Lecture "Experimental Physics I"

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Lecture 12

Systems of particles & Conservation of momentum

- Center of mass, and center-of-mass system
- Potential energy, velocity and momentum of a particle system
- Conservation of linear momentum
- Projectile with explosion
- Rocket propulsion

1) Multi-particle systems

So far, we described the motion of single isolated point masses under the influence of external forces. Even for planetary motion we considered for simplicity a single mass (planet) system that is moving in the stationary gravitational field of the sun.

Many **real systems are however multi-particle systems**, e.g. the moon on its trajectory around the earth has a sufficient mass to deflect the earth during its rotation, such that both **earth and moon are rotating** around a common point – their **joint center of mass** (see slides). In the following we want to work out important principles when dealing with more than one interacting particle.

An experimental motivation using two 'particles' is provided by the following setting:

Experiment with ballistic pendulum: An air-gun bullet is shot on a cube, which itself is suspended on a pendulum. We check energy conservation with the displacement of the pendulum, i.e. whether the initial kinetic energy of the bullet was fully transmitted into final potential energy of the pendulum We have the following numerical values:

Bullet: $v_B \;=\; 200\ \mathrm{m\ s^{-1}}, m_B \;=\; 0.5 \times 10^{-3}\ \mathrm{kg}$ which provides an initial kinetic energy:

$$E_{k,B} = \frac{m}{2}v^2 = \frac{5 \times 10^{-4}\ \mathrm{kg}}{2}\ 4 \times 10^4\ \mathrm{m^2 s^{-2}} \;=\; 10\ \mathrm{J}$$

Pendulum: for a lateral displacement $x = 8\ cm$ of the pendulum the potential energy is given as:

$$E_{pot,P} = (m_P + m_B)g\Delta h = m_p g\left(L - \sqrt{L^2 - x^2}\right)$$

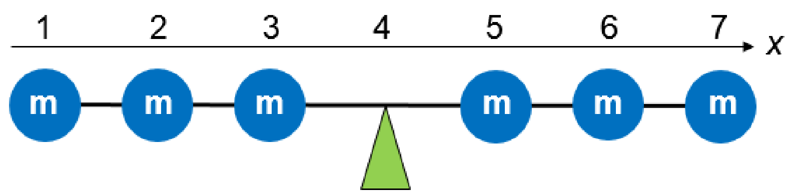
$$E_{pot,P} = 0.390\ \mathrm{kg}\ \cdot 9.81\ \mathrm{ms^{-2}}\left(1 - \sqrt{1^2 - 0.08^2}\right)\mathrm{m} = 3.7\ \mathrm{J}\ (1.000 - 0.997) = 12.5\ \mathrm{mJ}$$

The **pendulum motion has taken up only ~1/1000 of the initial kinetic energy**. The rest must have been "burned", i.e. released/dissipated as heat when the bullet was stopped by the pendulum. It is unfortunate that we cannot apply energy conservation in this case. Previously, it was a convenient way to calculate the state of a system without describing the exact history of the process. In the following we want therefore to **elaborate another conservation law, in particular linear momentum conservation**, which can be as useful as energy conservation.

2) Center of mass

To arrive at momentum conservation, we first define the center of mass, since this is a helpful coordinate to describe the motion of a particle system.

A) Definition

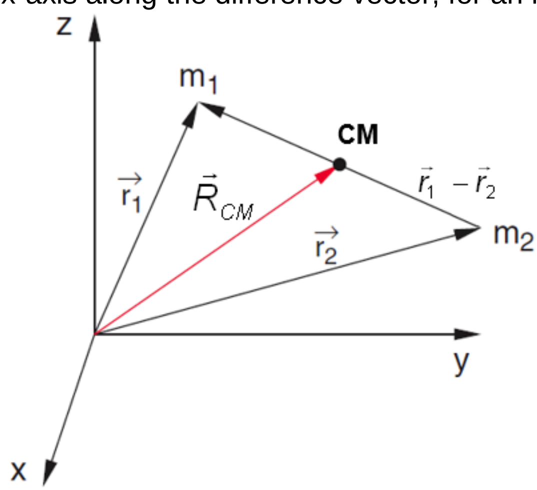


Let us first look at a linear **mass distribution and define its center** (similar to the mean of a probability distribution):

$$X_{CM} = \frac{\sum_i x_i m_i}{\underbrace{\sum_i m_i}_{M}}$$

where m_i are the masses of the individual mass points and x_i their locations along the x axis.

