MA1 – HW2, Stanislav IPSP 3720433

1. $$\lim_{n\to\infty}\frac{n^2+1}{(n-2)(2n+1)} = \left[x_n = \frac{n^2+1}{(n-2)(2n+1)} = \frac{n^2+1}{2n^2-3n-2} = \frac{1+n^{-2}}{2-3n^{-1}-2n^{-2}}\right] =$$

$$= \lim_{n\to\infty}\frac{n^{-2}+1}{-2n^{-2}-3n^{-1}+2} = \left[\begin{array}{l}\lim\limits_{n\to\infty}\left(1+\frac{1}{n^2}\right) = 1+\lim\limits_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\right) = 1+0^2 = 1 \\ \lim\limits_{n\to\infty}\left(2-\frac{3}{n}-\frac{2}{n^2}\right) = 2 \text{ by same reasoning}\end{array}\right] =$$

$$= \frac{\lim\limits_{n\to\infty}\left(1-\frac{1}{n^2}\right)}{\lim\limits_{n\to\infty}\left(2-\frac{3}{n}-\frac{2}{n^2}\right)} = \frac{1}{2}.$$

2. $$\lim_{n\to\infty}\frac{(2n+1)^4-(n-1)^4}{(2n+1)^4+(n-1)^4} = \lim_{n\to\infty}\frac{16n^4+4\cdot(2n)^3+6(2n)^2+4(2n)+1-[n^4-4n^3+6n^2-4n+1]}{16n^4+4(2n)^3+6(2n)^2+4(2n)+1+[n^4+4n^3+6n^2+4n+1]} =$$

$$= \lim_{n\to\infty}\frac{15n^4+an^3+bn^2+cn+d}{17n^4+An^3+Bn^2+Cn+D} = \left[\begin{array}{l}\text{some numbers} \\ a,\dots,d,\ A,\dots,D\end{array}\right] = \lim_{n\to\infty}\frac{15+a\frac{1}{n}+b\frac{1}{n^2}+\frac{c}{n^3}+\frac{d}{n^4}}{17+A\frac{1}{n}+B\frac{1}{n^2}+C\frac{1}{n^3}+\frac{D}{n^4}} =$$

$$= \frac{\lim\limits_{n\to\infty}\left(15+\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\frac{d}{n^4}\right)}{\lim\limits_{n\to\infty}\left(17+\frac{A}{n}+\frac{B}{n^2}+\frac{C}{n^3}+\frac{D}{n^4}\right)} = \cancel{\lim}\ \frac{15}{17}$$ (by same limit arithmetic as in t. 1).

3. $$\lim_{n\to\infty}\frac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2}-\sqrt{n^3+1}} \overset{\cdot\frac{1}{\sqrt{n^3}}}{=\!=} \lim_{n\to\infty}\frac{\frac{1}{\sqrt{n^3}}\left[\sqrt[4]{n^5+2}-\sqrt[6]{(n^2+1)^2}\right]}{\frac{1}{\sqrt{n^3}}\left[\sqrt[10]{(n^4+2)^2}-\sqrt{n^3+1}\right]} =$$

$$= \lim_{n\to\infty}\frac{\sqrt[4]{\frac{n^5+2}{n^6}}-\sqrt[6]{\frac{(n^2+1)^2}{n^9}}}{\sqrt[10]{\frac{(n^4+2)^2}{n^{15}}}-\sqrt{1+\frac{1}{n^3}}} = \lim_{n\to\infty}\frac{\sqrt[4]{\frac{1}{n}+\frac{2}{n^6}}-\sqrt[6]{\frac{1}{n^5}+\frac{1}{n^9}+\frac{2}{n^7}}}{\sqrt[10]{\frac{1}{n^7}+\frac{4}{n^{15}}+\frac{4}{n^{11}}}-\sqrt{1+\frac{1}{n^3}}} =$$

