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Theoretical Physics I

Problem 5.1 Vectors over complex numbers

a) $\vec{v}\cdot(\vec{c}\odot\vec{w}) = ((\vec{c}\odot\vec{w})\cdot\vec{v})^* = (c(\vec{w}\cdot\vec{v}))^* = c^*(\vec{w}\cdot\vec{v})^* =$

$= c^*(\vec{v}\cdot\vec{w})$

b) $\vec{v}\cdot\vec{v} = (\vec{v}\cdot\vec{v})^*$ $\quad \vec{v}\cdot\vec{v} = a+bi$

$\forall a,b \quad a+bi = a-bi \longleftrightarrow 2bi=0 \longleftrightarrow \underline{b=0}$

* c) $E = \{\hat{e}_1, \hat{e}_2 .. \hat{e}_d\}$ is ONB of $V \rightarrow \hat{e}_i\hat{e}_j = \delta_{ij}$

<u>Show</u>: $\forall \vec{v}\in V\ \exists! \{v_1, v_2, ..., v_d\}: \vec{v} = \sum_{i=1}^{d} v_i \odot \hat{e}_i$.

<u>Note</u>: for this task I assume that $\forall\vec{v}\ \exists\{v_1, ... v_d\}: \vec{v} = \sum_{i=1}^{d} v_i\odot\hat{e}_i$, and the goal is to show <u>uniqueness</u> of representation (otherwise d can be less than $\dim V$)

c.1) First let show $\{\hat{e}_i\}$ must be linearily independent, i.e. $\vec{0} = (v_1\odot\hat{e}_1)\oplus(v_2\odot\hat{e}_2)\oplus(v_3\odot\hat{e}_3), ..., \oplus(v_d\odot\hat{e}_d)$ has one solution $v_i = 0\ \forall i\in\{1,...d\}$

Proof by contradiction. Suppose $\exists\{v_{m_1}, v_{m_2}, ..., v_{m_n}\}$ $(n\le d)$, $v_{m_i}\neq 0$ (all other $v_j = 0$)

Then

$\vec{0} = (v_{m_1}\odot\hat{e}_{m_1})\oplus(v_{m_2}\odot\hat{e}_{m_2}), ..., \oplus(v_{m_n}\odot\hat{e}_{m_n}),$

$-v_{m_1}\odot\hat{e}_{m_1} = (v_{m_2}\odot\hat{e}_{m_2})\oplus ... \oplus(v_{m_n}\odot\hat{e}_{m_n}) \quad \Big|\cdot\frac{-1}{v_{m_1}}$

$\hat{e}_{m_1} = \left(-\frac{v_{m_2}}{v_{m_1}}\odot\hat{e}_{m_2}\right)\oplus ... \oplus\left(-\frac{v_{m_n}}{v_{m_1}}\odot\hat{e}_{m_n}\right).$

Now $\underline{\hat{e}_{m_1}\cdot\hat{e}_{m_1} = \delta_{m_1 m_1} = 1}$, at the same time

$\hat{e}_{m_1}\cdot\hat{e}_{m_1} = \hat{e}_{m_1}\cdot\left[\left(-\frac{v_{m_2}}{v_{m_1}}\odot\hat{e}_{m_2}\right)\oplus ... \oplus\left(-\frac{v_{m_n}}{v_{m_1}}\odot\hat{e}_{m_n}\right)\right] = \left(-\frac{v_{m_2}}{v_{m_1}}\right)^*\underbrace{(\hat{e}_{m_1}\cdot\hat{e}_{m_2})}_{0}\oplus ...$

$\oplus\left(-\frac{v_{m_n}}{v_{m_1}}\right)^*\underbrace{(\hat{e}_{m_1}\cdot\hat{e}_{m_n})}_{0} = 0.$ Contradiction. Speaking simple, if $\vec{v}$ can be expressed not uniquely — some of $\hat{e}_i$ can be dependent on others, which is impossible in basis.

c.2) Now using $\vec{0}$ is uniquely expressed as $(0\odot\hat{e}_1)\oplus(0\odot\hat{e}_2)\dots$,

suppose $\vec{v}\in V$, $\vec{v} = \sum_{i=1}^{d} v_i\odot\hat{e}_i = \sum_{i=1}^{d} w_i\odot\hat{e}_i$, $w_i \neq v_i$ for some i.

Then $\vec{v}-\vec{v} = \sum_{i=1}^{d}(v_i-w_i)\odot\hat{e}_i = \vec{0}$. But $v_i - w_i$ must be 0 according to part 1. So $v_i = w_i$, and representation of $\vec{v}\in V$ is always <u>unique</u>, in given basis.

d)
$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \oplus \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix} = \sum_{i=1}^{d}(v_i\odot\hat{e}_i)\oplus\sum_{j=1}^{d}(w_j\odot\hat{e}_j) = \sum_{i=1}^{d}(v_i\odot\hat{e}_i)\oplus(w_i\odot\hat{e}_i)$$

$$= \sum_{i=1}^{d}(v_i+w_i)\odot\hat{e}_i = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \dots \\ v_d+w_d \end{pmatrix}$$ — addition is isomorphic

$$c\odot\vec{v} = c\odot\sum_{i=1}^{d}(v_i\odot\hat{e}_i) = \sum_{i=1}^{d} c\odot(v_i\odot\hat{e}_i) = \sum_{i=1}^{d}(c\cdot v_i)\odot\hat{e}_i = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_d \end{pmatrix}$$

