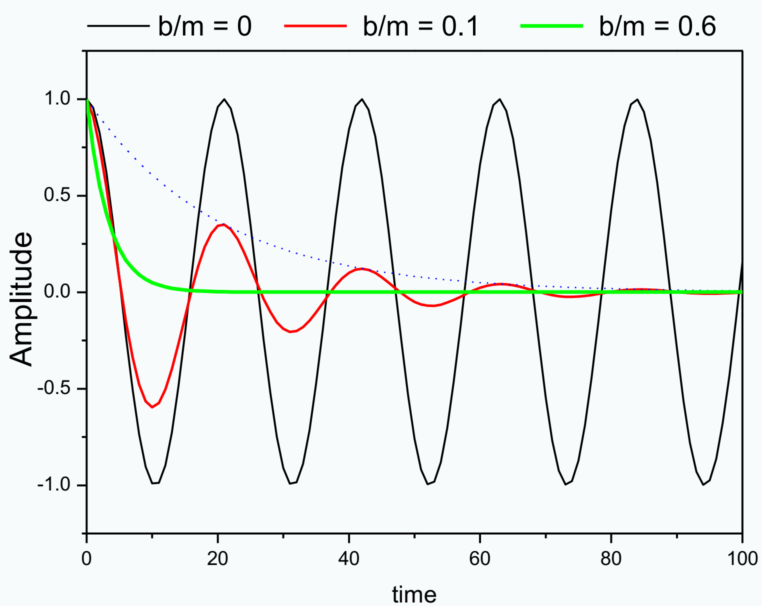
Damped oscillations

$-bv$



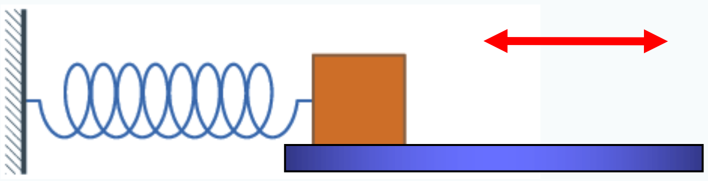
$$\boxed{A = A_0 e^{-\frac{b}{2m}t}}$$

$$\boxed{\omega = \omega_0\sqrt{1-\left(\frac{b}{2m\omega_0}\right)^2}}$$



Driven oscillations

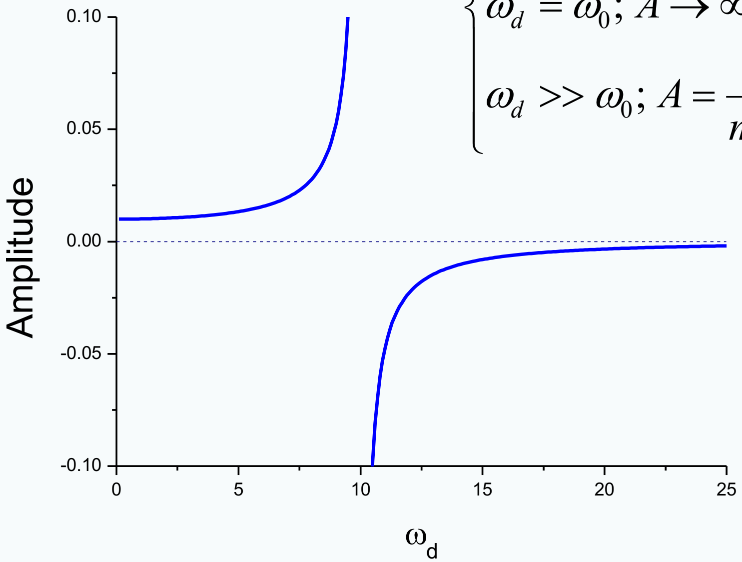
$F_0\cos(\omega_d t)$



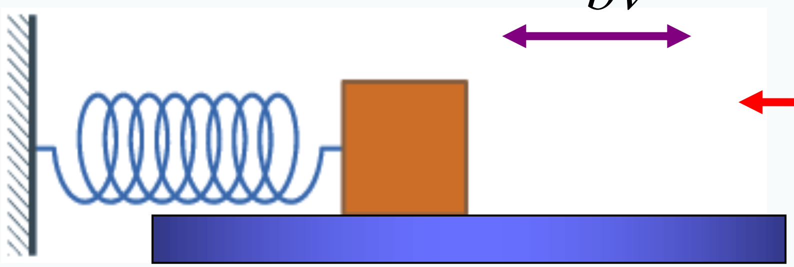
$$\boxed{A = \frac{F_0/m}{\omega_0^2-\omega^2}}$$

$$\boxed{\omega = \omega_d}$$

$$\begin{cases} \omega_d \ll \omega_0;\ A = \dfrac{F_0}{m\omega_0^2} \\ \omega_d = \omega_0;\ A \to \infty \\ \omega_d \gg \omega_0;\ A = \dfrac{F_0}{m\omega_d^2} \end{cases}$$