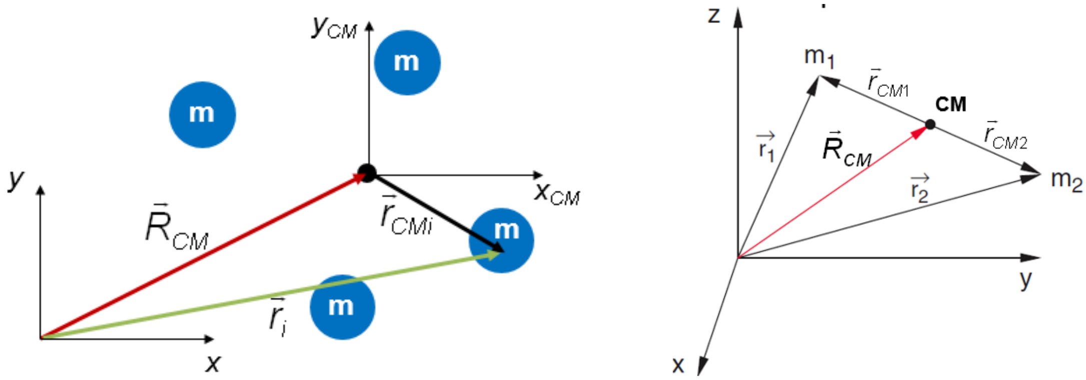
$$\sum_i m_i = M$$
is then the total mass of the system. For the depicted example we get (omitting length units):
$$X_{CM} = \frac{1\,m + 2\,m + 3\,m + 5\,m + 6\,m + 7\,m}{6\,m} = 4$$
which is directly in the center of the mass distribution, which makes also intuitively sense. Suspending the mass distribution at position 4 would lead to a balanced (equilibrium) situation.
The **center of mass can also be obtained by calculating the CMs of subsystems** and calculating the final center of mass from the masses and CMs of the sub-systems (**show on slides**). For example, having two subsystems one with the first two masses and one with the remaining two masses gives for their centers of mass:
$$X_{CM1} = \frac{1 \cdot m + 2 \cdot m}{2m} = 1.5 \quad and \quad X_{CM2} = \frac{3 \cdot m + 5 \cdot m + 6 \cdot m + 7 \cdot m}{4m} = 5.25$$
The joint center of mass of both subsystems is then again located at:
$$X_{CM} = \frac{\sum_j X_{CMj} M_j}{\sum_j M_j} = \frac{2m \cdot 1.5 + 4m \cdot 5.25}{2m + 4m} = 4$$
In **three dimensions** we use vectors for the mass positions and can write:
$$\vec{R}_{CM} = \frac{\sum_i \vec{r}_i m_i}{\sum_i m_i}$$
Where the CM components are obtained individually from the position components of the masses! With this equation we can **calculate the CM of two point masses** and rewrite the result a bit:
$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} = \frac{m_1\vec{r}_1 + (M - m_1)\vec{r}_2}{M}$$
which can be transformed to
$$\vec{R}_{CM} = \vec{r}_2 + \frac{m_1}{M}(\vec{r}_1 - \vec{r}_2)$$
i.e. the **center of mass lies on the difference vector that extends from $\vec{r}_2$** (see picture below). **As larger m_1, as closer is the CM to m_1**, which intuitively makes sense. We could also turn the coordinate system with the x-axis along the difference vector, for an intuitive understanding.



B) The center-of-mass system (CMS)

Often it is convenient to take the **center of mass as a new frame of reference**. This frame of reference is called the **center of mass system (CMS)**. The coordinates of the individual mass points $\vec{r}_i$ within the laboratory frame of reference (reference frame in which laboratory is at rest) get then transformed into relative coordinates$\vec{r}_{CMi}$ with respect to the center of mass:

$$\vec{r}_{CMi} = \vec{r}_i - \vec{R}_{CM}$$



For **two masses** the position vectors in the center of mass system point consequently into opposite direction, since all three masses lie on the same line (right side in picture above).

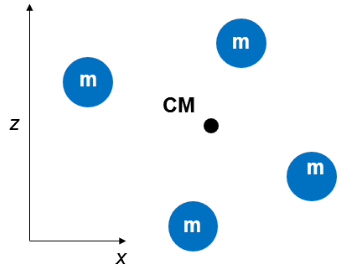
When we calculate the **CM within the center-of-mass system** we get:

$$\sum_i m_i \vec{r}_{CMi} = \sum_i m_i(\vec{r}_i - \vec{R}_{CM}) = \sum_i m_i\vec{r}_i - \vec{R}_{CM} \underbrace{\sum_i m_i}_{M} = 0$$

The last equation is zero following the definition of the center of mass. This provides a trivial result: The **center of mass in the center of mass system is located at its origin**. Nonetheless, the fact that the sum on the left side is zero, will be helpful for further calculations.

