Lecture "Experimental Physics I"
(Prof. Dr. R. Seidel)

Lecture 18

Elasticity of solids

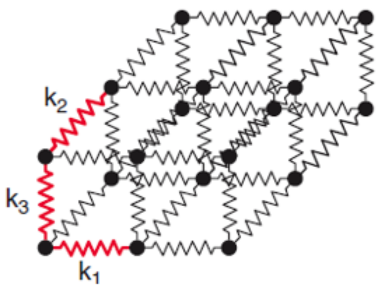
- Linear elasticity of solids
- Young’s modulus
- Poisson number & bulk modulus
- Shear modulus
- Bending of beams & second moment of area

1) Tensile deformation of solids

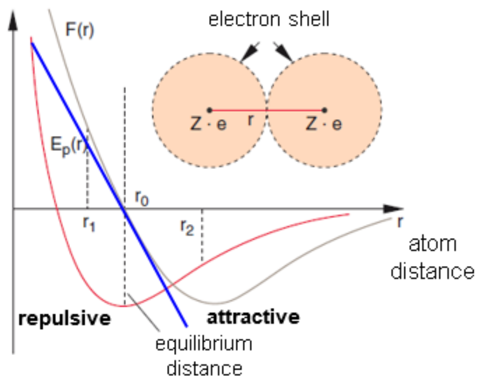
So far, we considered objects as mass points or completely rigid bodies, where we did not allow any deformations. Now we will consider the behavior of the material itself under the influence of external forces by looking at the occurring **deformations of solid bodies (shortly solids).**

A) Deformations at the atomic scale

Intuitive approach: We first look at the atoms that build up a solid. Atoms in a solid are often arranged on a crystal lattice. The effective **potential energy that each atom experiences has a minimum for the equilibrium atom-to-atom distance r_0** of a particular crystal type. There is an effective repulsive force for closer distances but an attraction for larger distances (see figure below), which together gives rise for a well-defined spacing between the atoms.



bead & spring model of crystalline solid



We can subject the effective potential of an atom to a Taylor-approximation, which provides:

$$E_p(r) = \sum_{n=0}^{\infty} \frac{(r-r_0)^n}{n!} \left(\frac{\partial^n E_{pot}}{\partial r^n}\right)_{r=r_0}$$

$$= E_p(r_0) + \frac{1}{2}(r-r_0)^2 \left(\frac{\partial^2 E_p}{\partial r^2}\right)_{r=r_0} + \frac{1}{6}(r-r_0)^3 \left(\frac{\partial^3 E_p}{\partial r^3}\right)_{r=r_0} + \cdots$$

In this equation we neglected the linear term, since the first derivative of $E_p(r)$ must be zero at r_0 where the energy has a minimum. Thus, the first order approximation of the potential is a quadratic function, i.e. a harmonic potential

The back-driving force on an displaced atom is calculated by $F(r) = -dE_p(r)/dr$ (see gray curve in figure above). Taking the partial derivatives as constants and differentiating only the polynomial terms, we thus get :

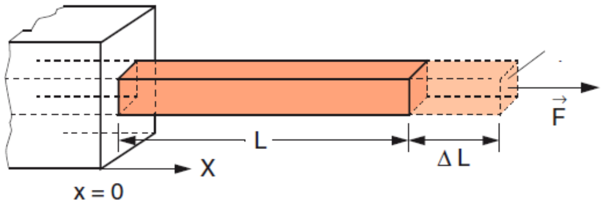
$$F(r) = -\frac{dE_p}{dr} = -\left(\frac{\partial^2 E_p}{\partial r^2}\right)_{r=r_0} (r-r_0) - \frac{1}{2}\left(\frac{\partial^3 E_p}{\partial r^3}\right)_{r=r_0} (r-r_0)^2 + \ldots$$

i.e. the force follows in 1st approximation Hooke's law in which $(r-r_0)$ represents the displacement from the equilibrium atomic distance.

An **external force that stretches a solid will therefore increasing linearly for small enough elongations** (until the deviation of the potential from a quadratic function becomes significant). Only at large displacements a nonlinear behavior will be see.

B) Deformations at the macroscopic scale

Now, we will have a look at macroscopic deformations of a solid, bearing in mind the atomic scale consideration. We define the following deformations:



Elastic deformation: The solid resumes its original shape after terminating the deformation, i.e. after returning to zero force.

Plastic deformation: The solid does not resume its original shape after the deformation force drops to zero.

Small enough deformations for any solid always elastic. The discussed linearity on the atomic scale for small elongations is preserved on the macroscopic scale. We thus get a type of Hooke's law for tensile deformations of solids, where the force that extends a solid grows linearly with the extension ΔL of the solid. One can write for the relation between force and extension:

$$F = E\,A\frac{\Delta L}{L}$$

where the proportionality constant $\boldsymbol{E}$ **is called the Young's modulus (or elastic modulus)** which is a material dependent constant with $[E] = N/m^2 = Pa$, which corresponds to a pressure. The other dependencies of the formula can be understood intuitively:

(i) The **force is proportional to the cross-sectional area $\boldsymbol{A}$** at which the force acts, since a proportionally larger number of atomic bonds is extended with increasing area.

