# MA1-HW7

Stanislav
3720433

① $\int_{-1}^{8}\sqrt[3]{x}\,dx = \int_{-1}^{8} x^{\frac13}\,dx = \frac{x^{\frac43}}{\frac43}\Big|_{-1}^{8} = \frac34\left[8^{\frac43}-(-1)^{\frac43}\right] = \frac{45}{4} = 11\frac14.$

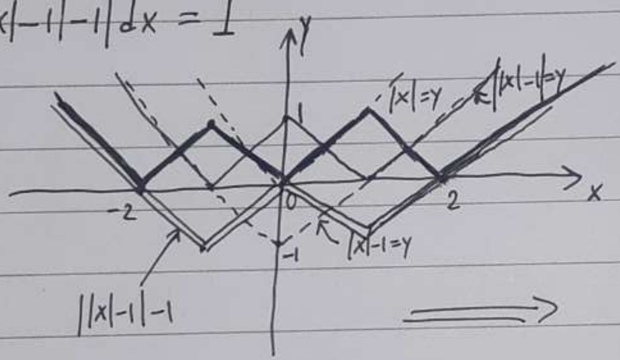
② $\int_0^{\ln 2} xe^{-x}\,dx = [\text{by parts}] = \int_0^{\ln 2} x\left(-e^{-x}\right)'dx = \left(-xe^{-x}\right)\Big|_0^{\ln 2} - \int_0^{\ln 2} -e^{-x}\,dx =$

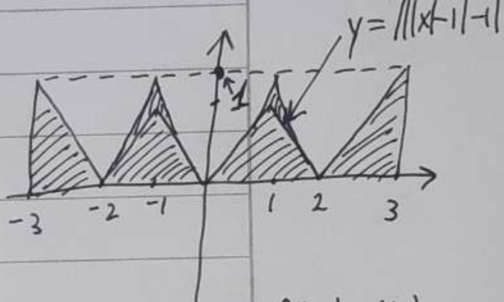
$= -\ln 2\, e^{-\ln 2} - e^{-x}\Big|_0^{\ln 2} = -\ln 2\, e^{-\ln 2} - \left[\underbrace{e^{-\ln 2}}_{\frac12} - 1\right] = -\frac{\ln 2}{2}+\frac12 = \frac12(1-\ln 2).$

③ $\int_{-1}^{1} \frac{dx}{x^2-x+1} = \int_{-1}^{1} \frac{dx}{\left(x-\frac12\right)^2+\left(\frac{\sqrt3}{2}\right)^2} = \frac{2}{\sqrt3}\operatorname{arctg}\left[\frac{2}{\sqrt3}\left(x-\frac12\right)\right]\Big|_{-1}^{1} = \frac{2}{\sqrt3}\left[\operatorname{arctg}\left(\frac{1\cdot 2}{2\sqrt3}\right) - \operatorname{arctg}\left(\frac{2}{\sqrt3}\cdot\left(-\frac32\right)\right)\right] =$

$= \frac{2}{\sqrt3}\left[\operatorname{arctg}\frac{1}{\sqrt3} - \operatorname{arctg}(-\sqrt3)\right] = \frac{2}{\sqrt3}\cdot\left[\frac{\pi}{6}+\frac{\pi}{3}\right] = \frac{\pi}{\sqrt3}.$

④ $\int_{-3}^{3} \left|\left||x|-1\right|-1\right|dx = I$



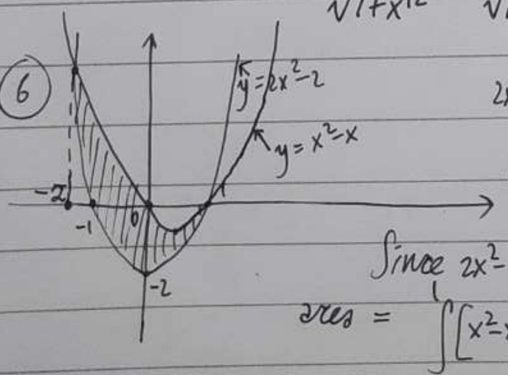

$f(-3)=f(3)=1$

$I = \left[\text{area of crossed figure, using geometric meaning of integral}\right] = 3\cdot\frac12\cdot 2 = \underline{3.}$

⑤ $\frac{d}{dx}\left[\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^4}}\right] = \left[\text{using } \frac{d}{dx}\int_{f(x)}^{g(x)} u(t)\,dt = u(g(x))\cdot g'(x) - u(f(x))\cdot f'(x)\right] = \frac{(x^3)'}{\sqrt{1+(x^3)^4}} - \frac{(x^2)'}{\sqrt{1+(x^2)^4}} =$

