6. Torques and Vectors

The Chapters 2.5–2.8 of my lecture notes provides the background to solve the following exercises. Your solution to the problems 6.1–6.4 should uploaded

to your Moodle account

as a PDF-file

by Wednesday, Nov 15, 10:30 (with a grace time till the start of the seminars).

The parts marked by $\star$ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

The self-test problems serve to check you understanding of vectors.

The bonus problem provides the solution to a problem that we introduced in class. It might take some extra effort to solve.

Problems

Problem 1. Determining the center of mass and the torque

Determine the the center of mass $\mathbf{Q}$ of bodies with the following mass density and shape. Determine also the torque $\mathbf{D}$ acting for the specified reference points.

a) A triangle in two dimensions with constant mass density $\rho = 1\,\mathrm{kg/m^2}$ and side length $6\,\mathrm{cm}$, $8\,\mathrm{cm}$, and $10\,\mathrm{cm}$.
Determine the torque acting for motion around the corner opposite of the longest side.
Bonus: What are the torques found for the other two corners?

b) A circle in two dimensions with center at position (a, b), radius $R = 5\,\mathrm{cm}$, and constant mass density $\rho = 1\,\mathrm{kg/m^2}$.
Hint: How do M, V and $\mathbf{Q}$ depend on the choice of the origin of the coordinate system?

c) A rectangle in two dimensions, parameterized by coordinates $0 \le x \le W$ and $0 \le y \le B$, and a mass density $\rho(x, y) = \alpha\, x$.

What is the torque for motion around the points $(0,0)$, $(W,0)$, $(0,B)$, and (B,W)?

d) A three-dimensional wedge with constant mass density $\rho = 1\,\mathrm{kg/m^3}$ that is parameterized by $0 \le x \le W$, $0 \le y \le B$, and $0 \le z \le H - Hx/W$.
What is the torque for motion around the points $(0,0)$, $(W,0)$, $(0,B)$, and (B,W)?
Discuss the relation to the result of part c).

e) A cube with edge length L. When its axes are aligned parallel to the axes $\hat{x}$, $\hat{y}$, $\hat{z}$, its density takes the form $\rho(x,y,z) = \beta\, z$.
What is the dimension of β in this case?

Discuss the torque exerted for motion around its corners.
This means: For each corner you determine the torque for motion around that corner. Do you recognize a pattern in the results? How can that be explained? How can it be exploited in the calculations?

Problem 2. Substitution with trigonometric and hyperbolic functions
Evaluate the following integrals by employing the suggested substitution, based on the substitution rule

$$\int_{q(x_1)}^{q(x_2)} \mathrm{d}q\, f(q) = \int_{x_1}^{x_2} \mathrm{d}x\, q'(x)\, f(q(x))$$

with a function $q(x)$ that is bijective on the integration interval $[x_1, x_2]$.

a) $\displaystyle\int_a^b \mathrm{d}q\, \frac{1}{\sqrt{1-q^2}}$ by substituting $q(x) = \sin x$

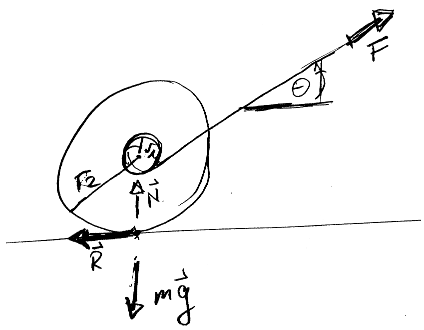
b) $\displaystyle\int_a^b \mathrm{d}q\, \frac{1}{\sqrt{1+q^2}}$ by substituting $q(x) = \sinh x$

c) $\displaystyle\int_a^b \mathrm{d}q\, \frac{1}{1+q^2}$ by substituting $q(x) = \tan x$

d) $\displaystyle\int_a^b \mathrm{d}q\, \frac{1}{1-q^2}$ by substituting $q(x) = \tanh x$

Problem 3. Walking a yoyo

The sketch to the right shows a yoyo of mass m standing on the ground. It is held at a chord that extends to the top right. There are four forces acting on the yoyo: gravity $m\mathbf{g}$, a normal force $\mathbf{N}$ from the ground, a friction force $\mathbf{R}$ at the contact to the ground, and the force $\mathbf{F}$ due to the chord. The chord is wrapped around an axle of radius r_1. The outer radius of the yoyo is r_2.

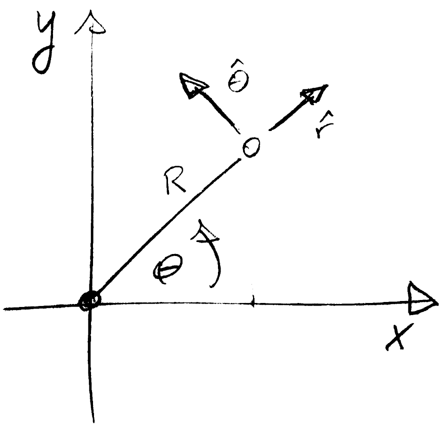


a) Which conditions must hold such that there is no net force acting on the center of mass of the yoyo?

b) For which angle θ will the torque vanish?

$\star$ c) Perform an experiment: What happens for larger and for smaller angels θ? How does the yoyo respond when you fix the height where you keep the chord and pull continuously?

