Mathematics 1. Selected proofs
Leipzig University, WiSe 2023/24, Dr. Tim Shilkin
Limit of a function

THEOREM. Definitions of the limit of a function according to Cauchy and Heine are equivalent.

We are given: $f : (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$, $y_0 \in \mathbb{R}$

PROOF of (Cauchy $\Longrightarrow$ Heine)

1. We assume the following is true:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta(\varepsilon) > 0 : \quad \forall\, x \in (a,b), \quad x \neq x_0 \qquad |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - y_0| < \varepsilon \quad \text{(C)}$$

   Take $\{x_n\}_{n=1}^{\infty} \subset (a,b)$: $x_n \to x_0$, $x_n \neq x_0$. We want to show that $f(x_n) \to y_0$.

   Assume $\varepsilon > 0$ is arbitrary. Applying (C) we obtain $\exists\, \delta = \delta(\varepsilon) > 0$ such that (C) holds.

2. $x_n \to x_0 \quad \Longrightarrow \quad \exists\, N(\delta) \in \mathbb{N}$: $\forall\, n \geq N(\delta) \quad |x_n - x_0| < \delta$.
3. (C) $\wedge\ |x_n - x_0| < \delta \quad \Longrightarrow \quad |f(x_n) - y_0| < \varepsilon$
4. In the proof above take the definition of $\lim\limits_{n\to\infty} f(x_n) = y_0$ in boxes. The following moments must be indicated (marked above with a blue color):

$$\boxed{\forall\, \varepsilon > 0} \qquad \boxed{\exists\, N(\delta(\varepsilon)) \in \mathbb{N} :} \qquad \boxed{\forall\, n \geq N(\delta(\varepsilon))} \qquad \boxed{|f(x_n) - y_0| < \varepsilon}$$

PROOF of (Heine $\Longrightarrow$ Cauchy)

5. We assume the following is true:

$$\forall\, \{x_n\}_{n=1}^{\infty} \subset (a,b) : \quad x_n \neq x_0 \qquad x_n \underset{n\to\infty}{\longrightarrow} x_0 \quad \Longrightarrow \quad f(x_n) \underset{n\to\infty}{\longrightarrow} f(x_0) \qquad \text{(H)}$$

   We want to show $\forall\, \varepsilon > 0 \;\; \exists\, \delta(\varepsilon) > 0$: $\quad |x - x_0| < \varepsilon \quad \Longrightarrow \quad |f(x) - y_0| < \varepsilon$

6. By contradiction, assume $\exists\, \varepsilon_0 > 0$: $\forall\, \delta > 0 \;\; \exists\, x_\delta \in (a,b)$: $\quad |x_\delta - x_0| < \delta \wedge |f(x_\delta) - y_0| \geq \varepsilon_0$
7. Choose $\delta > 0$ in the following way:

$$\begin{array}{lclll}
\text{Take } \delta = 1 & \Longrightarrow & \exists\, x_1 \in (a,b) : & |x_1 - x_0| < 1 & \wedge\ |f(x_1) - y_0| \geq \varepsilon_0 \\
\text{Take } \delta = \frac{1}{2} & \Longrightarrow & \exists\, x_2 \in (a,b) : & |x_2 - x_0| < \frac{1}{2} & \wedge\ |f(x_2) - y_0| \geq \varepsilon_0 \\
\text{Take } \delta = \frac{1}{3} & \Longrightarrow & \exists\, x_3 \in (a,b) : & |x_3 - x_0| < \frac{1}{3} & \wedge\ |f(x_3) - y_0| \geq \varepsilon_0 \\
\ldots & \ldots & \ldots \quad \ldots & \ldots \quad \ldots & \ldots \\
\text{Take } \delta = \frac{1}{n} & \Longrightarrow & \exists\, x_n \in (a,b) : & |x_n - x_0| < \frac{1}{n} & \wedge\ |f(x_n) - y_0| \geq \varepsilon_0 \\
\ldots & \ldots & \ldots \quad \ldots & \ldots \quad \ldots & \ldots
\end{array}$$

   We obtain $\{x_n\}_{n=1}^{\infty}$: $\forall\, n \in \mathbb{N} \quad |x_n - x_0| < \frac{1}{n} \ \wedge\ |f(x_n) - y_0| \geq \varepsilon_0$.

8. Contradiction:

$$\left.\begin{array}{rcl} |x_n - x_0| < \frac{1}{n} & \Longrightarrow & x_n \longrightarrow x_0 \\ |f(x_n) - y_0| \geq \varepsilon_0 & \Longrightarrow & f(x_n) \not\longrightarrow y_0 \end{array}\right\} \Longrightarrow \quad \text{This contradicts to (H) !!!}$$

IMPORTANT MOMENTS in the proofs:

1. A student knows the definition of the limit of a function according to Cauchy and knows what to do in principle.
2. A student knows the definition of a limit of a sequence and understands how to use it in the proof.
3. A student understands how to use the assumption of the theorem in the proof.
4. A student understands what is "the rigorous proof" concerning limits of a sequences.
5. A student knows the definition of the limit of a function according to Heine and knows what to do in principle.
6. A student knows the method by contradiction and is able to construct the negation of the statement with quantifiers correctly.
7. A student knows how to choose $\delta > 0$ to construct the sequence $\{x_n\}_{n=1}^{\infty}$.
8. A student obtains the contradiction with the assumption (H).

REMARK concerning marking of the theoretical proof on the final exam.

A theoretical proof on the final exam will be estimated from 8 points (most of computational problems are estimated from 5 points). It is just an occasion that the number of important moments in the proof above is exactly 8. In principal, the number of important moments in the proof can be arbitrary. In this case the number of important moments will be normalized to 8 points according to the following rule (with the rounding in the favour of a student):

- From 0% to < 12.5% important moments are mentioned — 0 point
- From 12.5% to < 25.0% important moments are mentioned — 1 point
- From 25.0% to < 37.5% important moments are mentioned — 2 point
- From 37.5% to < 50.0% important moments are mentioned — 3 point
- From 50.0% to < 62.5% important moments are mentioned — 4 point
- From 62.5% to < 75.0% important moments are mentioned — 5 point
- From 75.0% to < 87.5% important moments are mentioned — 6 point
- From 87.5% to < 100% important moments are mentioned — 7 point
- 100% important moments are mentioned — 8 point