Exam. 23 February

Problem 1. Prove that for any $x \geq 0$, $x \in \mathbb{R}$, one has

$$(1+x)^9 \geq 1 + 30x^2$$

Problem 2. Provide an example of a sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} a_n = 3, \qquad \sup_{n\geq 1} a_n = 5.$$

Problem 3. Compute the following limit

$$\lim_{n\to\infty} \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}$$

Problem 4. Compute the limit

$$\lim_{n\to\infty} \frac{n^2 2^n + 2n^2 5^n + 6^n}{3^n\left(n^7 + n^4\right) + 6^n + n^2 5^n}.$$

Problem 5. Prove that the equation

$$2^x = x^3 + 5$$

has at least two real roots.

Problem 6. Is the function

$$f(x) = \begin{cases} \sin\left(x^2\right)\sin\left(\frac{1}{x}\right), & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

differentiable at $x = 0$?

Problem 7. Compute the following limit

$$\lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{\ln(1+x) - x + \frac{x^2}{2}}.$$

Problem 8. Check the following series for convergence

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{3} - 1\right)$$

Problem 9. Compute the area of a region bounded by curves

$$y = e^x, \qquad x = 1, \qquad y = 5.$$

Problem 10. Find all local and global extrema of the following function

$$f(x) = x\ln^2(x), \qquad f : (0;3] \to \mathbb{R}.$$

Problem 11. Compute the following indefinite integral

$$\int \left(\frac{1-x}{x}\right)^2 dx.$$

Problem 12. Compute the intersection of the following three planes in $\mathbb{R}^3$:

$$x + 2y - z = 1, \qquad 2x + y = 0, \qquad 3x + 3y - z = 1$$