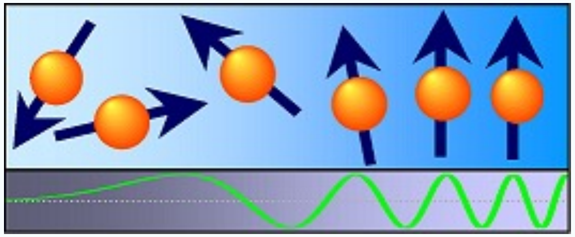
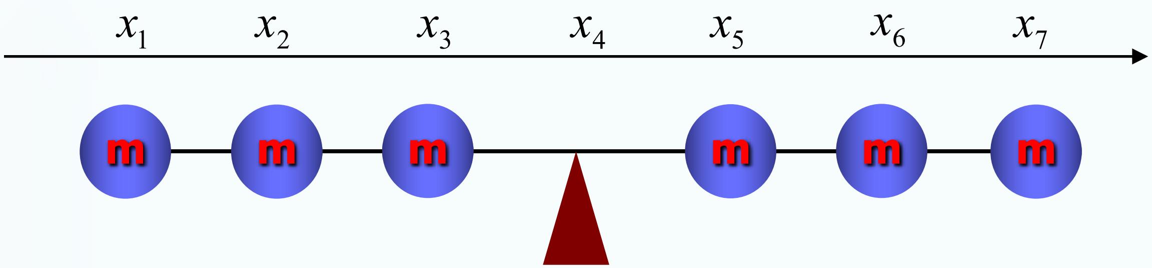
# Experimental Physics
# EP1 MECHANICS

## - System of Particles -
## - Conservation of Momentum -



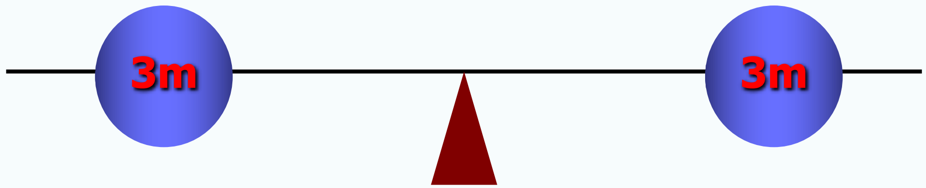
**https://www.physgeo.uni-leipzig.de/en/fbi/applied-magnetic-resonance**

# Center of mass



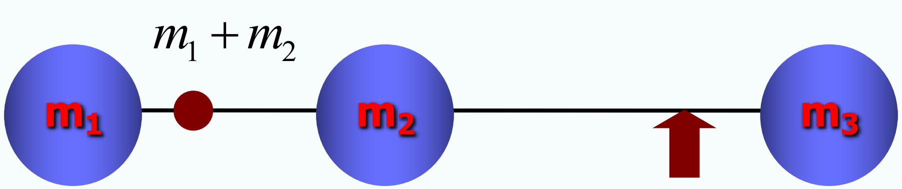

$$X_{cm}\sum_i m_i = \sum_i m_i x_i$$

$$X_{cm} = \frac{m\cdot 1 + m\cdot 7}{m+m} = 4$$



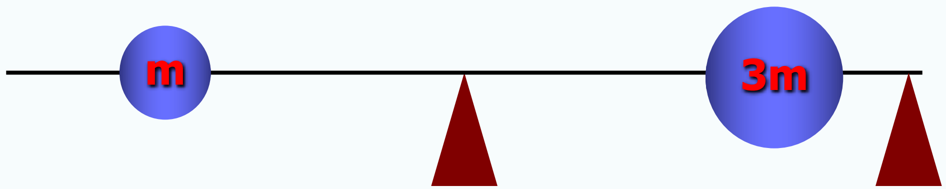

$$X_{cm} = \frac{1+2+6+7}{4} = 4$$

$$X^1_{cm} = \frac{1+2+3}{3} = 2 \qquad X^2_{cm} = \frac{5+6+7}{3} = 6 \qquad X_{cm} = \frac{\sum M_i X_{cm,i}}{\sum M_i} = \frac{3m(2+6)}{6m} = 4$$




$$\vec{R}_{cm}\sum_i m_i = \sum_i m_i \vec{r}_i$$

# Center of mass

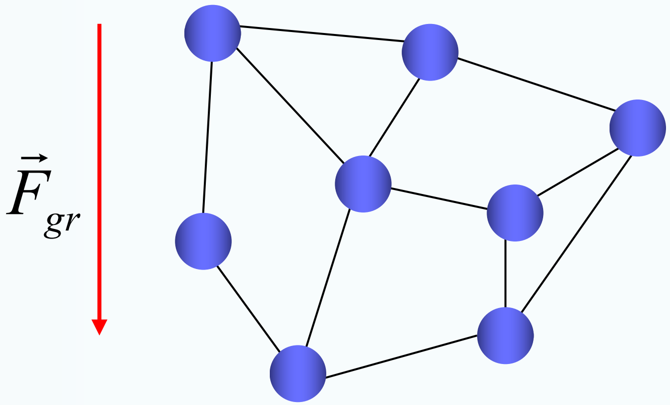



$$U_0 = \sum m_i g h_i$$

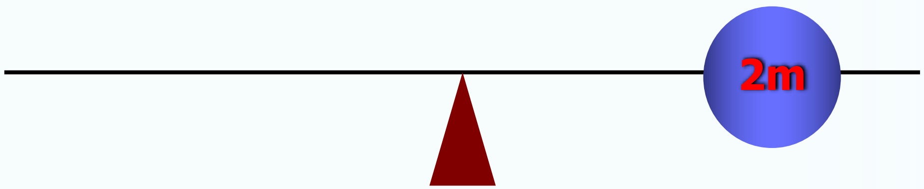
$$U_\theta = U_0 + \sum m_i g \Delta h_i$$

$$U_\theta = U_0 - 2mg\sin\theta$$

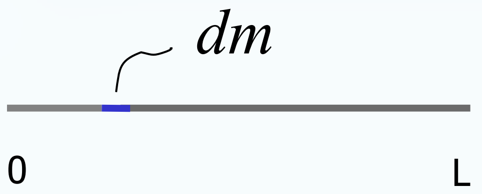
$$\frac{dU}{d\theta} = -2mg\cos\theta = 0$$




