Math for Students of Faculty of Physics
Mathematics 1
Linear Algebra and Calculus of Functions of One Variable

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References

The programm of the course "Mathematics 1. Linear Algebra and Calculus of Functions of One Variable" is standard and the related material can be found practically in any book devoted to the calculus of functions of one variable and the linear algebra.

Books recommended in the official syllabus:

https://www.physgeo.uni-leipzig.de/en/studying/courses-of-study/

- Serge Lang: Linear Algebra, Springer
- Serge Lang: A First Course in Calculus, Springer
- Kenneth A. Ross: Elementary Analysis, Springer
- Stephen Abbott: Understanding Calculus, Springer

Books I frequently use myself to prepare my lectures (to be completed):

Logic, sets, functions

- Keith Devlin, Sets, Functions, and Logic. An Introduction to Abstract Mathematics, CRC.
- Steven Galovich, Introduction to Mathematical Structures, HBJ, Academic Press.

Calculus of functions of one real variable

- Vladimir A. Zorich, Mathematical Analysis I, Springer.

Linear Algebra

- Steven J. Leon, Linear Algebra with Applications, Prentice Hall.

Contents

1. **Notation**

$\forall$ means "for all"

$\exists$ means "there exists"

$\exists!$ means "there exists a unique"

: means "such that"

:= means "denote by"

$\mathbb{N}$ is a set of all natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{Z}$ is a set of all integer numbers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Q}$ is a set of all rational numbers, $\mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z},\ n \in \mathbb{N} \right\}$

$\mathbb{R}$ is a set of all real numbers

an interval $(a, b) \subset \mathbb{R}$ is the set of all real x such that $a < x < b$

an interval $[a, b] \subset \mathbb{R}$ is the set of all real x such that $a \leq x \leq b$

$\operatorname{tg} x := \tan x$ is the tangent of x, $\quad \operatorname{ctg} x := \cot x$ is the cotangent of x

$\operatorname{arctg} y := \arctan y$ is the arctangent of y, $\quad \operatorname{arcctg} y := \operatorname{arccot} y$ is the arccotangent of y

1 One-dimensional calculus

1.1 Basic concepts

1. **Logic**

   - Logical operations: and, or, negation, implication, equivalence.
   - Negation of "and" and "or". Negation of an implication.
   - Equivalence of the implication to the contrapositive. Necessary and sufficient condition.

2. **Proof by contradiction**

   Example. $\sqrt{2}$ is not rational.

   Proof. By contradiction, assume $\sqrt{2}$ is rational. Then there exists integers n, $m \in \mathbb{Z}$:

   $$\sqrt{2} = \frac{n}{m}, \qquad \text{greatest common divisor of } n \text{ and } m \text{ is } 1.$$

   Taking a square of this relation we obtain

   $$2 = \frac{n^2}{m^2} \qquad \Longleftrightarrow \qquad n^2 = 2m^2$$

   $2m^2$ is even and hence n^2 even. But then n itself even, i.e.

   $$n = 2k \quad \text{for some} \quad k \in \mathbb{Z}.$$

   Then we obtain

   $$n^2 = 2m^2 \quad \Longleftrightarrow \quad (2k)^2 = 2m^2 \quad \Longleftrightarrow \quad 4k^2 = 2m^2 \quad \Longleftrightarrow \quad 2k^2 = m^2$$

   Then m^2 is even and hence m itself is even. As both n and m are even, this contradicts to the assumption that the greatest common divisor of n and m is 1. Hence our assumption that $\sqrt{2}$ is rational can not be true. Hence $\sqrt{2}$ is irrational. □

3. **Quantifiers**

   - sentences and predicates
   - $\forall$, $\exists$, $\exists!$ (there exists a unique)
   - Negation of the statement with quantifiers

   Example. Construct the explicit negation of the statement

   $$\Big(\forall\, n \in \mathbb{N} \;\; \exists\, x \in (1, +\infty) \;\; \forall\, y \in [0,1] \quad x + y > n^2\Big)$$

   Determine which of the statements (the original one or its negation) is true.

4. **Proof by induction**

$$\boxed{\Big(P(1)\text{ is true}\Big)\wedge\Big(\forall n\in\mathbb{N}\quad P(n)\implies P(n+1)\Big)\quad\implies\quad\forall n\in\mathbb{N}\quad P(n)\text{ is true}}$$

Example.

1) For any $n\in\mathbb{N}$ the number $5^{n+2}+6^{2n+1}$ is divisible by 31
2) Prove by induction that for any $n\in\mathbb{N}$ $\quad 1+2+\ldots+n=\frac{n(n+1)}{2}$

Proof.

1) Base of the induction $P(1)$. Let us verify that for $n=1$ the statement is true. Indeed,

$$5^{1+2}+6^{2\cdot1+1}=5^3+6^3=125+216=341 \text{ — is divisible by 31} \quad (341:31=11)$$

Inductional step $P(n)\Rightarrow P(n+1)$. Let us take some arbitrary $n\in\mathbb{N}$ and let us assume

$$5^{n+2}+6^{2n+1}\quad\text{is devisable by 31}\qquad\text{— we **assume** that this is true}$$

Basing on this information, let us try to derive that

$$5^{(n+1)+2}+6^{2(n+1)+1}=5^{n+3}+6^{2n+3}\quad\text{is divisible by 31}\qquad\text{— we **want** to prove this}$$

Indeed, we have

$$5^{n+3}+6^{2n+3}=5\cdot5^{n+2}+36\cdot6^{2n+1}=\underbrace{5\cdot5^{n+2}+6\cdot6^{2n+1}}_{\text{is divisible by 31}}+\underbrace{31\cdot6^{2n+1}}_{\text{is divisible by 31}}$$

So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n\in\mathbb{N}$. $\square$

2) Base of the induction $P(1)$. Let us verify that for $n=1$ the statement is true. Indeed,

$$1\;=\;\frac{1\cdot(1+1)}{2}\qquad\text{— is true}$$

Inductional step $P(n)\Rightarrow P(n+1)$. Let us assume that for some $n\in\mathbb{N}$

$$1+2+\ldots+n\;=\;\frac{n(n+1)}{2}\qquad\text{— we **assume** that this is true}$$

and basing on this information let us try to derive that

$$1+2+\ldots+n+(n+1)\;=\;\frac{(n+1)\big((n+1)+1\big)}{2}\qquad\text{— we **want** to prove this}$$

Indeed, using the inductional assumption we can transform

$$\underbrace{1+2+\ldots+n}_{=\frac{n(n+1)}{2}}+(n+1)\;=\;\frac{n(n+1)}{2}+(n+1)=(n+1)\Big(\frac{n}{2}+1\Big)\;=\;\frac{(n+1)(n+2)}{2}$$

So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n\in\mathbb{N}$. $\square$

5. **Sets**

- $x \in A$ and $B \subset A$ — elements and subsets
- Barber paradox, universal set X
- Equality of sets: $A = B \Leftrightarrow (A \subset B) \wedge (B \subset A)$, proofs of set identities
- Operations with sets: $A \cup B$, $A \cap B$, $A \setminus B$, $A' = cA = X \setminus A$
- Infinite union and intersection
- Cartesian product

Example. For any an indexed family of subsets $\{A_i\}_{i \in I}$ of some universal set X the following equalities hold:

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \setminus A_i) \qquad X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i)$$

6. **Relations**

- $a, b \in A$, $a \overset{\mathcal{R}}{\sim} b$
- Examples: $a \leq b$ (real numbers), $a \,\vdots\, b$ (integers), $a \perp b$ (lines on the plane) etc
- Reflexive, symmetric, transitive relations
- Equivalence relation is a relation which is reflexive, symmetric, transitive
- Formally: relation $\mathcal{R}$ is some subset of $A \times A$. Then $a \overset{\mathcal{R}}{\sim} b \Leftrightarrow (a, b) \in \mathcal{R}$

7. **Functions**

Definition. Let A and B be any non-empty sets.

- A *function* (or a *map*) $f : A \to B$ is a rule which associate with each member of A a unique element of B.
- A is called the *domain* of f (we denote the domain $D(f)$), and B is called *codomain*.
- Functions f and g are called *equal* if $D(f) = D(g)$ and $\forall\, a \in D(f)\ f(a) = g(a)$.
- g is an *extension* of f if $D(f) \subset D(g)$ and $\forall\, a \in D(f)\ f(a) = g(a)$.

8. **Domain and range of a function**

Definition. If $f : A \to B$ is a function then

$$\begin{aligned} D(f) &:= A \quad \text{is the } \textit{domain} \text{ of } f \\ R(f) &:= \left\{ b \in B \;\middle|\; \exists\, a \in A : \ b = f(a) \right\} \quad \text{is the } \textit{range} \text{ of } f \end{aligned}$$

If $f(a) = b$ then we say b is the *image* of a and a is a *pre-image* of b.

9. **Injection, surjection and bijection functions**

   Definition. Let $f : A \to B$ be a function.

   - f is *injection* $\iff$ f is one-to-one $\iff$ $\forall\, a_1,\, a_2 \in A$ if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$
   - f is *surjection* $\iff$ f is onto $\iff$ $R(f) = B$ $\iff$ $\forall\, b \in B\ \ \exists\, a \in A$: $f(a) = b$
   - f is a *bijection* $\iff$ f is both an injection and a surjection

10. **Countable sets**

   - The set A is *countable* $\iff$ $\exists$ a bijection $A \leftrightarrow \mathbb{N}$
   - The set of all rational numbers $\mathbb{Q}$ is countable

11. **Inverse function**

   Definition. A function $f : A \to B$ is *invertible* on A if f is an injection on A. If f is invertible on $D(f)$ then the *inverse function* to f is defined by

$$g := f^{-1}, \quad D(g) = R(f), \quad g(b) = a \quad \iff \quad b = f(a), \quad \forall\, a \in D(f), \quad \forall\, b \in R(f)$$

12. **Composition of functions**

   Definition. Assume functions $f : A \to E$ and $g : B \to C$ satisfy $R(f) \subset B$. A *composition* of f and g is the function $h : A \to C$ defined by

$$h := g \circ f, \qquad D(h) = D(f) = A, \qquad h(a) \;=\; g(f(a)), \qquad a \in A$$

13. **Real numbers**

   - Common sense concept: infinite decimal fractions
   - More formal: the real number can be associated with certain sequences of rational numbers $\{r_n\}$ which "approximate" (in a certain sense) this real number.
   - Axioms of $\mathbb{R}$: operations $x + y$ and $x \cdot y$ possess commutativity, associativity, existence on neutral element, existence of inverse, distributivity, relation $x \leq y$
   - Intervals (a, b), $[a, b)$, $[a, b]$, $(-\infty, b)$, $[a, +\infty)$ etc

14. **Bounded sets, lower and upper bounds**

   Definition.

   - $M \in \mathbb{R}$ is an *upper bound* of a set $A \subset \mathbb{R}$ $\iff$ $\forall\, x \in A \quad x \leq M$
   - $m \in \mathbb{R}$ is a *lower bound* of a set $A \subset \mathbb{R}$ $\iff$ $\forall\, x \in A \quad x \geq m$
   - $A \subset \mathbb{R}$ is *bounded from above* $\iff$ $\exists$ at least one upper bound of A
   - $A \subset \mathbb{R}$ is *bounded from below* $\iff$ $\exists$ at least one lower bound of A
   - $A \subset \mathbb{R}$ is *bounded* $\iff$ A is bounded both from above and from below

15. **Supremum and infimum**

DEFINITION.

$M \in \mathbb{R}$ is the *supremum* of a set $A \subset \mathbb{R} \iff$

1) M is an upper bound of A

2) if $M' \in \mathbb{R}$ is some other upper bound of A then $M \leq M'$

$m \in \mathbb{R}$ is the *infimum* of a set $A \subset \mathbb{R} \iff$

1) m is a lower bound of A

2) if $m' \in \mathbb{R}$ is some other lower bound of A then $m \geq m'$

NOTATION.

$\sup A$ is the supremum of A, $\sup A = +\infty \iff A$ is not bounded from above

$\inf A$ is the infimum of A, $\inf A = -\infty \iff A$ is not bounded from below

16. **Characterization of supremum and infimum using quantifiers**

THEOREM.

$M \in \mathbb{R}$ is the supremum of $A \subset \mathbb{R} \iff$

1) $\forall x \in A \quad x \leq M$

2) $\forall \varepsilon > 0 \quad \exists x_\varepsilon \in A\colon \quad M - \varepsilon < x_\varepsilon$

$m \in \mathbb{R}$ is the infimum of $A \subset \mathbb{R} \iff$

1) $\forall x \in A \quad m \leq x$

2) $\forall \varepsilon > 0 \quad \exists x_\varepsilon \in A\colon \quad x_\varepsilon < m + \varepsilon$

PROBLEM. Find the supremum and the infimum of the set $\left\{ \frac{2nm}{n^2+m^2} \;\middle|\; n, m \in \mathbb{N} \right\} \subset \mathbb{R}$

17. **Least upper bound axiom**

AXIOM OF THE SET $\mathbb{R}$.

