Problem 1: Stokes Friction $\mathbf{2+2}$ **Points**

A sphere with a radius of 1.5 cm and a mass of 10g is released at the bottom of a very long tube which is filled with oil (density $\rho = 850\,\frac{\text{kg}}{\text{m}^3}$, viscosity $\eta = 0.081\,\frac{\text{kg}}{\text{m}\cdot\text{s}}$).

a. What is the sphere's acceleration when it is released?
a. What is the maximum speed the sphere will reach after a while?

Solution

(a)

When the sphere is released, two forces (gravity and lifting force) contribute to the acceleration. The sphere floats upwards.

$ma = F_{lift} - G = \rho V g - mg$ (1P)

We got

$$a = \frac{\rho V g}{m} - g = \left(\rho \cdot \frac{4\pi r^3}{3m} - 1\right) \cdot g$$

$$= \left(850\,\frac{\text{kg}}{\text{m}^3} \cdot \frac{4\pi \cdot (0.015\text{ m})^3}{3 \cdot 0.01\text{ kg}} - 1\right) \cdot 9.81\,\frac{\text{m}}{\text{s}^2} = 1.98\,\frac{\text{m}}{\text{s}^2}$$

(1P)

(b)

According to Stoke's law, friction increase proportionally with velocity.

When the lifting force is equal to the sum of gravity and friction, the sphere reaches its maximum speed.

$G + f_r = F_{lift} \rightarrow mg + 6\pi\eta r \cdot u_{max} = \rho V g$

Solving for speed (1P):

$$u_{max} = \frac{\rho V g - mg}{6\pi\eta r}$$

$$= \frac{850\text{ kg/m}^3 \cdot \frac{4\pi}{3} \cdot (0.015\text{ m})^3 \cdot 9.81\text{ m/s}^2 - 0.01\text{ kg} \cdot 9.81\text{ m/s}^2}{6\pi \cdot 0.081\text{ kg/(s}\cdot\text{m)} \cdot 0.015\text{ m}}$$

$$= 0.86\text{ m/s (1P)}$$

Problem 2: Sinking Sphere **(X Points)**

A steel sphere (radius r) is immersed in water (viscosity η) and held at rest. At $t = 0$ the sphere is released and starts to sink.

b. Sketch the forces acting on the sphere right at the beginning of the decent.

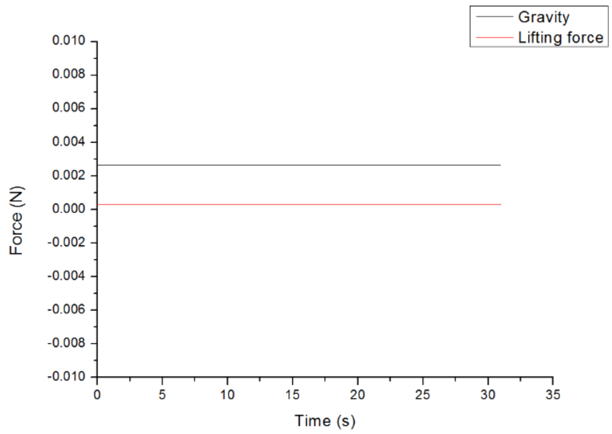
c. Sketch the forces acting on the sphere after it sunk for a quite a large distance.
d. Calculate the speed $v(t)$ of the sphere as a function of time by integration of the equation of motion.
e. After which time has the speed reached its $(1 - 1/e \approx 0.6321)$ of the terminal speed?

$\rho_{\text{steel}} = 8000\ \text{kg/m}^3$, $\rho_{\text{water}} = 1000\ \text{kg/m}^3$, $R = 2\ \text{mm}$, $\eta = 1\ \text{mPa s}$

Solution:

(a)

At the beginning of the decent, only two forces are applied to the steel sphere. Gravity and the lifting force at opposite direction. (see figure)

