Lecture "Experimental Physics I"

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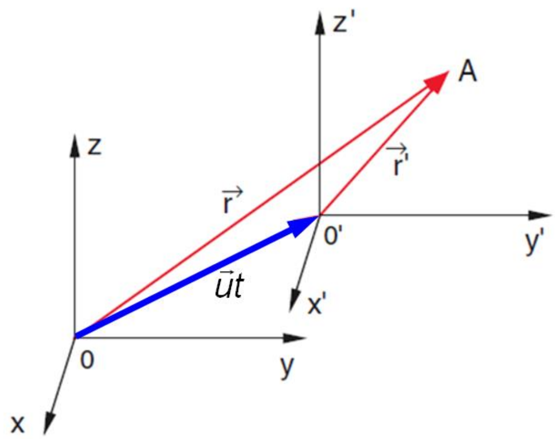
Lecture 8

Inertial and accelerated frames of reference

- Inertial frames and Galileo transformation
- Accelerated frames of references
- Rotating frames of reference
- Centripetal and Coriolis force

1) Inertial frames of reference

We discussed already beforehand the centrifugal force/acceleration and said that this is a pseudo-force/acceleration, since it is not occurring when looking onto a rotating system from the outside. In the following we want to understand this better by looking at reference frames and their relation to each other.



Consider two frames of reference (i.e. coordinate systems) - a system S with origin O and a system S' with origin O' - that **move with a speed $\vec{u} = const$ with respect to each other**. The origins of both frames of reference are thus separated by the vector $\vec{u}\,t$. When knowing the coordinates of a point A in within S we can thus write for the coordinates of A in S':

$$\vec{r}' = \vec{r} - \vec{u}t$$

Differentiation provides:

$$\frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} - \vec{u}$$

and replacing the derivatives by the velocities in the respective frames gives

$$\vec{v}' = \vec{v} - \vec{u}$$

i.e. the velocity in S' is the velocity in S minus the relative velocity, which is intuitively reasonable. Similarly we obtain the acceleration in S' by differentiating the velocity in S':

$$\vec{a}' = \frac{d\vec{v}'}{dt} = \frac{d\vec{v}}{dt} = \vec{a}$$

This provides that the **acceleration is equal in both reference frames**. Since acceleration is proportional to the acting forces, also the forces are identical. **Thus, observers in each reference frame experience the same laws of physics!** Formally, we define **inertial frames of reference** that we already know from Newton's laws as:

> **Non-accelerated frames of reference that move with constant speed with respect to each other** are called **Inertial frames of reference.**

Transformations between such inertial frames are described by the rules that we just derived and are summarized in the **Galileo transformation:**

$$\vec{r} = \vec{r}^{\,'} + \vec{u}t$$

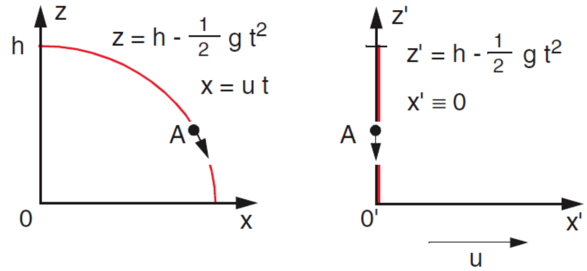
$$\vec{v} = \vec{v}^{\,'} + \vec{u} \qquad \Rightarrow \vec{a} = \vec{a}' \quad and \quad \vec{F} = \vec{F}'$$

$$t = t'$$

if $|\vec{u}| \ll c$

New is here only the condition *t* = *t'* for which we have to demand relative velocities that are much smaller than the speed of light *c*. The condition is not obeyed at relativistic velocities at which special relativity theory provides deviating times.

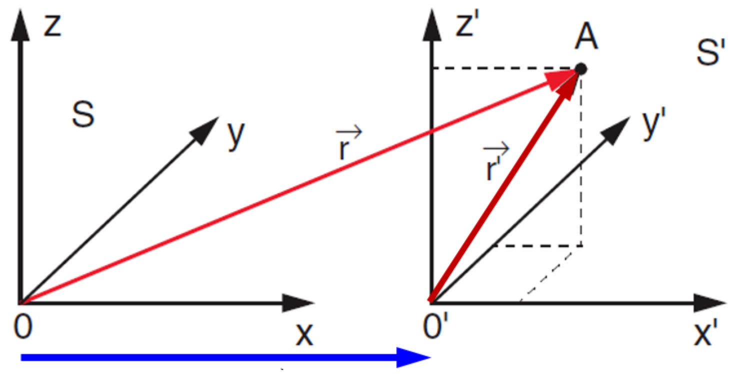
Example: Let us look at **projectile motion in a stationary frame of reference** and in a **frame of reference that moves horizontally with the constant horizontal velocity of the projectile**. In the stationary frame of reference the projectile is on a parabola track while in the moving frame of reference the observer sees that the projectile is only falling down in vertical direction. Nonetheless, both observers see only an acceleration of –g in the perpendicular direction and thus the same gravity force.



