9. Momentum and Conservation Laws

Your solution to the problems 9.1–9.4 should uploaded

to your Moodle account

as a PDF-file

by Wednesday, Dec 6, 10:30 (with a grace time till the start of the seminars).

The parts marked by $\star$ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

The self-test problems serve to check you understanding of vectors.

The bonus problem provides the solution to a problem that we introduced in class. It might take some extra effort to solve.

Problems

Problem 1. Center of mass and constants of motion

How do the expressions for the constants of motion behave when separating the center of mass motion and the relative motion, $\mathbf{q}_i(t) = \mathbf{Q}(t) + \mathbf{r}_i(t)$.

a) Assume that the system is moving in a gravitational field, and that the other forces on the particle arise from pair-wise conservative interactions that derive from a potential $\sum_{i<j} \Phi_{ij}\big(|\mathbf{r}_i - \mathbf{r}_j|\big)$. Show that the total energy can be written as

$$E = \frac{M}{2}\,\dot{\mathbf{Q}}^2 - M\mathbf{g}\cdot\mathbf{Q} + \sum_i \frac{m_i}{2}\,\dot{\mathbf{r}}_i^2 + \sum_{i<j} \Phi_{ij}\big(|\mathbf{r}_i - \mathbf{r}_j|\big)$$

and that this energy is conserved.

b) Show that the total angular momentum is conserved for a systems with the particles interactions adopted also in a), and an additional external force

$$\mathbf{F}_i = c\, m_i\, \mathbf{q}_i$$

acting on each particle i.

Bonus: How does that matter for the motion of Earth and Moon?

Problem 2. Motion in a harmonic central force field

A particle of mass m and at position $\mathbf{r}(t)$ is moving under the influence of a central force field

$$\mathbf{F}(\mathbf{r}) = -k\,\mathbf{r}\,.$$

a) We want to use the force to build a particle trap,[1] i.e. to make sure that the particle trajectories $\mathbf{r}(t)$ are bounded: Find a condition on k such that for all initial conditions there is a bound B with $|\mathbf{r}(t)| < B$ for all times t.

b) Determine the energy of the particle and show that energy conservation entails Newton's law, $\quad m\,\ddot{\mathbf{r}} = -k\,\mathbf{r}$.

c) Demonstrate that the angular momentum $\mathbf{L} = \mathbf{r}\times m\,\dot{\mathbf{r}}$ of the particle is conserved, too.

   Is this also true when considering a different origin of the coordinate system?
   Hint: The center of the force field does no longer coincide with the origin of the coordinate system in that case.

d) Conservation of the angular momentum $\mathbf{L}$ entails that the particle moves in a two-dimensional plane. We denote the position of the particle in this plane as (x_1, x_2). Show that $m\ddot{x}_i(t) = -k\,x_i(t)$ with $i \in \{1, 2\}$. Draw the trajectories in the phase space $(x_i, \dot{x}_i)$. What determines the form of the trajectories in the phase space?

e) Alternatively, we can describe the position of the particle in the plane by the complex number $z(t) = x_1(t) + \mathrm{i}\,x_2(t)$. Show that $m\,\ddot{z}(t) = -k\,z(t)$, and that this differential equation is solved by $z(t) = C_+\,\mathrm{e}^{\mathrm{i}\,\omega t} + C_+\,\mathrm{e}^{-\mathrm{i}\,\omega t}$. Here, the frequency ω is determined by the sytem parameters, while $C_\pm \in \mathbb{C}$ specify the intitial position and velocity of the particle. Determine ω.

f) Show that $z(t) = R\,\mathrm{e}^{\mathrm{i}\theta}\,\Big(\cos(\phi + \omega\,t) + \mathrm{i}\,A\,\sin(\phi + \omega\,t)\Big)$.

   Hint: Adopt the representation $C_\pm = R_\pm\,\mathrm{e}^{\mathrm{i}\,\theta_\pm}$ with $R_\pm, \theta_\pm \in \mathbb{R}$, and determine how $R_\pm, \theta_\pm$ are related to K, A, and ϕ.

[1] Particle traps with much more elaborate force fields, e.g. the Penning- and the Paul-trap, are used to fix particles in space for storage and use in high precision spectroscopy.

⋆ g) Show that the trajectories in the configuration space (x_1, x_2) are ellipses. How do R, A, and θ determine the shape of these trajectories? What is the role of ϕ?

