7. Energy and Conservation Laws

The Chapters 3.1–3.4.2 of my lecture notes provides the background to solve the following exercises. Your solution to the problems 7.1–7.3 should uploaded

to your Moodle account
as a PDF-file
by Wednesday, Nov 15, 10:30 (with a grace time till the start of the seminars).

The parts marked by $\star$ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

The self-test problems serve to check you understanding of vectors.

The bonus problem provides the solution to a problem that we introduced in class. It might take some extra effort to solve.

Problems

Problem 1. Contour lines and gradients

The contour lines of a function $f(x, y)$ are provided by a function $y(x)$ where the height function $H(x) = f(x, y(x))$ takes a constant value c. Alternatively this line can also be described by a function $x(y)$ where $h(y) = f(x(y), y)$ takes the same constant value c.

For instance the contour lines for $f_1(x, y) = x\, y$ are given by $y(x) = c/x$, or by $x(y) = c/y$.

a) Determine the gradient of $f_1(x, y)$.

   Sketch the contour lines of $f_1(x, y)$, and mark the gradient by arrows.

b) Determine the contour lines and the gradient of

$$f_2(x, y) = x^2 - y^2$$

   Sketch the contour lines of $f_2(x, y)$, and mark the gradient by arrows.

c) Express the functions $f_1(x,y)$ and $f_2(x,y)$ in terms of polar coordinates $x = R\cos\theta$ and $y = R\sin\theta$. What do the resulting expressions tell about the relation between the contour lines and gradients of $f_1(x,y)$ and $f_2(x,y)$?

$\star$ d) Express the function

$$f_3(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

in terms of polar coordinates $x = R\cos\theta$ and $y = R\sin\theta$. What does this expression tell about the contour lines of f_3?

Determine the gradient by employing $\nabla f_3 = \partial_R f_3\, \hat{R} + R^{-1}\partial_\theta f_3\, \hat{\theta}$.

Sketch the contour lines of $f_3(x,y)$, and mark the gradient by arrows.

Problem 2. Forces, potentials, and line integrals

A vector field $\mathbf{K}(x,y,z)$ is *conservative*, when the line integral of a path from $\mathbf{q}_I$ to $\mathbf{q}_F$ does not depend on the choice of path $\mathbf{q}(t)$, $t_I \le t \le t_F$ where $\mathbf{q}(t_I) = \mathbf{q}_I$ and $\mathbf{q}(t_F) = \mathbf{q}_F$. We will explore this statement now for three vector fields

$$\begin{aligned}
\mathbf{K}_1(x,y,z) &= \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right) \\
\mathbf{K}_2(x,y,z) &= \left(x^2+y^2,\ x^2-y^2,\ 0 \right) \\
\mathbf{K}_3(x,y,z) &= (x+y+z,\ x+y+z,\ x+y+z)
\end{aligned}$$

and the paths

$$\begin{aligned}
a:&\quad \mathbf{q}_a(t) = \big(x(t), y(t), z(t)\big) = \big(t/2, t,\ 0\big)\,, \\
b:&\quad \mathbf{q}_b(t) = \big(x(t), y(t), z(t)\big) = (t,\ 2t,\ 0)\,, \\
c:&\quad \mathbf{q}_c(t) = \big(x(t), y(t), z(t)\big) = (t^3,\ 2t^2,\ 0)\,.
\end{aligned}$$

from $\mathbf{q}_I = \mathbf{0}$ to $\mathbf{q}_F = (1,2,0)$.

a) Show that $\mathbf{K}_1 = \nabla|\mathbf{q}|$.
What does this imply for the result of the line integrals $W(\mathbf{q}) = -\int \mathrm{d}\mathbf{s}\cdot\mathbf{K}_1(\mathbf{q})$?

b) Interpret the vector field $\mathbf{K}_2$ as a force on a particle, and determine the work that must be performed to move the particle from the origin to the position $\mathbf{q}(t)$. How do the calculations differ for the paths a and b? What does this imply for the work $W(\mathbf{q})$ performed along the paths?

At time $t = 1$ both particles b and c reach the position $(1, 2, 0)$. Compare the work that is performed for motion along the three different paths. Is $\mathbf{K}_2$ a conservative force?

c) Repeat the discussion for the force $\mathbf{K}_3$.