2) Frames of reference with constant acceleration

Now let us look at the more interesting case where **two frames of reference move with constant acceleration** with respect to each, i.e. for the relative speed we have:

$$\frac{d\vec{u}}{dt} = \vec{a}_0$$



The displacement of the origins of the coordinate systems is described by vector $\vec{r}_{OO'}$ and given by our kinematic equation:

$$\vec{r}_{OO'} = \vec{u}_0 t + \frac{\vec{a}_0}{2} t^2$$

We will now take the viewpoint of **either**:

- an **Observer sitting at *O* in system *S*** or
- an **Observer sitting at *O'* in system *S'***

For the position, the velocity and the acceleration in either of the systems we can write:

$$\vec{r}\,' = \vec{r} - \vec{r}_{OO'} = \vec{r} - \left(\vec{u}_0 t + \frac{\vec{a}_0}{2} t^2\right)$$

By differentiating we obtain for the velocity and with another differentiation for the acceleration:

$$\vec{v}\,' = \vec{v} - (\vec{u}_o + \vec{a}_0 t)$$
$$\vec{a}\,' = (\vec{a} - \vec{a}_0)$$

Let *S* be a non-accelerated frame of reference. Then an acceleration in *S*' can only be described by subtracting the acceleration between the two reference frames $\vec{a}_0$ from the acceleration in *S*. When multiplying with the mass of the object of interest one obtains for the forces:

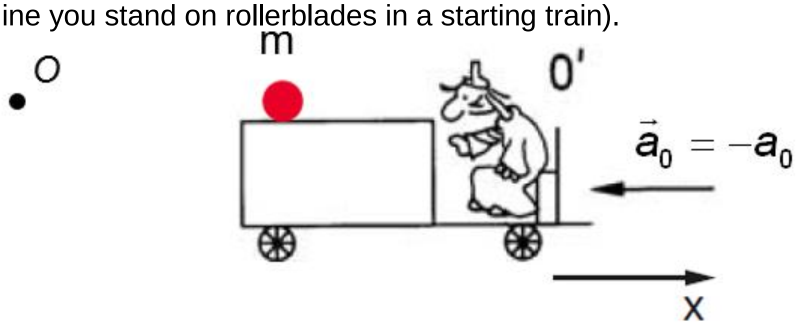
$$\vec{F}' = \vec{F} \underbrace{-m\vec{a}_0}_{\substack{\text{pseudo force} \\ \vec{F}_{ps}\text{acting in O'}}} = \vec{F} + \vec{F}_{ps}$$

$-m\vec{a}_0$ defines here a **pseudo (fictitious) force** to describe the reduced acceleration in S'. It is not a real force, since it is not present in the inertial frame of reference *S.* We call it **an inertial force**, since it **equals by magnitude the inertia of the object** and it is the "resistive" force of the object against a change in motion.

We will illustrate the concept of the inertial force in a number of examples:

A) Freely movable sphere on accelerated car

Let us first have a look onto a freely movable object within a car that is accelerated by $-a_0$ to the left (e.g. imagine you stand on rollerblades in a starting train).



The observer at O reports:

- the sphere is not moving, thus there is **no acting force on the sphere** (i.e. $a = 0$ and $F = 0$), while system **O' is moving with constant acceleration**

With the transformation from above we can write for the force in *S*':

$$F' = ma' = \underbrace{0}_{F} \underbrace{-m(-a_0)}_{F_{ps}} = ma_0$$

and thus

$$a' = a_0$$

The observer at O' thus reports:

- the sphere is approaching O' with constant acceleration $a' = a_0$, thus there is a **force $\boldsymbol{F' = ma' = ma_0 = F_{ps}}$ that accelerates *m*** with respect to *O*'

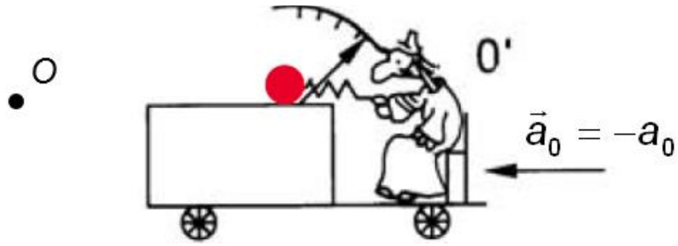
Thus, we have to introduce a fully **fictious force $\boldsymbol{F_{ps} = ma_0}$ to describe the motion in O'.**

Experiment: Movement of a free sphere on an accelerating track seen by an observer from the outside and an observer moving on the track.

