Lecture "Experimental Physics I"

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Lecture 16

Static equilibrium

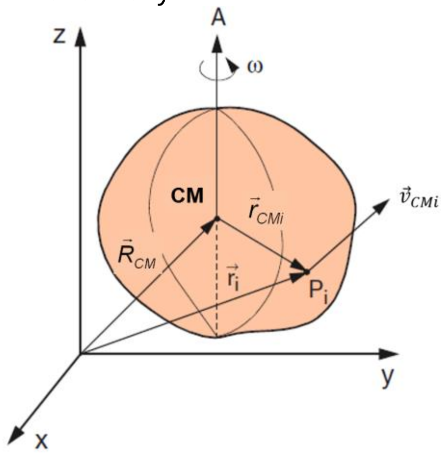
- Conditions for static equilibrium
- Lever law
- Ladder climbing problem

1) Static equilibrium

In the following we want to redefine mechanical equilibrium that we introduced for a point mass and extend it to a rigid body.

A) Motion of a rigid body

We first look again at the motion of a rigid body, where we said previously that it is fully described by the superposition of the rotation around its center of mass and the translation of the center of mass itself. Let us briefly derive this formally:



We look at a rigid body without a fixed rotation axis/attachment point and consider a small volume element at position $\vec{r}_i$. For its position with respect to the center of gravity we can write:

$$\vec{r}_{CMi} = \vec{r}_i - \vec{R}_{CM}$$

Differentiation by time provides then for the respective velocities:

$$\vec{v}_{CMi} = \vec{v}_i - \vec{V}_{CM}$$

For a rigid body, the distance of each volume element to the CM is constant, i.e.:

$$|\vec{r}_{CMi}| = \text{const.}$$

Differentiating the squared distance gives:

$$0 = \frac{d|\vec{r}_{CMi}|^2}{dt} = \frac{d(\vec{r}_{CMi} \cdot \vec{r}_{CMi})}{dt} = 2\vec{r}_{CMi} \cdot \underbrace{\frac{d\vec{r}_{CMi}}{dt}}_{\vec{v}_{CMi}}$$

Since this is a scalar multiplication, velocity and distance vector to the CM are perpendicular to each other:

$$\vec{v}_{CMi} \perp \vec{r}_{CMi}$$

This makes intuitively sense, since a volume element can within the body not move in radial direction away from the CM. It only can move in the perpendicular, i.e. tangential, direction, which corresponds to a rotation around the center of mass with an angular velocity ω, such that we can write:

$$\vec{v}_{CMi} = \vec{\omega} \times \vec{r}_{CMi}$$

One can now use the condition that the difference vector between any two points must be of constant length in time to show that $\vec{\omega}$ must be the same vector for every point of the body. Inserting the tangential velocity relation into the CM transformation provides for the motion of each volume element in the body according to:

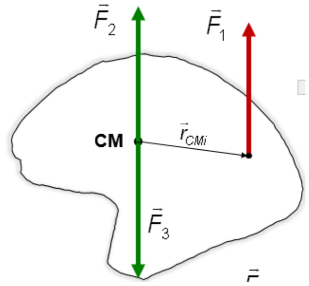
$$\vec{v}_i = \vec{V}_{CM} + \vec{\omega} \times \vec{r}_{CMi}$$

i.e. a **motion of a rigid body can always be dissected into a translation of the center of mass and a rotation of the body around the center of mass.**

Altogether, we have 6 degrees of freedom (parameters) for the movement of a rigid body - 3 degrees for the translation of the CM and another 3 degrees for the rotation around the different directions.

B) Rigid body under external force

Now, let us look at an external force $\vec{F}_1$ that is acting at distance $\vec{r}_{CM1}$ from the center of mass:



From particle systems we know that a net external force causes an accelerated translation of the center of mass motion.

Experiment: **Pulling at a rigid object** in an experiment shows, that **in addition to the translation there can also by a rotation of the object.**

To describe this process, we **separate translation and rotation** by adding two opposing forces that have the same magnitude and are parallel to $\vec{F}_1$ at the CM ($\vec{F}_1 = \vec{F}_2 = -\vec{F}_3$). The added force pair does not change the rigid body motion, since it cancels itself. From dissecting the contributions of the different forces, we can note:

- $\vec{F}_2$ causes a translational acceleration of the center of mass, but no rotation:

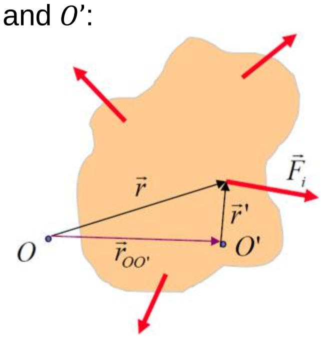
$$\vec{F}_2 = M\frac{d^2\vec{R}_{CM}}{dt^2}$$

- The force pair $\vec{F}_1/\vec{F}_3$ does not change the motion of the CM (since zero net force) but rather causes a torque around the CM that leads to a rotational acceleration around the CM.

$$\vec{\tau} = \vec{r}_{CM1} \times \vec{F}_1 = I\frac{d^2\varphi}{dt^2}$$

Any acting force on a rigid body can with this approach be split into a force acting on the center of mass (causing a translational acceleration of the CM) and a force pair with zero net force (causing only a rotational acceleration around CM).

For calculating the torque from different forces that act on a rigid body, we need always to define a reference point (rotation axis). Let us investigate how the torque changes when we calculate it for two different reference points O and O':



We hereby assume different forces $\vec{F}_i$ that cancel each other, such that:

$$\vec{F}_{net} = \sum_i \vec{F}_i = 0$$

The torque with respect to O' is given by:

$$\vec{\tau}_{net,O'} = \sum_i \vec{r}_i{}' \times \vec{F}_i = \sum_i (\vec{r}_i - \vec{r}_{OO'}) \times \vec{F}_i$$

$$= \sum_i \vec{r}_i \times \vec{F}_i - \sum_i \vec{r}_{OO'} \times \vec{F}_i$$

$$= \underbrace{\sum_i \vec{r}_i \times \vec{F}_i}_{\vec{\tau}_{net,O}} - \vec{r}_{OO'} \times \underbrace{\sum_i \vec{F}_i}_{0} = \vec{\tau}_{net,O}$$

i.e. the calculated torques are the same independent of the choice of the reference point. We can therefore say:

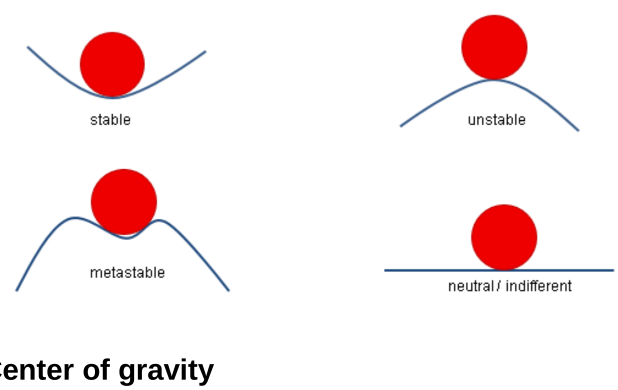
If the acting net force is zero, then the same net torque is obtained about any reference point. This finding will be a helpful tool for looking at static rigid bodies where all the forces cancel each other.

C) Equilibrium for rigid body

Given that any motion of a rigid body can be split into a translation of the CM and a rotation around the CM, we extend the our previous definition of equilibrium by the body rotation and say:

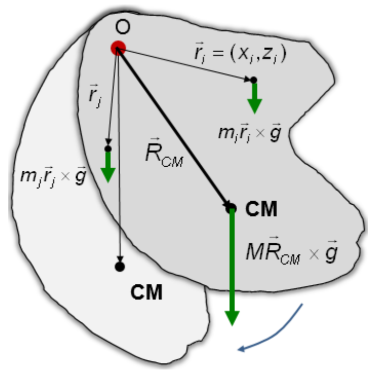
> **In equilibrium a rigid body shall not change its translation nor its rotation around the center of mass**.
>
> This requires that:
>
> 1) The **net external force must be zero**, to prevent a change in the translation of the object
>
> $$\vec{F}_{net} = \sum_i \vec{F}_i = 0$$
>
> 2) **The net external torque** (about any axis) **must be zero** to prevent a change of rotation
>
> $$\vec{\tau}_{net,O} = \sum_i \vec{r}_{iO} \times \vec{F}_i$$

Specifically, we call the **equilibrium static, if the body is at rest**, i.e. when it does not possess a translational nor an angular velocity. Remember that we distinguish the following types of equilibria:



D) Center of mass & Center of gravity

For translation we derived that the the sum of all external forces accelerates the motion of the center of mass. Thus, it appears that the total force acts at the center of mass. For torque we did not formally derive such an expression, but assumed before that e.g. the gravitational force of an object can be imagined to act at the center of mass (e.g. for rolling motion). In the following we want to briefly proof this assumption by calculating the torque that gravity produces at a rigid disk in the $z-x$ plane that is hold at a fixed anchor point at origin of coordinate system.



To this end we sum up all acting torques with respect to the anchor point:

$$|\vec{\tau}| = \sum_i |\vec{r}_i \times (m_i\vec{g})| = \sum_i m_i g|\vec{r}_i \times \hat{e}_z| = g\underbrace{\sum_i m_i x_i}_{M\cdot X_{CM}}$$

For the latter equality we used the fact that the torque is the product of the gravitational force and the effective radius, which is the x-coordinate of the mass. The sum equals (according to the definition) the total mass times the x coordinate of the center of mass. We can thus write:

$$\tau = MX_{CM}g$$

In vectoral form we can write

$$\vec{\tau} = M\vec{R}_{CM} \times \vec{g}$$

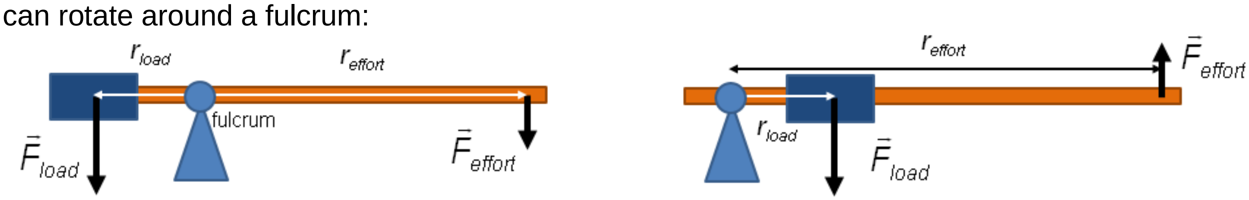
From this we can conclude that the torque due to gravity is the same torque as if all mass would be concentrated in the center of mass. Thus, the **center of mass equals the center of gravity.** (if g is constant over the body dimensions). For equilibrium we can write $\vec{\tau} = 0$. This demands that $X_{CM} = 0$, i.e. **in equilibrium the center of mass is located directly below the anchor point**

Experiment: Center of mass determination for different objects. We can use this finding to determine the center of mass of an object by attaching it at different anchor points and draw a line along plumb line. The intersections of all lines provides the center of mass.

2) Applications of static equilibrium conditions

A) Lever law (only on slides)

The most prominent example to illustrate the application of our equilibrium conditions are levers. They can be double-sided or single sided and typically comprise a rod as force transducer that can rotate around a fulcrum:



We can derive the effort force that is necessary to hold a given load using the previously derived equilibrium conditions. The first condition is that the **total force must be zero**. For the double-sided lever we can write:

$$F_y = 0 = F_{fulcrum} - F_{load} - F_{effort}$$

This provides that the force on the fulcrum equals the sum of load and effort force. Furthermore, the **total torque must be zero**. Calculating the torque with respect to the fulcrum gives:

$$\tau = 0 = \underbrace{r_{load} F_{load}}_{\tau_{load}} \underbrace{-r_{effort} F_{effort}}_{\tau_{effort}}$$

were we considered torques in counter-clockwise rotation direction to be positive. Transformation gives the well-known force transduction equation for levers:

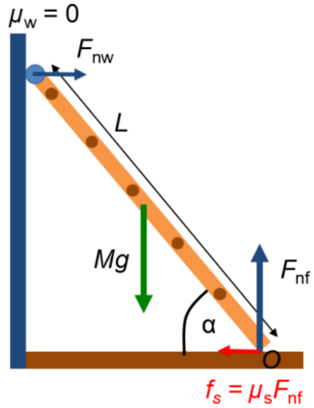
$$\frac{F_{effort}}{F_{load}} = \frac{r_{load}}{r_{effort}}$$

i.e. the **force ratio is the reciprocal value of the lever length ratio**.

Experiment: Force transduction at double- and single sided lever

B) The ladder problem

Let us further understand static equilibrium by calculating the angle at which we can stably lean a ladder onto a wall. We assume that there is no friction at the wall and only friction on the ground. In this case we have only four forces – the normal forces F_{nw} and F_{nf} at the wall and the floor, the friction $-f_s$ at the floor and the weight of the ladder $-Mg$:



For an angle at which the ladder is stably leaning at the wall, the equilibrium conditions must hold. We choose for convenience the ladder end on the ground as reference point O for the torque calculation, such that we can write for the force equilibrium:

$$\vec{F}_{net} = 0: \quad \begin{cases} F_{nw} - f_s = 0 \\ F_{nf} - Mg = 0 \end{cases}$$

and for the torque equilibrium considering the effective radii at which $-Mg$ and F_{nw} are acting:

$$\vec{\tau}_{net} = 0: \quad Mg\,\frac{L}{2}\cos\alpha - F_{nw}\,L\sin\alpha = 0$$

Transforming the torque equation and replacing F_{nw} by f_s from the first force equation gives:

$$f_s = F_{nw} = \frac{Mg\cos\alpha}{2\sin\alpha} = \frac{Mg}{2\tan\alpha}$$

Using the expression for the static friction and the second force equation provides an upper limit for the friction force:

$$f_s \le \underbrace{\mu_s F_{nf}}_{f_{s,max}} = \mu_s Mg$$

Inserting gives the relation:

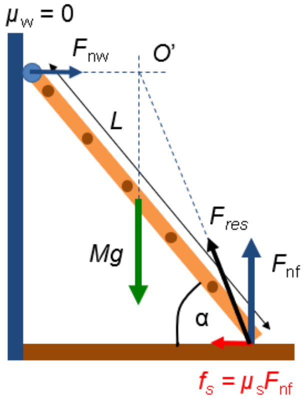
$$\frac{Mg}{2\tan\alpha} \le \mu_s Mg$$

such that we get after transformation the condition:

$$\tan\alpha \geq \frac{1}{2\mu_s} \qquad \text{or} \qquad \alpha \geq \underbrace{\arctan\frac{1}{2\mu_s}}_{\alpha_{crit}}$$

i.e. α must be larger than a minimum (critical) angle otherwise the ladder would slip (**see slides**).

Alternative approach (see slides): One can choose any reference point for the torque calculation, since the total force is zero. A convenient reference point is also O' at the intersection between gravity force vector and the wall force vector.



To get zero torque the net force at the bottom end of ladder must point towards O', i.e. its two components must form the same angle that is also provided by the ladder geometry. For the tangent of this angle (called θ) we can write:

$$\tan\theta = \frac{L\sin\alpha}{L/2\cos\alpha} = \frac{F_{nf}}{f_s}$$

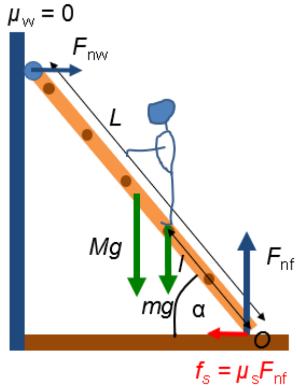
Using the second force equation from before this can be transformed to:

$$f_s = \frac{Mg}{2\tan\alpha}$$

Which is the same result as before.

C) The ladder climbing problem

To increase the complexity of the ladder problem, we now ask whether a ladder at the critical angle would slip, if we climb on it. We have here to consider the additional weight mg of a person that steps on it at distance l from the lower end of the ladder:



We thus get for the forces at static equilibrium:

$$\vec{F}_{net} = 0: \quad \begin{cases} F_{nw} - f_s = 0 \\ F_{nf} - Mg - mg = 0 \end{cases}$$

and for the torque:

$$\vec{\tau}_O = 0: \qquad Mg\frac{L}{2}\cos\alpha + mgl\cos\alpha - F_{nw}L\sin\alpha = 0$$

Transformation and combination with the first force equation provides:

$$f_s = F_{nw} = \frac{g}{2\tan\alpha}\left(M + m\frac{2l}{L}\right)$$

The second force equation gives the normal force at the lower ladder end

$$F_{nf} = (M+m)g$$

such that inserting the condition for the maximum friction provides:

$$f_s = \frac{g}{2\tan\alpha}\left(M + m\frac{2l}{L}\right) \leq \mu_s \underbrace{(M+m)g}_{F_{nf}}$$

From this we get:

$$\tan\alpha \geq \underbrace{\frac{1}{2\mu_s}}_{\substack{\tan\alpha_{crit} \\ \text{empty} \\ \text{ladder}}} \frac{M + m\frac{2l}{L}}{M+m}$$

Thus, the tangent of the critical angle of the empty ladder gets multiplied by the mass term at the right-hand side. We easily see that (see slides):

$$\frac{M + m\frac{2l}{L}}{M+m}\begin{cases} < 1 & \text{if } \mathrm{l} < \frac{\mathrm{L}}{2}, \text{ i.e. } \alpha_{cr} \downarrow \\ = 1 & \text{if } \mathrm{l} = \mathrm{L}/2, \text{ i.e. no change} \\ > 1 & \text{if } \mathrm{l} > \frac{\mathrm{L}}{2}, \text{ i.e. } \alpha_{cr} \uparrow \end{cases}$$

Thus, the ladder gets stabilized if standing in the lower half and destabilized if standing in the upper half. One can explain the latter by the larger lever length l that causes a larger force on the wall F_{nw} that counteracts the friction.

Experiment: Measurement of the critical angle of a ladder in absence of an additional load. Measurement of the critical angle for different positions of an additional load.

Lecture 16: Experiments

1) Pulling rigid body at off-center point results in a simultaneous translation and rotation
2) Center of mass determination using a plump bob (Lot)
3) Force transduction at double- and single sided lever
4) Measurement of the critical angle of a ladder in absence of an additional load. Measurement of the critical angle for different positions of an additional load.