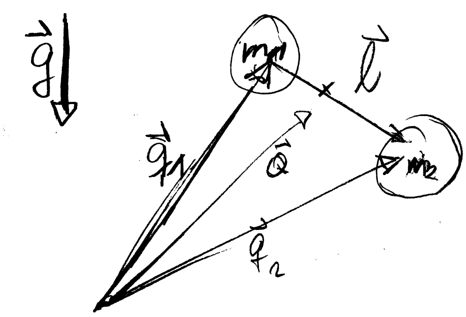
Theoretical Mechanics IPSP

Jürgen Vollmer, Universität Leipzig

13.2. Flight of a dumbbell



We explore the flight of a dumbbell under the influence of gravity $\boldsymbol{g}$ in our three-dimensional space. The dumbbell is idealized as two particles of masses m_1 and m_2. The positions $\boldsymbol{q_1}(t)$ and $\boldsymbol{q_2}(t)$ will be kept at an approximately constant distance $\ell_\star$ by a bar of negligible mass. We denote the center of mass of the dumbbell as $\boldsymbol{Q}$ and the relative coordinate as $\boldsymbol{\ell} = \boldsymbol{q_2} - \boldsymbol{q}_1$.

a) Center of mass

We express the relation between $(\boldsymbol{Q}, \boldsymbol{\ell})$ and the positions $\boldsymbol{q}_i, i \in \{1,2\}$ as $\boldsymbol{q}_i = \boldsymbol{Q} + \alpha_i\, \boldsymbol{\ell}$. Determine the real numbers $\alpha_i, i \in \{1,2\}$.

Solution:

The definition of the center of mass $\boldsymbol{Q}$ and of the distance vector $\boldsymbol{\ell}$ provide

$$\begin{aligned}(m_1+m_2)\,\boldsymbol{Q} &= m_1\left(\boldsymbol{Q}+\alpha_1\,\boldsymbol{\ell}\right) + m_2\left(\boldsymbol{Q}+\alpha_2\,\boldsymbol{\ell}\right) = (m_1+m_2)\,\boldsymbol{Q} + \left(m_1\,\alpha_1 + m_2\,\alpha_2\right)\boldsymbol{\ell} \\ \boldsymbol{\ell} &= -\left(\boldsymbol{Q}+\alpha_1\,\boldsymbol{\ell}\right) + \left(\boldsymbol{Q}+\alpha_2\,\boldsymbol{\ell}\right) = \left(-\alpha_1+\alpha_2\right)\boldsymbol{\ell}\end{aligned}$$

such that

$$\begin{aligned}1 &= -\alpha_1 + \alpha_2 && \Rightarrow \alpha_2 = 1+\alpha_1 \\ 0 &= m_1\,\alpha_1 + m_2\,\alpha_2 = m_1\,\alpha_1 + m_2\,(1+\alpha_1) \\ &= (m_1+m_2)\,\alpha_1 + m_2 && \Rightarrow \alpha_1 = -\frac{m_2}{m_1+m_2} \\ & && \Rightarrow \alpha_2 = \frac{m_1+m_2}{m_1+m_2} + \alpha_1 = \frac{m_1}{m_1+m_2}\end{aligned}$$

b) Kinetic and potential energy

Show that the kinetic energy and the potential energy of the dumbbell have the form

$$\begin{aligned}T &= \frac{M}{2}\,\dot{\boldsymbol{Q}}^2 + \frac{\mu}{2}\,\dot{\boldsymbol{\ell}}^{\,2}, \\ V &= -M\boldsymbol{g}\cdot\boldsymbol{Q} + \Phi(\ell)\end{aligned}$$

where $\Phi(\ell)$ is a potential that generates the force fixing the distance of the masses to the value of about $\ell_\star$.[1] How do M and μ depend on m_1 and m_2?