$$U = \sum m_i g h_i = g\sum m_i h_i = Mgh_{cm}$$

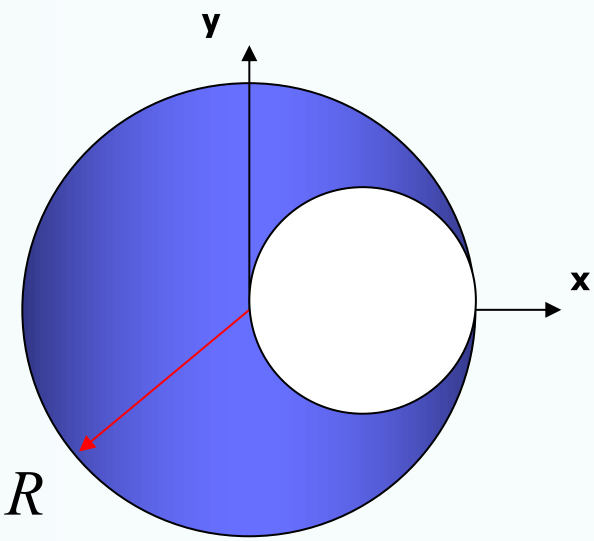


# Continuous bodies




$$\boxed{\vec{R}_{cm} = \frac{\int_0^L \vec{r}\,dm}{\int_0^L dm} = \frac{\int_0^L \vec{r}\rho(\vec{r})\,d\vec{r}}{\int_0^L \rho(\vec{r})\,d\vec{r}}}$$

$$X_{cm} = \frac{\int_0^L \rho_0 x\,dx}{\int_0^L \rho_0\,dx} = \frac{1}{2}\frac{\rho_0 L^2}{\rho_0 L} = \frac{L}{2}$$

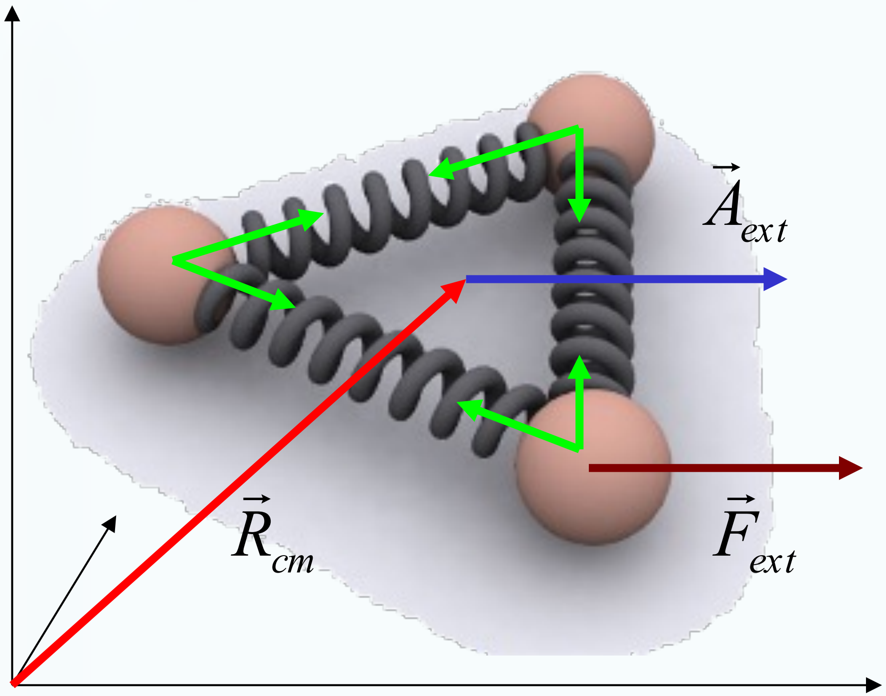



$$X_{cm} = \frac{\int_{-R}^{R} x\,dm(x)}{\int_{-R}^{R} dm(x)} = -\frac{R}{6}$$

**Negative mass:**

$$X_{cm} = \frac{M\cdot 0 + (-m)\cdot R/2}{M - m} = -\frac{R}{6}$$

# Motion of the center of mass




$$\vec{R}_{cm} M = \sum_i m_i \vec{r}_i$$

$$M \frac{d\vec{R}_{cm}}{dt} = \sum_i m_i \frac{d\vec{r}_i}{dt}$$

$$M \frac{d^2\vec{R}_{cm}}{dt^2} = \sum_i m_i \frac{d^2\vec{r}_i}{dt^2}$$

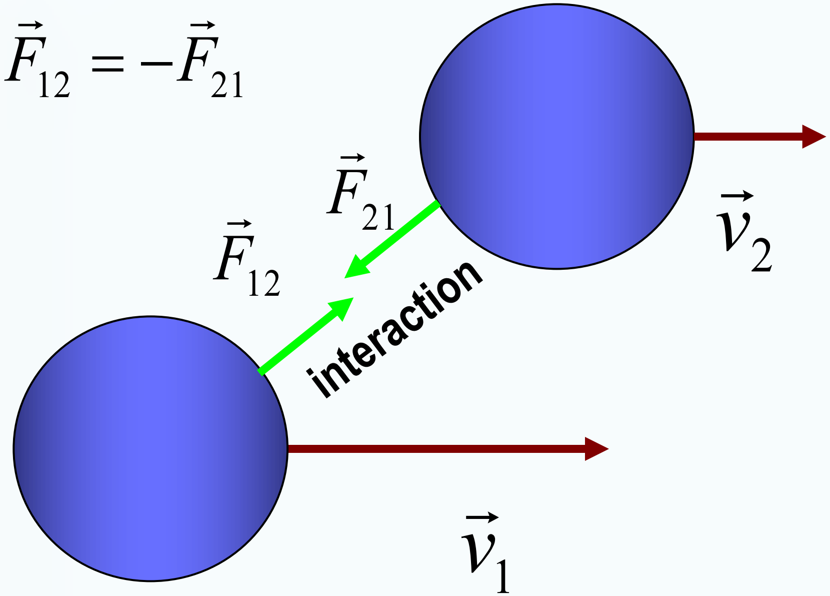
$$M\vec{A}_{cm} = \sum_i \vec{F}_i = \sum_i \left(\cancel{\vec{F}_{i,\mathrm{int}}} + \vec{F}_{i,\mathrm{ext}}\right)$$

$$\boxed{\vec{F}_{\mathrm{net,ext}} = M\vec{A}_{cm}}$$

**Under influence of an external force the center of mass of a system moves like a point particle with a mass equal to the total mass of the system and irrespective of internal forces acting within the system.**

