Homework 10 – Stanislav 3720433

Problem 1

a) $\frac{dy}{dx} = ye^x$, $y(-\ln e) = 3$

$$\int_3^y \frac{dy'}{y'} = \int_{-\ln e}^{x} e^{x'}dx'$$

$$\ln|y'|\Big|_3^y = e^{x'}\Big|_{-\ln e}^{x} \Rightarrow \ln|y| - \ln 3 = e^x - \underbrace{e^{-\ln e}}_{1}$$

$$\ln\left(\frac{|y|}{|3|}\right) = e^x - 1$$

$$|y| = 3e^{e^x-1} \Rightarrow \underline{y = 3e^{e^x-1}}$$

b) $\frac{dy}{dx} = \frac{\cos^2 y}{\sin^2 x}$ $\quad y(\pi/4) = 0$

$$\int \frac{dy}{\cos^2 y} = \int \frac{dx}{\sin^2 x}$$

$$\tan y = -\cot x + C$$

$$0 = -1 + C,\ C = 1 \Rightarrow \tan y = -\cot x + 1,$$

$$\underline{y = \arctan(1 - \cot x) + \pi n,\ n \in \mathbb{Z}}$$

c) $\frac{dy}{dx} = \frac{3x^2 y}{2y^2+1}$, $y(0) = 1$

$$\int \frac{2y^2+1}{y}dy = 3\int x^2 dx$$

$$\int 2y + \frac{1}{y}dy = y^2 + \ln|y| = x^3 + C,\ 1 = C,$$

$\boxed{y^2 + \ln|y| = x^3 + 1}$, solution curve $y^2 + \ln|y| - x^3 = 1$

d) $\frac{dy}{dx} = -\frac{1+y^3}{xy^2(1+x^2)}$, $y(1) = 2$

$$\int \frac{-y^2}{1+y^3}dy = \int \frac{dx}{x(1+x^2)} \qquad \left[\frac{1}{x} - \frac{x}{1+x^2} = \frac{1+x^2-x^2}{x(x^2+1)} = \frac{1}{x(x^2+1)}\right]$$

$$-\frac{1}{3}\ln|1+y^3| = \int \frac{1}{x} - \frac{x}{1+x^2}dx = \ln|x| - \frac{1}{2}\ln(1+x^2) + C,$$

$$-\frac{1}{3}\ln 9 = \underbrace{\ln 1}_{0} - \frac{1}{2}\ln 2 + C,\ C = \frac{1}{2}\ln 2 - \frac{1}{3}\ln 9 \text{ (no need, see below)}$$

So $-\frac{1}{3}\ln|1+y^3| = \ln|x| - \frac{1}{2}\ln(1+x^2) + \underbrace{\frac{1}{2}\ln 2 - \frac{1}{3}\ln 9}_{C}$

$$\ln\left|\left[(1+y^3)^{-\frac{1}{3}} x^{-1} (1+x^2)^{\frac{1}{2}}\right]\right| = C$$

$\left|(1+y^3)^{-\frac{1}{3}} x^{-1} (1+x^2)^{\frac{1}{2}}\right| = e^C$, so can always write

$$(1+y^3)^{-\frac{1}{3}} x^{-1} (1+x^2)^{\frac{1}{2}} = A \Rightarrow$$

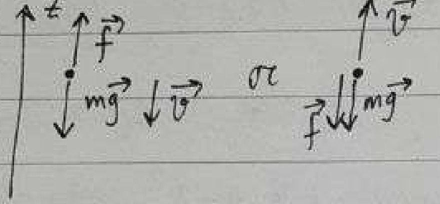
$$(1+2^3)^{-\frac{1}{3}} \cdot 1 \cdot 2^{\frac{1}{2}} = A = 9^{-\frac{1}{3}} \cdot \sqrt{2}.$$

$$\Rightarrow y = \sqrt[3]{-1 + \left(Ax(1+x^2)^{-\frac{1}{2}}\right)^{-3}} = \sqrt[3]{-1 + 9 \cdot 2^{-\frac{3}{2}} \cdot x^{-3}(1+x^2)^{\frac{3}{2}}}.$$

Problem 2 $\dot{v} = -g - Av^3$

a) $[A] = \frac{s}{m^2}$ (since $\frac{s}{m^2} \cdot \frac{m^3}{s^3} = \frac{m}{s^2} = [g]$)

b)



(opposite direction to $\vec{v}$)

c) $v_\infty = \sqrt[3]{\frac{g}{A}} \cdot C$ (because $\left[\sqrt[3]{\frac{g}{A}}\right] = \left(\frac{m}{s^2} \cdot \frac{m^2}{s}\right)^{\frac{1}{3}} = \frac{m}{s} = [v_\infty]$), $[C] = 1$

d) directly $\dot{v} = 0 = -g - Av^3$, so $v_\infty = \sqrt[3]{\frac{-g}{A}} < 0$

(note, sign info was hidden into C on dim. analysis)

e) Natural time scale is $T = (Ag^2)^{-\frac{1}{3}}$, $\left([(Ag^2)^{-\frac{1}{3}}] = s^{-\frac{1}{3}} m^{\frac{2}{3}} [g]^{-\frac{2}{3}} = \right.$ and vel. scale is v_∞. $= s^{-\frac{1}{3}} m^{\frac{2}{3}} \cdot m^{-\frac{2}{3}} \cdot s^{\frac{4}{3}} = s^1 = [t]$)

$$\frac{dv}{dt} = -g - Av^3$$

$\tilde{v}, \tau$ are dim-less: $\tilde{v} = \frac{v}{v_\infty}$, $\tau = \frac{t - t_0}{T}$.

