Lecture "Experimental Physics I"

(Prof. Dr. R. Seidel)

Lecture 29

Standing waves

- Standing waves in 1D and 2D
- Wave reflections at ends and media interfaces
- Huygen’s principle
- Reflection and refraction of waves

1) Standing waves in 1D

A) Interference of counter propagating waves

Now we will go back to linear waves and study what happens when they are counter- propagating, i.e. when they have opposing directions. A wave propagting towards increasing position x is described by a positive wave number:

$$\xi_1(x,t) = \xi_0 \cos(kx - \omega t)$$

A wave propagating towards decreasing position x is described by a negative wave number:

$$\xi_2(x,t) = \xi_0 \cos(-kx - \omega t - \varphi)$$

Superposition of the two waves provides with the corresponding trigonometric identity

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

the following product:

$$\xi = \xi_1 + \xi_2 = 2\xi_0 \underbrace{\cos(kx + \varphi/2)}_{\text{position}-\text{dep. amplitude}} \underbrace{\cos(\omega t + \varphi/2)}_{\text{stationary oscillation}}$$

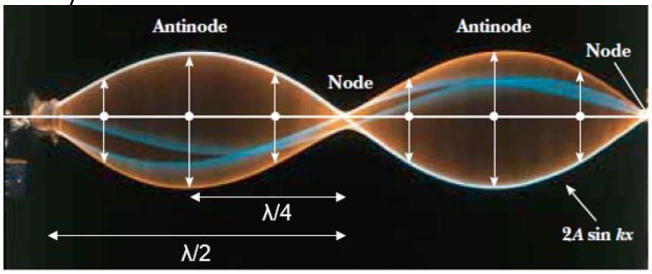
This formula is similar to the one we derived for the modes of the oscillating string. It consists of a position dependent amplitude and a simple hamonic and stationary oscillation. We can thus conclude:

Counterpropagating waves with equal amplitude and frequency cause **stationary oscillations with a position dependent amplitude** in form of a sinusoidal pattern. This is called a **standing wave** that (as for the string oscillations) **contains nodes with zero amplitude** and **anti-nodes with maximum** amplitude.

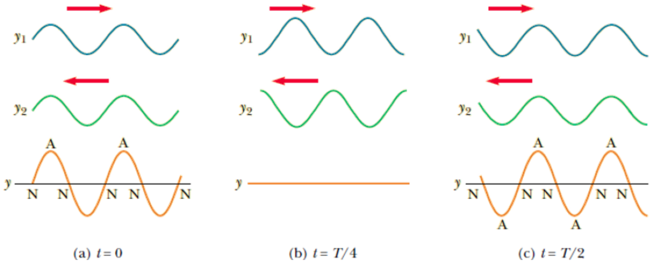
Anti-nodes are obtained when the total phase in the amplitude term equals a multiple of π:

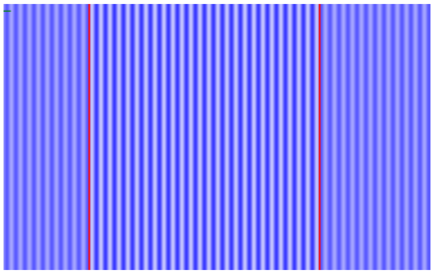
$$\frac{2\pi}{\lambda}x + \frac{\varphi}{2} = n\,\pi$$

ii.e. **anti-nodes as well as nodes occur every $\boldsymbol{\lambda/2}$ and the difference between a node to a neighboring anti-node is $\boldsymbol{\lambda/4}$**:



The formation of a standing wave by counterpropagating waves can be illustrated by graphically superimposing two counterpropagting waves for different fractions of the period (see below) and as well in simple animations (**see python animation & wave workshop animation with two line emitters**)





Counter propagating waves frequently occur due to reflections of waves at boundaries/walls where thus standing waves are obtained. This can be illustrated with:

Experiments:
- the wave machine with a fixed (reflecting) end that gets operated in resonance
- a helical string across lecture hall for which we have seen reflections before. Continuous wave excitation brings it into resonance such that standing waves are produced.
- the Ruben's tube: where standing sound wave causes sinusoidal modularion of the pressure amplitude and thus to a sinusoidal flame pattern

The standing wave pattern looks like our normal modes from the guitar, but **so far any wave frequency would give a standing wave pattern** on the guitar string such that one can not explain the discrete frequency spectrum of the normal modes. For this we have to consider the reflections of waves at the medium boundaries.

B) Reflection of a linear wave at a fixed or a free end

Let us look what happens to a wave when it gets reflected at a boundary.

- **Reflection at a fixed end:**

Let us first consider the reflection of a continuous wave at a **fixed end/boundary located at $\boldsymbol{x = 0}$**, where no displacement can take place.

