Lecture "Experimental Physics I"

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Lecture 9

Special relativity

- Frames of reference in special relativity
- Spacetime (Minkowski) diagrams
- Lorentz transformations
- Length contraction and time dilation
- Dynamics in special relativity

1) Frames of reference in special relativity

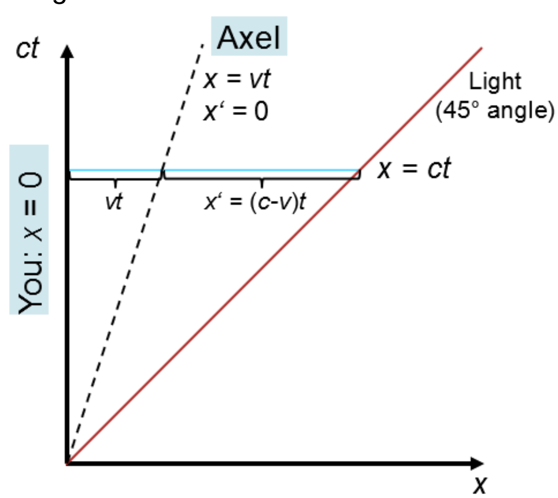
Before special relativity, it was of popular belief that light travels through a "non-viscous, mobile and incompressible" medium called the "ether". Only within the ether frame of reference, light would travel with its nominal speed. Many prominent figures in physics tried to develop an experiment to prove the theory of ether, which led to the Michelson-Morley experiment in 1895. However, it turned out that the experiment gave negative results, i.e. light was traveling with the same speed also in a moving frame of reference. This experiment was central of question the theory of the ether. To resolve the mystery, Einstein published his famous paper on special relativity in 1905. He based his thoughts on two postulates:

1) The laws of nature are the same in all frames of reference. Thus there can be no preferred ether frame.
2) It is a law of nature that light moves with velocity *c* (in all frames of reference).

He based his postulates on the Maxwell equations that predict a fixed speed for electromagnetic waves including light independent on the frame of reference. He was apparently not aware of the Michelson-Morley experiment.

A) Newtonian frames of reference

Let us first look again at our Newtonian frames of reference in particular at **inertial frames of reference (IRFs)** where (in the non-relativistic limit of small velocities) the Galileo transformations hold. Here we said previously that the same laws of physics hold in each inertial frame of reference.
Let us consider two IRFs a "resting" one (e.g. the lecture hall) and one that moves with relative velocity *v* e.g. a random person called Axel that walks in the lecture hall. The person shall always be in the origin of the moving coordinate system. We now are only interested in the *x*-coordinate (i.e. we consider only one dimension) and draw the relative motion of the IRFs as well as the movement of a light pulse emitted at $t = 0$ from position $x = 0$ in the resting IRF. Furthermore, we assume that at $t = 0$ the origins of both IRFs coincide, i.e. $x = x'$. We draw this situation in a **spacetime diagram**. In this diagram we multiply the time axis by *c* such that a light pulse shows up as a straight line with an angle of 45° to horizontal and vertical axis:



The dashed line shows the trajectory of the moving person in the resting frame of reference with

$$x = vt$$

This line corresponds however also to the **origin of the moving coordinate system** where the person resides. This can also be seen from the Galileo transformation:

$$x' = x - vt = 0$$

where $x' = 0$ follows from the equation for the dashed line from above. Furthermore, we have from the Galileo transformation that:

$$t = t'$$

i.e. all points on a line parallel to the x- axis have the same time in each coordinate system.
Let us now look at the **light pulse that** was emitted at $x = x' = 0$ at $t = 0$. After time t it has the position:

$$x = ct$$

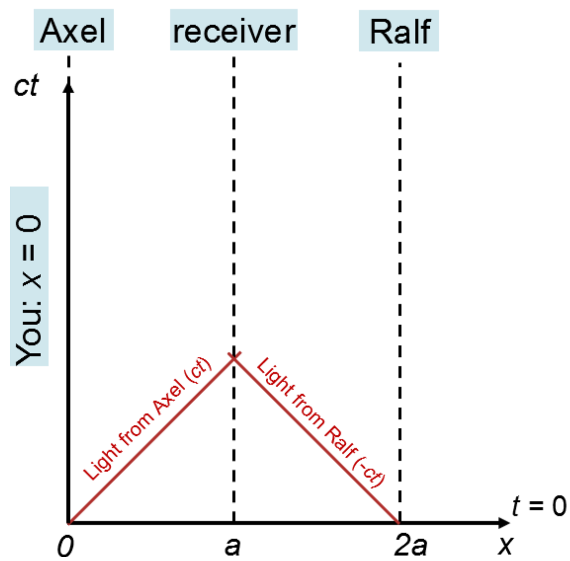
in the resting coordinate system and

$$x' = ct - vt = (c - v)t$$

in the moving coordinate system. We see from the latter equation that the light pulse would move in the moving coordinate system with a velocity $(c - v)$ being smaller than the speed of light. For a light pulse moving towards the negative direction with $-c$, we would in the moving coordinate system even get a velocity larger than the speed of light:

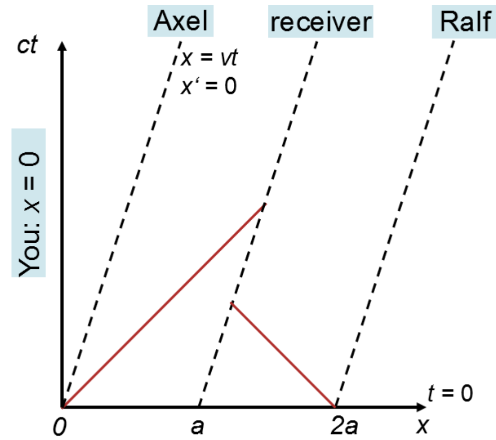
$$x' = (-c - v)t = -(c + v)t$$

These considerations are at odds with our previous statement that the speed of light is the same in all frames of reference.
To resolve this apparent contradiction, let us first have a look what **simultaneous** means in a resting frame of reference. To this end we let our initially moving person rest (Axel) and place a second person (Ralf) in the resting frame of reference at distance 2*a*. Halfway between Axel and Ralf we place a light detector. At time zero Axel and Ralf shall each emit a light pulse towards each other. They did this simultaneously, i.e. at exactly the same time, if the receiver detects both pulses at the same moment at time $t = a/c$. Using this principle one can **check whether Axel and Ralf have synchronized clocks**. In the spacetime diagram the situation looks the following:



B) Frames of reference in special relativity

Now we switch gears and let Axel, Ralf and the receiver move with velocity *v*. We again check for simultaneity, i.e. clock synchronization, by letting the two persons emit a light pulse at *t* = 0. In the spacetime diagram it looks the following:

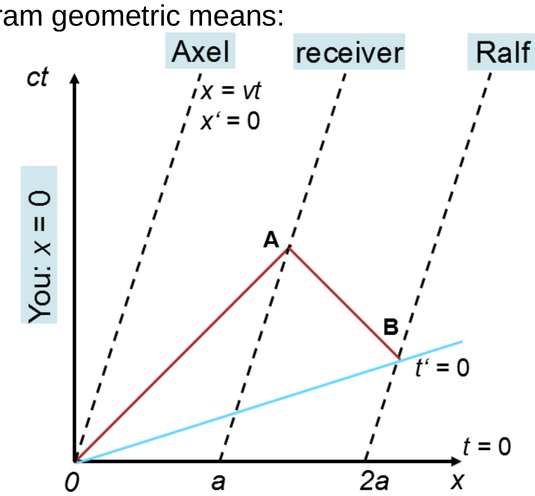


The resting observer in the lecture hall clearly sees that Ralf's pulse is reaching the detector earlier than Axel's pulse since the receiver moves towards the light pulse emitted by Ralf. This is, however, in apparent contradiction to the condition that *c* has the same value in each reference frame. In the moving frame of reference, Axel, Ralf and the receiver are resting and we have equal distances between receiver and both persons. Thus, the **receiver in the moving frame of reference must receive both light pulses at the same time!**.

The contradiction can be resolved by allowing:

$$t \neq t'$$

in contrast to the Galileo transformation. To this end we can find the position of the new $t' = 0$ axis in the spacetime diagram geometric means:



Let **A** be the point in the spacetime diagram where Axel's pulse hits the receiver. We can find the point **B** at which Ralf must have emitted his pulse for simultaneous arrival by plotting the trajectory of his light pulse back in time, i.e. a line with slope -1 from **A**. The crossing with the trajectory of Ralf gives then **B**. The origin (0) and B must thus lie on the $t' = 0$ axis of the moving frame of reference, which we thus can construct (see light blue line in plot).

Let us calculate in the following the location of point **B** in order to define the $t' = 0$ axis. We first have to locate point **A**. It is given by the intersection of Axel's light pulse described by:

$$x = ct$$

and the receiver trajectory given as the linear function:

$$x = vt + a$$

Since at $t' = 0$ the receiver is at $x = a$. Equating both equations gives:

$$ct_A = vt_A + a$$

such that the time when the light pulse arrives at the receiver is

$$t_A = \frac{a}{c - v}$$

Inserting into the equation for the light pulse gives for the x position of A:

$$x_A = \frac{ca}{c - v}$$

The location of **B** in the spacetime diagram is given by the intersection of the trajectory of Ralf:

$$x = vt + 2a$$

and the trajectory of the light pulse from Ralf passing **A**:

$$x = (-c)(t - t_A) + x_A$$

Note, one sees that this equation is correct, since the velocity is $-c$ and since the trajectory passes **A** at $t = t_A$. The latter can be seen by inserting t_A for which x becomes x_A. Equating both equations provides:

$$vt_B + 2a = (-c)(t_B - t_A) + x_A$$

Transformation and inserting the coordinates of A gives:

$$(c + v)t_B = ct_A + x_A - 2a = \frac{2ca}{c - v} - 2a = a\frac{2v}{c - v}$$

Such that we get:

$$t_B = a\frac{2v}{c^2 - v^2}$$

Most importantly $t_B \neq 0$ for any non-zero velocity, such that point **B** which is simultaneous with the origin in the moving frame **is not simultaneous with the origin on the resting frame**. Thus, we can say that the **simultaneity of two events depends on the frame of reference!**

Inserting t_B into the trajectory of Ralf then gives:

$$x_B = a\frac{2v^2}{c^2 - v^2} + 2a\frac{c^2 - v^2}{c^2 - v^2} = a\frac{2c^2}{c^2 - v^2}$$

The slope of the $t' = 0$ axis, which is the is x'-axis of the moving frame, is then given by:

$$\frac{t}{x} = \frac{t_B}{x_B} = \frac{v}{c^2}$$

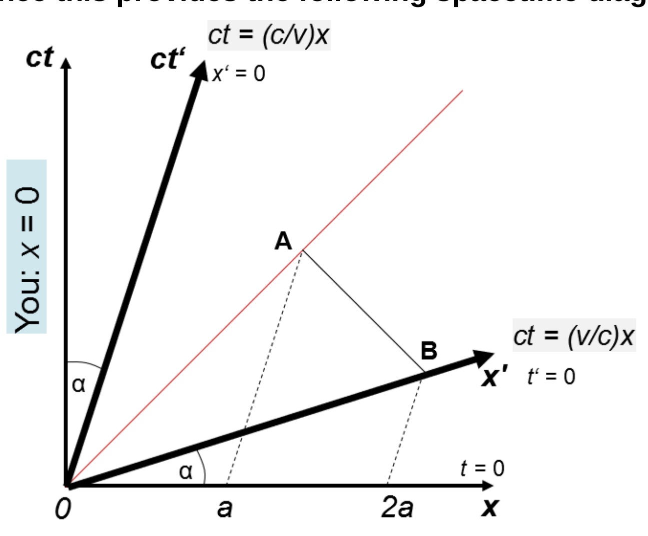
For the $ct - x$ plot this provides:

$$ct = \frac{v}{c}x$$

i.e. a slope of v/c. Similarly, the $x' = 0$ axis, is the t'-axis of the moving frame for which we have:

$$x = \frac{v}{c}ct$$

i.e. it has a slope of c/v in the $ct - x$ plot. In the **relativistic view with a constant speed of light in any frame of reference this provides the following spacetime diagram:**



