Theoretical Mechanics IPSP

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13.1. Vectors, derivatives, and phase-space portraits

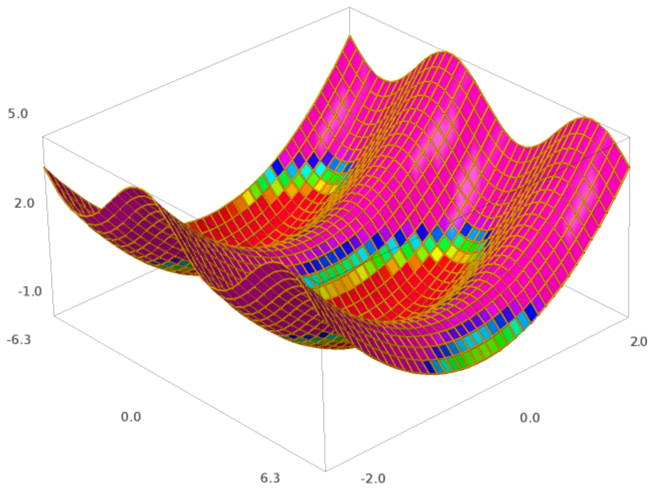
a) Contour lines

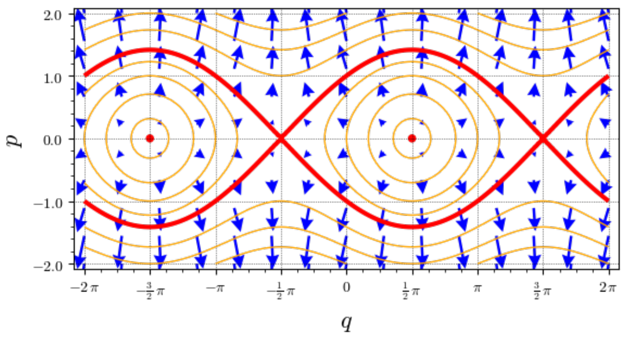
Contour lines in (q,p) are lines where a function $f(q,p)$ takes a constant value. Sketch the contour lines of

$$f(q,p) = \frac{p^2}{2} - \sin q \,,$$

in order to get an idea about the height profile of this function.

Solution:





The contours $f(q,p) = 1$ and the fixed points $f(q,p) = -1$ are marked in red. All other contour lines are marked by thin orange lines. The gradient of the function is indicated by blue arrows.

b) Conservative fields

A vector field $\mathbf{K}(x,y)$ is conservative if it can be written as a gradient of a potential $U(x,y)$. Which of the following vector fields are conservative:

$$\begin{aligned}
\mathbf{K}_1(x,y) &= (x+y, x+y) \\
\mathbf{K}_2(x,y) &= (x-y, x+y) \\
\mathbf{K}_3(x,y) &= \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)
\end{aligned}$$

Justify your answer!

Solution:

When $\mathbf{K} = -\nabla\Phi$ for some twice differentiable potential Φ, then

$$\partial_y K_x = -\partial_y \partial_x \Phi = -\partial_x \partial_y \Phi = \partial_x K_y$$

We will now test this condition of the given forces:

$$1. \qquad \begin{aligned} \partial_y K_{1,x} &= \partial_y (x+y) = 1 \\ \partial_x K_{1,y} &= \partial_x (x+y) = 1 \end{aligned}$$

Hence, $\mathbf{K}_1$ is conservative. Indeed, it can be written as $\mathbf{K}_1 = -\nabla \left(-\frac{(x+y)^2}{2} \right)$.

$$2. \qquad \begin{aligned} \partial_y K_{2,x} &= \partial_y (x-y) = -1 \\ \partial_x K_{2,y} &= \partial_x (x+y) = 1 \end{aligned}$$

Hence, $\mathbf{K}_2$ can not be conservative.

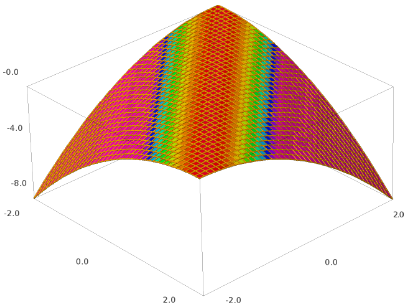
$$3. \qquad \begin{aligned} \partial_y K_{3,x} &= \partial_y \frac{x}{\sqrt{x^2+y^2}} &= -\frac{x\,y}{\left(x^2+y^2\right)^{3/2}} \\ \partial_x K_{3,y} &= \partial_x \frac{y}{\sqrt{x^2+y^2}} &= -\frac{x\,y}{\left(x^2+y^2\right)^{3/2}} \end{aligned}$$

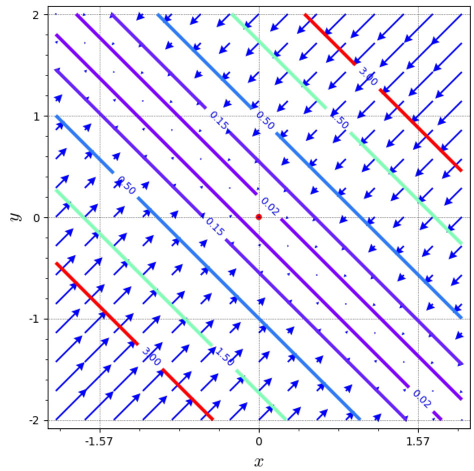
Hence, $\mathbf{K}_3$ is conservative. Indeed, it can be written as $\mathbf{K}_3 = -\nabla \left(-\sqrt{x^2+y^2} \right)$.

