Lecture "Experimental Physics I"
(Prof. Dr. R. Seidel)

Lecture 24

Damped and driven oscillations

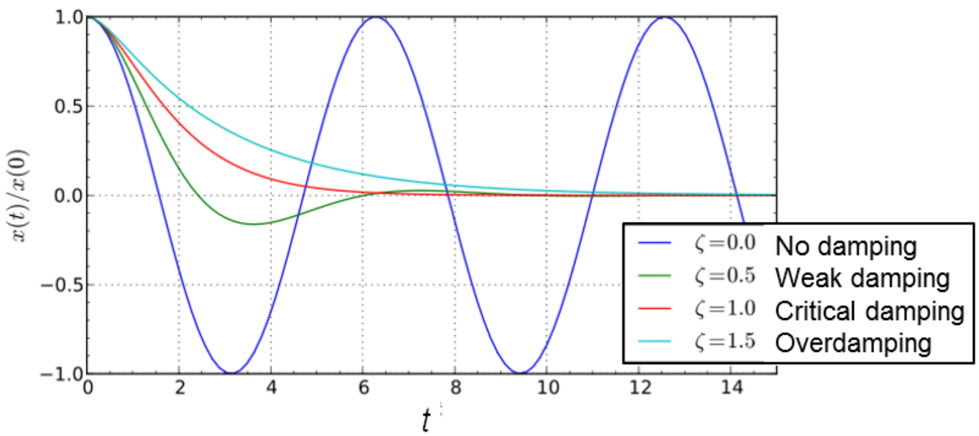
- Oscillation with damping
- Driven oscillations

1) Harmonic oscillator with damping

In the real world, oscillations are damped due to dissipation of energy (e.g. the friction in a swing), such that the amplitude of the oscillation becomes smaller over time. This can be undesired, but sometimes also wanted, e.g. in order to build shock absorbers or to prevent a closing door from oscillating. In the following we will study the physics of such damped oscillators.

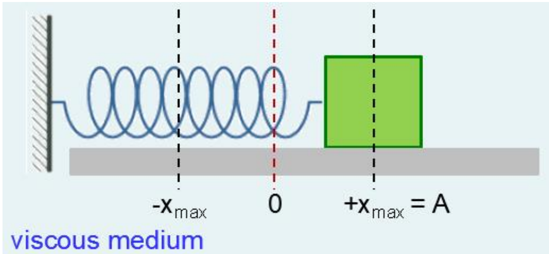
Experiment: To study damped oscillations in a simple setting, we look at a pendulum damped by an eddy current brake that allows to set different strengths of the damping.
From the experiment we can conclude:

- For **weak damping** one observes **oscillations with decreasing amplitude**
- For **strong damping** one observes **no oscillation** but a steady return to equilibrium



A) General approach

In the following we want to understand the experimentally observed behavior of damped oscillations mathematically. We consider as oscillator a mass on a spring. In order to include friction, we assume Stoke's friction, e.g. by placing the oscillator in a viscous fluid:



The force from Stokes friction acting on a sphere is given by

$$f_r = -\underbrace{6\pi\eta R}_{\gamma}\, v$$

with γ being the drag coefficient of the object. The friction acts in opposite direction to the velocity vector and adds up to the back-driving spring force:

$$-kx - \gamma v = ma$$

We can thus transform this equation to the usual oscillator equation with an additional first derivative of the position from the velocity-dependent friction:

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \underbrace{\frac{k}{m}}_{\omega_0{}^2}\, x = 0$$

where ${\omega_0}^2$ is the angular frequency for the undamped harmonic oscillator. The solution for the undamped harmonic oscillator was:

$$x(t) = A\cos(\omega t + \varphi)$$

An alternative representation could be formulated using complex numbers:

$$x = A\, e^{i(\omega t+\varphi)}$$

where the real part of x represented the oscillator position and the additional imaginary part comprised the phase information. We now apply a slightly different solution by omitting the complex symbol:

$$x(t) = c\, e^{\lambda t+\varphi_0}$$

Insertion into the differential equation then provides:

$$\lambda^2 + \frac{\gamma}{m}\lambda + {\omega_0}^2 = 0$$

Solving this quadratic equation yields:

$$\lambda_{+,-} = -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - {\omega_0}^2} \qquad (don't\ delete\ from\ board)$$

which can be transformed to:

$$\lambda_{+,-} = \omega_0\left[-\frac{\gamma}{2m\omega_0} \pm \sqrt{\left(\frac{\gamma}{2m\omega_0}\right)^2 - 1}\right] = \omega_0\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

with ζ (zeta)

$$\zeta = \frac{\gamma}{2m\omega_0}$$

being theso-called **damping constant** since it is proportional to the drag coefficient.

Overall, we get two possible values for λ. The full solution of a differential equation is always a linear combination of all possible solutions. Thus, we get for the oscillator position **(show on slides):**

$$x(t) = c_+ e^{\lambda_+ t+\varphi_+} + c_- e^{\lambda_- t+\varphi_-}$$

$$x(t) = c_+ e^{(-\zeta\omega_0 + \sqrt{(\zeta^2-1)})\,t+\varphi_+} + c_- e^{(-\zeta\omega_0 - \sqrt{(\zeta^2-1}\,)t+\varphi_-}$$

This can be transformed to:

$$x(t) = e^{-\zeta\omega_0 t}\left[c_+ e^{\sqrt{\zeta^2-1}\,t+\varphi_+} + c_- e^{-\sqrt{\zeta^2-1}\,t+\varphi_-}\right]$$

For sufficiently small damping, i.e. a small γ, the damping constant ζ can become smaller than 1, such that the discriminants in the square root terms are negative, while for large damping $\zeta > 1$ such that the discriminants would be positive. In the following we distinguish these different cases.