Both the **time and the position axis get transformed** for frames with different velocities. The axes are symmetric with respect to the trajectory of a light pulse in the positive direction. This means that both axes of the moving frame form the same angle with the corresponding axes of the resting frame with:

$$\beta = \tan\alpha = \frac{v}{c}$$

To simply the notation we will use in the following $\boldsymbol{\beta}$ for the ration $\boldsymbol{v/c}$.

With this transformation, time becomes relative, since the simultaneity and thus the time of an event depends on the frame of reference (consider that for event **B** we have $t' = 0$ but $t > 0$). Based on this **Hermann Minkowski** (a German mathematician and physicist with important contributions to special relativity) concluded:

“Space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality”

Thus, space and time form a unity and only together make sense, since only this allows a transformation to a different frame of reference. This **union of time and space is:**

- **4-dimensional** (x, y, z, t)
- called **spacetime or Minkowski space or world**
- points of spacetime are **events**
- Trajectories in spacetime are **world lines**

2) Lorentz transformations

A) Lorentz transformations

After acknowledging the union of space and time let us derive a quantitative transformation between frames of reference with relative velocity v, i.e. how to transform an event at x, y, z, t in the resting frame to x', y', z', t' in the moving frame. Einstein assumed that:

- spacetime is everywhere the same
- from which he implied that no event is different from any other
- or that the origin can be placed anywhere without that the laws of physics change…

Based on this one can derive the corresponding transformations. Our transformation shall provide that a point moving with $x = vt$ is at $x' = 0$ in the moving frame of reference. Also, the transformation shall be linear in position and in time (e.g. doubling *x* results in a doubling of *x*’). This can be provided by:

$$x' = (x - vt)f(v^2)$$

The term in the bracket is our Galileo transformation that ensures linearity. Multiplication with a transformation equation $f(v^2)$ ensures the velocity dependence of the transformation, which would unify relativistic and non-relativistic velocities. Furthermore, the v^2 dependence ensures that the transformation is not dependent on whether an object moves to the left or t the right, i.e. that there is no preferred direction in space. From our diagrams we know that for $t' = 0$:

$$t = \frac{v}{c^2}x = \frac{\beta}{c}x$$

We therefore can write for the time transformation in a similar manner:

$$t' = \left(t - \frac{\beta}{c}x\right)g(v^2)$$

Now let us consider a light pulse from the origin at $t = t' = 0$ that satisfies in the restringing frame:

$$x = ct$$

Since the speed of light is in each frame c, it must also satisfy:

$$x' = ct'$$

By inserting the transformations for x' and t' we get:

$$(x - vt)f(v^2) = (ct - \beta x)g(v^2) = (x - \beta ct)g(v^2) = (x - vt)g(v^2)$$

where we replaced x and t with $x = ct$. Thus, we get from this equation that

$$f(v^2) = g(v^2)$$

The next ingredient that Einstein used was that for the two reference frames it is indistinguishable which is the resting one. Assuming that system S' is resting then S moves with velocity $-v$ to it, such that we can than write for the coordinate transformation of x and t:

$$x = (x' + vt')f(v^2)$$

$$t = \left(t' + \frac{\beta}{c}x'\right)f(v^2)$$

Let us now plug the upper transformation for x' and t' into the new transformation for x:

$$x = \left(\underbrace{(x - vt)f(v^2)}_{x'} + v\underbrace{\left(t - \frac{\beta}{c}x\right)f(v^2)}_{t'}\right)f(v^2) = (x - vt + vt - \beta^2 x)f(v^2)^2$$

Transformation gives

$$1 = (1 - \beta^2)f(v^2)^2$$

From which we get:

$$f(v^2) = \frac{1}{\sqrt{1 - \beta^2}}, \quad with \quad \beta = \frac{|v|}{c}$$

With this we get for the coordinates inside the resting frame:

$$x' = \frac{x - vt}{\sqrt{1 - \beta^2}}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \beta^2}}$$

These are the famous **Lorentz transformations**. They were for the first time derived by the Dutch physicist Hendrik Lorentz. He assumed that moving matter would be squeezed in the motion direction to explain the invariance of the speed of light on the reference frame, which was however still rationalizing the ether theory.

The Lorentz transformations provide only noticeable corrections compared to the Galileo transformation when the velocity between frames is in the range of the speed of light. Imagine $v/c = 0.1$ (the incredible velocity of 30,000 km/s). In this case $v^2/c^2 = 0.01$ such that the reciprocal square root term gives us a value of 1.005, which is only slightly deviating from 1. Using the Taylor approximation when can approximate for small velocities:

$$\frac{1}{\sqrt{1 - \beta^2}} \approx 1 + \frac{1}{2}\beta^2$$

For example, a fast car goes with 100 m/s (360 km/h), where the reciprocal square root term would then equate to $1 + 6 \times 10^{-14}$ providing a practically negligible correction.

