5. Torques, Masses, and the Vector Product

The Chapters 2.5–2.8 of my lecture notes provides the background to solve the following exercises. Your solution to the problems 5.1–5.3 should uploaded

to your Moodle account

as a PDF-file

by Wednesday, Nov 8, 10:30 (with a grace time till the start of the seminars).

The parts marked by ⋆ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

The self-test problems serve to check you understanding of vectors.

The bonus problem provides the solution to a problem that we introduced in class. It might take some extra effort to solve.

Problems

Problem 1. Properties of the cross product

Verify the following properties of the cross product:

a) Two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are linearly independent unless their cross product vanishes

$$\forall \lambda \in \mathbb{R}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 : \quad \mathbf{b} = \lambda \mathbf{a} \Rightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0} \, .$$

b) When three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are linearly dependent, i.e. when one of them can be expressed as a linear combination of the other two, then their scalar triple product vanishes

1. $\forall \alpha, \beta \in \mathbb{R}, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 , \quad \mathbf{c} = \alpha \mathbf{a} + \beta \mathbf{b} : \qquad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0 \, ,$
2. $\forall \gamma, \delta \in \mathbb{R}, \quad \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 , \quad \mathbf{a} = \gamma \mathbf{b} + \delta \mathbf{c} : \qquad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0 \, ,$
3. $\forall \varepsilon, \varphi \in \mathbb{R}, \quad \mathbf{a}, \mathbf{c} \in \mathbb{R}^3 , \quad \mathbf{b} = \varepsilon \mathbf{a} + \varphi \mathbf{c} : \qquad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0 \, .$

Hint: There is a straightforward argument that 1. implies 2. and 3.

c) All vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ obey the identity:

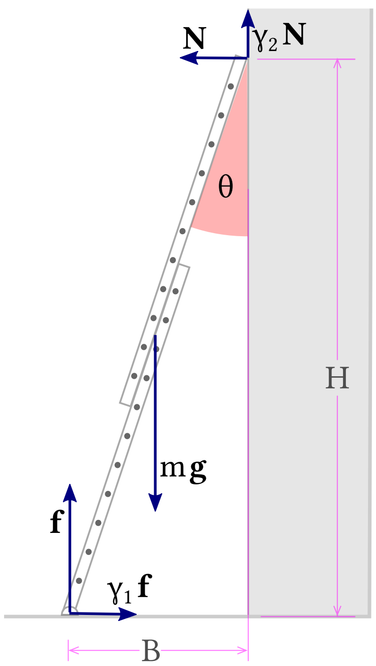
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$$

Remark: This identity is commonly denoted as bac-cab rule.
Hint: You may use that $\sum_k \varepsilon_{ijk}\, \varepsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl}$.

d) All vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$ obey the identity:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})\,.$$

Problem 2. Forces and Torques acting on a ladder



based on Original: Bradley – Vector: Sarang
[Public domain from wikimedia]

To the left you see the sketch to a ladder leaning to a wall. The angle from the downwards vertical to the ladder is denoted as θ. There is a gravitational force of magnitude Mg acting on the ladder that has a mass M. At the point where it leans to the wall there is a normal force $\mathbf{N}$ acting from the wall to the ladder. At the ladder feet there is a normal force $\mathbf{f}$ exerted by the ground, and a tangential friction force $\boldsymbol{F}_f$ of magnitude $F_f = \gamma_1 f$.

⋆ a) In principle there also is a friction force of magnitude $\gamma_2\, N$ acting at the contact from the ladder to the wall. Why is it admissible to neglect this force?
Remark: There are at least two good arguments.

b) Discuss the vertical and horizontal force balance for the ladder.[1] Is there a

[1]This means the conditions entailing that the components of the force representing the horizontal and vertical direction vanish.

unique solution?

Bonus: There will be solutions that do not match your physical intuition. What might be missing in the force balance?

c) Determine the torque acting on the ladder. Does it matter whether you consider the torque with respect to the contact point to the roof, the center of mass, or the foot of the ladder?

⋆ d) Where does the mass of the ladder enter the discussion? Do you see why?

e) Determine the threshold of sliding based on the balance of torques.
For metal feet on a wooden ground it takes a value of $\gamma_s \simeq 2$. For a slippery smooth ground it can be as small as $\gamma_s \simeq 0.3$. What does that tell about the range of angles where the ladder starts to slide?

f) The ladder will slide when the modulus of the friction force F_f exceeds a maximum value $\mu_S F_g$ where μ is the static friction coefficient for of the ladder feet on the ground. For metal feet on a wooden ground it takes a value of $\mu_S \simeq 2$. What does that tell about the angles θ where the ladder does not start to slide?

⋆ g) Why does a ladder commonly starts sliding when when a man has climbed to the top? Is there anything one can do against it? Is that even true, or just an urban legend?

Hint: In order to discuss this point you must augment the sketch with the gravitational force acting on the man. For a man of mass M this amounts to a force $M\mathbf{g}$ added at the position of the man on the ladder.

