HW-9

(1) $\int_0^1 \frac{\ln x}{\sqrt{x}}dx = \lim_{A\to 0+0}\int_A^1 \frac{\ln x}{\sqrt{x}}dx = \lim_{A\to 0+0}\left[2\sqrt{x}\ln x\Big|_A^1 - \int_A^1 2\underbrace{\sqrt{x}\cdot\frac{1}{x}}_{\frac{1}{\sqrt{x}}}dx\right] =$

$= \lim_{A\to 0+0}\left[0 - 2\sqrt{A}\ln A - 4\sqrt{x}\Big|_A^1\right] = \lim_{A\to 0+0}\left[\underset{\downarrow 0}{-2\sqrt{A}\ln A} - 4 + \underset{\downarrow 0}{4\sqrt{A}}\right] =$

$= -4$ $\quad\Big[$ here $\lim_{A\to 0}\sqrt{A}\ln A = \lim_{A\to 0+0}\frac{\ln\sqrt{A}^2}{1/\sqrt{A}} = 2\lim_{\sqrt{A}\to 0+0}\frac{\ln\sqrt{A}}{1/\sqrt{A}} =$

$= 2\lim_{A\to 0+0}\frac{\ln A}{1/A} = \left(\frac{\infty}{\infty}\right) = \left[\text{L'Hopitals rule}\right] = 2\lim_{A\to 0}\frac{1/A}{-1/A^2} =$

$= -2\lim_{A\to 0}A = 0\Big].$

(2) $\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}} = \int_0^1 \frac{dx}{\underbrace{(1+1-x)\sqrt{1-x}}_{f(x)}}$, here $f(x) \geq g(x) = \sqrt{1-x}^3$,

so $\frac{1}{f(x)} \leq \frac{1}{g(x)}$.

But $\int_0^1 \frac{dx}{(\sqrt{1-x})^3} = \int_0^1 \frac{dx}{(1-x)^{3/2}} = \lim_{A\to 1-0}\int_0^A \frac{dx}{(1-x)^{3/2}} =$

$= \lim_{A\to 1-0}\int_0^A (1-x)^{-3/2}dx = \lim_{A\to 1-0}\frac{(1-x)^{-1/2}}{-1/2}(-1)\Big|_0^A =$

$= \lim_{A\to 1-0} 2\left[\frac{1}{\sqrt{1-A}} - \frac{1}{1}\right] = -2$ exists, so $\int_0^1 \frac{dx}{(\sqrt{1-x})^3}$ converges.

But then by comparison test $\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}}$ converges.

(3) $\int_1^{+\infty} \frac{dx}{x^2+4x-5} = \lim_{\substack{A\to+\infty\\ B\to 1}}\int_B^A \frac{dx}{x^2+4x-5} = \left[\text{same as splitting } \lim_{B\to 1}\int_B^c \frac{dx}{x^2+4x-5} + \lim_{A\to\infty}\int_c^A \frac{dx}{x^2+4x-5} \text{ for } \forall c\in(1,+\infty)\right]$

$= \left[\frac{1}{x^2+4x-5} = \frac{1}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5} = \frac{1/6}{x-1} + \frac{-1/6}{x+5} = \frac{1}{6}\left[\frac{1}{x-1} - \frac{1}{x+5}\right]\right]$

$= \lim_{\substack{A\to+\infty\\ B\to 1}}\frac{1}{6}\int_B^A \frac{1}{x-1} + \frac{1}{x+5}dx = \lim_{\substack{A\to\infty\\ B\to 1}}\frac{1}{6}\ln(x-1) - \ln(x+5)\Big|_B^A = \lim_{\substack{A\to\infty\\ B\to 1}}\frac{1}{6}\ln\frac{x-1}{x+5}\Big|_B^A =$

$= \lim_{\substack{A\to\infty\\ B\to 1}}\frac{1}{6}\ln\frac{1-\frac{1}{x}}{1+\frac{5}{x}}\Big|_1^A = \lim_{A\to\infty}\underset{\downarrow 0}{\frac{1}{6}\ln\frac{1-\frac{1}{A}}{1+\frac{5}{A}}} - \lim_{\substack{A\to\infty\\ B\to 1}}\underset{\downarrow -\infty}{\frac{1}{6}\ln\frac{1-\frac{1}{B}}{1+\frac{5}{B}}} = -\infty$

diverges.

