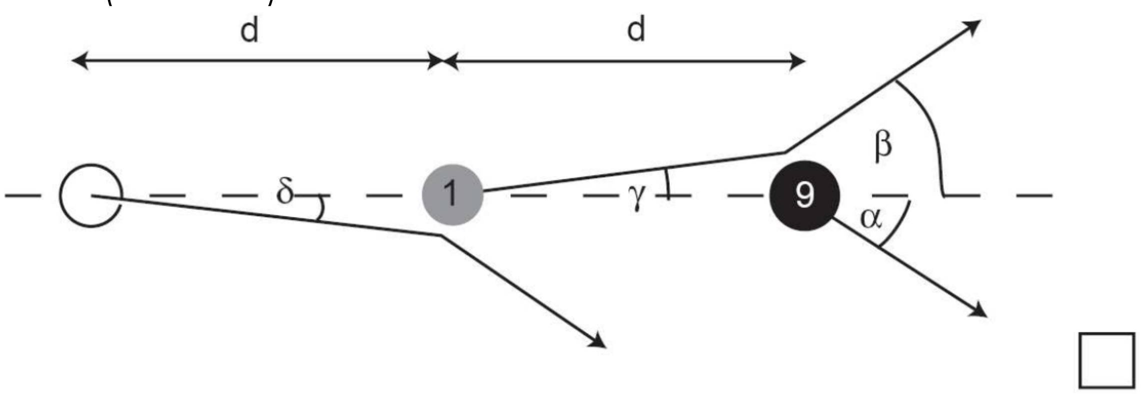
**Problem 1: Billiard Collision "9-Ball" 4 Points**

Three billiard balls lie on a straight line with the distance $d$ between the balls each. The first (white) ball is shot, hits the second ball (No. 1) which in turn strikes the third (No. 9). For the last ball (No. 9) to be pocketed the angle must have a value $\alpha = 30° \pm 2°$. The player decides to shoot the first ball under an angle $\delta = 1°$ $(d = \ 40 \text{ cm})$. Is this successful?




*(Hints. The angles $\delta$ and $\gamma$ are very small such that the following approximation can be used: $\gamma' \cong \sin\gamma$ and $\delta' \cong \sin\delta$. All angles have to be calculated in radians. Neglect friction forces.)*

Solution

The impact parameter for the white billiard ball hitting No. 1 is

$$b_1 = d \sin\delta \simeq d$$

This yields a scattering angle $\gamma'$ of ball No. 1 with respect to the line of the incoming white ball:

$$\sin\gamma' = \frac{b_1}{2R}$$

$$\gamma' \simeq \frac{b_1}{2R} \simeq \frac{d\delta}{2R}$$

since the incoming white ball moves under an angle δ with respect to the base line joining the three billiard balls at rest, the scattering angle γ of ball No. 1 with respect to the base line is given by

$$\gamma = \gamma' - \delta$$

$$\simeq \left(\frac{d}{2R} - 1\right)\delta = 5.67°$$

$$\gamma = \arcsin\left(\frac{d\sin\delta}{2R}\right) - \gamma$$

where the correct solution has also been given for completeness.

The impact parameter of ball No. 1 hitting ball No. 9 is given by

$$b_9 = d\sin\gamma \simeq d\gamma \simeq d\left(\frac{d}{2R} - 1\right)$$

Therefore, the scattering angle $\alpha'$ of ball No. 9 with respect to the line of motion of ball No. 1 is given by

$$\sin\alpha' = \frac{b_9}{2R} \simeq \frac{d}{2R}\left(\frac{d}{2R}-1\right)\delta$$

$$\alpha' \simeq \arcsin\left(\frac{d}{2R}\left(\frac{d}{2R}-1\right)\delta\right) = 41.25°$$

The scattering angle $\alpha$ of ball No. 9 with respect to the base line is

$$\alpha = \alpha' - \gamma = 35.6°$$

Therefore, the billiard player is unsuccessful. For completeness the exact solution for $\alpha'$ is given:

$$\alpha' = \arcsin\left[\left(\frac{d}{2R}\right)\sin\left(\arcsin\left(\frac{d\sin\delta}{2R}\right)-\delta\right)\right]$$

Comparison shows that the approximations are very good, i.e. the inaccuracy in the final angle is not more than $0.2°$.

**Problem 2: 3D Motion, Bottle out of Train** $\mathbf{2+2+2}$ **Points**

A bottle is thrown horizontally from a train at a right angle to the train's velocity. It hits the ground at a point that is $5\mathrm{m}$ below, $10\mathrm{m}$ afar (in perpendicular direction) and $25\mathrm{m}$ afar (in the direction along the velocity of the train) from the point of throw. (Air resistance is neglected)

a. Calculate the speed of the train,
a. the initial velocity of the bottle,
b. the velocity of the bottle as it hits the ground.