See also corresponding movie

B) Spring-tethered sphere on accelerated car

Now we let the sphere be tethered to the observer in *O*' by a spring. It accelerates with the car and the observer can measure the required force



The observer at O reports:

- the sphere is moving with acceleration $a' = -a_0$ driven by the force $F = -ma_0$ that is measured on the spring

Here we can write for the force balance in S':

$$\underbrace{F'}_{0} = m\underbrace{a'}_{0} = F - m(-a_0)$$

Therefore, we must have:

$$0 = \underbrace{-ma_0}_{F_{spring}} + \underbrace{ma_0}_{F_{ps}}$$

The observer in O' reports:

- the sphere is at rest, i.e. the total force is zero according to the equation above. Thus, the measured **force F at the spring must be balanced by the opposing pseudo force $F_{ps} = ma_0$**

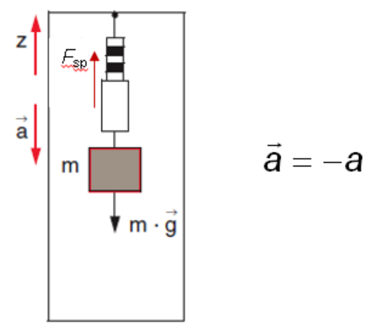
While the spring force F acts along the acceleration $-a_0$ (i.e. to the left), the pseudo force acts in the opposite direction. This **pseudo force is the resistance of the mass to get accelerated** and is thus equal to the inertia. It is therefore **called inertial force**. This is the force that "presses us" into the seat, when a car is accelerating.

C) Accelerating elevator

We now practice the forces in the different frames of reference further by looking at the forces that a mass on a spring experiences in an accelerating elevator. The acceleration a is given in the following as a positive number and just the accompanying sign defines the direction.

Downward acceleration:

We first look at downward acceleration of the elevator.



The two oberservers (*O* outside the elevator, *O*' inside the elevators) report:

O: The mass is accelerated with $-a$ by the acting spring and the gravitational force:

$$F_{spring} - mg = -ma$$

such that we get a force on the spring that is smaller than the weight:

$$F_{spring} = m(g - a)$$

O': The net force F' on the mass is zero, since there is no acceleration in O'. We write therefore

$$\underbrace{0}_{F'} = \underbrace{F_{spring} - mg}_{F} + F_{ps}$$

and we get for the pseudo force:

$$F_{ps} = -m(g-a) + mg = ma$$

The pseudo force $F_{ps} = ma$ is directed upward. It is the **resistance/inertial force of *m* against the downward acceleration and counteracts gravity**. It therefore leads to an apparent reduction of the weight at the spring scale.

<u>Upward acceleration</u>

Here the just the sign of the acceleration changes, such that we get:

O: $F_{spring} - mg = +ma$

$F_{spring} = m(g+a)$

i.e. the spring reports an apparent weight increase of the mass due to the additional inertia

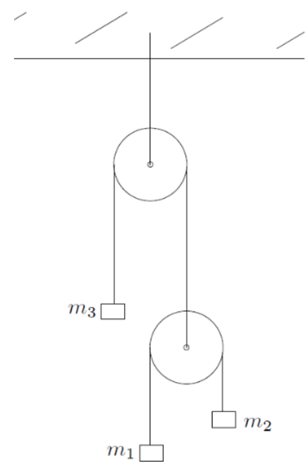
O': Inserting the new spring force into the equation from the downward acceleration gives:

$F_{ps} = -ma$

i.e. a corresponding downward oriented inertial force must be added in *O*', which makes the weight apparently heavier. (**see slides**)

Experiments:

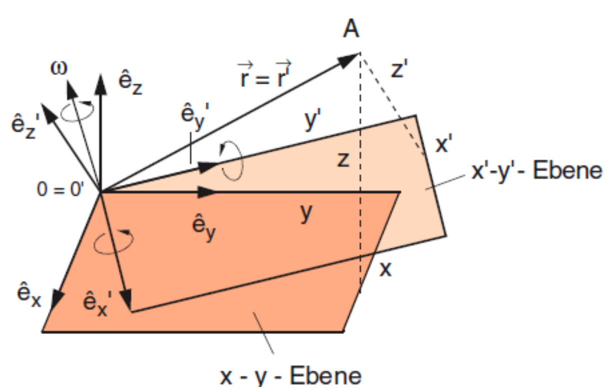
1) Inclined plane: a cuboid put with its short side on a wagon is falling if the wagon stands still. Once the wagon moves down the plane with const. acceleration the cuboid keeps standing. Explanation: In the non-moving reference frame, both the falling cuboid and the wagon accelerate at the same time which prevents the cuboid from falling. In the moving reference frame of the wagon, the inertial force from the inertia of the accelerating cuboidacts as a backdriving force that counteracts the gravity force that would let the cuboid fall.
2) Pseudoforce at „Poggendorf-Balance" (see picture)



3) Rotating frames of reference

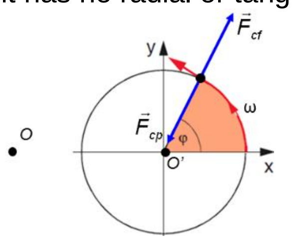
Another important set of frames of references that move with respect to each other are rotating frames of reference.

