Mathematics 1, FINAL EXAM Solutions
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1. **(5 points)** Find the oblique asymptotes of the following function as $x \to \pm\infty$

$$f(x) \;=\; x^2\left(\sqrt{x^2+5}-\sqrt{x^2+1}\right)$$

SOLUTION. Compute the slopes

$$k_1 \;=\; \lim_{x\to+\infty}\frac{f(x)}{x}, \qquad k_2 \;=\; \lim_{x\to-\infty}\frac{f(x)}{x}$$

We have

$$\frac{f(x)}{x} \;=\; x\left(\sqrt{x^2+5}-\sqrt{x^2+1}\right) \;=\; \frac{x(\sqrt{x^2+5}-\sqrt{x^2+1})(\sqrt{x^2+5}+\sqrt{x^2+1})}{\sqrt{x^2+5}+\sqrt{x^2+1}} \;=$$

$$= \;\left[\;(a-b)(a+b)=a^2-b^2\;\right] \;=\; \frac{4x}{\sqrt{x^2+5}+\sqrt{x^2+1}} \;=\; \frac{4\,\frac{x}{|x|}}{\sqrt{1+\frac{5}{x^2}}+\sqrt{1+\frac{1}{x^2}}}$$

Hence

$$k_1 \;=\; \lim_{x\to+\infty}\frac{f(x)}{x} \;=\; \frac{4}{1+1} \;=\; 2$$

$$k_2 \;=\; \lim_{x\to-\infty}\frac{f(x)}{x} \;=\; \frac{-4}{1+1} \;=\; -2$$

Let us compute the rise coefficients:

$$b_1 = \lim_{x\to+\infty}\left(f(x)-k_1x\right), \qquad b_2 = \lim_{x\to-\infty}\left(f(x)-k_2x\right)$$

We have

$$f(x)-k_1x \;=\; \frac{4x^2}{\sqrt{x^2+5}+\sqrt{x^2+1}}-2x \;=\; 2x\cdot\frac{2x-\sqrt{x^2+5}-\sqrt{x^2+1}}{\sqrt{x^2+5}+\sqrt{x^2+1}} \;=$$

$$= \;2x\cdot\frac{x-\sqrt{x^2+5}}{\sqrt{x^2+5}+\sqrt{x^2+1}} \;+\; 2x\cdot\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+5}+\sqrt{x^2+1}} \;=$$

$$= \;2x\cdot\frac{-5}{(\sqrt{x^2+5}+\sqrt{x^2+1})(x+\sqrt{x^2+5})} \;+\; 2x\cdot\frac{-1}{(\sqrt{x^2+5}+\sqrt{x^2+1})(x+\sqrt{x^2+1})}$$

Hence

$$b_1 = \lim_{x\to+\infty}\left(f(x)-k_1x\right) \;=$$

$$= \;\lim_{x\to+\infty}\frac{2}{x}\cdot\frac{-5}{\left(\sqrt{1+\frac{5}{x^2}}+\sqrt{1+\frac{1}{x^2}}\right)\left(1+\sqrt{1+\frac{5}{x^2}}\right)} \;+$$

$$+ \;\lim_{x\to+\infty}\frac{2}{x}\cdot\frac{-1}{\left(\sqrt{1+\frac{5}{x^2}}+\sqrt{1+\frac{1}{x^2}}\right)\left(1+\sqrt{1+\frac{1}{x^2}}\right)} \;=$$

$$= \;0 \;+\; 0 \;=\; 0$$

Similarly,

$$f(x) - k_2 x \;=\; \frac{4x^2}{\sqrt{x^2+5}+\sqrt{x^2+1}} + 2x \;=\; 2x \cdot \frac{2x + \sqrt{x^2+5} + \sqrt{x^2+1}}{\sqrt{x^2+5}+\sqrt{x^2+1}} \;=$$

$$=\; 2x \cdot \frac{x + \sqrt{x^2+5}}{\sqrt{x^2+5}+\sqrt{x^2+1}} \;+\; 2x \cdot \frac{x + \sqrt{x^2+1}}{\sqrt{x^2+5}+\sqrt{x^2+1}} \;=$$

$$=\; 2x \cdot \frac{5}{(\sqrt{x^2+5}+\sqrt{x^2+1})(\sqrt{x^2+5}-x)} \;+\; 2x \cdot \frac{1}{(\sqrt{x^2+5}+\sqrt{x^2+1})(\sqrt{x^2+1}-x)}$$

Hence

$$b_2 = \lim_{x \to -\infty} (f(x) - k_2 x) \;=\; 0 \;+\; 0 \;=\; 0$$

ANSWER.

