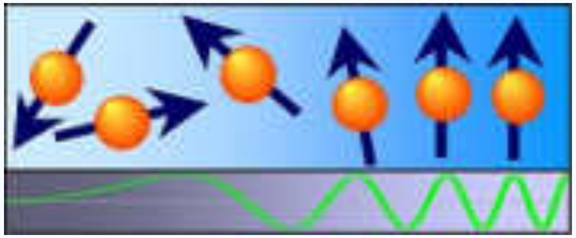
Experimental Physics
EP1 MECHANICS

- Energy and Work -



Rustem Valiullin

https://www.physgeo.uni-leipzig.de/en/fbi/applied-magnetic-resonance

Different forms of energy

- Kinetic
- Potential
- Mechanical
- Electromagnetic
- Chemical
- Internal (heat)
- … many more

Amalie Emmy Noether

Dimension: M L^2 T^{-2}

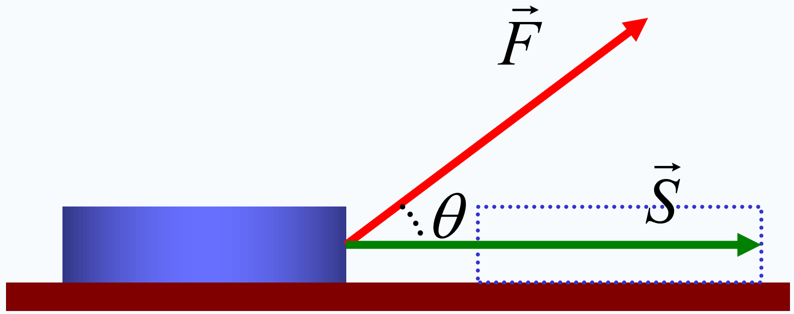
1 Joule = 1 kg m^2 s^{-2}

The invariance of physical systems with respect to time translation (in other words, that the laws of physics do not vary with time) gives the law of conservation of energy.

Work

$$\boxed{dW = \vec{F} \cdot d\vec{s} = F ds \cos\theta}$$

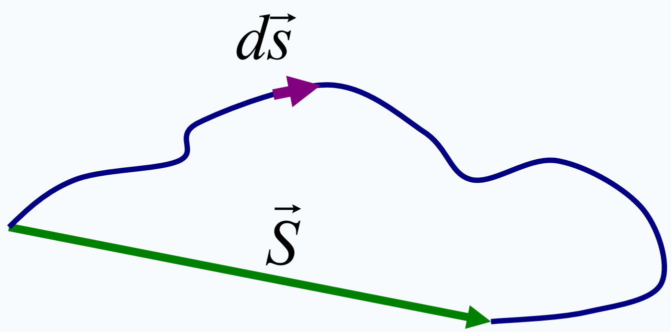
$$[W] = \frac{\mathrm{kg\,m}^2}{\mathrm{s}^2}$$



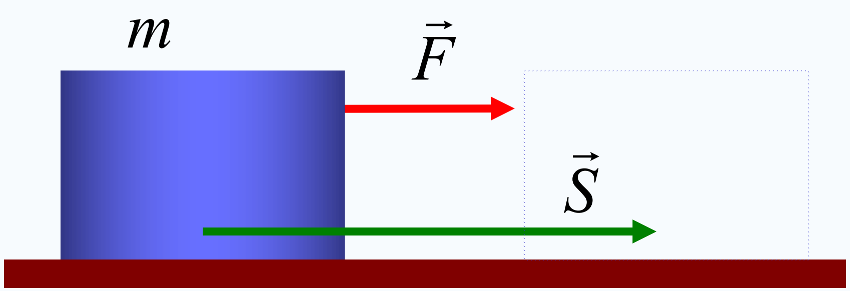
$$dW = \begin{cases} F ds, \theta = 0 \\ 0, \theta = \pi/2 \\ -F ds, \theta = \pi \end{cases}$$

$$\int dW = W = \int \vec{F} d\vec{s}$$

$$d\vec{s} = \hat{\mathrm{i}} dx + \hat{\mathrm{j}} dy + \hat{\mathrm{k}} dz$$



Kinetic energy



$$[E_k] = \frac{\text{kg}\,\text{m}^2}{\text{s}^2}$$

$$\boxed{E_k = \frac{mv^2}{2} - \text{kinetic energy}}$$

TABLE 7.1 Kinetic Energies for Various Objects

Object	Mass (kg)	Speed (m/s)	Kinetic Energy (J)
Earth orbiting the Sun	5.98×10^{24}	2.98×10^{4}	2.65×10^{33}
Moon orbiting the Earth	7.35×10^{22}	1.02×10^{3}	3.82×10^{28}
Rocket moving at escape speed[a]	500	1.12×10^{4}	3.14×10^{10}
Automobile at 55 mi/h	2000	25	6.3×10^{5}
Running athlete	70	10	3.5×10^{3}
Stone dropped from 10 m	1.0	14	9.8×10^{1}
Golf ball at terminal speed	0.046	44	4.5×10^{1}
Raindrop at terminal speed	3.5×10^{-5}	9.0	1.4×10^{-3}
Oxygen molecule in air	5.3×10^{-26}	500	6.6×10^{-21}

$$W = \int_A^B \vec{F} d\vec{s}$$

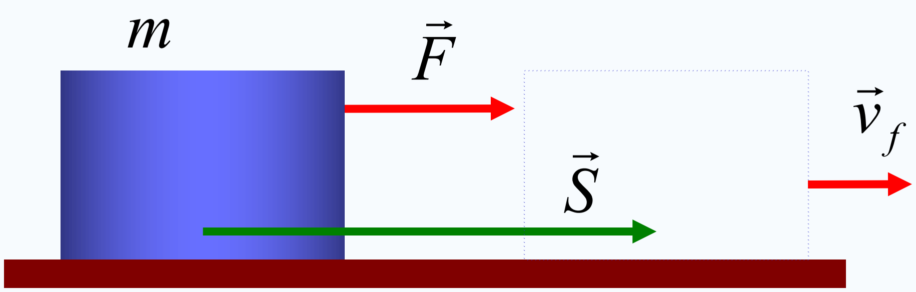
$$W = \int_A^B m \frac{d\vec{v}}{dt} d\vec{s} = \int_{t_A}^{t_B} m \frac{d\vec{v}}{dt} \vec{v} dt$$

$$\vec{v} d\vec{v} = \frac{1}{2} d\left(\vec{v} \cdot \vec{v}\right) = \frac{1}{2} dv^2$$

