Lecture "Experimental Physics I"
(Prof. Dr. R. Seidel)

Lecture 23

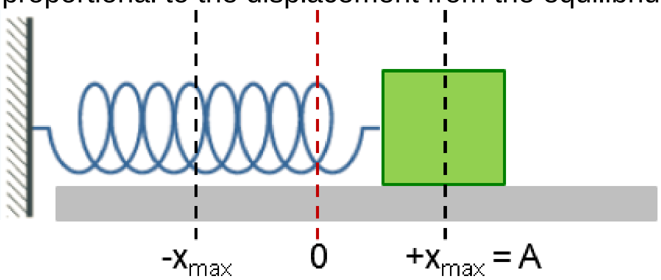
Oscillations

- Repetition simple harmonic motion
- Physical pendulum
- Representation of oscillations with complex numbers
- Fourier series and frequency spectrum of a signal/function

1) Simple harmonic motion

A) Oscillations of a mass on a spring

In the following we will first repeat our knowledge about simple harmonic oscillations. Remember the simple friction-free mechanical oscillator of a mass m on a spring (**see slides**), where the back-driving force was proportional to the displacement from the equilibrium position.



Its equation of motion was given by:

$$F_x = -kx = ma$$

which could be transformed to:

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This equation comprises the three conditions for any harmonic oscillator: the **acceleration** is **proportional** to **the negative displacement.** More generally we previously wrote:

$$\frac{d^2x}{dt^2} + \underbrace{\frac{k}{m}}_{\omega^2}x = 0$$

Here the term before the position x defined the **square of the angular frequency ω**. The solution to this equation was given by (**show only left half,** name the variables):

$$x(t) = \underbrace{A}_{\textbf{amplitude}} \cdot \cos(\overbrace{\underbrace{\omega}_{\textbf{angular frequency}} t + \underbrace{\varphi}_{\textbf{initial phase}}}^{\textbf{phase}}) = A \cdot \cos\left(2\pi \frac{t}{\underbrace{T}_{\textbf{period}}} + \varphi\right)$$

In addition to the angular frequency:

$$\omega = \sqrt{\frac{k}{m}}$$

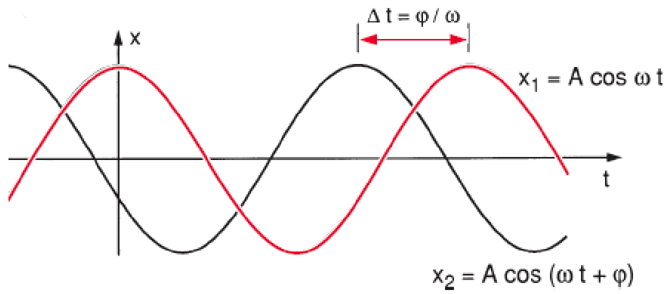
the oscillator solution is defined by the amplitude and the initial phase (**see slider animation**). The term within the cosine function is the total phase that increases linearly with time by ωt. During one **period *T*** of the oscillation the phase increases by 2π, since then the oscillator is again at the same position. Thus, $\omega T = 2\pi$ such that we get:

$$\omega = \frac{2\pi}{T}$$

The angular frequency is thus the phase change of the oscillation done per time. Replacing ω in the solution above provides the right side of the equation. We can rewrite the position equation to:

$$x(t) = A\cos(\omega t + \varphi_0) = A\ \cos[\,\omega(t + \underbrace{\varphi_0/\omega}_{\substack{\text{time} \\ \text{shift } t_0}})]$$

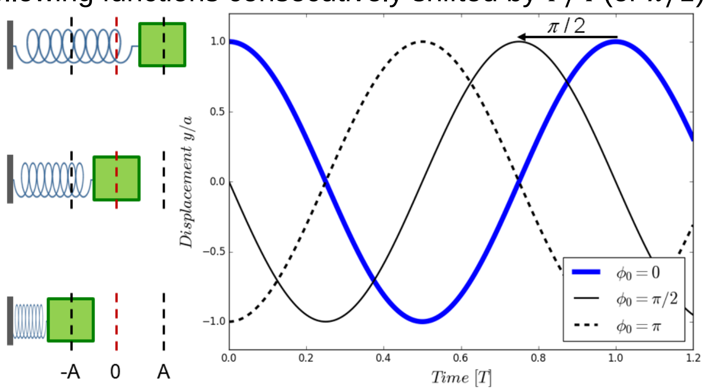
Thus, the **phase constant φ_0 shifts the cosine curve by φ_0/ω to the left along the time axis**.



The **initial phase φ_0** defines herby the "**start" condition at $t = 0$.** Common values are:

$$\begin{cases} \varphi_0 = 0 & (t_0 = 0) & \rightarrow x(0) = A \\ \varphi_0 = \pi/2 & (t_0 = T/4) & \rightarrow x(0) = 0 \\ \varphi_0 = \pi & (t_0 = T/2) & \rightarrow x(0) = -A \end{cases}$$

which provides the following functions consecutively shifted by $T/4$ (or $\pi/2$):



Knowing the time-dependent position of the oscillator provides by differentiation **the velocity and the acceleration** of the mass as function of time. Starting from

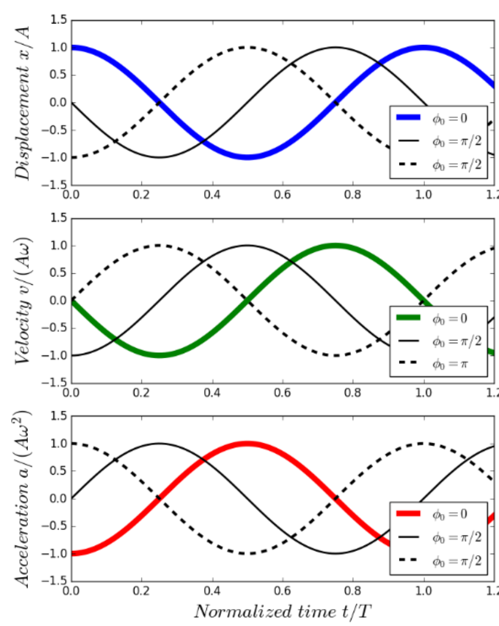
$$x(t) = A\cos(\omega t + \varphi_0)$$

we get the velocity:

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \varphi_0)$$

and from the velocity the acceleration

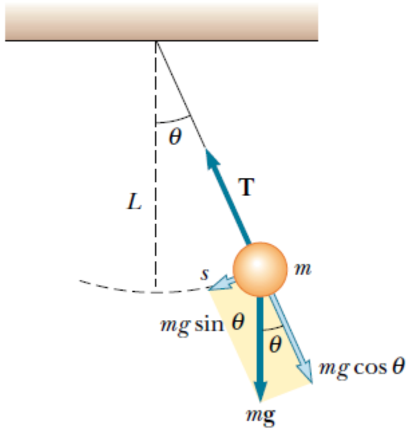
$$a(t) = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \varphi_0)$$



The velocity is phase shifted by $\pi/2$ (to the left, i.e. it acts earlier than the displacement), while the acceleration has exactly the opposite phase of the position (as reflected from negative sign of the initial equation of motion.)

After understanding the mass-spring system, we will have a look at other oscillators that in first approximation are all harmonic oscillators since they provide an oscillator equation of the same type.

B) Simple pendulum (point mass)



The simple pendulum is a **point mass** on a mass-free rigid rod. We derived previously an expression for the back-driving force and the associated acceleration:

$$\underbrace{-mg\sin\theta}_{F} = ma = m\underbrace{L\frac{d^2\theta}{dt^2}}_{a=L\cdot\alpha}$$

since the acceleration is given by the product of angular acceleration and rotation radius. For small angles we can approximate $\sin\theta \approx \theta$ and transform the equation into the following form:

$$\frac{d^2\theta}{dt^2} + \underbrace{\frac{g}{L}}_{\omega^2}\theta = 0$$

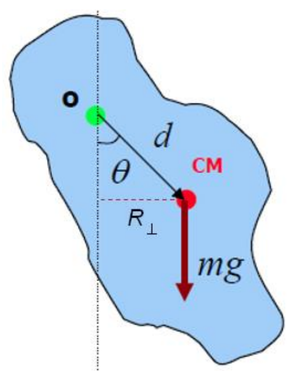
This is again the harmonic oscillator equation for which we already know the solution. The prefactor of the displacement angle provides the angular frequency and the period according to:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{L}}$$

and the time dependent displacement angle of the pendulum is given by

$$\theta(t) = \theta_{max}\cos(\omega t + \varphi_0)$$

C) Physical pendulum: pendulum motion of an extended object



A physical pendulum is a pendulum for which the extension of the oscillating mass matters, such that it cannot be approximated by a point mass anymore. We can derive its equation of motion by calculating the back-driving torque from the gravity force mg that acts at the center of mass at the effective radius $R_\perp = d\sin\theta$. The acting torque equals - in analogy to Newton's 2nd law - the moment of inertia times the angular acceleration. This provides:

$$\tau = -\underbrace{mg}_{F}\,\underbrace{d\sin\theta}_{R_\perp} = I\,\alpha = I\frac{d^2\theta}{dt^2}$$

In the small angle approximation $\sin\theta \approx \theta$, we can again convert this equation into the differential equation of a harmonic oscillator:

$$\frac{d^2\theta}{dt^2} + \underbrace{\frac{mgd}{I}}_{\omega^2}\theta = 0$$

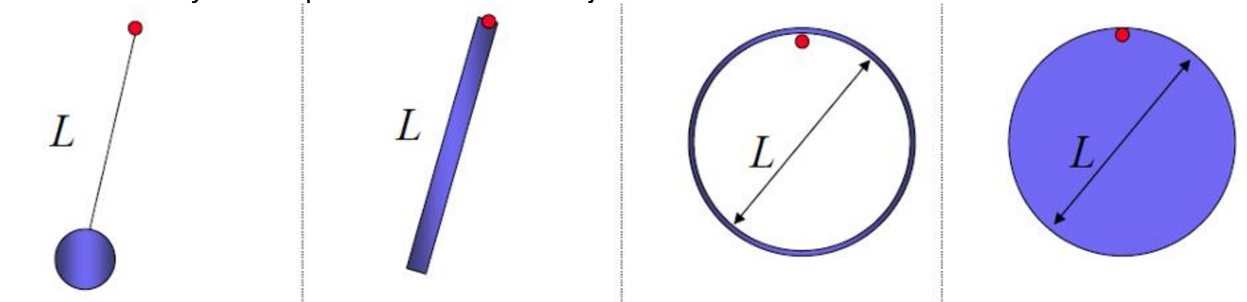
such that we obtain for angular frequency and period of the oscillation:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{mgd}{I}}$$

and the angular position as function of time is given as for the simple pendulum (see slide):

$$\theta(t) = \theta_{max}\,cos(\omega t + \varphi_0)$$

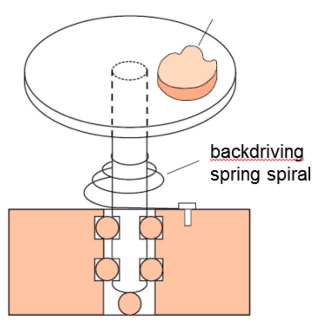
The moment of inertia contains the information about the pendulum shape. For a point mass we know that $I = m\,L^2$ and $d = L$ such that we obtain the period for the simple pendulum. For other geometries we must insert the moment of inertia known from before. The following table provides a little summary for the periods of different objects



$T = 2\pi\sqrt{L/g}$	$T = 2\pi\sqrt{\frac{2}{3}(L/g)}$	$T = 2\pi\sqrt{L/g}$	$T = 2\pi\sqrt{\frac{3}{4}(L/g)}$

D) Torsional pendulum

Another oscillator that we know from before is the torsional pendulum that undergoes rotary oscillations (**see slide**):



The backdriving torque that the spring exerts on the rotating body for an angular displacement φ is given by an analogon of Hooke's law:

$$\tau = -\kappa\varphi$$

where κ is the torsional rigidity ($[\kappa] = \mathrm{Nm}$). The equation of motion is calculated as for the physical pendulum using the moment of inertia of the system:

$$\tau = -\kappa\varphi = I\frac{d^2\varphi}{dt^2}$$

such that the oscillator equation becomes:

$$\frac{d^2\theta}{dt^2} + \underbrace{\frac{\kappa}{I}}_{\omega^2}\theta = 0$$

In analogy to before we get for the angular frequency and the period:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{\kappa}{I}}$$

From these examples we see that oscillators are quite abundant. They occur in many branches of physics and **understanding oscillator behavior is key for understanding the underlying physics.** For example, light-matter interaction can be in first approximation understood from the knowledge about oscillators.