In the limit of small velocities the square root term thus approaches 1. With this we get for the position transformation:

$$x' = x - vt$$

For the time transformation we have to consider that the second term in the numerator x/c describes the time that light needs to pass the distance x. This small time is multiplied by the small factor v/c, i.e. this would provide a negligible total correction for normally used times t at low velocities. We thus get:

$$t' = t$$

Thus, the Lorentz transformations converge to the Galileo transformations at low velocities, which is important to ensure consistency.
The Lorentz transformations apply only in the direction of the relative motion of the two frames. If we choose our coordinate system to have the motion only along x the Lorentz transformation for **3 dimensions becomes** (see slides):

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\beta^2}}$$

B) Nothing moves faster than light

Note that for $v > c$ the square root term in the transformation equations becomes imaginary, such that the Lorentz transformations would provide nonsense. Einstein resolved this paradox by additionally postulating:
No material system can move with a velocity greater than light relative to any other material system.
This is now a cornerstone of modern physics. It also states that no signal can travel faster than light, since even a photon is a material system.
The postulate that nothing moves faster than light can also be motivated by deriving how the velocity of an object translates from one frame into another. Let us consider that an object has velocity $\vec{u}$ in the resting frame S, such that its components are given as:

$$u_x = \frac{dx}{dt}\ ; \quad u_y = \frac{dy}{dt}\ ; \qquad u_z = \frac{dz}{dt}$$

The velocity components in a frame S' that moves with velocity *v* along the x direction are then given as:

$$u'_x = \frac{dx'}{dt'}; \quad u'_y = \frac{dy'}{dt'}\ ; \qquad u'_z = \frac{dz'}{dt'}$$

The differentials can be rewritten according to:

$$u'_x = \frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \frac{dx'}{dt}\Big/\frac{dt'}{dt}$$

Inserting the time derivatives of x' and t' using the Lorentz transformation provides:

$$u'_x = \frac{\frac{dx}{dt} - v}{\sqrt{1-\beta^2}}\Bigg/\frac{1 - \frac{v}{c^2}\frac{dx}{dt}}{\sqrt{1-\beta^2}}$$

Replacing the time derivative of x with u_x then gives:

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

In the same way one can obtain the velocity transformation for the y and z components:

$$u'_y = \frac{u_y\sqrt{1-\beta^2}}{1 - \frac{vu_x}{c^2}}; \qquad u'_z = \frac{u_z\sqrt{1-\beta^2}}{1 - \frac{vu_x}{c^2}}$$

They change since the time used in the differentiation is transformed.
Let us now consider a frame S' that moves (compared to the resting frame) along the x axis "to the left" with velocity $-v$ and a light pulse that moves with $u_x = c$ "to the right". The velocity of the light pulse in the moving frame S' is then given as:

$$u'_x = \frac{c-(-v)}{1-\frac{(-v)c}{c^2}} = c\frac{1+v/c}{1+v/c} = c$$

Thus, the light moves still with the speed of light in the moving frame showing that the Lorentz transformations are in agreement with the postulate the nothing is faster than light.

3) Length contraction and time dilation

A) Proper time and spacetime distance

Now we have a quantitative way to transform spacetime between different frames of reference. A consequence of this transformation is that also the unit length in each coordinate system gets transformed. To derive this, we first introduce a quantity that is invariant upon the transformation as found by Minkowski. One can show that for any space time event at x, t the following relation holds:

$$x'^2 - (ct')^2 = x^2 - (ct)^2 = s^2$$

i.e. s^2 is invariant for the transformation into any other frame. One can proof this easily by inserting the Lorentz transformations (not shown):

$$x'^2 - (ct')^2 = x' = \frac{(x-vt)^2}{1-\beta^2} - \frac{(ct-\beta x)^2}{1-\beta^2}$$

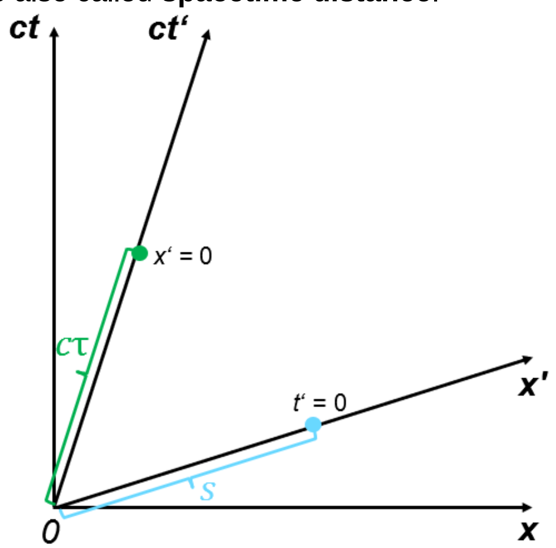
$$= \frac{x^2 - 2vtx + (vt)^2 - (ct)^2 + 2vtx - \left(\frac{v}{c}x\right)^2}{1-\beta^2}$$

$$= \frac{x^2(1-\beta^2) - (ct)^2(1-\beta^2)}{1-\beta^2} = x^2 - (ct)^2$$

The invariant value s has a two-fold physical meaning. This can be seen by setting $t' = 0$. With this we get a positive invariant

$$x'^2 = x^2 - (ct)^2 = s^2$$

It describes a point that sits at the $t' = 0$ axis in the moving frame S' (cyan point in figure). Thus, ***s* describes the physical distance of a point from the origin in the system where it occurs at zero time.** It is therefore also called **spacetime distance**.



