Lecture "Experimental Physics I"
(Prof. Dr. R. Seidel)

Lecture 26

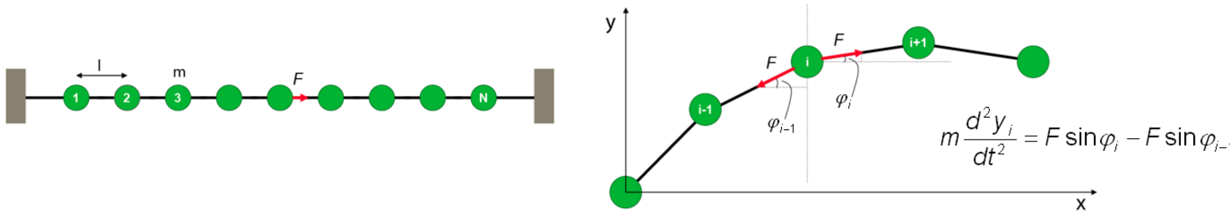
Normal modes in an oscillator chain

- Normal modes of an oscillator chain
- Dynamics of a string
- Superposition of oscillations in 2 dimensions

1) Normal modes of an oscillator chain

In the section before we looked at a coupled oscillator system consisting of just two oscillators. Now we will **look at a chain of N coupled harmonic oscillators**.

To this end, we consider a **string of N beads that is stretched by a force F**. The bead shall be held in place by $N+1$ string segments of equal length l. Displacing a single bead in vertical direction while keeping the others in place provides an oscillation of the displaced bead. Small orthogonal displacements of the whole string of beads can be considered as a collective displacement of the individual bead oscillators. For modelling the behavior of this string of beads, we allow the string to be slightly stretched while the initial stretching force F along the string shall remain roughly constant. This means the stress that is linear to the string stretching shall be small compared to the large prestress.



The force balance and thus the equation of motion for **bead $\boldsymbol{i}$** of the chain can be described using the forces from the adjacent two string linkers that connect the bead to its neighbors. The linker orientations shall be given by the angles φ_{i-1} and φ_i with respect to the equilibrium orientation of the string (see figure above). We then can write for the components of the net force and the inertia on bead i in the limit of small angles φ_i:

$$x:\quad F\underbrace{\cos\varphi_i}_{\approx 1} - F\underbrace{\cos\varphi_{i-1}}_{\approx 1} = m\frac{d^2x_i}{dt^2} \approx 0$$

since we have practically no longitudinal (along the string) but rather transverse (perpendicular to the string) displacements/accelerations. The latter are given by:

$$y:\quad F\sin\varphi_i - F\sin\varphi_{i-1} = m\frac{d^2y_i}{dt^2}$$

The sine of the linker angle can be expressed by the displacements of the adjacent beads:

$$\sin\varphi_i \approx \frac{\Delta y_i}{l} = \frac{y_{i+1}-y_i}{l}$$

assuming that the linker length remains roughly constant. Inserting this expression for $\sin\varphi_{i-1}$ and $\sin\varphi_i$ provides the following equation of motion:

$$\frac{d^2y_i}{dt^2} = \frac{F}{m}\left(\frac{y_{i+1}-y_i}{l} - \frac{y_i-y_{i-1}}{l}\right) = \omega_0^2(y_{i+1} - 2y_i + y_{i-1})$$

where we define the constant:

$$\omega_0^2 = \frac{F}{m}\frac{1}{l} = \frac{F}{\underbrace{m/l}_{\mu}}\frac{1}{l^2}$$

The ratio $\boldsymbol{\mu = m/l}$ **defines the linear mass density** of the chain (mass per length), such that we get:

$$\omega_0 = \sqrt{\frac{F}{\mu}\frac{1}{l}}$$

As discussed above, our system represents **N coupled harmonic oscillators**. In our example, we consider transversal displacement (perpendicular to the string). Alternatively, we could also

