Stanislav
Hubin
IPSP 3720433

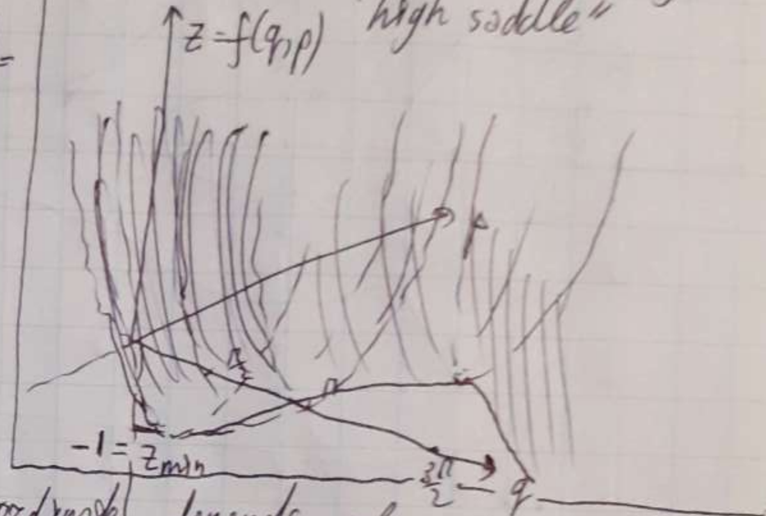
Theoretical Physics I
Exam Preparation

Prof. Dr.
Jürgen
Vollmer

Problem 13.1 Vectors, derivatives and phase-space portraits

a) $f(q,p) = \frac{p^2}{2} - \sin q$

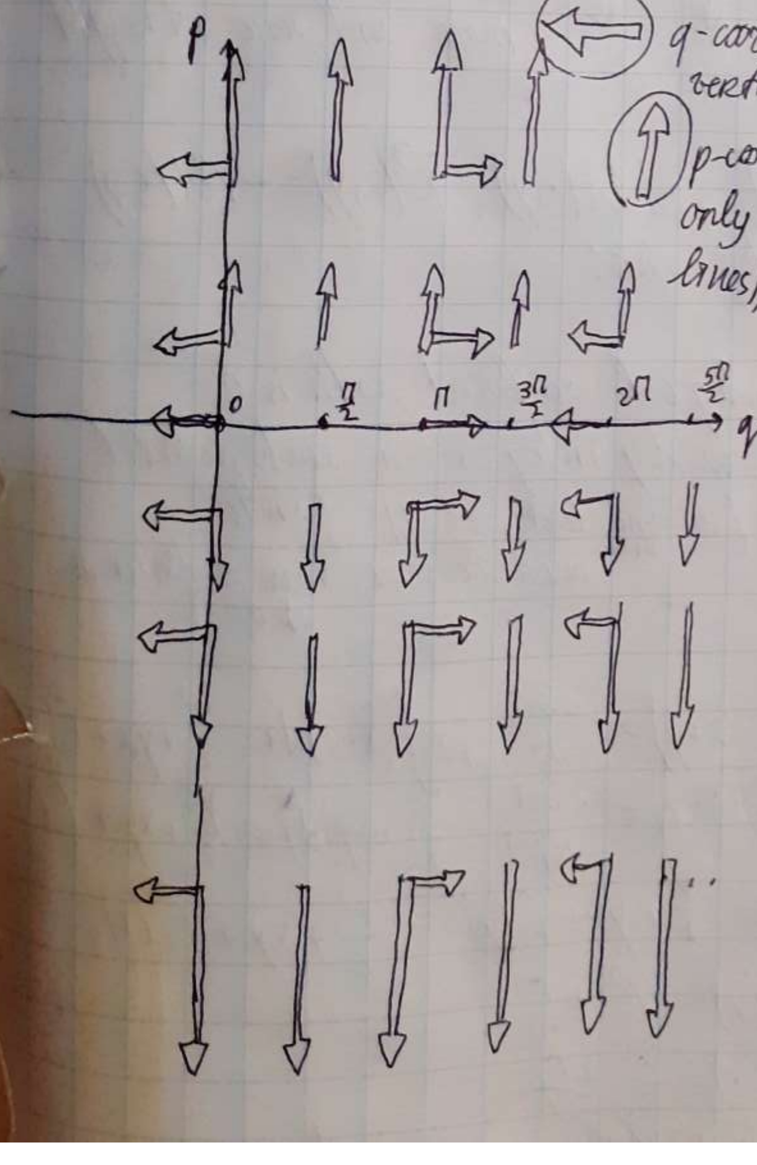
In 3D it is like „periodic infinitely high saddle"



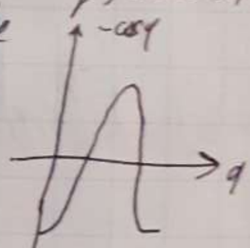
Gradient $\nabla f(q,p) = \begin{pmatrix}\partial q \\ \partial p\end{pmatrix} \cdot f(q,p) =$

$= \begin{pmatrix}\partial q \\ \partial p\end{pmatrix}\left(\frac{p^2}{2} - \sin q\right) = \begin{pmatrix}-\cos q \\ p\end{pmatrix}.$

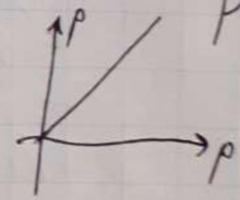
Plotting it in 2D by components:



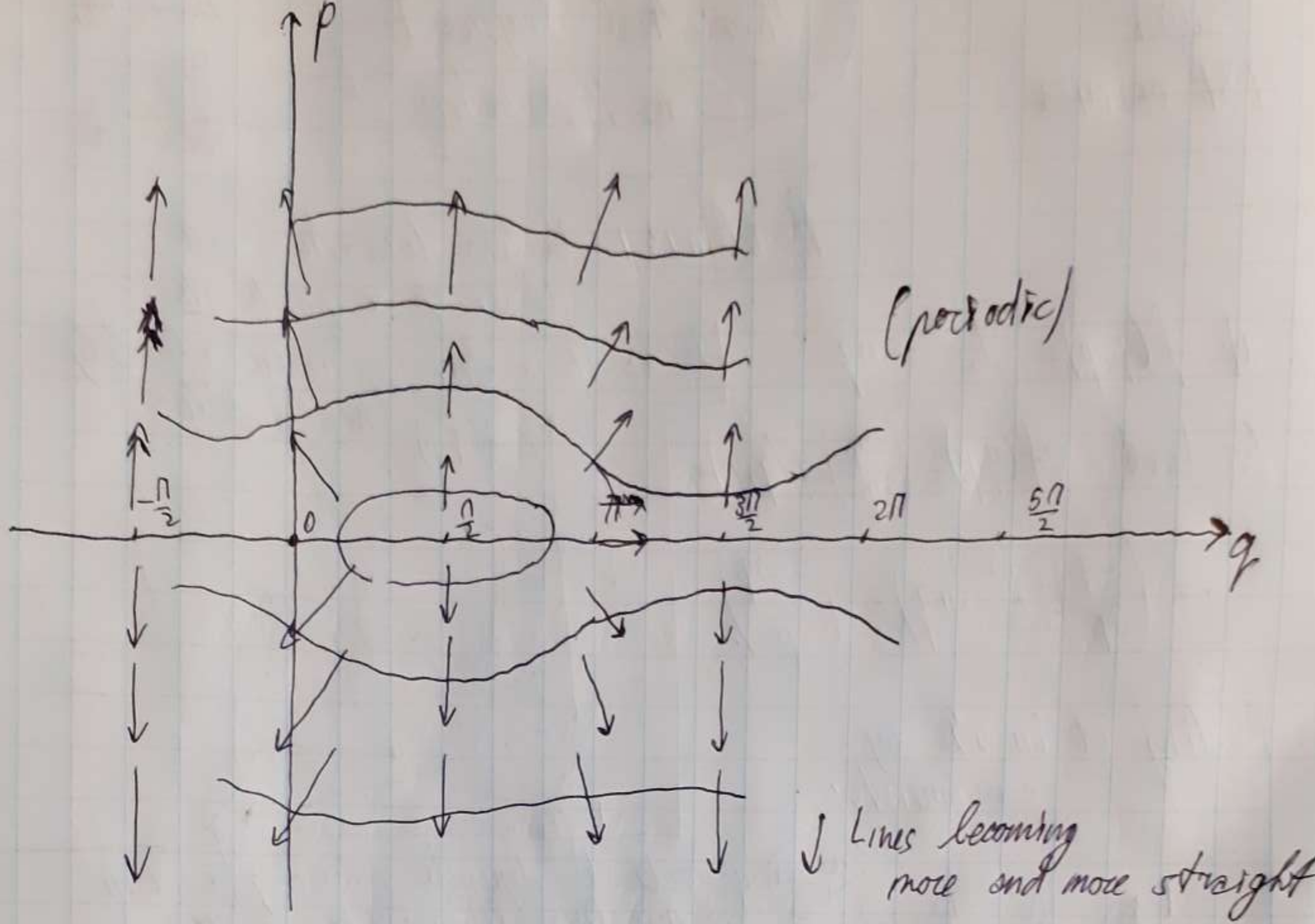
q-coordinate depends only on q; same on vertical lines, behaves like



