# 4. Vectors, Positions, and Velocities

Your solution to the problems 4.1–4.4 should be uploaded

to your Moodle account

as a PDF-file

by Saturday, Nov 05, 11pm (with a grace time till Sunday morning).

The parts marked by $\star$ and , and the additional exercises are suggestions for further exploration. They can be followed up in the seminars, but need not be submitted, and they will not be graded.

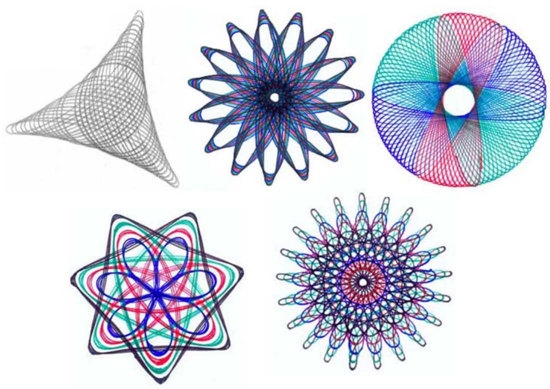
In the self-test exercises you check you background and understanding of the geometric summing of forces. You need not submit these exercises, but I strongly recommend that you take a look at these problems right away.

The bonus problems is an illuminating fun problems for those participants who are interested in additional challenges. Presently, you may safely ignore them. It might be worth, however, to take a look when you are preparing for the exam.

The problems marked by  take extra effort to solve. They should always be considered as Bonus problems.

## Problems

### Problem 4.1. Hypotrochoids, roulettes, and the spirograph



[Wikimedia Public domain]

A roulette is the curve traced by a point (called the generator or pole) attached to a disk or other geometric object when that object rolls without slipping along a fixed track. A pole on the circumference of a disk that rolls on a straight line generates a cycloid. A pole inside that disk generates a trochoid. If the disk rolls along the inside or outside of a circular track it generates a hypotrochoid. The latter curves can be drawn with a spirograph, a beautiful drawing toy based on gears that illustrates the mathematical concepts of the least common multiple (LCM) and the lowest common denominator (LCD).

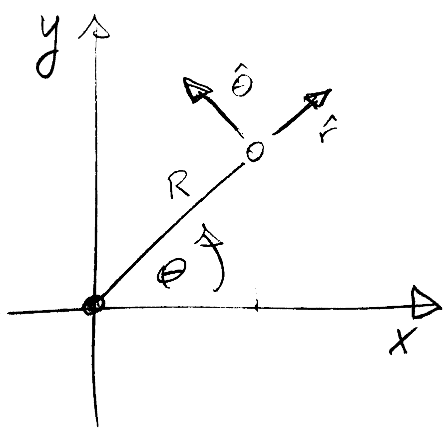
a) Consider the track of a pole attached to a disk with $n$ cogs that rolls inside a circular curve with $m > n$ cogs. Why does the resulting curve form a closed line? How many revolutions does the disk make till the curve closes? What is the symmetry of the resulting roulette? (The curves to the top left is an examples with three-fold symmetry, and the one to the bottom left has seven-fold symmetry.)

b) Adapt the description for the position of the retroreflector that we developed in the lecture[1] such that you can describe hypotrochoids, i.e. the position of the pen as function of how far the inner wheel has progressed.

⋆ c) Test your result by writing a program that plots the curves for given $m$ and $n$.
**Remark:** More explanation and a Sage Notebook that you can use to start this analysis will be given in the wiki.

d) Determine a general expression for the length of the roulette.
For which ratios of $d/r$ can you evaluate the expression by analogous calculations as performed for the cycloid?
How do the related roulettes look like?

**Problem 4.2. Motion on a circular track**




The position of a particle in the plane can be specified by Cartesian coordinates $(x, y)$ of by polar coordinates with basis vectors $\hat{\boldsymbol{r}}(\theta)$ and $\hat{\boldsymbol{\theta}}(\theta)$, that have the following representation in Cartesian coordinates (cf. the sketch to the left)

$$\hat{\boldsymbol{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} .$$

We will now explore the trajectory $\mathbf{q}(t)$ of a particle with mass $m$ that moves on a track with a fixed radius $R$.

