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EXERCISE 4321

Date of Issue 22nd of June 2020

Problem 1: Descent **5 Points**

Solution

Since the satellite orbits on a (quasi-)circular path, the force of the gravitational pull is balanced by the centripetal force (1). This means, that the velocity increases with decreasing height (1).

$$F_G = G\frac{mM}{r^2} = F_P = \frac{mv^2}{r} \qquad \to v(r) = \sqrt{\frac{GM}{r}}$$

Therefore, the kinetic energy will increase with decreasing height and decreasing total energy (1).

$$E_{\text{kin}}(r) = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GmM}{r}$$

Since the total energy is decreasing (as given in description) and the kinetic is increasing, the potential energy has to decrease (1) much stronger than the increase of the kinetic energy.

$$E_{\text{pot}}(r) = \left(-\int_{\infty}^{r} \overrightarrow{F_G}\cdot \mathrm{d}\vec{r} = \int_{\infty}^{r} G\frac{mM}{r^2}\mathrm{d}r\right) = -G\frac{mM}{r}$$

So, in principal, slowly decelerating a satellite actually accelerates it in the long run, since it will take a lower orbit which in turn releases a lot of potential energy which gets converted in kinetic energy. The energy conversion even outgrows the kinetic energy loss of the deceleration.

$$E_{\text{tot}} = E_{\text{pot}} + E_{\text{kin}} = -\frac{G}{2}\frac{mM}{r}$$

If everything is understandable, logically clear written and not filled with additional confusing information, then give (1). This point can also be given if an error is found in the explanation as long the rest is understandable.
Formulas and Sketches per se give 0 points, because not further needed for the explanation after force balance and velocity statement and are here just for completeness. Only if set in the right context formulas and sketches can also be given points to.

Problem 2: Constant Friction, Locomotive **3 Points**

Solution

The maximum force is given by friction:

$$F_{max} = m_{loc} \cdot g \cdot \mu$$

Acceleration of the train is:

$$a_{train} = \frac{F}{m_{loc} + m_{wag}}$$

Equations of motion:

$$x = \frac{a_{train}}{2} t^2 \text{ and } v_1 = a_{train} \cdot t_1 \text{ so } \mathrm{a_{train}} = \frac{v^2}{2x}$$

$$F_{train} = \frac{v^2}{2x}(m_{loc} + m_{wag})$$

This Force is now provided by friction, so

$$m_{loc} \cdot g \cdot \mu = \frac{v^2}{2x}(m_{loc} + m_{wag})$$

$$m_{loc}\left(g \cdot \mu - \frac{v^2}{2x}\right) = \frac{v^2}{2x} \cdot m_{wag}$$

$$m_{loc} = \frac{v^2}{2x \cdot g \cdot \mu - v^2} \cdot m_{wag} = 81.9\ \mathrm{t}$$

Note: A normal train obviously accelerates much slower. This also shows why trains have additional breaks on wagons, otherwise it would take very long to stop the train.

Problem 3: Plane and Volume Integrals **5 · 2 Points**

(a)

This problem might be best solved in polar coordinates. Thus $dxdy = rdrd$ and the ring-shaped region $4 \leq x^2 + y^2 \leq 9$ transforms into the condition $2 \leq r \leq 3$ and

$$0 \leq \varphi \leq 2\pi$$

$$\iint_D x^2\, \mathrm{d}A = \int_2^3 \mathrm{d}r\, r \int_0^{2\pi} \mathrm{d}\varphi\, r^2 \cos^2 \varphi$$

$$= \pi \int_2^3 \mathrm{d}r\, r^3 = \pi \frac{1}{4} r^4 \Big|_2^3 = \frac{65\pi}{4}$$

(b)

$$\int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^1 x\, y\, z\, \mathrm{dzdydx} = \int_0^1 \mathrm{d}x\, x \int_0^1 \mathrm{d}y\, y \int_0^1 \mathrm{d}z\, z$$

$$= \int_0^1 \mathrm{d}x\, x \int_0^1 \mathrm{d}y\, y \left[\frac{1}{2}z^2\right]_0^1$$
$$= \frac{1}{2}\int_0^1 \mathrm{d}x\, x \left[\frac{1}{2}y^2\right]_0^1 = \frac{1}{8}$$

(c)

$$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=\sqrt{x^2+y^2}}^{2} x\,y\,z\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^1 \mathrm{d}x\, x \int_0^1 \mathrm{d}y\, y \left[\frac{1}{2}z^2\right]^2_{\sqrt{x^2+y^2}}$$
$$= \frac{1}{2}\int_0^1 \mathrm{d}x\, x \int_0^1 \mathrm{d}y\, y[4 - x^2 - y^2]$$
$$= \frac{1}{2}\int_0^1 \mathrm{d}x\, x \left[2y^2 - \frac{1}{2}x^2y^2 - \frac{1}{4}y^4\right]_0^1$$
$$= \frac{1}{2}\int_0^1 \mathrm{d}x \left[\frac{7}{4}x - \frac{1}{2}x^3\right]$$
$$= \frac{1}{2}\left[\frac{7}{8}x^2 - \frac{1}{8}x^4\right]_0^1$$
$$= \frac{3}{8}$$