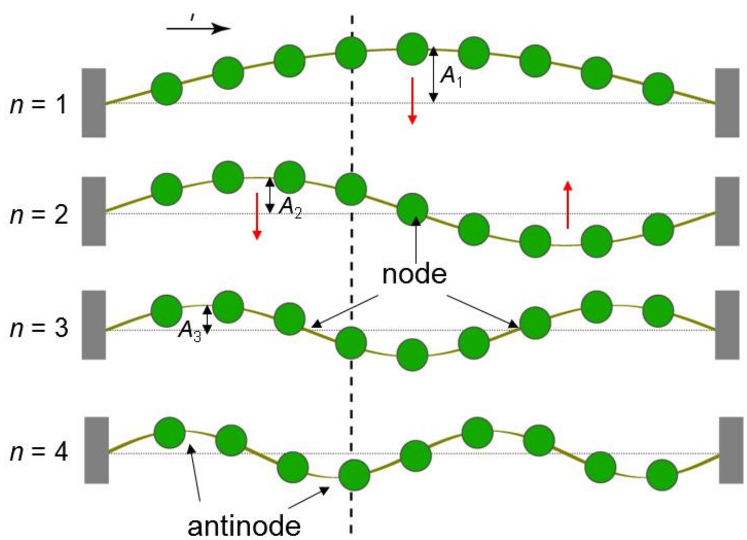
derive the same expression for longitudinal displacements of a chain of coupled masses and springs.
The motion of such a system can be **described by N normal modes** - since we now have N oscillators - as we will show in the following. By our definition **for each normal mode all beads must oscillate with the same frequency**: Thus, for a given mode we get for the displacement of a given bead:

$$y_i = A_{i,n} \cos \omega_n t$$

where **n represents the mode number** with $1 \leq n \leq N$ and i the bead number. This solution provides the same frequency but potentially a **different amplitude $A_{i,n}$ for each bead in the same mode**.
Due to the string attachment, the amplitude must be zero at positions $i = 0$ and $i = N + 1$ (where no beads are). It will be larger inbetween. Let us choose as a guess for the position-dependent amplitude a sinusoidal function that becomes zero at both positions.
The argument in the sine shall be zero at $i = 0$ to have a zero amplitude at that spot. Thus, the argument at position $i = N + 1$ must be a multiple of π also to ensure a zero amplitude at this spot. The resulting sine functions have thus the following form:

$$A_{i,n} = A_n \sin\left(n \frac{\pi\, i}{N+1}\right)$$

n corresponds to the number of (spatial) half-periods the amplitude function contains (see figure below). n shall also be our mode number and the factors A_n define the maximal amplitudes over the whole chain:



Our solution for the chain dynamics of mode n would thus look like:

$$y_i = \underbrace{A_n \sin\left(n \frac{\pi\, i}{N+1}\right)}_{\text{local amplitude}} \cos \omega_n t$$

where the prefactor before the cosine describes the local amplitude. Due to the fixed phase relation of the beads, our proposed solution corresponds essentially to sinusoidal curves with an oscillating amplitude in time (see slide).
These **normal modes** will have **node positions at which no displacement occurs** and **antinode positions where the amplitude is maximal**. Neighboring antinode positions move always with opposite phase.
We now will test whether our guessed solution $y_i = A_{i,n} \cos \omega_n t$ for the modes will solve our equation of motion. Inserting the solution provides (cosine terms already crossed off):

$$-\omega_n^2 A_{i,n} = \omega_0^2 (A_{i+1,n} - 2A_{i,n} + A_{i-1,n})$$

The angle sum identity of the sine function:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

allows to rewrite the local amplitudes at positions $i \pm 1$:

$$A_{i\pm 1,n} = A_n \sin\left(n\frac{\pi i \pm \pi)}{N+1}\right) = A_n \sin\left(\frac{n\,\pi\,i}{N+1}\right)\cos\left(\frac{n\,\pi}{N+1}\right) \pm A_n \cos\left(\frac{n\,\pi\,i}{N+1}\right)\sin\left(\frac{n\,\pi}{N+1}\right)$$

The right term of the sum will disappear when summing $A_{i+1,n} + A_{i-1,n}$ in the amplitude equation from before due to the opposite sign, such that we get:

$$A_{i+1,n} + A_{i-1,n} = 2A_n \sin\left(\frac{n\,\pi\,i}{N+1}\right)\cos\left(\frac{n\,\pi}{N+1}\right)$$