$$\frac{d(v_\infty \tilde{v})}{d(\tau T)} = -g - A(\tilde{v} v_\infty)^3$$

$$\left(\frac{v_\infty}{T}\right)\left(\frac{d\tilde{v}}{d\tau}\right) = -g - Av_\infty^3 (\tilde{v})^3$$

$\left[\sqrt[3]{\frac{-g}{A}} \cdot (Ag^2)^{\frac{1}{3}} = -g^{+\frac{1}{3}} \cdot g^{\frac{2}{3}} = -g, \text{ and } Av_\infty^3 = A \cdot \frac{-g}{A} = -g\right]$

$$-g \cdot \frac{d\tilde{v}}{d\tau} = -g - (-g)(\tilde{v})^3 \quad \Big| \cdot \frac{1}{-g}$$

$\frac{d\tilde{v}}{d\tau} = 1 - (\tilde{v})^3$, now forgetting about ~ write v as dimensionless:

$\boxed{\dot{v} = 1 - v^3}$ (here $\tilde{v}_\infty = 1$ on scale of $v_\infty < 0$)

*f) $$\int_{v_0}^{v} \frac{dv}{1 - v^3} = \int_0^\tau d\tau$$

$$\int_{v_0}^{v} \frac{dv}{(1-v)\underbrace{(1+v^2+v)}} = \tau, \quad \int_{v_0}^{v} \frac{dv}{(1-v)(v - e^{\frac{2\pi}{3}i})(v - e^{-\frac{2\pi}{3}i})} = \tau$$

$v^2 + v + 1 = 0$, $D = 1 - 4 = -3 < 0$, $v_{1,2} = \frac{-1 \pm i\sqrt{3}}{2}$ → $e^{\frac{2\pi}{3}i}$; $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ $\left(R = 1, \varphi = -\frac{2\pi}{3}\right)$, $= e^{-\frac{2\pi}{3}i}$

So $\frac{1}{(1-v)(v-e^{\frac{2\pi}{3}i})(v-e^{-\frac{2\pi}{3}i})} = \frac{a}{1-v} + \frac{b}{v-e^{\frac{2\pi}{3}i}} + \frac{c}{v-e^{-\frac{2\pi}{3}i}}$

[The idea is to use this (possible from polin. factoring) expansion, find a, b, c by solving linear equation system (or guess), and all integral will be $a\ln|\dots| + b\ln|\dots| + \dots$ etc.

~~$1 = a(v-e$~~ Set $e^{\frac{2\pi}{3}i} = \alpha$, $e^{-\frac{2\pi}{3}i} = \beta$, $\alpha, \beta \in \mathbb{C}$, $\underline{\alpha \cdot \beta = 1}$

$$\frac{1}{(1-v)(v-\alpha)(v-\beta)} = \frac{a}{1-v} + \frac{b}{v-\alpha} + \frac{c}{v-\beta}, \quad 1 = a(v-\alpha)(v-\beta) + b(1-v)(v-\beta) + c(1-v)(v-\alpha)$$

$$1 = a(v^2 - v\beta - \alpha v + \alpha\beta) + b(v - \beta - v^2 + v\beta) + c(v - \alpha - v^2 + v\alpha)$$

True for all possible v, so must

Easier with Kramer matrix $\rightarrow$

$$\begin{cases} a - b - c = 0 \\ -a\beta - a\alpha + b + b\beta + c + c\alpha = 0 \\ 1 = a\underbrace{\alpha\beta}_{1} - b\beta - c\alpha \end{cases} \quad \begin{cases} a - b - c = 0 \\ a - b\beta - c\alpha = 1 \\ -a\beta - a\alpha + b + b\beta + c + c\alpha = 0 \end{cases} \Longrightarrow$$

$$\begin{cases} a = b + c \\ b + c - b\beta - c\alpha = 1 \\ (b+c)(-\beta) + (b+c)(-\alpha) + b + b\beta + c + c\alpha = 0 \end{cases} \quad \begin{cases} a = b + c \quad (1) \\ b + c - b\beta - c\alpha = 1 \\ -\cancel{b\beta} - c\beta - b\alpha - \cancel{c\alpha} + b + \cancel{b\beta} + c + \cancel{c\alpha} = 0 \end{cases}$$

$\Downarrow$

$$-b\beta - c\alpha + c\beta + b\alpha = 1$$

$$b(\alpha - \beta) - c(\alpha - \beta) = 1$$

$$(b - c)(\alpha - \beta) = 1$$

$$b = c + \frac{1}{\alpha - \beta} \quad (2)$$

So $c + \frac{1}{\alpha-\beta} + c - \beta\left(c + \frac{1}{\alpha-\beta}\right) - c\alpha = 1$

$$2c(\alpha-\beta) + 1 - \beta(\alpha-\beta)c - \beta - c\alpha(\alpha-\beta) = \alpha - \beta$$

$$2c\alpha - 2c\beta + 1 - \cancel{\beta\alpha c} + \beta^2 c - \cancel{\beta} - c\alpha^2 + \cancel{c\alpha\beta} = \alpha - \cancel{\beta}$$

$$c\,(\underbrace{2\alpha - 2\beta + \beta^2 - \alpha^2}_{2(\alpha-\beta) - (\alpha-\beta)(\alpha+\beta)}) = \alpha - 1 \Longrightarrow \underline{c = \frac{\alpha - 1}{(\alpha-\beta)(2-(\alpha+\beta))}} \quad (3)$$

So $b = \frac{\alpha - 1 + 2 - (\alpha+\beta)}{(\alpha-\beta)(2-(\alpha+\beta))} = \underline{\frac{1-\beta}{(\alpha-\beta)(2-(\alpha+\beta))}}$, $a = \frac{\cancel{\alpha-\beta}}{\cancel{(\alpha-\beta)}(2-(\alpha+\beta))} = \underline{\frac{1}{2-(\alpha+\beta)}}$

Anyway, $a\ln\left|\frac{1-v}{1-v_0}\right| + b\ln\left|\frac{v-e^{\frac{2\pi}{3}i}}{v_0-e^{\frac{2\pi}{3}i}}\right| + c\ln\left|\frac{v-e^{-\frac{2\pi}{3}i}}{v_0-e^{-\frac{2\pi}{3}i}}\right| = \int_{v_0}^{v}\frac{dv}{\dots} =$

$= \tau$,

$$\ln\left|\frac{(1-v)^a(v-\alpha)^b(v-\beta)^c}{\underbrace{(1-v_0)^a(v_0-\alpha)^b(v_0-\beta)^c}_{K}}\right| = \tau, \quad \boxed{(1-v)^a(v-\alpha)^b(v-\beta)^c = \tau \cdot A}$$

of some sign depending on v_0 sign

May be some better way from here? Do not know.