The incident wave would be given by:

$$\xi_i(x,t) = \xi_i \cos(kx - \omega t)$$

The reflected wave would have a wave vector with opposite sign and may be phase shifted

$$\xi_r(x,t) = \xi_r \cos(-kx - \omega t - \varphi)$$

At the boundary $x = 0$ the total amplitude must always be zero, such that:

$$0 = \xi(0,t) = \xi_i(0,t) + \xi_r(0,t)$$

Inserting the plane wave solutions for the incident and the reflected wave provides:

$$0 = \xi_i \cos(-\omega t) + \xi_r \cos(-\omega t - \varphi)$$

and with $\cos(-x) = \cos(x)$:

$$\xi_i \cos \omega t = -\xi_r \cos(\omega t + \varphi)$$

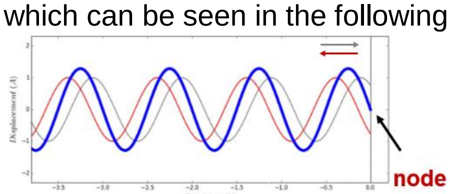
This equation must hold for all times, which is the case if $\varphi = 0$ and and $\xi_r = -\xi_i$. Thus, **upon reflection, the wave flips its amplitude in order to ensure a zero total amplitude**. Since amplitudes are typically given as positive values we can rewrite our result to

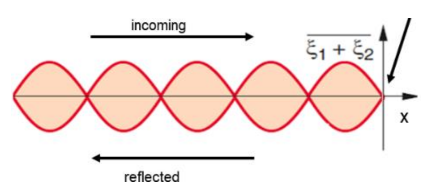
$$\xi_r \cos \omega t = -\xi_i \cos \omega t = \xi_i \cos(\omega t + \pi)$$

i.e. the **reflected wave has the same amplitude as the incident wave but experiences a phase shift of $\boldsymbol{\pi}$ upon reflection**. We have thus two counterpropagating waves with equal amplitude and therefore a **standing wave**. Using the formula from before we get for the displacement of the standing wave:

$$\xi = 2\xi_i \overbrace{\underbrace{\cos(kx + \pi/2)}_{-\sin(kx)}}^{\text{position−dep. amplitude}} \overbrace{\cos(\omega t + \pi/2)}^{\text{stationary oscillation}}$$

The position dependent amplitude of the resulting standing wave is given by a sine function, such that we have zero displacement and thus **a node at the reflection point $\boldsymbol{x = 0}$** and an **anti-node at $\boldsymbol{\lambda/4}$**, which can be seen in the following figures

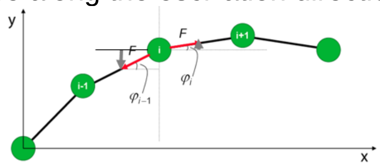




- **Reflection at a free end**

Wave reflections can also take place at a free end. Here the amplitude at the end is not constrained, however, the end does not support any forces along the oscillation direction (free end).

The general wave equations for the incoming and reflected waves are the same as for the fixed end. As boundary condition, we cannot use the amplitude anymore, but we can have a look at the force along the oscillation direction



For a prestressed string (or bead chain) the acting transversal force on a small mass element of length dx was given by:

$$F_{trans} \approx F(\underbrace{\varphi_i - \varphi_{i-1}}_{\Delta\varphi})$$

At a free chain end (at position zero) the transversal force on a mass element from the chain tension is not compensated anymore by the attachment such that only one side of the chain has to be considered :

$$F_{trans}(x=0) \approx -F\varphi_{x=0} = 0$$

For an infinitesimally small mass element this force would remain finite for a non-zero angle $\varphi_{x=0}$ and cause an infinite acceleration. Thus, we must demand a zero inclination of the string at a free end, such that

$$0 = \varphi_{x=0} \approx \left(\frac{\partial \xi}{\partial x}\right)_{x=0} = \frac{\partial}{\partial x}(\xi_1 + \xi_2) = -k\,\xi_i \sin(k\,0 - \omega t) + k\xi_r \sin(-k\,0 - \omega t - \varphi)$$

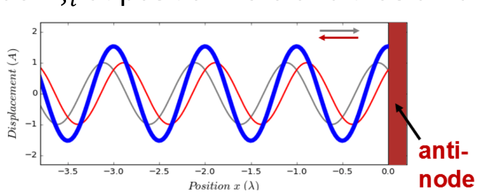
which can with $\sin(-x) = -\sin(x)$ be transformed to:

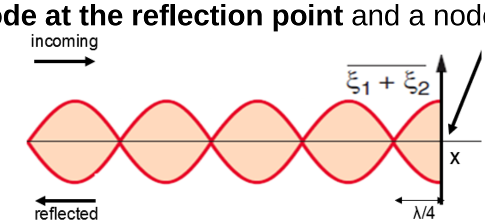
$$\xi_i \sin(\omega t) = \xi_r \sin(\omega t + \varphi)$$

This equation can only hold for all times if $\varphi = 0$ and and $\xi_i = \xi_r$. Thus, **upon reflection at a free end the wave keeps its amplitude and does not experience any phase shift**. The two counterpropagating waves with equal amplitude produce again a **standing wave**. Using the formula from before we get for the displacement of the standing wave:

$$\xi = 2\xi_i \overbrace{\cos kx}^{\text{pos.-dep.}} \quad \overbrace{\cos \omega t}^{\text{stationary osc.}}$$

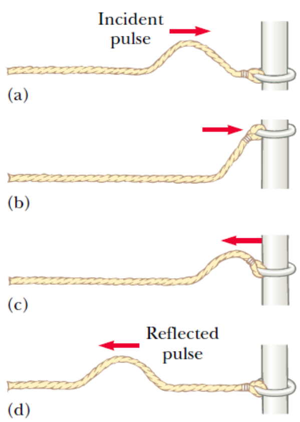
The position dependent amplitude is given by a cosine function, such that we have the maximum amplitude $2\xi_i$ at position zero and thus **an anti-node at the reflection point** and a node at $\lambda/4$:



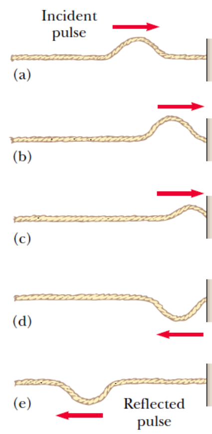


- **Pulse reflection at fixed and free end**

The phase change upon reflection is best seen for the reflection of a traveling wave pulse:



