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Theoretical Physics I
Homework 4

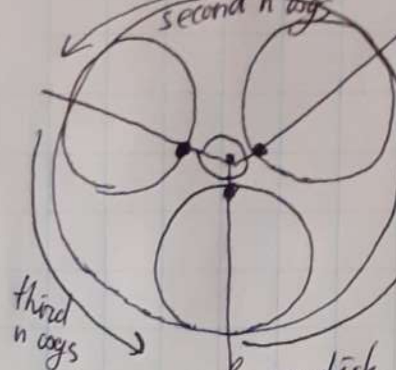
Problem 4.1 Hypotrochoids

a) Closed line: because if we take LCM(m,n) - the amount of cogs when both rotations/revolutions result in same position, after this distance the center of disk will be at the same position, and pole position on it will be same relative to center.

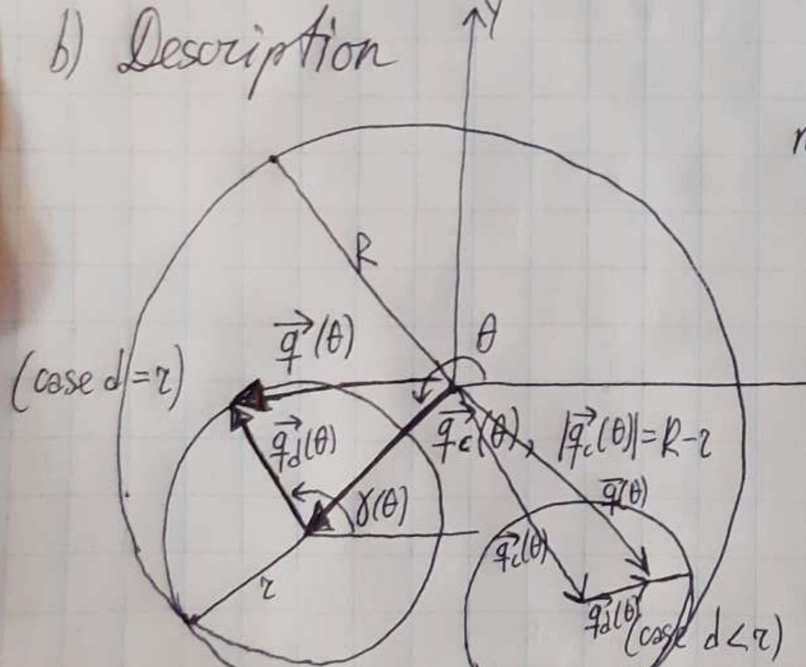
$N_{rev} = \frac{LCM(m,n)}{m}$

Symmetry $= \frac{LCM(m,n)}{n}$ -fold

for example



b) Description



m, n cogs measure circumference,

$\frac{m}{n} = \frac{2\pi R}{2\pi r} = \frac{R}{r} \longrightarrow r = \frac{n}{m}R.$

d is distance from point $\vec{q}_c(\theta)$ to $\vec{q}(\theta)$.

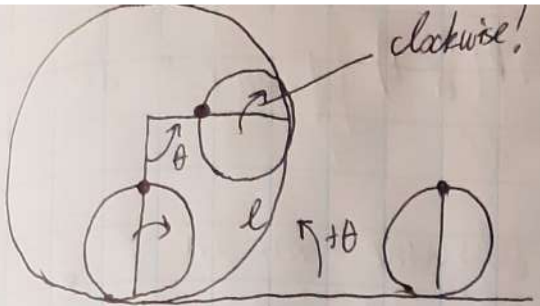
$\rho = \frac{d}{r}$,

So $d = \rho r = \rho \frac{n}{m} R$.

$$\vec{q}(\theta) = \vec{q}_c(\theta) + \vec{q}_d(\theta),$$

here $\vec{q}_c(\theta) = (R-r)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = R\left(1-\frac{n}{m}\right)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$

$\vec{q}_d(\theta) = d\begin{pmatrix}\cos\gamma \\ \sin\gamma\end{pmatrix}$, γ is rotation relative to disk, <u>must</u> find its relation to θ.



disk moved $\ell = \theta R$,

total rot. $= \frac{\ell}{2\pi r} = \frac{\theta R}{2\pi r}$, angle is

$\frac{\theta R}{2\pi r}\cdot(-2\pi) = -\frac{\theta R}{r}$

Since not straight line, but „curved" for θ, total is $-\frac{\theta R}{r}+\theta = \theta\left(1-\frac{R}{r}\right) = \gamma$.

So $\underline{\vec{q}(\theta)} = R\left(1-\frac{n}{m}\right)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + d\begin{pmatrix}\cos\theta\left(1-\frac{R}{r}\right)\\ \sin\theta\left(1-\frac{R}{r}\right)\end{pmatrix} = \underline{R\left(1-\frac{n}{m}\right)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + d\begin{pmatrix}\cos\theta\left(1-\frac{m}{n}\right)\\ \sin\theta\left(1-\frac{m}{n}\right)\end{pmatrix}}$

If disk center 2 equation is same with negative θ.

<u>General form:</u> $\vec{q}(\theta) = R\left(1-\frac{n}{m}\right)\begin{pmatrix}\cos(\theta_0+\theta)\\ \sin(\theta_0+\theta)\end{pmatrix} + \rho\frac{n}{m}R\begin{pmatrix}\cos\left\langle\left(1-\frac{m}{n}\right)(\theta+\gamma_0)\right\rangle\\ \sin\left\langle\left(1-\frac{m}{n}\right)(\theta+\gamma_0)\right\rangle\end{pmatrix}$.

R, n, m, ρ – parameters.

θ_0, γ_0 – initial conditions (0 below).

