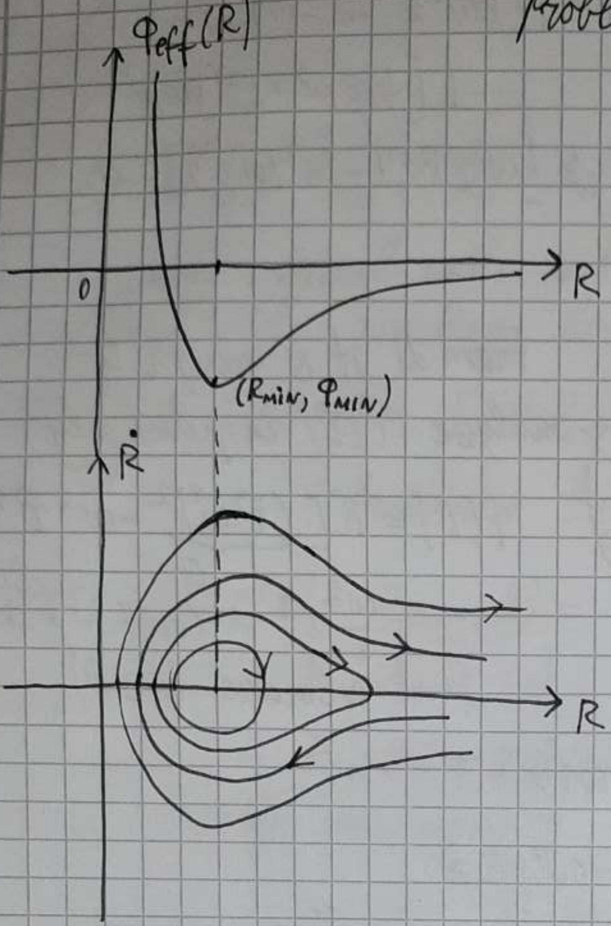
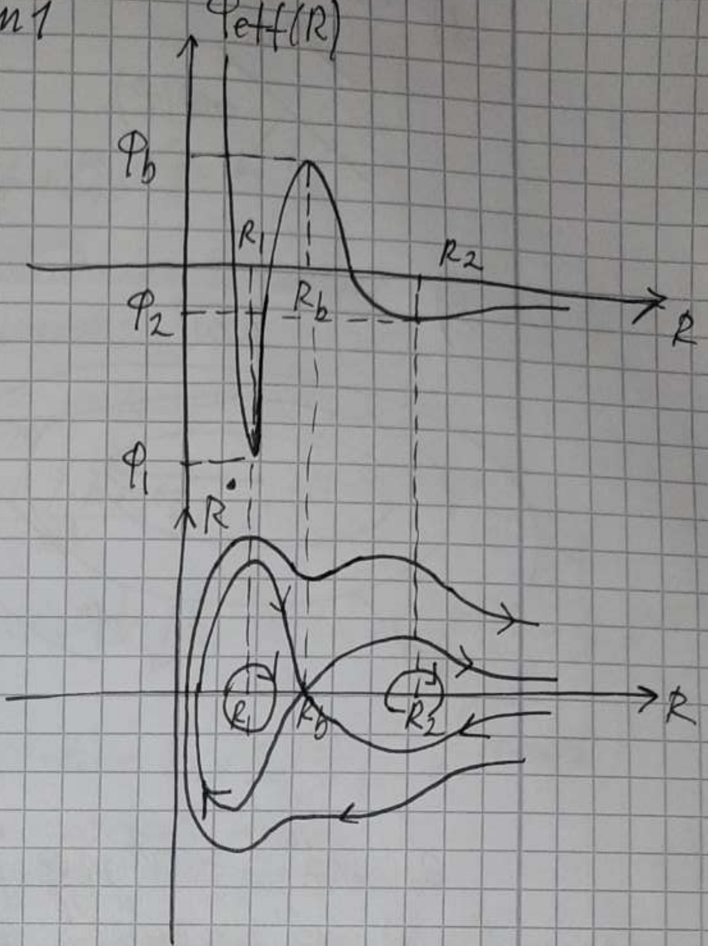
Problem 1





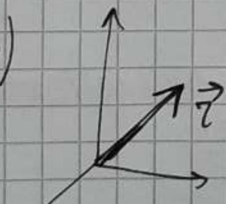
Problem 2

a) $\vec F(\vec r) = \underset{a}{\vec\omega}\times(\underset{b}{\vec\omega}\times\underset{c}{\vec r}) = \vec\omega(\vec\omega\vec r) - \vec r(\vec\omega\cdot\vec\omega) = f(\vec r)\vec\omega + c\,\vec r$, $\begin{cases} f(\vec r) = \vec\omega\vec r \\ c = -\omega^2 \end{cases}$

b) $\forall i\neq j$ $\frac{\partial F_i}{\partial r_j} = \frac{\partial F_j}{\partial r_i}$: $\frac{\partial}{\partial r_j}\left[\left(\sum_k \omega_k r_k\right)\omega_i - \omega^2 r_i\right] = \omega_i\omega_j$

$$\frac{\partial}{\partial r_i}\left[\left(\sum_k \omega_k r_k\right)\omega_j - \omega^2 r_j\right] = \omega_i\omega_j$$