$$W = \int_{t_A}^{t_B} \frac{m}{2} \frac{dv^2}{dt} dt$$

$$W = \frac{mv_B^2}{2} - \frac{mv_A^2}{2}$$

Work- kinetic energy theorem



$$\boxed{\Delta E_k = W}$$

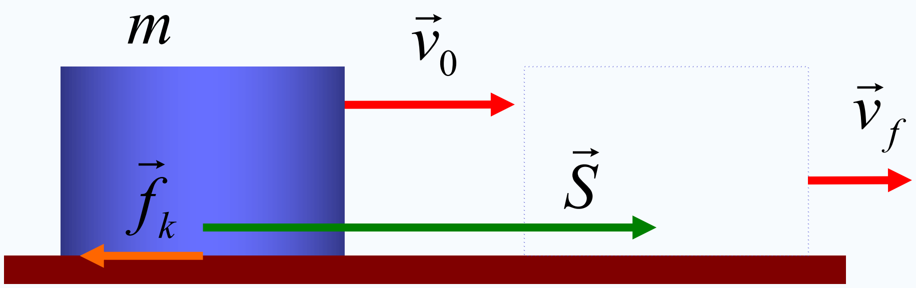
$$W = \int \vec{F} d\vec{s} = FS$$

$$W = \frac{mv_f^2}{2} - \frac{mv_0^2}{2} = \frac{mv_f^2}{2}$$

$$v_f = \sqrt{\frac{2FS}{m}}$$

Regenerative brake

Work done by kinetic friction



$$W_f = \int_A^B \vec{f}_k d\vec{s} = -f_k S$$

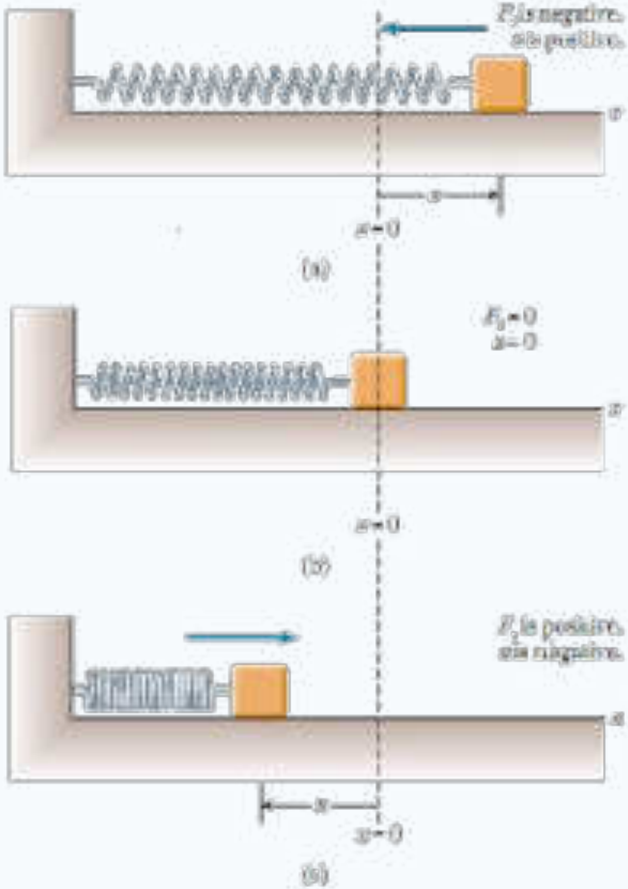
$$W_f = \int_A^B m \frac{d\vec{v}}{dt} d\vec{s} = \int_{t_A}^{t_B} m \frac{dv^2}{dt} dt$$

$$W_f = \frac{mv_f^2}{2} - \frac{mv_0^2}{2}$$

$$\boxed{\Delta E_k = -f_k S}$$

$$\boxed{E_{k,final} = E_{k,initial} - f_k S}$$

Work done by spring



$$F = -kx$$ **Hook's law**

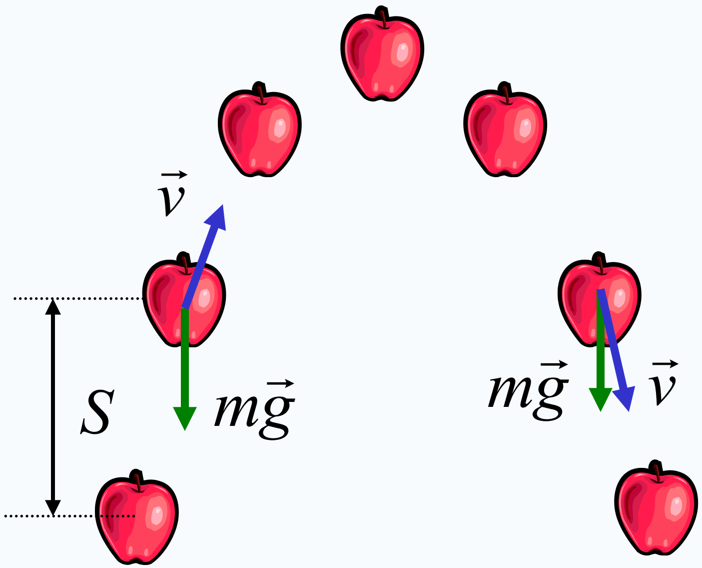
$$W_s = \int_{x_{initial}}^{x_{final}} F dx = \int_{x_{initial}}^{x_{final}} (-kx) dx$$

$$W_s = \int_{x_i=0}^{+x_f} (-kx) dx = -\frac{k}{2} x^2 \Big|_0^{x_f} = -\frac{k}{2} x_f^2$$

$$W_s = \int_{x_i=x_f}^{0} (-kx) dx = -\frac{k}{2} x^2 \Big|_{x_f}^{0} = \frac{k}{2} x_f^2$$

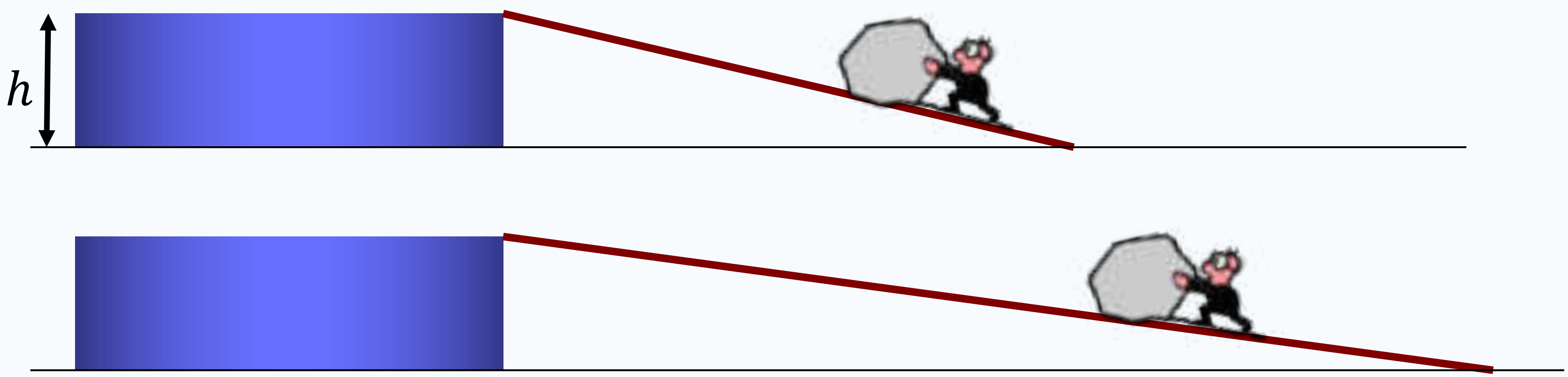
$$W_s = -\frac{k}{2} x^2 \Big|_{x_i}^{x_f} = -\frac{k}{2} x_f^2 - \frac{k}{2} x_i^2$$

