Lecture "Experimental Physics I"

(Prof. Dr. R. Seidel)

Lecture 28

Superposition of Waves

- Plane waves in 3D space & spherical waves
- Superposition of waves
- Standing waves in 1D and 2D

1) Waves in 3D

A) Wave equation and plane wave in 3D

So far we considered only linear plane waves in 1D given by:

$$\xi = \xi_0 \cos(kx - \omega t)$$

To get a plane wave that moves in any direction in 3D space, the (angular) wave number becomes now a vector, the so-called **wave vector**:

$$\vec{k} = k_x \hat{e}_x + k_y \hat{e}_y + k_z \hat{e}_z$$

whose **absolute value is the wave number** as also given by the wave length of the wave:

$$|\vec{k}| = 2\pi/\lambda$$

A plane wave along $\vec{k}$ is then described by:

$$\xi = \xi_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$$

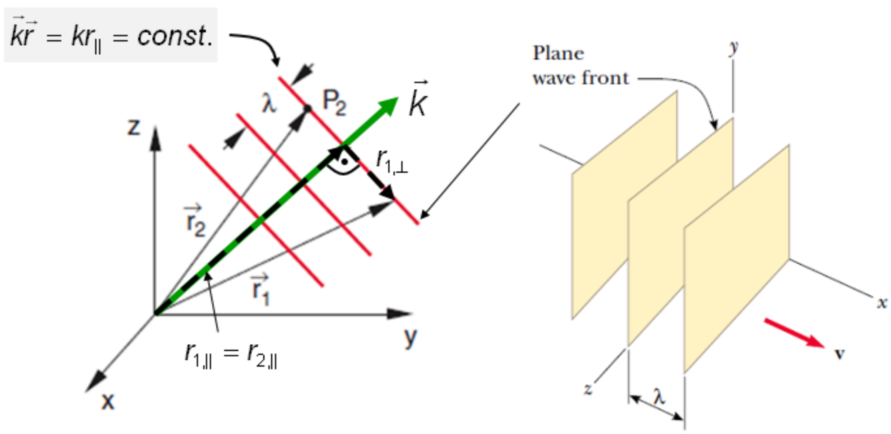
Where the displacement ξ and the amplitude ξ_0 can also be vectors. As in one dimension, for points of the wave with equal phase at $t = t_0$ we can write the relation:

$$\vec{k} \cdot \vec{r} - \omega t_0 = const.$$

Since we consider a defined moment in time, for these $\vec{r}$ the scalar product itself must be constant

$$const = \vec{k} \cdot \vec{r} = k r_{||}$$

This relationship comprises all vectors $\vec{r}$ that have the same parallel component along $\vec{k}$ while their component perpendicular to $\vec{k}$ can be arbitrary. All these points define thus a plane perpendicular to $\vec{k}$, which is called a **wave front** (see figure below). These planar wave fronts travel for increasing time in the direction of $\vec{k}$ such that the resulting wave is called plane wave. The periodicity of the plane wave pattern along $\vec{k}$ is again given by the wave length $\lambda = 2\pi/k$.



In complex notation such a plane wave is given by:

$$\vec{\xi} = \vec{\xi}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

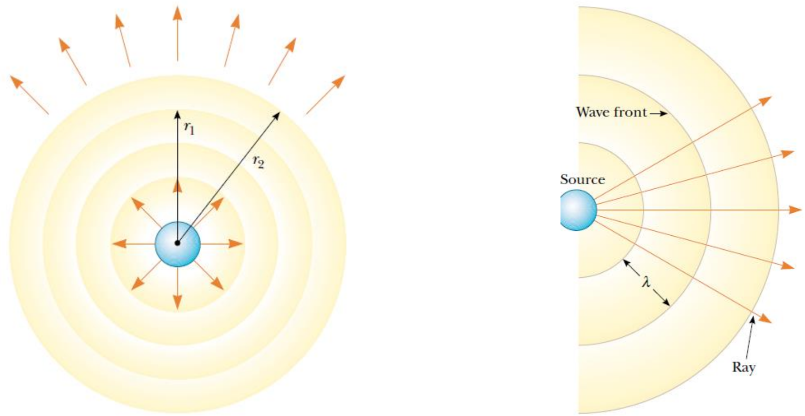
with $\vec{\xi}_0 = \vec{\xi}_{Re} + i\vec{\xi}_{Im}$ defining the phase at $t = 0$. The corresponding **wave equation in three dimensions** is provided by:

$$\Delta\vec{\xi} = \frac{1}{v_{Ph}^2} \frac{\partial^2 \vec{\xi}}{\partial t^2}$$

where the second position derivative becomes the **Laplace operator**. The displacement ξ is either a scalar or a vector. One can easily show that the plane wave equation in 3D is a solution to the wave equation in 3D.