(in other words $\exists\,\varphi(\vec r)$ $\frac{\partial^2\varphi}{\partial r_i\partial r_j} = \frac{\partial^2\varphi}{\partial r_j\partial r_i}$, or $\nabla\times\vec F = \vec 0$)

c) 

$\vec q = s\vec r$, $s\in[0,1]$, $\frac{d\vec q}{ds} = \vec r$

$$W = \int_{\vec q_i}^{\vec q_f}\vec F(\vec q)\,d\vec q = \int_{s=0}^{1}\left(\vec\omega(\vec\omega\cdot s\vec r) - s\vec r(\vec\omega\vec\omega)\right)\vec r\,ds = \int_{s=0}^{1} s\,\overbrace{\left[(\vec\omega\vec r)^2 - \omega^2\vec r^2\right]}^{A}ds = A\frac{s^2}{2}\Big|_0^1 = \frac{A}{2} = \underbrace{\varphi(\vec 0)}_{\text{set } 0} - \varphi(\vec r),$$

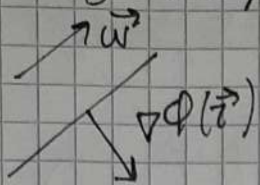
guess $\varphi(\vec r) = -\frac{1}{2}\left[(\vec\omega\vec r)^2 - \omega^2\vec r^2\right]$.

This trick assumed straight line path $\vec q$ for given $\vec r$, but this indeed is potential:

$$-\nabla\varphi(\vec r) = \frac{1}{2}\left[2(\vec\omega\vec r)\underbrace{\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}}_{\vec\omega} - \omega^2\cdot 2r\hat r\right] = (\vec\omega\vec r)\vec\omega - \omega^2\vec r = \vec F(\vec r).$$

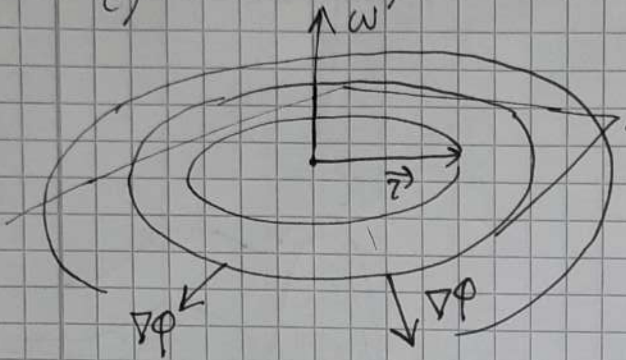
$[\nabla f(r) = f'(r)\cdot\hat r]$

d) Probably a simpler way is to check $\nabla\varphi$:



$$\nabla\varphi(\vec{r})\cdot k\vec{\omega} = -k\vec{F}(\vec{r})\cdot\vec{\omega} =$$
$$= -k\left[f(\vec{r})\vec{\omega} + c\vec{r}\right]\vec{\omega} =$$
$$= -k\left[\omega^2(\vec{\omega}\vec{r}) - \omega^2(\vec{\omega}\vec{r})\right] = 0.$$

e)



From d) it is enough to analyze $\varphi(\vec{r})$ in plane only

$$\varphi(\vec{r}) = K\Big[\underbrace{(\vec{r}\vec{\omega})^2}_{0} - \omega^2\vec{r}^{\,2}\Big] =$$
$$= \frac{\omega^2}{2}r^2 = \text{Const} \to \text{CL are circles.}$$

(and cylinders in 3-D)

Problem 3

a) $m\ddot{z} = \frac{4}{3}\pi R^3 g(\rho_f - \rho) - 6\pi\eta R\dot{z}$

(*) $m\ddot{z} + 6\pi\eta R(\dot{z} - \dot{z}_\infty) = 0$ when $\ddot{z} = 0$, $\dot{z} = \dot{z}_\infty = \dfrac{2R^2 g(\rho_f - \rho)}{9\eta}$ — terminal velocity

$$\ddot{z}(t) + \underbrace{\frac{6\pi\eta R}{m}}_{c_1}\dot{z} = \underbrace{\frac{6\pi\eta R}{m}\dot{z}_\infty}_{I}$$

$(c_0 = 0)$

b) From (*) Consider relative velocity inside moving frame

$$\dot{z}_{rel}(t) = \dot{z}(t) - \dot{z}_\infty, \quad \ddot{z}_{rel}(t) = \ddot{z}(t)$$

$$\ddot{z}_{rel}(t) + \underbrace{\frac{6\pi\eta R}{m}}_{C^{-1}}\dot{z}_{rel}(t) = 0$$

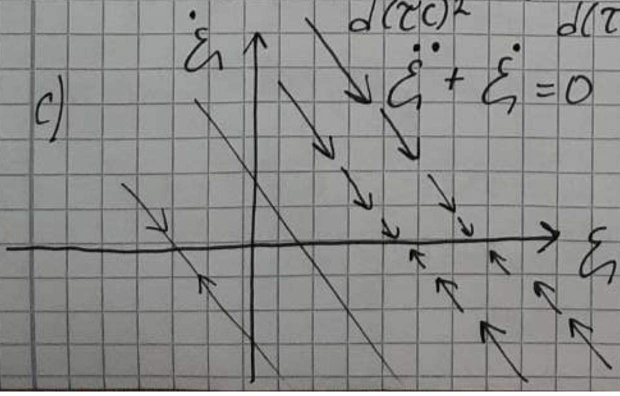
Non-dimensionalization:

$$\tau = \frac{t - t_0}{C}$$
$$\xi = \frac{z_{rel}}{R}$$

Now $\dfrac{d^2(\xi R)}{d(\tau C)^2} + C^{-1}\dfrac{d(\xi R)}{d(\tau C)} = 0 \;\Big|\cdot\dfrac{C^2}{R}$

$$\ddot{\xi} + \dot{\xi} = 0 \quad \text{w.r.t. moving frame}$$

c)



$$\ddot{\xi} = -\dot{\xi}$$

Traj. are straight lines reaching fixed position in the moving frame.

