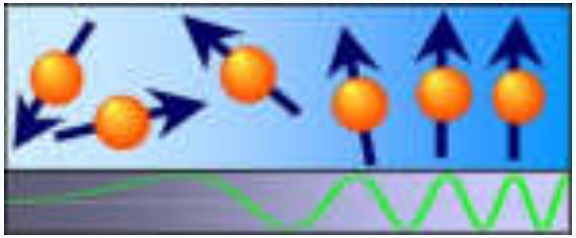
Experimental Physics
EP1 MECHANICS

- Gravity –



Rustem Valiullin

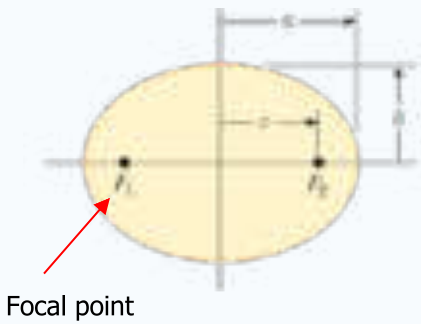
https://www.physgeo.uni-leipzig.de/en/fbi/applied-magnetic-resonance

Kepler's law

<u>Law 1</u>: All planets are moving in elliptical orbits with the Sun being in at one of the focuses.

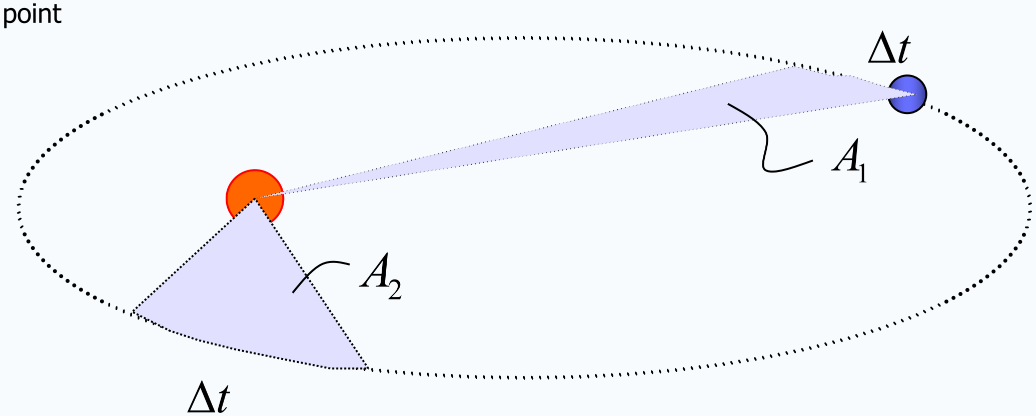
<u>Law 2</u>: A line joining the Sun and a planet sweeps out equal areas in equal times.

<u>Law 3</u>: The square of the period of a planet is proportional to the cube of the planet's distance from the Sun.



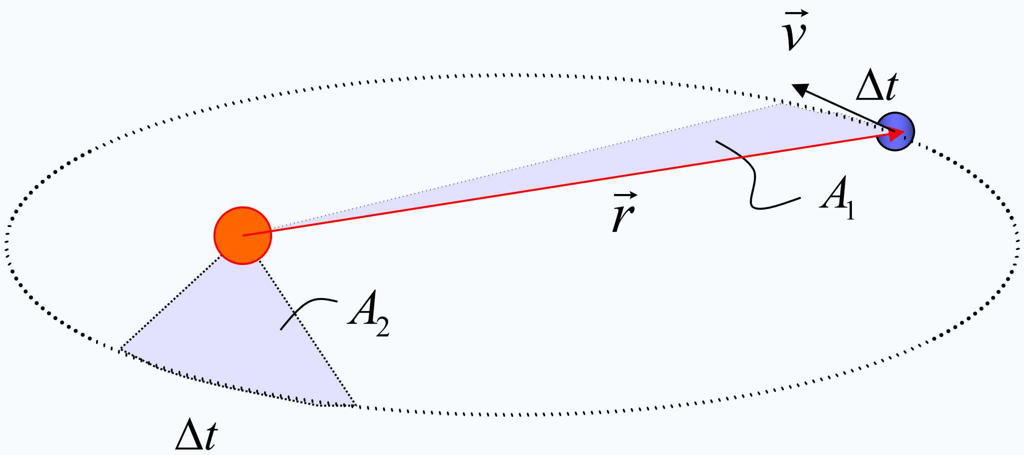
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$a^2 = b^2 + c^2$$