Solution:

Kinetic energy:

$$\begin{aligned}T &= \frac{m_1}{2}\,\dot{\boldsymbol{q}}_1^2 + \frac{m_2}{2}\,\dot{\boldsymbol{q}}_2^2 = \frac{m_1}{2}\left(\dot{\boldsymbol{Q}} - \frac{m_2}{m_1+m_2}\,\dot{\boldsymbol{\ell}}\right)^2 + \frac{m_2}{2}\left(\dot{\boldsymbol{Q}} + \frac{m_1}{m_1+m_2}\,\dot{\boldsymbol{\ell}}\right)^2 \\ &= \frac{m_1+m_2}{2}\,\dot{\boldsymbol{Q}}^2 + \frac{2}{2}\left(-\frac{m_1\,m_2}{m_1+m_2} + \frac{m_2\,m_1}{m_1+m_2}\right)\dot{\boldsymbol{Q}}\cdot\dot{\boldsymbol{\ell}} + \frac{1}{2}\left(\frac{m_1\,m_2^2}{(m_1+m_2)^2} + \frac{m_2\,m_1^2}{(m_1+m_2)^2}\right)\dot{\boldsymbol{\ell}}^2 \\ &= \frac{m_1+m_2}{2}\,\dot{\boldsymbol{Q}}^2 + \frac{1}{2}\,\frac{m_1\,m_2}{m_1+m_2}\,\frac{m_2+m_1}{m_1+m_2}\,\dot{\boldsymbol{\ell}}^2\end{aligned}$$

such that

$$M = m_1 + m_2 \qquad \text{and} \qquad \mu = \frac{m_1\,m_2}{m_1+m_2}$$

Potential energy:

$$V = -m_1 \boldsymbol{g} \cdot \boldsymbol{q}_1 - m_2 \boldsymbol{g} \cdot \boldsymbol{q}_2 + \Phi(\ell) = -\boldsymbol{g} \cdot \big(m_1\, \boldsymbol{q}_1 + m_2\, \boldsymbol{q}_2 \big) + \Phi(\ell) = -\boldsymbol{g} \cdot \big(M\, \boldsymbol{Q} \big) + \Phi(\ell)$$

c) Conservation laws

Show that

$$\ddot{\boldsymbol{Q}} = \boldsymbol{g}$$

How does the trajectory of the center of mass of the dumbbell look like when the dumbbell is thrown at time t_0 from a position $\boldsymbol{Q}_0$ with a velocity $\boldsymbol{V}_0$?

Solution:

The Langrangian takes the form

$$\mathcal{L} = \frac{M}{2}\, \dot{\boldsymbol{Q}}^2 + \frac{\mu}{2}\, \dot{\boldsymbol{\ell}}^{\,2} + M\boldsymbol{g} \cdot \boldsymbol{Q} - \Phi(\ell)$$

Consequenly, we find for the component Q_i of $\boldsymbol{Q}$

$$M\, \ddot{Q}_i = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{Q}_i} = \frac{\partial \mathcal{L}}{\partial Q_i} = M\, g_i$$

where g_i is the ith component of $\boldsymbol{g}$. Division by M and collecting the components into a vector provides the expresssion above. The center of mass follows the trajectory of free flight of a point particle in a gravitational field,

$$\boldsymbol{Q}(t) = \boldsymbol{Q}(t_0) + \dot{\boldsymbol{Q}}(t_0)\, (t - t_0) = \boldsymbol{Q}_0 + \boldsymbol{V}_0\, (t - t_0) + \frac{\boldsymbol{g}}{2}\, (t - t_0)^2$$

d) Center-of-mass motion

Show that

$$\mu\, \ddot{\boldsymbol{\ell}} = -\hat{\boldsymbol{\ell}}\, \frac{\mathrm{d}\Phi(\ell)}{\mathrm{d}\ell} \quad \text{with} \quad \hat{\boldsymbol{\ell}} = \frac{\boldsymbol{\ell}}{\ell}\,.$$

Solution:

We derive the EOM for the component ℓ_i of $\boldsymbol{\ell}$ by starting from the Euler-Lagrange equation

$$\mu \ddot{\ell}_i = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\ell}_i} = \frac{\partial \mathcal{L}}{\partial \ell_i} = -\frac{\partial \ell}{\partial \ell_i}\, \frac{\mathrm{d}\Phi(\ell)}{\mathrm{d}\ell}$$

with

$$\ell = \sqrt{\sum_j \ell_j^2} \quad \Rightarrow \quad \frac{\partial \ell}{\partial \ell_i} = \frac{\sum_j 2\, \ell_j \frac{\partial \ell_j}{\partial \ell_i}}{2\, \sqrt{\sum_k \ell_k^2}} = \frac{\ell_i}{\ell}$$

e) Relative motion in spherical coordinates

Show that the energy $E = \mu \dot{\boldsymbol{\ell}}^2/2 + \Phi(\ell)$ and the angular momentum $\boldsymbol{L} = \mu \boldsymbol{\ell} \times \dot{\boldsymbol{\ell}}$ are constants of the motion of the dumbbell.

