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Theoretical Physics I – HW 10

Problem 10.1 Terminal velocity for turbulent drag

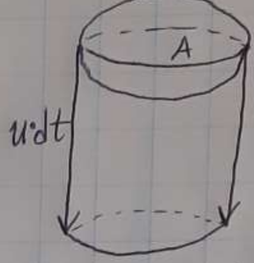
$$\boxed{\vec{F}_d = -\underbrace{\rho}_{\text{air dens.}} \frac{1}{2} \underbrace{|u|^2}_{\text{vel.}} \underbrace{C_d}_{\approx \frac{1}{2}} \underbrace{A}_{\text{cross-sect.}} \underbrace{\hat{u}}_{\frac{\vec{u}}{|\vec{u}|}}}$$

a) $[C_d]=1$, then

$$\frac{kg\cdot m}{s^2} = \frac{kg}{m^3}\cdot\left(\frac{m}{s}\right)^2\cdot m^2$$

$$\frac{kg\cdot m}{s^2} = \frac{kg\cdot m}{s^2} \quad \text{correct}$$

b)



Ball lost kinetik energy, transferring it to air.

$$\underbrace{dW}_{\text{during } dt} = \Delta E_k = F_d\cdot u\,dt = -\frac{dm\,u^2}{2}, \text{ so } F_d = \frac{-dm\,u^2}{2u\,dt} = -\frac{dm\,u}{2dt} =$$

$$= -\frac{\overbrace{A\cdot u\,dt}^{V}\,\rho\,u}{2dt} = -\frac{\rho A u^2}{2}, \quad \vec{F}_d = -\frac{\rho A|u|^2}{2}\hat{u}$$

up to constant C_d (drag coefficient).

c) $m\ddot{\vec{q}} = m\vec{g} + \vec{F}_d$, terminal: $\ddot{\vec{q}}=0$.

$$0 = mg - \frac{\rho}{2}u^2 C_d A$$

$$\boxed{u = \sqrt{\frac{2mg}{\rho C_d A}}} = \sqrt{\cancel{\frac{2\cdot mg\cdot 4}{\rho\cdot C_d\cdot \pi d^2}}} = \sqrt{\cancel{8g\cdot\rho}}$$

Note that ρ is air density, and m includes ball's, so do not cancel out.

$$u = \sqrt{\frac{2mg\cdot 4}{\rho C_d\cdot \pi d^2}} = \sqrt{\frac{8mg}{C_d \pi \rho d^2}} = \left[\begin{array}{ll} d = 42{,}7\cdot 10^{-3}\,m & C_d = \frac{1}{2} \\ m = 45\cdot 10^{-3}\,kg & \\ g = 10\,m\cdot s^{-2} & \rho_{air} \approx 1.2\,kg\cdot m^{-3} \end{array}\right]$$

$$= \sqrt{\frac{8 \cdot 45 \cdot 10^{-3}\,kg \cdot 10\,m\,s^{-2}}{\frac{1}{2}\pi \cdot 1.2\,kg\,m^{-3} \cdot 42.7^2 \cdot 10^{-6}\,m^2}} \approx \frac{m}{s} \cdot \sqrt{\frac{8 \cdot 45 \cdot 10^{-2}}{\pi \cdot 0.6 \cdot 1823 \cdot 10^{-6}}} \approx \frac{m}{s}\sqrt{\frac{8 \cdot 10^4}{24\pi}} \approx$$

$$\frac{45}{1823} \approx \frac{1}{40}$$

$$\frac{m}{s} \cdot \frac{1}{3} \cdot 10^2 \approx 33\ m/s.$$

d) $T \approx \frac{u}{g}$ (orders of m.) $\longrightarrow T \approx 3.3\ s$

Then $L \approx uT = \frac{u^2}{g} \longrightarrow L \approx \frac{33^2}{10}\,m = 90\,m.$

Problem 10.2 Damped oscillator PS

$\dot{x}(t) = v(t)$

$\dot{v}(t) = -\frac{k}{m}x(t) - \gamma v(t)$

a) $[k] = \frac{N}{m} = \frac{kg \cdot m}{s^2 \cdot m} = \frac{kg}{s^2}$

$[\gamma] = \left[\frac{\dot{v}}{v}\right] = \left[\frac{1}{t}\right] = \frac{1}{s}$

b) $\begin{pmatrix}\dot{x}\\ \dot{\dot{x}}\end{pmatrix}_{\begin{pmatrix}x_0\\0\end{pmatrix}} = \begin{pmatrix}0\\0\end{pmatrix}$ $\quad \begin{pmatrix}0\\ -\frac{k}{m}x_0 - \gamma \cdot 0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \longrightarrow \frac{k}{m}x_0 = 0 \longrightarrow x_0 = 0$

