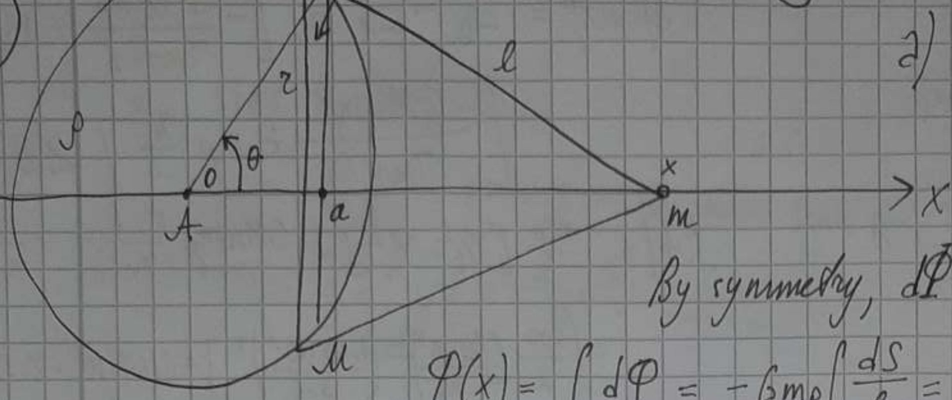
Gravity

(1)

a)



By symmetry, $d\Phi = -\frac{G\,dM\,m}{l}$ – potential from a ring at $(x, 0)$.

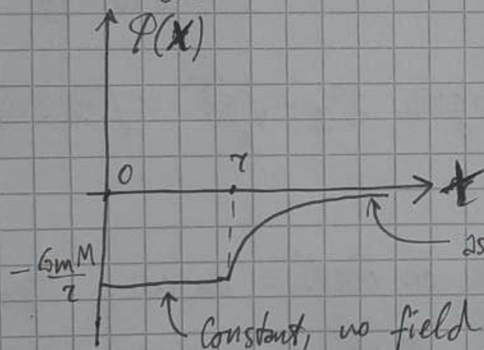
$$\Phi(x) = \int d\Phi = -Gm\rho\int\frac{dS}{l} = -Gm\rho\int\frac{dl\cdot 2\pi \overbrace{r\sin\theta}^{r\sin\theta =}}{l} =$$

$$= -Gm\rho\, 2\pi\int\frac{r\,d\theta\cdot r\sin\theta}{l} = \underbrace{-Gm\rho 2\pi r^2}_{K}\int\frac{\sin\theta\, d\theta}{\sqrt{r^2\sin^2\theta + (x - r\cos\theta)^2}} = K\int\frac{\sin\theta\, d\theta}{\sqrt{r^2 - 2xr\cos\theta + x^2}} =$$

$$= K\int_0^\pi\frac{\sin\theta\, d\theta}{\sqrt{A + B(-\cos\theta)}} = \left[\begin{matrix}-\cos\theta = u\\ du = \sin\theta\, d\theta\\ u(0) = -1,\ u(\pi) = 1\end{matrix}\right] = K\int_{-1}^{1}\frac{du}{\sqrt{A + Bu}} = \frac{K\cdot 2\sqrt{A+Bu}}{B}\Bigg|_{-1}^{1} =$$

$$= \frac{2K}{B}\left[\sqrt{A+B} - \sqrt{A-B}\right] = \frac{2K}{B}\left[\sqrt{x^2 + r^2 + 2xr} - \sqrt{x^2 + r^2 - 2xr}\right] = \frac{2K}{B}\left[|x + r| - |x - r|\right]$$

$$= -\underbrace{4\pi r^2\rho}_{M}\frac{mG}{2xr}[\ldots] = -\frac{GmM}{2xr}\cdot\begin{cases} x > r, & 2r\\ x < r, & 2x\end{cases} = \begin{cases} -\frac{GmM}{x} & \text{for } x > r\\ -\frac{GmM}{r} & \text{for } x < r\end{cases}$$



But $\vec{F}(\vec{x}) = -\nabla\Phi(\vec{x}) = \begin{cases} -\frac{GmM}{x^2}\hat{x} & \text{for } x > r,\ \hat{x} = \begin{pmatrix}1\\0\end{pmatrix}\\ \vec{0} & \text{for } x < r\end{cases}$

$-\frac{\partial}{\partial x}\Phi(x)$ in 1D

b) Everything except $\int$ borders is same.

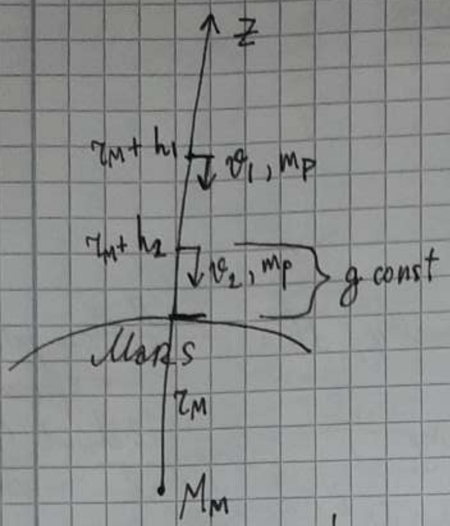
$$\Phi(x) = K\int_{-1}^{0}\frac{du}{\sqrt{A+Bu}} = \frac{2K}{B}\sqrt{A+Bu}\Bigg|_{-1}^{0} = \frac{2K}{B}\left[\sqrt{A} - \sqrt{A-B}\right] = \frac{2K}{B}\left[\sqrt{x^2+r^2} - |x - r|\right] =$$