I would use another way for ~~direct~~ solution

$$\int_{v_0}^{v} \frac{1}{1-v^3}\,dv = \tau, \quad \int_{v_0}^{v} \frac{dv}{(1-v)(1+v+v^2)} = \frac{1}{3}\int_{v_0}^{v} \frac{1}{1-v} + \frac{v+2}{1+v+v^2}\,dv \text{ (easily checked)}$$

$$= \frac{1}{3}\ln\left|\frac{1-v}{1-v_0}\right| + \frac{1}{3}\int_{v_0}^{v} \frac{v+\frac{1}{2}+\frac{3}{2}}{v^2+v+1}\,dv = \left[\begin{matrix} v^2+v+1 = u > 0 \\ \frac{1}{2}du = (v+\frac{1}{2})\,dv \end{matrix}\right] =$$

$$= \underbrace{\frac{1}{3}\ln\left|\frac{1-v}{1-v_0}\right| + \frac{1}{6}\int_{u_0}^{u}\frac{du}{u}}_{} + \frac{1}{2}\int_{v_0}^{v}\frac{dv}{v^2+v+1} = \underbrace{\frac{1}{3}\ln\left|\frac{1-v}{1-v_0}\right| + \frac{1}{6}\ln\left(\frac{v^2+v+1}{v_0^2+v_0+1}\right)}_{A} +$$

$$+ \frac{1}{2}\int_{v_0}^{v}\frac{dv}{(v+\frac{1}{2})^2+\frac{3}{4}} = A + \frac{1}{2}\cdot\int_{v_0}^{v}\frac{dv}{\frac{3}{4}\left(1+\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2\right)} = A + \frac{2}{3}\int_{v_0}^{v}\frac{dv}{1+\left(\frac{v+\frac{1}{2}}{\sqrt{3}/2}\right)^2} =$$

$$= \left[u = \frac{2}{\sqrt{3}}(v+\tfrac{1}{2})\right] = A + \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\int_{u_0}^{u}\frac{du}{1+u^2} = A + \frac{1}{\sqrt{3}}\left[\arctan(u) - \arctan(u_0)\right] =$$

$$= A + \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{2}{\sqrt{3}}(v+\tfrac{1}{2})\right) - \arctan\left(\frac{2}{\sqrt{3}}(v_0+\tfrac{1}{2})\right)\right].$$ Also indirect determination amounts to a contour line of $f(v,\tau)$...

Problem 3 Anharmonic oscillator

$$F(x) = -f\tanh\left(\frac{x}{L}\right)$$

a) $[f] = \frac{kg\cdot m}{s^2}$, $[L] = m$ – L is length scale, then time scale is $\sqrt{\frac{mL}{f}}$.

$m\ddot{x} = -f\tanh\left(\frac{x}{L}\right)$ is EOM

$$m\frac{d^2(\tilde{x}\cdot L)}{d\left(\frac{\sqrt{mL}}{\sqrt{f}}\cdot\tau\right)^2} = -f\tanh(\tilde{x})$$

$$\Downarrow$$

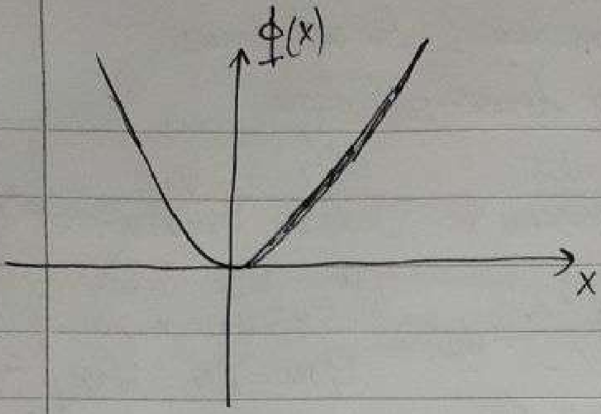
$$\frac{d^2\tilde{x}}{d\tau^2} = -\tanh\tilde{x} \text{ is dimensionless EOM.}$$

b) $\Phi(x) = -\int \ddot{x}\,dx = +\int \tanh x\,dx = \ln(\cosh x) + C$, set $C=0$.

$E = \frac{(\dot{x})^2}{2} + \ln(\cosh x)$, so $\dot{E} = \dot{x}\ddot{x} + \tanh x\cdot\dot{x} = \dot{x}(\underbrace{\ddot{x} + \tanh x}_{0}) = 0 \Longrightarrow$

$\Longrightarrow E$ is conserved (clear without calculation since only conserv. forces → general proof...)