<u>Solution</u>

(a)

Initial velocity: $\vec{v_0} = \begin{pmatrix} v_{0x} \\ v_{0y} \\ 0 \end{pmatrix}$

Initial position: $\vec{r_0} = \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix}$

Final position: $\vec{r_0} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$

Constant acceleration $\vec{g}$, so one derives:

$$\vec{v} = \vec{v}_0 - g\hat{e}_z t = \begin{pmatrix} v_{0x} \\ v_{0y} \\ -gt \end{pmatrix}$$

$$\vec{r} = \vec{r_0} + \vec{v_0}t - \frac{1}{2}g\hat{e}_z t^2 = \begin{pmatrix} v_{0x}t \\ v_{0y}t \\ z_0 - \frac{1}{2}gt^2 \end{pmatrix}$$

So, the time of the fall is:

$$t_1 = \sqrt{\frac{2z_0}{g}}$$

Thus, the initial velocity is:

$$v_{0x} = \frac{x_1}{t_1} = \sqrt{\frac{gx_1^2}{2z_0}} = 24.76\frac{\mathrm{m}}{\mathrm{s}} = 89.14\frac{\mathrm{km}}{\mathrm{h}}$$

(b)

$$v_{0y} = \frac{y_1}{t_1} = \sqrt{\frac{gy_1^2}{2z_0}} = 9.90\frac{\mathrm{m}}{\mathrm{s}} = 35.66\frac{\mathrm{km}}{\mathrm{h}}$$

$v_{0z} = 0$, thus

$$v_0 = \sqrt{v_{0x}^2 + v_{0y}^2 + v_{0z}^2} = 26.66\frac{\mathrm{m}}{\mathrm{s}} = 96.00\frac{\mathrm{km}}{\mathrm{h}}$$

(c)

$$v_0 = \sqrt{v_{0x}^2 + v_{0y}^2 + (gt_1)^2} = \sqrt{\frac{gx_1^2}{2z_0} + \frac{gy_1^2}{2z_0} + 2z_0 g} = \sqrt{\frac{g}{2z_0}}\sqrt{x_1^2 + y_1^2 + 4z_0^2}$$

$$= 28.44\frac{\mathrm{m}}{\mathrm{s}} = 102.4\frac{\mathrm{km}}{\mathrm{h}}$$

**Problem 3: Galilei Transformation 3 Points**

In an inertial system S a lamp is shining at coordinates $(x, y, z)$ = (940m, 236m, 204m) and time $t$ = 773s. What coordinates will someone prescribe to the lamp who passes by the system S at a constant speed of $(vx, vy, vz)$ = (2 $m$ $s$ , 1 $m$ $s$ , 0)? ?

gallilei transf.: $x' = x \pm v \cdot t$ , $t' = t$

$y' = y \pm v \cdot t$

$z' = z \pm v \cdot t$

$$S = \begin{pmatrix} 940 \\ 236 \\ 204 \end{pmatrix} m \quad v = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \frac{m}{s} \Rightarrow S' = \begin{pmatrix} 940 + 2 \cdot t \\ 236 + 1 \cdot t \\ 204 + 0 \cdot t \end{pmatrix} m$$

$$S' = \begin{pmatrix} 940+26 \\ 236+16 \\ 204 \end{pmatrix} m$$

**Problem 4: Galilei Invariance** 3 Points

Laplace: $\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$

transformation: $f(x,t) \to f(x', t') = f(x+v\cdot t \cdot t') = f(x+v\cdot t, t)$

shows only for $x$ (y & z works the same)

$$\to \frac{\partial^2}{\partial x^2} f = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'}\underbrace{\left(\frac{\partial}{\partial x}[x+v\cdot t]\right)}_{=1} + \underbrace{\frac{\partial f}{\partial t'}\cdot\underbrace{\frac{\partial}{\partial x}[\overset{=t'}{t}]}_{=0}}_{=0}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'}\right) + 0$$

$$= \frac{\partial^2 f}{\partial x'^2}\underbrace{\frac{\partial x'}{\partial x}}_{=1} + \frac{\partial^2 f}{\partial t'\partial x'}\underbrace{\frac{t'}{\partial x}}_{0}$$

$$= \frac{\partial^2 f}{\partial x'^2} \quad = \frac{\partial^2 f}{\partial x^2}$$