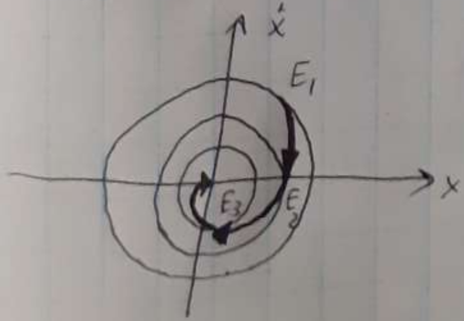
$(0,0)$ is fixed point, in PS.

c) $E = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2$

$$\frac{dE}{dt} = \frac{m}{2} \cdot 2\dot{x}\dot{x}' + \frac{k}{2} \cdot 2\dot{x}x = m\dot{x}\dot{x}' + k\dot{x}x = \dot{x}(kx + m\dot{x}') = m\dot{x}\left(\frac{k}{m}x + \dot{x}'\right) =$$

$$= m\dot{x}\left(\underline{\frac{k}{m}x} - \underline{\frac{k}{m}x} - \gamma \underbrace{v}_{\dot{x}}\right) = -m\gamma(\dot{x})^2.$$

E describes energy of non-damped oscillator which momentarily corresponds to given $(x, \dot{x})$, and evolution is in moving continuously from one such level to another, with losing mech. energy as heat:



$(E_1(t_1) > E_2(t_2 > t_1) > E_3(t_3 > t_2))$
$> 0\ (t_\infty > t_3)$

d) E_0 – initial energy.

$A=\sqrt{\frac{E_0}{k}}$, then $[A]=\sqrt{\frac{N\cdot m\cdot s^2}{kg}}=\sqrt{\frac{kg\cdot m^2 s^2}{s^2 kg}}=\sqrt{m^2}=m=[x]$

$T=\sqrt{\frac{m}{k}}$, then $[T]=\sqrt{\frac{kg\cdot s^2}{kg}}=\sqrt{s^2}=s=[t]$

Non-dimensionalization.

$\xi=\frac{x}{A}$, $\mathcal{E}=\frac{E}{E_0}$, $\zeta=\frac{vT}{A}=\frac{v}{(A/T)}$, $\tau=\frac{t-t_0}{T}$.

Show: $\dot{\mathcal{E}}=-c\,\zeta^2$.

1) Directly.

$$\frac{d\mathcal{E}}{d\tau}=-c\,\zeta^2$$

$$\frac{d\left(\frac{E}{E_0}\right)}{d\left(\frac{t}{T}\right)}=-c\cdot\frac{(\dot{x})^2}{(A/T)^2} \qquad \frac{T}{E_0}\cdot\underbrace{\frac{dE}{dt}}_{\text{from c)}}=-c\cdot\left(\frac{T}{A}\right)^2(\dot{x})^2$$

$$\frac{T}{E_0}\cdot\left(-m\gamma(\dot{x})^2\right)=-c\cdot\frac{m}{E_0}(\dot{x})^2 \quad \Big/\cdot\frac{E_0}{(\dot{x})^2 m}$$

$$\sqrt{\frac{m}{k}}\cdot(-\gamma)=-c, \quad \underline{c=+\sqrt{\frac{m}{k}}\,\gamma.}$$

2) Using dim. analysis

$[k]=\frac{kg}{s^2}$

$[\gamma]=\frac{1}{s}$

$[m]=kg$

$$[m^a k^b \gamma^d]=kg^{a+b}\cdot s^{-2b-d}=1$$

$$\begin{cases} a+b=0 \\ -2b-d=0 \end{cases}$$

Simplest solution is $a=1, b=-1, d=2$,

then $c=\frac{m}{k}\gamma^2$, in general

$\underline{c=\left(\frac{m}{k}\gamma^2\right)^n}$ for any $n\in\mathbb{Z}$ ($\underline{\text{in actual case } n=\frac{1}{2}}$), and $[c]=1$.

e) If $\gamma=0 \longrightarrow c=0$, $\dot{\mathcal{E}}=0$, $\mathcal{E}=\text{const}$.