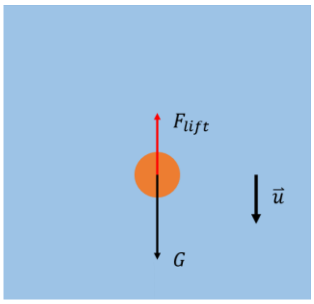


$$G = mg = \rho_{\text{steel}} V g = \rho_{\text{steel}} \cdot \frac{4}{3}\pi R^3 \cdot g$$
$$= 8000\ \text{kg/m}^3 \cdot \frac{4}{3}\pi \cdot (0.002\ m)^3 \cdot 9.81\ \text{m/s}^2$$
$$= 0.00263\ \text{N}$$

$$F_{\text{lift}} = \rho_{\text{water}} V g = \rho_{\text{water}} \cdot \frac{4}{3}\pi R^3 \cdot g$$
$$= 1000\ \text{kg/m}^3 \cdot \frac{4}{3}\pi \cdot (0.002\ m)^3 \cdot 9.81\ \text{m/s}^2$$
$$= 3.23 \cdot 10^{-4}\ \text{N}$$

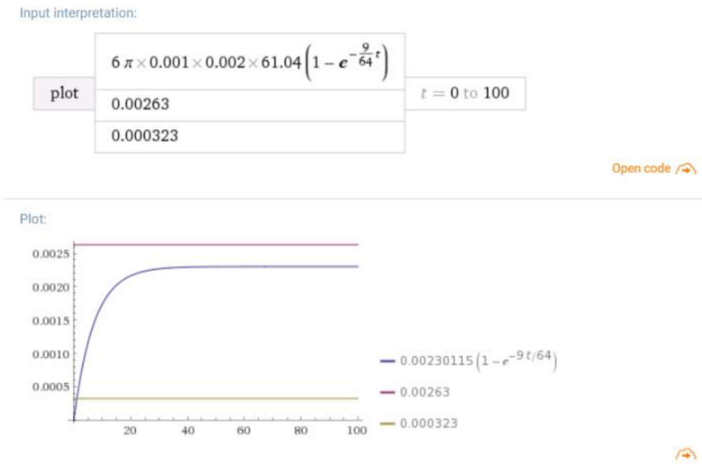
This yields,

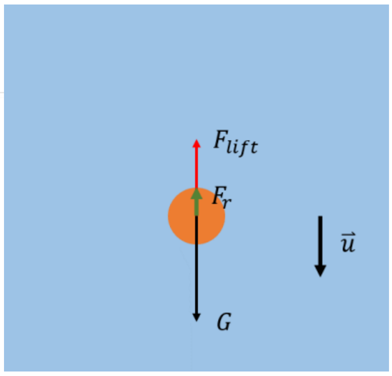
$$F = G - F_{\text{lift}} = \rho_{\text{steel}} \cdot \frac{4}{3}\pi R^3 \cdot g - \rho_{\text{water}} \cdot \frac{4}{3}\pi R^3 \cdot g$$
$$= (\rho_{\text{steel}} - \rho_{\text{water}}) \cdot \frac{4}{3}\pi R^3 \cdot g$$
$$= 7000\ \text{kg/m}^3 \cdot \frac{4}{3}\pi \cdot (0.002\ m)^3 \cdot 9.81\ \text{m/s}^2$$
$$= 0.0023\ \text{N}$$



(b)

After the sphere sunk for a large distance, three forces are applied.
Gravity and the lifting force do not change during this process. Friction force is proportional to the velocity, which can be know from the equation deviated from c).





(c)

Before the sphere reaches its terminal speed, it has the acceleration at the same direction as gravity.

$$ma = G - F_{\text{lift}} - f_r$$

This yield,

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{G - F_{\text{lift}} - f_r}{m} = \frac{(\rho_{\text{steel}} - \rho_{\text{water}})Vg - 6\pi\eta R \cdot v(t)}{\rho_{\text{steel}} V} =$$

$$= \frac{7000\ \mathrm{kg/m^3} \cdot \frac{4}{3}\pi \cdot (0.002\ \mathrm{m})^3 \cdot 9.81\ \mathrm{m/s^2}\ - 6\pi \cdot 0.001\ \mathrm{Pa\ s} \cdot 0.002\ \mathrm{m} \cdot v(t)}{8000\ \mathrm{kg/m^3} \cdot \frac{4}{3}\pi \cdot (0.002\ \mathrm{m})^3}$$