C) Potential energy within gravity field

The usefulness of the center of mass can be seen when calculating the **potential energy of a system of mass points** a bit:



It is given as the sum of the potential energies of the individual mass elements:

$$U = \sum_i m_i g z_i$$

Expressing the z positions by the sum of the CMS position and the relative position to the CMS $z_i = Z_{CM} + z_{CMi}$ gives:

$$U = g\sum_i m_i(Z_{CM} + z_{CMi}) = gZ_{CM}\sum_i m_i + g\underbrace{\sum_i m_i z_{CMi}}_{0}$$

The last term in the equation above is zero since the center of mass within the CMS is located at the origin (0 position). With this we get

$$U = MgZ_{CM}$$

The total **potential energy** of the potential energy is simply obtained **from product of total mass and the height of the center of mass**. Thus, **rotation about the CM does not change the potential energy,** since the height of the CM remains constant. **For rotations about the CM we have thus a neutral equilibrium.**

Experiment: Neutral equilibrium (with respect to rotation) for a rigid body that is supported at the center of mass.

D) Center of mass for continuous objects

In case of continuous objects (such as the object in the previous experiment), the sum in the definition of the center of mass simply transforms into an **integral over all volume elements times the local density**:

$$\vec{R}_{CM} = \frac{\int \vec{r} dm}{\int dm} = \frac{\int \vec{r}\, \rho(\vec{r}) dV}{\int \rho(\vec{r}) dV}$$

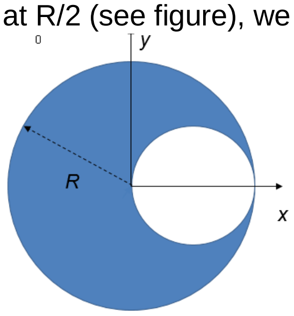
Example 1: Calculating the CM of a straight, **homogeneous rod of length *L***

We can write for a rod along the x axis using the integral formula:

$$X_{CM,rod} = \frac{\int_0^L x\, \rho_0 \underbrace{A\, dx}_{dV}}{\int_0^L \rho_0\, A\, dx} = \frac{\frac{1}{2}\rho_0 A\, L^2}{\rho_0 A\, L} = \frac{L}{2}$$

which provides the intuitive result that the CM of a rod is located at its geometric center.

Example 2: For calculating the **CM of a circular disk with a circular cavity** we can use the fact that we can calculate the center of mass from the masses and CMs of subsystems. The depicted disk with cavity is formed by the mass elements of the filled system (total mass *M*) minus the mass elements that need to be carved out to form the cavity. The minus sign for the latter can be seen as a negative mass.
The CM of the cavity system is thus obtained from the CM and the mass *M* of the full object and the CM of the cavity and its negative mass -*m*. For a disk (CM at zero) with a circular cavity (CM at R/2 (see figure), we can thus write:



$$X_{CM} = \frac{0 \cdot M - R/2\ m}{M - m} = \frac{R/2\ \rho\pi(R/2)^2}{\rho\pi R^2 - \rho\pi(R/2)^2} = \frac{R}{6}$$

2) Momentum conservation

A) Mass-point system with internal & external forces

After defining the center of mass of a particle system and defining the reference frame for the center-of-mass, we look now at the momentum of such a system. To this end we look at the definition of the center of mass and differentiate it by the time:

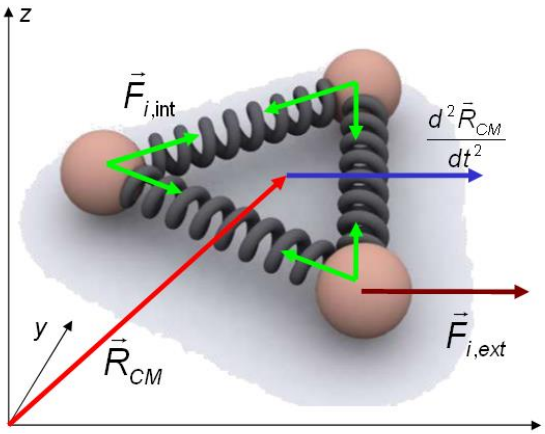
$$M\,\vec{R}_{CM} = \sum_i m_i \vec{r}_i$$

$$M\,\frac{d\vec{R}_{CM}}{dt} = \sum_i m_i \frac{d\vec{r}_i}{dt}$$

A second differentiation gives:

$$M\,\frac{d^2\vec{R}_{CM}}{dt^2} = \sum_i m_i \frac{d^2\vec{r}_i}{dt^2} = \sum_i \vec{F}_i$$

Assuming constant masses the terms in the sum with the second time derivatives correspond just to the **net forces acting on the individual particles**. Each net force on a particle can be **split into an external force**, e.g. from an external force field, **and into internal forces due to pairwise interactions between the particles**:



We can thus write for the equation above:

$$M\,\frac{d^2\vec{R}_{CM}}{dt^2} = \sum_i \vec{F}_i = \underbrace{\sum_i \vec{F}_{i,int}}_{0} + \sum_i \vec{F}_{i,ext}$$