$$\overset{x>r}{=} -\frac{GmM}{2xr}\cdot\left[\sqrt{\frac{x^2+r^2}{x^2}} - \frac{x}{x} + \frac{r}{x}\right],\quad F(x) = -\frac{\partial}{\partial x}\Phi(x) = -\frac{\partial}{\partial x}\left[-\frac{GmM}{2r}\left[\sqrt{1 + \left(\frac{r}{x}\right)^2} - 1 + \frac{r}{x}\right]\right]$$

$$= \frac{GmM}{2r}\left[-\frac{r}{x^2} + \frac{1}{\cancel{2}\sqrt{1 + \left(\frac{r}{x}\right)^2}}\cdot r^2(-\cancel{2}x^{-3})\right] = -\frac{GmM}{2r}\left[\frac{r}{x^2} + \frac{r^2}{x^3\sqrt{1 + (r/x)^2}}\right] = -\frac{GmM}{2r}\cdot$$

$$\cdot\left[\frac{r}{x^2}\right]\left(1 + \frac{r}{x\sqrt{1 + (r/x)^2}}\right) = -\frac{GmM}{2x^2}\left(1 + \frac{\lambda}{\sqrt{1 + \lambda^2}}\right) \text{ where } \lambda = r/x.$$

(2)



a) $A = E_2 - E_1 = \left[-\frac{GM_M m_p}{r_M+h_2} + m_p\frac{v_2^2}{2}\right] - \left[-\frac{GM_M m_p}{r_M+h_1} + m_p\frac{v_1^2}{2}\right] < 0$

$$|A| = \frac{m_p}{2}\left[v_1^2 - v_2^2\right] + GM_M m_p\left[\frac{1}{r_M+h_2} - \frac{1}{r_M+h_1}\right] =$$

$$= \left(\frac{600}{2}\left[\left(\frac{21\cdot10^3}{3.6}\right)^2 - \left(\frac{1.65\cdot10^3}{3.6}\right)^2\right] + 6.67\cdot10^{-11}\cdot6.39\cdot10^{23}\cdot 600\left[\frac{1}{3.39\cdot10^6+11\cdot10^3} - \frac{1}{3.39\cdot10^6+122.5\cdot10^3}\right]\right)\cdot J = \cancel{1.58\cdot10^{12}}\ 10^{10}\ J$$

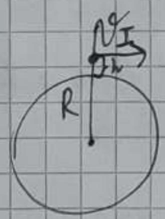
b) $\frac{v_2^2}{2} + gh_2 = \frac{v_{final}^2}{2} \Longrightarrow v_{final} = \sqrt{2gh_2 + v_2^2} = \sqrt{2\cdot3.69\cdot11\cdot10^3 + \left(\frac{1.65\cdot10^3}{3.6}\right)^2}\ \frac{m}{s} =$

(Mars atmosphere indeed ≈ 0 friction)

$= 540\ m/s.$

(3) a) $h\approx0,\quad \frac{mv_I^2}{R} = \frac{GmM}{R^2} = gm,\quad v_I = \sqrt{gR} = \sqrt{9.8\cdot6.4\cdot10^6}\ \frac{m}{s} = \cancel{2800}\ 7920\ \frac{m}{s}$

$\approx 7.9\ km/s$

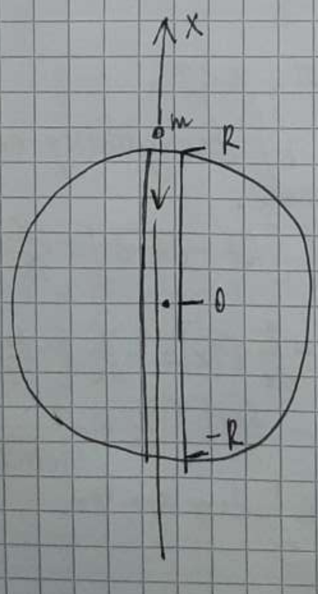


b) Energy on Earth = on ∞, $v_{II\infty\,min} = 0$

$-\frac{GmM}{R} + \frac{\cancel{m}v_{II}^2}{2} = 0 \Longrightarrow v_{II} = \sqrt{2GM/R} = \sqrt{2}\,v_I$

$\approx 11.2\ km/s$

(4)



$\dot{x}_0 = 0$

$x_0 = R$ From task 1 results – when object is at x, only

this part matters, and it attracts as

$$F_x = -\frac{GmM(x)}{R^2(x)} = -\frac{Gm\rho\cdot\frac{4}{3}\pi x^3}{x^2} =$$

$$= -\underbrace{Gm\rho\frac{4}{3}\pi}_{K}\,x = -Kx,$$

$m\ddot{x} = -Kx,\quad \ddot{x} = -\overbrace{\frac{K}{m}}^{w^2}x$ – harmonic motion!

$x(t) = A\cos(wt+\varphi_0)$

in this case $x(t) = R\cos\left(\sqrt{\tfrac{4}{3}\pi\rho G}\, t\right) = R\cos\left(\sqrt{g/R}\, t\right)$.

a) time to South $= \frac{T}{2} = \frac{\pi}{\omega} = \frac{\pi}{\sqrt{\frac{4}{3}\pi\rho G}} \approx \left[\begin{array}{l}\rho = \frac{M}{\frac{4}{3}\pi R^3} \\ \frac{GM}{R^3} = \frac{g}{R}\end{array}\right]$

$= \frac{\pi}{\sqrt{g/R}} = 2540\ s \approx 40$ min.

b) $v(0) = \dot{x}\left(\frac{T}{4}\right) = -R\sqrt{g/R}\,\underbrace{\sin\left(\omega\cdot\frac{\pi}{2\omega}\right)}_{1} = -\sqrt{gR} \approx 7{,}9\,km/s$

$(= v_I!)$