Lecture "Experimental Physics I"
(Prof. Dr. R. Seidel)

Lecture 17

Rotation about free axes - Tops

- Static & dynamic balance
- Tensors as “anisotropic proportionality constants”
- Moment of inertia in tensor form
- Movement of tops
- Nutation & Precession

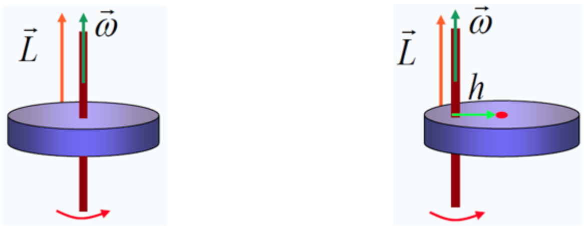
1) Balance of rotating masses

The balancing of rotating rigid bodies is important in technology to avoid vibrations. In particular, any torque that would try to tilt the rotation axis causes unwanted forces in the bearings etc.

Experiment: We look at a mass at the edge of a disk that is mounted on a horizontal rotation axis through the disk center. Spinning the system provides markable angular velocity changes due to the acting gravity.

To allow a smooth rotation we have to balance the system, such that gravity cannot influence it. This is called:

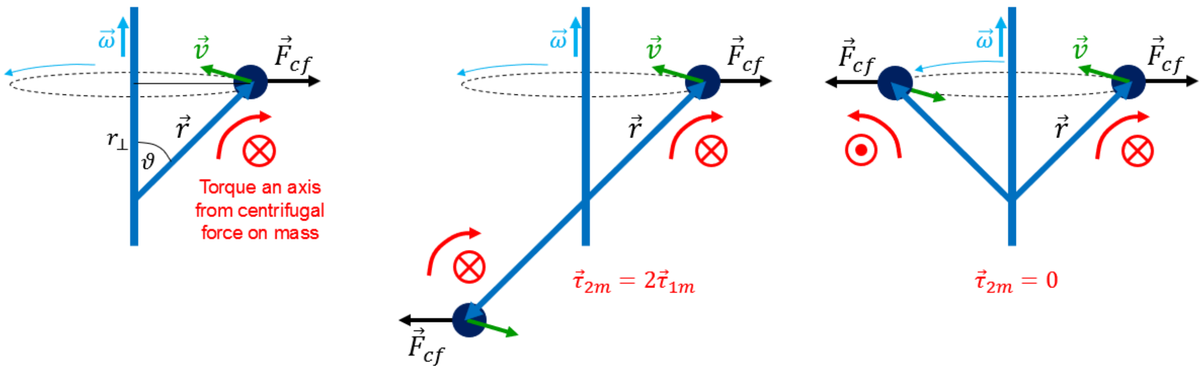
Static balance, which we have if the **center of gravity of an object is on the axis of rotation.** In this case gravity cannot drive on its own a rotation for any orientation of the rotation axis (e.g. horizontal).



balanced unbalanced

Experiment: The disk with the horizontal rotation axis from before is equipped with an additional mass on the opposite disk side to statically balance the system. There are two simple ways of doing this. The balancing mass can be below or above the disk. In the experiment (the rotation axis is either hold with the hands or put vertically on a soft support connected to an accelerometer) one finds that only the mirror symmetric system has negligible oscillations while the non-mirror-symmetric system experiences considerable oscillations of the rotation axis when spinning. Thus, we have to insure a second type of balance that appears only for the rotating object. This is called:

Dynamic balance where the **rotation does not produce any resultant net torque from the centrifugal forces on the rotation axis**. The depicted toy models below show different variants of static and dynamic (im)balance:



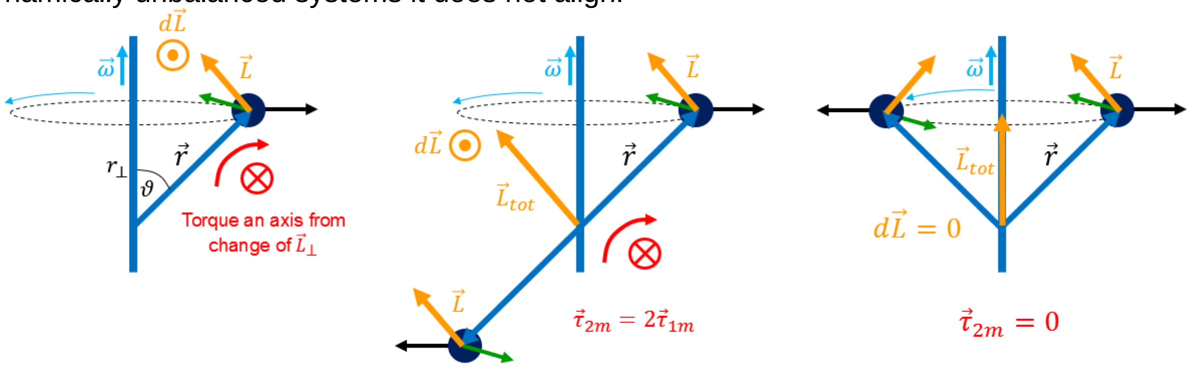
neither static nor dynamic
(non-zero force and torque)

only static, not dynamic
(only non-zero torque)

static and dynamic
(no force, no torque)

When rotating, we have in addition to the external (gravity) forces the centrifugal force. For the two systems on the left side, the centrifugal forces (black arrows) cause a torque on the rotation axis in perpendicular direction to it. Since the system is rotating the torque direction is changing, which causes the torque direction to oscillate. The left system is additionally also at static imbalance, since the rotation axis does not go through the CM. The right system is both in static and dynamic balance. Here, the torques of the two centrifugal forces cancel out.