$A \subset \mathbb{R}$ is bounded from above $\implies \exists$ the number $M \in \mathbb{R}$ which is the supremum of A

$A \subset \mathbb{R}$ is bounded from below $\implies \exists$ the number $m \in \mathbb{R}$ which is the infimum of A

REMARK. The least upper bound axiom does not hold for the set of rational numbers $\mathbb{Q}$:

$$A = \{ r \in \mathbb{Q} : \; r^2 < 2 \}, \quad A \text{ is bounded from above, BUT } \quad \nexists\, q \in \mathbb{Q} : \; q = \sup A$$

18. **Functions of a real variable**

DEFINITION. $f : D(f) \subset \mathbb{R} \to \mathbb{R}$

- Admissible set (maximal domain) of an algebraic expression

- Graph of a function of a real variable
- Range of a function of a real variable

19. **Monotone functions**

    Definition. Assume $f : [a, b] \to \mathbb{R}$.

    - f — $\nearrow$ on $[a, b] \iff \forall x_1, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) \leq f(x_2)$
    - f — $\uparrow$ on $[a, b] \iff \forall x_1, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) < f(x_2)$
    - f — $\searrow$ on $[a, b] \iff \forall x_1, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) \geq f(x_2)$
    - f — $\downarrow$ on $[a, b] \iff \forall x_1, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) > f(x_2)$

20. **Invertibility of strictly monotone functions**

    Theorem. A strictly monotone function $f : D(f) \subset \mathbb{R} \to \mathbb{R}$ in invertible on the interval of its monotonicity and the inverse function f^{-1} is strictly monotone in the same sense to f, i.e.

    - f — $\uparrow$ on $D(f) \implies f^{-1}$ — $\uparrow$ on $R(f)$
    - f — $\downarrow$ on $D(f) \implies f^{-1}$ — $\downarrow$ on $R(f)$

1.2 Limit of a sequence

1. **Definition of a sequence**

   Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ is a map (a function) $f : \mathbb{N} \to \mathbb{R}$, $x_n := f(n)$

2. **Informal notion helping to understand quantifiers in the definition of a limit**

   We say an interval (α, β) is a "feeder" for $\{x_n\}_{n=1}^{\infty} \iff \forall\, N \in \mathbb{N}\; \exists\, n \geq N$: $x_n \in (\alpha, \beta)$

   (i.e. the sequence $\{x_n\}_{n=1}^{\infty}$ "returns" to the interval (α, β) infinitely many times).

   We say an interval (α, β) is a "catcher" for $\{x_n\}_{n=1}^{\infty} \iff \exists\, N \in \mathbb{N}$: $\forall\, n \geq N$: $x_n \in (\alpha, \beta)$

   (i.e. once entering the interval (α, β), the sequence $\{x_n\}_{n=1}^{\infty}$ can never get out of it).

   We say the sequence $\{x_n\}_{n=1}^{\infty}$ is *bounded* $\iff$ $\exists\, M \geq 0$: $|x_n| \leq M$ $\forall\, n \in \mathbb{N}$.

   (i.e. there exists at least one "catcher" for a sequence $\{x_n\}_{n=1}^{\infty}$).

3. **Limit of a sequence**

   Definition. $a \in \mathbb{R}$ is the *limit* of a sequence $\{x_n\}_{n=1}^{\infty}$ (equivalently, the sequence $\{x_n\}_{n=1}^{\infty}$ *converges* to $a \in \mathbb{R}$) iff

   $$\forall\, \varepsilon > 0 \quad \exists\, N = N(\varepsilon) : \quad \forall\, n \geq N \qquad |x_n - a| < \varepsilon$$

   i.e. for any $\varepsilon > 0$ the interval $(a - \varepsilon, a + \varepsilon)$ is a "catcher" for a sequence $\{x_n\}_{n=1}^{\infty}$.

   Notation. $a = \lim\limits_{n\to\infty} x_n$ or $x_n \underset{n\to\infty}{\longrightarrow} a$.

4. **Infinitesimal sequences**

   Definition.

   A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is called *infinitesimal* $\iff$ $\lim\limits_{n\to\infty} \alpha_n = 0$

   A sequence $\{x_n\}_{n=1}^{\infty}$ is called *infinitely large* $\iff$ $\lim\limits_{n\to\infty} |x_n| = +\infty$ $\iff$

   $$\forall M > 0 \quad \exists N = N(M) : \quad \forall n \geq N \quad |x_n| \geq M$$

5. **Characterization of a limit in terms of infinitesimal**

   Theorem. Assume $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $a \in \mathbb{R}$. Then

   $$\exists \lim_{n\to\infty} x_n = a \quad \iff \quad \exists \text{ an infinitesimal } \{\alpha_n\}_{n=1}^{\infty} \text{ such that } \forall\, n \in \mathbb{N} \quad x_n = a + \alpha_n$$

6. **Properties of infinitesimal sequences**

1) $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are infinitesimal $\implies$ $\{\alpha_n + \beta_n\}_{n=1}^{\infty}$ is infinitesimal

2) $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal, $\{x_n\}_{n=1}^{\infty}$ is bounded $\implies$ $\{\alpha_n x_n\}_{n=1}^{\infty}$ is infinitesimal

3) $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal, $\alpha_n \neq 0$, $\forall\, n \in \mathbb{N}$ $\iff$ $\{\frac{1}{\alpha_n}\}_{n=1}^{\infty}$ is infinitely large

7. **Basic properties of limits**

THEOREM 1. The limit of a sequence, if exists, is unique.

THEOREM 2. $\exists \lim\limits_{n\to\infty} x_n = a$ (a is a finite real number) $\implies$ $\{x_n\}_{n=1}^{\infty}$ is bounded.

THEOREM 3. $\exists \lim\limits_{n\to\infty} x_n = a$, $\exists \lim\limits_{n\to\infty} y_n = b$, $\forall n \in \mathbb{N}$ $x_n \le y_n$ $\implies$ $a \le b$

THEOREM 4. $\forall n \in \mathbb{N}$ $x_n \le y_n \le z_n$, $\exists \lim\limits_{n\to\infty} x_n = a$, $\exists \lim\limits_{n\to\infty} z_n = a$ $\implies$ $\exists \lim\limits_{n\to\infty} y_n = a$

COROLLARY. $\{\beta_n\}_{n=1}^{\infty}$ is infinitesimal, $|\alpha_n| \le |\beta_n|$ $\implies$ $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal.

8. **Arithmetic operations with limits**

THEOREM 1. $\exists \lim\limits_{n\to\infty} x_n = a\ \exists \lim\limits_{n\to\infty} y_n = b \implies \exists \lim\limits_{n\to\infty} (x_n \pm y_n) = a \pm b$

THEOREM 2. $\exists \lim\limits_{n\to\infty} x_n = a\ \exists \lim\limits_{n\to\infty} y_n = b \implies \exists \lim\limits_{n\to\infty} (x_n y_n) = ab$

LEMMA. $\exists \lim\limits_{n\to\infty} y_n = b$ and $y_n \neq 0$, $b \neq 0 \implies \{\frac{1}{y_n}\}_{n=1}^{\infty}$ is bounded.

THEOREM 3. $\exists \lim\limits_{n\to\infty} x_n = a\ \exists \lim\limits_{n\to\infty} y_n = b$ and $y_n \neq 0$, $b \neq 0 \implies \exists \lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{a}{b}$

9. **Monotone sequences**

DEFINITION.

- $\{x_n\}_{n=1}^{\infty}$ — $\nearrow$ $\iff$ $\forall\, n \in \mathbb{N}$ $x_n \le x_{n+1}$
- $\{x_n\}_{n=1}^{\infty}$ — $\searrow$ $\iff$ $\forall\, n \in \mathbb{N}$ $x_n \ge x_{n+1}$

THEOREM.

1) $\{x_n\}_{n=1}^{\infty}$ — $\nearrow$ $\exists\, M \in \mathbb{R}$: $\forall\, n \in \mathbb{N}$ $x_n \le M$ $\implies$ $\exists \lim\limits_{n\to\infty} x_n \le M$
2) $\{x_n\}_{n=1}^{\infty}$ — $\searrow$ $\exists\, m \in \mathbb{R}$: $\forall\, n \in \mathbb{N}$ $x_n \ge m$ $\implies$ $\exists \lim\limits_{n\to\infty} x_n \ge m$

10. **Important examples**

1) $q > 1 \implies \lim\limits_{n\to\infty} \frac{n}{q^n} = 0$

2) $q \in \mathbb{R} \implies \lim\limits_{n\to\infty} \frac{q^n}{n!} = 0$

3) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

4) $a > 0 \implies \lim\limits_{n\to\infty} \sqrt[n]{a} = 1$

11. **Euler's number *e***

LEMMA 1. For all non-negative $a_1,\, a_2,\, a_3,\, \ldots a_n \in \mathbb{R}$

$$\underbrace{\sqrt[n]{a_1 a_2 a_3 \cdot \ldots \cdot a_n}}_{\text{Geometric mean}} \;\le\; \underbrace{\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}}_{\text{Arithmetic mean}}$$

LEMMA 2. Assume $a > 0$, $b > 0$. Then

$$\sqrt[n+1]{ab^n} \;\le\; \frac{a + nb}{n+1}$$

LEMMA 3. Denote $x_n := \left(1 + \frac{1}{n}\right)^n, \quad z_n := \left(1 - \frac{1}{n}\right)^n, \quad y_n := \left(1 + \frac{1}{n}\right)^{n+1}.$ Then

$$\forall\, n \in \mathbb{N} \qquad x_n < x_{n+1}, \qquad z_n < z_{n+1}, \qquad y_{n+1} < y_n$$

LEMMA 4. The following limits exist and coincide

$$\lim_{n\to\infty} x_n \;=\; \lim_{n\to\infty} y_n$$

DEFINITION. Euler's number e is defined by

$$e \;:=\; \lim_{n\to\infty} \left(1 + \tfrac{1}{n}\right)^n, \qquad (e \approx 2,718281828459045\ldots)$$

1.3 Subsequences

1. **Definition of a subsequence**

   Definition. Assume $f : \mathbb{N} \to \mathbb{R}$ is a sequence and assume $\varphi : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Denote by $h := f \circ \varphi$ the comosition $h(k) = f(\varphi(k))$. Then the sequence $h : \mathbb{N} \to \mathbb{R}$ is called a *subsequence* of the sequence $f : \mathbb{N} \to \mathbb{R}$.

   $$x_n := f(n), \qquad x_{n_k} = h(k), \qquad n_k := \varphi(k), \qquad n_1 < n_2 < \ldots < n_k < \ldots$$

   Notation. $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty} \quad \Longleftrightarrow \quad \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$

2. **Any subsequence of a convergent sequence converges to the same limit**

   Theorem. $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, $\exists \lim\limits_{n\to\infty} x_n = a \implies \exists \lim\limits_{k\to\infty} x_{n_k} = a$

3. **Nested interval theorem**

   Theorem. Assume a sequence of closed intervals $[a_n, b_n] \subset \mathbb{R}$ satisfies the properties

   1) $\forall\, n \in \mathbb{N} \quad [a_{n+1}, b_{n+1}] \subset [a_n, b_n]$
   2) $b_n - a_n \to 0 \quad \text{as} \quad n \to \infty$

   Then $\exists!\, c \in \mathbb{R}$: $c \in \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$

4. **Bolzano–Weierstrass theorem**

   Theorem. Each bounded sequence has a convergent subsequence:

   $$\{x_n\}_{n=1}^{\infty} \text{ is bounded } \implies \exists\, \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty} \text{ such taht } \exists\, a \in \mathbb{R} : \quad a = \lim_{k\to\infty} x_{n_k}$$

5. **Cauchy sequence**

   Definition. $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is a *Cauchy sequence* (or a *fundamental sequence*) if

   $$\forall\, \varepsilon > 0 \quad \exists\, N(\varepsilon) \in \mathbb{N} : \quad \forall\, n, m \geq N(\varepsilon) \qquad |x_n - x_m| < \varepsilon$$

   Theorem. Every fundamental sequence is convergent:

   $$\{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \text{ is a Cauchy sequence} \quad \Longleftrightarrow \quad \exists\, a \in \mathbb{R} : \quad x_n \xrightarrow[n\to\infty]{} a$$

1.4 Limit of a function

1. **Definition of the limit of a function according to Cauchy**

   Definition (C). Assume $f : (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$, $y_0 \in \mathbb{R}$. We write

   $$\lim_{x\to x_0} f(x) = y_0 \qquad \text{or} \qquad f(x) \underset{x\to x_0}{\to} y_0$$

   and say y_0 is the *limit* of $f(x)$ as x tends to x_0 iff

   $$\forall\, \varepsilon > 0 \;\; \exists\, \delta = \delta(\varepsilon) > 0 : \; \forall\, x \in (a,b) \quad (x \neq x_0) \;\wedge\; |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - y_0| < \varepsilon$$

   The set $\overset{\circ}{U}_\delta(x_0) := (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ is called a *deleted neighbourhood* of x_0.