*d) $\ddot{\varepsilon}_1 = -\dot{\varepsilon}_1$ $(\tau_0 = 0)$

$\dot{\zeta}_1 = -\zeta_1$

$\zeta_1 = \zeta_0 e^{-\tau}$, $\varepsilon_1(\tau) = \varepsilon_1(\tau_0) + \zeta_0 (1 - e^{-\tau})$

*e) To restore original units and coordinates I rewrite the res. equation & multiply - working backwards:

$$Z_{rel}(t) = \varepsilon_1(\tau) \cdot R = \varepsilon_1(0) \cdot R + \zeta_0 \cdot R \left(1 - e^{-\frac{t-t_0}{c}}\right) =$$

$$= Z_{rel}(t_0) + \frac{\dot{Z}_{rel}(t_0) \cdot R m}{6\pi\eta R^2}\left(1 - e^{-(t-t_0)\frac{6\pi\eta R}{m}}\right).$$

$$Z(t) = Z_{rel}(t) + \underbrace{Z_{frame}(t)}_{\dot{Z}_\infty (t-t_0)} = Z_{rel}(t_0) + \dot{Z}_\infty (t-t_0) + \frac{\dot{Z}_{rel}(t_0)}{\underbrace{6\pi\eta R}_{\mu}/m}\left(1 - e^{-(t-t_0)\frac{\mu}{m}}\right)$$

$$= Z_{rel}(t_0) + \dot{Z}_{rel}(t_0) \cdot (t-t_0) + \dot{Z}_{rel}(t_0)\frac{m}{\mu}\left(1 - e^{-(t-t_0)\frac{\mu}{m}}\right).$$

Exactly same equation, m represents inertia, μ represents speed decay.

Problem 4 - population (COVID !!!) „physics"

susceptible $\frac{d}{dt} S(t) = -\gamma S(t) I(t) + \beta R(t)$ γ - infection rate

infected $\frac{d}{dt} I(t) = \gamma S(t) I(t) - \rho I(t)$ ρ - recovery rate

recovered $\frac{d}{dt} R(t) = \rho I(t) - \beta R(t)$ β - losing immunity

a) $\frac{d}{dt}(S_{(t)} + I_{(t)} + R_{(t)}) = -\gamma \dot{S} I - \gamma S \dot{I} + \beta \dot{R} + \gamma \dot{S} I + \gamma S \dot{I} - \cancel{\rho} \dot{I} + \cancel{\rho} \dot{I} - \beta \dot{R} = 0.$

b) $R(t) = 1 - [S(t) + I(t)]$, $\tau = \gamma t$

$\dot{S} = -\gamma S I + \beta(1 - S - I)$

$\frac{dS}{d\tau} = -SI + \frac{\beta}{\gamma}(1 - S - I) = -I(\tau) \cdot \left(\frac{\beta}{\gamma} + S(\tau)\right) + \frac{\beta}{\gamma}(1 - S(\tau))$

$\frac{dI}{d\tau} = S(\tau) I(\tau) - \frac{\rho}{\gamma} I(\tau) = -I(\tau)\left[\frac{\rho}{\gamma} - S(\tau)\right]$

b, r

So $b = \frac{\beta}{\gamma} = \frac{\text{„losing immun." rate}}{\text{„infection" rate}}$,

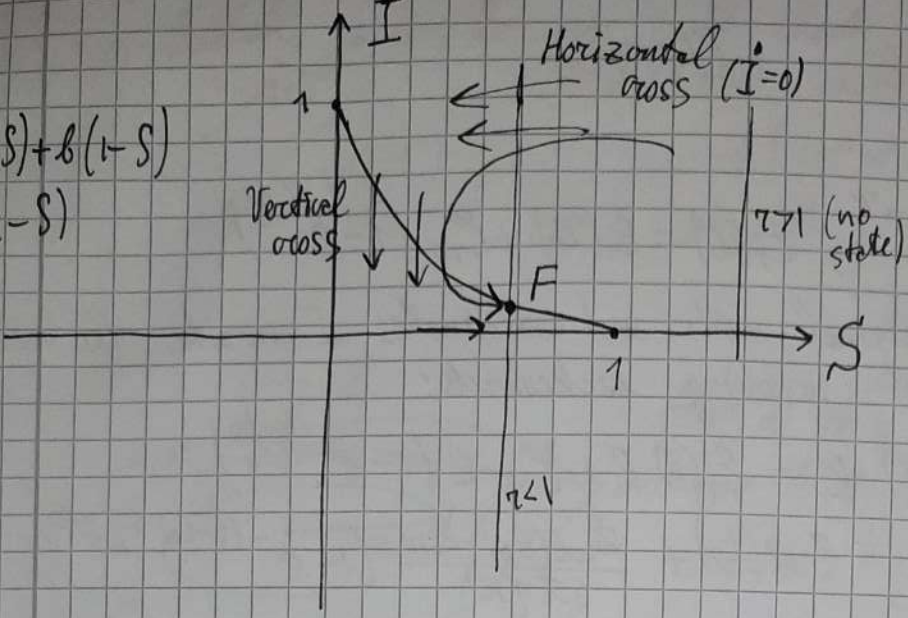
$r = \frac{\rho}{\gamma} = \frac{\text{„recovery" rate}}{\text{„infection" rate}}$