An alternative criterium for dynamic imbalance can be obtained by determining the (total) angular momentum of the rotating masses from the vector product of the connection vectors and the velocity vectors. Since the connection vectors are not perpendicular to the rotation axis, the angular momentum of each mass is tilted with respect to the rotation axis. For the fully balanced system (on the right) the total angular momentum aligns with the rotation axis, while for the dynamically unbalanced systems it does not align.



neither static nor dynamic (non-zero force and torque) | only static, not dynamic (only non-zero torque) | static and dynamic (no force, no torque)

Thus, we conclude that for a **dynamically unbalanced system the angular momentum does not align anymore with the angular velocity!** For the systems in **dynamic imbalance the angular momentum vector rotates particular its radial component**, i.e. it changes in time. According to the relation:

$$d\vec{\tau} = \frac{d\vec{L}}{dt}$$

The **change in angular momentum requires a torque that is provided by a torque in opposite direction on the rotation axis.** The torque calculated from the angular momentum change equals the torque calculated from the centrifugal forces.
The relationship between the angular velocity and the angular momentum for the unbalanced systems is still linear i.e. doubling one quantity doubles the other one. Strikingly, we see, however, that **angular momentum and angular velocity have different directions**.
We had before that the scalar moment of inertia provides the link between the two quantities:

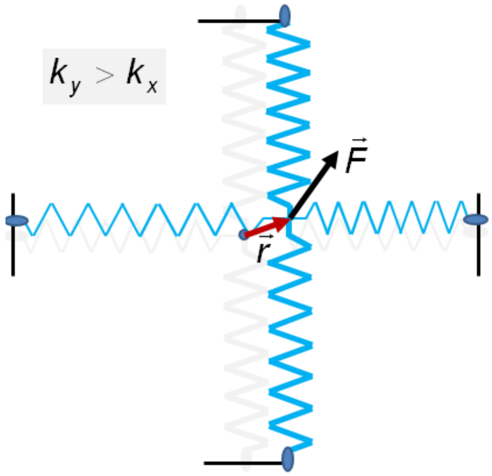
$$\vec{L} = \underbrace{I}_{scalar} \ \vec{\omega}$$

such that both vectors had the same orientation. To **allow different orientations** we now will write:

$$\vec{L} = \underbrace{\tilde{I}}_{tensor} \cdot \vec{\omega}$$

where $\tilde{I}$ **is a matrix – a so called tensor of rank 2** (2D matrix). The product represents a multiplication of a matrix with a vector. This is a linear relationship where the **direction if the initial angular velocity vector changes**.

2) Tensor description of an anisotropic spring system



Tensors are always used when we have an anisotropic system, i.e. if we have different proportionality constants for the different directions in space. Before we look at the tensor that describes the moment of inertia, we will first look at a more intuitive system, which is an anisotropic spring system. In two dimensions we shall have a spring constant k_x along x and a different spring constant k_y along y. If the system is displaced from the equilibrium position by $\vec{r}$ then the corresponding force is given by:

$$\vec{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} -k_x x \\ -k_y y \end{pmatrix}$$

due to the different spring constants for the x and y direction. This force does typically not point along the displacement (see sketch above). If one wants to express the force as the result of a product of $\vec{r}$ we have to use a diagonal matrix:

$$\vec{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \underbrace{\begin{pmatrix} -k_x & 0 \\ 0 & -k_y \end{pmatrix}}_{\text{spring constant tensor}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -k_x x + 0 \cdot y \\ 0 \cdot x + -k_y y \end{pmatrix} = \begin{pmatrix} -k_x x \\ -k_y y \end{pmatrix}$$

We have now an easy expression for the force with a spring constant $\tilde{k}$ that is a tensor in two- or three dimensions.

$$\vec{F} = -\tilde{k} \cdot \vec{r}$$

Only if we pull along one of the coordinate axes (called the principal axes of the system), the force points in the same direction as the displacement:

$$\vec{F}_x = \begin{pmatrix} -k_x & 0 \\ 0 & -k_y \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} -k_x x \\ 0 \end{pmatrix} \quad and \quad \vec{F}_y = \begin{pmatrix} -k_x & 0 \\ 0 & -k_y \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -k_y y \end{pmatrix}$$

Let us look at the potential energy of the system. It is given by:

$$E_{pot} = \frac{k_x}{2} x^2 + \frac{k_y}{2} y^2 = \frac{1}{2} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} k_x x \\ k_y y \end{pmatrix}}_{\text{scalar product}} = \frac{1}{2} \underbrace{\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{\text{matrix product}}$$

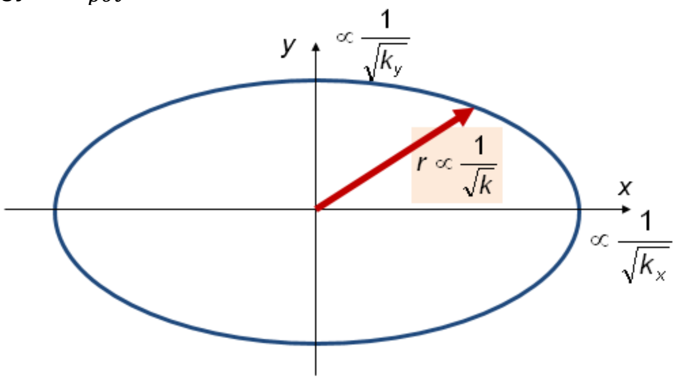
Such that we can write it in the short form:

$$E_{pot} = \frac{1}{2} \vec{r} \cdot \vec{F} = \frac{1}{2} \vec{r}^T \cdot \tilde{k} \cdot \vec{r}$$

It can thus also be described by a simple product in matrix representation. We now transform the potential energy by dividing it by E_{pot} to:

$$1 = \frac{x^2}{\underbrace{\left(\sqrt{2E_{pot}/k_x}\right)^2}_{a}} + \frac{y^2}{\underbrace{\left(\sqrt{2E_{pot}/k_y}\right)^2}_{b}}$$

This equation defines for a given E_{pot} an ellipse. The ellipse describes an isoenergy line. It corresponds to the distance that one has to displace the system in a given direction in order to reach a potential energy of E_{pot}.