2. **Definition of the limit of a function according to Heine**

   Definition (H). Assume $f : (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$, $y_0 \in \mathbb{R}$. We write

   $$\lim_{x\to x_0} f(x) = y_0 \qquad \text{or} \qquad f(x) \underset{x\to x_0}{\to} y_0$$

   and say y_0 is the *limit* of $f(x)$ as x tends to x_0 iff

   $$\forall\, \{x_n\}_{n=1}^{\infty} \subset (a,b) \setminus \{x_0\} : \qquad \lim_{n\to\infty} x_n = x_0 \quad \Longrightarrow \quad \lim_{n\to\infty} f(x_n) = y_0$$

3. **Equivalence of definitions**

   Theorem. Definition (C) $\Longleftrightarrow$ Definition (H)

4. **Arithmetic operations with limits of functions**

   Theorem 1. $\exists \lim\limits_{x\to x_0} f(x), \exists \lim\limits_{x\to x_0} g(x) \quad \Longrightarrow \quad \exists \lim\limits_{x\to x_0} (f(x) + g(x)) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x)$

   Theorem 2. $\exists \lim\limits_{x\to x_0} f(x), \exists \lim\limits_{x\to x_0} g(x) \quad \Longrightarrow \quad \exists \lim\limits_{x\to x_0} (f(x) \cdot g(x)) = \lim\limits_{x\to x_0} f(x) \cdot \lim\limits_{x\to x_0} g(x)$

   Lemma. $\lim\limits_{x\to x_0} g(x) > 0 \quad \Longrightarrow \quad \exists\, \delta > 0\text{:} \quad |x - x_0| < \delta \;\Rightarrow\; g(x) > 0$

   Theorem 3. $\exists \lim\limits_{x\to x_0} f(x), \exists \lim\limits_{x\to x_0} g(x) \neq 0 \quad \Longrightarrow \quad \exists \lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$

5. **One-sided limits**

   Definition. Assume $f : [a,b] \to \mathbb{R}$, $x_0 \in [a,b]$.

   $$\exists \lim_{x\to x_0 - 0} f(x) = y_0 \quad \Longleftrightarrow \quad \forall\, \varepsilon > 0 \;\; \exists\, \delta > 0\text{: } x \in (x_0 - \delta, x_0) \;\Rightarrow\; |f(x) - y_0| < \varepsilon$$
   $$\exists \lim_{x\to x_0 + 0} f(x) = y_0 \quad \Longleftrightarrow \quad \forall\, \varepsilon > 0 \;\; \exists\, \delta > 0\text{: } x \in (x_0, x_0 + \delta) \;\Rightarrow\; |f(x) - y_0| < \varepsilon$$

   Theorem. $\exists \lim\limits_{x\to x_0} f(x) = y_0 \quad \Longleftrightarrow \quad \exists \lim\limits_{x\to x_0 - 0} f(x) = \lim\limits_{x\to x_0 + 0} f(x) = y_0$

1.5 Indeterminate forms and important examples

1. **Types of indeterminate forms**

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty \quad 1^{\infty}, \quad \infty^0, \quad 0^0$$

2. **Auxiliary inequalities**

Theorem.

1) $0 < x < \frac{\pi}{2} \implies \sin x < x < \operatorname{tg} x, \quad x \in \mathbb{R} \implies |\sin x| \le |x|$
2) $x_1,\, x_2 \in \mathbb{R} \implies |\sin x_1 - \sin x_2| \le |x_1 - x_2|, \quad |\cos x_1 - \cos x_2| \le |x_1 - x_2|$
3) $\lim\limits_{x\to 0} \sin x = 0, \quad \lim\limits_{x\to 0} \cos x = 1$

3. **Properties of a^x and $\log_a x$ which are postulated for now**

We postulate the properties of the exponential and the logarithmic functions which actually come together with the definition of the exponential function and its inverse. We will return to discussion of these properties in the section "Continuity". These properties also allowed to use without further explanation when you solve homework problems.

Theorem. Assume $a > 0$ and $a \neq 1$. Then

1) $\forall \{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \quad x_n \to x_0 \quad \implies \quad a^{x_n} \to a^{x_0}$
2) $\forall \{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \quad x_n \to x_0, \quad x_0 > 0 \quad \implies \quad \log_a x_n \to \log_a x_0$

4. **Important examples**

1) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

2) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

3) $\lim\limits_{x\to +\infty} (1 + \frac{1}{x})^x = e \qquad \lim\limits_{x\to -\infty} (1 + \frac{1}{x})^x = e \qquad \lim\limits_{x\to 0} (1 + x)^{\frac{1}{x}} = e$

4) $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$

5) $\lim\limits_{x\to 0} \frac{a^x-1}{x} = \ln a$

6) $\lim\limits_{x\to 0} \frac{(1+x)^{\mu}-1}{x} = \mu$

1.6 Infinite limits and asymptotes

1. **Infinite limits, vertical asymptotes**

   Definition.

$$\lim_{x\to x_0-0} f(x) = +\infty \iff \forall\, M>0\ \ \exists\,\delta>0\colon\ x\in(x_0-\delta,x_0)\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to x_0+0} f(x) = +\infty \iff \forall\, M>0\ \ \exists\,\delta>0\colon\ x\in(x_0,x_0+\delta)\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to x_0-0} f(x) = -\infty \iff \forall\, M<0\ \ \exists\,\delta>0\colon\ x\in(x_0-\delta,x_0)\ \Rightarrow\ f(x)<M$$
$$\lim_{x\to x_0+0} f(x) = -\infty \iff \forall\, M<0\ \ \exists\,\delta>0\colon\ x\in(x_0,x_0+\delta)\ \Rightarrow\ f(x)<M$$

   Definition. The line $x=x_0$ is a *vertical asymptote* of the function $f:\mathbb{R}\to\mathbb{R}$ if at least one of the four statements is true:

$$\lim_{x\to x_0+0} f(x)=+\infty,\quad \lim_{x\to x_0-0} f(x)=+\infty,\quad \lim_{x\to x_0-0} f(x)=+\infty,\quad \lim_{x\to x_0-0} f(x)=-\infty$$

2. **Behavior of functions at infinity**

   Definition.

$$\lim_{x\to+\infty} f(x) = +\infty \iff \forall\, M>0\ \ \exists\, R>0\colon\ x>R\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to-\infty} f(x) = +\infty \iff \forall\, M>0\ \ \exists\, R<0\colon\ x<R\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to+\infty} f(x) = -\infty \iff \forall\, M<0\ \ \exists\, R>0\colon\ x>R\ \Rightarrow\ f(x)<M$$
$$\lim_{x\to-\infty} f(x) = -\infty \iff \forall\, M<0\ \ \exists\, R<0\colon\ x<R\ \Rightarrow\ f(x)<M$$

3. **Limits at infinity, horizontal asymptotes**

   Definition.

$$\lim_{x\to+\infty} f(x) = y_0 \iff \forall\,\varepsilon>0\ \ \exists\, R>0\colon\ x>R\ \Rightarrow\ |f(x)-y_0|<\varepsilon$$
$$\lim_{x\to-\infty} f(x) = y_0 \iff \forall\,\varepsilon>0\ \ \exists\, R<0\colon\ x<R\ \Rightarrow\ |f(x)-y_0|<\varepsilon$$

   Definition. The line $y=y_0$ is a *horizontal asymptote* of the function $f:\mathbb{R}\to\mathbb{R}$ if at least one of the two statements is true:

$$\lim_{x\to+\infty} f(x)=y_0 \quad\text{or}\quad \lim_{x\to-\infty} f(x)=y_0$$

4. **How to remember definitions of finite/infinite limits?**

$$y_0 = \lim_{x \to x_0} f(x)$$

x_0 is finite or infinite
y_0 is finite or infinite

	Values of $f(x)$	Values of x	When is used?
Small number	$\forall\, \varepsilon > 0$	$\exists\, \delta > 0$	For a finite value
Large number	$\forall\, M > 0$	$\exists\, R > 0$	For an infinite value

5. **Oblique asymptotes**

Definition. The line $y = kx + b$ is an *oblique asymptote* of the function $f : \mathbb{R} \to \mathbb{R}$ if at least one of the two statements is true:

$$\lim_{x \to +\infty} \Big(f(x) - kx - b \Big) = 0 \quad \text{or} \quad \lim_{x \to -\infty} \Big(f(x) - kx - b \Big) = 0$$

Remark. A horizontal asymptote is a particular case of an oblique asymptote with $k = 0$.

Theorem. $y = kx + b$ is a horizontal or an oblique asymptote of $f : \mathbb{R} \to \mathbb{R}$ as $x \to +\infty$ iff the following two limits exist:

1) $\exists \lim\limits_{x \to +\infty} \frac{f(x)}{x} = k$
2) $\exists \lim\limits_{x \to +\infty} (f(x) - kx) = b$

The similar statements hold as $x \to -\infty$.

1.7 Continuity

1. **Continuity of a function at a point**

   DEFINITION. $f:(a,b)\to\mathbb{R}$ is *continuous* at a point $x_0 \iff \lim\limits_{x\to x_0} f(x)=f(x_0)$

   REMARK. The following statements are equivalent:

   1) $f:(a,b)\to\mathbb{R}$ is continuous at x_0
   2) $\forall\,\varepsilon>0\ \ \exists\,\delta(\varepsilon)>0$: $\quad |x-x_0|<\delta \quad\Longrightarrow\quad |f(x)-f(x_0)|<\varepsilon$
   3) $\forall\ \{x_n\}_{n=1}^{\infty}\subset(a,b)$: $\quad x_n\to x_0 \quad\Longrightarrow\quad f(x_n)\to f(x_0)$

   DEFINITION. $f:(a,b)\to\mathbb{R}$ is continuous *on* $(a,b) \iff \forall\,x_0\in(a,b)\ \ f$ is continuous at x_0

2. **Stability of sign theorem**

   THEOREM. f is continuous at x_0, $f(x_0)>0 \implies \exists\,\delta>0$: $\forall\,x\in(x_0-\delta,x_0+\delta)\ \ f(x)>0$

3. **Arithmetic operations with continuous functions**

   THEOREM.

   1) f, g are continuous at $x_0 \implies f\pm g$ are continuous at x_0
   2) f, g are continuous at $x_0 \implies f\cdot g$ is continuous at x_0
   3) f, g are continuous at x_0, $\ g(x_0)\neq 0 \implies f/g$ is continuous at x_0

4. **Left and right continuity**

   DEFINITION.

   $f:[a,b]\to\mathbb{R}$ is *left continuous* at $x_0 \iff \lim\limits_{x\to x_0-0} f(x)=f(x_0)$

   $f:[a,b]\to\mathbb{R}$ is *right continuous* at $x_0 \iff \lim\limits_{x\to x_0+0} f(x)=f(x_0)$

   $$f:[a,b]\to\mathbb{R} \text{ is continuous } \textit{on } [a,b] \iff \begin{cases} f \text{ is continuous on } (a,b) \\ f \text{ is left continuous at } a \\ f \text{ is right continuous at } b \end{cases}$$

   THEOREM. f is continuous at $x_0\in(a,b) \iff f$ is left and right continuous at x_0.

