Re-take Exam. 30 March

Problem 1. Prove that for any $n \geq 2$, $n \in \mathbb{N}$, one has

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

Problem 2. Determine the supremum of the following set

$$\{\sqrt{n+1} - \sqrt{n} : n \in \mathbb{N}\}.$$

Problem 3. Provide an example of a positive sequence (a_n) such that $\sqrt[n]{a_n} \to 1$ as $n \to \infty$, but a_{n+1}/a_n does not tend to 1, as $n \to \infty$. (*clarifications*: $a_n > 0$ for any n. The sequence a_{n+1}/a_n must not have limit 1 ; it can have a different limit or no limit at all).

Problem 4. Compute the following limit

$$\lim_{n\to\infty} \frac{2 + n + 5n\sqrt{n}}{3 + 6\sqrt{n} + 2n\sqrt{n}}.$$

Problem 5. Check the following series for convergence

$$\sum_{n=1}^{\infty} \frac{n^{100}}{2^n + n^{101}}.$$

Problem 6. Provide an example of a continuous function $f : (1; 5) \to \mathbb{R}$ which is not differentiable at points $x = 2$ and $x = 4$, and is differentiable at all other points of $(1; 5)$.

Problem 7. Compute the following limit

$$\lim_{x\to 1} \frac{x + x^2 + x^3 + x^4 + x^5 - 5}{x - 1}.$$

Problem 8. Compute the following series

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!}$$

Problem 9. Find all local and global extrema of the following function

$$f(x) = x^3 e^x, \qquad f : [-10; 1] \to \mathbb{R}.$$

Problem 10. Compute the area of a region bounded by curves

$$y = x^2, \qquad y = x^3.$$

Problem 11. Determine all possible values of $\alpha \in \mathbb{R}$, $\alpha > 0$, such that the following improper integral is convergent

$$\int_0^1 \frac{3 + \frac{1}{\sqrt{x}}}{x^\alpha} dx$$

Problem 12. Let $u = (1, 1, 1)$ be a vector from $\mathbb{R}^3$. Provide an example of different vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ such that (u, v_1, v_2) and (u, v_3, v_4) are both linear bases of $\mathbb{R}^3$. In other words, provide two different pairs of vectors which together with $(1, 1, 1)$ form a linear basis of $\mathbb{R}^3$.