p-coordinate depends only on p (same on horiz. lines), behaves like



Now plot resulting ∇f and contour lines perpendicular to it:



b) $\vec{K}(x,y)$ is conservative iff $\exists\, U(x,y)$, $\vec{K}(x,y) = -\nabla U(x,y)$

Ways to check this:

- $\nabla \times \vec{K} = \vec{0}$
- its line integral on closed path is 0
- directly - show by integration such potential exists
- $\partial_x F_y = \partial_y F_x$ (so 2nd mixed deriv. of U must be same, same condition as 1 or 3)

Use way 3 here.

Case 1) $\vec{K}_1(x,y) = \begin{pmatrix} x+y \\ x+y \end{pmatrix}$, $(x+y) = -\frac{\partial U}{\partial x}$, $-\int (x+y)\,dx = \frac{x^2}{2} + yx + C_1(y)$

$(x+y) = -\frac{\partial U}{\partial y}$, $-\int (x+y)\,dy = \frac{y^2}{2} + xy + C_2(x)$

Now $U(x,y) = \frac{x^2}{2} + yx + C_1(y) = \frac{y^2}{2} + xy + C_2(x)$

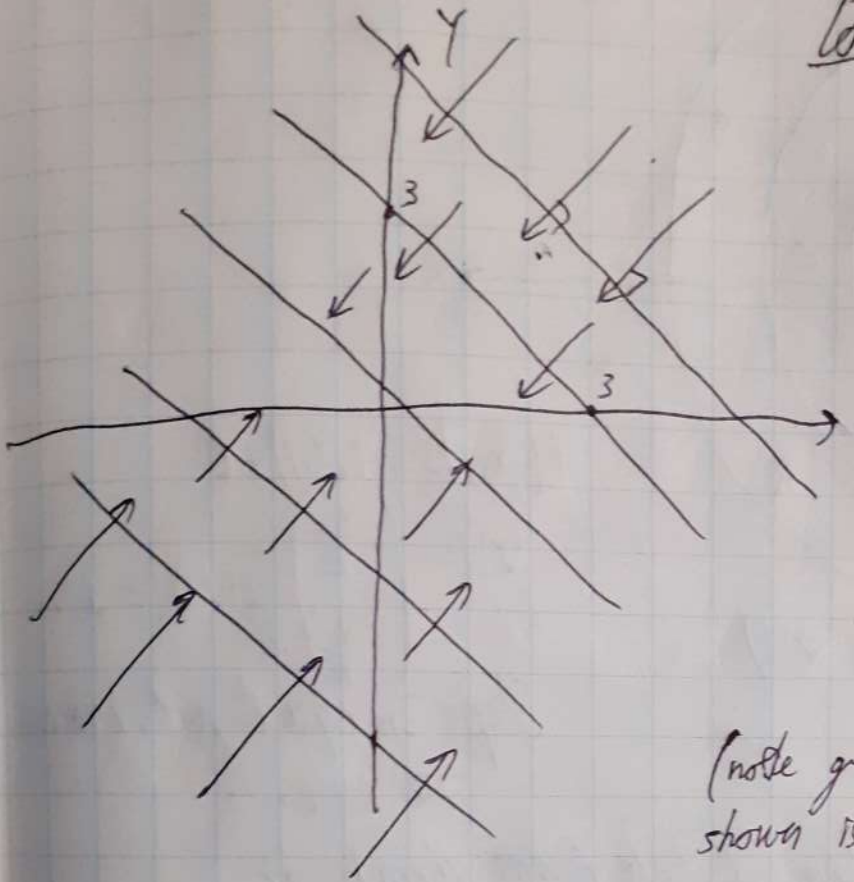
Take $C_1(y) = \frac{y^2}{2} + C$

$C_2(x) = \frac{x^2}{2} + C$

$U(x,y) = -\left(\frac{x^2}{2} + \frac{y^2}{2} + xy\right) + C$ up to const.

Check: $-\nabla U = -\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} U = \begin{pmatrix} x+y \\ x+y \end{pmatrix} = \vec{K}(x,y)$ ✓

Therefore potential is $U(x,y) = -\left[\frac{x^2}{2} + \frac{y^2}{2} + xy\right] + C.$



Contour lines:

$\frac{x^2}{2} + \frac{y^2}{2} + xy = const_1 \longleftrightarrow$

$(x+y)^2 = const_2 \longleftrightarrow$

$(x+y) = \pm\sqrt{const_2} = const_3,$

$y = const_3 - x \longrightarrow$ all possible lines slope -1.

! Due to $-$ sign potential drops with $const_3 \uparrow$.

(note gradient shown is opposite to force)

Case 2) $\vec{K_2}(x,y) = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$ In this case check $\partial_x K_{2y} = \partial_y K_{2x}$
$-1 \neq 1$, thus it is <u>not</u> conservative field, no potential.