Inserting this result as well as $A_{i,n}$ into the amplitude equation gives

$$-\omega_n^2 A_n \sin\left(\frac{n\,\pi i}{N+1}\right) = \omega_0^2 \underbrace{A_n\left[-2\sin\left(\frac{n\,\pi i}{N+1}\right) + 2\sin\left(\frac{n\,\pi i}{N+1}\right)\cos\left(\frac{n\,\pi}{N+1}\right)\right]}_{A_{i+1,n}-2A_{i,n}+A_{i-1,n}}$$

Simplification provides:

$$\omega_n^2 A_n = \omega_0^2 A_n \underbrace{\left(2 - 2\cos\left(\frac{n\,\pi}{N+1}\right)\right)}_{4\,sin^2(x/2)}$$

Using the double-angle identity $1 - \cos x \;=\; 2\sin^2(x/2)$, we get:

$${\omega_n}^2 = 4{\omega_0}^2 \sin^2\left(\frac{n\,\pi}{2(N+1)}\right)$$

and after further transformation:

$$\omega_n = 2\omega_0 \sin\left(\frac{n\,\pi}{2(N+1)}\right)$$

This has the following consequences:

1) Our guessed solution is indeed a solution to the equation of motion.
2) For each mode we have a different angular frequency $\boldsymbol{\omega_n}$ **which increases with increasing** $\boldsymbol{n}$. This means, that as shorter the spatial period of $A_{i,n}$ as faster is the oscillation.

This relation for $\boldsymbol{\omega_n}$ we will derive again later in solid state physics for the collective oscillation of atoms in a crystal lattice, which are called **phonons**.

2) Mode dynamics of a prestressed string

A) Modes of a prestressed string

We can use the derived solution from the bead chain to obtain similar modes for a **continuous string with linear mass density** $\boldsymbol{\mu}$ by letting the bead number approach infinity, i.e. $N \to \infty$. In this case, our position along the chain becomes continuous. If x is the chain position, we can express it using the bead number and the linker length:

$$x = i\,l$$

The ratio between bead number and total bead number equals then the ratio between x and total chain length:

$$\frac{i}{N+1} = \frac{x}{L}$$

The relation for the local amplitude becomes then:

$$A_{i,n} = A_n \sin\left(\frac{n\,\pi\,i}{N+1}\right) = A_n \sin\left(n\,\pi\frac{x}{L}\right) = A_n(x)$$

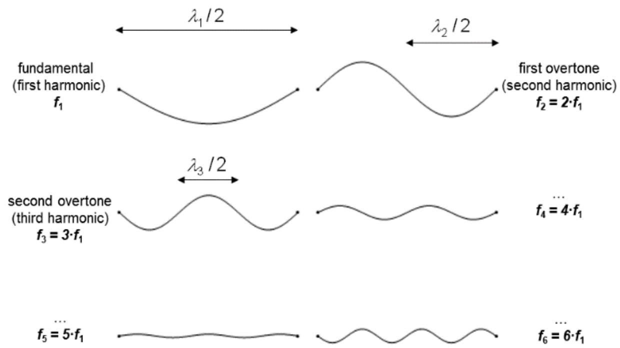
For all n, the local amplitude is zero at $x = 0$ and $x = L$. L/n is herby half the spatial period λ_n (later called wave length). The string length is thus an integer number of the half-period:

$$L = n\frac{\lambda_n}{2}$$

Using this relation to replace n/L in the equation before provides:

$$A_n(x) = A_n \sin\left(n\pi \frac{x}{L}\right) = \sin\left(\frac{2\pi}{\lambda_n} x\right)$$

This provides the following amplitude and oscillation pattern for the modes of the string: (**show animation!**)



With $N \to \infty$ the angular frequency of any finite mode $n \ll N$ is then given by:

$$\omega_n = 2\omega_0 \sin\left(\frac{\pi\, n}{2(N+1)}\right) \approx \frac{\pi}{N+1} n\, \omega_0, \quad (n \ll N)$$

Inserting the expression for $\omega_0 = \sqrt{F/\mu}/l$ provides:

$$\omega_n = \sqrt{\frac{F}{\mu}} \frac{1}{\underbrace{l\, N+1}_{L}} \frac{\pi}{} n = \overbrace{\sqrt{\frac{F}{\mu}} \frac{\pi}{L}}^{\omega_1} n = n\, \omega_1$$