Free end



Fixed end

In analogy to the continuous wave we see that at:

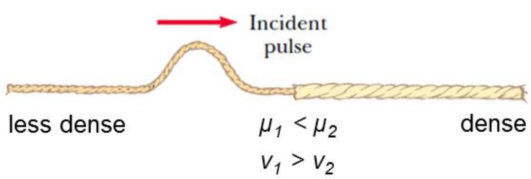
- at a **fixed end** the **pulse amplitude is inverted** after reflection, i.e. a **phase shift of $\boldsymbol{\pi}$** occurs (cosine goes to –cosine)
- at a **free end** the **pulse amplitude is maintained** after reflection, i.e. **no phase** shift occurs

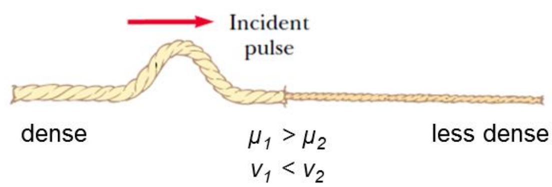
Experiment: reflection for closed and open end at wave machine

The **reflection at a fixed end can be viewed as the elastic reflection of a particle at a hard wall**, where its momentum gets inverted. Energy conservation demands the same wave energy before and after such that one has the same amplitudes for reflected and incident wave.

C) Reflection and transmission of a wave at denser or lighter media boundaries

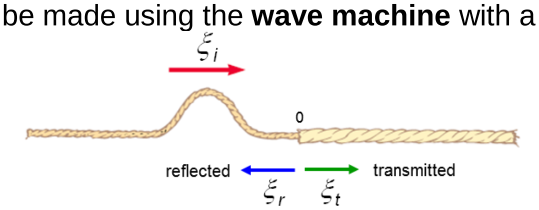
In addition to media ends, waves can encounter boundaries between dense and less dense media (e.g. sound waves entering a material or light entering a glass substrate). To understand the behavior at such a media boundary (in 1D), we look at a rope, composed of a lighter and a heavier section with linear mass densities of μ_1 and μ_2.





Since the phase velocity for a rope wave is given by $v_{Ph} = \sqrt{F/\mu}$ and the tension is the same throughout the rope, the phase velocities in the two sections are different, such that a higher velocity is obtained in the less dense medium.

Experiment: Let us have a look what happens at such a boundary using the **long spiral cord** with an additional mass in the center. We see that at the media boundary (lighter to heavier section) **both transmission and reflection of the incident wave are obtained** at the same time. The same observation can be made using the **wave machine** with a heavier element.



Derivation of wave transmission and reflection at a media boundary **(not part of lecture):**

From the experiment we learned that we have now to consier three waves – in incoming, a reflected and a transmitted wave. Considering an incoming plane wave, we can express the three waves in complex notation as:

$$\xi_i(x,t) = \xi_{i0} e^{i(k_i x - \omega_i t)}$$
$$\xi_r(x,t) = \xi_{r0} e^{i(-k_r x - \omega_r t)}$$
$$\xi_t(x,t) = \xi_{i0} e^{i(k_t x - \omega_t t)}$$

Only the reflceted plane wave has a negative wave number. Potential phase shifts of the waves upon reflection/transmission are comprised in the complex amplitudes of the waves.

Directly at the boundary that shall be placed at $x = 0$ we can formulate the following **continuity conditions**:

- the displacements at the boundary in medium 1 and 2 must be the same:
$$\overbrace{\xi_i(0,t) + \xi_r(0,t)}^{\text{medium 1}} = \overbrace{\xi_t(0,t)}^{\text{medium 2}}$$
- the transversal force at the boundary must be the same in medium 1 and 2, otherwise a finite force would accelerate an infinitessimally small mass element, i.e.
$$\overbrace{F\left(\frac{\partial}{\partial x}(\xi_i + \xi_r)\right)_{x=0}}^{\text{medium 1}} = \overbrace{F\left(\frac{\partial \xi_t}{\partial x}\right)_{x=0}}^{\text{medium 2}}$$

This provides for the condition of equal displacements at the boundary:

$$\xi_{i0} e^{i(-\omega_i t)} + \xi_{r0} e^{i(-\omega_r t)} = \xi_{t0} e^{i(-\omega_t t)}$$

Since this condition holds for all times, we must have the same frequencies for all waves , since sinusoidal functions are orthogonal to each other. This provides:

$$\omega_i = \omega_r = \omega_t = \omega$$

We have then however different wavenumbers for the two sections of the medium due to the different phase velocities:

$$k_i = k_r = k_1 = \frac{\omega}{v_1} = \frac{2\pi}{\lambda_1} \quad and \quad k_t = k_2 = \frac{\omega}{v_2} = \frac{2\pi}{\lambda_2}$$

This holds also for the **wavelengths, which are thus proportional to the phase velocity** in the media.

$$\lambda \propto v_{Ph}$$

For identical frequencies the displacement equation now provides:

$$\xi_{i0} + \xi_{r0} = \xi_{t0}$$

Now we can use the continuity of the transversal force at the boundary. Inserting the individual equations for the different waves and taking the derivatives provides:

$$i\xi_{i0} k_1 e^{-i\omega t} - i\xi_{r0} k_1 e^{-i\omega t} = i\xi_{t0} k_2 e^{-i\omega t}$$

and thus:

$$\xi_{i0} k_1 - \xi_{r0} k_1 = \xi_{t0} k_2$$

Replacing ξ_{r0} using the amplitude equation gives

$$\xi_{i0} k_1 - (\xi_{t0} - \xi_{i0}) k_1 = \xi_{t0} k_2$$

Transformation gives then the **amplitude of the transmitted wave**:

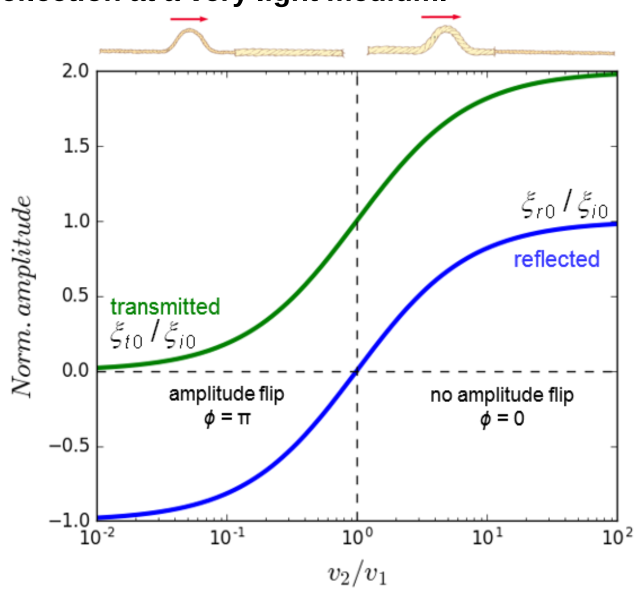
$$\xi_{t0} = \xi_{i0} \frac{2k_1}{k_1 + k_2} = \xi_{i0} \frac{2v_2}{v_1 + v_2}$$

Where we used $k = \omega / v$ to obtain the right side. Inserting the amplitude of the transmitted wave into the amplitude equation yields for the amplitude of the reflected wave:

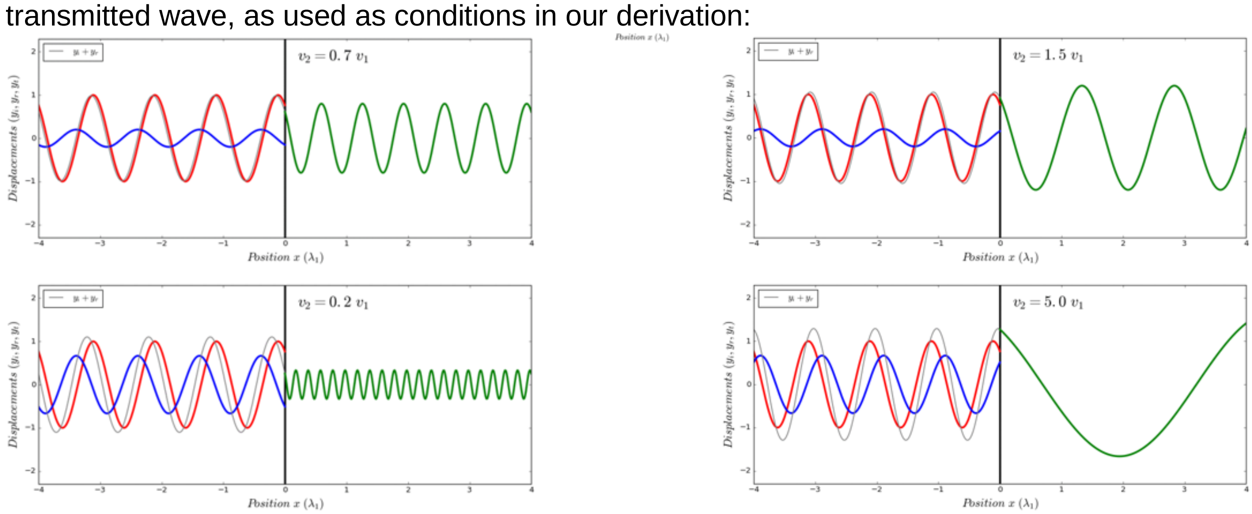
$$\xi_{r0} = \xi_{i0} \frac{k_1 - k_2}{k_1 + k_2} = \xi_{i0} \frac{v_2 - v_1}{v_1 + v_2}$$

When plotting these formulas, we see:

i) The **amplitude of the reflected wave is only zero if the phase velocity does not change**. Vice versa **any media boundary at which the phase velocity changes leads to a partial wave reflection**.
ii) The **reflected wave is phase shifted by π** (amplitude flip) compared to the incident wave upon **reflection at the denser medium. No phase shift occurs upon reflection at a less dense medium.**
iii) **Maximum transmission** and low reflection occur **for similar phase velocities**. **Maximum reflection** occurs for **strongly deviating phase velocities**.
iv) The **transmitted amplitude is zero** for reflection **at a very dense medium** and **twice** the initial amplitude **for reflection at a very light medium.**



