11. EOM and Recap

The Chapters 1–4 of my lecture notes provide the background to solve the following exercises. Your solution to the problems 11.1–11.4 should uploaded

to your Moodle account

as a PDF-file

by Wednesday, Jan 10, 10:30 (with a grace time till the start of the seminars).

The parts marked by ⋆ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

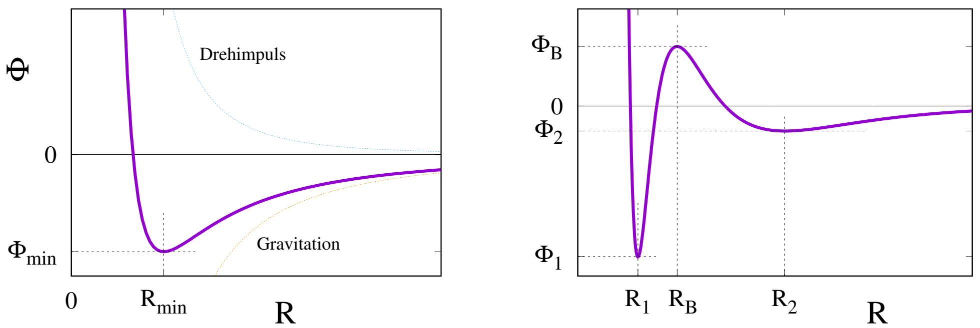
The self-test problems serve to check you understanding of vectors.

The bonus problem provides the solution to a problem that we introduced in class. It might take some extra effort to solve.

Problems

Problem 1. Phase-space portraits for the Kepler and the DLVO problem

The figures below show the effective potentials for the distance between two planets in the Kepler problem, and for the DLVO potential for the interaction of charged colloids.[1] Sketch the solutions for classical trajectories in these potentials in the phase space $(R, \dot{R})$.



[1] The DLVO theory predicts that there are two distinct avarage bond length for aggregates of two colloids. There is a strong bond of strength Φ_1 where the colloids have a small bond length R_1, and a weach bond of strength Φ_2 at a larger distance R_2. Between these two states there is an energy barrier of height Φ_B.

Problem 2. Conservative forces, potentials, and contour lines

We consider the force field

$$\mathbf{F}(\mathbf{r}) = \omega \times \big(\omega \times \mathbf{r}\big) \qquad \text{for} \quad \mathbf{r} \in \mathbb{R}^3$$

where ω is a constant vector in $\mathbb{R}^3$.

a) Show that the force can also be expressed as

$$\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\,\omega + c\,\mathbf{r}$$

where c is a real constant and $f(\mathbf{r})$ is a real-valued function of $\mathbf{r}$.

Determine c and $f(\mathbf{r})$.

b) Show that $\mathbf{F}$ is a conservative force.

c) Work out the line integral along the path $\mathbf{q} = s\,\mathbf{r}$, $0 \leq s \leq 1$ in order to show that the potential Φ associated to $\mathbf{F}$ takes the form

$$\Phi(\mathbf{r}) = K \;\big[(\mathbf{r}\cdot\omega)^2 - \omega^2\,\mathbf{r}^2\big]$$

where K is a constant factor. Determine K!

d) Show that $\Phi(\mathbf{r})$ is constant in every direction parallel to ω.

Hint: One possibility to approach this problem is to consider a line integral and look for a condition on the force such that the result does not change.

e) Determine the shape of the contour lines in a plane orthogonal to ω.

Sketch the contour lines and the gradient of the potential.

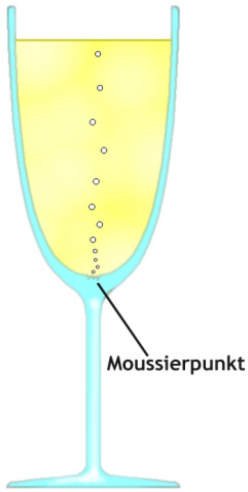
Problem 3. Bubbles rising in a fluid

Champaign owns its esprit from the sparkling of the rising air bubbles. When they arrive at the surface they burst and release odor and flavor. During its rise a gas bubble of radius R experiences an upward buoyant force due to Archimedes' principle

$$F_g = \frac{4\,\pi}{3}\, R^3\, (\varrho_f - \varrho_g)\, g$$

where ϱ_g and ϱ_f are the mass density of the gas and the surrounding fluid. Moreover, when it rises with a speed $\dot{z}(t)$ it also experiences a Stokes friction force

$$F_S = -6\,\pi\,\eta\,R\,\dot{z}(t)$$



a) Determine the EOM for the height of $z(t)$ of the gas bubble in the glass, and write the equation in the form

$$I = \ddot{z}(t) + c_1\,\dot{z}(t) + c_0\,z(t)\,.$$

How do the constant coefficients I, c_1 and c_0 depend on the parameters of the forces?