(ii) For a given force the ratio between the extension and the total length L in the elongation direction is a constant, since we get the **same relative extension for each bond**. For example, we get twice the extension when doubling the length of the solid.

Experiments:

- **Static tensile elongation of a steel wire** ($L = 7.07m, r = 0.05\,mm$). We check the linearity $\Delta L \propto F$ (Hooke's law) when stretching the wire and determine the Young's modulus of the wire. We stretch the wire with the weight of a mass of $m = 0.8\,kg$ providing an extension of $\Delta L \approx 33.7\,mm$. With this we can write:

$$E = \frac{F}{\pi r^2}\frac{1}{\Delta L/L} = \frac{0.8\text{ kg }\cdot 9.81\text{ m s}^{-2}}{3.14 \cdot 2.5\ 10^{-3}\text{mm}^2\ \underbrace{(0.0337/7.07)}_{\approx 0.005}} \approx \frac{8N}{\underbrace{3.14 \cdot 2.5}_{8}\ 10^{-12}m^2\ \cdot\ 5}$$

$$= 0.21 \cdot 10^{12}\text{Pa} = 210\text{ GPa}$$

- **Tensile oscillation of a steel wire** of the same material: Similar as for the spring we can also obtain the Young's modulus from the oscillation dynamics of the system. We measure for $m = 1.081\,kg$, a period $T = 8.58/20s$. Transformation of the equation from above provides the spring constant for wire elongations:

$$k = \frac{EA}{L}$$

With

$$T = 2\pi\sqrt{\frac{m}{k}}$$

we obtain

$$k = 4\pi^2\frac{m}{T^2} \approx 40\frac{1kg}{0.25s^2} = 160\frac{N}{m}$$

such that we get for the Young's modulus:

$$E = \frac{160N/m \cdot 7.1m}{3 \cdot 2.5 \cdot 10^{-9}m^2} \approx 1.6 \cdot 10^{-11}\frac{N}{m^2} = 160\ GPa$$

Overall the obtained values are in the range of Young's moduli obtained for different vaieties of steel:

Material	$E\ [10^9\ \mathrm{N/m^2}]$
Aluminium	71
Cast iron	64–181
Ferrite steel	108–212
Stainless steel	200
Copper	125
Tungsten	407
Lead	19
Fused silica	75
Water ice (−4 °C)	10

Transforming the equation from above provides a more universal form that does not contain the geometric parameters anymore:

$$\frac{F}{A} = E\frac{\Delta L}{L}$$

Defining the pressure equivalent

$$\sigma = \frac{F}{A}$$

as the **tensile stress** with $[\sigma] = [N/m^2] = Pa$ and the relative elongation

$$\varepsilon = \frac{\Delta L}{L}$$

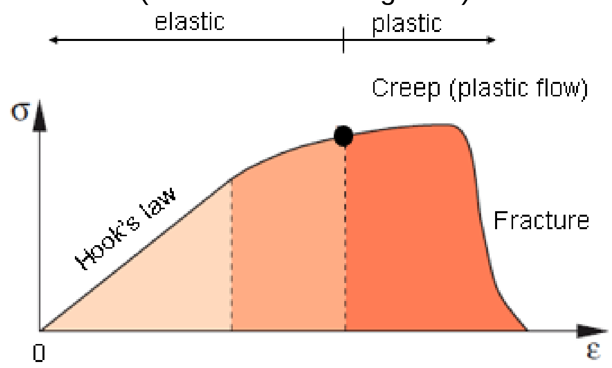
as the unitless **strain**, we get a universal relation for tensile elongations that does not contain the geometric parameters anymore**:**

$$\sigma = E\ \varepsilon$$

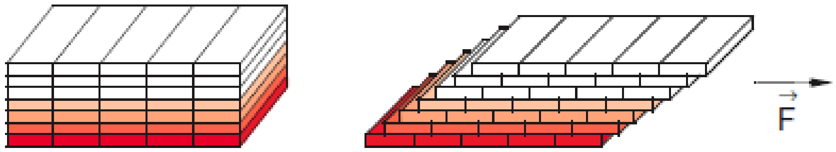
Experiment (not part of lecture): Watch out, the simple relations above become more complicated if the diameter of the object changes. For example, when pressing an elastic sphere onto a surface, a model developed by Hertz is applied. In this case the contact area increases with force $A \propto F^{2/3}$ and the force increases non-lineraly with the compression distance $F \propto d^{3/2}$.

Larger deformations of solids

At small deformations there is a linear relationship between stress and strain. For larger extensions the higher order Taylor terms cannot be neglected (since the interaction potential grows less steeply than the quadratic approximation) such that the stress typically increases less strongly for a given strain increase (central colored segment):

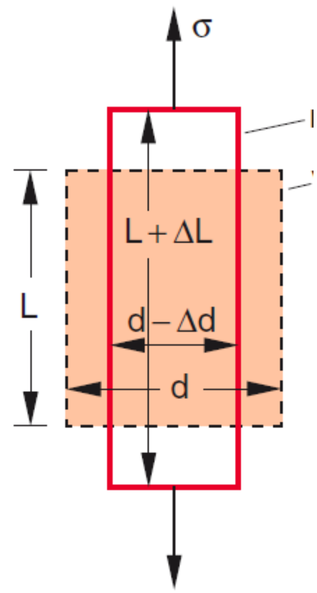


Extension over the so-called elastic limit point leads to plastic deformations, which is **called creep (or plastic flow).** Here, the atomic planes start to become displaced against each other, leading to irreversible deformations.