Thus, the angular frequency of the modes increases proportionally to the mode number. The first mode is called fundamental mode. Its frequency ω_1 is called the fundamental frequency and is given by:

$$\omega_1 = \sqrt{\frac{F}{\mu}} \frac{\pi}{L}$$

The following modes are called the 1st, 2nd, 3rd etc. overtones. Also, one calls these modes the 1st, 2nd etc harmonics starting from the fundamental mode. Expressing the string length as multiple of the half-period gives:

$$\omega_n = \sqrt{\frac{F}{\mu}} \frac{\pi}{\underbrace{n\, \lambda_n/2}_{L}} n = 2\pi \sqrt{\frac{F}{\mu}} \frac{1}{\lambda_n}$$

i.e. **the angular frequency of the modes is reciprocal to the spatial period of the mode.** The linear mode frequency is given as:

$$\frac{\omega_n}{2\pi} = f_n = \sqrt{\frac{F}{\mu}} \frac{1}{\lambda_n}$$

Thus, the product

$$\lambda_n f_n = \sqrt{\frac{F}{\mu}} = const$$

is a constant for a particular string and string tension. It defines a velocity, which – as we will see later – is the propagation velocity of any wave along the string.

Experiment: Excitation and visualization of a single mode on an elastic string connected to a driving oscillator. Resonance is only obtained, if the driving oscillator hits one of the normal mode frequencies! The frequency is proportional to the number of antinodes, i.e. the mode number.

Experiment: Frequency spectrum of a guitar string

- **Normal picking**: all modes are excited, i.e. we see only **multiples of the fundamental frequency**. A frequency doubling corresponds hereby exactly to one octave, i.e. the 1st, 2nd, 4th, 8th harmonics correspond to the same tone in music. The addition/mixture with the other harmonics + transient vibrations of the instrument make up the characteristic sound of the instrument

The motion of a string is indeed a superposition of discrete modes with equidistant frequencies!

- **Guitar harmonics**: The existence of nodes for particular string modes can be probed by putting a finger onto the string at half, third, quarter etc. positions of the string. Only particular modes with nodes at these positions get excited.
- Normal modes are also obtained from vocal folds and from a flute (sing and/or play flute).

Show slides from phone app + movie of string oscillations

B) String dynamics from mode superposition (see slides)

Why does one actually excite many modes when picking a string? Single mode excitation would in practice be very difficult, since the initial displacement of such a mode has to be sinusoidal with nodes being correctly placed at the corresponding positions. Typically, one has however a very initial different displacement, e.g. a triangular displacement when picking in the center. Such an initial displacement is achieved by a superposition of many different modes. This can be understood by looking again at our solution for the string displacement of a mode in time:

$$y_n(x,t) = \underbrace{A_n \sin\left(\frac{n\pi}{L}x\right)}_{\text{local amplitude}} \cos(n\omega_1 t)$$

It was the product of a position dependent amplitude and a harmonic oscillation. At **time zero we have maximal displacement, such that the mode shape becomes**

$$y_n(x,0) = \underbrace{A_n \sin\left(\frac{n\pi}{L}x\right)}_{\text{local amplitude}}$$

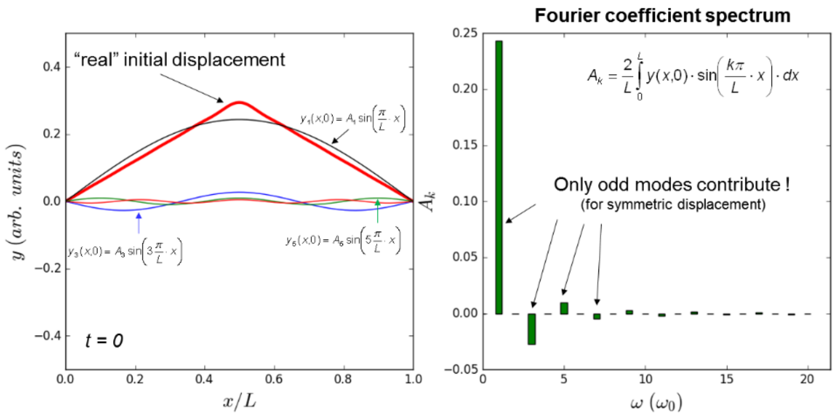
The modes define a set of sine functions. From Fourier analysis we learned that we can approximate any function in a given interval by a sum of sinusoidal functions whose period is an integer fraction of the interval length. Thus, any starting condition could be described by:

$$y(x,0) = \sum_k y_k(x,0) = \sum_k A_k \sin\left(\frac{k\pi}{L}x\right)$$

The Fourier coefficients A_k are then given by the overlap integral of our initial string displacement and the particular mode function:

$$A_k = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{k\pi}{L}x\right)dx$$

