Lecture "Experimental Physics I"

(Prof. Dr. R. Seidel)

Lecture 25

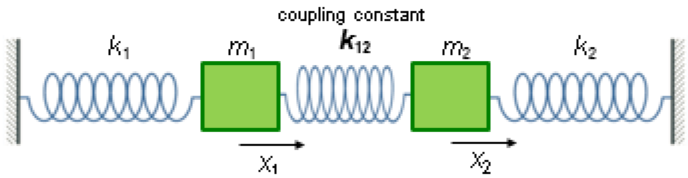
Coupled Oscillators

- Coupled oscillators
- Normal modes

1) Two coupled oscillators

An important extension of simple oscillators are systems of coupled oscillators. Coupling of two oscillating processes happens frequently in the physics world. Also, coupling between oscillators will bring us to the vibration of continuous objects and to waves, which can be simplified considered as a large arrays of coupled oscillators. The latter is of tremendous importance in acoustics, electrodynamics and quantum mechanics.

A **simple implementation** of two coupled oscillators is provided by two spring oscillators with a third spring (k_{12}) in between that somehow couples the oscillations of the individual oscillators:



Experiment: Such a configuration can be realized on the air track, where we observe a complex motion of the individual oscillators including an energy transfer, i.e. a differential excitation between both oscillators.

A) General solution

To describe the behavior of the coupled oscillator system we can write down the **equations of motion** – one for each oscillator. The displacement of each mass from the equilibrium position shall be x_i. The extension of the coupling spring is given by $x_2 - x_1$. Considering the direction of the force, which it exerts on each mass gives then the following two equations:

$$m_1\ddot{x}_1 = -k_1x_1 + k_{12}(x_2 - x_1)$$
$$m_2\ddot{x}_2 = -k_2x_2 - k_{12}(x_2 - x_1)$$

In both equations we have x_1 and x_2 such that this is a **coupled system of differential equations**. With two suitable linear combinations of the equations, they can be decoupled (e.g. using matrix notation and getting eigenvectors and eigenvalues) and solved.
We simplify the problem by **assuming that the spring constants and masses of the single oscillators are equal**, i.e. $k_1 = k_2 = k$ and $m_1 = m_2 = m$. Addition and subtraction of both equations then provides:

$$m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2)$$
$$m(\ddot{x}_1 - \ddot{x}_2) = -k(x_1 - x_2) - 2k_{12}(x_1 - x_2)$$

Now we define the position differences in the brackets as two new coordinates - the so called **normal coordinates:**

$$\xi^+ = \underbrace{½(x_1 + x_2)}_{\text{Center of mass motion}} \quad ; \quad \xi^- = \underbrace{½(x_1 - x_2)}_{\text{relative motion}}$$

ξ^+ describes the motion of the center of mass and **ξ^- the relative distance between the masses.** Inserting the new coordinates into the differential equation provides a decoupled equation set:

$$m\ddot{\xi}^+ = -k\xi^+$$
$$m\ddot{\xi}^- = -(k + 2k_{12})\xi^-$$

These equations **represent two simple oscillator equations.** For these we know that they are each solved by a single sinusoidal function:

$$\xi^+(t) = A_+\cos(\omega_+t + \varphi_+)$$
$$\xi^-(t) = A_-\cos(\omega_-t + \varphi_-)$$

For the solutions, we know the angular frequencies from the prefactor of the displacement in each differential equation, i.e.:

$$\omega_+ = \sqrt{k/m}$$
$$\omega_- = \sqrt{(k+2k_{12})/m}$$

Thus, the **center of mass motion occurs at ω_+, i.e. the simple oscillator frequency** and the **relative motion occurs at a higher frequency ω_-**.

These two different oscillations are called **normal modes.** The new coordinates $\xi^{+/-}$ **are called normal coordinates**. Within a given normal mode **all elements of the system move sinusoidally with the same frequency and with fixed phase relation (see also later).**

B) Inserting the start conditions

Using the definition of the normal coordinates, we can now calculate the real positions of the single oscillators according to:

$$x_1(t) = \xi^+ + \xi^- = \underbrace{C_1 \cos\omega_+ t + C_2 \sin\omega_+ t}_{A_+ \cos(\omega_+ t + \varphi_+)} + \underbrace{C_3 \cos\omega_- t + C_4 \sin\omega_- t}_{A_- \cos(\omega_- t + \varphi_-)}$$
$$x_2(t) = \xi^+ - \xi^- = C_1 \cos\omega_+ t + C_2 \sin\omega_+ t - C_3 \cos\omega_- t - C_4 \sin\omega_- t$$

where we expressed each normal mode osccilation already as a sum of a cosine and a sine function. The **oscillation of a single oscillator in the coupled system** is thus a **superposition of two harmonic oscillations with different frequencies!**

