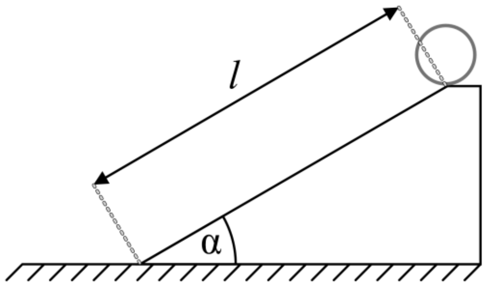
Problem 1: **Rolling Cylinders** **(X Points)**



Consider two long cylinders, one solid, the other one thin-walled, but both of outer radius R and total mass M. Both cylinders are placed on top of an incline of length l and an inclination angle α.

a. Which cylinder arrives first?
b. Calculate the acceleration of both cylinders.
c. What is the velocity of the cylinders when arriving at the end of the incline?
d. What are the values of the angular momenta of the cylinders when arriving at the end of the incline?

$R = 5\ \mathrm{cm}$, $m = 1\ \mathrm{kg}$, $l = 1\ \mathrm{m}$, $\alpha = 30°$

Solution:

(a)
We can consider the movement as the combination of rolling and rotation with respect to the center of mass. Also the rolling condition is fulfilled:

$$v = \omega \cdot R$$

Using energy conservation:

$$E_{\mathrm{kin}} = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2 = \frac{1}{2}(I + mR^2)\omega^2$$

The two cylinders have different moment of inertia.

$$I_{\mathrm{hollow}} = mR^2 \rightarrow E_{\mathrm{kin,hollow}} = \frac{1}{2}(mR^2 + mR^2)\omega_{\mathrm{hollow}}^2 = mR^2\omega_{\mathrm{hollow}}^2$$

$$I_{\mathrm{solid}} = \frac{1}{2} mR^2 \rightarrow E_{\mathrm{kin,solid}} = \frac{1}{2} \cdot \left(\frac{1}{2} mR^2 + mR^2\right) \omega_{\mathrm{solid}}^2 = \frac{3}{4} mR^2\omega_{\mathrm{solid}}^2$$

According to energy conservation law, during the rolling process, potential energy transfers to kinetic energy, this yields,

$$\Delta E_{\mathrm{pot}} = \Delta E_{\mathrm{kin}}$$

$$mgh = mR^2\omega_{\mathrm{hollow}}^2 = \frac{3}{4} mR^2\omega_{\mathrm{solid}}^2 \rightarrow \omega_{\mathrm{hollow}}^2 = \frac{3}{4}\omega_{\mathrm{solid}}^2$$

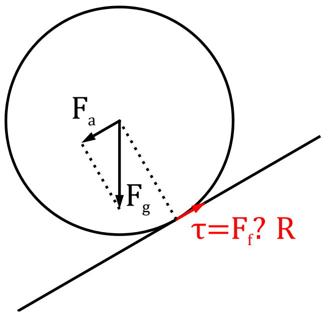
This shows, the solid cylinder will arrive first:

$$\omega_{\mathrm{hollow}} < \omega_{\mathrm{solid}}$$

(b)

We use β for the angular acceleration which is connected to the acceleration with:

$$\beta = \frac{a}{R}$$



As shown in the sketch we see that the resulting acceleration is a combination of Gravity pulling it down the incline ($F_a = m \cdot g \cdot \sin\alpha$) and a friction F_f which effectively generates a torque $\tau = F_f \times R = F_f \cdot R$ since it doesn't pull at the center of mass but at the surface of the cylinder:

$$F_{\text{tot}} = m \cdot a = F_a - F_f = m \cdot g \cdot \sin\alpha - \frac{\tau}{R}$$