$= \frac{3x^2}{\sqrt{1+x^{12}}} - \frac{2x}{\sqrt{1+x^8}}.$

⑥




$2x^2-2=x^2-x$

$x^2+x-2=0$

$x_1=-2,\ x_2=1$

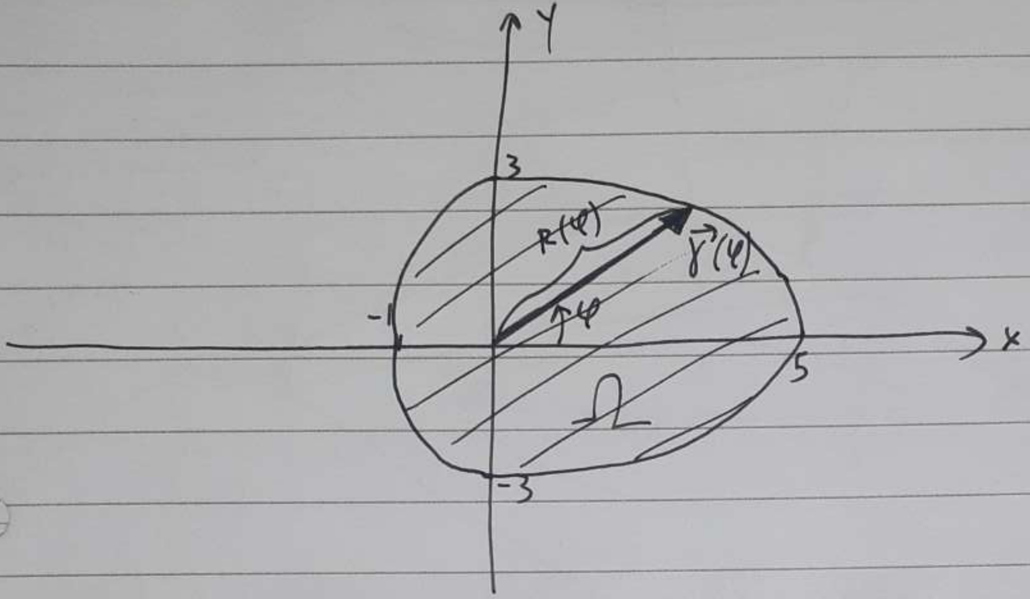
Since $2x^2-2 < x^2-x$ $\left(x^2+x-2<0\right)$ $\forall x\in(-2,1)$,

area $= \int_{-2}^{1}\left[x^2-x-(2x^2-2)\right]dx = \int_{-2}^{1} -x^2-x+2\,dx = -\frac{x^3}{3}-\frac{x^2}{2}+2x\Big|_{-2}^{1} = \left[-\frac13-\frac12+2\right] - \left[\frac83-\frac42-4\right] =$

$= \frac{-2-3+12-16+12+24}{6} = \frac{27}{6} = \underline{\frac92.}$

⑦ $\vec{\gamma} = \{(R,\varphi) : R = 3+2\cos\varphi,\ \varphi\in[0,2\pi]\}_{polar} =$

$$= \{(x,y) : x = R\cos\varphi,\ \varphi\in[0,2\pi],\ y = R\sin\varphi,\ R = 3+2\cos\varphi\}_{cartesian}$$




$$\vec{\gamma}(\varphi) = \begin{bmatrix} (3+2\cos\varphi)\cos\varphi \\ (3+2\cos\varphi)\sin\varphi \end{bmatrix} \text{ in cartesian, } \Omega = \{(R,\varphi) : R \le 3+2\cos\varphi,\ \varphi\in[0,2\pi]\} \text{ in polar}$$

$$S(\Omega) = \frac{1}{2}\int_{\varphi=0}^{2\pi} R(\varphi)^2\, d\varphi = \frac{1}{2}\int_{\varphi=0}^{2\pi}(3+2\cos\varphi)^2 d\varphi = \frac{1}{2}\int_{\varphi=0}^{2\pi} 9+4\cos^2\varphi+12\cos\varphi\, d\varphi =$$

$$= \frac{1}{2}\int_0^{2\pi} 9+\frac{4}{2}(1+\cos2\varphi)+12\cos\varphi\, d\varphi = \frac{1}{2}\Big[11\varphi+\underbrace{12\sin\varphi}_{\to 0}+\underbrace{\sin2\varphi}_{\to 0}\Big]_0^{2\pi} = \frac{11}{2}\cdot 2\pi = \underline{11\pi}.$$