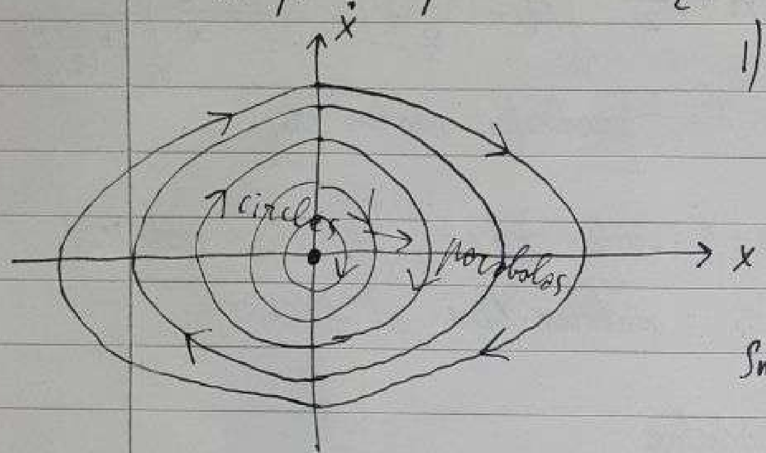
c) $\Phi(x) = \ln(\cosh(x))$



$\varphi(x) = \ln(\cosh x) = \ln\left(\frac{e^{x}+e^{-x}}{2}\right)$

even,
minima at $0\,(0)$,
for large x $\varphi(x) \approx \ln\left(\frac{e^{x}}{2}\right) = \ln\frac{1}{2} + x$
$\approx$ line $y=x$
for small x
$\approx y = ax^2$,
see below

In phase space: $\frac{(\dot{x})^2}{2} + \ln(\cosh x) = \text{const}$



1) $x\uparrow$, $\frac{(\dot{x})^2}{2} + kx = \text{const}$

$x = +A - \frac{(\dot{x})^2}{2} -$

$\approx$ parabolas on large x

2) $\frac{(\dot{x})^2}{2} + \ln\left(\frac{\cancel{1}+\cancel{x}+\frac{x^2}{2}+1-\cancel{x}+\frac{x^2}{2}}{2}\right) = \frac{(\dot{x})^2}{2} + \ln\left(1+\frac{x^2}{2}\right) \approx$

Small $x \to$ circles

$\approx \frac{(\dot{x})^2}{2} + \frac{x^2}{2}$

*d) for small oscillations $x \ll L$

$$\ddot{x} = -\tanh(x) \approx -\sum_{i=0}^{N} \frac{\tanh^{(i)}(0)\cdot x^i}{i!} = -\left(\frac{\overset{0}{\tanh(0)}\, x^0}{1} + \frac{x^1}{\underbrace{\cosh^2(0)\cdot 1!}_{1}} + \frac{-2x^2 \overset{0}{\sinh(0)}}{\cosh^3(0)\cdot 2!}\right) + \dots \approx -x.$$

$\Longrightarrow$ So $\begin{cases} \ddot{x} = -x \Rightarrow \text{harmonic}, \\ \Phi(x) = \frac{x^2}{2} \end{cases}$

(shown as $\smile$ near 0)
parabola

In non-dim. form $F = -f\cdot\frac{x}{L} = -\left(\frac{f}{L}\right)x$, $\frac{f}{L} \to k$

$m\ddot{x} = -\left(\frac{f}{L}\right)x$,

$\ddot{x} = -\left(\frac{f}{mL}\right)x$

$\frac{f}{mL} \to \omega^2$; $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{mL}{f}}$.

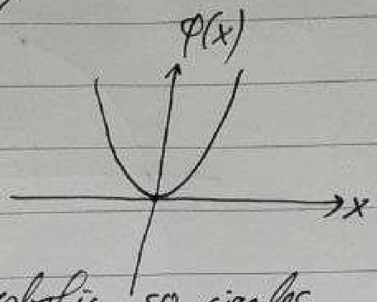
$\Longrightarrow$ So our non-dim scale $T = \sqrt{\frac{mL}{f}}$ was same by factor of 2π.

*e) Because for harmonic potential $\left[\frac{(\dot{x})^2}{2} + \frac{x^2}{2} = E\right]$ phase space trajectories are always circles (or ellipses up to different axis scaling). Here they are circles near $(0,0)$, but change to parabolic shape.

*g) $F_2(x) = -f\sinh(x/L) \Longrightarrow$

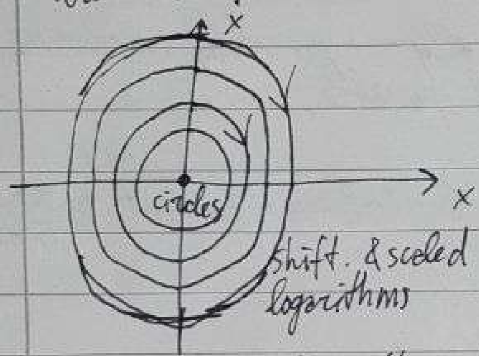
$\ddot{x} = -\sinh(x)$

$\varphi(x) = \cosh(x) + C$ ↓ set -1



for small $|x|$ also $\approx$ parabolic, so circles in phase space around 0.

but later $\varphi(x) \approx \exp(x)$, so $\frac{(\dot{x})^2}{2} + e^x = const$ means $x \approx k \cdot \ln$