Setting *s* to 1 describes then the unit length in the particular system. With this condition we get after transformation the following function for the position of the unit length:

$$ct = \sqrt{x^2 - s^2} = \sqrt{x^2 - 1}$$

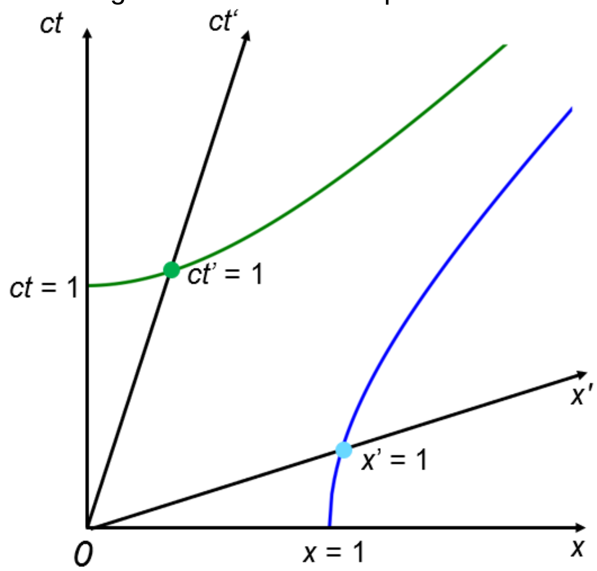
Vice versa we can also set $x' = 0$ for which we get a negative value for the invariant

$$-(ct')^2 = x^2 - (ct)^2 = -(c\tau)^2$$

This describes a point that sits at the x' $= 0$ axis, i.e. that is resting in the moving frame S' (green dot in figure above). τ is called the **proper time** and **represents the ticking of the clock in its resting frame**. Setting $c\tau = 1$ describes then the unit time in the particular system for which we get the equation:

$$ct = \sqrt{x^2 + 1}$$

Plotting the curves for the unit length and the unit time provides:

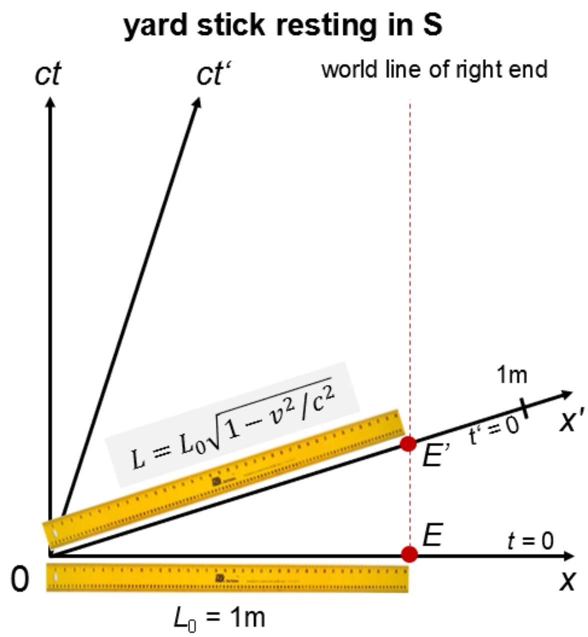


We see that the distance of the unit length/time from the origin increases with increasing velocity of the frame.

B) Length contraction

The different unit length in frames that move with different relative velocities to each other as well as the relativity of "simultaneous" provides a number of paradoxic phenomena.

Let us first consider a **yard stick that is resting in the resting frame S**. Its left end shall be at the origin and its right end at point E. The world line of the left and the right end are then vertical lines:



An observer in the resting frame S measures at *t* = 0 the length of the yard stick to be 1m. Now let us **consider a moving observer** (that rests in the moving frame) that passes by and also wants to measure the length. To get a precise value for the length, he measures the distance between both ends at exactly the same time, which is e.g. for him the time t' = 0. The right end is in this moment located at point E' in the spacetime diagram. From the figure we see that this measurement is not simultaneous for the resting frame. Also, we see (look for the 1m tick at x') that E' is located at a distance shorter than 1m from the origin. Thus, **the observer in S' that passes by the resting stick measures a shorter length** for the yard stick, which we call **length contraction.**

Let us derive a formula for the contraction. We know that *E*' is located at $x = L_0$ and also $t' = 0$. The latter condition provides according to the Lorentz transformation for t' that:

$$0 = t - \frac{v}{c^2}x$$

or

$$t = \frac{v}{c^2}x = \frac{v}{c^2}L_0$$

Inserting x and t into the Lorentz transformation gives for the length of the yard stick that the moving observer measures:

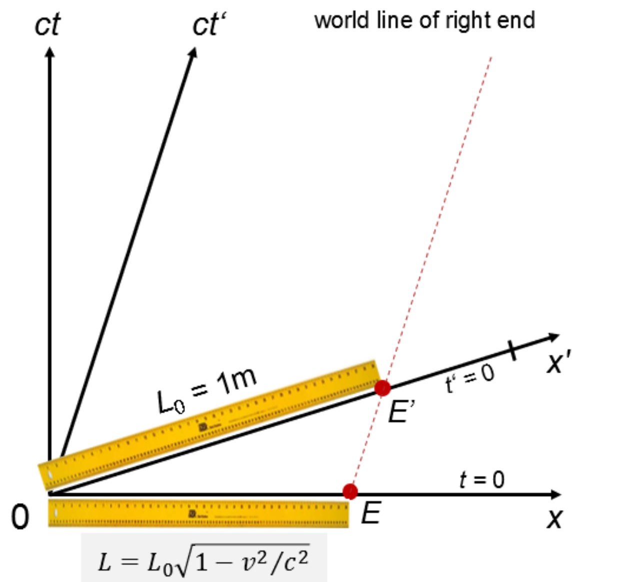
$$L = x' = \frac{L_0 - v\frac{v}{c^2}L_0}{\sqrt{1-\beta^2}} = L_0\frac{1-\beta^2}{\sqrt{1-\beta^2}}$$

Simplification gives then for the **length contraction**:

$$L = L_0\sqrt{1-\beta^2}$$

Now we change the perspective. We have a **moving yard stick that is resting in the moving frame S'**. An observer that moves with S' measures at t' = 0 a length of 1 m (from 0 to E' in figure below). **Which length would a resting observer measure?** To understand this, we have to consider that the left end of the yard stick moves at a world line described by $x = vt$ (i.e. x' =0) and the right end by $x = vt + L$ (see red dashed line in figure below):

yard stick resting in S' (moving against S)