# Conservation of momentum

$$\vec{F}_{12} = -\vec{F}_{21}$$




$$\vec{F}_{12} = \frac{d(m\vec{v}_2)}{dt} = \frac{d\vec{p}_2}{dt}$$

$$\frac{d\vec{p}_2}{dt} = -\frac{d\vec{p}_1}{dt} \Rightarrow \vec{p}_1 + \vec{p}_2 = const$$

**Linear momentum:** $\boxed{\vec{p} = m\vec{v}}$

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\vec{a} = \vec{F}_{\text{net}}$$

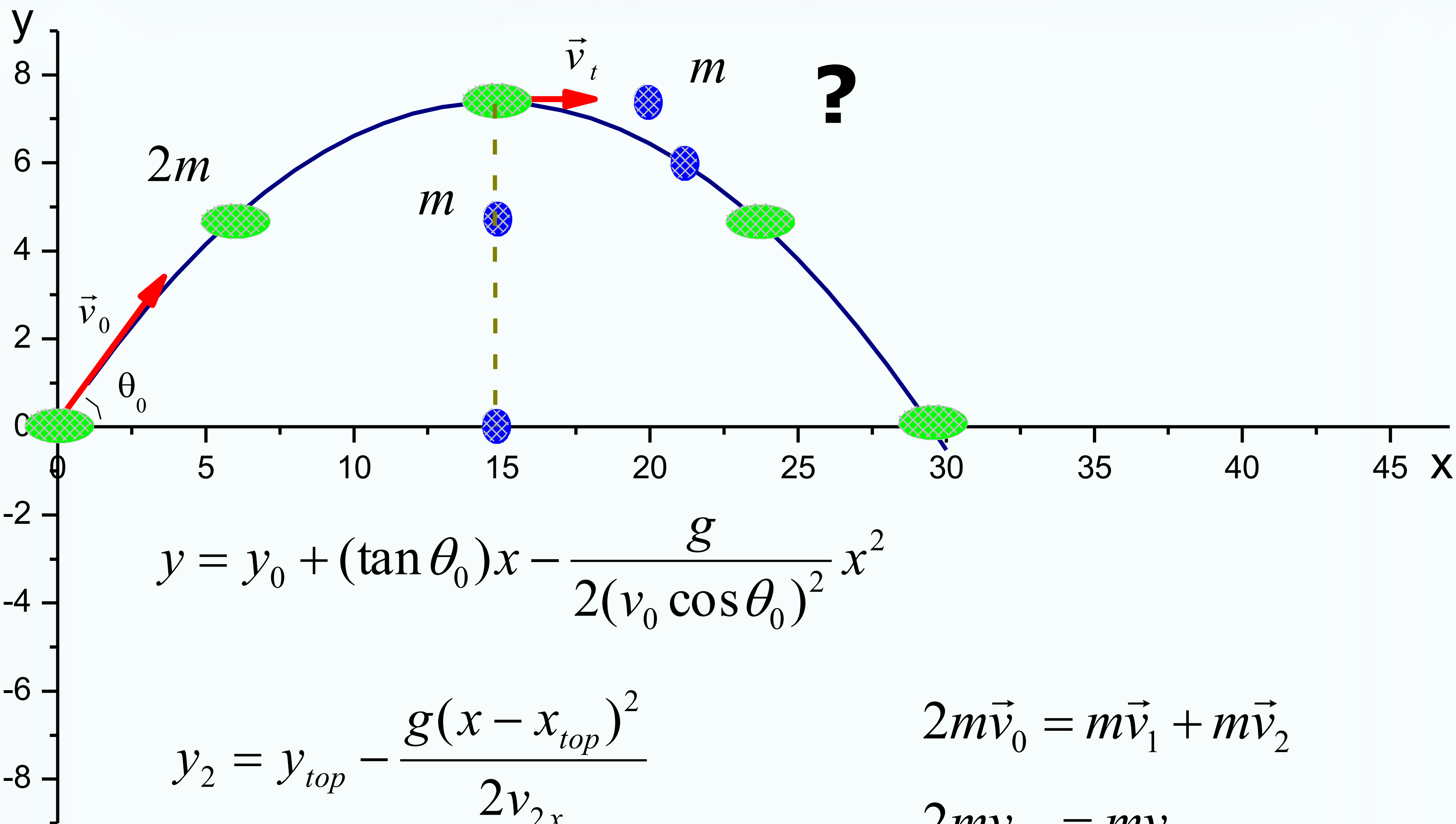
$$\vec{P} = \sum \vec{p}_i = \sum m_i \vec{v}_i = M\vec{V}_{cm}$$

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{V}_{cm}}{dt} = \vec{F}_{\text{net,ext}}$$