Born	December 27, 1571 Weil der Stadt near Stuttgart, Germany
Died	November 15, 1630) (aged 58) Regensburg, Bavaria, Germany
Residence	Württemberg; Styria; Bohemia; Upper Austria
Fields	Astronomy, astrology, mathematics and natural philosophy
Institutions	University of Linz
Alma mater	University of Tübingen
Known for	Kepler's laws of planetary motion Kepler conjecture

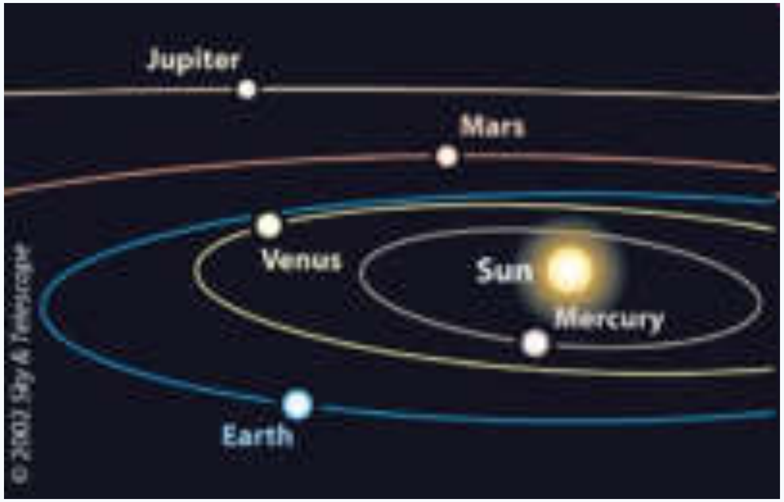
Second and third Kepler's law



$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \left(F \frac{\vec{r}}{r} \right) = 0$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d\left(\vec{r} \times m\vec{v}\right)}{dt} = 0$$

$$dA = \frac{1}{2}\left|\vec{r} \times d\vec{s}\right| = \frac{1}{2}\left|\vec{r} \times \vec{v}\,dt\right| = \frac{L}{2m}dt$$



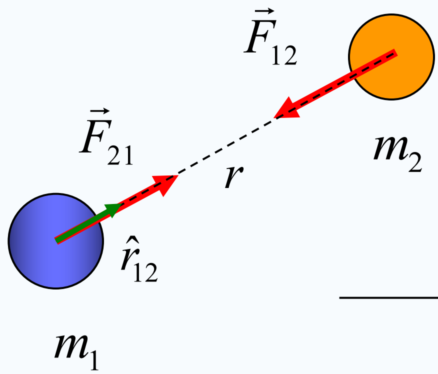
$$\frac{GmM}{r^2} = \frac{mv^2}{r} = mr\omega^2$$

$$\frac{GM}{4\pi^2} = \frac{r^3}{T^2} = \text{const}$$

T_{Venus} = 224.7 d **T_{Mars} = 686.98 d**

Newton's law of gravity

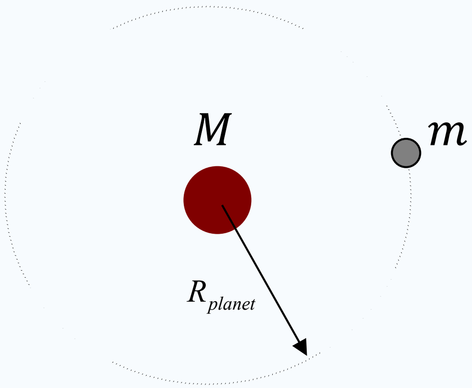
<u>1687</u>: Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.



$$\boxed{\vec{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\hat{r}_{12}} \qquad \hat{r}_{12} = \frac{\vec{r}}{r}$$

Universal gravitational constant: $G = 6.67\times 10^{-11}\ \mathrm{Nm^2kg^{-2}}$

$$a_{planet} = \frac{v^2}{R} = R\left(\frac{2\pi}{T}\right)^2 \qquad \frac{R^3}{T^2} = const \equiv K$$

