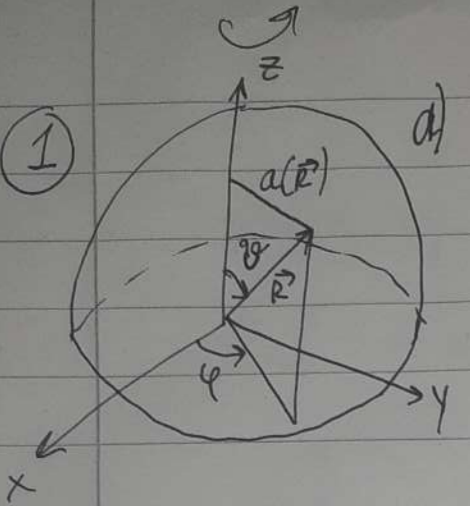
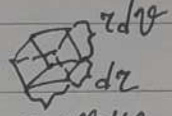
HW9 – Rotations Stanislav



① a) $\dot{I} = \int_V \underbrace{(a(\vec{R}))^2}_{\text{distance to } OZ} dm = \left[\begin{matrix}\text{const.}\\ \text{density}\end{matrix}\right] = \rho\int_V a^2 dV.$

In spherical coordinates 

$$dV = \underbrace{r^2}_{[M^2]}\sin\vartheta\, \underbrace{dr}_{[M]}\, d\vartheta\, d\varphi, \quad \vec{R} = \begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} r\sin\vartheta\cos\varphi\\ r\sin\vartheta\sin\varphi\\ r\cos\vartheta\end{bmatrix},$$

$$a^2 = x^2+y^2 = r^2\sin^2\vartheta \Longrightarrow$$

$$\dot{I} = \rho\int r^2\sin^2\vartheta\, dV = \rho\int_{r=0}^{R}\int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi} r^2\sin^2\vartheta\, r^2\sin\vartheta\, dr\, d\vartheta\, d\varphi =$$

$$= \rho\int_{r=0}^{R}\int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi} r^4\sin^3\vartheta\, d\vartheta\, d\varphi\, dr = \rho\int_0^R r^4 dr\int_{\varphi=0}^{2\pi} d\varphi\int_{\vartheta=0}^{\pi}\sin^3\vartheta\, d\vartheta =$$

$$= \rho\cdot\frac{R^5}{5}\cdot 2\pi\cdot\int_0^{\pi}\sin\vartheta(1-\cos^2\vartheta)\,d\vartheta = \begin{bmatrix} u=\cos\vartheta\\ du=-\sin\vartheta\, d\vartheta\end{bmatrix} = \frac{2}{5}\rho\pi R^5\int_1^{-1}(u^2-1)\,du =$$

$$= \frac{2}{5}\rho\pi R^5\cdot\left[\frac{u^3}{3}-u\right]\Big|_1^{-1} = \frac{2}{5}\rho\pi R^5\left[\frac{-1}{3}+1-\frac{1}{3}+1\right] = \frac{2}{5}\rho\pi R^5\cdot\frac{4}{3} =$$

$$= \left[\rho = \frac{M}{\frac{4}{3}\pi R^3}\right] = \frac{\frac{2}{5}MR^2\cdot\cancel{\frac{4}{3}\pi R^3}}{\cancel{\frac{4}{3}\pi R^3}} = \frac{2}{5}MR^2.$$

b) Assume 4 small spheres have negative mass: $\left(-\frac{M}{4^3}\right)$

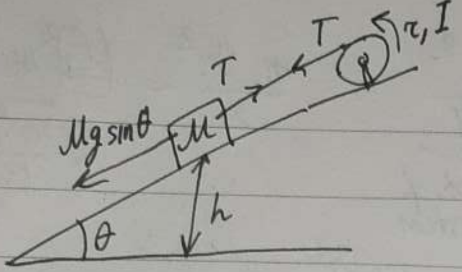
$$I_y = I_{(\text{all sphere})y} + 2I_{(\text{top})y} + 2I_{(\text{side})y} = \frac{2}{5}MR^2 + 2\cdot\frac{2}{5}\left(\frac{R}{4}\right)^2\cdot\frac{(-M)}{4^3} +$$

$$+2\cdot\Big[\underbrace{\frac{2}{5}\left(\frac{R}{4}\right)^2\frac{(-M)}{4^3}}_{\text{around CM of small}} + \underbrace{\frac{(-M)}{4^3}\cdot\left(\frac{3}{4}R\right)^2}_{\text{distance to } Oy}\Big] = MR^2\cdot\Big[\frac{2}{5} + 2\cdot\frac{2}{5}\cdot\frac{1}{4^2}\cdot\frac{(-1)}{4^3} +$$