Case 3) $\vec{K_3}(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$. Here by quick check it is clear that potential is $\underline{U(x,y) = -\sqrt{x^2+y^2} + C}$

✓ Check: $-\nabla U(x,y) = -\begin{pmatrix} \partial x \\ \partial y \end{pmatrix}\left(-\sqrt{x^2+y^2} + C\right) = \begin{pmatrix} \partial x \\ \partial y \end{pmatrix}\sqrt{x^2+y^2} = \begin{pmatrix} \frac{2x}{2\sqrt{x^2+y^2}} \\ \frac{2y}{2\sqrt{x^2+y^2}} \end{pmatrix} =$

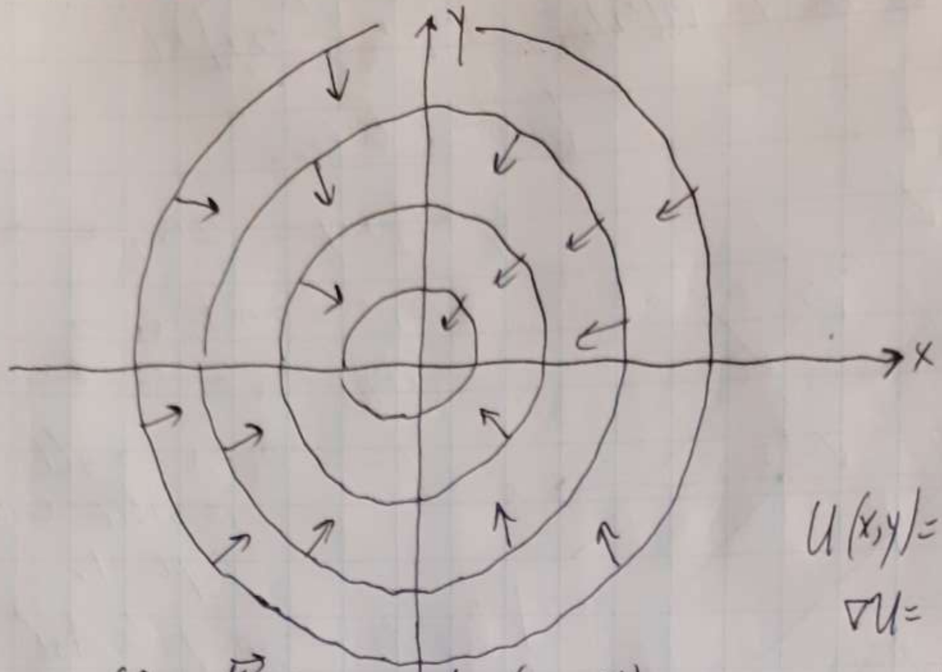
$= \begin{pmatrix} x \\ y \end{pmatrix} \cdot \frac{1}{r} = \vec{K_3}(\vec{r}),$

where $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$, $r = |\vec{r}|$.

The potential is

$U(x,y) = -R(x,y) + C,$

$R(x,y)$ is distance from origin.



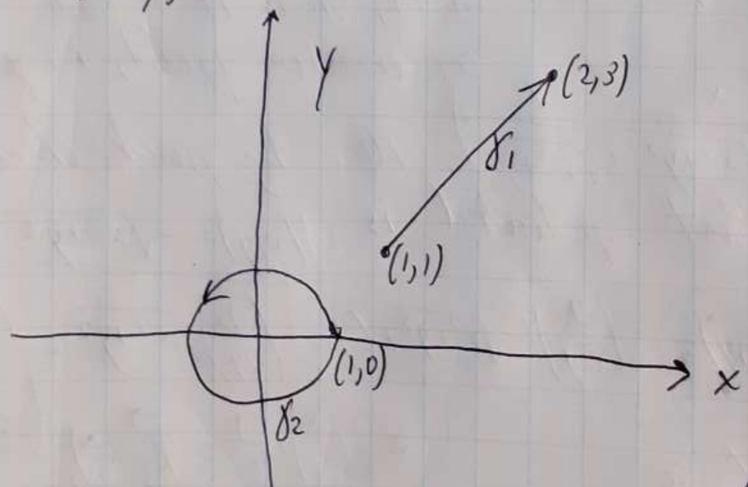
(force $\vec{K}_3$ is opposite to ∇U)

$U(x,y) = -r(x,y) + C$

$\nabla U = \begin{pmatrix} -\frac{x}{r} \\ -\frac{y}{r} \end{pmatrix}, \ |\nabla U| = 1.$

Steps in equal pot. increase are same.

Non-conservative among them (I take from them) is $\vec{K}_2 = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$. Work will depend on path, not just endpoints.



Now (see picture) 2 work integrals.

#1) $\int_{\gamma_1} \vec{F}\, d\vec{r} = W_1$.

Here path parametrization from $(1,1) \to (2,3)$ which is the shortest, is line $\vec{r}(t) = \begin{pmatrix} 1+t \\ 1+2t \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $t \in [0,1]$.

$$\int_{\vec{r}_1=(1,1)}^{\vec{r}_2=(2,3)} \begin{pmatrix} x(t)-y(t) \\ x(t)+y(t) \end{pmatrix} \underbrace{d\vec{r}}_{\dot{\vec{r}}\,dt} = \int_{t=0}^{1} \begin{pmatrix} 1+t-1-2t \\ 1+t+1+2t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} dt = \int_{t=0}^{1} \begin{pmatrix} -t \\ 2+3t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} dt$$

$$= \int_0^1 5t+4\, dt = \frac{5t^2}{2} + 4t \Big|_0^1 = \frac{5}{2} + 4 = \underline{6.5}$$

Note also that time parametrization is irrelevant, suppose $\vec{r}(t)=\begin{pmatrix}1+2t\\1+4t\end{pmatrix}$, $t\in(0,\frac{1}{2})$ — same path, twice speed, twice less time.
Will give $\displaystyle\int_{t=0}^{1/2}\vec{F}\,d\vec{r} = \int_0^{1/2}\begin{pmatrix}1+2t-1-4t\\1+2t+1+4t\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix}dt = \int_{t=0}^{1/2}\begin{pmatrix}-2t\\2+6t\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix}dt =$

$= \displaystyle\int_0^{1/2} 10t+8\,dt = 10t^2+8t\Big|_0^{1/2} = 4+2.5=\underline{6.5}$ as well.

#2) $\displaystyle\int_{\gamma_2}\vec{F}\,d\vec{r} = W_2$. Here the path counterclockwise is $\vec{r}(t)=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$, $t\in[0,2\pi]$, $\dot{\vec{r}}(t)=\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}$.

$\vec{F}(t)=\begin{pmatrix}x-y\\x+y\end{pmatrix}=\begin{pmatrix}\cos t-\sin t\\ \cos t+\sin t\end{pmatrix}$.

So $\displaystyle\int_{\gamma_2}\vec{F}\underbrace{d\vec{r}}_{\dot{\vec{r}}dt} = \int_0^{2\pi}\begin{pmatrix}\cos t-\sin t\\ \cos t+\sin t\end{pmatrix}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}dt = \int_0^{2\pi} -\cancel{\cos\cdot\sin}+\underbrace{\sin^2+\cos^2}_{1}+\cancel{\cos\sin}\,dt =$

$= \displaystyle\int_0^{2\pi}dt = \underline{2\pi}$

c) $\ddot{x}(t)=ax+bx^2+cx^3$, $a,b,c\in\mathbb{R}$. Analyze <u>second component of flow vectors</u> (tangent to trajectory)

<u>The first plot</u>

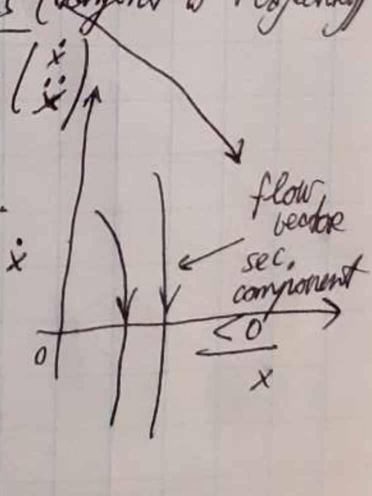
$\vec{v}=\begin{pmatrix}\dot{x}\\ \ddot{x}\end{pmatrix}$

