Lecture "Experimental Physics I"
(Prof. Dr. R. Seidel)

Lecture 30

Dispersion, wave packets & more

- Doppler effect
- Shock waves
- Huygen’s principle
- Reflection and refraction of waves
- Wave packets & dispersion
- Group velocity

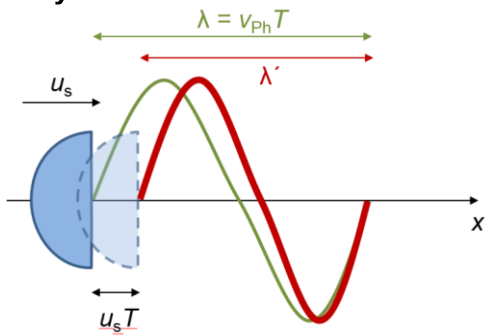
1) Waves of moving sources and observers

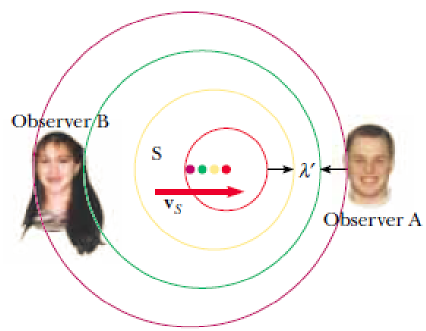
A) The Doppler effect

From our own experience we know that there is a frequency change when a “noisy” object passes by (imagine the car sound when standing at a highway). It starts with a higher frequency when the object is approaching and changes to lower frequencies when the object moves away. This frequency change is observed in everyday life for sound waves but also occurs for electromagnetic waves such as light. It is called the **Doppler-Effect**

Moving source with velocity u_s

Let us first try to understand the Doppler effect assuming a resting observer and a moving wave source, e.g. a loud speaker.
After the period T of the wave, the wave front moved by the distance $\lambda_0 = v_{Ph}T$. In this time, the source moved by the distance $u_S T$ such that the wave front that is 2π behind is not at position 0 but rather at $u_S T$. Thus the wave length decreased or increased depending on the motion direction (see figure below). This direction dependence can be also illustrated by drawing spherical wave fronts that get emitted from a moving source such that their centers become shifted with time. Thus, if the **object is moving towards the observer then the wave length shortens**, if it is **moving away from the observer the wave length increases**.





For a source moving towards the observer the wave length shortens by $u_S T$. Thus, we can write for the wave length of the wave from the moving source:

$$\lambda = (v_{Ph} - u_s)T = \frac{v_{Ph} - u_s}{f_0}$$

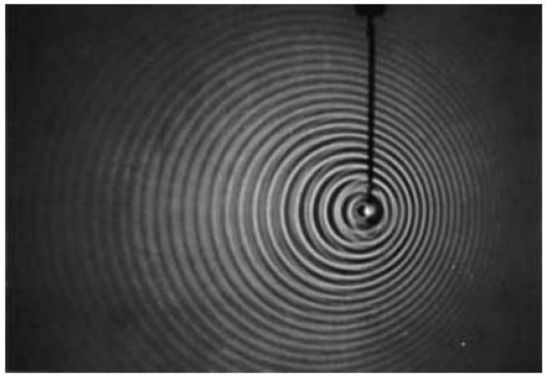
where f_0 is the wave frequency of the resting source. The frequency for the moving source can be calculated from the new wave length since the phase velocity of the medium is not dependent on the movement of the source:

$$f = \frac{v_{Ph}}{\lambda} = f_0 \frac{v_{Ph}}{v_{Ph} - u_s} = f_0 \frac{1}{1 - u_s/v_{Ph}}$$

From this equation we get for a source moving **towards the observer** with $u_s > 0$ a **frequency increase,** since λ decreases. For a source moving **away from the observer** with $u_s < 0$ we get **a frequency decrease** according to:

$$f = f_0 \frac{1}{1 + |u_s|/v_{Ph}}$$

since the wave length increases.
In agreement with our thoughts, one observes experimentally in a water tank (ripple tank) **compressed wave fronts before and expanded wave fronts distances behind** an object:



Experiments: The Doppler effect can be also experienced in other simple experiments:

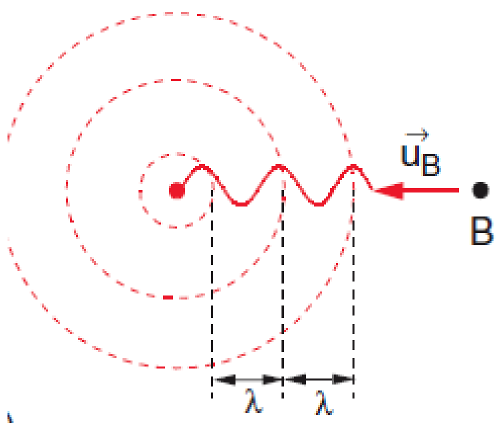
1) For a rotating pipe with open end one hears the sound frequency oscillating with the rotation frequency.
2) A quantitative measurement of the Doppler effect can be done with a moving wagon emitting ultrasound from a loudspeaker. The wagon moves away or approaches a microphone connected to a frequency analyzer:
   For the wagon movement we get: $v = 0.5\,m/10.8s = 0.046\,m/s$ such that we can calculate:

$$f_{approach} = 40.0058kHz\frac{1}{1-0.046/340} = 40.011(2)kHz$$
$$f_{retract} = 40.0058kHz\frac{1}{1+0.046/340} = 40.0004kHz$$

which agrees with the measured frequencies of 40.0111 kHz and 40.0005 kHz

Moving observer with velocity u_O

When the source is resting but the observer is moving with velocity u_O the frequency change is somewhat different. Here the wavelength of the wave does not change but rather an observer moving towards the source sees the wave fronts moving towards him/her with a phase velocity v^*_{Ph} that is increased by u_O:



The **increased frequency** for the altered phase velocity of the **observer moving towards the source** is thus given by:

$$f = \frac{v^*_{Ph}}{\lambda_0} = \frac{v_{Ph}+u_O}{\lambda_0} = \frac{v_{Ph}}{\lambda_0}(1+u_O/v_{Ph}) = f_0(1+u_O/v_{Ph})$$

For an observer moving away from the source the frequency is decreased according to:

$$f = f_0(1-|u_o|/v_{Ph})$$

When both **source and observer are moving** we can combine the frequency changes of both types of movement into one equation

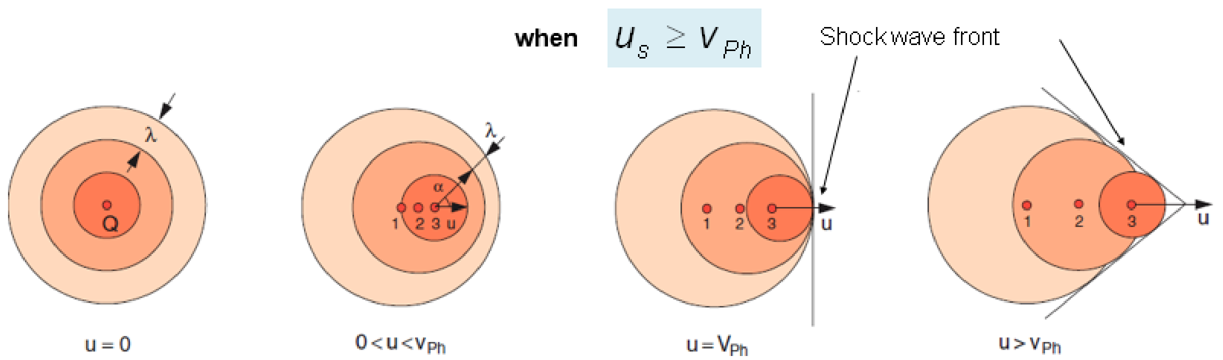
$$f = f_0 \frac{1 \pm u_o/v_{Ph}}{1 \mp u_s/v_{Ph}}$$

where the upper sign indicates an approach and the lower sign a movement away from the other object.

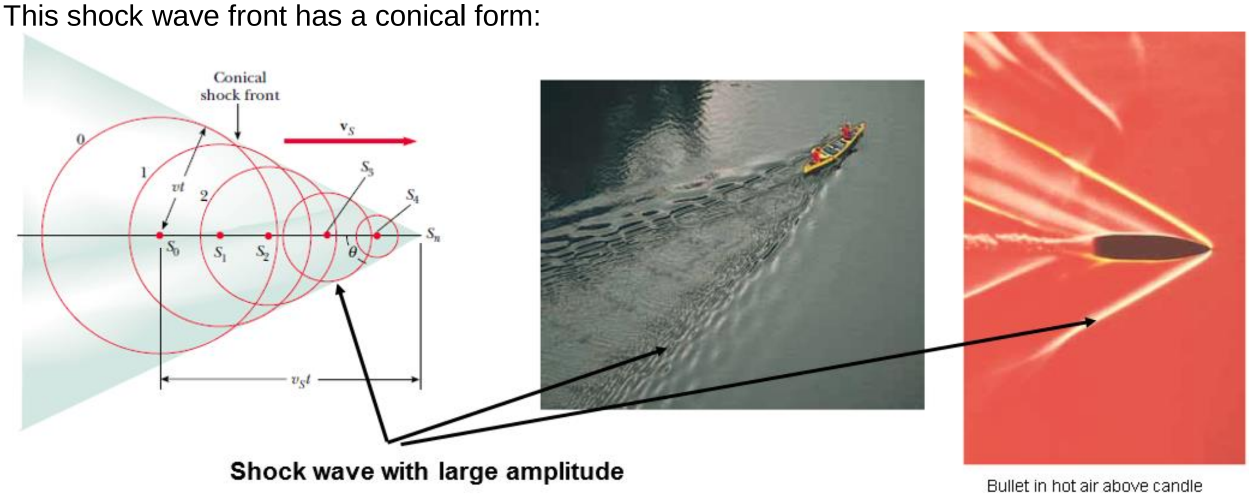
Note in case of electromagnetic waves:

- for light, only the relative velocity of observer and source matters since light is not coupled to a medium and special relativity holds
- due to the expansion of the universe, spectral lines emitted from stars appear typically red shifted. Using the Doppler shift one can study the universe expansion.
- in optical spectroscopy, spectral lines emitted from atoms appear to be broadened due to the thermal motion of the atoms and the resulting random Doppler shifts. This provides a resolution limit in spectroscopy

B) Shock waves



When a moving source is faster than the phase velocity of the emitted (sound) wave than the wave fronts superimpose each other in a shock wave front, which exhibits a sudden large amplitude (like a pulse). This is heard for example as the supersonic boom of aircrafts (see later). This shock wave front has a conical form:



Bullet in hot air above candle

According to the sketch above, the half opening angle of the shock wave front is given by:

$$\sin\theta = \frac{v_{Ph}}{u_s} = \frac{1}{M}$$

Where *M* is the so-called **Mach number,** i.e. the factor by which the source velocity is larger than the phase velocity. Supersonic speeds of aircrafts etc. are typically given by the Mach number that is the ration between object speed and speed of sound. (see supersonic boom video)

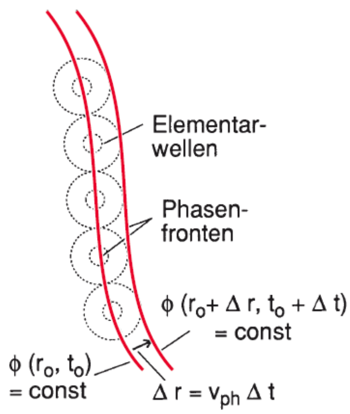
2) Huygens principle

So far, we looked at waves in one dimensions or waves that can freely propagate at least within a certain compartment. Now we want to introduce some basic principles that help to describe waves, in particular in presence of obstacles and extended media boundaries.

A) Huygens principle

One central principle that can explain many phenomena of waves in two- and three dimensions is the **Huygens principle**. It was formulated by **Christian Huygens in 1680** stating:

> "The propagation of waves in space can be described, by letting each point P on a wave front be a source of a new spherical wave (elementary wave). The full wave is then a superposition of an infinite number of spherical waves."

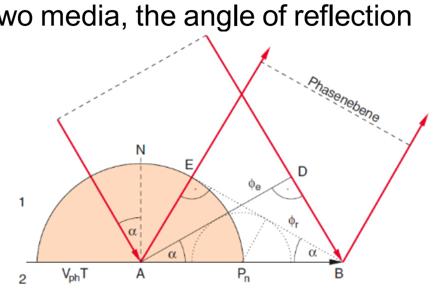


The **new wavefront at a time $t + \Delta t$ is then the tangent to all spherical waves that were emitted at time t.**
This principle can be illustrated by adding more and more emitters in the **wave workshop animation** (from 3 to 20 emitters and finally to an emitting line).
Huygens principle can conveniently be used to provide a simple explanation for a number of wave phenomena.

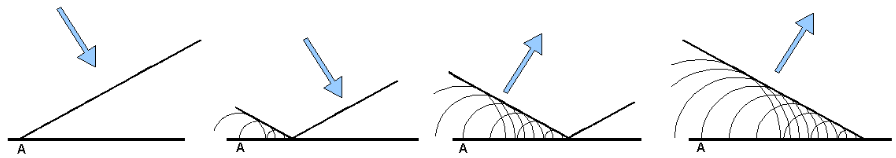
B) Reflection & refraction of waves

Similarly, to the one-dimensional case waves get split into a reflected and a transmitted component when encountering a medium with a different phase velocity.
For the **reflection of waves** on a media boundary in three-dimensions, we can formulate the **Law of reflection:**

> "When a plane wave is reflected at a plane boundary between two media, the angle of reflection is equal to the angle of incidence."



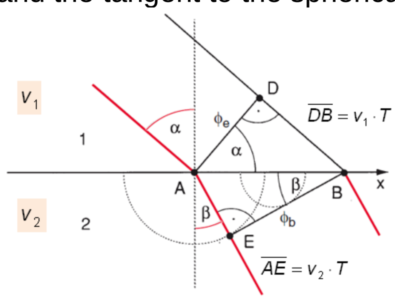
This can be easily understood from Huygens principle. Let us consider a plane wave that hits a reflecting surface between points A and B under an angle α to the surface normal. When the left edge of the wave front hits point A, a spherical wave expands from point A in all directions according to Huygen's principle. The right edge of the wave front is at point D and needs the time $T = \overline{DB}/v_{Ph}$ to reach point B that is located in normal direction to the wave front. The spherical wave from point a travels within T the distance $\overline{AE} = v_{Ph}T = \overline{DB}$, i.e the distance $\overline{DB}$. The new wave front is now given by the line through B and the tangent to the spherical wave front of A. The incoming wave front is defined by the rectangular triangle ABD and the outgoing wave front by the rectangular triangle ABE. Both share the same hypotenuse, another side of equal length ($\overline{AE = DB}$) and a right angle (90°) and are thus identical. Thus, the incident angle α must equal the angle of the reflected light.