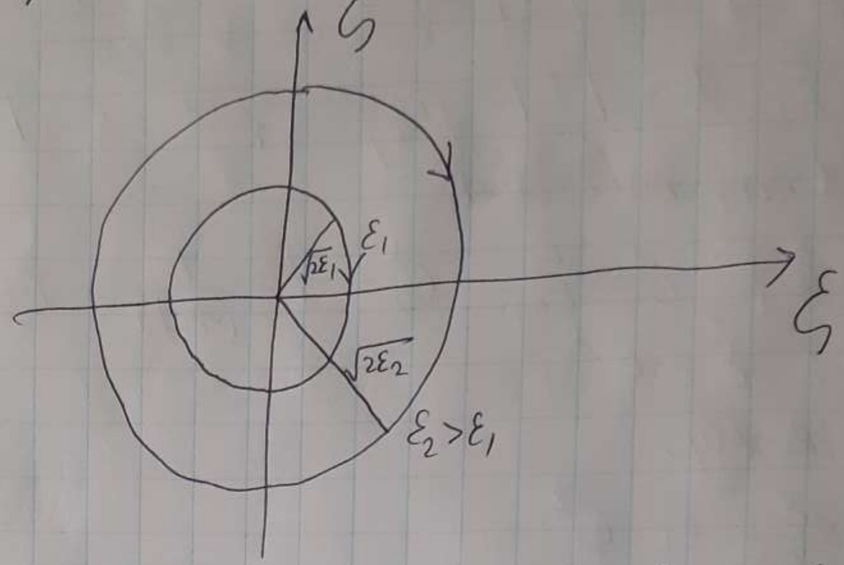
Show dimensionless energy $\mathcal{E}$ as $f(\xi,\zeta)$:

$$E=\frac{m}{2}(\dot{x})^2+\frac{k}{2}x^2$$

$$E=\frac{m}{2}\left(\frac{\zeta A}{T}\right)^2+\frac{k}{2}(\xi A)^2$$

$$E=\frac{\cancel{m}}{2}\cdot\frac{E_0}{\cancel{m}}\cdot\zeta^2+\frac{\cancel{k}}{2}\cdot\frac{E_0}{\cancel{k}}\cdot\xi^2, \qquad \frac{E}{E_0}=\boxed{\mathcal{E}=\frac{\zeta^2}{2}+\frac{\xi^2}{2}}$$

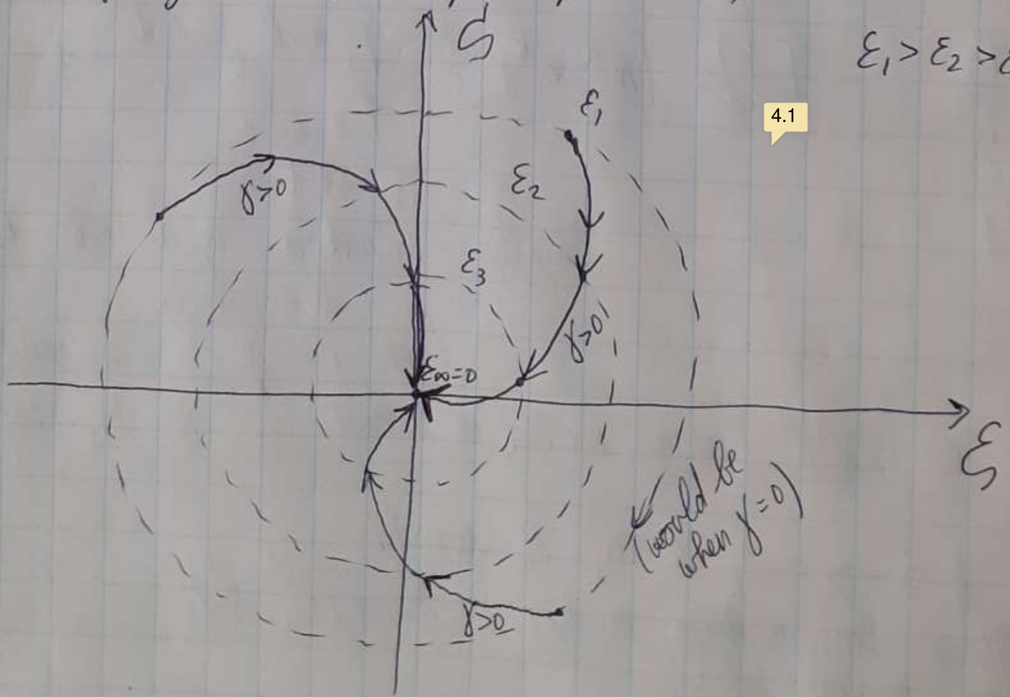
$2\mathcal{E} = \zeta'^2 + \zeta^2$ – circle with $R = \sqrt{2\mathcal{E}}$ ⟶ which is distance.



If $\gamma = 0$, initial position is any point, and trajectory is the circle through it centered in $(0,0)$.

