8. Momentum and Conservation Laws

The Chapters 3.1–3.4 of my lecture notes provides the background to solve the following exercises. Your solution to the problems 8.1–8.4 should uploaded

to your Moodle account
as a PDF-file
by Wednesday, Nov 29, 10:30 (with a grace time till the start of the seminars).

The parts marked by ⋆ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

The self-test problems serve to check you understanding of vectors.

The bonus problem provides the solution to a problem that we introduced in class. It might take some extra effort to solve.

Problems

Problem 1. Contour lines and gradients

a) Determine the gradients of the functions

$$A(\mathbf{q}) = A_0 \sin(\mathbf{q} \cdot \mathbf{q})\,, \qquad B(\mathbf{q}) = B_0 \sin(\mathbf{k} \cdot \mathbf{q})$$

where $\mathbf{k} \in \mathbb{R}^2$ is a constant vector and $\mathbf{q} = (x, y)$ a position in $\mathbb{R}^2$.

b) Determine the lines $y_c(x)$ where the functions take constant values. On a map these are contour lines. For a potential these are equipotential lines.

⋆ c) Show that the gradient is oriented vertically to the contour lines.
Hint: Consider a trajectory $\mathbf{q}_c(t) = \big(x(t), y_c(x(t))\big)$, and argue
1) that it describes the motion along a contour line,
2) that $\dot{\mathbf{q}}_c(t)$ is therefore aligned parallel to the contour line, and
3) that $\dot{\mathbf{q}}_c(t)$ is orthogonal to the gradient.

d) Sketch the contour lines of the functions, and mark the gradients by arrows.

Problem 2. Line integral along a helix

We consider the paths

$$\gamma_1 = \Big\{\mathbf{q}_1(t) = (1,\ 0,\ 2\pi\, t) \text{ with } 0 \le t \le 1\Big\}$$

$$\gamma_2 = \{\mathbf{q}_2(\theta) = (\cos\theta,\ \sin\theta,\ \theta) \ \text{ with } 0 \le \theta \le 2\pi\}$$

a) Verify that the paths have the same initial point and end point.

Sketch their form.

b) We consider now the force field $\mathbf{F}(x, y, z) = \Big(a\ \sin z\ ,\ a\ \cos z\ ,\ g\, z\Big)$.
Evaluate the line integral

$$W_i = \int_{\gamma_i} \mathrm{d}\mathbf{s} \cdot \mathbf{F}(x, y, z)\,.$$

Bonus: The result does not depend on a! Why?

c) Is $\mathbf{F}(x, y, z)$ a conservative force field?
Substantiate your argument, and provide conditions on the parameter values a and g, if required.

Problem 3. Elastic two-particle collisions

We consider the positions of two balls, $i \in \{1, 2\}$, with masses m_i and radii R_i at positions $\mathbf{q}_i \in \mathbb{R}^3$. In the beginning of the experiments, at time t_0, they have velocities $\dot{\mathbf{q}}_i(t_0) = \mathbf{v}_{0,i}$.

⋆ a) Determine the evolution of the center of mass $\mathbf{Q}(t)$ of the two balls. How does it evolve when the two balls are thrown on a playground, with gravitational acceleration $\mathbf{g} = -g\hat{z}$ acting?

b) Consider now the motion of the two balls relative to their center of mass (CM). Let these positions be $\mathbf{r}_i(t) = \mathbf{q}_i(t) - \mathbf{Q}(t)$ and the associated momentum be

$\mathbf{p}_i(t) = m_i\, \dot{\mathbf{r}}_i(t)$. Show that the following relations hold for the motion relative to the center of mass, irrespective of the choice of initial conditions

$$\mathbf{p}_{\text{rel}} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0} \qquad \text{and} \qquad \mathbf{L}_{\text{rel}} = (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{p}_2\,.$$

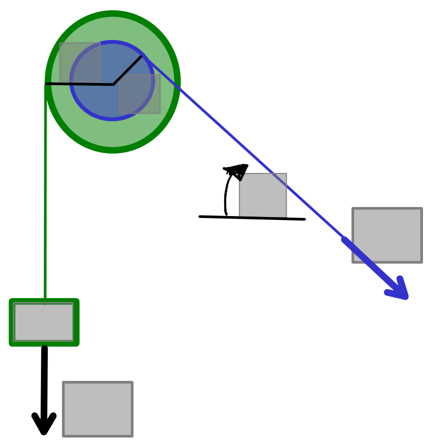
c) Compare the result of (b) with the relations that we discussed in the lecture for the case of two disks that were not subjected to an external gravitational force. How do the results obtained in the lecture carry over to the present system?
Hint: Most effectively this is answered by showing that for all times the vectors $\mathbf{r}_2$ and $\mathbf{p}_2$ are orthogonal to $\mathbf{L}_1 = \mathbf{r}_1 \times \mathbf{p}_1$, and that therefore $\mathbf{r}_1$, $\mathbf{p}_1$, $\mathbf{r}_2$, and $\mathbf{p}_2$ will always lie in a plain.

⋆ d) How do the trajectories look like before the collision (in the CM system and for an observer standing on the playground).