2) Alternative representation of harmonic motion

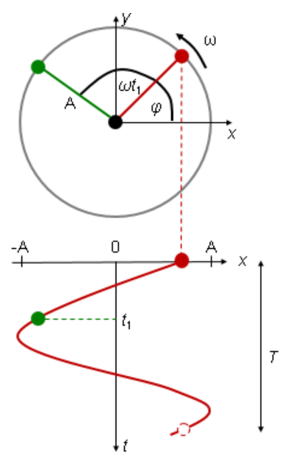
To describe oscillators that are more complex than the simple harmonic case, we will in the following expand our mathematical toolbox and develop alternative ways to represent harmonic motion.

A) Harmonic motion as projection of a circular motion

When we look at the equation for the position of the harmonic oscillator:

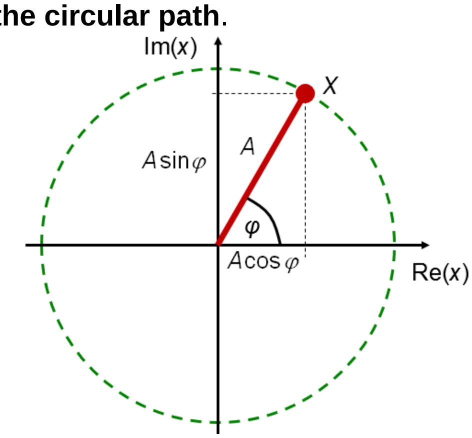
$$x(t) = A\cos\overbrace{(\omega t + \varphi_0)}^{\substack{\text{angular pos.}\\ \text{of circ. motion}}}$$

we note that the **total phase** can be seen as the **angle of a circular motion** that starts at φ_0. In this case the **oscillator position is the projection of the circular motion onto one coordinate axis**. According to the figure we can further state that if our position equation is expressed by the cosine function then it is the projection of the circular motion onto the x-axis. If the position is expressed with the sine function, then it is the projection onto the y-axis (**see slide**)



B) Complex numbers

Considering the projection from before we go one step further and put the circular motion into complex space. The **position *x* of the oscillator shall now be a complex number that represents its position on the circular path**.



Thus, *x* is provided by:

$$x = \underbrace{Re(x)}_{\text{former x}} + i\,\underbrace{Im(x)}_{\text{former y}}$$

where its real and its imaginary part are the former x and y coordinates respectively. With the amplitude A of the oscillation and the associated (phase) angle φ, we can express both components as:

$$Re(x) = A\cos\varphi$$
$$Im(x) = A\sin\varphi$$

Thus, x becomes:

$$x = A\cos\varphi + i\,A\sin\varphi = A(\cos\varphi + i\sin\varphi)$$

The term in brackets is nothing else than the **Euler formula**, thus leading to

$$x = A\,e^{i\,\varphi}$$

This is called the **polar representation of a complex number**, since the **coordinates are the angular position** in the complex space as well as the "radius" A, which is nothing else as than the **absolute value of the complex number**:

$$|x|^2 = x \cdot x^* = (Re(x) + i\,Im(x)) \cdot (Re(x) - i \cdot Im(x)) = Re(x)^2 + Im(x)^2$$

where *x** is the complex conjugate of x. Inserting real and imaginary part gives then

$$|x|^2 = A^2\cos^2\varphi + A^2\sin^2\varphi = A^2$$

The polar (phase) angle is according to the figure provided by the real and imaginary part of x:

$$\tan\varphi = \frac{Im(x)}{Re(x)}$$

For the **oscillator the phase increases linearly with time**. Thus, the corresponding circular motion is noted as:

$$x = Ae^{i(\omega t + \varphi_0)}$$

This equation can be transformed to:

$$x = e^{i\omega t}Ae^{i\varphi_0} = \underbrace{A\,e^{i\varphi_0}}_{\substack{\text{complex} \\ \text{amplitude } A_0}}\; e^{i\omega t} = A_0 e^{i\omega t}$$

There are two consequences of this equation:

- The **term $e^{i\omega t}$ turns $x(0)$ by the angle ωt in the complex plane**. In general, we can note that multiplication of a complex number C with a complex number that has an absolute value of 1 (such as $e^{i\omega t}$), turns C in the complex plane but does not change its absolute value
- Using a **complex amplitude we can incorporate the initial phase angle directly into the amplitude**, since the initial phase is given by imaginary and real part of $x(0)$

The **useful feature of describing the oscillator position by a complex number** is that:

- The **real world oscillator position is directly provided by the real part of x** (since it is the projection of x onto the real axis):

  $$x_{(real)} = Re(x) = Re\left[Ae^{i(\omega t + \varphi_0)}\right] = A\cos(\omega t + \varphi_0)$$

- In addition to the oscillator position we **know at each time also the phase of the oscillator,** since we have also the imaginary part of *x*:

  $$\tan\varphi = \frac{Im(x)}{Re(x)}$$

  and can thus **also calculate the amplitude of the oscillator**. Thus, we have a very comprehensive description of our oscillator motion

Importantly, the **complex oscillator notation of x solves also the oscillator equation**, as one can easily show. We start with the original differential equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Inserting of the complex position x provides:

$$A_0\underbrace{(i\omega)(i\omega)}_{-\omega^2}e^{i\omega t}+\omega^2A_0e^{i\omega t}=0$$

Overall, we will see the usefulness of this approach later in many fields of physics. Again, it is important to note that the **complex notation** of x **contains the real oscillator position** in the real part of x **as well as the current phase** in the ratio of imaginary and real part.