If $\gamma > 0$, initial position is again point on one of those circles, and trajectory goes as spiral from outer circles to lower, losing energy. When $\tau \gg 1$, it ends in origin and stays there (origin is fixed point). Here:

$\mathcal{E}_1 > \mathcal{E}_2 > \mathcal{E}_3 > \mathcal{E}_\infty = 0$



* f) $x(t) = x_0 \sin(\omega t - \varphi) e^{-t/t_c}$ (damping oscillations)

Way 1 – put $x(t)$ inside EOM: $m\ddot{x} + \gamma m \dot{x} + kx = 0$ and check, it's a bit stupid, but generic and relatively simple in this case approach:

$$\dot{x} = x_0 \omega \cos(\omega t - \varphi) e^{-t/t_c} - \frac{x_0}{t_c} \sin(\omega t - \varphi) e^{-t/t_c} =$$
$$= x_0 e^{-t/t_c} \left[\omega \cos(\omega t - \varphi) - \frac{1}{t_c} \sin(\omega t - \varphi)\right]$$

$$\ddot{x} = x_0 \left(-\frac{1}{t_c}\right) e^{-\frac{t}{t_c}} \left[\omega \cos(\omega t - \varphi) - \frac{1}{t_c} \sin(\omega t - \varphi)\right] +$$
$$+ x_0 e^{-t/t_c} \left[-\omega^2 \sin(\omega t - \varphi) - \frac{\omega}{t_c} \cos(\omega t - \varphi)\right]$$

EOM:

$$m\Big[\overbrace{-\frac{x_0}{t_c} \cancel{e^{-t/t_c}} \left[\omega\cos(\omega t - \varphi) - \frac{1}{t_c}\sin(\omega t - \varphi)\right] + \cancel{x_0 e^{-t/t_c}} \left[-\omega^2 \sin(\omega t - \varphi) - \frac{\omega}{t_c}\cos(\omega t - \varphi)\right]}^{\ddot{x}}\Big] =$$
$$= -k \underbrace{\left[\cancel{x_0 e^{-t/t_c}}\right] \sin(\omega t - \varphi)}_{x} - \gamma m \cdot \cancel{x_0 e^{-t/t_c}} \underbrace{\left[\omega\cos(\omega t - \varphi) - \frac{\sin(\omega t - \varphi)}{t_c}\right]}_{\dot{x}}$$

Matching sin and cos coefficients, dividing by $x_0 e^{-t/t_c}$:

$$\cos(\omega t - \varphi) \cdot \left[-\frac{2m\omega}{t_c} + \gamma m \omega\right] = \sin(\omega t - \varphi) \cdot \left[-\frac{m}{t_c^2} + m\omega^2 - k + \frac{\gamma m}{t_c}\right]$$

Equal $\forall t \to$ set to zero.

$$-\frac{2m\omega}{t_c} = -\gamma m \omega \longrightarrow \gamma = \frac{2}{t_c}, \boxed{t_c = \frac{2}{\gamma}}$$

$\cancel{\frac{m}{t_c^2} - k + \frac{\gamma m}{t_c} = m\omega^2}$ $\left(\omega\left[\gamma - \frac{2}{t_c}\right] = 0,\ \omega \text{ is not } 0\right)$

$$-\frac{m}{t_c^2} + m\omega^2 - k + \frac{\gamma m}{t_c} = 0,$$

$$\omega^2 = \frac{k}{m} + \frac{1}{t_c^2} - \frac{\gamma}{t_c}, \quad \omega = \sqrt{\frac{k}{m} + \frac{1}{t_c^2} - \frac{\gamma}{t_c}} = \sqrt{\frac{k}{m} + \left(\frac{\gamma^2}{4} - \frac{\gamma^2}{2}\right)} =$$

$\cancel{\omega = \sqrt{\frac{k}{m} + \frac{1}{t_c^2} - \frac{\gamma m}{t_c}} = \sqrt{\frac{k}{m} - \frac{\gamma^2 m}{2}}}$

$$\boxed{= \sqrt{\frac{k}{m} - \frac{\gamma^2}{4}}}$$

non-damped part / damping

Way 2 By solving EOM:

$m\ddot{x} + \gamma m \dot{x} + kx = 0 \qquad \ddot{x} + \gamma \dot{x} + \frac{k}{m} x = 0 \qquad r^2 + \gamma r + \frac{k}{m} = 0$ (oscil. will give $D<0$)

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - \frac{4k}{m}}}{2}, \quad \omega = \sqrt{-\gamma^2 + \frac{4k}{m}}, \text{ which}$$

?

is 2 times more than with consideration way 1.

$t_c = \frac{\gamma}{2}$ (real part).