In reality, defects of the crystal lattice play an important role for plastic deformations, such as dislocations and other lattice defects. At even higher stress **fracture** takes place.

Experiment: Stretching of different objects using a motor and a force sensor allows to record their stress-strain behavior: (i) A spring yields linear behavior, while (ii) a polymerstring (e.g. Haribo Apfelring) shows a strongly non-linear behavior. At even larger deformations we see creep and finally rupture of the object.



C) Transverse contraction - Poisson effect

When deforming a solid with a force pair normal to two opposing surfaces there is typically a compression or an expansion along the transverse direction. Typical **materials tend to expand in transverse directions to the direction of compression**. In contrast, typical **materials tend to contract in the transverse directions to the direction of stretching.**

This can be seen in **experiments**:

- A rubber cord that is stuck in an aperture of a plate becomes mobile after pulling at its ends due to transverse contraction.
- An illustration for transverse contraction is the deformation of a circle to an ellipse by pulling on it (OHP-model)

The extend of transverse compression/expansion is a material property and is described by the **Poisson's ratio:**

$$\boldsymbol{\nu} = -\frac{\Delta d}{d} \Big/ \frac{\Delta L}{L}$$

It is the negative ratio of the transverse strain divided by the axial strain, i.e. it represents the **relative transverse contraction per relative elongation**. Since the transverse strain is typically negative for positive axial strain, the **Poisson's ratio is typically positive.**

Let us derive the volume change for axial stretching considering a **cuboid with quadratic base** in the limit of small axial stretching:

$$\Delta V = \underbrace{(d+\Delta d)^2(L+\Delta L)}_{after\ stretching} - \underbrace{d^2L}_{before}$$

Transformation gives

$$\Delta V = d^2\Delta L + 2Ld\Delta d + \underbrace{(L\Delta d^2 + 2d\Delta d\Delta L + \Delta L\Delta d^2)}_{corners\ with\ negligable\ volume}$$

Terms with two small length changes (corresponding to corners of the elongation volume) can be neglected for low strain. Division by the initial volume $V = d^2L$ provides the relative volume change:

$$\frac{\Delta V}{V} \approx \frac{\Delta L}{L} + 2\frac{\Delta d}{d} = \frac{\Delta L}{L}\left(1 + 2\frac{\Delta d}{d} \Big/ \frac{\Delta L}{L}\right)$$

such that we obtain with the definition of the Poisson's ratio, a formula for the relative volume change which is independent of the geometry of the object:

$$\frac{\Delta V}{V} = \varepsilon(1-2\nu) = \frac{\sigma}{E}(1-2\nu)$$

For typical materials the volume increases when stretching. Thus, in this case the Poisson's ratio must be $\nu < 0.5$. A big exceptionare **auxetic materials** where $\nu < 0$. These materials expand transversally when stretched.

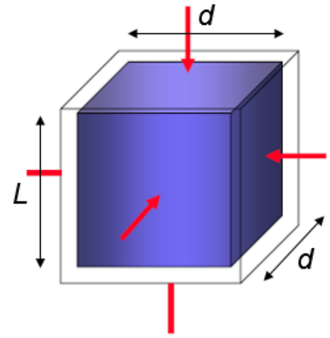
Experiment: Toy model for an auxetic material, which expands when stretched ($\nu < 0$).

2) Bulk compression of solids

Now we consider a compression with homogeneous pressure p from all sides using a cuboid with quadratic base.

Experiment: Hydrostatic pressure on a balloon (in a vacuum chamber) illustrates very nicely the concept of a bulk compression that acts from all sides (though a highly compressible gas instead of a solid is used).

To describe a bulk compression using the knowledge we acquired so far, we first consider the vertical extension of a cuboid with length L and quadratic cross-section of edge length d:



Due to the acting homogeneous pressure from all sides, we have for the vertical extension the superposition of two effects:
Normal strain, i.e. a vertical compression along L due to the external pressure p, which is equal to the (negative) normal stress acting on the surface:

$$p = -\sigma = -E\frac{\Delta L_{comp}}{L}$$

Transverse strain, i.e. an expansion, along L due to the normal compression along d for which we can write:

$$\nu = -\frac{\Delta L_{trans}}{L} \Big/ \frac{\Delta d}{d}$$

The normal strain from the pressure in the lateral direction (along d) is by analogy given as:

$$p = -E\frac{\Delta d}{d}$$

Using this equation to replace $\Delta d/d$ in the equation for the transverse strain and transformation gives:

$$\frac{\Delta L_{trans}}{L} = -\nu\frac{\Delta d}{d} = +\nu\frac{p}{E}$$

The total strain in axial direction is provided by the normal strain and twice the transverse expansion to consider the compression along each of the two lateral directions:

$$\frac{\Delta L}{L} = \frac{\Delta L_{comp}}{L} + 2\frac{\Delta L_{trans}}{L} = -\frac{p}{E}(1-2\nu)$$