Due to Newton's 3rd law each internal force on a particle causes an oppositely oriented force on a second particle. These pairwise forces cancel each other out, such that we are left with just the external forces. With $\vec{F}$ being the net external force we can therefore write:

$$\vec{F} = \sum_i \vec{F}_{i,ext} = M \cdot \frac{d^2\vec{R}_{CM}}{dt^2}$$

This means: **Under the influence of an external force the center of mass of a system moves like a point-like particle with a mass equal to the total mass of the system and irrespective of internal forces acting within the system.**

With other words one can say: **The action of an external force on a particle system is fully described by considering its effect on the center of mass**

B) Conservation of momentum

Now we look again at the time derivatives from above, since they contain also the linear momentum $\vec{p} = m\,\vec{v}$. Rewriting of the equation after the first time derivative provides:

$$M\,\frac{d\vec{R}_{CM}}{dt} = \sum_i m_i \frac{d\vec{r}_i}{dt} \quad \rightarrow \quad M\,\vec{V}_{CM} = \sum_i m_i \vec{v}_i$$

This can be rewritten as:

$$\vec{P}_{CM} = \sum_i \vec{p}_i$$

i.e. the total momentum of a particle system (sum of all individual momenta) **equals the momentum of the center of mass**. The force equation from above can be rewritten as:

$$\vec{F} = M\frac{d^2\vec{R}_{CM}}{dt^2} = \frac{d\left(M\,\vec{V}_{CM}\right)}{dt}$$

This connects now the total momentum change to the net external force:

$$\vec{F} = \frac{d\vec{P}_{CM}}{dt} = \frac{d(\sum_i \vec{p}_i)}{dt} = \sum_i \frac{d\vec{p}_i}{dt}$$

The center of mass of a particle system moves as a single body with total mass *M* at which the total external force is acting. This is an astonishingly simply formula, since we do make any constraints on which individual particle the force is acting.

In addition, it has a fundamental consequence in **case there are no external forces:**

$$0 = \frac{d\vec{P}_{CM}}{dt} = \sum_i \frac{d\vec{p}_i}{dt}$$

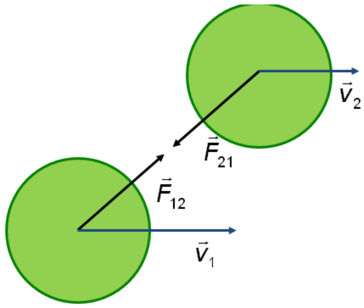
If the change of the total momentum per time is zero, it remains constant in time, such that in absence of external force:

$$\vec{P}_{CM} = \sum_i \vec{p}_i = \sum_i m\vec{v}_i = const.$$

An isolated system (in absence of an external net force) has a constant total momentum. The vectorial character of this equation demands that momentum conservation holds component-wise, i.e. **momentum conservation holds for each coordinate axis (along x,y and z)**. In three dimensions one has thus 3 equations as condition for momentum conservation.

Next to energy conservation this is the **most important conservation law!** Consequently, **in the absence of external forces the velocity of the center of mass does not change**, due to mass conservation.

We took here a very general approach to obtain momentum conservation for a multi particle system. A simpler route is just to look at **two interacting particles in the absence of an external force** **(show only on slides)**:



According to Newton's 3rd law a force that acts on particle 1 due to particle 2 causes an oppositely oriented force with the same magnitude on particle 2 (remember apple and earth ☺):

$$\vec{F}_{12} = -\vec{F}_{21}$$

Replacing the force on each particle by the change in momentum per time and transformation provides:

$$\frac{d\vec{p}_2}{dt} = -\frac{d\vec{p}_1}{dt} \qquad or \qquad 0 = \frac{d}{dt}(\vec{p}_1 + \vec{p}_2)$$

and we arrive again at momentum conservation:

$$\vec{p}_1 + \vec{p}_2 = const.$$

Experiment: Let us check whether momentum conservation can explain the result from the ballistic pendulum experiment.

From the (kinetic) energy of the pendulum, we can calculate its velocity directly after the collision:

$$E_{k,p} = \frac{m}{2} {v_p}^2 = 12.5\, mJ$$

such that we get:

$$v_p = \sqrt{2E_{k,p}/m)} = \sqrt{2 \cdot 0.0125 \cdot kg \cdot m \cdot s^{-2}/0.39kg} = 0.25m/s$$

We now can calculate the **momentum of the bullet before collision and the momentum of the pendulum after the collision**:

$$p_B = 200 \text{ m s}^{-1} \cdot 0.5 \times 10^{-3} \text{ kg} \;=\; 0.1 \text{ kg m s}^{-1}$$
$$p_P = \;(0.39 + 0.5 \times 10^{-3})\text{kg} \;\cdot\; 0.25 \text{ m s}^{-1} \;=\; 0.098 \text{ kg m s}^{-1}$$

Within error both momenta are equal, which agrees with momentum conservation. Thus, momentum conservation is in this case the law, which helps us to calculate the pendulum velocity after the collision.