- $y = \;\; 2x$ is the oblique asymptote of $f(x)$ as $x \to +\infty$
- $y = -2x$ is the oblique asymptote of $f(x)$ as $x \to -\infty$

2. **(5 points)** Find all inflection points of the following function

$$f(x) \;=\; x \,\ln(1+x^2)$$

SOLUTION. We know that if $c \in \mathbb{R}$ is an inflection point of f then $f''(c) = 0$. Find $f'(x)$:

$$f'(x) \;=\; \ln(1+x^2) \;+\; x \cdot \frac{2x}{1+x^2} \;=\; \ln(1+x^2) \;+\; 2 \;-\; \frac{2}{1+x^2}$$

Find $f''(x)$:

$$f''(x) \;=\; \frac{2x}{1+x^2} \;+\; \frac{2}{(1+x^2)^2} \cdot 2x \;=\; \frac{2x(1+x^2+2)}{(1+x^2)^2} \;=\; \frac{2x(3+x^2)}{(1+x^2)^2}$$

Solve $f''(x) = 0$

$$\frac{2x(3+x^2)}{(1+x^2)^2} \;=\; 0 \qquad \Longleftrightarrow \qquad x = 0$$

Check that f change its convexity at $x = 0$

$$x \in (-\infty, 0) \quad \Longrightarrow \quad f''(x) \;<\; 0 \quad \Longrightarrow \quad f \text{ is concave on } (-\infty, 0)$$

$$x \in (0, +\infty) \quad \Longrightarrow \quad f''(x) \;>\; 0 \quad \Longrightarrow \quad f \text{ is convex on } (-\infty, 0)$$

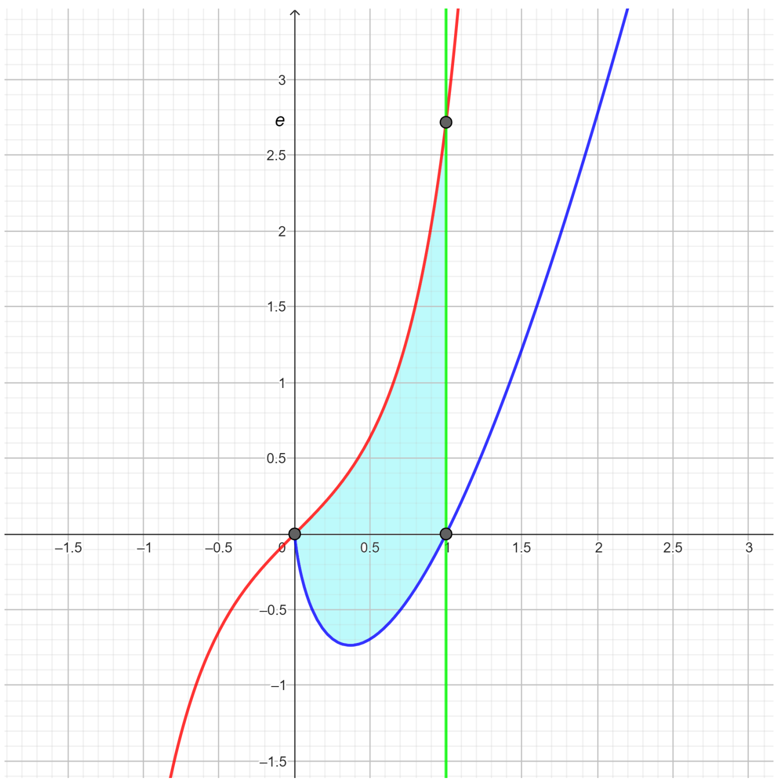
Hence $x = 0$ is an inflection point of $f(x)$.

ANSWER. $x = 0$ is the only inflection point of $f(x)$.

3. **(5 points)** Find the area of a plane figure bounded by the following three lines given in Cartesian coordinates ($0 < x < 1$):

$$y \,=\, x\, e^{x^2}, \qquad y \,=\, 2x \ln x, \qquad x \,=\, 1$$

SOLUTION.