For a given displacement direction (described by angle ϑ) the potential energy as function of the displacement r becomes:

$$E_{pot} = \frac{k_x}{2}(r\cos\vartheta)^2 + \frac{k_y}{2}(r\sin\vartheta)^2 = \frac{1}{2}\underbrace{(k_x\cos^2\vartheta + k_y\sin^2\vartheta)}_{k} r^2$$

The term in the bracket corresponds to an effective spring constant for the particular direction. For a given E_{pot} the required displacement can thus be expressed as:

$$r = \sqrt{x^2 + y^2} = \sqrt{2E_{pot}/k} \propto 1/\sqrt{k}$$

The required displacement in a given direction of the ellipse is thus proportional to $1/\sqrt{k}$. Thus, by looking at the isoenergy ellipse one can directly understand the symmetry of the system. The semiminor axis is hereby the stiff direction and the semimajor axis is hereby the soft direction due to the $1/\sqrt{k}$ dependency.

Each tensor has such associated isoenergy ellipses (an ellipsoid in three dimensions), even if the physical objects we look at are very complicated in shape.

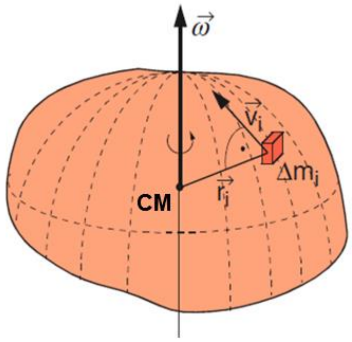
3) Rotating about free axes

Up to now we have only discussed rotations of rigid bodies about space-fixed axes. Now we will extend our considerations to rigid objects that can rotate about a free axis, which might change its direction in space. We will treat at first the case that no external forces act on the body. Such rigid bodies rotating about free axes are called spinning tops or gyroscopes.

In the force/torque free case, the CM is at rest or undergoes uniform translational motion. The only dynamic movement is therefore the rotation around the CM. We will see in the following, that the space-orientation of free axes typically changes with time and the motion of an arbitrary point of the rigid body might perform a complicated trajectory.

A) Tensor description of the moment of inertia

In order to calculate the motion about free axes, we first determine the dependence of the moment of inertia on the direction of a rotation axis that runs through the CM. To obtain the moment of inertia we will calculate the moment of inertia as function of the angular velocity.



While for a rotation of a rigid body, the angular velocity $\vec{\omega}$ is parallel to the particular rotation axis, $\vec{L}$ is defined with respect to a specific point, in our case the center of mass. As we saw before $\vec{\boldsymbol{\omega}}$ **and $\vec{L}$ don't have to be aligned.**

The angular momentum $\vec{L}$ of a mass segment dmi with respect to the CM is given by:

$$d\vec{L}_i = dm_i(\vec{r}_i \times \vec{v}_i) = dm_i(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i))$$

since we have only a tangential velocity component from the rotation. One can transform the double cross product to scalar products using the relation (**see slides**):

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

With this, one obtains:

$$d\vec{L}_i = [(r_i{}^2\vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\vec{r}_i]dm_i$$

The total angular momentum is then obtained by integration:

$$\vec{L} = \int [(r^2\vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r}]dm$$

Let us look at the z component of the resulting vector:

$$L_z = \int \left[(x^2 + y^2 + z^2)\omega_z - (x\omega_x + y\omega_y + z\omega_z)z\right]dm$$

Simplification gives:

$$L_z = \int \left[\underbrace{(x^2 + y^2)}_{"I_{zz}"}\omega_z - \underbrace{xz}_{"I_{zx}"}\omega_x - \underbrace{yz}_{"I_{zy}"}\omega_y\right]dm = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

The z-component of $\vec{L}$ contains all components of $\vec{\omega}$. The term in brackets looks like the squared distance of a point from the z-axis, thus the integral over this term provides the moment of inertia for a rotation around z multiplied by ω_z. The other terms provide moment of inertia contributions to the z-components of $\vec{L}$ for rotation about the other axes. Naming these moment of inertias as indicated in the equation above provides than a simplified notion (see term on right hand side).

We can derive very similar expressions for the other components of $\vec{L}$ (**see slides**),

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

off axis ("unbalanced") momentum for rotation around z

on axis momentum for rotation around z

with

$$I_{xx} = \int (r^2 - x^2)\,\mathrm{d}m$$
$$I_{xy} = I_{yx} = -\int xy\,\mathrm{d}m$$
$$I_{yy} = \int (r^2 - y^2)\,\mathrm{d}m$$
$$I_{yz} = I_{zy} = -\int yz\,\mathrm{d}m$$
$$I_{zz} = \int (r^2 - z^2)\,\mathrm{d}m$$
$$I_{xz} = I_{zx} = -\int xz\,\mathrm{d}m$$

where each diagonally located term (red ovals) gives the on-axis angular momentum for the rotation around the particular component of the angular velocity (e.g., $I_{zz}\omega_z$ provides the z component of $\vec{L}$ for a rotation around the z-axis component with I_{zz} being the previously

determined moment of inertia for rotation around z). Each non-diagonal term (blue ovals) provides an off-axis angular momentum, which causes the on-axis torque when rotating a dynamically unbalanced mass system.
These three equations can be conveniently written in matrix notation, where the matrix is a tensor of rank 2 called the **inertial tensor**:

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

It is a symmetric tensor, since the it can be mirrored across the diagonal ($I_{xz} = I_{zx}, \ldots$). In tensor notation we write for the relationship between angular momentum and angular velocity:

$$\vec{L} = \tilde{I} \cdot \vec{\omega}$$

As for the spring system, the tensor provides that angular momentum and velocity are typically not aligned to each other.
Let us now determine the kinetic energy in tensor representation. Into the kinetic energy only the tangential velocity at the closest distance $r_\perp$ to the rotation axis (i.e. perpendicular to $\vec{\omega}$) enters:

$$E_{rot} = \int \frac{1}{2} \underbrace{(r_\perp \omega)}_{v_t}{}^2 \, dm_i = \int \frac{1}{2} {r_\perp}^2 \vec{\omega} \cdot \left| \frac{1}{{r_\perp}^2} \vec{r_\perp} \times \vec{v} \right| dm_i = \frac{1}{2} \vec{\omega} \cdot \underbrace{\int |\vec{r_\perp} \times \vec{v}| dm_i}_{L_\parallel}$$

In the transformation above we replaced one $\vec{\omega}$ by the crossproduct between $\vec{r_\perp}$ and $\vec{v}$, which points along $\vec{\omega}$. The resulting integral provides then the angular momentum $L_\parallel$ along the rotation axis. The projection of $\vec{L}$ onto the direction of $\vec{\omega}$ can be expressed by scalar multiplication between the two vectors, such that we get:

$$E_{rot} = \frac{1}{2} \vec{\omega} \cdot \vec{L}$$

and finally:

$$E_{rot} = \frac{1}{2} \vec{\omega}^T \cdot \tilde{I} \cdot \vec{\omega}$$

which is a corresponding expression to the spring system. In matrix representation this provides (**see slide**):

$$E_{rot} = \frac{1}{2} \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

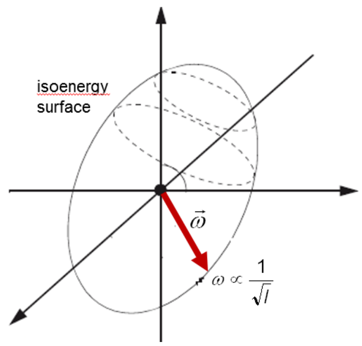
and thus:

$$E_{rot} = \frac{1}{2} \left({\omega_x}^2 I_{xx} + {\omega_y}^2 I_{yy} + {\omega_z}^2 I_{zz} \right) + \omega_x \omega_y I_{xy} + \omega_x \omega_z I_{xz} + \omega_y \omega_z I_{yz}$$

This equation can be transformed to:

$$1 = \frac{{\omega_x}^2}{\left(\sqrt{2E_{rot}/I_{xx}}\right)^2} + \frac{{\omega_y}^2}{\left(\sqrt{2E_{rot}/I_{yy}}\right)^2} + \frac{{\omega_z}^2}{\left(\sqrt{2E_{rot}/I_{zz}}\right)^2} + \frac{\omega_x \omega_y}{\left(\sqrt{E_{rot}/I_{xy}}\right)^2} + \frac{\omega_x \omega_z}{\left(\sqrt{E_{rot}/I_{xz}}\right)^2} + \frac{\omega_y \omega_z}{\left(\sqrt{E_{rot}/I_{yz}}\right)^2}$$

This is the equation of an ellipsoid, the so-called inertial ellipsoid, whose principal axes are not aligned with the coordinate system axes.



It describes again an isoenergy surface, i.e. it tells how large ω must be in a given direction to reach the particular kinetic energy E_{rot}. Similarly, as for the spring system, we now can define the effective moment of inertia I for a give $\vec{\omega}$. With

$$E_{rot} = \frac{I}{2}\omega^2$$

we obtain for the length of ω to be:

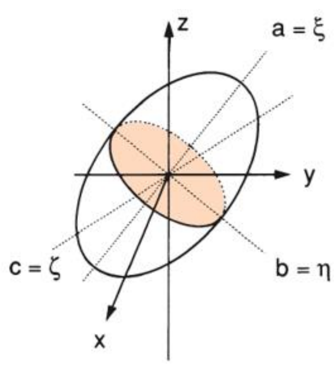
$$\omega = \sqrt{\frac{2E_{rot}}{I}} \propto \frac{1}{\sqrt{I}}$$

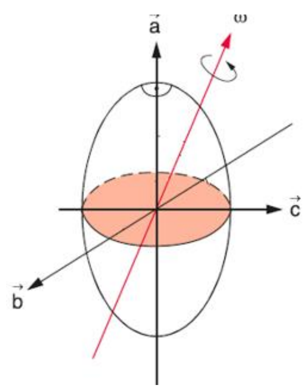
The **ellipsoid represents** thus **the directional dependence of the moment of inertia**. Thus the inertial ellipsoid (or the inertial tensor) can be used to obtain the moment of inertia for any rotation axis. For a low moment of inertia around a rotation axis ω is large and vice versa, which explains the reciprocal relation between the two quantities.

Interestingly the **inertial ellipsoid is always an ellipsoid non-depending how "weird/complicated" the object looks like.**

B) Inertial tensor in the principal axis system

It is convenient (simpler) to choose the symmetry (principal) axes of the inertial ellipsoid as new coordinate system.