Transmitted wave components that enter a medium experience a **change in their propagation direction**, which we call **refraction**. Refraction occurs due to the changed phase velocity, which we can also understand using Huygens principle using the same setting as before:
When the left edge of a wave front hits point A (see sketch below), a time T is required such that the right edge of the wave front at point D also reaches the media boundary at point B located in the normal direction to the wave front. The distance $\overline{DB}$ is set by the distance $\overline{AB}$ and the incidence angle α. Using the phase velocity in the initial medium v_1 and T, we can formulate:

$$\overline{DB} = \overline{AB}\ \sin\alpha = v_1 T$$

During time T, a hemispherical wave expands from point A inside the medium with phase velocity v_2. During this time the hemispherical wave from A expands by $\overline{AE} = v_2 T$. The new wave front is now given by the line through B and the tangent to the spherical wave front of A.



Considering the triangle ABE that defines the wave front of the refracted wave at angle β we can write

$$\overline{AE} = \overline{AB}\ \sin\beta = v_2 T$$

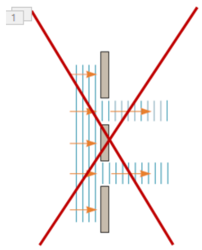
Division of the two equations provides the well-known **Snell's law** of geometric optics:

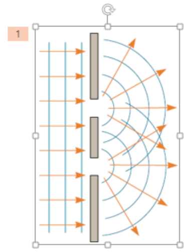
$$\frac{sin\,\alpha}{sin\,\beta} = \frac{v_1}{v_2}$$

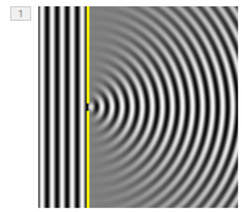
It relates incidence and refraction angle to the ratio of the phase velocities in the two media.

C) Diffraction

Huygen's principle demands that one cannot "cut-off" plane waves laterally, but rather that after passing an obstacle, part of the wave is moving "around the corner". This is called diffraction, which blurs the wave front and redirects it leading to interesting effects



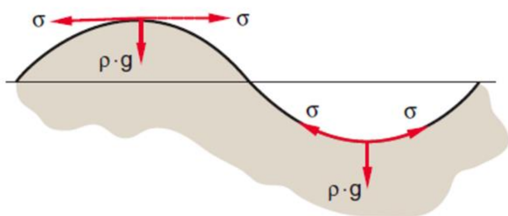




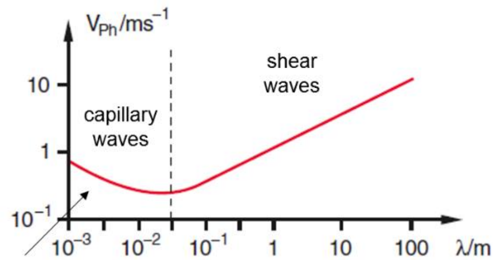
3) Dispersion, wave packets & group velocity

A) Dispersion

For waves on water surfaces, surface tension and gravity provide different contributions to the back-driving force that depend on the curvature and thus on the wave length of the wave excursion. This leads to a **phase velocity that depends on the wave length**



$$v_{\mathrm{Ph}} = \sqrt{\left(\frac{g\cdot\lambda}{2\pi} + \frac{2\pi\sigma}{\varrho\cdot\lambda}\right)\cdot\tanh\left(\frac{2\pi h}{\lambda}\right)}$$



A wave length-dependent phase velocity is called **dispersion** and the known relation for the phase velocity

$$v_{Ph} = v_{Ph}(\lambda) = \lambda f$$
$$v_{Ph} = v_{Ph}(k) = \frac{\omega}{k}$$

becomes the **DISPERSION RELATION.** If we have a wavelength-independent phase velocity the wave number depends linearly on the frequency. For a non-constant phase velocity, we have a non-linear relation.