c) Show everything in (S, I) space -

$0 \leq S + I \leq 1$ are valid states

$\dot{S} = -I(b+S) + b(1-S)$

$\dot{I} = -I(\tau - S)$



Nullclines

$\dot{I} = 0 \rightarrow I = 0$ or $S = \tau$

$\dot{S} = 0 \rightarrow I = b\,\frac{1-S}{b+S} = b\left(-1 + \frac{b+1}{b+S}\right)$ hyperbola crossing at $(S=1, I=0)$ & $(S=0, I=1)$

d) Fixed point $(1,0)$ – nothing interesting, no one infected

Fixed point $F = \left(\tau;\ b\left(-1+\frac{b+1}{b+\tau}\right)\right)$ ← possible only for $\tau < 1$ when recovery rate does not „win"

2 parameters control the system

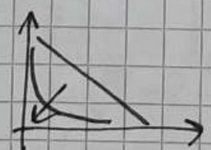
(main) τ – shows whether final state F is towards „no infected" (vert. line →) or „all infected" (vert. line ←)

b – accounts for $R(\tau)$ not shown here

$b\uparrow$

Meaning less part left for R („immun. loss")

$b\downarrow$



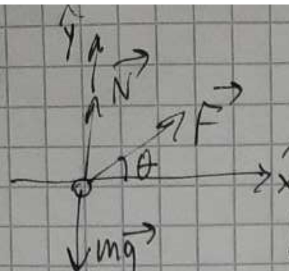
*e) The idea is to represent f, g as linear function of S, I using 1st part of Taylor:

$$\frac{dS}{d\tau} = \underbrace{f(F)}_{0} + \nabla f(F) \cdot \begin{pmatrix} S - S_F \\ I - I_F \end{pmatrix}$$

and similarly for $\frac{dI}{d\tau}$, only $\nabla g(F)$ changes.

TBD.

Problem 5



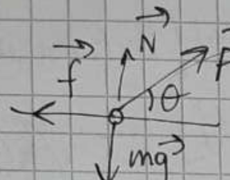
$\tan\theta = \frac{3}{4}$

a) Duck moves along $\hat{x}$ so that

$$\begin{cases} m\ddot{x} = \vec{F}\cdot\hat{x} = F\cos\theta \\ 0 = N + \vec{F}\cdot\hat{y} + mg(-1) \end{cases} \qquad \begin{cases} m\ddot{x} = F\cos\theta \\ N + F\sin\theta = mg \end{cases}$$

$\dot{x} = \frac{F}{m}\cos\theta\, t$, $x = \frac{F}{m}\cos\theta \frac{t^2}{2}$ $(t_0 = 0)$

b) Additional force $\rightarrow$ smaller $\ddot{x}$ up to uniform motion



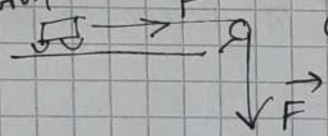
$|\vec{f}|_{max} = \mu N$

$$\begin{cases} F\cos\theta - \mu N = m\ddot{x} \\ F\sin\theta + N = mg \end{cases} \Longrightarrow x(t) = \left(-\mu g + \frac{F}{m}(\mu\sin\theta + \cos\theta)\right)\frac{t^2}{2}$$

c) direction will change, F and N and f change, $\ddot{x}$ not const

Problem 6

Idea here: get EOM by Newton

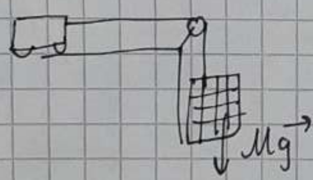


a, b) $a = \frac{F}{m}$

$v(t) = \frac{F}{m}t$, $x(t) = \underbrace{x_0}_{0} + \int_0^t v(t')dt' = \frac{F}{m}\frac{t^2}{2}$

c) Same as F

d)



$E = E_k + E_{pot} = \frac{m+M}{2}v^2 + Mgh$, $\dot{E} = 0$

$h < 0$, $|h(t)| = x(t)$ for car

$\dot{E} = \frac{M+m}{2}\cdot 2v\dot{v} + Mg\underbrace{\dot{h}}_{-v} = 0$

$(M+m)\dot{v} - Mg = 0$

$\dot{v} = \frac{Mg}{M+m}$ — it is less than before (same causing force, as car's mg is compensated by N, but masses of both of them)

Idea here: get EOM by $\frac{dE}{dt} = 0$

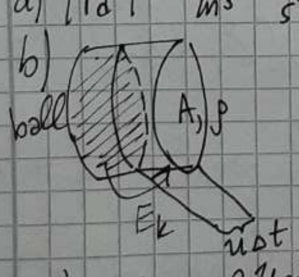
Problem 7

Idea here: derive turb. friction simply

$\vec{F_d} = -\frac{\rho|u|^2}{2}C_d A\hat{u}$

a) $|\vec{F_d}| = \frac{kg}{m^3}\cdot\frac{m^2}{s^2}\cdot m^2 = \frac{kg\cdot m}{s^2} = N$ ✓

b)



ball loses E_k $\rightarrow$ giving to air, $\vec{f_d}$ does negative work.

