10. Momentum and Conservation Laws

The Chapters 4.1–4.5 of my lecture notes provides the background to solve the following exercises. Your solution to the problems 10.1–10.4 should uploaded

to your Moodle account

as a PDF-file

by Wednesday, Dec 13, 10:30 (with a grace time till the start of the seminars).

The parts marked by $\star$ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

The self-test problems serve to check you understanding of vectors.

The bonus problem provides the solution to a problem that we introduced in class. It might take some extra effort to solve.

Problems

Problem 1. Solving ODEs by separation of variables

Determine the solutions of the following ODEs

$\star$a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y\,\mathrm{e}^x$ such that $y(-\ln \mathrm{e}) = 3$

b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos^2 y}{\sin^2 x}$ such that $y(\pi/4) = 0$

c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3\,x^2\,y}{2y^2+1}$ such that $y(0) = 1$

d) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1+y^3}{x\,y^2\,(1+x^2)}$ such that $y(1) = 2$

Hint: Show and use that $\dfrac{1}{x\,(1+x^2)} = \dfrac{1}{x} - \dfrac{x}{1+x^2}$

Problem 2. Dimensional analysis and non-dimensionalization

We consider a droplet that is falling and deforming in response to the attacking gravity g and the friction forces. The deformation increases the effective cross section of the droplet such that we find the following equation of motion for its falling velocity v,

$$\dot{v} = -g - A\, v^3 \tag{10.1}$$

a) What is the dimension of A?

b) Why is it necessary that A takes a positive value?

c) Assume that the terminal velocity v_∞ of the droplet only depends on g and A. Adopt dimensional analysis to determine v_∞.
Remark: The terminal velocity is the constant falling velocity that will be adopted for $t \gg (A\, g^2)^{-1/3}$.

d) Determine v_∞ directly from Eq. (10.1). Discuss the relation of this result to the prediction of dimensional analysis.

e) Introduce now dimensionless units based on the reference velocity v_∞, and a time scale T. For which choice of T does the equation of motion take the form

$$\dot{v} = 1 - v^3$$

⋆f) Adopt separation of variables to determine $v(t)$.
Hint: Observe that $\dfrac{1}{1-v^3} = \dfrac{a}{1-v} + \dfrac{b}{v - \mathrm{e}^{2\mathrm{i}\pi/3}} + \dfrac{b}{v - \mathrm{e}^{-2\mathrm{i}\pi/3}}$ for an appropriate choice of a, b, and c.

Problem 3. Anharmonic oscillators

We consider a particle of mass m that takes a position x on a one-dimensional track. Its motion is subjected to a force

$$F(x) = -f\ \tanh \frac{x}{L}$$

a) What are the dimensions of the constant paramters f and L in the definition of the force?
Determine a length and a time scale based on the system parameters m, f, and L.
Provide the EOM of the particle, and the non-dimensional version of the EOM.

b) Determine the dimensionless potential energy $\Phi(x)$ of the particle in the force field $F(x)$.
Provide also the dimensionless total energy and show that it is conserved.

c) Sketch the dimensionless potential and the phase-space portrait.
Hint: Argue and use that the trajectories in phase space amount to contour lines of the energy.

$\star$ d) Perform a Taylor approximation of the force and of the potential for small amplitude oscillations $x \ll L$. This will show that the force is harmonic in the limit of small amplitude oscillations, i.e. $F \simeq -k\,x$ for sufficiently small x.

How is Hook's constant k related to f and L?

What is the period of small oscillations? How is it related to T?

$\star$ e) How does the emergence of anharmonicity show up in the shape of trajectories in phase space?

f) Perform a numerical integration of the trajectories in order to explore how anharmonicity effects the period of the oscillations.

