Problem 1: **Geostationary Satellite** **3 + 3 Points**

A geostationary satellite is on a circular orbit around Earth above the equator. Later on, the satellite collides with some debris and loses 50 % of its speed. The debris does not deflect the satellite's direction of motion.

a. Before the collision: Calculate the height of the satellite above the ground and its speed.
b. After the collision: Will the satellite collide with the earth? Provide a calculation.

Hint: It may help to calculate the smallest possible orbit around the earth for the satellite.

Solution

For part (a) the word "geostationary" is key. From the perspective of an observer on the earth, a geostationary satellite "stays" above a certain spot of the earth. From perspective of an outside observer, e.g. moon or sun, this means that while the earth is rotating and the spot is moving away from its initial position, the satellite has to follow the movement in order to "stay" at the same position. Or in simple mathematical language: the angular velocity ω of earth and satellite has to be the equal.
The geostationary orbit was already calculated in a previous exercise, however, in a reversed way. One has to relate the gravitational pull with the centripetal force:

$$F_G = F_{\mathrm{CP}}$$

$$G\frac{mM_{\mathrm{E}}}{r^2} = m\omega^2 r$$

$$\rightarrow\ r = \sqrt[3]{\frac{M_{\mathrm{E}}G}{\omega^2}}\,.$$

The radius r is here measured from the centre of the earth and the angular velocity of the earth $\omega = 2\pi/T_{24}$ has to be plugged in. One has to subtract the radius of the earth to retrieve the asked height above the ground

$$R_{\mathrm{ground}} = \sqrt[3]{\frac{M_{\mathrm{E}}GT_{24}^2}{4\pi^2}} - R_{\mathrm{E}}\,.$$

The solution to (b) is already given in the test exam solution. In fact, this sub-task is identical to the comet task of the test exam. However, one can simply check for a special case for a guaranteed collision. The smallest possible orbit would be $r = R_{\mathrm{E}}$. One can easily calculate the angular momentum and total energy of this orbit:

$$F_G = F_{CP}$$

$$G\frac{mM_\mathrm{E}}{r^2} = \frac{mv^2}{r}$$

$$v_\mathrm{low} = \sqrt{\frac{GM_\mathrm{E}}{R_\mathrm{E}}}$$

$$L = \vec{r} \times \vec{p} = m\sqrt{GM_\mathrm{E}r}\,, \qquad \rightarrow \frac{L_\mathrm{low}}{m} = \sqrt{GM_\mathrm{E}R_\mathrm{E}}$$

$$E = \frac{1}{2}mv^2 - G\frac{mM_\mathrm{E}}{r} = \frac{m}{2r^2}\left(\frac{L}{m}\right)^2 - G\frac{mM_\mathrm{E}}{r}\,, \qquad \rightarrow \frac{E_\mathrm{low}}{m} = +\frac{1}{2}\frac{GM_\mathrm{E}}{R_\mathrm{E}}$$

One can check first whether both the total energy and the angular momentum is larger than for the smallest possible orbit. If only one is smaller than for the minimal orbit, the satellite will crash at some point on its trajectory. However, even if both are larger than the minimum orbit, it is not guaranteed that the satellite will not crash! Unfortunately, we used values where both are larger and the satellite will still crash…

$$L_\mathrm{sat} = 64.728 \cdot 10^9\ \mathrm{Js}\,,$$

$$L_\mathrm{low} = 50.624 \cdot 10^9\ \mathrm{Js}\,,$$

$$\frac{E_\mathrm{tot,sat}}{m} = -8.7821 \cdot 10^6\,\frac{\mathrm{J}}{\mathrm{kg}}\,,$$

$$\frac{E_\mathrm{tot,low}}{m} = -34.341 \cdot 10^6\,\frac{\mathrm{J}}{\mathrm{kg}}\,.$$

If the satellite would lose ¾ of its speed instead of only ½ after collision, it would have been an easy and short calculation.

Problem 2: Gravity, Flying to Mars **$2 + 2 + 1$ Points**

A space probe ($m_P = 600\ \mathrm{kg}$) is flying to the Mars (radius $r_M = 3390\ \mathrm{km}$, mass $M_M = 6.39 \cdot 10^{23}\ \mathrm{kg}$). At a height $h_1 = 122.5$ km over the surface of the Mars the speed of the probe is $v_1 = 21\,000\ \mathrm{km/h}$. The parachute is supposed to open at a height of $h_2 = 11$ km and a speed of $v_2 = 1650\ \mathrm{km/h}$.

a. How much work has been done to decelerate the probe to this point? (Gravity is not constant here).
b. Unfortunately, the parachute doesn't open. What is the speed of impact? (Neglect any friction, assume a constant gravity of $g = 3.69$ m/s^2.)
c. In another attempt the parachute actually does open. It decelerates the probe with a speed dependent Force $F_p = c_w \cdot v$, with $c_w = 450$ kg/s. What is the terminal speed? (Assume there is enough time to reach the terminal speed)

Solution

(a)

Total energy of the system (assuming $U_{gravity}(\infty) = 0$):

$$E_{total} = U_{gravity}(r) + E_{kinetics}(v) = -G\frac{M_m \cdot m_p}{r} + \frac{m \cdot v^2}{2}$$

At the h1 full energy is:

$$E_1 = -\,G\frac{M_m \cdot m_p}{r_m + h_1} + \frac{m \cdot {v_1}^2}{2}$$

At the h2 full energy is:

$$E_2 = -\,G\frac{M_m \cdot m_p}{r_m + h_2} + \frac{m \cdot {v_2}^2}{2}$$

Change in the full energy show the work done by external forces to decelerate the probe:

$$W = \int \vec{F} \cdot \overrightarrow{dr} = E_2 - E_1 = -G\frac{M_m \cdot m_p}{r_m + h_2} + G\frac{M_m \cdot m_p}{r_m + h_1} + \frac{m \cdot {v_2}^2}{2} - \frac{m \cdot v_1^2}{2}$$
$$\approx 6.674 \cdot 10^{-11} \cdot 6.39 \cdot 10^{23} \cdot 600$$
$$\cdot \left(-\frac{1}{3390000 + 11000} + \frac{1}{3390000 + 122500}\right) + 300 \cdot (458^2 - 5833^2)$$
$$\approx -2.39 \cdot 10^8 - 1.01 \cdot 10^{10} \approx$$
$$\approx -1.04 \cdot 10^{10}\, J$$

(b)

E_{total} is conserved in this case, so $\Delta U_{gravity} = -\Delta E_{kinetics} \rightarrow$

$$mg\Delta h = \frac{m \cdot {v_{impact}}^2}{2} - \frac{m \cdot {v_2}^2}{2} \rightarrow$$
$${v_{impact}}^2 = 2g\Delta h + {v_2}^2 \approx 2 \cdot 3.69 \cdot 11000 + 458^2 \rightarrow$$
$$v_{impact} = \sqrt{2g\Delta h + {v_2}^2} \approx 539.4\ \frac{m}{s}$$

(c)

The steady state will be reached when $\overrightarrow{F_p} = -\overrightarrow{F_{grav}}$ so the resulting force is 0.

Then, $c_w \cdot v = m \cdot g \rightarrow v_{terminal} = \frac{m \cdot g}{c_w} = 4.92\ \frac{m}{s}$

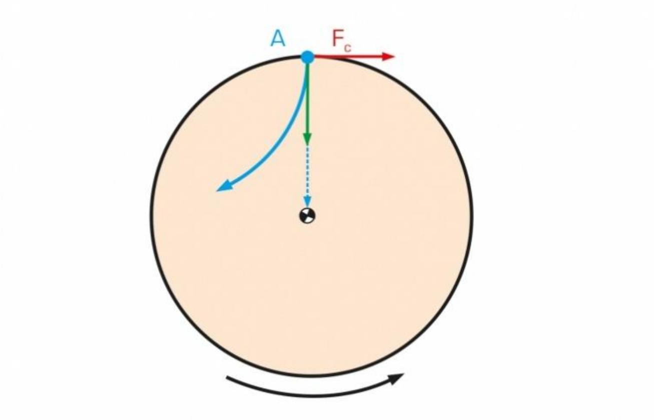
Rotating disk & Coriolis force **1+3 points**

We consider a rotating disk with a diameter of 4.0 m. The angular velocity of the disk is exactly such that the centrifugal acceleration at the edge of the disk corresponds to twice the acceleration due to gravity.
Now a body of mass 1.5 kg moves with the constant speed 1 m/s on the disk in the direction of the center of the disk (axis of rotation).

a) Determine the direction in which the Coriolis force acts with respect to the direction of rotation of the disk!
b) Calculate the Coriolis force acting on the body!