Work done by gravitation

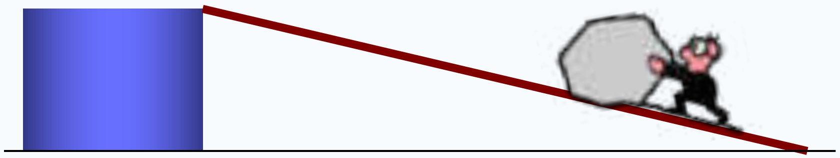


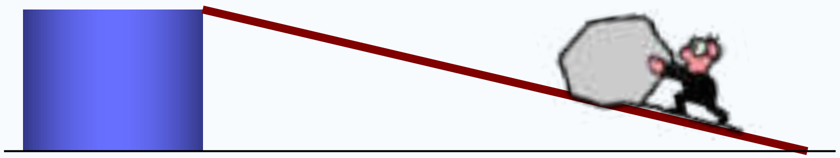
$$W_g = \int_A^B m\vec{g} \cdot d\vec{s} = mgS\cos\theta$$

$$\begin{cases} W_{g,u} = -mgh \\ W_{g,d} = mgh \end{cases}$$



Power





$$P_{av} = W / \Delta t$$

$$P = \frac{dW}{dt}$$

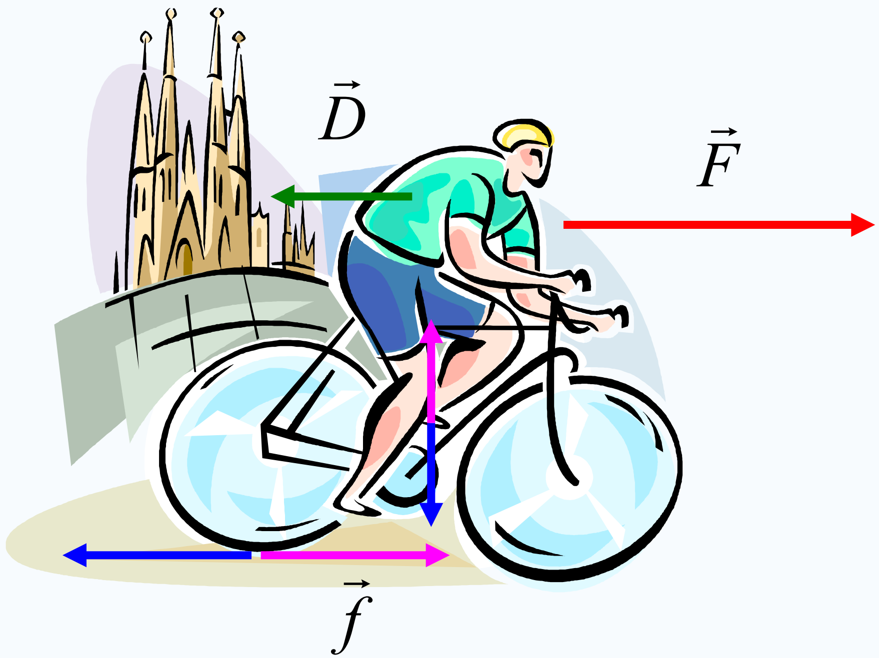
$$[P] = \mathrm{Watt} = \frac{\mathrm{kg\,m}^2}{\mathrm{s}^3}$$

1 Watt = 1 J/s
1 Horsepower = 1 hp = 746 W
1 kilowatt-hour = 1 kWh = 3.6×10^{6} J

$$P = \frac{d(\vec{F}\vec{s})}{dt} = \vec{F}\frac{d\vec{s}}{dt} + \cancel{\vec{s}\,\frac{d\vec{F}}{dt}} = \vec{F}\cdot\vec{v}$$

For constant force

Physics of bicycle



v, km/h	P, W
20	90
40	?

$$\vec{f} - \vec{D} = m\vec{a}$$

For the constant velocity case

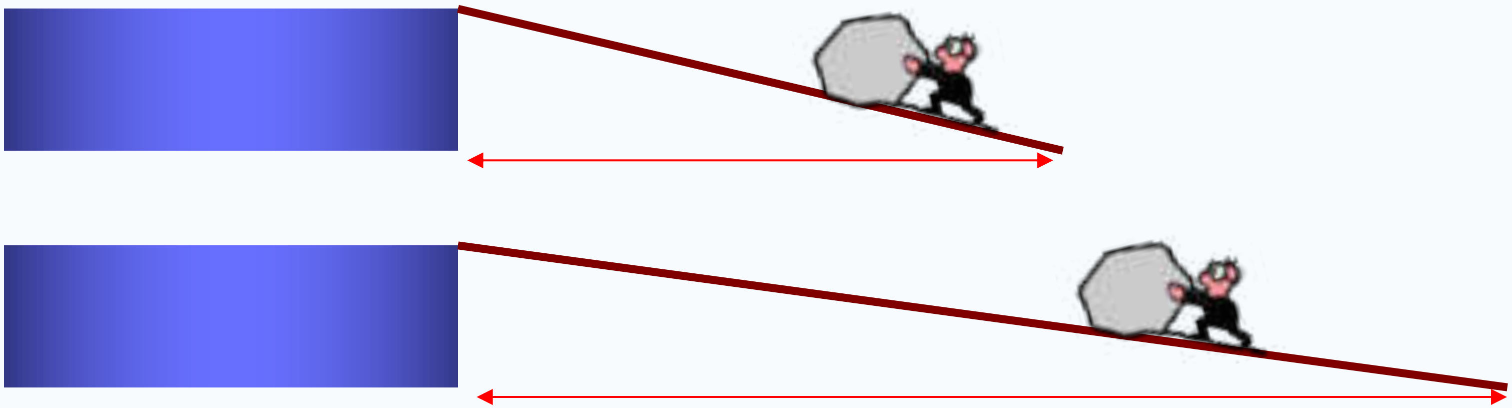
$$f = D = kv^2$$

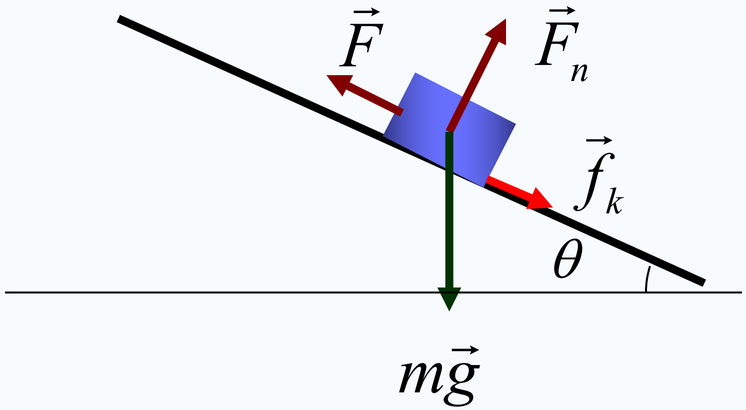
$$P = fv = kv^3$$

$$\frac{P_1}{P_2} = \left(\frac{v_1}{v_2}\right)^3$$

1 Horsepower = 1 hp = 746 W

Taking account of friction





$$-F_n + mg\cos\theta = 0$$

$$f_k = \mu_k F_n$$

$$W = FL = \mu_k mgL\cos\theta$$

To remember!