Let us assume that the observer in S measures the length of the stick at t = 0, i.e. he measures the distance from 0 to E. The (t' = 0) axis is again described by (**not shown in lecture**):

$$t = \frac{v}{c^2}x$$

E' is located at the intersection between this axis and the world line of the right end. Inserting the latter equation into the equation for the world line gives us for E':

$$x = \frac{v^2}{c^2}x + L$$

and the x-coordinates of E' is

$$x = \frac{L}{1 - \beta^2}$$

Using the Lorentz transformation and inserting the relation for *t* one gets:

$$L_0 = x' = \frac{x - v\frac{v}{c^2}x}{\sqrt{1 - \beta^2}} = \frac{\frac{L}{1 - \beta^2}(1 - \beta^2)}{\sqrt{1 - \beta^2}}$$

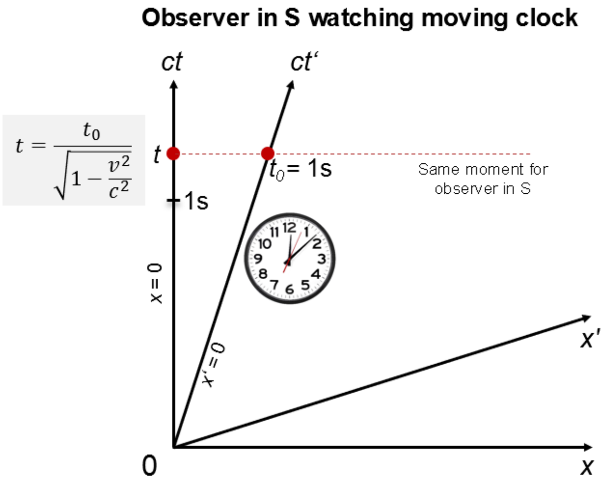
With this we get again that

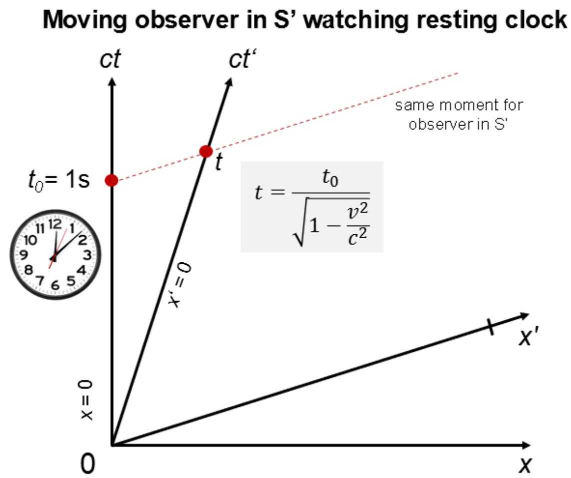
$$L = L_0\sqrt{1 - \beta^2}$$

Thus, the resting observer measures a shorter length for the moving yard stick. It makes sense that we got in both cases the same result, since the contraction was always seen by an observer that moved relative to the yard stick, while the motion direction did not matter (due to the square in the equation).

C) Time dilation

An addition to different lengths seen by moving observers also the clocks appear differently fast in the different frames. This effect is called **time dilation** and works similarly as length contraction. Let us imagine a **clock moving with velocity v relative to the resting frame S**. The clock is thus stationary in the moving frame S'. A **resting observer now follows the moving clock until it reaches a given time t_0** and then looks how much time has passed in his stationary frame. This situation is depicted by the red line in the left figure below:





We know that for the clock at $t' = t_0$, the time in the resting frame is *t*, furthermore we know that the position of the clock with respect to the resting frame is given by

$$x = vt$$

Using the Lorentz transformation for the time we can thus write:

$$t' = t_0 = \frac{t - \frac{v}{c^2}vt}{\sqrt{1-\beta^2}} = \frac{t - \beta^2 t}{\sqrt{1-\beta^2}}$$

Simplification then provides:

$$t = \frac{t_0}{\sqrt{1-\beta^2}}$$

i.e. the **resting observer sees that more time has passed for him compared to the moving clock in S'**.

We can now also **consider a resting clock in the stationary frame S and a moving observer** that is stationary in S'. The moving observer follows again at which time in S' the resting clock reaches the time t_0. For the moving observer the time *t* when he/she looks at the clock is described by the red line in the right figure above. It is parallel to the x', i.e. the t' = 0 axis. One sees intuitively that t_0 appears to be smaller than *t*. Doing an exact calculation (similarly as for the moving yard stick), provides the same formula for the time dilation as before:

$$t = \frac{t_0}{\sqrt{1-\beta^2}}$$

Again it does not matter which is the resting frame (which can anyway not be determined) but only whether the clock is moving relative to the frame the observer. For the observer more time passes compared to the moving clock.