B) Harmonic oscillator with weak damping

Let us first consider **weak damping** with $\boldsymbol{\zeta < 1}$. The discriminant is in this case negative, such that taking the square rout provides an imaginary number, such that we can write:

$$\omega_0\sqrt{\zeta^2 - 1} = \omega_0\sqrt{-1}\sqrt{1-\zeta^2} = i\omega_0\sqrt{1-\zeta^2} = i\omega$$

with

$$\omega = \omega_0\sqrt{1-\zeta^2}$$

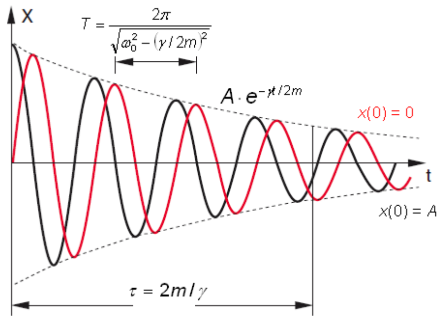
Inserting into the derived solution gives:

$$x(t) = e^{-\zeta\omega_0 t}[\underbrace{\underbrace{c_+ e^{i(\omega t+\varphi_+)}}_{Re()=c_+\cos[\omega(t+t_{0+})]} + \underbrace{c_- e^{-i(\omega t+\varphi_-)}}_{Re()=c_-\cos[\omega(t-t_{0-})]}}_{A\cos(\omega t+\varphi)}]$$

The exponential terms with imaginary arguments are nothing else than harmonic oscillator solutions. Taking their real parts and using $\cos -x = \cos x$, we get two cosine functions with the same frequency ω but different phases and amplitudes that together give again a sinusoidal function of the same frequency, such that we can write for the real part of $x(t)$:

$$x_{(real)}(t) = Re(x) = A\, e^{-\zeta\omega_0 t}\cos(\omega t + \varphi)$$

Plotting yields:



This is the expected harmonic oscillation with decaying amplitude. Its **angular frequency $\omega = \omega_0\sqrt{1-\zeta^2}$ is close to ω_0 for weak damping but decreases for increasing damping**. This is due to the damping slowing down the return to the equilibrium position. The decay is provided by an exponential envelope for the harmonic function. The term in the exponent:

$$\tau = \frac{1}{\zeta\omega_0} = \frac{T_0}{2\pi\zeta}$$

defines the characteristic damping time τ at which the amplitude has decreased by a factor of *e* with where T_0 being the period of the undamped oscillator. For weak damping ($\zeta \ll 1$), the decay time τ is considerably larger than T_0 but it approaches T_0 for stronger damping. One can also say that we get an oscillation with decaying amplitude as solution, if the characteristic damping time is not small compared to the period of the (undamped) oscillator.

As before the initial amplitude A and the phase φ define the starting conditions (see cosine and sine function in the plot).

C) Harmonic oscillator with strong damping:

Let us now consider **strong damping** with $\boldsymbol{\zeta > 1}$. In this case the damping time from above would considerably faster than the period of the undamped oscillator. The discriminant is in this case positive such that the exponents $\lambda_{+,-}$ become real values:

$$\lambda_{+,-} = \omega_0\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

Our oscillator solution becomes thus a sum of two exponential decays (double exponential decay):

$$x(t) = \underbrace{c_+ e^{-\omega_0\left(\zeta - \sqrt{\zeta^2-1}\right)t}}_{\text{slow phase}} + \underbrace{c_- e^{-\omega_0\left(\zeta + \sqrt{\zeta^2-1}\right)t}}_{\text{fast phase}}$$

Since $\zeta > \sqrt{\zeta^2 - 1}$, the first term of the sum represents a slow exponential decay, while the second term represents a decay on a faster time scale. **Such a system is called an overdamped system, since it returns to equilibrium without oscillating.** To understand the contributions of the two exponentials better, let us look at different start conditions (**only on slides**). For

- $x(0) = A, v(0) = 0,$ we obtain from inserting $t = 0$ into $x(t)$ and $\dot{x}(t)$:

$$x(0) = A = c_+ + c_-$$

$$\dot{x}(0) = 0 = -c_+\omega_0(\zeta - \sqrt{\zeta^2 - 1}) - c_-\omega_0(\zeta + \sqrt{\zeta^2 - 1})$$

  Transformation then gives:

$$c_+ = \frac{A}{2}\left(1 + \frac{1}{\sqrt{1 - 1/\zeta^2}}\right), \quad c_- = \frac{A}{2}\left(1 - \frac{1}{\sqrt{1 - 1/\zeta^2}}\right)$$

