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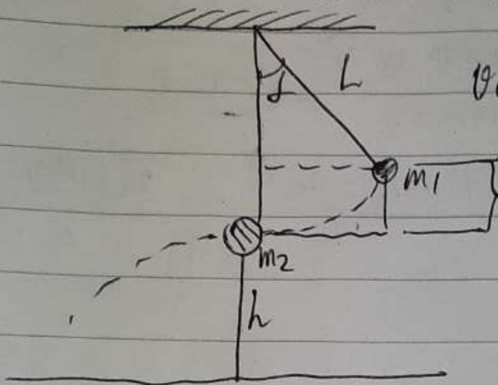
EP-HW7
Conservation laws, center of mass

(1)



$\vartheta(t=0)=0.$

$$h = L - L\cos\alpha = L(1-\cos\alpha) \approx L\left(1-\left(1-\frac{\alpha^2}{2}\right)\right) = \frac{L\alpha^2}{2},$$

a) Time before collision.

For small angles pendulum equation: $-mg\alpha = mL\alpha'' \longrightarrow$

$$\alpha'' + \underbrace{\frac{g}{L}}_{\omega^2}\alpha = 0$$

$$\alpha = \alpha_0\cos(\omega t),$$

$$\Longrightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/L}} = 2\pi\sqrt{\frac{L}{g}},$$

Collision happens at $\frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{L}{g}}.$

b) Speeds ~~after~~ before collision.

way1: $\alpha' = -\omega\alpha_0\sin(\omega t)$, $v = L\alpha'(t)$,

$|v| = L|\alpha'(t)|$, at $\frac{T}{4}$,

$$\left|v\left(\tfrac{T}{4}\right)\right| = L\cdot\omega\alpha_0\sin\left(\omega\frac{T}{4}\right) =$$

$$= L\omega\alpha_0\sin\frac{\pi}{2} = L\omega\alpha_0 = \boxed{\sqrt{gL}\,\alpha_0}$$

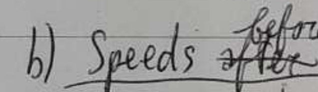
way2:

Energy conservation

$$m_1 g h = \frac{m_1 v^2}{2} \Longrightarrow \text{(see picture)}$$

$$v = \sqrt{2gh} = \sqrt{2g\cdot\frac{L\alpha_0^2}{2}} = \boxed{\sqrt{gL}\,\alpha_0}\ \text{(same)}.$$

Speeds after collision: ~~let~~ ($v_{i,in}$ means before, $v_{i,out}$ after, all are signed components in $\hat{x}$ direction)



From momentum & energy conservation:

$$\begin{cases} m_1 v_{1,in} = m_1 v_{1,out} + m_2 v_{2,out} \\ \dfrac{m_1 v_{1,in}^2}{2} = \dfrac{m_1 v_{1,out}^2}{2} + \dfrac{m_2 v_{2,out}^2}{2} \end{cases}$$

$$\div \uparrow \begin{cases} m_1 v_{1,in} = m_1 v_{1,out} + m_2 v_{2,out} \Longrightarrow m_1(v_{1,in} - v_{1,out}) = m_2 v_{2,out} & (1) \\ m_1(v_{1,in}^2 - v_{1,out}^2) = m_2 v_{2,out}^2 & (2) \end{cases}$$

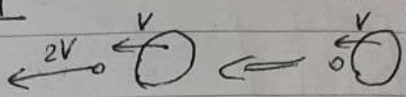
$$v_{1,in} + v_{1,out} = v_{2,out} \xRightarrow{\text{(to 1)}} m_1 v_{1,in} = m_1 v_{1,out} + m_2(v_{1,in} + v_{1,out})$$

$$\Longrightarrow \boxed{v_{1,out} = \frac{v_{1,in}(m_1 - m_2)}{m_1 + m_2}}$$

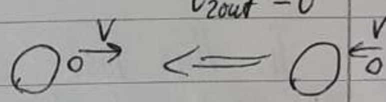
$$\boxed{v_{2,out} = \frac{2m_1 v_{1,in}}{m_1 + m_2}}$$

Special cases:

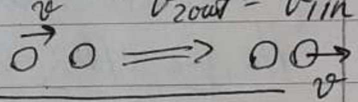
1) $m_1 \gg m_2$, $v_{1out} = v_{1in}$
$v_{2out} = 2v_{1in}$