5. **Types of discontinuity**

   DEFINITION. Assume $x_0\in(a,b)$, $\ f:(a,b)\setminus\{x_0\}\to\mathbb{R}$

   - x_0 is a *removable discontinuity* $\iff \exists$ finite $\lim\limits_{x\to x_0\pm 0} f(x)$, $\ \lim\limits_{x\to x_0-0} f(x)=\lim\limits_{x\to x_0+0} f(x)$

- x_0 is a *jump discontinuity* $\iff$ $\exists$ both finite $\lim\limits_{x\to x_0\pm 0} f(x)$, $\lim\limits_{x\to x_0-0} f(x) \neq \lim\limits_{x\to x_0+0} f(x)$
- $x = x_0$ is a *vertical asymptote* $\iff$ $\exists \lim\limits_{x\to x_0\pm 0} f(x) = \pm\infty$ (at least one of limits)
- x_0 may be a discontinuity of no above type $\iff$ $\nexists \lim\limits_{x\to x_0\pm 0} f(x)$ $\quad (f(x) = \sin\frac{1}{x})$

6. **Continuity of some elementary functions**

   - $f(x) = x^n$ is continuous on $\mathbb{R}$, $n \in \mathbb{N}$
   - $f(x) = \frac{1}{x^n}$ is continuous on $\mathbb{R}\setminus\{0\}$, $n \in \mathbb{N}$
   - $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is continuous on $\mathbb{R}$
   - $f(x) = \sin x$, $g(x) = \cos x$ are continuous on $\mathbb{R}$
   - $f(x) = \operatorname{tg} x$ is continuous at $x \neq \frac{\pi}{2} + \pi k$, $g(x) = \operatorname{ctg} x$ is continuous at $x \neq \pi k$, $k \in \mathbb{Z}$
   - $f(x) = x^p$, $p \in (0, +\infty)$ is continuous on $[0, +\infty)$
   - $f(x) = a^x$, $a > 0$, is continuous on $\mathbb{R}$

7. **Continuity of a composition of functions**

   Theorem. $f : (a, b) \to \mathbb{R}$, $g : (c, d) \to \mathbb{R}$, $x_0 \in (a, b)$, $f(x_0) \in (c, d)$

   $$f \text{ is continuous at } x_0, \quad g \text{ is continuous at } f(x_0) \quad \Longrightarrow \quad g \circ f \text{ is continuous at } x_0$$

8. **Two intermediate value theorems (Bolzano–Cauchy theorems)**

   Theorem 1. f is continuous on $[a, b]$, $f(a) \le 0$, $f(b) \ge 0$ $\Longrightarrow$ $\exists\, c \in [a, b]$: $f(c) = 0$

   Theorem 2. f is continuous on $[a, b]$, $f(a) = y_1$, $f(b) = y_2$, $y_1 \le y_2$ $\Longrightarrow$ $[y_1, y_2] \subset R(f)$

9. **Continuity of the inverse function**

   Theorem. Assume $f : (a, b) \to \mathbb{R}$ is continuous and strictly monotone on (a, b). Then f is invertible on (a, b) and f^{-1} is continuous on $R(f)$.

10. **Continuity of some elementary inverse functions**

    - $f(x) = \sqrt[n]{x}$ is continuous on $D(f)$, $n \in \mathbb{N}$
    - $f(x) = \log_a x$, $a > 0$, $a \neq 1$, is continuous on $(0, +\infty)$
    - $f(x) = \arcsin x$, $g(x) = \arccos x$ are continuous on $[-1, 1]$
    - $f(x) = \operatorname{arctg} x$, $g(x) = \operatorname{arcctg} x$ are continuous on $\mathbb{R}$

11. **Two extreme value theorems (Weierstrass theorems)**

THEOREM 1. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\Longrightarrow$ f is bounded on $[a,b]$, i.e.

$$\exists\, M > 0 : \qquad \forall\, x \in [a,b] \qquad |f(x)| \le M$$

THEOREM 2. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\Longrightarrow$ f achieves on $[a,b]$ its maximum and minimal values, i.e. $\exists\, c_1,\, c_2 \in [a,b]$ such that

$$f(c_1) = \inf_{x \in [a,b]} f(x) \qquad \text{and} \qquad f(c_2) = \sup_{x \in [a,b]} f(x)$$

12. **Uniform continuity**

DEFINITION. A function $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* on $D(f)$ iff

$$\forall\, \varepsilon > 0 \quad \exists\, \delta(\varepsilon) > 0 : \quad \forall\, x_1, x_2 \in D(f) \qquad |x_1 - x_2| < \delta \quad \Longrightarrow \quad |f(x_1) - f(x_2)| < \varepsilon$$

13. **Heine–Cantor theorem**

THEOREM. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\Longrightarrow$ f is uniformly continuous on $[a,b]$.

1.8 Differentiability and derivatives

1. **Main definition**

   DEFINITION. Assume $f:(a,b)\to\mathbb{R},\ \ x_0\in(a,b)$. A function f is *differentiable at* a point x_0 iff there exists a finite limit

$$f'(x_0):=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

   The number $f'(x_0)$ is called the *derivative* of a function f at x_0. Another notation

$$\frac{df}{dx}(x_0)\ :=\ f'(x_0)$$

   - We say f is *differentiable on* (a,b) if for any $x_0\in(a,b)$ the function f is differentiable at x_0.
   - We say f is *continuously differentiable* on (a,b) if f is differentiable on (a,b) and the function f' is continuous on (a,b).

2. **Physical and geometric meaning of the derivative**

   - From a physical point of view the derivative of a function is the instantaneous speed (or the rate of change) of the function.
   - From a geometrical point of view the derivative of a function f at a point x_0 is equal to the slope of the tangent line to the graph of the function at the point x_0.

3. **Differentiable function is continuous**

   THEOREM. $f:(a,b)\to\mathbb{R}$ is differentiable at $x_0\in(a,b)\quad\Longrightarrow\quad f$ is continuous at x_0

   REMARK. The inverse is false: $f(x)=|x|$ is continuous but non-differentiable at 0.

4. **Basic properties of derivatives**

   THEOREM . Assume $f,\ g:(a,b)\to\mathbb{R}$ are differentiable at $x\in(a,b)$. Then $f\pm g$ and fg are differentiable at x. If additionally $g(x)\neq 0$ then f/g is also differentiable at x. Besides,

   1) $(f\pm g)'(x)=f'(x)\pm g'(x)$

   2) $(cf)'(x)=cf'(x)$

   3) $(fg)'(x)=f'(x)g(x)+f(x)g'(x)$

   4) $\left(\frac{f}{g}\right)'(x)=\dfrac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$

5. Derivatives of elementary functions

1. $(C)' = 0$
2. $(x)' = 1$
3. $(x^m)' = mx^{m-1}, \quad m \in \mathbb{Z}, \quad x \in \mathbb{R} \quad (x \neq 0 \;\text{ if }\; m < 0)$
4. $(x^r)' = rx^{r-1}, \quad r \in \mathbb{R}, \quad x > 0$
5. $(e^x)' = e^x, \qquad (a^x)' = a^x \ln a, \quad a > 0, \quad x \in \mathbb{R}$
6. $(\ln|x|)' = \dfrac{1}{x}, \quad x \neq 0, \qquad (\log_a|x|)' = \dfrac{1}{x \ln a}, \qquad a > 0, \quad a \neq 1$
7. $(\sin x)' = \cos x, \quad (\cos x)' = -\sin x$
8. $(\operatorname{tg} x)' = \dfrac{1}{\cos^2 x}, \quad (\operatorname{ctg} x)' = -\dfrac{1}{\sin^2 x}$

1.9 Chain rule

1. **Differential of a function**

   DEFINITION. Assume $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$. A linear map $l_{x_0} : \mathbb{R} \to \mathbb{R}$,

   $$l_{x_0}(h) \; = \; k_{x_0} h, \qquad \forall\, h \in \mathbb{R}, \qquad (k_{x_0} \in \mathbb{R})$$

   is called the *differential* of f at a point x_0 iff there exists a number $h_0 > 0$ and a function $\alpha_{x_0} : (-h_0, h_0) \to \mathbb{R}$ such that

   $$\lim_{h \to 0} \alpha_{x_0}(h) = 0,$$

   $$f(x_0 + h) = f(x_0) + k_{x_0} h + \alpha_{x_0}(h) h, \qquad \forall\, |h| < h_0$$

   NOTATION. $\quad l_{x_0} := df(x_0) \quad \Longleftrightarrow \quad df(x_0)(h) \; := \; l_{x_0}(h) \; = \; k_{x_0} h, \quad \forall\, h \in \mathbb{R}$

2. **When the differential of a function is well-defined?**

   THEOREM. Assume $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$. The differential of f at x_0 exists if and only if f is differentiable at x_0. Moreover, if $df(x_0)(h) = k_{x_0} h$ is the differential of f at x_0 then

   $$k_{x_0} = f'(x_0) \qquad \Longleftrightarrow \qquad df(x_0)(h) = f'(x_0) h, \qquad \forall\, h \in \mathbb{R}$$

   where $f'(x_0)$ is the derivative of f at x_0.

3. **What is the difference between the derivative $f'(x_0)$ and the differential $df(x_0)$?**

   It is well-known that any linear map $l : \mathbb{R} \to \mathbb{R}$ has the form

   $$l(x) = kx, \qquad x \in \mathbb{R}$$

   i.e. any linear map from $\mathbb{R}$ to $\mathbb{R}$ can be characterized in the unique way by the coefficient $k \in \mathbb{R}$ which is called a *slope* of the linear map $l(x)$. The theorem above shows that the slope of the differential $df(x_0)$ is equal to $f'(x_0)$. So, the differential and the derivative of a function are related in the same way as a linear transformation $l(x) = kx$ is related to its slope $k \in \mathbb{R}$.

   NOTATION. Very often people write $df(x_0) = f'(x_0) dx$ meaning that $f'(x_0)$ is a slope of a linear function $df(x_0)$ and dx stands for the identity linear map, i.e. $dx(h) = h$.

4. **Chain rule**

   THEOREM. Assume $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$, $\; g : (c, d) \to \mathbb{R}$, $\; R(f) \subset (c, d)$, $\; x_0 \in (a, b)$. If f is differentiable at x_0 and g is differentiable at $y_0 = f(x_0)$ then the composition $\varphi := g \circ f$ is differentiable at x_0 and

   $$\varphi'(x_0) = g'(y)\Big|_{y = f(x_0)} \cdot f'(x_0)$$

5. **Inverse function rule**

THEOREM. Assume $f : (a,b) \to \mathbb{R}$ is strictly monotone and continuous (so the inverse function f^{-1} exists, f^{-1} is strictly monotone in the same sense to f and continuous on $R(f)$). Assume additionally f is differentiable at $x_0 \in (a,b)$ and $f'(x_0) \neq 0$. Then the function $g := f^{-1}$ is differentiable at $y_0 := f(x_0)$ and

$$g'(y_0) = \frac{1}{f'(x_0)}$$

6. **Derivatives of inverse trigonometric functions**

9. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}, \qquad (\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

10. $(\operatorname{arctg} x)' = \dfrac{1}{1+x^2}, \qquad (\operatorname{arcctg} x)' = -\dfrac{1}{1+x^2}$

1.10 The basic theorems of differential calculus

1. **Fermat's theorem (or Fermat's lemma)**

   THEOREM. Assume $f : (a,b) \to \mathbb{R}$ has a local extremum (maximum or minimum) on the interval (a,b) at some internal point $c \in (a,b)$, i.e.

   $$\begin{array}{rlcl} \text{either} & f(c) = \max\limits_{x\in(a,b)} f(x) & \iff & \forall\, x \in (a,b) \quad f(x) \le f(c) \\ \text{or} & f(c) = \min\limits_{x\in(a,b)} f(x) & \iff & \forall\, x \in (a,b) \quad f(x) \ge f(c) \end{array}$$

   If f is differentiable at c then $f'(c) = 0$.

2. **Rolle's theorem**

   THEOREM. Assume $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Assume that $f(a) = f(b)$. Then there exists $c \in (a,b)$ such that $f'(c) = 0$.