= [$\frac{1}{n} = \{f_n\}_{n=1}^{\infty}$ is infinitesimal, then
$\frac{1}{n^2}$ is also infinitesimal, and for example $\frac{1}{n}+\frac{2}{n^6}$ also. Then $\sqrt[4]{\frac{1}{n}+\frac{2}{n^6}}$ is also infinitesimal, because:
– it is bounded below by 0
– it is $\searrow$ $\left(\frac{1}{n+1}+\frac{2}{(n+1)^6} < \frac{1}{n}+\frac{2}{n^6} \Longrightarrow \sqrt[4]{\frac{1}{n+1}+\frac{2}{(n+1)^6}} < \sqrt[4]{\frac{1}{n}+\frac{2}{n^6}}\right.$
So it has limit a.
Then $a^4 = \lim\limits_{n\to\infty}\left(\frac{1}{n}+\frac{2}{n^6}\right) = 0$, meaning $a=0$. With same reasoning all roots have limits 0, except $\lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n^3}} = 1$] =

$$= \frac{\lim\limits_{n\to\infty}\sqrt[4]{\frac{1}{n}+\frac{2}{n^6}}-\lim\limits_{n\to\infty}\sqrt[6]{\frac{1}{n^5}+\frac{1}{n^9}+\frac{2}{n^7}}}{\lim\limits_{n\to\infty}\sqrt[10]{\frac{1}{n^7}+\frac{4}{n^{15}}+\frac{4}{n^{11}}}-\lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n^3}}} = \frac{0-0}{0-1} = \frac{0}{-1} = 0.$$

4. $$\lim_{n\to\infty}\frac{\sqrt{n+2}-\sqrt{n-2}}{\sqrt{n+1}-\sqrt{n-1}} = \lim_{n\to\infty}\frac{(\sqrt{n+2}-\sqrt{n-2})(\sqrt{n+2}+\sqrt{n-2})(\sqrt{n+1}+\sqrt{n-1})}{(\sqrt{n+1}-\sqrt{n-1})(\sqrt{n+2}+\sqrt{n-2})(\sqrt{n+1}+\sqrt{n-1})} =$$

$$= \lim_{n\to\infty}\left[\frac{(n+2)-(n-2)}{(n+1)-(n-1)}\cdot\frac{\sqrt{n-1}+\sqrt{n+1}}{\sqrt{n-2}+\sqrt{n+2}}\right] = 2\lim_{n\to\infty}\underbrace{\frac{\sqrt{n-1}+\sqrt{n+1}}{\sqrt{n-2}+\sqrt{n+2}}}_{x_n} = 2\lim_{n\to\infty}x_n.$$

$$\underbrace{\sqrt{1-\frac{3}{n+2}}}_{y_n:=} = \frac{\sqrt{n-1}}{\sqrt{n+2}} = \frac{2\sqrt{n-1}}{2\sqrt{n+2}} < \frac{\sqrt{n-1}+\sqrt{n+1}}{2\sqrt{n+2}} < \boxed{x_n = \frac{\sqrt{n-1}+\sqrt{n+1}}{\sqrt{n-2}+\sqrt{n+2}}} < \frac{2\sqrt{n+1}}{2\sqrt{n-2}} = \underbrace{\sqrt{1+\frac{3}{n-2}}}_{z_n:=}$$

* Now show y_n has limit and find that:

1) $y_{n+1} = \sqrt{1-\frac{3}{n+3}} > y_n = \sqrt{1-\frac{3}{n+2}}$, so $\{y_n\}_{n=1}^{\infty} - \nearrow$

2) $\forall n\in\mathbb{N}\ \ y_n < 1$

$\Longleftrightarrow \{y_n\}_{n=1}^{\infty}$ bound above and $\uparrow \Longrightarrow$ has limit y.