$\boxed{\text{c) Pyth. program in .pdf} \rightarrow}$

d) length can be evaluated for given θ. Length of one closed line means $\underset{\text{outer angle}}{\theta} \in [0, N_{rev}] = \left[0, \frac{LCM(m,n)}{m}\right]$.

$$L = \int_{t_{start}}^{t_{end}} |\dot{\vec{q}}(t)|\,dt, \quad \vec{q}(\theta) = R\left(1-\frac{n}{m}\right)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + \rho\frac{n}{m}R\begin{pmatrix}\cos\theta\left(1-\frac{m}{n}\right)\\ \sin\theta\left(1-\frac{m}{n}\right)\end{pmatrix}$$

transform to θ

$$\dot{\vec{q}}(\theta) = R\left(1-\frac{n}{m}\right)\begin{pmatrix}-\sin\theta\cdot\dot\theta\\ \cos\theta\cdot\dot\theta\end{pmatrix} + \rho\frac{n}{m}R\begin{pmatrix}-\left(\sin\theta\left(1-\frac{m}{n}\right)\right)\cdot\dot\theta\\ \left(\cos\theta\left(1-\frac{m}{n}\right)\right)\cdot\dot\theta\end{pmatrix}\left(1-\frac{m}{n}\right) =$$

$$= \underline{R\left(1-\frac{n}{m}\right)}\begin{pmatrix}-\sin\theta\cdot\dot\theta\\ \cos\theta\cdot\underline{\dot\theta}\end{pmatrix} + \underline{\rho R\left(\frac{n}{m}-1\right)}\begin{pmatrix}-\sin\left\langle\theta\left(1-\frac{m}{n}\right)\right\rangle\dot\theta\\ \cos\left\langle\theta\left(1-\frac{m}{n}\right)\right\rangle\underline{\dot\theta}\end{pmatrix} =$$

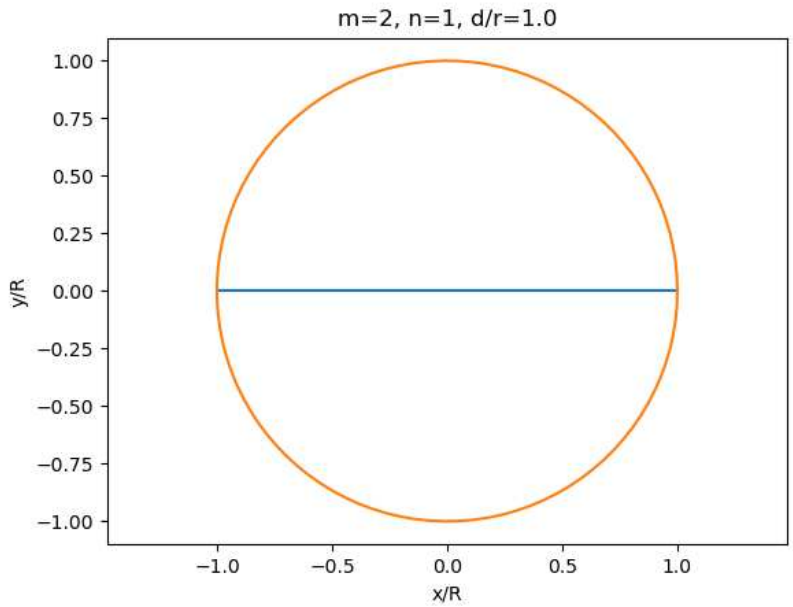
$$= R\dot\theta\underbrace{\left(\frac{n}{m}-1\right)}_{<0}\cdot\begin{pmatrix}\sin\theta - \rho\sin\left\langle\theta\left(1-\frac{m}{n}\right)\right\rangle\\ -\cos\theta + \rho\cos\left\langle\theta\left(1-\frac{m}{n}\right)\right\rangle\end{pmatrix}, \quad |\dot{\vec{q}}(\theta)| = R|\dot\theta|\underbrace{\left(\frac{m}{n}-1\right)}_{>0}\sqrt{\begin{matrix}\sin^2\theta+\rho^2\sin^2(\dots)+\\ -2\rho\sin\theta\sin(\dots)+\\ +\cos^2\theta+\rho^2\cos^2(\dots)+\\ -2\rho\cos\theta\cos(\dots)\end{matrix}} =$$

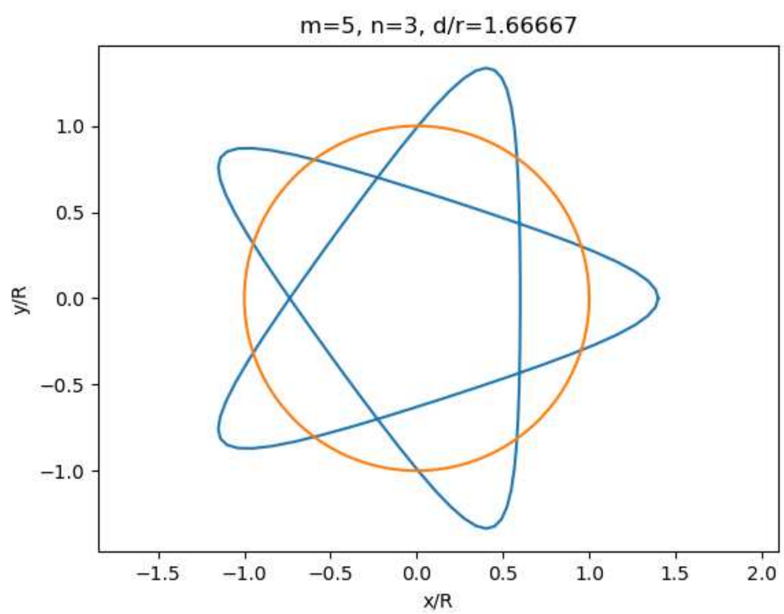
$$= R\dot\theta\left(\frac{m}{n}-1\right)\sqrt{1+\rho^2-2\rho\cos\left(\theta-\theta\left(1-\frac{m}{n}\right)\right)} = R\dot\theta\left(\frac{m}{n}-1\right)\sqrt{1+\rho^2-2\rho\left(\theta\frac{m}{n}\right)}.$$