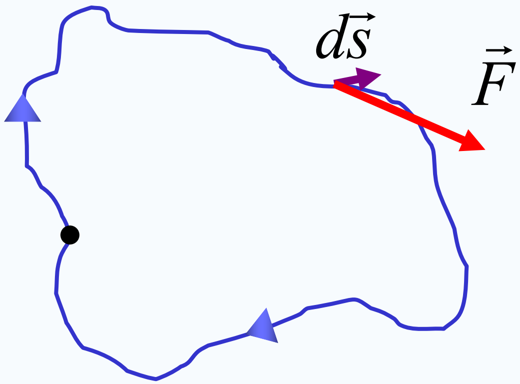
- **Work** is energy transferred to/from an object by applying a force to this object.
- **Kinetic energy** is associated with the motion of an object. Change in the kinetic energy is due to work done on the object.
- **Frictional**, **gravitational** and **spring** forces can do work.
- **Power** is the rate at which work is done on an object.

Conservative forces

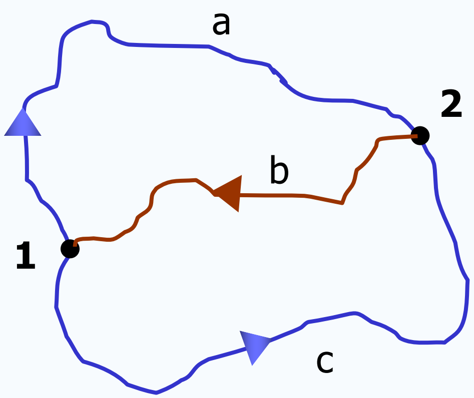
$$dW = \vec{F} \cdot d\vec{s}$$

$$W = \oint \vec{F}_{\text{conservative}} d\vec{s} = 0$$

Loop integral: integration over a closed curve or loop



$$\begin{cases} W = 0 = W_{12}^{a} + W_{21}^{b} \\ W = 0 = W_{12}^{c} + W_{21}^{b} \end{cases} \Rightarrow W_{12}^{a} = W_{12}^{c}$$



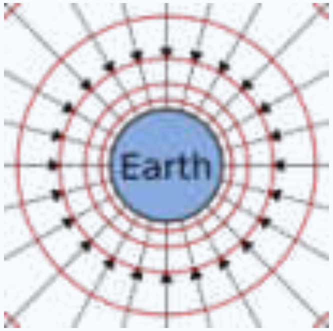
- **Position-dependent forces**

$$\boxed{dW = \vec{F} \cdot d\vec{s} = -dU}$$

Field forces

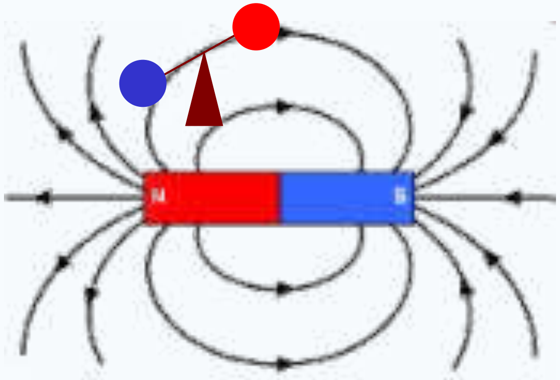
$$dW = \vec{F} \cdot d\vec{s} = m\frac{dv}{dt}vdt = d\left(\frac{mv^2}{2}\right) = dE_k$$

$$dW_{gr} = -mgdh = -d(mgh) = -dU_{gr}$$



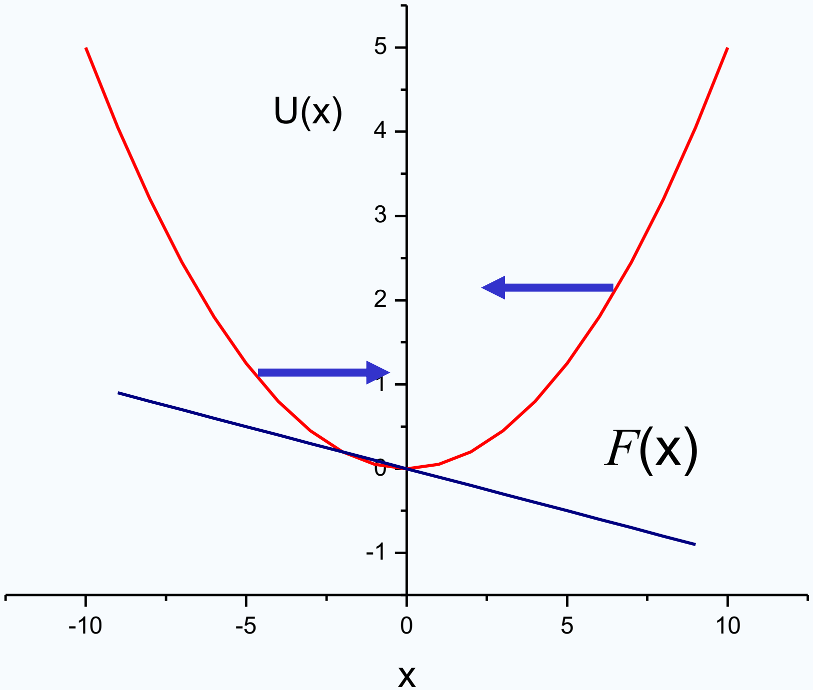
$$dW_{sp} = -kxdx = -d\left(\frac{kx^2}{2}\right) = -dU_{sp}$$

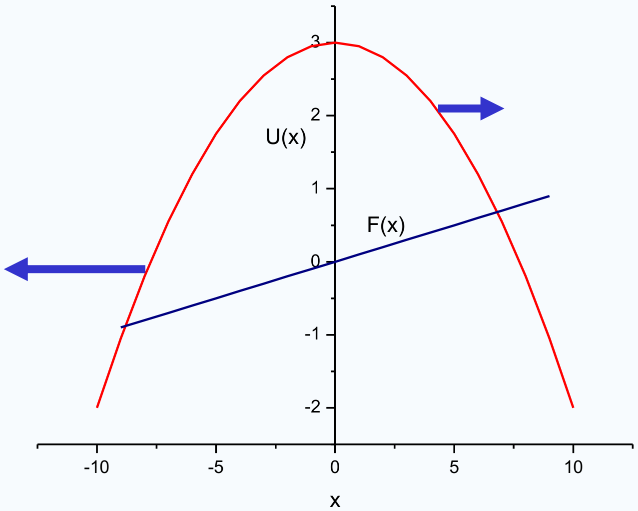
$$dW_C = \frac{kq_1q_2}{r^2}dr = -d\left(\frac{kq_1q_2}{r}\right) = -dU_C$$