General case: damping and driving forces



$$\omega_0^2 \equiv \frac{k}{m}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega_d t)$$

$$\boxed{x = A(t)\cos(\omega_d t - \delta)}$$

Via initial conditions:

$$A\left(\omega_0^2 - \omega^2\right) + \frac{b}{m}\frac{dA}{dt} + \frac{d^2A}{dt^2} - \frac{F_0}{m} = 0$$

Therefore, we get similar transient behavior:

$$A = A_0 e^{-\frac{b}{2m}t}$$

$$\omega^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2 - \frac{F_0}{mA_0}\exp\left\{\frac{b}{2m}t\right\}$$

We are looking now for the steady-state solution:

$$\frac{dA}{dt} = 0 \quad \Rightarrow \quad A = const$$

Damping and driving: steady-state solution

$$\boxed{x = A\cos(\omega_d t - \delta)}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega_d t)$$

$$\frac{dx}{dt} = -A\omega_d \sin(\omega_d t - \delta)$$

$$\frac{d^2x}{dt^2} = -A\omega_d^2 \cos(\omega_d t - \delta)$$

$$A(\omega_0^2 - \omega_d^2)\cos(\omega_d t - \delta) - A\frac{b}{m}\omega_d \sin(\omega_d t - \delta) = \frac{F_0}{m}\cos(\omega_d t)$$

The solution for A must be valid for any instants (long enough) of time! Let us choose $\omega_d t = \pi/2$ and 2π:

$$\boxed{A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega_d^2)^2 + b^2\omega_d^2}}}$$

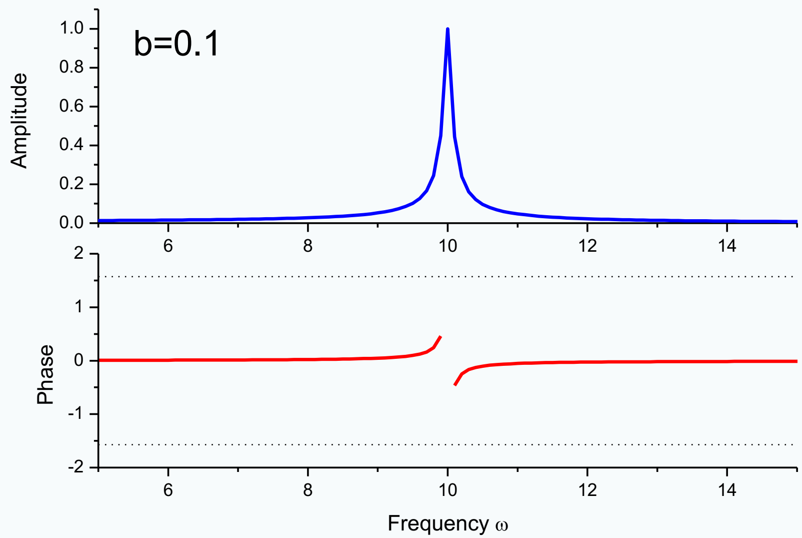
$$\boxed{\tan\delta = \frac{b\omega_d}{m(\omega_0^2 - \omega_d^2)}}$$