C) Sum of cosine and sine

Another useful way to rewrite harmonic motion which is somewhat analogous to the complex notations is obtained by rewriting the position using the angle sum identity:

$$x(t)=A\cos(\omega t+\varphi)=A\cos\varphi\cos\omega t-A\sin\varphi\sin\omega t=C_1\cos\omega t+C_2\sin\omega t$$

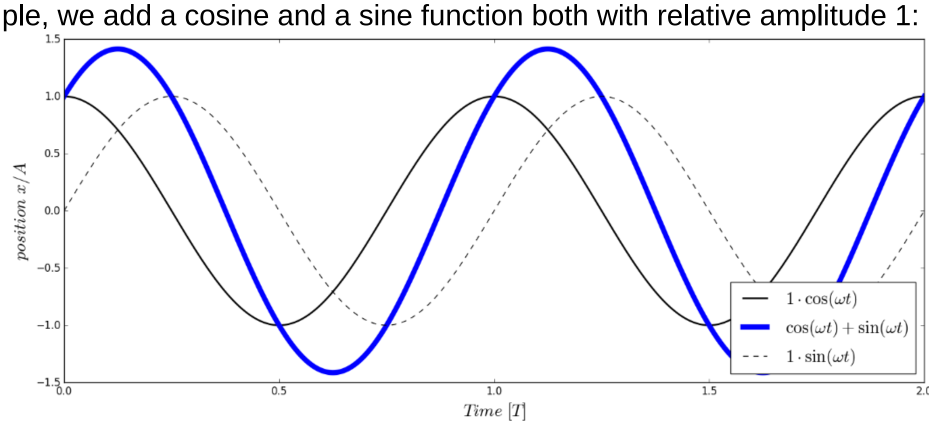
Thus, the harmonic oscillator motion for any initial phase can be expressed as superposition (sum) of a sinus and a cosine function of equal frequency. The cosine is the solution for a start at $x(0)=A$ while the sine is the solution for a start at $x(0)=0$. Any other starting condition can be derived from such a superposition. Cosine and sine are also called orthogonal functions. From above we see that:

$$C_1=A\cos\varphi$$
$$C_2=-A\sin\varphi$$

Squaring and adding as well as dividing both equations provides:

$$A=\sqrt{C_1^2+C_2^2}$$
$$\tan\varphi=-\frac{C_2}{C_1}$$

As an example, we add a cosine and a sine function both with relative amplitude 1:

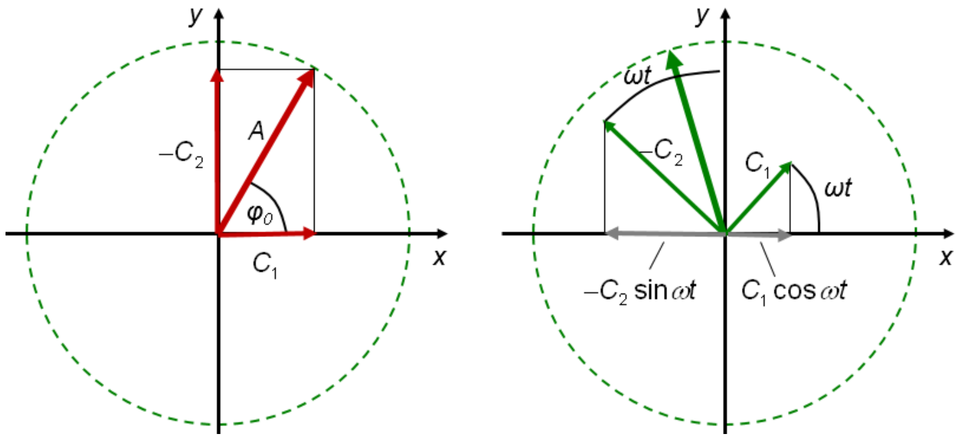


The resulting function can be described by

$$x(t)=\sqrt{1^2+1^2}\cos(\omega t+\arctan(-1/1))=\sqrt{2}\cos\left(\omega t-\frac{\pi}{4}\right)$$

which matches the blue function in the plot.

In the representation of the circular motion C_1 and $-C_2$ correspond to two orthogonal vectors (called phasors, see figure below). The **position on the circle is the vector sum of the two orthogonal axial vectors with length $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$**. At $t=0$, C_1 points along the x- and C_2 along the y-axis. After time t, the vector rectangle has turned by the angle ωt. $C_1\cos\omega t$ is then the projection of $\vec{C}_1$ onto the x-axis, whereas $-C_2\, sin\,\omega\, t$ is the projection of $\vec{C}_2$ vector onto the x-axis.



The real position $x(t)$ of the oscillator is then provided by the sum of the two projections:

$$x(t) = C_1 \cos \omega\, t + C_2 \sin \omega\, t$$

which provides the original equation.

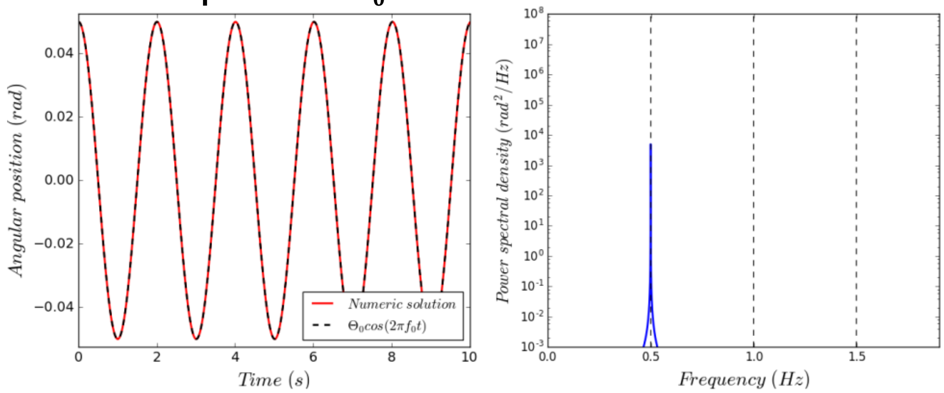
3) Fourier series and frequency spectrum of a signal/function