$\begin{cases}\text{point } (-1,0):\ -a+b-c=0 \longrightarrow \underline{b=0.}\\ \text{point } (1,0):\ a+b+c=0\end{cases}$

From picture $\underline{c<0}$

Then a must $\underline{be>0.}$

$\underline{a=-c.}$



Take $p>0$, $\ddot{x}(t)=px-px^3$, simplifying scale, $\underline{\ddot{x}(t)=x-x^3 \text{ is EOM}}$

The second plot (similar analyses of flow vector vertical components $\ddot{x}$)

$$\left\{\begin{array}{l} a+b+c=0 \\ -a+b-c<0 \end{array}\right\} \; b<0$$

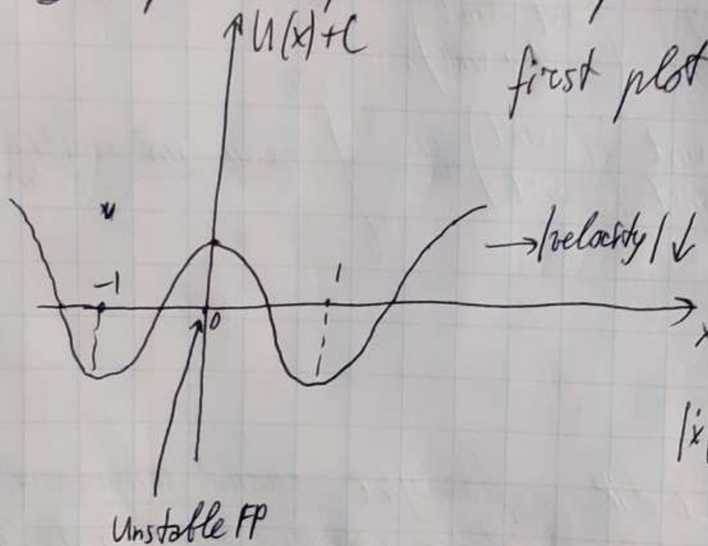
$c<0$ again from picture,

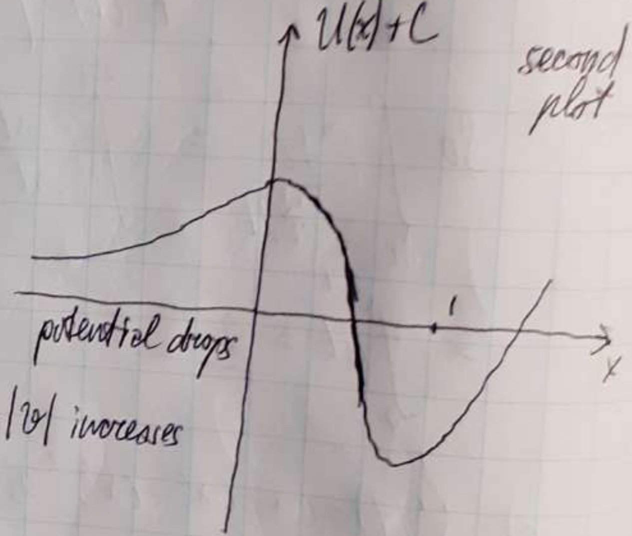
but then say at $\left(\frac{1}{2}\right)$, $\frac{a}{2}+\underbrace{\frac{b}{4}+\frac{c}{8}}_{<0}>0$

Then $a>0$.

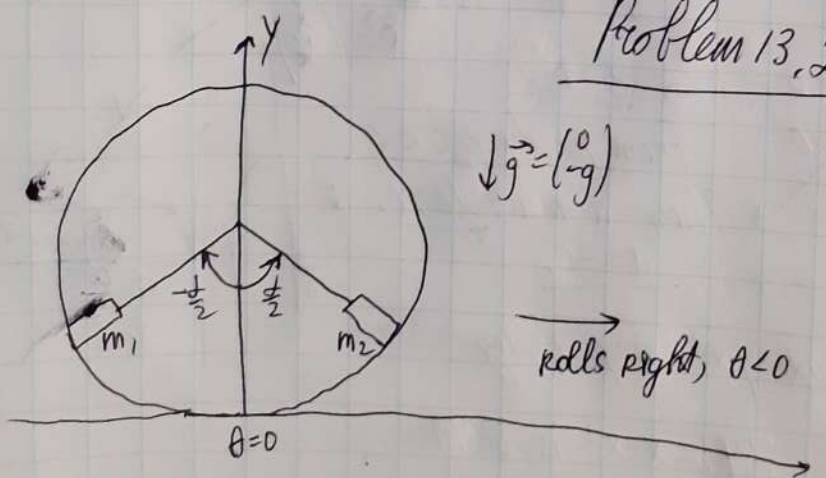
$>, <, <$.

Interpret as position in potential, sketch potentials.





Problem 13.2



$\downarrow \vec{g}=\begin{pmatrix}0\\-g\end{pmatrix}$

α is const angle between magnets.

$\longrightarrow$ rolls right, $\theta<0$

d) position of centre of can: $\vec{M}(\theta)=\begin{pmatrix}-R\theta\\R\end{pmatrix}$

Magnets in its frame: $\vec{D}_1(\theta)=\begin{pmatrix}R\sin\left(-\frac{\alpha}{2}+\theta\right)\\-R\cos\left(-\frac{\alpha}{2}+\theta\right)\end{pmatrix}$, $\vec{D}_2(\theta)=\begin{pmatrix}R\sin\left(\frac{\alpha}{2}+\theta\right)\\-R\cos\left(\frac{\alpha}{2}+\theta\right)\end{pmatrix}$

Then $\vec{q}_1(\theta) = \vec{M}(\theta) + \vec{D}_1(\theta) = R\begin{pmatrix}-\theta\\ 1\end{pmatrix} + R\begin{pmatrix}\sin(-\frac{\alpha}{2}+\theta)\\ -\cos(-\frac{\alpha}{2}+\theta)\end{pmatrix} = R\begin{pmatrix}-\theta+\sin(-\frac{\alpha}{2}+\theta)\\ 1-\cos(-\frac{\alpha}{2}+\theta)\end{pmatrix},$

$\dot{\vec{q}}_1(\theta) = R\begin{pmatrix}-\dot\theta+\cos(-\frac{\alpha}{2}+\theta)\dot\theta\\ \sin(-\frac{\alpha}{2}+\theta)\cdot\dot\theta\end{pmatrix} = R\dot\theta\begin{pmatrix}-1+\cos(-\frac{\alpha}{2}+\theta)\\ \sin(-\frac{\alpha}{2}+\theta)\end{pmatrix}$, by symmetry

$\dot{\vec{q}}_2(\theta) = R\dot\theta\begin{pmatrix}-1+\cos(\frac{\alpha}{2}+\theta)\\ \sin(\frac{\alpha}{2}+\theta)\end{pmatrix}.$

Then $T_1 = \frac{m}{2}(\dot{\vec{q}}_1)^2 = \frac{m}{2}R^2\dot\theta^2\left[(-1+\cos(\theta-\frac{\alpha}{2}))^2+\sin^2(\theta-\frac{\alpha}{2})\right] =$