In the principal axis system there are no mixed terms in the ellipsoid equation, providing essentially "an ellipse equation with an additional dimension":

$$1 = \frac{{\omega_a}^2}{\left(\sqrt{2E_{rot}/I_a}\right)^2} + \frac{{\omega_b}^2}{\left(\sqrt{2E_{rot}/I_b}\right)^2} + \frac{{\omega_c}^2}{\left(\sqrt{2E_{rot}/I_c}\right)^2}$$

The mixed terms corresponded to the off-diagonal elements in the inertial tensor. Therefore, we have as for the spring system a tensor that has only diagonal entries, which are called **principle moments of inertia**:

$$\tilde{I} = \begin{pmatrix} I_a & 0 & 0 \\ 0 & I_b & 0 \\ 0 & 0 & I_c \end{pmatrix}$$

For an **angular velocity along one of the principal axes** we obtain an **angular momentum pointing in the same direction** (as for spring system) which is the **product of the angular velocity and principle moment** as for a fixed axis**:**

$$\vec{L} = \tilde{I}\{\omega_a, 0{,}0\} = \{I_a\omega_a, 0{,}0\} \text{ etc. (for b,c)}$$

The calculation of the diagonal inertial tensor corresponds thus to the Eigenvalue problem of a matrix. Since the tensor is symmetric ($\tilde{I} = \tilde{I}^T$), linear algebra tells that there is always a principal axis system in which the tensor is diagonal. A principal axis vector must fulfill the relation from above:

$$\tilde{I}\vec{\omega}_{a(,b,c)} = I_{a(,b,c)}\vec{\omega}$$

The vectors that fulfill this relation are orthogonal to each other and are called the Eigenvectors of that matrix, while the principle moments $I_{a(,b,c)}$ are the corresponding Eigenvalues. With the diagonal matrix the angular momentum becomes:

$$\vec{L} = \begin{pmatrix} L_a \\ L_b \\ L_c \end{pmatrix} = \begin{pmatrix} I_a & 0 & 0 \\ 0 & I_b & 0 \\ 0 & 0 & I_c \end{pmatrix} \begin{pmatrix} \omega_a \\ \omega_b \\ \omega_c \end{pmatrix} = \begin{pmatrix} I_a\omega_a \\ I_b\omega_b \\ I_c\omega_c \end{pmatrix}$$

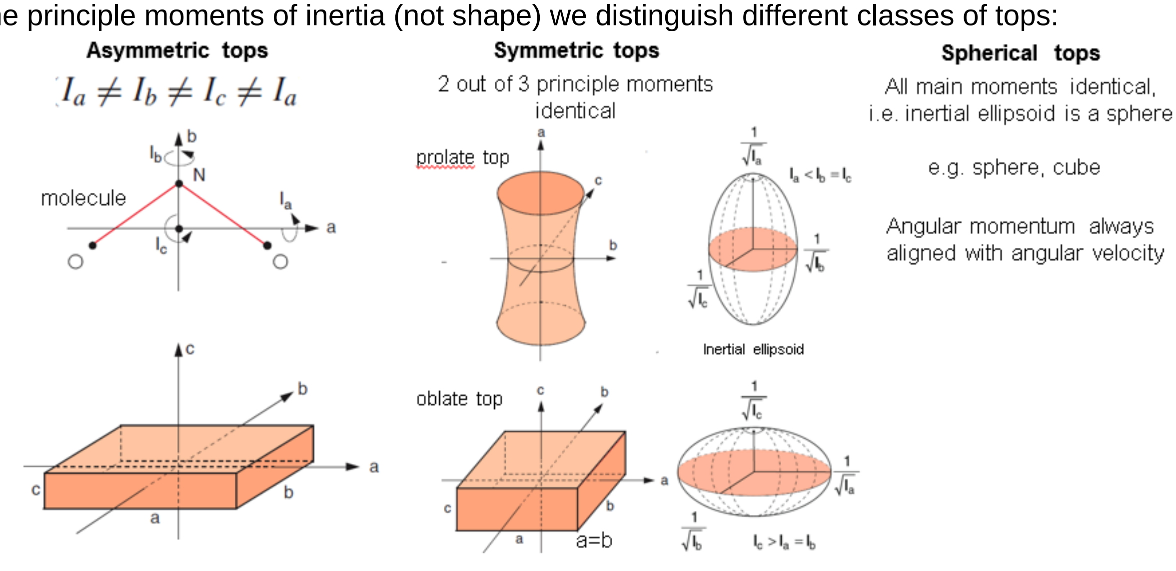
i.e. each component contains only the corresponding angular velocity component. Nonetheless, $\vec{L}$ and $\vec{\omega}$ are not aligned to each other, unless the rotation axis is one of the principle axes. If we use

$$I_{a(,b,c)}\omega_{a(,b,c)} = L_{a(,b,c)}$$

we can write for the **kinetic energy** without the mixed terms (**show slide**):

$$E_{rot} = \frac{1}{2}({\omega_a}^2 I_a + {\omega_b}^2 I_b + {\omega_c}^2 I_c) = \frac{{L_a}^2}{2I_a} + \frac{{L_b}^2}{2I_b} + \frac{{L_c}^2}{2I_c}$$

With the considerations from above we now define and classify tops: A **top is a rigid body with a free rotation axis whose movement is determined by one fixed point in space.** Based on the principle moments of inertia (not shape) we distinguish different classes of tops:



C) Free rotational axes

Let us first consider a top in absence of any external forces/torques. The absence of torque demands that the angular momentum is constant. If the rotation axis does not align with the angular momentum it would like to “move” the angular momentum. Since this is forbidden the rotation axis has to change.