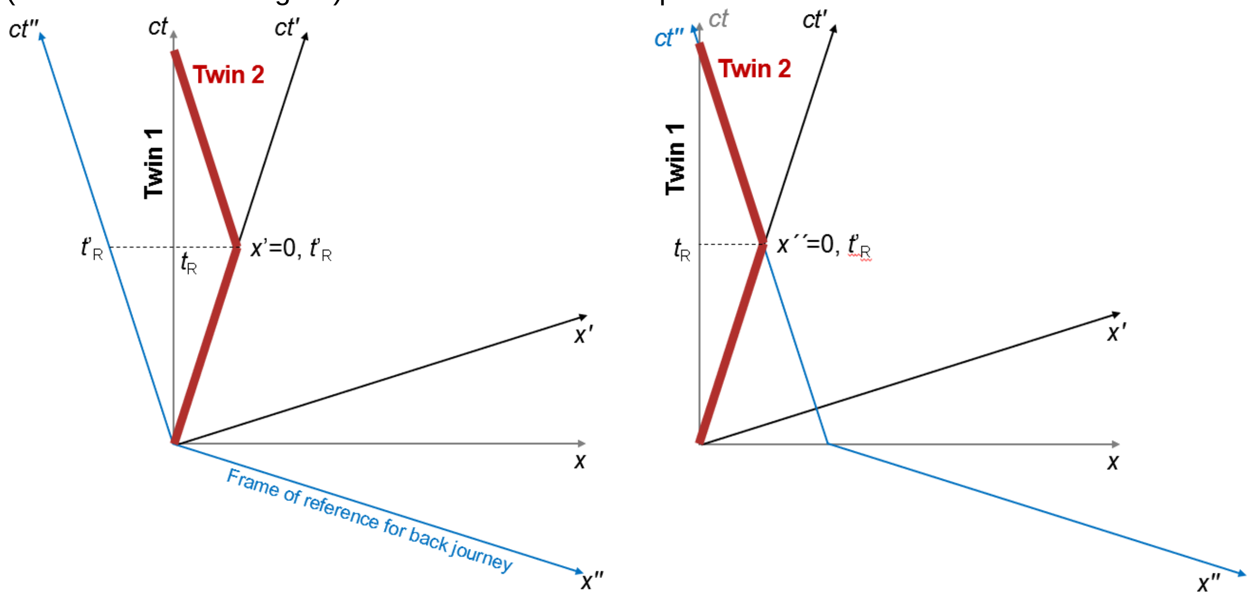
Twin Paradox

An interesting paradox (at least at first glance) about the time dilation is the **twin paradox**. Imagine two twins. Twin 1 stays at home while twin 2 goes on a journey with fast velocity along *x.* After a while twin 2 switches the direction and returns home. He actually finds that **due to time dilation twin 1 is now older** than him.

The paradoxical aspect of this story is that we said before that the **time dilation is like motion a matter of the prespective.** From the perspective of twin 1, twin 2 should age less quickly, while from the perspective of twin 2, twin 1 should age less quickly. It is all a matter of the viewpoint.

In reality **twin 1 ages really more quickly than twin 2**. To understand this paradox one has to consider that the **motion of the twins is actually distinguishable**. Twin 1 remains all the time in an inertial frame of reference. **Twin 2** undergoes acceleration twice and **even worse he/she switches even the frame of reference!**

To understand that the switching of the reference frame has a particularly large effect, we draw the motion of the twins in a space time diagram. Twin 1 remains all the time at $x = 0$ (at the ct axis), while twin 2 follows the trajectory $x = vt$, which is the $x' = 0$ axis. After time t_R twin 1 sees that twin 2 has switched the direction, which corresponds to time t_R' to twin 2. Assuming the same speed for the backward motion the trajectory now occurs with the negative of the original slope (see left side in the figure) until twin 2 reaches the position $x = 0$:



The backward journey is described by the additional frame S’’ with the velocity -v as depicted in blue in the figure. Due to the same magnitude of the velocity the unit scales of S´and S´´ are the same. The return of twin 2 must happen also in S’’ at $t'' = t'_R$, since for the quick moment of the direction reversal the clock for twin 2 has not changed. Twin 2 will, however, be always at zero position in S’’ after the reversal. Therefore, we have to shift the coordinate system of S’’, such that it takes $x'' = 0$ and $t'' = t'_R$ at the reversal point. This corresponds to a simple shift to the right as done in the picture above on the right side. At the crossing point between the ct and the new ct´´ axis, twin 2 has returned. Due to the crossing the time in each frame is not relative anymore, since the two clocks are next to each other.

For twin 1 a time of $T_1 = 2t_R$ has passed, while for twin 2 a time of $T_2 = 2t_R'$ has passed. Using this result, the return point is located at:

$$x_R = vt_R$$

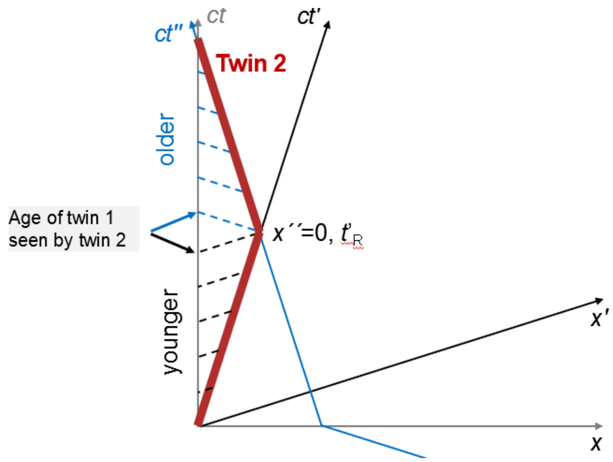
we get with the Lorentz transformation:

$$t_R{}' = \frac{t_R - \frac{v}{c^2} v t_R}{\sqrt{1-\beta^2}} = t_R\sqrt{1-\beta^2}$$

Thus, the time that has passed for twin 2 relative to twin 1 is indeed reduced, which provides the same result as the for the time dilation:

$$T_2 = T_1\sqrt{1-\beta^2}$$

The crucial step for this time difference is the direction reversal. We can understand this by asking how old twin 1 appears to twin 2. Before the direction reversal this is given by the crossings of the black dashed lines (lines of equal time in S') with the ct axis, i.e. twin 1 appears younger to twin 2:



After the direction reversal, the slope of the lines with equal time in S'' suddenly changes and now twin 1 appears from one moment to the other older.

The different twin aging can be experimentally proven. For example, two clocks are synchronized in Paris and transported by a fast plane (such as the concorde with *v* = 2400 km/h). One of them is then transported to New York. The traveling clock would then show after the flight time of 3 hours of -8.9ns. Even during a hike to a high mountain at which the tangential velocity from the earth rotation is larger than on the ground the time dilation can be shown (see slides).
More extreme consequence has the time dilation for fast elementary particles, such as muons with relativistic velocities (0.994 c) that are created by cosmic radiation. The moving mouns exhibit a strongly increased life time (45-fold) compared to "resting" ones.