Similarly, we obtain for the total strain along each of the two lateral directions:

$$\frac{\Delta d}{d} = \frac{\Delta d_{comp}}{d} + 2\frac{\Delta d_{trans}}{d} = -\frac{p}{E}(1-2\nu)$$

From this we can calculate the relative volume change according to the previously derived equation:

$$\frac{\Delta V}{V} = \frac{\Delta L}{L} + 2\frac{\Delta d}{d} = -\frac{3p}{E}(1-2\nu)$$

$\Delta V/V$ defines the **volume strain**, which according to this equation is proportional to the applied pressure which represents the “bulk stress”. We can transform this equation to:

$$p = -K\frac{\Delta V}{V} = -\frac{E}{3(1-2\nu)}\frac{\Delta V}{V}$$

where we define $\boldsymbol{K}$ **as the bulk modulus** corresponding to the required pressure per relative volume reduction. According to our derivation, K is given by:

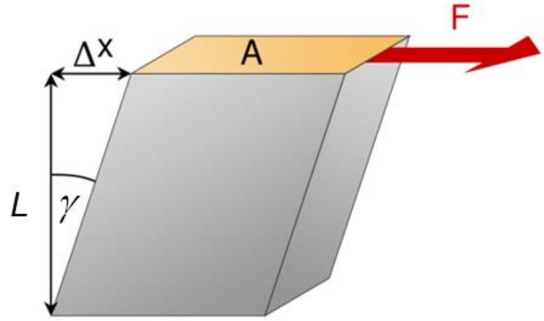
$$\frac{1}{K} = \kappa = \frac{3}{E}(1-2\nu)$$

where we call $\boldsymbol{\kappa}$ **the compressibility**, that provides the relative volume reduction per applied pressure.

3) Shear deformation of solids

So far, we considered only forces acting normal to a given surface. A second major type of **stress acts parallel to the surface and causes a so-called shear deformation** (see figure below).

Experiment: Model of a shear deformation, OHP



Such a stress is called **shear stress** (being coplanar, tangential stress) and is analogously given by the tangential force per area:

$$\tau = \frac{F}{A}$$

Shear stress causes an angular deformation γ of the object which is called the shear strain. It is thus given by:

$$\tan\gamma = \frac{\Delta x}{L}$$

which for small shear strain becomes:

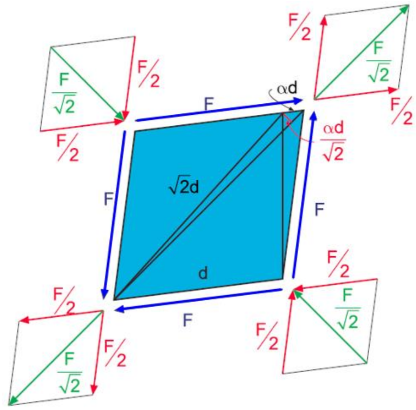
$$\gamma \approx \frac{\Delta x}{L}$$

i.e. it is the lateral surface displacement normalized by the thickness of the object in normal direction to the surface. For small γ, one can describe the shear stress again by a linear function of the shear strain:

$$\tau = G\,\gamma$$

with $\boldsymbol{G}$ **being the shear modulus.**

A shear deformation can be considered as a simultaneous stretch deformation in one direction and a contraction in the orthogonal direction. This can be seen when regrouping the acting forces on a sheared cube:



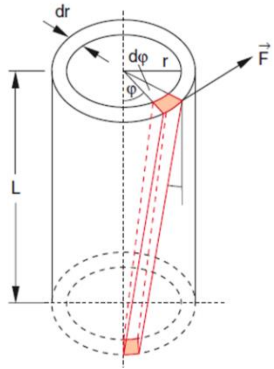
The original blue forces cause the shear deformation as discussed before. Note that we have here 4 forces. The additional force pair ensures a zero torque and would in the normal shear configuration be exerted by the surfaces that are shearing against each other. When splitting each of the 4 forces into halves we get the red corner forces. Adding the two corner forces at each corner into one effective force provides a stretching along one diagonal and a compression along the perpendicular diagonal.
Treating the shear deformation as two tensile deformations (similarly as done for the bulk modulus) provides the following mutual relations between G, E, K and ν in case of an isotropic material (i.e. the elastic properties are direction independent, **show on slides**):

$$\frac{E}{2G} = 1 + \nu, \qquad \frac{E}{3K} = 1 - 2\nu, \qquad \frac{2G}{3K} = \frac{1 - 2\nu}{1 + \nu}$$

Thus, two quantities are sufficient to calculate the other quantities. The following table provides an overview over the moduli for the different deformations:

Material	E [10^9 N/m^2]	G [10^9 N/m^2]	K [10^9 N/m^2]	ν
Aluminium	71	26	74	0.34
Cast iron	64–181	25–71	48–137	0.28
Ferrite steel	108–212	42–83	82–161	0.28
Stainless steel	200	80	167	0.3
Copper	125	46	139	0.35
Tungsten	407	158	323	0.29
Lead	19	7	53	0.44
Fused silica	75	32	38	0.17
Water ice (−4 °C)	10	3.6	9	0.33

<u>Twisting of solids as shear deformation</u>:
A special type of shear deformation is the **twisting of an object**, such as a cylindrical homogeneous wire as we will see in the following. Envision a wire of radius R that is composed of concentrically arranged hollow cylinders of radii r and wall thicknesses dr.