Experiment: We have two sliders (ice skaters) on an air track that repel each other through opposing magnets after the connection between the two sliders gets released. We measure the speeds of both sliders and use for them different masses. Calculating the total momentum demonstrates again momentum conservation:

$$0 = P_{x\,CM} = \sum_i p_{xi} = -m_1 v_{x1} + m_2 v_{x2}$$

Transformation provides that that the velocities scale inversely proportional with their masses.

$$\frac{m_1}{m_2} = \frac{v_2}{v_1}$$

C) Momentum within the CMS

Let us calculate the momentum within the CMS. Within the CMS we found before that the center of mass is located at zero, such that:

$$\sum_i m_i \vec{r}_{CMi} = 0$$

By differentiation we obtain from this the total momentum in the CMS:

$$0 = \sum_i \frac{d(m_i \vec{r}_{CMi})}{dt} = \sum_i m_i \underbrace{\frac{d\vec{r}_{CMi}}{dt}}_{\vec{v}_{CMi}} = \sum_i \vec{p}_{CMi} = 0$$

Thus, the sum of all momenta in the CMS is always zero! The disappearance of the momentum makes sense, since we showed before that the **total momentum of a particle system is already fully comprised within the momentum of the CM**.

This can also be formally shown with the definition of the CMS **(not part of lecture)**. Starting the vector transformation into the CM $\vec{r}_i = \vec{r}_{CMi} + \vec{R}_{CM}$ and differentiating it gives:

$$\vec{v}_i = \vec{v}_{CMi} + \vec{V}_{CM} \quad \text{and also} \quad m_i \vec{v}_i = m_i \vec{v}_{CMi} + m_i \vec{V}_{CM}$$

Summation over all masses then provides:

$$\underbrace{\sum_i m_i \vec{v}_i}_{\sum_i \vec{p}_i} = \underbrace{\sum_i m_i \vec{v}_{CMi}}_{\sum_i \vec{p}_{CMi} = 0} + \underbrace{\sum_i m_i \vec{V}_{CM}}_{M\vec{V}_{CM} = \vec{P}_{CM}}$$

D) Kinetic energy within the CMS

Now we look at the kinetic energy and try to express it with the center-of-mass coordinates. The total kinetic energy is given by:

$$\sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{v}_{CMi} + \vec{V}_{CM}\right)^2 = \sum_i \frac{1}{2} m_i v_{CMi}^2 + \sum_i \frac{1}{2} m_i V_{CM}^2 + \underbrace{\sum_i m_i \vec{v}_{CMi} \cdot \vec{V}_{CM}}_{0\, \vec{V}_{CM} = 0}$$

The last term in the sum vanishes since the total momentum in the CMS is zero. Thus, we obtain:

$$E_k = E_{kCM} + \frac{1}{2} M V_{CM}{}^2$$

The kinetic energy E_k in the laboratory frame of reference equals the sum of the kinetic energy E_{kCM} within the center of mass system and the kinetic energy (translational energy) of the center of mass itself.

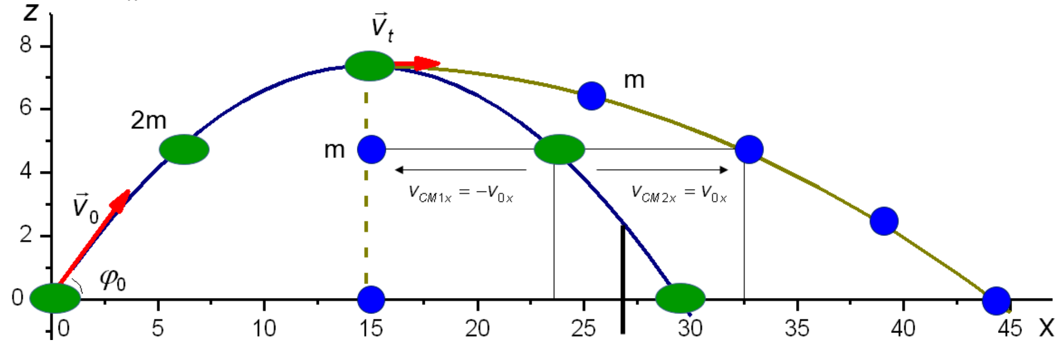
Both, the momentum as well as the energy equation are very helpful, since one can **look separately on the center of mass motion that is influenced only by external forces and the motion within the CMS that is influenced only by internal forces**. In the absence of external forces, the kinetic energy of the CM remains constant and only the kinetic energy for relative movements within the CMS can change.

