

Math for Students of Faculty of Physics
Mathematics 1
Linear Algebra and Calculus of Functions
of One Variable

Dr. Tim Shilkin

Leipzig University, Winter Semester 2023/24

January 22, 2024

References

The programm of the course “Mathematics 1. Linear Algebra and Calculus of Functions of One Variable” is standard and the related material can be found practically in any book devoted to the calculus of functions of one variable and the linear algebra.

Books recommended in the official syllabus:

<https://www.physgeo.uni-leipzig.de/en/studying/courses-of-study/>

- Serge Lang: Linear Algebra, Springer
- Serge Lang: A First Course in Calculus, Springer
- Kenneth A. Ross: Elementary Analysis, Springer
- Stephen Abbott: Understanding Calculus, Springer

Books I frequently use myself to prepare my lectures (to be completed):

Logic, sets, functions

- Keith Devlin, Sets, Functions, and Logic. An Introduction to Abstract Mathematics, CRC.
- Steven Galovich, Introduction to Mathematical Structures, HBJ, Academic Press.

Calculus of functions of one real variable

- Vladimir A. Zorich, Mathematical Analysis I, Springer.

Linear Algebra

- Steven J. Leon, Linear Algebra with Applications, Prentice Hall.

Contents

1	One-dimensional calculus	5
1.1	Basic concepts	5
1.2	Limit of a sequence	11
1.3	Subsequences	14
1.4	Limit of a function	15
1.5	Indeterminate forms and important examples	16
1.6	Infinite limits and asymptotes	17
1.7	Continuity	19
1.8	Differentiability and derivatives	22
1.9	Chain rule	24
1.10	The basic theorems of differential calculus	26
1.11	Investigation of functions using derivatives	27
1.12	Primitive function and indefinite integral	31
1.13	Basic methods for finding primitives	33
1.14	Definite integral	38
1.15	Properties of definite integrals	42
1.16	Applications of definite integrals	44
1.17	Comparison of infinitesimals and Taylor's formula	48
1.18	Improper integrals	51
1.19	Infinite series	54
2	Matrices and determinants	57
2.1	Gaussian elimination method	57
2.2	Matrix algebra	63
2.3	Square matrices	66
2.4	Matrices and linear systems	68

1. Notation

\forall means “for all”

\exists means “there exists”

$\exists!$ means “there exists a unique”

$:$ means “such that”

$:=$ means “denote by”

\mathbb{N} is a set of all natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$

\mathbb{Z} is a set of all integer numbers, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

\mathbb{Q} is a set of all rational numbers, $\mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N} \right\}$

\mathbb{R} is a set of all real numbers

an interval $(a, b) \subset \mathbb{R}$ is the set of all real x such that $a < x < b$

an interval $[a, b] \subset \mathbb{R}$ is the set of all real x such that $a \leq x \leq b$

$\operatorname{tg} x := \tan x$ is the tangent of x , $\operatorname{ctg} x := \cot x$ is the cotangent of x

$\operatorname{arctg} y := \arctan y$ is the arctangent of y , $\operatorname{arcctg} y := \operatorname{arccot} y$ is the arccotangent of y

1 One-dimensional calculus

1.1 Basic concepts

1. Logic

- Logical operations: and, or, negation, implication, equivalence.
- Negation of “and” and “or”. Negation of an implication.
- Equivalence of the implication to the contrapositive. Necessary and sufficient condition.

2. Proof by contradiction

EXAMPLE. $\sqrt{2}$ is not rational.

PROOF. By contradiction, assume $\sqrt{2}$ is rational. Then there exists integers $n, m \in \mathbb{Z}$:

$$\sqrt{2} = \frac{n}{m}, \quad \text{greatest common divisor of } n \text{ and } m \text{ is } 1.$$

Taking a square of this relation we obtain

$$2 = \frac{n^2}{m^2} \quad \Longleftrightarrow \quad n^2 = 2m^2$$

$2m^2$ is even and hence n^2 even. But then n itself even, i.e.

$$n = 2k \quad \text{for some } k \in \mathbb{Z}.$$

Then we obtain

$$n^2 = 2m^2 \quad \Longleftrightarrow \quad (2k)^2 = 2m^2 \quad \Longleftrightarrow \quad 4k^2 = 2m^2 \quad \Longleftrightarrow \quad 2k^2 = m^2$$

Then m^2 is even and hence m itself is even. As both n and m are even, this contradicts to the assumption that the greatest common divisor of n and m is 1. Hence our assumption that $\sqrt{2}$ is rational can not be true. Hence $\sqrt{2}$ is irrational. \square

3. Quantifiers

- sentences and predicates
- $\forall, \exists, \exists!$ (there exists a unique)
- Negation of the statement with quantifiers

EXAMPLE. Construct the explicit negation of the statement

$$\left(\forall n \in \mathbb{N} \quad \exists x \in (1, +\infty) \quad \forall y \in [0, 1] \quad x + y > n^2 \right)$$

Determine which of the statements (the original one or its negation) is true.

4. Proof by induction

$$\left(P(1) \text{ is true } \right) \wedge \left(\forall n \in \mathbb{N} \quad P(n) \implies P(n+1) \right) \implies \forall n \in \mathbb{N} \quad P(n) \text{ is true}$$

EXAMPLE.

- 1) For any $n \in \mathbb{N}$ the number $5^{n+2} + 6^{2n+1}$ is divisible by 31
- 2) Prove by induction that for any $n \in \mathbb{N}$ $1 + 2 + \dots + n = \frac{n(n+1)}{2}$

PROOF.

- 1) Base of the induction $P(1)$. Let us verify that for $n = 1$ the statement is true. Indeed,

$$5^{1+2} + 6^{2 \cdot 1 + 1} = 5^3 + 6^3 = 125 + 216 = 341 \text{ — is divisible by 31 } (341 : 31 = 11)$$

Inductional step $P(n) \Rightarrow P(n+1)$. Let us take some arbitrary $n \in \mathbb{N}$ and let us assume

$$5^{n+2} + 6^{2n+1} \text{ is divisible by 31} \quad \text{— we assume that this is true}$$

Basing on this information, let us try to derive that

$$5^{(n+1)+2} + 6^{2(n+1)+1} = 5^{n+3} + 6^{2n+3} \text{ is divisible by 31} \quad \text{— we want to prove this}$$

Indeed, we have

$$5^{n+3} + 6^{2n+3} = 5 \cdot 5^{n+2} + 36 \cdot 6^{2n+1} = \underbrace{5 \cdot 5^{n+2} + 6 \cdot 6^{2n+1}}_{\text{is divisible by 31}} + \underbrace{31 \cdot 6^{2n+1}}_{\text{is divisible by 31}}$$

So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n \in \mathbb{N}$. \square

- 2) Base of the induction $P(1)$. Let us verify that for $n = 1$ the statement is true. Indeed,

$$1 = \frac{1 \cdot (1+1)}{2} \quad \text{— is true}$$

Inductional step $P(n) \Rightarrow P(n+1)$. Let us assume that for some $n \in \mathbb{N}$

$$1 + 2 + \dots + n = \frac{n(n+1)}{2} \quad \text{— we assume that this is true}$$

and basing on this information let us try to derive that

$$1 + 2 + \dots + n + (n+1) = \frac{(n+1)((n+1)+1)}{2} \quad \text{— we want to prove this}$$

Indeed, using the inductional assumption we can transform

$$\underbrace{1 + 2 + \dots + n}_{= \frac{n(n+1)}{2}} + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1) \left(\frac{n}{2} + 1 \right) = \frac{(n+1)(n+2)}{2}$$

So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n \in \mathbb{N}$. \square

5. Sets

- $x \in A$ and $B \subset A$ — elements and subsets
- Barber paradox, universal set X
- Equality of sets: $A = B \Leftrightarrow (A \subset B) \wedge (B \subset A)$, proofs of set identities
- Operations with sets: $A \cup B$, $A \cap B$, $A \setminus B$, $A' = cA = X \setminus A$
- Infinite union and intersection
- Cartesian product

EXAMPLE. For any an indexed family of subsets $\{A_i\}_{i \in I}$ of some universal set X the following equalities hold:

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \setminus A_i) \quad X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i)$$

6. Relations

- $a, b \in A$, $a \mathrel{\mathcal{R}} b$
- Examples: $a \leq b$ (real numbers), $a : b$ (integers), $a \perp b$ (lines on the plane) etc
- Reflexive, symmetric, transitive relations
- Equivalence relation is a relation which is reflexive, symmetric, transitive
- Formally: relation \mathcal{R} is some subset of $A \times A$. Then $a \mathrel{\mathcal{R}} b \Leftrightarrow (a, b) \in \mathcal{R}$

7. Functions

DEFINITION. Let A and B be any non-empty sets.

- A *function* (or a *map*) $f : A \rightarrow B$ is a rule which associate with each member of A a unique element of B .
- A is called the *domain* of f (we denote the domain $D(f)$), and B is called *codomain*.
- Functions f and g are called *equal* if $D(f) = D(g)$ and $\forall a \in D(f) f(a) = g(a)$.
- g is an *extension* of f if $D(f) \subset D(g)$ and $\forall a \in D(f) f(a) = g(a)$.

8. Domain and range of a function

DEFINITION. If $f : A \rightarrow B$ is a function then

$$D(f) := A \text{ is the domain of } f$$
$$R(f) := \{ b \in B \mid \exists a \in A : b = f(a) \} \text{ is the range of } f$$

If $f(a) = b$ then we say b is the *image* of a and a is a *pre-image* of b .

9. Injection, surjection and bijection functions

DEFINITION. Let $f : A \rightarrow B$ be a function.

- f is *injection* $\iff f$ is one-to-one $\iff \forall a_1, a_2 \in A$ if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$
- f is *surjection* $\iff f$ is onto $\iff R(f) = B \iff \forall b \in B \exists a \in A: f(a) = b$
- f is a *bijection* $\iff f$ is both an injection and a surjection

10. Countable sets

- The set A is *countable* $\iff \exists$ a bijection $A \leftrightarrow \mathbb{N}$
- The set of all rational numbers \mathbb{Q} is countable

11. Inverse function

DEFINITION. A function $f : A \rightarrow B$ is *invertible* on A if f is an injection on A . If f is invertible on $D(f)$ then the *inverse function* to f is defined by

$$g := f^{-1}, \quad D(g) = R(f), \quad g(b) = a \iff b = f(a), \quad \forall a \in D(f), \quad \forall b \in R(f)$$

12. Composition of functions

DEFINITION. Assume functions $f : A \rightarrow E$ and $g : B \rightarrow C$ satisfy $R(f) \subset B$. A *composition* of f and g is the function $h : A \rightarrow C$ defined by

$$h := g \circ f, \quad D(h) = D(f) = A, \quad h(a) = g(f(a)), \quad a \in A$$

13. Real numbers

- Common sense concept: infinite decimal fractions
- More formal: the real number can be associated with certain sequences of rational numbers $\{r_n\}$ which “approximate” (in a certain sense) this real number.
- Axioms of \mathbb{R} : operations $x + y$ and $x \cdot y$ possess commutativity, associativity, existence on neutral element, existence of inverse, distributivity, relation $x \leq y$
- Intervals (a, b) , $[a, b)$, $[a, b]$, $(-\infty, b)$, $[a, +\infty)$ etc

14. Bounded sets, lower and upper bounds

DEFINITION.

- $M \in \mathbb{R}$ is an *upper bound* of a set $A \subset \mathbb{R}$ $\iff \forall x \in A \quad x \leq M$
- $m \in \mathbb{R}$ is a *lower bound* of a set $A \subset \mathbb{R}$ $\iff \forall x \in A \quad x \geq m$
- $A \subset \mathbb{R}$ is *bounded from above* $\iff \exists$ at least one upper bound of A
- $A \subset \mathbb{R}$ is *bounded from below* $\iff \exists$ at least one lower bound of A
- $A \subset \mathbb{R}$ is *bounded* $\iff A$ is bounded both from above and from below

15. Supremum and infimum

DEFINITION.

$M \in \mathbb{R}$ is the *supremum* of a set $A \subset \mathbb{R}$ \iff

- 1) M is an upper bound of A
- 2) if $M' \in \mathbb{R}$ is some other upper bound of A then $M \leq M'$

$m \in \mathbb{R}$ is the *infimum* of a set $A \subset \mathbb{R}$ \iff

- 1) m is a lower bound of A
- 2) if $m' \in \mathbb{R}$ is some other lower bound of A then $m \geq m'$

NOTATION.

$\sup A$ is the supremum of A , $\sup A = +\infty \iff A$ is not bounded from above

$\inf A$ is the infimum of A , $\inf A = -\infty \iff A$ is not bounded from below

16. Characterization of supremum and infimum using quantifiers

THEOREM.

$M \in \mathbb{R}$ is the supremum of $A \subset \mathbb{R}$ \iff

- 1) $\forall x \in A \quad x \leq M$
- 2) $\forall \varepsilon > 0 \quad \exists x_\varepsilon \in A: \quad M - \varepsilon < x_\varepsilon$

$m \in \mathbb{R}$ is the infimum of $A \subset \mathbb{R}$ \iff

- 1) $\forall x \in A \quad m \leq x$
- 2) $\forall \varepsilon > 0 \quad \exists x_\varepsilon \in A: \quad x_\varepsilon < m + \varepsilon$

PROBLEM. Find the supremum and the infimum of the set $\left\{ \frac{2nm}{n^2+m^2} \mid n, m \in \mathbb{N} \right\} \subset \mathbb{R}$

17. Least upper bound axiom

AXIOM OF THE SET \mathbb{R} .

$A \subset \mathbb{R}$ is bounded from above $\implies \exists$ the number $M \in \mathbb{R}$ which is the supremum of A

$A \subset \mathbb{R}$ is bounded from below $\implies \exists$ the number $m \in \mathbb{R}$ which is the infimum of A

REMARK. The least upper bound axiom does not hold for the set of rational numbers \mathbb{Q} :

$$A = \{ r \in \mathbb{Q} : r^2 < 2 \}, \quad A \text{ is bounded from above, BUT } \nexists q \in \mathbb{Q} : q = \sup A$$

18. Functions of a real variable

DEFINITION. $f : D(f) \subset \mathbb{R} \rightarrow \mathbb{R}$

- Admissible set (maximal domain) of an algebraic expression

- Graph of a function of a real variable
- Range of a function of a real variable

19. Monotone functions

DEFINITION. Assume $f : [a, b] \rightarrow \mathbb{R}$.

- f — \nearrow on $[a, b] \iff \forall x_1, x_2 \in [a, b] \ x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
- f — \uparrow on $[a, b] \iff \forall x_1, x_2 \in [a, b] \ x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- f — \searrow on $[a, b] \iff \forall x_1, x_2 \in [a, b] \ x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$
- f — \downarrow on $[a, b] \iff \forall x_1, x_2 \in [a, b] \ x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

20. Invertibility of strictly monotone functions

THEOREM. A strictly monotone function $f : D(f) \subset \mathbb{R} \rightarrow \mathbb{R}$ is invertible on the interval of its monotonicity and the inverse function f^{-1} is strictly monotone in the same sense to f , i.e.

- f — \uparrow on $D(f) \implies f^{-1}$ — \uparrow on $R(f)$
- f — \downarrow on $D(f) \implies f^{-1}$ — \downarrow on $R(f)$

1.2 Limit of a sequence

1. Definition of a sequence

DEFINITION. A sequence $\{x_n\}_{n=1}^{\infty}$ is a map (a function) $f : \mathbb{N} \rightarrow \mathbb{R}$, $x_n := f(n)$

2. Informal notion helping to understand quantifiers in the definition of a limit

We say an interval (α, β) is a “feeder” for $\{x_n\}_{n=1}^{\infty} \iff \forall N \in \mathbb{N} \exists n \geq N: x_n \in (\alpha, \beta)$
 (i.e. the sequence $\{x_n\}_{n=1}^{\infty}$ “returns” to the interval (α, β) infinitely many times).

We say an interval (α, β) is a “catcher” for $\{x_n\}_{n=1}^{\infty} \iff \exists N \in \mathbb{N}: \forall n \geq N: x_n \in (\alpha, \beta)$
 (i.e. once entering the interval (α, β) , the sequence $\{x_n\}_{n=1}^{\infty}$ can never get out of it).

We say the sequence $\{x_n\}_{n=1}^{\infty}$ is *bounded* $\iff \exists M \geq 0: |x_n| \leq M \quad \forall n \in \mathbb{N}$.
 (i.e. there exists at least one “catcher” for a sequence $\{x_n\}_{n=1}^{\infty}$).