Looking at a vertical segment of such a shell reveals (see sketch above) that **twisting a single hollow cylinder by an angle φ gives a similar displacement as for a shear deformation.**

Experiment: Model for a shear deformation during object twisting (shadow projection)

The lateral displacement along the cylinder arc is given by $\Delta x = r\varphi$ for a cylinder length of length L. If $r\varphi \ll L$, the shear strain is given by the small angle approximation:

$$\gamma = \frac{r\varphi}{L}$$

which depends on the radius r, segments close to the axis are less sheared compared to more distal segments. We therefore get for the shear stress on the thin cylinder shell:

$$\tau = G\,\gamma = G\,\frac{r\varphi}{L}$$

The force that is required for such a twist/shear deformation of a thin hollow cylinder is given as:

$$dF = \tau\, dA = G\frac{r\varphi}{L}2\pi r\, dr$$

which for twisting translates into a torque (now we use a different symbol for the torque to avoid a variable clash):

$$d\Gamma = rdF = G\frac{\varphi}{L}2\pi r^3\, dr$$

Now the total torque required to twist the full cylinder is the sum of all torques that are needed to twist each hollow cylinder, which is obtained by integration:

$$\Gamma = \int d\Gamma = \int_0^R G\frac{\varphi}{L}2\pi r^3\, dr = \frac{G}{L}\frac{\pi}{2}R^4\,\varphi$$

Thus, the torque required for a given angular displacement increases linearly with the twisting angle! This allows to consider the wire as a torsional spring as done in the past. The **back-driving torque** is then given by:

$$\Gamma_{back} = -k_{tor}\,\varphi$$

with k_{tor} being the twist rigidity, which is given according to the equation above by:

$$k_{tor} = \frac{G}{L}\underbrace{\frac{\pi}{2}R^4}_{\substack{\text{polar moment}\\ \text{of area}}}$$

The twist rigidity is proportional to the shear modulus and inversely proportional to the wire length. The term on the righthand side is a **geometry factor of the cross-sectional area**. It is called **polar moment of area** and describes the material distribution in the cross-section (see later).

Experiments: Determination of the shear modulus of a steel rod and a steel wire

- Determination of the shear modulus from a static torsional deformation of a steel rod ($d = 2\,mm$) by a tangential force acting at distance R from the rod axis:

$$k_{tor} = \frac{F\,R}{\Delta\varphi} = \frac{0.147\ \mathrm{Nm}}{36°/360°\ 2\pi} = 0.25\,\frac{\mathrm{Nm}}{\mathrm{rad}}$$

The shear modulus is then given as

$$G = \frac{2k_{tor}L}{\pi R^4} = \frac{2\cdot 0.25\ \mathrm{Nm}\cdot 0.5\mathrm{m}}{3.14\cdot(10^{-3}\mathrm{m})^4} = 8\cdot 10^{10}\,\frac{\mathrm{N}}{\mathrm{m}^2} = 80\ \mathrm{GPa}$$

- Determination of the shear modulus of a steel wire ($r = 0.05\ \mathrm{mm}, L = 0.57\ \mathrm{m}$) from a dynamic torsional oscillation of a cylindrical weight ($R = 1.5\ \mathrm{cm}, m = 0.34\ \mathrm{kg}$) attached to the wire end. We measure in the experiment a period of about $T = 32.4\ \mathrm{s}$. From

  $$T = 2\pi\sqrt{\frac{I}{k_{tor}}}$$

  we obtain

  $$k_{tor} = 4\pi^2\frac{I}{T^2} \approx 40\cdot\frac{1/2\cdot 0.34\ \mathrm{kg}\cdot(1.5\ \mathrm{cm})^2}{(32.4\ \mathrm{s})^2} = 1.4\cdot 10^{-6}\,\frac{\mathrm{Nm}}{\mathrm{rad}}$$

  and we obtain for the shear modulus:

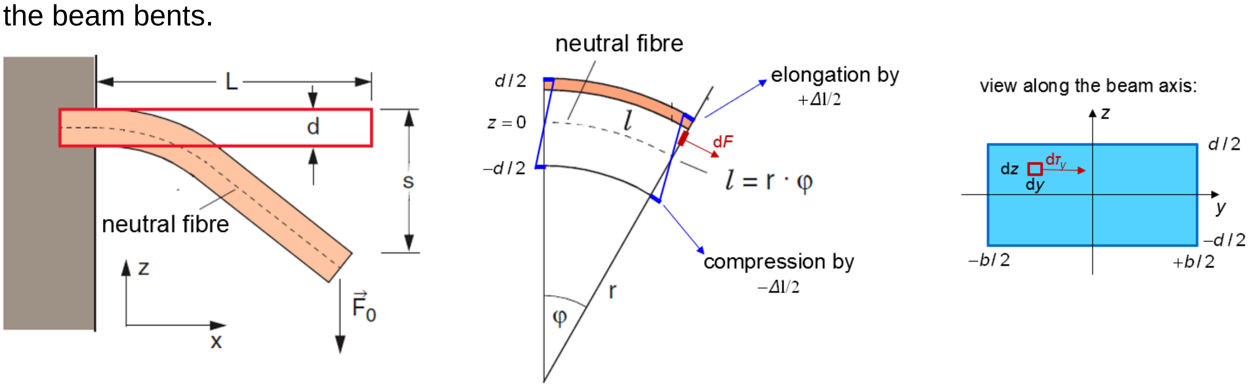
  $$G = \frac{2k_{tor}L}{\pi R^4} = \frac{2\cdot 1.4\cdot 10^{-6}\ \mathrm{Nm}\cdot 0.57\mathrm{m}}{3.14\cdot(5\cdot 10^{-5}\mathrm{m})^4} = 8\cdot 10^{10}\,\frac{\mathrm{N}}{\mathrm{m}^2} = 80\ \mathrm{GPa}$$

4) Bending of beams

After considering twisting of objects as shear deformations we will now look at bending of objects, in particular beams. We will see that bending can be described as multiple parallel tensile deformations.

A) Beam equation

For technical constructions, the bending of beams, bars etc. is of tremendous importance. Here we look at a simply example: a **cantilevered beam with a terminal force**. This is a beam which is **anchored at one end with a fixed orientation.** At its other end there shall be a **constant force F_0 acting in perpendicular direction** to the beam. We now want to determine how strongly the beam bents.



To this end, we look at a **short segment with length *l*** of the beam where we can define a **local bending radius *r*** at its center. While the lower part of the beam becomes compressed, the upper part becomes extended. Only the **neutral line** (or fibre) does not change its length. For a rectangular beam cross-section the neutral line is due to symmetry reasons located in the middle of the beam along the *z* direction. The neutral line can thus be defined through the whole beam.

Experiment: Visualization of neutral line of a suspended beam under load using birefringence induced by the tension

We now associate an angle φ to the arc formed by l, such that together with the bending radius we can write $l = r\varphi$. The length change for a line segment at distance z from the neutral line can then be written as:

$$\Delta l = (r+z)\varphi - r\varphi = z\varphi = z\frac{l}{r}$$

A given line segment experiences thus a tensile stress (upper half) or a compression stress (lower half) of:

$$\sigma = E\frac{\Delta l}{l} = E\frac{z}{r}$$

To achieve the combined compression/stretching stresses, a torque about the neutral fibre from the normal forces dF in the cross-section is required. This torque points along y, i.e. into the drawing plane in the figure. The torque per line segment with cross section $dz\,dy$ is (see figure):

$$d\tau_y = z\,dF = z\,\sigma dA = \frac{E}{r}z^2\,dz\,dy$$

The total torque is then obtained by a double integration over the whole beam cross section:

$$\tau_y = \int d\tau_y = \frac{E}{r}\underbrace{\int\int z^2 dydz}_{= I\,(\text{cross section dep.})}$$

The double integral I is a factor that is only dependent on the geometry of the cross-sectional area. The derived expression provides the so-called **beam equation**:

$$\tau_y = EI\frac{1}{r} = EI\,\kappa$$

where we define

$\boldsymbol{\kappa = 1/r}$	**as local curvature of the beam**
$\boldsymbol{I}$	**as second moment of area** describing the geometry of the cross-sectional area
$\boldsymbol{EI}$	**as flexural rigidity**, i.e. the resistance of the beam material & cross section geometry against bending

B) Second moment of area

With z being the distance to the neutral line, the **second moment of area** is according to our derivation given by:

$$I = \int\int z^2\,dydz$$

The quadratic dependence on the distance to the neutral line (z^2) provides that there is an increased resistance against bending as more the area is distributed away from the neutral line in analogy to the moment of inertia.

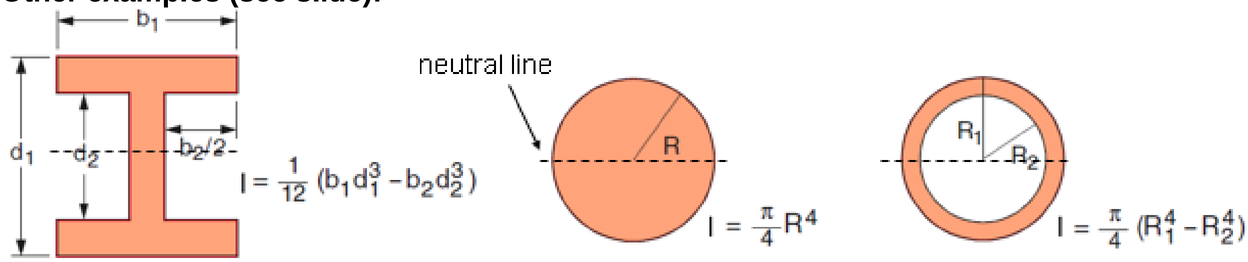
Example: The second moment of area for a beam with rectangular cross section bent along z is thus given as

$$I = \int_{-d/2}^{d/2}\int_0^b z^2 dy\,dz = \int_{-d/2}^{d/2} b\,z^2 dz = \frac{b}{3}z^3\Big|_{-d/2}^{d/2} = \frac{bd^3}{3\cdot 8} + \frac{bd^3}{3\cdot 8}$$

such that we get:

$$I = \frac{b}{12}d^3$$

Other examples (see slide):