$= \frac{m(R\dot\theta)^2}{2}\left[1+\cos^2+\sin^2-2\cos(\theta-\frac{\alpha}{2})\right] = m(R\dot\theta)^2(1-\cos(\theta-\frac{\alpha}{2}))$

$T_2 = (\text{similarly}) = m(R\dot\theta)^2(1-\cos(\theta+\frac{\alpha}{2})).$

$V_1 = mg\,\vec{q}_1\cdot\hat{e}_y = mgR(1-\cos(-\frac{\alpha}{2}+\theta))$

$V_2 = mg\,\vec{q}_2\cdot\hat{e}_y = mgR(1-\cos(\frac{\alpha}{2}+\theta))$

Then $L = T-V = (T_1+T_2)-(V_1+V_2) = m(R\dot\theta)^2\left[2-\cos(\theta+\frac{\alpha}{2})-\cos(\theta-\frac{\alpha}{2})\right] -$

$- mgR\left[2-\cos(\theta+\frac{\alpha}{2})-\cos(\theta-\frac{\alpha}{2})\right] =$

$= (2-\cos(\theta+\frac{\alpha}{2})-\cos(\theta-\frac{\alpha}{2}))\cdot\left[m(R\dot\theta)^2-mgR\right].$

Now $\frac{\partial L}{\partial\theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)$,

$\left[\sin(\theta+\frac{\alpha}{2})+\sin(\theta-\frac{\alpha}{2})\right]\cdot\left[m(R\dot\theta)^2-mgR\right] = \frac{d}{dt}(2mR^2\dot\theta)$

$m\left[(R\dot\theta)^2-gR\right]\left[\sin(\theta+\frac{\alpha}{2})+\sin(\theta-\frac{\alpha}{2})\right] = 2mR^2\ddot\theta \quad \Big| \cdot\frac{1}{mR^2}$

$(\dot\theta^2-\frac{g}{R})\cdot 2\sin\theta\cos\frac{\alpha}{2} = 2\ddot\theta$

$(\dot\theta^2-\frac{g}{R})\sin\theta\cdot\cos\frac{\alpha}{2} = \ddot\theta$ — EOM

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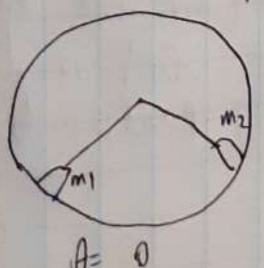
b) Eq. position $\longleftrightarrow$ $\begin{pmatrix}\dot\theta\\ \ddot\theta\end{pmatrix} = \vec{0}$, $-\frac{g}{R}\sin\theta\underbrace{\cos\frac{\alpha}{2}}_{\neq 0} = 0$

Then $\sin\theta = 0$, $\theta = \pi n$, meaning $\theta\in\{0,\pi\}$.

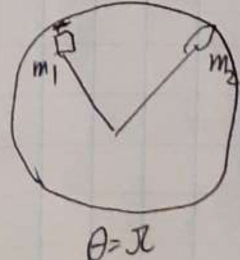
Physically:

Stable

$U\downarrow$



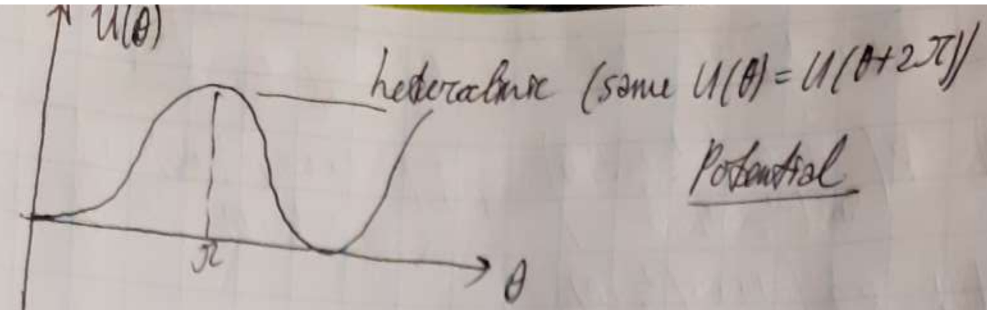
$\theta = 0$



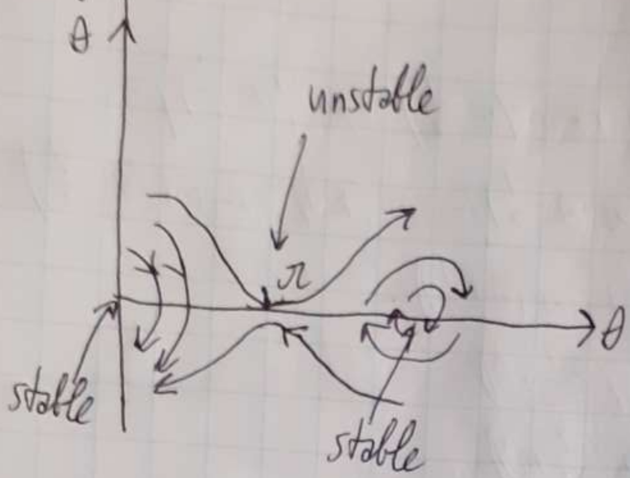
$\theta = \pi$

Unstable

$U\uparrow$



Potential

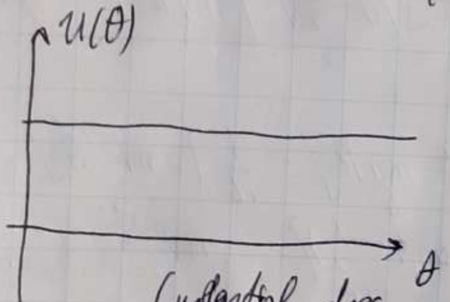


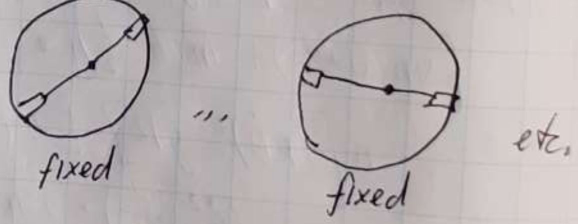
Phase space

c) $\mathcal{L} = \pi \rightarrow$ any θ is fixed point, potential

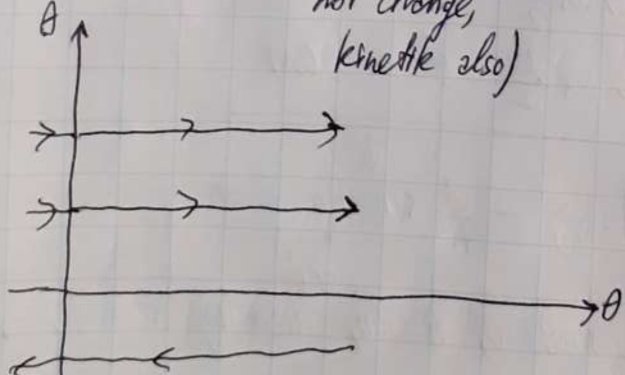
$$V(\theta) = V_1 + V_2 = mgR\left(2 - \cos\left(-\frac{\pi}{2}+\theta\right) + \cos\left(\frac{\pi}{2}+\theta\right)\right) =$$

$$= mgR(2 - \sin\theta + \sin\theta) = 2mgR = \underline{\text{const}}$$





(potential does not change, kinetic also)



d) for $\mathcal{L} = 0$,

$$\left(\dot{\theta}^2 - \frac{g}{R}\right)\sin\theta = \ddot{\theta}$$

!!! (not sure why my EOM is different from proposed, either model or calculation error)

$[\omega] = \frac{1}{s}$

$[g] = \frac{m}{s^2}$ $\quad [\omega] = \left[\frac{g}{R}\right]^{\frac{1}{2}}$

$[R] = m$

$\omega = \sqrt{\frac{g}{R}}$,

$(\dot{\theta}^2 - \omega^2)\sin\theta = \ddot{\theta}$

$\ddot{\theta} - \dot{\theta}^2 \sin\theta + \omega^2 \sin\theta = 0$. $\quad \omega = \sqrt{\frac{g}{R}}$, does not depend on m.