Denote

$$f(x) \,=\, x\, e^{x^2}, \qquad g(x) \,=\, 2x \ln x$$

Then

$$S \,=\, \int_0^1 \Big(f(x) - g(x)\Big)\, dx \,=\, \int_0^1 x\, e^{x^2}\, dx \,-\, \int_0^1 2x \ln x\, dx$$

Compute antiderivative of $f(x)$

$$\int x\, e^{x^2}\, dx \,=\, \left[\begin{array}{c} y = x^2 \\ dy = 2x dx \end{array}\right] \,=\, \frac{1}{2} \int e^y\, dy \,=\, \frac{1}{2}\, e^y + c \,=\, \frac{1}{2}\, e^{x^2} + c$$

Compute antiderivative of $g(x)$

$$\int 2x\, \ln x\, dx \,=\, \int \ln x\, d(x^2) \,=\, \text{[by parts]} \,=\, x^2\, \ln x \,-\, \int x^2\, d(\ln x) \,=$$

$$=\, x^2\, \ln x \,-\, \int x^2 \cdot \frac{1}{x}\, dx \,=\, x^2\, \ln x \,-\, \int x\, dx \,=\, x^2\, \ln x \,-\, \frac{x^2}{2} + c$$

Take

$$F(x) = \frac{1}{2}\, e^{x^2}, \qquad G(x) = x^2 \ln x - \frac{x^2}{2}$$

By the Newton-Leibniz formula we obtain

$$\int_0^1 x\, e^{x^2}\, dx = F(1) - F(0) = \frac{e^1}{2} - \frac{e^0}{2} = \frac{e}{2} - \frac{1}{2}$$

Note that

$$\lim_{x \to +0} \Big(x^2 \ln x \Big) = 0$$

Indeed,

$$\lim_{x \to +0} \Big(x^2 \ln x \Big) = \left[\begin{array}{c} y = \frac{1}{x} \\ x \to +0 \;\Leftrightarrow\; y \to +\infty \end{array} \right] = -\lim_{y \to +\infty} \frac{\ln y}{y^2} =$$

$$= [\text{L'Hopital}] = -\lim_{y \to +\infty} \frac{(\ln y)'}{(y^2)'} = -\lim_{y \to +\infty} \frac{\frac{1}{y}}{2y} = -\lim_{y \to +\infty} \frac{1}{2y^2} = 0$$

Hence the improper integral

$$\int_0^1 2x \ln x\, dx \quad \text{is convergent}$$

and

$$\int_0^1 2x \ln x\, dx = \lim_{c \to +0} \int_c^1 2x \ln x\, dx = G(1) - \underbrace{\lim_{c \to +0} G(c)}_{=0} = 1^2 \cdot \underbrace{\ln 1}_{=0} - \frac{1^2}{2} = -\frac{1}{2}$$

Hence

$$S = \left(\frac{e}{2} - \frac{1}{2} \right) - \left(-\frac{1}{2} \right) = \frac{e}{2}$$

Answer. $S = \dfrac{e}{2}$

4. **(5 points)** Find all values of the parameter $\alpha \in \mathbb{R}$ such that the following improper integral is convergent. Explain your answer. (Here $\operatorname{arctg} x := \arctan x$ is the arctangent of x).

$$\int_0^1 \frac{dx}{(\operatorname{arctg} x)^\alpha}$$

Solution. Denote

$$f(x) = \operatorname{arctg} x$$

By the Taylor formula we have

$$f(x) \;=\; f(0) \;+\; \frac{f'(0)}{1!}\,x \;+\; o(x), \qquad \text{as} \qquad x \to 0$$

and hence

$$\operatorname{arctg} x \;=\; x + o(x), \qquad \text{as} \qquad x \to 0$$

Note that

$$\lim_{x\to 0} \frac{o(x)}{x} \;=\; 0 \qquad \exists\, \delta > 0: \quad \forall\, x \in (0,\delta) \qquad \left|\frac{o(x)}{x}\right| < \frac{1}{2} \quad \Longrightarrow \quad -\frac{x}{2} \;<\; o(x) \;<\; \frac{x}{2}$$

Hence

$$\frac{x}{2} \;<\; \operatorname{arctg} x \;<\; \frac{3x}{2}, \qquad \forall\, x \in (0,\delta)$$

Note that

$$\int_0^1 \frac{dx}{(\operatorname{arctg} x)^\alpha} \;=\; \int_0^\delta \frac{dx}{(\operatorname{arctg} x)^\alpha} \;+\; \underbrace{\int_\delta^1 \frac{dx}{(\operatorname{arctg} x)^\alpha}}_{\text{is convergent}}$$

The second integral is convergent for all $\alpha \in \mathbb{R}$ as its integrand is continuous on $[\delta, 1]$. For the first integral we can use the comparison test:

$$\frac{2^\alpha}{3^\alpha x^\alpha} \;\le\; \frac{1}{(\operatorname{arctg} x)^\alpha} \;\le\; \frac{2^\alpha}{x^\alpha}, \qquad \forall\, x \in (0,\delta)$$

$$\alpha \ge 1 \quad \Longrightarrow \quad \int_0^\delta \frac{dx}{x^\alpha} \;\; \text{is divergent} \quad \Longrightarrow \quad \int_0^\delta \frac{dx}{(\operatorname{arctg} x)^\alpha} \;\; \text{is divergent}$$

$$\alpha < 1 \quad \Longrightarrow \quad \int_0^\delta \frac{dx}{x^\alpha} \;\; \text{is convergent} \quad \Longrightarrow \quad \int_0^\delta \frac{dx}{(\operatorname{arctg} x)^\alpha} \;\; \text{is convergent}$$

ANSWER.