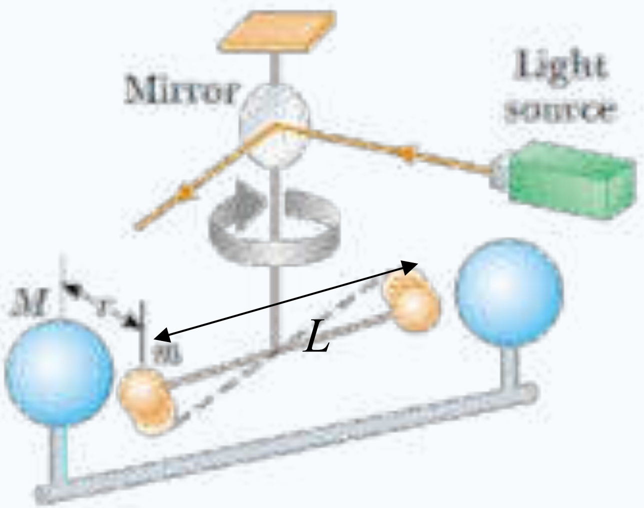


$$F_{cf} = m\frac{4\pi^2 K}{R^2} \qquad F_g = -m\frac{4\pi^2 K}{R^2}$$

$$F_g = -G\frac{mM}{R^2} = -\frac{4\pi^2 K}{M}\frac{mM}{R^2}$$

Measuring the gravitational constant

The **Cavendish experiment**, performed in 1797–98 by British scientist Henry Cavendish (not exactly shown below)



$$F = G\frac{mM}{r^2}$$

$$\tau = 2F\frac{L}{2} \qquad \tau = k\theta$$

$$T = 2\pi\sqrt{\frac{m}{k_s}} \qquad T = 2\pi\sqrt{\frac{I}{k}}$$

$$I = \sum ml^2 = 2m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{2}$$

$$k\theta = FL = \frac{GmM}{r^2}L \qquad T = 2\pi\sqrt{\frac{mL^2}{2}\frac{\theta r^2}{GmML}} = 2\pi\sqrt{\frac{L\theta r^2}{2GM}}$$

$$\boxed{G = \frac{2\pi^2 L\theta r^2}{MT^2}}$$

Cavendish's value for the Earth's density, 5.448 g cm^{-3}

$\Rightarrow$ G = 6.74 × 10^{-11} m^3 kg^{-1} s^{-2}

Our "Cavendish experiment"