φ is not related to parameters k, m, γ which is clear from both approaches. It is determined by $(x_0, \dot{x}_0)$ only.

Problem 10.3 Bubbles in fluid

$$F_g = \frac{4\pi}{3}R^3(\rho_f - \rho_g)g$$

$$F_S = -6\pi\eta R\dot{z}(t)$$

a) $m\ddot{z} = \frac{4\pi}{3}R^3(\rho_f - \rho_g)g - 6\pi\eta R\dot{z}$ — EOM

$$\underbrace{\frac{4\pi}{3}R^3(\rho_g - \rho_f)\frac{g}{m}}_{I} = \ddot{z} + \underbrace{\frac{6\pi\eta R}{m}}_{c_1}\dot{z} + \underbrace{0}_{c_0}\cdot z$$

6.1

See dependencies of I, c_1, c_0 above ↑

b) length and time scale, $\cancel{\zeta = \frac{d\xi}{d\tau}}$ → so that $0 = \dot{\zeta}(\tau) + \xi(\tau)$

Using hint, reference velocity will be terminal.

$$\frac{4\pi}{3m}R^3(\rho_f - \rho_g)g - \frac{6\pi\eta R}{m}\dot{z}_t = 0 \quad (\text{then } \ddot{z} = 0 \text{ when } t \to \infty)$$

$$(\dot{z})_t = \frac{4\pi R^3(\rho_f - \rho_g)g}{3 \cdot 6\pi\eta R} = \boxed{\frac{2}{9}\frac{R^2(\rho_f - \rho_g)g}{\eta}}$$

6.2

$$(\dot{z})_t = \frac{2}{9}\cdot\frac{R^2 g}{\eta}(\rho_f - \rho_g) \cdot | \cdot 6\pi\eta R$$

restoring force

$$\underbrace{6\pi\eta R(\dot{z})_t}_{\frac{4\pi}{3}R^3(\rho_f - \rho_g)g} = m\ddot{z} + 6\pi\eta R\dot{z} \qquad \dot{z} - (\dot{z})_t = \dot{\tilde{z}} \text{ (relative to moving frame);}$$

$$0 = m\ddot{z} + 6\pi\eta R\dot{\tilde{z}}$$

$$\ddot{\tilde{z}} = \frac{d}{dt}(\dot{\tilde{z}}) = \frac{d}{dt}(\dot{z} - (\dot{z})_t) =$$

$$= \frac{d\dot z}{dt} - \underbrace{\frac{d(const)}{dt}}_{0} = \frac{d\dot z}{dt} = \ddot z.$$

(denote just as $z, \dot z, \ddot z$ later)

So $0 = m\ddot z + 6\pi\eta R\dot z$

$\xi = \frac{z}{R}$ — length scale

$\tau = \frac{t-t_0}{T}$. Must determine T.

$$0 = m\frac{d^2(\xi R)}{d(\tau T)^2} + 6\pi\eta R\frac{d(\xi R)}{d(\tau T)}$$

$$0 = \frac{mR}{T^2}\cdot\frac{d^2\xi}{d\tau^2} + \frac{6\pi\eta R^2}{T}\cdot\frac{d\xi}{d\tau} \quad \Big/ \cdot\frac{T}{R}$$

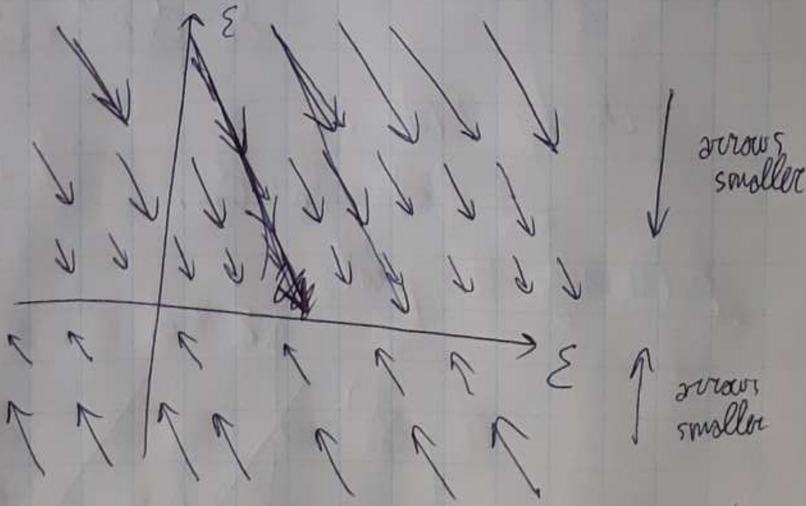
$$0 = \frac{m}{T}\cdot\frac{d^2\xi}{d\tau^2} + 6\pi\eta R\cdot\frac{d\xi}{d\tau}$$

$$\frac{m}{T} = 6\pi\eta R \rightarrow T = \frac{m}{6\pi\eta R}$$ — time scale

Then $0 = \frac{d^2\xi}{d\tau^2} + \frac{d\xi}{d\tau}$, $\quad 0 = \ddot\xi(\tau) + \dot\xi(\tau)$