2) $m_1 \ll m_2$, $v_{1out} = -v_{1in}$
$v_{2out} = 0$



3) $m_1 = m_2$, $v_{1out} = 0$
$v_{2out} = v_{1in}$



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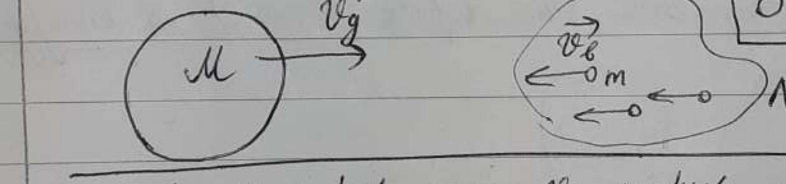


$M = 400\,kg$, $v_g = 40\,km/h$, $m = 10\,g$, $v_b = 400\,km/h$, ~~how~~ $N - ?$ (M stops)
$\to v_b/2$ E losses - ?

1) Momentum everywhere conserved, E not.
Now if balls attack M one after another („shots"), suppose after n balls green ball has $v_g(n)$. Then:

$$M v_g(n) - m v_b = -m\frac{v_b}{2} + M v_g(n+1)$$

$$M(v_g(n+1) - v_g(n)) = m v_b\left(-\frac{1}{2}\right)$$

$$v_g(n+1) = v_g(n) + \frac{-m}{2M} v_b\ ,\quad v_g(0) = v_g.$$

After N blue balls:

$$v_g(N) = 0 = v_g + N\left(\frac{-m}{2M}\right) v_b \Longrightarrow \boxed{N = \frac{2M v_g}{m v_b}} =$$

$$= \frac{2 \cdot 400\,kg \cdot 40\,km/h}{10\,g \cdot 400\,km/h} = \frac{800 \cdot 10^3}{10 \cdot 10} = \underline{8000 \text{ balls!}}$$

P.S. Interestingly, if all balls attack M in parallel, the result is same (since M does not change and v_b always halved):

$$M v_g - N m v_b = -N m \frac{v_b}{2} \Longrightarrow N = \frac{2M v_g}{m v_b}.$$

2) Energy losses:

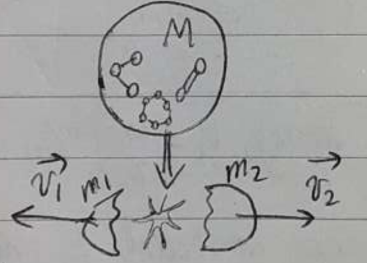
$$\frac{MV_g^2}{2} + N\frac{mV_8^2}{2} + \overset{<0}{A_{loses}} = N\frac{m(V_8/2)^2}{2}$$ (bullets lose 3/4 of their energy, but „energy deposited" is total loss)

$$|A_{loses}| = \frac{MV_g^2}{2} + \frac{NmV_8^2}{2}\cdot\frac{3}{4} = \frac{400kg\cdot 40^2 km^2/h^2}{2} + \frac{3}{4}\cdot\frac{(8000)\cdot 10\cdot 10^{-3}kg\cdot 400^2 (km/h)^2}{2} =$$

$$= \begin{bmatrix} km/h = \\ = \frac{1}{3.6}\, m/s \end{bmatrix} = \left[\frac{400\cdot 40^2}{2}\cdot\frac{1}{3.6^2} + \frac{3}{4}\cdot\frac{8000}{2}\cdot\left[10\cdot 10^{-3}\cdot 400^2\cdot\frac{1}{3.6^2}\right]\right] J \approx 395k\,J.$$

(same as for 4 tonn elephant falls from 10m ≈ 3 floor building)

③ In this task unlike the last E_k of system is increased (in last task it → heat), raising from potential energy of chemical interactions [similar as big compressed spring]. The E_k after explosion (since before it was 0) is energy gain.



$(m_1\vec{v_1} + m_2\vec{v_2} = \vec{0})$

Then $\begin{cases} m_1 v_1 = m_2 v_2 \\ \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = E \\ m_1 = k m_2 \end{cases}$

Find k to maximize (for any part i)

a) E_{ki}

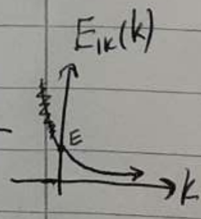
b) v_i

c) $p_i = m_i v_i$

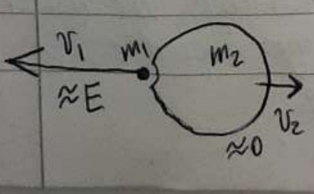
a) Maximize energy:

$$k v_1 = v_2, \quad \underbrace{\frac{k m_2 v_1^2}{2}}_{E_{1k}} + \underbrace{\frac{m_2 k^2 v_1^2}{2}}_{E_{2k}} = E_{1k} + kE_{1k} = (k+1)E_{1k} = E$$

$$E_{1k}(k) = \frac{E}{k+1}$$



To maximize k should be 0, means m_1 is as small as possible, $m_2 \approx M$. Then $E_k(m_1) \approx E$.