**The law of conservation of momentum:**

$$\boxed{\vec{P} = \sum m_i \vec{v}_i = M\vec{V}_{cm} = const}$$

# Projectile with explosion




$$y = y_0 + (\tan\theta_0)x - \frac{g}{2(v_0\cos\theta_0)^2}x^2$$

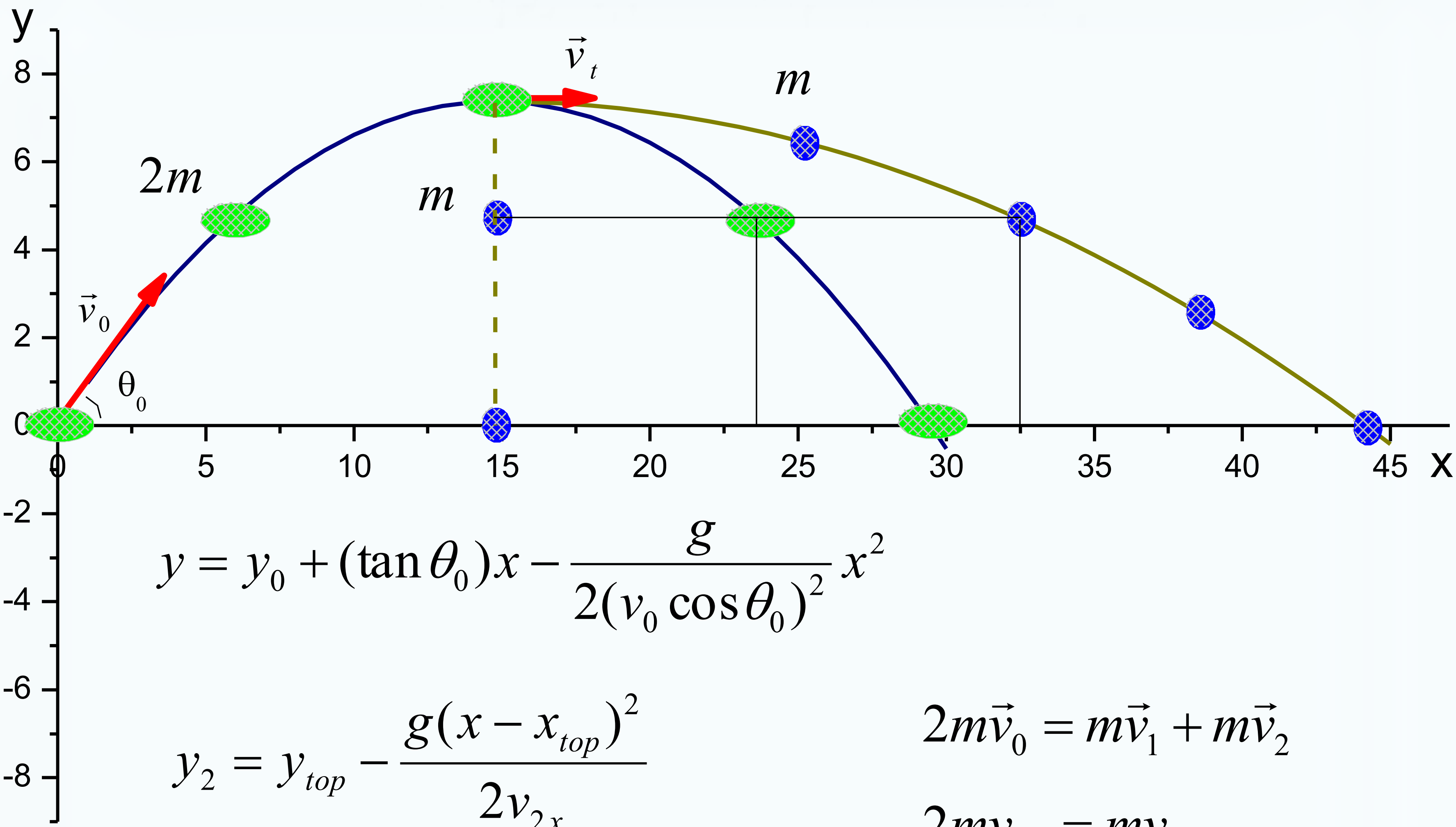
$$y_2 = y_{top} - \frac{g(x - x_{top})^2}{2v_{2x}}$$

$$2m\vec{v}_0 = m\vec{v}_1 + m\vec{v}_2$$

$$2mv_{0x} = mv_{2x}$$

$$v_{2x} = 2v_{0x}$$

# Projectile with explosion




$$y = y_0 + (\tan\theta_0)x - \frac{g}{2(v_0\cos\theta_0)^2}x^2$$

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$$2m\vec{v}_0 = m\vec{v}_1 + m\vec{v}_2$$

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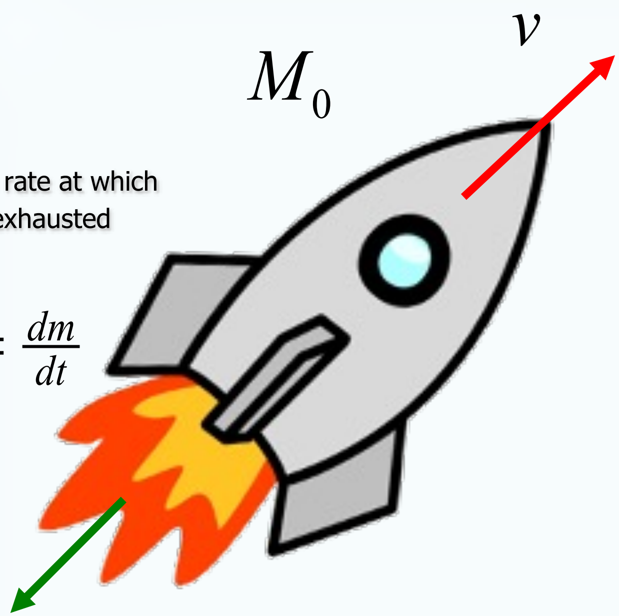
$$v_{2x} = 2v_{0x}$$