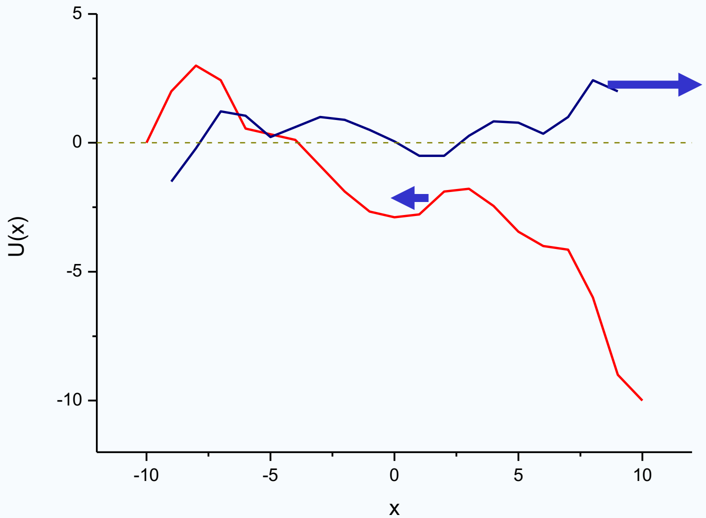


Equilibrium

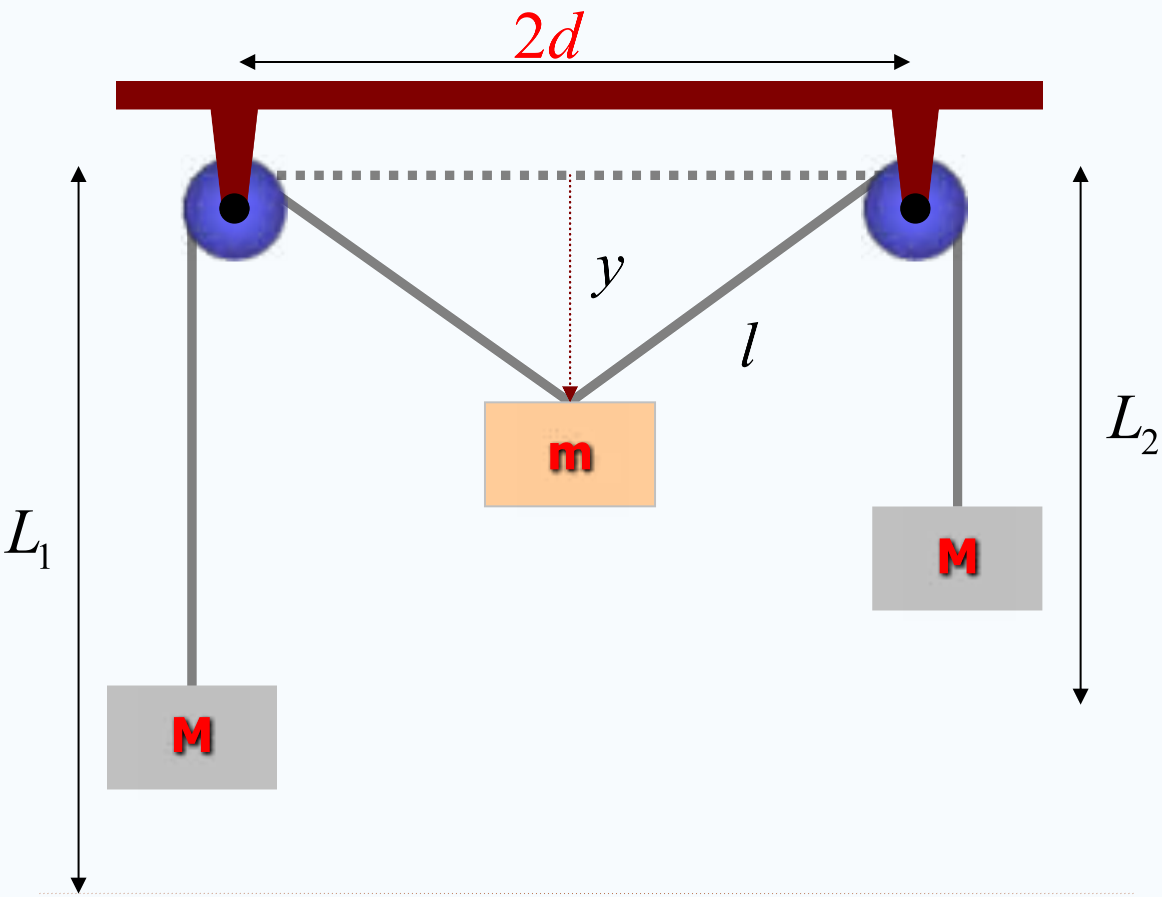
$$U_{sp} = \frac{kx^2}{2} \qquad F = -\frac{dU}{dx}$$