3. Limit of a sequence

DEFINITION. $a \in \mathbb{R}$ is the *limit* of a sequence $\{x_n\}_{n=1}^{\infty}$ (equivalently, the sequence $\{x_n\}_{n=1}^{\infty}$ *converges* to $a \in \mathbb{R}$) iff

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) : \quad \forall n \geq N \quad |x_n - a| < \varepsilon$$

i.e. for any $\varepsilon > 0$ the interval $(a - \varepsilon, a + \varepsilon)$ is a “catcher” for a sequence $\{x_n\}_{n=1}^{\infty}$.

NOTATION. $a = \lim_{n \rightarrow \infty} x_n$ or $x_n \xrightarrow{n \rightarrow \infty} a$.

4. Infinitesimal sequences

DEFINITION.

A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is called *infinitesimal* $\iff \lim_{n \rightarrow \infty} \alpha_n = 0$

A sequence $\{x_n\}_{n=1}^{\infty}$ is called *infinitely large* $\iff \lim_{n \rightarrow \infty} |x_n| = +\infty \iff$

$$\forall M > 0 \quad \exists N = N(M) : \quad \forall n \geq N \quad |x_n| \geq M$$

5. Characterization of a limit in terms of infinitesimal

THEOREM. Assume $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $a \in \mathbb{R}$. Then

$$\exists \lim_{n \rightarrow \infty} x_n = a \iff \exists \text{ an infinitesimal } \{\alpha_n\}_{n=1}^{\infty} \text{ such that } \forall n \in \mathbb{N} \quad x_n = a + \alpha_n$$

6. Properties of infinitesimal sequences

- 1) $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ are infinitesimal $\implies \{\alpha_n + \beta_n\}_{n=1}^{\infty}$ is infinitesimal
- 2) $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal, $\{x_n\}_{n=1}^{\infty}$ is bounded $\implies \{\alpha_n x_n\}_{n=1}^{\infty}$ is infinitesimal
- 3) $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal, $\alpha_n \neq 0, \forall n \in \mathbb{N} \iff \{\frac{1}{\alpha_n}\}_{n=1}^{\infty}$ is infinitely large

7. Basic properties of limits

THEOREM 1. The limit of a sequence, if exists, is unique.

THEOREM 2. $\exists \lim_{n \rightarrow \infty} x_n = a$ (a is a finite real number) $\implies \{x_n\}_{n=1}^{\infty}$ is bounded.

THEOREM 3. $\exists \lim_{n \rightarrow \infty} x_n = a, \exists \lim_{n \rightarrow \infty} y_n = b, \forall n \in \mathbb{N} \ x_n \leq y_n \implies a \leq b$

THEOREM 4. $\forall n \in \mathbb{N} \ x_n \leq y_n \leq z_n, \exists \lim_{n \rightarrow \infty} x_n = a, \exists \lim_{n \rightarrow \infty} z_n = a \implies \exists \lim_{n \rightarrow \infty} y_n = a$

COROLLARY. $\{\beta_n\}_{n=1}^{\infty}$ is infinitesimal, $|\alpha_n| \leq |\beta_n| \implies \{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal.

8. Arithmetic operations with limits

THEOREM 1. $\exists \lim_{n \rightarrow \infty} x_n = a \exists \lim_{n \rightarrow \infty} y_n = b \implies \exists \lim_{n \rightarrow \infty} (x_n \pm y_n) = a \pm b$

THEOREM 2. $\exists \lim_{n \rightarrow \infty} x_n = a \exists \lim_{n \rightarrow \infty} y_n = b \implies \exists \lim_{n \rightarrow \infty} (x_n y_n) = ab$

LEMMA. $\exists \lim_{n \rightarrow \infty} y_n = b$ and $y_n \neq 0, b \neq 0 \implies \{\frac{1}{y_n}\}_{n=1}^{\infty}$ is bounded.

THEOREM 3. $\exists \lim_{n \rightarrow \infty} x_n = a \exists \lim_{n \rightarrow \infty} y_n = b$ and $y_n \neq 0, b \neq 0 \implies \exists \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{a}{b}$

9. Monotone sequences

DEFINITION.

- $\{x_n\}_{n=1}^{\infty} \text{ — } \nearrow \iff \forall n \in \mathbb{N} \ x_n \leq x_{n+1}$
- $\{x_n\}_{n=1}^{\infty} \text{ — } \searrow \iff \forall n \in \mathbb{N} \ x_n \geq x_{n+1}$

THEOREM.

- 1) $\{x_n\}_{n=1}^{\infty} \text{ — } \nearrow \exists M \in \mathbb{R}: \forall n \in \mathbb{N} \ x_n \leq M \implies \exists \lim_{n \rightarrow \infty} x_n \leq M$
- 2) $\{x_n\}_{n=1}^{\infty} \text{ — } \searrow \exists m \in \mathbb{R}: \forall n \in \mathbb{N} \ x_n \geq m \implies \exists \lim_{n \rightarrow \infty} x_n \geq m$

10. Important examples

$$1) \quad q > 1 \implies \lim_{n \rightarrow \infty} \frac{n}{q^n} = 0$$

$$2) \quad q \in \mathbb{R} \implies \lim_{n \rightarrow \infty} \frac{q^n}{n!} = 0$$

$$3) \quad \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$4) \quad a > 0 \implies \lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$$

11. Euler's number e

LEMMA 1. For all non-negative $a_1, a_2, a_3, \dots, a_n \in \mathbb{R}$

$$\underbrace{\sqrt[n]{a_1 a_2 a_3 \cdot \dots \cdot a_n}}_{\text{Geometric mean}} \leq \underbrace{\frac{a_1 + a_2 + a_3 + \dots + a_n}{n}}_{\text{Arithmetic mean}}$$

LEMMA 2. Assume $a > 0, b > 0$. Then

$$\sqrt[n+1]{ab^n} \leq \frac{a + nb}{n+1}$$

LEMMA 3. Denote $x_n := \left(1 + \frac{1}{n}\right)^n$, $z_n := \left(1 - \frac{1}{n}\right)^n$, $y_n := \left(1 + \frac{1}{n}\right)^{n+1}$. Then

$$\forall n \in \mathbb{N} \quad x_n < x_{n+1}, \quad z_n < z_{n+1}, \quad y_{n+1} < y_n$$

LEMMA 4. The following limits exist and coincide

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$$

DEFINITION. Euler's number e is defined by

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n, \quad (e \approx 2,718281828459045\dots)$$

1.3 Subsequences

1. Definition of a subsequence

DEFINITION. Assume $f : \mathbb{N} \rightarrow \mathbb{R}$ is a sequence and assume $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ is strictly increasing. Denote by $h := f \circ \varphi$ the composition $h(k) = f(\varphi(k))$. Then the sequence $h : \mathbb{N} \rightarrow \mathbb{R}$ is called a *subsequence* of the sequence $f : \mathbb{N} \rightarrow \mathbb{R}$.

$$x_n := f(n), \quad x_{n_k} = h(k), \quad n_k := \varphi(k), \quad n_1 < n_2 < \dots < n_k < \dots$$

NOTATION. $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty} \iff \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$

2. Any subsequence of a convergent sequence converges to the same limit

THEOREM. $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, $\exists \lim_{n \rightarrow \infty} x_n = a \implies \exists \lim_{k \rightarrow \infty} x_{n_k} = a$

3. Nested interval theorem

THEOREM. Assume a sequence of closed intervals $[a_n, b_n] \subset \mathbb{R}$ satisfies the properties

$$1) \quad \forall n \in \mathbb{N} \quad [a_{n+1}, b_{n+1}] \subset [a_n, b_n]$$

$$2) \quad b_n - a_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

Then $\exists! c \in \mathbb{R} : c \in \bigcap_{n=1}^{\infty} [a_n, b_n]$

4. Bolzano–Weierstrass theorem

THEOREM. Each bounded sequence has a convergent subsequence:

$$\{x_n\}_{n=1}^{\infty} \text{ is bounded} \implies \exists \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty} \text{ such that } \exists a \in \mathbb{R} : a = \lim_{k \rightarrow \infty} x_{n_k}$$

5. Cauchy sequence

DEFINITION. $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is a *Cauchy sequence* (or a *fundamental sequence*) if

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in \mathbb{N} : \quad \forall n, m \geq N(\varepsilon) \quad |x_n - x_m| < \varepsilon$$

THEOREM. Every fundamental sequence is convergent:

$$\{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \text{ is a Cauchy sequence} \iff \exists a \in \mathbb{R} : x_n \xrightarrow[n \rightarrow \infty]{} a$$

1.4 Limit of a function

1. Definition of the limit of a function according to Cauchy

DEFINITION (C). Assume $f : (a, b) \rightarrow \mathbb{R}$, $x_0 \in (a, b)$, $y_0 \in \mathbb{R}$. We write

$$\lim_{x \rightarrow x_0} f(x) = y_0 \quad \text{or} \quad f(x) \xrightarrow{x \rightarrow x_0} y_0$$

and say y_0 is the *limit* of $f(x)$ as x tends to x_0 iff

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 : \forall x \in (a, b) \quad (x \neq x_0) \wedge |x - x_0| < \delta \implies |f(x) - y_0| < \varepsilon$$

The set $\overset{\circ}{U}_\delta(x_0) := (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ is called a *deleted neighbourhood* of x_0 .

2. Definition of the limit of a function according to Heine

DEFINITION (H). Assume $f : (a, b) \rightarrow \mathbb{R}$, $x_0 \in (a, b)$, $y_0 \in \mathbb{R}$. We write

$$\lim_{x \rightarrow x_0} f(x) = y_0 \quad \text{or} \quad f(x) \xrightarrow{x \rightarrow x_0} y_0$$

and say y_0 is the *limit* of $f(x)$ as x tends to x_0 iff

$$\forall \{x_n\}_{n=1}^\infty \subset (a, b) \setminus \{x_0\} : \quad \lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} f(x_n) = y_0$$

3. Equivalence of definitions

THEOREM. Definition (C) \iff Definition (H)

4. Arithmetic operations with limits of functions

$$\text{THEOREM 1.} \quad \exists \lim_{x \rightarrow x_0} f(x), \exists \lim_{x \rightarrow x_0} g(x) \implies \exists \lim_{x \rightarrow x_0} (f(x) + g(x)) = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$$

$$\text{THEOREM 2.} \quad \exists \lim_{x \rightarrow x_0} f(x), \exists \lim_{x \rightarrow x_0} g(x) \implies \exists \lim_{x \rightarrow x_0} (f(x) \cdot g(x)) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x)$$

$$\text{LEMMA.} \quad \lim_{x \rightarrow x_0} g(x) > 0 \implies \exists \delta > 0 : |x - x_0| < \delta \implies g(x) > 0$$

$$\text{THEOREM 3.} \quad \exists \lim_{x \rightarrow x_0} f(x), \exists \lim_{x \rightarrow x_0} g(x) \neq 0 \implies \exists \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)}$$

5. One-sided limits

DEFINITION. Assume $f : [a, b] \rightarrow \mathbb{R}$, $x_0 \in [a, b]$.

$$\exists \lim_{x \rightarrow x_0 - 0} f(x) = y_0 \iff \forall \varepsilon > 0 \quad \exists \delta > 0 : x \in (x_0 - \delta, x_0) \implies |f(x) - y_0| < \varepsilon$$

$$\exists \lim_{x \rightarrow x_0 + 0} f(x) = y_0 \iff \forall \varepsilon > 0 \quad \exists \delta > 0 : x \in (x_0, x_0 + \delta) \implies |f(x) - y_0| < \varepsilon$$

$$\text{THEOREM.} \quad \exists \lim_{x \rightarrow x_0} f(x) = y_0 \iff \exists \lim_{x \rightarrow x_0 - 0} f(x) = \lim_{x \rightarrow x_0 + 0} f(x) = y_0$$

1.5 Indeterminate forms and important examples

1. Types of indeterminate forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad 1^\infty, \quad \infty^0, \quad 0^0$$

2. Auxiliary inequalities

THEOREM.

- 1) $0 < x < \frac{\pi}{2} \implies \sin x < x < \operatorname{tg} x, \quad x \in \mathbb{R} \implies |\sin x| \leq |x|$
- 2) $x_1, x_2 \in \mathbb{R} \implies |\sin x_1 - \sin x_2| \leq |x_1 - x_2|, \quad |\cos x_1 - \cos x_2| \leq |x_1 - x_2|$
- 3) $\lim_{x \rightarrow 0} \sin x = 0, \quad \lim_{x \rightarrow 0} \cos x = 1$

3. Properties of a^x and $\log_a x$ which are postulated for now

We postulate the properties of the exponential and the logarithmic functions which actually come together with the definition of the exponential function and its inverse. We will return to discussion of these properties in the section “Continuity”. These properties also allowed to use without further explanation when you solve homework problems.

THEOREM. Assume $a > 0$ and $a \neq 1$. Then

- 1) $\forall \{x_n\}_{n=1}^\infty \subset \mathbb{R} \quad x_n \rightarrow x_0 \implies a^{x_n} \rightarrow a^{x_0}$
- 2) $\forall \{x_n\}_{n=1}^\infty \subset \mathbb{R} \quad x_n \rightarrow x_0, \quad x_0 > 0 \implies \log_a x_n \rightarrow \log_a x_0$

4. Important examples

- 1) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
- 2) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$
- 3) $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e \quad \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = e \quad \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e$
- 4) $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$
- 5) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$
- 6) $\lim_{x \rightarrow 0} \frac{(1+x)^\mu - 1}{x} = \mu$

1.6 Infinite limits and asymptotes

1. Infinite limits, vertical asymptotes

DEFINITION.

$$\begin{aligned}
 \lim_{x \rightarrow x_0-0} f(x) = +\infty &\iff \forall M > 0 \exists \delta > 0: x \in (x_0 - \delta, x_0) \Rightarrow f(x) > M \\
 \lim_{x \rightarrow x_0+0} f(x) = +\infty &\iff \forall M > 0 \exists \delta > 0: x \in (x_0, x_0 + \delta) \Rightarrow f(x) > M \\
 \lim_{x \rightarrow x_0-0} f(x) = -\infty &\iff \forall M < 0 \exists \delta > 0: x \in (x_0 - \delta, x_0) \Rightarrow f(x) < M \\
 \lim_{x \rightarrow x_0+0} f(x) = -\infty &\iff \forall M < 0 \exists \delta > 0: x \in (x_0, x_0 + \delta) \Rightarrow f(x) < M
 \end{aligned}$$

DEFINITION. The line $x = x_0$ is a *vertical asymptote* of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ if at least one of the four statements is true:

$$\lim_{x \rightarrow x_0+0} f(x) = +\infty, \quad \lim_{x \rightarrow x_0-0} f(x) = +\infty, \quad \lim_{x \rightarrow x_0-0} f(x) = +\infty, \quad \lim_{x \rightarrow x_0-0} f(x) = -\infty$$

2. Behavior of functions at infinity

DEFINITION.

$$\begin{aligned}
 \lim_{x \rightarrow +\infty} f(x) = +\infty &\iff \forall M > 0 \exists R > 0: x > R \Rightarrow f(x) > M \\
 \lim_{x \rightarrow -\infty} f(x) = +\infty &\iff \forall M > 0 \exists R < 0: x < R \Rightarrow f(x) > M \\
 \lim_{x \rightarrow +\infty} f(x) = -\infty &\iff \forall M < 0 \exists R > 0: x > R \Rightarrow f(x) < M \\
 \lim_{x \rightarrow -\infty} f(x) = -\infty &\iff \forall M < 0 \exists R < 0: x < R \Rightarrow f(x) < M
 \end{aligned}$$

3. Limits at infinity, horizontal asymptotes

DEFINITION.

$$\begin{aligned}
 \lim_{x \rightarrow +\infty} f(x) = y_0 &\iff \forall \varepsilon > 0 \exists R > 0: x > R \Rightarrow |f(x) - y_0| < \varepsilon \\
 \lim_{x \rightarrow -\infty} f(x) = y_0 &\iff \forall \varepsilon > 0 \exists R < 0: x < R \Rightarrow |f(x) - y_0| < \varepsilon
 \end{aligned}$$

DEFINITION. The line $y = y_0$ is a *horizontal asymptote* of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ if at least one of the two statements is true:

$$\lim_{x \rightarrow +\infty} f(x) = y_0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = y_0$$

4. How to remember definitions of finite/infinite limits?

$$y_0 = \lim_{x \rightarrow x_0} f(x)$$

x_0 is finite or infinite

y_0 is finite or infinite

	Values of $f(x)$	Values of x	When is used?
Small number	$\forall \varepsilon > 0$	$\exists \delta > 0$	For a finite value
Large number	$\forall M > 0$	$\exists R > 0$	For an infinite value

5. Oblique asymptotes

DEFINITION. The line $y = kx + b$ is an *oblique asymptote* of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ if at least one of the two statements is true:

$$\lim_{x \rightarrow +\infty} (f(x) - kx - b) = 0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} (f(x) - kx - b) = 0$$

REMARK. A horizontal asymptote is a particular case of an oblique asymptote with $k = 0$.

THEOREM. $y = kx + b$ is a horizontal or an oblique asymptote of $f : \mathbb{R} \rightarrow \mathbb{R}$ as $x \rightarrow +\infty$ iff the following two limits exist:

$$1) \quad \exists \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = k$$

$$2) \quad \exists \lim_{x \rightarrow +\infty} (f(x) - kx) = b$$

The similar statements hold as $x \rightarrow -\infty$.

1.7 Continuity

1. Continuity of a function at a point

DEFINITION. $f : (a, b) \rightarrow \mathbb{R}$ is *continuous* at a point $x_0 \iff \lim_{x \rightarrow x_0} f(x) = f(x_0)$

REMARK. The following statements are equivalent:

- 1) $f : (a, b) \rightarrow \mathbb{R}$ is continuous at x_0
- 2) $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0: |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$
- 3) $\forall \{x_n\}_{n=1}^\infty \subset (a, b): x_n \rightarrow x_0 \implies f(x_n) \rightarrow f(x_0)$

DEFINITION. $f : (a, b) \rightarrow \mathbb{R}$ is continuous *on* $(a, b) \iff \forall x_0 \in (a, b)$ f is continuous at x_0

2. Stability of sign theorem

THEOREM. f is continuous at x_0 , $f(x_0) > 0 \implies \exists \delta > 0: \forall x \in (x_0 - \delta, x_0 + \delta) f(x) > 0$

3. Arithmetic operations with continuous functions

THEOREM.

- 1) f, g are continuous at $x_0 \implies f \pm g$ are continuous at x_0
- 2) f, g are continuous at $x_0 \implies f \cdot g$ is continuous at x_0
- 3) f, g are continuous at x_0 , $g(x_0) \neq 0 \implies f/g$ is continuous at x_0

4. Left and right continuity

DEFINITION.

$f : [a, b] \rightarrow \mathbb{R}$ is *left continuous* at $x_0 \iff \lim_{x \rightarrow x_0-0} f(x) = f(x_0)$

$f : [a, b] \rightarrow \mathbb{R}$ is *right continuous* at $x_0 \iff \lim_{x \rightarrow x_0+0} f(x) = f(x_0)$

$f : [a, b] \rightarrow \mathbb{R}$ is continuous *on* $[a, b] \iff \begin{cases} f \text{ is continuous on } (a, b) \\ f \text{ is left continuous at } a \\ f \text{ is right continuous at } b \end{cases}$

THEOREM. f is continuous at $x_0 \in (a, b) \iff f$ is left and right continuous at x_0 .

5. Types of discontinuity

DEFINITION. Assume $x_0 \in (a, b)$, $f : (a, b) \setminus \{x_0\} \rightarrow \mathbb{R}$

- x_0 is a *removable discontinuity* $\iff \exists$ finite $\lim_{x \rightarrow x_0 \pm 0} f(x)$, $\lim_{x \rightarrow x_0-0} f(x) = \lim_{x \rightarrow x_0+0} f(x)$

- x_0 is a *jump discontinuity* $\iff \exists$ both finite $\lim_{x \rightarrow x_0 \pm 0} f(x)$, $\lim_{x \rightarrow x_0 - 0} f(x) \neq \lim_{x \rightarrow x_0 + 0} f(x)$
- $x = x_0$ is a *vertical asymptote* $\iff \exists \lim_{x \rightarrow x_0 \pm 0} f(x) = \pm\infty$ (at least one of limits)
- x_0 may be a discontinuity of no above type $\iff \nexists \lim_{x \rightarrow x_0 \pm 0} f(x)$ ($f(x) = \sin \frac{1}{x}$)

6. Continuity of some elementary functions

- $f(x) = x^n$ is continuous on \mathbb{R} , $n \in \mathbb{N}$
- $f(x) = \frac{1}{x^n}$ is continuous on $\mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$
- $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is continuous on \mathbb{R}
- $f(x) = \sin x$, $g(x) = \cos x$ are continuous on \mathbb{R}
- $f(x) = \operatorname{tg} x$ is continuous at $x \neq \frac{\pi}{2} + \pi k$, $g(x) = \operatorname{ctg} x$ is continuous at $x \neq \pi k$, $k \in \mathbb{Z}$
- $f(x) = x^p$, $p \in (0, +\infty)$ is continuous on $[0, +\infty)$
- $f(x) = a^x$, $a > 0$, is continuous on \mathbb{R}

7. Continuity of a composition of functions

THEOREM. $f : (a, b) \rightarrow \mathbb{R}$, $g : (c, d) \rightarrow \mathbb{R}$, $x_0 \in (a, b)$, $f(x_0) \in (c, d)$

f is continuous at x_0 , g is continuous at $f(x_0)$ $\implies g \circ f$ is continuous at x_0

8. Two intermediate value theorems (Bolzano–Cauchy theorems)

THEOREM 1. f is continuous on $[a, b]$, $f(a) \leq 0$, $f(b) \geq 0$ $\implies \exists c \in [a, b]$: $f(c) = 0$

THEOREM 2. f is continuous on $[a, b]$, $f(a) = y_1$, $f(b) = y_2$, $y_1 \leq y_2$ $\implies [y_1, y_2] \subset R(f)$

9. Continuity of the inverse function

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ is continuous and strictly monotone on (a, b) . Then f is invertible on (a, b) and f^{-1} is continuous on $R(f)$.

10. Continuity of some elementary inverse functions

- $f(x) = \sqrt[n]{x}$ is continuous on $D(f)$, $n \in \mathbb{N}$
- $f(x) = \log_a x$, $a > 0$, $a \neq 1$, is continuous on $(0, +\infty)$
- $f(x) = \arcsin x$, $g(x) = \arccos x$ are continuous on $[-1, 1]$
- $f(x) = \operatorname{arctg} x$, $g(x) = \operatorname{arcctg} x$ are continuous on \mathbb{R}

11. Two extreme value theorems (Weierstrass theorems)

THEOREM 1. $[a, b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b] \implies f$ is bounded on $[a, b]$, i.e.

$$\exists M > 0 : \quad \forall x \in [a, b] \quad |f(x)| \leq M$$

THEOREM 2. $[a, b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b] \implies f$ achieves on $[a, b]$ its maximum and minimal values, i.e. $\exists c_1, c_2 \in [a, b]$ such that

$$f(c_1) = \inf_{x \in [a, b]} f(x) \quad \text{and} \quad f(c_2) = \sup_{x \in [a, b]} f(x)$$

12. Uniform continuity

DEFINITION. A function $f : D(f) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is *uniformly continuous* on $D(f)$ iff

$$\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 : \quad \forall x_1, x_2 \in D(f) \quad |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$$

13. Heine–Cantor theorem

THEOREM. $[a, b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b] \implies f$ is uniformly continuous on $[a, b]$.

1.8 Differentiability and derivatives

1. Main definition

DEFINITION. Assume $f : (a, b) \rightarrow \mathbb{R}$, $x_0 \in (a, b)$. A function f is *differentiable at a point* x_0 iff there exists a finite limit

$$f'(x_0) := \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The number $f'(x_0)$ is called the *derivative* of a function f at x_0 . Another notation

$$\frac{df}{dx}(x_0) := f'(x_0)$$

- We say f is *differentiable on* (a, b) if for any $x_0 \in (a, b)$ the function f is differentiable at x_0 .
- We say f is *continuously differentiable* on (a, b) if f is differentiable on (a, b) and the function f' is continuous on (a, b) .

2. Physical and geometric meaning of the derivative

- From a physical point of view the derivative of a function is the instantaneous speed (or the rate of change) of the function.
- From a geometrical point of view the derivative of a function f at a point x_0 is equal to the slope of the tangent line to the graph of the function at the point x_0 .

3. Differentiable function is continuous

THEOREM. $f : (a, b) \rightarrow \mathbb{R}$ is differentiable at $x_0 \in (a, b) \implies f$ is continuous at x_0

REMARK. The inverse is false: $f(x) = |x|$ is continuous but non-differentiable at 0.

4. Basic properties of derivatives

THEOREM . Assume $f, g : (a, b) \rightarrow \mathbb{R}$ are differentiable at $x \in (a, b)$. Then $f \pm g$ and fg are differentiable at x . If additionally $g(x) \neq 0$ then f/g is also differentiable at x . Besides,

- 1) $(f \pm g)'(x) = f'(x) \pm g'(x)$
- 2) $(cf)'(x) = cf'(x)$
- 3) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- 4) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

5. Derivatives of elementary functions

1. $(C)' = 0$

2. $(x)' = 1$

3. $(x^m)' = mx^{m-1}, \quad m \in \mathbb{Z}, \quad x \in \mathbb{R} \quad (x \neq 0 \text{ if } m < 0)$

4. $(x^r)' = rx^{r-1}, \quad r \in \mathbb{R}, \quad x > 0$

5. $(e^x)' = e^x, \quad (a^x)' = a^x \ln a, \quad a > 0, \quad x \in \mathbb{R}$

6. $(\ln |x|)' = \frac{1}{x}, \quad x \neq 0, \quad (\log_a |x|)' = \frac{1}{x \ln a}, \quad a > 0, \quad a \neq 1$

7. $(\sin x)' = \cos x, \quad (\cos x)' = -\sin x$

8. $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}, \quad (\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}$

1.9 Chain rule

1. Differential of a function

DEFINITION. Assume $f : (a, b) \rightarrow \mathbb{R}$ and $x_0 \in (a, b)$. A linear map $l_{x_0} : \mathbb{R} \rightarrow \mathbb{R}$,

$$l_{x_0}(h) = k_{x_0}h, \quad \forall h \in \mathbb{R}, \quad (k_{x_0} \in \mathbb{R})$$

is called the *differential* of f at a point x_0 iff there exists a number $h_0 > 0$ and a function $\alpha_{x_0} : (-h_0, h_0) \rightarrow \mathbb{R}$ such that

$$\lim_{h \rightarrow 0} \alpha_{x_0}(h) = 0,$$

$$f(x_0 + h) = f(x_0) + k_{x_0}h + \alpha_{x_0}(h)h, \quad \forall |h| < h_0$$

NOTATION. $l_{x_0} := df(x_0) \iff df(x_0)(h) := l_{x_0}(h) = k_{x_0}h, \quad \forall h \in \mathbb{R}$

2. When the differential of a function is well-defined?

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ and $x_0 \in (a, b)$. The differential of f at x_0 exists if and only if f is differentiable at x_0 . Moreover, if $df(x_0)(h) = k_{x_0}h$ is the differential of f at x_0 then

$$k_{x_0} = f'(x_0) \iff df(x_0)(h) = f'(x_0)h, \quad \forall h \in \mathbb{R}$$

where $f'(x_0)$ is the derivative of f at x_0 .

3. What is the difference between the derivative $f'(x_0)$ and the differential $df(x_0)$?

It is well-known that any linear map $l : \mathbb{R} \rightarrow \mathbb{R}$ has the form

$$l(x) = kx, \quad x \in \mathbb{R}$$

i.e. any linear map from \mathbb{R} to \mathbb{R} can be characterized in the unique way by the coefficient $k \in \mathbb{R}$ which is called a *slope* of the linear map $l(x)$. The theorem above shows that the slope of the differential $df(x_0)$ is equal to $f'(x_0)$. So, the differential and the derivative of a function are related in the same way as a linear transformation $l(x) = kx$ is related to its slope $k \in \mathbb{R}$.

NOTATION. Very often people write $df(x_0) = f'(x_0)dx$ meaning that $f'(x_0)$ is a slope of a linear function $df(x_0)$ and dx stands for the identity linear map, i.e. $dx(h) = h$.

4. Chain rule

THEOREM. Assume $f : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}$, $g : (c, d) \rightarrow \mathbb{R}$, $R(f) \subset (c, d)$, $x_0 \in (a, b)$. If f is differentiable at x_0 and g is differentiable at $y_0 = f(x_0)$ then the composition $\varphi := g \circ f$ is differentiable at x_0 and

$$\varphi'(x_0) = g'(y) \Big|_{y=f(x_0)} \cdot f'(x_0)$$

5. Inverse function rule

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ is strictly monotone and continuous (so the inverse function f^{-1} exists, f^{-1} is strictly monotone in the same sense to f and continuous on $R(f)$). Assume additionally f is differentiable at $x_0 \in (a, b)$ and $f'(x_0) \neq 0$. Then the function $g := f^{-1}$ is differentiable at $y_0 := f(x_0)$ and

$$g'(y_0) = \frac{1}{f'(x_0)}$$

6. Derivatives of inverse trigonometric functions

$$9. (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$10. (\operatorname{arctg} x)' = \frac{1}{1+x^2}, \quad (\operatorname{arcctg} x)' = -\frac{1}{1+x^2}$$

1.10 The basic theorems of differential calculus

1. Fermat's theorem (or Fermat's lemma)

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ has a local extremum (maximum or minimum) on the interval (a, b) at some internal point $c \in (a, b)$, i.e.

$$\begin{aligned} \text{either} \quad f(c) &= \max_{x \in (a, b)} f(x) && \iff \quad \forall x \in (a, b) \quad f(x) \leq f(c) \\ \text{or} \quad f(c) &= \min_{x \in (a, b)} f(x) && \iff \quad \forall x \in (a, b) \quad f(x) \geq f(c) \end{aligned}$$

If f is differentiable at c then $f'(c) = 0$.

2. Rolle's theorem

THEOREM. Assume $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) . Assume that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

3. Lagrange's theorem

THEOREM. Assume $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) . Then there is a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

4. Cauchy's theorem

THEOREM. Assume functions $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b) . Assume $g'(x) \neq 0$ for any $x \in (a, b)$. Then there is a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

5. Functions with identically zero derivatives are constants

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ is differentiable on (a, b) and $f'(x) = 0$ for any $x \in (a, b)$. Then f is equal to a constant, i.e.

$$\exists y_0 \in \mathbb{R} : \quad f(x) = y_0 \quad \forall x \in (a, b)$$

1.11 Investigation of functions using derivatives

1. Investigation of the monotonicity of functions

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ is differentiable on (a, b) . Then

- 1) $f'(x) \geq 0 \quad \forall x \in (a, b) \iff f \text{ — } \nearrow$ (is non-decreasing) on (a, b)
- 2) $f'(x) \leq 0 \quad \forall x \in (a, b) \iff f \text{ — } \searrow$ (is non-increasing) on (a, b)
- 3) $f'(x) > 0 \quad \forall x \in (a, b) \implies f \text{ — } \uparrow$ (is strictly increasing) on (a, b)
- 4) $f'(x) < 0 \quad \forall x \in (a, b) \implies f \text{ — } \downarrow$ (is strictly decreasing) on (a, b)

2. Conditions for extreme in terms of the first derivative

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ is differentiable on (a, b) . Then and assume $c \in (a, b)$ satisfies $f'(c) = 0$. Then

- 1) if $f'(x) \leq 0 \quad \forall x \in (a, c)$ and $f'(x) \geq 0 \quad \forall x \in (c, b)$ then c is the point of minimum of f
- 2) if $f'(x) \geq 0 \quad \forall x \in (a, c)$ and $f'(x) \leq 0 \quad \forall x \in (c, b)$ then c is the point of maximum of f

3. Higher order derivatives

DEFINITION. Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable on (a, b) . If the function $g = f' : (a, b) \rightarrow \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ then the function f is called *twice differentiable* at x_0 . In this case the derivative of the function $g = f'$ at x_0 is called the *second derivative* of f at x_0 .

$$f''(x_0) := g'(x_0) = \lim_{x \rightarrow x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

The higher-order derivatives are defined similarly:

$$f''' = (f'')', \quad \dots, \quad f^{(k)} = (f^{(k-1)})', \quad \dots$$

ALTERNATIVE NOTATION.

$$\frac{d^2 f}{dx^2}(x) = f''(x), \quad \frac{d^3 f}{dx^3}(x) = f'''(x), \quad \dots \quad \frac{d^k f}{dx^k}(x) = f^{(k)}(x)$$

4. Conditions for extreme in terms of the second derivative

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ is differentiable on (a, b) . Let f be twice differentiable at $c \in (a, b)$ and $f'(c) = 0$. Then

- 1) $f''(c) > 0 \implies c$ is a point of a local min of $f \iff \exists \delta > 0: f(c) = \min_{x \in (x_0 - \delta, x_0 + \delta)} f(x)$
- 2) $f''(c) < 0 \implies c$ is a point of a local max of $f \iff \exists \delta > 0: f(c) = \max_{x \in (x_0 - \delta, x_0 + \delta)} f(x)$

5. Convex functions

DEFINITION. $f : [a, b] \rightarrow \mathbb{R}$ is *convex* on $(a, b) \iff \forall x_1, x_2 \in (a, b), \forall \lambda \in [0, 1]$

$$f((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)f(x_1) + \lambda f(x_2)$$

$f : [a, b] \rightarrow \mathbb{R}$ is *strictly convex* on $(a, b) \iff \forall x_1 \neq x_2 \in (a, b), \forall \lambda \in (0, 1)$

$$f((1 - \lambda)x_1 + \lambda x_2) < (1 - \lambda)f(x_1) + \lambda f(x_2)$$

$f : [a, b] \rightarrow \mathbb{R}$ is *concave* on $(a, b) \iff \forall x_1, x_2 \in (a, b), \forall \lambda \in [0, 1]$

$$f((1 - \lambda)x_1 + \lambda x_2) \geq (1 - \lambda)f(x_1) + \lambda f(x_2)$$

$f : [a, b] \rightarrow \mathbb{R}$ is *strictly concave* on $(a, b) \iff \forall x_1 \neq x_2 \in (a, b), \forall \lambda \in (0, 1)$

$$f((1 - \lambda)x_1 + \lambda x_2) > (1 - \lambda)f(x_1) + \lambda f(x_2)$$

6. Inequalities for slopes of secants

THEOREM.

1) f is convex on $(a, b) \iff \forall x_1, x_2 \in (a, b), x_1 < x_2, \forall x \in (x_1, x_2)$

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}$$

2) f is strictly convex on $(a, b) \iff \forall x_1, x_2 \in (a, b), x_1 < x_2, \forall x \in (x_1, x_2)$

$$\frac{f(x) - f(x_1)}{x - x_1} < \frac{f(x_2) - f(x)}{x_2 - x}$$

3) Let f be differentiable on (a, b) . Then f is convex on $(a, b) \iff \forall x_1, x_2 \in (a, b)$

$$x_1 < x_2 \implies f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2)$$

7. Investigation of convexity using the first derivative

THEOREM. Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable on (a, b) .

1) f is convex on $(a, b) \iff f' \text{ — } \nearrow \text{ on } (a, b)$

2) f is concave on $(a, b) \iff f' \text{ — } \searrow \text{ on } (a, b)$

3) f is strictly convex on $(a, b) \iff f' \text{ — } \uparrow \text{ on } (a, b)$

4) f is strictly concave on $(a, b) \iff f' \text{ — } \downarrow \text{ on } (a, b)$

8. Investigation of convexity using the second derivative

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ is twice differentiable on (a, b) .

- 1) $\forall x \in (a, b) \quad f''(x) \geq 0 \iff f$ is convex on (a, b)
- 2) $\forall x \in (a, b) \quad f''(x) \leq 0 \iff f$ is concave on (a, b)
- 3) $\forall x \in (a, b) \quad f''(x) > 0 \implies f$ is strictly convex on (a, b)
- 4) $\forall x \in (a, b) \quad f''(x) < 0 \implies f$ is strictly concave on (a, b)

9. Tangent line to the graph of a function

DEFINITION. Assume $f : (a, b) \rightarrow \mathbb{R}$ is differentiable at $x_0 \in (a, b)$. The straight line given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is called the *tangent* to the function $y = f(x)$ (or to the graph of f) at the point x_0 .

10. Convexity and tangent lines

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ is differentiable on (a, b) .

- 1) f is convex on $(a, b) \iff$ for any $x_0 \in (a, b)$ the graph of the function f lies above the tangent to f at the point x_0 , i.e.

$$\forall x \in (a, b) \quad f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

- 2) f is concave on $(a, b) \iff$ for any $x_0 \in (a, b)$ the graph of the function f lies below the tangent to f at the point x_0 , i.e.

$$\forall x \in (a, b) \quad f(x) \leq f(x_0) + f'(x_0)(x - x_0)$$

- 3) f is strictly convex on $(a, b) \iff \forall x_0 \in (a, b), \forall x \in (a, b),$

$$x \neq x_0 \implies f(x) > f(x_0) + f'(x_0)(x - x_0)$$

- 4) f is strictly concave on $(a, b) \iff \forall x_0 \in (a, b), \forall x \in (a, b),$

$$x \neq x_0 \implies f(x) < f(x_0) + f'(x_0)(x - x_0)$$

11. Inflection points

DEFINITION. Assume $f : (a, b) \rightarrow \mathbb{R}$. The point $c \in (a, b)$ is called the *inflection point* of the function f if $\exists \delta > 0$, such that on the intervals $(c - \delta, c)$ and $(c, c + \delta)$ the function f changes convexity (was convex, became concave, or vice versa).

12. Necessary condition for an inflection point

THEOREM. Let $f : (a, b) \rightarrow \mathbb{R}$ be twice differentiable on (a, b) and let $c \in (a, b)$ be the inflection point of f . Then

$$f''(c) = 0.$$

1.12 Primitive function and indefinite integral

1. Primitive function (antiderivative)

DEFINITION. $F : (a, b) \rightarrow \mathbb{R}$ is called a *primitive function* (or an *antiderivative*) of the function $f : (a, b) \rightarrow \mathbb{R}$ on the interval (a, b) if F is differentiable on (a, b) and

$$F'(x) = f(x), \quad \forall x \in (a, b).$$

2. Which functions possess antiderivatives?

This is a non-trivial question. In the section “Riemann integral” we will show that at least any *continuous* function possesses antiderivatives on the interval of its continuity.

Discussing antiderivatives below we will always assume that all functions involved in the formulas possess the corresponding antiderivatives by assumption and we will focus only on the derivation of relations between these antiderivatives.

3. Properties of primitives

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ has a primitive on (a, b) . Note that in this case f has infinitely many other primitives on (a, b) , but any two primitives of f can differ only by a constant:

- 1) if $F(x)$ is a primitive of $f(x)$ on (a, b) then for any $c \in \mathbb{R}$ the function $F(x) + c$ is also a primitive of $f(x)$ on (a, b)
- 2) if F_1 and F_2 are two primitives of f on (a, b) then there exists $c \in \mathbb{R}$ such that

$$\forall x \in (a, b) \quad F_1(x) - F_2(x) = c$$

4. Indefinite integral

DEFINITION. The set $\{F(x) + C\}$ of all primitives of the function $f : (a, b) \rightarrow \mathbb{R}$, defined on the interval (a, b) , is called the indefinite integral of f on (a, b) and is denoted by

$$\int f(x) dx$$

5. Properties of the indefinite integral

THEOREM.

$$\int (\alpha f(x) + \beta g(x)) dx = \alpha \int f(x) dx + \beta \int g(x) dx, \quad \forall \alpha, \beta \in \mathbb{R}$$

$$\frac{d}{dx} \int f(x) dx = f(x), \quad \int g'(x) dx = g(x) + C$$

6. List of simplest antiderivatives

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
2. $\int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int a^x dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
5. $\int \sin x dx = -\cos x + C, \quad \int \cos x dx = \sin x + C$
6. $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C, \quad \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$
7. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C \quad (a \neq 0)$
8. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C \quad (a > 0)$

7. Antiderivatives computed by standard methods (see the next section)

1. $\int \operatorname{tg} x dx = -\ln|\cos x| + C, \quad \int \operatorname{ctg} x dx = \ln|\sin x| + C$
2. $\int \frac{x dx}{a^2 \pm x^2} = \pm \frac{1}{2} \ln|a^2 \pm x^2| + C$
3. $\int \frac{x dx}{\sqrt{a^2 \pm x^2}} = \pm \sqrt{a^2 \pm x^2} + C \quad (a > 0)$
4. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C \quad (a > 0)$
5. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C \quad (a \neq 0)$
6. $\int \ln x dx = x(\ln x - 1) + C$
7. $\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$
8. $\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} + C, \quad \int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$
9. $\int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C$

8. Non-elementary integrals (can not be computed in terms of elementary functions)

1. $\operatorname{Ei}(x) = \int \frac{e^x}{x} dx$ (exponential integral)
2. $\operatorname{Si}(x) = \int \frac{\sin x}{x} dx, \quad \operatorname{Ci}(x) = \int \frac{\cos x}{x} dx$ (sine and cosine integrals)
3. $\operatorname{S}(x) = \int \sin x^2 dx, \quad \operatorname{C}(x) = \int \cos x^2 dx$ (Fresnel integral)
4. $\Phi(x) = \int e^{-x^2} dx$ (Gaussian integral)
5. $\operatorname{li}(x) = \int \frac{dx}{\ln x}$ (logarithmic integral)

1.13 Basic methods for finding primitives

1. Notation for the differential of a function

$$df(x) = f'(x) dx$$

2. Change of variables

THEOREM. If $f : (a, b) \rightarrow \mathbb{R}$ and $\varphi : (\alpha, \beta) \rightarrow (a, b)$ is differentiable on (α, β) then

$$\int \underbrace{f(\varphi(x))}_y \underbrace{\varphi'(x) dx}_{dy} = \left[\begin{array}{l} y = \varphi(x) \\ dy = \varphi'(x) dx \end{array} \right] = \int f(y) dy \Big|_{y=\varphi(x)}$$

ABBREVIATED NOTATION.

$$\int f(\varphi(x)) d\varphi(x) = \int f(y) dy \Big|_{y=\varphi(x)}$$

EXAMPLES.

- 1) $\int \frac{dx}{x+a} = \ln|x+a| + C$
- 2) $\int \cos(ax) dx = \frac{1}{a} \sin(ax) + C, \quad a \neq 0$
- 3) $\int \operatorname{tg} x dx = -\ln|\cos x| + C, \quad \int \operatorname{ctg} x dx = \ln|\sin x| + C$
- 4) $\int \frac{x}{x^2+a} dx = \frac{1}{2} \ln|x^2+a| + C, \quad a \in \mathbb{R}$
- 5) $\int \frac{x}{\sqrt{x^2+a}} dx = \sqrt{x^2+a} + C, \quad a \in \mathbb{R}$
- 6) $\int \frac{dx}{\sqrt{x^2+a}} = \ln|x + \sqrt{x^2+a}| + C, \quad a \in \mathbb{R}$

PROOFS.

1.

$$\begin{aligned} \int \frac{dx}{x+a} &= \int \frac{d(x+a)}{x+a} = \left[\begin{array}{l} y = x+a \\ dy = dx \end{array} \right] = \int \frac{dy}{y} = \ln|y| + C = \\ &= \left[\begin{array}{l} y = x+a \end{array} \right] = \ln|x+a| + C \end{aligned}$$

2.

$$\begin{aligned} \int \cos(ax) dx &= \frac{1}{a} \int \cos(ax) d(ax) = \left[\begin{array}{l} y = ax \\ dy = a dx \end{array} \right] = \frac{1}{a} \int \cos y dy = \frac{1}{a} \sin y + C = \\ &= \left[\begin{array}{l} y = ax \end{array} \right] = \frac{1}{a} \sin(ax) + C \end{aligned}$$

3.

$$\begin{aligned}\int \operatorname{tg} x \, dx &= \int \frac{\sin x}{\cos x} \, dx = - \int \frac{d(\cos x)}{\cos x} \, dx = \left[\begin{array}{l} y = \cos x \\ dy = -\sin x \, dx \end{array} \right] = \\ &= - \int \frac{dy}{y} = -\ln |y| + C = \left[y = \cos x \right] = -\ln |\cos x| + C\end{aligned}$$

4.

$$\begin{aligned}\int \frac{x}{x^2 + a} \, dx &= \frac{1}{2} \int \frac{d(x^2 + a)}{x^2 + a} = \left[\begin{array}{l} y = x^2 + a \\ dy = 2x \, dx \end{array} \right] = \frac{1}{2} \int \frac{dy}{y} = \frac{1}{2} \ln |y| + C = \\ &= \left[y = x^2 + a \right] = \frac{1}{2} \ln |x^2 + a| + C\end{aligned}$$

5.

$$\begin{aligned}\int \frac{x}{\sqrt{x^2 + a}} \, dx &= \frac{1}{2} \int \frac{d(x^2 + a)}{\sqrt{x^2 + a}} = \left[\begin{array}{l} y = x^2 + a \\ dy = 2x \, dx \end{array} \right] = \frac{1}{2} \int \frac{dy}{\sqrt{y}} = \sqrt{y} + C = \\ &= \left[y = x^2 + a \right] = \sqrt{x^2 + a} + C\end{aligned}$$

6.

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + a}} &= \int \frac{1}{x + \sqrt{x^2 + a}} \frac{x + \sqrt{x^2 + a}}{\sqrt{x^2 + a}} \, dx = \int \frac{1}{x + \sqrt{x^2 + a}} \left(1 + \frac{x}{\sqrt{x^2 + a}} \right) \, dx = \\ &= \left[\begin{array}{l} y = x + \sqrt{x^2 + a} \\ dy = \left(1 + \frac{x}{\sqrt{x^2 + a}} \right) \, dx \end{array} \right] = \int \frac{dy}{y} = \ln |y| + C = \\ &= \left[y = x + \sqrt{x^2 + a} \right] = \ln(x + \sqrt{x^2 + a}) + C\end{aligned}$$

3. Decomposition of an integrand into a sum of elementary functions

THEOREM. If $f = f_1 + f_2$ on (a, b) then

$$\int f(x) \, dx = \int f_1(x) \, dx + \int f_2(x) \, dx$$

EXAMPLES.

$$1) \int \frac{x+a}{x+b} \, dx = x + (a-b) \ln |x+b| + C$$

$$2) \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C, \quad a \neq 0$$

PROOFS.

1.

$$\frac{x+a}{x+b} = \frac{x+b-b+a}{x+b} = 1 + \frac{a-b}{x+b}$$

$$\int \frac{x+a}{x+b} dx = \int dx + (a-b) \int \frac{dx}{x+b} = x + (a-b) \ln|x+b| + C$$

2.

$$\begin{aligned} \frac{1}{a^2-x^2} &= \frac{1}{2a} \cdot \frac{(a-x) + (a+x)}{a^2-x^2} = \frac{1}{2a} \left(\frac{1}{a+x} + \frac{1}{a-x} \right) \\ \int \frac{dx}{a^2-x^2} &= \frac{1}{2a} \int \left(\frac{1}{a+x} + \frac{1}{a-x} \right) dx = \frac{1}{2a} (\ln|a+x| - \ln|a-x|) + C = \\ &= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C \end{aligned}$$

4. Integrations by parts

THEOREM.

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

ABBREVIATED NOTATION.

$$\int f dg = fg - \int g df$$

EXAMPLES.

- 1) $\int \ln x dx = x(\ln x - 1) + C$
- 2) $\int x \cos x dx = x \sin x + \cos x + C$
- 3) $\int x e^x dx = x e^x - e^x + C$
- 4) $\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$
- 5) $\int e^x \sin x dx = \frac{1}{2}(\sin x - \cos x) e^x + C, \quad \int e^x \cos x dx = \frac{1}{2}(\sin x + \cos x) e^x + C,$

PROOFS.

1.

$$\int \ln x dx = [\text{by parts}] = x \ln x - \int x d(\ln x) = x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C$$

2.

$$\int x \cos x dx = \int x d(\sin x) = [\text{by parts}] = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

3.

$$\int x e^x dx = \int x d(e^x) = [\text{by parts}] = x e^x - \int e^x dx = x e^x - e^x + C$$

4.

$$\begin{aligned}\int x \ln x \, dx &= \int \ln x \, d\left(\frac{x^2}{2}\right) = [\text{by parts}] = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \underbrace{d(\ln x)}_{\frac{1}{x} dx} = \\ &= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C\end{aligned}$$

5.

$$\begin{aligned}\int e^x \sin x \, dx &= \int \sin x \, d(e^x) = [\text{by parts}] = \sin x \, e^x - \int e^x \underbrace{d(\sin x)}_{\cos x \, dx} = \\ &= \sin x \, e^x - \int e^x \cos x \, dx = \sin x \, e^x - \int \cos x \, d(e^x) = [\text{again by parts}] = \\ &= \sin x \, e^x - \cos x \, e^x + \int e^x \underbrace{d(\cos x)}_{-\sin x \, dx} = \sin x \, e^x - \cos x \, e^x - \int e^x \sin x \, dx\end{aligned}$$

So, we obtain

$$\int e^x \sin x \, dx = \frac{\sin x - \cos x}{2} + C$$

5. Method of substitution

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ and $\varphi : (\alpha, \beta) \rightarrow (a, b)$ is differentiable and invertible on (α, β) . Then

$$\int f(x) \, dx = \left[\begin{array}{l} x = \varphi(t) \\ dx = \varphi'(t) \, dt \end{array} \right] = \int f(\varphi(t)) \varphi'(t) \, dt \Big|_{t=\varphi^{-1}(x)}$$

EXAMPLES.

$$1) \int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$$

PROOFS.

1.

$$\begin{aligned}\int \frac{dx}{(1-x^2)^{3/2}} &= \left[\begin{array}{l} x = \sin t \\ dx = \cos t \, dt \\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{array} \right] = \int \frac{\cos t}{(1-\sin^2 t)^{3/2}} \, dt = \int \frac{dt}{\cos^2 t} = \\ &= \operatorname{tg} t + C = [t = \arcsin x] = \operatorname{tg}(\arcsin x) + C = \\ &= \left[\begin{array}{l} \operatorname{tg} t = \frac{\sin t}{\sqrt{1-\sin^2 t}} \\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{array} \right] = \frac{x}{\sqrt{1-x^2}} + C\end{aligned}$$

6. Using trigonometrical formulas

Power-reduction formulas:

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Double angle formulas:

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

EXAMPLE.

$$\begin{aligned} 1) \int \sin^2 x \, dx &= \frac{x}{2} - \frac{\sin 2x}{4} + C, & \int \cos^2 x \, dx &= \frac{x}{2} + \frac{\sin 2x}{4} + C \\ 2) \int \frac{dx}{\sin x} &= \ln \left| \operatorname{tg} \frac{x}{2} \right| + C, & \int \frac{dx}{\cos x} &= -\ln \left| \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| + C \end{aligned}$$

PROOFS.

1.

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx = \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x \, dx = \frac{x}{2} - \frac{1}{4} \sin 2x + C$$

2.

$$\begin{aligned} \int \frac{dx}{\sin x} &= \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \frac{1}{2} \int \frac{dx}{\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \cdot \cos^2 \frac{x}{2}} = \frac{1}{2} \int \frac{dx}{\operatorname{tg} \frac{x}{2} \cdot \cos^2 \frac{x}{2}} = \\ &= \left[\begin{array}{l} y = \operatorname{tg} \frac{x}{2} \\ dy = \frac{dx}{2 \cos^2 \frac{x}{2}} \end{array} \right] = \int \frac{dy}{y} = \ln |y| + C = \left[y = \operatorname{tg} \frac{x}{2} \right] = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C \end{aligned}$$

$$\cos x = \sin \left(\frac{\pi}{2} - x \right)$$

$$\begin{aligned} \int \frac{dx}{\cos x} &= \int \frac{dx}{\sin \left(\frac{\pi}{2} - x \right)} = \left[\begin{array}{l} y = \frac{\pi}{2} - x \\ dy = -dx \end{array} \right] = - \int \frac{dy}{\sin y} = -\ln \left| \operatorname{tg} \frac{y}{2} \right| + C = \\ &= \left[y = \frac{\pi}{2} - x \right] = -\ln \left| \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| + C \end{aligned}$$

1.14 Definite integral

1. A partition of an interval

A *partition* of an interval $[a, b]$ is a finite sequence of numbers $T = \{x_i\}_{i=0}^N$ such that

$$a = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = b$$

The *mesh* $\lambda(T)$ of a partition T is the length of the longest interval in this partition:

$$\lambda(T) := \max_{i=1, \dots, N} \Delta x_i, \quad \Delta x_i := x_i - x_{i-1}$$

Let $T = \{x_i\}_{i=0}^N$ be a partition of $[a, b]$. Choose a finite set of points $\xi = \{\xi_i\}_{i=1}^N$ so that

$$\xi_i \in [x_{i-1}, x_i], \quad i = 1, \dots, N$$

The pair (T, ξ) is called a *partition with distinguished points* or a *tagged partition* of $[a, b]$.

2. Riemann sum

DEFINITION. Assume $f : [a, b] \rightarrow \mathbb{R}$ and let (T, ξ) be a tagged partition of $[a, b]$, where

$$T = \{x_i\}_{i=0}^N, \quad \xi = \{\xi_i\}_{i=1}^N, \quad \xi_i \in [x_{i-1}, x_i].$$

The *Riemann sum* of the function f corresponding to the tagged partition (T, ξ) is the sum

$$\sigma(T; \xi) := \sum_{i=1}^N f(\xi_i) \Delta x_i, \quad \Delta x_i := x_i - x_{i-1}$$

3. Riemann integral

DEFINITION. A function $f : [a, b] \rightarrow \mathbb{R}$ is called *Riemann-integrable* on $[a, b]$ if there exists a finite limit

$$\exists I \in \mathbb{R} : \quad I = \lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^N f(\xi_i) \Delta x_i,$$

as the mesh $\lambda(T)$ of the partition tends to zero, and this limit does not depend neither on the method of partitioning the interval $[a, b]$ by points $T = \{x_i\}_{i=0}^N$, nor on the choice of points ξ_i in the intervals $[x_{i-1}, x_i]$. If such limit exists, then the number I is called the *definite integral* (or the *Riemann integral*) of the function f over the interval $[a, b]$ and is denoted by

$$\int_a^b f(x) dx := I$$

4. Refinement of the definition of the Riemann integral

The definition of the integral above is an informal one: we know only the definition of a function depending on some variable. But the Riemann sum is not a function of a *mesh* of the partition, since it also depends on the choice of points x_j and ξ_j . The rigorous definition of the integral can be done only using ε - δ formalism: the function f is called *integrable* on $[a, b]$ if there exists a number $I \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any partition (T, ξ) of the interval $[a, b]$ with $T = \{x_i\}_{i=0}^N$ and $\xi = \{\xi_i\}_{i=1}^N$, $\xi_i \in [x_{i-1}, x_i]$, the following implication holds:

$$\lambda(T) < \delta \quad \implies \quad \left| I - \sum_{i=1}^N f(\xi_i) \Delta x_i \right| < \varepsilon$$

5. The geometric meaning of the definite integral

- For a non-negative function f , the Riemann sum is the sum of the areas of rectangles with bases $[x_{i-1}, x_i]$ and heights $f(\xi_i)$.
- For a non-negative function f , the definite integral is equal to the area of the figure bounded by the graph of the function f , the axis OX , and the parallel lines $x = a$, $x = b$.
- For an sign-indefinite function f , the definite integral is equal to the algebraic sum of the areas of the figures formed by the parts of the graph f lying above the axis OX (taken with a “plus” sign) and the ones lying below the axis OX (taken with a “minus” sign).

6. Example of a non-integrable function

In the next section we will show that any continuous or piecewise-continuous function is integrable. A typical example of a non-integrable function is the following function (which is discontinuous at every point):

$$f(x) = \begin{cases} 1, & x \in [a, b] \cap \mathbb{Q} \\ 0, & x \in [a, b] \setminus \mathbb{Q} \end{cases}$$

7. Integral with “upside down” limits

$$a < b \quad \implies \quad \int_b^a f(x) \, dx \stackrel{\text{def}}{=} - \int_a^b f(x) \, dx$$

8. Integrable functions are bounded

THEOREM. f is integrable on $[a, b] \implies f$ is bounded on $[a, b]$

9. Darboux sums

DEFINITION. Assume $f : [a, b] \rightarrow \mathbb{R}$ is bounded on $[a, b]$ and $T = \{x_j\}_{j=1}^N$ is a partition of the interval $[a, b]$, i.e.

$$a = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = b$$

Denote

$$m_j = \inf_{x \in [x_{j-1}, x_j]} f(x), \quad M_j = \sup_{x \in [x_{j-1}, x_j]} f(x), \quad j = 1, \dots, N.$$

The *lower* and the *upper Darboux sums* corresponding to the partition T are

$$s(T) := \sum_{j=1}^N m_j \Delta x_j, \quad S(T) := \sum_{j=1}^N M_j \Delta x_j, \quad \Delta x_j := x_j - x_{j-1}.$$

10. Refinement and common refinement of partitions

DEFINITION. A partition $\tilde{T} = \{y_j\}_{j=1}^N$ is the *refinement* of a partition $T = \{x_i\}_{i=1}^m$ if all points included in T are also included in \tilde{T} , i.e. $x_i = y_{n_i}$ for some $n_1 < n_2 < \dots < n_m$.

The partition $T_1 \cdot T_2$ is the *common refinement* of partitions T_1 and T_2 if it is formed by all points included in T_1 or T_2 .

The common refinement of two partitions is a refinement of each of the original partitions.

11. Properties of the Darboux sums

- 1) for any tagged partition the Riemann sum $\sigma(T, \xi)$ is in between the corresponding Darboux sums $s(T)$ and $S(T)$: $\forall T = \{x_j\}_{j=1}^N, \forall \xi = \{\xi_j\}_{j=1}^N \quad s(T) \leq \sigma(T, \xi) \leq S(T)$
- 2) if T_1 is a refinement of T then $s(T_1) \geq s(T), \quad S(T_1) \leq S(T)$
- 3) \forall partitions T_1 and $T_2 \implies s(T_1) \leq S(T_2)$
- 4) for any fixed partition $T = \{x_j\}_{j=1}^N$ the infimum and supremum of all possible values of Riemann's sums $\sigma(T; \xi)$ are equal respectively to the lower and upper Darboux sums $s(T)$ and $S(T)$

$$s(T) = \inf_{\xi} \sigma(T; \xi), \quad S(T) = \sup_{\xi} \sigma(T; \xi)$$

where the infimum is taken over all sets of points $\xi = \{\xi_j\}_{j=1}^N$ such that $\xi_j \in [x_{j-1}, x_j]$.

12. The basic condition for Riemann integrability

THEOREM. Assume f is bounded on $[a, b]$. Then f is integrable on $[a, b]$ if and only if $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ such that for any partition T of $[a, b]$ the following implication holds

$$\lambda(T) < \delta \implies S(T) - s(T) < \varepsilon,$$

where $\lambda(T)$ is the mesh of the partition T .

13. Oscillation and the Darboux sums

NOTATION. For any partition T of the interval $[a, b]$ we have

$$S(T) - s(T) = \sum_{j=1}^N \omega_j(f) \Delta x_j, \quad \omega_j(f) := M_j - m_j, \quad \Delta x_j := x_j - x_{j-1}$$

The number $\omega_j(f)$ is called the *oscillation* of f over the interval $[x_{j-1}, x_j]$.

14. Continuous functions are integrable

THEOREM. Assume f is continuous on $[a, b]$. Then f is integrable on $[a, b]$.

1.15 Properties of definite integrals

1. Basic properties

1. f, g are integrable on $[a, b]$, $\alpha, \beta \in \mathbb{R} \implies \alpha f + \beta g$ is integrable on $[a, b]$,

$$\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

2. $a < c < b \implies f$ is integrable on $[a, b] \Leftrightarrow f$ is integrable on $[a, c]$ and $[c, b]$,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

3. f, g are integrable on $[a, b]$, $f(x) \leq g(x)$, $\forall x \in [a, b] \implies$

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$

4. f is integrable on $[a, b] \implies |f|$ is integrable on $[a, b]$ and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

5. f is integrable on $[a, b]$, $m \leq f(x) \leq M$, $\forall x \in [a, b] \implies$

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

6. f is continuous on $[a, b] \implies \exists c \in [a, b]$:

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx \quad (\text{Mean value theorem})$$

7. f is continuous on $[a, b]$, g is integrable and non-negative on $[a, b] \implies \exists c \in [a, b]$:

$$\int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx$$

2. Fundamental theorem of calculus

THEOREM. Assume $f : [a, b] \rightarrow \mathbb{R}$ is integrable on $[a, b]$. Define $\Phi : [a, b] \rightarrow \mathbb{R}$,

$$\Phi(x) := \int_a^x f(t) dt, \quad x \in [a, b] \quad \text{— the integral with variable upper limit.}$$

If f is continuous on $[a, b]$ then Φ is differentiable on $[a, b]$ and

$$\forall x_0 \in (a, b) \quad \Phi'(x_0) = f(x_0).$$

COROLLARY. $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b] \implies f$ has a primitive on (a, b) .

3. Newton–Leibniz formula

THEOREM. Let f be continuous on $[a, b]$ and let F be any primitive of f on (a, b) . Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

4. Integration by parts in a definite integral

THEOREM. Let f, g be differentiable on $[a, b]$ and f', g' be continuous on $[a, b]$. Then

$$\int_a^b f(x)g'(x) dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x) dx.$$

ABBREVIATED NOTATION.

$$\int_a^b fg' dx = fg \Big|_a^b - \int_a^b f'g dx$$

5. Change of variables in the Riemann integral

THEOREM. Assume $\varphi : [\alpha, \beta] \rightarrow [a, b]$ is differentiable bijection of $[\alpha, \beta]$ onto $[a, b]$, φ' is continuous and strictly positive on $[\alpha, \beta]$. Assume f is continuous on $[a, b]$. Then the function $(f \circ \varphi) \cdot \varphi'$ is continuous on $[\alpha, \beta]$ and

$$\int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t) dt = \left[\begin{array}{l} x = \varphi(t) \\ dx = \varphi'(t) dt \end{array} \right] = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x) dx$$

1.16 Applications of definite integrals

1. Area of a plane figure

DEFINITION. A subset of the plane F is called *Jordan measurable* if for any $\varepsilon > 0$ there exist two polygons F_1 and F_2 such that

$$F_1 \subset F \subset F_2 \quad \text{and} \quad S(F_2 \setminus F_1) < \varepsilon$$

(here for every polygon P on the plane we denote by $S(P)$ its area). For the Jordan measurable figure F its area is defined

$$S(F) = \sup_{F_1 \subset F} S(F_1) = \inf_{F \subset F_2} S(F_2)$$

where the supremum is taken over all polygons F_1 contained in F , and the infimum is taken over all polygons F_2 containing F . (These supremum and infimum are equal, since the figure is Jordan measurable).

2. Area of a curved trapezoid

THEOREM.

1) $f : [a, b] \rightarrow \mathbb{R}$, $f(x) \geq 0$, $\forall x \in [a, b]$, and assume $\mathcal{F} \subset \mathbb{R}^2$ is given by

$$\mathcal{F} = \{ (x, y) \in \mathbb{R}^2 : x \in [a, b], 0 \leq y \leq f(x) \}$$

Then

$$S_{\mathcal{F}} = \int_a^b f(x) dx$$

2) $f, g : [a, b] \rightarrow \mathbb{R}$, $f(x) \geq g(x)$, $\forall x \in [a, b]$, and assume $\mathcal{F} \subset \mathbb{R}^2$ is given by

$$\mathcal{F} = \{ (x, y) \in \mathbb{R}^2 : x \in [a, b], g(x) \leq y \leq f(x) \}$$

Then

$$S_{\mathcal{F}} = \int_a^b (f(x) - g(x)) dx$$

3. Polar coordinate on the plane

Converting from the polar coordinates r and φ to the Cartesian coordinates x and y :

$$\begin{cases} x = r \cos \varphi & r \in [0, +\infty) \\ y = r \sin \varphi & \varphi \in [0, 2\pi) \end{cases}$$

Converting from the Cartesian coordinates x and y to the polar coordinates r and φ :

$$\begin{cases} r = \sqrt{x^2 + y^2} & \varphi \in [0, \pi], \quad \text{if } y \geq 0 \\ \operatorname{tg} \varphi = \frac{y}{x} & \varphi \in (\pi, 2\pi), \quad \text{if } y < 0 \end{cases}$$

4. Area of a curved sector

THEOREM. $f : [\alpha, \beta] \rightarrow [0, +\infty)$, $0 \leq \alpha < \beta \leq 2\pi$, and assume $\mathcal{F} \subset \mathbb{R}^2$ is given by

$$\mathcal{F} := \{ (r, \varphi) : \varphi \in [\alpha, \beta], 0 \leq r \leq f(\varphi) \}$$

Then

$$S_{\mathcal{F}} = \frac{1}{2} \int_{\alpha}^{\beta} (f(\varphi))^2 d\varphi$$

5. Curves on the plane

DEFINITION. Let $\gamma : [a, b] \rightarrow \mathbb{R}^2$, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be continuous on $[a, b]$

- The mapping $\gamma : [a, b] \rightarrow \mathbb{R}^2$ is called a *parameterized plane curve*.
- γ is *regular* if γ is differentiable on $[a, b]$, γ' is continuous on $[a, b]$ and

$$|\gamma'(t)| := \sqrt{(\gamma'_1(t))^2 + (\gamma'_2(t))^2} \neq 0, \quad \forall t \in [a, b].$$

- If $T = \{t_j\}_{j=0}^N$ is some partition of $[a, b]$, i.e. $a = t_0 < t_1 < t_2 < \dots < t_N = b$, then a broken line consisting of segments $[\gamma(t_{j-1}), \gamma(t_j)]$ is said to be *inscribed* in the curve γ . The *length* of a broken line corresponding to the partition T is

$$\sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})|$$

where

$$|\gamma(t_j) - \gamma(t_{j-1})| := \sqrt{(\gamma_1(t_j) - \gamma_1(t_{j-1}))^2 + (\gamma_2(t_j) - \gamma_2(t_{j-1}))^2}$$

- A curve γ is called *rectifiable* if there is a limit on the lengths of all broken lines inscribed in this curve as the mesh of the partition tends to zero. In this case, the *length* is

$$l_{\gamma} = \lim_{\lambda(T) \rightarrow 0} \sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})|$$

- More precisely, a curve is called rectifiable if there is a number $l_{\gamma} \in \mathbb{R}$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that for any partition $T = \{t_j\}_{j=1}^N$ of the segment $[a, b]$ the implication is true

$$\lambda(T) < \delta \quad \implies \quad \left| l_{\gamma} - \sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})| \right| < \varepsilon$$

Here $\lambda(T) = \max_{j=1, \dots, N} (t_j - t_{j-1})$ is the mesh of the partition T .

6. Length of a curve

THEOREM.

1) Length of a parameterized curve

Any regular parameterized curve $\gamma : [a, b] \rightarrow \mathbb{R}^2$ is rectifiable and its length is

$$l_\gamma = \int_a^b |\gamma'(t)| dt := \int_a^b \sqrt{(\gamma'_1(t))^2 + (\gamma'_2(t))^2} dt$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $\gamma'(t) = (\gamma'_1(t), \gamma'_2(t))$.

2) Length of the curve that is the graph of the function

$f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ and f' is continuous on $[a, b]$,

$$\Gamma := \{ (x, y) \in \mathbb{R}^2 : x \in [a, b], y = f(x) \}$$

Then

$$l_\Gamma = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

3) Length of the curve, specified in polar coordinates

Let (r, φ) be polar coordinates on the plane \mathbb{R}^2 , let $f : [\alpha, \beta] \rightarrow [0, +\infty)$ be continuously differentiable on $[\alpha, \beta]$, $0 \leq \alpha < \beta \leq 2\pi$, and

$$\Gamma := \{ (r, \varphi) : \varphi \in [\alpha, \beta], r = f(\varphi) \}$$

Then

$$l_\Gamma = \int_\alpha^\beta \sqrt{(f(\varphi))^2 + (f'(\varphi))^2} d\varphi$$

7. Generalized Cavalieri's principle (1635)

For any solid of revolution (such as a barrel)

$$\text{Volume of the barrel} \approx \sum_j \left(\text{Area of a crossection of a barrel} \right)_j \cdot \left(\text{Vertical height} \right)_j$$

$$\text{Area of a side surface} \approx \sum_j \left(\text{Length of a crossection of a barrel} \right)_j \cdot \left(\text{Vertical \underline{length}} \right)_j$$

8. The volume of a solid of revolution

THEOREM. Assume $f : [a, b] \rightarrow \mathbb{R}$ is non-negative and continuous on $[a, b]$ and

$$\Omega := \left\{ (x, y, z) \in \mathbb{R}^3 : x \in [a, b], y^2 + z^2 \leq f^2(x) \right\}$$

Then the volume $V(\Omega)$ of a solid of revolution Ω is equal to

$$V(\Omega) = \pi \int_a^b f^2(x) dx$$

9. The area of the side surface of a circular cone

THEOREM. The side surface of a circular cone whose base radius is R and height H :

$$S_{R,H} = \pi R \sqrt{R^2 + H^2}$$

For a truncated cone:

$$S_{R,H} - S_{r,h} = \pi \sqrt{1 + \operatorname{tg}^2 \alpha} \cdot (RH - rh) = 2\pi \sqrt{1 + \operatorname{tg}^2 \alpha} \cdot \frac{R+r}{2} \cdot (H-h),$$

where $\operatorname{tg} \alpha = \frac{r}{h} = \frac{R}{H}$.

10. The area of the side surface of a solid of revolution

THEOREM. Assume $f : [a, b] \rightarrow \mathbb{R}$ is non-negative and differentiable on (a, b) , f' is continuous on $[a, b]$ and

$$\Sigma := \left\{ (x, y, z) \in \mathbb{R}^3 : x \in [a, b], y^2 + z^2 = f^2(x) \right\}$$

Then the area of the side surface $S(\Sigma)$ the corresponding solid of revolution is equal to

$$S(\Sigma) = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx$$

1.17 Comparison of infinitesimals and Taylor's formula

1. L'Hôpital's (actually Bernoulli's) rule for an indeterminate form $\frac{0}{0}$

THEOREM. Let $f, g : (a, b) \rightarrow \mathbb{R}$ be differentiable on (a, b) and $g'(x) \neq 0, \forall x \in (a, b)$. Assume

$$1) \quad \exists c \in (a, b): \quad f(c) = g(c) = 0$$

$$2) \quad \exists k \in \mathbb{R}: \quad \exists \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = k$$

Then there exists the limit

$$\exists \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = k.$$

2. Other versions of the L'Hôpital rule

There are also versions of the L'Hôpital rule which suit to the following cases:

- Limit at infinity: $f, g : [a, +\infty) \rightarrow \mathbb{R}, \quad \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} g(x) = 0,$

$$\exists k \in \mathbb{R} : \quad \exists \lim_{x \rightarrow +\infty} \frac{f'(x)}{g'(x)} = k \quad \implies \quad \exists \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = k$$

- Indeterminate form $\frac{\infty}{\infty}$: $f, g : (a, b) \rightarrow \mathbb{R}, \quad c \in (a, b), \quad \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow c} g(x) = +\infty,$

$$\exists k \in \mathbb{R} : \quad \exists \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = k \quad \implies \quad \exists \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = k$$

- Version of the above statements with $k = +\infty$ or $k = -\infty$
- Version of the above statements with $x \rightarrow -\infty$ instead of $x \rightarrow +\infty$

All these statements can be proved similarly to the proof of the first statement and we omit their proofs in our course. Informally we can summarize the L'Hôpital rules as follows:

The limit of a ratio of functions is equal to the limit of the ratio of their derivatives if the latter exists.

3. Taylor's formula

THEOREM. $f : [a, b] \rightarrow \mathbb{R}$ is differentiable $(n+1)$ times on $(a, b) \implies \forall x, x_0 \in (a, b)$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1}(x),$$

where

$$R_{n+1}(x) := \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t)(x-t)^n dt \quad \text{--- the integral form of the remainder}$$

If $f^{(n+1)}$ is continuous on $[a, b]$ then there exists $\xi \in [x_0, x]$ such that

$$R_{n+1}(x) := \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1} \quad \text{--- the Lagrange form of the remainder}$$

COROLLARY. $[a, b]$ is bounded, $f^{(n+1)}$ is continuous on $[a, b] \implies R_{n+1}(x) = O(|x-x_0|^{n+1})$

4. O - asymptotic notation

Assume $f, g : (a, b) \rightarrow \mathbb{R}$ (we allow here a and b be infinite), $x_0 \in (a, b)$. Then

- $g = O(f)$ as $x \in (a, b) \iff \exists M > 0: \forall x \in (a, b): |g(x)| \leq M |f(x)|$
- $g = O(f)$ as $x \rightarrow x_0 \iff \exists M > 0, \delta > 0: \forall x \in (x_0 - \delta, x_0 + \delta): |g(x)| \leq M |f(x)|$
- $g = O(f)$ as $x \rightarrow +\infty \iff \exists M > 0, R > 0: \forall x \geq R: |g(x)| \leq M |f(x)|$
- $g = O(f)$ as $x \rightarrow -\infty \iff \exists M > 0, R < 0: \forall x \leq R: |g(x)| \leq M |f(x)|$

5. o - asymptotic notation

Assume $f, g : (a, b) \rightarrow \mathbb{R}$, $-\infty \leq a \leq x_0 \leq b \leq +\infty$.

- $g = o(f)$ as $x \rightarrow x_0 \iff \exists \alpha : (a, b) \rightarrow \mathbb{R}: \lim_{x \rightarrow x_0} \alpha(x) = 0 \text{ and } g(x) = \alpha(x)f(x)$

6. Properties of O - and o - symbols

THEOREM. Assume $f, g : (a, b) \rightarrow \mathbb{R}$, $-\infty \leq a \leq x_0 \leq b \leq +\infty$.

1) for $x \in (a, b)$ we have

- $O(f) + O(f) = O(f)$
- $O(f) \cdot O(g) = O(fg)$

2) for $x \rightarrow x_0$ we have

- $g = o(f) \implies g = O(f)$
- $o(f) + o(f) = o(f)$
- $o(f) \cdot O(g) = o(fg)$

3) in particular, for $x \rightarrow 0$ we have

- $O(x^n) \cdot O(x^m) = O(x^{n+m})$
- $o(x^n) \cdot O(x^m) = o(x^{n+m})$
- $O(x^n) = o(x^m)$, if $n > m$

7. Asymptotics of the reminder in Taylor's expansion

THEOREM. Assume $f : (a, b) \rightarrow \mathbb{R}$ is differentiable $(n + 1)$ times on (a, b) and $f^{(n+1)}$ is continuous on (a, b) . Then for any $x_0 \in (a, b)$

$$R_{n+1}(x) = O((x - x_0)^{n+1}) \quad \text{as } x \rightarrow x_0,$$

where we denote by $R_{n+1}(x)$ the reminder in the Taylor formula for f .

8. Taylor expansions of some elementary functions

THEOREM. The following asymptotics holds as $x \rightarrow 0$:

- 1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$
- 2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + o(x^{2n})$
- 3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$
- 4) $(1 + x)^r = 1 + \frac{r}{1!} x + \frac{r(r-1)}{2!} x^2 + \frac{r(r-1)(r-2)}{3!} x^3 + \dots + \frac{r(r-1)\dots(r-n+1)}{n!} x^n + o(x^n)$
- 5) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$

1.18 Improper integrals

1. Improper integral of an unbounded function

DEFINITION. Assume $f : [a, b) \rightarrow \mathbb{R}$ is unbounded near $x = b$ and integrable over $[a, c]$ for any $c \in [a, b)$. We say the *improper integral* of f over $[a, b)$ is *convergent* if there exists a finite limit

$$\int_a^b f(x) dx = \lim_{c \rightarrow b-0} \int_a^c f(x) dx,$$

Otherwise we say the corresponding improper integral is *divergent*. For $f : (a, b] \rightarrow \mathbb{R}$ which is unbounded near $x = a$ the improper integral is defined in a similar way:

$$\int_a^b f(x) dx = \lim_{c \rightarrow a+0} \int_c^b f(x) dx,$$

2. Improper integral over an unbounded interval

DEFINITION. Assume $f : [a, +\infty) \rightarrow \mathbb{R}$ is integrable over $[a, c]$ for any $c \in [a, +\infty)$. We say the *improper integral* of f over $[a, +\infty)$ is *convergent* if there exists a finite limit

$$\int_a^{+\infty} f(x) dx = \lim_{c \rightarrow +\infty} \int_a^c f(x) dx,$$

Otherwise we say the corresponding improper integral is *divergent*. For $f : (-\infty, b] \rightarrow \mathbb{R}$ the improper integral is defined in a similar way:

$$\int_{-\infty}^b f(x) dx = \lim_{c \rightarrow -\infty} \int_c^b f(x) dx,$$

3. Cauchy's criterion for convergence of improper integrals

THEOREM.

1) Assume $F : [a, b) \rightarrow \mathbb{R}$. Then

$$\exists \lim_{c \rightarrow b-0} F(c) \iff \forall \varepsilon > 0 \exists \delta > 0 : \forall c_1, c_2 \in (b - \delta, b) \quad |F(c_1) - F(c_2)| < \varepsilon.$$

2) In particular, for $f : [a, b) \rightarrow \mathbb{R}$

$$\int_a^b f(x) dx \text{ is convergent} \iff \forall \varepsilon > 0 \exists \delta > 0 : \forall c_1, c_2 \in (b - \delta, b) \quad \left| \int_{c_1}^{c_2} f(x) dx \right| < \varepsilon.$$

3) Assume $F : [a, +\infty) \rightarrow \mathbb{R}$. Then

$$\exists \lim_{c \rightarrow +\infty} F(c) \iff \forall \varepsilon > 0 \exists R > 0 : \forall c_1, c_2 \in [R, +\infty) \quad |F(c_1) - F(c_2)| < \varepsilon.$$

4) In particular, for $f : [a, b) \rightarrow \mathbb{R}$

$$\int_a^{+\infty} f(x) dx \text{ is convergent} \iff \forall \varepsilon > 0 \exists R > 0 : \forall c_1, c_2 \in [R, +\infty) \quad \left| \int_{c_1}^{c_2} f(x) dx \right| < \varepsilon.$$

4. Absolute convergence of improper integrals

DEFINITION. Assume $f : [a, b) \rightarrow \mathbb{R}$ is integrable over $[a, c]$ for any $c \in [a, b)$ and either the interval $[a, b)$ is finite and f is unbounded near $x = b$ or $b = +\infty$. We say the improper integral of f over the interval $[a, b)$ *converges absolutely* if the improper integral of $|f|$ over $[a, b)$ is convergent:

$$\int_a^b |f(x)| dx < +\infty.$$

5. Absolutely convergent improper integral is convergent

THEOREM. If the improper integral of f over $[a, b)$, $b \leq +\infty$, converges absolutely then it converges:

$$\int_a^b |f(x)| dx \text{ is convergent} \implies \int_a^b f(x) dx \text{ is convergent}.$$

6. Necessary and sufficient condition for absolute convergence of improper integrals

THEOREM. The improper integral of f over $[a, b)$, $b \leq +\infty$, converges absolutely if and only if the primitive of $|f|$ defined by

$$F(c) := \int_a^c |f(x)| dx, \quad c \in [a, b)$$

is bounded on the interval $[a, b)$.

7. Comparison test for absolute convergence of improper integrals

THEOREM. Assume $f, g : [a, b) \rightarrow \mathbb{R}$ are integrable over $[a, c]$ for any $c \in [a, b)$ and either the interval $[a, b)$ is finite and f is unbounded near $x = b$ or $b = +\infty$. Assume

$$0 \leq f(x) \leq g(x) \quad \forall x \in [a, b)$$

If the improper integral of g over $[a, b)$ is convergent then the improper integral of f over $[a, b)$ is also convergent:

$$\begin{aligned} \int_a^b g(x) \, dx \text{ is convergent} &\implies \int_a^b f(x) \, dx \text{ is convergent} \\ \int_a^b f(x) \, dx \text{ is divergent} &\implies \int_a^b g(x) \, dx \text{ is divergent} \end{aligned}$$

8. Dirichlet–Abel test for non-absolute convergence of improper integrals

THEOREM. Assume $f, g : [a, +\infty) \rightarrow \mathbb{R}$ satisfy the conditions

- 1) f is continuous on $[a, +\infty)$ and its primitive F is bounded on $[a, +\infty)$
- 2) g is non-negative, non-increasing, differentiable on $[a, +\infty)$, g' is continuous on $[a, +\infty)$
- 3) $\lim_{x \rightarrow +\infty} g(x) = 0$

Then the improper integral of fg over $[a, +\infty)$ is convergent:

$$\int_a^{+\infty} f(x)g(x) \, dx \text{ is convergent.}$$

9. There are improper integrals that are convergent, but not absolutely convergent

$$\begin{aligned} \int_1^{+\infty} \frac{\sin x}{x} \, dx, \quad \int_1^{+\infty} \frac{\cos x}{x} \, dx &\text{ are convergent non-absolutely,} \\ \int_1^{+\infty} \frac{|\sin x|}{x} \, dx, \quad \int_1^{+\infty} \frac{|\cos x|}{x} \, dx &\text{ are divergent} \end{aligned}$$

10. Improper integrals with more than one singularity

- If $f : (a, b) \rightarrow \mathbb{R}$ is unbounded near both ends of the interval (a, b) then we split the interval (a, b) by some point $c \in (a, b)$ so that f possess only one singularity on each of the intervals $(a, c]$ and $[c, b)$ and define the improper integral of f over (a, b) as the sum of improper integrals of f over $(a, c]$ and $[c, b)$.
- If $f : (a, +\infty) \rightarrow \mathbb{R}$ is unbounded near $x = a$ then we split the interval $(a, +\infty)$ by some point $c \in (a, +\infty)$ and define the improper integral of f over $(a, +\infty)$ as the sum of improper integrals of f over $(a, c]$ and $[c, +\infty)$.
- If $f : (a, b) \rightarrow \mathbb{R}$ is unbounded near an internal point $c \in (a, b)$ then we define the improper integral of f over (a, b) as the sum of improper integrals of f over (a, c) and (c, b) .

1.19 Infinite series

1. Convergent and divergent series

DEFINITION. Assume $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is a sequence. We say

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \text{ is convergent} &\iff \exists S \in \mathbb{R} : S = \lim_{n \rightarrow \infty} \sum_{j=1}^n a_j \\ \sum_{n=1}^{\infty} a_n \text{ is divergent} &\iff \text{there is no finite limit of } \sum_{j=1}^n a_j \text{ as } n \rightarrow \infty\end{aligned}$$

2. Cauchy's criterion for convergence

THEOREM. The following conditions are equivalent:

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \iff \forall \varepsilon > 0 \exists N = N(\varepsilon) : \forall n \geq m \geq N(\varepsilon) \sum_{j=m}^n a_j < \varepsilon$$

3. A necessary condition for convergence of a series

$$\text{THEOREM. } \sum_{n=1}^{\infty} a_n \text{ is convergent} \implies \lim_{n \rightarrow \infty} a_n = 0$$

4. Absolute convergence of series

$$\text{DEFINITION. } \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent} \iff \sum_{n=1}^{\infty} |a_n| \text{ is convergent.}$$

5. Absolutely convergent series is convergent

THEOREM. If the series converges absolutely then it converges:

$$\sum_{n=1}^{\infty} |a_n| \text{ is convergent} \implies \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

6. Comparison test for series with non-negative terms

THEOREM. Assume $0 \leq a_n \leq b_n \quad \forall n \in \mathbb{N}$. Then

$$\begin{aligned}1) \quad \sum_{n=1}^{\infty} b_n \text{ is convergent} &\implies \sum_{n=1}^{\infty} a_n \text{ is convergent} \\ 2) \quad \sum_{n=1}^{\infty} a_n \text{ is divergent} &\implies \sum_{n=1}^{\infty} b_n \text{ is divergent}\end{aligned}$$

7. M-test (majorant test) of convergence

THEOREM. Assume $\{b_n\}_{n=1}^{\infty}$ is non-negative and $\{a_n\}_{n=1}^{\infty}$ satisfies

$$|a_n| \leq b_n, \quad \forall n \in \mathbb{N}$$

Then

$$\sum_{n=1}^{\infty} b_n \text{ is convergent} \implies \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent.}$$

8. Cauchy's test for the absolute convergence

THEOREM. Assume $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and assume there exists a finite limit

$$\alpha := \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

Then

- 1) $\alpha < 1 \implies \sum_{n=1}^{\infty} a_n$ is absolutely convergent;
- 2) $\alpha > 1 \implies \sum_{n=1}^{\infty} a_n$ is divergent;
- 3) there exist both absolutely convergent and divergent series with $\alpha = 1$.

9. D'Alembert's test for the absolute convergence

THEOREM. Assume $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and assume there exists a finite limit

$$\beta := \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then

- 1) $\beta < 1 \implies \sum_{n=1}^{\infty} a_n$ is absolutely convergent;
- 2) $\beta > 1 \implies \sum_{n=1}^{\infty} a_n$ is divergent;
- 3) there exist both absolutely convergent and divergent series with $\beta = 1$.

10. Leibniz's test for alternating series

THEOREM. Assume $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ satisfies

- 1) $0 \leq a_{n+1} \leq a_n, \quad \forall n \in \mathbb{N}$
- 2) $\lim_{n \rightarrow \infty} a_n = 0$

Then

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{is convergent.}$$

11. Integral test for the absolute convergence

THEOREM. Assume $f : [1, +\infty) \rightarrow \mathbb{R}$ is continuous on $[1, +\infty)$ and

1) $f \geq 0$ on $[1, +\infty)$

2) f — \searrow on $[1, +\infty)$

Then

$$\sum_{n=1}^{\infty} f(n) \quad \text{and} \quad \int_1^{+\infty} f(x) dx \quad \text{converges or diverges simultaneously.}$$

12. There exist infinite series that converge but do not converge absolutely

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \quad \text{is convergent,} \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{is divergent}$$

2 Matrices and determinants

2.1 Gaussian elimination method

1. System of linear equations

We consider the $m \times n$ system of linear algebraic equations of the following form:

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3 \\ \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m \end{array} \right. \quad (*)$$

Here $x_1, x_2, x_3, \dots, x_n$ are unknowns, $a_{jk}, j = 1, \dots, m, k = 1, \dots, n$ and $b_1, b_2, b_3, \dots, b_m$ are given (real or complex) numbers.

By a solution of an $m \times n$ system, we mean an ordered n -tuple of numbers (x_1, x_2, \dots, x_n) that satisfies all the m equations of the system.

We say the system $(*)$ is *consistent* if there exists at least one solution to this system. We say the system $(*)$ is *inconsistent* if there is no solutions to this system.

2. Geometrical interpretation of a 2×2 system

Consistent, inconsistent and uniquely solvable systems:

$$\left\{ \begin{array}{l} x_1 + x_2 = 3 \\ x_1 - x_2 = 1 \end{array} \right. \quad \left\{ \begin{array}{l} x_1 + x_2 = 2 \\ 2x_1 + 2x_2 = 6 \end{array} \right. \quad \left\{ \begin{array}{l} x_1 + x_2 = 2 \\ 2x_1 + 2x_2 = 4 \end{array} \right.$$

3. Overdetermined and underdetermined linear systems

A linear system $(*)$ is said to be overdetermined if there are more equations than unknowns (i.e. $m > n$). Overdetermined systems are usually (but not always) inconsistent.

A linear system $(*)$ is said to be underdetermined if there are fewer equations than unknowns (i.e. $m < n$). Although it is possible for underdetermined systems to be inconsistent, they are usually consistent with infinitely many solutions.

4. Equivalent systems

DEFINITION. Two systems of equations involving the same variables are said to be *equivalent* if they have the same solution set.

5. Elementary operations with linear systems

THEOREM. Applying any of the following operations with the equation of a linear system we obtain the new linear system which is equivalent to the original one:

- I. Interchange equations in the system.
- II. Multiply both sides of some equation by a nonzero number.
- III. Replace some equation by its sum with a multiple of another equation.

6. The systems in the strict triangular form are easy to solve

We say an $n \times n$ systems is in *strict triangular form* if in the k -th equation the coefficients of the first $k - 1$ variables are all zero and the coefficient of x_k is nonzero ($k = 1, \dots, n$).

The systems in the strict triangular form are easy to solve by the back substitution:

$$\begin{aligned} \begin{cases} x_1 - 3x_2 + 2x_3 &= 7 \\ 2x_2 - 5x_3 &= 1 \\ x_3 &= 3 \end{cases} &\implies \begin{cases} x_1 - 3x_2 + 2x_3 &= 6 \\ x_2 &= \frac{1}{2}(1 + 5 \cdot 3) = 8 \\ x_3 &= 3 \end{cases} \\ &\implies \begin{cases} x_1 &= 7 + 3 \cdot 8 - 2 \cdot 3 = 25 \\ x_2 &= 8 \\ x_3 &= 3 \end{cases} \implies \begin{cases} x_1 &= 25 \\ x_2 &= 8 \\ x_3 &= 3 \end{cases} \end{aligned}$$

So, our strategy is: applying the elementary operations with the equations I, II and III to try to reduce our system to the equivalent system which has the strict triangular form.

7. Coefficient matrix, augmented matrix

To avoid extra writing we organize the coefficient of the linear system (*) as an $m \times n$ array

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

We will call this array the *coefficient matrix* of the system (*).

To include the right-hand side of the system (*) into consideration we can build an *augmented matrix* corresponding to the system (*):

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{array} \right)$$

8. Elementary row operations with matrices

Now the elementary operations I, II and III with the equations in the linear system (*) corresponds to the following elementary operations with the rows of the augmented matrix of the system (*):

- I. Interchange rows.
- II. Multiply a row by a nonzero real number.
- III. Replace a row by its sum with a multiple of another row.

9. Gaussian elimination

We want to try to the linear system (*) to an equivalent system in a triangular form.

- 1) Assume the first column of the coefficient matrix contain at least one non-zero entry. We can fix this element and refer to this element as a *pivot* element of the first column of the coefficient matrix. The row of an augmented matrix containing the pivot element is referred as the *pivotal row*.

Interchanging (if necessary) the rows of the augmented matrix (i.e. applying to the augmented matrix the elementary row operation I) we can move the pivotal row as the new first row.

Multiplying the pivotal row of the augmented matrix by an appropriate nonzero number and subtracting the results from each of the remaining $m - 1$ rows (i.e. applying to the augmented matrix the elementary row operation III) we can obtain zeros in the first entries of all rows from 2 to m .

- 2) At the second step, a pivot element is chosen from the nonzero entries in column 2, rows 2 through m (if there are no non-zero entries in among them then we just switch to the next column and look for a pivotal element there).

The row containing the pivot is then interchanged with the second row of the matrix and is used as the new pivotal row.

Multiples of the pivotal row are then subtracted from the remaining $m - 2$ rows so as to eliminate all entries below the pivot in the second column.

- etc) The same procedure is repeated for all the rest columns of the coefficient matrix until either there is no more rows under the next pivotal row or all rows of the coefficient matrix below the next pivotal row consist on zero entries.

Note that at the second step row 1 and column 1 remain unchanged, at the third step the first two rows and first two columns remain unchanged, and so on.

What we obtain at the end is we reduce the original augmented matrix to some matrix which has the *row echelon form* (see the definition below). The algorithm described above is called *Gaussian elimination*.

10. A matrix in the row echelon form

DEFINITION. A matrix is said to be in the *row echelon form* if

- The first nonzero entry in each row is 1.
- If row k does not consist entirely of zeros, the number of leading zero entries in row $k + 1$ is greater than the number of leading zero entries in row k .
- If there are rows whose entries are all zero, they are below the rows having nonzero entries.

A matrix is said to be in the *reduced row echelon form* if additionally

- The first nonzero entry in each row is the only nonzero entry in its column.

11. Solving of the linear systems by the Gaussian elimination method

THEOREM. With the help of row operations I, II, and III any linear system (*) can be transformed to an equivalent linear system whose augmented matrix is in the row echelon form.

12. Analysis of consistency and the unique solvability of a linear system from the row echelon form

THEOREM.

- 1) The system (*) is consistent if and only if after the Gaussian reduction of the augmented matrix of the system (*) to a row echelon form the non-pivotal rows of the coefficient matrix (i.e. those rows of the coefficient matrix whose all entries are zeros) have also zero entries in the right-hand side column.
- 2) The system (*) is uniquely solvable if and only if after the Gaussian reduction of the augmented matrix of the system (*) to a row echelon form and elimination of those rows of the augmented matrix which are identically zeros (both in the coefficient matrix and in the right-hand side column) the coefficient matrix has a strictly triangular form.

13. Lead variables, free variables

DEFINITION. If the matrix of a linear system is in the row echelon form then the variables corresponding to the columns containing the pivotal elements are called the *lead variables* and the variables corresponding to all other columns are called *free variables*.

14. Gauss-Jordan reduction

If the system in the row echelon form is consistent and contains free variables, it is convenient to continue the elimination process and simplify the form of the reduced matrix even further.

- 1) First we take the last pivotal row and using the elementary row operation III transform into zeros all entries above in the pivotal element in this row.
- 2) Then we take the previous pivotal row and do all the same with the entries of the column of the pivotal element in this row.

etc) The same procedure is repeated for all the rest pivotal rows until the first one.

What we obtain at the end is the *reduced row echelon form* of the augmented matrix of the system. The process of using of elementary row operations I, II, and III to transform a linear system into reduced row echelon form is called *Gauss-Jordan reduction*.

15. Solving of the underdetermined consistent linear systems

THEOREM.

- 1) If the linear system in the reduced row echelon form is consistent and contains free variables then we can assign arbitrary values to free variables and represent lead variables in the unique way via right and sides and free variables.
- 2) Underdetermined consistent linear systems have infinitely many solutions.
- 3) Any solution of such system can be specified by the particular choice of the values of parameters corresponding to the free variables.

16. Example

PROBLEM. Solve the following underdetermined system:

$$\begin{aligned}x_1 + 7x_2 + 9x_3 + 4x_4 &= 2 \\2x_1 + 2x_2 + 3x_3 + 5x_4 &= 4 \\5x_1 + 3x_2 + 5x_3 + 12x_4 &= 10\end{aligned}$$

SOLUTION.

$$\begin{aligned}&\left(\begin{array}{cccc|c}1 & 7 & 9 & 4 & 2 \\2 & 2 & 3 & 5 & 4 \\5 & 3 & 5 & 12 & 10\end{array}\right) \left[\begin{array}{l} \text{use } 1 \text{ as a pivotal entry} \\ \text{multiply the 1st row by } -2 \\ \text{and add to the 2nd row} \end{array} \right] \\&\left(\begin{array}{cccc|c}1 & 7 & 9 & 4 & 2 \\0 & -12 & -15 & -3 & 0 \\5 & 3 & 5 & 12 & 10\end{array}\right) \left[\begin{array}{l} \text{use } 1 \text{ as a pivotal entry} \\ \text{multiply the 1st row by } -5 \\ \text{and add to the 3rd row} \end{array} \right] \\&\left(\begin{array}{cccc|c}1 & 7 & 9 & 4 & 2 \\0 & -12 & -15 & -3 & 0 \\0 & -32 & -40 & -8 & 0\end{array}\right) \left[\begin{array}{l} \text{intermediate step:} \\ \text{multiply the 2nd row by } -\frac{1}{3} \\ \text{multiply the 3rd row by } -\frac{1}{8} \end{array} \right] \\&\left(\begin{array}{cccc|c}1 & 7 & 9 & 4 & 2 \\0 & 4 & 5 & 1 & 0 \\0 & 4 & 5 & 1 & 0\end{array}\right) \left[\begin{array}{l} \text{use } 4 \text{ as a pivotal entry} \\ \text{multiply the 2nd row by } -1 \\ \text{and add to the 3rd row} \end{array} \right] \\&\left(\begin{array}{cccc|c}1 & 7 & 9 & 4 & 2 \\0 & 4 & 5 & 1 & 0 \\0 & 0 & 0 & 0 & 0\end{array}\right) \left[\begin{array}{l} \text{intermediate step:} \\ \text{eliminate the identically zero row} \end{array} \right] \\&\left(\begin{array}{cccc|c}1 & 7 & 9 & 4 & 2 \\0 & 4 & 5 & 1 & 0\end{array}\right) \left[\begin{array}{l} \text{intermediate step:} \\ \text{multiply the 2nd row by } \frac{1}{4} \\ \text{to obtain the row echelon form} \end{array} \right]\end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 7 & 9 & 4 & 2 \\ 0 & 1 & \frac{5}{4} & \frac{1}{4} & 0 \end{array} \right) \quad \left[\begin{array}{l} \text{Gauss-Jordan reduction} \\ \text{multiply the 2nd row by } -7 \\ \text{and add to the 1st row} \end{array} \right]$$

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & \frac{9}{4} & 2 \\ 0 & 1 & \frac{5}{4} & \frac{1}{4} & 0 \end{array} \right) \quad \left[\begin{array}{l} \text{this is the reduced} \\ \text{row echelon form} \end{array} \right]$$

$$\text{Lead variables: } x_1, x_2 \quad \left[\begin{array}{l} \text{correspond to 1st and 2nd columns} \\ \text{containing pivotal elements} \end{array} \right]$$

$$\text{Free variables: } x_3, x_4 \quad \left[\begin{array}{l} \text{correspond to 3rd and 4th columns} \\ \text{which do not contain pivotal elements} \end{array} \right]$$

$$\begin{array}{rcl} x_1 & = & 2 - \frac{1}{4}x_3 - \frac{9}{4}x_4 \\ x_2 & = & -\frac{5}{4}x_3 - \frac{1}{4}x_4 \end{array} \quad \left[\begin{array}{l} \text{represent lead variables} \\ \text{as functions of free variables} \end{array} \right]$$

$$\begin{array}{rcl} x_3 & = & \alpha \\ x_4 & = & \beta \end{array} \quad \left[\begin{array}{l} \text{assign arbitrary values (parameters)} \\ \text{to the free variables} \end{array} \right]$$

$$\begin{array}{rcl} x_1 & = & 2 - \frac{1}{4}\alpha - \frac{9}{4}\beta \\ x_2 & = & -\frac{5}{4}\alpha - \frac{1}{4}\beta \end{array} \quad \left[\begin{array}{l} \text{represent lead variables} \\ \text{as functions of arbitrary parameters} \end{array} \right]$$

ANSWER. Any solution of the system is given by

$$\begin{array}{rcl} x_1 & = & 2 - \frac{1}{4}\alpha - \frac{9}{4}\beta \\ x_2 & = & -\frac{5}{4}\alpha - \frac{1}{4}\beta \\ x_3 & = & \alpha \\ x_4 & = & \beta \end{array} \quad \text{where } \alpha, \beta \in \mathbb{R} \text{ are arbitrary}$$

17. Homogeneous systems

DEFINITION. The system $(*)$ is said to be *homogeneous* if all right-hand sides are zeros, i.e.

$$b_1 = b_2 = \dots = b_m = 0$$

THEOREM.

- 1) Homogeneous system are always consistent.
- 2) Underdetermined homogeneous system of always has a nontrivial solution.

2.2 Matrix algebra

1. Matrix, augmented matrix

In general, a $m \times n$ *matrix* is the $m \times n$ array whose entries are numbers (real or complex). The entry denoted by a_{jk} stands in the j -th row and k -th column of the array:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

Given a $m \times n$ matrix $A = (a_{jk})$ and a $m \times r$ matrix $B = (b_{jl})$ we can always build a $m \times (n + r)$ *augmented* matrix $(A|B)$ which is

$$\left(\begin{array}{ccccc|cccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_{11} & b_{12} & \dots & b_{1r} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_{21} & b_{22} & \dots & b_{2r} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_{31} & b_{32} & \dots & b_{3r} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_{m1} & b_{m2} & \dots & b_{mr} \end{array} \right)$$

Two $m \times n$ matrices A and B are said to be equal iff $a_{ij} = b_{ij}$ for each i and j .

2. Row vectors and column vectors

A *scalar* is a 1×1 matrix. We identify this matrix with a number which is the only entry of this matrix (so, essentially a ‘scalar’ means a ‘number’):

$$(a) = a, \quad a \in \mathbb{R}$$

A *row vector* of order n is a $1 \times n$ matrix:

$$\mathbf{a} = (a_1, a_2, a_3, \dots, a_n), \quad a_j \in \mathbb{R}$$

A *column vector* (or just a *vector*) of order m is a $m \times 1$ matrix:

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ b_m \end{pmatrix}, \quad b_j \in \mathbb{R}$$

We denote by \mathbb{R}^m the set of all column vectors of order m with real entries.

3. Linear operations with matrices

DEFINITION. Assume $A = (a_{jk})$ and $B = (b_{jk})$ are $m \times n$ matrices and $\alpha \in \mathbb{R}$ is a scalar.

- The sum $C = A + B$ is the $m \times n$ matrix $C = (c_{jk})$ whose entries are

$$c_{jk} := a_{jk} + b_{jk}, \quad j = 1, \dots, m, \quad k = 1, \dots, n.$$

- The product $C = \alpha A$ is the $m \times n$ matrix $C = (c_{jk})$ whose entries are

$$c_{jk} := \alpha a_{jk}, \quad j = 1, \dots, m, \quad k = 1, \dots, n.$$

- The zero $m \times n$ matrix \mathbb{O} is the matrix whose all entries are zeros.
- We denote by $-A := (-1)A$ the *additive inverse* of A and define $A - B := A + (-B)$

4. Properties of linear operations with matrices

THEOREM.

- 1) $A + B = B + A$
- 2) $(A + B) + C = A + (B + C)$
- 3) $A + \mathbb{O} = A$
- 4) $A + (-A) = \mathbb{O}$
- 5) $(\alpha\beta)A = \alpha(\beta A)$
- 6) $1 \cdot A = A$
- 7) $(\alpha + \beta)A = \alpha A + \beta A$
- 8) $\alpha(A + B) = \alpha A + \alpha B$

Further properties:

- 9) $0 \cdot A = \mathbb{O}$
- 10) $\alpha \cdot \mathbb{O} = \mathbb{O}$

5. Matrix product

DEFINITION.

- If \mathbf{a} is a row vector of order n and \mathbf{x} is a column vector of the same order n then the *product* of a row vector by a column vector is a scalar

$$\mathbf{a} \cdot \mathbf{x} = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{pmatrix} = a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n$$

- Assume A is an $m \times n$ matrix and \mathbf{x} is a column vector of the order n then the *product* of a matrix by a column vector is a column vector of order m

$$A\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \dots & \dots & \dots & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}$$

In other words, the product of a matrix $A = (a_{jk})$ by a column vector $\mathbf{x} = (x_k)$ is the column vector $\mathbf{b} = (b_j)$ whose entries are the products of rows of the matrix A by the column \mathbf{x}

$$b_i = \sum_{k=1}^n a_{ik}x_k = (i\text{-th row of } A) \cdot \mathbf{x}$$

- Assume A is an $m \times n$ matrix, B is an $n \times l$ matrix. The the matrix product of $A = (a_{ik})$ by $B = (b_{kj})$ is the $m \times l$ matrix $C = (c_{ij})$ whose entries are

$$c_{ij} := \sum_{k=1}^n a_{ik}b_{kj} = (i\text{-th row of } A) \cdot (j\text{-th column of } B)$$

- Just as in ordinary algebra, if an expression involves both multiplication and addition and there are no parentheses to indicate the order of the operations, multiplications are carried out before additions. This is true for both scalar and matrix multiplications.

6. Properties of matrix product

THEOREM.

- 1) $\alpha(AB) = (\alpha A)B$
- 2) $(AB)C = A(BC)$
- 3) $A(B + C) = AB + AC$
- 4) $(A + B)C = AC + BC$

7. Transpose of a matrix

DEFINITION. An $n \times m$ matrix B is called a *transpose* of an $m \times n$ matrix A if $b_{ij} = a_{ji}$ for any $i = 1, \dots, n$ and $j = 1, \dots, m$. A transpose of A is denoted by A^T .

THEOREM.

- 1) $(A^T)^T = A$
- 2) $(\alpha A)^T = \alpha A^T$
- 3) $(A + B)^T = A^T + B^T$
- 4) $(AB)^T = B^T A^T$

2.3 Square matrices

1. Algebra of matrices

The product AB of two $n \times n$ matrices A and B is again a $n \times n$ matrix. So, the matrix product of two square matrix returns a matrix of the same type. Attention! To the contrast to scalars, the matrix product in general is not commutative:

In general:

$$AB \neq BA$$

2. Commuting matrices

DEFINITION. The *commutator* of two $n \times n$ matrices A and B is the matrix

$$[A, B] := AB - BA$$

Matrices A and B *commute* if $AB = BA$ or, equivalently, $[A, B] = \mathbb{O}$.

3. Power of a matrix

DEFINITION. For a $n \times n$ matrix A we can define the k -th power of A :

$$A^k = \underbrace{A \cdot A \cdot \dots \cdot A}_k$$

THEOREM. For any $n \times n$ matrix A and for any $k, m \in \mathbb{N}$

$$A^m A^k = A^k A^m = A^{k+m}$$

4. Identity matrix

DEFINITION. The $n \times n$ identity matrix I is a $n \times n$ matrix whose diagonal entries are equal to 1's and all other elements are zeros.

$$I = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Note that for any $n \times n$ matrix A

$$AI = IA = A$$

5. Inverse matrix

DEFINITION. An $n \times n$ matrix A is called *nonsingular* (or *invertible*) if there exists an $n \times n$ matrix $B \in \mathbb{M}^{n \times n}$ such that

$$AB = BA = I$$

The matrix B is called an *multiplicative inverse* (or simply *inverse*) of A and is denoted A^{-1} . An $n \times n$ matrix A is called *singular* if it does not have a multiplicative inverse.

THEOREM. If the inverse matrix exists, it is unique.

6. Inverse of the matrix product

THEOREM. Assume square matrices A and B are invertible. Then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

7. Symmetric and skew-symmetric matrices

DEFINITION. An $n \times n$ matrix A is called *symmetric* if

$$A^T = A$$

An $n \times n$ matrix A is called *skew symmetric* if

$$A^T = -A$$

8. Upper triangular, lower triangular and diagonal matrices

DEFINITION. Assume A is $n \times n$ matrix whose elements are $A = (a_{ij})$

- A is called *upper triangular* if $a_{ij} = 0$ for any $i < j$
- A is called *lower triangular* if $a_{ij} = 0$ for any $i > j$
- A is called *diagonal* if $a_{ij} = 0$ for any $i \neq j$

THEOREM. Assume A and B are $n \times n$ matrices. Then

- A and B are upper triangular $\implies AB$ is upper triangular
- A and B are lower triangular $\implies AB$ is lower triangular
- A and B are diagonal $\implies AB$ is diagonal

9. LU factorization of a matrix

DEFINITION. The factorization of a square matrix A into a product

$$A = LU$$

of a lower triangular matrix L and an upper triangular matrix U is called an *LU factorization*.

2.4 Matrices and linear systems

1. Linear system in the matrix form

Using the notion of matrix multiplication we can rewrite the linear system (*) in the form

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array} \right. \iff A\mathbf{x} = \mathbf{b} \quad (*)$$

where A is a given $m \times n$ coefficient matrix, $\mathbf{b} \in \mathbb{R}^m$ is a given vector and $\mathbf{x} \in \mathbb{R}^n$ is the unknown vector.

2. Linear combination of the vectors

DEFINITION. If $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are (row or column) vectors of order m and c_1, c_2, \dots, c_n are scalars, then the vector

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots + c_n\mathbf{a}_n$$

is said to be a *linear combination* of the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$.

3. Consistency for linear systems

THEOREM. Let A be $m \times n$ matrix whose columns are $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$:

$$A = (\mathbf{a}_1 \mid \mathbf{a}_2 \mid \dots \mid \mathbf{a}_n)$$

Then for any $\mathbf{x} \in \mathbb{R}^n$ the product $A\mathbf{x}$ is the linear combination of columns of the matrix A :

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n$$

COROLLARY. A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if \mathbf{b} can be written as a linear combination of the column vectors of A .

4. Equivalent systems

THEOREM. A is a $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$ and M is a $m \times m$ non-singular matrix. Then the systems

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad MA\mathbf{x} = M\mathbf{b}$$

are equivalent, i.e. if $\mathbf{x} \in \mathbb{R}^n$ is a solution of one of these systems then it is also a solution of the another one. In the matrix form:

$$(A \mid \mathbf{b}) \iff (MA \mid M\mathbf{b}) \quad \text{if } M \text{ is non-singular}$$

5. Elementary matrices

DEFINITION. We define elementary $m \times m$ matrices of the following types

- 1) Type I elementary matrix is a $m \times m$ matrix obtained from the $m \times m$ identity matrix by interchanging of some its two rows;

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad m = 3$$

- 2) Type II elementary matrix is a $m \times m$ matrix obtained by multiplying some row of the $m \times m$ identity matrix by a nonzero constant.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha \neq 0$$

- 3) Type III elementary matrix is a $m \times m$ matrix obtained from the $m \times m$ identity matrix by adding a multiple of one row to another row.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}$$

6. Inverse of an elementary matrix

THEOREM. If E is an elementary matrix then E is nonsingular and E^{-1} is an elementary matrix of the same type.

Type	EA	Algebraic form of E	E^{-1}
Type I	Interchange two rows of A	Obtained from I by interchanging of it's two rows	$E^{-1} = E$
Type II	Multiply a row of A by a scalar α	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Type III	Replace a row of A by it's sum with a multiple of another row	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha & 1 \end{pmatrix}$

7. Multiplication by an elementary matrix

THEOREM. Assume A is an $m \times n$ matrix and E is an $m \times m$ elementary matrix. Then multiplication of A by E from the left has the effect of performing the same operation with rows of A , i.e.

- 1) if E is the elementary matrix of type I obtained from the $m \times m$ identity matrix by interchanging its i -th and j -th rows then the matrix EA is obtained from A by interchanging its i -th and j -th rows;
- 2) if E is the elementary matrix of type II obtained from the $m \times m$ identity matrix by multiplication of its i -th row by α then the matrix EA is obtained from A by multiplication of its i -th row by α ;
- 3) if E is the elementary matrix of type III obtained from the $m \times m$ identity matrix by adding a multiple of j -th row to i -th row then the matrix EA is obtained from A by adding a multiple of j -th row of A to i -th row of A .

Moreover, if E is an $n \times n$ elementary matrix then multiplication of A by E from the right has the effect of performing the same operation with columns of A .

8. Row equivalent matrices

DEFINITION. Assume A and B are two $m \times n$ matrices. We say the matrix B is *row equivalent* to A if there exists a finite sequence E_1, E_2, \dots, E_k of $m \times m$ elementary matrices such that

$$B = E_k E_{k-1} \dots E_1 A$$

REMARK. Row equivalence is an equivalence relation on the set of $m \times n$ matrices.

9. Matrix form of the Gaussian–Jordan elimination

THEOREM. Every $m \times n$ matrix A is row equivalent to some $m \times n$ matrix U in the reduced row echelon form:

$$\exists \text{ elementary } m \times m \text{ matrices } E_1, E_2, \dots, E_k : \quad E_k E_{k-1} \dots E_1 A = U,$$

$$A\mathbf{x} = \mathbf{b} \quad \Longleftrightarrow \quad U\mathbf{x} = E_k E_{k-1} \dots E_1 \mathbf{b}$$

10. Equivalent conditions for nonsingularity

THEOREM. Assume A is a square matrix. Then the following conditions are equivalent:

- 1) A is nonsingular.
- 2) $A\mathbf{x} = \mathbf{0}$ has only trivial solution $\mathbf{0}$.
- 3) A is row equivalent to I .

11. Equivalent conditions for the unique solvability of linear systems

THEOREM. Assume A is an $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$. The system of n linear equations in n unknowns $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A is nonsingular.

12. Finding the inverse matrix by the Gaussian method

THEOREM. Assume A is nonsingular and E_1, E_2, \dots, E_k are elementary matrices. Then

$$\text{if } E_k E_{k-1} \dots E_1 A = I \quad \text{then} \quad E_k E_{k-1} \dots E_1 I = A^{-1},$$

i.e. the same series of elementary row operations that transform nonsingular A into I will transform also I into A^{-1} :

$$(A|I) \longrightarrow (I|A^{-1})$$

13. Example

PROBLEM. Find the matrix A^{-1} inverse to the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

SOLUTION.

$$\begin{aligned} & \left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array} \right) & \left[\begin{array}{l} \text{use } \color{red}{1} \text{ as a pivotal entry} \\ \text{multiply the 1st row by } -1 \\ \text{and add to the 2nd row} \end{array} \right] \\ & \left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array} \right) & \left[\begin{array}{l} \text{use } \color{red}{1} \text{ as a pivotal entry} \\ \text{multiply the 1st row by } -2 \\ \text{and add to the 3rd row} \end{array} \right] \\ & \left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & \color{blue}{1} & -2 & 0 & 1 \end{array} \right) & \left[\begin{array}{l} \text{use } \color{red}{1} \text{ as a pivotal entry} \\ \text{multiply the 3rd row by } -3 \\ \text{and add to the 2nd row} \end{array} \right] \\ & \left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & -3 \\ 0 & 0 & \color{blue}{1} & -2 & 0 & 1 \end{array} \right) & \implies A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{pmatrix} \end{aligned}$$

Verification:

$$A^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{--- OK}$$

$$\text{ANSWER.} \quad A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{pmatrix}$$

14. Finding LU factorization of a square matrix

THEOREM. The $n \times n$ matrix A can be transformed into the $n \times n$ matrix U in the row echelon form with the help of elementary matrices E_1, E_2, \dots, E_k of Type III only (i.e. without interchanging rows and multiplication of rows by non-zero numbers):

$$E_k E_{k-1} \dots E_1 A = U$$

Define the matrix

$$L = E_1^{-1} E_2^{-1} \dots E_k^{-1}$$

Then the matrix L is lower triangular with all diagonal entries equal to 1 and the matrices L and U provide an LU -factorization of the matrix A :

$$A = LU,$$

where L is lower triangular and U is upper triangular.

15. Solving of consistent underdetermined systems

THEOREM. Assume the linear system $A\mathbf{x} = \mathbf{b}$ is consistent and row equivalent to the undetermined system

$$U\mathbf{x} = \mathbf{c} \quad (**)$$

where U is $m \times n$ matrix ($m < n$) in the reduced row echelon form. Assume the system $(**)$ contains $k = n - m$ free variables and denote by $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \dots, \mathbf{u}^{(k)}$ the columns of the matrix U corresponding to free variables (i.e. each $\mathbf{u}^{(i)} \in \mathbb{R}^m$). Then any solution of the equivalent systems $(*)$ and $(**)$ can be represented in the form

$$\mathbf{x} = \hat{\mathbf{c}} + \underbrace{\alpha_1 \hat{\mathbf{u}}^{(1)} + \alpha_2 \hat{\mathbf{u}}^{(2)} + \dots + \alpha_k \hat{\mathbf{u}}^{(k)}}_{\text{general solution of homogeneous system}}$$

where

- $\hat{\mathbf{c}} \in \mathbb{R}^n$ is the *extended column* corresponding to $\mathbf{c} \in \mathbb{R}^m$, i.e. $\hat{\mathbf{c}}$ is a column vector of order n whose entries matching to free variables are all zeros and entries matching to lead variables are equal to the corresponding entries of \mathbf{c}
- $\hat{\mathbf{u}}^{(i)} \in \mathbb{R}^n$ is the *extended column* corresponding to $\mathbf{u}^{(i)} \in \mathbb{R}^m$, i.e. $\hat{\mathbf{u}}^{(i)}$ is a column vector of order n whose entry matching to i -th free variable is equal to -1 , entries matching to other free variables are zeroes, and entries matching to free variables are all zeros and entries matching to lead variables are equal to the corresponding entries of $\mathbf{u}^{(i)}$
- $\alpha_1, \alpha_2, \dots, \alpha_k$ are arbitrary scalars.