— scalar mult. is also isomorphic

e)
$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix} = \left(\sum_{i=1}^{d} v_i\odot\hat{e}_i\right)\left(\sum_{j=1}^{d} w_j\odot\hat{e}_j\right) = \sum_{i,j=1}^{d}(v_i\odot\hat{e}_i)(w_j\odot\hat{e}_j) =$$

$$= \sum_{i,j=1}^{d} v_i\left[\hat{e}_i\cdot(w_j\odot\hat{e}_j)\right] \overset{*}{=} \sum_{i,j=1}^{d} v_i\left[w_j^*\underbrace{(\hat{e}_i\cdot\hat{e}_j)}_{\delta_{ij}}\right] = \sum_{i,j=1}^{d} v_i w_j^*\delta_{ij} =$$

$$= \sum_{i=1}^{d} v_i w_i^*.$$

$$\forall\vec{v}\in V\ \ \vec{v}\cdot\vec{v} = \sum_{i=1}^{d} v_i v_i^* = \sum_{i=1}^{d} |v_i| \underbrace{e^{\varphi_i\cdot i}\ |v_i| e^{-\varphi_i\cdot i}}_{1} = \left(v_i = |v_i| e^{\varphi_i\cdot i} \text{ here}\right)$$

$$= \sum_{i=1}^{d}|v_i|^2 \geq 0,$$ equality means $v_i = 0\ \forall i$, $\vec{v}=\vec{0}$.

Problem 5.2 Complex Numbers and 2D-vectors

Note: For all tasks I use properties $zz^* = |z|^2$; $z = |z|e^{\varphi i}$; $(z_1 \bullet z_2)^* = z_1^* \bullet z_2^*$, where $\bullet$ can be $\cdot, +, -$; $\sin\varphi = -\sin(-\varphi)$, $\cos\varphi = \cos(-\varphi)$.

a) $\forall z_1, z_2,\ z_1 = x_1 + iy_1,$
$z_2 = x_2 + iy_2,$

$$(z_1+z_2)(z_1+z_2)^* = (z_1+z_2)(z_1^*+z_2^*) = z_1z_1^* + z_2z_2^* + z_1z_2^* + z_2z_1^* =$$

$$= |z_1|^2 + |z_2|^2 + |z_1||z_2|e^{(\varphi_1-\varphi_2)i} + |z_2||z_1|e^{(\varphi_2-\varphi_1)i} = |z_1|^2 + |z_2|^2 +$$

$$+ |z_1||z_2|\big(\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2) + \cos(\varphi_2-\varphi_1) + i\sin(\varphi_2-\varphi_1)\big) =$$

opposite

$$= \underline{|z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\varphi_1-\varphi_2),\ \in \mathbb{R}.}$$

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b) $\vec{q_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\vec{q_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, $(\vec{q_1}+\vec{q_2})\cdot(\vec{q_1}+\vec{q_2}) = |\vec{q_1}|^2 + |\vec{q_2}|^2 + 2\vec{q_1}\vec{q_2} =$

$$= |\vec{q_1}| + |\vec{q_2}| + 2|\vec{q_1}||\vec{q_2}|\cos\angle(\vec{q_1}, \vec{q_2}).$$

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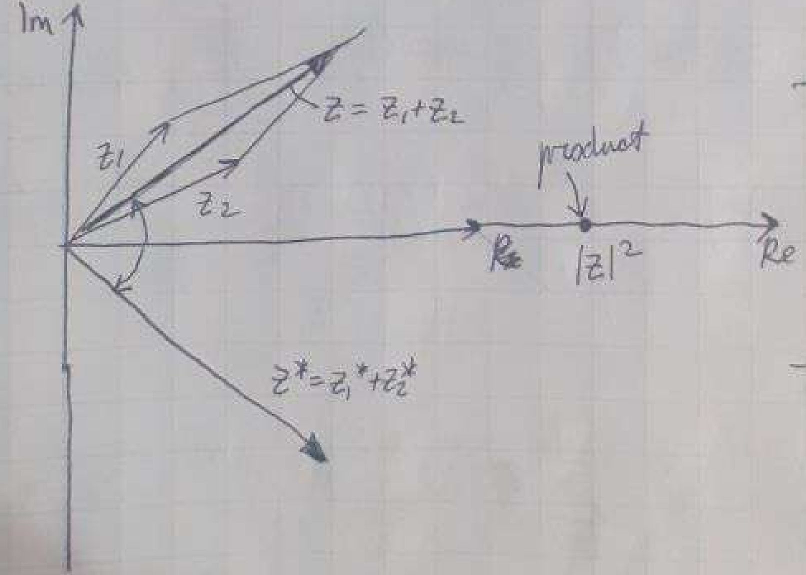
Behaviour on $\mathbb{R}^2$ and $\mathbb{C}$ is similar, but not isomorphic



— In case of complex multiplication of conjugates there is scaling, and two rotations undoing each other.