Problem 3. Motion of two particles with harmonic interaction

We consider the motion of two particles of masses m_1 and m_2 at positions $\mathbf{q}_1(t)$ and $\mathbf{q}_2(t)$. They interact with a harmonic force, such that

$$m_1 \ddot{\mathbf{q}}_1(t) = -k \left(\mathbf{q}_1(t) - \mathbf{q}_2(t)\right),$$
$$m_2 \ddot{\mathbf{q}}_2(t) = -k \left(\mathbf{q}_2(t) - \mathbf{q}_1(t)\right).$$

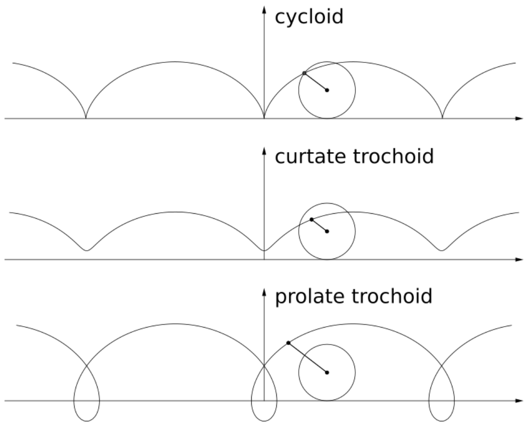
a) Discuss the evolution of the center of mass of this two particle system. How does it evolve in time, and how does the evolution depend on the initial condition $\big(\mathbf{q}_1(t_0), \mathbf{q}_2(t_0), \dot{\mathbf{q}}_1(t_0), \dot{\mathbf{q}}_2(t_0)\big)$?

b) Determine the equations of motion of the relative motion $\mathbf{R}(t) = \mathbf{q}_1(t) - \mathbf{q}_2(t)$. Discuss their solution by adapting the results of Problem 2.

⋆ c) How to trajectories of the two particles with harmonic interaction differ from those of the Kepler problem?

Problem 4. Retroreflector paths on bike wheels

The more traffic you encounter when it becomes dark the more important it becomes to make your bikes visible. Retroreflectors fixed in the spokes enhance the visibility to the sides. They trace a path of a curtate trochoid that is characterized by the ratio ρ of the reflectors distance d to the wheel axis and the wheel radius r.



[Wikimedia CC BY 4.0 (modified)]

A small stone in the profile traces a cycloid ($\rho = 1$). Animations of the trajectories are provided in the Wiki. It also provides a Jupyter Notebook that allows you to explore the parameter dependence of the tracks.

A trochoid is most easily described in two steps: Let $\mathbf{M}(\theta)$ be the position of the center of the disk, and $\mathbf{D}(\theta)$ the vector from the center to the position $\mathbf{q}(\theta)$ that we follow (i.e. the position of the retroreflector) such that

a) The point of contact of the wheel with the street at the initial time t_0 is the origin of the coordinate system. Moreover, we single out one spokes and denote the change of its angle with respect to its initial position as θ. Note that negative angles θ describe forward motion of the wheel!

   Sketch the setup and show that

$$\mathbf{M}(\theta) = \begin{pmatrix} -r\theta \\ r \end{pmatrix}\,, \qquad \mathbf{D}(\theta) = \begin{pmatrix} -d\,\sin(\varphi+\theta) \\ d\,\cos(\varphi+\theta) \end{pmatrix}\,.$$

   What is the meaning of φ in this equation?

b) The length of the track of a trochoid can be determined by integrating the modulus of its velocity over time, $L = \int_{t_0}^{t} \mathrm{d}t\ |\dot{\mathbf{q}}(\theta(t))|$. Show that therefore

$$L = r \int_0^\theta \mathrm{d}\theta\ \sqrt{1+\rho^2+2\rho\,\cos(\varphi+\theta)}$$

   How does ρ depend on r and d in this expression?

c) Consider now the case of a cycloid and use $\cos(2x) = \cos^2 x - \sin^2 x$ to show that the expression for L can then be written as

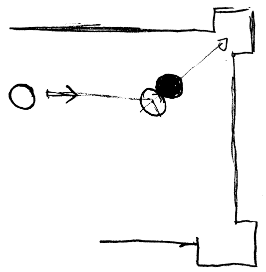
$$L = 2\,r \int_0^\theta \mathrm{d}\theta\ \left|\cos\frac{\varphi+\theta}{2}\right|$$

   How long is one period of the track traced out by a stone picked up by the wheel profile?

Self Test

Problem 5. Collisions on a billiard table

The sketch to the right shows a billiard table. The white ball should be kicked (i.e. set into motion with velocity $\mathbf{v}$), and hit the black ball such that it ends up in pocket to the top right. What is tricky about the sketched track? What might be a better alternative?