(**Exercise**: look at moving wagons connected by a spring or at an explosion, i.e. a sudden release and consider the energies and momenta of the CM and within the CMS, where the splitting of the energies will be very helpful)

3) Applying momentum conservation

A) Projectile with explosion

We now want to apply the formulas from above by looking first at the motion of an exploding projectile. The projectile shall start at $z(t=0)$ and $\vec{v}(t=0)=\vec{v}_{\mathbf{0}}$ moving under the influence of constant gravity. At its maximum it explodes and releases two equal halves of mass m. Right after the explosion both halves have zero vertical velocity ($v_{1z}=v_{2z}=0$) and half 1 shall fall straight to the ground ($v_{1x}=0$):



What can we tell about the motion of half 2?

We know from kinematics that the motion of a **projectile without explosion** can be described by a **superposition of a translation with constant velocity v_{0x} along x and a free fall with initial velocity v_{0z} along z** and. From this one obtains the typical parabola of the projectile motion (**see slide**):

$$z(x)=z_{max}-\frac{g}{2v_{0x}}(x-x_{max})^2$$

This **path is also identical to the path of the center of mass after explosion**, since the external force changes only the motion of the center of mass:

$$\vec{F}=2m\frac{d^2\vec{R}_{CM}}{dt^2}$$

Now we look at the problem from two different perspectives:

<u>Motion after explosion in the lab frame of reference</u>

Along z both halves undergo a free fall identical to the center of mass, since they did not obtain any velocity along the z- direction during the explosion. We can thus write for the z-position of both halves:

$$z_1(t)=z_2(t)=Z_{CM}(t)=z_{max}-\frac{g}{2}(t-t_{max})^2$$

Along x we can use momentum conservation since $F_x=0$:

$$\text{const.}=2mv_{0x}=m\underbrace{v_{1x}}_{0}+mv_{2x}$$

and we obtain:

$$v_{2x}=2v_{0x}$$

The second half travels in horizontal direction twice as fast as the CM, i.e. it travels twice the distance from x_{max} compared to the CM. Inserting this increased velocity into the projectile parabola solution provides:

$$z_2(x)=z_{max}-\frac{g}{4v_{0x}}(x-x_{max})^2$$

since both halves started after the explosion from position (x_{max},z_{max})

Motion after explosion in the center-of-mass system

Before and also after the explosion the CM travels with v_{0x} along x and undergoes a free fall along *z*, i.e. in the same way as the unexploded projectile. For the motion of the halves after explosion we can write within the CMS:

- *z*-direction:
  $v_{CM1z} = v_{CM2z} = 0$ and $z_{CM1} = z_{CM2} = 0$
  since no internal force/momentum transfer occurs along z.
- *x*-direction: for which we have zero total momentum within the CMS
  $0 = mv_{CM1x} + mv_{CM2x}$
  such that we get:
  $$v_{CM2x} = -v_{CM1x}$$
  i.e. within the CMS both halves travel with equal speed in opposite directions along x.

Furthermore, we know that in the lab system half 1 must have zero velocity:

$$0 = v_{1x} = v_{CM1x} + \underbrace{v_{0x}}_{V_{CMx}}$$

such that

$$v_{CM1x} = -v_{0x}$$

This provides that the second half travels with v_{0x} within the CMS:

$$v_{CM2x} = -v_{CM1x} = v_{0x}$$

We can furthermore calculate the gain in mechanical energy from the explosion.

$$E = E_{kCM} + \underbrace{\frac{1}{2}MV_{CM}{}^{2} + Mgh_{CM}}_{\text{const}}$$

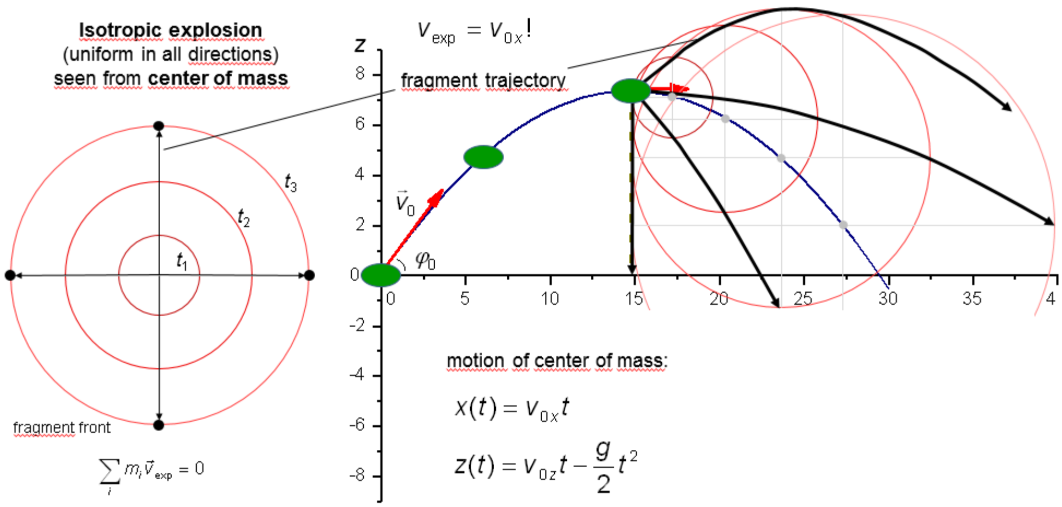
For the CM we have energy conservation, since it moves within the external gravity field like the intact projectile. The change in mechanical energy is therefore only arising from the gained speed during the explosion within the CMS:

$$\Delta E = E_{kCM} = \frac{m}{2}v_{CM1x}{}^{2} + \frac{m}{2}v_{CM2x}{}^{2} = mv_{0x}{}^{2}$$