Mapping the potential field

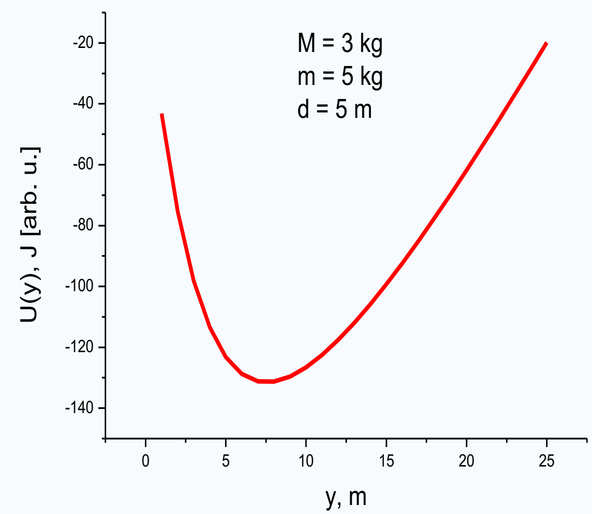


$$U(y)-?$$

$$l^2+y^2=d^2$$

$$\Delta L=l-d$$

y_{eq} **= 7.54 m**



$$U_0=Mg(L_1-L_2)+mgL_1$$

$$U_y=2Mg\Delta L-mgy+U_0$$

$$y_{eq}=d\,\frac{m/2M}{\sqrt{1-m^2/4M^2}}$$

Conservation of mechanical energy

$$W_{\text{total}} = \int \vec{F} \cdot d\vec{s} = -\Delta U = \Delta E_k$$

$$\Delta E_k + \Delta U = 0$$

$$\boxed{E \equiv E_k + U = const}$$

Mechanical energy

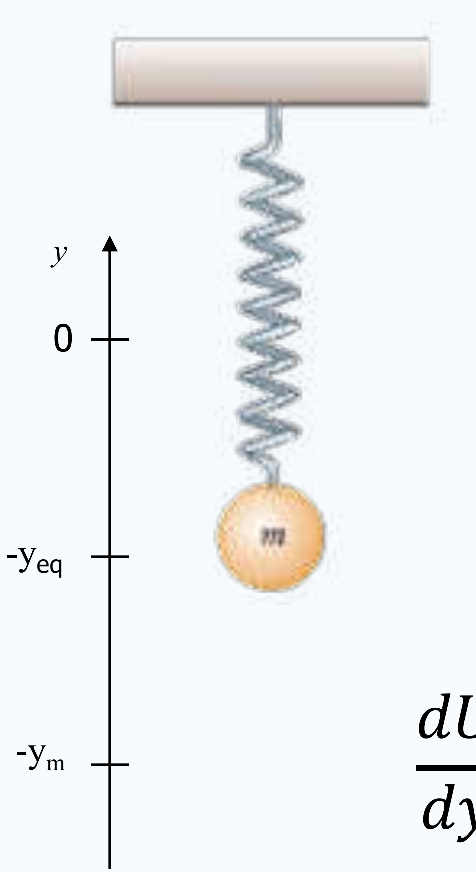
$$\vec{F} = \vec{F}_{nc} + \sum \vec{F}_i \qquad W_{\text{total}} = \int \vec{F} \cdot d\vec{s} = \int \vec{F}_{nc} \cdot d\vec{s} + \sum \int \vec{F}_i d\vec{s}$$

$$W_{total} = W_{nc} + \sum W_i = W_{nc} - \sum \Delta U_i = \Delta E_k$$

$$W_{nc} = \sum \Delta U_i + \Delta E_k = \Delta E$$

Work done by nonconservative forces is equal to change of mechanical energy.

A mass on a spring under gravitation



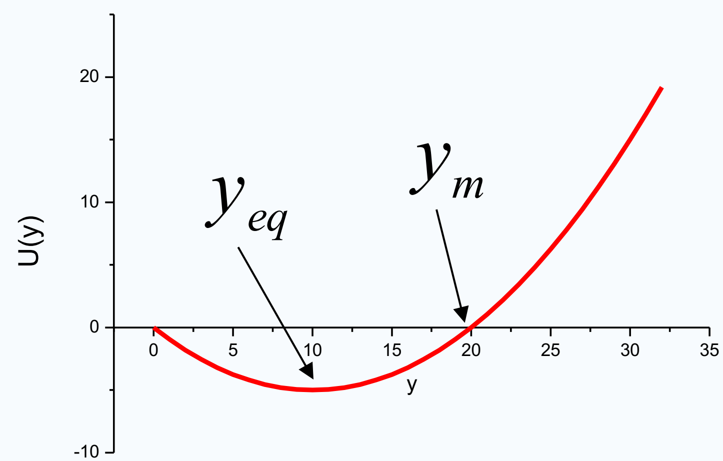
$$E_{initial} = E_k + \sum U_i = 0$$

$$E = \frac{1}{2}mv^2 + mgy + \frac{1}{2}ky^2 = 0$$

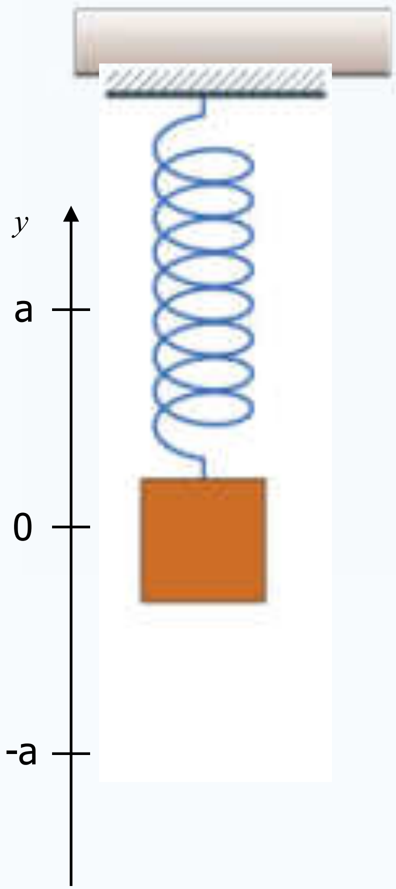
$$mgy_m + \frac{1}{2}ky^2 = 0 \;\Rightarrow\; y_m = -\frac{2mg}{k}$$

$$\frac{dU}{dy} = mg + ky \quad = 0$$

$$y_{eq} = -\frac{mg}{k}$$



A mass on a spring: dynamics (no gravity!)



$$E = \frac{mv^2}{2} + U(y) \Rightarrow v = \sqrt{\frac{2(E - U(y))}{m}}$$

$$v = \frac{dy}{dt} = \sqrt{\frac{2(E - U(y))}{m}} \Rightarrow \frac{dy}{\sqrt{2(E - U(y))}} = \frac{1}{\sqrt{m}} dt$$

$$U(y) = \frac{ky^2}{2} \qquad E = ? \qquad E = \frac{ka^2}{2}$$

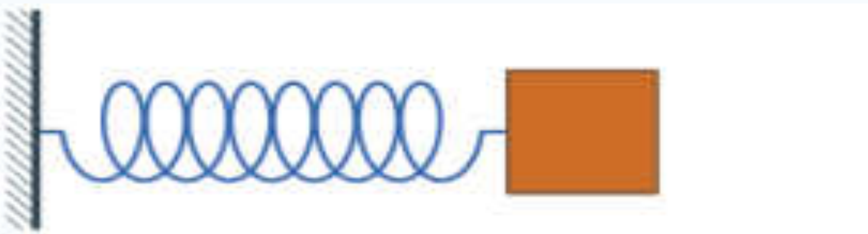
$$\frac{dy}{\sqrt{a^2 - y^2}} = \sqrt{\frac{k}{m}} dt \qquad \int_{-a}^{y'} \frac{dy}{\sqrt{a^2 - y^2}} = \int_0^t \sqrt{\frac{k}{m}} dt$$

$$\int_{3\pi/2}^{\arcsin y'/a} \frac{a \cos\theta d\theta}{\sqrt{a^2 - a^2 \sin^2\theta}} = \arcsin\frac{y'}{a} - \frac{3\pi}{2} = \sqrt{\frac{k}{m}} t$$