Now we replace the torque $\tau = I \cdot \beta$ and apply the rolling condition:

$$m \cdot a = m \cdot g \cdot \sin\alpha - \frac{I \cdot \beta}{R} = m \cdot g \cdot \sin\alpha - \frac{I \cdot a}{R^2}$$

$$\to m \cdot g \cdot \sin\alpha = \frac{I \cdot a}{R^2} + m \cdot a$$

$$\to a = \frac{g \cdot \sin\alpha}{1 + \frac{I}{mR^2}}$$

For the hollow cylinder,

$$a_{\text{hollow}} = \frac{g \cdot \sin\alpha}{1 + \frac{I_{\text{hollow}}}{mR^2}} = \frac{g \cdot \frac{1}{2}}{1 + 1} = \frac{g}{4}$$

$$= \frac{1}{4} \cdot 9.81 \text{ m/s}^2 = 2.45 \text{ m/s}^2$$

For the solid cylinder,

$$a_{\text{solid}} = \frac{g \cdot \sin\alpha}{1 + \frac{I_{\text{solid}}}{mR^2}} = \frac{g \cdot \frac{1}{2}}{1 + \frac{1}{2}} = \frac{g}{3}$$

$$= \frac{1}{3} \cdot 9.81 \text{ m/s}^2 = 3.27 \text{ m/s}^2$$

(c)

Since the movement down the incline is a constant acceleration one can derive:

$$x = \frac{a}{2} t^2, \qquad v = a \cdot t \to v = \sqrt{2ax}$$

We got,

$$v_{\text{hollow}} = \sqrt{2a_{\text{hollow}} L} = \sqrt{2 \cdot \frac{g}{4} \cdot L} = \sqrt{0.5 \cdot 9.81 \text{ m/s}^2 \cdot 1 \text{ m}} = 2.21 \text{ m/s}$$

$$v_{\text{solid}} = \sqrt{2a_{\text{solid}} L} = \sqrt{2 \cdot \frac{g}{3} \cdot L} = \sqrt{\frac{2}{3} \cdot 9.81 \text{ m/s}^2 \cdot 1 \cdot \text{m}^2/\text{s}^2} = 2.56 \text{ m/s}$$

(d)

$$L_{\text{hollow}} = I_{\text{hollow}}\omega_{\text{hollow}} = m \cdot R^2 \cdot \frac{v_{\text{hollow}}}{R} = m \cdot R \cdot v_{\text{hollow}}$$

$$= 1\ \text{kg} \cdot 0.05\ \text{m} \cdot 2.21\ \text{m/s} = 0.111\ \text{kg}\frac{\text{m}^2}{\text{s}}$$

$$L_{\text{solid}} = I_{\text{solid}}\omega_{\text{solid}} = \frac{1}{2}mR^2 \cdot \omega_{\text{solid}} = \frac{1}{2}m \cdot R \cdot v_{\text{hollow}}$$

$$= \frac{1}{2} \cdot 1\ \text{kg} \cdot 0.05\frac{\text{m}}{\text{s}} \cdot 2.56\ \text{m/s} = 0.064\ \text{kg}\frac{\text{m}^2}{\text{s}}$$

Problem 2: Particle Collision **2 + 2 Points**

In a particle collider protons (mass $m_p = 1.67 \cdot 10^{-27}\text{kg}$) are accelerated to a kinetic energy of 5GeV. Calculate the energy that is set free if,

e. A proton from the beam hits a stationary proton.
f. A proton from the beam hits a proton from another particle beam which propagates in the opposing direction.

Hint: Both collisions are perfectly inelastic.

Solution

The collision is inelastic, so only momentum conservation holds:

Stationary Proton

Before collision

$$P = p_1 = m_p \cdot v;\ \ p_2 = 0;\ \ E_{kin} = \frac{m_p}{2} \cdot v^2$$

After the collision:

$$p'_1 = p'_2 = m_p \cdot v';\ \ E'_{kin} = 2 \cdot \frac{m_p}{2} \cdot v'^2$$

Since

$$P' = p'_1 + p'_2 = 2 \cdot m_p \cdot v' = m_p \cdot v = P \rightarrow v' = \frac{v}{2}$$