Let us look at circular motion and consider two coordinate systems (*x, y, z*) and (*x',y',z'*) with corresponding unit vectors that are connected at their origins. **System S' rotates with angular velocity ω with respect to system S**.



A) Centrifugal force

Let us first repeat something that we know from before by considering an **object that is fixed in S'**, i.e. it has no radial or tangential motion in it such that $v' = 0$



For such a system the observers report:

O: object is moving on a circular track, since it experiences a centripetal acceleration/force towards the circle center
$\vec{F} = \vec{F}_{cp} = m\omega^2 r(-\hat{e}_r)$

O': object is not moving, i.e. $F' = 0$

Using the transformation equation from above for an accelerated frame of reference we get:

$$\underbrace{0}_{\vec{F}'} = \underbrace{m\omega^2 r(-\hat{e}_r)}_{\vec{F}} + \vec{F}_{ps}$$

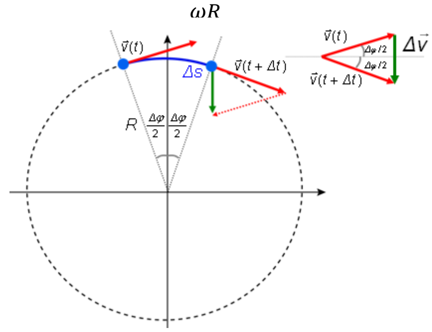
Thus:

$$\vec{F}_{ps} = \vec{F}_{cf} = m\omega^2 r\hat{e}_r$$

i.e. within S' we have to introduce the **centrifugal pseudo force** that is **always directed outwards** non-depending on the rotation direction! It counteracts the centripetal force, such that the object does not move in *O*'. The centrifugal force arises from the resistance of the mass to undergo circular motion.

The simple way to derive the centripetal acceleration was to look at the change of the tangential velocity during uniform circular motion, which was equal to the tangential velocity times the angular displacement (**see slides**):

$$\Delta v_t = 2v_t \cdot sin(\,\Delta\varphi/2) \approx 2v_t \cdot \Delta\varphi/2 = \underbrace{v_t}_{\omega R}\,\Delta\varphi$$



The centripetal acceleration was then provided by:

$$a_{cp} = \frac{\Delta v_t}{\Delta t} = \omega R\frac{\Delta\varphi}{\Delta t} = \omega^2 r$$

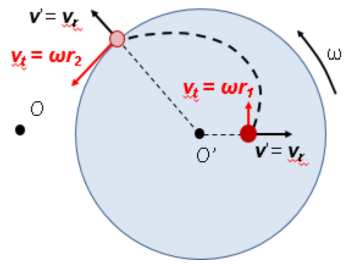
B) Coriolis force

Now we make the situation more complicated and assume that the object is moving in O' on a straight track away from O' with $v' = const$.

The observers report:

O: the object is moving on a spiral-like track

O’: the object is moving with constant velocity away from O‘

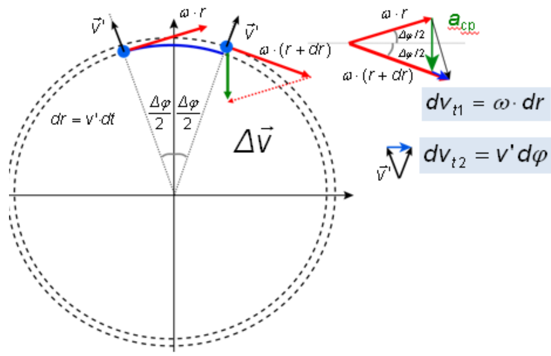


In O we need different forces to obtain the spiral-like trajectory:

1) The **centripetal acceleration in radial direction**, which changes the direction but not the magnitude of the tangential velocity
2) Additional **accelerations in tangential direction** particularly:
    - a change of the magnitude of v_t (linear increase with radius due to constant ω)
    - a change of the direction of radial velocity v'

The additional force to obtain the accelerations in tangential direction is **called Coriolis force**. In *O*’ the **Coriolis force** is compensated by a **corresponding counteracting pseudo force.**

To derive a formula for the Coriolis force including both contributions, we make a similar drawing as for the centripetal force:



We look again at two time points during the motion separated by time dt and angle $d\varphi$. We separately look at the tangential velocity and the radial velocity

Tangential velocity: The tangential velocity is given as

$$v_t = r\omega$$

For a motion in radial direction by distance dr, the tangential velocity thus increases by

$$dv_{t1} = \omega(r + dr) - \omega r = \omega dr$$

For small $d\varphi$, the velocity increase (see blue arrow in figure) occurs in the tangential direction and adds up with the centripetal velocity change to the total velocity change of the tangential velocity

Radial velocity: As for the centripetal velocity change the change of the radial velocity is given by:

$$dv_{t2} = v'd\varphi$$

As can be seen from the figure it also occurs in the tangential direction.