$$s \approx l\alpha$$

$$m\frac{d^2 s}{dt^2} \approx \frac{mMG}{s_0^2}$$

$$t = 0, \quad s = 0$$

$$t = 0, \quad v_0 = 0$$

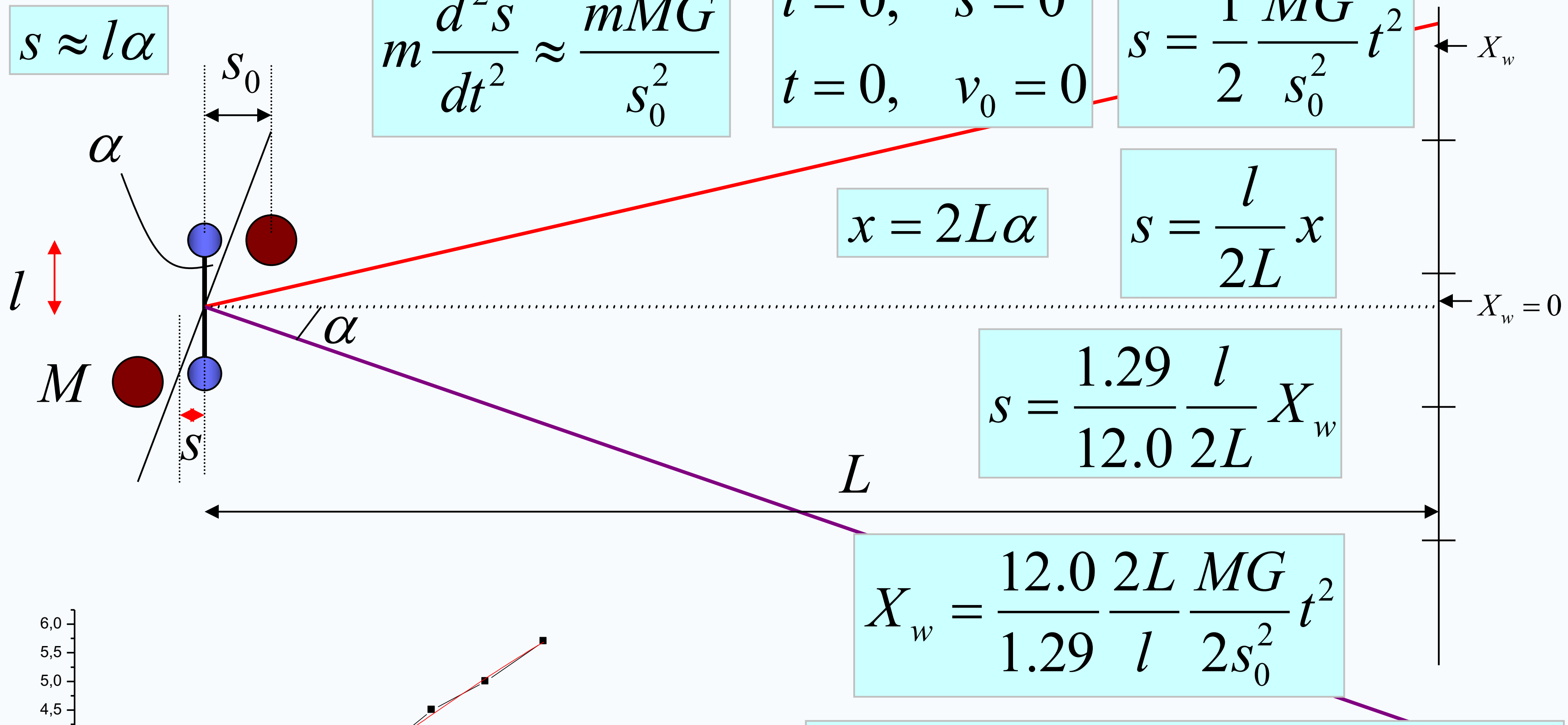
$$s = \frac{1}{2}\frac{MG}{s_0^2}t^2$$

$$x = 2L\alpha$$

$$s = \frac{l}{2L}x$$

$$s = \frac{1.29}{12.0}\frac{l}{2L}X_w$$

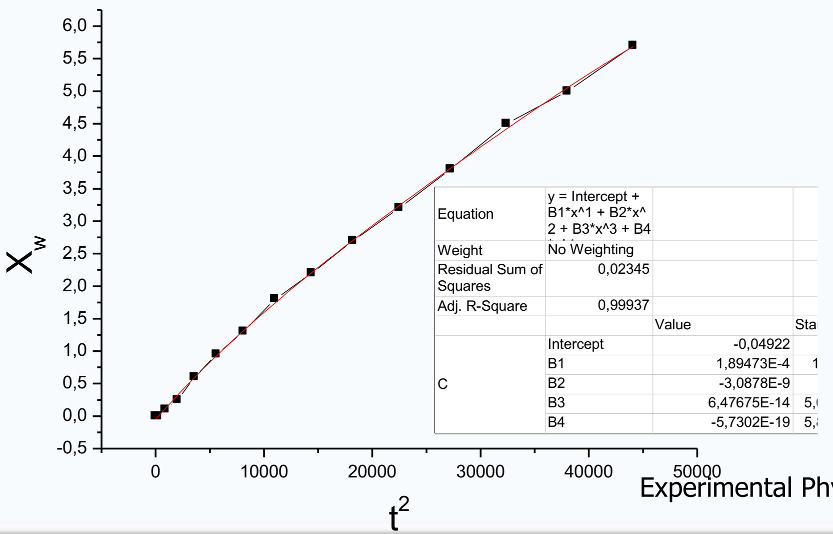
$$X_w = \frac{12.0}{1.29}\frac{2L}{l}\frac{MG}{2s_0^2}t^2$$



$$X_w = \frac{12.0}{1.29}\frac{2\times 19.4}{0.1}\frac{10.2}{2\times(0.08)^2}Gt^2 = 28.76\times 10^5 Gt^2$$

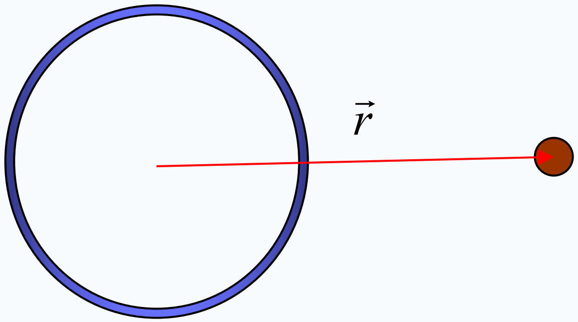
$$28.76\times 10^5 G = 1.89\times 10^{-4}$$