So:

$$E'_{kin} = 2 \cdot \frac{m_p}{2} \cdot v'^2 = 2 \cdot \frac{m_p}{2} \cdot \left(\frac{v}{2}\right)^2 = \frac{1}{2}\frac{m}{2}v^2 = \frac{E_{kin}}{2} \rightarrow \Delta E = \frac{E_{kin}}{2} = 2.5\ \text{GeV}$$
$$= 4 \cdot 10^{-10}\ \text{J}$$

Moving Proton

Before collision:

$$P = 0\,, \qquad p_1 = m_p \cdot v\,, \qquad p_2 = -m_p \cdot v,$$

$$E_{kin} = 2 \cdot \frac{m_p}{2} \cdot v^2 = 10\ \text{GeV} = 1.6 \cdot 10^{-9}\ \text{J}$$

After the collision:

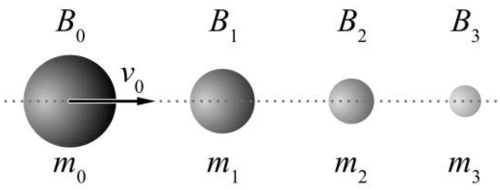
$$P' = p'_1 = p'_2 = 0; \;\; E'_{kin} = 0$$

So:

$$\Delta E = E_{kin} = 10 \text{ GeV} = 4 \cdot 10^{-10}\,\text{J}$$

So, the Energy set free is much bigger in a head on collisions, then when one target is stationary.

Problem 3: Consecutive Collisions **4 + 2 Points**



Given is the one-dimensional, procedural, elastic collision process provided in figure aside. The balls B_1, B_2 and B_3 are at rest, $v_1 = v_2 = v_3 = 0$. Ball B_0 has a speed of $v_0 = 1\,\text{m/s}$ before colliding with ball B_1. The mass of the next neighbouring ball is one third of the previous ball: $m_0 = 3m_1$, $m_1 = 3m_2$, $m_2 = 3m_3$.

g. Consider the first collision. Calculate the speed after the collision v'_0 and v'_1 of B_0 and B_1.
h. What is the speed v'_3 of B_3 after its collision with B_2?

Solution

One has to combine the conservation of momentum and the conservation of energy to get the velocities. A non-chaotic nomenclature is advised.

$$p_0 = p'_0 + p'_1$$
$$m_0 v_0 = m_0 v'_0 + m_1 v'_1$$
$$3m_1 v_0 = 3m_1 v'_0 + m_1 v'_1$$
$$3v_0 = 3v'_0 + v'_1$$
$$\rightarrow \boldsymbol{v'_1 = 3(v_0 - v'_0)}$$
$$E_0 = E'_0 + E'_1$$
$$\frac{1}{2} m_0 v_0^2 = \frac{1}{2} m_0 {v'_0}^2 + \frac{1}{2} m_1 {v'_1}^2$$
$$3m_1 v_0^2 = 3m_1 {v'_0}^2 + m_1 {v'_1}^2$$
$$3v_0^2 = 3{v'_0}^2 + {v'_1}^2$$
$$\rightarrow \boldsymbol{{v'_1}^2 = 3\left(v_0^2 - {v'_0}^2\right)}$$

Combining both equations yield the searched velocities. (3rd binomial formula)

$$\boldsymbol{{v'_1}^2 = 3\left(v_0^2 - {v'_0}^2\right)} = \underbrace{3(v_0 - v'_0)}_{v'_1}(v_0 + v'_0) = v'_1(v_0 + v'_0)$$
$$\rightarrow v'_1 = v_0 + v'_0$$
$$v'_1 = v'_1$$
$$3(v_0 - v'_0) = v_0 + v'_0$$
$$\rightarrow \boldsymbol{2v'_0 = v_0}$$

$$\rightarrow \boldsymbol{v_1' = \frac{3}{2} v_0}$$

For the n-th collision the velocity can be calculated recursively ($n = 3$):