The total acceleration in the tangential direction, i.e. the **Coriolis acceleration** is thus given by:

$$a_c = a_t = \frac{dv_{t1}}{dt} + \frac{dv_{t2}}{dt} = \omega \underbrace{\frac{dr}{dt}}_{v'} + v' \underbrace{\frac{d\varphi}{dt}}_{\omega} = 2v'\omega$$

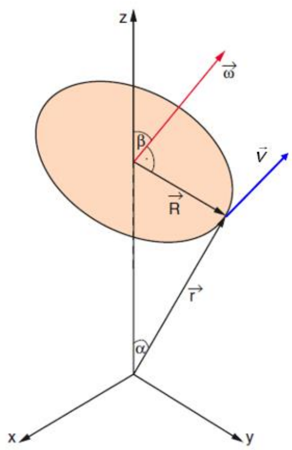
where we used the relations $v' = \mathrm{d}r/\mathrm{d}t$ and $\omega = \mathrm{d}\varphi/\mathrm{d}t$.

The **total acceleration** is the sum of centripetal and Coriolis acceleration:

$$\vec{a} = \underbrace{\omega^2 r(-\hat{e}_r)}_{\vec{a}_{cp}} + \underbrace{2v'\omega\hat{e}_t}_{\vec{a}_c}$$

C) The angular velocity as vector

The formula above is only correct for a radial motion in O'. To arrive at a more general expression we first redefine the angular velocity.



For more advanced considerations, we define the angular velocity of an object that moves with velocity $\vec{v}$ with respect to any desired reference point at distance $\vec{R}$ as a vector:

$$\vec{\omega} = \frac{1}{R^2}\vec{R} \times \vec{v}$$

$\vec{\omega}$ has a number of properties according to our previous definition of the vector product:

1) $\vec{\omega}$ is perpendicular to $\vec{R}$ and $\vec{v}$, i.e. $\vec{\omega}$ is thus normal to the rotation plane and points along the rotation axis
2) The direction of the rotation is given by right hand
3) The absolute value of $\vec{\omega}$ is given by

$$|\vec{\omega}| = \frac{1}{R^2} Rv \sin\alpha = \frac{v_\perp}{R} = \left(\frac{v_t}{R}\right)$$

where $v_\perp$ is the velocity component that is perpendicular to $\vec{R}$, i.e. the tangential velocity. Thus, the absolute value of $\vec{\omega}$ is provided only by the tangential component of the velocity normalized by *R*, which agrees with our previous definition. The advantage of the vectorial form of $\vec{\omega}$ is that in addition to its magnitude, we define also the direction of the rotation axis. $\vec{\omega}$ is thus a typical axial vector.

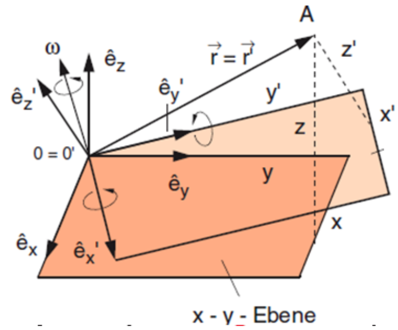
Example: The earth has an $\vec{\omega}$ that points from south to north pole.

With some vector calculus one can show that the tangential velocity is correspondingly given by:

$$\vec{v}_t = \vec{\omega} \times \vec{R}$$

D) General form of acceleration in rotating reference frame

Using the vector representation of $\vec{\omega}$ one can derive a general formula when transforming the acceleration in a stationary frame of reference S into a rotating frame of reference S'.



The **transformation is then given by** (see below or Demtröder engl. version, page 85)

$$\vec{a}' = \vec{a} + \underbrace{2(\vec{v}' \times \vec{\omega})}_{\vec{a}_c} + \underbrace{\vec{\omega} \times (\vec{r} \times \vec{\omega})}_{\vec{a}_{cf}}$$

When one measures in O a linear acceleration $\vec{a}$ one has to add in S' **two additional pseudo "acceleration" terms**. These terms look similar to our simple derivations and are respectively called **Coriolis and Centrifugal acceleration/force**:

$$\vec{a}_c = 2(\vec{v}' \times \vec{\omega}) \qquad \vec{F}_c = 2m(\vec{v}' \times \vec{\omega})$$

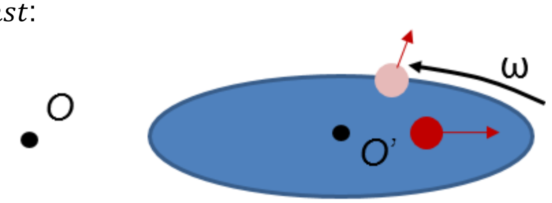
$$\vec{a}_{cf} = \vec{\omega} \times (\vec{r} \times \vec{\omega}) \qquad \vec{F}_{cf} = m\vec{\omega} \times (\vec{r} \times \vec{\omega})$$

Note, **both terms act in the plane of rotation**, since the force is perpendicular to $\vec{\omega}$ due to the vector product. The Coriolis force occurs for any velocity $\vec{v}'$ in the rotating frame of reference. It can thus also point in a non-tangential direction!