— In case of $\mathbb{R}^2$ multip. of vector with itself there is just squaring of length. Both give real value.

c) $z_1 z_2^* = |z_1|e^{\varphi_1 i} \cdot |z_2|e^{-\varphi_2 i} = |z_1||z_2|e^{(\varphi_1-\varphi_2)i} = |z_1||z_2| \cdot$
$\cdot \big(\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)\big)$

$\vec{q_1}\vec{q_2} = |\vec{q_1}||\vec{q_2}|\cos(\theta_1-\theta_2)$, where $\theta_i = \angle(\vec{q_i}, \hat{e}_x)$.

If $\varphi_1 = \varphi_2$ for z_1 and z_2, $z_2 = z_1$, $z_1 z_1^* = |z_1|^2 \in \mathbb{R}$.
Otherwise $z_1 z_2^* \notin \mathbb{R}$ in general, but $\vec{q_1} \cdot \vec{q_2} \in \mathbb{R}$.

d) $(\vec{q_1} + \vec{q_2})(\vec{q_1} + \vec{q_2}) = \vec{0}$, $\theta = \angle(\vec{q_1}, \vec{q_2})$, $q_i = |\vec{q_i}|$

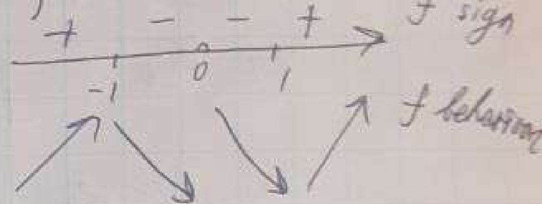
$$q_1^2 + q_2^2 + 2 q_1 q_2 \cos\theta = 0 \quad \Big| \cdot \frac{1}{q_1 q_2}$$

$$\frac{q_1}{q_2} + \frac{q_2}{q_1} + 2\cos\theta = 0. \quad \text{Let } \frac{q_1}{q_2} = \rho$$

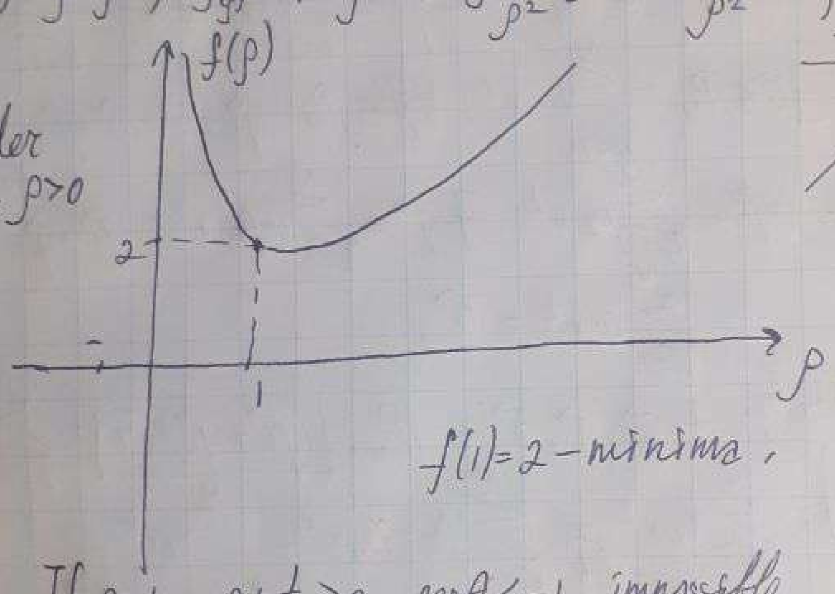
$$\underline{\rho + \frac{1}{\rho} + 2\cos\theta = 0}$$ (another choice is $\vec{q_1} = \vec{q_2} = \vec{0}$)

Let's analyze:

$f(\rho) = \rho + \rho^{-1}$, $f'(\rho) = 1 - \rho^{-2} = \frac{\rho^2 - 1}{\rho^2} = \frac{(\rho+1)(\rho-1)}{\rho^2}$ $f'(\rho) = 0$,



Consider only $\rho > 0$

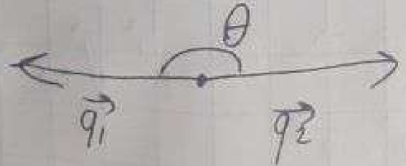


$f(1) = 2$ – minima,

Side note: another way is to use inequality
$$\frac{\sum_{i=1}^{n} a_i}{n} \geq \sqrt[n]{\prod_{i=1}^{n} a_i}$$
for $n=2$, $\rho + \frac{1}{\rho} \geq 2\sqrt{\rho \cdot \frac{1}{\rho}} = 2$

If $\rho > 1$, $\rho + \frac{1}{\rho} > 2$, $\cos\theta < -1$, impossible.
So the only solution is then $\rho = 1$; $\cos\theta = -1$, $\theta = \pi$.

<u>Conclusion</u> So $(\vec{q_1} + \vec{q_2})^2 = 0$ is possible only if → both $\vec{0}$
→ they have equal length and $\theta = \pi$.