*e) (for this task I assume proposed EOM, otherwise meaningless)

$$0 = 2\ddot{\theta}(1-\cos\theta) + \dot{\theta}^2 \sin\theta + 2\omega^2 \sin\theta \xrightarrow{\theta \to x} \ddot{x} = \frac{\omega^2}{2}x$$

Let $x = \cos\left(\frac{\theta}{2}\right)$, then $\dot{x} = -\frac{\dot{\theta}}{2}\sin\left(\frac{\theta}{2}\right)$

$$\ddot{x} = -\frac{\ddot{\theta}}{2}\sin\frac{\theta}{2} - \frac{\dot{\theta}}{2}\cdot\cos\left(\frac{\theta}{2}\right)\frac{\dot{\theta}}{2} = -\frac{\ddot{\theta}}{2}\sin\left(\frac{\theta}{2}\right) - \frac{(\dot{\theta})^2}{4}\cos\frac{\theta}{2} \quad (*)$$

Now

$$2\ddot{\theta}(1-\cos\theta) + \dot{\theta}^2 \sin\theta + 2\omega^2 \sin\theta = 0$$

$$2\ddot{\theta}\cdot 2\sin^2\frac{\theta}{2} + (\dot{\theta}^2 + 2\omega^2)2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0 \quad \Big/ \cdot \frac{1}{2\sin\frac{\theta}{2}}$$

$$2\ddot{\theta}\sin\frac{\theta}{2} + (\dot{\theta}^2 + 2\omega^2)\cos\frac{\theta}{2} = 0 \quad \Big/ \cdot -\frac{1}{4}$$

$$-\frac{\ddot{\theta}}{2}\sin\frac{\theta}{2} - \frac{(\dot{\theta}^2 + 2\omega^2)}{4}\cos\frac{\theta}{2} = 0$$

$$\underbrace{-\frac{\ddot{\theta}}{2}\sin\frac{\theta}{2} - \frac{(\dot{\theta})^2\cos\frac{\theta}{2}}{4}}_{\ddot{x} \text{ from } (*)} = \frac{\omega^2}{2}\underbrace{\cos\frac{\theta}{2}}_{x}$$

So $\ddot{x} = \frac{\omega^2}{2}x$

f) $\ddot{x} - \frac{\omega^2}{2}x = 0$, solve using char. equation.

$\lambda^2 - \frac{\omega^2}{2} = 0$

$\lambda = \pm\frac{\omega}{\sqrt{2}}$, $\quad x(t) = C_1 e^{\frac{\omega}{\sqrt{2}}t} + C_2 e^{-\frac{\omega}{\sqrt{2}}t}$

$\dot{x}(t) = \frac{\omega}{\sqrt{2}}\left[C_1 e^{\frac{\omega}{\sqrt{2}}t} - C_2 e^{-\frac{\omega}{\sqrt{2}}t}\right]$

Initial conditions then are:
$$\begin{cases} C_1 + C_2 = x(0) \\ \frac{\omega}{\sqrt{2}}[C_1 - C_2] = \dot{x}(0) \end{cases}$$

Now $\dot{x}(0)=0$, $x(0)=\cos(\theta_0/2)$.

Then $(\nabla)\ \begin{cases} C_1+C_2=\cos\frac{\theta_0}{2} \\ \frac{\omega}{\sqrt{2}}(C_1-C_2)=0 \end{cases} \longrightarrow C_1=C_2=C$

$C=\frac{1}{2}\cos\frac{\theta_0}{2}$ (from (∇)) and then from $(\square)$

$$x(t)=C\left[e^{\frac{\omega}{\sqrt{2}}t}+e^{-\frac{\omega}{\sqrt{2}}t}\right]=\frac{1}{2}\cos\frac{\theta_0}{2}\left[e^{\frac{\omega}{\sqrt{2}}t}+e^{-\frac{\omega}{\sqrt{2}}t}\right]=$$

$$=\underline{\cos\frac{\theta_0}{2}\cdot\cosh\left(\frac{\omega t}{\sqrt{2}}\right)=x(t)}$$

If needed can be rewritten in terms of $\theta(t)$:

$$\cos\left(\frac{\theta}{2}\right)=\cos\frac{\theta_0}{2}\cdot\cosh\left(\frac{\omega t}{\sqrt{2}}\right)$$

$$\underline{\theta(t)=2\arccos\left(\cos\frac{\theta_0}{2}\cdot\cosh\left(\frac{\omega t}{\sqrt{2}}\right)\right)}.$$

* g) In this case do just similarly as f), but in initial conditions $\dot{x}(0)\neq 0$, ~~nothing for this~~.

$x=\cos\left(\frac{\theta}{2}\right)$

$\ddot{x}=\frac{\omega^2}{2}x \longrightarrow x(t)=C_1e^{\frac{\omega}{\sqrt{2}}t}+C_2e^{-\frac{\omega}{\sqrt{2}}t}$

$\dot{x}(t)=\frac{\omega}{\sqrt{2}}\left[C_1e^{\frac{\omega}{\sqrt{2}}t}-C_2e^{-\frac{\omega}{\sqrt{2}}t}\right]$

Init. conditions are:

$$\begin{cases} C_1+C_2=\cos\left(\frac{\pi/2}{2}\right)=\frac{1}{\sqrt{2}}=\ |x(0) \\ \frac{\omega}{\sqrt{2}}(C_1-C_2)=-\frac{\sin\left(\frac{\theta_0}{2}\right)}{2}\dot{\theta}=-\frac{\sin\left(\frac{\pi}{4}\right)}{2}\cdot(-\Omega)=\frac{\Omega}{2\sqrt{2}}\ \left|=\dot{x}(0)\right. \end{cases}$$

$$\longrightarrow$$

$$\begin{cases} C_1+C_2=\frac{1}{\sqrt{2}} \\ \omega(C_1-C_2)=\frac{\Omega}{2} \end{cases} \longrightarrow \begin{array}{l} C_1+C_2=\frac{1}{\sqrt{2}} \\ C_1-C_2=\frac{\Omega}{2\omega} \end{array}$$

$$2C_1=\frac{1}{\sqrt{2}}+\frac{\Omega}{2\omega},\quad \boxed{\begin{array}{l} C_1=\frac{1}{2\sqrt{2}}+\frac{\Omega}{4\omega} \\ C_2=-\frac{\Omega}{4\omega}+\frac{1}{2\sqrt{2}} \end{array}}$$