[1] In the wiki you find a link to an excercise and solution that summarizes the steps taken in the derivation.

a) Verify that $\hat{\boldsymbol{\theta}} = \partial_\theta \hat{\boldsymbol{r}}$ and $\partial_\theta^2 \hat{\boldsymbol{r}} = -\hat{\boldsymbol{r}}$.

Please provide a geometric interpretation of this result!

b) The position of the particle can be specified as $\mathbf{q}(t) = R\,\hat{\boldsymbol{r}}(\theta(t))$. Determine $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ based on these equations.

Verify your result by performing the same calculation in Cartesian coordinates.

c) Consider the motion at a constant angular speed, $\theta(t) = \omega t$, and show that the acceleration in this setting takes the form $\ddot{\mathbf{q}} = -R\,\omega^2\,\hat{\boldsymbol{r}}(\omega t)$.

Verify that this amounts to an acceleration that is perpendicular to the velocity.

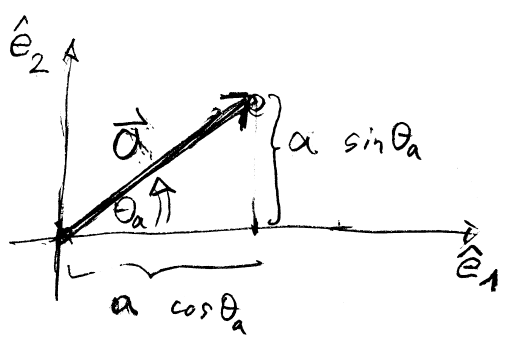
What does this imply for the absolute value of the velocity?
**Hint:** Discuss the time derivative of $\mathbf{v}^2$.

## Problem 4.3. Geometric and algebraic form of the inner product

The sketch to the right shows a vector $\mathbf{a}$ in the plane, and its representation as a linear combination of two orthonormal vectors $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$,

$$\mathbf{a} = a\,\cos\theta_a\,\hat{\boldsymbol{e}}_1 + a\,\sin\theta_a\,\hat{\boldsymbol{e}}_2$$

Here, $a$ is the length of the vector $\mathbf{a}$, and $\theta_1 = \angle(\hat{\boldsymbol{e}}_1, \mathbf{a})$.




a) Analogously to $\mathbf{a}$ we will consider another vector $\mathbf{b}$ with a representation

$$\mathbf{b} = b\,\cos\theta_b\,\hat{\boldsymbol{e}}_1 + b\,\sin\theta_b\,\hat{\boldsymbol{e}}_2$$

Employ the rules of inner products, vector addition and multiplication with scalars to show that

$$\mathbf{a}\cdot\mathbf{b} = a\,b\,\cos(\theta_a - \theta_b)$$

**Hint:** Work backwards, expressing $\cos(\theta_a - \theta_b)$ in terms of $\cos\theta_a$, $\cos\theta_b$, $\sin\theta_a$, and $\sin\theta_b$, and observe that $\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}$.

b) As a shortcut to the explicit calculation of a) one can introduce the coordinates $a_1 = a\cos\theta_a$ and $a_2 = a\sin\theta_a$, and write $\mathbf{a}$ as a tuple of two numbers. Proceeding analogously for $\mathbf{b}$ one obtains

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

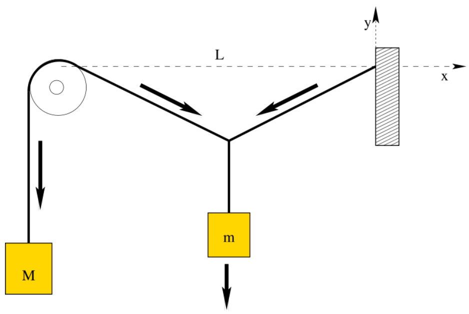
How will the product $\mathbf{a}\cdot\mathbf{b}$ look like in terms of these coordinates?

**Remark:** You may use that the result of the calculations must no depend on the orientation of the unit vectors. How does this freedom of choice simplify this argument?

