Mathematics 1. Selected proofs
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Investigation of functions using derivatives. Convexity

THEOREM. Assume $f : (a,b) \to \mathbb{R}$ is twice differentiable on (a,b). Then

$$f \ \text{ is convex on } (a,b) \qquad \Longleftrightarrow \qquad f'' \geq 0 \ \text{ on } (a,b)$$

PROOF $\boxed{\Longrightarrow}$

1. Assume f is convex on (a,b). Use the definition of convexity:

$$\forall\, x_1,\, x_2 \in (a,b), \quad x_1 < x_2, \quad \forall\, \lambda \in [0,1] \qquad f\big((1-\lambda)x_1 + \lambda x_2\big) \leq (1-\lambda) f(x_1) + \lambda f(x_2)$$

2. Take arbitrary $x_1 < x < x_2$ and choose $\lambda \in (0,1)$ in a specific way:

$$\lambda := \frac{x - x_1}{x_2 - x_1} \qquad \Longrightarrow \qquad 1 - \lambda = \frac{x_2 - x}{x_2 - x_1}, \qquad (1-\lambda)x_1 + \lambda x_2 = x$$

3. Proof the inequality for slopes of the secants:

$$x_1 < x < x_2 \qquad \Longrightarrow \qquad \frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}$$

Indeed,

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \leq (1-\lambda) f(x_1) + \lambda f(x_2) \quad \Longleftrightarrow$$

$$f(x) \ \leq \ \tfrac{x_2 - x}{x_2 - x_1}\, f(x_1) + \tfrac{x - x_1}{x_2 - x_1}\, f(x_2) \quad \Longleftrightarrow$$

$$\underbrace{\tfrac{x - x_1}{x_2 - x_1}\, f(x) + \tfrac{x_2 - x}{x_2 - x_1}\, f(x)}_{=\ f(x)} \ \leq \ \tfrac{x_2 - x}{x_2 - x_1}\, f(x_1) + \tfrac{x - x_1}{x_2 - x_1}\, f(x_2) \quad \Longleftrightarrow$$

$$\tfrac{x_2 - x}{x_2 - x_1} \Big(f(x) - f(x_1)\Big) \ \leq \ \tfrac{x - x_1}{x_2 - x_1} \Big(f(x_2) - f(x)\Big) \quad \Longleftrightarrow$$

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}$$

(You can omit everything in blue color if you find it routine).

4. Proof $f' - \nearrow$ on (a,b). Take any $x_1,\, x_2 \in (a,b)$ such that $x_1 < x_2$.

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}, \quad x \to x_1 + 0 \quad \Longrightarrow \quad f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}, \quad x \to x_2 - 0 \quad \Longrightarrow \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2)$$

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2), \ \ \forall\, x_1 < x_2 \quad \Longrightarrow \quad f' - \nearrow \text{ on } (a,b) \quad \Longrightarrow \quad f'' \geq 0 \text{ on } (a,b)$$

PROOF $\boxed{\Longleftarrow}$

5. Assume $f'' \geq 0$ on (a,b) $\quad\Longrightarrow\quad$ f' — $\nearrow$ on (a,b).

6. Proof the inequality for slopes of the secants:

$$x_1 < x < x_2 \qquad \Longrightarrow \qquad \frac{f(x)-f(x_1)}{x-x_1} \leq \frac{f(x_2)-f(x)}{x_2-x}$$

   Indeed, from the Lagrange theorem we obtain

$$\begin{aligned} \exists\, c_1 \in (x_1, x): &\qquad \frac{f(x)-f(x_1)}{x-x_1} = f'(c_1) \\ \exists\, c_2 \in (x, x_2): &\qquad \frac{f(x_2)-f(x)}{x_2-x} = f'(c_2) \end{aligned}$$

$$c_1 \in (x_1, x), \quad c_2 \in (x, x_2) \quad \Longrightarrow \quad c_1 < c_2$$

$$f \text{ — } \nearrow \text{ on } (a,b) \quad \Longrightarrow \quad \frac{f(x)-f(x_1)}{x-x_1} = f'(c_1) \leq f'(c_2) = \frac{f(x_2)-f(x)}{x_2-x}$$

7. Take arbitrary $x_1 < x_2$ and $\lambda \in (0,1)$ and choose $x \in (x_1, x_2)$ in a specific way:

$$x = (1-\lambda)x_1 + \lambda x_2 \qquad \Longrightarrow \qquad \lambda = \frac{x-x_1}{x_2-x_1}, \qquad 1-\lambda = \frac{x_2-x}{x_2-x_1}$$

8. Verify the definition of convexity for f:

$$\begin{aligned} \frac{f(x)-f(x_1)}{x-x_1} &\leq \frac{f(x_2)-f(x)}{x_2-x} \quad \Longleftrightarrow \\ \tfrac{x_2-x}{x_2-x_1}\Big(f(x)-f(x_1)\Big) &\leq \tfrac{x-x_1}{x_2-x_1}\Big(f(x_2)-f(x)\Big) \quad \Longleftrightarrow \\ \underbrace{\tfrac{x-x_1}{x_2-x_1}\, f(x) + \tfrac{x_2-x}{x_2-x_1}\, f(x)}_{=\,f(x)} &\leq \tfrac{x_2-x}{x_2-x_1}\, f(x_1) + \tfrac{x-x_1}{x_2-x_1}\, f(x_2) \quad \Longleftrightarrow \\ f(x) &\leq \tfrac{x_2-x}{x_2-x_1}\, f(x_1) + \tfrac{x-x_1}{x_2-x_1}\, f(x_2) \quad \Longleftrightarrow \\ f\big((1-\lambda)x_1 + \lambda x_2\big) &\leq (1-\lambda)f(x_1) + \lambda f(x_2) \end{aligned}$$

   (You can omit everything in blue color if you find it routine). $\square$