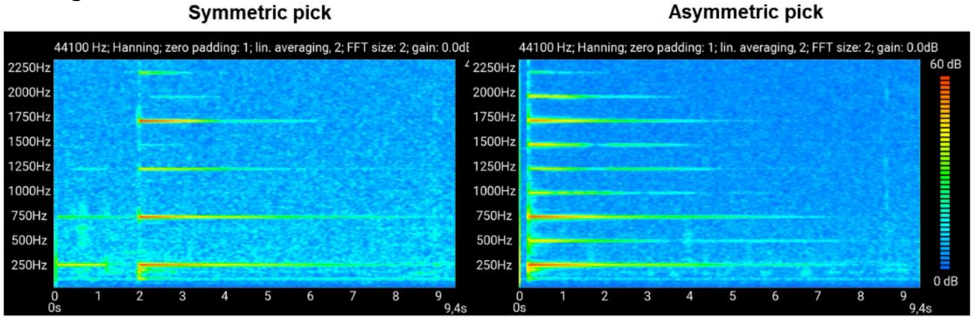
Thus, the **Fourier coefficient spectrum will provide us with the excitation, i.e. the initial amplitudes of all individual modes!** This is illustrated by the Fourier series of a symmetric triangular displacement:



Due to the symmetry, only odd modes have a non-zero Fourier coefficient, i.e. amplitude. If we know the fundamental frequency we know that each mode will oscillate with angular frequency $\omega_n = n\,\omega_1$ in time. Thus, using the initial amplitudes we can describe the dynamics of the whole string in time for any starting condition. The **string dynamics is the superposition of the modes that are excited during the initial displacement!** Using this superposition, one can visualize the string oscillation in an **animation**.

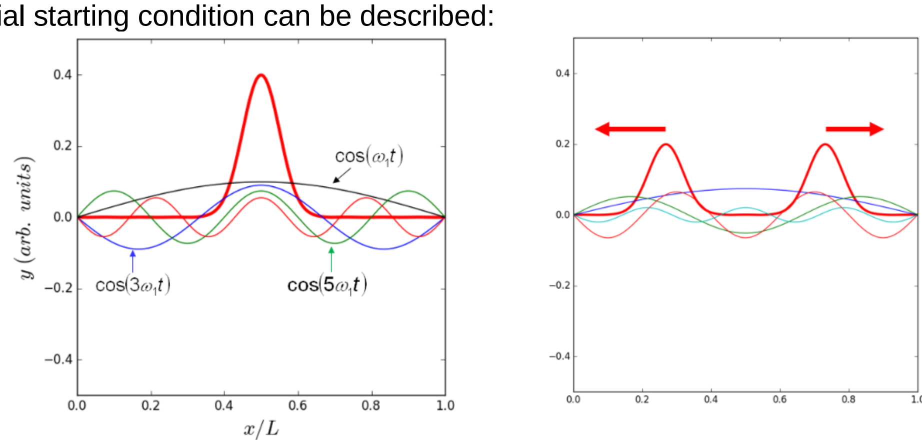
For a symmetric triangular displacement only the odd modes will thus contribute, while an asymmetric pick will excite all modes. This is actually seen in the frequency spectrum of the guitar string, confirming our idea

Symmetric pick | **Asymmetric pick**



Only odd modes | All modes

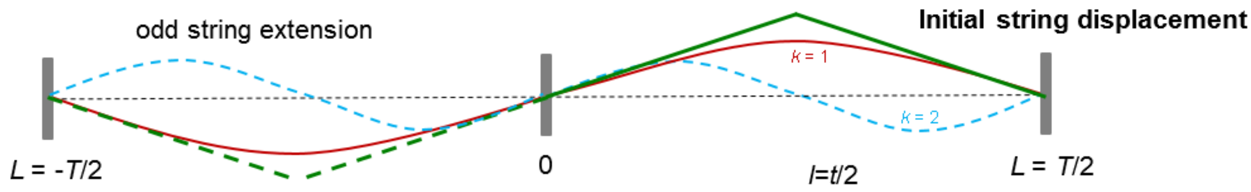
Now we test this idea further by applying a **local Gaussian-shaped pick**. With a Fourier series also this initial starting condition can be described:



When simulating the dynamics, we experience a surprise. The **initial pick splits into to two pulses that propagate away from each other**. At the boundaries they get reflected by inverting their amplitudes and continue this way by periodically bouncing between the boundaries. **The two pulse are actually waves** (with a pulse shape). It is quite surprising that we get these propagating waves despite **superpositioning oscillating modes that are only stationary**. We will discuss in the following lectures waves and how we can explain this observation including the pulse splitting.

C) Analytical solution for initial triangular string displacement (not shown)

Let us have again a look at an initial symmetric triangular displacement in order to find an analytic solution. To get a full period of the fundamental mode we extend the string virtually to the negative side:



This **function is odd** and thus for all Fourier coefficients for the cosine we can write:

$$A_k = 0$$

When replacing *T/2* with *L* in the original definition of the Fourier series, the sine coefficients are obtained from:

$$B_k = \frac{2}{2L}\int_{-L}^{L} y(x,0)\sin\left(\frac{k\pi}{L}x\right)dx$$

Every sine with even k (i.e. $k = 2,4,6,\ldots$) has a node at $x = L/2$ and is odd with respect to $L/2$. Since our displacement is even with respect to $L/2$, the even Fourier coefficients vanish in this case:

$$B_k = 0\,; \quad n = \ 0,2,4,6,\ldots$$

The integral for the remaining coefficients is the same for each quarter of the interval $(-L, L)$, such that we can write after inserting a triangular pick with maximum displacement d:

$$B_{k=2n-1} = \frac{4}{L}\int_0^{L/2} \underbrace{\frac{x}{L/2}d}_{y(x,0)}\sin\left(\frac{k\pi}{L}x\right)dx = \frac{8d}{L^2}\underbrace{\int_0^{L/2} x\sin\left(\frac{k\pi}{L}x\right)dx}_{sin\left(\frac{\pi}{2}k\right)\left(\frac{L}{\pi k}\right)^2}$$

This integral can be solved by integrating by parts and considering that k is odd:

$$\int_0^{\frac{L}{2}} x\sin\left(\frac{k\pi}{L}x\right)dx = \underbrace{-\frac{L}{\pi k}\cos\left(\frac{k\pi}{L}x\right)x\Big|_0^{\frac{L}{2}}}_{0} + \int_0^{\frac{L}{2}}\frac{L}{\pi k}\cos\left(\frac{k\pi}{L}x\right)dx = \left(\frac{L}{\pi k}\right)^2 sin\left(\frac{\pi}{L}kx\right)\Big|_0^{\frac{L}{2}}$$

$$= \sin\left(\frac{\pi}{2}k\right)\left(\frac{L}{\pi k}\right)^2$$

The sine of an odd multiple of *k* is either 1 or -1, such that we get for the odd Fourier coefficients after inserting the integral solution:

$$B_k = \frac{8d}{(\pi k)^2}; k = 1,5,9,13,\ldots; \qquad\qquad B_k = -\frac{8d}{(\pi k)^2}; k = 3,7,11,\ldots$$

For a **symmetrically displaced string** we thus have **only contributions from the odd harmonics** (1,3,5,..). The **amplitude of the higher modes decreases with $1/k^2$**, thus the higher modes are less exited.

Interestingly, when analyzing the measured acoustic signal from a string that was symmetrically picked, we actually see that even the time trajectory of the microphone signal looks triangular. According to our mode composition of the oscillating string, the full shape of the string over time is given by

$$y(x,t) = \sum_{k=1,3,5,\ldots} B_k(x)\cos(\omega_k t) = \sum_{k=1,3,5,\ldots} B_k \sin\left(\frac{\pi}{L} xk\right)\cos(\omega_1 t\; k)$$

At zero time this was a triangle (show on slides)

$$y(x,0) = \sum_{k=1,3,5,\ldots} B_k \sin\left(\frac{\pi}{L} x\; k\right)$$

If we just consider the time dependence of the string center, we get

$$y\left(\frac{L}{2},t\right) \sum_{k=1,3,} B_k \sin\left(\frac{\pi}{2} k\right)\cos(\omega_1 tk) = \sum_{k=1,3,} \pm B_k \cos(\omega_1 t\; k) = \sum_{k=1,3,} \frac{8d}{(\pi k)^2}\cos(\omega_1 t\; k)$$

The alternating sign on the right side cancels the sign alteration of B_k. This is infact a triangular time signal described by a Fourier series of cosine functions (see Lecture 23) which explains the measurement. From the animation before we have actually seen that the motion of the string center produces a triangular time signal.