- The double-T beam has a large part of its area distant to the neutral axis and is therefore an efficient way to obtain a rigid beam with minimal material use. It is therefore frequently used in constructions.
- Beam with circular cross section:

$$I = \frac{R^2}{4}\pi R^2 = \frac{R^2}{4}A$$

- Hollow circular beam:

$$I = \frac{1}{4}(R_1^2 + R_2^2)\pi(R_1^2 - R_2^2) = \frac{1}{4}(R_1^2 + R_2^2)A$$

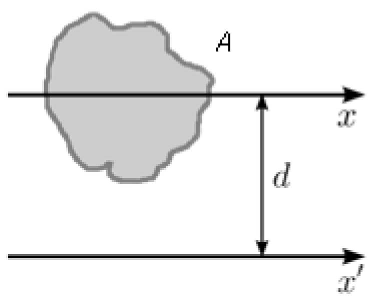
with thin wall approximation $R_1 \approx R_2 = R$:

$$I = \frac{R^2}{2}A$$

Thus, a hollow beam is more rigid than a filled beam with equal cross-sectional area, allowing an increased stiffness for the same amount of material used.

Parallel axis theorem

Due to similar mathematical form compared to the moment of inertia there is also a parallel axis theorem for the moment of area:



With:

- I_x being the known second moment of area about a neutral line x of area A and
- $I_{x'}$ being the unknown second moment of area about a different neutral line x' (e.g. in a composite geometry of two cylindrical rods)

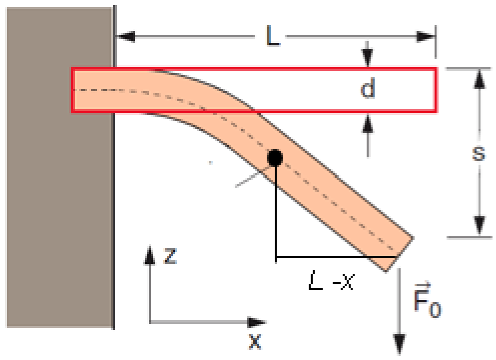
we can write:

$$I_{x'} = I_x + Ad^2$$

where A is the total area of the beam cross section and d is the distance between the two neutral lines.

C) Bending of cantilevered beam

In the following we will apply the beam equation to the example of a beam that is suspended at one end only. This is called a cantilevered beam. It extrudes in normal direction from a support on which it is rigidly mounted.



A force F_0 acting at the beam end **causes a torque on the beam cross-section at position *x*** of $\tau_y = F_0(L-x)$ in the limit of a small beam deflection. We want to find the curve of the deformed beam in form of a function $z(x)$. The locally acting torque creates the local line segment expansion/compression whose torque dependence we calculated above. We thus can write:

$$\tau_y = F_0(L-x) = EI\frac{1}{r}$$

such that we obtain the local bending radius. From differential geometry follows a relationship between local bending radius and the derivatives of the beam curve:

$$\frac{1}{r} = \frac{z''(x)}{(1+z'(x)^2)^{\frac{3}{2}}}$$

For small beam deflections we have $z'(x) \ll 1$, such that:

$$\frac{1}{r} \approx |z''(x)|$$

For our geometry the 2nd derivative of the beam is negative, such that we get:

$$-z''(x) = \frac{F_0}{EI}(L-x)$$

Integration provides then the 1st derivative of the beam curve:

$$z'(x) = -\frac{F_0}{EI}\left[C_1 + Lx - \frac{1}{2}x^2\right]$$

A second integration provides the beam curve itself

$$z(x) = -\frac{F_0}{EI}\left[C_2 + C_1 x + \frac{1}{2}Lx^2 - \frac{1}{6}x^3\right]$$

In this case we obtained indefinite integrals, where C_1 and C_2 are the associated integration constants. These define the beam position and the slope of the beam at $x = 0$, where the beam is mounted. Using the (boundary) conditions $\mathrm{z}(0) = 0$ and $\mathrm{z}'(0) = 0$ corresponding to a left beam end at position zero with an horizontal orientation, we obtain by inserting $C_1 = 0$ and $C_2 = 0$. This provides finally:

$$z(x) = -\frac{F_0}{EI}\left[\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right]$$

