Exam

Instructions for working on this exam

1. There are 120 min to execute the exam. In total there are 60 points + 30 bonus points. You pass the exam with 25 points. For the best grade, 1.0, you need 45 points. The points for the different sub-tasks are provided on the next page. If you want to go through the full exam, there will be about 1 min/point.

2. In the present exam there are two exercises that address skills that we need to solve physical problems, and subsequently you will discuss in some depth two physical problems.

3. The best strategy to attack the exam is to strive to collect as many points as possible. Do not try to solve the full exam. Work on things that you can solve — ignore anything where you get stuck.

4. Each exercise should be solved on a separate set of sheets. Please write neatly and leave space on the margin for remarks and indicating credits.

5. Bonus problems are marked by (*). Often they involve a tricky argument that might not be immediately obvious. I recommend that you first work on the other exercises. Only attack the *-problems in the end when there is still time, or when you immediately see a fast and straightforward solution.

6. Explain carefully what you are doing. What do you assume? What do you intend to show? We will only give points when we understand what you intend to do.

7. You may also use my lecture notes and one sheet of A4 paper with hand-written notes that you prepared to take the exam. You must not use other resources; in particular no calculators and algebra programs.

Declaration of independence (written examinations)

For the exam of the course:	Theoretical Physics I. Theoretical Mechanics
Name of the examiner:	Prof. Dr. Jürgen Vollmer
Date:	18 February 2022

Note: If unauthorized aids are used and/or the exam is not written independently or in the case of obvious cooperation with fellow students or obvious copying, this will be treated as an attempt to cheat!

I confirm with my signature that I have prepared the answers/solutions of the examination task(s) independently and have not used any unauthorized aids. I did not ask any other person for help in preparing the answers/solutions and I did not copy from the answers/solutions of other students.

name, given name

matriculation number

1 (a)	(b)	(c)	$*[d]$	$*[e]$	2 (a)	$*[b]$	(c)	$*[d]$
1	8	5	3	3	3	3	4	2

3 (a)	(b)	(c)	(d)	(e)	$*[f]$	4 (a)	(b)	(c)	(d)	(e)	(f)	$*[g]$
3	3	3	5	3	7	4	3_{+2}	4_{+2}	3	5	3_{+2}	6

total points

grade

Problem 1. Non-dimensionalization and phase-space portraits
We consider the EOM

$$m\,\ddot{x}(t) = a\,x(t) - b\,x^3(t)$$

where $x(t)$ is the position of a particle of mass m.

a) What are the dimensions of a and b?

b) We consider the case where a and b take positive values. Determine a length scale L and a time scale T that provide dimensionless coordinates

$$\tau = (t - t_0)/T \qquad\qquad \xi(\tau) = x(t)/L$$

such that the EOM takes the dimensionless form

$$\ddot{\xi}(\tau) = \xi(\tau) - \xi^3(\tau)$$

Determine the dimensionless energy $\mathcal{E}$, and sketch the phase-space portrait.

c) We consider the case where a takes a negative and b takes a positive value. Determine a length scale L and a time scale T that provide dimensionless coordinates

$$\tau = (t - t_0)/T \qquad\qquad \xi(\tau) = x(t)/L$$

such that the EOM takes the dimensionless form

$$\ddot{\xi}(\tau) = -\xi(\tau) - \xi^3(\tau)$$

Determine the dimensionless energy $\mathcal{E}$, and sketch the phase-space portrait.
Remark: Beware the different sign of the linear term. How does it come about?

$\star$ d) Verify that the minimal dimensionless energy takes the value $-1/2$ and 0 for positive and negative a, respectively. How can it change *discontinuously* when a changes sign?

$\star$ e) Linearize the dimensionless EOM for positions $\xi = \xi_c + \epsilon$ close to the stable fixed points ξ_c of the dimensionless EOM, i.e. determine a constant c such that

$$\ddot{\epsilon} \simeq c\,\epsilon \qquad \text{for} \quad |\epsilon| \ll 1$$

How does the value of c influence the shape of the phase-space portrait in the vicinity of the fixed points?

Problem 2. Conservative forces, potentials, and contour lines

We consider a particle with unit mass that resides at a dimensionless position $\boldsymbol{q}(t)$ and moves under the influence of a dimensionless force

$$\boldsymbol{F} = \boldsymbol{A} \times \dot{\boldsymbol{q}} + \boldsymbol{B}$$

Note that this force depends on the velocity $\dot{\boldsymbol{q}}$ of the particle, but not on its position! We will be interested in the work, W, performed by this force, when the particle is moved from the origin to a position $\boldsymbol{x}$.

a) Determine the work performed for motion along the path $\gamma = \{\boldsymbol{q}(t) = t\,\boldsymbol{x} \mid 0 \leq t \leq 1\}$.

⋆ b) Show that

$$W(\boldsymbol{x}) = -\boldsymbol{B} \cdot \boldsymbol{x}$$

irrespective of the path taken from the origin to $\boldsymbol{x}$.

c) Sketch the contour lines and the gradient of $W(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^2$.

⋆ d) Is this $\boldsymbol{F}$ a conservative force?
If yes: Provide an argument to support your conclusion.
If no: Can you think of a special case where it would be conservative?