Solution:

energy:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mu\, \dot{\boldsymbol{\ell}} \cdot \ddot{\boldsymbol{\ell}} + \frac{\mathrm{d}\ell}{\mathrm{d}t}\, \frac{\mathrm{d}\Phi(\ell)}{\mathrm{d}\ell} = -\dot{\boldsymbol{\ell}} \cdot \hat{\boldsymbol{\ell}}\, \frac{\mathrm{d}\Phi(\ell)}{\mathrm{d}\ell} + \frac{\boldsymbol{\ell} \cdot \dot{\boldsymbol{\ell}}}{\ell}\, \frac{\mathrm{d}\Phi(\ell)}{\mathrm{d}\ell} = 0$$

angular momentum:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \mu\, \dot{\boldsymbol{\ell}} \times \dot{\boldsymbol{\ell}} + \mu\, \dot{\boldsymbol{\ell}} \times \ddot{\boldsymbol{\ell}} = \mu\, \dot{\boldsymbol{\ell}} \times \left(-\hat{\boldsymbol{\ell}}\, \frac{\mathrm{d}\Phi(\ell)}{\mathrm{d}\ell} \right) = \boldsymbol{0}$$

since the cross product vanishes for parallel vectors.

f) Parameterization of relative motion

Discuss the evolution of $\boldsymbol{\ell}$ in terms of spherical coordinates (r, θ, ϕ) that are chosen such that initially $\boldsymbol{\ell}$ and $\dot{\boldsymbol{\ell}}$ lie in the equatorial plane, $\theta = \pi/2$ of the coordinate system:

- Show that ϕ is a cyclic coordinate. How is the associated conservation law $\mu\,\ell^2\,\dot\phi$ related the $\boldsymbol{L}$?
- Show that $\theta = \pi/2$ is a fixed point of the θ dynamics. How is this fixed point related to the conservation of $\boldsymbol{L}$?

Solution:

In polar coordinates the vector $\boldsymbol{\ell}$ and its time derivative $\dot{\boldsymbol{\ell}}$ take the form

$$\begin{aligned}\boldsymbol{\ell} &= \ell\,\hat{\boldsymbol{r}}(\theta,\phi) \\ \dot{\boldsymbol{\ell}} &= \dot\ell\;\hat{\boldsymbol{r}}(\theta,\phi) + \ell\,\dot\theta\;\hat{\boldsymbol{\theta}}(\theta,\phi) + \ell\;\sin\theta\,\dot\phi\;\hat{\boldsymbol{\phi}}(\theta,\phi)\end{aligned}$$

such that the resulting contribution to the kinetic energy takes the form

$$\frac{\mu}{2}\,\dot{\boldsymbol{\ell}}^2 = \frac{\mu}{2}\left(\dot\ell^2 + \ell^2\,\dot\theta^2 + \ell^2\;\sin^2\theta\,\dot\phi^2\right)$$

The angle ϕ is a cyclic coordinate such that

$$L = \mu\,\ell^2\,\dot\phi = \text{const}$$

Here, L is the absolute value of the angular momentum $\boldsymbol{L}$. The EOM for θ takes the form

$$\mu\,\ell^2\,\ddot\theta = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{\partial\mathcal{L}}{\partial\theta} = \mu\,\ell^2\;\sin\theta\;\cos\theta\,\dot\phi^2 = \frac{L^2}{2\,\mu\,\ell^2}\;\sin(2\theta)$$

In the admissible range $0 \le \theta \le \pi$ this dynamics has fixed points for $\theta \in \{0, \pi/2, \pi\}$. Hence, the initial condition of $\boldsymbol{\ell}$ selects a fixed point of the θ dynamics such that $\theta = \pi/2$ for all times. The motion proceeds in the equatorial plane of our coordinate system, and the direction of $\boldsymbol{L}$ remains vertical to this plane. The EOM for ℓ takes the form

$$\mu\ddot\ell = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\ell} = \frac{\partial\mathcal{L}}{\partial\ell} = \mu\ell\,\dot\theta^2 + \mu\ell\;\sin\theta\,\dot\phi^2 - \frac{\mathrm{d}\Phi(\ell)}{\mathrm{d}\ell} = \frac{L^2}{\mu\,\ell^3} - \frac{\mathrm{d}\Phi(\ell)}{\mathrm{d}\ell}$$