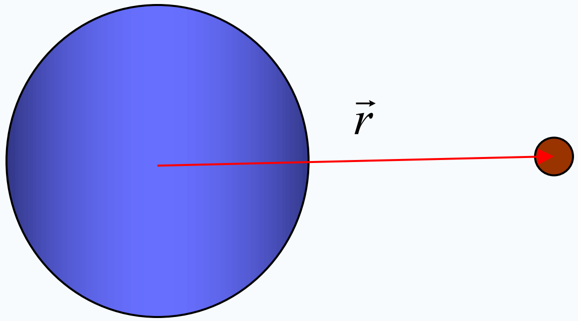
$$\boxed{G = 6.57\times 10^{-11}\ \mathrm{m^3 kg^{-1} s^{-2}}}$$



Equation	y = Intercept + B1*x^1 + B2*x^2 + B3*x^3 + B4		
Weight	No Weighting		
Residual Sum of Squares	0,02345		
Adj. R-Square	0,99937		
		Value	Sta
C	Intercept	-0,04922	
	B1	1,89473E-4	1
	B2	-3,0878E-9	
	B3	6,47675E-14	5,
	B4	-5,7302E-19	5,

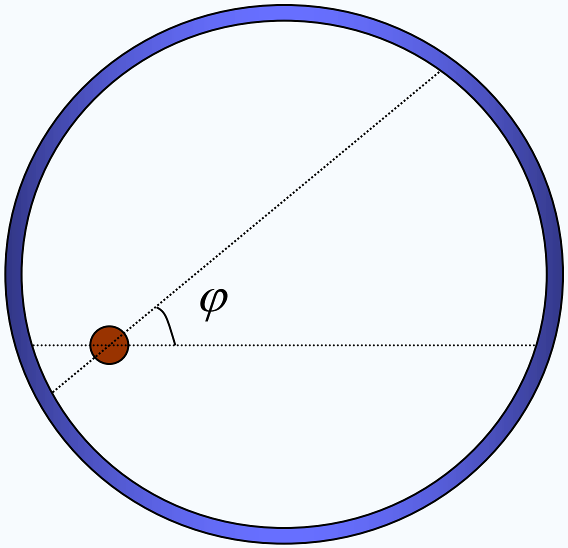
Gravitational force: extended objects

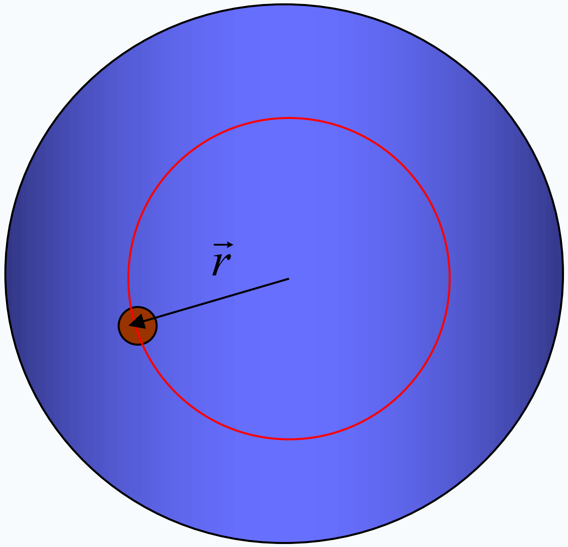




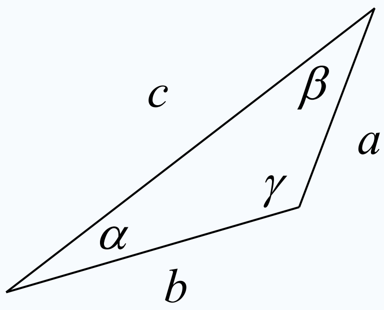
$$\begin{cases} \vec{F}_g = -\dfrac{GmM}{r^2}\hat{\mathrm{r}} & \mathrm{r} > \mathrm{R} \\ \vec{F}_g = 0 & \mathrm{r} < \mathrm{R} \end{cases}$$

$$\begin{cases} \vec{F}_g = -\dfrac{GmM}{r^2}\hat{\mathrm{r}} & \mathrm{r} > \mathrm{R} \\ \vec{F}_g = -\dfrac{GmMr}{R^3}\hat{\mathrm{r}} & \mathrm{r} < \mathrm{R} \end{cases}$$

