HW-12

(1)
$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 2 \\ -1 & 3 & 2 \\ -1 & 1 & 6 \end{vmatrix} = (-1)\begin{vmatrix} \boxed{0 \quad 1 \quad 1} \\ -1 \quad 3 \quad 1 \\ 2 \quad 2 \quad 6 \end{vmatrix} = \left[\begin{matrix}\text{cof} \\ \text{expansion}\end{matrix}\right]$$

$+ \cdot(-1)$ -1 + transpose

$$= (-1)\left[1(-1)[-6-2] + 1\cdot 1\cdot[-1\cdot 2 - 3\cdot 2]\right] = 0.$$

(2)
$$\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 & 0 \\ 1 & 6 & 2 \\ 1 & 1 & -2 \end{vmatrix} = 2\cdot 1\begin{vmatrix} 2 & 0 \\ -2 & 3 \end{vmatrix} -$$

$$- 1\cdot(-1)\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 2\cdot 6 + \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 12 - 4 = 8.$$

(3)
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & x \\ 1 & x & 3 \end{pmatrix}, \quad \nexists A^{-1} \iff \det A = 0 \iff$$

$$\iff 0 = 1\begin{vmatrix} 9 & x \\ x & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & x \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 9 \\ 1 & x \end{vmatrix} = 27 - x^2 - (3-x) +$$
$$+ x - 9 = 15 - x^2 + 2x.$$

$$\iff 15 - x^2 + 2x = 0$$
$$x^2 - 2x - 15 = 0$$
$$\underline{x \in \{5, -3\}}$$

(4) $\det A = 4$,
$\det B = 5$, 3×3 matrices.

$$\det(2A^{-1}B^2) = 2^3 \det(A^{-1})\det(\underbrace{B^2}_{B\cdot B}) = 8\cdot\frac{1}{\det A}\cdot\det B\cdot\det B =$$
$$= \frac{8}{4}\cdot 5^2 = 50.$$

(5)
$$\begin{vmatrix} 1001 & 1002 & 1003 & 1004 \\ 1002 & 1003 & 1001 & 1002 \\ 1001 & 1001 & 1001 & 999 \\ 1001 & 1000 & 998 & 999 \end{vmatrix} = \left[\begin{matrix}\text{subtract} \\ \text{first column} \\ \text{from others}\end{matrix}\right] =$$

$$= \begin{vmatrix} 1001 & 1 & 2 & 3 \\ 1002 & 1 & -1 & 0 \\ 1001 & 0 & 0 & -2 \\ 1001 & -1 & -3 & -2 \end{vmatrix} = 1001 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & -2 \\ -1 & -3 & -2 \end{vmatrix} - 1002 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ -1 & -3 & -2 \end{vmatrix} + 1001 \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & -3 & -2 \end{vmatrix} -$$

$$-1001 \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 1001 \cdot (-1)^{2+3}(-2) \begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} -$$

$$-1002(-1)^{2+3}(-2) \begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix} + 1001 \cdot \left[(-1)^{2+1} \cdot 1 \begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix} + (-1)^{2+2}(-1) \begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} \right]$$

$$-1001 \cdot (-1)^{3+3}(-2) \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = 2002(-4) - 2004(-1) + 1001\big((-1) \cdot 5 -$$

$$-1(-2+3)\big] + 2002(-1-2) = -6006 - \underbrace{7 \cdot 2002 + 2004}_{-12010} = \underline{-18016}$$

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$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

E^{-1}

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Since $(E_1 E_2 E_3 E_4)^{-1} = E_4^{-1} E_3^{-1} E_2^{-1} E_1^{-1}$)

With elementary matrix properties then $\Longrightarrow$

$$A^{-1} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{pmatrix}}_{E_4^{-1} E_3^{-1}} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{E_2^{-1} E_1^{-1}} = \cancel{\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -3 & 1 \end{pmatrix}} \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 7 & -3 & 1 \end{pmatrix}$$

$$\text{So } A^{-1} = \cancel{\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -3 & 1 \end{pmatrix}} \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 7 & -3 & 1 \end{pmatrix}$$

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<u>Way 1</u> $\underbrace{\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}}_{A} X = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{B} \Longrightarrow \underbrace{\underbrace{A^{-1} A}_{I} X}_{X} = A^{-1} B \Longrightarrow$

$$\Longrightarrow X = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}^T \cdot \frac{1}{2-3} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

<u>Way 2</u> – easier: any $\begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, so solution is obvious.

$$X=\begin{pmatrix}1&1\\0&0\end{pmatrix}$$

(8) $$A=\begin{pmatrix}1&3&1\\2&1&1\\-2&2&-1\end{pmatrix}$$

1) $$\det A=\begin{vmatrix}1&3&1\\2&1&1\\-2&2&-1\end{vmatrix}=\begin{vmatrix}1&3&1\\2&1&1\\ \boxed{0\;\;3\;\;0}\end{vmatrix}=3\cdot\underbrace{(-1)^{3+2}\cdot\begin{vmatrix}1&1\\2&1\end{vmatrix}}_{1}=3,$$