Here we used that $\dot\theta = 0, \sin\theta = 1$, and $\dot\phi = L/(\mu\ell^2)$. Henceforth, we assert that the right-hand side of this equation vanishes such that $\dot\ell = \ddot\ell = 0$. In this case the dynamics of $\dot\phi$ is solved by

$$\phi(t) = \phi_0 + \frac{L}{\mu\ell^2}\,(t - t_0)$$

where the integration constants ϕ_0 are L are determined by the initial conditions.

g) Solution of the equation of motion

Provide the position of the masses $\boldsymbol{q}_1(t)$ and $\boldsymbol{q}_2(t)$ for the initial conditions provided in c), some fixed $\boldsymbol{\Omega}$, and $\boldsymbol{\ell}_0$.

Solution:

The initial conditions for $\boldsymbol{q}_1(t_0)$, $\boldsymbol{q}_2(t_0)$ and their velocities provide the angular momentum

$$\boldsymbol{L} = \mu\,(\boldsymbol{q}_2(t_0) - \boldsymbol{q}_1(t_0)) \times (\dot{\boldsymbol{q}}_2(t_0) - \dot{\boldsymbol{q}}_1(t_0))$$

The length of this vector provides the constant rotation frequency $\dot\phi = |\boldsymbol{L}|\,/\mu\ell^2$. It orientation defines a plane, and $\hat{\boldsymbol{r}}(\pi/2, \phi(t))$ defines a direction in that plane. The angle ϕ will be measured with respect to the initial orientation of $\boldsymbol{\ell}$ such that

$$\boldsymbol{\ell} = \ell\,\hat{\boldsymbol{r}}(\pi/2, (t - t_0)\,\dot\phi)$$

and consequently

$$\begin{aligned}\boldsymbol{q}_1 &= \boldsymbol{Q}(t) - \frac{m_2}{M}\boldsymbol{\ell}(t) = \boldsymbol{Q}_0 + \boldsymbol{V}_0\,(t - t_0) + \frac{\boldsymbol{g}}{2}\,(t - t_0)^2 - \frac{m_2}{M}\,\ell\,\hat{\boldsymbol{r}}(\pi/2, (t - t_0)\,\dot\phi) \\ \boldsymbol{q}_2 &= \boldsymbol{Q}(t) + \frac{m_1}{M}\boldsymbol{\ell}(t) = \boldsymbol{Q}_0 + \boldsymbol{V}_0\,(t - t_0) + \frac{\boldsymbol{g}}{2}\,(t - t_0)^2 + \frac{m_1}{M}\,\ell\,\hat{\boldsymbol{r}}(\pi/2, (t - t_0)\,\dot\phi)\end{aligned}$$

1)
Here, $\ell_\star$ is the length of the bar, when no forces are acting, and the potential counteracts centrifugal forces such that ℓ always takes a value very close to $\ell_\star$.

Discussion

Mohammed Zakaria Hasan AL-Obaidi, 2022/01/22 02:40, 2022/01/22 02:40

In practice, a well-built dumbbell would have equal masses on both ends. Can we make that assumption for this problem (m1=m2)?

Thanks, Mohammed

Jürgen Vollmer, 2022/01/23 17:42

No: as discussed in the question hour the present version also accounts for drum sticks and similar objects.

Seyed-Mostafa Moussavi, 2022/01/24 21:28, 2022/01/24 23:34

Hello dear all

Sorry for the late comment. Now I'm going over the questions. Is it better to say:
"$\Phi(\ell)$ is a potential that will generate the force fixing the distance of the masses to the value $\ell_\star$" ?

I mean Φ has different values depending on the independent variable ℓ (i.e. the length of the bar) and has a minimum value at a special length that we call it $\ell_\star$ (i.e. the unstretched length of the bar).

Regards
S.M Moussavi

Jürgen Vollmer, 2022/01/24 23:39

This is indeed a clear way to put it. I introduced $\ell_\star$ and added a footnote that is explicitly describing what $\ell_\star$ refers to.