A) Non-harmonic periodic oscillation

So far, we only looked at harmonic periodic oscillations, i.e. the oscillator position in time was described by a sinusoidal function with a single (angular) frequency. Let us have a **look at a non-harmonic oscillation**, where the solution is not sinusoidal anymore. Such an oscillation can be obtained with a simple pendulum for large initial displacements. In this case we cannot approximate the sine of the angle with the angle itself, such that our differential equation becomes:

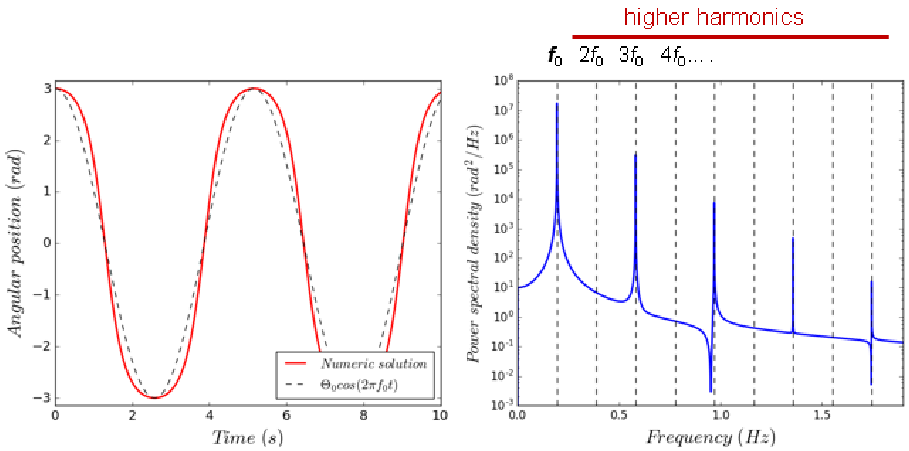
$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\, \theta = 0$$

This equation can be solved by numeric integration, since it gives us the acceleration of the pendulum for a given displacement. We first check the numeric solution for a pendulum length of 1 m and a **small initial displacement** $\boldsymbol{\theta_0 = 0.05}$ **rad**:



We obtain a solution that overlaps perfectly with the cosine function as expected from our previous analytic solution. When calculating the frequency spectrum (called power spectral density), we see that the oscillation occurs at a single frequency of about 0.5 Hz. This gives a pure sinusoidal, pure harmonic tone if converted into an acoustic signal.

When simulating a **large initial displacement** $\boldsymbol{\theta_0 = 3}$ **rad**, such that the pendulum makes almost a full turn, we see an increased period but most importantly we see considerable deviations from the sinusoidal trajectory; the curve is broadened at the turning points



These deviations are also seen in the frequency spectrum, where we now have in **addition to the base frequency f_0** of the oscillation also **higher frequency contributions at multiples of f_0**. These multiples of f_0 are called **higher harmonics**.

Experiment: Using a ruler as a rod-shaped pendulum that is connected to a sensor that measures the angular displacement we can study the angular position for small and large displacements and calculate the frequency spectrum. For small displacement we get a peak at a single frequency, while for large displacements we get a peak at the fundamental frequency as provided by the period as well as up to two higher harmonics at 3-fold and 5-fold the base frequency. As predicted by the simulation

Experiment: Using an electronic frequency generator one can generate a sinusoidal as well as an alternating rectangular periodic voltage output. Feeding the voltage into an oscilloscope and that calculates the frequency spectrum provides a single frequency for the sinusoidal output but the fundamental frequency as well as a large range of higher harmonics as we saw for the pendulum.

Overall one can state that**, any oscillation that deviates from a pure sinusoidal is a superposition of multiple oscillations with different frequency!**

B) Fourier series development of a periodic function

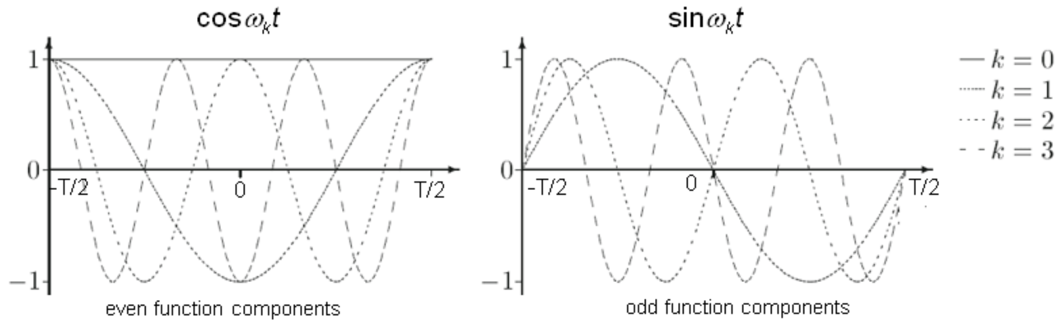
From the simulation and the experiments we saw that any non-harmonic periodic signal comprises multiple oscillations at frequencies of nf_0 with $(n = 1, 2, \ldots)$. This empiric finding is due to a fundamental principle that says that **every function that is limited to an interval of $-T/2 < x < T/2$ or that is periodic with period T can be expressed as a Fourier series** (also harmonic series), being the **sum of orthogonal cosine and sine functions:**

$$f(t) = \sum_{k=0}^{\infty} (A_k \cos \omega_k t + B_k \sin \omega_k t)$$

with **discrete angular frequencies that are multiples of the fundamental frequency $\omega_1 = 2\pi/T$ (that is set by the interval length):**

$\omega_k = k\,\omega_1$,

i.e. $0, 2\pi/T, 4\pi/T, 6\pi/T, \ldots$ with $B_0 = 0$. For $k = 0$ we have a constant function. For $k > 0$, the selected sinusoidal functions contain exactly 1, 2, 3 … full periods within the interval of length T.



