Exam. 23 February. Solutions

Problem 1. Prove that for any $x \geq 0$, $x \in \mathbb{R}$, one has

$$(1+x)^9 \geq 1 + 30x^2$$

Solution: By binomial's theorem, one has

$$(1+x)^9 = \sum_{k=0}^{9} \binom{9}{k} x^k$$

Since x is nonnegative, each summand in the expansion is also nonnegative, and one has

$$(1+x)^9 \geq \binom{9}{0} + \binom{9}{2} x^2 = 1 + 36x^2 \geq 1 + 30x^2.$$

Problem 2. Provide an example of a sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} a_n = 3, \qquad \sup_{n\geq 1} a_n = 5.$$

Solution: For example, $a_1 = 5$ and $a_n = 3$, for all $n \geq 2$. Then 5 is the maximum, thus, also a supremum, and the sequence obviously converges to 3.

Problem 3. Compute the following limit

$$\lim_{n\to\infty} \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}$$

Solution: One has

$$1 \leq \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + 1 + \cdots + 1 \leq n.$$

One has $\lim_{n\to\infty} \sqrt[n]{1} = 1$ and $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Thus, by "Sandwich" theorem, one has

$$\lim_{n\to\infty} \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}} = 1$$

Problem 4. Compute the limit

$$\lim_{n\to\infty} \frac{n^2 2^n + 2n^2 5^n + 6^n}{3^n\left(n^7 + n^4\right) + 6^n + n^2 5^n}.$$

Solution: Divide both numerator and denominator by 6^n. One obtains

$$\frac{n^2(2/6)^n + 2n^2(5/6)^n + 1}{(3/6)^n\left(n^7 + n^4\right) + 1 + n^2(5/6)^n}$$

From lectures we know that each individual summand both in numerator and denominator has a limit. Thus, by arithmetics of limits, this expression converges to

$$\frac{0 + 0 + 1}{0 + 1 + 0} = 1.$$

Problem 5. Prove that the equation

$$2^x = x^3 + 5$$

has at least two real roots.

Solution: Let $f(x) := 2^x - x^3 - 5$, and note that $f(-2) = \frac{1}{4} + 8 - 5 > 0$, $f(0) = 1 - 0 - 5 < 0$, and $f(10) = 1024 - 1000 - 5 > 0$. Also note that $f(x)$ is a continuous function. Therefore, by Intermediate Value Theorem, intervals $(-2; 0)$ and $(0; 10)$ contain at least one real root of the equation. Thus, the equation has at least two real roots.

Problem 6. Is the function

$$f(x) = \begin{cases} \sin\left(x^2\right)\sin\left(\frac{1}{x}\right), & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

differentiable at $x = 0$?

Solution: Let us check the definition of the derivative:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{\sin\left(x^2\right)\sin\left(\frac{1}{x}\right) - 0}{x} = \lim_{x\to 0} \left(\frac{\sin\left(x^2\right)}{x^2}\right) \lim_{x\to 0} \left(x \sin\left(\frac{1}{x}\right)\right)$$

We have $\lim_{x\to 0}\left(\frac{\sin(x^2)}{x^2}\right) = 1$ (fact from lectures up to change of variables $y = x^2$), and $\lim_{x\to 0}\left(x\sin\left(\frac{1}{x}\right)\right) = 0$, since this is the limit of the product of a function converging to 0 and a bounded function.

Thus, by arithmetics of limits, one has

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = 0,$$

which means that the function is differentiable at 0.

Problem 7. Compute the following limit

$$\lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{\ln(1+x) - x + \frac{x^2}{2}}.$$

Solution: By Taylor's theorem, the limit in question equals

$$\lim_{x\to 0} \frac{\left(x - \frac{x^3}{6} + o(x^4)\right) - x(1 - \frac{x^2}{2} + o(x^3))}{\left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right) - x + \frac{x^2}{2}} = \lim_{x\to 0} \frac{\frac{x^3}{3} + o(x^3)}{\frac{x^3}{3} + o(x^3)} = \lim_{x\to 0} \frac{\frac{1}{3} + o(1)}{\frac{1}{3} + o(1)} = 1.$$

Problem 8. Check the following series for convergence

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{3} - 1\right)$$

Solution: From lectures we know that

$$3 > e > \left(1 + \frac{1}{n}\right)^n, \qquad \text{for any } n \in \mathbb{N}$$

Therefore, we have

$$\sqrt[n]{3} - 1 > \sqrt[n]{\left(1 + \frac{1}{n}\right)^n} - 1 = \frac{1}{n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, thus, by Comparison Test, the series $\sum_{n=1}^{\infty} \left(\sqrt[n]{3} - 1\right)$ is also divergent.