Then $x(t) = \left(\frac{\Omega}{4\omega} + \frac{1}{2\sqrt{2}}\right) e^{\frac{\omega}{\sqrt{2}} t} + \left(-\frac{\Omega}{4\omega} + \frac{1}{2\sqrt{2}}\right) e^{-\frac{\omega}{\sqrt{2}} t} =$

$$= \frac{\Omega}{2\omega} \cdot \frac{e^{\frac{\omega}{\sqrt{2}}t} - e^{-\frac{\omega}{\sqrt{2}}t}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{e^{\frac{\omega}{\sqrt{2}}t} + e^{-\frac{\omega}{\sqrt{2}}t}}{2} = \frac{\Omega}{2\omega} \sinh\left(\frac{\omega}{\sqrt{2}}t\right) + \frac{1}{\sqrt{2}} \cosh\left(\frac{\omega}{\sqrt{2}}t\right) \rightarrow$$

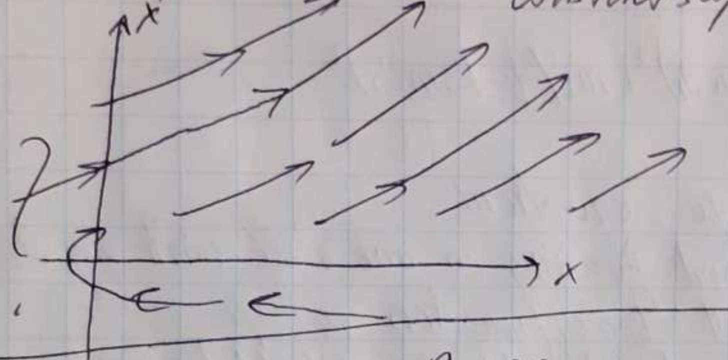
can be further reduced to one hyperbolic function ...

Relation between x and $\dot{x}$

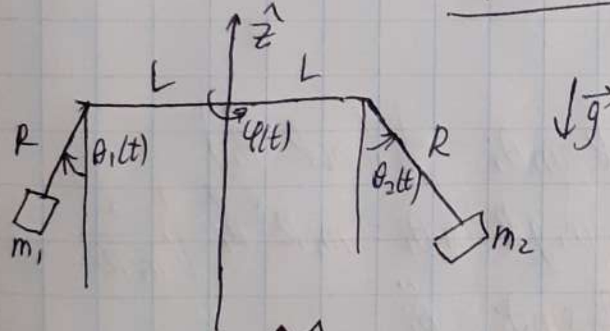
* for phase space:

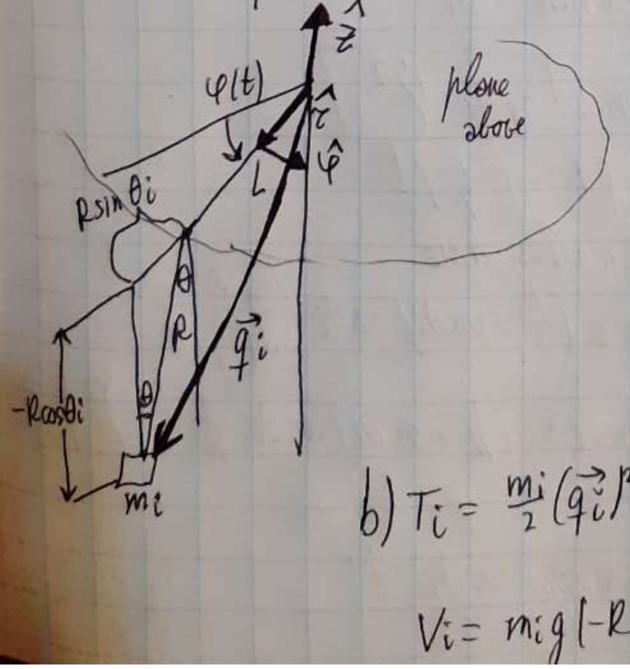
After long time $e^{-\frac{\omega}{\sqrt{2}}t} \rightarrow 0$,

$\dot{x} \approx \frac{\omega}{\sqrt{2}} x$, lines with constant slope > 0.



Problem 13.3 Freely rotating carousel





a) $\vec{q}_i = (L + R\sin\theta_i)\hat{r} - R\cos\theta_i \hat{z}$

$$\dot{\vec{q}}_i = (\underbrace{\dot{L}}_{0} + R\cos\theta_i \dot{\theta}_i)\hat{r} + (R\sin\theta_i + L)\hat{\varphi}\dot{\varphi} + R\sin\theta_i \dot{\theta}_i \hat{z} =$$
$$= R\dot{\theta}_i \cos\theta_i \hat{r} + (L + R\sin\theta_i)\dot{\varphi}\hat{\varphi} + R\dot{\theta}_i \sin\theta_i \hat{z}.$$

b) $T_i = \frac{m_i}{2}(\dot{\vec{q}}_i)^2 = \frac{m_i}{2}\left[(R\dot{\theta}_i\cos\theta_i)^2 + (L + R\sin\theta_i)^2\dot{\varphi}^2 + (R\dot{\theta}_i)^2\sin^2\theta_i\right] = \frac{m_i}{2}\left[R^2\dot{\theta}_i^2 + (L + R\sin\theta_i)^2\dot{\varphi}^2\right]$

$V_i = m_i g(-R\cos\theta_i)$ (0 on level of candelier)

$$L = T_1 + T_2 - (V_1 + V_2) = \frac{m_1}{2}\left[R^2\dot\theta_1^2 + (L+R\sin\theta_1)^2\dot\varphi^2\right] +$$
$$+ \frac{m_2}{2}\left[R^2\dot\theta_2^2 + (L+R\sin\theta_2)^2\dot\varphi^2\right] + m_1 g R\cos\theta_1 + m_2 g R\cos\theta_2.$$
(*)

c) $\frac{\partial L}{\partial \varphi} = 0$ (no explicit dependency on φ) $\longrightarrow$ then

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) = 0, \quad \frac{\partial L}{\partial \dot\varphi} = const,$$

$$\frac{\partial L}{\partial \dot\varphi} = K = \frac{m_1}{2}(L+R\sin\theta_1)^2 \cdot 2\dot\varphi + \frac{m_2}{2}(L+R\sin\theta_2)^2 \cdot 2\dot\varphi =$$
$$= \underline{\left[m_1(L+R\sin\theta_1)^2 + m_2(L+R\sin\theta_2)^2\right]}\dot\varphi \quad (*)$$

Then $f(\theta_1, \theta_2) = m_1(L+R\sin\theta_1)^2 + m_2(L+R\sin\theta_2)^2$

d) From Euler-Lagrange equation:

[from (*) and focusing on one half with any θ_i:
equivalent Lagr. function is
$$L_i = \frac{m_i}{2}\left[R^2\dot\theta_i^2 + (L+R\sin\theta_i)^2\dot\varphi^2\right] + m_i g R\cos\theta_i]$$

$$\frac{\partial L}{\partial \theta_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_i}\right)$$

$$\frac{m_i}{2}\cdot 2(L+R\sin\theta_i)R\cos\theta_i\,\dot\varphi^2 - m_i g R\sin\theta_i = \frac{d}{dt}(m_i R^2\dot\theta_i)$$

$$m_i R(L+R\sin\theta_i)\cos\theta_i\,\dot\varphi^2 - m_i g R\sin\theta_i = m_i R^2\ddot\theta_i \quad \Big| \cdot \frac{1}{m_i R^2}$$

$$\left(\frac{L}{R} + \sin\theta_i\right)\cos\theta_i \cdot \dot\varphi^2 - \frac{g}{R}\sin\theta_i = \ddot\theta_i$$

$$\boxed{\text{Now } \ddot\theta_i = -\frac{g}{R}\sin\theta_i + \left(\frac{L}{R} + \sin\theta_i\right)\cos\theta_i \cdot \dot\varphi^2}$$