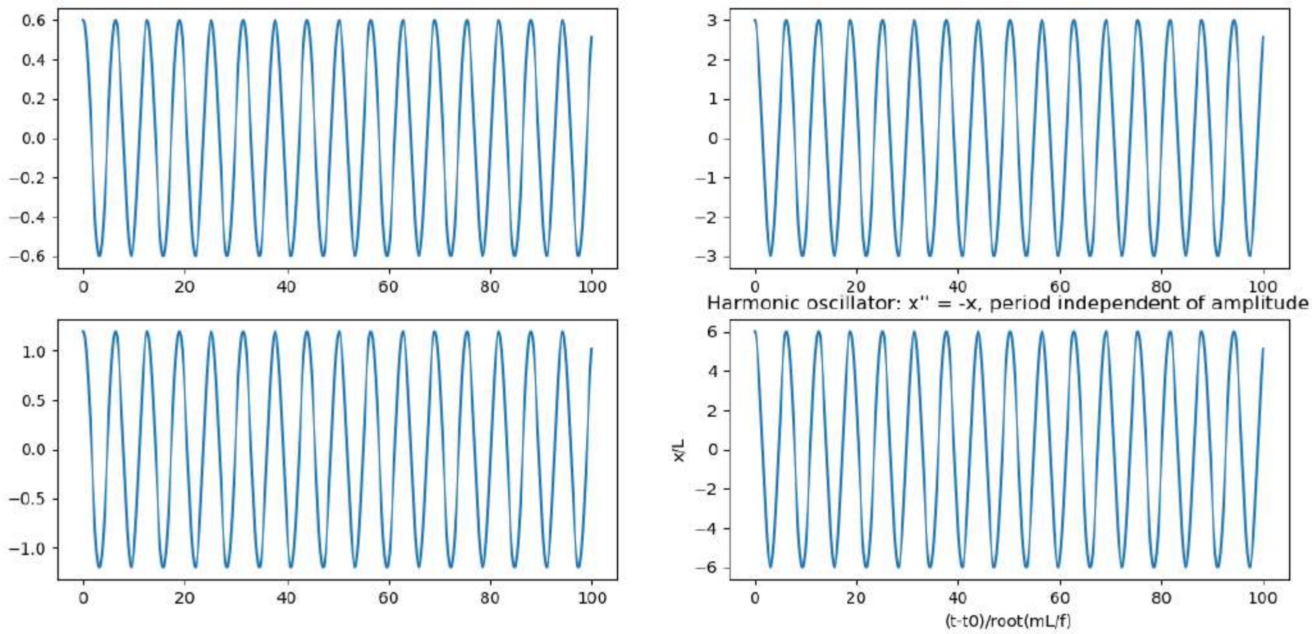
*f) See a script, the main idee is:

1) for harmonic T does not depend on amplitude ($\ddot{x} = -x \to x = A\cos(\tau + \varphi_0)$, $T = 2\pi$)

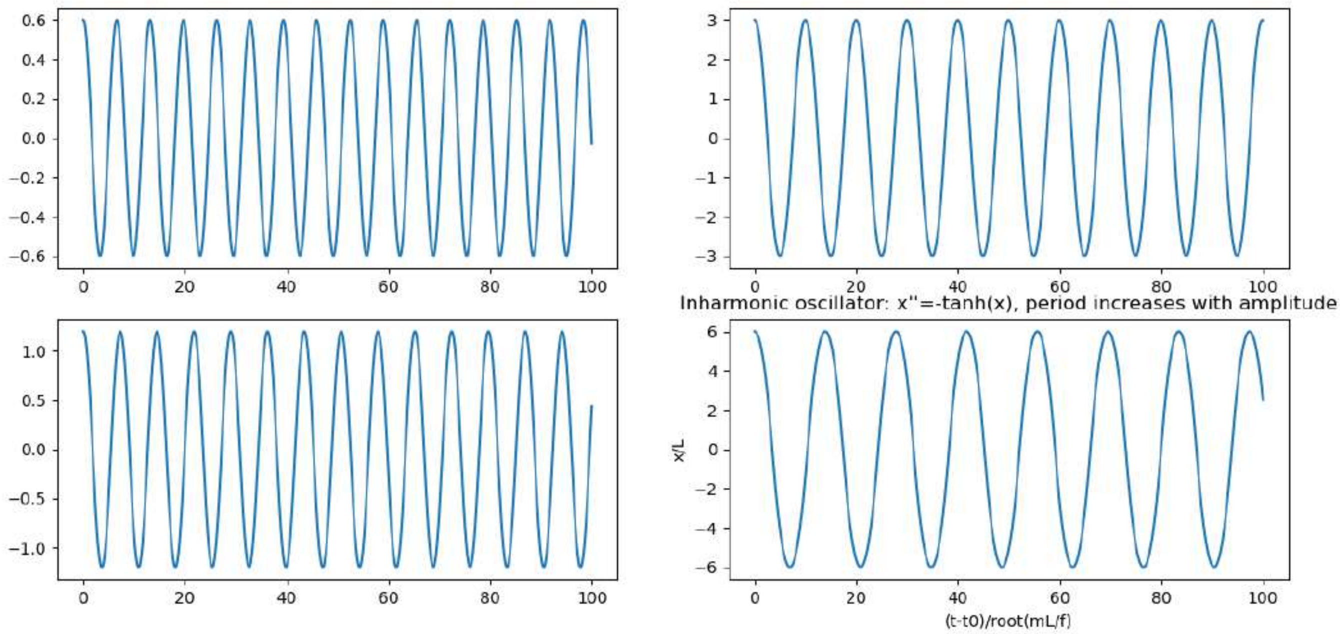
inharmonic

2) for $\ddot{x} = -\tanh(x) \approx -\frac{e^x - e^{-x}}{e^x + e^{-x}} \approx \frac{-1}{const}$ | T increases with amplitude (const. force on increasing distance, like free fall from higher height)