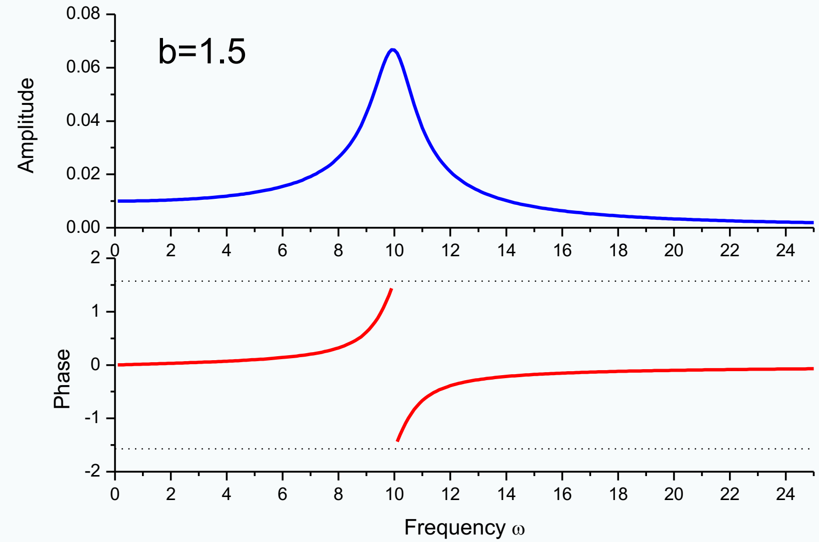
$$Am(\omega_0^2 - \omega_d^2)\sin(\delta) - Ab\omega_d \cos(\delta) = 0$$

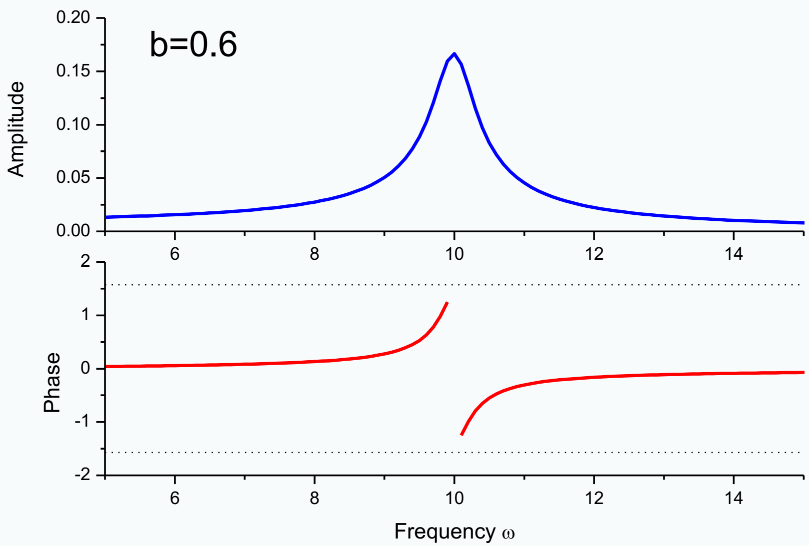
$$Am(\omega_0^2 - \omega_d^2)\cos(\delta) + Ab\omega_d \sin(\delta) = F_0$$

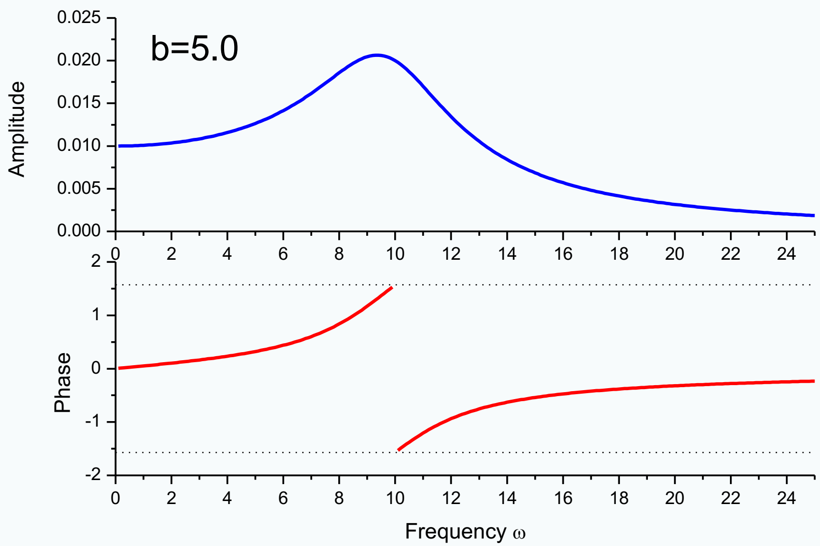
$$\begin{cases} \cos(\frac{\pi}{2} + \delta) = -\sin(\delta) \\ \sin(\frac{\pi}{2} + \delta) = \cos(\delta) \end{cases}$$

Some examples (F_0=1, ω_0=10, m=1)