Thus, we have a doubling of the kinetic energy along the x-direction after the explosion.

Isotropic explosion

We can easily transfer the principles from before to an isotropic explosion. Within the CMS, the explosion products shall be sent with initial speed v_{0x} isotropically in all directions:



Within the center of mass system, we see concentric rings around the origin that expand with constant velocity $v_{exp} = v_{0x}$. This condition provides the zero total momentum within the CMS.

In the laboratory frame of reference, we have to center the expanding rings on the position of the center of mass at that time. We know that both the center of mass as well as the backward directed fragment have the same speed. The latter does thus not undergo a lateral displacement. All the rings have thus the vertical line through the explosion position as tangent. We can thus draw the fragment positions at different times and even draw trajectories of individual fragments that were ejected in a particular direction (see figure).

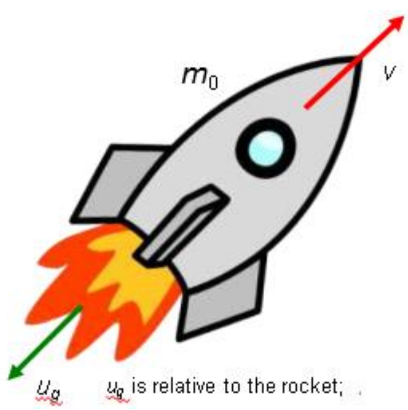
The trajectory for a particle being ejected during the explosion in a given directions is given by the superposition of the motion of the center of mass and the uniform motion of the particle within the CMS:

$$\vec{r}(t) = \begin{pmatrix} v_{0x} t \\ v_{0z} t - \frac{g}{2} t^2 \end{pmatrix} + v_{exp} t\, \hat{e}_r = \begin{pmatrix} v_{0x} t \\ v_{0z} t - \frac{g}{2} t^2 \end{pmatrix} + v_{exp} t \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

with θ being the angle of the particle trajectory with respect to the horizontal direction in the CMS.

B) Rocket propulsion

How does a rocket accelerate in free space, i.e. without any atmosphere (that e.g. planes are using) or any gravity? **Momentum conservation** tells us that in order to accelerate (gain momentum) the rocket has to eject exhaust at high velocity (high negative momentum) behind itself similar to the recoil of guns. In parallel, we have however to consider that during fuel burning the rocket loses weight.



Let us assume that the exhaust is released at velocity u_g relative to the rocket with a constant mass release per time R:

$$R = \frac{dm_{ex}}{dt} = const.$$

The time course of the (accumulated) mass release can be obtained by separation of variables and integration:

$$\underbrace{\int_0^{m_{ex}(t)} dm_{ex}}_{m_{ex}(t)} = \underbrace{\int_0^t R dt}_{Rt}.$$

Solving the integral provides a linear increase of the released mass with time, such that the total mass of the rocket is decreasing from the initial mass m_0 according to

$$m(t) = m_0 - m_{ex}(t) = m_0 - Rt$$

Let v and m be the velocity and mass of the rocket at a given time. The mass dm is relased per time dt, for which momentum conservation must hold. To use this, we compare the momentum at time t with the momentum at time $t + dt$, such that we can write:

$$\underbrace{(m + dm)v}_{at\ time\ t} = \underbrace{\overbrace{(v - u_g)dm}^{exhaust\ release} + \overbrace{m(v + dv)}^{new\ rocket\ momentum}}_{at\ time\ t+dt}$$

The first term on the right site is the momentum of the released exhaust of mass dm (in the lab frame) and the second term the new momentum of the rocket, for which the mass reduced by dm but the velocity increased by dv. Simplification of the equation terms provides for the time interval dt:

$$0 = -u_g dm + m(t)dv$$

Since this occurs per time interval d*t* we can write for the relation between mass and velocity change per time:

$$m(t)\frac{dv}{dt} = u_g \frac{dm}{dt} = \underbrace{u_g R}_{thrust}$$

The right part is called **thrust** which is the product of the rate of mass release and the exhaust velocity, both of which enter the transferred momentum. Separation of the variables yields:

$$dv = \frac{u_g R}{m(t)} dt$$

Inserting the time-dependent mass and integration provides:

$$\int_0^v dv = \int_0^t \frac{u_g R}{m_0 - Rt} dt$$

The integral on the right side can be solved by substitution ($z = m_0 - Rt$) and we obtain:

$$v = -\frac{u_g R}{R} \ln(m_0 - Rt)|_0^t$$

$$v = -u_g[\ln(m_0 - Rt) - \ln m_0]$$

From which we finally get a logarithmic dependency for the velocity:

$$v = -u_g \ln\left(1 - \frac{Rt}{m_0}\right)$$

The obtained function is a logarithmic function that is mirrored and shifted by the time m_0/R to the right at which the rocket mass becomes zero (see figure).