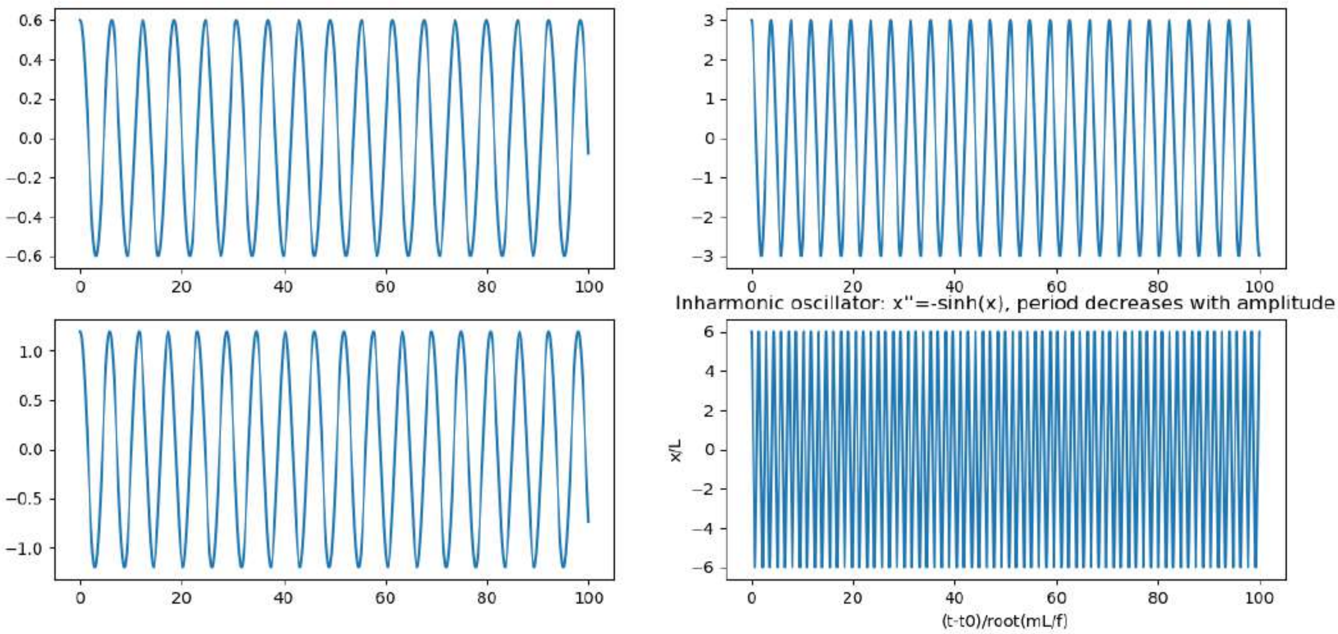
3) for $\ddot{x} = -\sinh(x) \approx -e^x$ T decreases with amplitude, as force (acceleration) grows exponentially w.r.t. to it



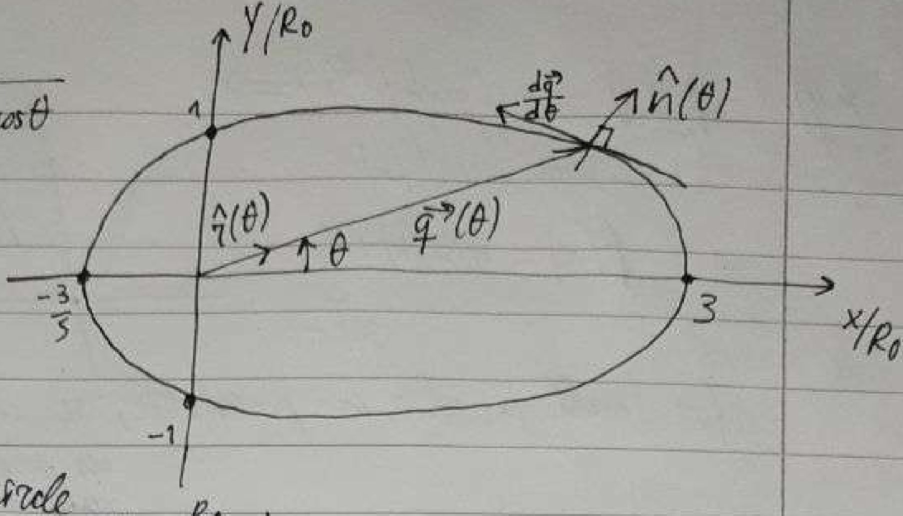
Harmonic oscillator: x'' = -x, period independent of amplitude
x/L
(t-t0)/root(mL/f)



Inharmonic oscillator: x''=-tanh(x), period increases with amplitude
x/L
(t-t0)/root(mL/f)


Inharmonic oscillator: x''=-sinh(x), period decreases with amplitude
x/L
(t-t0)/root(mL/f)

Problem 4 $R(\theta) = \dfrac{R_0}{1 - \varepsilon\cos\theta}$



a) $\theta = 0$, $\frac{R(\theta)}{R_0} = 3$, so

$1 - \varepsilon\underbrace{\cos\theta}_{1} = \frac{1}{3}$

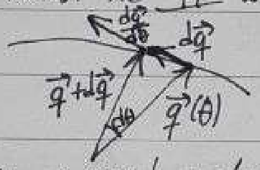
$\underline{\varepsilon = \frac{2}{3}}$.

If $\varepsilon = 0$ it is circle $\left(R(\theta) = \frac{R_0}{1-0}\right)$

For $\varepsilon \in (0,1)$ – ellipse.

At $\varepsilon = 1$ for $\theta = 0$ $R(\theta) \to \infty$ – ellipse $\to$ parabola, no closed $\Longrightarrow$ no wheel possible.

b) First way 1) $\hat{n}(\theta)$ must be $\perp$ to wheel $\Longrightarrow$ ~~tangent~~ parallel to $d\vec{q}/d\theta$



2) $\hat{n}(\theta)$ is a unit vector.

$\Longrightarrow$ 1) $\frac{d\vec{q}}{d\theta} = \frac{d}{d\theta}\left(\hat{r}(\theta)\cdot R(\theta)\right) = \frac{dR(\theta)}{d\theta}\cdot\hat{r}(\theta) + R(\theta)\frac{d\hat{r}(\theta)}{d\theta} =$

$= \frac{-R_0\varepsilon(-\sin\theta)}{(1-\varepsilon\cos\theta)^2}(-1)\,\hat{r}(\theta) + R(\theta)\cdot\hat{\theta}(\theta) = \begin{bmatrix} \frac{-R_0\varepsilon\sin\theta}{(1-\varepsilon\cos\theta)^2} \\ \frac{R_0}{1-\varepsilon\cos\theta}\end{bmatrix}$ (polar).