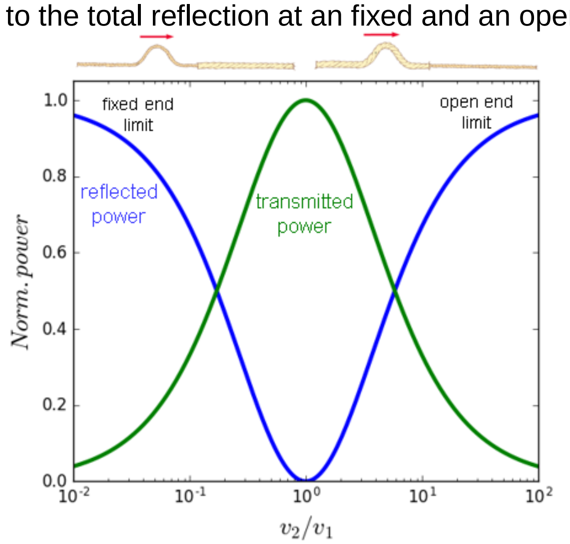
This is nicely illustrated in simulations using the obtained amplitudes, where also the different wavelength in the two media are visualized. The sum of the displacements of incident and reflected wave (gray line in plot) equals in magnitude and slope the displacement of the transmitted wave, as used as conditions in our derivation:



It is somewhat surprising that the amplitude of the transmitted wave almost doubles at the boundary from a dense to a very light medium, since energy conservation must hold, i.e. that incident power must equal the sum of reflected and transmitted power. This can be understood using the formula for the power propagated by a mechanical wave, which was proportional to the medium density. A light medium can despite a large amplitude transport only little power. Transforming the power equation using the relationship ${v_{Ph}}^2 = F/\mu$ for a wave in a prestressed string yields:

$$P = I\,A = \frac{1}{2} v_{Ph} \underbrace{\rho\,A}_{\mu} {\xi_0}^2 \omega^2 = \frac{1}{2} F^2 \omega^2 \frac{{\xi_0}^2}{v_{Ph}}$$

Plotting the normalized power confirms energy conservation and shows that **for strongly deviating phase velocities almost all power is in the reflected wave** and almost no power is in the transmitted wave. These limits, i.e. complete reflection at an infinitely dense or an infinitely light medium, correspond to the total reflection at an fixed and an open end, respectively.



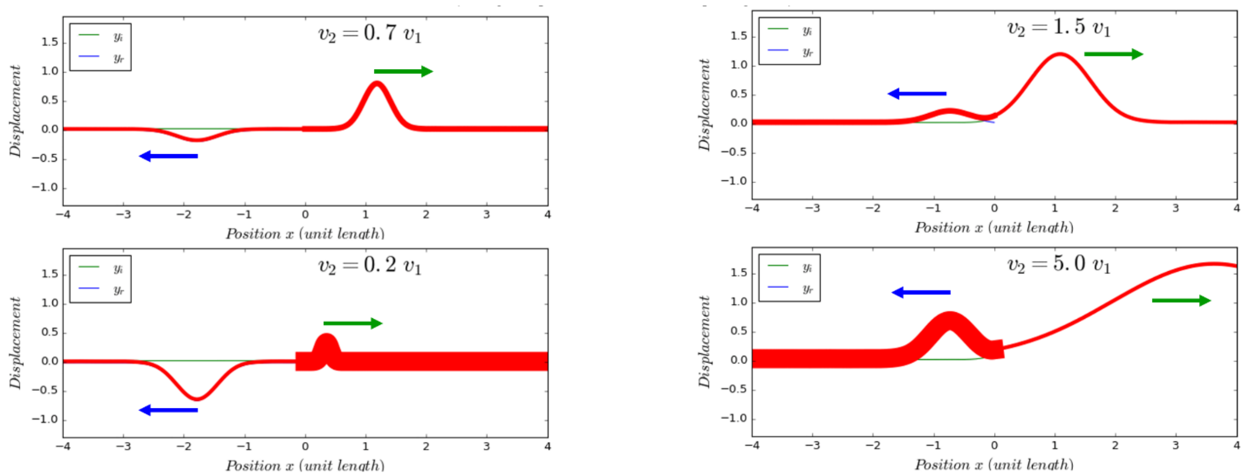
We can formally proof energy conservation by inserting the obtained amplitudes into the power formula and summing up reflected and transmitted power:

$$P_r + P_t = \frac{1}{2} F^2 \omega^2 \frac{{\xi_{0i}}^2}{(v_1+v_2)^2} \left(\frac{(v_1-v_2)^2}{v_1} + 2 \right) = \frac{1}{2} F^2 \omega^2 \frac{{\xi_{0i}}^2}{(v_1+v_2)^2} \left(\frac{(v_1-v_2)^2}{v_1} + \frac{2{v_2}^2}{v_2} \right)$$

$$P_r + P_t = \frac{1}{2} F^2 \omega^2 \frac{{\xi_{0i}}^2}{(v_1+v_2)^2} \left(\frac{(v_1+v_2)^2}{v_1} \right) = \frac{1}{2} F^2 \omega^2 \frac{{\xi_{0i}}^2}{v_1} = P_i$$

Overall, wave reflection and transmission at a boundary can be well understood when comparing it to the central **elastic collision of a moving with a resting particle.** In fact, the amplitude formulas have the same form as for the collision when replacing the initial amplitude with the initial particle velocity and the phase velocities with the particle masses. Full energy transmission in an elastic collision (as in pool billiards) can only be achieved for spheres of equal masses. If a smaller mass collides with a larger mass, its velocity gets inverted, corresponding to the flipped amplitude of the reflected wave. For a larger mass hitting a smaller mass the colliding particles maintains its direction (no amplitude flip for reflected wave), while the initially resting particle gains twice the speed.