$$+2\cdot\left(\frac{2}{5}\cdot\frac{1}{4^2}\cdot\frac{(-1)}{4^3} + \frac{(-1)}{4^3}\cdot\left(\frac{3}{4}\right)^2\right)\Big] = MR^2\left[\frac{2}{5} + \frac{-4}{5\cdot4^5} + 2\cdot\left(\frac{-2}{5\cdot4^5} - \frac{9}{4^5}\right)\right] =$$

$$= MR^2\cdot\frac{2\cdot4^5-4-4-18\cdot5}{5\cdot4^5} = MR^2\,\frac{2048-8-90}{5\cdot1024} = MR^2\cdot\frac{1950}{5120} = \frac{195}{512}MR^2.$$

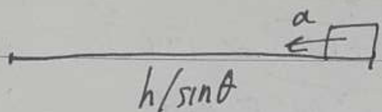
(2)



1) $$\begin{cases} Mg\sin\theta - T = Ma & \text{(N. 2nd law)} \\ \mathcal{E}\cdot I = T\cdot r & \text{(N. 2nd law for rotation)} \\ \uparrow \text{ ang. acceleration} & \\ a = \mathcal{E}\cdot r & \text{(kinematik connection for rope)} \end{cases}$$

$$\Longrightarrow Mg\sin\theta - \frac{a}{r}\cdot\frac{I}{r} = Ma,$$

$$a\left(M + \frac{I}{r^2}\right) = Mg\sin\theta \Longrightarrow a = \frac{Mg\sin\theta}{M + \frac{I}{r^2}}$$ ($\to g\sin\theta$ ~~when block~~ wheel has no mass)

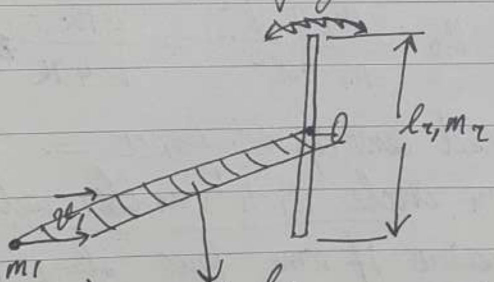
2) 

$$\frac{h}{\sin\theta} = \frac{at^2}{2} \Longrightarrow$$

$$t = \sqrt{\frac{2h}{a\sin\theta}} = \sqrt{\frac{2h\left(M + \frac{I}{r^2}\right)}{Mg\sin^2\theta}} = \sqrt{\frac{2h\left(M + \frac{I}{r^2}\right)}{Mg\sin^2\theta}} =$$

$$= \sqrt{\frac{2h}{g\sin^2\theta}\left[1 + \frac{I}{Mr^2}\right]},$$ ← time to fall if no friction

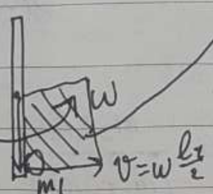
(3)



a) Angular momentum is conserved around O (any forces through axle $\to \tau = 0$; others are internal; no gravity in plane)

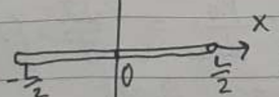
1) $m_1 v_1 \frac{l_r}{2} = \dot{I}\omega$

(ang. mom. of point m_1) (ang. mom. of system in rotat. terms)



$\dot{I}$ = total mom. of inertia around O =

$= I_{rod_0} + I_{m_{1_0}}$.

2) 

for a thin rod:

$$I = \rho\int_{-L/2}^{L/2} x^2\,dx = \rho\,\frac{x^3}{3}\Big|_{-L/2}^{L/2} = \frac{\rho}{3}\cdot 2\cdot x^3\Big|_0^{L/2} =$$

$$= \frac{2\rho}{3}\cdot\frac{L^3}{8} = \frac{\rho L^3}{12} = \underline{\frac{ML^2}{12}}.$$

3) So $$m_1 v_1 \frac{l_r}{2} = \Big[\underbrace{\frac{M_r l_r^2}{12}}_{\text{rod}} + \underbrace{m_1\left(\frac{l_r}{2}\right)^2}_{\text{bullet}}\Big]\cdot\omega \Longrightarrow$$

$$\omega = \frac{m_1 v_1 \frac{l_r}{2}}{\frac{M_r l_r^2}{12} + m_1\left(\frac{l_r}{2}\right)^2} = \frac{0.01\,kg\cdot 200\,m\cdot s^{-1}\cdot 0.2\,m}{\frac{1\,kg\cdot 0.16\,m^2}{12} + 0.01\,kg\cdot 0.04\,m^2} \approx 29\,s^{-1}$$ ($\simeq 4.6\,Hz$)