⋆ c) How do the arguments in a) and b) change for $D$ dimensional vectors that are represented as linear combinations of a set of orthonormal basis vectors $\hat{\boldsymbol{e}}_1, \dots, \hat{\boldsymbol{e}}_D$?

d) What changes when the basis vectors are not normalized?
What when they are not even orthogonal?

## Problem 4.4. Mounting street lanterns and trolley systems




A latern of mass $m$ is mounted in a wall at the right side of a street, and at the right side one uses a roll and a counterweight of mass $M$ (see sketch). This setup is also widely used for the hanging of cables in trolley systems (as shown in the photo to the left).

a) Label the forces in the sketch: Denote the weight of the mass $M$ as $\mathbf{F}_M$; the weight of the lamp as $\mathbf{F}_m$; the force along the suspension cable to the left of the lamp as $\mathbf{F}_1$ and the force to the right as $\mathbf{F}_2$.

b) The suspension of the roll is exerting a force $\mathbf{F}_R$ on the roll such that it does not move. How can this force be expressed in terms of the forces introduced in a)? Determine the force graphically and add it to your sketch.

c) Let the masses of the lamp and the counterweight be $m = 15\,\mathrm{kg}$ and $M = 80\,\mathrm{kg}$, respectively. Determine the angle $\alpha$ between the horizontal the suspension cable, when the lamp is positioned right in the middle between the wall and the roll. Determine $|\mathbf{F}_R|$.

d) The angle $\alpha$ is a function of the mass ratio $m/M$. Why is this not unexpected?

Determine the function $\alpha(m, M)$, and sketch the angle as function of the ratio $m/M$. What happens when $m > 2M$?
**Hint:** Try it! The setup can easily build at home with a wire and two weights. Photos and descriptions of measurements of $\alpha(m, M)$ will be published in the wiki, and awarded by bonus points.

⋆ e) What is the maximum admissible mass when the wall anchor can support a mximum load of $14.0\,\mathrm{kN}$? Which value does the angle $\alpha$ take in that case?

⋆ f) Why is it a bad idea to stretch a wire horizontally between two wall anchors, and then use it to support a lamp or some other heavy object?

What is the advantage of the trolley system with the roll – in particular, when the lamp gains substantial additional weight after a freezing-rain shower?

## Self Test

### Problem 4.5. Towing a stone

Three Scottish muscleman[2] try to tow a stone with mass $M = 20\mathrm{cwt}$ from a field. Each of them gets his own rope, and he can act a maximal force of $300\,\mathrm{lb\,g}$ as long as the ropes run in directions that differ by at least 30°.

[2] In highland games one still uses Imperial Units. A hundredweight (cwt) amounts to eight stones (stone) that each have a mass of 14 pounds(lb). A pound-force (lb g) amounts to the gravitational force acting on a pound. One can solve this problem without converting units.

a) Sketch the forces acting on the stone and their sum. By which ratio is the force exerted by three men larger than that of a single man?

b) The stone counteracts the pulling of the men by a static friction force $\mu M g$, where $g$ is the gravitational acceleration. What is the maximum value that the friction coefficient $\mu$ may take when the men can move the stone?

## Bonus Problem

### Problem 4.6. Planetary Conjunctions

Every now and then there are planetary conjunctions, where two or even several planets appear in a very close vicinity on the sky. The conjunction of Mercury and Venus appearing above the Moon is shown to the right. Upcoming events are listed on in-the-sky.org. A funny feature of conjunctions is that the times between subsequent conjunctions vary a lot. For instance there have been Mercury-Venus conjunctions on September 13 and October 30, 2019, and then it lasted till May 22, 2020.

6 March 2008[ESO/Y. Beletsky [CC BY 4.0]

a) Determine the time evolution of the angle of sight of the planet, as they are observed at midnight.

b) Discuss the monotonicity of the evolution of the planet position on the night sky.
**Hint:** Plot the angle as a function of time, and have a look at how it evolves!

c) What does this dependences imply about the time intervals between subsequent conjunctions of a pair of stars? Can you derive a recusion relation for the times?

d) Plot subsequent times as function of one another. What do you observe?
**Remark:** High-precision data reveal further interesting periodic structures. See for instance Graham Jones article in skyandtelescope.org.