2) $$\operatorname{Cof} A=\begin{pmatrix}(-1)^{1+1}\cdot\begin{vmatrix}1&1\\2&-1\end{vmatrix}, & (-1)^{1+2}\cdot\begin{vmatrix}2&1\\-2&-1\end{vmatrix}, & (-1)^{1+3}\begin{vmatrix}2&1\\-2&2\end{vmatrix},\\ (-1)^{2+1}\begin{vmatrix}3&1\\2&-1\end{vmatrix}, & (-1)^{2+2}\begin{vmatrix}1&1\\-2&-1\end{vmatrix}, & (-1)^{2+3}\begin{vmatrix}1&3\\-2&2\end{vmatrix},\\ (-1)^{3+1}\begin{vmatrix}3&1\\1&1\end{vmatrix}, & (-1)^{3+2}\begin{vmatrix}1&1\\2&1\end{vmatrix} & (-1)^{3+3}\begin{vmatrix}1&3\\2&1\end{vmatrix}\end{pmatrix}=$$

$$=\begin{pmatrix}+(-3)&-0&+6\\-(-5)&+1&-8\\+(2)&-(-1)&+(-5)\end{pmatrix}=\begin{pmatrix}-3&0&6\\5&1&-8\\2&1&-5\end{pmatrix}.$$

3) $$\Longrightarrow A^{-1}=\frac{1}{\det A}(\operatorname{Cof} A)^T=\frac{1}{3}\begin{pmatrix}-3&5&2\\0&1&1\\6&-8&-5\end{pmatrix}=\boxed{\begin{pmatrix}-1&\frac{5}{3}&\frac{2}{3}\\0&\frac{1}{3}&\frac{1}{3}\\2&-\frac{8}{3}&-\frac{5}{3}\end{pmatrix}}$$

Verify: $$AA^{-1}=\frac{1}{3}\begin{pmatrix}1&3&1\\2&1&1\\-2&2&-1\end{pmatrix}\cdot\begin{pmatrix}-3&5&2\\0&1&1\\6&-8&-5\end{pmatrix}=$$

$$=\frac{1}{3}\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\mathbb{I}.$$

In all cases $A\vec{x}=\vec{b} \underset{\exists A^{-1}}{\Longrightarrow} \vec{x}=A^{-1}\vec{b}$ 9

$$\begin{cases} x_1+3x_2+x_3=3 \\ 2x_1+x_2+x_3=6 \\ -2x_1+2x_2-x_3=9 \end{cases} \longrightarrow \vec{x}=\frac{1}{3}\begin{pmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}=\frac{1}{3}\begin{pmatrix} 39 \\ 15 \\ -75 \end{pmatrix}=\begin{pmatrix} 13 \\ 5 \\ -25 \end{pmatrix}$$

$$\text{Sys. } 2 \Longrightarrow \vec{x}=\frac{1}{3}\begin{pmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{pmatrix}\cdot\begin{pmatrix} 9 \\ 3 \\ 6 \end{pmatrix}=\frac{1}{3}\begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\text{Sys. } 3 \Longrightarrow \vec{x}=\frac{1}{3}\begin{pmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}=\frac{1}{3}\begin{pmatrix} 33 \\ 12 \\ -51 \end{pmatrix}=\begin{pmatrix} 11 \\ 4 \\ -17 \end{pmatrix}.$$

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$$\begin{cases} 2x_1+x_2-3x_3=0 \\ 4x_1+5x_2+x_3=8 \\ -2x_1-x_2+4x_3=2 \end{cases} \quad \det A=\begin{vmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{vmatrix}=2\cdot(21)-1(16+2)-3\cdot(-4+10)=42-18-18=6.$$

$$\det A^{(1)}=\begin{vmatrix} 0 & 1 & -3 \\ 8 & 5 & 1 \\ 2 & -1 & 4 \end{vmatrix}=-1(32-2)-3(-8-10)=-30+54=24$$

$$\det A^{(2)}=\begin{vmatrix} 2 & 0 & -3 \\ 4 & 8 & 1 \\ -2 & 2 & 4 \end{vmatrix}=2(32-2)-3(8+16)=60-72=-12$$

$$\det A^{(3)}=\begin{vmatrix} 2 & 1 & 0 \\ 4 & 5 & 8 \\ -2 & -1 & 2 \end{vmatrix}=2(10+8)-1(8+16)=36-24=12$$

$$\forall i \quad x_i=\frac{\det A^{(i)}}{\det A} \Longrightarrow \underline{x_1=4,\ x_2=-2,\ x_3=2.}$$