$$= \frac{9 \cdot (61.04\ \mathrm{m/s} - v(t))}{64\ \mathrm{s}}$$

Now we substitute $v'(t) = 61.04\ \mathrm{m/s} - v(t)$, this yield,

$$\frac{9 \cdot v'(t)}{64\ \mathrm{s}} = -\frac{\mathrm{d}v'(t)}{\mathrm{d}t}$$

$$\int \frac{\mathrm{d}v'(t)}{v'(t)} = \int -\frac{9}{64\ \mathrm{s}}\mathrm{d}t$$

$$\ln v'(t) + C = -\frac{9}{64\ \mathrm{s}}t$$

$$C = -\ln 61.04\ \mathrm{m/s}$$

This yield:

$$v'(t) = \mathrm{Exp}\left[-\frac{9}{64\ \mathrm{s}}t + \ln 61.04\ \mathrm{m/s}\right] = e^{-\frac{9}{64}t + \ln 61.04\ \mathrm{m/s}} = 61.04\ \mathrm{m/s} \cdot e^{-\frac{9}{64\ \mathrm{s}}t}$$

$$v(t) = 61.04\ \mathrm{m/s} - v'(t) = 61.04\ \mathrm{m/s} \cdot \left(1 - e^{-\frac{9}{64\ \mathrm{s}}t}\right).$$

(d)

When the sum of friction and lifting force is equal to gravity, the sphere reaches to its terminal speed.

$$G = f_r + F_{\text{lift}} \to \rho_{\text{steel}} V g = mg = 6\pi\eta R \cdot u_{\text{terminal}} + \rho_{\text{water}} V$$

Solving for speed: (Optional)

$$u_{\text{terminal}} = \frac{(\rho_{\text{steel}} - \rho_{\text{water}}) \cdot \frac{4}{3}\pi R^3 \cdot g}{6\pi\eta R} = \frac{2 \cdot (\rho_{\text{steel}} - \rho_{\text{water}}) \cdot R^2 \cdot g}{9\eta}$$
$$= \frac{2 \cdot 7000\ \text{kg/m}^3 \cdot (0.002\ m)^2 \cdot 9.81\ \text{m/s}^2}{9 \cdot 0.001\ \text{Pa s}} = 61.04\ \text{m/s}$$

Or, when t reaches infinite, from the expression for $v(t)$, we get

$$u_{\text{terminal}} = 61.04\ \text{m/s}$$
$$v(t) = \left(1 - \frac{1}{e}\right) u_{\text{terminal}} = u_{\text{terminal}} \cdot \left(1 - e^{-\frac{9}{64\ \text{s}}t}\right)$$

we get

$$e^{-\frac{9}{64\ \text{s}}t} = \frac{1}{e},$$
$$t = \frac{64\ \text{s}}{9} = 7.11\ s$$

Problem 3: Beer keg $\mathbf{(2+2+4+1\ Points)}$

A large keg of height H and cross-sectional area A_1 is filled with root beer. The top is open to the atmosphere. There is a spigot opening of area A_2, which is much smaller than A_1, at the bottom of the keg.

a) Show that when the height of the root beer is h, the speed of the root beer leaving the spigot is approximately $\sqrt{2gh}$
b) Show that if $A_2 \ll A_1$, the rate of change of the height h of the root beer is given by $\frac{dh}{dt} = -\frac{A_2}{A_1} \cdot \sqrt{2gh}$.
c) Find h as a function of time if $h = H$ at $t = 0$.
d) Find the total time needed to drain the keg if $H = 2.00\ \text{m}$, $A_1 = 0.8\ \text{m}^2$, and $A_2 = 1 \cdot 10^{-4} A_1$. Assume laminar non viscous flow.