$-(u\Delta t)F_d = \underbrace{(u\Delta t)A}_{V}\,\rho\frac{u^2}{2}(-1)K$ ← the higher, the more losses

c) $Mg = \rho\frac{u_\infty^2}{2}C_d\pi\frac{d^2}{4} \Longrightarrow F_d = K\frac{\rho A}{2}u^2$

$u_\infty = \sqrt{\frac{8Mg}{\pi C_d d^2\rho}}$ = [calculate]

d) $L \approx \frac{u_\infty^2}{g}$ roughly

$t \approx \frac{u_\infty}{g}$.

Idea here dim. analysis gives order of magnitude guesses

Problem 9

$$\dot{x}(t) = v(t)$$
$$\dot{v}(t) = -\frac{k}{m}x(t) - \gamma v(t)$$

a) $[k] = \left[\frac{N}{m}\right] = kg\cdot s^{-2}$, $[\gamma] = \frac{N\cdot s}{kg\cdot m} = \frac{kg\cdot m\cdot s^{-1}}{kg\cdot m} = s^{-1}$

b) $\dot{v}(t)=0$, $v(t)=0 \to x(t)=0$ is rest position (no grav.)

c) $E = \frac{m\dot{x}(t)^2}{2} + \frac{k\,x(t)^2}{2}$

$$\frac{dE}{dt} = \frac{m}{\cancel{2}}\cdot\cancel{2}\cdot\dot{x}(t)\ddot{x}(t) + \frac{k}{\cancel{2}}\cdot\cancel{2}\cdot x(t)\dot{x}(t) = \dot{x}(t)\left[kx(t) + \underbrace{m\ddot{x}(t)}_{m\dot{v}(t)}\right] =$$
$$= \dot{x}(t)\cdot\left[\cancel{kx(t)} + (\cancel{-kx(t)}) - m\gamma\,\dot{x}(t)\right] = -m\gamma(\dot{x}(t))^2.$$

(non-conservat. system)

But E gives the current contour line in phase space $\to$ evolution of dynamics is fall to lower contour lines until $E=0$.

d) $A=\sqrt{E_0/k}$, $\left[\sqrt{E_0/k}\right] = \left[\sqrt{\frac{m}{k}\dot{x}^2 + x^2}\right] = [x] = m$

$\mathcal{E} = \frac{E}{E_0}$, $T=\sqrt{m/k}$, $\left[\sqrt{m/k}\right] = \left[\left(\frac{\dot{v}}{x}\right)^{-\frac{1}{2}}\right] = s$ } Non-dim.

$\xi = \frac{x}{A}$, $\tau = \frac{t-t_0}{T}$, $\mathcal{E} = \frac{E}{E_0}$

$\to \dot{\xi} = \frac{vT}{A}$.

$$E_0\mathcal{E} = \frac{m}{2}(\dot{x}(t))^2 + \frac{k}{2}(x(t))^2 = \frac{m}{2}\left(\frac{d(A\xi)}{d(\tau T)}\right)^2 + \frac{k}{2}(A\xi)^2 \quad [A^2 = E_0/k]$$
$$\cancel{A^2}k\mathcal{E} = \frac{m\cancel{A^2}}{2T^2}\cdot\dot{\xi}^2 + \frac{k\cancel{A^2}}{2}\xi^2$$

[$c=\sqrt{\frac{m}{k}}\gamma$ simplest suggest. from. dim. anal.]

$$\mathcal{E} = \frac{\dot{\xi}^2}{2} + \frac{\xi^2}{2}$$

Show dissipation:

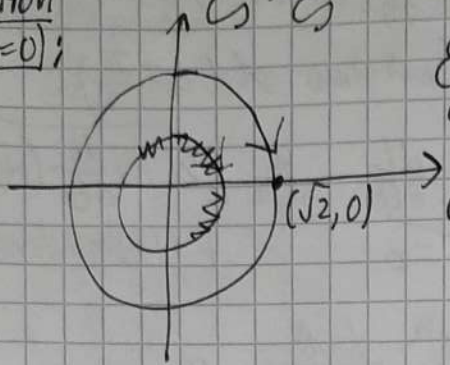
$$\dot{\mathcal{E}} = \dot{\xi}\ddot{\xi} + \xi\dot{\xi} = \dot{\xi}[\xi + \ddot{\xi}]$$

[From EOM using same scales:
$\ddot{x} = -\frac{k}{m}x - \gamma\dot{x} \to \frac{1}{T^2}\ddot{\xi} = -\frac{1}{T^2}\xi - \frac{\gamma}{T}\dot{\xi} \to \ddot{\xi} = -\xi - (T\gamma)\dot{\xi}$]

$$\dot{\mathcal{E}} = \dot{\xi}[\cancel{\xi} - \cancel{\xi} - (T\gamma)\dot{\xi}] = -\underbrace{(T\gamma)}_{c}\dot{\xi}^2$$

e) $\dot{\mathcal{E}} = -\underset{\substack{\downarrow \\ T\gamma}}{c}\,\varsigma^2$