⑧ $\vec{q}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t^2 \\ \frac{1}{3}t^3 \end{bmatrix},\ t\in[0,1],\ l - ?$

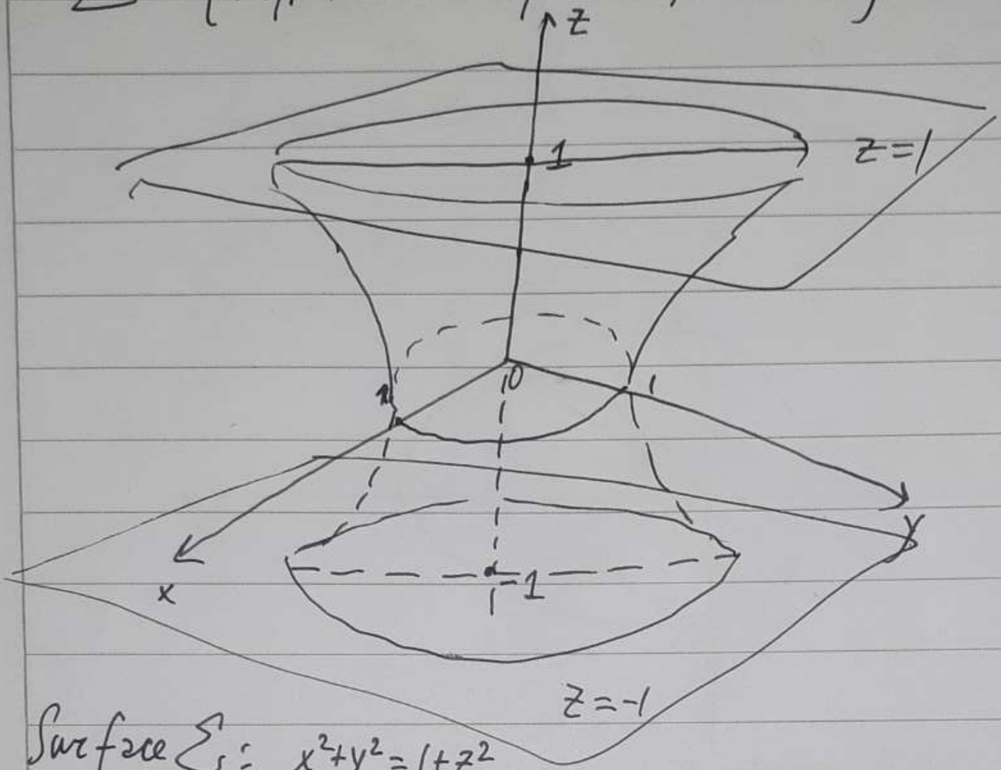
$$l = \int_{t_{start}}^{t_{end}} |\dot{\vec{q}}(t)|\, dt = \int_0^1 |(\dot{x},\dot{y})|\, dt = \begin{bmatrix} \dot{x} \rightleftarrows \frac{dx}{dt} & \dot{\vec{q}}(t) \rightleftarrows \frac{d\vec{q}}{dt} \\ \dot{y} \rightleftarrows \frac{dy}{dt} & \end{bmatrix} =$$

$$= \int_0^1 |(t,t^2)|\, dt = \int_0^1 \sqrt{t^2+(t^2)^2}\, dt = \int_0^1 \sqrt{t^2+t^4}\, dt = \int_0^1 t\sqrt{1+t^2}\, dt =$$

$$= \begin{bmatrix} 1+t^2 = u, & u(0)=1 \\ du = 2t\,dt, & u(1)=2 \end{bmatrix} = \frac{1}{2}\int_1^2 \sqrt{u}\, du = \frac{1}{2}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^2 = \frac{1}{3}u^{3/2}\Big|_1^2 = \frac{1}{3}\left[2^{3/2}-1\right] =$$

$$= \underline{\frac{2\sqrt{2}-1}{3}}.$$

⑨ $\Sigma = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2-z^2=1, -1 \le z \le 1\}$

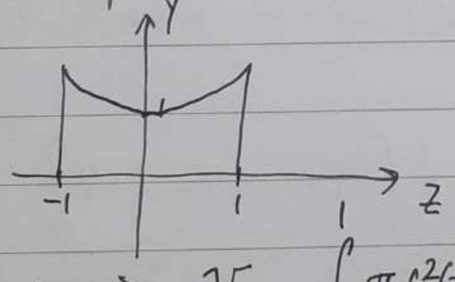



Surface $\Sigma$: $\underbrace{x^2+y^2}_{\text{distance}^2 \text{ to } OZ} = 1+z^2$

Solid of revolution is obtained by rotating (for example)

function $y = f(z) = \sqrt{1+z^2}$

round $OZ$ (taking $x=0$).