3) Max. momentum:

$\cancel{E_{ki} = \frac{m_i v_i^2}{2} = \frac{p_i^2}{2m_i}, \; p_i=\sqrt{2m_iE_{ki}}}$

$p=\sqrt{\frac{2E\,km_2}{1+k}}$

$kv_1=v_2$

$\frac{m_1v_1^2}{2}+\frac{m_2v_2^2}{2}=E$

$\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2} = [p_1=p_2!]\ \frac{p^2}{2}\left[\frac{1}{km_2}+\frac{1}{m_2}\right]=E$ not constant

$p=\sqrt{\frac{2E\cdot km_2}{k+1}}=\sqrt{\frac{k}{k+1}}\cdot C,$

To maximize momentum must maximize $\sqrt{\frac{k}{k+1}}\to$

maximize $\frac{k}{k+1}=f(k)$

$f(k)=\frac{k}{k+1}$

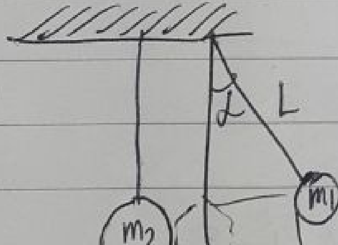
m_2 not const, incorrect argument $\Longrightarrow p_{max}$ when $m_1=m_2$

$\frac{df}{dk}=\frac{k+1-k}{(k+1)^2}=\frac{1}{(k+1)^2}>0$

Maxima when $k\to\infty$ (also series $\frac{1}{2}, \frac{2}{3}, \dots$ $\lim_{k\to\infty}\frac{k}{k+1}=1$).

So in all kases divide into ∞...

4.



elastic

a) $v_{1in}=\sqrt{gL}\,\alpha$ (see task 1a) before hit

Momentum & energy preserved, then (see task 1.b)

$$v_{1out}=\frac{v_{1in}(m_1-m_2)}{m_1+m_2}$$

$$v_{2out}=\frac{2v_{1in}m_1}{m_1+m_2}$$

New energy for m_1: $\frac{m_1v_{1out}^2}{2}=\frac{m_1}{2}\frac{(m_1-m_2)^2}{(m_1+m_2)^2}v_{1in}^2=\frac{m_1(m_1-m_2)^2}{2(m_1+m_2)^2}gL\alpha^2,$

~~Energy~~ Energy cons. for m_1:

$$E_k=E_p=m_1g(1-\cos\alpha_{new})L$$

$$\frac{\cancel{m_1}}{2}\left(\frac{m_1-m_2}{m_1+m_2}\right)^2 g\cancel{L}\alpha^2=\cancel{m_1}g\cancel{L}(1-\cos\alpha_{new})$$

2) Similarly $\frac{m_2 v_{2out}^2}{2} = \frac{m_2}{2}\cdot\frac{4v_{1,in}^2 m_1^2}{(m_1+m_2)^2} = m_2\cdot \cancel{2g}\cancel{L}\,\alpha^2\left(\frac{m_1}{m_1+m_2}\right)^2 =$

$= E_{pot} \approx m_2 g L \frac{\alpha_{2out}^2}{2}$, $\boxed{\alpha_{2out} = \frac{2m_1}{m_1+m_2}\alpha}$.

Special cases: $m_1 \ll m_2$, $\alpha_{1out} = -\alpha$, $\alpha_{2out} = 0$ etc. (as in task 1 but for angles.

b) Each collision happening at $T/2$,

$f = \frac{2}{T} = \sqrt{\frac{g}{L}}\cdot\frac{1}{\pi}$.

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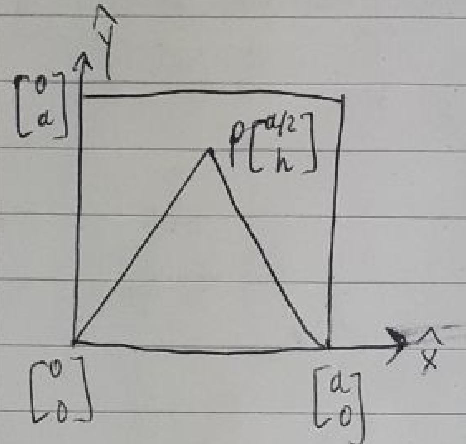
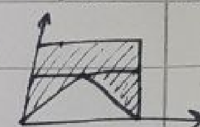


Center of mass – can be also done by

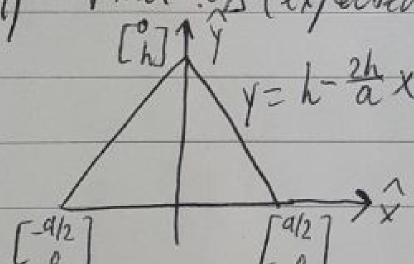


P is CM then.

