Exam

Instructions for working on this exam

1. There are 180 min to execute the exam.
2. Each exercise should be solved on a separate set of sheets. Please write neatly and leave space on the margin for remarks and indicating credits.
3. Do not use pens that can be erased, and **never write in red!**
4. Explain carefully what you are doing. What do you assume? What do you intend to show? We will only give points when we understand what you intend to do.
5. There are four exercises that address skills needed to solve physical problems, and subsequently you will discuss in some depth three physics problems.
6. Bonus problems are marked by (*). Often they involve a tricky argument that might not be immediately obvious. I recommend that you first work on the other exercises. Only attack the *-problems in the end when there is still time, or when you immediately see a fast and straightforward solution.
7. In total there are 90 points + 50 bonus points. You pass the exam with 30 points. The points for the different sub-tasks are provided on the next page.
8. You may use one sheet of A4 paper with hand-written notes that you prepared to take the exam. You must not use other resources; in particular no calculators and algebra programs.
9. The best strategy to attack the exam is to strive to collect as many points as possible. Do not try to solve the full exam. Work on things that you can solve — ignore anything where you get stuck.

I declare that I feel fit to take the exam today. I confirm with my signature that I have prepared the answers/solutions of the examination task(s) independently, and have not used any unauthorized aids. I did not ask any other person for help in preparing the answers, and I did not copy from the solutions of others.

family name	given name	matriculation number

1				2				3				
*[a]	(b)	(c)	(d)	(a)	(b)	(c)	*[d]	(a)	(b)	*[c]	*[d]	(e)
4	1	3	4+1	2	1	3	3	3	4	2	4	4+3

4					5									6	
(a)	(b)	(c)	(d)	(e)	(a)	*[b]	(c)	(d)	(e)	*[f]	*[g]	(h)	*[i]	*[a]	*[b]
2	4	2	1	4	2	2	3	3	4	4	6	4+1	1	1	2

								7							
*[c]	(d)	(e)	(f)	(g)	*[h]	(i)	(j)	(a)	*[b]	(c)	(d)	(e)	(f)	(g)	*[h]
2	4	2	2	5	6	2	3	3	1	3	3	2	2+2	5	5

total points

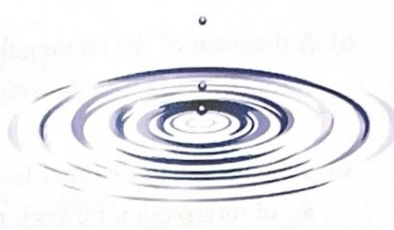
grade

Problem 1. Contour lines

We consider waves on a water surface with a height profile

$$H(x,y,t) = A\,\frac{\sin\left(k\,\sqrt{x^2+y^2} - \omega\,t\right)}{x^2+y^2}$$

where (x,y) denotes a position on the surface and t time.



Source: Collection of water ripples in pluspng.com

⋆ a) Note that H is a function of $R = \sqrt{x^2+y^2}$. Provide monotonic upper and lower bounds for $H(R,t)$, and sketch the radial profile $H(R,t)$ together with its bounds.

b) Determine the gradient of $R = \sqrt{x^2+y^2}$.

c) Determine the gradient of $H(x,y,t)$ for a given time t.
Hint: It might pay off to use the chain rule and the result of b).

d) Determine the contour lines of $H(x,y,t)$ for a given value of t.

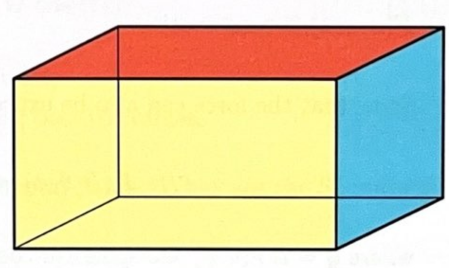
Sketch the contour lines, and mark the gradients of $H(x,y,t)$ by arrows in your plot.

Bonus: How do the contour lines change in time?

Problem 2. Rectangular cuboids

A cuboid is a body with six faces, and the opposite faces of a rectangular cuboid are identical rectangles. Hence, with appropriate coordinates its vertices $\boldsymbol{q}_i$ with $i \in \{1 \cdots 8\}$ are located at the positions

$$\boldsymbol{q}_i \in \left\{ \begin{pmatrix} s_1\, a \\ s_2\, b \\ s_3\, c \end{pmatrix} \quad \text{with} \quad s_1,\ s_2,\ s_3 \in \{\pm 1\} \right\}$$



source: A2569875, Public domain, via Wikimedia Commons

a) Mark the origin, the positive coordinate axes of the right-handed coordinate system, and a, b, c in the sketch.

b) A diagonal of the rectangular cuboid connects to opposite vertices. Why do all vertices intersect in the origin?

c) The rectangular cuboid has four diagonals. Determine the cosine of the angle θ_{ij} of intersection for each pair ij of diagonals.