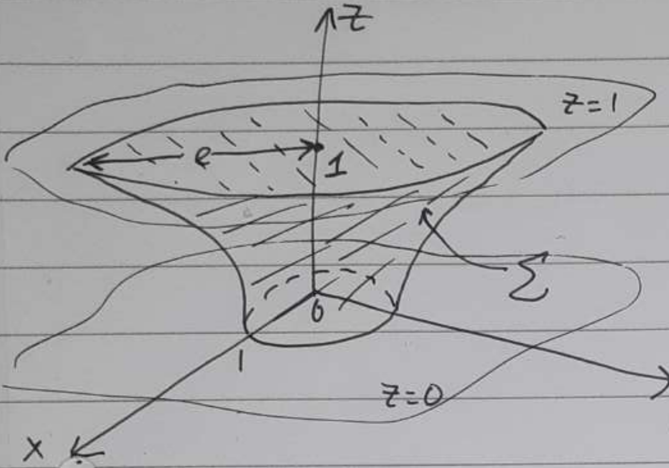



$$\Longrightarrow V = \int_{-1}^{1} \pi f^2(z)\,dz = \pi \int_{-1}^{1} (1+z^2)\,dz = 2\pi \int_0^1 (1+z^2)\,dz =$$

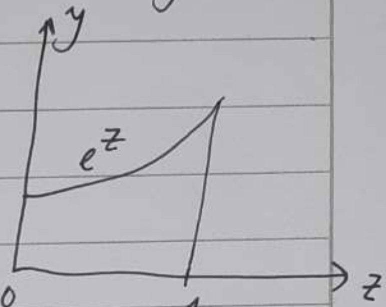
$$= 2\pi \left[z + \frac{z^3}{3}\right]_0^1 = 2\pi \cdot \frac{4}{3} = \underline{\frac{8\pi}{3}}.$$

(10) $\Sigma = \{(x,y,z)\in\mathbb{R}^3 : z = \ln\sqrt{x^2+y^2},\ z\in[0,1]\}$

$\left[\, z=\ln\sqrt{x^2+y^2} \iff e^z=\sqrt{x^2+y^2},\quad y=e^z \text{ for } x=0,\ y>0,\ z\in[0,1]. \,\right]$




Surface obtained when rotating (set $x=0$) $y=e^z$ around $OZ$.




$$S(\Sigma) = 2\pi\int_0^1 f(z)\sqrt{1+(f'(z))^2}\,dz = 2\pi\int_0^1 e^z\sqrt{1+e^{2z}}\,dz = \left[\begin{array}{l} \cancel{1+e^{2z}=u} \quad u=e^z \\ \cancel{du=2e^{2z}dz} \quad du=e^z dz \\ u(0)=1,\ u(1)=e \end{array}\right] =$$

$$= 2\pi\int_1^e \sqrt{1+u^2}\,du = \left[\begin{array}{l} u=sh(t),\ t=sh^{-1}(u) \\ du=ch(t)dt, \\ t(1)=sh^{-1}(1),\ t(e)=sh^{-1}(e) \end{array}\right] =$$

$$= 2\pi\int_{sh^{-1}(1)}^{sh^{-1}(e)} \sqrt{\underbrace{1+sh^2t}_{ch^2t}}\,ch(t)\,dt = 2\pi\int_{sh^{-1}(1)}^{sh^{-1}(e)} ch^2(t)\,dt = 2\pi\int_{sh^{-1}(1)}^{sh^{-1}(e)} \frac{1+ch(2t)}{2}\,dt = \pi\left[t+\frac{sh(2t)}{2}\right]_{sh^{-1}(1)}^{sh^{-1}(e)} =$$

$$\left[\begin{array}{l} sh(2t)= \\ =2sh(t)ch(t) \end{array}\right] = \pi\left[sh^{-1}(e)-sh^{-1}(1)+sh(sh^{-1}e)\,ch(sh^{-1}e)-sh(sh^{-1}1)\,ch(sh^{-1}1)\right] = \left[\begin{array}{l} ch(sh^{-1}x)= \\ =\sqrt{1+(sh(sh^{-1}x))^2}= \\ =\sqrt{1+x^2} \end{array}\right]$$

$$= \pi\left[sh^{-1}(e)-sh^{-1}(1)+e\sqrt{1+e^2}-\sqrt{2}\right].$$