# Rocket propulsion




$R$ is the rate at which gas is exhausted

$$R = \frac{dm}{dt}$$

$u_g$ is relative to the rocket;

$$\vec{P} = \sum m_i \vec{v}_i = const$$

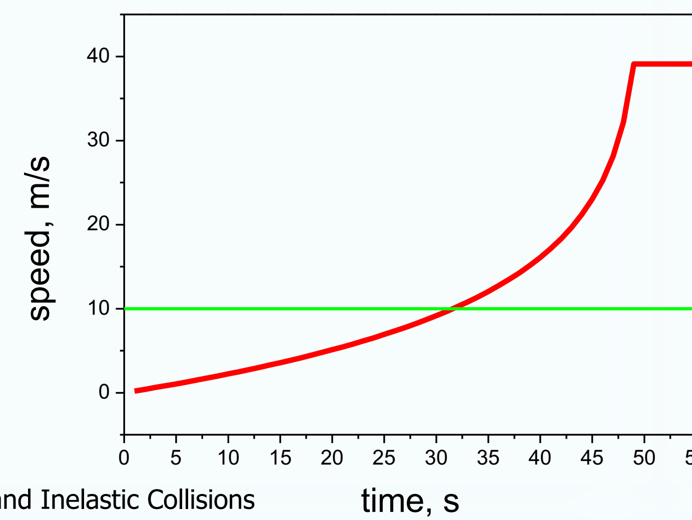
$$\overbrace{mv}^{\text{At an arbitrary time } t} = \underbrace{(v - u_g)dm + (m - dm)(v + dv)}_{\text{At a time } t+dt}$$

$$m\frac{dv}{dt} = u_g \frac{dm}{dt} = Ru_g \text{ - the thrust} \quad \boxed{+\vec{F}_{ext}}$$

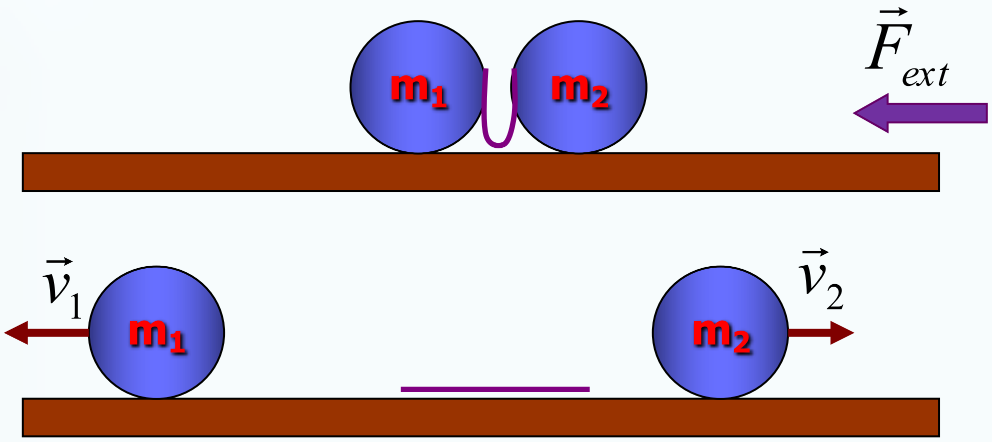
$$m = M_0 - \int_0^t dm = M_0 - Rt$$

$$\int_{v_i}^{v_f} dv = Ru_g \int_0^t (M_0 - Rt)^{-1} dt$$

$$\boxed{v_f = v_i + u_g \ln\left(\frac{M_0}{M_0 - Rt}\right)}$$



# Skaters on ice




$$\sum m\vec{v}_i = 0$$

$$m_1 v_1 = m_2 v_2$$

$$m_1 \frac{x_1}{t} = m_2 \frac{x_2}{t}$$

$$\vec{F}_{ext} = m\vec{A}_{cm} \Rightarrow V_{cm} \qquad m_1\left(-v_1 - V_{cm}\right) + m_2\left(v_2 - V_{cm}\right) = -(m_1 + m_2)V_{cm}$$

$$\vec{u} = \vec{v} + \vec{V}_{cm} \qquad -m_1 v_1 + m_2 v_2 = 0$$

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$$E_k = \frac{1}{2}\sum m_i u_i^2 = \frac{1}{2}\sum m_i\left(v_i + V_{cm}\right)^2 = \frac{1}{2}\sum m_i v_i^2 + \frac{1}{2}\sum m_i V_{cm}^2 + V_{cm}\sum m_i v_i$$

$$E_k = \frac{1}{2} M V_{cm}^2 + E_{k,\mathrm{rel}}$$

$E_{\mathbf{k,rel}}$ **is reference-independent**

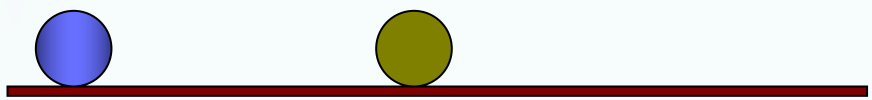
# To remember!

- **The <u>center of mass</u> of a system moves like a point-like object having the total system mass.**

- **If there is no net force acting upon the system, then the <u>linear momentum</u> is conserved.**

- **The <u>total kinetic energy</u> of the system can be written as the sum of the kinetic energy associated with the center of mass plus the kinetic energy associated with the particle movements in the center-of-mass reference system.**

# Elastic collision



$$\boxed{\begin{aligned} E_{k,total} &= const \\ P_{total} &= const \end{aligned}}$$

$$E_{k,total} = const$$

$$\tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2 = \tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2$$

$$m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) = m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f})$$

$$P_{total} = const$$

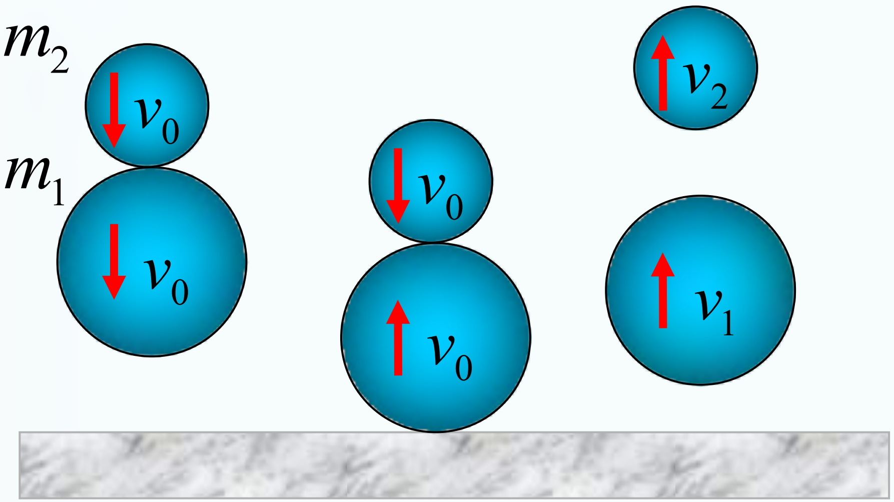
$$\boxed{m_1v_{1f} + m_2v_{2f} = m_1v_{1i} + m_2v_{2i}}$$

$$m_2(v_{2f} - v_{2i}) = m_1(v_{1i} - v_{1f})$$

$$v_{2f} + v_{2i} = v_{1i} + v_{1f} \quad \Rightarrow \quad \boxed{v_{2f} - v_{1f} = -(v_{2i} - v_{1i})}$$

Relative speed of approach is equal to relative speed of recession.