Problem 3. Motion on a helix

We consider a pearl of mass m that is moving without friction on a helix where it can take up positions

$$\boldsymbol{q}(\theta) = c\,\theta\,\hat{\boldsymbol{z}} + R\,\hat{\boldsymbol{r}}(\theta)\,, \qquad \theta \in \mathbb{R}$$

and we explore the evolution of $\theta(t)$ in the presence of gravity $\boldsymbol{g}$.

a) The vectors $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{r}}(\theta)$ are two orthogonal unit vectors of a cylindrical coordinate system. The third vector will be $\hat{\boldsymbol{\theta}}(\theta) = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}}(\theta)$. Express the θ derivatives of $\hat{\boldsymbol{r}}(\theta)$ and $\hat{\boldsymbol{\theta}}(\theta)$ in terms of the basis vectors $\big(\hat{\boldsymbol{z}}, \hat{\boldsymbol{r}}(\theta), \hat{\boldsymbol{\theta}}(\theta)\big)$.

b) Sketch the helix and indicate where you see c and R.

c) Determine the kinetic energy and the potential energy of the particle for the case where $\boldsymbol{g} = -g\,\hat{\boldsymbol{z}}$.

d) Determine the EOM for $\theta(t)$.
Sketch the solutions in the phase space $(\theta, \dot{\theta})$.

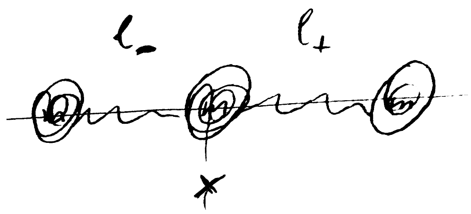
e) Determine $\theta(t)$ for the case where at time t_0 the particle is released with zero velocity at an angle θ_0.

$\star$ f) Turn the helix now by $\pi/2$ such that gravity is acting along $\hat{\boldsymbol{r}}(\theta_g)$ for a *fixed* angle θ_g.
1. Show that the axis $\hat{z}$ of the helix is aligned vertically to gravity.
2. How does this turn change the kinetic energy and the potential energy?
3. Determine the EOM. Do you recognize this ODE?

Problem 4. Springs in line

We consider three particles with identical masses m that reside at the positions $x_1(t)$, $x_2(t)$, and $x_3(t)$ on a one-dimensional track. The particles $(1,2)$ and $(2,3)$ are connected by identical Hookian springs with a spring constant k and rest length ℓ. No further forces are acting.

a) We adopt a parameterization in terms of x, l_- and l_+, where $x_1 = x - l_-$, $x_2 = x$, and $x_3 = x + l_+$ (see the sketch to the right).
Determine the kinetic energy and the potential energy.



b) Observe that x is a cyclic observable. Determine the related conservation law, and verify that it entails

$$\ddot{x} = c\,\left(\ddot{l}_+ - \ddot{l}_-\right)$$

Determine the value of c.
Bonus: What does the conservation law tell about the center-of-mass motion of the three particles?

c) Show that l_+ and l_- follow the EOMs

$$\ddot{l}_+ + c\,\left(\ddot{l}_+ - \ddot{l}_-\right) = -\omega^2\ (l_+ - \ell)$$
$$\ddot{l}_- - c\,\left(\ddot{l}_+ - \ddot{l}_-\right) = -\omega^2\ (l_- - \ell)$$

How does ω depend on the system parameters k, m, ℓ.

Bonus: Determine the dependence of ω on k, m, ℓ also by dimensional analysis.

d) Introduce the dimensionless variables $\Delta = (l_+ - l_-)/\ell$ and $\Sigma = (l_+ + l_- - 2\,\ell)/\ell$, and the dimensionless time $\omega\,(t - t_0)$. Show that they follow the dimensionless EOM

$$\ddot{\Delta}(\tau) = -3\,\Delta \qquad \text{and} \qquad \ddot{\Sigma}(\tau) = -\Sigma$$

where the dots denote here derivatives with respect to τ.

e) Determine the solution of $\Delta(\tau)$ and $\Sigma(\tau)$ for the initial condition $\dot{l}_-(t_0) = \dot{l}_+(t_0) = 0$ and $l_-(t_0) = l_+(t_0) = \ell + L$.
How would x evolve in this case?

f) Determine the solution of $\Delta(\tau)$ and $\Sigma(\tau)$ for the initial condition $\dot{l}_-(t_0) = \dot{l}_+(t_0) = 0$, $l_-(t_0) = \ell$, and $l_+(t_0) = \ell + L$.
Bonus: How does $x(t)$ evolve in this case.

$\star$ g) Verify that for the latter initial condition

$$l_\pm = \ell + \frac{L}{2}\,\Big(\cos\big((\Omega + \epsilon)\,\tau\big) \pm \cos\big((\Omega - \epsilon)\,\tau\big)\Big)$$

Determine the real numbers Ω and ϵ.

Expand the cosine function by trigonometric relations to show that

$$l_+(t) = \ell + L\ \cos(\Omega\,\tau)\ \cos(\epsilon\,\tau)$$
$$l_-(t) = \ell - L\ \sin(\Omega\,\tau)\ \sin(\epsilon\,\tau)$$

The energy of a Hookian spring amounts to k times the vibration amplitude squared. Assume that ϵ is much smaller than Ω, such that it can be interpreted as a modulation of the vibration amplitude. How is the vibrational energy distributed then among the two springs, and how does the distribution evolve in time?