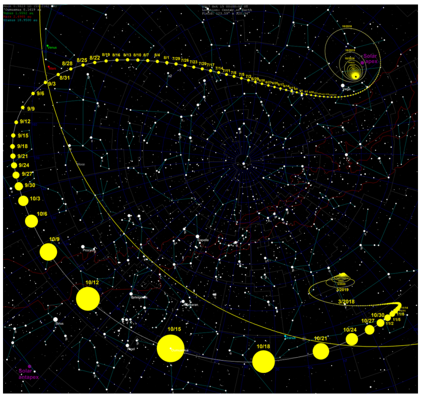
   Determine the potential when the force is conservative.
   Instruction: Determine a candidate for a potential based on the result of the line integral along the path

$$d : \quad \mathbf{q}_d(t) = \big(x_d\, t, y_d\, t, z_d\, t\big)$$

   for $t \in [0, 1]$ and some fixed values $(x_d,\, y_d,\, z_d)$. Verify your result by determining the gradient of the potential.

d) It is instructive to explore why the approach adopted in c) does not provide a potential for the force $\mathbf{K}_2$.
   Determine a candidate for a potential based on the result of the line integral of $\mathbf{K}_2$ along the path d.
   What happens when you take the gradient of this function?

Problem 3. 'Oumuamua



On 19 October 2017 astronomers at the Haleakala Observatory in Hawaii discovered 'Oumuamua, the first interstellar object observed in our solar system. It approached the solar system with a speed of about $v_I = 26\,\mathrm{km/s}$ and reached a maximum speed of $v_P = 87.71\,\mathrm{km/s}$ at its perihelion, i.e., upon closest approach to the Sun on 9 September 2017.

a) Show that at the perihelion the speed and 'Oumuamua's smallest distance to the Sun, D, obey the relation[1]

$$\frac{v_P^2 - v_I^2}{2} = \frac{M_S\, G}{D}$$

[1] Hint: Consider energy conservation and note that the potential energy of a body of mass m at a distance D from the Sun is $\Phi(D) = -m\, M_S\, G/D$.

while for the Earth we always have[2]

$$\frac{4\pi^2 R}{T^2} \simeq \frac{M_S\, G}{R^2}$$

Here, M_S is the mass of Sun, R is the Earth-Sun distance, and $T = 1\,\text{year}$ is the period of Earth around Sun.

b) Show that this entails that $\dfrac{D}{R} = \dfrac{2\,v_E^2}{v_P^2 - v_I^2}$, where $v_E = 2\pi R/T$ is the speed of Earth around Sun.

c) Use the relation obtained in (b) to determine D in astronomical units, and compare your estimate with the observed value $D = 0.25534(7)\text{AU}$.

Self Test

Problem 4. Running mothers

In the Clara Zetkin Park one regularly encounters blessings[3] of dozens of mothers jogging in the park while pushing baby carriages. Troops of kangaroo mothers rather carry their youngs in pouches.

a) Estimate the energy consumption spend in pushing the carriages as opposed to carrying the newborn.
   The carriages suffer from friction. Let the friction coefficient be $\gamma = 0.3$.
   When carrying the baby the kangaroo must lift it up in every jump and the associated potential energy is dissipated.

b) How does the running speed matter in this discussion?

c) How does the mass of the babies/youngs make a difference?

[2] Hint: Assume that the Earth orbit is circular, and explore Newton's 2nd law.
[3] Look up "terms of venery" if you ever run out of collective nouns.

Bonus Problem

Problem 5. Coulomb potential and external electric forces

We consider the Hydrogen atom to be a classical system as suggested by the Sommerfeld model. Let the proton be at the center of the coordinate system and the electron at the position $\mathbf{r}$. The interaction between the proton and the electron is described by the Coulomb potential $\alpha/|\mathbf{r}|$. In addition to this interaction there is a constant electric force acting, that is described by the potential $\mathbf{F} \cdot \mathbf{r}$. Altogether the motion of the electron is therefore described by the potential

$$U = -\frac{\alpha}{|\mathbf{r}|} - \mathbf{F} \cdot \mathbf{r}$$

a) Sketch the system and the relevant parameters.

b) Which force is acting on the particle?

   How do its equation of motion look like?

c) Verify that the energy is conserved.

d) Show that also the following quantity is a constant of motion,

$$I = \mathbf{F} \cdot (\dot{\mathbf{r}} \times \mathbf{L}) - \alpha \, \frac{\mathbf{F} \cdot \mathbf{r}}{|\mathbf{r}|} + \frac{1}{2} \, (\mathbf{F} \times \mathbf{r})^2$$

   Here $\mathbf{L}$ is the angular momentum of the particle with respect to the origin of the coordinate system.