3. **Lagrange's theorem**

   THEOREM. Assume $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there is a point $c \in (a,b)$ such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

4. **Cauchy's theorem**

   THEOREM. Assume functions $f,\, g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). Assume $g'(x) \neq 0$ for any $x \in (a,b)$. Then there is a point $c \in (a,b)$ such that

   $$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

5. **Functions with identically zero derivatives are constants**

   THEOREM. Assume $f : (a,b) \to \mathbb{R}$ is differentiable on (a,b) and $f'(x) = 0$ for any $x \in (a,b)$. Then f is equal to a constant, i.e.

   $$\exists\, y_0 \in \mathbb{R} : \qquad f(x) = y_0 \qquad \forall\, x \in (a,b)$$

1.11 Investigation of functions using derivatives

1. **Investigation of the monotonicity of functions**

   Theorem. Assume $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Then

   1) $f'(x) \geq 0 \;\; \forall x \in (a, b) \iff f - \nearrow$ (is non-decreasing) on (a, b)
   2) $f'(x) \leq 0 \;\; \forall x \in (a, b) \iff f - \searrow$ (is non-increasing) on (a, b)
   3) $f'(x) > 0 \;\; \forall x \in (a, b) \implies f - \uparrow$ (is strictly increasing) on (a, b)
   4) $f'(x) < 0 \;\; \forall x \in (a, b) \implies f - \downarrow$ (is strictly decreasing) on (a, b)

2. **Conditions for extreme in terms of the first derivative**

   Theorem. Assume $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Then and assume $c \in (a, b)$ satisfies $f'(c) = 0$. Then

   1) if $f'(x) \leq 0 \; \forall\, x \in (a, c)$ and $f'(x) \geq 0 \; \forall\, x \in (c, b)$ then c is the point of minimum of f
   2) if $f'(x) \geq 0 \; \forall\, x \in (a, c)$ and $f'(x) \leq 0 \; \forall\, x \in (c, b)$ then c is the point of maximum of f

3. **Higher order derivatives**

   Definition. Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). If the function $g = f' : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ then the function f is called *twice differentiable* at x_0. In this case the derivative of the function $g = f'$ at x_0 is called the *second derivative* of f at x_0.

   $$f''(x_0) \;:=\; g'(x_0) \;=\; \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

   The higher-order derivatives are defined similarly:

   $$f''' = (f'')', \quad \ldots, \quad f^{(k)} = (f^{(k-1)})', \quad \ldots$$

   Alternative notation.

   $$\frac{d^2 f}{dx^2}(x) = f''(x), \qquad \frac{d^3 f}{dx^3}(x) = f'''(x), \quad \ldots \quad \frac{d^k f}{dx^k}(x) = f^{(k)}(x)$$

4. **Conditions for extreme in terms of the second derivative**

   Theorem. Assume $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Let f be twice differentiable at $c \in (a, b)$ and $f'(c) = 0$. Then

   1) $f''(c) > 0 \implies c$ is a point of a local min of $f \;\Leftrightarrow\; \exists\, \delta > 0\colon\; f(c) = \min\limits_{x \in (x_0 - \delta, x_0 + \delta)} f(x)$
   2) $f''(c) < 0 \implies c$ is a point of a local max of $f \;\Leftrightarrow\; \exists\, \delta > 0\colon\; f(c) = \max\limits_{x \in (x_0 - \delta, x_0 + \delta)} f(x)$

5. **Convex functions**

DEFINITION. $f : [a, b] \to \mathbb{R}$ is *convex* on $(a, b) \iff \forall\, x_1,\, x_2 \in (a, b),\ \forall\, \lambda \in [0, 1]$

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \;\le\; (1-\lambda)f(x_1) + \lambda f(x_2)$$

$f : [a, b] \to \mathbb{R}$ is *strictly convex* on $(a, b) \iff \forall\, x_1 \ne x_2 \in (a, b),\ \forall\, \lambda \in (0, 1)$

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \;<\; (1-\lambda)f(x_1) + \lambda f(x_2)$$

$f : [a, b] \to \mathbb{R}$ is *concave* on $(a, b) \iff \forall\, x_1,\, x_2 \in (a, b),\ \forall\, \lambda \in [0, 1]$

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \;\ge\; (1-\lambda)f(x_1) + \lambda f(x_2)$$

$f : [a, b] \to \mathbb{R}$ is *strictly concave* on $(a, b) \iff \forall\, x_1 \ne x_2 \in (a, b),\ \forall\, \lambda \in (0, 1)$

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \;>\; (1-\lambda)f(x_1) + \lambda f(x_2)$$

6. **Inequalities for slopes of secants**

THEOREM.

1) f is convex on $(a, b) \iff \forall\, x_1,\, x_2 \in (a, b),\ \ x_1 < x_2,\ \ \forall\, x \in (x_1, x_2)$

$$\frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x_2) - f(x)}{x_2 - x}$$

2) f is strictly convex on $(a, b) \iff \forall\, x_1,\, x_2 \in (a, b),\ \ x_1 < x_2,\ \ \forall\, x \in (x_1, x_2)$

$$\frac{f(x) - f(x_1)}{x - x_1} < \frac{f(x_2) - f(x)}{x_2 - x}$$

3) Let f be differentiable on (a, b). Then f is convex on $(a, b) \iff \forall\, x_1,\, x_2 \in (a, b)$

$$x_1 < x_2 \quad \Longrightarrow \qquad f'(x_1) \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le f'(x_2)$$

7. **Investigation of convexity using the first derivative**

THEOREM. Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b).

1) f is convex on $(a, b) \iff f' - \nearrow$ on (a, b)
2) f is concave on $(a, b) \iff f' - \searrow$ on (a, b)
3) f is strictly convex on $(a, b) \iff f' - \uparrow$ on (a, b)
4) f is strictly concave on $(a, b) \iff f' - \downarrow$ on (a, b)

8. **Investigation of convexity using the second derivative**

THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is twice differentiable on (a, b).

1) $\forall\, x \in (a, b) \quad f''(x) \geq 0 \iff f$ is convex on (a, b)
2) $\forall\, x \in (a, b) \quad f''(x) \leq 0 \iff f$ is concave on (a, b)
3) $\forall\, x \in (a, b) \quad f''(x) > 0 \implies f$ is strictly convex on (a, b)
4) $\forall\, x \in (a, b) \quad f''(x) < 0 \implies f$ is strictly concave on (a, b)

9. **Tangent line to the graph of a function**

DEFINITION. Assume $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$. The straight line given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is called the *tangent* to the function $y = f(x)$ (or to the graph of f) at the point x_0.

10. **Convexity and tangent lines**

THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b).

1) f is convex on $(a, b) \iff$ for any $x_0 \in (a, b)$ the graph of the function f lies above the tangent to f at the point x_0, i.e.

$$\forall x \in (a, b) \quad f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

2) f is concave on $(a, b) \iff$ for any $x_0 \in (a, b)$ the graph of the function f lies below the tangent to f at the point x_0, i.e.

$$\forall x \in (a, b) \quad f(x) \leq f(x_0) + f'(x_0)(x - x_0)$$

3) f is strictly convex on $(a, b) \iff \forall\, x_0 \in (a, b), \forall\, x \in (a, b)$,

$$x \neq x_0 \quad \implies \quad f(x) > f(x_0) + f'(x_0)(x - x_0)$$

4) f is strictly concave on $(a, b) \iff \forall\, x_0 \in (a, b), \forall\, x \in (a, b)$,

$$x \neq x_0 \quad \implies \quad f(x) < f(x_0) + f'(x_0)(x - x_0)$$

11. **Inflection points**

DEFINITION. Assume $f : (a, b) \to \mathbb{R}$. The point $c \in (a, b)$ is called the *inflection point* of the function f if $\exists\, \delta > 0$, such that on the intervals $(c - \delta, c)$ and $(c, c + \delta)$ the function f changes convexity (was convex, became concave, or vice versa).

12. **Necessary condition for an inflection point**

THEOREM. Let $f : (a, b) \to \mathbb{R}$ be twice differentiable on (a, b) and let $c \in (a, b)$ be the inflection point of f. Then

$$f''(c) = 0.$$

1.12 Primitive function and indefinite integral

1. **Primitive function (antiderivative)**

   DEFINITION. $F : (a,b) \to \mathbb{R}$ is called a *primitive function* (or an *antiderivative*) of the function $f : (a,b) \to \mathbb{R}$ on the interval (a,b) if F is differentiable on (a,b) and

   $$F'(x) = f(x), \qquad \forall\, x \in (a,b).$$

2. **Which functions possess antiderivatives?**

   This is a non-trivial question. In the section "Riemann integral" we will show that at least any *continuous* function possesses antiderivatives on the interval of its continuity.

   Discussing antiderivatives below we will always assume that all functions involved in the formulas possess the corresponding antiderivatives by assumption and we will focus only on the derivation of relations between these antiderivatives.

3. **Properties of primitives**

   THEOREM. Assume $f : (a,b) \to \mathbb{R}$ has a primitive on (a,b). Note that in this case f has infinitely many other primitives on (a,b), but any two primitives of f can differ only by a constant:

   1) if $F(x)$ is a primitive of $f(x)$ on (a,b) then for any $c \in \mathbb{R}$ the function $F(x)+c$ is also a primitive of $f(x)$ on (a,b)
   2) if F_1 and F_2 are two primitives of f on (a,b) then there exists $c \in \mathbb{R}$ such that

   $$\forall\, x \in (a,b) \qquad F_1(x) - F_2(x) = c$$

4. **Indefinite integral**

   DEFINITION. The set $\{F(x)+C\}$ of all primitives of the function $f : (a,b) \to \mathbb{R}$, defined on the interval (a,b), is called the indefinite integral of f on (a,b) and is denoted by

   $$\int f(x)\ dx$$

5. **Properties of the indefinite integral**

   THEOREM.

   $$\int \Big(\alpha f(x) + \beta g(x)\Big)\, dx \;=\; \alpha \int f(x)\, dx \;+\; \beta \int g(x)\, dx, \qquad \forall\, \alpha, \beta \in \mathbb{R}$$

   $$\frac{d}{dx}\int f(x)\ dx \;=\; f(x), \qquad \int g'(x)\, dx \;=\; g(x) + C$$

6. **List of simplest antiderivatives**

    1. $\int x^n \; dx = \frac{x^{n+1}}{n+1} + C$
    2. $\int x^r \; dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
    3. $\int \frac{dx}{x} = \ln|x| + C$
    4. $\int a^x \; dx = \frac{a^x}{\ln a} + C \quad (a > 0,\, a \neq 1)$
    5. $\int \sin x \; dx = -\cos x + C, \quad \int \cos x \; dx = \sin x + C$
    6. $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C, \quad \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$
    7. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C \quad (a \neq 0)$
    8. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C \quad (a > 0)$

7. **Antiderivatives computed by standard methods (see the next section)**

    1. $\int \operatorname{tg} x \; dx = -\ln|\cos x| + C, \quad \int \operatorname{ctg} x \; dx = \ln|\sin x| + C$
    2. $\int \frac{x\,dx}{a^2 \pm x^2} = \pm\frac{1}{2} \ln|a^2 \pm x^2| + C$
    3. $\int \frac{x\,dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2} + C \quad (a > 0)$
    4. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C \quad (a > 0)$
    5. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C \quad (a \neq 0)$
    6. $\int \ln x \; dx = x(\ln x - 1) + C$
    7. $\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$
    8. $\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C, \quad \int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$
    9. $\int \frac{dx}{\sin x} = \ln\left|\operatorname{tg} \frac{x}{2}\right| + C$

8. **Non-elementary integrals (can not be computed in terms of elementary functions)**

    1. $\mathrm{Ei}(x) = \int \frac{e^x}{x} \; dx$ (exponential integral)
    2. $\mathrm{Si}(x) = \int \frac{\sin x}{x} \; dx, \quad \mathrm{Ci}(x) = \int \frac{\cos x}{x} \; dx$ (sine and cosine integrals)
    3. $\mathrm{S}(x) = \int \sin x^2 \; dx, \quad \mathrm{C}(x) = \int \cos x^2 \; dx$ (Fresnel integral)
    4. $\Phi(x) = \int e^{-x^2} \; dx$ (Gaussian integral)
    5. $\mathrm{li}(x) = \int \frac{dx}{\ln x}$ (logarithmic integral)

1.13 Basic methods for finding primitives

1. **Notation for the differential of a function**

$$df(x) \; = \; f'(x)\,dx$$

2. **Change of variables**

THEOREM. If $f : (a,b) \to \mathbb{R}$ and $\varphi : (\alpha,\beta) \to (a,b)$ is differentiable on (α,β) then

$$\int f(\underbrace{\varphi(x)}_{y})\,\underbrace{\varphi'(x)\,dx}_{dy} \; = \; \left[\begin{array}{c} y = \varphi(x) \\ dy = \varphi'(x)\,dx \end{array}\right] \; = \; \int f(y)\,dy\,\Big|_{y=\varphi(x)}$$

ABBREVIATED NOTATION.

$$\int f(\varphi(x))\,d\varphi(x) \; = \; \int f(y)\,dy\,\Big|_{y=\varphi(x)}$$

EXAMPLES.

1) $\int \frac{dx}{x+a} \; = \; \ln|x+a| + C$
2) $\int \cos(ax)\,dx \; = \; \frac{1}{a}\,\sin(ax) + C, \qquad a \neq 0$
3) $\int \operatorname{tg} x\; dx = -\ln|\cos x| + C, \qquad \int \operatorname{ctg} x\; dx = \ln|\sin x| + C$
4) $\int \frac{x}{x^2+a}\,dx = \frac{1}{2}\ln|x^2+a| + C, \qquad a \in \mathbb{R}$
5) $\int \frac{x}{\sqrt{x^2+a}}\,dx = \sqrt{x^2+a} + C, \qquad a \in \mathbb{R}$
6) $\int \frac{dx}{\sqrt{x^2+a}} = \ln\left|x + \sqrt{x^2+a}\right| + C, \qquad a \in \mathbb{R}$

PROOFS.