Problem 6. Keeping the Moon at a distance

Something goes wrong at the farewell party for the settlers of the new Moon colony *Sleeping Beauty 1* such that the extremely annoyed evil fairy *Valeluna* switches off gravity for the Moon. Luckily the good fairy *Veloxadriutrica* is present at the party. She cannot undo the curse but offers to strip all protons from all water-molecules in a bucket of water that you give to her, and hide them on Moon. The Coulomb attraction between electrons on Earth and protons on Moon can then undo the damage.

a) How much water would you give to her?

b) What will happen to the Earth-Moon system when you are off by 20%, by a factor of two, or even by an order of magnitude?
   Hint: The idea is that you discuss the motion for an initial condition where Earth and Moon are at their present position and move with their present velocity, while the gravitational force is changed by a the specified factor.

Problem 7. Inelastic collisions, ballistics, and cinema heroes

We first discuss a few CSI techniques to investigate firearms. Then we wonder how cinema heroes shoot.

a) The velocity of a projectile can be determined by investigating its impact into a wooden block (mass M) that is fixed to a rotor arm that lets it move horizontally on a circular track with radius ℓ. We choose our coordinates such that the block moves in the $(x - y)$-plane and that the rotor axis is at the origin of the coordinate system. The angle θ describes the angle of the arm with respect to the positive x axis.

We set up the experiment such that initially $\theta = 0$ while the projectile approaches the wooden block along a trajectory parallel to the y-axis. At time t_0 it hits the center of the block. The projectile has mass m and velocity $\mathbf{v}_0$. Sketch the setup.

b) What is the angular momentum of the projectile, the wooden block and the total angular momentum before the impact of the bullet? How does it change upon impact? What is the angular momentum after the impact? What is the angular speed of the rotor after the collision? What does this tell about the speed of the projectile?

c) Determine the kinetic energy of the projectile and of the rotor after the impact. What might be the origin of the energy difference?

d) The title of Stanley Kubrick's movie *Full Metal Jacket* refers to full metal jacket bullets, i.e. projectiles as they were used in the M16 assault rifle used in the Vietnam war. Its bullets have a mass of 10 g and they set a 1 kg wooden block revolving at a 1 m arm into a 8 Hz motion. What is the velocity of the bullets?

e) Alternatively one can preform this measurement by shooting the bullet into a swing where a wooded block of mass M is attached to ropes of length L. Initially it is at rest. Consider momentum conservation to determine its velocity immediately after impact. What does this tell about the kinetic energy immediately after the impact, and what about the maximum height of reached by the swing in its subsequent motion?

Let L be 0.2 m. Which mass is required to let the swing go up to the height of its spindle?

⋆ f) In the continuous fire mode the M16 has a rate of fire of 700–950 rounds/min. What does this tell about the recoil of the rifle? What do you think now about the Rambo shooting scene that you can find here on YouTube?

Problem 8. Conic sections and the trajectories in the Kepler problem

In the lecture we showed that the shape of the trajectories of the Kepler problem are described by the following equation

$$R(\theta) = \frac{R_0}{1 + \varepsilon\ \cos\theta} \tag{9.1}$$

In this problem we explore why these shapes are denoted as cone section.

a) Let c be a positive real constant. Sketch the points $(x, y, z) \in \mathbb{R}^3$ where $z^2 = c^2\,(x^2 + y^2)$. How does c change the shape of the surface?

b) Let H and m be real numbers. Sketch the points $(x, y, z) \in \mathbb{R}^3$ where $z = H + m\,y$. How do H and m influence the form and position of this surface?

c) We introduce polar coordinates,

$$x = R\ - \sin\theta \qquad\qquad y = R\ \cos\theta$$

and explore which points (x, y, z) are located in the intersection of the cone and the surface.
Find a parameterization of the problem such that $R(\theta)$ takes the form specified above. How do R_0 and ε depend on the parameters H, c and m?