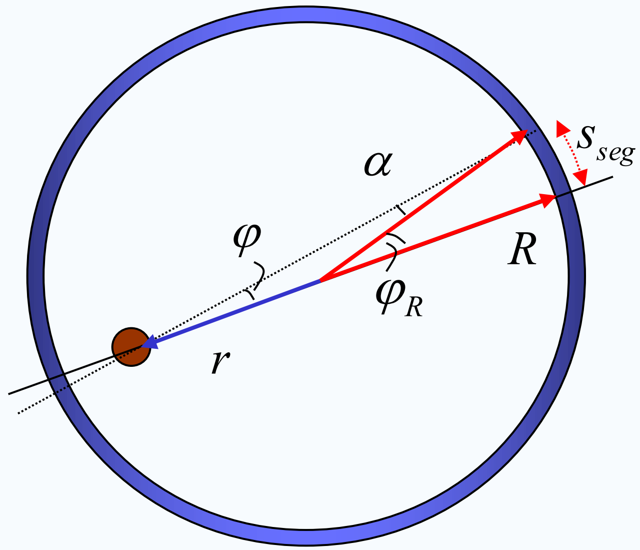




An object within a spherical shell



$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$



$$s_{seg} = R\varphi_R$$

$$\frac{\sin\varphi}{R} = \frac{\sin\alpha}{r} \qquad \Rightarrow \frac{\varphi}{R} \approx \frac{\alpha}{r}\Big|_{\text{small angles}}$$

$$\varphi + \alpha + (180° - \varphi_R) = 180°$$

$$\varphi_R = \alpha + \varphi = \varphi(1 + r/R)$$

$$s_{seg} = \varphi(R + r)$$

$$\Rightarrow m_{seg} \sim A_{seg} \sim (R+r)^2 = \text{distance}^2$$

$$F = K\frac{m_{seg}}{r^2} = K'$$ **- the same from both sides**

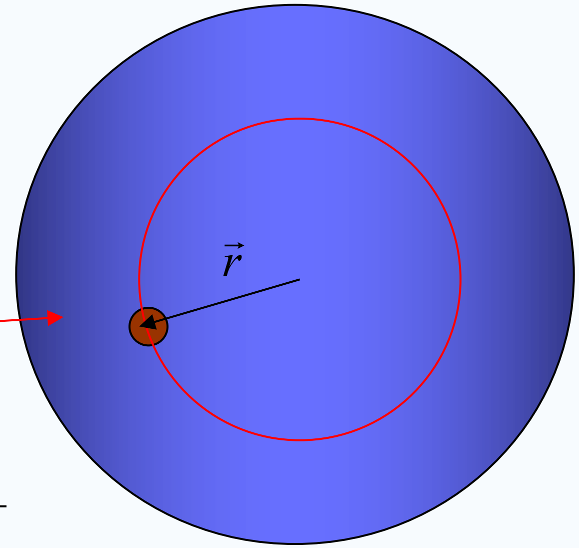
An object within a planet

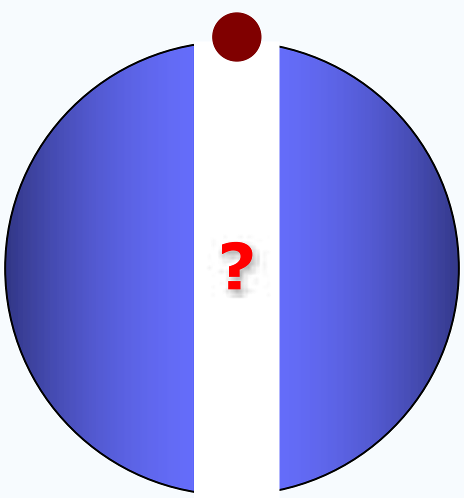
$$F = \frac{GmM(r)}{r^2}$$

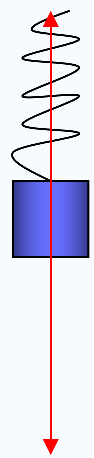
$$\begin{cases} M(r) = \frac{4}{3}\rho\pi r^3 \\ M_{tot} = \frac{4}{3}\rho\pi R^3 \end{cases}$$