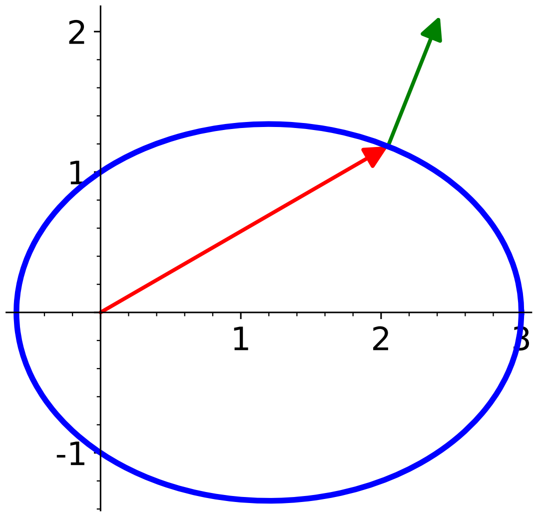
$\star$ g) What is changing in the present discussion when one considers the force

$$F_2(x) = -f\ \sinh(x/L)$$

Problem 4. Potential energy of a cart with a wobbly wheel

We consider a cart of mass M that is running on wobbly wheels. To explore the consequences of the wobbling on the potential energy we assume that the wheels take the form of ellipses, and that they both have the same form. Adopting polar coordinates the form is described by $\mathbf{q} = R(\theta)\,\hat{\boldsymbol{r}}(\theta)$ (red arrow in the sketch to the right) with

$$R(\theta) = \frac{R_0}{1 - \epsilon\, \cos\theta}$$



a) In the sketch the length scales are made dimensionless based on R_0. The red arrow denotes a vector from the wheel axle to its surface. Add a label indicating θ for this position, and specify which value of ϵ was adopted.
How does the shape change when adopting another value of ϵ: For which value will you obtain a circle? For which values of ϵ does one obtain ellipses?

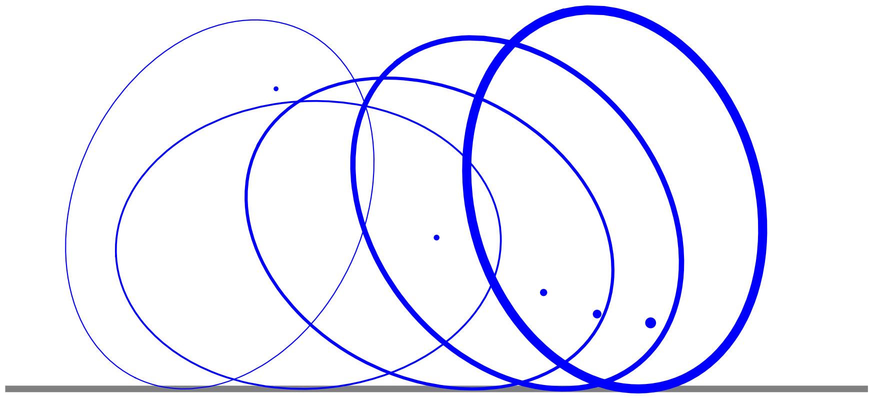
b) The green arrow shows the arrow $\hat{\boldsymbol{n}}(\theta)$ that is orthogonal to the wheel surface at position θ. Show that it takes the following form

$$\hat{\boldsymbol{n}}(\theta) = L\,\left(\hat{\boldsymbol{r}}(\theta) + \frac{c\,\sin\theta}{1 - \epsilon\cos\theta}\,\hat{\boldsymbol{\theta}}(\theta)\right)$$

Determine L and c in this expression.

Hint: 1. Argue that $\mathbf{v}(\theta) = \mathrm{d}\mathbf{q}(\theta)/\mathrm{d}\theta$ is tangential to the wheel surface.
2. Show that $\hat{\boldsymbol{n}}(\theta) \cdot \mathbf{v}(\theta) = 0$.

c) Let θ be the position where the wheel is touching the ground:



How does the potential energy of the cart depend on the orientation θ of the wheel?

Hint: You may assume that the mass M of the cart is dominated by its axle.

d) What is the smallest force required to hold the cart at a given position θ?
What will be the direction of this force?
What happens when you push in another direction?

e) Sketch the potential and the associated phase-space portrait in the $(\theta, \dot{\theta})$-plane for the motion of the cart.