$|\vec{q_1}| = |\vec{q_2}|$

Problem 5.3 Polynomial vector space

$\mathbb{P}_N = \{\vec{p} = (\sum_{k=0}^{N} p_k x^k), p_k \in \mathbb{R}, k \in \{0, \dots N\}\}$

In this task usage of $+, \cdot$ is clear from context

a) Prove that $(\mathbb{P}_N, \mathbb{R}, +, \cdot)$ is VS, given such def:

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$\forall \vec{p}, \vec{q} \in \mathbb{P}_N, c \in \mathbb{R}$:

$c \cdot \vec{p} = \sum_{k=0}^{N} (c p_k) x^k$, $\vec{p} + \vec{q} = \sum_{k=0}^{N} (p_k + q_k) x^k$,

(*) $(\mathbb{P}_N, +)$ is commutative group.

(**) Closure: $\forall \vec{p}, \vec{q} \in \mathbb{P}_N$: $\vec{p} + \vec{q} = \sum_{k=1}^{N} (p_k + q_k) x^k = \sum_{k=1}^{N} r_k x^k \in \mathbb{P}_N$

(**) Neutral: $\exists \vec{0} \in \mathbb{P}_N = \sum_{k=1}^{N} 0 \cdot x^k$, $\forall \vec{p} \in \mathbb{P}_N$: $\vec{p} + \vec{0} = \vec{0} + \vec{p} = \sum_{k=1}^{N} (0 + p_k) x^k = \vec{p}$.

(**) Inverse: $\forall \vec{p} = \sum_{k=1}^{N} p_k x^k \ \exists \vec{q} = \sum_{k=1}^{N} (-p_k) x^k$: $\vec{p} + \vec{q} = \vec{q} + \vec{p} =$

$= \sum_{k=1}^{N} 0 \cdot x^k = \vec{0}$

(**) Associativity: $\forall \vec{p}, \vec{q}, \vec{r} \in \mathbb{P}_N$: $\boxed{(\vec{p} + \vec{q}) + \vec{r}} = \left(\sum_{k=1}^{N} p_k x^k + \sum_{k=1}^{N} q_k x^k\right) + \sum_{k=1}^{N} r_k x^k \underset{\text{def}+}{=} \sum_{k=1}^{N} (p_k + q_k) x^k + \sum_{k=1}^{N} r_k x^k \underset{\text{def}+}{=}$

$= \sum_{k=1}^{N} ((p_k + q_k) + r_k) x^k \underset{\text{associativity of } \mathbb{R} \text{ field}}{=} \sum_{k=1}^{N} (p_k + (q_k + r_k)) x^k \underset{\text{def}+}{=} \vec{p} + \sum_{k=1}^{N} (q_k + r_k) x^k =$

$= \boxed{\vec{p} + (\vec{q} + \vec{r})}$

(**) Commutativity: $\forall \vec{p}, \vec{q} \in \mathbb{P}_N$: $\vec{p} + \vec{q} \underset{\text{def}}{=} \sum_{k=1}^{N} (p_k + q_k) x^k \underset{\mathbb{R}\text{-field}}{=}$

$= \sum_{k=1}^{N} (q_k + p_k) x^k = \vec{q} + \vec{p}$.

(*) $\forall r_1, r_2 \in \mathbb{R}, \vec{p} \in \mathbb{P}_N$:

$r_1 \cdot (r_2 \vec{p}) \underset{\text{def}}{=} r_1 \sum_{k=1}^{N} (r_2 p_k x^k) \underset{\text{def}}{=} \sum_{k=1}^{N} r_1 (r_2 p_k x^k) \underset{\mathbb{R}\text{-field}}{=} \sum_{k=1}^{N} (r_1 r_2) p_k x^k =$

$\underset{\text{def}}{=} (r_1 r_2) \vec{p}$.

(*) $\forall \tau_1, \tau_2 \in \mathbb{R}, \vec{p} \in \mathbb{P}_N$;

$$(\tau_1+\tau_2)\vec{p} = \sum_{k=1}^{N}(\tau_1+\tau_2)p_k x^k = \sum_{k=1}^{N}\tau_1 p_k x^k + \sum_{k=1}^{N}\tau_2 p_k x^k = \tau_1\vec{p} + \tau_2\vec{p}$$

(*) $\forall \tau \in \mathbb{R}, \vec{p}, \vec{q} \in \mathbb{P}_N$: $\tau\cdot(\vec{p}+\vec{q}) = \tau\sum_{k=1}^{N}(p_k+q_k)x^k = \sum_{k=1}^{N}\tau(p_k+q_k)x^k =$

$$= \sum_{k=1}^{N}(\tau p_k + \tau q_k)x^k = \tau\sum_{k=1}^{N}p_k x^k + \tau\sum_{k=1}^{N}q_k x^k = \tau\vec{p} + \tau\vec{q}.$$

Therefore, $\mathbb{P}_N$ is vector space.

b) $\vec{p}\cdot\vec{q} = \int_0^1 dx \left(\sum_{k=0}^{N}p_k x^k\right)\left(\sum_{j=0}^{N}q_j x^j\right)$ is valid scalar product done for this VS. Prove.

