Mathematics 1. Selected proofs
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Definite integral. Basic condition of integrability

Theorem. Assume f is bounded on $[a,b]$. Then f is integrable on $[a,b]$ if and only if $\forall\, \varepsilon > 0$ $\exists\, \delta(\varepsilon) > 0$ such that for any partition T of $[a,b]$ the following implication holds

$$\lambda(T) < \delta \quad \Longrightarrow \quad S(T) - s(T) < \varepsilon,$$

where $\lambda(T)$ is the mesh of the partition T.

Proof $\boxed{\Longrightarrow}$

1. Use the definition of the integrable function:

$$\exists\, I \in \mathbb{R}: \quad \forall\, \varepsilon > 0 \;\; \exists\, \delta\left(\tfrac{\varepsilon}{3}\right) > 0: \;\; \forall \text{ tagged partition } (T,\xi): \;\; T = \{x_j\}, \;\; \xi = \{\xi_j\}, \;\; \xi_j \in [x_{j-1}, x_j]$$
$$\lambda(T) < \delta \quad \Longrightarrow \quad I - \tfrac{\varepsilon}{3} < \sigma(T,\xi) < I + \tfrac{\varepsilon}{3}$$

2. Use the property of the Darboux sums:

$$s(T) = \inf_{\xi} \sigma(T;\xi), \quad I - \tfrac{\varepsilon}{3} < \sigma(T,\xi) < I + \tfrac{\varepsilon}{3} \quad \Longrightarrow \quad I - \tfrac{\varepsilon}{3} \le s(T) < I + \tfrac{\varepsilon}{3}$$
$$S(T) = \sup_{\xi} \sigma(T;\xi), \quad I - \tfrac{\varepsilon}{3} < \sigma(T,\xi) < I + \tfrac{\varepsilon}{3} \quad \Longrightarrow \quad I - \tfrac{\varepsilon}{3} < S(T) \le I + \tfrac{\varepsilon}{3}$$

3. Use the fact that both numbers $s(T)$ and $S(T)$ lay in the interval $[I - \frac{\varepsilon}{3}, I + \frac{\varepsilon}{3}]$:

$$\lambda(T) < \delta\left(\tfrac{\varepsilon}{3}\right) \quad \Longrightarrow \quad I - \tfrac{\varepsilon}{3} \le s(T) \le S(T) \le I + \tfrac{\varepsilon}{3} \quad \Longrightarrow \quad S(T) - s(T) \le \tfrac{2\varepsilon}{3} < \varepsilon.$$

Proof $\boxed{\Longleftarrow}$

4. Use the property if the Darboux sums:

$$\forall \text{ partitions } T \text{ and } T_0 \text{ of } [a,b] \qquad s(T) \;\le\; S(T_0) \quad \Longrightarrow \quad \begin{array}{c}\text{the set } \{s(T)\}_T \\ \text{is bounded from above}\end{array}$$

5. Define the number $I \in \mathbb{R}$. Denote

$$\text{Least upper bound axiom} \quad \Longrightarrow \quad \exists\, I := \sup\left\{ s(T) \;\middle|\; T \;\text{ is a partition of } [a,b] \right\}$$

where the supremum is taken over all possible partitions T of the interval $[a,b]$

6. Use the property of the sup (the least upper bound less or equal than some upper bound):

$$\forall \text{ partitions } T,\, T_0 \qquad s(T) \;\le\; S(T_0) \quad \Longrightarrow \quad \underbrace{\sup_T s(T)}_{\text{the least u.b.}} \;\le\; \underbrace{S(T_0)}_{\text{some u.b.}} \quad \Longrightarrow \quad I \le S(T_0)$$

7. Use the fact that the partition T_0 is also arbitrary:

$$\forall \, \text{partition } T_0 \qquad I \,\le\, S(T_0) \quad \Longrightarrow \quad \forall \, \text{partition } T \quad s(T) \le I \,\le\, S(T)$$

8. Use assumption and the definition of the Riemann integral:

$$\forall \varepsilon > 0 \quad \exists \, \delta(\varepsilon) > 0 : \quad \forall \text{ partition } T = \{x_j\}_{j=1}^N \quad \lambda(T) < \delta \quad \Longrightarrow \quad S(T) - s(T) < \varepsilon$$

Let $\xi = \{\xi_j\}_{j=1}^N$, $\xi_j \in [x_{j-1}, x_j]$. Then $\forall$ tagged partition (T, ξ) if $\lambda(T) < \delta(\varepsilon)$ then

$$\Longrightarrow \qquad \left. \begin{array}{c} s(T) \le \sigma(T,\xi) \le S(T) \\ s(T) \le I \le S(T) \\ S(T) - s(T) < \varepsilon \end{array} \right\} \quad \Longrightarrow \quad |I - \sigma(T,\xi)| \,\le\, \varepsilon$$

Use the definition of the Riemann integral: the lined colored in blue imply

$$\exists \, I \;=\; \int_a^b f(x)\, dx$$