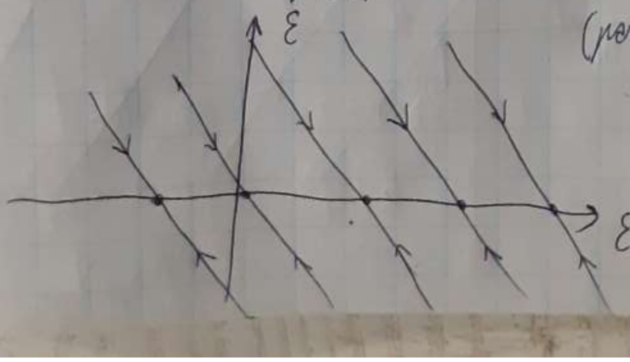
d $\quad \ddot\xi = -\dot\xi$

$$\vec v(\tau) = \begin{pmatrix}\dot\xi\\ \ddot\xi\end{pmatrix} = \begin{pmatrix}\dot\xi\\ -\dot\xi\end{pmatrix}$$



(velocity field) — no curving!

Solutions $\vec r(\tau) = \begin{pmatrix}\xi\\ \dot\xi\end{pmatrix}$ will be respective straight lines (parallel lines, slope -1)



all motion will end with $\dot\xi = 0$ as terminal velocity of the frame of reference chosen

* d) Solutions.

$$0 = \ddot{\varepsilon} + \dot{\varepsilon} \qquad \dot{\varepsilon} = \zeta$$

$$0 = \dot{\zeta} + \zeta$$

$$\dot{\zeta} = -\zeta$$

$$\frac{d\zeta}{d\tau} = -\zeta$$

$$\int \frac{d\zeta}{\zeta} = -\int d\tau + C$$ (currently without borders, any antiderivative + initial conditions)

$$e^{\ln(\zeta)} = e^{-\tau + C}$$

$$\boxed{\zeta = \zeta_0 e^{-\tau}}$$

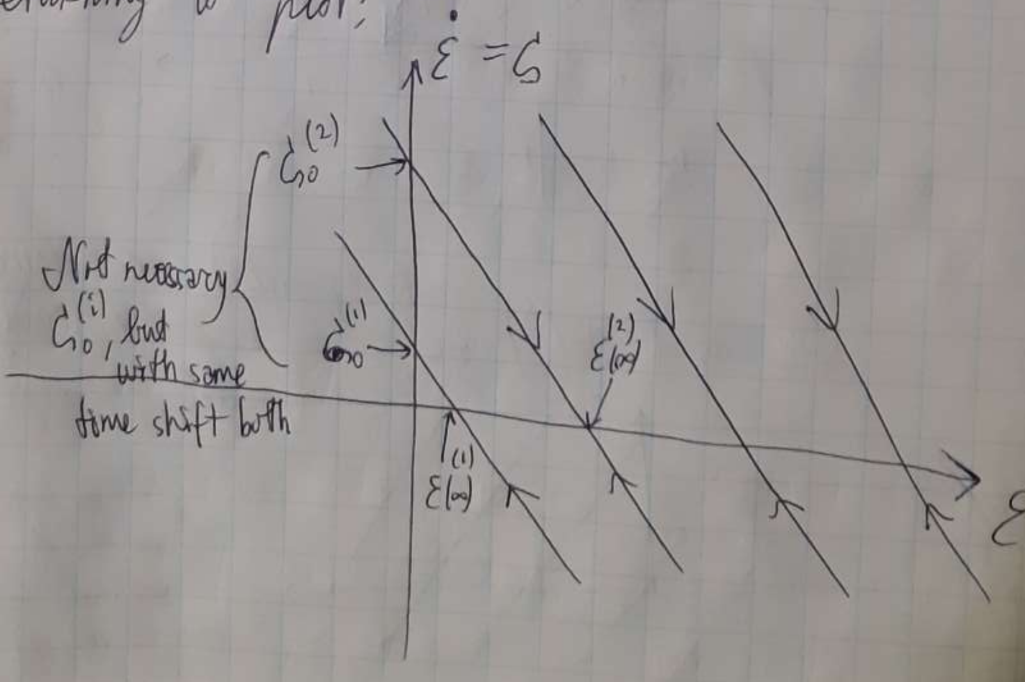
Then $\varepsilon(\tau) = \varepsilon_0 + \int_0^{\tau} \zeta_0 e^{-\tau'} d\tau' = \varepsilon_0 - \left(\zeta_0 e^{-\tau'}\right)\Big|_0^{\tau} =$

$$= \varepsilon_0 - \zeta_0 e^{-\tau} + \zeta_0 = \boxed{\varepsilon_0 + \zeta_0(1 - e^{-\tau}) = \varepsilon(\tau)}.$$