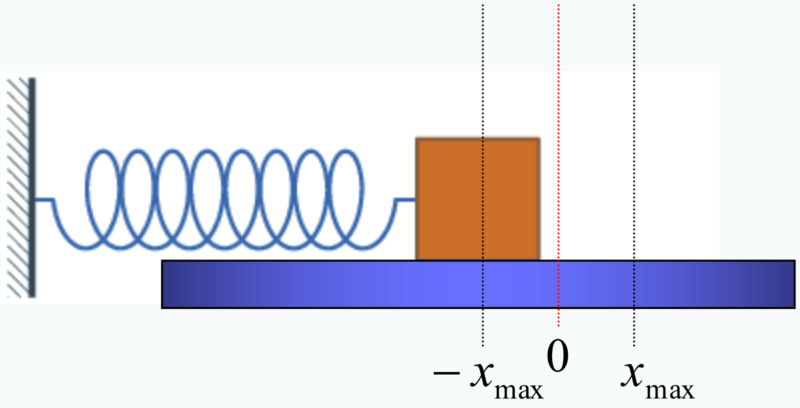








Energy of a simple harmonic oscillator



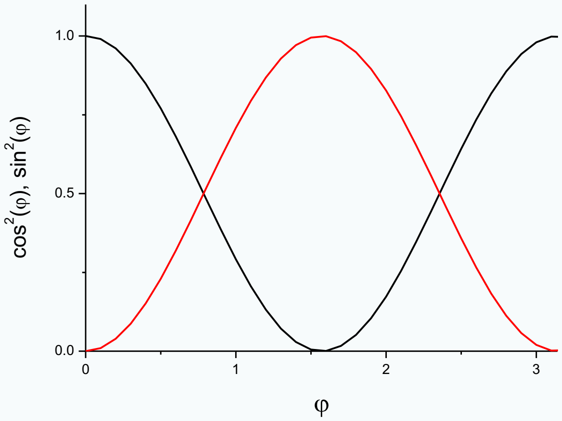
$$E = E_k + U = const$$

$$U = \tfrac{1}{2} kx^2 = \tfrac{1}{2} kx_{max}^2 \cos^2(\omega_0 t)$$

$$E_k = \tfrac{1}{2} mv^2 = \tfrac{1}{2} mx_{max}^2 \omega_0^2 \sin^2(\omega_0 t)$$

We look for the average values of U and E_k over one period:

$$\overline{U} = \tfrac{1}{2} kx_{max}^2 \overline{\cos^2(\omega_0 t)} \qquad \overline{E_k} = \tfrac{1}{2} mx_{max}^2 \omega_0^2 \overline{\sin^2(\omega_0 t)}$$



$$\overline{\cos^2(\omega_0 t)} = 1/2$$

$$\overline{\sin^2(\omega_0 t)} = 1/2$$

$$\overline{U} = \tfrac{1}{2}\left(\tfrac{1}{2} kx_{max}^2\right)$$

$$\overline{E_k} = \tfrac{1}{2}\left(\tfrac{1}{2} mx_{max}^2 \omega_0^2\right)$$

$$\overline{E_k} = \overline{U}$$

$$\boxed{E = m\overline{v^2} = k\overline{x^2}}$$

Energy dissipation in a weakly damped oscillator

$$\Delta E = W$$

- energy dissipation is equal to work done against damping force.

$$W = \int F dx \qquad x = vt \Rightarrow dx = vdt + tdv \qquad F = -bv$$

$$W = -b\int v^2 dt - b\int vtdv \xrightarrow{\text{one period}} W = -b\int_0^T v^2 dt = -b\overline{v^2}T$$

$$\Delta E_T = -b\overline{v^2}T = -\frac{b}{m}ET \qquad \Rightarrow \qquad \frac{\Delta E_T}{E} = -\frac{b}{m}T$$

Quality factor:

$$Q \equiv 2\pi\frac{E}{|\Delta E_T|} = 2\pi\frac{m}{bT} \qquad Q \to \infty$$

with decreasing damping constant and with increasing mass

For weak damping:

$$E = E_0 \exp\left\{-\frac{b}{m}t\right\} = E_0 \exp\left\{-\frac{t}{\tau}\right\} \qquad \tau = \frac{m}{b}$$

To remember!

- **If acceleration of an object is proportional to its negative displacement, then the object performs harmonic oscillations.**

- **In this case, the amplitude and angular frequency are independent. Anharmonicity makes them dependent.**

- **Damping force causes the amplitude to decay and also changes the frequency.**

- **Driving force affects the amplitude, making it dependent on the frequency of the applied driving force.**

- **Upon passing the resonance frequency, the phase constant changes by π.**