Bonus Problem

Problem 9. Collision with an elastic bumper

Consider two balls of radius R with masses m_1 and m_2 that are moving along a line. The positions of their center of mass will be denoted as $x_1(t)$ and $x_2(t)$. Hence, they touch when $x_1(t) + R = x_2(t) - R$, and they do not feel each other when $x_1(t) + R < x_2(t) - R$. The distance between their centers takes the value $x_2(t) - x_1(t) = 2R + d(t)$. Here, $d(t) > 0$ when the balls do not touch. Moreover, when they run into each other, the balls can slightly be deformed such that $d(t) < 0$. In that case they experience harmonic repulsive forces $\pm k\,d(t)$, respectively.

a) During the collision of the two balls Newton's equations take the form

$$m_1\,\ddot{x}_1(t) = k\,d(t) \qquad\qquad m_2\,\ddot{x}_2(t) = -k\,d(t)$$

Show that this implies

$$\ddot{d}(t) = -\omega^2\,d(t)$$

for some positive constant ω. How does ω depend on the spring constant k and on the masses m_1 and m_2?

b) Let $d(t) = -d_M\,\sin\bigl(\omega\,(t - t_0)\bigr)$ describe the deformation of the balls for a collision at $t = t_0$, and contact in the time interval $t_0 \le t \le t_R$. Verify that it is a solution of the equation of motion. At which time t_R will the particles release (i.e. there is no overlap any longer)? What is the maximum potential energy stored in the harmonic potential?

c) We consider initial conditions where particle 1 arrives with a constant velocity v_0 from the left, and particle 2 is at rest. What is the total kinetic energy in this situation? Assume that at most a fraction α of the kinetic energy is transferred to potential energy.
What is the relation between v_0 and the maximum deformation d_M?
How can α be expressed in terms of v_0, d_M, k, and the masses?

d) The velocity of the two particles at times $t_0 \le t \le t_R$ can now be obtained by solving the integrals

$$m_i\,\dot{x}_i(t) = m_i\,\dot{x}_i(t_0) - (-1)^i \int_{t_0}^{t} \mathrm{d}t'\;k\,d(t')\,, \quad \text{with } i \in \{1,2\}$$

Why does this hold?
Which values does $\dot{x}_i(t_0)$ take?

Solve the integral and show that for $t_0 \le t \le t_R$

$$\dot{x}_1(t) = v_0\,\left[1 + \sqrt{\alpha\beta}\,\Bigl(\cos(\omega\,(t - t_0)) - 1\Bigr)\right]$$
$$\dot{x}_2(t) = -v_0\,\frac{m_1}{m_2}\,\sqrt{\alpha\beta}\,\Bigl(\cos(\omega\,(t - t_0)) - 1\Bigr)$$

How does β depend on the masses?

e) Verify that at release we have

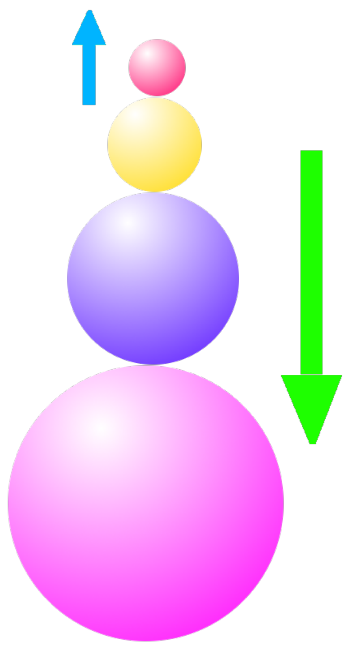
$$\dot{x}_1(t_R) = v_0 \left(1 - 2\sqrt{\alpha\beta}\right)$$
$$\dot{x}_2(t_R) = v_0 \, \frac{2\, m_1}{m_2} \, \sqrt{\alpha\beta}$$

Verify that these expressions comply to momentum conservation.
Verify that the kinetic energies before and after the collision agree iff $\alpha = \beta = m_2/(m_1 + m_2)$.

⋆ f) What does this imply for particles of identical masses, $m_1 = m_2$? How does your result fit to the motion observed in Newton's cradle? What does it tell about the assumption of instantaneous collisions of balls that is frequently adopted in theoretical physics?

Problem 10. Galilean cannon

In the margin we show a sketch of a Galilean cannon. Assume that the mass mass ratio of neighboring balls is always two, and that they perform elastic collisions.



a) Initially they are stacked exactly vertically such that their distance is negligible. Let the distance between the ground and the lowermost ball be 1 m. How will the distance of the balls evolve prior to the collision of the lowermost ball with the ground?

b) After the collision with the ground the balls will move up again. Determine the maximum height that is reached by each of the balls.

Hint: Argue that the velocity of the lowermost ball reverses upon collision. Subsequently, use energy and momentum conservation to determine the velocity of the second ball. With an informed choice of notation the velocity of the other balls follows iteratively, and the height of the uppermost ball is then again determined by energy conservation.