Check: $\varepsilon(0) = \varepsilon_0$

$\varepsilon(\infty) = \varepsilon_0 + \zeta_0$.

Returning to plot:



Always end up on horizontal axis, but how far depends on initial velocity.

* e) ...

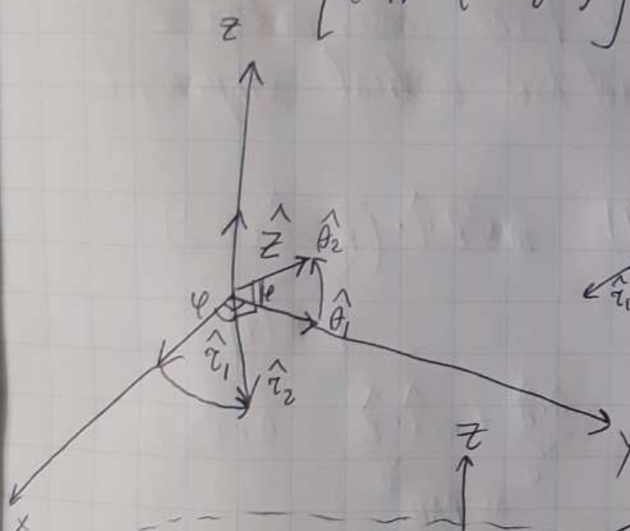
Problem 10.4 Motion in magnetic field

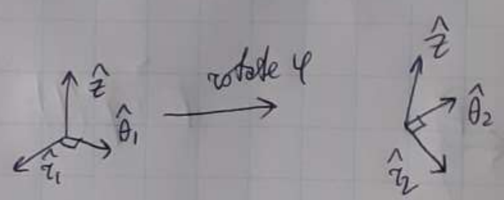
$$\vec{q}(t) = R\hat{r}(\omega t) + vt\hat{z}, \quad \hat{r}(\omega t) = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix} \quad \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

a) Determine $\hat{\theta}$ so that form RNB$(\hat{r}, \hat{\theta}, \hat{z})$.

$\hat{r}\cdot(\hat{\theta}\times\hat{z}) = \hat{\theta}\cdot(\hat{z}\times\hat{r}) = 1$, must hold.

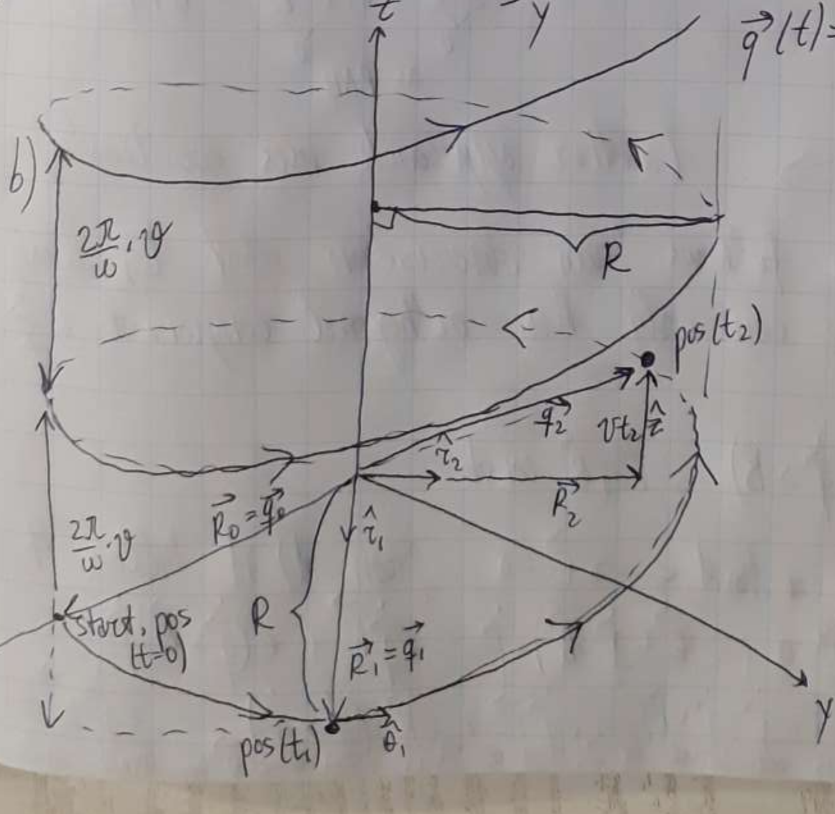
$$\hat{\theta}\cdot\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix}\right] = \hat{\theta}\cdot\begin{bmatrix} -\sin\omega t \\ \cos\omega t \\ 0 \end{bmatrix} = 1, \quad \hat{\theta} = \begin{bmatrix} -\sin\omega t \\ \cos\omega t \\ 0 \end{bmatrix}$$