B) Spherical waves

An oscillating spherical body whose radius expands and contracts sinusoidally with time produces a **spherical (sound) wave**, where the wave propagates isotropically in all directions. For symmetry reasons **the wave fronts must be spherical planes** in an isotropic medium, since in each direction the points that have the same phase expand with the same phase velocity.



The **wave propagates along the radial wave vector $\vec{k}$, that is perpendicular to the wave front,** such that we have here a radial propagation of the wave**.** To indicate the **local propagation direction for non-planar waves we use rays. Rays point along the wave vector and are perpendicular to the wave fronts.** For spherical waves, the rays are radial lines pointing outward from the source (see figure above)**.**

In cartesian coordinates, the wave equation for a spherical wave is the same as the wave equation in three dimensions introduced above. For spherical waves, one typically switches, however, into spherical coordinates, where the Laplace operator looks slightly modified. A good guess for a continuous spherical wave in spherical coordinates is a rotationally invariant function that depends only on the radial distance r and the time:

$$\xi(r,t) = f(r)\cos(kr - \omega t)$$

This equation defines spherical wave fronts with radius *r*. For a phase difference 2π they are separated by distance λ. Due to energy conservation, the total power that is propagated in radial direction by each wavefront must be the same. From before we know the flux/intensity of a sound wave, i.e. the power per area that passes a given area:

$$I = \frac{P}{A} = \frac{1}{2}\rho{\xi_0}^2\omega^2 v_{Ph} = \frac{1}{2}\rho f(r)^2\omega^2 v_{Ph}$$

Applying that through each spherical shell of radius r around the wave origin the same power has to pass, we can write:

$$const = P = I\,4\pi r^2 = 2\pi r^2\,\rho\,f(r)^2\omega^2\,v_{Ph}$$

Thus, the **intensity or likewise the squared amplitude $\boldsymbol{f(r)^2}$ must decrease with $\mathbf{1/r^2}$**, which agrees with our everyday experience that the sound volume is rapidly decreasing with distance from the source. From the equation above we can formulate the condition:

$$r\,f(r) = const$$

which yields

$$f(r) = \frac{const}{r} = \frac{\xi_0}{r}$$

This provides for **a spherical wave**:

$$\xi(r,t) = \frac{\xi_0}{r}\cos(kr - \omega t)$$

This is a solution to the wave equation (in spherical coordinates), which one can prove by inserting (see **animation on slide**)

2) Interference: Superposition of waves

For waves we can also formulate a **superposition principle**:

> When two or more waves move in the same linear medium, the **net displacement in the medium at any point equals the sum of the displacements caused by the individual waves**. The superposition of waves (typically of the same frequency) is **called interference**.

This is a cause of the **linearity of the wave equation.** The displacement $\xi(\vec{r},t)$ and all its derivatives appear only as simple linear terms but not as part of a higher order function. Thus, if $\xi_1(\vec{r},t)$ and $\xi_2(\vec{r},t)$ are solutions to the wave equation then any linear combination $c_1\xi_1(\vec{r},t) + c_2\xi_2(\vec{r},t)$ is also a solution and thus a wave.

A) Linear (1D) waves with equal direction

Let us look at the **superposition of two linear waves** that propagate in the **same direction with equal frequency** and thus equal wave number. They shall further have equal amplitudes but can have different initial phases:

$$\xi_1(x,t) = \xi_0 \cos(kx - \omega t)$$
$$\xi_2(x,t) = \xi_0 \cos(kx - \omega t + \varphi)$$

The superposition principle provides for the combined wave:

$$\xi = \xi_1 + \xi_2 = \xi_0[\cos(kx - \omega t) + \cos(kx - \omega t + \varphi)]$$

Using the trigonometric identity:

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2}$$

this can be transformed to:

$$\xi = \underbrace{2\xi_0 \cos(\varphi/2)}_{\text{effective amplitude}}\ \underbrace{\cos(kx - \omega t + \varphi/2)}_{\text{single plane wave}}$$