⋆ d) The cuboid has six pairs of diagonals and three distinct angles θ_{ij}. What is the relation between the pairs of diagonals that share the same intersection angle θ_{ij}?

Problem 3. Line integrals

We consider the force field

$$\boldsymbol{F}(x, y, z) = \begin{pmatrix} y \sin\alpha + x \cos\alpha \\ y \cos\alpha - x \sin\alpha \\ z \end{pmatrix}$$

where $\boldsymbol{q} = (x, y, z)$ is a position in $\mathbb{R}^3$.

a) Evaluate the line integral from the south pole to the north pole of the unit sphere along the path $\boldsymbol{q}(t) = (0,\ 0,\ t)$ with $-1 < t < 1$ (path i.).

b) Determine the potential of the force field $\boldsymbol{F}(x, y, z)$ for all values of α where $\boldsymbol{F}(x, y, z)$ is conservative.

⋆ c) Show that the force can also be expressed as

$$\boldsymbol{F}(R, \theta, \phi) = \frac{f\,R}{L}\,\hat{\boldsymbol{r}}(\theta, \phi - \alpha)$$

where $\boldsymbol{q} = R\,\hat{\boldsymbol{r}}(\theta, \phi)$ are spherical coordinates.

★ d) We consider a path with $R = R_0$ =const, where ϕ can be expressed as function of θ, i.e., $\phi(t) = \phi(\theta(t))$. Show the that the line integral

$$W = -\int_{t_I}^{t_F} \mathrm{d}t\ \dot{\boldsymbol{q}}(t) \cdot \boldsymbol{F}(\boldsymbol{q}(t))$$

can then be expressed as an integral over θ,

$$W = f(\alpha) \int_{c_1}^{c_2} \mathrm{d}\theta\ \sin(2\theta) + g(\alpha) \int_{c_1}^{c_2} \mathrm{d}\theta\ \frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}\ \sin^2\theta$$

and determine the integration boundaries c_1 and c_2, and the functions $f(\alpha)$ and $g(\alpha)$ in this expression.

e) Evaluate the line integral from the south pole to the north pole of the unit sphere along the paths where

ii. ϕ =constant

iii. $\phi(\theta) = \dfrac{\cos\theta}{\sin\theta}$

Hint: Beware the result of d).

Bonus: How do the paths i.–iii. look like?

Problem 4. Linear inhomogeneous ordinary differential equation

We consider the linear inhomogeneous ordinary differential equation

$$\dot{x}(t) + \gamma\, x(t) = A\,\Omega\, \cos(\Omega\, t) \tag{4.1}$$

Here, $x(t)$ is a position on a one-dimensional track.
The real parameters γ, A, and Ω take constant, positive values.

a) We specify the position x in inch and time in hours. What are the SI units of the parameters γ, A and Ω.

b) Verify that the ODE (4.1) has a special solution of the form

$$x_s(t) = K\, \mathrm{e}^{-\mathrm{i}\phi}\, \mathrm{e}^{\mathrm{i}\Omega t} + c.c.$$

where c.c. denote the complex conjugate of the first term.

How do the real constants K and ϕ depend on the parameters γ, A, and Ω.

c) Determine the general homogeneous solution $x_h(t)$ of the ODE.

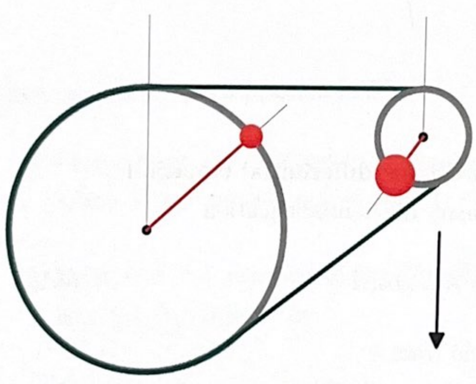
d) We are looking for a solution that starts at time t_0 at the position x_0. Is this information sufficient to uniquely specify the trajectory?

If yes: why?

If not: which information is missing?

e) Provide the evolution $x(t)$ of a trajectory that starts at the position $x(t_0) = x_0$.

Problem 5. Masses on Wheels



We consider two wheels of radii R_1 and R_2 (gray circles) with their axles (black dots) fixed to a wall such that they can rotate in a vertical plane. Their motion is coupled by a rope (sketched in green). On each wheel we attach a weight, with masses m_1 (small red circle) and m_2 (large red circle), respectively. Their positions θ_1 and θ_2 specify the deviation from the vertically upwards direction. Gravity is acting vertically downwards (black arrow).

a) Mark the masses, radii, angles, and gravity in the sketch.