(4) $\int_1^{+\infty} \frac{x\cos(x^2)dx}{1+x} = \int_1^{+\infty} f(x)g(x)dx$ where 1) $f(x) = x\cos(x^2)$ is continuous at $[1,+\infty)$, its primitive

$F(x) = \frac{1}{2}\sin(x^2) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ is bounded

2) $g(x) = \frac{1}{1+x}$ is non-negative, $\searrow$, differentiable on $[1, +\infty)$,

$g'(x) = -\frac{1}{(1+x)^2}$ is continuous on $[1, +\infty)$.

3) $\lim\limits_{x \to +\infty} g(x) = \lim\limits_{x \to \infty} \frac{1}{1+x} = 0.$

Therefore by Dirichle-Abel test $\int$ converges.

(5) $\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + \dots + \frac{1}{(4n+1)\cdot(4n+5)} + \dots = \sum\limits_{n=0}^{\infty} \frac{1}{(4n+1)(4n+5)}$

$S_N = \sum\limits_{n=0}^{N} \frac{1}{(4n+1)(4n+5)} = \sum\limits_{n=0}^{N} \frac{1}{4}\left[\frac{1}{4n+1} - \frac{1}{4n+5}\right] = \frac{1}{4}\left[1 - \frac{1}{4N+5}\right]$

[for example $\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \dots = \frac{1}{4}\left(\frac{1}{1} - \frac{1}{5}\right) + \frac{1}{4}\left(\frac{1}{5} - \frac{1}{9}\right) + \dots = \frac{1}{4}\left(1 - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{9}} + \dots\right)$]

So $\sum\limits_{n=0}^{\infty} \frac{1}{(4n+1)(4n+5)} = \frac{1}{4} \lim\limits_{N \to \infty} \left[1 - \frac{1}{4N+5}\right] = \frac{1}{4}.$

(6) $\sum\limits_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \left(\text{see geometric series derivation in old HW}\right) = \frac{\frac{3}{5}}{1 - \frac{3}{5}} = \frac{3}{2}.$

(7) $\sum\limits_{n=1}^{\infty} n^2 e^{-\sqrt{n}}$ has same convergence as integral $\int\limits_{1}^{+\infty} x^2 e^{-\sqrt{x}} dx$ $\quad (x^2 e^{-\sqrt{x}} \geq 0 \searrow)$

But $\int x^2 e^{-\sqrt{x}} dx = \left[\begin{array}{l} u = -\sqrt{x}, \\ du = -\frac{dx}{2\sqrt{x}} \Rightarrow dx = 2u\,du \\ x^2 = u^4 \end{array}\right] = \int 2u \cdot u^4 e^u du = \left[\text{continue pattern, but show convergence}\right] = 2\int u^5 e^u du = [\text{by parts}] = 2\left[e^u u^5 - \int e^u \cdot 5u^4 du\right] \overset{\text{by parts}}{=} 2\left[e^u u^5 - \right.$

$\left. - 5 \cdot \left(e^u u^4 - \int e^u \cdot 4u^3 du\right)\right] \overset{bp}{=} \cancel{2\left[e^u u^5 - 5\right.}$ [by parts] $= \dots$

$= \left[\text{continue pattern, easily shown}\right] = \dots = 2\left[e^u u^5 - 5(e^u u^4 - 4(e^u u^3 - 3(e^u u^2 - 2(e^u u - u)))))\right]$

$+ C = F(x)$, $u(1) = -1$, $u \xrightarrow{x \to \infty} -\infty$.