- $\alpha \ge 1 \quad \Longrightarrow \quad \int_0^1 \frac{dx}{(\operatorname{arctg} x)^\alpha}$ is divergent
- $\alpha < 1 \quad \Longrightarrow \quad \int_0^1 \frac{dx}{(\operatorname{arctg} x)^\alpha}$ is convergent

5. **(5 points)** Determine for what value of the parameter α the system is CONSISTENT and SOLVE the system for this value of α:

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 4 \\ x_1 + x_2 + 2x_3 + 3x_4 &= 8 \\ 2x_1 + 4x_2 + 5x_3 + 10x_4 &= 20 \\ 3x_1 + x_2 + 5x_3 + 5x_4 &= \alpha \end{aligned}$$

Solution:

$$\left(\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 2 & 3 & 8 \\ 2 & 4 & 5 & 10 & 20 \\ 3 & 1 & 5 & 5 & \alpha \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 4 \\ 0 & 2 & 1 & 4 & 4 \\ 0 & 6 & 3 & 12 & 12 \\ 0 & 4 & 2 & 8 & \alpha-12 \end{array}\right) \to$$

$$\to \left(\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 4 \\ 0 & 2 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha-20 \end{array}\right) \quad \text{— row echelon form}$$

So, the system is consistent if and only if $\alpha = 20$. In this case we delete the rows of zeros and find a reduced row echelon form of our matrix:

$$\left(\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 4 \\ 0 & 2 & 1 & 4 & 4 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 0 & \frac{3}{2} & 1 & 6 \\ 0 & 1 & \frac{1}{2} & 2 & 2 \end{array}\right) \quad \text{— reduced row echelon form}$$

Then assign x_3 and x_4 as free variables and find

$$x_1 = 6 - \tfrac{3}{2}x_3 - x_4, \qquad x_2 = 2 - \tfrac{1}{2}x_3 - 2x_4,$$

$$x_3,\ x_4 \quad \text{— are free variables}$$

Answer: System is consistent for $\alpha = 20$. In this case solution is the vector

$$\begin{pmatrix} 6 - \frac{3}{2}\lambda - \mu \\ 2 - \frac{1}{2}\lambda - 2\mu \\ \lambda \\ \mu \end{pmatrix},$$

where λ and μ are arbitrary numbers.

6. **(5 points)** Find the inverse of the following matrix A:

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 3 & 2 & 0 \\ -1 & 3 & 1 \end{pmatrix}$$

Solution: Let us compute the inverse matrix with the help of the cofactor matrix:

$$A^{-1} = \frac{1}{\det A}\left(\operatorname{Cof} A\right)^T = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T$$

First, we evaluate $\det A$ by direct computation:

$$\det A = 1 \cdot 2 \cdot 1 + 3 \cdot 3 \cdot 1 + (-1) \cdot 4 \cdot 0 -$$

$$-1 \cdot 2 \cdot (-1) - 3 \cdot 4 \cdot 1 - 1 \cdot 0 \cdot 3 =$$
$$= 2 + 9 + 0 - (-2) - 12 - 0 = 1$$

Next, we find the cofactor matrix Cof A computing cofactors A_{ij} (which are the determinants of 2×2 matrices) by mental arithmetic:

$$\text{Cof } A = \begin{pmatrix} 2 & -3 & 11 \\ -1 & 2 & -7 \\ -2 & 3 & -10 \end{pmatrix}$$

Answer:

$$A^{-1} = \frac{1}{\det A} \left(\text{Cof } A \right)^T = \begin{pmatrix} 2 & -1 & -2 \\ -3 & 2 & 3 \\ 11 & -7 & -10 \end{pmatrix}$$

7. **(8 points)** Write the rigourous proofs of the following two extreme value theorems:

Theorem 1. If a function $f : [a,b] \to \mathbb{R}$ is continuous on the closed and bounded interval $[a,b]$ then f is bounded on this interval.