Wave behavior at boundaries can be even better be visualized by looking at pulse waves that can be constructed by a set of sinusoidal waves (see later chapters):



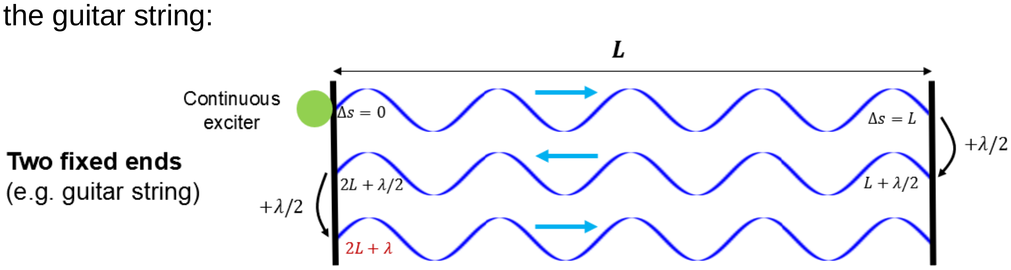
Again, we see that at a boundary **from a lighter to a denser** medium we have **amplitude inversion for the reflection wave** (phase shift of π) upon reflection. At the boundary **from a denser to a lighter** medium we have **no amplitude inversion** (no phase shift).

This principle is quite general. Very similar phenomena happen to light which is an electromagnetic wave at boundaries to denser/less dense optical media. This causes e.g. the unwanted reflection at the surfaces of glasses.

Experiment: Electromagnetic wave reflection at an end of a coaxial cable, where a shortcut corresponds to a fixed end and not-connected ends to a free end for the voltage. One sees for the voltage of the reflected pulse an amplitude flip for the fixed and no amplitude flip for the open end. An impedance matching end by connecting a properly chosen resistor at the end (that acts like a damping element) avoids any reflection.

D) (Resonant) standing waves

For the guitar string oscillation (**two fixed ends**) we previously found selected oscillation modes. These can be seen as a standing wave of an incoming wave that due to reflections at the string ends propagates multiple times forward and backward. The presence of distinct mode frequencies can be explained by resonant conditions for which all the different forward and backward propagating waves are constructively interfering.
Let us consider a wave that starts at one end. **A resonant mode occurs only if the wave - after being reflected at the distal end and then again at the original end - constructively interferes with the original incoming wave**. To calculate the phase shift between the double reflected and the incoming wave, one has to consider the phase shifts that may occur upon reflection.
Let us have a look at such **a resonator that contains 2 fixed ends** as we have for example for the guitar string:



The total path difference of the double reflected wave compared to the incoming wave corresponds to twice the resonator length, since it travels forward & backward in the resonator plus twice $\lambda/2$ for the phase jump of π at each fixed end:

$$\Delta s = \underbrace{2L}_{\substack{forward\ and \\ backward\ path}} + \underbrace{\lambda/2}_{\substack{right \\ end}} + \underbrace{\lambda/2}_{\substack{left \\ end}} = 2L + \lambda$$

We have **constructive interference** between incoming and double reflected wave **und thus resonance**, if the total path difference equals a multiple of the wavelength:

$$\Delta s = 2L + \lambda = n\lambda$$

Transformation provides:

$$L = (n-1)\frac{\lambda}{2} \qquad (n \geq 2)$$

i.e. we have **resonance when the resonator length equals multiples of $\boldsymbol{\lambda/2}$**, i.e. even multiples of $\lambda/4$. This is the same condition we found for the string vibration modes. Thus, the **normal modes of a string can be seen as constructively interfering planes waves that bounce back and forth within the resonator**. The standing wave pattern is actually produced by the interference of the forward with the backward waves.

We have **destructive interference** if the total path difference corresponds to odd multiples of the half wave length

$$\Delta s = 2L + \lambda = (2n+1)\frac{\lambda}{2}$$

Destructive interference is thus obtained for odd multiples of $\boldsymbol{\lambda/4}$:

$$L = (2n-1)\frac{\lambda}{4}$$

In this case the waves that bounce forth and back in the resonator cancel each, such that the wave amplitude is quenched and no energy can be taken up by the resonator (similar as for a driven oscillation far away from the resonance frequency).

We see that only a discrete set of lengths gives us fully constructively interfering waves, such that we obtain a **discrete spectrum of resonance frequencies** in agreement with the discrete mode spectrum. For a string with two fixed ends we get according to our considerations resonant standing waves if the half wave length is an integer fraction of the distance:

$$L = n\frac{\lambda_n}{2}$$

Furthermore, we know the relation for the phase velocity:

$$v_{Ph} = \sqrt{\frac{F}{\mu}} = \lambda_n f_n$$

And we get for the resonance frequencies:

$$f_n = \sqrt{\frac{F}{\mu}}\frac{1}{\lambda_n} = \sqrt{\frac{F}{\mu}}\frac{1}{2L}n$$

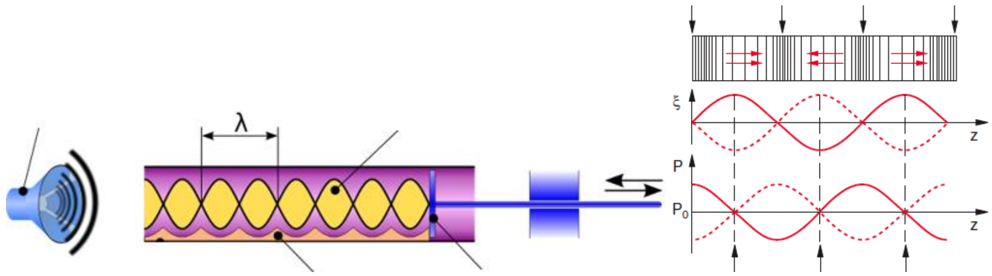
This is exactly the **same solution as for the normal modes of the oscillating string** where the frequency is linear to the mode number. This provides a number of important conclusions:

Note that **often only the resonant case is called standing wave**, since in the non-resonant case no standing waves with significant amplitude are obtained.