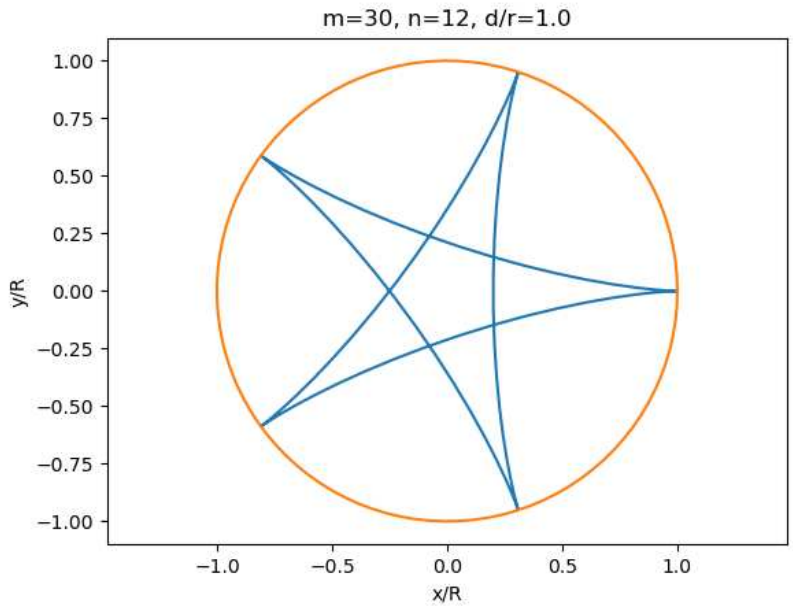
$$L = \int_0^{\theta_{max} = \frac{LCM(m,n)}{m}\cdot 2\pi} R\left(\frac{m}{n}-1\right)\sqrt{1+\rho^2-2\rho\underset{\cos}{}\left(\theta\frac{m}{n}\right)}\;\underbrace{d\theta}_{\dot\theta dt}.$$

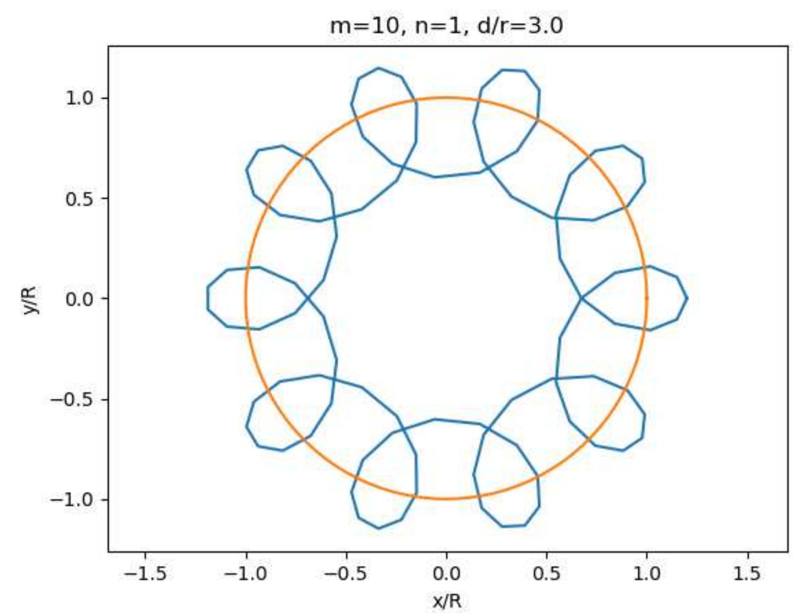
Check for $\rho=1$ (point on edge of disk):

```
# usage:
# launch in command terminal
# $ python hypotrochoid.py <m> <n> <d/r>

# specifying sizes of large and small circles, and ratio where pole on small circle is

import sys
import numpy as np
import matplotlib.pyplot as plt
import math

R=1

m=int(sys.argv[1])
n=int(sys.argv[2])
ro=float(sys.argv[3])

def LCM(m,n):
    return m*n/GCD(m,n)

def GCD(m,n):
    if n==0:
        return m
    else:
        return GCD(n, m%n)

theta = np.linspace(0, LCM(m,n)/m * 2*math.pi, 100)
x = R*(1-n/m)*np.cos(theta) + ro*(n/m)*R*np.cos((1-m/n)*theta)
y = R*(1-n/m)*np.sin(theta) + ro*(n/m)*R*np.sin((1-m/n)*theta)

plt.plot(x, y)

# external circle to compare
theta = np.linspace(0, 2*math.pi, 100)
x = R*np.cos(theta)
y = R*np.sin(theta)
plt.plot(x, y)
plt.axis('equal')
plt.ylabel('y/R')
plt.xlabel('x/R')
plt.title("m=" + str(m) + ", n=" + str(n) + ", d/r=" + str(ro))

plt.show()
```


m=2, n=1, d/r=1.0
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y/R
−1.0
−0.5
0.0
0.5
1.0
x/R


m=5, n=3, d/r=1.66667
1.0
0.5
0.0
−0.5
−1.0
y/R
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x/R