Rotation about a stable axis of a force/torque free top thus requires that the angular momentum and angular velocity align. Alignment is given if:

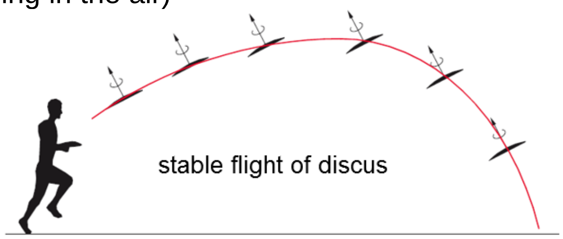
• $I_a = I_b = I_c$, i.e. we have a spherical top for any rotation direction or

• if the body rotates about one of his principal axes, such that $\vec{\omega} = (\omega_a, 0{,}0)$, $(0, \omega_b, 0)$ or $(0{,}0, \omega_c)$ or
• if for a symmetric top the rotation axis through the CM is perpendicular to the symmetry axis of the top

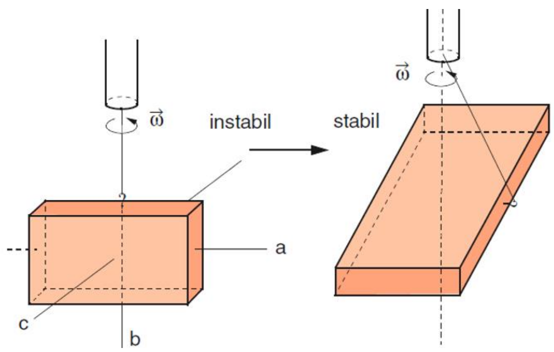
The principal axes of a body are therefore also **called free axes because the body can freely rotate about them as it would be fixed even if it is not fixed by an additional mount**.

Examples:

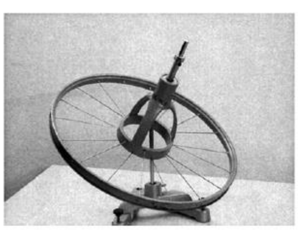
- Stabilized flight of a discus due to angular momentum conservation (provides better aero-dynamics and even gliding in the air)

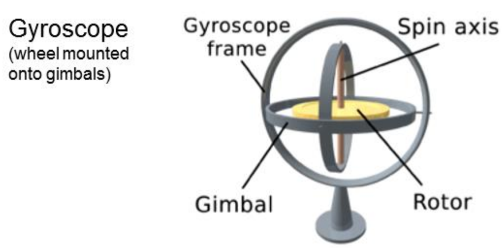


- Fully stable rotation is only possible about the axes of the smallest and the largest moment of inertia.



- Tops that are suspended torque-free at the center of mass (plate on a rod, gyroscope, see experiments and slide) keep their orientation due to angular momentum conservation despite of attempts to tilt them.


top suspended at center of mass



e.g. used in navigation, such as attitude indicator in aircrafts

- Spinning coin

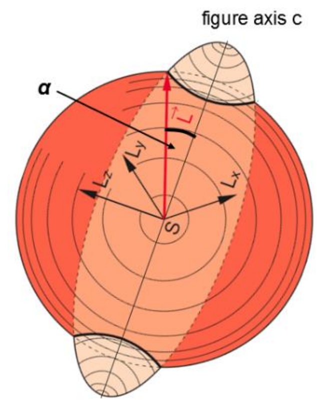
D) The torque-free symmetric top

We have so far looked at the rotation of a top around a stable (free) axis which is one of the principle axes. Now let us distort the rotation of the top from the stable axis by giving it a little push (**Experiment**). The wiggling motion that we see is called **nutation** and we will try in the following to understand its origin. To this end we look at a **symmetric, torque-free top**, i.e.

$$I_a = I_b$$

From $\vec{\tau} = 0$, we have angular momentum conservation $\vec{L} = const$. In addition, we also have energy conservation for the rotation. Both conservation laws provide the following two equations (show on slides):

$$L_x{}^2 + L_y{}^2 + L_z{}^2 = const = C_1; \qquad \frac{L_a{}^2}{2I_a} + \frac{L_b{}^2}{2I_b} + \frac{L_c{}^2}{2I_c} = const = C_2$$



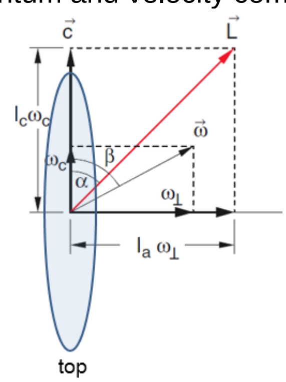
The former describes a sphere in the angular momentum space, the latter is an ellipsoid that is oriented along the main principal axis (called figure axis). Both conditions must be full-filled such that the constant angular momentum vector must lie at the intersection between the sphere and ellipsoid. Thus, the ellipsoid (and thus the figure axis) can only rotate around $\vec{L}$.

This provides that there is a **fixed angle $\boldsymbol{\alpha}$ between angular momentum and figure axis.** Since $\vec{L} = const$ in space, we have a constant component L_c of the angular momentum along the figure axis *c* according to:

$$L_c = L\cos\alpha = I_c\omega_c = const$$

Since the angular momentum along c is given by $I_c\omega_c$, we have also a constant angular velocity along the figure axis, i.e. $\omega_c = const$.

Now let us look at the angular momentum and velocity components perpendicular to the c axis:



The perpendicular angular momentum component $L_\perp$can according to the figure be expressed as:

$$L_\perp = L\sin\alpha = \sqrt{L_a^2 + L_b^2} = I_a \underbrace{\sqrt{I_a{}^2\omega_a^2 + I_a{}^2\omega_b^2}}_{\omega_\perp} = I_a \underbrace{\sqrt{\omega_a^2 + \omega_b^2}}_{\omega_\perp} = const.$$

Since I_a is constant, it follows that also the perpendicular component of ω is constant:

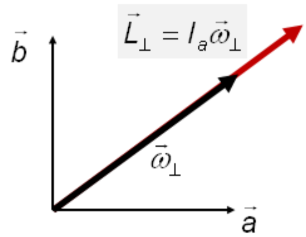
$$\omega_\perp = \sqrt{\omega_a^2 + \omega_b^2 = const}$$

Together with $\omega_c = const$, we have thus a **constant angular velocity magnitude and a fixed angle $\boldsymbol{\beta}$ between figure axis and angular velocity**. Note, however that ω_a and ω_b are not constant. Only their vector sum has a constant magnitude.