$\lim_{n\to\infty}(1-\frac{3}{n+2}) = \lim_{n\to\infty} y_n^2 = [\text{now can use}] = \lim_{n\to\infty} y_n\cdot\lim_{n\to\infty} y_n = y^2 = 1,$

then $\underline{y=1}$ ($y\neq -1$ as growing positive seq.)

* Now show z_n has limit and find that:

1) $z_{n+1} = \sqrt{1+\frac{3}{n-1}} < \sqrt{1+\frac{3}{n-2}} = z_n$, so $\{z_n\}_{n=1}^{\infty} - \searrow$

2) $\forall n\in\mathbb{N}\ \ z_n \geq 1$

$\Longleftrightarrow \{z_n\}_{n=1}^{\infty}$ bound. below and $\downarrow \Longrightarrow$ has limit z.

$\lim_{n\to\infty}(1+\frac{3}{n-2}) = \lim_{n\to\infty} z_n^2 = [\text{now can use}] = \lim_{n\to\infty} z_n\cdot\lim_{n\to\infty} z_n = z^2 = 1,$

then $\underline{z=1}$ ($z\neq -1$ as positive, bounded below seq. by 1)

* $\lim_{n\to\infty} y_n = 1 \wedge \lim_{n\to\infty} z_n = 1 \wedge \forall n\in\mathbb{N}\ \ y_n\leq x_n\leq z_n \Longrightarrow$

by squeeze theorem (2 policemen) $\lim_{n\to\infty} x_n = 1$.

Therefore $\underline{2\lim_{n\to\infty} x_n = 2}$.

5. $$\lim_{n\to\infty}\underbrace{\frac{5^{2n+1}}{n!}}_{d_n}\underbrace{\sin\frac{n^2+1}{2n}}_{a_n} = \left[\begin{array}{l}|\{a_n\}|\leq 1 \Longrightarrow \{a_n\}_{n=1}^{\infty} \text{ is bounded}\\ \lim_{n\to\infty}\frac{5^{2n+1}}{n!} = 5\lim_{n\to\infty}\frac{25^n}{n!} = 0 \ \left(\text{theorem } \lim_{n\to\infty}\frac{q^n}{n!}=0\right)\end{array}\right]$$

$= 0$ (by theorem of product of bounded and infinitesimal).

6. $\lim\limits_{n\to\infty} \dfrac{\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}}{\frac{1}{3}+\frac{1}{9}+\dots+\frac{1}{3^n}} = \left[\begin{array}{l}\text{deriving sum for geom. series} \\ S_n = b+bq+\dots+bq^{n-1} \\ qS_n = bq+bq^2+\dots+bq^n \end{array} \Longrightarrow S_n = \dfrac{b(q^n-1)}{q-1}\right] =$

$= \lim\limits_{n\to\infty}\left[\dfrac{\frac{1}{2}((\frac{1}{2})^n-1)}{-\frac{1}{2}}\cdot\left(\dfrac{\frac{1}{3}((\frac{1}{3})^n-1)}{-\frac{2}{3}}\right)^{-1}\right] = \lim\limits_{n\to\infty}\left(\dfrac{(\frac{1}{2})^n-1}{(\frac{1}{3})^n-1}\cdot 2\right) = \left[\begin{array}{l}\text{lim. arithmetic} \\ \text{from previous} \\ \text{tasks}\end{array}\right] =$

$= 2.$

7. $\lim\limits_{n\to\infty} \dfrac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!} \overset{\cdot\frac{1}{(n+2)!}}{=\!=} \lim\limits_{n\to\infty} \dfrac{1+\frac{1}{n+2}}{1-\frac{1}{n+2}} = \dfrac{\lim\limits_{n\to\infty}(1+\frac{1}{n+2})}{\lim\limits_{n\to\infty}(1-\frac{1}{n+2})} = \left[\begin{array}{l}\text{lim. arithmetic} \\ \text{from previous} \\ \text{tasks}\end{array}\right] =$

$= \dfrac{1}{1} = 1.$

8. $\lim\limits_{n\to\infty} (n!)^{\frac{1}{2(n!)}} = \lim\limits_{n\to\infty}\sqrt{(n!)^{\frac{1}{n!}}}$. Sequence $a_n = (n!)^{\frac{1}{n!}}$ is a subsequence of $x_n = n^{\frac{1}{n}}$, so $\boxed{\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1}$ (theorem).