(*) Commutativity (since $\int_0^1 dx \dots \in \mathbb{R}$, we must prove $\vec{p}\vec{q} = \vec{q}\vec{p}$, forget on *)

$$\vec{p}\vec{q} = \int_0^1 dx \left(\sum_{k=0}^{N}q_j x^j\right)\left(\sum_{k=0}^{N}p_k x^k\right) = \vec{q}\vec{p} \ \checkmark$$

(*) linearity in first arg.

$$(c\vec{p})\cdot\vec{q} = \int_0^1 dx \left(\sum_{k=0}^{N}(cp_k)x^k\right)\sum_{j=0}^{N}q_j x^j = \int_0^1 dx\cdot c\sum_{k=0}^{N}p_k x^k\sum_{j=0}^{N}q_j x^j =$$

$$= c\int_0^1 dx \sum p_k x^k \sum q_j x^j = c(\vec{p}\cdot\vec{q}).$$

(*) distributivity: $\forall \vec{p}, \vec{q}, \vec{\tau} \in \mathbb{P}_N$:

$$(\vec{p}+\vec{q})\cdot\vec{\tau} = \int_0^1 dx \sum_{k=1}^{N}(p_k+q_k)x^k\sum_{j=1}^{N}\tau_j x^j = \int_0^1 dx\left[\left(\sum_{k=1}^{N}p_k x^k\right)\left(\sum_{j=1}^{N}\tau_j x^j\right) + \right.$$

$$\left. + \left(\sum_{k=1}^{N}q_k x^k\right)\left(\sum_{j=1}^{N}\tau_j x^j\right)\right] = \int_0^1 dx\left(\sum_{k=1}^{N}p_k x^k\right)\left(\sum_{j=1}^{N}\tau_j x^j\right) + \int_0^1 dx\left(\sum_{k=1}^{N}q_k x^k\right)\left(\sum_{j=1}^{N}\tau_j x^j\right) =$$

$$= \vec{p}\cdot\vec{\tau} + \vec{q}\cdot\vec{\tau}.$$

(*) positivity: $\vec{p}\cdot\vec{p} = \int_0^1 dx \sum p_k x^k \sum p_k x^k = \int_0^1 dx \left(\underbrace{\sum p_k x^k}_{f(x)}\right)^2 = \int_0^1 f(x)^2 dx \geq 0,$

equal when $\vec{p} = \vec{0}$ (although I am not sure there is no other $\vec{p}$, giving $\vec{p}\cdot\vec{p} = 0$)

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c) $\vec{b_0} = (1)$, $\vec{b_1} = (x)$, $\vec{b_2} = (x^2)$ — base for $\mathbb{P}_2$.

c.1) $\forall \vec{p} \in \mathbb{P}_2 : \exists x_{0,1,2} : \vec{p} = x_0 \vec{b_0} + x_1 \vec{b_1} + x_2 \vec{b_2}$. Proof.

$\forall \vec{p} \in \mathbb{P}_2 : \vec{p} = p_0 x^0 + p_1 x^1 + p_2 x^2$ for some $p_{0,1,2}$, by def. of $\mathbb{P}_2$.

Since $x^i = \vec{b_i}$, $\vec{p} = p_0 \vec{b_0} + p_1 \vec{b_1} + p_2 \vec{b_2}$, just take $x_i = p_i$.

c.2) But $x_i \neq \vec{p} \cdot \vec{b_i}$ because $\vec{b_{0,1,2}}$ do not form ONB.

For instance $\vec{b_0} \cdot \vec{b_1} = \int_0^1 dx \cdot 1 \cdot x = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} \neq 0$, $\vec{b_0} \not\perp \vec{b_1} \to$

no ONB.

d) $\hat{e_0} = (1)$, $\hat{e_1} = \sqrt{3}(2x-1)$, $\hat{e_2} = \sqrt{5}(6x^2 - 6x + 1)$. Prove: $\{\hat{e_0}, \hat{e_1}, \hat{e_2}\}$ are ONB.

(*) $\hat{e_0} \cdot \hat{e_0} = \int_0^1 1 \cdot 1\, dx = 1 \cdot 1 = 1$

(*) $\hat{e_1} \cdot \hat{e_1} = \int_0^1 (\sqrt{3})^2 (2x-1)^2 dx = \frac{3}{2} \frac{(2x-1)^3}{3}\Big|_0^1 = \frac{3}{2}\left[\frac{1}{3} - \frac{(-1)}{3}\right] = \frac{3}{2} \cdot \frac{2}{3} = 1$

(*) $\hat{e_2} \cdot \hat{e_2} = \int_0^1 (\sqrt{5})^2 (6x^2 - 6x + 1)^2 dx = 5 \int_0^1 36x^4 + 36x^2 + 1 - 72x^3 + 12x^2 - 12x\, dx =$

using $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$

$= 5\left(\frac{36x^5}{5} + \frac{36x^3}{3} + x - \frac{72x^4}{4} + \frac{12x^3}{3} - \frac{12x^2}{2}\right)\Big|_0^1 =$

$= 5\left(\frac{36}{5} + 12 + 1 - 18 + 4 - 6\right) = 5\left(7\frac{1}{5} - 7\right) = 5 \cdot \frac{1}{5} = \underline{1}$

