4. Vectors

The Chapters 2.5–2.8 of my lecture notes provides the background to solve the following exercises. Your solution to the problems 4.1–4.3 should uploaded

to your Moodle account

as a PDF-file

by Wednesday, Nov 1, 10:30 (with a grace time till the start of the seminars).

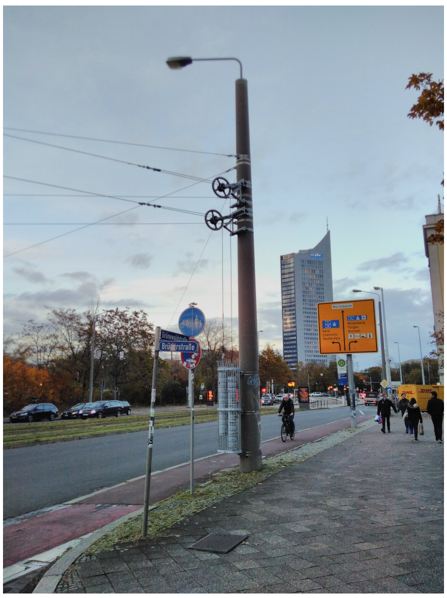
The parts marked by ⋆ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

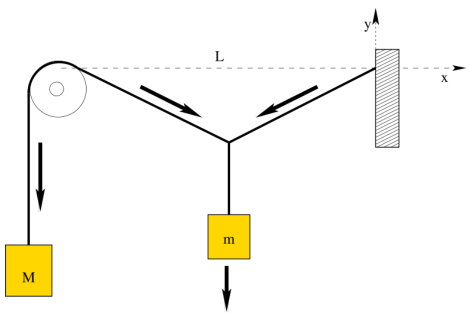
The self-test problems serve to check you understanding of vectors.

The bonus problem provides the solution to a problem that we introduced in class. It might take some extra effort to solve.

Problems

Problem 1. Mounting street lanterns and trolley systems





A latern of mass m is mounted in a wall at the right side of a street, and at the left side one uses a roller and a counterweight of mass M (see sketch). This setup is also widely used for the hanging of cables in trolley systems (as shown in the photo to the left).

a) Label the forces in the sketch: Denote the weight of the mass M as $\mathbf{F}_M$; the weight of the lamp as $\mathbf{F}_m$; the force along the suspension cable to the left of the lamp as $\mathbf{F}_1$ and the force to the right as $\mathbf{F}_2$.

b) The suspension of the roll is exerting a force $\mathbf{F}_R$ on the roll such that it does not move. How can this force be expressed in terms of the forces introduced in a)? Determine the force graphically and add it to your sketch.

c) Let the masses of the lamp and the counterweight be $m = 15\,\mathrm{kg}$ and $M = 80\,\mathrm{kg}$, respectively. Determine the angle α between the horizontal the suspension cable, when the lamp is positioned right in the middle between the wall and the roll. Determine $|\mathbf{F}_R|$.

d) The angle α is a function of the mass ratio m/M. Why is this not unexpected?

   Determine the function $\alpha(m, M)$, and sketch the angle as function of the ratio m/M. What happens when $m > 2M$?
   Hint: Try it! The setup can easily build at home with a wire and two weights. Photos and descriptions of measurements of $\alpha(m, M)$ will be published in the wiki, and awarded by bonus points.

$\star$ e) What is the maximum admissible mass when the wall anchor can support a mximum load of 14.0 kN? Which value does the angle α take in that case?

$\star$ f) Why is it a bad idea to stretch a wire horizontally between two wall anchors, and then use it to support a lamp or some other heavy object?

   What is the advantage of the trolley system with the roll – in particular, when the lamp gains substantial additional weight after a freezing-rain shower?

Problem 2. Fibonacci Sequences form a vector space

A mathematical sequence $\left(a_n\right)_{n\in\mathbb{N}}$ assigns a number $a_n \in \mathbb{R}$ to every $n \in \mathbb{N}$. In particular a *Fibonacci sequence,* is a sequence where $a_{n+2} = a_n + a_{n+1}$ for all $n > 1$.

a) Show that the Fibonacci sequences F form a vector space when one adopts the following binary operations
$\forall \mathbf{a}, \mathbf{b} \in \mathsf{F} : \quad \mathbf{a} + \mathbf{b} = \big(a_n + b_n\,,\ n \in \mathbb{N}\big)$
$\forall \alpha \in \mathbb{R}\ \forall \mathbf{a} \in \mathsf{F} : \alpha\, \mathbf{a} = \big(\alpha\, a_n\,,\ n \in \mathbb{N}\big)$