It makes sense to select only frequencies that are multiples of the fundamental frequency, since each subsequent interval of length T contains the same curve shape thus the functions of the Fourier series should also have the same shape in there. The sum of sine and cosine function at a given frequency sets the phase of the resulting sinusoidal function at that frequency:

$$f(t) = \sum_{k=0}^{\infty} \overbrace{(A_k \cos \omega_k t + B_k \sin \omega_k t)}^{C_k \cos(\omega_k t + \varphi_k)}$$

A_k and B_k are the so-called Fourier coefficients. These coefficients are the same for the periodic function with period T AND for the function limited to $-T/2$ and $T/2$. Selecting the right Fourier coefficients defines the final shape of the function.

One says that the cosine functions harbor the even, i.e. mirror symmetric function components with $f(x) = f(-x)$ and the sine functions the odd function components with $f(x) = -f(-x)$. Thus, an **even function contains only non-zero Fourier coefficients A_k** and an **odd function contains only non-zero Fourier coefficients B_k**.

Experiment: We try now to represent a triangular periodic function by cosine functions and a rectangular periodic function by sine functions using a function generator that produces sinusoidal alternating voltages. When adding higher order frequency contributions, the desired shape of the function is more and more approximated.

In order to compose a desired periodic function, the Fourier coefficients must be known. **The Fourier-coefficients are given by an integral representing the overlap of $f(t)$ with the particular harmonic base function:**

$$A_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_k t)\,dt \quad (k \neq 0)\,; \qquad A_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$B_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega_k t)\,dt \quad (k \neq 0)\,; \qquad B_0 = 0$$

The **integral calculates the overlap between the harmonic function and *f*(*t*)**, i.e. how much of this harmonic function is comprised within $f(t)$. **A_0 is the simple avarage of the function within the interval of length T**, it thus defines a vertical shift of the function.

:

Interlude derivation of Fourier coefficients (not shown):

The normalization, i.e. the prefactor $2/T$ of the Fourier coefiicients can be verified by considering that:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos \omega_{k'} t \ \cos \omega_k t \, dt = \frac{T}{2}\delta_{kk'}$$

while

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \omega_{k'} t \ \cos \omega_k t \, dt = 0$$

Inserting the Fourier series for $f(t)$ into the equation for the Fourier coefficient gives A_k

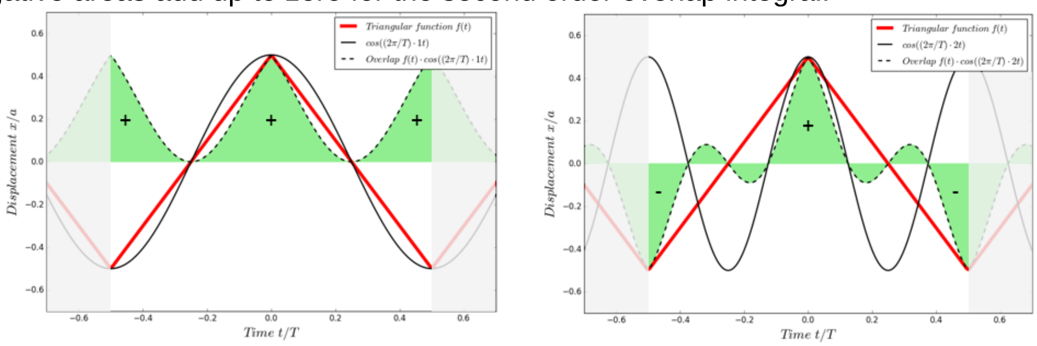
$$A_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_k t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2}\sum_{k'=0}^{\infty}\left(A_{k'}\cos\omega_k t + B_{k'}\sin\omega_k t\right)\cos(\omega_k t)\,dt$$

Splitting the cosine and sine terms and using the integral solutions from above gives then:

$$A_k = \frac{2}{T}\sum_{k\prime=0}^{\infty}\left(A_{k'}\underbrace{\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos\omega_{k'}t\cos\omega_k t\,dt}_{\frac{T}{2}\delta_{kk'}} + B_{k'}\underbrace{\int_{-\frac{T}{2}}^{\frac{T}{2}}\sin\omega_{k'}t\cos\omega_k t\,dt}_{0}\right) = A_k$$

The verification of B_k follows similar steps.

To understand the calculation of the Fourier coefficients better, we look at a plot of the **triangular function** together with the **product between this function and the harmonic function of the particular order** (see figure below). The first order cosine function describes already quite well the shape of the symmetric triangle. It is positive when the $f(t)$ is positive and negative when $f(t)$ is negative. Thus, the product between both functions is always positive and the overlap integral gives a large positive value. The second order cosine is however positive at the interval boundaries, where $f(t)$ is negative. The overlap product is therefore negative. Overall the positive and negative areas add up to zero for the second order overlap integral:



We now approximate the **triangular function $f(t)$ with peak-to-peak-amplitude A** with **a Fourier series** of increasing order. We let $f(t)$ be symmetric in time. It is thus an even function such that all sine terms vanish:

$$B_k = 0$$

If it oscillates around zero than it has a zero mean such that

$$A_0 = 0$$

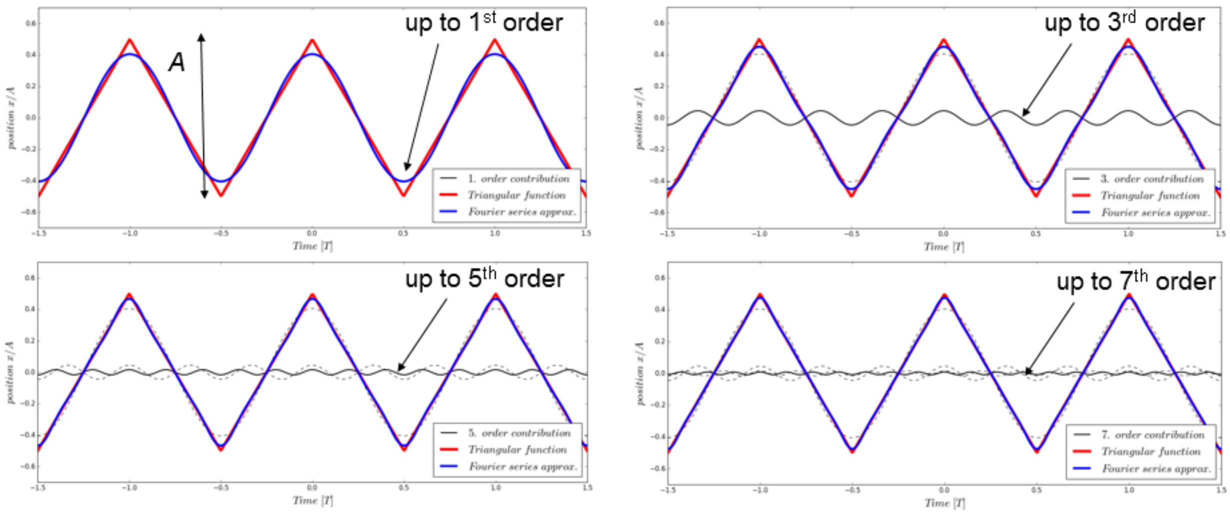
The other A_k are obtained by calculating the overlap integral. They are given by:

$$A_k = A\frac{4}{(\pi k)^2}$$

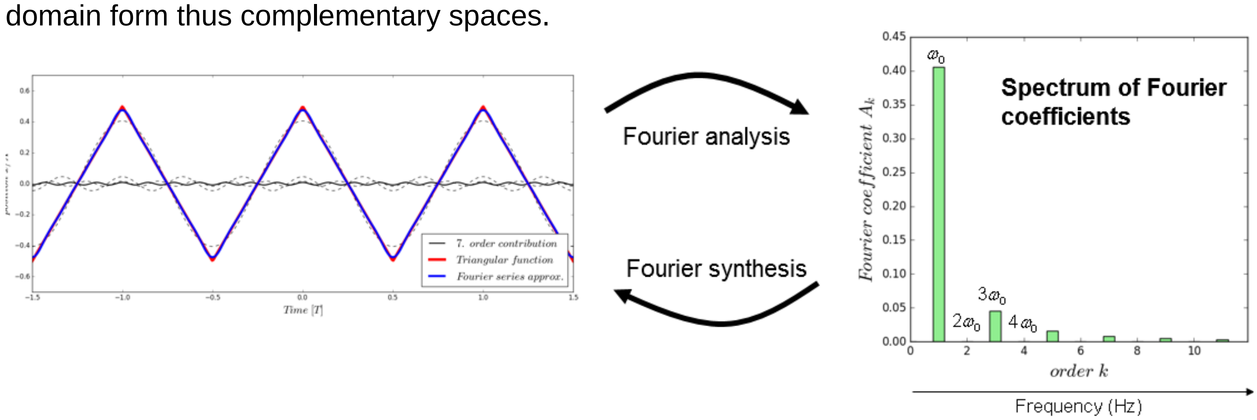
if k is odd and $A_k = 0$ if k is even. The resulting Fourier series approximation of the triangular function is thus given as:

$$f(t) = \frac{4A}{\pi^2}\left(\cos\omega t + \frac{1}{3^2}\cos 3\omega t + \frac{1}{5^2}\cos 5\omega t + \cdots\right)$$

The triangular shape of the function is already well approximated for not too high orders of cosine functions (**show slider animation and/or slide**). This is due to the $1/k^2$ dependence of the higher order (higher frequency) cosine functions, whose amplitude thus rapidly decreases.



Using the integration recipe one can obtain for **any periodic function a set Fourier coefficients**, which is called **Fourier analysis**. The **Fourier coefficients define a discrete frequency spectrum** since the frequency space is limited to discrete values of $k\omega_0 \; = \; k \, 2\pi/T$.
$f(t)$ and its Fourier coefficient spectrum contain the same information, since one can obtain $f(t)$ from the spectrum by **Fourier synthesis** and vice versa. The time domain and the frequency domain form thus complementary spaces.



C) Fourier transformation: continuous Fourier analysis

So far, we looked at periodic signals/oscillations with a known period. Often the period of a signal may be hard to guess (e.g. due to noise) or it may be non-periodic at all (e.g. stochastic). In this case we use a continuous frequency space instead of a discrete frequency space in order to describe any function (even nonperiodic) as a superposition of orthogonal functions.
To do so, we first build **complex Fourier coefficients** out of our two types of Fourier coefficients (A_k, B_k).

$$C_k = A_k + iB_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(-\omega_k t)dt + i\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(-\omega_k t)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-i\omega_k t}dt$$

The resulting complex Fourier series contains the same information in analogy to the complex-number notation of oscillations.
For mathematical symmetry (not discussed further) we extend our frequency range hereby to positive and negative ω_k such that the 2 in the prefactor is gone and we also use a negative sign

in the sine function. We thus obtain a complex discrete frequency spectrum C_k, where the real parts A_k are the cosine and the imaginary parts B_k are the sine Fourier coefficients.
We furthermore do not assume any periodicity but extend our integration interval T to infinity. For $T \to \infty$,the fundamental frequency that defines the distance between the neighboring frequencies in the Fourier spectrum approaches zero:

$$\omega_1 = \frac{2\pi}{T} \to 0$$

Thus, we go **from a discrete to a continuous frequency spectrum**. The limit

$$\lim_{T\to\infty}(TC_k) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \ = F(\omega)$$

is **called the Fourier-Transform of *f(t)***. It represents the **frequency spectrum of $f(t)$**. It is a complex spectrum that provides the following information:

- $|F(\omega)|$ represents the amplitude (per frequency interval) of the signal at the given frequency
- $\frac{Im(F(\omega))}{Re(F(\omega))} = tan(\ \varphi(\omega))$ represents the phase of the signal at the given frequency
- $Re(\ F(\omega))$ and $Im(\ F(\omega))$ are the even and odd components of $f(t)$ at the particular frequency. An even function that is symmetric along the time axis has therefore only a real Fourier transform and a function that is odd has only an imaginary Fourier transform.
- $G(\omega) = |F(\omega)|^2$ provides the square amplitude that is proprtional to the total energy per frequency interval at the given frequency. It is related to the **Power (spectrum)** which is the typical representation of a frequency spectrum, since e.g. for sound the detectable energy at a given frequency matters most.