1.

$$\int \frac{dx}{x+a} \; = \; \int \frac{d(x+a)}{x+a} \; = \; \left[\begin{array}{c} y = x+a \\ dy = dx \end{array}\right] \; = \; \int \frac{dy}{y} \; = \; \ln|y| + C \; = \; = \; \left[\; y = x+a \;\right] \; = \; \ln|x+a| + C$$

2.

$$\int \cos(ax)\,dx \; = \; \frac{1}{a}\int \cos(ax)\,d(ax) \; = \; \left[\begin{array}{c} y = ax \\ dy = a\,dx \end{array}\right] \; = \; \frac{1}{a}\int \cos y\,dy \; = \; \frac{1}{a}\,\sin y + C \; = \; = \; \left[\; y = ax \;\right] \; = \; \frac{1}{a}\,\sin(ax) + C$$

3.

$$\int \operatorname{tg} x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{d(\cos x)}{\cos x}\, dx = \left[\begin{array}{c} y = \cos x \\ dy = -\sin x\, dx \end{array} \right] =$$
$$= -\int \frac{dy}{y} = -\ln|y| + C = \left[\ y = \cos x\ \right] = -\ln|\cos x| + C$$

4.

$$\int \frac{x}{x^2+a}\, dx = \frac{1}{2}\int \frac{d(x^2+a)}{x^2+a} = \left[\begin{array}{c} y = x^2+a \\ dy = 2x\, dx \end{array} \right] = \frac{1}{2}\int \frac{dy}{y} = \frac{1}{2}\ln|y| + C =$$
$$= \left[\ y = x^2+a\ \right] = \frac{1}{2}\ln|x^2+a| + C$$

5.

$$\int \frac{x}{\sqrt{x^2+a}}\, dx = \frac{1}{2}\int \frac{d(x^2+a)}{\sqrt{x^2+a}} = \left[\begin{array}{c} y = x^2+a \\ dy = 2x\, dx \end{array} \right] = \frac{1}{2}\int \frac{dy}{\sqrt{y}} = \sqrt{y} + C =$$
$$= \left[\ y = x^2+a\ \right] = \sqrt{x^2+a} + C$$

6.

$$\int \frac{dx}{\sqrt{x^2+a}} = \int \frac{1}{x+\sqrt{x^2+a}}\,\frac{x+\sqrt{x^2+a}}{\sqrt{x^2+a}}\, dx = \int \frac{1}{x+\sqrt{x^2+a}}\left(1+\frac{x}{\sqrt{x^2+a}}\right) dx =$$
$$= \left[\begin{array}{c} y = x+\sqrt{x^2+a} \\ dy = \left(1+\frac{x}{\sqrt{x^2+a}}\right) dx \end{array} \right] = \int \frac{dy}{y} = \ln|y| + C =$$
$$= \left[\ y = x+\sqrt{x^2+a}\ \right] = \ln(x+\sqrt{x^2+a}) + C$$

3. **Decomposition of an integrand into a sum of elementary functions**

THEOREM. If $f = f_1 + f_2$ on (a, b) then

$$\int f(x)\ dx = \int f_1(x)\ dx + \int f_2(x)\ dx$$

EXAMPLES.

1) $\int \frac{x+a}{x+b}\, dx = x + (a-b)\ln|x+b| + C$

2) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad a \neq 0$

PROOFS.

1.

$$\frac{x+a}{x+b} = \frac{x+b-b+a}{x+b} = 1 + \frac{a-b}{x+b}$$

$$\int \frac{x+a}{x+b}\,dx \;=\; \int dx \;+\; (a-b)\int \frac{dx}{x+b} \;=\; x+(a-b)\ln|x+b|+C$$

2.

$$\frac{1}{a^2-x^2} \;=\; \frac{1}{2a}\cdot\frac{(a-x)+(a+x)}{a^2-x^2} \;=\; \frac{1}{2a}\left(\frac{1}{a+x}+\frac{1}{a-x}\right)$$

$$\int \frac{dx}{a^2-x^2} \;=\; \frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx \;=\; \frac{1}{2a}\big(\ln|a+x|-\ln|a-x|\big)+C \;=$$

$$= \frac{1}{2a}\,\ln\left|\frac{a+x}{a-x}\right|+C$$

4. **Integrations by parts**

THEOREM.

$$\int f(x)g'(x)\,dx \;=\; f(x)g(x) \;-\; \int f'(x)g(x)\,dx$$

ABBREVIATED NOTATION.

$$\int f\,dg \;=\; fg \;-\; \int g\,df$$

EXAMPLES.

1) $\int \ln x\;dx = x(\ln x - 1)+C$
2) $\int x\;\cos x\;dx = x\sin x + \cos x + C$
3) $\int x\,e^x\;dx = xe^x - e^x + C$
4) $\int x\;\ln x\;dx = \frac{x^2}{2}\;\ln x \;-\; \frac{x^2}{4}+C$
5) $\int e^x \sin x\,dx = \frac{1}{2}(\sin x - \cos x)\,e^x + C,\qquad \int e^x\cos x\,dx = \frac{1}{2}(\sin x+\cos x)\,e^x+C,$

PROOFS.

1.

$$\int \ln x\,dx \;=\; [\text{ by parts }] \;=\; x\ln x \;-\; \int x\,d(\ln x) \;=\; x\ln x \;-\; \int x\,\frac{1}{x}\,dx \;=\; x\ln x - x + C$$

2.

$$\int x\;\cos x\;dx \;=\; \int x\,d(\sin x) \;=\; [\text{ by parts }] \;=\; x\sin x - \int \sin x\,dx \;=\; x\sin x+\cos x + C$$

3.

$$\int x\,e^x\;dx \;=\; \int x\,d(e^x) \;=\; [\text{ by parts }] \;=\; xe^x - \int e^x\,dx \;=\; xe^x - e^x + C$$

4.

$$\int x\,\ln x\,dx = \int \ln x\,d\left(\tfrac{x^2}{2}\right) = [\text{ by parts }] = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\underbrace{d(\ln x)}_{\frac{1}{x}\,dx} =$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

5.

$$\int e^x \sin x\,dx = \int \sin x\,d(e^x) = [\text{ by parts }] = \sin x\,e^x - \int e^x\,\underbrace{d(\sin x)}_{\cos x\,dx} =$$

$$= \sin x\,e^x - \int e^x \cos x\,dx = \sin x\,e^x - \int \cos x\,d(e^x) = [\text{ again by parts }] =$$

$$= \sin x\,e^x - \cos x\,e^x + \int e^x\,\underbrace{d(\cos x)}_{-\sin x\,dx} = \sin x\,e^x - \cos x\,e^x - \int e^x \sin x\,dx$$

So, we obtain

$$\int e^x \sin x\,dx = \frac{\sin x - \cos x}{2} + C$$

5. **Method of substitution**

THEOREM. Assume $f : (a,b) \to \mathbb{R}$ and $\varphi : (\alpha,\beta) \to (a,b)$ is differentiable and invertible on (α,β). Then

$$\int f(x)\,dx = \left[\begin{array}{c} x = \varphi(t) \\ dx = \varphi'(t)\,dt \end{array}\right] = \int f(\varphi(t))\varphi'(t)\,dt\,\Big|_{t=\varphi^{-1}(x)}$$

EXAMPLES.

1) $\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$

PROOFS.

1.

$$\int \frac{dx}{(1-x^2)^{3/2}} = \left[\begin{array}{c} x = \sin t \\ dx = \cos t\,dt \\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{array}\right] = \int \frac{\cos t}{(1-\sin^2 t)^{3/2}}\,dt = \int \frac{dt}{\cos^2 t} =$$

$$= \operatorname{tg} t + C = [\ t = \arcsin x\ ] = \operatorname{tg}(\arcsin x) + C =$$

$$= \left[\begin{array}{c} \operatorname{tg} t = \frac{\sin t}{\sqrt{1-\sin^2 t}} \\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{array}\right] = \frac{x}{\sqrt{1-x^2}} + C$$

6. **Using trigonometrical formulas**

Power-reduction formulas:

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Double angle formulas:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

Example.

1) $\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C,\qquad \int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$
2) $\int \frac{dx}{\sin x} = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C,\qquad \int \frac{dx}{\cos x} = -\ln\left|\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C$

Proofs.

1.

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

2.

$$\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{2}\int \frac{dx}{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\cos^2\frac{x}{2}} = \frac{1}{2}\int \frac{dx}{\operatorname{tg}\frac{x}{2}\cdot\cos^2\frac{x}{2}} =$$

$$= \left[\begin{array}{c} y = \operatorname{tg}\frac{x}{2} \\ dy = \frac{dx}{2\cos^2\frac{x}{2}} \end{array}\right] = \int \frac{dy}{y} = \ln|y| + C = \left[\, y = \operatorname{tg}\frac{x}{2}\,\right] = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(\frac{\pi}{2}-x\right)} = \left[\begin{array}{c} y = \frac{\pi}{2} - x \\ dy = -dx \end{array}\right] = -\int \frac{dy}{\sin y} = -\ln\left|\operatorname{tg}\frac{y}{2}\right| + C =$$

$$= \left[\, y = \frac{\pi}{2} - x\,\right] = -\ln\left|\operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + C$$

1.14 Definite integral

1. **A partition of an interval**

   A *partition* of an interval $[a, b]$ is a finite sequence of numbers $T = \{x_i\}_{i=0}^N$ such that

   $$a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b$$

   The *mesh* $\lambda(T)$ of a partition T is the length of the longest interval in this partition:

   $$\lambda(T) := \max_{i=1,\ldots,N} \Delta x_i, \qquad \Delta x_i := x_i - x_{i-1}$$

   Let $T = \{x_i\}_{i=0}^N$ be a partition of $[a, b]$. Choose a finite set of points $\xi = \{\xi_i\}_{i=1}^N$ so that

   $$\xi_i \in [x_{i-1}, x_i], \qquad i = 1, \ldots, N$$

   The pair (T, ξ) is called a *partition with distinguished points* or a *tagged partition* of $[a, b]$.

2. **Riemann sum**

   Definition. Assume $f : [a, b] \to \mathbb{R}$ and let (T, ξ) be a tagged partition of $[a, b]$, where

   $$T = \{x_i\}_{i=0}^N, \qquad \xi = \{\xi_i\}_{i=1}^N, \qquad \xi_i \in [x_{i-1}, x_i].$$

   The *Riemann sum* of the function f corresponding to the tagged partition (T, ξ) is the sum

   $$\sigma(T; \xi) := \sum_{i=1}^N f(\xi_i)\, \Delta x_i, \qquad \Delta x_i := x_i - x_{i-1}$$

3. **Riemann integral**

   Definition. A function $f : [a, b] \to \mathbb{R}$ is called *Riemann-integrable* on $[a, b]$ if there exists a finite limit

   $$\exists\, I \in \mathbb{R} : \qquad I = \lim_{\lambda(T)\to 0} \sum_{i=1}^N f(\xi_i)\, \Delta x_i,$$

   as the mesh $\lambda(T)$ of the partition tends to zero, and this limit does not depend neither on the method of partitioning the interval $[a, b]$ by points $T = \{x_i\}_{i=0}^N$, nor on the choice of points ξ_i in the intervals $[x_{i-1}, x_i]$. If such limit exists, then the number I is called the *definite integral* (or the *Riemann integral*) of the function f over the interbal $[a, b]$ and is denoted by

   $$\int_a^b f(x)\, dx := I$$

4. **Refinement of the definition of the Riemann integral**

The definition of the integral above is an informal one: we know only the definition of a function depending on some variable. But the Riemann sum is not a function of a *mesh* of the partition, since it also depends on the choice of points x_j and ξ_j. The rigorous definition of the integral can be done only using ε–δ formalism: the function f is called *integrable* on $[a,b]$ if there exists a number $I \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any partition (T,ξ) of the interval $[a,b]$ with $T = \{x_i\}_{i=0}^{N}$ and $\xi = \{\xi_i\}_{i=1}^{N}$, $\xi_i \in [x_{i-1}, x_i]$, the following implication holds:

$$\lambda(T) < \delta \qquad \Longrightarrow \qquad \left| \, I - \sum_{i=1}^{N} f(\xi_i)\Delta x_i \, \right| \; < \; \varepsilon$$

5. **The geometric meaning of the definite integral**

   - For a non-negative function f, the Riemann sum is the sum of the areas of rectangles with bases $[x_{i-1}, x_i]$ and heights $f(\xi_i)$.
   - For a non-negative function f, the definite integral is equal to the area of the figure bounded by the graph of the function f, the axis OX, and the parallel lines $x = a$, $x = b$.
   - For an sign-indefinite function f, the definite integral is equal to the algebraic sum of the areas of the figures formed by the parts of the graph f lying above the axis OX (taken with a “plus” sign) and the ones lying below the axis OX (taken with a “minus” sign).