The constants C_i are defined by the start conditions. For simplification, we assume in the following only initial displacements but zero initial velocities as start conditions:

$$x_1(0) = x_{10}\,; \qquad v_1(0) = \dot{x}_1(0) = 0$$
$$x_2(0) = x_{20}\,; \qquad v_2(0) = \dot{x}_2(0) = 0$$

Inserting the starting conditions into our solution from above provides:

$$x_{10} = C_1 + C_3; \qquad 0 = \omega_+ C_2 + \omega_- C_4$$
$$x_{20} = C_1 - C_3\,; \qquad 0 = \omega_+ C_2 - \omega_- C_4$$

since only the cosine terms are non-zero at $t = 0$. Addition and subtraction of either the two position or the two vleocity equations provides:

$$C_1 = (x_{10} + x_{20})/2\,; \qquad C_2 = 0$$
$$C_3 = (x_{10} - x_{20})/2\,; \qquad C_4 = 0$$

such that we finally get for initial displacements x_{10} and x_{20} and zero initial velocities:

$$x_1(t) = \tfrac{1}{2}(x_{10} + x_{20}) \cos\omega_+ t + \tfrac{1}{2}(x_{10} - x_{20}) \cos\omega_- t$$
$$x_2(t) = \underbrace{\tfrac{1}{2}(x_{10} + x_{20})}_{\xi^+(0)} \cos\omega_+ t - \underbrace{\tfrac{1}{2}(x_{10} - x_{20})}_{\xi^-(0)} \cos\omega_- t$$

Note that the amplitudes of the individual sinusoidal terms correspond in this case to the initial displacments of the two normal modes.

C) Symmetric normal mode ξ^+

In the following we want to look at different start conditions. Let us first test conditions, where we have an oscillation with just a single frequency, i.e. **pure normal mode oscillations**. To this end we start only with an initial displacement of a single normal mode, particularly the ξ^+ mode. This is achieved by an equal initial displacement of both masses such that **only the center of mass is displaced**:

$$x_{10} = x_0$$
$$x_{20} = x_0$$

such that $\xi^+(0) = x_0$ and $\xi^-(0) = 0$. Inserting in our solution eliminates the ω_- terms,such that we get:

$$x_1(t) = x_0 \cos\omega_+ t$$
$$x_2(t) = x_0 \cos\omega_+ t$$

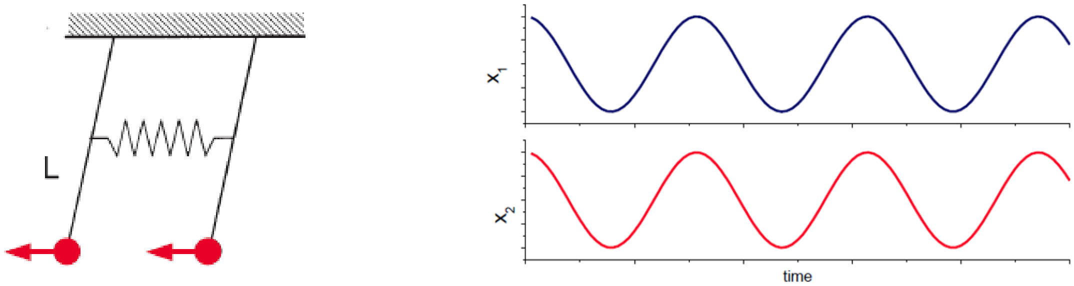
i.e. each mass oscillates only at ω_+. This excites only the first normal mode, which can be easily checked:

$$\xi^+ = ½(x_1 + x_2) = x_0 \cos \omega_+ t$$
$$\xi^- = ½(x_1 - x_2) = 0$$

i.e. we **only have a center of mass motion**

The coupled oscillator motion corresponds in this case to an **in-phase oscillation of both masses with the simple oscillator frequency.** Both masses move symetrically, such that this mode is called **symmetric mode**.

$$\omega_+ = \sqrt{k/m}$$



Experiment: This mode can be easily demonstrated using the previously used coupled oscillator and an equal initial displacement of both masses.