Theorem 2. If a function $f : [a,b] \to \mathbb{R}$ is continuous on the closed and bounded interval $[a,b]$ then f must attain its maximum and a minimum on this interval.

Theorem 1. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\Longrightarrow$ f is bounded on $[a,b]$, i.e.

$$\exists\, M > 0 : \qquad \forall\, x \in [a,b] \qquad |f(x)| \leq M$$

Proof.

1. Proof by contradiction:
   Assume $\forall\, M > 0 \quad \exists\, x_M \in [a,b]$: $\quad |f(x_M)| > M$
2. Construct a sequence $\{x_n\}_{n=1}^{\infty} \subset [a,b]$

   Take $M = 1 \quad \Longrightarrow \quad \exists\, x_1 \in [a,b]$: $\quad |f(x_1)| > 1$
   Take $M = 2 \quad \Longrightarrow \quad \exists\, x_2 \in [a,b]$: $\quad |f(x_2)| > 2$
   Take $M = 3 \quad \Longrightarrow \quad \exists\, x_3 \in [a,b]$: $\quad |f(x_3)| > 3$
   ...
   Take $M = n \quad \Longrightarrow \quad \exists\, x_n \in [a,b]$: $\quad |f(x_n)| > n$
   ...
   So, we obtain $\{x_n\}_{n=1}^{\infty} \subset [a,b]$: $\quad \forall\, n \in \mathbb{N} \quad |f(x_n)| > n$

3. Use Bolzano–Weierstrass theorem:
$\{x_n\}_{n=1}^{\infty}$ is bounded $\implies$ $\exists$ a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$, $\exists\, c \in \mathbb{R}$: $x_{n_k} \to c$
One can pass to the limit in the inequality: $a \le x_{n_k} \le b \implies a \le c \le b \implies c \in [a,b]$
4. Use continuity to obtain a contradiction:
$x_{n_k} \to c$, f is continuous on $[a,b] \implies f(x_{n_k}) \to f(c)$
Convergent sequence is bounded $\implies \exists\, L > 0$: $\forall\, k \in \mathbb{N}$ $|f(x_{n_k})| \le L$
$\forall\, k \in \mathbb{N}$ $|f(x_{n_k})| > n_k \to \infty$ — this contradicts to the boundedness of $\{f(x_{n_k})\}_{k=1}^{\infty}$!!!

THEOREM 2. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\implies$ f achieves on $[a,b]$ its maximum and minimal values, i.e. $\exists\, c_1, c_2 \in [a,b]$ such that

$$f(c_1) = \inf_{x\in[a,b]} f(x) \qquad \text{and} \qquad f(c_2) = \sup_{x\in[a,b]} f(x)$$

PROOF. Let us prove that f achieves its maximum. The proof for the minimum is analogous.

5. Function which is continuous on a closed bounded interval is bounded:
f is continuous on $[a,b] \implies f$ is bounded on $[a,b] \implies \exists\, M \in \mathbb{R}$: $M = \sup_{x\in[a,b]} f(x)$
6. Use the characterization of supremum using the quantifiers:

$\forall\, \varepsilon > 0\ \exists\, x_\varepsilon \in [a,b]$: $M - \varepsilon < f(x_\varepsilon) \le M$
Take $\varepsilon = 1$ $\exists\, x_1 \in [a,b]$: $M - 1 < f(x_1) \le M$
Take $\varepsilon = \frac{1}{2}$ $\exists\, x_2 \in [a,b]$: $M - \frac{1}{2} < f(x_2) \le M$
Take $\varepsilon = \frac{1}{3}$ $\exists\, x_3 \in [a,b]$: $M - \frac{1}{3} < f(x_3) \le M$
...
Take $\varepsilon = \frac{1}{n}$ $\exists\, x_n \in [a,b]$: $M - \frac{1}{n} < f(x_n) \le M$
...
So, we obtain $\{x_n\}_{n=1}^{\infty} \subset [a,b]$: $\forall\, n \in \mathbb{N}$ $M - \frac{1}{n} < f(x_n) \le M$
Two policemen theorem $\implies f(x_n) \to M$
7. Use Bolzano–Weierstrass theorem:
$\{x_n\}_{n=1}^{\infty}$ is bounded $\implies$ $\exists$ a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$, $\exists\, c \in [a,b]$: $x_{n_k} \to c$
8. Use continuity of f and uniqueness of the limit:
f is continuous at $c \in [a,b]$, $x_{n_k} \to c \implies f(x_{n_k}) \to f(c)$
$f(x_{n_k}) \to M$, $f(x_{n_k}) \to f(c) \implies f(c) = M$

□