- $x(0) = 0;\ v(0) = A\omega_0$

  This represents the same initial mechanical energy as before and we obtain:

  $$x(0) = 0 = c_+ + c_-$$

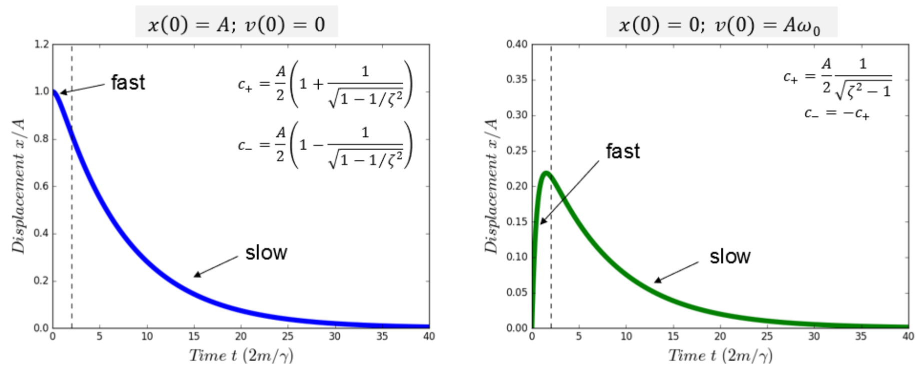
  which gives $c_+ = -c_-$ such that we get:

  $$v(0) = A\omega_0 = -c_+\omega_0\left(\zeta - \sqrt{\zeta^2 - 1}\right) + c_+\omega_0(\zeta + \sqrt{\zeta^2 - 1}) = 2c_+\omega_0\sqrt{\zeta^2 - 1}$$

  This provides:

  $$c_+ = \frac{A}{2}\frac{1}{\sqrt{\zeta^2 - 1}}; \qquad c_- = -\frac{A}{2}\frac{1}{\sqrt{\zeta^2 - 1}}$$

Plotting the resulting functions for a damping constant of $\zeta = 2$ yields:



During the fast phase a steady state velocity is reached, i.e. if started with an initial displacement the mass first accelerates until friction stops the acceleration. If started with an initial velocity, the velocity is slowed down in the fast phase by the friction and later on by the spring until the direction reverses. After reaching a steady state, the system returns to the equilibrium position with the slow time constant.

For **extremely strong damping $\zeta \gg 1$**, we can Taylor-expand the square root term within α around ζ^2. The Taylor expansion of a square root is given as:

$$\sqrt{x + \Delta x} \approx \sqrt{x} + \frac{1}{2\sqrt{x}}\Delta x$$

Thus,

$$\sqrt{\zeta^2 - 1} \approx \zeta - \frac{1}{2\zeta}$$

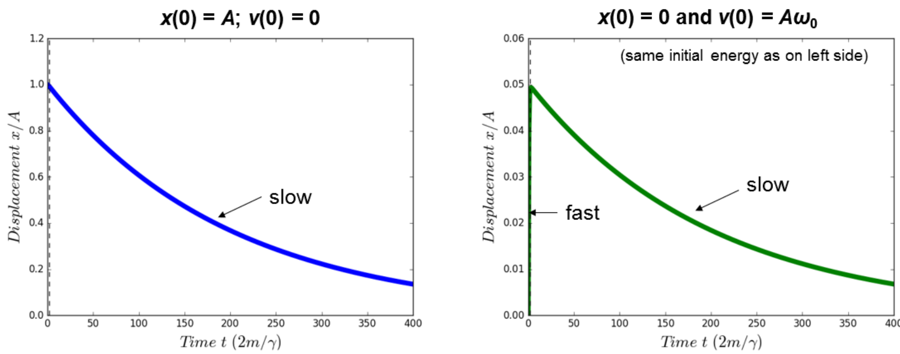
Our solution from above then becomes this way:

$$x(t) = \underbrace{c_+ e^{-(\omega_0/2\zeta)t}}_{\text{slow phase}} + \underbrace{c_- e^{-\omega_0\left(\overbrace{2\zeta - 1/2\zeta}^{\approx 2\zeta}\right)t}}_{\text{fast phase}}$$

Inserting $\zeta = \gamma/2m\omega_0$ and $-\omega_0^2 = -k/m$ provides:

$$x(t) = c_+ e^{-(k/\gamma)t} + c_- e^{-(\gamma/m)t}$$

In this extreme case the time constants have a simple form. The short time scale m/γ is the time at which an object with an initial velocity is **slowed down in a viscous medium**. At the much, much slower time scale γ/k the oscillator returns to its equilibrium position. This process is called **slow creep** and is only determined by the spring that is driving back the object against the viscous drag. Notably, it is independent of the mass, i.e. the inertia, which do not play a role at this time scale.



The loss/gain of the initial velocity is much faster than the creep, such that is practically occurring instantaneous.
This is similar to the low Reynolds-Number world in which microorganisms and biomolecules operate. There are no resonances for motions of such systems. A deformed protein such as a motor protein will therefore creep back to its equilibrium position. **Show kinesin movie**

D) Harmonic oscillator with critical damping

If $\boldsymbol{\zeta = 1}$**,** we have a **critically damped system.** In this case the angular frequency of the weakly damped oscillator approaches zero. We now get only a single time constant for our exponential solution of the oscillator equation:

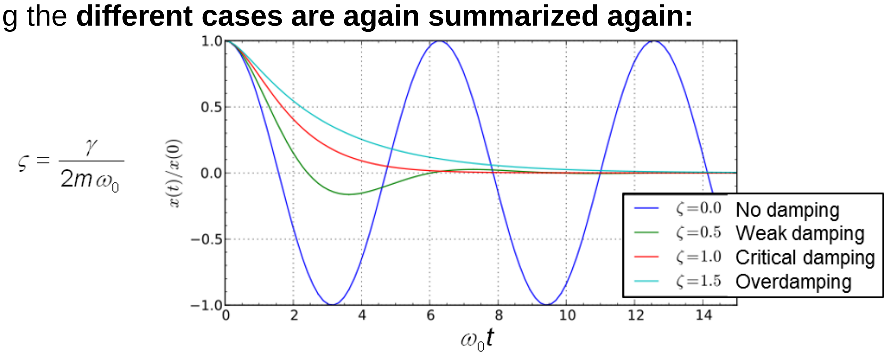
$$\lambda = -\zeta\omega_0 = -\frac{\gamma}{2m}$$