4) Dynamics in special relativity

A) Momentum in special relativity

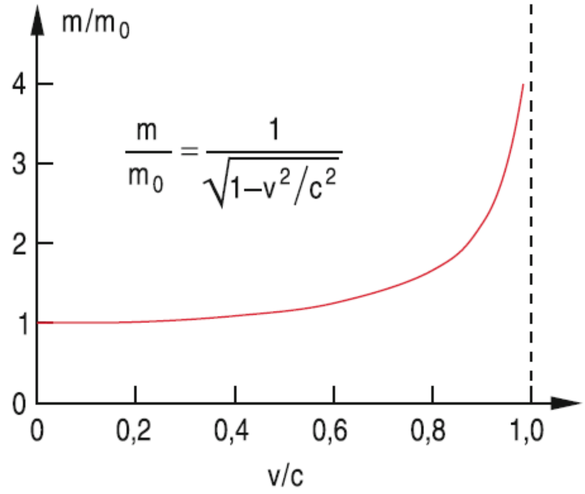
Using the fact that momentum is conserved during coordinate transformation perpendicular to the relative motion of two inertial frames and combining it with the relativistic transformation of velocities derived above one can show (see Demtröder, Mechanics, p 119) that the momentum of a particle moving with velocity *v* in a frame of reference is given as:

$$p = m(v)v = \frac{m_0}{\sqrt{1-\beta^2}} v$$

where

$$m(v) = \frac{m_0}{\sqrt{1-\beta^2}}$$

is the mass of the moving particle and m_0 is the so-called **rest-mass** (at zero velocity). At velocities approaching the speed of light the mass of the moving particle approaches infinity (see figure below). This means one would need infinite energy to reach the speed of light being in agreement with the postulate that it cannot be exceeded.

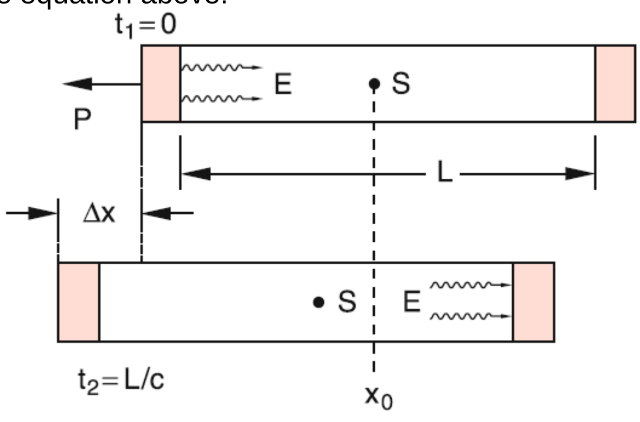


B) Relativistic energy

From the equation one sees that the form of the kinetic energy of a particle must be different at relativistic speeds compared to the classical result. Einstein derived the relativistic energy using the momentum of light (i.e. photons) that was already known at that time to be:

$$p = \frac{E}{c}$$

For a light "portion", i.e. a certain number of photons with total energy *E*. He then made a thought experiment using a box with length L and mass M. At time t = 0 a light pulse with energy E shall be emitted from the left side of the box which travels with the velocity of light c to the right with a momentum given by the equation above:



Using Newtons 3rd law (actio equals reactio), an opposite amount of momentum is therefore transferred to the box itself such that it starts travelling with velocity -v to the left. We can thus write for the momentum of the box:

$$-Mv = -p = -\frac{E}{c}$$

The light requires a time $\Delta t = L/c$ to travel through the box. During that time the box gets displaced by:

$$\Delta x = -v\Delta t = -\frac{EL}{Mc^2}$$

When the photons get absorbed at the right side of the box they now transfer the opposite momentum to the box, such that the box motion stops.

Now comes the crucial step: We will learn in the subsequent lectures that for an initially resting system the position of the center of mass is not changing in absence of external forces. The position change of the center of mass can be calculated from the displacement of the box and of the photons as:

$$\Delta x_{CM} = \frac{M\Delta x + mL}{M+m} = 0$$

In order to have a zero shift, one had to introduce here a mass m for the photons (corresponding to their momentum). Inserting Δx in the equation now gives:

$$-M\frac{EL}{Mc^2} + mL = 0$$

Transformation then provides the famous equation:

$$E = mc^2$$

Each mass *m* is thus correlated to the energy $E = mc^2$. **Mass and energy are proportional to each other**. For the mass we have in this case to insert the relativistic mass, such that:

$$E = \frac{m_0c^2}{\sqrt{1-\beta^2}} = \underbrace{m_0c^2}_{E_{rest}} + \underbrace{m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)}_{E_{kin}}$$

On the right side we split this equation into a velocity independent term, so so-called rest energy (which is always present) and a velocity dependent term, that becomes zero at zero velocity, which is nothing else than the **kinetic energy**. Thus the kinetic energy is hidden in the relativistic mass increase of the formula $E = mc^2$.

For small velocities the term in brackets can be approximated by a Taylor series:

$$\frac{1}{\sqrt{1-\beta^2}} - 1 \approx 0 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \cdots$$

Considering only the square term gives then for the kinetic energy

$$E_{kin} \approx \frac{1}{2}m_0v^2$$

in agreement with the classical result.

Another important relation for the relativistic energy is

$$E = {m_0}^2c^4 + m^2v^2c^2$$

Which one can proof by inserting. Replacement of the velocity on the right side by the relativistic momentum now gives and important relation between relativistic energy and momentum:

$$E = {m_0}^2c^4 + p^2c^2$$

Lecture 9: Experiments

N.A.