Take <u>time scale</u> $t = \tau\sqrt{\frac{R}{g}}$, $[t] = \left[\frac{R}{g}\right]^{\frac{1}{2}}$

$\tau \rightarrow$ dimensionless time

$$\frac{d^2\theta_i}{d\left(\sqrt{\frac{R}{g}}\tau\right)^2} = \frac{g}{R}\cdot\frac{d^2\theta_i}{d\tau^2} = -\frac{g}{R}\sin\theta_i + \left(\frac{L}{R} + \sin\theta_i\right)\cos\theta_i\left(\frac{d\varphi}{d\left(\sqrt{\frac{R}{g}}\tau\right)}\right)^2$$

$$\frac{g}{R}\cdot\frac{d^2\theta_i}{d\tau^2} = -\frac{g}{R}\sin\theta_i + \left(\frac{L}{R} + \sin\theta_i\right)\cos\theta_i\,\frac{g}{R}\left(\frac{d\varphi}{d\tau}\right)^2 \Big| \cdot \frac{R}{g}$$

$\dot\theta_i$ now is w.r.t. τ.

$$\underline{\ddot\theta_i = -\sin\theta_i + \left(\frac{L}{R} + \sin\theta_i\right)\cos\theta_i \cdot \dot\varphi^2}$$

Now since $\dot{\varphi} = \dfrac{K}{f(\theta_1, \theta_2)}$ from (*),

$$\ddot{\theta}_i = -\sin\theta_i + \left(\frac{L}{R} + \sin\theta_i\right) \cdot \frac{K^2}{f^2(\theta_1, \theta_2)} \cdot \cos\theta_i$$

must match

$$\ddot{\theta}_i = -\sin\theta_i + \frac{\cos\theta_i}{f^2(\theta_1,\theta_2)} \left[k^2(\lambda + \sin\theta_i)\right]$$

where $K^2\left[\frac{L}{R} + \sin\theta_i\right] = k^2\left[\lambda + \sin\theta_i\right]$

Therefore $\underline{k = K \ (\text{or } -K)}$, $\underline{\lambda = \frac{L}{R}}$ dimensionless ratio of lengths.

No dependency on mass.

So finally EOM is $\underline{\ddot{\theta}_i = -\sin\theta_i + \dfrac{k^2}{f(\theta_1,\theta_2)^2}\cos\theta_i\,(\lambda + \sin\theta_i)}$

*e) no time for this yet

f) $m_2 = 0 \longrightarrow f(\theta_1,\theta_2) = m_1(L + R\sin\theta_1)^2 = m_1(\lambda R + R\sin\theta_1)^2 =$

$$= \underbrace{m_1 R^2}_{p}(\lambda + \sin\theta_1)^2 = p\,(\lambda + \sin\theta_1)^2$$

Then $\ddot{\theta}_1 = -\sin\theta_1 + \dfrac{\cos\theta_1\, k^2(\lambda + \sin\theta_1)}{f(\theta_1,\theta_2)^2} = -\sin\theta_1 + \dfrac{\cos\theta_1\, k^2(\lambda+\sin\theta_1)}{p^2(\lambda + \sin\theta_1)^4} =$

$$= -\sin\theta_1 + \frac{\cos\theta_1 \cdot K^2}{(\lambda + \sin\theta_1)^3}.$$

Then $K = \frac{k}{p}$, where $k = K$ (see (*)), $p = m_1 R^2$.

$\nu = 3$ here.

*g) no time for this yet

h) Using EOM: $\ddot{\theta}_1 = -\sin\theta_1 + \dfrac{K^2}{(\lambda + \sin\theta_1)^3}\cos\theta_1$

$\lambda \gg 1$, then $\ddot{\theta}_1 = -\sin\theta_1 + C\cos\theta_1$

$$\ddot{\theta}_1 = \frac{\sqrt{C^2+1}}{\sqrt{C^2+1}}\left(-\sin\theta_1 + C\cos\theta_1\right) = \sqrt{C^2+1}\left(\underbrace{\frac{-1}{\sqrt{C^2+1}}}_{\cos\theta_0}\sin\theta_1 + \underbrace{\frac{C}{\sqrt{C^2+1}}}_{\sin\theta_0}\cos\theta_1\right) =$$

$$= \sqrt{C^2+1}\,\sin(\theta_0 - \theta_1) = \ddot{\theta}_1$$

$\underline{\ddot{\theta}_1 = \underset{>0}{B}\sin(\theta_0 - \theta_1)}$ Take $\theta_0 - \theta_1 = \tilde{\theta}_1$ (shifted angle, dynamics is same)

$) \cdot \frac{R}{g}$

$\underline{\ddot{\theta}_1 = B\sin\theta_1.}$ Not solved directly, but can be transformed.

$$\ddot{\theta}_1 = B\sin\theta_1 \;\Big|\cdot 2\dot{\theta}_1$$

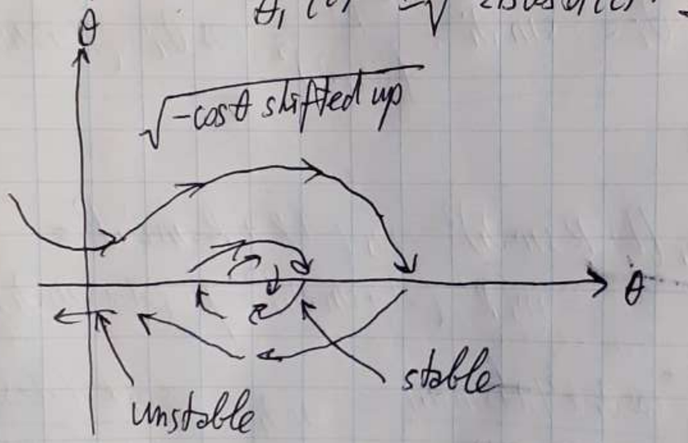
$$2\ddot{\theta}_1\,\dot{\theta}_1 = B\sin\theta_1\cdot 2\dot{\theta}_1$$

$$\frac{d}{dt}\left(\dot{\theta}_1^2\right) = -2B\frac{d}{dt}\left(\cos\theta_1\right) \;\Big|\int dt$$

$$\dot{\theta}_1^2(t) - \dot{\theta}_1^2(0) = -2B\left[\cos\theta_1(t) - \cos\theta_1(0)\right]$$

$$\dot{\theta}_1^2(t) = -2B\cos\theta_1(t) + \underbrace{2B\cos\theta_1(0) + \dot{\theta}_1^2(0)}_{\text{initial conditions (IC)}}$$

$$\dot{\theta}_1^2(t) = \pm\sqrt{-2B\cos\theta_1(t) + IC}$$



(If B was negative would be stable when $\theta = 0$, unstable later)

i)