Experiments:

- Standing wave of a **helical spring**: a mechanical example of a standing longitudinal wave with two fixed ends

- Standing wave on a **wire ring**: another mechanical standing wave for transversal excursions but in this case with periodic boundary conditions, such that the ring length should equal always $\lambda/2$.
- **Kundt's tube**: To directly reveal the standing wave pattern one can use the Kundt's tube. Here the cork powder distribution in a tube is influenced by a standing sound wave. At the position of large amplitude (of air molecules) the powder overcomes the static friction and settles down at positions of low amplitude (pressure anti-nodes). Displacement and pressure nodes/antinodes are shifted by λ/4 according to our earlier derivation.



The distance between the nodes corresponds to $\lambda/2$, such that the speed of sound can be obtained from the applied frequency of 2 kHz according to $v_{Ph} = \lambda f = 340\ m/s$

If our resonator has also **open/free ends** (for pipes and other wind instruments) **the resonance conditions change:**

- For **two open ends** we have no phase shifts upon reflection such that the total path difference becomes:

  $$\Delta s = 2L$$

  As before we get constructive interference if $\Delta s = 2L = n\lambda$. Transformation provides:

  $$L = n\frac{\lambda}{2} \qquad (n \geq 1)$$

  i.e. we have again **resonance when the resonator length equals multiples of $\boldsymbol{\lambda/2}$**. The wavelength thus decreases with $1/n$, such that the frequency increases with n, resulting in a similar overtone spectrum as for the guitar.
- For **one open and one closed end,** we have to consider only one additional phase shift occurring at the closed end. For the path difference and the resonance condition we can thus write:

  $$\Delta s = 2L + \lambda/2 = n\lambda$$

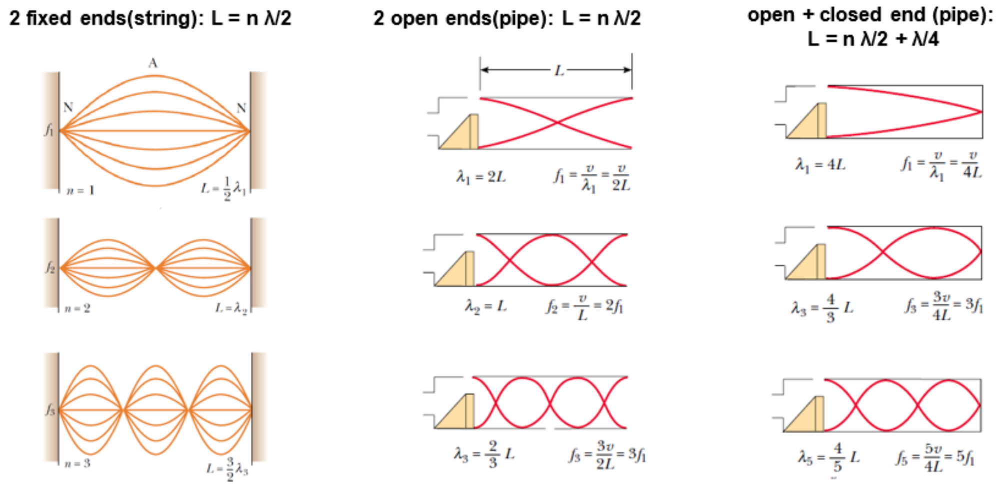
  Transformation provides:

  $$L = n\frac{\lambda}{2} - \frac{\lambda}{4} \qquad (n \geq 0)$$

  We thus **get resonance** if:

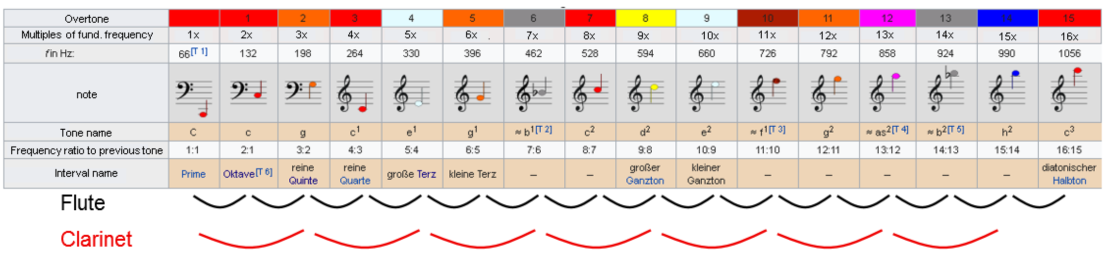
  $$L = \frac{1}{4}\lambda, \frac{3}{4}\lambda, \frac{5}{4}\lambda, \frac{7}{4}\lambda, \ldots$$

  In this case, the overtones correspond thus to 3,5,7,… times the fundamental frequency, such that the first overtone occurs at the triple frequency of the fundamental frequency.
  These end dependent resonance conditions can be understood by drawing the corresponding node and anti-node positions. In real life, they can be observed on different music instruments (see below).