In case of a plane wave with a single (sharp) frequency, we do not care about dispersion, since we can precisely describe the phase velocity of the plane wave. This changes when we consider pulse waves, where it is unclear which wave number or frequency they have

Experiment: Mechanical pulse through an aluminum rod, that gets reflected at the ends, which is an acoustic wave inside the rod. One measures the pulse oscillation with a piezoelectric sensor at the rod end. For the pulse velocity we get:

$$v_{Pulse} = \frac{2L}{T} = 2\times 1.3\frac{m}{0.5}ms = 5200\, m/s$$

(nominal value for speed of sound in Al is 5100 m/s)

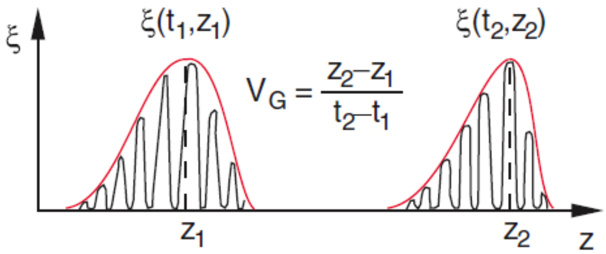
If the measured oscillation is actually a standing wave, we have for the fundamental mode:

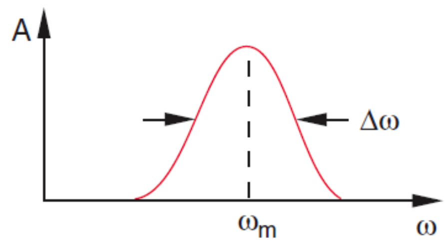
$$\frac{\lambda}{2} = L \quad \text{and} \quad v_{Phase} = \lambda f = 2L/T$$

in agreement with the calculation above.

In the following we want to understand pulse propagation and we will see that a **pulse is a superposition of waves of different wave numbers/frequencies, where dispersion changes the way how the pulse propagates**.

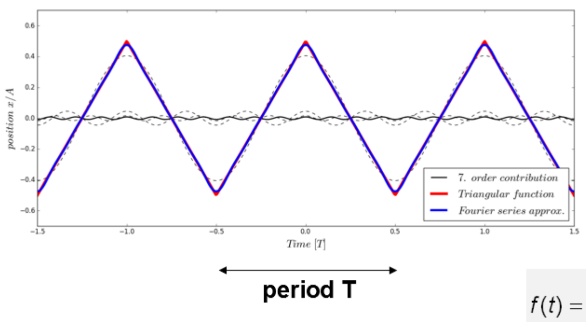
B) Static "pulse"

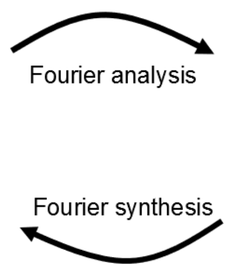




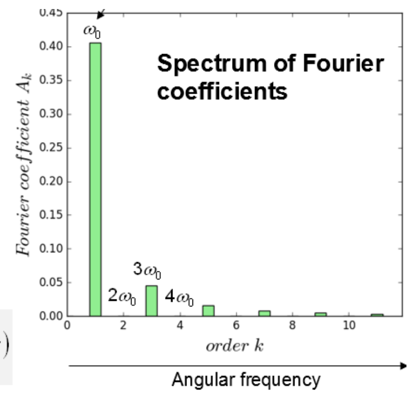
A pulse is a wave whose displacement is limited to a short region that travels in time. For simplicity let's look first at a pulse at $t = 0$. From Fourier series development and Fourier transform we know that we can **represent such pulse** by a **superposition of an infinite number of harmonic functions with different wave numbers**

REMEMBER: Fourier series (show slide)





$$f(t) = \sum_{k=0}^{\infty} (A_k \cos\omega_k t + B_k \sin\omega_k t)$$



Any function in a given (time) interval can be represented by a superposition of cosine and sine functions who fit exactly into the interval with an integer multiple of their periods:

$$f(t) = \sum_{n=0}^{\infty} (A_n \cos\omega_n t + B_n \sin\omega_n t)$$

for any signal between –T/2 and T/2, with $\omega_n = n\,(2\pi/T)$. The Fourier coefficients A_n and B_n are given by the overlap integrals of the sinusoidal functions with the signal (**see slide**).
In complex numbers we could express this as:

$$\xi(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\omega_n t}, \quad with \quad C_n = \frac{1}{T}\int_{-T/2}^{T/2} \xi(x) e^{-i\omega_n t} dt$$

Now let the interval T go to infinity and get this way a continuous Fourier coefficient spectrum, which is called the **(forward) Fourier transformation:**

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

According to the Fourier series we can describe the displacement $\xi(x)$ by an integral of these "Fourier coefficients":