b) Determine the length and time scale, and a constant reference velocity such that the dimensionless position ξ of the bubble, and its dimensionless velocity $\zeta = \mathrm{d}\xi/\mathrm{d}\tau$ are described by the following homogeneous ODE

$$0 = \ddot{\xi}(\tau) + \dot{\xi}(\tau) \tag{11.1}$$

Hint: The choice of a constant reference velocity implies here that one selects the coordinate system that moves upwards with a constant velocity such that the bubble is at rest in this reference system at late times.

c) Sketch the velocity field in the phase space and some solutions of Equation (11.1).

⋆ d) Determine the solution of the dimensionless velocity $\zeta(\tau)$ and the position $\xi(\tau)$.

⋆ e) Determine $z(t)$ by substituting the definitions of the dimensionless units in the result for $\xi(\tau)$.

Compare your result to Eq. (4.3.3b) of my lecture notes.
What is the relation between these expressions?

Problem 4. The SIR model

Arguably the SIR model is the starting point of modeling infection dynamics. It splits a population into three groups of people: persons susceptible to the disease S, infected I, and those who recovered R and are immune. This approach adopts the following equations of motion

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} S(t) &= -\gamma\, S(t)\, I(t) && + \beta\, R(t) \\
\frac{\mathrm{d}}{\mathrm{d}t} I(t) &= \gamma\, S(t)\, I(t) - \rho\, I(t) \\
\frac{\mathrm{d}}{\mathrm{d}t} R(t) &= \qquad\qquad\quad\ \rho\, I(t) - \beta\, R(t)
\end{aligned}$$

Here, γ is the rate that a susceptible person in infected upon meeting an infected person, ρ is the rate of recovery from the disease, and *beta* the rate of loos of immunity are recovery. As a consequence, immune people turn back into susceptible.

a) Show that the overall population is conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(S(t) + I(t) + R(t) \Big) = 0\,.$$

b) Given that S, I, and R refer to fraction of the total population we will henceforth stipulate that $S(t) + I(t) + R(t) = 1$. Use this condition to eliminate $R(t)$ in the evolution, and incorporate a dimensionless time $\tau = \gamma\, t$. This will lead to the ODE

$$\begin{aligned}
\dot{S}(\tau) &= -I(\tau)\, \big(b + S(\tau)\big) + b\, \big(1 - S(\tau)\big) \\
\dot{I}(\tau) &= -I(\tau)\, \big(r - S(\tau)\big)
\end{aligned}$$

where the dot denotes the derivative with respect to τ. It involves the dimensionless parameters b and r. How do they depend on the rates γ, β, and ρ.

c) The conservation law allows us to discuss the evolution of the SIR model in the two dimensional phase space (S, I). Which portion of the (S, I) plane represents valid initial conditions of the dynamics?

Sketch the lines in the phase space where $\dot{S} = 0$ and where $\dot{I} = 0$. These lines are called nullclines. How does the phase-space flow cross the nullclines? Mark the phase-space flow by arrows in your sketch.

d) Fixed points of the dynamics lie at intersections of the nullclines. Which fixed points arise in the SIR dynamics? How do the cases $0 < r < 1$ and $r > 1$ differ?

$\star$ e) Linearize the EOM for trajectories that approach the vicinity of a fixed point, and discuss the parameter dependence of their stability. Provide sketches of the different cases.

Compare your results with those provided by this interactive Sage worksheet.

Self Test

Problem 5. Pulling a duck

A child is pulling a toy duck with a force of $F = 5\,\mathrm{N}$. The duck has a mass of $m = 100\,\mathrm{g}$ and the chord has an angle $\theta = \pi/5$ with the horizontal.[2]

a) Describe the motion of the duck when there is no friction.
   In the beginning the duck is at rest.

b) What changes when there is friction with a friction coefficient of $\gamma = 0.2$, i.e. a horizontal friction force of magnitude $-\gamma m g$ acting on the duck.

c) Is the assumption realistic that the force remains constant and will always act in the same direction? What might go wrong?