$\hat{n}(\theta)\cdot\frac{d\vec{q}}{d\theta} = 0 \Longrightarrow \begin{bmatrix} L \\ \frac{cL\sin\theta}{1-\varepsilon\cos\theta}\end{bmatrix}\cdot\begin{bmatrix} \frac{-R_0\varepsilon\sin\theta}{(1-\varepsilon\cos\theta)^2} \\ \frac{R_0}{1-\varepsilon\cos\theta}\end{bmatrix} = (\hat{r}, \hat{\theta}$ are orthonormal basis$) =$

$= \frac{-LR_0\varepsilon\sin\theta + LR_0c\sin\theta}{(1-\varepsilon\cos\theta)^2} = 0 \Longrightarrow \boxed{c = \varepsilon}$.

2) Now $|\hat{n}(\theta)| = 1$, $L\sqrt{1 + \frac{\varepsilon^2\sin^2\theta}{(1-\varepsilon\cos\theta)^2}} = L\sqrt{\frac{1+\varepsilon^2\cos^2-2\varepsilon\cos\theta+\varepsilon^2\sin^2\theta}{(1-\varepsilon\cos\theta)^2}} =$

$= L\sqrt{\frac{1+\varepsilon^2-2\varepsilon\cos\theta}{(1-\varepsilon\cos\theta)^2}} = 1 \Longrightarrow \boxed{L = \frac{1-\varepsilon\cos\theta}{\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta}}} = L(\theta)$

Second way Imagine ellipse as contour line for function $R(\theta)\cdot(1-\varepsilon\cos\theta)$ with value R_0.

Then $\hat{n}(\theta)$ must be directed along the gradient!
(and „out" since R_0 grows out)

$$\nabla\Big[\underbrace{R(\theta)\cdot(1-\varepsilon\cos\theta)}_{f(R,\theta)}\Big] = \begin{bmatrix} \partial f/\partial R \\ \frac{1}{R}\cdot \partial f/\partial\theta \end{bmatrix}_{polar} = \begin{bmatrix} 1-\varepsilon\cos\theta \\ \frac{R}{R}\cdot\varepsilon\sin\theta \end{bmatrix}, \quad \hat{n} = \frac{\nabla f}{|\nabla f|} =$$

derivation

$$= \begin{bmatrix} 1-\varepsilon\cos\theta \\ \varepsilon\sin\theta \end{bmatrix}\cdot\frac{1}{\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta}}.$$

in polar coordinates:

$$\frac{d\varphi}{dt} = \underbrace{\frac{\partial\varphi}{\partial r}\cdot\frac{dr}{dt}+\frac{\partial\varphi}{\partial\theta}\cdot\frac{d\theta}{dt}}_{\text{full derivative}} =$$

$$= \underbrace{\nabla\varphi\cdot\dot{\vec{q}}}_{\text{represent. using }\nabla\varphi},$$

where $\dot{\vec{q}} = \frac{d}{dt}(r\hat{r}) =$

$$= \frac{dr}{dt}\cdot\hat{r} + \dot{\theta}r\hat{\theta},$$

$$\Longrightarrow \begin{cases} \frac{\partial\varphi}{\partial r}\cdot\frac{dr}{dt} = \nabla\varphi_r\cdot\frac{dr}{dt} \\ \frac{\partial\varphi}{\partial\theta}\cdot\frac{d\theta}{dt} = \nabla\varphi_\theta\cdot\frac{d\theta}{dt}\cdot r \end{cases}$$

$$\Longrightarrow \nabla\varphi_r = \frac{\partial\varphi}{\partial r}, \quad \nabla\varphi_\theta = \frac{\partial\varphi}{\partial\theta}\cdot\frac{1}{r},$$

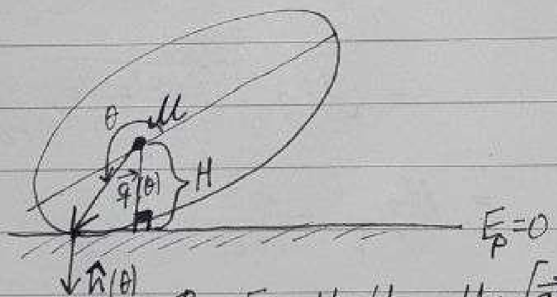
$$\Longrightarrow \nabla\varphi = \begin{bmatrix} \partial\varphi/\partial r \\ \frac{1}{r}\cdot\partial\varphi/\partial\theta \end{bmatrix}$$

$$\Longrightarrow \text{So } \hat{n}(\theta) = \hat{r}(\theta)\cdot\frac{1-\varepsilon\cos\theta}{\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta}} + \hat{\theta}(\theta)\cdot\frac{\varepsilon\sin\theta}{\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta}} =$$

$$= (\text{must}) = L\,\hat{r}(\theta) + \frac{Lc\sin\theta}{1-\varepsilon\cos(\theta)}\,\hat{\theta}(\theta).$$

$$\Longrightarrow \text{So } L = \frac{1-\varepsilon\cos\theta}{\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta}}, \quad c = \varepsilon$$ (same as with $\dot{\vec{q}}$ analysis)

c)

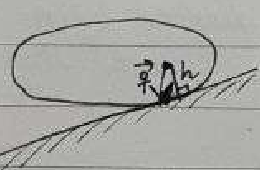


$$\varphi = E_p = MgH = Mg\cdot\left[\vec{q}(\theta)\cdot\hat{n}(\theta)\right] =$$

$$= Mg\cdot\begin{bmatrix} \frac{R_0}{1-\varepsilon\cos\theta} \\ 0 \end{bmatrix}\cdot\begin{bmatrix} \frac{1-\varepsilon\cos\theta}{\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta}} \\ \dots \end{bmatrix} =$$