Solution:

a) 1 point



The direction of the Coriolis force acts opposite to the direction of rotation of the disk.

b) 3 points

$$F_{Cor} = 2m(v' \times \omega) = 2m \cdot |v'| \cdot |\omega| \cdot \sin(\theta) = 2m \cdot |v'| \cdot |\omega| \quad (1)$$

$|v'|$ and $|\omega|$ are perpendicular

v' - velocity of the object relative to the rotating reference frame

ω - angular velocity, of the rotating reference frame relative to the inertial frame

$$\omega = \frac{v_d}{r} \quad (2)$$

v_d - velocity at the edge of the disk

$$2F_G = F_{Cen}$$

$$2mg = \frac{m \cdot {v_d}^2}{r}$$

$$v_d = \sqrt{2gr} \quad (3)$$

(2) & (3) in (1) $F_{Cor} = 2m \cdot v' \cdot \frac{\sqrt{2gr}}{r}$

$$F_{Cor} = 2 \cdot 1{,}5\ kg \cdot 1\frac{m}{s} \cdot \frac{6{,}26\frac{m}{s}}{2m} = 9{,}39N$$

Elastic head-on collision **4+1+1 points**

In the center-of-mass reference frame a particle with mass m_1 and momentum p_1 makes an elastic head-on collision with a second particle of mass m_2 and momentum $p_2 = -p_1$. After the collision its momentum is p'_1.

a) Write the total kinetic energy in terms of m_1, m_2, and p_1 and the total final energy in terms of m_1, m_2, and p'_1.
b) Show that $p'_1 = \pm p_1$.
c) If $p'_1 = -p_1$, the particle is merely turned around by the collision and leaves with the speed it had initially. What is the situation for the $p'_1 = +p_1$ solution?

Solution

91 •• In the center-of-mass reference frame a particle with mass m_1 and momentum p_1 makes an elastic head-on collision with a second particle of mass m_2 and momentum $p_2 = -p_1$. After the collision its momentum is p_1'. Write the total kinetic energy in terms of m_1, m_2, and p_1 and the total final energy in terms of m_1, m_2, and p_1', and show that $p_1' = \pm p_1$. If $p_1' = -p_1$, the particle is merely turned around by the collision and leaves with the speed it had initially. What is the situation for the $p_1' = +p_1$ solution?

Picture the Problem The total kinetic energy of a system of particles is the sum of the kinetic energy of the center of mass and the kinetic energy relative to the center of mass. The kinetic energy of a particle of mass m is related to its momentum according to $K = p^2/2m$.

Express the total kinetic energy of the system:

$$K = K_{rel} + K_{cm} \quad (1)$$

Relate the kinetic energy relative to the center of mass to the momenta of the two particles:

$$K_{rel} = \frac{p_1^2}{2m_1} + \frac{p_1^2}{2m_2} = \frac{p_1^2(m_1+m_2)}{2m_1m_2}$$

Express the kinetic energy of the center of mass of the two particles:

$$K_{cm} = \frac{(2p_1)^2}{2(m_1+m_2)} = \frac{2p_1^2}{m_1+m_2}$$

Substitute in equation (1) and simplify to obtain:

$$K = \frac{p_1^2(m_1+m_2)}{2m_1m_2} + \frac{2p_1^2}{m_1+m_2} = \frac{p_1^2}{2}\left[\frac{m_1^2+6m_1m_2+m_2^2}{m_1^2m_2+m_1m_2^2}\right]$$

In an elastic collision:

$$K_i = K_f$$
$$= \boxed{\frac{p_1^2}{2}\left[\frac{m_1^2+6m_1m_2+m_2^2}{m_1^2m_2+m_1m_2^2}\right]}$$
$$= \boxed{\frac{p_1'^2}{2}\left[\frac{m_1^2+6m_1m_2+m_2^2}{m_1^2m_2+m_1m_2^2}\right]}$$

Simplify to obtain:

$$(p_1')^2 = (p_1)^2 \Rightarrow \boxed{p_1' = \pm p_1}$$

and if $p_1' = +p_1$, the particles do not collide.