Sketch the contour lines and the gradients for a case where the field is conservative.

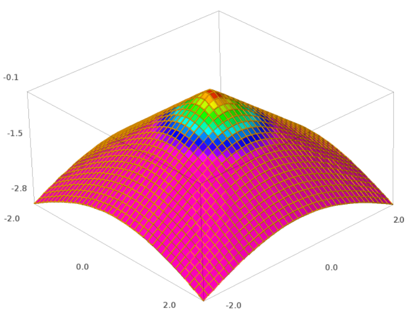
Solution:

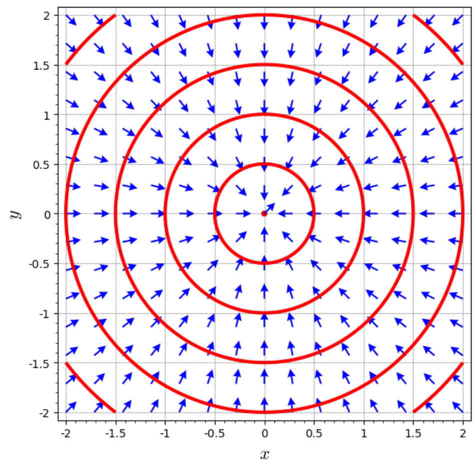
Potential $U_1(x, y) = -(x^2 + y^2)/2$





Potential $U_3(x, y) = -\sqrt{x^2 + y^2}$

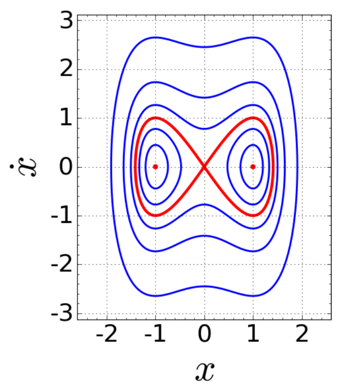


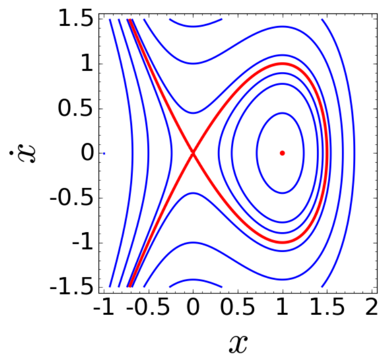


c) Interpretation of phase portraits

The graphs below show phase portraits of differential equations of the form

$$\ddot{x}(t) = a\,x + b\,x^2 + c\,x^3 \qquad \text{mit} \quad a, b, c \in \mathbb{R}$$





Discuss whether the respective constants a, b and c are positive, negative or whether they vanish.

Solution:

left

$b = 0$ due to the left-right symmetry
$a > 0$ since the origin is repulsive
$c < 0$ to have a bounded potential

right

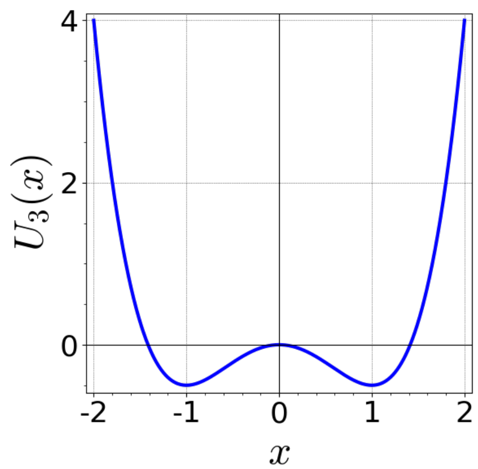
$a > 0$ since the origin is repulsive
$b < 0$ and $c = 0$ since potential approaches $-\infty$ at left and ∞ at right

d) Effective potentials

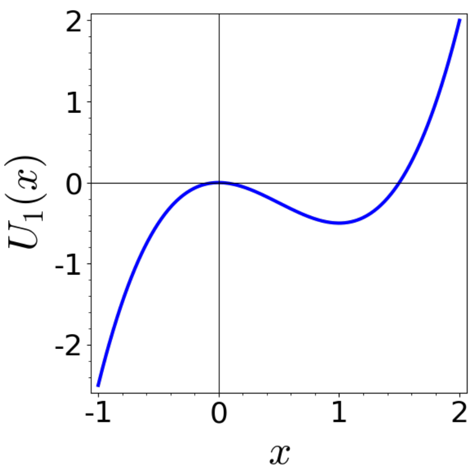
Interpret x as the position of a particle in a potential, and sketch the potentials that result in the given phase portraits.

Solution:

$a = 2, b = 0, c = -2$



$a = 0, b = 0, c = -1$



Discussion