However, for a complete solution of the differential equation, one needs a second constant (like a second integration constant when integrating the equation). One can find that a relation of the form

$$x(t) = e^{-\zeta\omega_0 t}[c_1 t + c_2]$$

solves our equation in this case, which can be verified by inserting. This equation provides again an exponential decay for long times scales
Critical damping represents the fastest non-oscillatory return to equilibrium. It is therefore relevant in technology (e.g. for constructing optimal shock absorbers in a car, damping mechanisms at doors etc.).

In the following the **different cases are again summarized again:**



2) The driven harmonic oscillator

Let us now consider a harmonic oscillator with an external periodic driving force. Understanding such a driven oscillator is very important for many fields in physics. For example, it is important for the design of shock absorbers in vehicles, it is used to understand how light interacts with the

electrons of atoms and molecules and brings them into excited states. Also, it explains why light can get diffracted in optically dense media.
A driven oscillator can be intuitively imagined by a swing, where one periodically gives the swing a kick. A central question is the best frequency and the best phase that is necessary to drive the swing. To this end we first have an empirical look.

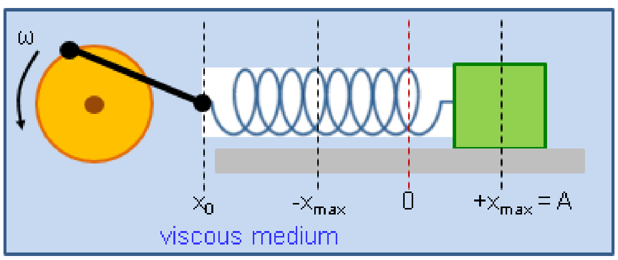
Experiments: Harmonic driving forces acting on oscillators

- Driven spring oscillator. One sees that at a particular frequency the oscillator is driven to maximum amplitude. This state is called **resonance**.
- Driven torsional spring oscillator. One observes that:
  - (i) below the resonance frequency the driving force and oscillator are in phase.
  - (ii) in resonance a 90° phase shift is observed
  - (iii) above resonance a 180° phase shift occurs.

In addition, we see that there is an **initial transient period** with a very different oscillation followed by a **stationary period** at which the **oscillator moves with the same frequency as the driving force.**

A) Driven oscillator solution in real space

Let us now mathematically describe the motion of a driven oscillator. It can be constructed from a damped oscillator at which an additional periodic driving force is acting. In the experiment this was realized by an eccentric mechanism that drove the oscillator.



The left spring end is moved by the eccentric mechanism according to:

$$\Delta x_0(t) = A_0 \cos \omega t$$

where A_0 is the radius of the eccentric attachment. The total back-driving force from the spring on the object at position x is:

$$F_{spring} = -k(x - \Delta x_0) = -kx + \underbrace{kA_0}_{F} \cos \omega t$$

since for a positive Δx_0 the spring extension is reduced. Thus, we have an external **single harmonic driving force of amplitude *F*** on the left spring end. Adding the external force to the equation of motion of the damped oscillator yields:

$$-\gamma v - kx + F \cos \omega t = ma$$

We thus get by transformation:

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \underbrace{\frac{k}{m}}_{\omega_0{}^2} x = \frac{F}{m} \cos \omega t$$

where ω_0 is again the angular frequency for the undamped harmonic oscillator. The difference to the non-driven damped oscillator is that we have now a non-zero driving force on the right side. In the stationary case, the experiment indicated the a simple harmonic oscillation as solution:

$$x(t) = A \cos(\omega t + \varphi)$$

Inserting this equation into the equation of motion provides:

$$A(\omega_0{}^2 - \omega^2) \cos(\omega t + \varphi) - \frac{\gamma}{m} A\omega \sin(\omega t + \varphi) = \frac{F}{m} \cos \omega t$$

Due to the angle sums inside the sine and cosine functions this equation looks rather complicated. However, we can transform the equation and describe both functions as sum of a $\cos\omega t$ and a$\sin\omega t$ using the angle sum identities (**see slide**):

$$\cos(\omega t+\varphi)=\cos\omega t\ \cos\varphi-\sin\omega t\sin\varphi$$
$$\sin(\omega t+\varphi)=\cos\omega t\ \sin\varphi+\sin\omega t\cos\varphi$$

Inserting these equations provides several terms that either contain $\cos\omega t$ or $\sin\omega t$. Grouping the terms provides:

$$\cos\omega t\underbrace{\left[A(\omega_0^2-\omega^2)\cos\varphi-\frac{A\omega\gamma}{m}\sin\varphi-\frac{F}{m}\right]}_{=0}+\sin\omega t\underbrace{\left[-A({\omega_0}^2-\omega^2)\sin\varphi-\frac{A\omega\gamma}{m}\cos\varphi\right]}_{=0}=0$$