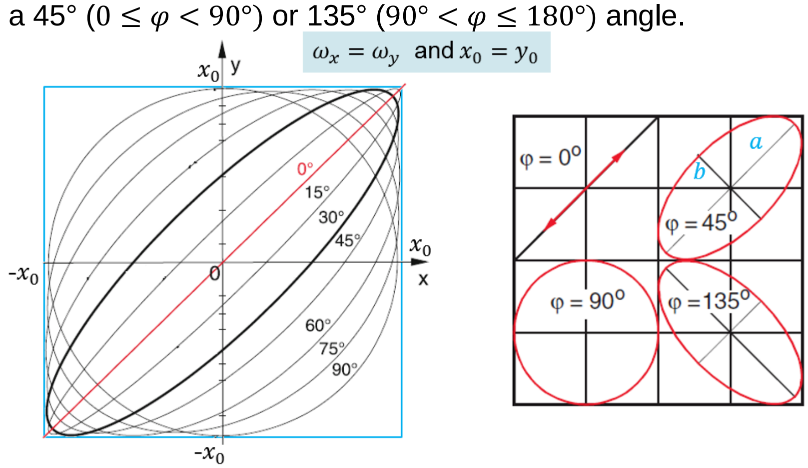
3) Superposition of oscillations in two dimensions

To finalize our view on oscillations we will have a look at the superposition of oscillations of an object in two dimensions in absence of any coupling. We hereby assume that the object undergoes one oscillation in one and another oscillation in a perpendicular direction. The position of the object is then described from the two components:

$$x = x_0\ \cos(\omega_x t)$$
$$y = y_0\ \cos(\omega_y t + \varphi)$$

The resulting geometric pattern is typically quite chaotic. For specific ratios between the two frequencies that can be expressed from the ratio of two integer numbers (i.e. a rational ratio), we get distinct geometric patterns, since after a finite number of oscillations the same pattern reoccurs. These geometric patterns are called **Lissajous curves**.

Let us first consider **equal frequencies** $\boldsymbol{\omega_a = \omega_b = \omega}$ **and equal amplitudes** $\boldsymbol{x_0 = y_0}$ but **different phases**. This provides in elliptically shaped curves that are fit into a square with an edge length of $2x_0$. Due to the symmetry of the ellipse, this “fit” provides that the semimajor axis is oriented at either a 45° ($0 \le \varphi < 90°$) or 135° ($90° < \varphi \le 180°$) angle.



The phase difference φ determines the eccentricity of the ellipse (i.e. the ratio between semiminor and semimajor axis b and a). To intuitively understand this let us consider different cases:

For $\boldsymbol{\varphi = 0}°$ the trajectory is given by

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} x_0 \\ x_0 \end{pmatrix}}_{\vec{r}_0} \cos\omega t$$

which defines on oscillation along the vector (x_0, x_0) at 45° angle. This defines a simple line where the seminar axis $b = 0$.

For $\boldsymbol{\varphi = \pm 90}$° the trajectory is given by

$$\vec{r} = \begin{pmatrix} x_0 \cos \omega t \\ \mp x_0 \sin \omega t \end{pmatrix} = x_0 \begin{pmatrix} \cos \omega t \\ \mp \sin \omega t \end{pmatrix}$$

which defines simple circular motion. In this case the semimajor and semiminor axis are equal ($a = b = x_0$).

For the **other phase angles**, the semiminor axis increases with increasing phase angle from 0 until it reaches $a = b$ at $\boldsymbol{\varphi = 90}$°. Subsequently the former semimajor axis decreases and now becomes the seminar axis, such that a tilt angle of 135° is obtained.