Problem 6. Car on an air-cushion

We consider a car of mass $m = 20\,\mathrm{g}$ moving – to a very good approximation without friction – on an air-cushion track. There is a string attached to the car that moves over a roller and hangs vertically down on the side opposite to the car.

a) Sketch the setup and the relevant parameters.

b) Which acceleration is acting on the car when the string is vertically pulled down with a force of $F = 2\,\mathrm{N}$. Determine the velocity $v(t)$ and its position $x(t)$.

c) Determine the force acting on a 200 g chocolate bar, in order to get a feeling for the size of the force that was considered in (b).

[2]For this angle one has $\tan\theta \approx 3/4$.

d) Now we fix the chocolate bar at the other side of the string. The equation can then be obtained based on energy conservation

$$E = E_{\text{kin}} + E_{\text{pot}} = \frac{m+M}{2}\, v^2 + Mgh = \text{const},$$

where M is the mass of the chocolate bar. Is the acceleration the same or different as in the cases (b) and (c)? Provide an argument for your conclusion.

Problem 7. Terminal velocity for turbulent drag

A golf ball falling in air experiences a drag force

$$\mathbf{F}_d = -\frac{\rho |u|^2}{2} c_d A \, \hat{\boldsymbol{u}}$$

where A is the cross section of the ball, ρ the density of air, $c_d \simeq 0.5$ is the drag coefficient, $\mathbf{u}$ the velocity of the golf ball, and $\hat{\boldsymbol{u}}$ a dimensionless vector pointing in the direction of $\mathbf{u}$.

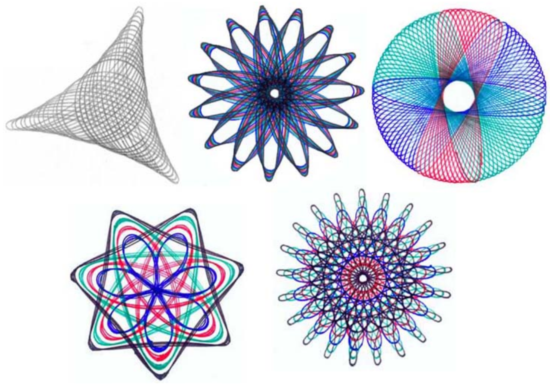
a) The drag coefficient is a dimensionless number that depends on the shape of the object that experiences drag. For the rest the expression for the drag force follows from dimensional analysis. Verify this claim.

b) A slightly more informed derivation of $\mathbf{F}_d$ is based on a discussion of the kinetic energy of the air. It asserts that drag arises because the ball has to push air out of its way. When moving it has to push air out of the way at a rate $A\,u$. The air was at rest initially and must move roughly with a velocity u to get out of the way. Subsequently, its kinetic energy is lost. Check out, how this leads to the expression provided for $\mathbf{F}_d$.

c) What is the terminal velocity of a golf ball that is falling out of the pocket of a careless hang glider?
Note: A golf ball has a mass $M \simeq 45\,\text{g}$, and a diameter $D \simeq 42.7\,\text{mm}$.

d) Use dimensional analysis to estimate the distance after which the ball acquires its terminal velocity, and how long it takes to reach the velocity.

Problem 8. Hypotrochoids, roulettes, and the spirograph



[Wikimedia Public domain]

A roulette is the curve traced by a point (called the generator or pole) attached to a disk or other geometric object when that object rolls without slipping along a fixed track. A pole on the circumference of a disk that rolls on a straight line generates a cycloid. A pole inside that disk generates a trochoid. If the disk rolls along the inside or outside of a circular track it generates a hypotrochoid. The latter curves can be drawn with a spirograph, a beautiful drawing toy based on gears that illustrates the mathematical concepts of the least common multiple (LCM) and the lowest common denominator (LCD).

a) Consider the track of a pole attached to a disk with n cogs that rolls inside a circular curve with $m > n$ cogs. Why does the resulting curve form a closed line? How many revolutions does the disk make till the curve closes? What is the symmetry of the resulting roulette? (The curves to the top left is an examples with three-fold symmetry, and the one to the bottom left has seven-fold symmetry.)

b) Adapt the description for the position of the retroreflector that we developed in the lecture[3] such that you can describe hypotrochoids, i.e. the position of the pen as function of how far the inner wheel has progressed.

⋆ c) Test your result by writing a program that plots the curves for given m and n.
Remark: More explanation and a Sage Notebook that you can use to start this analysis will be given in the wiki.

d) Determine a general expression for the length of the roulette.
For which ratios of d/r can you evaluate the expression by analogous calculations as performed for the cycloid?
How do the related roulettes look like?