(*) $\hat{e_0} \cdot \hat{e_1} = \int_0^1 \sqrt{3}\,(2x-1)\, dx = \sqrt{3}\,(x^2 - x)\Big|_0^1 = 0$

(*) $\hat{e_0} \cdot \hat{e_2} = \sqrt{5} \int_0^1 6x^2 - 6x + 1\, dx = \sqrt{5}\Big(\underbrace{\frac{6x^3}{3} - \frac{6x^2}{2} + x}_{2-3+1}\Big)\Big|_0^1 = 0$

(*) $\hat{e_1} \cdot \hat{e_2} = \int_0^1 \sqrt{15}\,(2x-1)(6x^2 - 6x + 1)\, dx = \sqrt{15} \int_0^1 12x^3 - 12x^2 + 2x - 6x^2 + 6x - 1\, dx =$

$= \sqrt{15}\left[\frac{12x^4}{4} - \frac{12x^3}{3} + \frac{2x^2}{2} - \frac{6x^3}{3} + \frac{6x^2}{2} - x\right]_0^1 = \sqrt{15}\,[3 - 4 + 1 - 2 + 3 - 1] = 0$

So $\hat{e}_i \hat{e}_j = \delta_{ij} \leftrightarrow$ orthonormal.

e) Now will show not just orthonormal but basis — can reach any $\vec{p}$:

$\forall \vec{p} \in \mathbb{P}_2$: $p_0 x^0 + p_1 x^1 + p_2 x^2 = c_0(1) + c_1(\sqrt{3}(2x-1)) + c_2(\sqrt{5}(6x^2-6x+1))$

for some c_0, c_1, c_2. Goal — find $\{c_0, c_1, c_2\}$ for given p_0, p_1, p_2. This will be coordinates of this polyn. in our basis.

$$c_0 + 2\sqrt{3}c_1 x^1 - \sqrt{3}c_1 + 6\sqrt{5}c_2 x^2 - 6\sqrt{5}c_2 x + c_2\sqrt{5} = p_0 x^0 + p_1 x^1 + p_2 x^2$$

$$\begin{cases} c_0 - \sqrt{3}c_1 + c_2\sqrt{5} = p_0 \\ 2\sqrt{3}c_1 - 6\sqrt{5}c_2 = p_1 \\ 6\sqrt{5}c_2 = p_2 \end{cases} \longrightarrow \begin{aligned} c_2 &= \frac{p_2}{6\sqrt{5}} \\ c_1 &= \frac{p_1 + p_2}{2\sqrt{3}} \\ c_0 &= \frac{p_1 + p_2}{2} - \frac{p_2}{6} + p_0 \end{aligned}$$

Hence $\{\hat{e}_0, \hat{e}_1, \hat{e}_2\}$ is ~~really~~ ONB on $\mathbb{P}_2$.

Finally for all ONBs $\vec{p} = c_0\hat{e}_0 + c_1\hat{e}_1 + \dots + c_N\hat{e}_N$,

$$\vec{p}\cdot\hat{e}_j = \sum_{i=0}^{N} c_i \underbrace{\hat{e}_i \hat{e}_j}_{\delta_{ij}} = \hat{e}_j, \quad \cancel{\text{So } \vec{p} = (\vec{p}\cdot\hat{e}_0)\hat{e}_0}$$

So $\vec{p} = (\vec{p}\cdot\hat{e}_0)\hat{e}_0 + (\vec{p}\cdot\hat{e}_1)\hat{e}_1 + (\vec{p}\cdot\hat{e}_2)\hat{e}_2 \quad \forall p \in \mathbb{P}_2$.

*f) $\boxed{\hat{n}_0 = (cx),\ \hat{n}_1 = ?}$ — give ONB on $\mathbb{P}_1$.

Set $\hat{n}_1 = a + bx$ in general way.

part 1) $\forall \vec{p} \in \mathbb{P}_1$: $p = p_0 x^0 + p_1 x^1 = v_0 \hat{n}_0 + v_1 \hat{n}_1$ for some v_0, v_1.

$$\cancel{p=}\ p_0 + p_1 x = v_0 c x + a v_1 + b v_1 x = a v_1 + (v_0 c + b v_1)x$$

$$\begin{cases} p_0 = a v_1 \\ p_1 = v_0 c + v_1 b \end{cases} \quad \left.\begin{aligned} v_1 &= \frac{p_0}{a} \\ v_0 &= \frac{p_1 - v_1 b}{c} \end{aligned}\right\} \text{ unique } v_0, v_1 \text{ exist if } a \text{ and } c \neq 0.$$

part 2) Now satisfy orthonorm.