The length of the semimajor and -minor axes can be determined considering their inclination of 45° or 135°. For the ellipse the absolute values of the x and y coordinates at these positions equal ($x_{a,b} = \pm y_{a,b}$). Thus,

$$x_0 \cos \omega t_{ab} = \pm x_0 \cos(\omega t_{ab} + \varphi) = \pm x_0 (\cos \omega t_{ab} \cos \varphi - \sin \omega t_{ab} \sin \varphi)$$

This can be transformed to

$$\cos \omega t_{ab} (1 \mp \cos \varphi) = \mp \sin \omega t_{ab} \sin \varphi = \mp \sqrt{1 - \cos^2 \omega t_{ab}} \sin \varphi$$
$$\cos^2 \omega t_{ab} [(1 \mp \cos \varphi)^2 + \sin^2 \varphi] = \sin^2 \varphi$$
$$\cos^2 \omega t_{ab} [2 \mp 2 \cos \varphi] = \sin^2 \varphi$$

such that we get

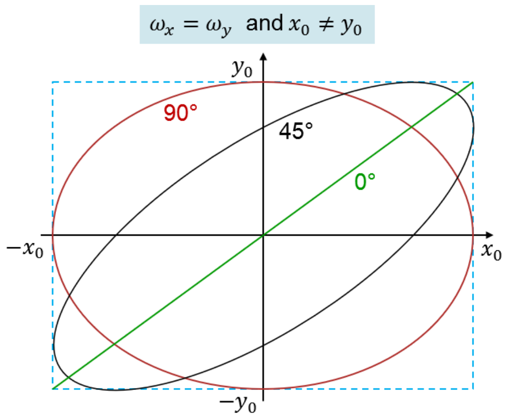
$$\cos \omega t_{ab} = \frac{\sin \varphi}{\sqrt{2[1 \mp \cos \varphi]}}$$
$$a = \sqrt{2} x_0 \cos \omega t_a = \frac{a \sin \varphi}{\sqrt{1 - \cos \varphi}}$$
$$b = \sqrt{2} x_0 \cos \omega t_b = \frac{a \sin \varphi}{\sqrt{1 + \cos \varphi}}$$

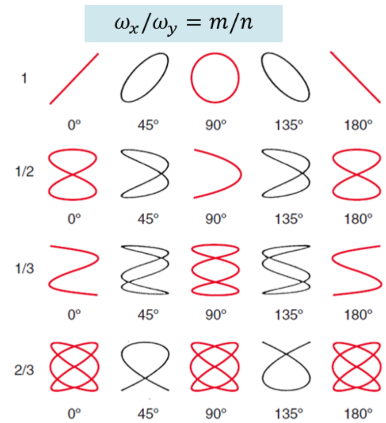
For **unequal amplitudes $\boldsymbol{x_0 \neq y_0}$** but equal frequencies $\omega_a = \omega_b = \omega$, the elliptic curves are correspondingly stretched along the vertical direction, such that the ellipses now fit into the corresponding rectangle (see below). The tilt angle of the ellipse becomes no dependent on the phase shift φ, decreasing from $\varphi_{tilt} = \arctan(y_0/x_0)$ at $\varphi = 0$ to $\varphi_{tilt} = 0$ at $\varphi = 90°$ (see figure below).

For **other frequency ratios**, typically, chaotic time-variant curves are obtained. Only if the **ratio $\boldsymbol{\omega_x/\omega_y = m/n}$ is a rational number** (i.e. m, n being integers), we get closed, time-invariant curves. This is due to the fact that after m periods of the oscillation with ω_x and one completes m periods of the oscillation with ω_y:

$$m\, T_x = m \frac{2\pi}{\omega_x} = m \frac{2\pi}{\omega_y} \frac{n}{m} = n\, T_y$$

Thus, the curve starts again. m and n define the number of antinodes of the Lissajous patterns in the corresponding directions.





Experiment: Superimposing two orthogonal oscillating voltages at an oscilloscope to deflect the electron beam shows formation of such ellipses at equal frequencies and more complicated figures at other frequency ratios

Lecture 26: Experiments

1) Excitation and visualization of a single mode on a string connected to a driving oscillator.
2) Frequency spectrum as overlay of the different modes of a guitar string (normal picking, guitar harmonics, symmetric picking)
3) Lissajous curves