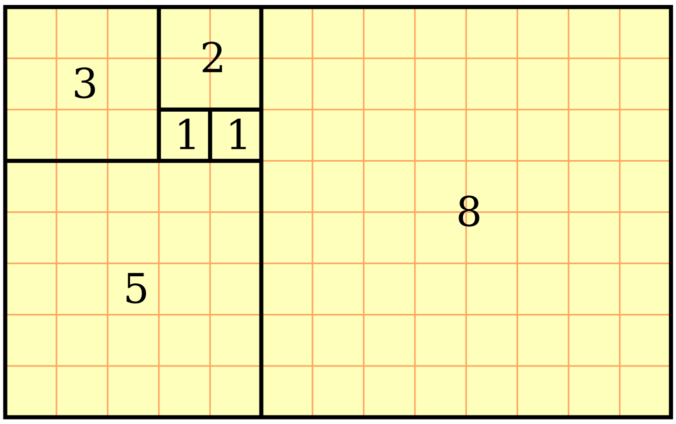
b) Show that there are exactly two Fibonacci sequences $\mathbf{f}^{\pm} = \big(f_n^{\pm}\,,\ n \in \mathbb{N}\big)$ with

$$\forall n \in \mathbb{N} : \ f_n^{\pm} = \gamma_{\pm}^{n}$$

where $\gamma_{\pm}$ are suitable real numbers.
Determine $\gamma_{\pm}$.

⋆ c) Show that the two vectors $\mathbf{f}^{+}$ and $\mathbf{f}^{-}$ form a basis of the vector space F.

d) The most famous Fibonacci sequence starts with the numbers 1 and 1: $\mathbf{a} = (1, 1, 2, 3, 5, 8, \dots)$. Squares with these side length can be arranged as follows to form a reactangle:



Determine the area of the rectangle that is obtained when following this construction till $n = 1729$.

Remark: The Hardy-Ramanujan number, 1729, is the smallest number that can be represented in two different ways as the sum of two positive cubic numbers.

Hint: Determine the coordintes $\alpha_{\pm}$, such that $\mathbf{a} = \alpha_{+}\, \mathbf{f}^{+} + \alpha_{-}\, \mathbf{f}^{-}$. They imply that $a_{1729} = \alpha_{+}\, \gamma_{+}^{1729} + \alpha_{-}\, \gamma_{-}^{1729}$. Which value does this provide for the area?

Problem 3. Polynomials are a vector space

We consider the set of polynomials $\mathbb{P}_N$ of degree N with real coefficients $p_k,\ k \in \{0, \dots, N\}$,

$$\mathbb{P}_N := \left\{ \mathbf{p} = \left(\sum_{k=0}^{N} p_k\, x^k \right) \quad \text{with} \;\; p_k \in \mathbb{R},\ k \in \{0, \dots, N\} \right\}$$

⋆ a) Demonstrate that $(\mathbb{P}_N, \mathbb{R}, +, \cdot)$ is a vector space when one adopts the operations

$$\forall \quad \mathbf{p} = \left(\sum_{k=0}^{N} p_k\, x^k \right) \in \mathbb{P}_N\,, \quad \mathbf{q} = \left(\sum_{k=0}^{N} q_k\, x^k \right) \in \mathbb{P}_N\,, \text{ and } \; c \in \mathbb{R} \; :$$

$$\mathbf{p} + \mathbf{q} = \left(\sum_{k=0}^{N} (p_k + q_k)\, x^k \right) \qquad \text{and} \qquad c \cdot \mathbf{p} = \left(\sum_{k=0}^{N} (c\, p_k)\, x^k \right) .$$

b) Demonstrate that

$$\mathbf{p} \cdot \mathbf{q} = \left(\int_0^1 \mathrm{d}x \left(\sum_{k=0}^{N} p_k\, x^k \right) \left(\sum_{j=0}^{N} q_j\, x^j \right) \right) ,$$

establishes an inner product on this vector space.

c) Demonstrate that the three polynomials $\mathbf{b}_0 = (1)$, $\mathbf{b}_1 = (x)$ und $\mathbf{b}_2 = (x^2)$ form a base of the vector space $\mathbb{P}_2$: For each polynomial $\mathbf{p} \in \mathbb{P}_2$ there are real numbers $x_k,\ k \in \{0, 1, 2\}$, such that $\mathbf{p} = x_0\, \mathbf{b}_0 + x_1\, \mathbf{b}_1 + x_2\, \mathbf{b}_2$. However, in general we have $x_i \neq \mathbf{p} \cdot \mathbf{b}_i$. Why is that?
Hint: Is this an orthonormal base?

d) Demonstrate that the three vectors $\hat{\mathbf{e}}_0 = (1)$, $\hat{\mathbf{e}}_1 = \sqrt{3}\,(2\,x - 1)$ and $\hat{\mathbf{e}}_2 = \sqrt{5}\,(6\,x^2 - 6\,x + 1)$ are orthonormal.