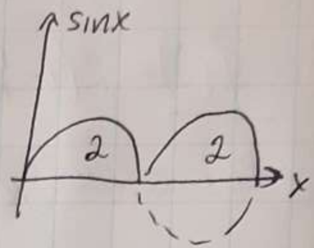

m=30, n=12, d/r=1.0
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y/R
−1.0
−0.5
0.0
0.5
1.0
x/R


m=10, n=1, d/r=3.0
1.0
0.5
0.0
−0.5
−1.0
y/R
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x/R

$$L_{p=1} = R\left(\frac{m}{n}-1\right)\int_0^{\theta_{max}} \sqrt{2\left(1-\cos\left(\theta\frac{m}{n}\right)\right)}\,d\theta = \sqrt{2}R\left(\frac{m}{n}-1\right)\int_0^{\theta_{max}}\sqrt{2\sin^2\left(\frac{\theta m}{2n}\right)}\,d\theta =$$

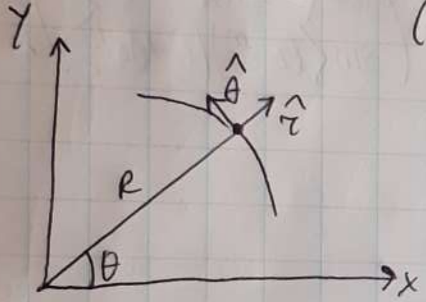
$$= 2R\left(\frac{m}{n}-1\right)\int_0^{\theta_{max}} \left|\sin\left(\theta\frac{m}{2n}\right)\right|d\theta.$$ Assume $m=2n$

$$\theta_{max} = \frac{L(M(2n,n))}{2n}\,2\pi = 2\pi, \quad L = 2R\int_0^{2\pi}|\sin\theta|\,d\theta = \underline{8R}.$$



Problem 4.2 Circular track

$(R = const)$



$$\hat{r} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \qquad \hat{\theta} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

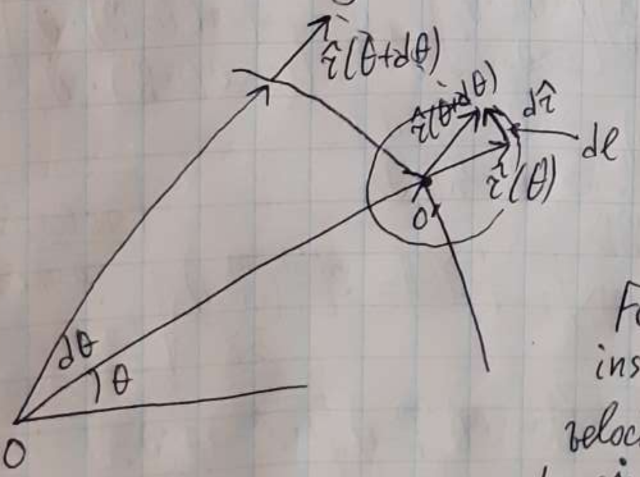
(Since $\hat{i},\hat{j}$ make ONB, $\left.\begin{array}{l}\hat{r}\hat{\theta} = -\sin\cos+\sin\cos=0,\\ \hat{r}\hat{r} = \cos^2+\sin^2=1\\ \hat{\theta}\hat{\theta} = \sin^2+\cos^2=1\end{array}\right\}$ $\hat{r},\hat{\theta}$ orthonormal vectors)

a) $\partial_\theta \hat{r} = \frac{d}{d\theta}\hat{r} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} = \hat{\theta}$

$\partial^2_\theta \hat{r} = \frac{d}{d\theta}(\hat{\theta}) = \begin{pmatrix}-\cos\theta\\ -\sin\theta\end{pmatrix} = -\hat{r}.$

Velocity is directed as $\hat{\theta}$, so tangentially, acceleration as $-\hat{r}$ — to center.

Geometry:



In the limit $(\dot{\hat{r}})$ becomes $\perp$, and since it's on unit circle O' $\left|\frac{d\hat{r}}{d\theta}\right| \approx \left|\frac{dl}{d\theta}\right| = |\hat{r}(\theta)| = 1.$

For $\ddot{\hat{r}}$ similar reasoning but instead of this plot — plot velocity vectors in their own „space", and $(\dot{\hat{\theta}}) = \ddot{\hat{r}}$ will be $\perp$ to $\hat{\theta}$, so $\uparrow\uparrow$ to original $\hat{r}$.
(can also say — each different. rotates $\frac{\pi}{2}$)

b) $\vec{q}(t) = R\,\hat{r}(\theta(t))$, $\dot{\vec{q}}, \ddot{\vec{q}}$?

Polaric way $\dot{\vec{q}}(\theta(t)) = \frac{d\vec{q}}{d\theta}\cdot\frac{d\theta}{dt} = \frac{d}{d\theta}(R\cdot\hat{r}(\theta))\cdot\dot{\theta} = R\hat{\theta}\dot{\theta} = R\dot{\theta}\hat{\theta}.$

$\ddot{\vec{q}}(\theta(t)) = (R\dot{\theta}\hat{\theta})^{\cdot} = R\left(\ddot{\theta}\hat{\theta} + \dot{\theta}\left(\dot{\hat{\theta}}(t)\right)\right) = R\left(\ddot{\theta}\hat{\theta} + \dot{\theta}\frac{d\hat{\theta}}{d\theta}\cdot\frac{d\theta}{dt}\right) =$

$= R(\ddot{\theta}\hat{\theta} + \dot{\theta}(-\hat{r})\dot{\theta}) = R(\ddot{\theta}\hat{\theta} - (\dot{\theta})^2\hat{r}) = R\ddot{\theta}\hat{\theta} - R(\dot{\theta})^2\hat{r}.$

Cartesian way $\vec{q}(\theta(t)) = \begin{pmatrix} R\cos\theta(t) \\ R\sin\theta(t)\end{pmatrix}$, $\dot{\vec{q}}(\theta(t)) = \begin{pmatrix} -R\sin\theta(t)\cdot\dot{\theta} \\ R\cos\theta(t)\cdot\dot{\theta}\end{pmatrix} = R\dot{\theta}\hat{\theta}$ exactly

$\ddot{\vec{q}}(\theta(t)) = R\begin{pmatrix} -\cos\theta(t)\cdot(\dot{\theta})^2 \boxed{-\sin\theta(t)\,\ddot{\theta}} \\ -\sin\theta(t)(\dot{\theta})^2 \boxed{+\cos\theta(t)\,\ddot{\theta}}\end{pmatrix} = \underbrace{-R(\dot{\theta})^2\hat{r} + R\ddot{\theta}\hat{\theta}}_{\text{same}}.$

split

c) $\theta(t) = \omega t$, $\dot{\theta} = \omega$, $\ddot{\theta} = 0$.