Both forces are pseudoforces that can be real measurable forces in the corresponding frame of reference. In the following we will briefly apply this formula and check it with our previous approaches.

Coriolis force from general formula

1) We first consider an object moving on **straight radial track in S'** from the origin with $\vec{v}' = const$:



O: the object is moving on a spiral-like track

O': the object is moving with constant velocity & zero acceleration, such that $F' = 0$

Our transformation of the accelerations provides:

$$0 = \vec{a} + \underbrace{2(\overbrace{v'\hat{e}_r}^{\vec{v}'} \times \vec{\omega})}_{2v'\omega\cdot(-\hat{e}_t)} + \underbrace{\vec{\omega} \times (\overbrace{\vec{r} \times \vec{\omega}}^{r\omega\cdot(-\hat{e}_t)})}_{\omega^2 r\,\hat{e}_r}$$

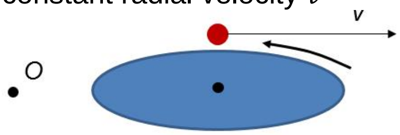
With this we get for the acceleration in S in agreement with our previous result:

$$\vec{a} = 2v'\omega\,\hat{e}_t + \omega^2 r\,(-\hat{e}_r)$$

In *O* the motion has a Coriolis acceleration in tangential and a centripetal acceleration in negative radial direction. In O' the Coriolis force and the centripetal force are compensated by corresponding counteracting pseudo forces.

Example: The pseudoforces in S' can be felt when walking on a rotating disk. We experience additional forces that are not due to the centrifugal force but the Coriolis force.

2) We now consider an object moving on a **straight radial track in S** from the origin with constant radial velocity v



Seen from *O*

Seen from *O'*

O: object is moving with constant velocity & zero acceleration such that $\vec{a} = 0$.

O': point is moving on a spiral-like track being a superposition of a circular path around O' with $-\omega$ and a radial motion with v. The velocity in O' is thus given by the two components: $\vec{v}' = v\,\hat{e}_r - \vec{\omega} \times \vec{r}$

We can then write for the acceleration in S':

$$\vec{a}' = 0 + 2(\underbrace{\vec{v}'}_{v\,\hat{e}_r - \vec{\omega}\times\vec{r}} \times \vec{\omega}) + \vec{\omega} \times (\vec{r} \times \vec{\omega})$$

Simplification then gives:

$$\vec{a}' = 2v\,\underbrace{\hat{e}_r \times \vec{\omega}}_{\omega\,(-\hat{e}_t)} \underbrace{-2(\vec{\omega} \times \vec{r}) \times \vec{\omega} + \vec{\omega} \times (\vec{r} \times \vec{\omega})}_{-\vec{\omega}\times(\vec{r}\times\vec{\omega})} = 2v\omega\,(-\hat{e}_t) - \underbrace{\vec{\omega} \times (\vec{r} \times \vec{\omega})}_{\omega^2 r\cdot\hat{e}_r}$$

such that we finally get:

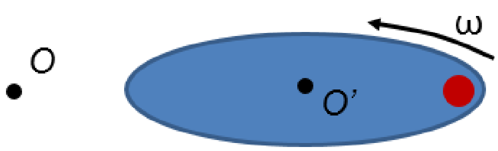
$$\vec{a}' = 2v\omega \cdot (-\hat{e}_t) + \omega^2 r(-\hat{e}_r)$$

In O' the point experiences a centripetal acceleration as well as a tangential acceleration against the rotation direction. The tangential component is exactly the Coriolis force from before, i.e. it turns the radial component of the velocity vector and increases the radial distance, such that we get a spiral like motion.

Centrifugal force from general formula (not part of lecture)

In the following we test our formula for objects that are resting in O or in O'

<u>Object that is fixed in O' → v' = 0</u>



O: point is moving on circular track experiencing a centripetal acceleration of $\omega^2 r$

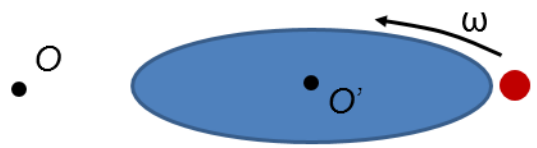
O': point is not moving such that $F' = 0$

Inserting into the above equations gives:

$$0 = \underbrace{\vec{a}}_{F_{cp}=\omega^2 r\cdot(-\hat{e}_r)} + 2(0 \times \vec{\omega}) + \underbrace{\vec{\omega} \times (\vec{r} \times \vec{\omega})}_{F_{cf}=\omega^2 r\cdot\hat{e}_r} = 0$$

The centrifugal force component is always directed outwards non-depending on the rotation direction! It is compensated by the previously determined centripetal acceleration, such that the object does not move in O' (real forces here)

<u>Object that is fixed in O</u>



O: point is not moving → zero acceleration
O': point is moving on circular path around O' with angular velocity –ω!