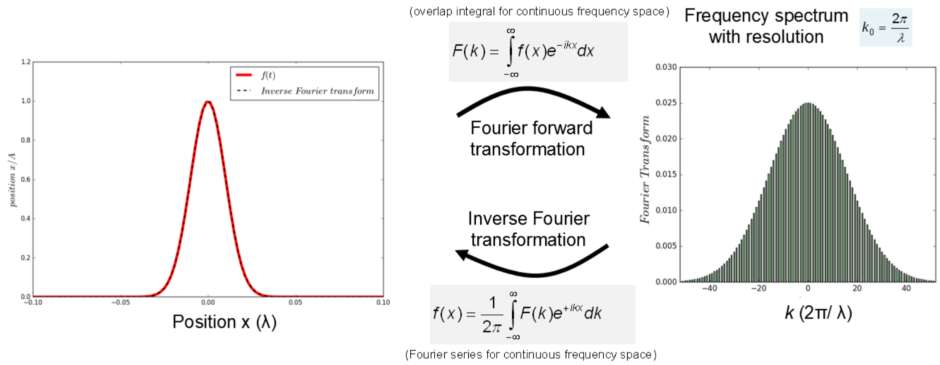
$$\xi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{+i\omega t} d\omega$$

which is called the **inverse Fourier transformation**. The factor 2π comes into play since we integrate over ω and not over the linear wave number.

The Fourier transform connects the **time domain reversibly with the frequency domain ($t \rightarrow \omega$)**. Similarly, we can connect the **spatial domain (position x) with the spatial frequency domain, i.e. the wavenumber k**. Fourier transform and inverse Fourier transform then become:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx\,, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{+ikx} dk$$

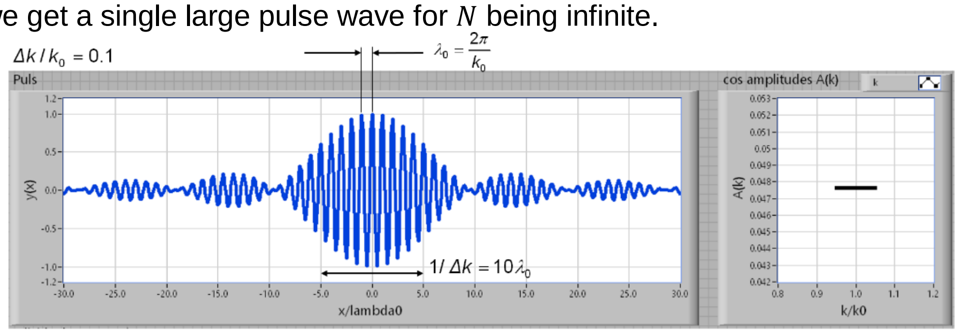
$F(k)$ is the wave number spectrum of $f(x)$, i.e. how much of a given "spatial frequency" is present in a signal, while the inverse Fourier transform is the synthesis of $f(x)$ from an infinite set of harmonic functions:



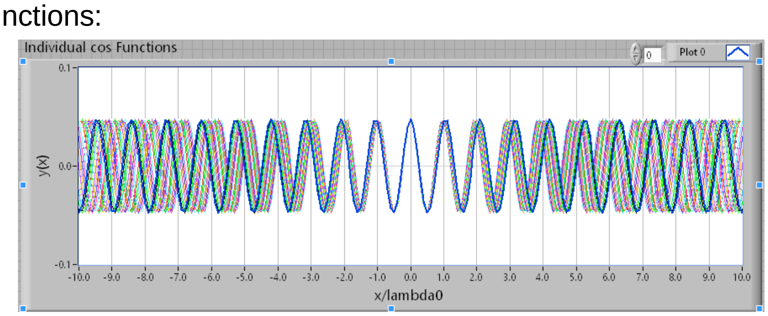
Let us try to understand this in practice by constructing stationary pulses from a set of sinusoidal functions (see **Labview animation**). To this end we add an increasing number of cosine functions picked from a fixed wave number interval $\boldsymbol{k_0 \pm \Delta k/2}$. For $2N + 1$ cosine functions we get:

$$y(x) = A \sum_{n=-N}^{n=N} \cos\left[\left(k_0 + \frac{n}{N}\frac{\Delta k}{2}\right) x\right]$$

For $N = 2$ we get the beat pattern that we know from the coupled oscillators, For $N > 2$ we get "beats" with large amplitudes periodically separated by "beats" with small amplitude. With larger N we get an increasing number of small beats and an increased distance between the large beats, such that we get a single large pulse wave for N being infinite.

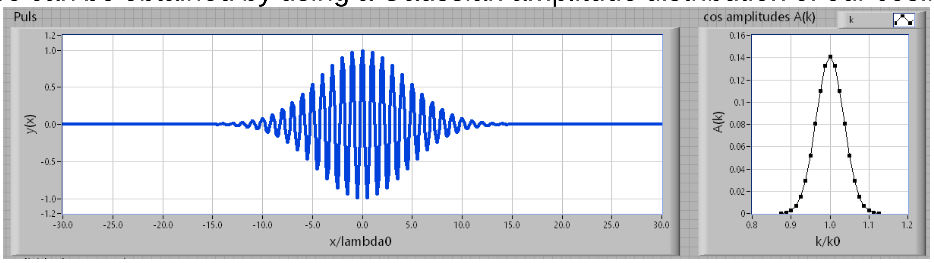


The periodicity of the wave inside the pulse is given by the average wave number according to $\lambda_0 = 2\pi k_0$while the pulse width is $\Delta x \sim 1/\Delta k$. The pulse width can be understood by plotting the involved cosine functions:

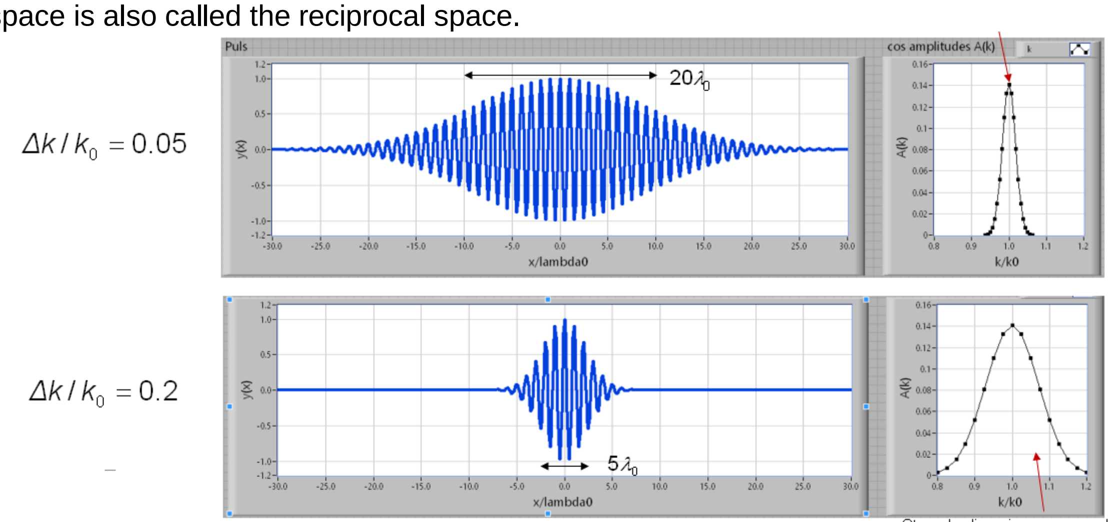


Only around $x = 0$ the functions are in phase and add up constructively. For larger x the cosine functions get fully dephased due to the different periodicities, such that they average out. The side lobes (small beats), however, always remain, if all cosine functions have the same amplitude.

A better pulse can be obtained by using a Gaussian amplitude distribution of our cosine functions:



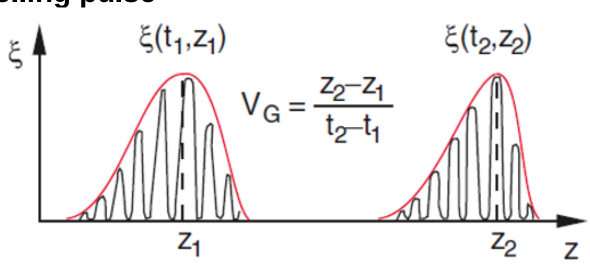
The superposition of **harmonic functions with wave numbers $\boldsymbol{k}$ in interval $\boldsymbol{k_0 \pm \Delta k/2}$** results in a **pulse of width $\boldsymbol{\Delta x \sim 2\pi/\Delta k}$**. Due to the reciprocal relationship between Δx and Δk the k –space is also called the reciprocal space.



Intuitively, one can understand this by considering that for little diverging wave numbers the dephasing occurs at a larger length scale. In contrast, a broader wave number distribution leads to a dephasing of the cosine functions on shorter length scale.

C) Wave packet as moving pulse

A **wave packet is a travelling pulse**



i.e. a short range **pulse that is obtained by the superposition of plane waves** with wave numbers k in the intervall $km \pm \Delta k/2$:

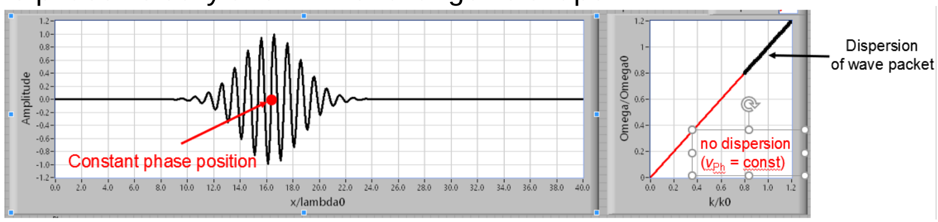
$$\xi(x,t) = \int_0^{\infty} A(k) e^{i(kx-\omega t)} dk$$