The deflection of the beam end we get by inserting $x = L$, which provides:

$$s = z(L) = -\frac{F_0}{3EI}L^3$$

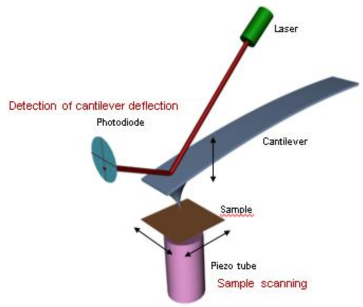
We see that the beam deflection increases with the 3rd power of beam length. Inserting the moment of inertia for a rectangular beam gives:

$$s = -\frac{4F_0}{E}\frac{L^3}{\underbrace{b\,d^3}_{\text{rectangul. beam}}}$$

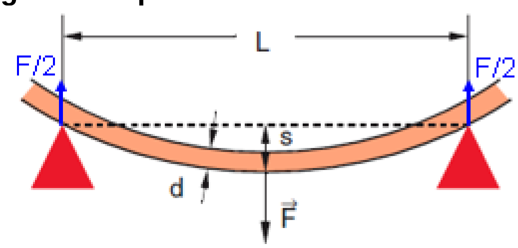
The deflection decreases with the 3rd power of the beam thickness. Making the beam wider only provides a simple proportionality. For the same cross-sectional area, it is thus beneficial to use a high but narrow beam to minimize bending (e.g. a wooden board in an upright orientation).

Experiment: Bending of cantilevered rod. We can show the proportionality of the deflection with force and the scaling of deflection with 3rd power of length

Cantilevered beams are widely used in **atomic force microscopy** (AFM, see slide) which is a nanoimaging technique based on mechanical forces:



Bending of a suspended beam



Let us finally consider a beam that is suspended at both ends at which it is allowed to undergo a free rotation. A force F shall act in the beam center. For small deflections we can again apply our solution to the beam equation. Considering just one half of the beam of length $L/2$, we have the same boundary conditions as for the single sided beam, particularly a zero derivative of the beam path in the center. At each end at distance $L/2$, a force of $F/2$ acts upwards, since the total force splits equally between the two supports. To get the center deflection of the suspended beam we can simple insert $L/2$ and $F/2$ into the formula for the curve of the single-sided beam. This provides:

$$s = -\frac{F}{3EI}\frac{L^3}{16}$$

It provides a 16-fold reduction of the deflection compared to the cantilevered beam with single-site attachment.

Experiment: Comparison of bending of suspended filled and hollow cylindrical beams made from the same amount of material (mass) per length. The hollow cylinder is deflected less due to its higher second moment of area.

DNA origami multihelix bending: see slides

Introduction to the concept of DNA origami nanostructures and their measurements with magnetic tweezers. Stretching experiments can be used to evaluate the bending rigidity.

The measured bending rigidity agrees well with the 2^{nd} moment of area prediction, while the torsional rigidity fails the so-called polar moment of inertia prediction, since the origami does not have a sufficient cylinder symmetry.
The mechanical properties of such complicated objects can be simulated using finite element simulation software.

Lecture 18: Experiments

- Static tensile elongation of a steel wire (*L* = 7.07m, *r* = 0.05 mm). Check of the linearity $L \propto F$ (Hooke's law) when stretching the wire. Determine the Young's modulus of the wire with a mass of *m* = 0.8 kg providing an extension of *ΔL* = ≈33.7 mm from which one gets $E = 0.21 \cdot 10^{12}\mathrm{Pa} = 210\ \mathrm{GPa}$
- Tensile oscillation of a steel wire of the same material: Similar as for the spring we can also obtain the Young's modulus from the dynamics of the system. We measure for m = 1.081kg, T = 8.58/20 s and get $E = \omega^2 Lm/A = 2.05 \times 10^{11}\ \mathrm{Pa}$
- Stretching of different objects using a motor and a force sensor allows to record their stress-strain behavior: (i) A spring yields linear behavior, while (ii) a polymerstring (e.g. Haribo Apfelring) shows a strong non-linear behavior. At even larger deformations we see creep and finally rupture of the object

- Transverse contraction of a rubber cord

- Model for transverse contraction: deformation of a circle into an ellipse (OHP-Model)

- Model for an auxetic material ($\nu < 0$)

- (Hertz model for the compression of a sphere on a surface, A prop. F^2/3)

- Hydrostatic compression of a balloon (in vacuum chamber)

- Model for a shear deformation (OHP)

- Model for a shear deformation during object twisting (shadow projection)

- Determination of the shear modulus of a steel rod by applying a static torque (after the torsional deformation the lever arm and the applied weight are perpendicular to each other, *τ* = 0.147Nm, φ = 36◦, L = 0.5m, d = 2mm); G = 7.5x10^10 Pa

- Determination of the shear modulus of a steel strong (r = 0.05mm, L = 0.57m) from a torsional oscillation of an attached cylindric weight (R =1.5 cm, m = 0.337kg); from T = 32.4 s we get k = 1.42x10^−6 Nm and a shear modulus of G = 8.3x10^10 Pa. With E = 2.10x10^11 Pa we get ν = (E/2G) − 1 = 0.27

- Visualization of neutral line of a suspended beam under load using birefringence induced by the tension

- Bending of cantilevered rod. We can show the proportionality of the deflection with force and the scaling of deflection with 3rd power of length

- Comparison of bending of double-side supported cylindrical filled and hollow beams made from same amount of material (mass) per length