The cosine and the sine functions are orthogonal to each other, i.e. the sine cannot be used to describe the cosine by a linear combination and vice versa. To ensure that this equation can hold for all times, the only possibility is that both terms in brackets must be zero.
From the $\sin\omega t$ term on the right we get:

$$A\sin\varphi=-\frac{\omega\gamma/m}{\omega_0^2-\omega^2}A\cos\varphi$$

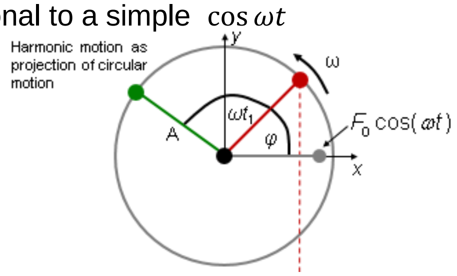
This provides us directly the initial phase:

$$\tan\varphi=-\frac{\omega\gamma/m}{\omega_0^2-\omega^2}=\left(\frac{\gamma}{m\omega_0^2}\frac{\omega}{(\omega/\omega_0)^2-1}\right)=\frac{2\zeta\omega/\omega_0}{(\omega/\omega_0)^2-1}$$

which can be transformed to the relation on the right side using the definition of the damping constant $\zeta=\gamma/(2m\omega_0)$. Inserting the obtained expression for $A\sin\varphi$ into the left $\cos\omega t$ term and setting it to zero provides an expression for $A\cos\varphi$:

$$A\cos\varphi=\frac{F}{m}\frac{(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+(\omega\gamma/m)^2}$$

This is the **part of the amplitude that oscillates in phase** with the external driving force, since the driving force was proportional to a simple $\cos\omega t$



Correspondingly, inserting the expression for $A\cos\varphi$ into the equation for $A\sin\varphi$ provides an expression for $A\sin\varphi$, which is the **part of the amplitude that oscillates with a phase shift of 90° relative to the driving force:**

$$A\sin\varphi=-\frac{F}{m}\frac{\omega\gamma/m}{(\omega_0^2-\omega^2)^2+(\omega\gamma/m)^2}$$

The total squared amplitude is provided by adding the squares of the in- and the out-of-phase amplitude components, such that we get:

$$A=\sqrt{(A\sin\varphi)^2+(A\cos\varphi)^2}=\frac{F/m}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma/m)^2}}$$

Using $\omega_0^2=k/m$ and $\zeta=\gamma/(2m\omega_0)$, this can be transformed to:

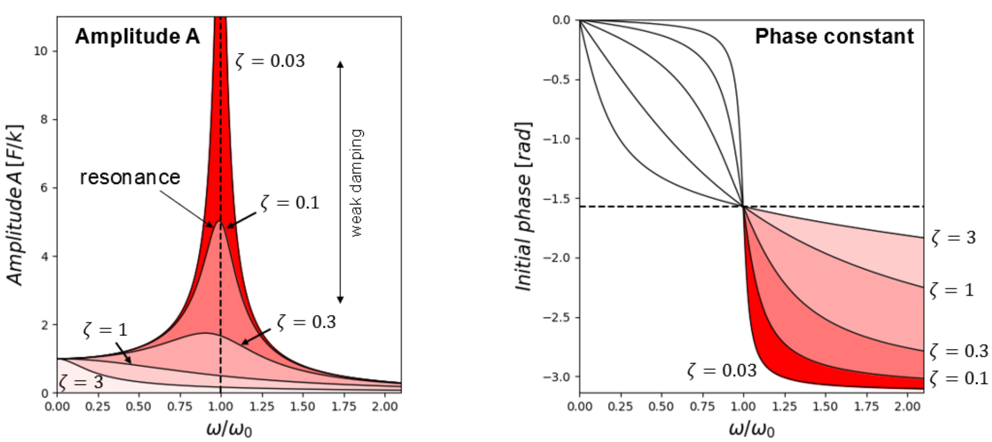
$$A(\omega)=\frac{F/k}{\sqrt{(1-(\omega/\omega_0)^2)^2+(2\zeta\omega/\omega_0)^2}}$$

For the obtained solution, we do not have to make assumptions about the value of ζ, such that it **holds for weak as well as strong damping**.

B) Resonance and beyond

Let us now evaluate the derived formula for the frequency dependent amplitude $A(\omega)$ of the driven oscillation by inspecting different frequency limits:

- At **low driving frequencies $\omega \ll \omega_0$** we can neglect the contribution of the terms with ω in the equation above and the amplitude becomes:
  $$A_{\omega\to 0} \approx \frac{F}{k}$$
  It practically reflects the stationary case, where in agreement with Hookes' law the displacement is proportional to the applied force .
- At **large driving frequencies $\omega \gg \omega_0$** the $((\omega/\omega_0)^2)^2$- term in the root of the amplitude equation dominates. The amplitude approaches zero in a spring-constant independent manner:
  $$A_{\omega\to\infty} \approx \frac{F}{k}\frac{\omega_0^2}{\omega^2} = \frac{F}{m}\frac{1}{\omega^2} \propto \frac{F}{m}T^2 \to 0$$
  At these frequencies the inertia of the mass prevents any larger displacement. F/m decribes the acceleration. Due to the very short period, the acceleration occurs in one direction only for a very short time and instantaneously flips sign such pulling back the mass in the other direction. The **amplitude is proportional to the squared period**, as for a constantly accelerated motion, with a period that approaches zero.
- **Resonance:** For weak damping, the **amplitude becomes maximal** at some intermediate frequency. This case (we have seen in the experiment) **is called resonance.** The maximum is obtained when the term in the root becomes minimal. Setting its derivative with respect to ω to zero provides for the maximum position:
  $$0 = -4(1-(\omega_{Re\,s}/\omega_0)^2)\omega_{Re\,s}/\omega_0^2 + 8\zeta^2\omega_{Re\,s}/\omega_0^2$$
  Transformation gives then:
  $$\omega_{Res} = \omega_0\sqrt{1-2\,\zeta^2}$$
  This is **even lower than the frequency of the damped harmonic oscillator**. As weaker the damping as closer the resonance frequency and the frequency of the damped oscillator are to ω_0, since the ζ^2 can be neglected. The amplitude at ω_0 is:
  $$A(\omega_0) = \frac{F/k}{\sqrt{0+(2\zeta\omega_0/\omega_0)^2}} = \frac{F}{2k\zeta}$$
  F/k would correspond to the static displacment of a spring. Division by 2ζ provides, that the amplitude strongly increases with decreasing damping factor ζ becoming potentially much larger than the static spring displacment of a corresponding force. This way one can drive the oscillator to very large ampolitudes:



For zero damping $\zeta = 0$, A becomes infinite, since the energy introduced by the external driving force is not dissipated anymore. This is called **resonance catastrophe.**

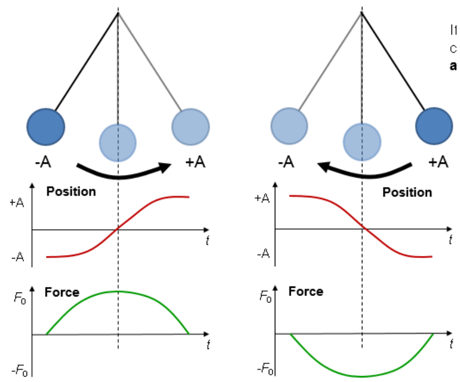
Movies: Tacoma bridge, 3rd longest bridge at that time & washing machine

Note, that also for **overdamping (ζ > 1) and critical damping (ζ =1)** we get harmonic solutions. However, in this case we **do not have a resonance peak** anymore.
To understand the oscillator behavior at resonance better, we now look at the phase shift $\varphi(\omega)$ (see plot above) that is given by:

$$\tan\varphi = \frac{2\zeta\omega/\omega_0}{(\omega/\omega_0)^2 - 1}$$

It grows for $\omega \le \omega_0$ from 0 to $-\pi/2$ and for $\omega \ge \omega_0$ from $-\pi/2$ to $-\pi$. The phase is always negative, i.e. the driven oscillation limps behind the driving force. We again can distinguish the different cases:

- At **low driving frequencies $\boldsymbol{\omega \ll \omega_0}$** the oscillator displacement has enough time to tightly follow the force (inertia does not play a role) such that the **phase approaches zero** as it is expected for Hookean displacement.
- At $\boldsymbol{\omega \approx \omega_0}$, which is **typically close to resonance,** the oscillator displacement limps behind the force by $-\pi/2$. Plotting the oscillator displacement and the force one sees that in this way the force acts always along the motion direction of the oscillator and thus does always positive work. We will see later that we can best pump in energy in the oscillator under this condition, such that the maximimum amplitude is obtained.
- **Experiment:** The phase shift by $-\pi/2$ can be demonstrated when driving a swing (long pendulum):



- At **large driving frequencies $\boldsymbol{\omega \gg \omega_0}$** the oscillator amplitude is so small, such that the back-driving spring force does not matter anymore. The moving mass only returns because the external driving force pulls it in the opposite direction. Thus, displacement and external force have opposite signs leading to the phase shift of $-\pi$.

C) Solution with complex numbers (Not part of lecture)

A more elegant way to solve the oscillator equation can be found by using complex number notation for the oscillator position *x*. We said before that in addition to the real oscillator position given by the real part of x, both the imaginary and real part define the oscillator phase. In the polar representation of complex numbers, the oscillator position becomes (**show first on slides**):

$$z = Ae^{i(\omega t+\varphi)} = \underbrace{A_0}_{Ae^{i\varphi}}\, e^{i\omega t}$$

A_0 is hereby a complex amplitude that contains both the oscillator amplitude and the initial phase according to:

$$A = |A_0|, \qquad \tan\varphi = \frac{Im(A_0)}{Re(A_0)}$$

The force is given by:

$$F(t) = Fe^{i\omega t}$$

Its amplitude is only real due to the zero initial phase. The driven oscillator equation thus becomes:

$$\ddot{z} + \frac{\gamma}{m}\dot{z} + {\omega_0}^2 z = \frac{F}{m}e^{i\omega t}$$

Inserting the complex number solution provides:

$$(-\omega^2 + {\omega_0}^2)A_0 e^{i\omega t} + i\frac{\gamma}{m}\omega A_0 e^{i\omega t} = \frac{F}{m}e^{i\omega t}$$

Dividing by $e^{i\omega t}$ and further transformation provides immediately the complex amplitude of the solution:

$$A_0 = \frac{F/m}{({\omega_0}^2 - \omega^2) + i\omega\gamma/m}$$

We now transform this result further, such that we can identify the real and the imaginary part of the amplitude. To this end, we multiply numerator and denominator with the complex conjugate of the denominator, which such that we get a real denominator:

$$A_0 = \frac{F/m}{({\omega_0}^2 - \omega^2) + i\omega\gamma/m} \cdot \frac{({\omega_0}^2 - \omega^2) - i\omega\gamma/m}{({\omega_0}^2 - \omega^2) - i\omega\gamma/m} = \frac{F/m \cdot ({\omega_0}^2 - \omega^2 - i\omega\gamma/m)}{({\omega_0}^2 - \omega^2)^2 + (\omega\gamma/m)^2}$$

The real amplitude and the initial phase are then given by:

$$A^2 = Re(A_0)^2 + Im(A_0)^2 = \frac{(F/m)^2[({\omega_0}^2 - \omega^2) + (\omega\gamma/m)^2]}{[({\omega_0}^2 - \omega^2)^2 + (\omega\gamma/m)^2]^2} = \frac{(F/m)^2}{({\omega_0}^2 - \omega^2)^2 + (\omega\gamma/m)^2}$$

$$\tan\varphi = \frac{Im(A_0)}{Re(A_0)} = -\frac{\omega\gamma/m}{\omega_0^2 - \omega^2}$$

which is our previous result.

D) Energy balance of the driven harmonic oscillator

To understand resonance even better, we have a look at the energy balance of the oscillator, in particular the work done by the external force. To this end we multiply our equation of motion by the velocity $\dot{x}$:

$$m\ddot{x}\,\dot{x} + \gamma\dot{x}\,\dot{x} + kx\,\dot{x} = F(t)\,\dot{x} = dW/dt$$

The right hand side equals the power or the work done per time d*t*, since

$$\frac{dW}{dt} = \frac{Fdx}{dt} = F\dot{x}$$

Interlude – work by external force is fully dissipated by the friction (not shown):

The terms with different derivatives can be rewritten such that regrouping the equation provides:

$$0 = \frac{d}{dt}\left(\underbrace{\overbrace{\frac{m}{2}\dot{x}^2}^{E_{kin}} + \overbrace{\frac{k}{2}x^2}^{E_{pot}}}_{\text{const. for stationary case}}\right) = -\underbrace{\gamma\dot{x}\,\dot{x}}_{\text{dissip. P}} + \underbrace{F(t)\dot{x}}_{\text{ext. power}}$$

The left side is the change in the total mechanical energy of the system. For the stationäry solution it is constant, as we already saw for the simple harmonic oscillator without damping. Thus, the left-hand-side of the equation equals zero zero.

The first term on the right side equals the friction force times the velocity. It is the power loss of the system by friction. The second term is the power put into the system by the external force.

Thus, in the stationary phase, the power put into the system by the external force is directly disspated/consumed by the friction, which makes intuitely sense.

With the expression for the work by the external force from above, we can calculate the work done per period of oscillation, since we know $F(t)$ and $v(t)$:

$$W = \int_0^T F(t)\frac{dx}{dt}dt = \int_0^T F\cos\omega t\,[-A\omega\sin(\omega t+\varphi)]\,dt$$

Using the angle sum identity from before, we can transform this equation to:

$$W = -F\omega\int_0^T \cos\omega t\,(A\sin\varphi\ \cos\omega t + A\cos\varphi\ \sin\omega t)dt$$

Separating the $\cos\omega t$ and $\sin\omega t$ terms and carrying out the integrazion gives:

$$W = -F\omega\,A\,sin\,\varphi\underbrace{\int_0^T \underbrace{\cos^2(\omega t)}_{\frac{1-cos\,2\omega t}{2}}dt}_{\frac{T}{2}} - F\omega\,A\cos\varphi\underbrace{\int_0^T \underbrace{\sin\omega t\cos\omega t}_{sin(2\omega t)/2}dt}_{0}$$

The integral on the right side becomes zero, since it contains the integral of a sine function over two full periods. The integral on the left-hand side gives $T/2$ due to its constant term. The average power is then given by:

$$P = \frac{W}{T} = -\frac{1}{2}F\omega\underbrace{A\sin\varphi}_{Im\,A_0} = \left(-\frac{1}{2}F\omega\frac{-A^2\omega\,\gamma/m}{F/m}\right) = \frac{\gamma\omega^2A^2}{2}$$

Thus, the **power** that is taken up by the oscillator is **proportional to the drag coefficient and to the squares of amplitude and frequency**.
Importantly, **only the $A\,sin\,\varphi$ component of the amplitude, i.e. the $-\pi/2$ shifted, orthogonal component of the amplitude (imaginary part of the complex amplitude) dissipates energy**. **At ω_0, i.e. near resonance,** we are close to the maximum amplitude and have only an orthogonal/imaginary amplitude part since $\varphi = -\pi/2$:

Resonance is thus the most efficient way to pump energy into the system, since the force acts always in the direction of motion (as we saw before).
The in-phase component $A\cos\varphi$ (real part of the complex amplitude) has in contrast a balanced **periodic uptake and release of energy** (one pushes the swing as much as one gets pushed by the swing). It dominates at very low and very high frequencies where thus only little energy is dissipated.
The spilt into in-phase and orthogonal component of a driven oscillations, i.e. real and complex part of the complex amplitude, is an important tool that is used in many branches of physics, e.g. when measuring elastic properties of materials that dissipate energy during the deformation or importantly the interaction of electromagnetic waves with matter, where the dielectric constant will become a complex number to express polarization of the medium + the losses (absorbance) of the wave (light).
Inserting the derived equation for the amplitude of the driven oscillator into the power equation from above provides (**not shown in lecture, final result on slide**):

$$P = \frac{\gamma\omega^2 A^2}{2} = \frac{\gamma\omega^2}{2k^2}\frac{F^2}{(1-(\omega/\omega_0)^2)^2 + (2\zeta\omega/\omega_0)^2}$$

which gives after transformation:

$$P = \frac{F^2}{2}\frac{\gamma/k^2}{(1-(\omega/\omega_0)^2)^2/\omega^2 + (2\zeta/\omega_0)^2} = \frac{\gamma{\omega_0}^2}{k^2(2\zeta)^2}\frac{F^2/2}{(1-(\omega/\omega_0)^2)^2/(2\zeta\omega/\omega_0)^2 + 1}$$

using ${\omega_0}^2 = k/m$ and $2\zeta = \gamma/(m\omega_0)$ this can be further transformed to:

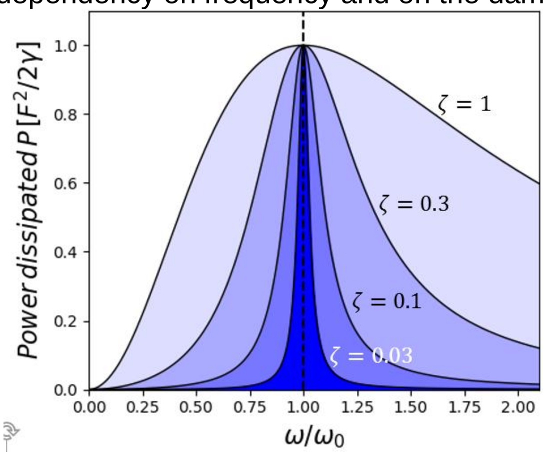
$$P = \frac{F^2}{2\gamma}\frac{1}{(1-(\omega/\omega_0)^2)^2/(2\zeta\omega/\omega_0)^2 + 1}$$

The **power uptake has a maximum at $\boldsymbol{\omega = \omega_0}$** at which the only frequency-dependent term of the sum in the denominator becomes zero. The maximum power is given by:

$$P = \frac{F^2}{2\gamma}$$

Thus, the maximum power uptake depends only at the driving force amplitude and the friction coefficient. Transforming the power equation further, we get:

Plotting reveals the power dependency on frequency and on the damping coefficient:



As expected there is a pronounced maximum at ω_0. With increasing damping, the resonance peak of the power becomes broader.

E) Transient phase

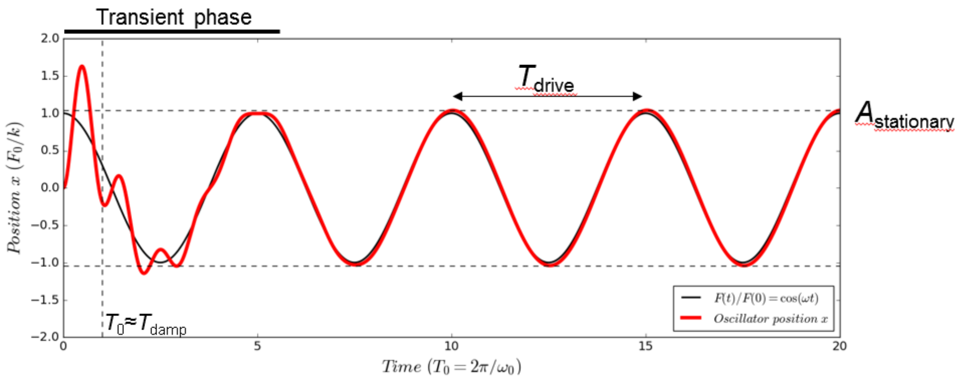
So far, only the stationary phase of the oscillator, where it reached a constant amplitude was considered. When starting the driving force, there is however a transient phase during which the oscillator reaches its steady state amplitude.

The full solution of the oscillator equation is provided by the sum of our stationary solution and the solution for the damped oscillator without driving force (**show on slides**).

$$x(t) = \underbrace{A_{damp}e^{-\zeta\omega_0 t}\cos(\omega_{damp}t + \varphi_{damp})}_{\text{transient part with } \omega_{damp} \text{ of damped oscillator}} + \underbrace{A_{drive}\cos(\omega_{drive}t + \varphi_{drive})}_{\text{stationary part with } \omega_{drive} \text{ of external force}}$$

Inserting the additional contribution just adds a zero to the driving force in the differential equation, such that the new equation also solves the driven oscillator equation of motion.

This is seen in the experiment as an initial transient phase where oscillations occur at frequency ω_{damp} on top of the oscillations with ω_{drive} (see **animation and slide** for the numeric solution in case of weak damping $\zeta = 0.1$ and slow driving $\omega_{drive} = 0.2\ \omega_0$ or fast driving $\omega_{drive} = 5\ \omega_0$).



The time constant of the exponential decay of the transient phase is given by the reciprocal factor of the time in the exponential, for which we can write for the weak damping of $\zeta = 0.1$:

$$\tau_{decay} = \frac{1}{\zeta\omega_0} = \frac{10}{2\pi}T_0$$

Thus, the transient phase decays over just a few periods of the damped oscillator. As smaller the damping is as longer the transient phase persists.

Lecture 24: Experiments

1) Pendulum damped by eddy current brake to apply differently strong damping
2) Driven spring oscillator. One sees that at a given frequency it is driven into resonance.
3) Driven torsional spring oscillator to observes the phase of the oscillator below, at and above the resonance frequency
4) Swing (long pendulum with mass) to illustrate the 90° phase shift to optimally drive the swing in resonance