$$\vec{q}(t) = \begin{bmatrix} R\cos\omega t \\ R\sin\omega t \\ vt \end{bmatrix}$$

b)



$\frac{2\pi}{\omega}$ is period →

$\frac{2\pi}{\omega}\cdot v$ is step size between two stairs.

10.1

c) $\vec{q}(t)=\begin{bmatrix}R\cos(\omega t)\\ R\sin(\omega t)\\ vt\end{bmatrix}$ $\quad\dot{\vec{q}}(t)=\begin{bmatrix}-\omega R\sin(\omega t)\\ \omega R\cos(\omega t)\\ v\end{bmatrix}$ - velocity (*)

$|\dot{\vec{q}}(t)|=\boxed{\sqrt{\omega^2R^2+v^2}}$ - speed (constant!)

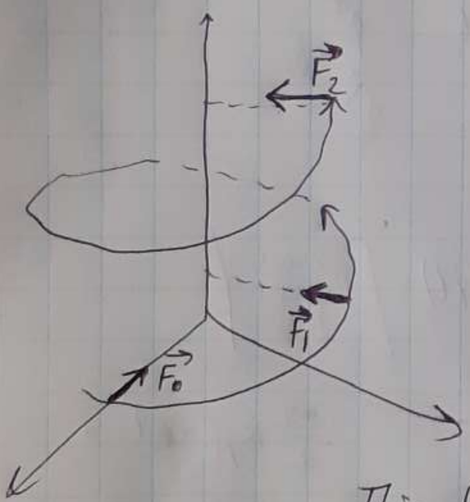
Kinetik energy $E_k=\frac{m(\dot{\vec{q}})^2}{2}=\boxed{\frac{m(\omega^2R^2+v^2)}{2}}$ → vertical direction

tangential direction

10.2

d) $\vec{F}=m\ddot{\vec{q}}(t)=m\begin{bmatrix}-\omega^2R\cos(\omega t)\\ -\omega^2R\sin(\omega t)\\ 0\end{bmatrix}=-m\omega^2R\,\hat{r}.$

Force causes centripetal curving, but does not have vertical component.



$$\vec{F}\cdot\dot{\vec{q}}(t)=-m\omega^2R\,\hat{r}\cdot(\underbrace{R\omega\hat{\theta}+v\hat{z}}_{\text{from }(*)})=$$

$$=\dots\underbrace{\hat{r}\cdot\hat{\theta}}_{0}+\dots\underbrace{\hat{r}\cdot\hat{z}}_{0}=0+0=0.$$

as RNB

(another approach uses coordinates)

This follows from observation that E_k=const, so force can only have orthogonal component to $\dot{\vec{q}}$.

* e) $\vec{F}=e(\vec{E}+\dot{q}\times B)$, $\vec{E},\vec{B}$ const

Speed is constant $\longleftrightarrow \vec{E}=0$, from d) it follows that

$|\dot{\vec{q}}|$=const $\longleftrightarrow \vec{F}\cdot\dot{\vec{q}}=0$, $e(\vec{E}+\dot{\vec{q}}\times\vec{B})\cdot\dot{\vec{q}}=e\dot{\vec{q}}\vec{E}+e(\dot{\vec{q}}\times\vec{B})\dot{\vec{q}}=$

$=e\dot{\vec{q}}\vec{E}+e\dot{\vec{q}}(\dot{\vec{q}}\times\vec{B})=e\dot{\vec{q}}\vec{E}+e\vec{B}\underbrace{(\dot{\vec{q}}\times\dot{\vec{q}})}_{\vec{0}}=e\dot{\vec{q}}\vec{E}=0,$

so $\vec{E}=\vec{0}$ (it can't be always orthogonal to $\dot{\vec{q}}$, since it is ~~constant~~ in time, unlike $\dot{\vec{q}}$).

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