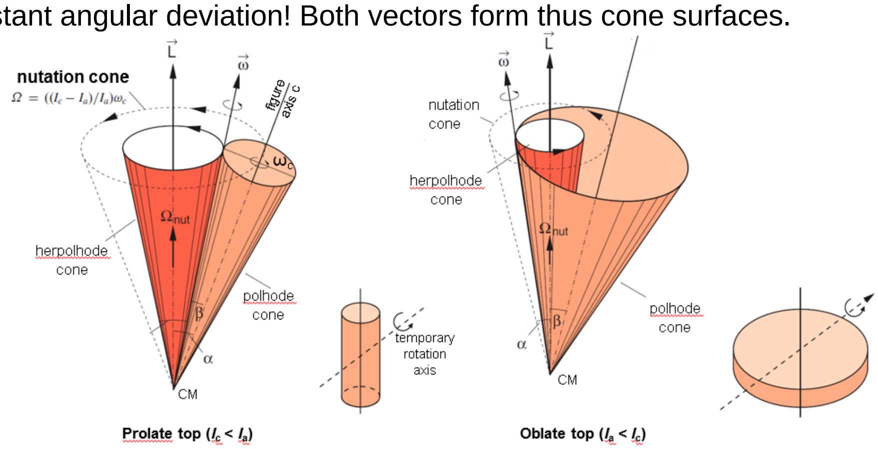
Let us look at the components of $\vec{L}$ and $\vec{\omega}$ perpendicular to the figure axis more closely. We can write down the angular momentum in the principle axis system and split it into a parallel and a perpendicular component to the figure axis:

$$\vec{L} = \begin{pmatrix} I_a\omega_a \\ I_a\omega_b \\ I_c\omega_c \end{pmatrix} = I_c \underbrace{\begin{pmatrix} 0 \\ 0 \\ \omega_c \end{pmatrix}}_{\vec{\omega}_c} + I_a \underbrace{\begin{pmatrix} \omega_a \\ \omega_b \\ 0 \end{pmatrix}}_{\vec{\omega}_\perp} = I_c\vec{\omega}_c + I_a\vec{\omega}_\perp = \vec{L}_c + \vec{L}_\perp$$

The perpendicular component $\vec{L}_\perp$ points therefore in the same direction as $\vec{\omega}_\perp$. Looking along the figure axis (from above), this provides the following picture:



Therefore, **figure axis, angular velocity and angular momentum are in one plane** for the symmetric top. While **ω_c is the angular velocity of the top about the figure axis**, **$\vec{\omega}_\perp$ constantly tilts the figure axis around $\vec{L}$**.
This tilting of the figure axis causes a migration of the figure axis and thus even the angular velocity vector/rotation axis (that lies in the same plane) around the constant angular momentum vector at constant angular deviation! Both vectors form thus cone surfaces.



The **cone of the figure axis is called nutation cone.** The **cone of the rotation axis is called herpolhode cone**. The **rotation axis** about which the top is rotating **is thus traveling within the top**. Since both vectors are in one plane one can construct a third cone called **polhode cone** (body fixed cone) that is centered around the figure axis and rolls along the surface of the herpolhode cone. The rotation axis is here located at the boundary between these two cones.
The angular frequency at which the rotation along these cones occurs is given by:

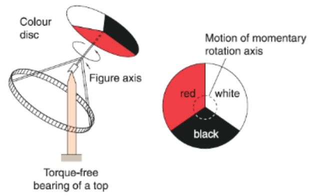
$$\Omega = ((I_c - I_a)/I_a)\,\omega_c$$

which can be derived by additional calculations (see Euler equations in Demtröder). Depending on whether the top is prolate or oblate the cones look slightly different due to the relative location of $\vec{L}$ and $\vec{\omega}$ with respect to the figure axis (see figure above).

Experiments:

- Nutation of a gyroscope where one nicely sees the circular path of the gyroscope top during nutation. Friction causes the return to the stable free axis.

- The movement of the rotation axis within the top can be visualized with a top that has colored sections:

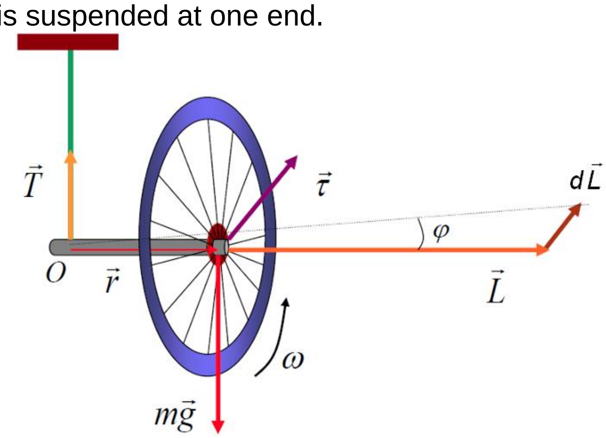


E) Precession (top with external torque)

So far, we looked at torque free tops. When looking at tops (like the normal toy top) that are not suspended at the center of mass, we have an additional torque acting on the top due to gravity:

Experiment: Precession of a spinning wheel suspended on one side.

Such tops undergo a **rotation about the suspension point that is called precession**. To derive a formula for the precession frequency, let us consider a spinning wheel with a horizontal rotation axis at which the wheel is suspended at one end.