<u>No friction ($c=0$)</u>:

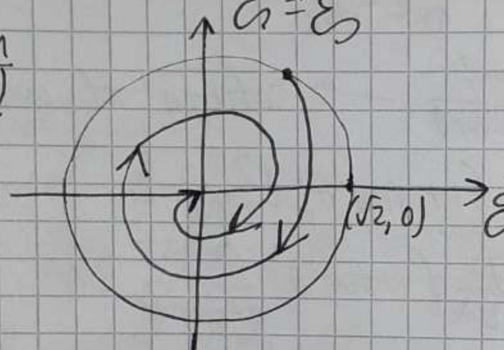


$\underset{\text{const}}{\mathcal{E}} = \frac{\varepsilon^2}{2} + \frac{\varsigma^2}{2}$ — circles, dist. $= \sqrt{2\mathcal{E}}$

Since $\mathcal{E}_0 = 1$ (We choose the scale in such way! $E = \mathcal{E} E_0$)

There is always 1 circle, distance $\sqrt{2}$.

<u>Friction ($c>0$)</u>



Infinitely long „motion" to $(0,0)$.

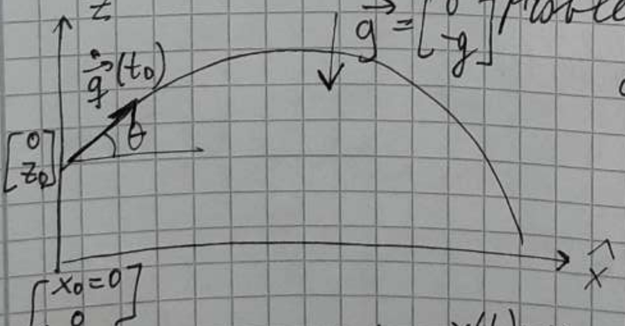
*f) $\ddot{x}(t) = -\frac{k}{m}x(t) - \gamma \dot{x}(t)$

Assume $x(t) = x_0 \sin(\omega t - \varphi) e^{-t/t_c}$

1) $\omega = \sqrt{\frac{k}{m}}$, $t_c = \frac{1}{\gamma}$ — by dim. analysis or direct checking

2) φ is not determined by k, m, γ, but by $x(0)$, $\dot{x}(0)$

Problem 10 — projectile motion (with non-trivial stuff)



a) EOM: $\ddot{\vec{q}} = \vec{g}$

$\ddot{\vec{q}}(t) = \begin{bmatrix}\ddot{x}(t)\\ \ddot{z}(t)\end{bmatrix} = \begin{bmatrix}0\\-g\end{bmatrix} \xRightarrow{\int dt} \dot{\vec{q}}(t) = \begin{bmatrix}\dot{x}_0\\ \dot{z}_0 - g(t-t_0)\end{bmatrix}$

$\xRightarrow{\int dt} \vec{q}(t) = \begin{bmatrix}x(t)\\z(t)\end{bmatrix} = \begin{bmatrix}x_0 + \dot{x}_0(t-t_0)\\ z_0 + \dot{z}_0(t-t_0) - \frac{g(t-t_0)^2}{2}\end{bmatrix}$

$\Longrightarrow t - t_0 = \frac{x(t) - x_0}{\dot{x}_0}$, take $x_0 = 0$, $q_0 = v$,

b, *c) $z(x) = z_0 + \frac{\dot{z}_0}{\dot{x}_0}(x) - \frac{g}{2}\left(\frac{x}{\dot{x}_0}\right)^2$ — parabola, $\dot{x}_0 = v\cos\theta$, $\dot{z}_0 = v\sin\theta$. $\Longrightarrow$

$$z(x) = H + \tan\theta \cdot x - \frac{g}{2v^2}\frac{x^2}{\cos^2\theta} = H + \tan\theta \cdot x - \frac{g}{2v^2}(1+\tan^2\theta)x^2$$

$z(L) = 0$,

$$(*) \quad H + \tan\theta \cdot L(\theta) - \frac{g}{2v^2}(1+\tan^2\theta)L^2(\theta) = 0 \quad \Big/ \frac{d}{d\theta}$$

L max/min when $\frac{dL}{d\theta} = 0$ (minima at $\theta = \pm\frac{\pi}{2}$)

$$\frac{1}{\cos^2\theta}\cdot L(\theta) + \tan\theta \underbrace{\frac{dL}{d\theta}}_{=0} - \frac{g}{2v^2}\Big[\overbrace{2L(\theta)\frac{dL}{d\theta}(1+\tan^2\theta)}^{=0} + L(\theta)^2 \cdot 2\tan\theta \cdot \frac{1}{\cos^2\theta}\Big] = 0$$

$$\frac{\cancel{L(\theta)}}{\cancel{\cos^2\theta}} - \frac{g}{v^2} L_{max}^{\cancel{2}} \frac{\tan\theta}{\cancel{\cos^2\theta}} = 0$$