# Astroblaster



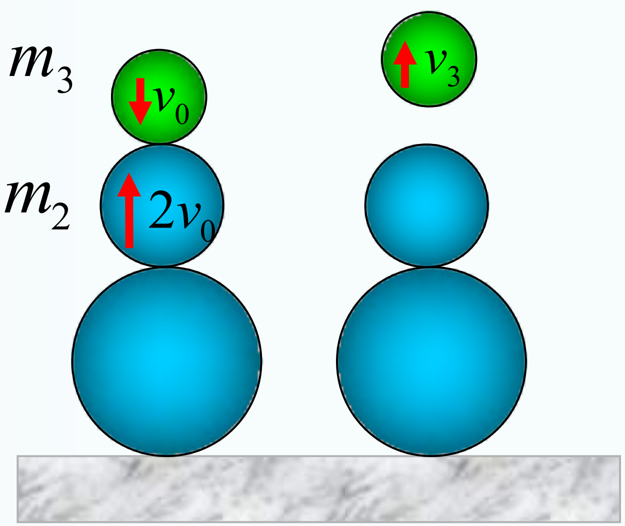

$$m_1 v_0 - m_2 v_0 = m_1 v_1 + m_2 v_2$$

$$\boxed{v_{2f} - v_{1f} = -(v_{2i} - v_{1i})}$$

$$v_2 - v_1 = -(-v_0 - v_0) = 2v_0$$

$$v_1 = \frac{m_1 - 3m_2}{m_1 + m_2} v_0$$

$$\boxed{\begin{array}{l} m_1 = 3m_2 \\ v_1 = 0;\ v_2 = 2v_0 \end{array}}$$




$$2m_2 v_0 - m_3 v_0 = m_2 v_2 + m_3 v_3$$

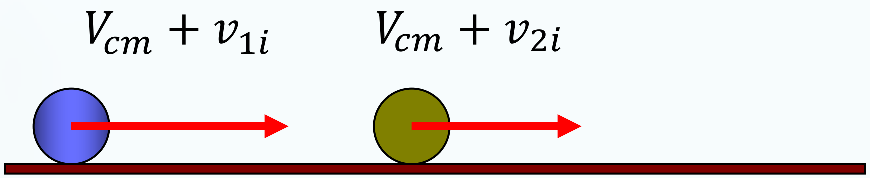
$$v_3 - v_2 = -(-v_0 - 2v_0) = 3v_0$$

$$v_2 = 2v_0 \frac{m_2 - 2m_3}{m_2 + m_3}$$

$$\boxed{\begin{array}{l} m_2 = 2m_3 \\ v_2 = 0;\ v_3 = 3v_0 \end{array}}$$

$$h_3/h_0 = (v_3/v_0)^2 = 9$$

# Center-of-mass reference frame




$$\vec{P} = \sum m_i \vec{v}_i = M\vec{V}_{cm} = const$$

$$\boxed{p_{1i} = -p_{2i}}$$

$$E_{k,i} = \frac{m_1 v_{1i}^2}{2} + \frac{m_2 v_{2i}^2}{2} = \frac{p_{1i}^2}{2m_1} + \frac{p_{2i}^2}{2m_2} = p_{1i}^2\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)$$

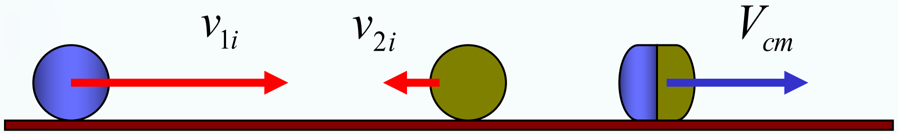
No external forces!

$$E_{k,f} = \frac{m_1 v_{1f}^2}{2} + \frac{m_2 v_{2f}^2}{2} = \frac{p_{1f}^2}{2m_1} + \frac{p_{2f}^2}{2m_2} = p_{1f}^2\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)$$

Elastic collision $\Rightarrow$ $p_{1i}^2 = p_{1f}^2 \Rightarrow \begin{cases} p_{1f} = \pm p_{1i} \\ p_{2f} = \pm p_{2i} \end{cases}$ $\begin{cases} v_{1f} = \pm v_{1i} \\ v_{2f} = \pm v_{2i} \end{cases}$

In CoM reference frame the velocities of each object are simply reversed by collision.

# Perfectly inelastic collision

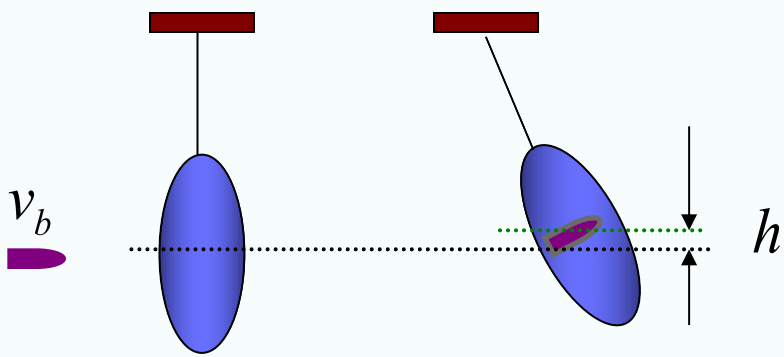



$$\sum m_i v_i = MV_{cm}$$

$$E_{k,i} = \frac{m_1 v_{1i}^2}{2} + \frac{m_2 v_{2i}^2}{2} = \frac{p_{1i}^2}{2m_1} + \frac{p_{2i}^2}{2m_2} \qquad E_{k,f} = \frac{m_1 + m_2}{2} V_{cm}^2 = \frac{p^2}{2(m_1 + m_2)}$$

$$\boxed{v_{2i} = 0} \qquad E_{k,f} = \frac{2m_1 E_{k,i}}{2(m_1 + m_2)} = \frac{m_1}{m_1 + m_2} \frac{m_1 v_{1i}^2}{2}$$

$$\boxed{\begin{aligned} v_{1i} &= -v_{2i} \\ m_1 &= m_2 \end{aligned}} \qquad E_{k,f} = 0$$