Experiment: From the lecture about Newton's law we know the experiment with the **water rocket**. Let's have a look at a video of this experiment and try to describe the acceleration of the rocket with the derived rocket equation.

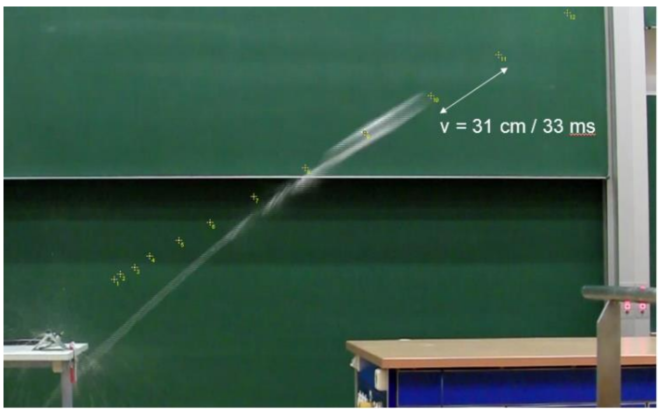
From the video (**show**) we can extract a time of 0.27 s at which the 100 ml water inside the bottle were ejected through the outlet. This gives us the rate R at which mass is ejected:

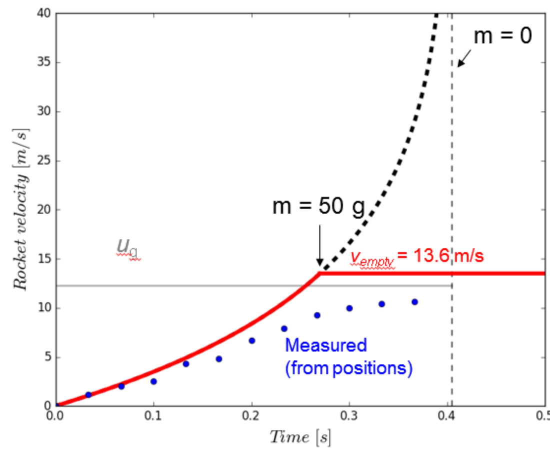
$$R = \frac{\Delta m}{\Delta t} = \frac{0.1\ kg}{0.27\ s} = 0.37\ kg\ s^{-1}$$

Knowing the outlet radius r_{outlet} and thus the cross sectional area A_{outlet}, we can also estimate the velocity of the ejected water, since it is an incompressible fluid. We get for $r_{outlet} = 0.31 cm$ (**see slide**):

$$u_g = \frac{\Delta V / A_{outlet}}{\Delta t} = \frac{\Delta V / (\pi r_{outlet}^2)}{\Delta t} = \frac{100\ cm^3}{\pi (0.31\ cm)^2\ 0.27\ s} = 12.3\ m\ s^{-1}$$

With $m_0 = 0.05\ kg\ (bottle) + 0.1\ kg\ (water) = 0.15\ kg$ we can plot the expected time course of the velocity increase:





The velocity would in principle go to infinity if the remaining mass approaches zero (i.e. when $Rt = m_0$), since we have the constant thrust but an increasingly lighter rocket. However, at 0.27 s the bottle is empty but still has its own weight of 50 g. Due to the finite rest mass, the velocity reaches thus a finite value. We obtain here a final velocity of 13.5 m/s, this would according to the equation of projectile motion provide a distance (if shot at 45° angle) of 19 m (plus 2 m added for the initial acceleration phase), which is the correct order of magnitude. Evaluation of the rocket positions in the video provides a velocity of about 10 m/s, which is a very good agreement with our theoretical result given the simplicity of our estimates! One sees from the experimental data that there is still some acceleration after 0.27 s due to the ejected air in the bottle.

Lecture 12: Experiments

1. Ballistic pendulum: Look at energy and momentum conservation
2. Neutral equilibrium for a rigid body that is supported at the center of mass.
3. Two sliders (ice skaters) on an air track that repel each other through opposing magnets after the connection between the two gets released. We measure the speeds of both sliders and use for them different masses. Calculating the total momentum demonstrates momentum conservation.
4. Video: Propulsion of a water rocket and application of the rocket equation to the rocket velocity extracted from the video.