* (!) Sequence $\sqrt{a_n}$ is bounded and monotone in correct direction $\Longleftarrow\!\!\Longrightarrow$ a_n is bounded/monotone.

need to show

1) $\dfrac{a_{n+1}}{a_n} = \dfrac{n!^{\frac{1}{(n+1)!}}(n+1)^{\frac{1}{(n+1)!}}}{n!^{\frac{1}{n!}}} = n^{\frac{1-(n+1)}{(n+1)!}}(n+1)^{\frac{1}{(n+1)!}} = n!^{\frac{-n}{(n+1)!}}(n+1)^{\frac{1}{(n+1)!}}$

compare

$n!^{\frac{-n}{(n+1)!}}(n+1)^{\frac{1}{(n+1)!}} \vee 1$

$(n+1)^{\frac{1}{(n+1)!}} \vee n!^{\frac{n}{(n+1)!}} \;\Big|\; {}^{(n+1)!}$

$n+1 \vee (n!)^n$

$< \; (\forall n>1, n\in\mathbb{N}) \Longrightarrow \sqrt{\dfrac{a_{n+1}}{a_n}} < 1$

$\Longrightarrow \{\sqrt{a_n}\}_{n=1}^{\infty} \searrow$

2) $\forall n\in\mathbb{N}\ \sqrt{a_n} > 0 \Longrightarrow$ bounded below

$1) \wedge 2) \Longrightarrow \{\sqrt{a_n}\}_{n=1}^{\infty}$ has limit $:= L$.

* $\lim\limits_{n=\infty} a_n = 1 = \lim\limits_{n=\infty}(\sqrt{a_n})^2 = \left[\begin{array}{l}\text{now can} \\ \text{do this}\end{array}\right] = \lim\limits_{n\to\infty}\sqrt{a_n}\cdot\lim\limits_{n\to\infty}\sqrt{a_n} = L^2$,

$L = 1$ since positive. $\Longrightarrow$

$\lim\limits_{n\to\infty} (n!)^{\frac{1}{2(n!)}} = 1.$

9. $\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{2n} = \lim_{n\to\infty}\left(\left(1+\frac{1}{n+1}\right)^{n}\right)^2 = \lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)}\right)^2 = \left[\begin{array}{l}\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1} = e, \\ \text{subsequence of } \left\{\left(1+\frac{1}{n}\right)^{\frac{1}{n}}\right\}_{n=1}^{\infty};\end{array}\right] =$

$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right) = 1$ by lim. arithmetic

$= \frac{e^2}{1^2} = e^2.$

10. $\lim_{n\to\infty} \sqrt[n]{2^n+6^n} = \lim_{n\to\infty} x_n$

$6 = \sqrt[n]{6^n} < x_n = \sqrt[n]{2^n+6^n} < \sqrt[n]{6^n+6^n} = \sqrt[n]{2\cdot 6^n} = 6\cdot\sqrt[n]{2}$

$6 \xrightarrow{\infty} 6$, $\quad 6\cdot\sqrt[n]{2} \xrightarrow{\infty} 6\cdot 1$ (theorem $\lim_{n\to\infty}\sqrt[n]{a} = 1$) $= 6$

Since $\lim_{n\to\infty} 6 = 6$, $\lim_{n\to\infty} 6\sqrt[n]{2} = 6$, and $\sqrt[n]{2^n+6^n} \in (6, 6\sqrt[n]{2})$,
by 2-policemen theorem $\underline{\lim_{n\to\infty} x_n = 6}$