The velocity in O' is thus the tangential velocity of the object in that system pointing in opposite direction of the tangential velocity of the disk.

$$\vec{v}\,' = -\vec{\omega} \times \vec{r}\,'$$

Inserting provides:

$$\vec{a}\,' = 0 + 2(\underbrace{(-\vec{\omega} \times \vec{r}\,')}_{\vec{v}\,'=\vec{v}_t'} \times \vec{\omega}) + \vec{\omega} \times (\vec{r} \times \vec{\omega})$$

Further transformation yields (by 2-fold swapping the order of the vector product and $\vec{r} = \vec{r}\,'$):

$$\vec{a}\,' = \underbrace{-2\vec{\omega} \times (\vec{r}\,' \times \vec{\omega})}_{a_c=2\omega^2 r\,(-\hat{e}_r)} + \underbrace{\vec{\omega} \times (\vec{r} \times \vec{\omega})}_{a_{cf}=\omega^2 r \cdot \hat{e}_r}$$

$$\vec{a}\,' = \underbrace{-\vec{\omega} \times (\vec{r} \times \vec{\omega})}_{-\omega^2 r\,\hat{e}_r}$$

Here the **Coriolis acceleration is directed in radial direction inwards** thus twice compensating the centrifugal acceleration. This creates a net centripetal acceleration in O' that is the pseudo force that forces the object on a circular path (not real force).

E) Examples for the Coriolis force

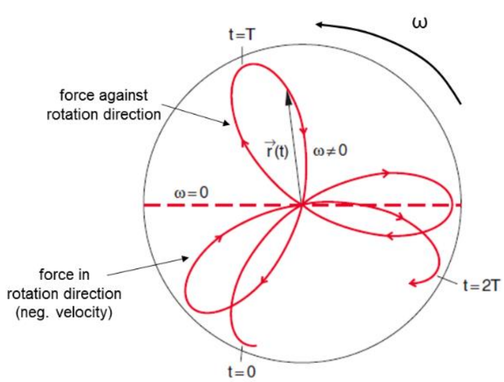
We can nicely see the action of the Coriolis force in real world examples

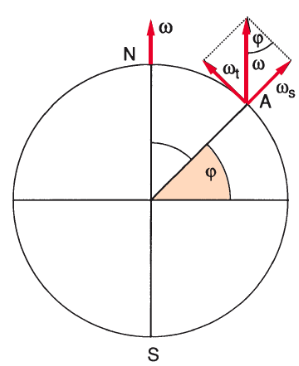
Experiments (see slides):

1) Figures drawn by a pendulum above a rotating disk: Here we have a (not real) pseudoforce acting against the rotation direction (outward motion) and in the rotation direction (inward motion). In **corresponding videos** the perspectives of an observer that rotates with the disk and of a resting observer are shown.

2) Foucault pendulum: Clockwise rotation every 11° per hour, making a full circle in 32.7 hours Effective angular velocity depends on geographic latitude φ. The effective angular velocity along pendulum axis that enters the cross product can be obtained by vector components thus providing

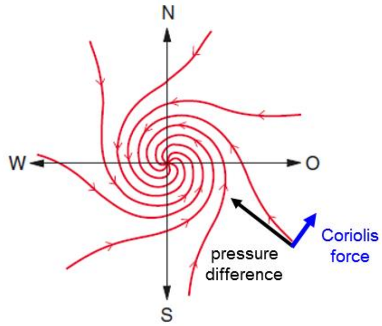
$$\omega_s = \omega \cdot \sin\varphi.$$

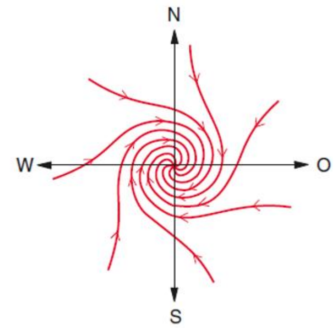




Atmospheric air flows:

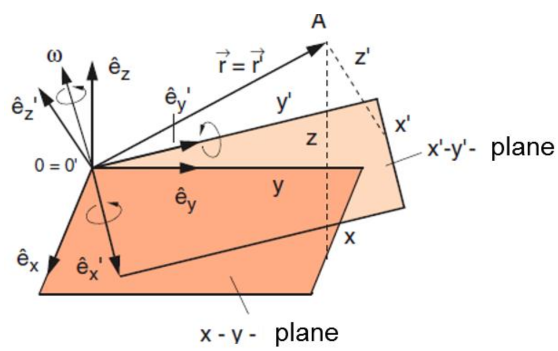
Atmospheric air flows are strongly influenced by the Coriolis force, such as in deep pressure areas where spiral like flow profiles are obtained. On the northern hemisphere the Coriolis force pushes the air flow to the right (when seen from above) while on the southern hemisphere to the left, since here the angular velocity vector has the opposite direction.