Problem 4. Motion on a circular track



The position of a particle in the plane can be specified by Cartesian coordinates (x, y) or by polar coordinates with basis vectors $\hat{\boldsymbol{r}}(\theta)$ and $\hat{\boldsymbol{\theta}}(\theta)$, that have the following representation in Cartesian coordinates (cf. the sketch to the left)

$$\hat{\boldsymbol{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} .$$

We will now explore the trajectory $\mathbf{q}(t)$ of a particle with mass m that moves on a track with a fixed radius R.

a) Verify that $\hat{\boldsymbol{\theta}} = \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{\boldsymbol{r}}$ and $\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\hat{\boldsymbol{r}} = -\hat{\boldsymbol{r}}$.

Hint: Express $\hat{\boldsymbol{r}}$ as $\hat{\boldsymbol{r}} = \cos\theta\ \hat{\boldsymbol{x}} + \sin\theta\ \hat{\boldsymbol{y}}$, and employ the sum rule for differentiation.

b) Show that $\frac{\mathrm{d}}{\mathrm{d}\theta}(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}) = 2\,\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\theta}}$ and that $\frac{\mathrm{d}}{\mathrm{d}\theta}(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}) = -2\,\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\theta}}$.
Bonus: How can this observation be used to provide a geometric interpretation of the result of a).

c) The position of the particle can be specified as $\mathbf{q}(t) = R\,\hat{\boldsymbol{r}}(\theta(t))$. Determine $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ based on these equations.
Instruction: Adopt the chain rule of differentiation: $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{r}}\big(\theta(t)\big) = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\,\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{\boldsymbol{r}}\big(\theta(t)\big)$

Verify your result by performing the same calculation in Cartesian coordinates, i.e. based on the representation of $\hat{\boldsymbol{r}}$ provided in a).

d) Consider the motion at a constant angular speed, $\theta(t) = \omega t$, and show that the acceleration in this setting takes the form $\ddot{\mathbf{q}} = -R\,\omega^2\,\hat{\boldsymbol{r}}(\omega t)$.

Verify that this amounts to an acceleration that is perpendicular to the velocity.

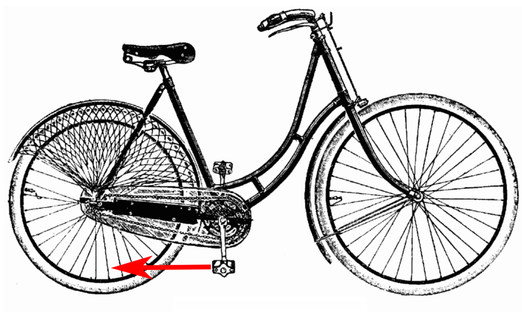
What does this imply for the absolute value of the velocity?

⋆ e) In addition to the acceleration determined in d) also gravity is acting on a race track. What does this imply for curves in bike races, bobsled races and in skate parks?

Self Test

Problem 5. Where will the bike go?

Consider the picture of the bicycle to the right. The red arrow indicates a force that is acting on the paddle in backward direction.
Will the bicycle move forwards or backwards?
Take a bike and do the experiment!



adapted from original: Damenfahrrad (Otto Lueger, 1904) [Public domain, wikimedia]

Bonus Problem

Problem 6. Surface area of a hypersphere

A hypersphere is a generalization of a sphere to d-dimensional space. The surface of a d-dimensional hypersphere with radius R comprises all points $\mathbf{q} \in \mathbb{R}^d$ with $|\mathbf{q}| = R$.
For $d = 2$ this is a circles and its surface "area" amounts to $A_2 = 2\pi R$.
For $d = 3$ this is a normal sphere with area $A_3 = 4\pi R^2$.
In general the surface area can be written as $A_d = S_d\, R^d$ where S_d is the surface area of the d-dimensional unit sphere.
We will adopt hyper-spherical coordinates to evaluate the d-dimensional Gaussian integral

$$G_d = \int_{\mathbb{R}^d} \mathrm{d}^d q \ \exp(-\mathbf{q}^2) = \int_{-\infty}^{\infty} \mathrm{d}q_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}q_d \ \exp(-\sum_{i=1}^{d} q_i^2)$$

a) Proof that $G_d = (G_1)^d$.

b) Consider the case $d = 2$, and introduce polar coordinates to show that

$$\left[\int_{-\infty}^{\infty} \mathrm{d}q \ \exp(-q^2)\right]^2 = 2\pi \int_0^{\infty} \mathrm{d}q \ q \ \exp(-q^2)$$

Solve the integral on the right-hand-side by adopting the substitution $w = q^2$. What does this tell about G_1?

c) Rewrite the integral for $d = 3$ in the form

$$\pi^{3/2} = G_1^3 = G_3 = S_3 \int_0^{\infty} \mathrm{d}q\, q^2 \ \exp(-q^2)$$

and use partial integration to show that this entails $S_3 = 4\pi$.

⋆ d) Generalize the previous argument to the d-dimensional case, and adopt induction to proof that

$$S_d = \frac{d\, \pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)}$$

where $\Gamma(x)$ is the Γ-Function. It takes the values $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$, and it obeys the recursion relation $\Gamma(x+1) = x\,\Gamma(x)$.