$$1 - \frac{g}{v^2} L_{max} \tan\theta = 0$$

$$L_{max} = \frac{v^2}{g\tan\theta} \longrightarrow \text{return at our } (*)$$

$$H + \tan\theta \cdot L_{max} - \frac{g}{2v^2}(1+\tan^2\theta)L^2_{max} = 0$$

$$H + \cancel{\tan\theta}\frac{v^2}{g\cancel{\tan\theta}} - \frac{g}{2v^2}(1+\tan^2\theta)\frac{v^4}{g^2\tan^2\theta} = 0$$

$$gH + v^2 - \frac{v^2}{2}\left(1 + \frac{1}{\tan^2}\right) = 0$$

$$\tan\theta = \sqrt{\frac{1}{-1 + \frac{2}{v^2}(gH+v^2)}} = \frac{1}{\sqrt{1+\frac{2gH}{v^2}}}$$ – "best" angle if no friction / Earth curvature.

(if $H = 0$ → from ground, $\theta = \arctan 1 = \frac{\pi}{4}$)

d) friction – TBD

Problem 11

TBD.

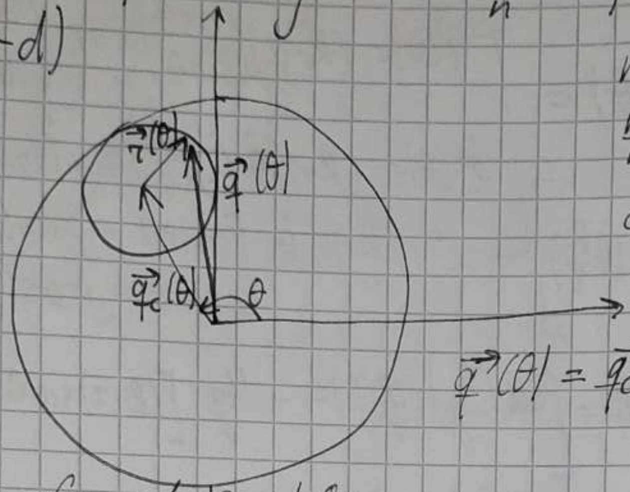
Problem 8 — hypotrochoids etc.

a) Curve is always closed since after $LCM(m,n)$ cogs the point will be at same spot after period

$N_{revolution} = LCM(m,n)$ (w.r.t. outer circle)

Symmetry $= \frac{LCM(m,n)}{n}$ -fold

b-d)



m, n cogs measure circumf.

$$\frac{m}{n} = \frac{2\pi R}{2\pi r} = \frac{R}{r}$$

$$d = |\vec{q_c}(\theta) - \vec{q}(\theta)|$$

$$\rho = \frac{d}{r}$$

reference rotation

$$\vec{q}(\theta) = \vec{q_c}(\theta) + \vec{r}(\theta) = (R-r)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + d\begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix}$$

inner rotation

Connect γ and θ:

disk moved $l = \theta R$,

$$N_{rot} = \frac{l}{2\pi r} = \frac{\theta R}{2\pi r}, \quad \gamma = \frac{\theta R}{2\pi r}\underbrace{(-2\pi)}_{\text{other direction}} \overset{+\theta}{=} -\theta\frac{R}{r} + \theta$$ (due to rotation on curved line, $+\theta$).

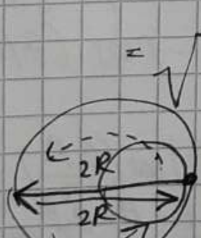
Length of one closed curve: $\theta \in \left[0, 2\pi\frac{LCM(m,n)}{m}\right]$

$$\vec{q}\,'(\theta) = (R-r)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} + \rho r\begin{pmatrix}-\sin(\underbrace{\theta(1-\tfrac{R}{r})}_{\text{take 1}})\\ \cos(\theta(1-\tfrac{R}{r}))\end{pmatrix}\left(1-\frac{R}{r}\right) =$$

$$= (R-r)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} - \rho(R-r)\begin{pmatrix}-\sin(\theta(1-\tfrac{R}{r}))\\ \cos(\theta(1-\tfrac{R}{r}))\end{pmatrix} = (R-r)\begin{bmatrix}-\sin\theta + \rho\sin\ldots\\ \cos\theta - \rho\cos(\theta\ldots\end{bmatrix}$$

$$|\vec{q}\,'(\theta)| = (R-r)\sqrt{\underbrace{\sin^2\theta + \cos^2\theta}_{1} + \underbrace{\rho^2\sin^2\ldots + \rho^2\cos^2\ldots}_{\rho^2} \underbrace{- 2\rho\sin\theta\sin\ldots}_{-2\rho\cos\theta\cos(\ldots)}} =$$

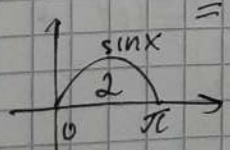
$$= \sqrt{1 + \rho^2 - 2\rho\cos\left(\theta\frac{R}{r}\right)}\,(R-r).$$ Take $\frac{R}{r} = 2$, $\rho = 1$,



length must be $4R$.