$\hat{n}_0 \cdot \hat{n}_0 = 1 \longrightarrow \int_0^1 (cx)^2 dx = \frac{c^2x^3}{3}\Big|_0^1 = \frac{c^2}{3} = 1, \ \underline{c^2 = 3}$

$\hat{n}_1 \cdot \hat{n}_1 = 1 \longrightarrow \int_0^1 (a+bx)^2 dx = \frac{1}{b}\frac{(a+bx)^3}{3}\Big|_0^1 = \frac{1}{3b}\left((a+b)^3 - a^3\right) = 1,$

$\underline{(a+b)^3 - a^3 = 3b}$

$\hat{n}_0 \cdot \hat{n}_1 = 0 \longrightarrow \int_0^1 (cx)(a+bx)dx = \int_0^1 acx + bcx^2 dx = \frac{acx^2}{2} + \frac{bcx^3}{3}\Big|_0^1 = \underline{\frac{ac}{2} + \frac{bc}{3} = 0},$

Thereby:

$$\begin{cases} c^2 = 3 \\ (a+b)^3 - a^3 = 3b \\ \frac{ac}{2} + \frac{bc}{3} = 0 \quad \Big| \cdot \frac{b}{c} \\ a \neq 0 \end{cases} \leftrightarrow \begin{cases} c^2 = 3 \\ (a+b)^3 - a^3 = 3b \\ 3a + 2b = 0 \\ a \neq 0 \end{cases} \leftrightarrow \begin{cases} c^2 = 3 \\ \left(\frac{b}{3}\right)^3 + \frac{8b^3}{27} = 3b \\ a = -\frac{2b}{3} \neq 0 \end{cases} \leftrightarrow \begin{cases} c^2 = 3 \\ b^2 = 9 \\ a = -\frac{2b}{3} \end{cases}$$

$$\begin{cases} c = \pm\sqrt{3} \\ \left[\begin{array}{l} b = 3, a = -2 \\ b = -3, a = 2 \end{array}\right. \end{cases} \quad (a,b,c) \in \{(2,-3,\sqrt{3}), (2,-3,-\sqrt{3}), (-2,3,\sqrt{3}), (-2,+3,-\sqrt{3})\}$$

4 Possibilities:

$c \in \{-\sqrt{3}, \sqrt{3}\}$

$\hat{n}_1 = a + bx$, where $(a,b) \in \{(-2,3),(2,-3)\}$.

Problem 5.4 Systems of linear equations

In this task usage of $\cdot$ and $+$ is clear from context

a) $(\mathbb{L}_M, \mathbb{R}, +, \cdot)$ is VS if $+, \cdot$ defined:

$\forall \vec{p} = (p_0 = p_1x_1 + p_2x_2 + \dots + p_Mx_M) \in \mathbb{L}_M$

$\vec{q} = (q_0 = q_1x_1 + q_2x_2 + \dots + q_Mx_M) \in \mathbb{L}_M,$

$c \in \mathbb{R};$

We decided:
$$\begin{cases} \vec{p} + \vec{q} = (p_0 + q_0 = (p_1+q_1)x_1 + (p_2+q_2)x_2 + \dots + (p_M+q_M)x_M) \\ c\vec{p} = (cp_0 = cp_1x_1 + cp_2x_2 + \dots + cp_Mx_M) \end{cases}$$

Proof that VS.

(*) $(\mathbb{L}_M, +)$ is commutative group:

** Closure: $\forall \vec{p}, \vec{q} \in \mathbb{L}_M$: $\vec{p}+\vec{q} = (p_1+q_1)x_1 + (p_2+q_2)x_2 + \dots + (p_M+q_M)x_M =$
$= c_1 x_1 + c_2 x_2 + \dots + c_M x_M \in \mathbb{L}_M$

** Neutral: $\vec{0} = 0x_1 + 0x_2 + \dots + 0x_M \in \mathbb{L}_M$, and
$\forall \vec{p} \in \mathbb{L}_M$: $\vec{p}+\vec{0} = \vec{0}+\vec{p} = (p_1+0)x_1 + \dots + (p_M+0)x_M = \vec{p}$

** Inverse element:

$$\forall \vec{p} \in \mathbb{L}_M,\ \vec{p} = \left(p_0 = \sum_{i=1}^{M} p_i x_i\right)\ \exists \vec{q} = \left(-p_0 = \sum_{i=1}^{M}(-p_i)x_i\right),$$
$$\vec{p}+\vec{q} = \vec{q}+\vec{p} = \sum_{i=1}^{M}(p_i - p_i)x_i = \vec{0}.$$

** Associativity: $\forall \vec{p}, \vec{q} \in \mathbb{L}_M$, $(\vec{p}+\vec{q})+\vec{r} = \sum_{i=1}^{M}(p_i+q_i)x_i + \sum_{i=1}^{M} r_i x_i =$
$$\overset{def+}{=} \sum_{i=1}^{M}((p_i+q_i)+r_i)x_i \overset{assoc.\ on\ \mathbb{R}}{=} \sum_{i=1}^{M}(p_i+(q_i+r_i))x_i = \text{(similar reasoning)}$$
$$= \vec{p}+(\vec{q}+\vec{r}).$$

** Commutativity: $\forall \vec{p}, \vec{q} \in \mathbb{L}_M$, $\vec{p}+\vec{q} \overset{def+}{=} \sum_{i=1}^{M}(p_i+q_i)x_i \overset{com.\ on\ \mathbb{R}}{=}$
$$= \sum_{i=1}^{M}(q_i+p_i)x_i \overset{def+}{=} \vec{q}+\vec{p}.$$