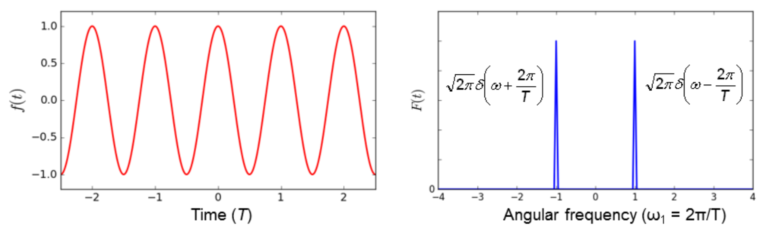
Applying the Fourier transform to periodic functions, we obtain continuous frequency spectra that resemble the spectra of the discrete Fourier coefficients, since only for sinusoidal functions matching the periodicity of $f(t)$ the Fourier integral becomes non-zero:

Cosine function

$$f(t) = \cos\left(\frac{2\pi}{T}t\right)$$

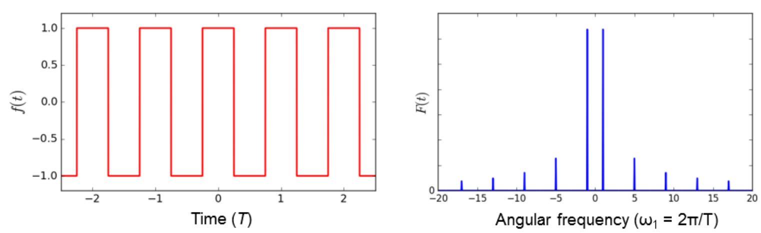
Dirac delta-function

$$F(\omega) = \sqrt{2\pi}\,\frac{\delta(\omega - 2\pi/T) + \delta(\omega + 2\pi/T)}{2}$$



Rectangular function

$$F(\omega) \propto A_k\,\frac{\delta(\omega - k\cdot 2\pi/T) + \delta(\omega + k\cdot 2\pi/T)}{2}$$



The discrete peaks in the spectra are Dirac delta functions. They are infinitely high, since any amplitude is put into a infinitesimally small frequency interval.
The simple cosine function has two single frequency peaks at $\pm 2\pi/T$. The rectangular function has peaks at the same fundamental frequency as well as at some of the higher harmonics as we also saw for the Fourier analysis of a rectangular voltage signal.

We now can also obtain Fourier transforms (frequency spectra) for non-periodic functions, such as a Gaussian and a symmetric exponential that yield a gaussian and a Lorentzian function, respectively:

Gaussian function

$$f(t) = e^{-\frac{t^2}{2\tau^2}}$$

$$F(\omega) = \tau \cdot e^{-\frac{\tau^2\omega^2}{2}}$$

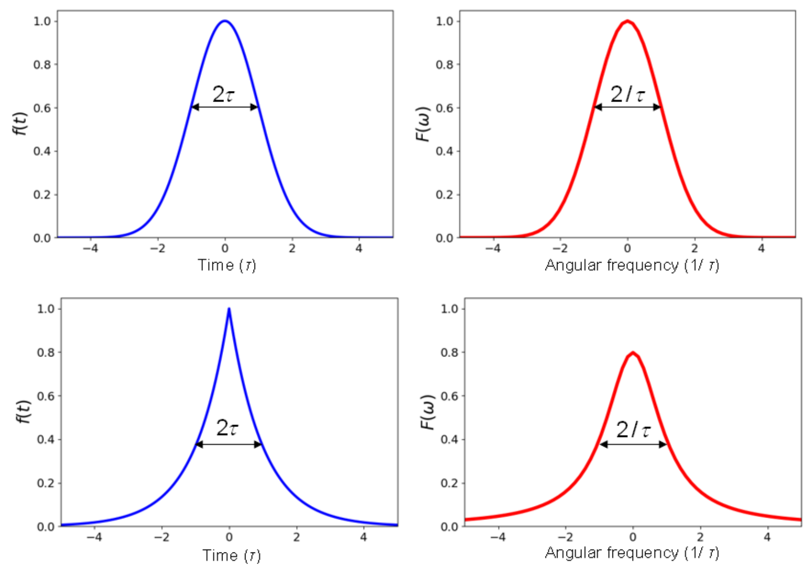
Gaussian

Symmetric exponential

$$f(t) = e^{-|t|/\tau}$$

$$F(\omega) = \sqrt{\frac{2}{\pi}} \cdot \tau \frac{1}{1+\tau^2\omega^2}$$

Lorentzian



Interestingly, their spectra are continuous functions, i.e. they comprise a whole frequency range. The **Fourier transform of a Gaussian with standard deviation τ is again a Gaussian with standard deviation $1/\tau$** (as angular frequency). The **exponential function with decay time τ has a Lorentzian function as Fourier transform that decays to half the maximum value at $1/\tau$.**

Thus, **as narrower the time pulse described by these functions is, as larger is the covered frequency range** and vice versa.

A good book illustrating the basis of Fourier analysis and Fourier transform: **Fourier transform for pedestrians** by Tilmann Butz (see slide)**.**

Lecture 23: Experiments

1) Animation: Harmonic oscillation as projection of a circular motion onto the x- or the y-axis
2) Tracking of Pendulum for large displacements and calculation of frequency spectrum
3) Fourier analysis of sinusoidal, rectangular and triangular periodic voltage signals from a frequency generator. Fourier analysis is done by an oscilloscope.
4) Fourier-Synthesis using a PC-application