$$\Big|_{-a}^{y'} y = a \sin\theta \Big|_{3\pi/2}^{\arcsin y'/a}$$

$$\boxed{y = a \sin\left(\sqrt{k/m} \cdot t + \theta_0\right)}$$

Harmonic oscillator



$$E_k + U = \text{const}$$

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \text{const}$$

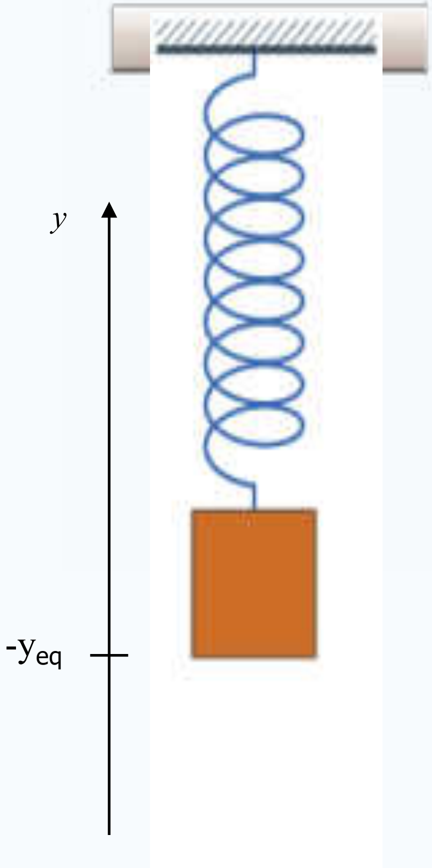
$$kx\cancel{\frac{dx}{dt}} + m\cancel{v}\frac{dv}{dt} = 0$$

$$\ddot{x} + \omega_0^2 x = 0$$

$$\omega_0^2 \equiv \sqrt{k/m}$$

$$x = x_0 \cos(\omega_0 t + \delta)$$

Harmonic oscillator with gravity



$$E = \frac{1}{2}mv^2 + mgy + \frac{1}{2}ky^2 = \text{const}$$

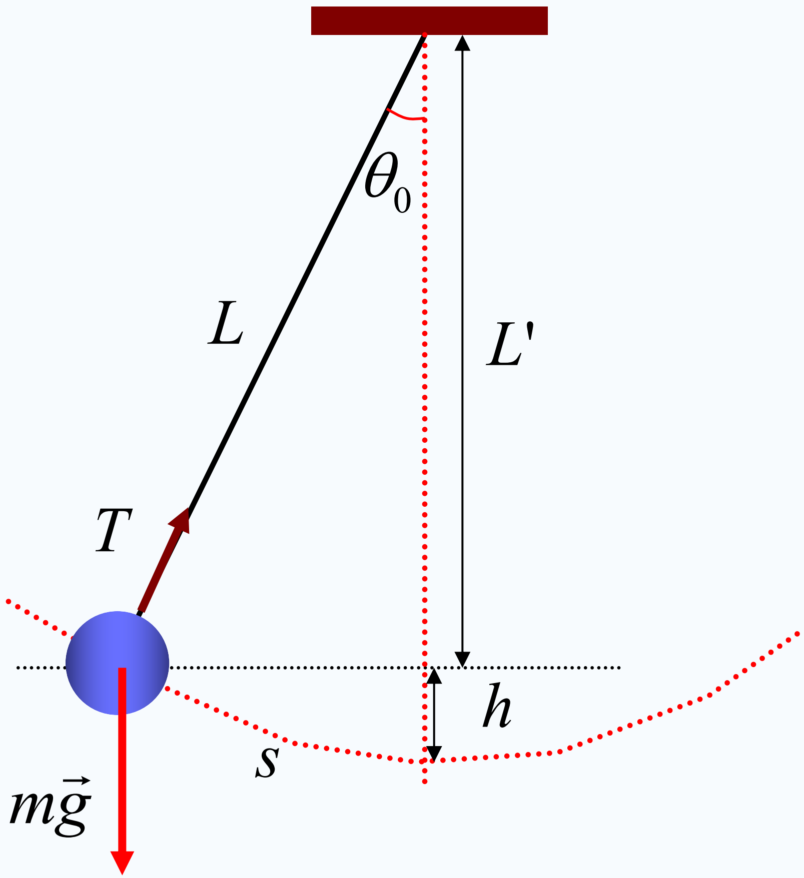
$$s = y - y_{eq}$$

$$\ddot{s} + \omega_0^2 s = 0$$

$$y = y_{eq} + y_0\cos(\omega_0 t + \delta)$$

$$y_{eq} = -\frac{mg}{k}$$

Physics of pendulum



$$E_i = mgh + 0 \qquad E_f = 0 + mv^2/2$$

$$h + L' = L \qquad L' = L\cos\theta_0$$

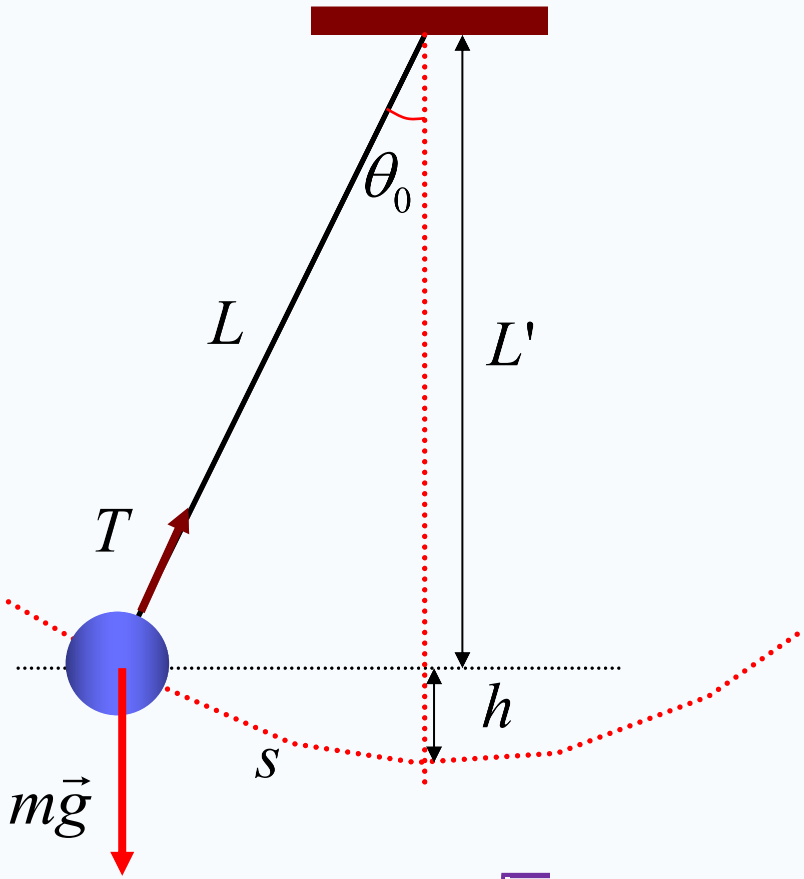
$$v^2 = 2gh = 2gL(1-\cos\theta_0)$$

$$T - mg = mv^2/L = 2g(1-\cos\theta_0)$$

$$mg\sin\theta = m\frac{dv}{dt} \qquad s = L\theta$$

$$v = \frac{ds}{dt} = L\frac{d\theta}{dt} \qquad \frac{dv}{dt} = \frac{dv}{d\theta}\frac{d\theta}{dt} \qquad \frac{dv}{d\theta}\frac{v}{L} = g\sin\theta$$