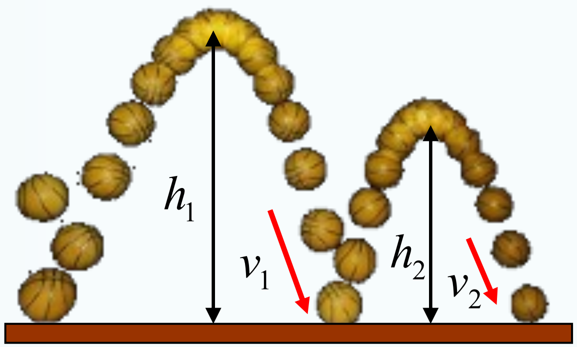



$$E_{k,f} = \frac{m_1}{m_1 + m_2} \frac{m_1 v_{1i}^2}{2} = (m_1 + m_2) gh$$

# Coefficient of restitution

$e$ = 1 – elastic collision
$e$ = 0 – perfectly inelastic collision

$$\boxed{v_{2f} - v_{1f} = -e \cdot (v_{2i} - v_{1i})}$$




$$v_{plate} = 0 \Rightarrow v_2 = ev_1$$

$$e = \frac{v_2}{v_1} = \frac{\sqrt{mgh_1}}{\sqrt{mgh_2}} = \sqrt{\frac{h_1}{h_2}}$$

Table tennis
Stainless table
$e$ = 0.92

Bat-bat COR
$e$ < 0.5

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$$\Delta E_k = f(e) - ? \qquad E_{k,i} = \frac{p^2}{M} \qquad E_{k,f} = \frac{e^2 p^2}{M} \qquad \Delta E_k = E_{k,i}(e^2 - 1)$$

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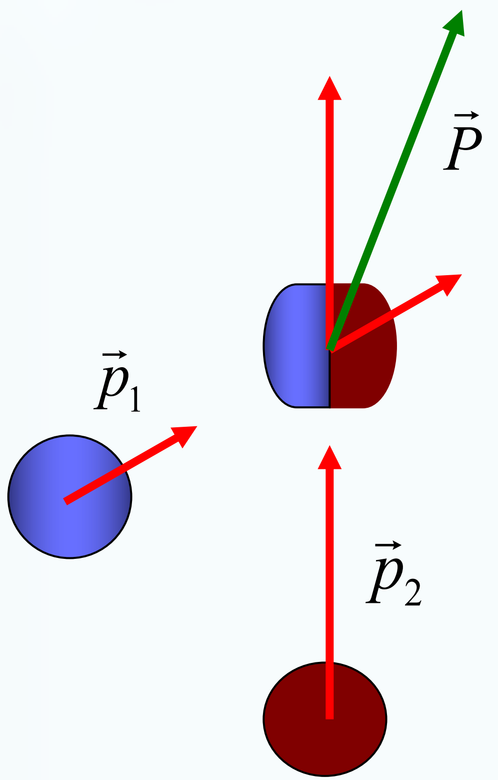
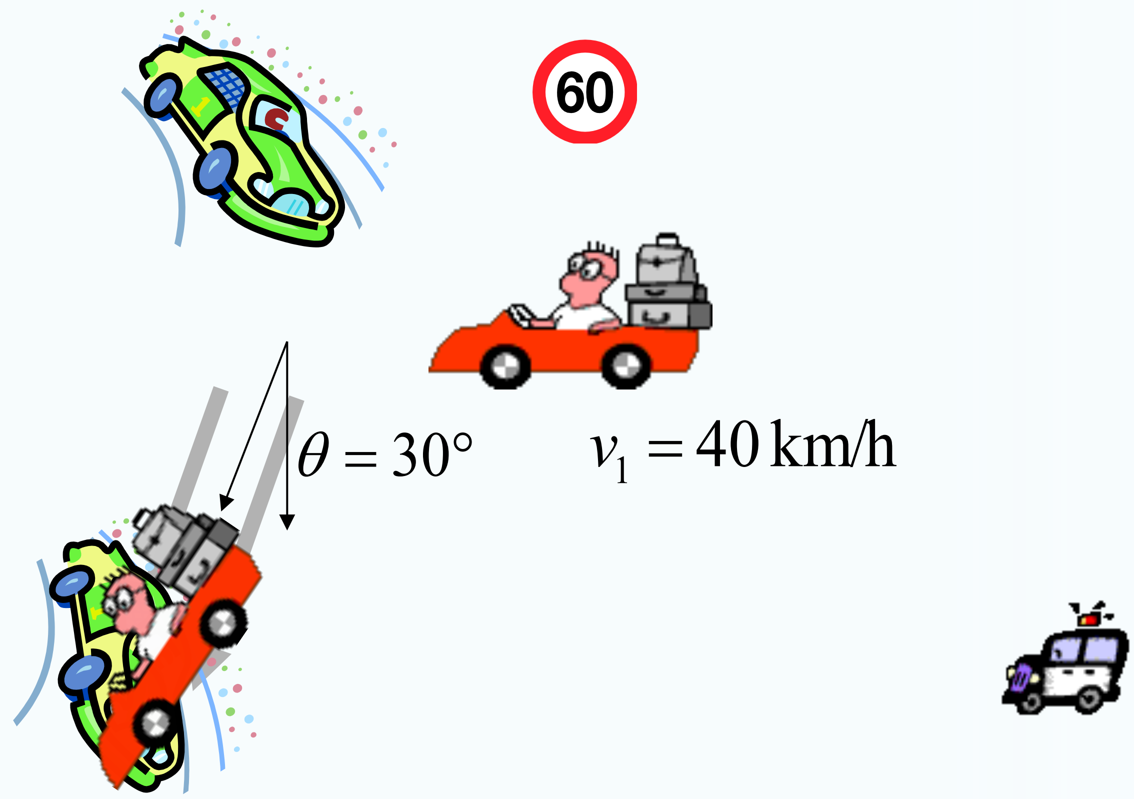
$$\vec{I} = \int_{t_i}^{t_f} \vec{F} dt$$ - impulse