$\ddot{\vec{q}} = \underbrace{R\cdot 0\cdot\hat{\theta}}_{0} - R(\underbrace{\dot{\theta}}_{\omega})^2\hat{r} = -\omega^2 R\,\hat{r}$

$\ddot{\vec{q}}\cdot\dot{\vec{q}} = R\dot{\theta}\hat{\theta}\cdot(-\omega^2\hat{R}) = -\omega^2 R\dot{\theta}\,(\underbrace{\hat{\theta}\cdot\hat{R}}_{0}) = 0 \to$ orthogonal

Abs. value of velocity.

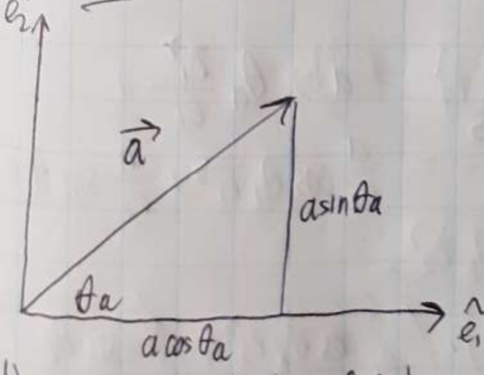
$\frac{d(\vec{v}^2)}{dt} = 2\vec{v}\cdot\frac{d\vec{v}}{dt} = \underbrace{2\dot{\vec{q}}\cdot\ddot{\vec{q}}}_{0} = 0$, $(\vec{v})^2 = v^2$ const, $\boxed{v \text{ const.}}$

$|\ddot{\vec{q}}| = \omega^2 R$ also const,

$|v| = |R\dot{\theta}\underbrace{\hat{\theta}}_{1}| = R|\omega| = \sqrt{\omega^2 R^2} = \sqrt{|\ddot{\vec{q}}|R}$,

$|\ddot{\vec{q}}| = \frac{v^2}{R}$, all const.

Problem 4.3 In. product



a) Let $\vec{b} = b\cos\theta_b\,\hat{e}_1 + b\sin\theta_b\,\hat{e}_2$,

$\vec{a}\cdot\vec{b} = (a\cos\theta_a\,\hat{e}_1 + a\sin\theta_a\,\hat{e}_2)\cdot(b\cos\theta_b\,\hat{e}_1 + b\sin\theta_b\,\hat{e}_2) =$ [use distributivity, and $\hat{e}_i\hat{e}_j = \delta_{ij}$]

$= ab\cos\theta_a\cos\theta_b + ab\sin\theta_a\theta_b =$

$= ab\cos(\theta_a - \theta_b)$, [formula for cos we covered when Euler identity & isomorphism on $\mathbb{C}$]

b) $\vec{a} = \begin{pmatrix} a\cos\theta_a \\ a\sin\theta_a\end{pmatrix} = \begin{pmatrix} a_1 \\ a_2\end{pmatrix}$

then $\vec{a}\cdot\vec{b} = \underline{a_1 b_1 + a_2 b_2}$.

$\vec{b} = \begin{pmatrix} b\cos\theta_b \\ b\sin\theta_b\end{pmatrix} = \begin{pmatrix} b_1 \\ b_2\end{pmatrix}$, Since does not depend, it is enough to calculate $\sum_{i=1}^{2} a_i b_i$ in any orthn. basis, and then $\sum a_i^{(e)} b_i^{(e)} = \sum a_i^{(n)} b_i^{(n)}$ etc.

c) Let $\theta_{ai} = \angle(\vec{a}, \hat{e}_i)$, $\theta_{bi} = \angle(\vec{b}, \hat{e}_i)$

$$\vec{a}\cdot\vec{b} = \sum_{i=1}^{D}(a\cos\theta_{ai}\cdot\hat{e}_i)\sum_{i=1}^{D}(b\cos\theta_{bi}\,\hat{e}_i) = \sum_{i,j=1}^{D} ab\cos\theta_{ai}\cos\theta_{bi}\,\delta_{ij}$$

$$= \sum_{i=1}^{D} ab\cos\theta_{ai}\cos\theta_{bi}\,.$$

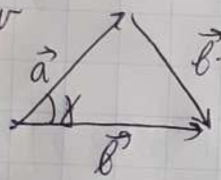
Denoting $a\cos\theta_{ai} = a_i$,
$b\cos\theta_{bi} = b_i$,

$$\vec{a}\vec{b} = \begin{pmatrix} a_1 \\ \vdots \\ a_D \end{pmatrix}\cdot\begin{pmatrix} b_1 \\ \vdots \\ b_D \end{pmatrix} = \sum_{i=1}^{D} a_i b_i.$$

d) Special bases. (show on $\mathbb{R}^2$ example)

See Chapter 2 exploration p.35

I rely that on $\mathbb{R}^D$ always $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\angle(\vec{a},\vec{b})$, from cosine Law



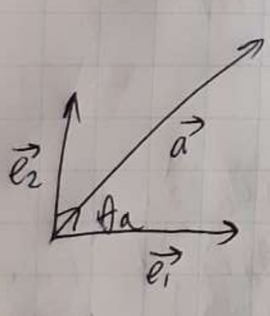
$|(\vec{b}-\vec{a})|^2 = (\vec{b}-\vec{a})^2 = \vec{b}^2 + \vec{a}^2 - 2\vec{a}\vec{b} =$
$a^2 + b^2 - 2\vec{a}\vec{b}$,

but also
$a^2 + b^2 - 2ab\cos\gamma$, $\vec{a}\vec{b} = ab\cos\gamma$.