Experiments:

- Frequency spectrum of a flute and a clarinet: A **flute has 2 open ends** such that the overtone spectrum shows every overtone with the first overtone being one octave higher. The **clarinet has one open and one closed end**, such that only the odd harmonics are appearing. The first overtone has a 3-fold larger frequency and is thus a different tone than the fundamental frequency, which characteristically influences the sound of the clarinet:



Overtone		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Multiples of fund. frequency	1x	2x	3x	4x	5x	6x	7x	8x	9x	10x	11x	12x	13x	14x	15x	16x
f in Hz:	66[T 1]	132	198	264	330	396	462	528	594	660	726	792	858	924	990	1056
note																
Tone name	C	c	g	c^1	e^1	g^1	≈ b^1[T 2]	c^2	d^2	e^2	≈ f^1[T 3]	g^2	≈ as^2[T 4]	≈ b^2[T 5]	h^2	c^3
Frequency ratio to previous tone	1:1	2:1	3:2	4:3	5:4	6:5	7:6	8:7	9:8	10:9	11:10	12:11	13:12	14:13	15:14	16:15
Interval name	Prime	Oktave[T 6]	reine Quinte	reine Quarte	große Terz	kleine Terz	–	–	großer Ganzton	kleiner Ganzton	–	–	–	–	–	diatonischer Halbton

Flute

Clarinet

We demonstrate the different overtone spectra using a pipe with two open ends. Its fundamental mode has a wavelength of $\lambda = 2L$. Closing the outlet end with the hand provides the clarinet spectrum. Acoustically, this can be heard since the fundamental mode has now a wavelength of $\lambda = 4L$, such that its frequency is half the frequency of the open pipe. This represents a shift by one octave to lower frequencies.

- **High voice when inhaling helium**: our voice sound is primarily influenced by allowing certain overtones to form standing waves. When inhaling helium the voice becomes much higher, since the low frequencies have due to the increased speed of sound a much larger wave length, such that they cannot form standing waves anymore. Only the higher frequencies with lower wave lengths are in resonance and are heard.
- **Overtone singing**: see movie, is a technique to emphasize particular overtones on top a fundamental tone, such that two melodies can be sung in parallel.

2) Standing waves in 2D

So far, we looked only at standing waves/modes in one dimension. Modes are however also possible in two- or three dimensions. A typical two-dimensional example is a membrane that is stretched within a rectangular frame of dimensions $a \times b$ (e.g. a drum), such that we have **zero membrane displacement at:**

$$x = 0, \qquad x = a, \qquad y = 0, \qquad y = b.$$

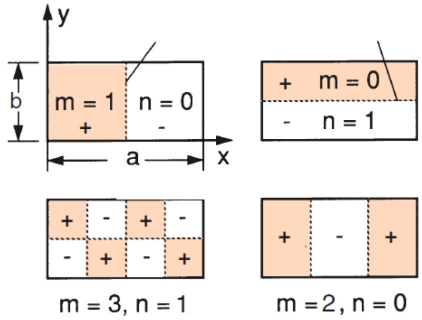
The modes are standing waves and thus must be solutions to the two-dimensional wave equation:

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = \frac{1}{v_{Ph}^2}\frac{\partial^2 \xi}{\partial t^2}$$

Modes that solve the equation for the given boundary conditions, i.e. the **normal modes,** have the form:

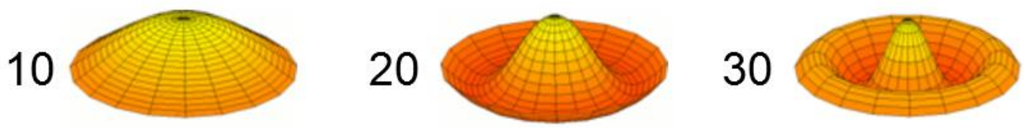
$$\xi_{m,n}(x,y) = \xi_0 \sin\frac{(m+1)\pi x}{a}\sin\frac{(n+1)\pi y}{b}\cos\omega_{mn}t$$

Where m and n are the number of node lines perpendicular to x any y. Each normal mode can be seen as **superposition of two counterpropagating waves, one for each coordinate axis. The following amplitude patterns are thus produced:**



A section parallel to either the x or the y axis provides exactly the simple sinusoidal normal modes of a string. Again, **any other vibration** can be expressed as **superposition of the normal modes**

In case of circular or spherical symmetry the modes are so called spherical harmonics (e.g. **normal modes of a drum**)



Similar modes are used to describe electrons as standing waves around atoms, since Schrödinger's equation is also something like a wave equation.

Experiment:

- Chladni sound patterns are a famous visualization of the node patterns of 2D plates in case of "open" boundary conditions
- Chladni sound patterns on corpus of a guitar excited with sound from a loud speaker (see slide)

Lecture 29: Experiments

1) Wave machine operated in resonance
2) Helical string across lecture hall with visualization of wave propagation can be brought into resonance such producing standing waves
3) Ruben's tube: standing sound wave causes sinusoidal modulation of pressure amplitude
4) Reflection for closed and open end at wave machine
5) Long helical string with mass in center at which partial reflection occurs (local distortion)
6) Wave machine where one element has additional masses such that partial reflection and transmission occurs.
7) Electromagnetic wave reflection at an end of a coaxial cable, where a shortcut corresponds to a fixed end, not-connected ends to a free end and an impedance matching end to a pure translation.
8) Standing wave of a helical spring
9) Kundt's tube: cork powder in a tube with a standing sound wave at position of large amplitude (of air molecules) the static friction is overcome and less powder is found. Displacement and pressure nodes/antinodes are shifted by $\lambda/4$ according to our earlier derivation
10) Standing wave on a wire ring
11) Frequency spectrum of a flute (2 open ends) and a clarinette (1 open, 1 closed end) using a pipe
12) Mickey-mouse voice after inhaling helium
13) Chladni sound patterns: node patterns in 2D for "open" boundary
14) Chladni sound patterns on corpus of a guitar excited with sound from a loud speaker