$$\vec{I} = \int_{t_i}^{t_f} \frac{d\vec{p}}{dt} dt = \vec{p}_f - \vec{p}_i$$

# Inelastic collision in 2D

$$\vec{P} = \sum m_i \vec{v}_i = M\vec{V}_{cm}$$

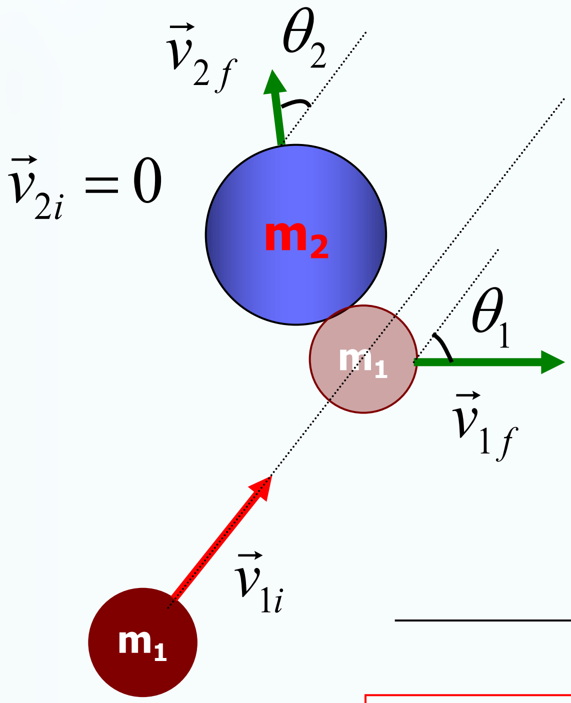



$$\theta = 30^\circ \qquad v_1 = 40\ \text{km/h}$$

$$v_2 = \frac{v_1}{\tan\theta} = 70\ \text{km/h}$$

# Elastic collisions in 2D



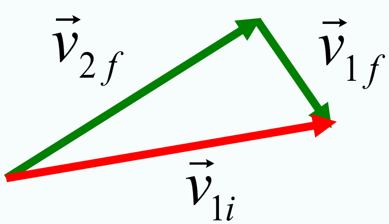

$$\vec{P} = \sum m_i \vec{v}_i = const$$

$$\begin{cases} P_x = m_1 v_{1x,i} = m_1 v_{1x,f} \cos\theta_1 + m_2 v_{2x,f} \cos\theta_2 & (1) \\ P_y = 0 = m_1 v_{1x,f} \sin\theta_1 + m_2 v_{2x,f} \sin\theta_2 & (2) \end{cases}$$

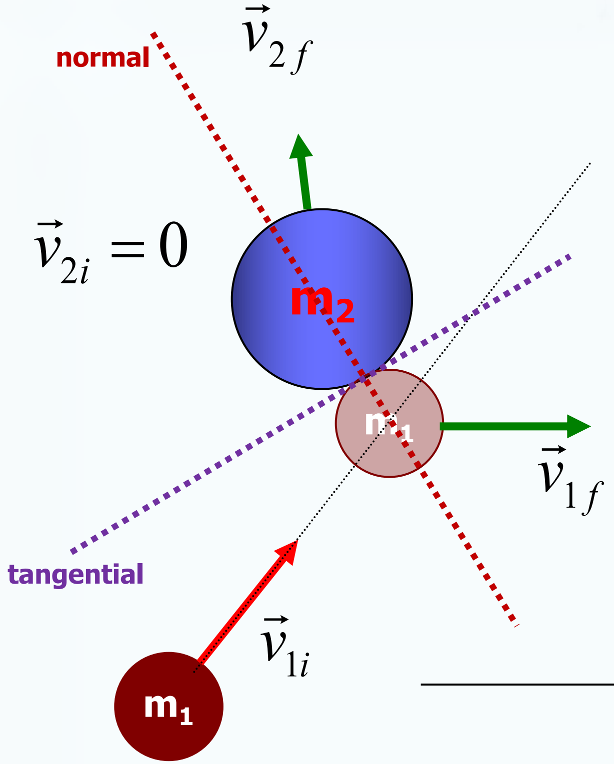
$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \quad (3)$$

$\boxed{m_1 = m_2}$

$$\vec{v}_{1i} = \vec{v}_{1f} + \vec{v}_{2f} \qquad v_{1i}^2 = v_{1f}^2 + v_{2f}^2$$



# Elastic collisions in 2D




$$\vec{P} = \sum m_i \vec{v}_i = const$$

1 $$m_1 v_{1i,n} = m_1 v_{1f,n} + m_2 v_{2f,n}$$

2 $$m_1 v_{1i,t} = m_1 v_{1f,t} + m_2 v_{2f,t}$$

3 $$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$

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$$v_{1i,t} = v_{1f,t} \qquad v_{2i,t} = v_{2f,t}$$

**otherwise frictional forces would violate the energy conservation**

$$m_1 v_{1i,n} = m_1 v_{1f,n} + m_2 v_{2f,n}$$

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$

# To remember!

- During inelastic collisions the total kinetic energy is not conserved. It is conserved during elastic collision.
- In a perfectly inelastic collision the bodies stick together and move with the velocity of COM.
- In the COM reference frame, upon 1D elastic collision the bodies change their velocities to the opposite ones.
- For non-central collisions one has to consider conservation of all components of the linear momentum.