3. Groups and Numbers

Chapter 2.1–2.4 of my lecture notes provides the background to solve the following exercises. Your solution to the problems 3.1–3.4 should uploaded

to your Moodle account

as a PDF-file

by Wednesday, Oct 25, 10:30 (with a grace time till the start of the seminars)

The parts marked by ⋆ and are bonus problems suggestions for further exploration. The latter problems might take some extra effort to solve. Bonus problems need not be submitted and they will not be graded, but they can be followed up in the seminars. This also applies to the Bonus Problems 5. and 6.

Problems

Problem 1. Groups with three elements

Let $\mathcal{G}$ be a group with three elements $\{n, l, r\}$, where n is the neutral element.

a) Show that there only is a single choice for the result of the group operations $a \circ b$ with $a, b \in \mathcal{G}$. Provide the group table.

b) Verify that the group describes the rotations of an equilateral triangle that interchange the positions of the angles.

c) Show that there is a bijective map $\mathrm{m} : \{n, l, r\} \to \{0, 1, 2\}$ with the following property:

$$\forall a, b \in \mathcal{G} | \mathrm{m}(a \circ b) = \big(\mathrm{m}(a) + \mathrm{m}(b)\big) \mathrm{mod} 3 \,.$$

We say that the group $\mathcal{G}$ is isomorphic to the natural numbers with addition addition modulo 3.[1]

[1] The natural number modulo n amount to n classes that represent the remainder of the numbers after division by n. For instance, for the natural numbers modulo two the 0 represents even numbers, and the 1 odd numbers. Similarly, for the natural numbers modulo three the 0 represents numbers that are divisible by three, and for the sum of 2 and 2 modulo 3 one obtains $(2+2) \mathrm{mod} 3 = 4 \mathrm{mod} 3 = 1$.

Problem 2. A Field with three elements

Let $\oplus$ and $\odot$ denote summation and multiplication modulo three.

a) Verify that $\big(\{0,\ 1,\ 2\}, \oplus, \odot\big)$ is a field.

$\star$ b) What about $\{0,\ 1,\ 2,\ 3\}$ with summation and multiplication modulo four.

c) Consider $\{0,\ 1,\ 2, \dots, p-1\}$ with summation and multiplication modulo $p \in \mathbb{N}$. For which p is this a field?

Problem 3. A group where the operation is a cross sum

We explore the set $M = \{1, 2, 4, 5, 7, 8\}$, and combine the elements by multiplication and subsequently taking the cross sum. For example $3 \circ 5 = 6$, because the product of 3 and 5 is $3 \cdot 5 = 15$, and the cross sum of 15 is $1 + 5 = 6$. For larger numbers we repeatedly take the sum of the digits until we arrive at a single-digit number. For instance, $7 \circ 8 = 2$, because $7 \cdot 8 = 56$ with cross sum $5 + 6 = 11$, and eventually we obtain $1 + 1 = 2$. We will show now that $(M, \circ)$ is a group, and discuss its relation to the dihedral group D_3.

a) Verify that the operation $\circ$ on M is commutative.

b) Fill in the group table for the operation $\circ$ on M. How do you see in this table that the operation is commutative?

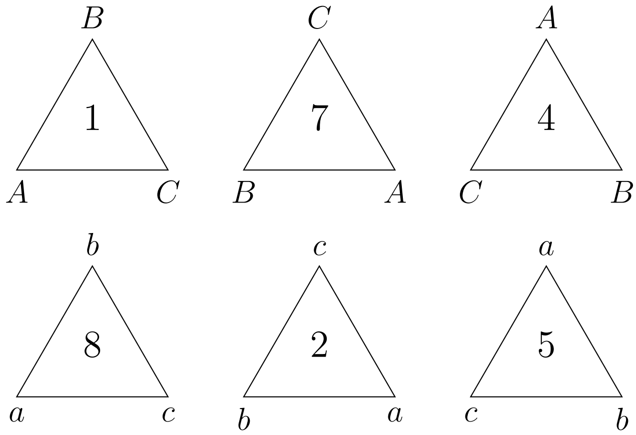
$\circ$	1	2	4	5	7	8
1						
2						
4						
5						
7						
8						

c) Verify that the set M is closed with respect to the operation $\circ$. Besides by checking the group table, this can also be shown by a direct calculation. How?
Hint: Observe the divisibility rule for 3 and 9.

d) What is the neutral element of this group?

e) What are the inverse elements for the other elements of M?

f) We arrange the elements graphically as follows:



Demonstrate that the element 7 amounts to a 120° rotation of the triangles. Henceforth, we denote it as d.
Verify that the element 8 swaps between small and capital letters at the vertices. Henceforth, we denote it as s.

g) How can the elements of the group be expressed in terms of s and d?

h) Use the commutatively of the operation and the representation of the group elements obtained in (g) to proof associativity.

Problem 4. Inequalities in the Complex Plain
Sketch the sets of those complex numbers, $z \in \mathbb{C}$, in the complex plain that obey the following inequalities

a) $|z - \mathrm{i}| < 2$

b) $1 \leq |z| < 3$

c) $\dfrac{3\pi}{4} < \arg z < \pi$

d) $\mathrm{Im}\big(z^2\big) > \mathrm{Im} z$

e) $\mathrm{Im}\big(z^2\big) \geq 2$

Here $\arg z \in [0, 2\pi[$ ("argument of z") denotes the angle between the positive real axis and the straight line from the origin to the point z.

Bonus Problems

Problem 5. Complex multiple exponentiation
One easily checks that $1^{1^1} = (1^1)^1 = 1$ and $-1^{-1^{-1}} = (-1^{-1})^{-1} = -1$.
What about $\mathrm{i}^{\mathrm{i}^{\mathrm{i}}}$ and $\big(\mathrm{i}^{\mathrm{i}}\big)^{\mathrm{i}}$?

Problem 6. Algebraic number fields

Consider the set $\mathbb{K} = \mathbb{Q} + \mathrm{I}\mathbb{Q}$ with $\mathrm{I}^2 \in \mathbb{Q}$. We define the operations $+$ and $\cdot$ in analogy to those of the complex numbers (cf. Example 2.13 of the lecture notes):
For $z_1 = x_1 + \mathrm{I}y_1$ and $z_2 = x_2 + \mathrm{I}y_2$ we have $x_1, y_1, x_2, y_2 \in \mathbb{Q}$ and

$$\begin{aligned}
\forall z_1, z_2 \in \mathbb{K} : & z_1 + z_2 = (x_1 + x_2) + \mathrm{I}\,(y_1 + y_2) \\
& z_1 \cdot z_2 = (x_1\, x_2 + \mathrm{I}^2\, y_1 y_2) + \mathrm{I}\,(x_1\, y_2 + y_1\, x_2) \\
\forall c \in \mathbb{Q}, z = (x + \mathrm{I}y) \in \mathbb{K} : & cz = c\,x + \mathrm{I}\,c\,y
\end{aligned}$$

a) Let I be a rational number, $\mathrm{I} \in \mathbb{Q}$. Show that $\mathbb{K} = \mathbb{Q}$.

b) Consider $\mathrm{I} = \sqrt{2}$. Show that $\mathbb{K}$ is a field that is different from $\mathbb{Q}$.

c) Consider $\mathrm{I} = \sqrt{8}$. Show that this is a field! What is its relation to the case $\mathrm{I} = \sqrt{2}$?

d) Find the general rule: For which natural numbers n does $\mathrm{I} = \sqrt{n}$ provide a non-trivial field?
Remark: Non-trivial means here that the field differs from fields encountered for smaller values of n.