$$v_n' = (3/2)^n \cdot v_0\,.$$

Problem 4: The Spinning Disk **$3 + 1 + 2 + 1 + 2$ Points**

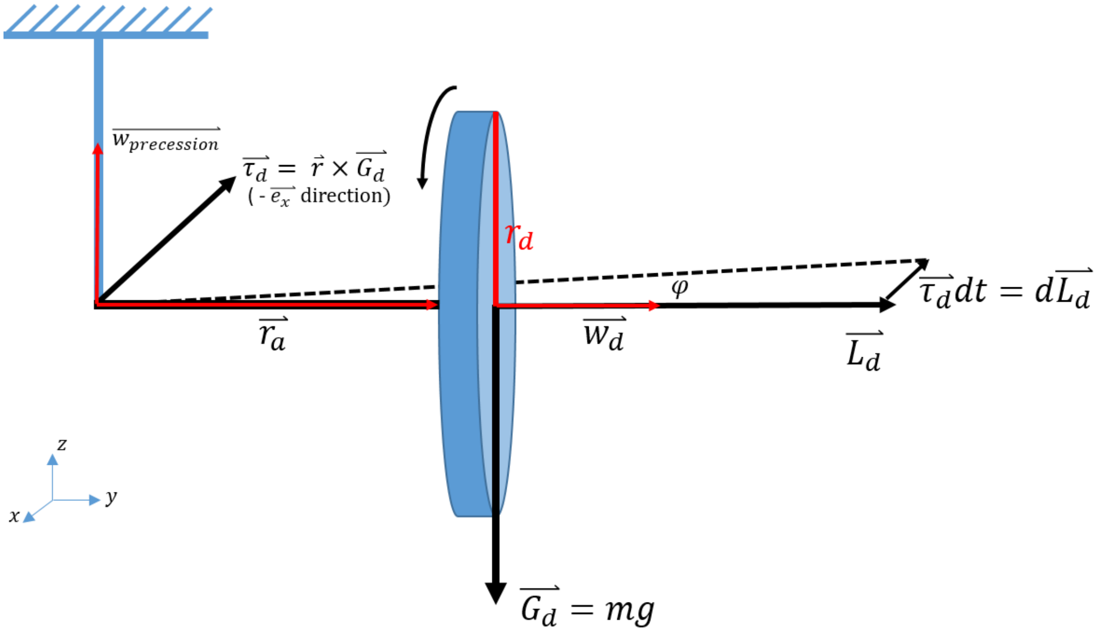
A solid disc with a radius of $r = 28\ \mathrm{cm}$ is centrally placed on an axis with a length of $l = 50\ \mathrm{cm}$ and a mass of $m_a = 100g$. The other end of the axis is attached to a rope. The disk has a mass of $m_d = 500\mathrm{g}$ and spins with $\omega = 12\,Hz$ (12 revolutions per second). At $t = 0$ the axis is parallel to the y-axis so that $\vec{\omega} = \omega \cdot \vec{e}_y$. The rope is parallel to the z-axis so that gravity points in $-\vec{e}_z$ direction.

i. Make a big, detailed sketch of the whole system. Include a coordinate system and the vectors of the angular momentum of the spinning wheel, the force and torque caused by gravity, the angular velocity of the wheel and the angular velocity of the precession movement.
j. Calculate the angular momentum of the wheel.
k. Calculate the angular speed of the precession movement
l. Calculate the time that the axis and the wheel need to make a full turn around the axis of the precession movement.
m. Calculate the magnitude and direction of the angular momentum of the precession movement.

Hint: Make sure to use the coordinate system provided. There is a helpful video here: https://youtu.be/n5bKzBZ7XuM

Solution

(a)



Points: 1P for the correct coordinate system. 1P for correctly displaying all the asked vectors (0.5P if up to two are missing). 1P if the Drawing is complete and understandable.