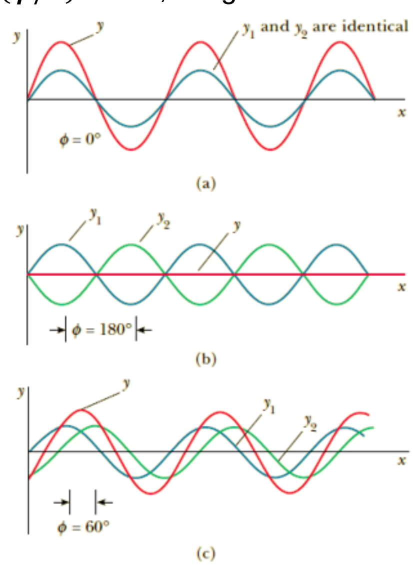
This equation represents a single plane wave moving with the same frequency and wavelength as both individual waves but an **effective amplitude of $2\xi_0\cos(\varphi/2)$**. Thus, we get:

- **a maximum amplitude of $2\xi_0$ when $\varphi = 0, \pm 2\pi, \pm 4\pi \ldots$,** i.e. when both waves are shifted by multiples of a full wavelength, such that their maxima (and minima) coincide. This case is called **constructive interference.**
- **a minimum amplitude of 0, when $\varphi = \pm\pi, \pm 3\pi, \pm 5\pi \ldots$,** i.e. when both waves are shifted by odd multiples of half a wavelength such that the maximum of one wave coincides with the minimum of the other wave and vice versa. In this case the displacements cancel, which we call **destructive interference.** For destructive interference the wave sources cannot release any energy into a wave
- **amplitudes between 0 and $2\xi_0$ for other than the mentioned phases differences**

constructive interference

destructive interference

other phases



B) Interference in 2D and 3D

Interference of waves in two and three dimensions also obeys the simple superposition principle. Let us look at two spherical waves that are released from point sources sources Q1 and Q2 with with zero initial phases

$$\xi_1(r_1,t) = \frac{\xi_0}{r_1}\cos(kr_1 - \omega t)$$
$$\xi_2(r_1,t) = \frac{\xi_0}{r_2}\cos(kr_2 - \omega t)$$

As we learned before, **constructive interference** occurs at points where the total phase difference $\Delta\varphi$ between the two waves becomes **multiples of 2π,** since in this case the maxima and the minima of both waves always coincide. For a point P located at distances r_1 and r_2 from either source (see figure below), the maximum condition gives:

$$\Delta\varphi = (kr_1 - \omega t) - (kr_2 - \omega t) = \frac{2\pi}{\lambda}(r_1 - r_2) = 2\pi\, n$$

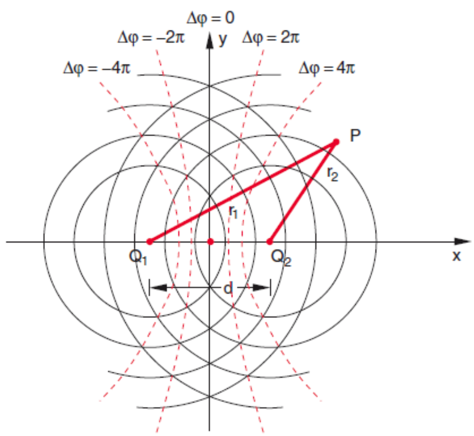
with $n = 0, \pm 1, \pm 2, \ldots$ Expressing the wave number using the wave length and transformation provides that constructive interference is obtained where the path difference becomes **multiples of the wavelength**

$$r_2 - r_1 = n\,\lambda$$

Destructive interference occurs at positions where the total phase difference $\Delta\varphi$ corresponds to odd multiples of π corresponding to a path difference of an odd multiple of half the wavelength:

$$r_2 - r_1 = (n + 1/2)\lambda$$

Each condition for a fixed phase difference (such as $\Delta\varphi = 2\pi\, n$) provides a hyperbola in a plane that contains both sources. Remember, for a hyperbola the difference in distance to the two focal points is a constant in contrast to an ellipse where this condition was holding for the sum of the distances. In three dimensions a hyperboloid is formed. The wave sources are in thus the focal points of the hyperbolas:



Interference can be nicely visualized by animations and experiments:

- Overhead projector overlays of two sets of equally spaced concentric rings illustrate the interference patterns of spherical waves
- a **wave workshop animation** (http://www.jsingler.de/apps/waveworkshop/index.php) allows to visualize displacement and amplitude and amplitude for two interfering spherical waves
- **Experiment:** For counterpropagating sound waves one can measure amplitude maxima and minima with a microphone.
- **Experiment:** When interfering a sound wave with its reflection at the wall, one (students in the audience) can hear the different sound levels from interference by moving the head to a slightly different position

Lecture 28: Experiments

1) Overhead projector overlays of two concentric ring sets to illustrate interference of spherical waves.
2) Intensity pattern of counterpropagating sound waves from two loudspeakers measured with a microphone
3) Interference of sound wave with its reflection at the wall, students can hear the different sound levels by moving the head