$$\boxed{\vec{F} = -\frac{GmM_{tot}r}{R^3}\hat{\mathrm{r}}}$$

Does not contribute



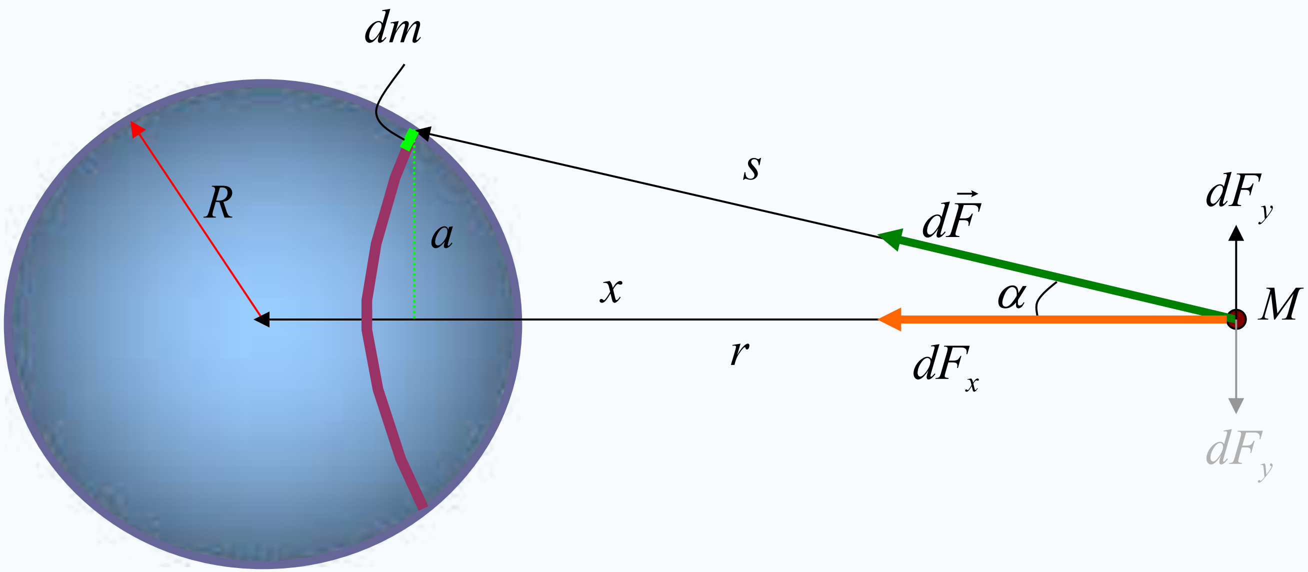




$$F = -kx = ma$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Interaction with ring



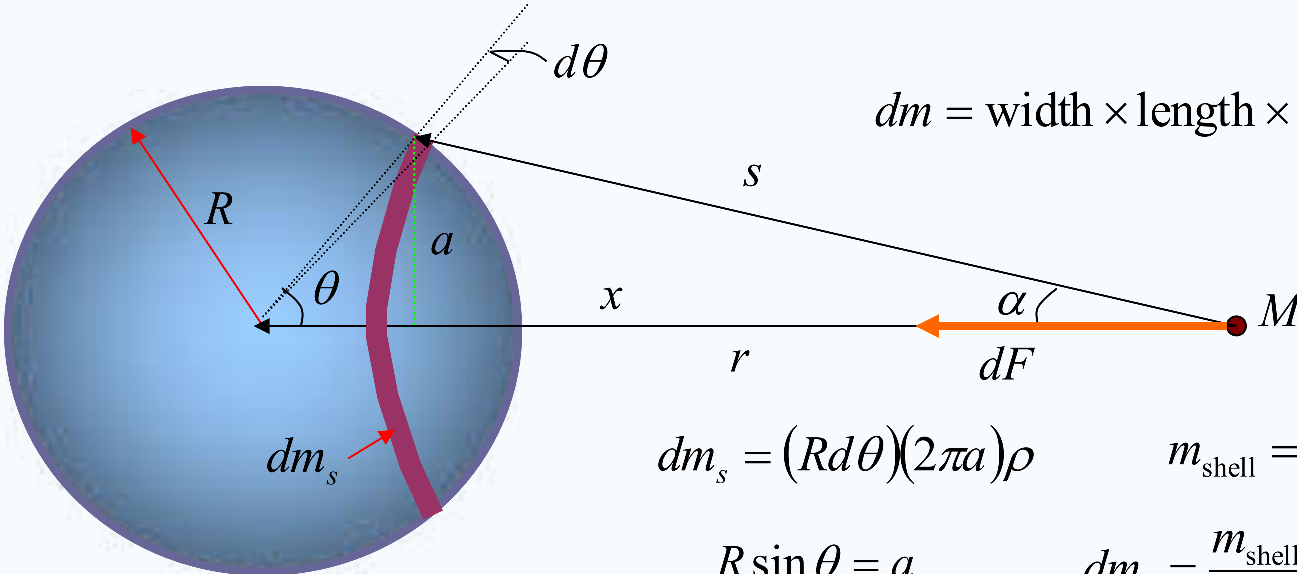
$$\boxed{dF = \frac{GM}{s^2}dm} \qquad dF_x = -\frac{GM}{s^2}dm\cos\alpha$$