The angular velocity of the spinning wheel shall be much higher than the angular velocity of the precession:

$$\omega_{preces.} \ll \omega$$

To understand the precession, we first calculate the torque that the gravity exerts on the wheel axis. For simplicity, we assume a mass-free axis, such that all mass would be concentrated in the wheel. The torque is then given by:

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times m\vec{g}$$

and points into the figure plane (in figure above). According to

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

This causes within the time interval dt an angular momentum change $d\vec{L}$ in the same direction, i.e. the angular momentum gets tilted into the figure plane and the axis undergoes a precession in the horizontal plane. For the absolute values of torque and angular momentum change, we can write:

$$\tau = \frac{dL}{dt} = \frac{L\,d\varphi}{dt} = L\omega_{preces.}$$

Inserting the magnitude of the torque $\tau = rmg$ and the angular momentum of the wheel $L = I\,\omega$, we get:

$$\omega_{preces.} = \frac{\tau}{L} = \frac{rmg}{I\ \omega}$$

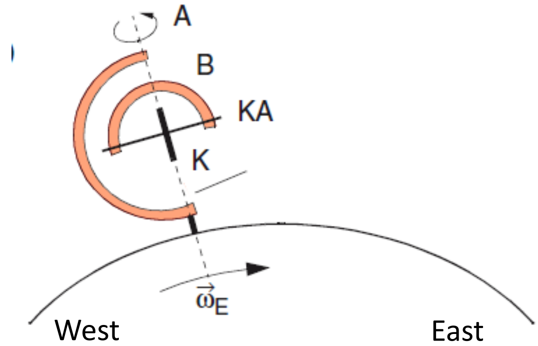
The precession frequency thus increases for an increasing torque (wheel weight) and decreases of the angular velocity of the wheel is increased.

Experiments: As a model of a top with external torque we use a gyroscope and obtain precession by attaching a mass at the end of the rotation axis. We test the derived formula using different applied torques and different angular velocities of the gyroscope

In quantum physics you will see that in **nuclear magnetic resonance (NMR) experiments**, a similar precession of the nuclear atom spins occurs in an external magnetic field. This precession has for a given magnetic field a constant frequency called the **Larmor frequency**, which forms the basis of the NMR detection

F) Applications:

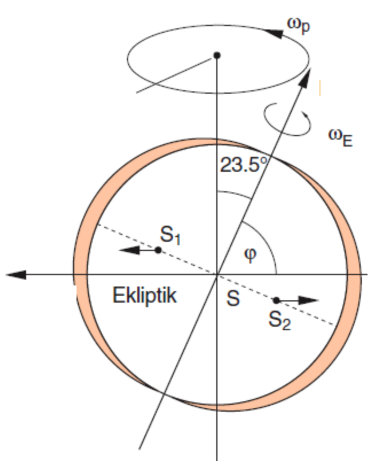
Gyrocompass

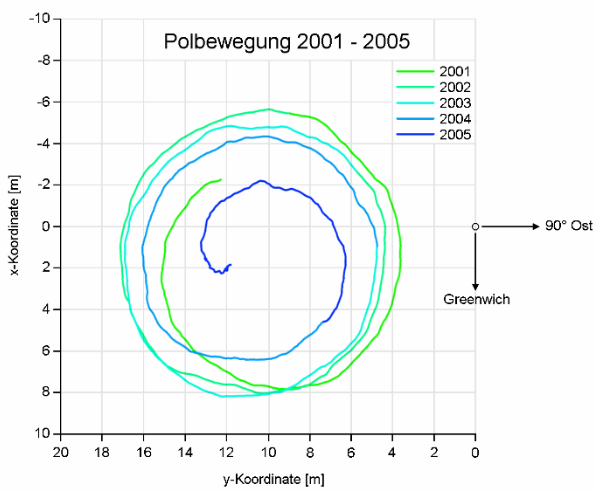


A gyrocompass uses the precession of a spinning top due to the rotation of the earth ω_E. It has one stable axis (A) that cannot change its orientation with respect to the earth surface. Therefore, axis A is coupled to the rotation of the earth ω_E. Rotation of the earth changes the angular momentum of the top when its rotation axis (KA) lies within the rotation plane of the earth (west-east direction). This induces a torque $\vec{\tau}$ on the top along axis A, that turns the top around A (towards north-south direction). **When the rotation axis (KA) is oriented in N-S direction, the torque will become 0** and the system will be stable.

Experiments: Using a gyroscope with one fixed axis placed on a rotatable disk we simulate the rotation of earth. Due to the resulting torque, the gyrocompass aligns vertically upon disk rotation.

Precession & Nutation of earth





The earth (in good approximation) can be looked at as a slightly **oblate symmetric top ($I_a = I_b < I_c$).** Due to centrifugal forces, the equator diameter is ~43 km bigger than the pol diameter. Due to the tilt of the rotation axis of the earth axis with respect to the ecliptic ($\varphi = 90° - 23.5° = 66.5°$) the CMs for the two halfes of erath (S_1 and S_2) are located on different sides of the ecliptic. Therefore, the gravitation force of the sun as well as the centrifugal force acting on S_1 and S_2 are slightly different. This provides a net torque on earth axis and thus a precession of the earth axis with $\boldsymbol{\omega_{precession} \sim 2600\ years}$. Furthermore, the figure axis c and the angular velocity axis $\vec{\omega}$ of the earth due not point in the same direction. As discussed for the torque free top, this leads to a nutation. Its period is given as:

$$T_{nutation} = T_{Earth}\ \frac{I_a}{I_c - I_a} = 305\ days$$

Euler disk

Lecture 17: Experiments

1) Rotating disk with masses to illustrate static and dynamic balance. System with rotation axis put horizontally (hand held) and system with axis put vertically on a soft support connected to an accelerometer to measure dynamic imbalance.
2) Tensor: 2D spring system to illustrate misalignment of force and displacement
3) Torque free top: Fixed angular momentum of a top that is suspended at the center of mass (disk on a rod), Gyroscope
4) Torque free top: nutation of disk on a rod and of gyroscope. Movement of the rotation axes illustrated by disk with 3 different color sections
5) Top with external torque: Precession at the bicycle-wheel top
6) Top with external torque: Precession of gyroscope for different applied torques and different angular velocities of the gyroscope
7) Gyroscope: Model of the gyrocompass