⋆ b) Argue that the presence of the rope implies that $\dot\theta_1(t) = \rho\,\dot\theta_2(t)$, and that this entails

$$\theta_1 = \rho\,\theta_2(t) - \alpha$$

What is the interpretation of α in the latter equation, and how is ρ related to the system parameters?
Why is it admissible to assume that $0 < \rho \leq 1$?

c) Show that the kinetic energy of the system can be expressed as

$$T = k_T\,\dot\theta_1^2$$

where the constant k_T depends only on the system parameters. Determine k_T.

d) Determine the potential energy of the system, and use the result of b) to express θ_2 in terms of θ_1. This will provide a Lagrange function that depends only on the coordinate θ_1 and its time derivative.

e) Derive the EOM for θ_1 and introduce dimensionless units to show that

$$\ddot\theta_1 = \sin\theta_1 + \mu\,\sin\left(\frac{\theta_1}{\rho} - \varphi\right)$$

Which dimensionless time has been adopted here?
How do the constants μ and φ depend on the system parameters?

⋆ f) Show that for $\mu < \rho$ the EOM has exactly one stable and one unstable point in each period of $\sin\theta_1$.

Hence, the phase space looks similar to the one of the mathematical pendulum. What will actually be the difference?

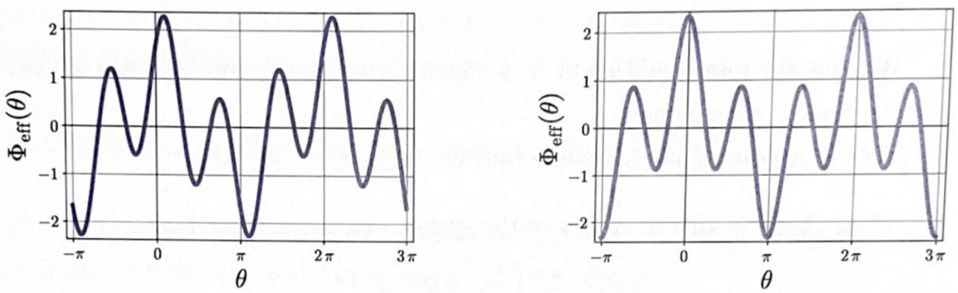
⋆ g) We specialize now to the case $\rho = 1/3$, $\mu = 10$, and $\varphi = 0$. Use the trigonometric relation for $\sin(3\,\theta)$ to show that

$$\ddot\theta_1 = a\,\sin\theta\,\left(\cos^2\theta - b\right)$$

with $0 < b < 1$. Determine the constants a and b.

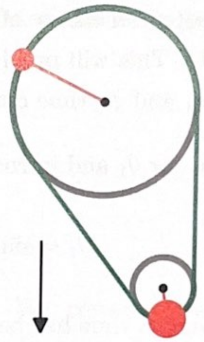
Determine the fixed points of the dynamics, and specify which of them are stable and unstable.

h) The following two plots show the effective potential for $\rho = 1/3$ and $\mu = 4$. For φ we adopted the values $0°$ and $60°$, respectively. How do the potentials differ? Provide the phase-space plots for the two cases.



Bonus: How do you recognize which one refers to $\varphi = 0°$?

⋆ i) What changes in this discussion when the device is turned around such that the big wheel is located above the small wheel:



Problem 6. Rolling in a torus

We consider the motion of a point particle that moves without friction on the surface of a torus. In 3D Cartesian coordinates its shape is described by a circle in the (x, z) plane that is rotated around the $\hat{\boldsymbol{z}}$ axes. We denote the radius of the circle as R, and place its center at the position $D\,\hat{\boldsymbol{x}}$. There is a gravitational acceleration $\boldsymbol{g} = -g\,\hat{\boldsymbol{z}}$ acting on the particle.

⋆ a) Why do we need that $D > R$?

⋆ b) We describe positions in the (x, y) plane via polar coordinates with unit vector $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}} = \partial_\phi \hat{\boldsymbol{\rho}}$. Together with $\hat{\boldsymbol{z}} = \hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}}$ they form the right-handed orthonormal basis vectors of cylindrical coordinates.

Moreover, we adopt spherical coordinates based on $\hat{\boldsymbol{r}}(\theta,\phi) = \sin\theta\, \hat{\boldsymbol{\rho}}(\phi) + \cos\theta\, \hat{\boldsymbol{z}}$.

i. Verify that indeed $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} = \partial_\phi \hat{\boldsymbol{\rho}}$

ii. Determine how the derivative $\partial_\phi \hat{\boldsymbol{\phi}}$ can be expressed as linear combination of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$.