6. **Example of a non-integrable function**

In the next section we will show that any continuous or piecewise-continuous function is integrable. A typical example of a non-integrable function is the following function (which is discontinuous at every point):

$$f(x) \; = \; \begin{cases} 1, & x \in [a,b] \cap \mathbb{Q} \\ 0, & x \in [a,b] \setminus \mathbb{Q} \end{cases}$$

7. **Integral with “upside down” limits**

$$a < b \qquad \Longrightarrow \qquad \int\limits_{b}^{a} f(x)\, dx \;\stackrel{\text{def}}{=}\; -\int\limits_{a}^{b} f(x)\, dx$$

8. **Integrable functions are bounded**

Theorem. f is integrable on $[a,b] \;\Longrightarrow\; f$ is bounded on $[a,b]$

9. **Darboux sums**

Definition. Assume $f : [a,b] \to \mathbb{R}$ is bounded on $[a,b]$ and $T = \{x_j\}_{j=1}^N$ is a partition of the interval $[a,b]$, i.e.

$$a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b$$

Denote

$$m_j = \inf_{x \in [x_{j-1}, x_j]} f(x), \qquad M_j = \sup_{x \in [x_{j-1}, x_j]} f(x), \qquad j = 1, \ldots, N.$$

The *lower* and the *upper Darboux sums* corresponding to the partition T are

$$s(T) := \sum_{j=1}^N m_j \, \Delta x_j, \qquad S(T) := \sum_{j=1}^N M_j \, \Delta x_j, \qquad \Delta x_j := x_j - x_{j-1}.$$

10. **Refinement and common refinement of partitions**

Definition. A partition $\tilde{T} = \{y_j\}_{j=1}^N$ is the *refinement* of a partition $T = \{x_i\}_{i=1}^m$ if all points included in T are also included in $\tilde{T}$, i.e. $x_i = y_{n_i}$ for some $n_1 < n_2 < \ldots < n_m$.

The partition $T_1 \cdot T_2$ is the *common refinement* of partitions T_1 and T_2 if it is formed by all points included in T_1 or T_2.

The common refinement of two partitions is a refinement of each of the original partitions.

11. **Properties of the Darboux sums**

1) for any tagged partition the Riemann sum $\sigma(T, \xi)$ is in between the corresponding Darboux sums $s(T)$ and $S(T)$: $\forall\, T = \{x_j\}_{j=0}^N$, $\forall\, \xi = \{\xi_j\}_{j=1}^N \quad s(T) \le \sigma(T, \xi) \le S(T)$
2) if T_1 is a refinement of T then $\quad s(T_1) \ge s(T), \quad S(T_1) \le S(T)$
3) $\forall$ partitions T_1 and $T_2 \quad \Longrightarrow \quad s(T_1) \le S(T_2)$
4) for any fixed partition $T = \{x_j\}_{j=1}^N$ the infimum and supremum of all possible values of Riemanns' sums $\sigma(T; \xi)$ are equal respectively to the lower and upper Darboux sums $s(T)$ and $S(T)$

$$s(T) = \inf_{\xi} \sigma(T; \xi), \qquad S(T) = \sup_{\xi} \sigma(T; \xi)$$

where the infimum is taken over all sets of points $\xi = \{\xi_j\}_{j=1}^N$ such that $\xi_j \in [x_{j-1}, x_j]$.

12. **The basic condition for Riemann integrability**

Theorem. Assume f is bounded on $[a,b]$. Then f is integrable on $[a,b]$ if and only if $\forall\, \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0$ such that for any partition T of $[a,b]$ the following implication holds

$$\lambda(T) < \delta \quad \Longrightarrow \quad S(T) - s(T) < \varepsilon,$$

where $\lambda(T)$ is the mesh of the partition T.

13. Oscillation and the Darboux sums

NOTATION. For any partition T of the interval $[a, b]$ we have

$$S(T) - s(T) = \sum_{j=1}^{N} \omega_i(f)\,\Delta x_j, \qquad \omega_j(f) := M_j - m_j, \qquad \Delta x_j := x_j - x_{j-1}$$

The number $\omega_j(f)$ is called the *oscillation* of f over the interval $[x_{j-1}, x_j]$.

14. Continuous functions are integrable

THEOREM. Assume f is continuous on $[a, b]$. Then f is integrable on $[a, b]$.

1.15 Properties of definite integrals

1. **Basic properties**

   1. f, g are integrable on $[a,b]$, $\alpha,\ \beta \in \mathbb{R} \implies \alpha f + \beta g$ is integrable on $[a,b]$,

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx \;=\; \alpha \int_a^b f(x)\,dx \;+\; \beta \int_a^b g(x)\,dx$$

   2. $a < c < b \implies f$ is integrable on $[a,b] \Leftrightarrow f$ is integrable on $[a,c]$ and $[c,b]$,

$$\int_a^b f(x)\;dx \;=\; \int_a^c f(x)\,dx \;+\; \int_c^b f(x)\,dx$$

   3. f, g are integrable on $[a,b]$, $f(x) \le g(x)$, $\forall x \in [a,b] \implies$

$$\int_a^b f(x)\,dx \;\le\; \int_a^b g(x)\,dx$$

   4. f is integrable on $[a,b] \implies |f|$ is integrable on $[a,b]$ and

$$\Big|\int_a^b f(x)\,dx\Big| \;\le\; \int_a^b |f(x)|\,dx$$

   5. f is integrable on $[a,b]$, $m \le f(x) \le M$, $\forall\, x \in [a,b] \implies$

$$m\,(b-a) \;\le\; \int_a^b f(x)\,dx \;\le\; M\,(b-a)$$

   6. f is continuous on $[a,b] \implies \exists\, c \in [a,b]$:

$$f(c) \;=\; \frac{1}{b-a}\int_a^b f(x)\,dx \qquad \text{(Mean value theorem)}$$

   7. f is continuous on $[a,b]$, g is integrable and non-negative on $[a,b] \implies \exists\, c \in [a,b]$:

$$\int_a^b f(x)g(x)\,dx \;=\; f(c)\int_a^b g(x)\,dx$$

2. **Fundamental theorem of calculus**

THEOREM. Assume $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$. Define $\Phi : [a,b] \to \mathbb{R}$,

$$\Phi(x) := \int\limits_a^x f(t)\,dt, \quad x \in [a,b] \qquad \text{— the } \textit{integral with variable upper limit.}$$

If f is continuous on $[a,b]$ then Φ is differentiable on $[a,b]$ and

$$\forall\, x_0 \in (a,b) \qquad \Phi'(x_0) = f(x_0).$$

COROLLARY. $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b] \implies f$ has a primitive on (a,b).

3. **Newton–Leibniz formula**

THEOREM. Let f be continuous on $[a,b]$ and let F be any primitive of f on (a,b). Then

$$\int\limits_a^b f(x)\,dx = F(b) - F(a).$$

4. **Integration by parts in a definite integral**

THEOREM. Let f, g be differentiable on $[a,b]$ and f', g' be continuous on $[a,b]$. Then

$$\int\limits_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int\limits_a^b f'(x)g(x)\,dx.$$

ABBREVIATED NOTATION.

$$\int\limits_a^b fg'\,dx = fg\Big|_a^b - \int\limits_a^b f'g\,dx$$

5. **Change of variables in the Riemann integral**

THEOREM. Assume $\varphi : [\alpha,\beta] \to [a,b]$ is differentiable bijection of $[\alpha,\beta]$ onto $[a,b]$, φ' is continuous and strictly positive on $[\alpha,\beta]$. Assume f is continuous on $[a,b]$. Then the function $(f \circ \varphi) \cdot \varphi'$ is continuous on $[\alpha,\beta]$ and

$$\int\limits_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt = \left[\begin{array}{c} x = \varphi(t) \\ dx = \varphi'(t)\,dt \end{array}\right] = \int\limits_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

1.16 Applications of definite integrals

1. **Area of a plane figure**

   Definition. A subset of the plane F is called *Jordan measurable* if for any $\varepsilon > 0$ there exist two polygons F_1 and F_2 such that

   $$F_1 \subset F \subset F_2 \qquad \text{and} \qquad S(F_2 \setminus F_1) < \varepsilon$$

   (here for every polygon P on the plane we denote by $S(P)$ its area). For the Jordan measurable figure F its area is defined

   $$S(F) \;=\; \sup_{F_1 \subset F} S(F_1) \;=\; \inf_{F \subset F_2} S(F_2)$$

   where the supremum is taken over all polygons F_1 contained in F, and the infimum is taken over all polygons F_2 containing F. (These supremum and infimum are equal, since the figure is Jordan measurable).

2. **Area of a curved trapezoid**

   Theorem.

   1) $f : [a, b] \to \mathbb{R},\;\; f(x) \geq 0,\;\; \forall\, x \in [a, b],$ and assume $\mathcal{F} \subset \mathbb{R}^2$ is given by

   $$\mathcal{F} \;=\; \{\, (x, y) \in \mathbb{R}^2 : \, x \in [a, b],\;\; 0 \leq y \leq f(x) \,\}$$

   Then

   $$S_{\mathcal{F}} \;=\; \int\limits_a^b f(x)\, dx$$

   2) $f, g : [a, b] \to \mathbb{R},\;\; f(x) \geq g(x),\;\; \forall x \in [a, b],$ and assume $\mathcal{F} \subset \mathbb{R}^2$ is given by

   $$\mathcal{F} \;=\; \{\, (x, y) \in \mathbb{R}^2 : \, x \in [a, b],\;\; g(x) \leq y \leq f(x) \,\}$$

   Then

   $$S_{\mathcal{F}} \;=\; \int\limits_a^b \big(f(x) - g(x)\big)\, dx$$

3. **Polar coordinate on the plane**

   Converting from the polar coordinates r and φ to the Cartesian coordinates x and y:

   $$\begin{cases} x \;=\; r\,\cos\varphi & \quad r \in [0, +\infty) \\ y \;=\; r\,\sin\varphi & \quad \varphi \in [0, 2\pi) \end{cases}$$

   Converting from the Cartesian coordinates x and y to the polar coordinates r and φ:

   $$\begin{cases} r \;=\; \sqrt{x^2 + y^2} & \quad \varphi \in [0, \pi], \quad \text{if} \quad y \geq 0 \\ \operatorname{tg}\varphi \;=\; \dfrac{y}{x} & \quad \varphi \in (\pi, 2\pi), \quad \text{if} \quad y < 0 \end{cases}$$

4. **Area of a curved sector**

THEOREM. $f : [\alpha, \beta] \to [0, +\infty)$, $0 \le \alpha < \beta \le 2\pi$, and assume $\mathcal{F} \subset \mathbb{R}^2$ is given by

$$\mathcal{F} := \{ (r, \varphi) : \varphi \in [\alpha, \beta], \ 0 \le r \le f(\varphi) \}$$

Then

$$S_{\mathcal{F}} = \frac{1}{2} \int\limits_a^b (f(\varphi))^2 \, d\varphi$$

5. **Curves on the plane**

DEFINITION. Let $\gamma : [a, b] \to \mathbb{R}^2$, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be continuous on $[a, b]$

- The mapping $\gamma : [a, b] \to \mathbb{R}^2$ is called a *parameterized plane curve.*
- γ is *regular* if γ is differentiable on $[a, b]$, γ' is continuous on $[a, b]$ and

$$|\gamma'(t)| := \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2} \ne 0, \qquad \forall \, t \in [a, b].$$

- If $T = \{t_j\}_{j=0}^N$ is some partition of $[a, b]$, i.e. $a = t_0 < t_1 < t_2 < \ldots < t_N = b$, then a broken line consisting of segments $[\gamma(t_{j-1}), \gamma(t_j)]$ is said to be *inscribed* in the curve γ. The *length* of a broken line corresponding to the partition T is

$$\sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})|$$

where
$$|\gamma(t_j) - \gamma(t_{j-1})| := \sqrt{(\gamma_1(t_j) - \gamma_1(t_{j-1}))^2 + (\gamma_2(t_j) - \gamma_2(t_{j-1}))^2}$$

- A curve γ is called *rectifiable* if there is a limit on the lengths of all broken lines inscribed in this curve as the mesh of the partition tends to zero. In this case, the *length* is

$$l_\gamma = \lim_{\lambda(T) \to 0} \sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})|$$

- More precisely, a curve is called rectifiable if there is a number $l_\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that for any partition $T = \{t_j\}_{j=1}^N$ of the segment $[a, b]$ the implication is true

$$\lambda(T) < \delta \qquad \Longrightarrow \qquad \left| \, l_\gamma - \sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})| \, \right| < \varepsilon$$

Here $\lambda(T) = \max\limits_{j=1,\ldots,N} (t_j - t_{j-1})$ is the mesh of the partition T.

6. **Length of a curve**

THEOREM.

1) <u>Length of a parameterized curve</u>

Any regular parameterized curve $\gamma : [a,b] \to \mathbb{R}^2$ is rectifiable and its length is

$$l_\gamma \;=\; \int_a^b |\gamma'(t)|\,dt \;:=\; \int_a^b \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}\,dt$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t))$.