(*a*) Apply Bernoulli's equation to the beer at the top of the keg and at the spigot:

$$P_1 + \rho_{\text{beer}} g h_1 + \tfrac{1}{2}\rho_{\text{beer}} v_1^2 = P_2 + \rho_{\text{beer}} g h_2 + \tfrac{1}{2}\rho_{\text{beer}} v_2^2$$

or, because $v_1 \approx 0$, $h_2 = 0$, $P_1 = P_2 = P_{\text{at}}$, and $h_1 = h$,

$$gh = \tfrac{1}{2} v_2^2 \Rightarrow v_2 = \boxed{\sqrt{2gh}}$$

(*b*) Use the continuity equation to relate v_1 and v_2:

$$A_1 v_1 = A_2 v_2$$

Substitute $-dh/dt$ for v_1 and $\sqrt{2gh}$ for v_2 to obtain:

$$-A_1 \frac{dh}{dt} = A_2 \sqrt{2gh}$$

Solving for dh/dt yields:

$$\frac{dh}{dt} = \boxed{-\frac{A_2}{A_1}\sqrt{2gh}}$$

(*c*) Separate the variables in the differential equation to obtain:

$$-\frac{A_1/A_2}{\sqrt{2g}} \frac{dh}{\sqrt{h}} = dt$$

Express the integral from H to h and 0 to t:

$$-\frac{A_1/A_2}{\sqrt{2g}} \int_H^h \frac{dh}{\sqrt{h}} = \int_0^t dt$$

Evaluate the integral to obtain:

$$-\frac{2A_1/A_2}{\sqrt{2g}}\left(\sqrt{H} - \sqrt{h}\right) = t$$

Solving for h gives:

$$h = \boxed{\left(\sqrt{H} - \frac{A_2}{2A_1}\sqrt{2g}\, t\right)^2}$$

(*d*) Solve $h(t)$ for the time-to-drain t':

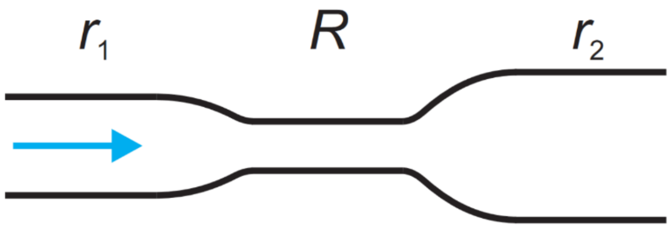
$$t' = \frac{A_1}{A_2}\sqrt{\frac{2H}{g}}$$

Substitute numerical values and evaluate t'

$$t' = \frac{A_1}{1.00 \times 10^{-4} A_1}\sqrt{\frac{2(2.00\,\text{m})}{9.81\,\text{m/s}^2}} = 6.39 \times 10^3\,\text{s} = \boxed{1\text{h}\ 46\ \text{min}}$$

Problem 4: Pipe with changing radius (5 Points)

In the image below you can see water flowing from left to right through a pipe with changing cross section. The radius r_1 of the left pipe section is twice the radius of the narrowed section: $2R = r_1$. The radius of the right section is $r_2 = 3R$. Furthermore, in the middle narrow section the water flows with a velocity of $v = 0.5\,\frac{\text{m}}{\text{s}}$. Calculate the energy that is needed to move $0.4\ \text{m}^3$ water from the left pipe section to the right.



Lsg:

First off, we calculate the flow velocities in the left and-right hand side segments using the continuity equation:

$$A_1 v_1 = 2\pi r_1^2 v_1 = Av = 2\pi R^2 v \rightarrow v_1 = \frac{R^2}{r_1^2} v = \frac{R^2}{4R^2} = \frac{1}{4} v \qquad (1P)$$

$$v_2 = \frac{R^2}{r_2^2} v = \frac{1}{9} v \qquad (1P)$$

To calculate the work done, we evaluate the change of kinetic energy of the water while going from the segment 1 to 2 and from 2 to 3:

$$\Delta E_{kin,12} = \frac{1}{2} m v^2 - \frac{1}{2} m v_1^2 = \frac{1}{2} m (v^2 - v_1^2) \qquad (1P)$$

$$\Delta E_{kin,23} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v^2 = \frac{1}{2} m (v_2^2 - v^2) \qquad (1P)$$

In conclusion:

$$\Delta E_{kin} = \frac{1}{2} m (v^2 - v_1^2 + v_2^2 - v^2) = \frac{1}{2} m (v_2^2 - v_1^2) = \frac{1}{2} m \left(\frac{1}{9^2} - \frac{1}{4^2}\right) v^2$$

$$= -2{,}50\mathrm{J} \qquad (1P)$$