(d)

$$\iint_R (x + y^2)\,\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_{x^2}^{x} \mathrm{d}y\,(x + y^2)$$
$$= \int_0^1 \mathrm{d}x \left[xy + \frac{1}{3}y^3\right]_{x^2}^{x}$$
$$= \int_0^1 \mathrm{d}x \left[x^2 - \frac{2}{3}x^3 - \frac{1}{3}x^6\right]$$
$$= \left[\frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{21}x^7\right]_0^1$$
$$= \frac{5}{42}$$

(e)

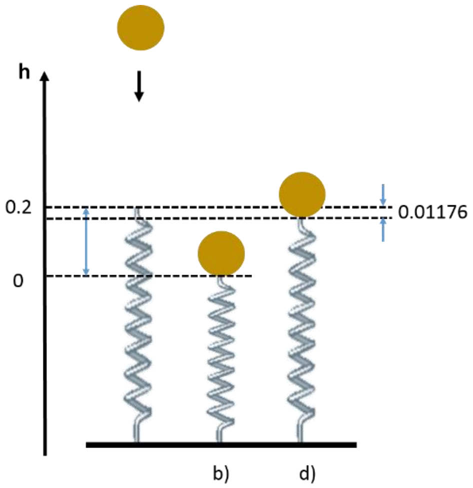
$z = 0$ is the xy-plane, $z = 1 - \sqrt{x^2 + y^2}$ is a cone intersecting the xy-plane at $\sqrt{x^2 + y^2} = 1$. (The solution $z = 1 + \sqrt{x^2 + y^2}$ has no intersection with the xy-plane and is not appropriate.) It is favorable to treat the problem in cylindrical coordinates.

$$\iiint_T [(x + y)^2 - z]\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \mathrm{d}r \int_0^{2\pi} r\,\mathrm{d}\varphi \int_0^{1-r} \mathrm{d}z\,[x^2 + 2xy + y^2 - z]$$
$$= \int_0^1 \mathrm{d}r\, r \int_0^{1-r} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\varphi\,[r^2 + r^2\sin(2\varphi) - z]$$
$$= 2\pi \int_0^1 \mathrm{d}r\, r \int_0^{1-r} \mathrm{d}z\,[r^2 - z]$$

$$= 2\pi \int_0^1 \mathrm{d}r\, r \left[r^2(1-r) - \frac{1}{2}(1-r)^2\right]$$
$$= 2\pi \int_0^1 \mathrm{d}r \left[-\frac{1}{2}r + r^2 + \frac{1}{2}r^3 - r^4\right]$$
$$= 2\pi \left[-\frac{1}{4} + \frac{1}{3} + \frac{1}{8} - \frac{1}{5}\right]$$
$$= \frac{\pi}{60}$$

Problem 4: Spring under gravity **2 + 1 + 1 + 1 Points**

(a)



(b)

When the mass reaches its lowest position, let's assume at this position the potential energy is zero.

$$E = mgh + \frac{1}{2}mv^2 + \frac{1}{2}D\Delta h^2$$
$$D = 2.5\frac{\mathrm{N}}{\mathrm{cm}} = 250\frac{\mathrm{N}}{\mathrm{m}}$$

According to energy conservation law, the energy at the position just before the mass touch the spring should be equal to the energy at the lowest position. So, we got

$$E = 0.3 \cdot 9.8 \cdot 0.2 + \frac{1}{2} \cdot 0.3 \cdot {v_1}^2 + 0 = 0 + 0 + \frac{1}{2} \cdot 250 \cdot 0.2^2 \,.$$

We got

$$v_1 = 5.42\frac{m}{s}.$$

(c)

Even when the mass is doubled, the velocity of the mass just before touching the spring won't change.

With the energy conservation law similar to the question b),

$$\mathrm{E} = \mathrm{mgh} + \frac{1}{2}mv^2 + \frac{1}{2}D\Delta h^2$$

$$\mathrm{E} = 0.6 \cdot 9.8 \cdot \Delta h + \frac{1}{2} \cdot 0.6 \cdot {v_1}^2 + 0 = 0 + 0 + \frac{1}{2} \cdot 250 \cdot \Delta h^2$$

We got,

$$125\Delta h^2 - 5.88\Delta h - 8.822 = 0$$

$$\Delta h = \frac{5.88 \pm \sqrt{5.88^2 + 4 * 125 * 8.822}}{250} = 0.29\,, -0.24$$

So, the distance is $29\ \mathrm{cm}$.

(d)

At this point, the mass is sitting on the spring and the system reaches its equilibrium state. With Hooke's law:

$$\mathrm{F} = \mathrm{D}\Delta x = mg\,,$$

we got

$$\Delta x = \frac{mg}{D} = 0.3 * \frac{9.8}{250} = 0.01176\, m = 1.176\ \mathrm{cm}\,.$$