[3] In the wiki you find a link to an excercise and solution that summarizes the steps taken in the derivation.

Problem 9. Damped oscillator in phase space

We consider the EOM

$$\dot{x}(t) = v(t)$$
$$\dot{v}(t) = -\frac{k}{m}\, x(t) - \gamma\, v(t)$$

that describes the damped motion of a particle of mass m that is attached to a spring with spring constant k.

a) What are the SI units of the spring constant k and the damping constant γ?

b) When a particle is placed, with zero velocity, at a rest posion x_0, then it will remain at the rest position. In other words, the rest position $(x_0, 0)$ is a fixed point of the EOM. Determine the rest position of the particle at the spring.

c) Let $E = \frac{m}{2}\, \dot{x}^2(t) + \frac{k}{2}\, x^2(t)$ be the energy of the particle.

   Show that $\frac{\mathrm{d}E}{\mathrm{d}t} = -m\, \gamma\, \dot{x}^2(t)$.

   Hence, E is not conserved in this system.
   What is the rational to call it *the energy of the particle*?
   In other words: What does E describe?

d) We start the oscillator with energy E_0. Show that $A = \sqrt{E_0/k}$ is a length scale and that $T = \sqrt{m/k}$ is a time scale.

   Consider now the dimensionless position $\xi = x/A$, velocity $\zeta = vT/A$, energy $\mathcal{E} = E/E_0$, and time $\tau = t/T$. Show that

$$\dot{\mathcal{E}} = -c\, \zeta^2\,,$$

   How does the constant c depend on the parameters of the EOM: m, k, and γ. (This should be a result of your derivation of the equation.)

   Bonus: How would you determine c by dimensional analysis?

e) Sketch the evolution of $\big(\xi(\tau), \zeta(\tau)\big)$ in phase space when $\gamma = 0$.
   How is the distance from the origin related to the energy of the oscillator? What does this imply for the admissible initial positions of the trajectory in phase space?
   The time derivative of $\mathcal{E}$ is strictly negative when $\gamma > 0$. What does this imply for the evolution of the trajectory in phase space? Where will it end for large times $\tau \gg 1$?

⋆ f) Assume that the solution of the EOM takes the form

$$x(t) = x_0 \, \sin(\omega t - \varphi) \, \mathrm{e}^{-t/t_c}$$

How should ω and t_c be related in that case to k, m, und γ?
Is there also a condition on φ?

Bonus: Can you answer the question concerning φ *without* performing a calculation?

Bonus Problems

Problem 10. Maximum distance of flight

There is a well-known rule that one should through a ball at an angle of roughly $\theta = \pi/4$ to achieve a maximum width.

a) Solve the equation of motion of the ball thrown in x direction with another velocity component in vertical z direction. Do not consider friction in this discussion, and verify that the ball will then proceed on a parabolic trajectory in the (x, z) plane.

b) Well-trained shot put pushers push the put with an initial angle substantially smaller than $\pi/4$, i.e., they provide more forward than upward thrust. Verify that this is a good idea when the height H of the release point of the trajectory over the ground is noticeable as compared to the length L between the release point and touchdown, i.e. when H/L is not small.

c) What is the optimum choice of θ for the shot put?

d) Consider now friction:

- Is it relevant for the conclusions on throwing shot puts?
- Is it relevant for throwing a ball?
- How much does it impact the maximum distance that one can reach in a gun shot?

Problem 11. Separation of variables for a non-autonomous ODE
We consider the ODE

$$y'(x) = \frac{x}{y}$$

It illustrates problems that may arise concerning the existence and uniqueness of the solutions of ODEs.

a) What is the phase space of this ODE? State the ODE as a two-dimensional, first order ODE in terms of its phase-space variables.

b) Sketch the direction field in phase space.

c) Find the solution of the ODE for ICs (x_0, y_0) with $y_0 \neq 0$ and

   i. $x_0 < 0$ and $x_0 < y_0 < -x_0$

   ii. $x_0 > 0$ and $x_0 > y_0 > -x_0$

   iii. other ICs with $|x_0| \neq |y_0|$

   iv. $|x_0| = |y_0|$

d) Determine the largest interval of values $x \in \mathbb{R}$ where the solutions $y(x)$ obtained in b) are defined.

e) Is the function $y(x) = |x|$ a solution of the ODE? If in doubt: Where do you see problems for this solution?