Physics of pendulum



$$E = \frac{1}{2}mv^2 + mgh = \text{const}$$

$$s \approx L\theta \qquad h \approx s\theta = \frac{1}{2}L\theta^2$$

$$v\dot{v} + g\dot{h} = 0$$

$$L\dot{\theta}L\ddot{\theta} + gL\theta\dot{\theta} = 0$$

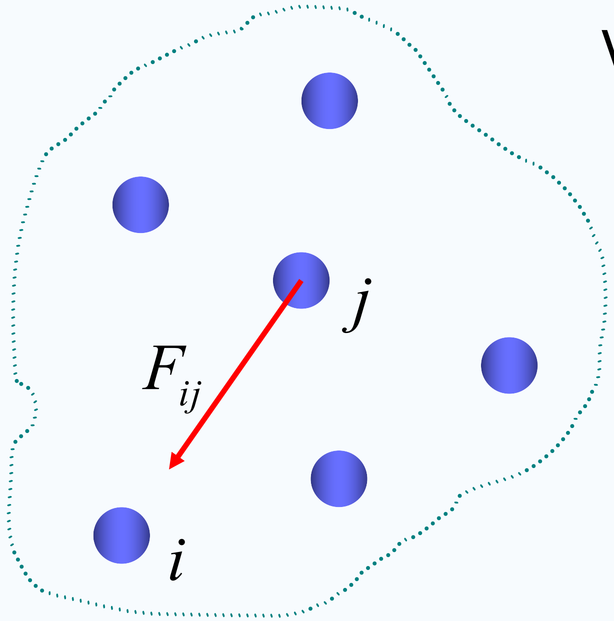
$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

$$\omega_0 = \sqrt{\frac{g}{L}}$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$\theta = \theta_0 \cos(\omega_0 t)$$

The virial theorem



Virial: $-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{F}_{ij}\cdot\mathbf{r}_i$

$$-\frac{1}{2}\sum_{i=1}^{N}\mathbf{F}_i\cdot\mathbf{r}_i$$

$$\overline{E_k} = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\overline{\mathbf{F}_{ij}\cdot\mathbf{r}_i}$$

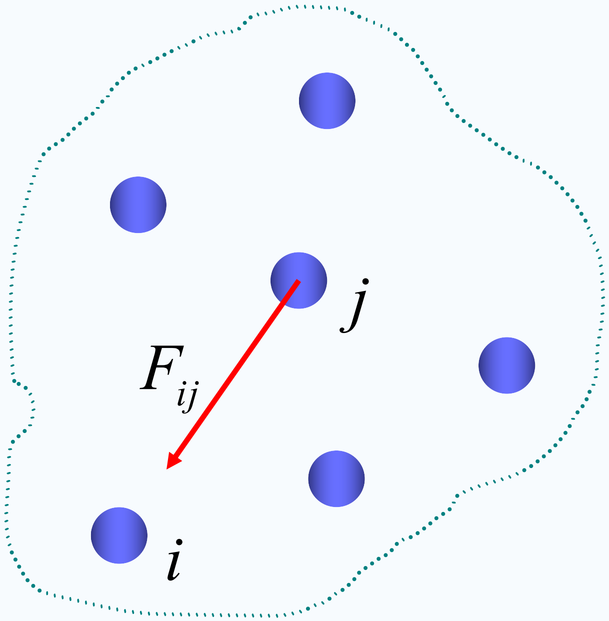
Time average

$$\overline{f(t)} = \frac{1}{T}\int_0^T f(t)dt$$

$$\langle E_k\rangle = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\langle\mathbf{F}_{ij}\cdot\mathbf{r}_i\rangle$$

Ensemble average

The virial theorem: gravitational field

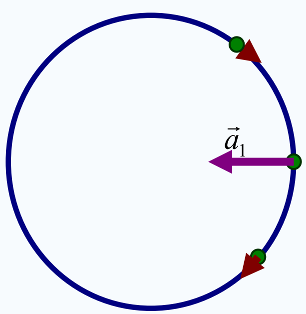


$$\overline{E_k} = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\overline{\mathbf{F}_{ij}\cdot\mathbf{r}_i}$$

$$F_{ij} = \frac{Gm_i m_j}{r_{ij}^2} \qquad U_{ij} = -\frac{Gm_i m_j}{r_{ij}}$$

$$\boxed{2\overline{E}_k = -\overline{U}}$$

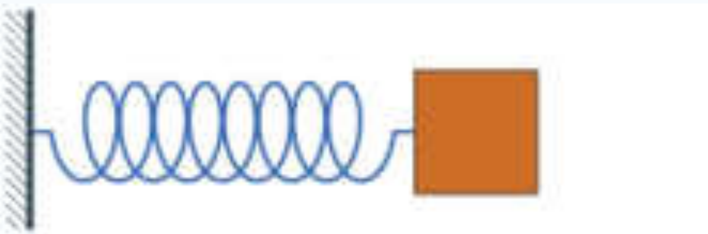
$$\vec{F} = m\vec{a}$$



$$F = \frac{GmM}{R^2} = \frac{mv^2}{R} \qquad \frac{mv^2}{2} = \frac{GmM}{2R}$$

$$U = -\frac{GmM}{R}$$

The virial theorem: Selected examples



$$-\frac{1}{2}(-kx)x = \frac{1}{2}kx^2 = U_{sp}$$

$$\overline{E_k} = \overline{U_{sp}}$$

$$\overline{U_{sp}} = \frac{\int_0^T U_{sp}dt}{\int_0^T dt} = \frac{k}{4}a^2$$

$$2\overline{E}_k = -\overline{U}$$

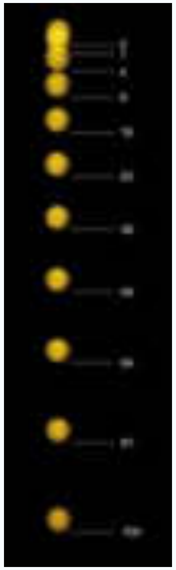
gravity

$$\overline{E_k} = \overline{U_{sp}}$$

spring

$$U \propto x^n$$

$$\overline{E_k} = \frac{n}{2}\overline{U}$$



$$\overline{E_k} - ?$$

$$\overline{E_k} = \frac{1}{2}\overline{U}$$

$$\overline{E_k} = \frac{1}{2}mgh$$

To remember!

- Work done by <u>conservative forces</u> along a closed trajectory is zero.
- Under action of conservative forces <u>mechanical energy</u> of the object does not change.
- Mechanical energy is <u>sum</u> of kinetic and potential energies.
- Work done by <u>nonconservative forces</u> equals the change of mechanical energy.