Note that v_{Ph} is not necessarily a constant but a function of k according to the dispersion relation:

$$v_{Ph}(k) = \omega/k$$

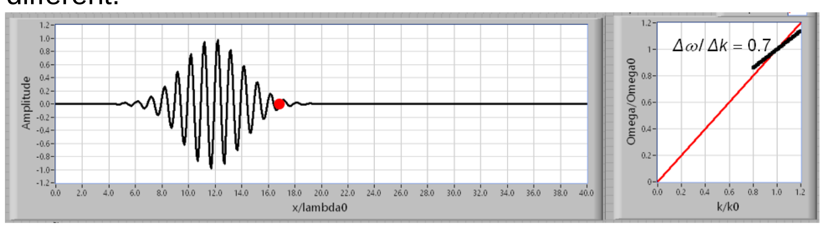
We can then distinguish (as shown by animations):

- **no dispersion** where all individual waves have the same phase velocity and the pulse will travel with the phase velocity and will not change its shape

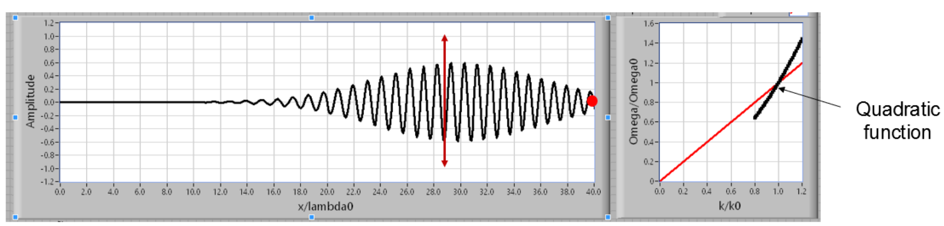


- **dispersion** individual plane wave components travel with a different velocity. In this case the **pulse will move with a different speed than the phase and may also change its shape**.

  If the **dispersion relation is linear** then the pulse shape is invariant but phase and pulse velocity are different:



  If the **dispersion relation is non-linear** then the pulse shape changes. In particular, the wave packet "smears out", i.e. delocalizes. This is actually the case for all material waves, such as elementary particles (electrons, protons etc) and forms the basis for Heisenberg's uncertainty principle:



The velocity of the wave packet is dependent on the dispersion relation. It is called **group velocity** and **describes the motion of the center of the wave packet**. For $\Delta k \ll k$, one can show that the **group velocity** is given by:

$$v_G = \frac{d\omega}{dk}$$

This explains also the results of the animations shown above.

Example: (not shown) An intuitive example for the group velocity is the superposition of two waves with slightly different wave numbers and frequencies. This provides a **travelling "beat wave"**. The individual wave functions are given by

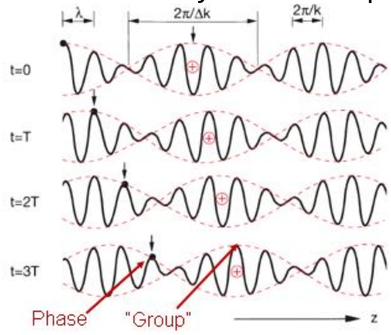
$$\xi_1 = A\cos(k_1 x - \omega_1 t)$$
$$\xi_2 = A\cos(k_2 x - \omega_2 t)$$

The superposition of the two waves gives after applying the trigonometric identity:

$$\xi = \xi_1 + \xi_2 = 2A \cdot cos\left(\underbrace{\frac{k_1 + k_2}{2}}_{k_m} x - \underbrace{\frac{\omega_1 + \omega_2}{2}}_{\omega_m} t\right) \cdot cos\left(\underbrace{\frac{k_1 - k_2}{2}}_{\Delta k/2} x - \underbrace{\frac{\omega_1 - \omega_2}{2}}_{\Delta\omega/2} t\right)$$

This can be seen as:

- a **wave with high spatial frequency $\boldsymbol{k = k_m}$** moving with phase velocity $v_{Ph} = \omega_m / k_m$
- a **modulation of the wave amplitude at low spatial frequency $\boldsymbol{k = \Delta k/2}$** moving with group velocity $v_G = \Delta\omega/\Delta k$. This describes the velocity of the beat pattern



With $v_1 = \omega_1/k_1$ and $v_2 = \omega_2/k_2$, we get for the group velocity

$$v_G = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{v_1 k_1 - v_2 k_2}{k_1 - k_2}$$

If $v_1 = v_2$, we get:

$$v_G = v_1 = v_{Ph}$$

Thus, **if the phase velocity is independent of the wavelength, phase and group velocity are equal.** Otherwise the "group" (beat pattern) moves at a different velocity than the phase.

Lecture 30: Experiments

1) Doppler effect: Rotating pipe where one hears the sound frequency oscillating with the ritation frequency

2) Doppler effect quantitative measurement: wagon with loudspeaker that moves away or approaches observer (microphone)

3) Pulse through an aluminum rod, that gets reflected at the ends. v_{Puls} = 2 L / T = 2*1.3 m/0.5 ms = 5200 m/s (table value 5100 m/s). If standing wave, fundamental mode: $\lambda/2 = L$; $v_{Phase} = \lambda f = 2L/T$