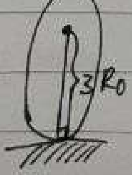
$$= Mg\frac{R_0}{\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta}}.$$

Special cases $\theta = \frac{\pi}{2}$



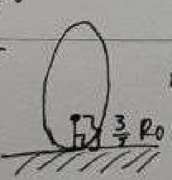
$$E_p = \frac{MgR_0}{\sqrt{1+\varepsilon^2}}$$

$\theta = 0$ (max. E_p)



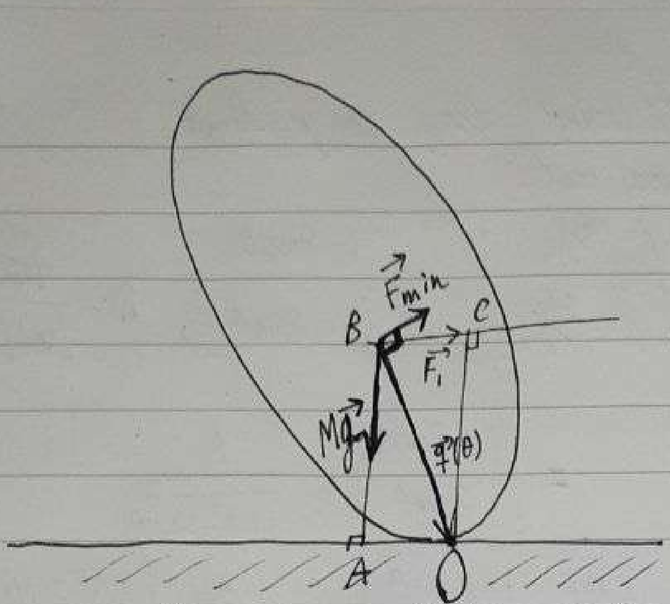
$$E_p = \frac{MgR_0}{\sqrt{1+\varepsilon^2-2\varepsilon}} = \frac{MgR_0}{1-\varepsilon} = \frac{MgR_0}{1-\frac{2}{3}} = 3MgR_0 \ \checkmark$$

$\theta = \pi$ (min. E_p)



$$E_p = \frac{MgR_0}{\sqrt{1+\varepsilon^2+2\varepsilon}} = \frac{MgR_0}{1+\varepsilon} = \frac{3}{5}MgR_0 \ \checkmark$$

*d)



$OB = |\vec{q}| = \dfrac{R_0}{1-\varepsilon\cos\theta}$

$AB = h = \dfrac{R_0}{\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta}}$ (see c)),

$OA = \sqrt{OB^2 - AB^2}$

Consider contact point O: torques balanced when $(-\vec{q}) \times M\vec{g} = (-\vec{q}) \times \vec{F}$, or

$OA \cdot Mg = OB \cdot F_{min}$ ($\vec{F}_{min} \perp OB$, otherwise $OC < OB$, $F_1 > F_{min}$)

$$F_{min} = \frac{OA \cdot Mg}{OB} = Mg \cdot \underbrace{\frac{1-\varepsilon\cos\theta}{R_0}}_{\frac{1}{OB}} \cdot \sqrt{\frac{R_0^2}{(1-\varepsilon\cos\theta)^2} - \frac{R_0^2}{(1+\varepsilon^2-2\varepsilon\cos\theta)}} =$$

$$= Mg\sqrt{\frac{(1+\varepsilon^2 - 2\varepsilon\cos\theta) - (1-\varepsilon\cos\theta)^2}{(1-\varepsilon\cos\theta)^2(1+\varepsilon^2-2\varepsilon\cos\theta)}} \leftarrow (1-\varepsilon\cos\theta)$$

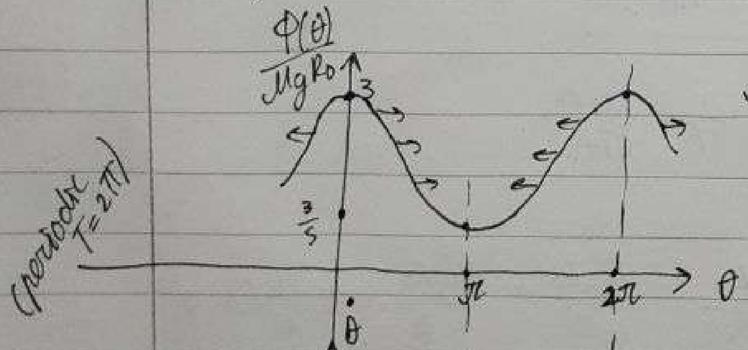
$$= Mg\sqrt{\frac{\varepsilon^2 - \varepsilon^2\cos^2\theta}{1+\varepsilon^2-2\varepsilon\cos\theta}} = \frac{Mg\,\varepsilon\sin\theta}{\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta}}$$

Special cases $\theta = 0/\pi \rightarrow F_{min} = 0$ (stable/unstable equilibrium)

e) $\dfrac{\Phi(\theta)}{MgR_0} = \dfrac{1}{\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta}}$

Minima when $1+\varepsilon^2-2\varepsilon\cos\theta$ is max $(\theta=\pi)$, $\dfrac{\varphi_{min}}{MgR_0} = \dfrac{1}{1+\varepsilon} = \dfrac{3}{5}$

Maxima when $1+\varepsilon^2-2\varepsilon\cos\theta$ is min $(\theta=0)$, $\dfrac{\varphi_{min}}{MgR_0} = \dfrac{1}{1-\varepsilon} = 3$



Then phase space plot: