Lecture "Experimental Physics I"
(Prof. Dr. R. Seidel)

Lecture 15

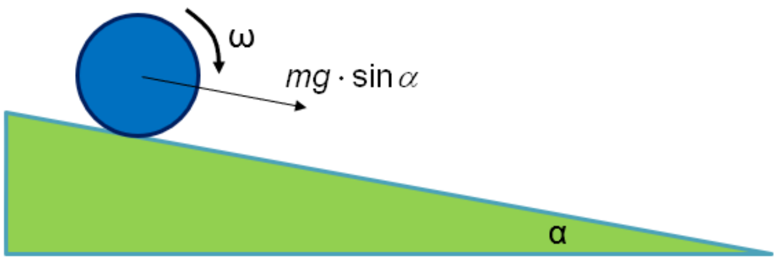
Angular momentum

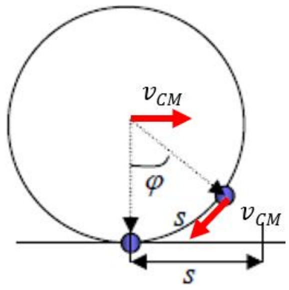
- Torque driven motion
- Angular momentum
- Angular momentum conservation

1) Torque driven motion

In the following we will look at a few applications where torque influences or drives the motion of objects.

A) Rolling motion

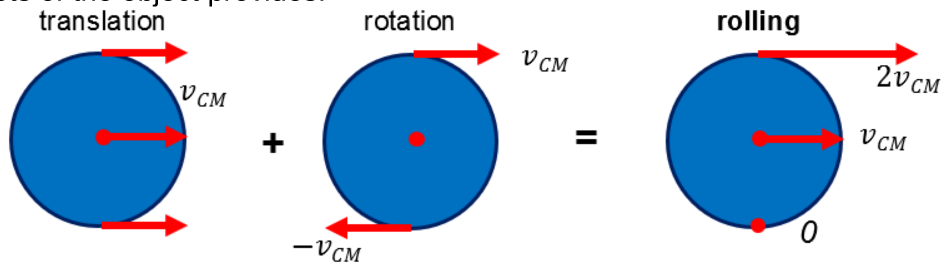




We first look at an object with cylindrical symmetry that is rolling down an inclined plane. Key to understand this motion is to consider that there shall be **no slip between object and surface**. When the object moves by distance ds the arc length on the periphery must thus also be ds:

$$v_{CM} = \frac{ds}{dt} = \frac{ds_{arc}}{dt} = R\frac{d\varphi}{dt} = v_{(t)}$$

Thus, the cylinder undergoes a translation with velocity v_{CM} and at the same time a rotation with tangential velocity v_{CM} at the periphery. **Superimposing the translation and the rotation** at different spots of the object provides:



We have zero velocity at the contact line (since there is no slip), a velocity for the center of mass of v_{CM} and a velocity $2v_{CM}$ for the periphery at the opposite side of the attachment line. This velocity profile is nothing else than a rotation around the contact line in which the rotation axis is constantly moving.

For the equation of motion, we have just to consider the torque around the contact line and set it equal to the inertia due to the rotary motion around this rotation axis:

$$\tau = mg\sin\alpha\, R = I\alpha = (I_{CM} + MR^2)\frac{a}{R}$$

where we used the parallel axis theorem to obtain the moment of inertia for the rotation around the contact line. Transformation provides the linear acceleration of the cylinder:

$$a = \frac{mg\sin\alpha\ R^2}{I_{CM} + mR^2} = \frac{g\ \sin\alpha}{1 + I_{CM}/mR^2}$$

The denominator is a reduction factor compared to the acceleration in case of pure translational motion, since we have to additionally accelerate the rotation of the object. We get for the acceleration of a:

- **solid cylinder**

$$I_{CM} = \frac{1}{2}mR^2 \qquad \Rightarrow \quad a = \frac{g\ \sin\alpha}{1 + 1/2} = \frac{2}{3}g\ \sin\alpha$$

- **hollow cylinder with thin walls**:

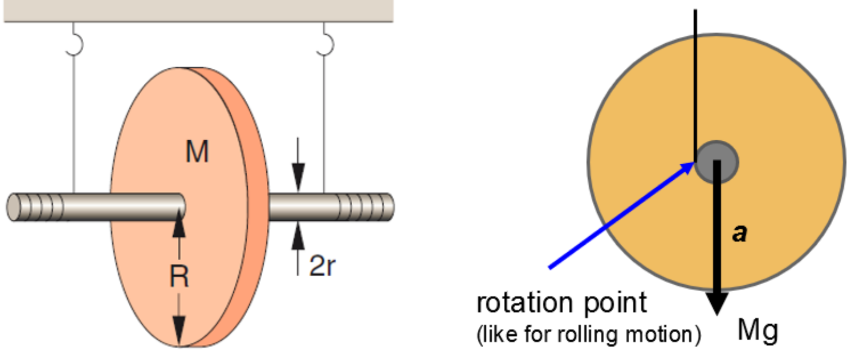
$$I_{CM} = mR^2 \qquad \Rightarrow \quad a = \frac{g\ \sin\alpha}{1+1} = \frac{1}{2} g\, sin\, \alpha$$

The hollow cylinder is resisting stronger the acceleration since its mass distribution is more distal to the object axis such that it has a higher moment of inertia.

Experiment: We qualitatively compare the rolling motion of a filled and a hollow cylinder on an inclined plane to confirm the obtained equations. Notably the acceleration does not depend on the object length nor the mass, such that any filled cylinder that has the same radius has the same acceleration in our experiment.

B) Maxwell's wheel

The Maxwell wheel is a toy that "falls" down with a much smaller acceleration than the free fall, since considerable work needs to be done to accelerate the rotation of the wheel in addition to its translation. At the lowest point it continues to spin in the same direction since it has stored rotational kinetic energy and winds itself up again.



Similar to the rolling motion, we have here a rotation around the tangent point of the cord, such that we can write:

$$\tau = Mg\, r = I\alpha = I\frac{a}{r}$$

The wheel is composed out of a heavy disk and a much lighter rod such that we can approximate the moment of inertia with the moment of the disk. Considering that the rotation occurs around the cord attachment, we can write for the moment of inertia using the parallel axis theorem:

$$I = I_{CM} + Mr^2 = \underbrace{\frac{1}{2}MR^2}_{disk} + Mr^2 = Mr^2\left(1 + \frac{R^2}{2r^2}\right)$$

Inserting provides:

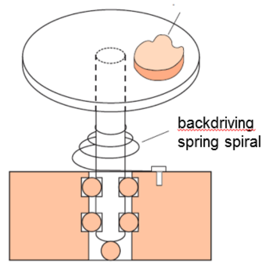
$$a = \frac{gMr^2}{I} = \frac{g}{1 + R^2/(2r^2)}$$

This yields a significant reduction of the free fall acceleration. The large difference arises due to the additional acceleration of the rotational motion of the large disk. In fact, the rotational kinetic energy by far exceeds the translational kinetic energy.

Experiment: Demonstration of the Yo-Yo effect on a Maxwell wheel

C) Rotary oscillations

Like linear oscillations, e.g. for a mass-spring-system, one can have rotary oscillations. An example is a rotary table with a back-driving spring (**see below**) or steel string with an attached mass that can undergo torsional displacements.



The back-driving torque as function of the angular displacement can be in first approximation described by a linear relationship:

$$\tau = -\kappa\varphi$$

where we call κ the **torsional rigidity** with $[\kappa] = Nm$. In absence of additional external forces, an angular displacement causes a backdriving angular acceleration of the rotary table with an associated moment of inertia I. For the equation of motion, we can thus write:

$$\tau = -\kappa\varphi = I\frac{d^2\varphi}{dt^2}$$

which can be transformed to:

$$0 = \frac{d^2\varphi}{dt^2} + \underbrace{\frac{\kappa}{I}}_{\omega^2}\varphi$$

This equation has the same form as the differential equation of the harmonic oscillator. From before we know that is equation is solved by a sinusoidal function:

$$\varphi(t) = \varphi_0 \sin(\omega t + \theta_0)$$

Thus, the system undergoes a harmonic rotary oscillation at a single frequency. In the differential equation, the prefactor of the angle equals the squared angular frequency of the oscillation, yielding:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{\kappa}{I}}$$

Note, that ω is here not the angular velocity of the rotary table but the angular frequency of the oscillation.

Experiment: We will use such a rotary oscillation to proof the distance dependence of the moment of inertia. As oscillator we use a dumbbell arrangement of two equal masses on a rod that is mounted on a vertical torsionally constraint steel string. The square of the oscillation period should be proportional to the moment of inertia of this arrangement:

$$T^2 \propto I \approx MR^2$$

where M is the total mass of the dumbbell weights and R is the distance of each mass from the steel string.

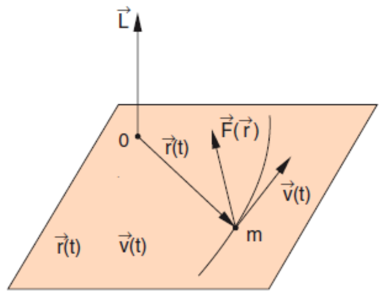
Experiment: We test the parallel axis theorem using a rotary-table oscillator. We first obtain the period for a mass M mounted at the center of the rotary disk and then displace the mass from the center by distance $R.$ According to the parallel axis theorem, the moment of inertia und thus the squared period of the oscillation should increase with the square of the distance of the mass from the center:

$$T^2 \propto I_{CM} + MR^2$$

2) Angular momentum

A) Definition & conservation of the angular momentum

We already defined previously the angular momentum for planetary motion. We saw that it was a conserved quantity as emerging from Kepler's 3rd law. We now want to expand our knowledge about the angular momentum and derive an equivalent to Newton's 2nd law $\vec{F} = d\vec{p}/dt$ for rotational motions. With this, we will be able to define the torque as a change of angular momentum. Let us first look at a point-like mass m with velocity $\vec{v}$ at position $\vec{r}$ from a given (rotation) center 0:



If a force is acting on the mass point, we can write for the torque on the particle acting around point 0:

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \frac{d(m\vec{v})}{dt}$$

We now expand the right side of the cross product by adding **a term that equals zero** in particular the cross product of the velocity vector with itself and apply the product rule for differentiation in the opposite direction such that $\vec{r}$ can be placed inside of the derivative:

$$\vec{\tau} = m\left(\vec{r} \times \frac{d\vec{v}}{dt} + \underbrace{\frac{d\vec{r}}{dt} \times \vec{v}}_{0}\right) = \frac{d}{dt}(m\vec{r} \times \vec{v}) = \frac{d}{dt}(\underbrace{\vec{r} \times \vec{p}}_{\vec{L}})$$

By **defining the angular momentum** as:

$$\vec{L} = m\vec{r} \times \vec{v} = \vec{r} \times \vec{p}$$

we get:

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt}$$

Thus, the **net external torque equals the change of the angular momentum per time** (in analogy to Newton's 2nd law). In absence of any torque have thus an **conservation of the angular momentum**:

$$0 = \frac{d\vec{L}}{dt}$$

In absence of an external torque, the angular momentum will be conserved.

Let us briefly check whether this expression for torque agrees with angular momentum conservation during planetary motion. We can write for the torque acting on the planet from the gravity force if the sun is in the origin of the coordinate system:

$$\vec{\tau} = \vec{r} \times \left(-G\frac{mM}{r^2}\underbrace{\frac{\vec{r}}{r}}_{\hat{e}_r}\right) = -G\frac{mM}{r^3}\vec{r} \times \vec{r} = 0 = \frac{d\vec{L}}{dt}$$

Thus, **in any central force field** (where the force points along the radial vector), **the angular momentum remains conserved**!

B) Angular momentum and angular velocity

So far, the definition of the angular momentum contains still the quantities from translational motion. We will now rewrite the definition to obtain an expression for the angular momentum based on quantities describing rotations. To this end we transform the relation we derived for the angular velocity:

$$\vec{\omega} = \frac{1}{r^2}\vec{r} \times \vec{v} \quad \Rightarrow \quad \vec{r} \times \vec{v} = r^2 \vec{\omega}$$

The angular momentum contains the same vector product, such that insertion into the definition of $\vec{L}$ gives:

$$\vec{L} = m\,\vec{r} \times \vec{v} = \underbrace{mr^2}_{I \text{ of point mass}} \vec{\omega}$$

Thus the angular momentum of a point mass is proportional to the vector of the angular velocity. The proprtionality factor is the nothing else than the moment of inertia of the point mass. Thus, we obtain:

$$\vec{L} = I\vec{\omega}$$

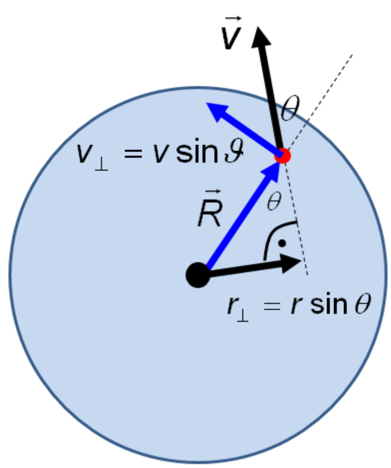
This is an analogoes expression as for the linear momentum $\vec{p} = m\vec{v}$, confirming that the angular momentum is an equivalent to the linear momentum in case of rigid body rotations. For a point mass, both **$\vec{\omega}$ and $\vec{L}$ point along the rotation axis**, i.e. they are **perpendicular to the plane of rotation.** The absolute value of the angular momentum can be written as

$$\left|\vec{L}\right| = m \cdot |\vec{r} \times \vec{v}| = m\,r\,v\,sin\,\theta$$

According to the sketch below this is equivalent to:

$$\left|\vec{L}\right| = m\,r\,v_\perp = m \cdot r_\perp \cdot v$$

where $v_\perp$ is the tangential velocity component (perpendicular to $\vec{r}$) and $r_\perp$ is the radial component of $\vec{r}$ perpendicular to $\vec{v}$.



Like for the torque, the angular momentum is either provided by the **full distance times the tangential component of the velocity or the full velocity times the effective radius** of the motion path**.**

From the latter it is obvious that **any point mass passing by a reference point on a straight trajectory has a conserved angular momentum in absence of force**, since the effective radius remains constant.

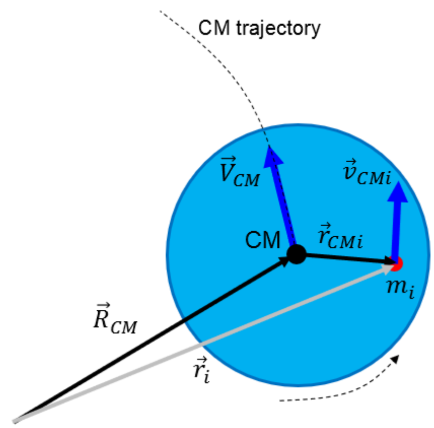
Important notes (on slide):

- The angular momentum is always defined with respect to a given point in space! It can be any point in space for which always the conservation law holds!

- Angular momentum calculations are done by components due to the vectorial character of $\vec{L}$.

C) Angular momentum with respect to the CM

So far, we just looked at the angular momentum of a point mass. In the following we will make sure that our equations can also be used for systems of point masses and rigid bodies. We start by looking at a system of mass points (e.g. planets moving around the sun). The center of mass shall have a give trajectory and the mass points shall have an individual trajectory around the center of mass:



The angular momentum with respect to a given point in space is then given by the sum:

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i = \sum_i m_i (\vec{R}_{CM} + \vec{r}_{CMi}) \times (\vec{V}_{CM} + \vec{v}_{CMi})$$

which can be transformed to:

$$\vec{L} = \vec{R}_{CM} \times \vec{V}_{CM} \sum_i m_i + \vec{R}_{CM} \times \underbrace{\sum_i m_i \vec{v}_{CMi}}_{0} + \underbrace{\left(\sum_i m_i \vec{r}_{CMi}\right)}_{0} \times \vec{V}_{CM} + \sum_i m_i \vec{r}_{CMi} \times \vec{v}_{CMi}$$

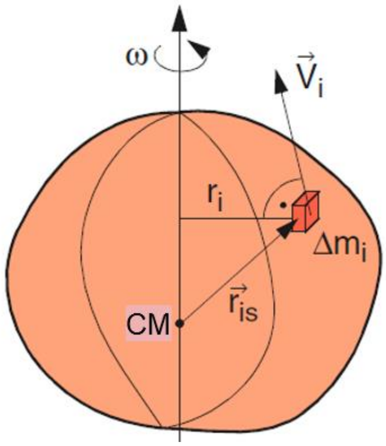
The two terms in the center of the equation become zero, since one defines the position of the center-of-mass within the CMS and the other the total momentum within the CMS. Thus, we get:

$$\vec{L} = M\,\vec{R}_{CM} \times \vec{V}_{CM} + \sum_i m_i\,\vec{r}_{CMi} \times \vec{v}_{CMi}$$

The angular momentum of a mass system is the angular momentum of the center of mass moving around the chosen reference point plus the angular momentum for the rotation of the system around its center of mass.
An example is the total angular momentum of the earth, which contains the angular momentum for the annual orbiting of the center of mass of the earth around the sun as well as the angular momentum of the daily rotation of the earth around its center of mass

D) Angular momentum of a rigid body

Now we make use of further constraints that act in a rigid body to derive a simple expression for its angular momentum about a given rotation axis, such as the center of mass:



Let us dissect a rigid body into small mass elements Δm_i. The angular momentum along a fixed rotation axis for mass segment Δm_i is given by:

$$\vec{L}_i = \Delta m_i \underbrace{\vec{r}_{i_\perp} \times \vec{v}_i}_{\vec{r}_{i_\perp}^{\,2}\vec{\omega}}$$

where we used the fact that for a rigid body all mass segments rotate with the same angular velocity (in contrast to the different planets in our solar system). The total angular momentum is then provided by:

$$\vec{L} = \sum_i \vec{L}_i = \left(\sum_i \Delta m_i \vec{r}_{i_\perp}^{\,2}\right)\vec{\omega}$$

The term in the bracket is nothing else than the moment of inertia of the object

$$\vec{L} = I\vec{\omega}$$

which is the same expression we had for a point mass and thus convenient to take.
Watch out: This is only valid for the angular momentum about a fixed rotation axis or in case of a freely spinning body for the rotation about certain (distinguished) rotation axes. Generally, the angular velocity does not align with the angular momentum when as we will see later when discussing tops.

Combining the two previous expressions allow us now to calculate the total angular momentum of a rigid body on more complex trajectories, (such as the total angular momentum of the earth on its path around the sun):

$$\vec{L} = M\vec{R}_{CM} \times \vec{V}_{CM} + I_{CM}\vec{\omega}$$

For an object that rotates around an axis at a fixed distance a to the center of mass we can write:

$$\vec{L} = M\vec{a} \times \vec{V}_{CM} + I_{CM}\vec{\omega} = M\,a^2\vec{\omega} + I_{CM}\vec{\omega} = (Ma^2 + I_{CM})\vec{\omega}$$

in agreement with the parallel axis theorem. This shows that all our definitions are consistent.

D) Angular momentum conservation for rigid bodies

Using the same derivation as for the point mass one can extend the principle of angular momentum conservation also to rigid bodies. We had that

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

which in absence of external torque provides angular momentum conservation:

$$0 = \frac{d\vec{L}}{dt}$$

i.e. **when no torque is acting on a rigid body then its angular momentum remains conserved.**

In the following we will study a couple of **experiments** that show such an angular momentum conservation:

Pirouette:
For a pirouette (e.g. by an ice skater) one starts rotating with a radially expanded mass distribution. When the mass distribution is suddenly narrowed, the angular velocity is simultaneously increasing.

One can easily explain the increase in the angular velocity by momentum conservation, which demands equal initial and final angular momentum:

$$L_i = I_i\omega_i = I_f\omega_f = L_f$$

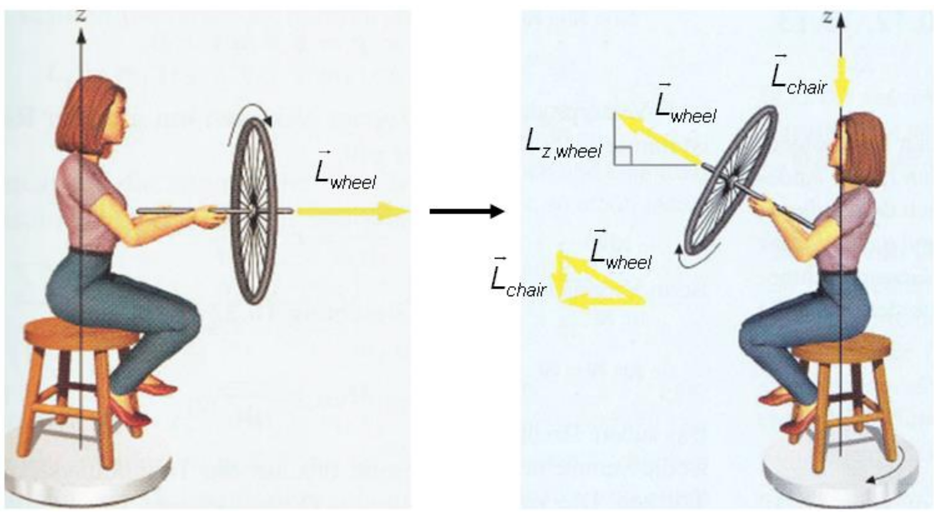
Transformation of the equation provides the final angular velocity:

$$\omega_f = \frac{I_i}{I_f}\omega_i > \omega_i$$

where we get an increase of the final angular velocity, since the initial moment of inertia (more extended mass distribution) is larger than the final moment of inertia ($I_i > I_f$).

Angular momentum transfer by a spinning wheel

A spinning wheel with horizontal rotation axis is passed to a person sitting on a rotary chair. The person is now lifting the wheel and brings its rotation axis into a vertical position. One observes that the person on the chair is now spinning in the opposite direction of the wheel rotation until friction stops it.



One can easily explain the experiment considering angular momentum conservation along the vertical (z-)direction. The angular momentum along the z-axis is zero before lifting the wheel. Since the chair can **freely rotate around the z-axis there is no torque acting around this axis, such that the angular momentum along z must be conserved**. This provides:

$$L_z = const. = \underbrace{0}_{\text{before}} = \underbrace{L_{z,wheel} + L_{chair}}_{\text{after turning}}$$

When we lift the wheel then its angular momentum gains an increasing z-component. This must be compensated with an increasing negative angular momentum of the person and the chair. When the **wheel axis is vertical** then its whole angular momentum is along the z-direction and we get:

$$L_{chair} = -L_{wheel}$$

Obviously, only the angular momentum along z is conserved, since the horizontal angular momentum decreases from L_{wheel} to 0 during that process. For the horizontal direction there can be a non-zero **horizontal torque that tries to tilt the rotation axis** of the chair. This is best seen for the almost vertical wheel axis. When tilting, the change in momentum is directed horizontally and causes a horizontal torque that is felt by the hands (seen by own experience with a turning bicycle wheel).

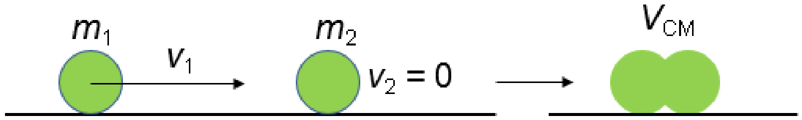
Rotary recoil

The same principle as before can be used to explain rotary recoil. A person on a rotary chair starts to drive a wheel with its axis along the z-direction using a drilling machine. The person is then turning in the opposite direction of the drilling machine. Since the angular momentum along z was zero before driving, it must be zero afterwards if the chair is freely rotatable. Thus, the **person and the chair must have opposite angular momentum after the wheel acceleration**.
One can also explain this experiment by looking at the torque. The change in **angular momentum of the wheel is due to the torque** from the drilling machine. Due to Newton's 3rd law there must be an **oppositely oriented torque that the wheel exerts onto the drilling machine** and thus the person gets equivalently accelerated.

Movie "Gravity": (see slides) This movie provides some stunning examples of angular and linear momentum conservation. If nothing stops the persons from moving/rotating they will keep moving and rotating and rotating and rotating...

Inelastic rotary collision

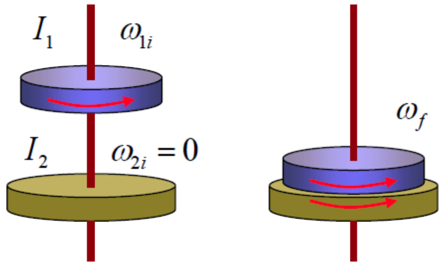
If collisions involve rotating objects (particularly inelastic collisions) we have also to look at angular momentum conservation. For a central perfect inelastic collision we had before: (see slides)



with V_{CM} being the velocity of the center of mass, i.e. the final velocity of the collision object:

$$V_{CM} = \frac{m_1}{m_1 + m_2} v_1$$

A perfect rotary inelastic collision can, for example, occur between one rotating and one non-rotating disk that get into tight contact and finally spin at the same angular velocity:



Angular momentum conservation gives in this case:

$$L_z = const = I_1\omega_{1i} = I_1\omega_f + I_2\omega_f$$

From which the final angular velocity can be obtained after transformation:

$$\omega_f = \frac{I_1}{I_1 + I_2}\omega_{1i}$$

which has the equivalent form to the central inelastic collision for translational motion.

Lecture 15: Experiments

1) Qualitatively comparison of the rolling motion of a filled and a hollow cylinder on an inclined plane. Demonstration that the acceleration does not depend on the object length nor the mass, such that any filled cylinder that has the same radius has the same acceleration in our experiment.
2) Maxwell's wheel – Yo-Yo
3) Torsion pendulum: rotary oscillator made of an dumbbell arrangement of two equal masses on a rod that is centrally mounted on a torsionally-constraint steel wire. Recording the oscillation period as function of the mass and the distance from the rotation axis is used to validate the formula for the moment of inertia
4) Oscillations of a rotary table to test the parallel axis theorem. A large mass on the table is more and more displaced from the rotation axis. We demonstrate that the squared oscillation period is proportional to the squared displacement of the mass from the rotation axis.
5) Rotary chair: Angular momentum conservation during a pirouette
6) Rotary chair: Angular momentum transfer when tilting a spinning wheel
7) Rotary chair: Rotary recoil when powering a wheel with a drilling machine