2) <u>Length of the curve that is the graph of the function</u>

$f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ and f' is continuous on $[a,b]$,

$$\Gamma \;:=\; \{\, (x,y) \in \mathbb{R}^2 :\; x \in [a,b],\; y = f(x) \,\}$$

Then

$$l_\Gamma \;=\; \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

3) <u>Length of the curve, specified in polar coordinates</u>

Let (r,φ) be polar coordinates on the plane $\mathbb{R}^2$, let $f : [\alpha,\beta] \to [0,+\infty)$ be continuously differentiable on $[\alpha,\beta]$, $0 \le \alpha < \beta \le 2\pi$, and

$$\Gamma \;:=\; \{\, (r,\varphi) :\; \varphi \in [\alpha,\beta],\; r = f(\varphi) \,\}$$

Then

$$l_\Gamma \;=\; \int_\alpha^\beta \sqrt{(f(\varphi))^2 + (f'(\varphi))^2}\,d\varphi$$

7. **Generalized Cavalieri's principle (1635)**

For any solid of revolution (such as a barrel)

$$\text{Volume of the barrel} \;\approx\; \sum_j \Big(\text{Area of a crossection of a barrel}\Big)_j \cdot \Big(\text{Vertical height}\Big)_j$$

$$\text{Area of a side surface} \;\approx\; \sum_j \Big(\text{Length of a crossection of a barrel}\Big)_j \cdot \Big(\text{Vertical }\underline{\text{length}}\Big)_j$$

8. **The volume of a solid of revolution**

THEOREM. Assume $f : [a,b] \to \mathbb{R}$ is non-negative and continuous on $[a,b]$ and

$$\Omega \;:=\; \Big\{ (x,y,z) \in \mathbb{R}^3 : \; x \in [a,b], \;\; y^2 + z^2 \leq f^2(x) \Big\}$$

Then the volume $V(\Omega)$ of a solid of revolution Ω is equal to

$$V(\Omega) \;=\; \pi \int\limits_a^b f^2(x)\,dx$$

9. **The area of the side surface of a circular cone**

THEOREM. The side surface of a circular cone whose base radius is R and height H:

$$S_{R,H} \;=\; \pi\, R\, \sqrt{R^2 + H^2}$$

For a truncated cone:

$$S_{R,H} - S_{r,h} \;=\; \pi \sqrt{1 + \operatorname{tg}^2 \alpha} \cdot (RH - rh) \;=\; 2\pi \sqrt{1 + \operatorname{tg}^2 \alpha} \cdot \frac{R + r}{2} \cdot (H - h),$$

where $\operatorname{tg}\alpha \;=\; \frac{r}{h} \;=\; \frac{R}{H}$.

10. **The area of the side surface of a solid of revolution**

THEOREM. Assume $f : [a,b] \to \mathbb{R}$ is non-negative and differentiable on (a,b), f' is continuous on $[a,b]$ and

$$\Sigma \;:=\; \Big\{ (x,y,z) \in \mathbb{R}^3 : \; x \in [a,b], \;\; y^2 + z^2 = f^2(x) \Big\}$$

Then the area of the side surface $S(\Sigma)$ the corresponding solid of revolution is equal to

$$S(\Sigma) \;=\; 2\pi \int\limits_a^b f(x)\,\sqrt{1 + (f'(x))^2}\,dx$$

1.17 Comparison of infinitesimals and Taylor's formula

1. **L'Hôpital's (actually Bernoulli's) rule for an indeterminate form $\frac{0}{0}$**

   THEOREM. Let $f, g : (a,b) \to \mathbb{R}$ be differentiable on (a,b) and $g'(x) \neq 0$, $\forall\, x \in (a,b)$. Assume

   1) $\exists\, c \in (a,b)$: $\quad f(c) = g(c) = 0$
   2) $\exists\, k \in \mathbb{R}$: $\quad \exists \lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = k$

   Then there exists the limit
   $$\exists \lim_{x\to c} \frac{f(x)}{g(x)} = k.$$

2. **Other versions of the L'Hôpital rule**

   There are also versions of the L'Hôpital rule which suit to the following cases:

   - Limit at infinity: $\quad f,\, g : [a, +\infty) \to \mathbb{R}, \quad \lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} g(x) = 0,$
   $$\exists\, k \in \mathbb{R} : \qquad \exists \lim_{x\to+\infty} \tfrac{f'(x)}{g'(x)} = k \qquad \Longrightarrow \qquad \exists \lim_{x\to+\infty} \tfrac{f(x)}{g(x)} = k$$
   - Indeterminate form $\frac{\infty}{\infty}$: $\quad f,\, g : (a,b) \to \mathbb{R}, \quad c \in (a,b), \quad \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to c} g(x) = +\infty,$
   $$\exists\, k \in \mathbb{R} : \qquad \exists \lim_{x\to c} \tfrac{f'(x)}{g'(x)} = k \qquad \Longrightarrow \qquad \exists \lim_{x\to c} \tfrac{f(x)}{g(x)} = k$$
   - Version of the above statements with $k = +\infty$ or $k = -\infty$
   - Version of the above statements with $x \to -\infty$ instead of $x \to +\infty$

   All these statements can be proved similarly to the proof of the first statement and we omit their proofs in our course. Informally we can summarize the L'Hôpital rules as follows:

The limit of a ratio of functions is equal to the limit of the ratio of their derivatives if the latter exists.

3. **Taylor's formula**

   THEOREM. $f : [a,b] \to \mathbb{R}$ is differentiable $(n+1)$ times on $(a,b) \quad \Longrightarrow \quad \forall\, x, x_0 \in (a,b)$
   $$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_{n+1}(x),$$

where

$$R_{n+1}(x) := \frac{1}{n!} \int\limits_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \; dt \quad \text{— the integral form of the remainder}$$

If $f^{(n+1)}$ is continuous on $[a,b]$ then there exists $\xi \in [x_0, x]$ such that

$$R_{n+1}(x) := \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1} \quad \text{— the Lagrange form of the remainder}$$

COROLLARY. $[a,b]$ is bounded, $f^{(n+1)}$ is continuous on $[a,b]$ $\implies$ $R_{n+1}(x) = O(|x-x_0|^{n+1})$

4. **O– asymptotic notation**

   Assume $f,\, g : (a,b) \to \mathbb{R}$ (we allow here a and b be infinite), $x_0 \in (a,b)$. Then

   - $g = O(f)$ as $x \in (a,b) \iff \exists\, M > 0$: $\forall\, x \in (a,b)$: $|g(x)| \le M\,|f(x)|$
   - $g = O(f)$ as $x \to x_0 \iff \exists\, M > 0, \delta > 0$: $\forall\, x \in (x_0-\delta, x_0+\delta)$: $|g(x)| \le M\,|f(x)|$
   - $g = O(f)$ as $x \to +\infty \iff \exists\, M > 0,\ R > 0$: $\forall\, x \ge R$: $|g(x)| \le M\,|f(x)|$
   - $g = O(f)$ as $x \to -\infty \iff \exists\, M > 0,\ R < 0$: $\forall\, x \le R$: $|g(x)| \le M\,|f(x)|$

5. **o– asymptotic notation**

   Assume $f,\, g : (a,b) \to \mathbb{R}$, $-\infty \le a \le x_0 \le b \le +\infty$.

   - $g = o(f)$ as $x \to x_0 \iff \exists\, \alpha : (a,b) \to \mathbb{R}$: $\lim\limits_{x \to x_0} \alpha(x) = 0$ and $g(x) = \alpha(x) f(x)$

6. **Properties of O– and o– symbols**

   THEOREM. Assume $f,\, g : (a,b) \to \mathbb{R}$, $-\infty \le a \le x_0 \le b \le +\infty$.

   1) for $x \in (a,b)$ we have
      - $O(f) + O(f) = O(f)$
      - $O(f) \cdot O(g) = O(fg)$
   2) for $x \to x_0$ we have
      - $g = o(f) \implies g = O(f)$
      - $o(f) + o(f) = o(f)$
      - $o(f) \cdot O(g) = o(fg)$
   3) in particular, for $x \to 0$ we have
      - $O(x^n) \cdot O(x^m) = O(x^{n+m})$
      - $o(x^n) \cdot O(x^m) = o(x^{n+m})$
      - $O(x^n) = o(x^m)$, if $n > m$

7. **Asymptotics of the reminder in Taylor's expansion**

THEOREM. Assume $f : (a,b) \to \mathbb{R}$ is differentiable $(n+1)$ times on (a,b) and $f^{(n+1)}$ is continuous on (a,b). Then for any $x_0 \in (a,b)$

$$R_{n+1}(x) \;=\; O\left((x-x_0)^{n+1}\right) \qquad \text{as} \quad x \to x_0,$$

where we denote by $R_{n+1}(x)$ the reminder in the Taylor formula for f.

8. **Taylor expansions of some elementary functions**

THEOREM. The following asymptotics holds as $x \to 0$:

1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + o(x^n)$
2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + o(x^{2n})$
3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$
4) $(1+x)^r = 1 + \frac{r}{1!}\,x + \frac{r(r-1)}{2!}\,x^2 + \frac{r(r-1)(r-2)}{3!}\,x^3 + \ldots + \frac{r(r-1)\ldots(r-n+1)}{n!}\,x^n + o(x^n)$
5) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$

1.18 Improper integrals

1. **Improper integral of an unbounded function**

   DEFINITION. Assume $f : [a, b) \to \mathbb{R}$ is unbounded near $x = b$ and integrable over $[a, c]$ for any $c \in [a, b)$. We say the *improper integral* of f over $[a, b)$ is *convergent* if there exists a finite limit

   $$\int_a^b f(x)\, dx = \lim_{c \to b-0} \int_a^c f(x)\, dx,$$

   Otherwise we say the corresponding improper integral is *divergent*. For $f : (a, b] \to \mathbb{R}$ which is unbounded near $x = a$ the improper integral is defined in a similar way:

   $$\int_a^b f(x)\, dx = \lim_{c \to a+0} \int_c^b f(x)\, dx,$$

2. **Improper integral over an unbounded interval**

   DEFINITION. Assume $f : [a, +\infty) \to \mathbb{R}$ is integrable over $[a, c]$ for any $c \in [a, +\infty)$. We say the *improper integral* of f over $[a, +\infty)$ is *convergent* if there exists a finite limit

   $$\int_a^{+\infty} f(x)\, dx = \lim_{c \to +\infty} \int_a^c f(x)\, dx,$$

   Otherwise we say the corresponding improper integral is *divergent*. For $f : (-\infty, b] \to \mathbb{R}$ the improper integral is defined in a similar way:

   $$\int_{-\infty}^b f(x)\, dx = \lim_{c \to -\infty} \int_c^b f(x)\, dx,$$

3. **Cauchy's criterion for convergence of improper integrals**

   THEOREM.

   1) Assume $F : [a, b) \to \mathbb{R}$. Then

   $$\exists \lim_{c \to b-0} F(c) \iff \forall \varepsilon > 0\ \exists \delta > 0 : \ \forall c_1, c_2 \in (b - \delta, b) \quad |F(c_1) - F(c_2)| < \varepsilon.$$

   2) In particular, for $f : [a, b) \to \mathbb{R}$

   $$\int_a^b f(x)\, dx \text{ is convergent} \iff \forall \varepsilon > 0\ \exists \delta > 0 : \ \forall c_1, c_2 \in (b - \delta, b) \quad \left| \int_{c_1}^{c_2} f(x)\, dx \right| < \varepsilon.$$

3) Assume $F : [a, +\infty) \to \mathbb{R}$. Then

$$\exists \lim_{c \to +\infty} F(c) \quad \Longleftrightarrow \quad \forall \varepsilon > 0 \;\; \exists R > 0 : \;\; \forall c_1, c_2 \in [R, +\infty) \quad |F(c_1) - F(c_2)| < \varepsilon.$$

4) In particular, for $f : [a, b) \to \mathbb{R}$

$$\int_a^{+\infty} f(x)\,dx \text{ is convergent} \;\; \Longleftrightarrow \;\; \forall \varepsilon > 0 \;\exists R > 0 : \;\; \forall c_1, c_2 \in [R, +\infty) \qquad \Big| \int_{c_1}^{c_2} f(x)\,dx \Big| < \varepsilon.$$

4. **Absolute convergence of improper integrals**

Definition. Assume $f : [a, b) \to \mathbb{R}$ is integrable over $[a, c]$ for any $c \in [a, b)$ and either the interval $[a, b)$ is finite and f is unbounded near $x = b$ or $b = +\infty$. We say the improper integral of f over the interval $[a, b)$ *converges absolutely* if the improper integral of $|f|$ over $[a, b)$ is convergent:

$$\int_a^b |f(x)|\,dx \; < \; +\infty.$$

5. **Absolutely convergent improper integral is convergent**

Theorem. If the improper integral of f over $[a, b)$, $b \leq +\infty$, converges absolutely then it converges:

$$\int_a^b |f(x)|\,dx \;\; \text{is convergent} \qquad \Longrightarrow \qquad \int_a^b f(x)\,dx \;\; \text{is convergent.}$$