⋆ c) Argue that every position $\hat{\boldsymbol{q}}$ on the torus can be described as

$$\boldsymbol{q} = D\, \hat{\boldsymbol{\rho}}(\phi) + R\, \hat{\boldsymbol{\theta}}(\theta,\phi)$$

Why is it necessary in this case that not only ϕ but also θ takes values in $(0,\, 2\pi)$?

d) Determine the velocity $\dot{\boldsymbol{q}}$ and show that the kinetic energy takes the form

$$T = C\, \left(\dot{\theta}^2 + (\Delta + \sin\theta)^2\, \dot{\phi}^2\right)$$

What is the dimension of the constant C in this expression?
How do C and Δ depend on the system parameters R, D, m, and g?

e) Determine the potential energy of the particle, and provide $\mathcal{L}$.

f) Note that ϕ is a cyclic variable. Determine the associated conserved quantity, K.

g) Determine the EOM for θ, and use the conserved quantity K to eliminate $\dot{\phi}$ in this equation. Introduce then appropriate dimensionless quantities to show that the dimensionless second time derivative of θ takes the form

$$\ddot{\theta} = \sin\theta + \frac{\kappa\, \cos\theta}{(1 + \rho\, \sin\theta)^3}$$

Which dimensionless time has been adopted here?
How does κ depend on the system parameters?

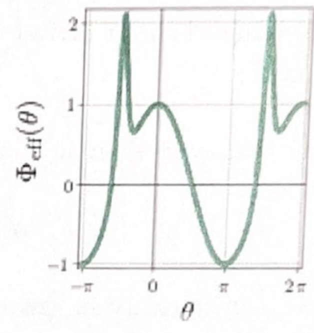
⋆ h) Show the the fixed points of the motion are obtained by the solutions of

$$1 + \rho\, \sin\theta = -\left(\frac{\kappa}{\tan\theta}\right)^{1/3}$$

Sketch the two sides of this equation and show that for small holes of the torus (i.e., R close to D) and small values of κ a second stable fixed point emerges at the upper inner side of the torus.
What is the physical origin of this fixed point?

i) Determine the effective potential of the motion.

j) The plot to the right shows the effective potential for $\kappa = 0.01$ and $\rho = 0.95$. Provide the phase-space plot for these parameter values where you clearly mark the heteroclinic and homoclinic trajectories.



Problem 7. Collective Swimming

We consider two swimmers that move in a circular domain such that their positions can be specified by the angles θ_i with $i = 1$ and 2, respectively. They swim with preferred velocities Ω_i and they have a preference to stay close to each other. Hence, we describe their motion as

$$\ddot{\theta}_i = -\gamma\left(\dot{\theta}_i - \Omega_i\right) - \frac{2k}{f_i}\,\sin\frac{\theta_i - \theta_{3-i}}{2}$$

Note that θ_{3-i} is a short hand for the position of the other swimmer; $3 - i$ takes the value of 2 and 1 for $i = 1$ and 2, respectively.

a) To gain insight into the first contributions to the acceleration, we consider the case $k = 0$.
Sketch the evolution of the speed $\dot{\theta}_1(t)$ and its position $\theta_1(t)$, and discuss how they are effected by γ and Ω_1.

⋆ b) What is the motivation to adopt the sine term to model the preference to stay close? What does this assume about the signs of k, f_1 and f_2.

c) We introduce a position ψ in the middle of the swimmers

$$\psi = f_1\,\theta_1 + f_2\,\theta_2$$

Determine $\ddot{\phi}$, and argue that $\dot{\psi}$ decays exponentially to a constant value, similar to the case of a single swimmer discussed in a). Determine the relaxation rate of this decay, and the asymptotic speed.

d) We denote the distance between the swimmers as $\phi = (\theta_2 - \theta_1)/2$. Show that the EOM for ϕ amounts to an ODE of the form

$$\ddot{\phi} + a\,\dot{\phi} + \sin\phi = c \tag{7.1}$$

How do the constants a, and c in this equation depend on the system parameters γ, Ω_i, k and f_i.

e) Which distance ϕ_∞ will the swimmers adopt for long times, and under which conditions is ϕ_∞ small?

f) Determine the fixed points of the EOM, (7.1).
Discuss their stability!

Bonus: In which circumstances will there be no fixed points? How would the motion of the two swimmers look like in that case?

g) Assume that $a = 0$ in Eq. (7.1). Determine the potential that describes the motion in this case, and sketch the phase space plot for a situation where the system has fixed points.

⋆ h) 1. How does the phase-space plot of g) look like for a situation where there are no fixed points?

2. How does the phase-space plot of g) change when a takes a small positive value?