Consider Δ to have negative mass, „eating" part of mass of all $\square$.

Then $\vec{r}_{CM} = \frac{\vec{r}_\square m_\square + \vec{r}_\Delta(-m_\Delta)}{m_\square - m_\Delta}$ where $m_i > 0$.

1) Find $\vec{r}_\Delta$ (expected $\begin{bmatrix}a/2\\h/3\end{bmatrix}$):



in new coordinates (expected $\begin{bmatrix}0\\h/3\end{bmatrix}$):

$\vec{r} = \frac{1}{M}\cdot\iint_{x\,y}\vec{r}\,dm = \frac{1}{M}\int_{x=-\frac{a}{2}}^{+\frac{a}{2}}\int_{y=0}^{\text{line }(y=f(x))}\begin{bmatrix}x\\y\end{bmatrix}\rho\,dy\,dx$,

where $M = \rho S = \rho\cdot\frac{1}{2}\cdot ah$.

By symmetry (or since x is odd function), $\frac{1}{M}\iint_{x=-\frac{a}{2}}^{x=+\frac{a}{2}} x\,dm = 0$.

By symmetry (or since y is even function), $\frac{1}{M}\iint_{y=0\ x=-\frac{a}{2}}^{x=+\frac{a}{2}\ \text{line } y=f(x)} y\,dm = \frac{2}{M}\int_{x=0}^{a/2}\int_{y=0}^{h-\frac{2h}{a}x} y\,\rho\,dy\,dx =$

$= \frac{4}{ah}\int_0^{a/2}\frac{(h-\frac{2h}{a}x)^2}{2}dx = \frac{2h}{a}\int_0^{a/2}(1-\frac{2}{a}x)^2dx =$

$= \frac{2h}{a}\left(-\frac{a}{2}\right)\frac{(1-\frac{2x}{a})^3}{3}\Big|_0^{a/2} = \frac{h}{3}\left(\frac{2x}{a}-1\right)^3\Big|_0^{a/2} = 0-\left(-\frac{h}{3}\right) = \frac{h}{3}$;

$\Longrightarrow$ so $\vec{r} = \begin{bmatrix}0\\h/3\end{bmatrix}$.

$\vec{r}_\Delta$ in original coordinates is $\begin{bmatrix}a/2\\0\end{bmatrix} + \vec{r} = \begin{bmatrix}a/2\\h/3\end{bmatrix}$.

2) $\vec{r}_\square = \begin{bmatrix}a/2\\a/2\end{bmatrix}$ (by symmetry or trivial calculation).

3) Then $\vec{r}_{CM} = \dfrac{\begin{bmatrix} a/2 \\ a/2 \end{bmatrix} \cdot \cancel{\rho} a^2 + \begin{bmatrix} a/2 \\ h/3 \end{bmatrix} \cdot \cancel{\rho} \cdot a \cdot h \frac{1}{2}(-1)}{\cancel{\rho} a^2 - \rho a h \frac{1}{2}} =$

$$= \frac{\begin{bmatrix} a/2 \\ a/2 \end{bmatrix} a + \begin{bmatrix} a/2 \\ h/3 \end{bmatrix} (-h/2)}{a - h/2} = \begin{bmatrix} a^2/2 - ah/4 \\ a^2/2 - h^2/6 \end{bmatrix} \cdot \frac{1}{a - h/2} = \begin{bmatrix} a/2 \\ h \end{bmatrix}$$

Then $\dfrac{a^2}{2} - \dfrac{h^2}{6} = h\left(a - \dfrac{h}{2}\right)$, $\quad \dfrac{h^2}{3} - ha + \dfrac{a^2}{2} = 0$

$$D = a^2 - \frac{2}{3} a^2 = \frac{a^2}{3},$$

$$h_{1,2} = \frac{a \pm \frac{a}{\sqrt{3}}}{\frac{2}{3}} = \frac{3}{2} a \left[1 \pm \frac{1}{\sqrt{3}}\right]$$

Since $h < a$, $h = \frac{3}{2} a \left(1 - \frac{1}{\sqrt{3}}\right) = \boxed{\frac{3 - \sqrt{3}}{2} a}$.