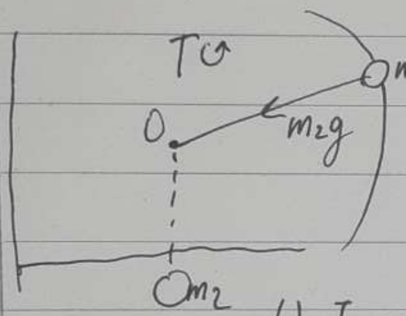
b) $\underbrace{\Delta E}_{\text{dissipated}} = E_1 - E_2 = \underbrace{\frac{m_1 v_1^2}{2}}_{\text{kinetik of bullet}} - \underbrace{\frac{L^2}{2I}}_{\text{rotat. of system}} = \frac{m_1 v_1^2}{2} - \frac{\left(\frac{l_r}{2}\right)^2 m_1^2 v_1^2}{2\left(\frac{m_r l_r^2}{12} + \frac{m_1 l_r^2}{4}\right)} = \frac{m_1 v_1^2}{2} -$

$- \frac{m_1^2 v_1^2}{8\left(\frac{m_r}{12} + \frac{m_1}{4}\right)} = \frac{m_1 v_1^2}{2} - \frac{m_1^2 v_1^2}{\frac{2m_r}{3} + 2m_1} = \frac{m_1 v_1^2}{2} \cdot \left(1 - \frac{m_1}{\frac{m_r}{3} + m_1}\right) =$

$= \frac{m_1 v_1^2}{2} \cdot \underbrace{\left(\frac{m_r}{m_r + 3m_1}\right)}_{\text{dissip. fraction}}$

So $\Delta E = \frac{0.01\,kg \cdot 200^2\, m^2 s^{-2}}{2} \cdot \underbrace{\left(\frac{1}{1+0.03}\right)}_{\text{almost all } E \text{ lost}} \approx 194\, J.$

(4)



a) N. 2nd law:

$m_2 g = m_1 \omega_0^2 R_0 \Rightarrow R_0 = \frac{m_2 g}{m_1 \omega_0^2} = \boxed{\frac{m_2 g T^2}{m_1 \cdot 4\pi^2}} \approx$

$\approx \frac{2 \cdot 10 \cdot 9}{1 \cdot 4 \cdot 10} M = \underline{4.5 M}.$

b) Increase of m_2 — stronger centripetal force, m_1 will not be able to stay on „orbit" R_0 — so it will move closer, but since pulling tension is "central → angular momentum around O will be conserved! → $\omega_0 R_0^2 = \omega_{new} R_{new}^2$ always (no friction).

[also, changes E_k = -changes of E_p of m_2, E also cons.]

c) $\begin{cases} R_{new}^2 \omega_{new} = R_0^2 \omega_0 & \text{(A.M. conservation)} \\ (m_2 + \Delta m) g = m_1 \omega_{new}^2 R_{new} & \text{(2nd N. law for new system)} \end{cases}$

$\Downarrow$

$\begin{cases} R_{new}^4 \omega_{new}^2 = R_0^4 \omega_0^2 \\ m_1 \omega_{new}^2 R_{new} = (m_2 + \Delta m) g \end{cases}$

$\frac{R_{new}^3}{m_1} = \frac{R_0^4 \omega_0^2}{(m_2+\Delta m) g}, \quad R_{new} = \sqrt[3]{\frac{R_0^4 \omega_0^2 m_1}{(m_2+\Delta m) g}} = R_0 \sqrt[3]{\frac{m_2 \cancel{g}}{(m_2+\Delta m)\cancel{g}}} = R_0 \sqrt[3]{\frac{m_2}{m_2+\Delta m}} =$

$= \frac{R_0}{\sqrt[3]{1 + \frac{\Delta m}{m_2}}}$ (smaller) $= \frac{4.5\, M}{\sqrt[3]{1.5}} \approx 3.9 M$

And $\omega_{new} = \frac{R_0^2 \omega_0}{R_{new}^2} = \left(\frac{R_0}{R_{new}}\right)^2 \omega_0 = \left(1 + \frac{\Delta m}{m_2}\right)^{\frac{2}{3}} \omega_0$ (larger) $= (1.5)^{\frac{2}{3}} \cdot \frac{2\pi}{3} s^{-1} \approx 2.74 s^{-1}.$