<u>Orthogonal, not orthonormal basis!</u>

$\exists\,\hat{e}_i, \hat{e}_j$, $\hat{e}_i\hat{e}_j = \delta_{ij}$,
$\vec{e}_1 = e_1\cdot\hat{e}_1$, $\vec{e}_2 = e_2\cdot\hat{e}_2$

our basis



$$\vec{a} = a\cos\theta_a\cdot\hat{e}_1 + a\sin\theta_a\,\hat{e}_2 = a\cos\theta_a\frac{\vec{e}_1}{e_1} + a\sin\theta_a\frac{\vec{e}_2}{e_2} = \frac{a}{e_1}\cos\theta_a\vec{e}_1 + \frac{a}{e_2}\sin\theta_a\vec{e}_2,$$

$$\vec{b} = \frac{b}{e_1}\cos\theta_b\vec{e}_1 + \frac{b}{e_2}\sin\theta_b\vec{e}_2.$$

Then $\vec{a}\cdot\vec{b} = \left(\frac{a}{e_1}\cos\theta_a\vec{e}_1 + \frac{a}{e_2}\sin\theta_a\vec{e}_2\right)\cdot\left(\frac{b}{e_1}\cos\theta_b\vec{e}_1 + \frac{b}{e_2}\sin\theta_b\vec{e}_2\right) =$ 0

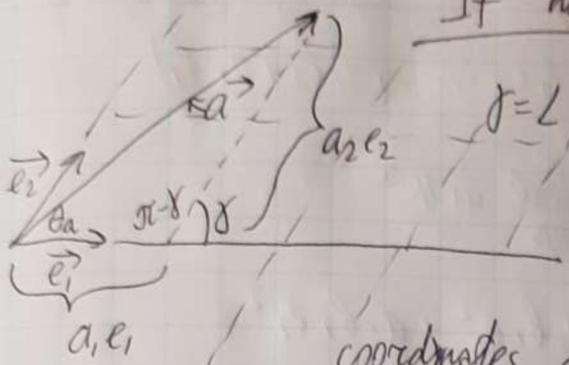
$$= \frac{a}{\cancel{e_1}}\cos\theta_a\cdot\frac{b}{\cancel{e_1}}\cos\theta_b\cdot\cancel{e_1}^2 + \frac{a}{\cancel{e_2}}\sin\theta_a\cdot\frac{b}{\cancel{e_2}}\sin\theta_b\,\cancel{e_2}^2 = ab\cos(\theta_a - \theta_b)$$

same as in any

basis! Denote $a_1 = \frac{a\cos\theta_a}{e_1}$, $a_2 = \frac{a\sin\theta_a}{e_2}$, then

$\vec{a}\vec{b} = a_1 b_1 e_1^2 + a_2 b_2 e_2^2$.

If not orthogonal b.



$\gamma = \angle(\vec{e}_1, \vec{e}_2)$

Sine law:

$$\frac{a}{\sin(\pi-\gamma)} = \frac{a_2 e_2}{\sin\theta_a} = \frac{a_1 e_1}{\sin(\gamma-\theta_a)} \to$$

coordinates $a_1 = \frac{a\sin(\gamma-\theta_a)}{(\sin\gamma)e_1}$, $a_2 = \frac{a\sin\theta_a}{(\sin\gamma)e_2}$.

Special cases: $\gamma = \frac{\pi}{2}$ (orthogonal),

$a_1 = \frac{a\cos\theta_a}{e_1}$, $a_2 = \frac{a\sin\theta_a}{e_2}$ ✓

$\gamma = \frac{\pi}{2} \wedge e_1 = e_2 = 1$ (orthonormal)

$a_1 = a\cos\theta_a$, $a_2 = a\sin\theta_a$ ✓

General formula for inner product:

$$\vec{a}\vec{b} = (a_1\vec{e}_1 + a_2\vec{e}_2)\cdot(b_1\vec{e}_1 + b_2\vec{e}_2) = \boxed{a_1 b_1 e_1^2 + a_2 b_2 e_2^2 + e_1 e_2\cos\gamma\cdot(a_1 b_2 + a_2 b_1)} =$$

= (now check its value is same) $= \frac{a\sin(\gamma-\theta_a)}{e_1\sin\gamma}\cdot\frac{b\sin(\gamma-\theta_b)}{e_1\sin\gamma}e_1^2 +$

$$+ \frac{a\sin\theta_a}{e_2\sin\gamma}\cdot\frac{b\sin\theta_b}{e_2\sin\gamma}e_2^2 + e_1 e_2\cos\gamma\left(\frac{a\sin(\gamma-\theta_a)}{e_1\sin\gamma}\cdot\frac{b\sin\theta_b}{e_2\sin\gamma} + \frac{a\sin\theta_a}{e_2\sin\gamma}\cdot\frac{b\sin(\gamma-\theta_b)}{e_1\sin\gamma}\right) = ab\left[\frac{\sin(\gamma-\theta_a)\sin(\gamma-\theta_b)}{\sin^2\gamma} + \frac{\sin\theta_a\sin\theta_b}{\sin^2\gamma} + \frac{\cos\gamma\sin(\gamma-\theta_a)\sin\theta_b}{\sin^2\gamma} + \frac{\cos\gamma\sin\theta_a\sin(\gamma-\theta_b)}{\sin^2\gamma}\right] =$$

must be $\cos(\theta_a-\theta_b)$!