⋆ e) How do the momenta of the particles change in an elastic collision? How do the trajectories look like after the collision (in the CM system and for an observer standing on the playground).

Problem 4. Differential pulley

We consider a differential pulley (cf. sketch to the right) where a load of mass M is attached to the green chord that is released from the outer roller of the pulley. It has radius R. We hold the pulley at rest by pulling with a force $\mathbf{F}$ at the blue chord that is released from the inner roller of radius r. This line will have an angle α with respect to the horizontal: For $\alpha = \pi/2$ we are pulling downwards, for $\alpha = 0$ straight to the right. The rollers are fixed to one another and attached to a common axle.



a) Add the labels into the gray fields in the sketch. Fields with a thick border refer to vectors; those with thin borders to scalar quantities.

b) Which torques are acting on the rollers?
Which force $\mathbf{F}$ is needed such that the total torque $\mathbf{T}$ on the differential pulley vanishes?

c) Determine the change in the potential energy of the load when the blue chord is reeled in by a length ℓ. What does this imply for the magnitude of the force $\mathbf{F}$.

d) Assume that $r = 10\,\mathrm{cm}$, $R = 30\,\mathrm{cm}$ and $M = 30\,\mathrm{kg}$. Would you hold the chord for a moment when somebody is offering it to you? Motivate your decision!

e) Which force $\mathbf{F}_A$ on the axle *must* act in addition to $\mathbf{F}$ and $m\,\mathbf{g}$ such that it does not move?

   Hint: Determine which force $\mathbf{F}_A$ is needed in addition to the ones acting along the chords such that there is no net force acting on the pulley.
   Why does $\mathbf{F}_A$ not contribute to the torque?
   Determine the absolute value of $\mathbf{F}_A$ as function of α, and plot $|\mathbf{F}_A|/Mg$ as function of α for $R/r = 3$.

⋆ f) Consider a case where the axle of the pulley is poorly greased. As the consequence the pulley only moves when the torque induced by the forces along the chords exceeds a value $\varepsilon\, R\, |\mathbf{F}_A|$.
   What is the smallest value of ε such that the pulley does not move when you do not pull, $\mathbf{F} = \mathbf{0}$?

   Due to the friction there is an interval of absolute values of the forces, $\mathbf{F}$, where the net torque vanishes. Determine the limits of the interval for $\alpha = \pi/2$ and a friction coefficient $0 \leq \varepsilon < 1$.
   What happens to the limits of the interval when ε approaches 1? Provide a physical interpretation for your observation.

Self Test

Problem 5. Contour lines and gradients

a) Determine the contour lines and the gradients of

$$f_1(x, y) = \mathrm{e}^x \, \mathrm{e}^{-y^2}$$

Sketch the contour lines of $f_1(x, y)$, and mark the gradients by arrows.

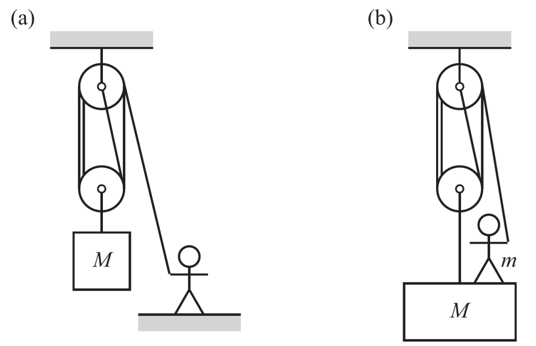
* b) Determine the contour lines and the gradients of

$$f_2(x, y) = \sin x \; \sin y + \cos x \; \cos y$$

Sketch the contour lines of $f_2(x, y)$, and mark the gradients by arrows.
Hint: Observe the trigonometric sum rules.

Problem 6. Tackling tackles and pulling pulleys

The sketch to the right shows two tackles where a person of mass m is hauling a weight of mass M. We do not consider effects of friction and the mass of the rope.



a) Which forces are required to hold the balance in sketch (a) and (b)?

b) Let the sketched person and the weight have masses of 75 kg and 300 kg, respectively. Which power is required when to haul the line at a speed of 1 m/s.
Hint: The power is defined here as the change of (a) $M g \, z(t)$ and (b) $(M + m) \, g \, z(t)$, per unit time, respectively.

c) Determine the change of the potential energy of the system. Which contributions does it involve?

Bonus Problem

Problem 7. The length of a spiral

The phase space plot of a trajectory of the damped, harmonic oscillator with initial excitation $q(t_0) = A$ and zero velocity amounts to a spiral

$$\begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} = A\, \mathrm{e}^{-\gamma\,(t-t_0)/2} \begin{pmatrix} \cos\bigl(\omega(t-t_0)\bigr) \\ \sin\bigl(\omega\,(t-t_0)\bigr) \end{pmatrix}$$

a) At which point does the spiral start at $t = t_0$? How long does it take till the oscillator arrives at its asymptotic point? Where does the spiral end then?

b) What is the length L of this spiral in phase space? How long does it take to draw the line with a plotter whose pen moves with a constant speed v?