Indeed $L = \int_0^{2\pi} |\vec{q}\,'(\theta)|\,d\theta = \frac{R}{2}\int_0^{2\pi}\sqrt{2 - 2\cos(2\theta)}\,d\theta =$

$$= \frac{R}{\sqrt{2}}\sqrt{2}\underbrace{\int_0^{2\pi}|\sin\theta|\,d\theta}_{4} = 4R.$$



*) Now assume gravity acts, given initial conditions $\theta_0, \dot{\theta}_0$ — constrained rotation. Is it possible to find $\theta(t)$ by Lagrange?

1DOF $\theta(t)$

Lagrange function: $\mathcal{L}(\theta, \dot{\theta}, t) = T(\dot{\theta}) - V(\theta) =$

$$\left[\begin{aligned} \vec{q}(\theta) &= (R-r)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + \rho r \begin{pmatrix}\cos(\theta(1-\frac{R}{r}))\\ \sin(\theta(1-\frac{R}{r}))\end{pmatrix} \\ \dot{\vec{q}}(\theta) &= (R-r)\dot{\theta}\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} + \rho\dot{\theta}(r-R)\begin{bmatrix}-\sin(\dots)\\ \cos(\dots)\end{bmatrix} \\ &= (R-r)\dot{\theta}\begin{bmatrix}-\sin\theta + \rho\sin(\dots)\\ \cos\theta - \rho\cos(\dots)\end{bmatrix}\end{aligned}\right]$$

$$= \left[|\dot{\vec{q}}(\theta)|^2 = (R-r)^2\dot{\theta}^2 \cdot (1+\rho^2 - 2\rho\cos(\theta\tfrac{R}{r})) \right]$$

$$= \frac{m}{2}|\dot{\vec{q}}(\theta)|^2 + M\vec{g}\,\vec{q}(\theta) = \left[M\begin{pmatrix}0\\ -g\end{pmatrix}\cdot\vec{q}(\theta) = -Mg\cdot[(R-r)\sin\theta + \rho r\sin(\dots)] \right]$$

$$= \frac{m}{2}(R-r)^2\dot{\theta}^2(1+\rho^2-2\rho\cos(\theta\tfrac{R}{r})) - Mg[(R-r)\sin\theta + \rho r\sin(\dots)]$$

Then Euler-Lagrange equation: $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} \implies \frac{d}{dt}\left[m(R-r)^2(1+\rho^2-2\rho\cos(\theta\frac{R}{r}))\dot{\theta}\right] =$

$$= m(R-r)^2\left[(1+\rho^2-2\rho\cos(\theta\tfrac{R}{r}))\ddot{\theta} + (\dot{\theta})^2 2\rho\sin(\theta\tfrac{R}{r})\tfrac{R}{r}\right] =$$

$$= \frac{m}{\cancel{2}}(R-r)^2\dot{\theta}^2\cdot\cancel{2}\rho\sin(\theta\tfrac{R}{r})\tfrac{R}{r} - mg\left[(R-r)\cos\theta + \rho r\cos[\theta(1-\tfrac{R}{r})]\cdot[1-\tfrac{R}{r}]\right]$$

$$\implies (R-r)^{\cancel{2}}[1+\rho^2-2\rho\cos(\theta\tfrac{R}{r})]\ddot{\theta} = -(R-r)^{\cancel{2}}\rho\sin(\theta\tfrac{R}{r})\tfrac{R}{r}(\dot{\theta})^2 - g\left[\cancel{(R-r)}\cos\theta - \rho\cancel{(R-r)}\cos[\theta(1-\tfrac{R}{r})]\right]$$

$$(R-r)[1+\rho^2-2\rho\cos(\theta\tfrac{R}{r})]\ddot{\theta} = -(R-r)\rho\sin(\theta\tfrac{R}{r})\tfrac{R}{r}(\dot{\theta})^2 - g[\cos\theta - \rho\cos(\theta(1-\tfrac{R}{r}))]$$

Take $\rho = 1$ to get rid of this

$$(R-r)\cancel{2}\cdot 2\sin^{\cancel{2}}(\tfrac{\theta R}{2r})\ddot{\theta} = -(R-r)\cdot\cancel{2}\sin(\tfrac{\theta R}{2r})\cos(\tfrac{\theta R}{2r})\tfrac{R}{r}(\dot{\theta})^2 + g\cancel{2}\cdot\sin\frac{\theta+\theta-\theta\frac{R}{r}}{2}\cancel{\sin\frac{\theta R}{2r}}$$

$$(R-r)2\sin(\tfrac{\theta R}{2r})\ddot{\theta} = -(R-r)\cos(\tfrac{\theta R}{2r})\tfrac{R}{r}(\dot{\theta})^2 + g\sin(\theta - \tfrac{\theta R}{2r})$$

$$\ddot{\theta} = \frac{1}{2}\frac{g\sin(\theta-\frac{\theta R}{2r})}{(R-r)\sin(\frac{\theta R}{2r})} - \cot(\tfrac{\theta R}{2r})\cdot\tfrac{R}{2r}(\dot{\theta})^2$$

Set $R = Kr$

$$\ddot{\theta} = \frac{g}{2}\frac{\sin(\theta-\frac{\theta K}{2})}{(K-1)r\sin(\frac{\theta K}{2})} - \cot(\tfrac{\theta K}{2})\cdot\tfrac{K}{2}(\dot{\theta})^2$$

Special case $K=2$:

$$\ddot{\theta} = -\cot\theta \cdot (\dot{\theta})^2$$