D) Anti-symmetric normal mode ξ^-

Now we start only with an initial displacement of the ξ^- mode, which is achieved by an equal initial displacement of both masses on opposite directions such that only **the relative distance is changed** but **the center of mass is not displaced**:

$$x_{10} = x_0$$
$$x_{20} = -x_0$$

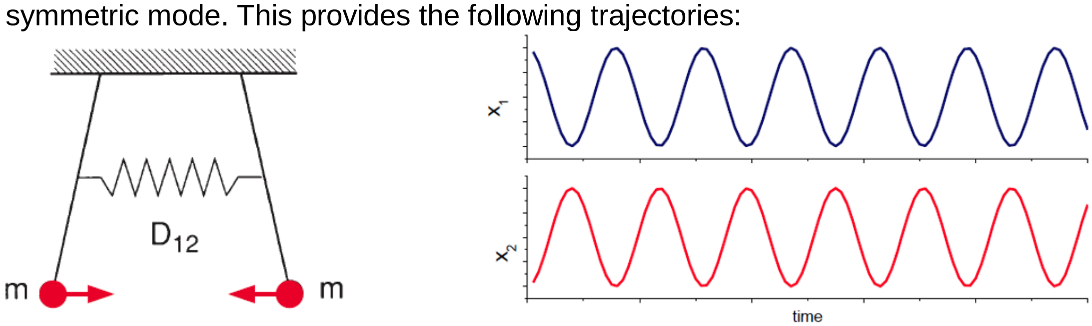
such that $\xi^+(0) = 0$ and $\xi^-(0) = x_0$. Inserting into the solution eliminates the ω_+ terms, which gives:

$$x_1(t) = x_0 \cos \omega_- t$$
$$x_2(t) = -x_0 \cos \omega_- t = x_0 \cos(\omega_- t + \pi)$$

i.e. each mass oscillates only at ω_-. Due to the opposite sign of $x_1(t)$ and $x_2(t)$, both masses move always in opposite directions, such that this mode is called **anti-symmetric mode**. This corresponds to a phase shift between the two oscillations of of π or $T/2$. Since

$$\omega_- = \sqrt{(k + 2k_{12})/m}$$

The oscillation in the anti-symmetric mode occurs at a higher frequency compared to the symmetric mode. This provides the following trajectories:



The solution in this case excites only the second normal mode, which can be quaickly checked:

$$\xi^+ = ½(x_1 + x_2) = 0$$

$$\xi^- = ½(x_1 - x_2) = x_0 \cos \omega_- t$$

i.e. we have **only a realtive motion of both masses.** In summary we can verify that **in a normal mode, all parts** of the system **move sinusoidally with the same frequency and a fixed phase relation.**

E) Excitation of both normal modes

Any other motion of the coupled oscillator can be described as a superposition of the two normal mode oscillations, i.e. it is a superposition of a center-of-mass and a relative motion. To demonstrate this, we choose as start condition an initial displacement for only one of the oscillators:

$$x_{10} = x_0$$
$$x_{20} = 0$$

This changes the center of mass position as well as the position difference, such that both modes should get excited. Inserting the start conditions into the solution provides that the oscillation of each mass contains equal contributions from both frequencies:

$$x_1(t) = ½x_0(\cos \omega_+ t + \cos \omega_- t)$$
$$x_2(t) = ½x_0(\cos \omega_+ t - \cos \omega_- t)$$

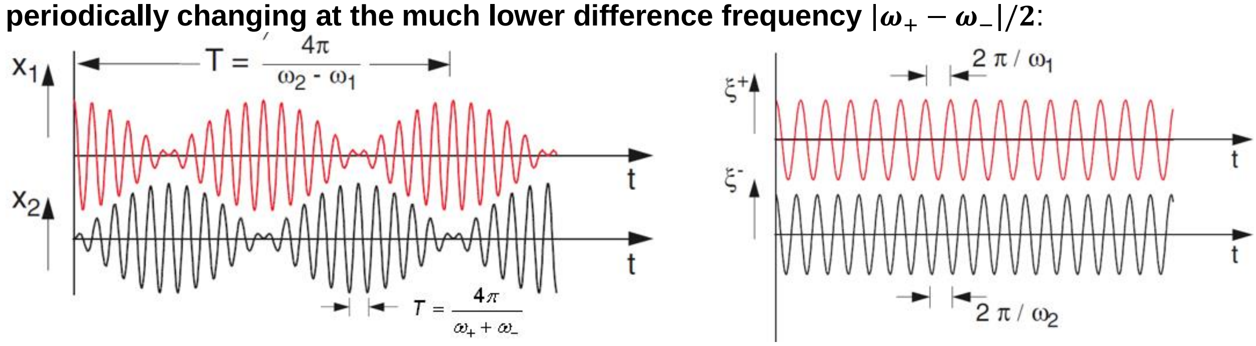
Here both normal modes are equally excited:

$$\xi^+ = ½(x_1 + x_2) = ½x_0 \cos \omega_+ t$$
$$\xi^- = ½(x_1 - x_2) = ½x_0 \cos \omega_- t$$