(b)

$$\overrightarrow{L_d} = I_d \overrightarrow{\omega_d} = \frac{1}{2} m_d r_d^2 \overrightarrow{\omega_d} = \frac{1}{2} * 0.5 \text{ kg} * (0.28 \text{ m})^2 * 12 \text{ Hz} * 2\pi$$

$$= 1.48 \frac{\text{kg m}^2}{\text{s}}$$

(c)

If the wheel is on the end of the axis ():

$$\overrightarrow{\tau_d} = \frac{d\overrightarrow{L_d}}{dt} = \frac{\overrightarrow{L_d} \cdot d\varphi_p}{dt} = \overrightarrow{L_d} \times \overrightarrow{\omega_p} = L_d \cdot \omega_p$$

With the index p for the precession and d for the movement of the wheel

By definition: $\overrightarrow{\tau_d} = \vec{r} \times \vec{F} = \vec{r} \times m\vec{g}$. The axis can be viewed as pulling with its mass at the centre of the axis so: $\tau_d = l \cdot m_d g$ and $\tau_a = \frac{l}{2} \cdot m_a g$

So, in total, we got, $\overrightarrow{\omega_p} = \frac{l \cdot m_d g + \frac{l}{2} \cdot m_a g}{\overrightarrow{L_d}}$ (1P)

$$= \frac{0.5 \text{ m} \cdot 0.5 \text{ kg} \cdot 9.81 \text{ms}^{-2} + 0.25 \text{ m} \cdot 0.1 \text{ kg} \cdot 9.81 \text{ms}^{-2}}{1.48 \frac{\text{kg m}^2}{\text{s}}}$$

$$\omega_p = 1.82 \frac{\text{rad}}{\text{s}} \quad (1P)$$

If the wheel is at the centre of the axis ($r_a = \frac{l}{2}$)

$$\overrightarrow{\omega_p} = \frac{\frac{l}{2} \cdot m_d g \, + \frac{l}{2} \cdot m_a g}{\overrightarrow{L_d}} \quad (1P)$$

$$= \frac{0.25\text{ m} * 0.5\text{ kg} * 9.81\text{ms}^{-2} + 0.25\text{ m} * 0.1\text{ kg} * 9.81\text{ms}^{-2}}{1.48\ \frac{\text{kg m}^2}{\text{s}}}$$

$$\omega_p = 0.99 \frac{\text{rad}}{\text{s}} \quad (1P)$$

(d)

$T = \frac{2\pi}{\overrightarrow{\omega_d}} = 3.45\ s$ (edge) or 6.32 s (centre)

(e)

The Moment of inertia for the disc around its symmetry axis in z-direction:

$I_{cm,d} = I_z = \frac{1}{2} I_y = \frac{1}{4} m_d r_d^2$ (Perpendicular axis theorem)

So, the total angular momentum is:

$$\overrightarrow{L_d} = I\overrightarrow{\omega_d} = \left(I_{cm,d} + m_d r_a^2 + I_a\right)\overrightarrow{\omega_d}$$

For the disc at the edge:

$$\overrightarrow{L_d} = \left(\frac{1}{4} m_d r_d^2 + m_d l^2 + \frac{1}{3} m_a l^2\right)\overrightarrow{\omega_d}$$

$$= \left(\frac{1}{4} \cdot 0.5\text{ kg} \cdot (0.28\text{ m})^2 + 0.5\text{ kg} \cdot (0.5\text{ m})^2 + \frac{1}{3} \cdot 0.1\text{ kg} \cdot (0.5\text{ m})^2\right)\overrightarrow{\omega_d}$$

$$= 0.26\ \frac{\text{kg m}^2}{\text{s}} \overrightarrow{e_z}$$

For the disc at the centre:

$$\overrightarrow{L_d} = \left(\frac{1}{4} m_d r_d^2 + m_d \left(\frac{l}{2}\right)^2 + \frac{1}{3} m_a l^2\right)\overrightarrow{\omega_d}$$

$$= \left(\frac{1}{4} \cdot 0.5\text{ kg} \cdot (0.28\text{ m})^2 + 0.5\text{ kg} \cdot (0.25\text{ m})^2 + \frac{1}{3} \cdot 0.1\text{ kg} \cdot (0.5\text{ m})^2\right)\overrightarrow{\omega_d}$$

$$= 0.049\ \frac{\text{kg m}^2}{\text{s}} \overrightarrow{e_z}$$

(0.5P for the value, 0.5P for the direction)