Mathematics 1. Selected proofs
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Mean value formula and fundamental theorem of calculus

THEOREM. $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b] \implies \exists\, c \in [a,b]$:

$$f(c) \;=\; \frac{1}{b-a} \int\limits_a^b f(x)\,dx$$

PROOF.

1. Use the extreme value theorem:

$$f \text{ is continuous on } [a,b] \quad \implies \quad f \text{ is bounded on } [a,b]$$

Denote

$$m \;:=\; \inf_{x\in[a,b]} f(x), \qquad M \;:=\; \sup_{x\in[a,b]} f(x) \qquad \implies \qquad R(f) \;=\; [m,M]$$

2. Use the property of the definite integral:

$$\forall\, x \in [a,b] \qquad m \;\le\; f(x) \;\le\; M \qquad \implies \qquad m\,(b-a) \;\le\; \int\limits_a^b f(x)\,dx \;\le\; M\,(b-a)$$

3. Use the intermediate value theorem:

$$y_0 \;:=\; \frac{1}{b-a} \int\limits_a^b f(x)\,dx \qquad \implies \qquad m \;\le\; y_0 \;\le\; M \qquad \implies \qquad y_0 \in R(f)$$

$$f \text{ is continuous on } [a,b] \quad \implies \quad \exists\, c \in [a,b]: \quad f(c) = y_0 \quad \implies \quad f(c) \;=\; \frac{1}{b-a} \int\limits_a^b f(x)\,dx$$

THEOREM. Assume $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$. Define $\Phi : [a,b] \to \mathbb{R}$,

$$\Phi(x) \;:=\; \int\limits_a^x f(t)\;dt, \quad x \in [a,b] \qquad \text{— the } \textit{integral with variable upper limit.}$$

If f is continuous on $[a,b]$ then Φ is differentiable on $[a,b]$ and

$$\forall\, x_0 \in (a,b) \qquad \Phi'(x_0) \;=\; f(x_0).$$

PROOF.

4. Use the property of the definite integral:

$$\forall\, x \in [a,b], \quad x > x_0 \qquad \Longrightarrow \qquad \frac{\Phi(x) - \Phi(x_0)}{x - x_0} \;=\; \frac{1}{x - x_0} \int\limits_{x_0}^{x} f(t)\, dt$$

5. Use the mean value formula:

$$f \;\text{ is continuous on } [x_0, x] \quad \Longrightarrow \quad \exists\, c_x \in [x_0, x] : \qquad \frac{1}{x - x_0} \int\limits_{x_0}^{x} f(t)\, dt \;=\; f(c_x)$$

6. Use the two policemen theorem and continuity of f:

$$x_0 \le c_x \le x \qquad \Longrightarrow \qquad c_x \underset{x \to x_0}{\longrightarrow} x_0 \qquad \overset{f \text{ is continuous}}{\Longrightarrow} \qquad f(c_x) \underset{x \to x_0}{\longrightarrow} f(x_0)$$

7. Use the definition of the derivative

$$\exists\, \Phi'(x_0) \;=\; \lim_{x \to x_0} \frac{\Phi(x) - \Phi(x_0)}{x - x_0} \;=\; \lim_{x \to x_0} f(c_x) \;=\; f(x_0)$$

8. Consider the case $x < x_0$:

$$x < x_0 \quad \Longrightarrow \quad \frac{\Phi(x) - \Phi(x_0)}{x - x_0} \;=\; \frac{1}{x - x_0} \int\limits_{x_0}^{x} f(t)\, dt \;=\; \frac{1}{x_0 - x} \int\limits_{x}^{x_0} f(t)\, dt \;\overset{c_x \in [x, x_0]}{=}\; f(c_x) \;\to\; f(x_0)$$