The equation above can be converted using the sum rule for the cosine into

$$x_1(t) = ½x_0(\cos \omega_+ t + \cos \omega_- t) = x_0 \cos[½(\omega_+ + \omega_-)t] \cos[½(\omega_+ - \omega_-)t]$$
$$x_2(t) = ½x_0(\cos \omega_+ t - \cos \omega_- t) = x_0 \sin[½(\omega_+ + \omega_-)t] \sin[½(\omega_+ - \omega_-)t]$$

This is a product between two sinusoidal functions. It describes an **oscillation of each mass at the average frequency** $(\boldsymbol{\omega_+ + \omega_-})/\mathbf{2}$ **of the two modes** but with an **amplitude that is periodically changing at the much lower difference frequency** $|\boldsymbol{\omega_+ - \omega_-}|/\mathbf{2}$:



The modulation of the amplitude is called the “**beat**”. It can for example be experienced in an **experiment** when listening to two oscillating and slightly mistuned (guitar) strings.
The phase shift between the amplitude modulation of the two oscillators ensures **energy conservation**: Energy is transmitted between the oscillators but the total energy remains constant. The **time to transmit the energy** from one oscillator and back is given by $\mathbf{2\pi}/|\boldsymbol{\omega_+ - \omega_-}|$.

Experiments:

- Coupled pendulum: **Excitation of individual modes and composite case**. Tracking of pendulum positions allows to calculate the frequency spectrum. The frequency spectrum returns only the frequencies of both normal modes. In agreement with the equations above, the fast oscillation calculated from the period in the trajectories corresponds to $(f_+ + f_-)/2$.

There are also other coupled systems, where the coupling is less clear:

- Coupled linear and torsional oscillation of a large mass on a strong spring. Both motions are coupled via a twist-stretch coupling, i.e. when elongating the spring also a torsion around the spring axis is created and vice versa when twisting the spring it tends to shorten, such that a tensile stress is created.
- Coupled pendulum and linear oscillation of a mass on a spring. In addition to the linear oscillation a mass-on-a-spring system is additionally excited to undergo a pendulum motion. The pendulum motion

F) Energy transfer between the oscillators (only on slides)

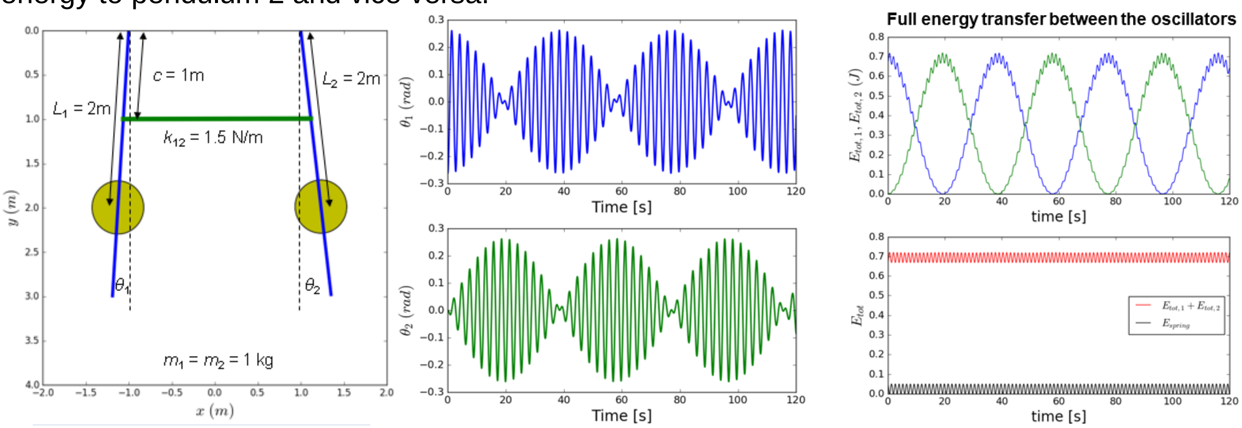
We said before that during the coupled oscillation the energy gets periodically transferred from one oscillator to the other and vice versa. To check this, we numerically solve the coupled oscillator problem for two identical pendula that are coupled by a spring for the depicted configuration below. When initially displacing only one pendulum this yields the same beating pattern as before, i.e. we equally exit the symmetric and the anti-symmetric normal mode. Energy is stored in three components of the system: the two pendula and the spring. The mechanical energy of the each pendulum is given by the sum of potential and kinetic energy:

$$E_{tot,i} = \underbrace{mgL_i(1-\cos\theta_i)}_{E_{pot,i}} + \underbrace{\frac{m}{2}(L_i\dot{\theta}_i)^2}_{E_{kin,i}}$$

The mechanical energy of the spring is given by:

$$E_{spring} = \frac{k_{12}}{2}(c\sin\theta_1 - c\sin\theta_2)^2$$

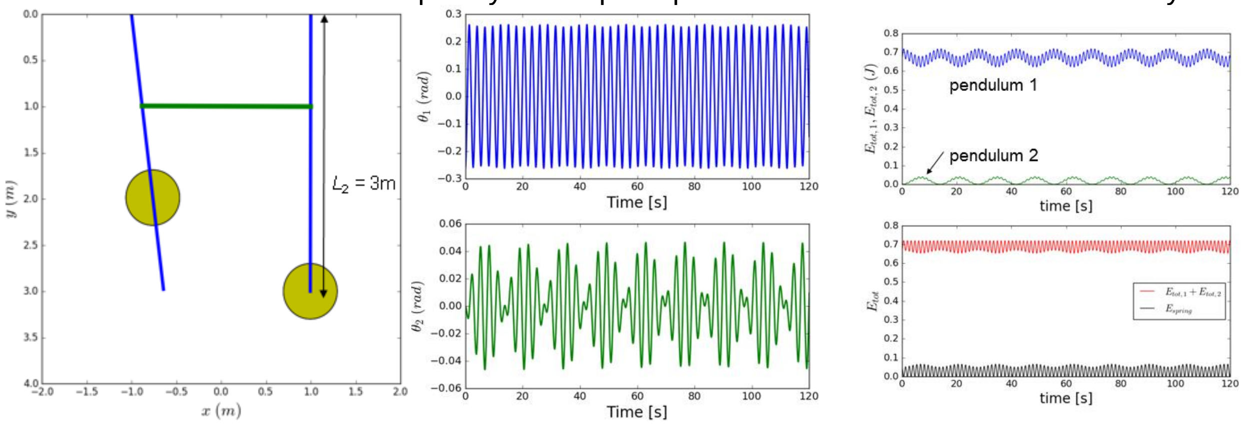
Plotting the mechanical energy of each pendulum shows, that pendulum 1 really transfers all its energy to pendulum 2 and vice versa:



Adding the mechanical energy of both pendula shows that the total energy is practically conserved except a little oscillatory variation that can be explained by the energy that is additionally stored in the spring due to the excitation of the anti-symmetric normal mode.
For the symmetric oscillator system we say that **both oscillators are resonant,** because they can efficiently **exchange energy.** This is due to the **same resonance frequency**, such that the oscillation of one pendulum can drive the oscillation of the other pendulum.

The situation is markedly different for two pendula that are not equal (see figure below):
When initially exciting pendula 1 there is still a periodic excitation of pendulum 2. However, the energy transfer is only partial and pendula 1 retains at all time large part of its mechanical energy. For the **asymmetric oscillator system both pendula are not in resonance**. In general, efficient energy transfer in physics typically requires resonant conditions.

Interestingly, both pendula alone have different frequencies but when coupled they oscillate with the same fast frequency. This can be explained by the fact that also in this case we have two normal modes with different frequency that superimposed describe the full motion of the system.



Lecture 25: Experiments

1) Coupled oscillator system with 3 springs and 2 masses on the air track (only qualitative demo)
2) Beat between two slightly mistuned strings
3) Coupled pendulum: Excitation of individual modes and composite case. Tracking of pendulum positions allows to calculate the frequency spectrum. The frequency spectrum returns only the frequencies of both normal modes, while the fast oscillation seen in the trajectories corresponds to $(f_+ + f_-)/2$
4) Coupled linear and torsional oscillation of a large mass on a strong spring
5) Coupled pendulum and linear oscillation of a mass on a spring that is additionally excited to undergo a pendulum motion