Problem 3. Determining the volume and the mass

Determine the mass M, and the area or volume V of bodies with the following mass density and shape.
The purpose of this exercise is that you determine the volume (or surface area) by working out the integrals over the area. The well-known relations from geometry may only be used to check the result.

a) A triangle in two dimensions with constant mass density $\rho = 1\,\mathrm{kg/m^2}$ and side length $6\,\mathrm{cm}$, $8\,\mathrm{cm}$, and $10\,\mathrm{cm}$.
Hint: Determine first the angles at the corners of the triangle. Decide then about a convenient choice of the coordinate system (position of the origin and direction of the coordinate axes).

b) A circle in two dimensions with center at position (a, b), radius $R = 5\,\mathrm{cm}$, and constant mass density $\rho = 1\,\mathrm{kg/m^2}$.
Hint: How do M, V and $\mathbf{Q}$ depend on the choice of the origin of the coordinate system?

c) A rectangle in two dimensions, parameterized by coordinates $0 \le x \le W$ and $0 \le y \le B$, and a mass density $\rho(x, y) = \alpha\, x$.
What is the dimension of α in this case?

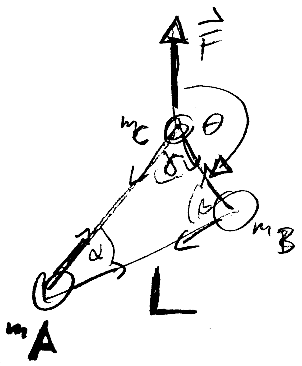
d) A three-dimensional wedge with constant mass density $\rho = 1\,\mathrm{kg/m^3}$ that is parameterized by $0 \le x \le W$, $0 \le y \le B$, and $0 \le z \le H - Hx/W$.
Discuss the relation to the result of part b).

e) A cube with edge length L. When its axes are aligned parallel to the axes $\hat{x}$, $\hat{y}$, $\hat{z}$, it density takes the form $\rho(x, y, z) = \beta\, z$.
What is the dimension of β in this case?

Self Test

Problem 4. How does a triangle hang?

We consider a triangle with masses m_A, m_B, and m_C concentrated at its three corners, that have angles α, β, and γ. The triangle is suspended from a chord attached at corner C. The side between A and B has length L. We will now determine the angle θ from the vertical to the side BC, the upward force due to the suspension, and the forces acting along the sides.
(See the sketch to the right.)



a) The forces and the angles do not depend on L. Why is that?

b) Set up the force balance for each of the masses m_A, m_B, and m_C.

c) Rewrite the result of b) into overall force balances for the horizontal force components and the vertical force components. These equations should take the form of linear combinations of vectors, that we can write down a priori, and coefficients involving the forces and the angle θ.

d) Determine the force $\mathbf{F}$.
How does $\mathbf{F}$ depend on the masses, and on the geometry and size of the triangle? Provide a physical argument why this dependence should hold. and the dependence of the mass ratio $\mu = m_A/(m_A + m_B)$ on the angles.

e) Determine the dependence of the mass ratio $\mu = m_A/(m_A + m_B)$ on the angles.

   Use this result to discuss the dependence of θ on the geometry and the mass distribution:

   - What is the valid range of values for μ?
   - What is the range of physically meaningful values for θ, for a given geometry of the triangle?
   - Check your ideas based on three plots of $\mu(\theta)$ with fixed $\gamma \in \{\pi/3, \pi/2, 2\pi/3\}$, respectively. Plot several lines in each plot that represent the dependence for different choices of β.

Problem 5. Volume and surface of solids of revolution

The surface of a solid of revolution can be obtained by rotating some function $f(x)$ around the x axis. For instance, the function $\sqrt{R^2 - x^2}$ with $-R \leq x \leq R$ describes a sphere of radius R. The volume V and the surface O of a solid of revolution are given by the integrals

$$V = \pi \int \mathrm{d}x \; (f(x))^2 \qquad S = 2\,\pi \int \mathrm{d}x \; f(x) \, \sqrt{1 + \big(f'(x)\big)^2}$$

a) Sketch the function $f(x) = \sqrt{R^2 - x^2}$ and verify the the solid of revolution is indeed a sphere.

b) Determine the volume and the surface of the sphere by evaluating the integrals provided above.

Bonus Problems

Problem 6. Can you paint this funnel?

We consider the solid of revolution that is described by $f(x) = x^{-1}$ with $x \geq 1$.

a) Sketch the function $f(x)$. It describes a funnel of infinite length, or a "horn".

b) Show that the funnel has the volume π.

c) Determine the surface $S(L)$ of the piece of the funnel with $x \in \{1, L\}$. How does the function $S(L)$ look like in the limit $L \to \infty$? What does this imply for the surface area of the funnel?

d) Since the surface area of the funnel is infinite, one would expect that it can not painted with a finite amount of paint. On the other hand, its volume is finite such that it can be filled with a finite amount of paint. Wouldn't that imply that the surface is painted? How can that be?

Problem 7. Balancing a Stick

We consider a very thin, semi-infinite infinite stick (i.e., one that goes off to infinity in one direction) that balances on a support located at a distance ℓ from its end.

a) Sketch the problem and argue that it can be dealt with as a one-dimensional problem where the mass distribution of the stick is described by a function $\rho(x)$. Here x is the distance along the stick, measured starting from its end (i.e., $x = 0$ at the end).

b) Assume that the stick has a 1D mass density $\rho(x) = \rho_0/(1+x)^3$. Determine the mass of the stick, and the position ℓ of the support.

⋆ c) Consider now a stick with the following property: You can cut it at an arbitrary position, and the resulting semi-infinite piece will always balance at the same position ℓ. Determine the mass density $\varrho(x)$ of this stick.
Hint: Consider an ansatz of the form $\rho(x) = \rho_0 \, \mathrm{e}^{-\gamma x}$.