Hence, $(\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ form an orthonormal base of $\mathbb{P}_2$.

e) Demonstrate that every vector $\mathbf{p} \in \mathbb{P}_2$ can be written as a scalar combination of $(\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$,

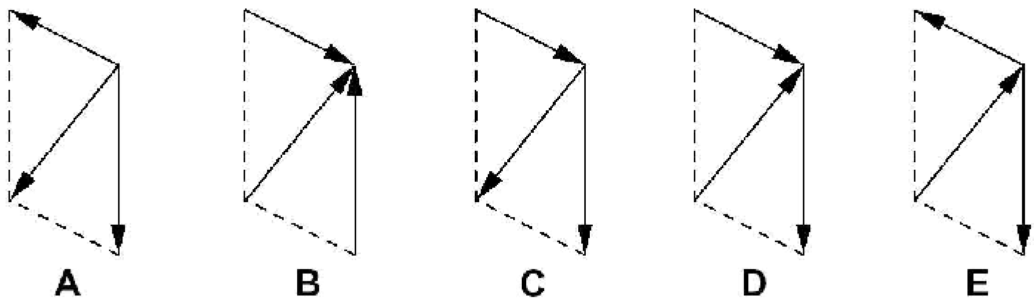
$$\mathbf{p} = (\mathbf{p} \cdot \hat{\mathbf{e}}_0)\, \hat{\mathbf{e}}_0 + (\mathbf{p} \cdot \hat{\mathbf{e}}_1)\, \hat{\mathbf{e}}_1 + (\mathbf{p} \cdot \hat{\mathbf{e}}_2)\, \hat{\mathbf{e}}_2\,.$$

⋆ f) Find a constant c and a vector $\hat{\mathbf{n}}_1$, such that $\hat{\mathbf{n}}_0 = (c\,x)$ and $\hat{\mathbf{n}}_1$ form an orthonormal basis of $\mathbb{P}_1$.

Self Test

Problem 4. Particles at rest

There are three forces acting on the center of mass of a body. In which cases does it stay at rest?



Problem 5. Towing a stone

Three Scottish muscleman[1] try to tow a stone with mass $M = 20$cwt from a field. Each of them gets his own rope, and he can act a maximal force of 300 lb g as long as the ropes run in directions that differ by at least 30°.

a) Sketch the forces acting on the stone and their sum. By which ratio is the force exerted by three men larger than that of a single man?

b) The stone counteracts the pulling of the men by a static friction force $\mu M g$, where g is the gravitational acceleration. What is the maximum value that the friction coefficient μ may take when the men can move the stone?

Problem 6. Linear dependence of three vectors in 2D

In the lecture I pointed out that every vector $\mathbf{v} = (v_1, v_2)$ of a two-dimensional vector space can be represented as a *unique* linear combination of two linearly independent vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{v} = \alpha\,\mathbf{a} + \beta\,\mathbf{b}$$

In this exercise we revisit this statement for $\mathbb{R}^2$ with the standard forms of vector addition and multiplication by scalars.

[1] In highland games one still uses Imperial Units. A hundredweight (cwt) amounts to eight stones (stone) that each have a mass of 14 pounds(lb). A pound-force (lb g) amounts to the gravitational force acting on a pound. One can solve this problem without converting units.

a) Provide a triple of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{v}$ such that $\mathbf{v}$ can *not* be represented as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

b) To be specific we will henceforth fix

$$\mathbf{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Determine the numbers α and β such that

$$\mathbf{v} = \alpha\,\mathbf{a} + \beta\,\mathbf{b}$$

Hint: Verify that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. How would you determine α and β when the vectors $\mathbf{a}$ and $\mathbf{b}$ had unit length? What changes when they have a different length?

c) Consider now also a third vector

$$\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and find two different choices for (α, β, γ) such that

$$\mathbf{v} = \alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma\mathbf{c}$$

What is the general constraints on (α, β, γ) such that $\mathbf{v} = \alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma\mathbf{c}$. What does this imply on the number of solutions?

d) Discuss now the linear dependence of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ by exploring the solutions of

$$\mathbf{0} = \alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma\mathbf{c}$$

How are the constraints for the null vector related to those obtained in part c)?

Bonus Problems

Problem 7. Switching off 7 lamps

In the lecture we discussed how the states (on=1, off=0) of 7 lamps can be represented by a 7-tuple of of elements of the field $(\{0,1\}, +\text{mod } 2, \cdot\text{mod } 2)$.
Pushing one of the buttons next to a lamp is represented by adding a vector $\mathbf{s}_i$ where the components i, $i-1$, and $i+1$ take the value 1, and all other components are zero (with the understanding that we identify component 0 as 7, and 8 as 1, repectively).

a) Show that every state is self inverse: For all vectors $\mathbf{v}$ of the vector space we have $\mathbf{v} \oplus \mathbf{v} = \mathbf{0}$.

b) Show that the vectors $\mathsf{B} = \{\mathbf{s}_i, i = 1, \cdots, 7\}$ form a basis of the vector space.

c) Determine how the natural basis vectors $\mathbf{b}_i$, where only the ith basis vector has a non-vanishing component also for entry i, can be represented in terms of the basis vectors in B.

d) Determine how a vector $\mathbf{v}$ is represented as linear combination of the basis vectors B.
Hint: Start from the representation in terms of the natural basis, and insert the result of c).

e) Show that one can switch of any configuration of the lamps by observing the following rule: For each buttom you consider the lamb next the buttom, its two neightbors and the two lambs on the opposite side. You push the button iff this number is odd for the considered initial state.