$$= \frac{ab}{\sin^2\gamma}\Big[(\sin\gamma\cos\theta_a - \cos\gamma\sin\theta_a)(\sin\gamma\cos\theta_b - \cos\gamma\sin\theta_b) + \sin\theta_a\sin\theta_b + \cos\gamma\cdot(\sin\gamma\cos\theta_a - \cos\gamma\sin\theta_a)\sin\theta_b + \cos\gamma\sin\theta_a(\sin\gamma\cos\theta_b - \cos\gamma\sin\theta_b)\Big] =$$

$$= \frac{ab}{\sin^2\gamma}\Big[\underline{\sin^2\gamma\cos\theta_a\cos\theta_b} - \cancel{\sin\gamma\cos\gamma\cos\theta_a\sin\theta_b} - \cancel{\sin\gamma\cos\gamma\cos\theta_b\sin\theta_a} + \cos^2\gamma\sin\theta_a\sin\theta_b + \sin\theta_a\sin\theta_b + \cancel{\cos\gamma\sin\gamma\cos\theta_a\sin\theta_b} - \cancel{\cos^2\gamma\sin\theta_a\sin\theta_b} + \cancel{\cos\gamma\sin\gamma\sin\theta_a\cos\theta_b} - \cos^2\gamma\sin\theta_a\sin\theta_b\Big] = \frac{ab}{\sin^2\gamma}\Big[\cancel{\sin^2\gamma}(\cos\theta_a\cos\theta_b + \sin\theta_a\sin\theta_b)\Big] =$$

$\to (1-\cos^2\gamma)\sin\theta_a\sin\theta_b$

$= ab\cos(\theta_a - \theta_b) \rightarrow$ still some number!
(calculation are terrible, do not know how to simplify)

In coordinates formula is not scalar product but (see in start):

$$\underbrace{\vec{a}\vec{b} = ab\cos(\theta_a - \theta_b)}_{\text{always in } \mathbb{R}^D} = \underbrace{a_1 b_1 e_1^2 + a_2 b_2 e_2^2 + e_1 e_2 \cos\gamma\,(a_1 b_2 + a_2 b_1)}_{\text{~~general~~ always in } \mathbb{R}^2}$$

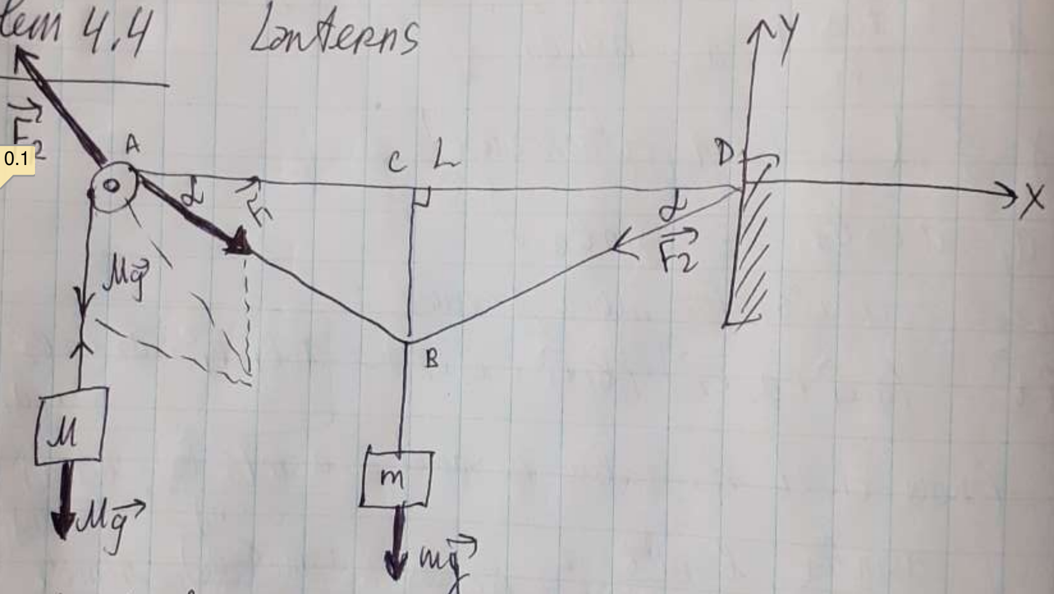
Special case:

$\gamma = \frac{\pi}{2}$ (orthogonal); $\vec{a}\vec{b} = a_1 b_1 e_1^2 + a_2 b_2 e_2^2$

$\gamma = \frac{\pi}{2}$, $e_1 = e_2 = 1$; $a_1 b_1 + a_2 b_2$ (orthonormal)

Problem 4.4 Lanterns

a)



~~Important for all tasks:~~

~~Lantern~~ Roll is stationary, so

b) $\vec{F}_R + M\vec{g} + \vec{F}_1 = \vec{0}$, $\qquad \vec{F}_R = -(M\vec{g} + \vec{F}_1)$

($M\vec{g}$: transmitted from M)

c) Since m is on the middle, triangles ABC and DBC are equal, $\angle A = \angle D = \alpha$.

$\underline{F_1 = F_2}$ $\xrightarrow{\text{reason 1}}$ $m\vec{g} + \vec{F}_1 + \vec{F}_2 = \vec{0}$,

$mg_x + F_{1x} + F_{2x} = 0$,

$0 + F_1\cos\alpha - F_2\cos\alpha = 0 \rightarrow F_1 = F_2$

reason 2: from symmetry, if we

look at the picture from other side, it looks same. There is no way to distinguish F_1 and F_2.