F) Derivation of the general form of acceleration in rotating reference frame

(not part of lecture)



The position and velocities of an object in each coordinate system is described by the vectors:
Observer in O:

$$\vec{r}(t) = x(t)\hat{e}_x + y(t)\hat{e}_y + z(t)\hat{e}_z$$

$$\vec{v}(t) = \frac{dx}{dt}\hat{e}_x + \frac{dy}{dt}\hat{e}_y + \frac{dz}{dt}\hat{e}_z$$

& Observer in O‘:

$$\vec{r}'(t) = x'(t)\hat{e}_x{}' + y'(t)\hat{e}_y{}' + z'(t)\hat{e}_z{}'$$

$$\vec{v}'(t) = \frac{dx'}{dt}\hat{e}_x{}' + \frac{dy'}{dt}\hat{e}_y{}' + \frac{dz'}{dt}\hat{e}_z{}'$$

$\vec{r}'(t) = \vec{r}(t)$ since it is the same object/vector. Identity is provided if on expresses the unit vectors of one system with the unit vectors of the other

Now let’s express the velocity v in O by the velocity in O‘ by adding the position change of the unit vectors of O’:

$$\vec{v}(x', y', z') = \left(\frac{dx'}{dt}\hat{e}_x{}' + \frac{dy'}{dt}\hat{e}_y{}' + \frac{dz'}{dt}\hat{e}_z{}'\right) + x'(t)\frac{d\hat{e}_x'}{dt} + y'(t)\frac{d\hat{e}_y'}{dt} + z'(t)\frac{d\hat{e}_z'}{dt}$$

The unit vectors undergo a circular motion with constant angular velocity **ω** around O’=O for which one can write according to the definition $\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$. Thus,

$$\frac{d\hat{e}_x^{'}}{dt} = \omega \times \hat{e}_x^{'} \qquad \frac{d\hat{e}_y^{'}}{dt} = \omega \times \hat{e}_y^{'} \qquad \frac{d\hat{e}_z^{'}}{dt} = \omega \times \hat{e}_z^{'}$$

Thus we can write:

$$\vec{v}(x', y', z') = \left(\frac{dx'}{dt}\hat{e}_x{}' + \frac{dy'}{dt}\hat{e}_y{}' + \frac{dz'}{dt}\hat{e}_z{}'\right) + \vec{\omega} \times \left(x'(t)\hat{e}_x{}' + y'(t)\hat{e}_y{}' + z'(t)\hat{e}_z{}'\right)$$

And by replacing we get:

$$\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}' = \vec{v}' + \vec{\omega} \times \vec{r}$$

Equivalently we get for the acceleration in O if expressed by the coordinates of O‘ and the unit vectors of O’ in coordinates of O:

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}(x', y', z')}{dt} \\ &= \left(\frac{d^2x'}{dt^2}\hat{e}_x{}' + \frac{d^2y'}{dt^2}\hat{e}_y{}' + \frac{d^2z'}{dt^2}\hat{e}_z{}'\right) + \left(\frac{dx'}{dt}\frac{d\hat{e}_x'}{dt} + \frac{dy'}{dt}\frac{d\hat{e}_y'}{dt} + \frac{dz'}{dt}\frac{d\hat{e}_z'}{dt}\right) \\ &\quad + \vec{\omega} \times \frac{d\vec{r}}{dt} \end{aligned}$$

This transforms according to the arguments from above:

$$\vec{a} = \frac{d^2\vec{r}'}{dt^2} + \vec{\omega} \times \frac{d\vec{r}'}{dt} + \omega \times \frac{d\vec{r}}{dt}$$

And we get:

$$\vec{a} = \vec{a}' + \ \vec{\omega} \times \vec{v}' + \omega \times \vec{v}$$

Replacing $\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}$ provides:

$$\vec{a} = \vec{a}' + 2\,(\vec{\omega} \times \vec{v}') + \omega \times (\vec{\omega} \times \vec{r})$$

And we obtain the previously introduced formula:

$$\vec{a} = \vec{a}' + \underbrace{2(\vec{\omega} \times \vec{v}')}_{\vec{a}_c} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\vec{a}_{cf}}$$

Lecture 8: Experiments

1) Movement of a free sphere on an accelerating track seen by an observer from the outside and an observer moving on the track.
2) Video: Movement of a free sphere on an accelerating track seen by an observer moving on the track.
3) Inclined plane: a cuboid put with its short side on a wagon is falling if the wagon stands still. Once the wagon moves down the plane with const. acceleration the cuboid keeps standing.
4) Pseudoforce, Poggendorf-Balance
5) Coriolis force: Sideward deviation on rotating disk
6) Figures drawn by a pendulum above a rotating disk: Here we have a (not real) pseudoforce acting against the rotation direction (outward motion) and in the rotation direction (inward motion)
7) Video: Figures drawn by a pendulum above a rotating disk imaged by stationary and rotating camera
8) Foucault pendulum: Clockwise rotation every 11° per hour, making a full circle in 32.7 hours