Problem 9. Compute the area of a region bounded by curves

$$y = e^x, \qquad x = 1, \qquad y = 5.$$

Solution: The area in question has three "corners" obtained as pairwise intersections of the curves: $(1, e)$, $(\ln 5, 5)$, and $(1, 5)$. In order to compute it, we can (for example), first to compute the area of a rectangle $(1, 0)$, $(\ln(5), 0)$, $(\ln(5), 5)$, $(1, 5)$, and then subtract from it $\int_1^{\ln(5)} e^x dx$.

The area of rectangle described above equals $5(\ln(5) - 1)$, and

$$\int_1^{\ln(5)} e^x dx = e^{\ln(5)} - e^1 = 5 - e$$

Therefore, the area in question equals

$$5(\ln(5) - 1) - (5 - e) = 5\ln(5) + e - 10$$

Problem 10. Find all local and global extrema of the following function

$$f(x) = x\ln^2(x), \qquad f : (0; 3] \to \mathbb{R}.$$

Solution: We have

$$f'(x) = \ln^2(x) + 2\ln(x) = \ln(x)(\ln(x) + 2).$$

Therefore, one has $f'(x) = 0$ if and only if $x = 1, e^{-2}$, and the function $f'(x)$ is positive in the interval $(0; e^{-2})$, negative in the interval $(e^{-2}; 1)$, and positive in the interval $(1; 3]$. Therefore, $x = e^{-2}$ is a point of a local maximum, and $x = 1$ is a point of a local minimum.

In order to determine the global minimum and maximum, notice first that $f(x) > 0$ for all points of $(0; 3]$ except of $x = 1$, because $x > 0$ everywhere in our region, and $\ln^2(x) > 0$ also holds everywhere except of $x = 1$. Therefore, $f(1) = 0$ is the global minimum.

$f(3) = 3\ln^2(3) > 3$, while $f(e^{-2}) = 4e^{-2} < 1$. Also notice that $f(x)$ is increasing in the interval $(0; e^{-2})$, so $f(x) < f(e^{-2}) < 1$ for all $x \in (0; e^{-2})$. Therefore, $f(3) = 3\ln^2(3)$ is the global maximum.

Problem 11. Compute the following indefinite integral

$$\int \left(\frac{1-x}{x}\right)^2 dx.$$

Solution: The integral in question is equal to

$$\int \left(\frac{1}{x^2} - \frac{2}{x} + 1\right) dx = \frac{-1}{x} - 2\ln(x) + x + C, \qquad \text{for } x > 0.$$

Problem 12. Compute the intersection of the following three planes in $\mathbb{R}^3$:

$$x + 2y - z = 1, \qquad 2x + y = 0, \qquad 3x + 3y - z = 1$$

Solution: One needs to solve a system of linear equations given by three equations above. One can do this via a general method of Gaussian elimination, or for smaller systems like this one, in many other ways. Let us illustrate a way which does not use the Gaussian elimination.

Notice that the third equation is the sum of the first two, so the intersection of the first two planes will also lie in the third plane. Next, plug $y = -2x$ to the first equation: One gets $-3x - z = 1$. Let $z = t$ be a free variable, then $x = \frac{-1-t}{3}$ and $y = \frac{2+2t}{3}$.

Answer: $x = \frac{-1-t}{3}$, $y = \frac{2+2t}{3}$, $z = t$, $t \in \mathbb{R}$. This is a line in $\mathbb{R}^3$. Note that the same line (and the same answer) can be written in many different ways.