Further, $F_1 = F_2 = F$, act on same rope and must equal Mg (mass of rope/wire not given, can neglect and it acts only as transmitter).

Then:

c.1) $2F\sin\alpha = 2Mg\sin\alpha = mg$, (equilibr. for lantern)

$$2M\sin\alpha = m, \quad \boxed{\alpha = \arcsin\left(\frac{m}{2M}\right)}.$$

c.2) Equilibr. for roll: $F_{Ry} = Mg + \underbrace{Mg}_{F_1}\cos\left(\frac{\pi}{2}-\alpha\right) = Mg + Mg\sin\alpha = Mg(1+\sin\alpha)$,

$$F_{Rx} = -F_1\cos\alpha = -Mg\cos\alpha.$$

Then $|\vec{F_R}| = \sqrt{F_{Rx}^2 + F_{Ry}^2} = Mg\sqrt{\cos^2\alpha + (1+\sin\alpha)^2} = Mg\sqrt{2+2\sin\alpha} = \boxed{\sqrt{2}Mg\sqrt{1+\sin\alpha}}$

(special cases: $\alpha = \frac{\pi}{2}$ F_R $F_{Ry} = 2Mg$, $F_{Rx} = 0$

Mg $\alpha = 0$, $F_{Ry} = Mg$ ~~$2\dots$~~ ✓

$F_{Rx} = -Mg$)

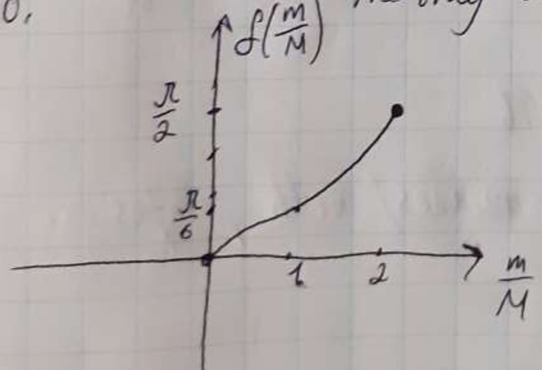
Actual values:

$$\alpha = \arcsin\left(\frac{15\,kg}{2\cdot 80\,kg}\right) = \arcsin\left(\frac{3\,kg}{32\,kg}\right) \approx \arcsin\frac{1}{10}, \text{ almost horizontal}$$

$$|\vec{F_R}| = \sqrt{2}\cdot 80\,kg\cdot 10\,\frac{N}{kg}\cdot\sqrt{1+\frac{15}{160}} \approx 800\sqrt{2}\,N \approx \text{weight of big man.}$$

d) $f = 1\arcsin\left(\frac{m}{2M}\right)$, – it should be dimensionless, and this is the only way with given data, up to f, and $\frac{1}{2}$, and 1.

≥ 0.

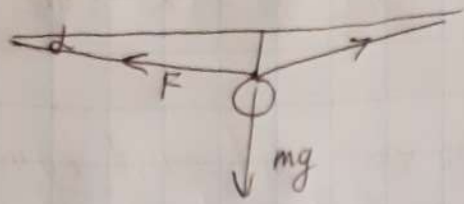


If $m > 2M$, it should break.

e) $F_{max} = Mg_{max} = 14\,kN$, $m_{max} = 2M_{max} = 2\frac{F_{max}}{g} = \frac{2\cdot 14\cdot 10^3\,N}{g} = 28\cdot 10^2\,kg \approx 3$ tonn.

f becomes $\frac{\pi}{2}$.

f)



$mg = 2F\sin\alpha \approx 2\alpha F$ for small angles,

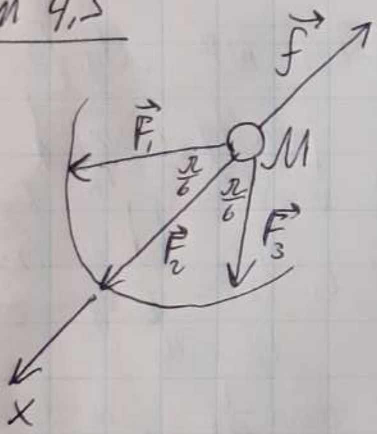
then $F = \frac{mg}{2\alpha}$ ↑↑ becomes $> F_{max}$ described above, and ropes break.

With the roll we can adjust α, remedying what was described above.

I promise I will try to do experiment for d, but very unhandy I did not find suitable tools yet.

— Self Test

Problem 4.5



Max. force will be exactly at required angle.

a) $F + 2F\cdot\frac{\sqrt{3}}{2} = (\sqrt{3}+1)F$ total force, ≈ 3 times more

b) $(\sqrt{3}+1)F = \mu Mg$,

$\mu = \frac{(\sqrt{3}+1)F}{Mg}$

1 cwt = 8 st = 8·14 lb ≈ 56 kg

Actual values $\mu \approx \frac{3\cdot 300\cdot\frac{1}{2}\cdot\cancel{10}\,\cancel{N}}{1120\,\cancel{kg}\cdot\cancel{10}\,\frac{\cancel{km}}{s^2}} \approx \frac{450}{1120} = \frac{45}{112} =$ [illegible] $\approx \frac{3}{7}$.

Bonus problem – problem 4.6

To discuss on seminars / with professor.

Index of comments

6.1 4.2: 11.5/12

6.2 3.5/4

7.1 4/4

7.2 4/4

8.1 Good try, perhaps think about it in these lines :
Let the first two basis vectors, eˆ1 and eˆ2, span this plane. Then the coordinates of all other vectors are zero, and the same expressions are found as in the 2D case.
5/5 +1 bonus :)
Also nice plots in the earlier exercises :D

10.1 4.4a: 4/4

10.2 4.4b: 3/3

11.1 4.4c: 4/4

11.2 4.4d: 3/3