But $\int\limits_{1}^{+\infty} x^2 e^{-\sqrt{x}} dx = \lim\limits_{A \to +\infty} F(x)\Big|_1^A = \lim\limits_{A \to \infty} \left(\underset{0}{F(A)} - \underset{\text{some number}}{F(1)}\right)$ exists.

(since $e^u \xrightarrow{u \to -\infty} 0$)

By integral test, since

$\int\limits_{1}^{+\infty} x^2 e^{-\sqrt{x}} dx$ converges $\Longrightarrow \sum\limits_{n=1}^{\infty} n^2 e^{-\sqrt{n}}$ converges

(8) $\sum\limits_{n=1}^{\infty} a_n$ where $a_n = \frac{2^n \cdot n!}{n^n}$.

$\lim\limits_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n \to \infty} \left(\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!}\right) = \lim\limits_{n \to \infty} \underbrace{\frac{2(n+1)}{(n+1)}}_{\to 1} \cdot \left(\frac{n}{n+1}\right)^n =$

$= 2 \lim\limits_{n \to \infty} \left(\frac{1}{1 + \frac{1}{n}}\right)^n = \frac{2}{\lim\limits_{\infty}\left(1 + \frac{1}{n}\right)^n} = \frac{2}{e} < 1.$

Therefore by D'Alembert test series converges.

(9) $\sum_{n=1}^{\infty} \underbrace{\frac{n!}{n^{\sqrt{n}}}}_{a_n}$

~~But $\lim_{n\to\infty} a_n \neq 0$, because $\lim_{n\to\infty} \frac{n!}{n^{\sqrt{n}}} \neq 0$ (proof?)~~

~~$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{(n!)^{1/n}}{n^{1/\sqrt{n}}} = \lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{(\sqrt{n}^{\frac{1}{\sqrt{n}}})^2} =$~~

But $\lim_{n\to\infty} a_n \neq 0$, because $\lim_{n\to\infty} \frac{n!}{n^{\sqrt{n}}} \neq 0$ consider

$n! = 1 \cdot 2 \cdot \ldots \cdot n$ ~~for even n (for odd~~ (analogous argumentation for odd)

$f(n) = \underbrace{1 \cdot n}_{\geq n} \cdot \underbrace{2 \cdot (n-1)}_{\geq n} \cdot 3(n-2) \cdot \ldots \cdot \underbrace{\frac{n}{2}\left(\frac{n}{2}+1\right)}_{\geq n}$

$\forall n \geq 2$, because $\frac{n}{2}\left(\frac{n}{2}+1\right) = \frac{n^2}{4} + \frac{n}{2}$, $\frac{n^2}{4} + \frac{n}{2} - n = \frac{n^2}{4} - \frac{n}{2} \geq 0$ when $\frac{n}{2} - 1 \geq 0$.

Then $f(n) \geq n^{\frac{n}{2}} \geq n^{\sqrt{n}}$ ($\frac{n}{2} \geq \sqrt{n}$, $\sqrt{n} \geq 2$ $\forall n \geq 4$)

$\Longrightarrow \lim_{n\to\infty} \frac{f(n)}{n^{\sqrt{n}}} = \lim_{n\to\infty} \frac{n!}{n^{\sqrt{n}}} \geq 1 \neq 0$.

Necessary condition violated.

Series diverges.

(10) $\sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{\sqrt{n}}{n+100}}_{a_n}$

$\forall n \in \mathbb{N}\ a_n \geq 0$, $a_n = \cancel{\frac{1}{\frac{n}{\sqrt{n}} + \frac{100}{\sqrt{n}}}}$ decreasing sequence (shown from $\frac{\sqrt{n+1}}{n+101} \cdot \frac{n+100}{\sqrt{n}} < 1$ or prev. HW's),

$\lim a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n} + \frac{100}{\sqrt{n}}} = 0 \Longrightarrow \sum_{n=1}^{\infty} (-1) a_n$ converges by Leibnitz test.