Lecture "Experimental Physics I"

(Prof. Dr. R. Seidel)

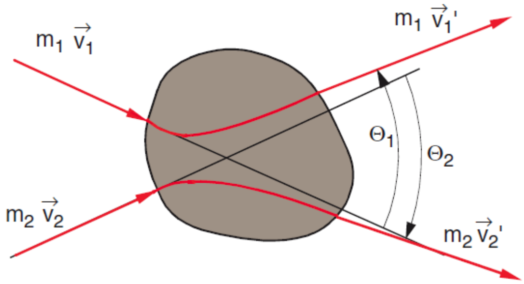
Lecture 13

Collisions

- Elastic and inelastic collisions in 1D
- Astroblaster
- Elastic and inelastic collisions in 2D

1) Collisions

In this lecture we want to better understand what happens during collisions of two particles and how we can describe the underlying processes.
The general scheme of a collision process are two particles each possessing an initial velocity that interact within a certain distance range and subsequently continue with resulting final velocities.



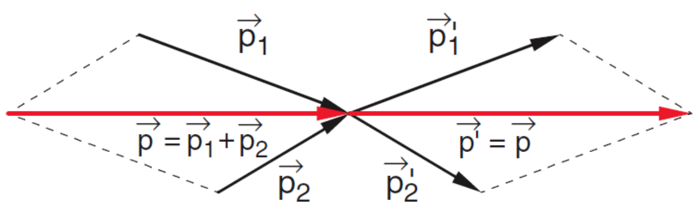
For such a two particle collision, we can write down the equations for the conservation of the momentum (**see slides**):

$$\underbrace{m_1 v_1}_{p_1} + \underbrace{m_2 v_2}_{p_2} = \underbrace{m_1 v_1'}_{p_1'} + \underbrace{m_2 v_2'}_{p_2'}$$

as well as the energy, where we consider also a possible energy dissipation:

$$\underbrace{\left(\frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2\right)}_{E_{k1}+E_{k2}} = \underbrace{\left(\frac{m_1}{2} {v_1'}^2 + \frac{m_2}{2} {v_2'}^2\right)}_{E'_{k1}+E'_{k2}} + \underbrace{Q}_{\substack{heat \\ released}}$$

The momentum will for such a collision process always be conserved, since during the collision process only internal forces between the two particles act. Thus, we can draw the following vector diagram for the momenta before and after collision:



Based on the dissipated energy we distinguish the following classes of collisions:

Systematics of collisions

$Q = 0$: elastic collision – conservation of kinetic energy
$Q > 0$: inelastic collision – reduced kinetic energy
$Q < 0$: superelastic collision – increased kinetic energy

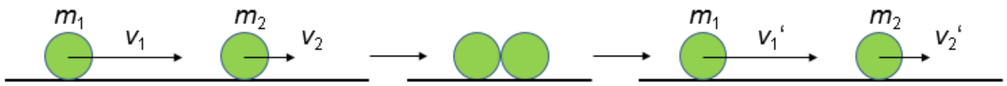
Experiment: The concept of elastic and inelastic collisions can be nicely seen at the collision of a steel sphere falling onto different supports. We have an:

- An almost fully elastic collision when the steel sphere falls onto a steel anvil, such that the sphere gets reflected to almost its initial height (the small energy loss is due to the excitations of anharmonic oscillations inside the anvil).
- a strongly inelastic collision when the sphere is falling onto a brass or lead support, where the reflection almost completely vanishes.

2) Collisions in 1D (central collisions)

We first have a look at collisions in one dimension, which is equivalent to central collisions, where all velocities vectors are parallel to each other. We consider two particles (1 and 2) with corresponding masses and velocities.

A) Elastic collisions



For a fully elastic collision no energy is dissipated, i.e. we can write down the equations for energy and momentum conservation for the two particle system:

Energy conservation: $E_{k,tot} = const$

$$\frac{m_1}{2}v_1^2 + \frac{m_2}{2}v_2^2 = \frac{m_1}{2}v_1^{'2} + \frac{m_2}{2}v_2^{'2}$$

which can be transformed to

$$m_2\left(v_2^2 - v_2^{'2}\right) = m_1\left(v_1^{'2} - v_1^2\right)$$

and

$$m_2(v_2 - v_2^{'})(v_2 + v_2^{'}) = m_1(v_1^{'} - v_1)(v_1^{'} + v_1)$$

Momentum conservation: $P_{tot} = const$

$$m_1v_1 + m_2v_2 = m_1v_1^{'} + m_2v_2^{'}$$

which can be transformed to

$$m_2(v_2 - v_2^{'}) = m_1(v_1^{'} - v_1)$$

Division of both transformed equations provides:

$$v_2 + v_2^{'} = v_1 + v_1^{'}$$

Being equivalent to

$$v_1 - v_2 = -(v_1^{'} - v_2^{'})$$

i.e. the **relative velocity of the two particles before the collision equals the negative relative velocity after the collision**, i.e. if particle 1 was faster before the collision, particle 2 becomes faster after the collision. This is a useful relation that later on simplifies calculations.

Let us now calculate the velocity $v_1^{'}$ of the mass 1 after the collision by transforming the momentum conservation realtion:

$$m_1v_1^{'} = m_1v_1 + m_2v_2 - m_2v_2^{'}$$

Replacing $v_2^{'}$ using the derived relation between the velocities gives:

$$m_1v_1^{'} = m_1v_1 + m_2v_2 - m_2(v_1 - v_2 + v_1^{'})$$

Further transformation provides

$$(m_1 + m_2)v_1^{'} = (m_1 - m_2)v_1 + 2m_2v_2$$

such that we finally obtain:

$$v_1^{'} = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2}$$

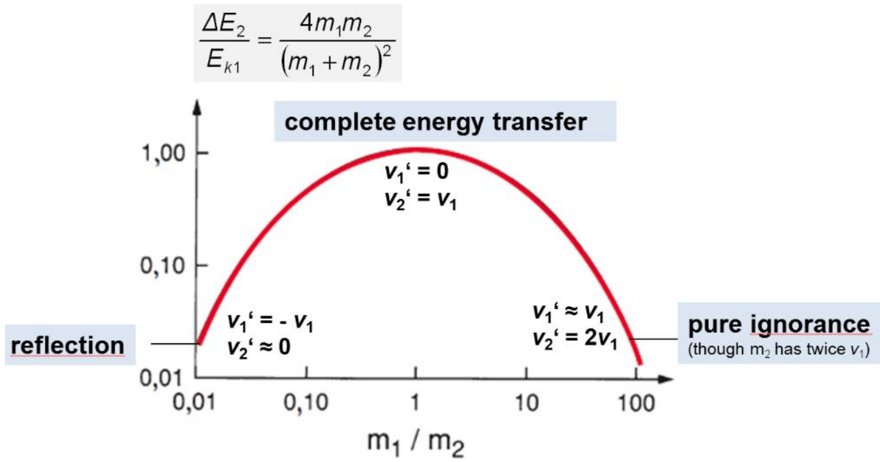
Inserting this result into the velocity relation yields

$$v_2^{'} = \frac{2m_1v_1 + (m_2 - m_1)v_2}{m_1 + m_2}$$

Often in collision processes the **velocity of the second mass is 0**. For the specific case $v_2 = 0$ we get:

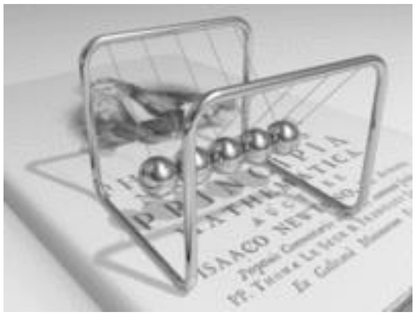
$$v_1^{'} = \frac{(m_1 - m_2)}{m_1 + m_2} v_1$$
$$v_2^{'} = \frac{2m_1}{m_1 + m_2} v_1$$
Before the collision, all the kinetic energy is concentrated on mass 1. After collision the transferred energy to mass 2 is in this case simply given by:
$$\Delta E_2 = \frac{m_2}{2} v_2^{',2} = \frac{m_2}{2}\left(\frac{2m_1}{m_1 + m_2} v_1\right)^2 = \frac{4m_1 m_2}{(m_1 + m_2)^2} E_{k1}$$

which provides the following plot for the relative transferred energy



To understand this equation/plot it is informative to distinguish different cases:

- **<u>Equal masses</u>** $\boldsymbol{m_1 = m_2}$:
  Here $\Delta E_2 = E_{k1}$, i.e. the whole energy is transferred to the second mass! Inserting into the velocity equations provides:
  $$v_1^{'} = \frac{m - m}{m + m} v_1 = 0, \quad v_2^{'} = \frac{2m}{m + m} v_1 = v_1$$
  Experiment: Newton's cradle with just two spheres
  Experiment: whole Newton's cradle (see Wikipedia article for precise calculation)



- **Light mass collides with heavy mass** $\boldsymbol{m_1 \ll m_2}$:

  Experiment: Newton's cradle with unequal spheres
  Experiment: Reflection of steel sphere at hard block

  Here one sees that $\boldsymbol{m_1}$ **gets reflected** while $\boldsymbol{m_2}$ gets hardly deflected!
  The energy transfer approaches zero (do not show in detail):

$$\frac{\Delta E_2}{E_{k1}} = 4\underbrace{\frac{dm}{dm+M}}_{\approx dm/M}\underbrace{\frac{M}{M+dm}}_{\approx 1} \approx 4\frac{dm}{M} \to 0$$

For the velocities we get:

$$v_1^{'} = \frac{dm - M}{dm + M} v_1 \approx \frac{-M}{M} v_1 = -v_1$$

$$v_2^{'} \approx \frac{2dm}{dm + M} v_1 \approx 0$$

Calculating the momentum transfer for both particles yields:

$$\Delta p_1 \approx \underbrace{m_1(-v_1)}_{p_1^{'}} - \underbrace{m_1 v_1}_{p1} = -2m_1 v_1 = -2p_1$$

$$\Delta p_2 = m_2 v_2^{'} - 0 = m_2 \frac{2m_1}{m_1 + m_2} v_1 \approx 2m_1 v_1 = 2p_1$$

Though there is hardly kinetic energy transfer onto m_2 twice the original momentum is transferred to it, since all momentum changes must cancel each other out.
$\boldsymbol{\Delta p_{1,2} = \pm 2p_1}$ is the maximum possible momentum transfer since energy conservation must also be obeyed.

- **Heavy mass collides with light mass $\boldsymbol{m_1 \gg m_2}$**:

  Experiment: Newton's cradle with unequal spheres

  Here one sees that $\boldsymbol{m_1}$ **ignores the collision**, while $\boldsymbol{m_2}$ **becomes faster than $\boldsymbol{m_1}$.** The energy transfer approaches again zero:

$$\frac{\Delta E_2}{E_{k1}} \to 0$$

  For the velocities we get:

$$v_1' = \frac{M - dm}{M + dm} v_1 \approx \frac{M}{M} v_1 = v_1$$

$$v_2' = \frac{2M}{M + dm} v_1 \approx 2v_1$$

  energetically m_1 ignores m_2, though v_2 is non-zero.

  Experiment: Doubled velocity v_2' for light car on air track that is hit by a heavy car. The heavy car loses only little velocity.

B) Astroblaster (or Galilean canon)

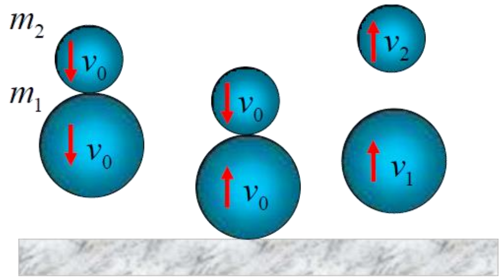
The astroblaster also called Galilean cannon was (re)invented by the American astrophysicist Stirling Colgate. He wanted to show, where the enormous kinetic energy during a supernova explosion comes from.
It is a wonderful illustration of sequential elastic collisions between unequal spheres. It is a toy that comprises a stack of balls, starting with a large, heavy ball at the base of the stack and progresses up to a small, lightweight ball at the top.

Experiment: Let an astroblaster fall down and admire that the smallest sphere jumps much higher than the original fall height.

There is even a Guiness-Book world record for the tallest launch of an astroblaster of 11.5 m (https://wlna-webservice.gannettdigital.com/articleservice/view/1685694002/michigan-state-spartans/24.3.57/iphone), which does not appear to be so high…

2-sphere astroblaster:

The astroblaster experiment can only be explained by the fact that the other spheres transferred kinetic energy to the top sphere during the collision.
To better grasp this idea, we first look at a two sphere astroblaster. Important for the understanding is to realize that first the heavier sphere 1 is inverting its velocity (due to its reflection at the surface), while sphere 2 has still the downward oriented free fall velocity:



We can thus write down for the **momentum conservation** upon the collision:

$$m_1 v_0 - m_2 v_0 = m_1 v'_1 + m_2 v'_2$$

From the **formula for the relative velocities** from before (that included also energy conservation)**:**

$$v_1 - v_2 = -(v_1' - v_2')$$

we get:

$$v_2' - v_1' = v_0 - (-v_0) = 2v_0$$
$$v_2' = 2v_0 + v_1'$$

Inserting this relation into momentum conservation relation provides:

$$v_1' = \frac{m_1 - 3m_2}{m_1 + m_2} v_0$$

We get full transmission of the kinetic energy to m_2 when $v_1' = 0$, i.e.

$$m_1 = 3m_2$$

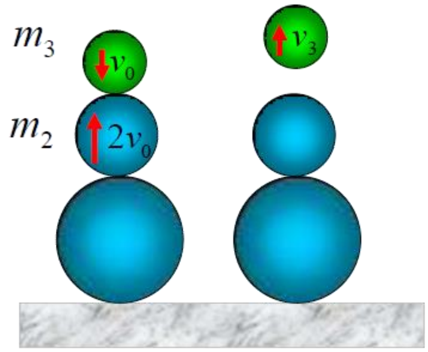
With this, we achieve a **speed doubling for the light sphere**:

$$v'_{2,max} = 2v_0 + v_1' = 2v_0$$

the complete transfer of the 3-fold larger kinetic energy of m_2 provides a 4-fold kinetic energy increase of m_1.

3-sphere astroblaster **(see only on slides):**

Now we can extend this principle to 3 spheres by considering that after collision with m_1 sphere 2 moves with $2v_0$ upwards, while sphere 3 moves before the collision with v_0 downwards:



Momentum conservation provides:

$$m_2 2v_0 - m_3 v_0 = m_2 v'_2 + m_3 v'_3$$

With the equation for the **inversion of relative velocity** we get:

$$v_3' - v_2' = v_2 - v_3 = 2v_0 - (-v_0) = 3v_0$$

and thus:

$$v_3^{'} = 3v_0 + v_2^{'}$$

In case of **full transmission of the kinetic energy to the third sphere** we then have:

$$v'_2 = 0\,; \quad v'_3 = 3v_0$$

Thus, we have for the last sphere a three-fold increase of the free fall velocity and thus a 9-fold increase of its kinetic energy and jump height. Inserting the velocity equation into the momentum equation gives:

$$m_2 2v_0 - m_3 v_0 = m_2 v'_2 + m_3(3v_0 + v'_2)$$

and after transformation:

$$v_2^{'} = \frac{2m_2 - 4m_3}{m_2 + m_3} v_0$$

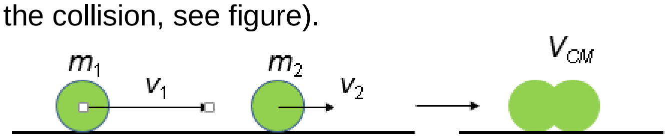
Thus, the condition for full transmission of kinetic energy to m_3 is:

$$m_2 = 2m_3$$

This is the most energy efficient solution, where ideally all energy gets transferred to the last mass. If one makes the lower masses much more heavy than the corresponding upper masses than one can even get a larger velocity increase (where the initial upward velocity gets inverted).

C) Perfect inelastic collision

After studying elastic collisions, we want to look at a perfect inelastic collision, in which the two particles travel together after the collision (i.e. there is no elastic component of the deformation process that underlies the collision, see figure).



Experiments: Perfect inelastic collision between two wagons on the air track **+** collision between two sand bags mounted in a pendulum configuration

Here we have **only momentum conservation**, since we do not know how much kinetic energy is converted into heat during the collision

$$\underbrace{\sum_i m_i v_i}_{before} = \underbrace{MV_{CM}}_{after} = P = \text{const}$$

The momentum conservation provides a simple equation, since after collision both particles are unified in the center of mass. Inserting the total mass $M = m_1 + m_2$ provides:

$$V_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

With this we can calculate the produced heat during the collision from the reduction of the kinetic energy (remember the 99.9% energy loss for the bullet):

$$Q = \underbrace{\left(\frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2\right)}_{E_{k1}+E_{k2}} - \underbrace{\frac{m_1 + m_2}{2}\left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right)^2}_{\frac{M}{2}V_{CM}^2}$$

This equation can be simplified (**not part of lecture**):

$$Q = \frac{1}{2}\left[(m_1 v_1^2 + m_2 v_2^2) - \frac{1}{m_1 + m_2}(m_1 v_1 + m_2 v_2)^2\right]$$

$$Q = \frac{1}{2(m_1 + m_2)}[(m_1{}^2 v_1^2 + m_1 m_2 v_1^2 + m_1 m_2 v_2^2 + m_2{}^2 v_2^2) - (m_1{}^2 v_1^2 + 2m_1 m_2 v_1 v_2 + m_2{}^2 v_2^2)]$$

$$Q = \frac{1}{2(m_1 + m_2)}[m_1 m_2 v_1^2 - 2m_1 m_2 v_1 v_2 + m_1 m_2 v_2^2]$$

and we get:

$$Q = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(v_1 - v_2)^2$$

for the **external heat that is released** (added if negative). We can again distinguish different cases (only slides):

- Collision onto an object at rest with $\boldsymbol{v_2} = \mathbf{0}$:

  $$V_{CM} = \frac{m_1}{m_1 + m_2} v_1$$

  $$Q = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_1^2 = \frac{m_2}{m_1 + m_2} E_{k1}$$

  i.e. if $m_2 \gg m_1$ (e.g. ballistic pendulum) almost all kinetic energy gets converted into heat

- Head-on collision of two objects with opposite velocities with $\boldsymbol{v_2} = -\boldsymbol{v_1}$ and $\boldsymbol{m_1} = \boldsymbol{m_2}$**:** for which we get:

  $$V_{CM} = \frac{mv_1 - mv_1}{2m} = 0$$

  since both momenta add up to zero such that all kinetic energy gets converted into heat:

  $$Q = \frac{1}{2}\frac{mm}{m + m}(v_1 - (-v_1))^2 = \left(\frac{1}{4} 4\, m v_1^2\right) = E_{tot}$$

These equations explain the **terrible impact of head-on collisions and crashes with immobile stationary objects (walls, trees etc.)**, where all energy goes into the destructive process.

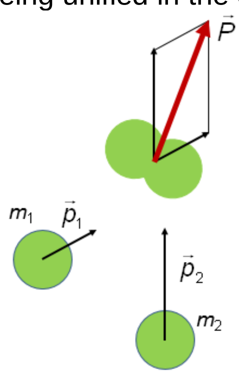
Movie: Car crash advertisement

3) Collisions in in two and three dimensions

For looking at collisions in 2D or in 3D, we can use an analogous approach as for 1D. The momentum conservation holds now for each vector component, i.e. each direction of the coordinate system. Thus, in 3D we get 3 equations momentum conservation while energy conservation provides still a single equation.

A) Perfect inelastic collision

A perfect inelastic collision can this way be very simply described. We have only momentum conservation with the collision object being unified in the center of mass after the collision.

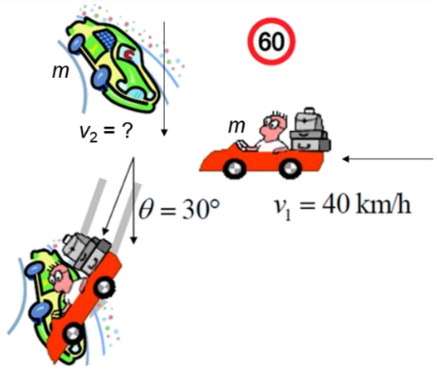


Momentum conservation thus provides:

$$\vec{P} = \sum_i m_i \, \vec{v}_i = M\vec{V}_{CM}$$

Thus, if all initial velocities are known, then the final velocity of the center of mass is fully determined.

Example: Two cars of equal mass shall approach each other at a crossroad in perpendicular directions. After a crash leading to a perfect inelastic collision, they move as a joint collision object at an angle of 30° with respect to the direction of the fast car (see slide). The slower car had a speed of 40 km/h. Can we tell whether the 2nd car obeyed the speed limit of 60 km/h?



We chose a convenient coordinate system along the motion direction of each car and write down the momentum equation:

$$\vec{P} = m\begin{pmatrix} v_1 \\ 0 \end{pmatrix} + m\begin{pmatrix} 0 \\ v_2 \end{pmatrix} = m\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2m\vec{V}_{CM}$$

We get then for the velocity of the collided cars:

$$\vec{V}_{CM} = \begin{pmatrix} v_1/2 \\ v_2/2 \end{pmatrix}$$

The angular direction of the final velocity is given by:

$$\tan\theta = \frac{V_{cmx}}{V_{cmy}} = \frac{v_1/2}{v_2/2}$$

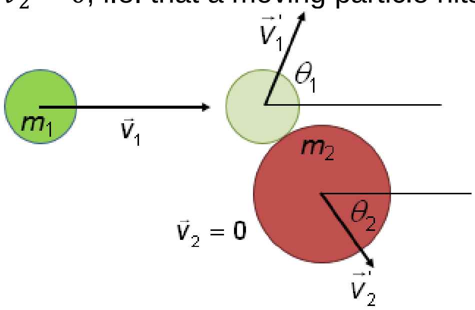
such that we get for the velocity of the 2nd car:

$$v_2 = \frac{v_1}{\tan\theta} = \frac{\sqrt{3}/2}{1/2} v_1 = \sqrt{3}\, v_1 = 69\ km/h$$

Thus, the 2nd car was violating the speed limit.

B) Perfect elastic collision

In case of a perfect elastic collision in 2D, we still have two separately moving objects after the collision. However, we can additionally employ energy conservation to extract possible solutions Let us calculate a perfect elastic collision between two particles and for simplicity assume that $v_2 = 0$, i.e. that a moving particle hits a resting one:



Momentum conservations has to be considered in this case by components:

$$\vec{P} = \sum_i m_i \vec{v}_i = \sum_i m_i \vec{v}_i' = \text{const.}$$

With θ_1 and θ_2 being the angles of the velocity vectors after the collision with respect to the direction of the incoming particle we can write:

$$\begin{vmatrix} \text{x:} \\ \text{y:} \end{vmatrix} \begin{cases} m_1 v_{1x} = m_1 v_{1x}' + m_2 v_{2x}' = m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2 \\ 0 \quad\quad = m_1 v_{1y}' + m_2 v_{2y}' = m_1 v_1' \sin\theta_1 + m_2 v_2' \sin\theta_2 \end{cases}$$

Energy conservation demands:

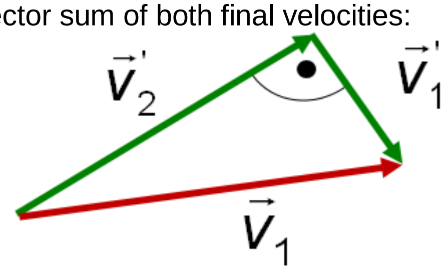
$$\frac{m_1}{2} v_1^2 = \frac{m_1}{2} v_1'^2 + \frac{m_2}{2} v_2'^2$$

These are 3 equations, but we have 4 unknown variables (both final speeds and both angles). Thus, we must know one additional variable (e.g. one of the resulting angles or details of the impact itself). Below we will see that one can also use additional knowledge about the collision process to solve the problem at least numerically.

Let us first consider the **special case $\boldsymbol{m_1 = m_2 = m}$.** Here we can divide the upper equations by m such that momentum conservation yields:

$$\vec{v}_1 = \vec{v}_1{}' + \vec{v}_2{}'$$

, i.e. the initial velocity is the vector sum of both final velocities:

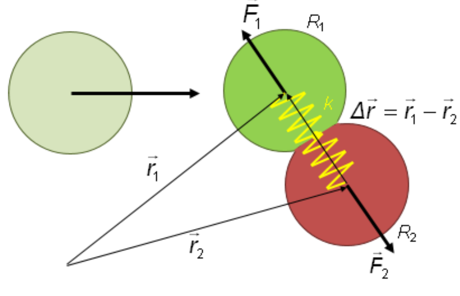


Energy conservation yields

$$v_1{}^2 = v_1'^2 + v_2'^2$$

According to the Pythagorean theorem, v_1' must be perpendicular to v_2' to fullfill this equation for the triangle that is formed by the three velocity vectors. From this we can conclude that **for equal masses, both particles will fly in perpendicular directions away from each other**.

Numeric simulation of an elastic collision in 2D between two spherical particles:
To fully predict an elastic collision process in 2D, e.g. as we have in pool billiards, we must have additional knowledge about the interaction between the two colliding particles, i.e. the forces that occur during the elastic deformation of the particles during the collision. Only this will finally allow to extract the deflections after the collision.
To simplify life, we will in the following assume that we have two spheres and that there is no friction occurring for sliding movements of the two surfaces relative to each other. This means we have **only elastic forces perpendicular to the surfaces**, that act **along the line that connects the two sphere centers**.



We then can describe the elastic forces that occur once the two spheres get in contact. It will be **zero if there is no contact** and will after the contact **increase with decreasing distance between the sphere centers**. We will here use a linear (Hookean) force increase, since this is the first order approximation for such a process. We can thus write for the force onto sphere 1 using $\Delta\vec{r} = \vec{r}_1 - \vec{r}_2$ as the difference vector between the two sphere centers:

$$\vec{F}_1 = m_1 \frac{d^2\vec{r}_1}{dt} = \begin{cases} 0 & \text{if } \Delta r > R_1 + R_2 \\ k \cdot [\Delta r - (R_1 + R_2)] \cdot \overbrace{\frac{\Delta\vec{r}}{\Delta r}}^{\hat{e}_{\Delta\vec{r}}} & \text{if } \Delta r \leq R_1 + R_2 \end{cases}$$

k is here the spring constant that results from the material properties and the radius of the two spheres. The difference term between Δr and $R_1 + R_2$ ensures that the force is zero when the spheres just touch and only increases for smaller distances. The ratio on the right-hand site defines the direction of the force. It is the unit vector of $\Delta\vec{r}$ and is thus directed away from sphere 2 towards sphere 1. From the geometry but also from Newton's third law the force onto sphere 2 is simply:

$$\vec{F}_2 = m_2 \frac{d^2\vec{r}_2}{dt} = -\vec{F}_1$$

i.e. the unit vector flips just the sign. It is important to note that k is always the effective spring constant of the interactions of both spheres even if the spheres have a different stiffness. It is calculated by a serial connection of two springs and for the sphere center movement it does matter which of the two particles would be softer than the other particle (just the deformation is different ☺).

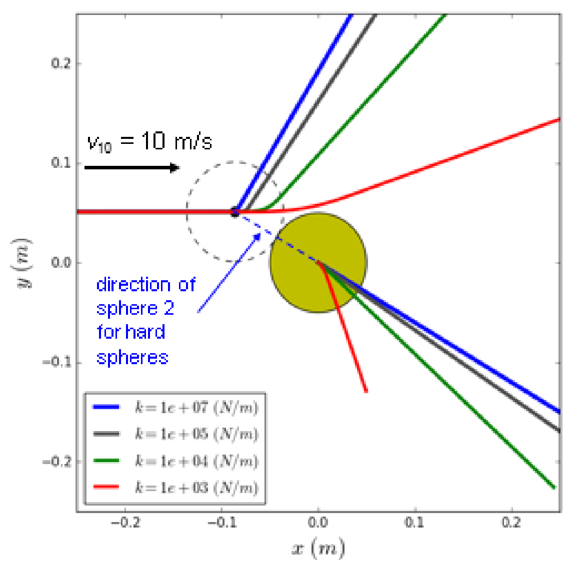
From the force equations, we can calculate the position and time dependent accelerations for both spheres and can thus stepwise integrate the two velocity and position vectors for small time intervals Δt. Since we are in 2D we have in total 8 parallel equations that need to be integrated on the computer.

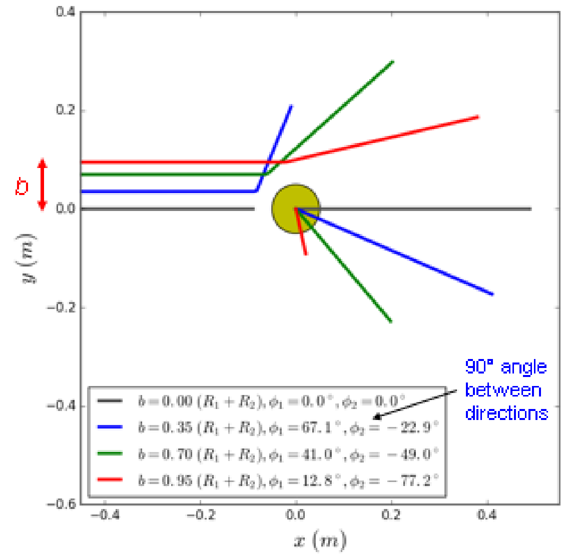
It is informative to play with different settings of the collision process:

- central collision for hard and soft sphere, where one sees that the interaction time depends on the stiffness
- non-central collision for different spring constants (different stiffness of spheres), where one sees that the final direction of both sphere depends on the effective stiffness.
- non-central collision with different impact parameters *b* which is the fractional distance of the initial path of sphere 1 from the position of sphere 2. As more decentralized the collision, as less sphere 1 changes its velocity

When looking at the obtained trajectories for different spring constants one sees **that for hard spheres, sphere 2 follows the line between the first contact point and its sphere center**. This is sensible, since the interaction is short and thus restricted to this direction such that also the momentum transfer occurs in this direction. For softer spheres, the sphere centers approach each other considerable, such that the direction of the repulsive force changes during the collision process providing a deflection of sphere 2 at large angles. This is also seen in the curvature of the obtained trajectories.

When looking again at the trajectories for different impact parameters (hard spheres, plotted for limited times to see the final velocities) one sees nicely that for spheres of equal mass the deflection of both spheres is always perpendicular to each other and that for increasing *b* sphere 1 is less and less affected.

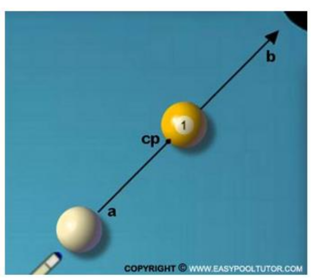


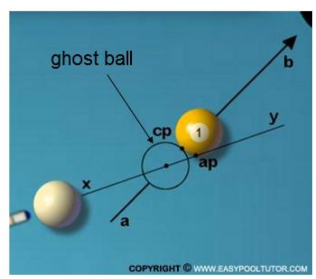


Similar plots are obtained in atomic physics for the scattering of charged particles in a Coulomb potential (e.g. of atomic nuclei), where it is called **Rutherford scattering**. In this case we have a different interaction potential but the same physical principle.

Experiment: Billiards with two coins. We can use two coins to look at such a collision process in real. If the coins are equally large their deflection occurs perpendicular to each other.

Physics of billiards:
Our knowledge which we now gained on the elastic collisions in 2D is routinely used by **pool billiards** players. They determine the initial contact point using the **ghost sphere technique** by putting the cue from the pocket in the billiard table over the center of the target ball. The incoming ball has to hit the target ball at this contact point to get the desired direction. *Watch out*: the contact point is not the aiming point for non-central collisions.

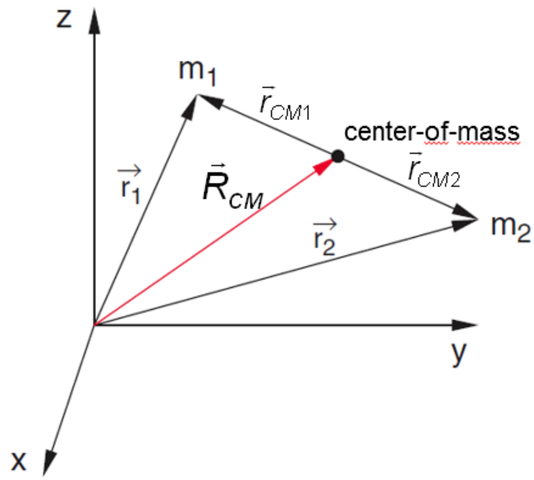


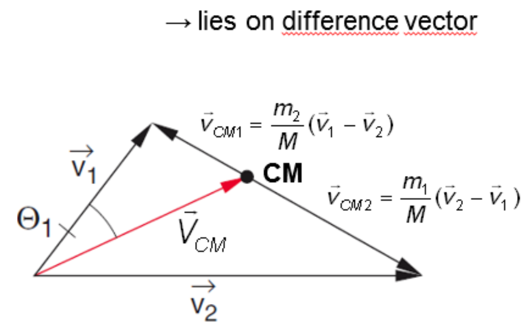


Movie: Interactions & dissipated energy between pool balls and pool cue

C) Elastic collisions in CMS (not part of lecture?)

Looking at elastic collisions between two particles in 2D within the center-of-mass system is surprisingly simple. This we will study in the following by remembering that the center of mass of two point masses lies on the difference vector:





$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} = \vec{r}_2 + \frac{m_1}{M}(\vec{r}_1 - \vec{r}_2)$$

with $M = m_1 + m_2$ and $\vec{r}_i = \vec{R}_{CM} + \vec{r}_{CMi}$. Differentiation of this equation gives for the velocity vector of the center of mass:

$$\vec{V}_{CM} = \vec{v}_2 - \frac{m_1}{M}(\vec{v}_1 - \vec{v}_2)$$

Thus, also the end of $\vec{\boldsymbol{V}}_{\boldsymbol{CM}}$ **lies on the difference vector of both velocities and the velocities in the CMS have opposite directions**. (Note that the position vectors have different directions than the velocity vectors).

Opposite velocity vectors in the CMS make sense, since the total **momentum in the CMS adds up to zero** before and after the collision:

$$\vec{p}_{CM1} = -\vec{p}_{CM2} \quad \text{and} \quad \vec{p}'_{CM1} = -\vec{p}'_{CM2}$$

Now we look at energy conservation in the CMS for the collision. First, we rewrite the kinetic energy of a single point mass using the linear momentum:

$$E_k = \frac{m}{2}v^2 = \frac{m}{2}\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m}$$

The kinetic energy of a particle system was the sum of the kinetic energy of the center of mass and the kinetic energy within the center of mass. Since for a collision we have no external forces, the kinetic energy of the center of mass is conserved und thus also **the kinetic energy within the CMS is conserved**

$$\underbrace{E_k = \frac{p_{CM1}^2}{2m_1} + \frac{\overbrace{p_{CM2}^2}^{p_{CM1}^2}}{2m_2}}_{\text{before}} \underbrace{= \frac{p'^2_{CM1}}{2m_1} + \frac{\overbrace{p'^2_{CM2}}^{p'^2_{CM1}}}{2m_2} = E'_k}_{\text{after}}$$

Using that before and after collision the momenta of both particles have equal magnitudes, we can rewrite the energy equation to:

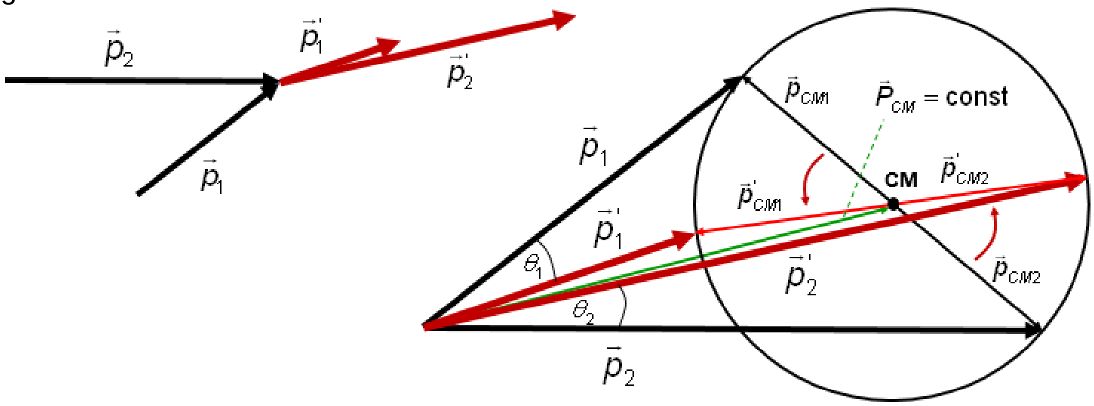
$$p_{CM1}^2\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right) = p'^2_{CM1}\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)$$

and divide by the common mass term. This yields:

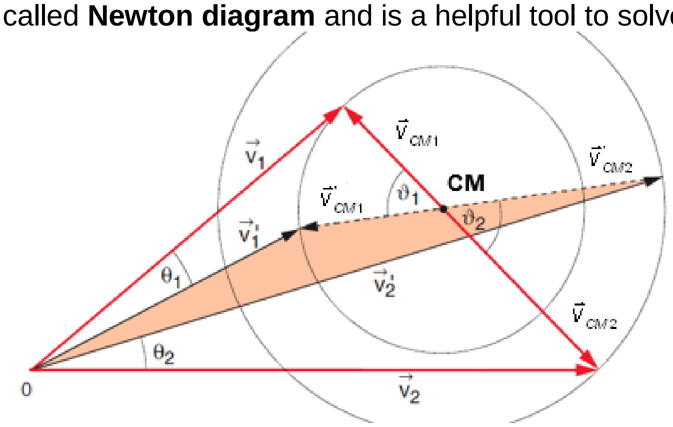
$$p_{CM1}^2 = p'^2_{CM1} \quad \Rightarrow \begin{cases} |\vec{p}_{CM1}| = |\vec{p}'_{CM1}| \\ |\vec{p}_{CM2}| = |\vec{p}'_{CM2}| \end{cases} \quad \Rightarrow \begin{cases} |\vec{v}_{CM1}| = |\vec{v}'_{CM1}| \\ |\vec{v}_{CM2}| = |\vec{v}'_{CM2}| \end{cases}$$

In the CMS each mass retains its momentum und thus its original kinetic energy. In the CMS the velocity vectors of each object are simply turned by the same angle.

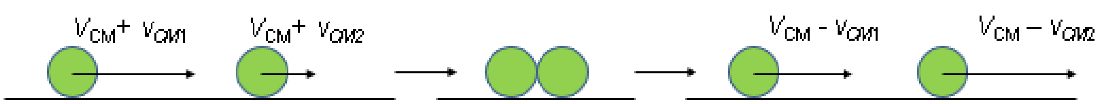
From this one can construct a vector diagram (**see slides**) and get the resulting momenta from geometric considerations:



A similar diagram one can also draw for the velocity vectors, just that here two circles (spheres in 3D) have to be drawn, since the opposing velocity vectors do not have the same absolute value. This is called **Newton diagram** and is a helpful tool to solve collision problems



In **one dimension** one can only use 0° (trivial, no collision) or 180° turns of the CMS vectors. Therefore, the velocities in the CMS of each object are simply reversed by collision.



This can be also seen from the previously used equations where in 1D either the velocity stays the same or changes its sign in order to keep the same absolute value (see slides):

$$p_{CM1}^2 = p_{CM1}^{'2} \quad \Rightarrow \begin{cases} \vec{p}_{CM1} = -\vec{p}_{CM1}^{\,'} \\ \vec{p}_{CM2} = -\vec{p}_{CM2}^{\,'} \end{cases} \quad \Rightarrow \begin{cases} \vec{v}_{CM1} = -\vec{v}_{CM1}^{\,'} \\ \vec{v}_{CM2} = -\vec{v}_{CM2}^{\,'} \end{cases}$$

Lecture 13: Experiments

1. Steel sphere falling onto an anvil covered with different materials (steel, brass, lead)
2. Newton's cradle with two spheres (Pendulum) for $m_1 = m_2$, $m_1 \gg m_2, m_1 \ll m_2$
3. Newton's cradle with many spheres
4. Elastic collision of a steel sphere at a hard block provides reflection i.e. momentum inversion
5. Collision of two sliders with $m_1 \gg m_2$ on an airtrack. Velocity tracking provides that m_2 moves with $2v_1$ after the collision
6. "Astroblaster"
7. Inelastic collision between two sliders at the air track (get tethered to each other by Velcro) and between two sandbags each on a pendulum
8. Billards with two coins: perpendicular deflection after collision