(*) $\forall c_1, c_2 \in \mathbb{R}, \vec{p} \in \mathbb{L}_M$: $c_1 \cdot (c_2\vec{p}) \overset{def.}{=} c_1 \sum_{i=1}^{M} c_2 p_i x_i =_{def} \sum_{i=1}^{M} c_1(c_2 p_i x_i) =$
$$\overset{assoc.\ on\ \mathbb{R}}{=} \sum_{i=1}^{M}(c_1 c_2)p_i x_i \overset{def.}{=} (c_1 c_2)\sum_{i=1}^{M} p_i x_i = (c_1 c_2)\vec{p}.$$

(*) $\forall c_1, c_2 \in \mathbb{R}, \vec{p} \in \mathbb{L}_M$:
$$(c_1+c_2)\vec{p} \underset{def}{=} \sum_{i=1}^{M}(c_1+c_2)p_i x_i \underset{\mathbb{R}\ field}{=} \sum_{i=1}^{M}(c_1 p_i x_i + c_2 p_i x_i) =$$
$$= c_1 \sum_{i=1}^{M} p_i x_i + c_2 \sum_{i=1}^{M} p_i x_i = c_1\vec{p} + c_2\vec{p}.$$

(*) $\forall c \in \mathbb{R}, \vec{p}, \vec{q} \in \mathbb{L}_M$:
$$\underbrace{c \cdot (\vec{p}+\vec{q}) = c\sum_{i=1}^{M}(p_i+q_i)x_i = \sum_{i=1}^{M} c(p_i+q_i)x_i}_{def\ for\ \mathbb{L}_M} = \underbrace{\sum_{i=1}^{M}(cp_i x_i + cq_i x_i)}_{field\ properties\ of\ \mathbb{R}} =$$
$$= c\sum_{i=1}^{M} p_i x_i + c\sum_{i=1}^{M} q_i x_i = c\vec{p} + c\vec{q}.$$

Therefore $\mathbb{L}_M$ is Vector Space.

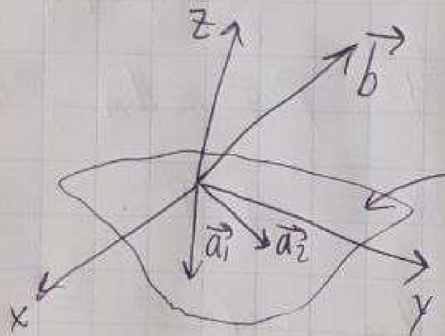
Operations with Gauss method take equations (vectors in $\mathbb{L}_M$) and, since closure, produce another vectors in $\mathbb{L}_M$. They try to do it in such way that resulting equations will be linearily independent vectors, to reduce redundand information how possible. I do not know though how this ensures reaching solution. (except just simplify everything)

b) $$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{N2} \end{pmatrix} x_2 + \dots + \begin{pmatrix} a_{1M} \\ a_{2M} \\ \vdots \\ a_{NM} \end{pmatrix} x_M$$

$$\vec{b} = \vec{a_1} x_1 + \vec{a_2} x_2 + \dots + \vec{a_M} x_M$$

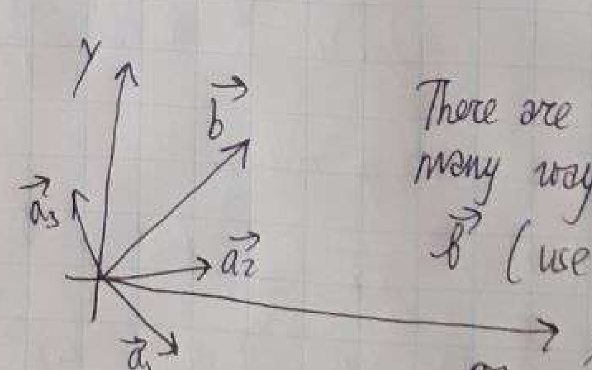
Here it is not row (equation) but column $\mathbb{R}^N$ vector space. We want to find coefficients $x_1, \dots x_M$ in linear combination $\vec{a_1}, \dots \vec{a_M} \rightarrow$ to get $\vec{b}$.

1) If $N > M$;
(example $N=3$, $M=2$)
↓
No solutions



$\vec{a_1}, \vec{a_2}$ are basis for plane, no possible way to reach $\vec{b}$ from them,

2) If $N < M$
(example $N=2$, $M=3$)
↓
Infinitely many



There are infinitely many ways to reach $\vec{b}$ (use $\vec{a_2}$ and $\vec{a_3}$ only, or $\vec{a_1}$ and $\vec{a_2}$ etc.)

3) $N = M$
Expected one solution if equations are linearily independent in $\mathbb{L}_M$.

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3.2	2/2
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4.2	5
5.1	great work!
6.1	as the integrand, f(x)^2, is non negative the integral is exactly zero iff f(x)^2=0 which is the case iff f(x) = 0 if there would be a non negative function thats non zero the integral would be greater than zero from monoticity of the integral
7.1	4/6 linear indedepency missing
8.1	6/6
8.2	e) Okay, great proof, but you could calculate the coefficients and express it finally for the given information in your task. 4/6