Mathematics 1. Selected proofs
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Derivative. Fermat's and Rolle's theorems

1. Statement of Fermat's theorem:

   THEOREM 1. Assume $f : (a, b) \to \mathbb{R}$ has a local extremum (maximum or minimum) on the interval (a, b) at some internal point $c \in (a, b)$, i.e.

   $$\exists\, c \in (a,b): \quad \forall\, x \in (a,b) \quad f(x) \le f(c) \qquad \Big(\text{or} \quad \forall\, x \in (a,b) \quad f(x) \le f(c) \Big)$$

   If f is differentiable at c then $f'(c) = 0$.

   PROOF. Assume $\forall\, x \in (a, b) \;\; f(x) \le f(c)$. The case of minimum is similar.

2. Use the characterization of the limit in terms of one-sided limits:

   $$\exists\, f'(x) \;=\; \lim_{x\to c} \tfrac{f(x)-f(c)}{x-c} \;=\; \lim_{x\to c-0} \tfrac{f(x)-f(c)}{x-c} \;=\; \lim_{x\to c+0} \tfrac{f(x)-f(c)}{x-c}$$

3. Compute the limit from the left:
   $\forall\, x \in (a, c), \quad f(x) \le f(c) \quad \Rightarrow \quad \tfrac{f(x)-f(c)}{x-c} \ge 0 \quad \Rightarrow \quad \lim\limits_{x\to c-0} \tfrac{f(x)-f(c)}{x-c} \ge 0 \quad \Rightarrow \quad f'(c) \ge 0$
   Compute the limit from the right:
   $\forall\, x \in (c, b), \quad f(x) \le f(c) \quad \Rightarrow \quad \tfrac{f(x)-f(c)}{x-c} \le 0 \quad \Rightarrow \quad \lim\limits_{x\to c+0} \tfrac{f(x)-f(c)}{x-c} \le 0 \quad \Rightarrow \quad f'(c) \le 0$
   Compare limits from the left and from the right:

   $$f'(c) \ge 0 \quad \text{and} \quad f'(c) \le 0 \quad \Longrightarrow \quad f'(c) = 0$$

4. Statement of Rolle's theorem:

   THEOREM 2. Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Assume that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

5. PROOF. Use the extreme value theorem:

   $$\exists\, c_1,\, c_2 \in [a,b]: \qquad f(c_1) = \inf_{x\in[a,b]} f(x), \qquad f(c_2) = \sup_{x\in[a,b]} f(x)$$

6. Consider the case $f(x) = const$:

   $$f(c_1) = f(c_2) \quad \Longrightarrow \quad \forall\, x \in [a,b] \quad f(x) = f(a) = f(b) \quad \Longrightarrow \quad \forall\, x \in (a,b) \quad f'(x) = 0$$

7. Consider the case $f(x) \neq const$:

$$f(c_1) \neq f(c_2) \quad \Longrightarrow \quad \text{at least one of the points } c_1 \text{ and } c_2 \text{ is different from } a \text{ and } b$$
$$\text{denote by } c \text{ those of } c_1 \text{ and } c_2 \text{ for which } c \neq a \text{ and } c \neq b \quad \Longrightarrow \quad c \in (a,b)$$

8. Use Fermat's theorem:

   Assume $\quad c \in (a,b), \quad f(c) = \sup\limits_{x\in[a,b]} f(x) \quad \Rightarrow \quad \forall\, x \in (a,b) \quad f(x) \leq f(c) \quad \overset{\text{Fermat}}{\Longrightarrow} \quad f'(c) = 0$

   The case $\quad c \in (a,b), \quad f(c) = \inf\limits_{x\in[a,b]} f(x) \quad$ is similar.