$$F_x = -\int \frac{GM\cos\alpha}{s^2}dm = \frac{GMm_{\text{ring}}}{s^2}\cos\alpha \qquad s^2 = a^2 + x^2 \qquad \cos\alpha = \frac{x}{s}$$

integration over the ring

$$\boxed{F_x = \frac{GMm_{\text{ring}}}{s^2}\frac{x}{s} = \frac{GMm_{\text{ring}}x}{\left(a^2+x^2\right)^{3/2}}}$$

An object outside a hollow sphere



$$dm = \text{width} \times \text{length} \times \text{density}$$

$$dm_s = (Rd\theta)(2\pi a)\rho \qquad m_{\text{shell}} = 4\pi R^2 \rho$$

$$R\sin\theta = a \qquad dm_s = \frac{m_{\text{shell}}}{2}\sin\theta d\theta$$

$$dF_x = -\frac{GM}{s^2}\cos\alpha \frac{1}{2} m_{\text{shell}} \sin\theta d\theta$$

Now we have to integrate over all angles θ, but keep in mind that s and α are functions of θ.

$$s^2 = r^2 + R^2 - 2rR\cos\theta \qquad \Rightarrow sds = rR\sin\theta d\theta$$

$$R^2 = s^2 + r^2 - 2rs\cos\alpha \qquad \Rightarrow \cos\alpha = \frac{s^2 + r^2 - R^2}{2rs}$$

$$\boxed{dF_x = -\frac{GMm_{\text{shell}}}{4r^2 R}\left(1 + \frac{r^2 - R^2}{s^2}\right)ds}$$

$$\begin{cases} \theta = 0 & s = r - R \\ \theta = \pi & s = r + R \end{cases}$$ **integration limits**

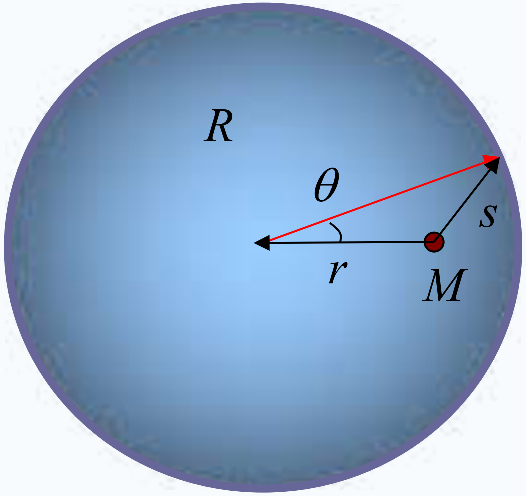
Extended objects: results

Hollow sphere:

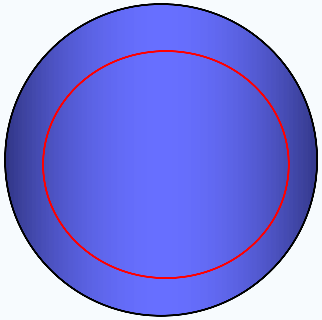
$$F_x = -\frac{GMm_{\text{shell}}}{4r^2R}\int_{r-R}^{r+R}\left(1+\frac{r^2-R^2}{s^2}\right)ds = -\frac{GMm_{\text{shell}}}{4r^2R}\left(s\Big|_{r-R}^{r+R} - \frac{r^2-R^2}{s}\Bigg|_{r-R}^{r+R}\right) = \boxed{-\frac{GMm_{\text{shell}}}{r^2}}$$

A **What happens if we move the mass M into the spherical shell?** **(integration limits)**

$$F_x = -\frac{GMm_{\text{shell}}}{4r^2R}\int_{R-r}^{R+r}\left(1+\frac{r^2-R^2}{s^2}\right)ds = -\frac{GMm_{\text{shell}}}{4r^2R}\left(s\Big|_{R-r}^{R+r} - \frac{r^2-R^2}{s}\Bigg|_{R-r}^{R+r}\right) = 0$$



B **What if object is not empty?** **(integrate over all shells)**



$$F_x = -\int \frac{GM}{r^2}\,dm_{\text{shell}}$$

$$F_x = -\frac{GM}{r^2}\int dm_{\text{shell}}$$

What causes tides

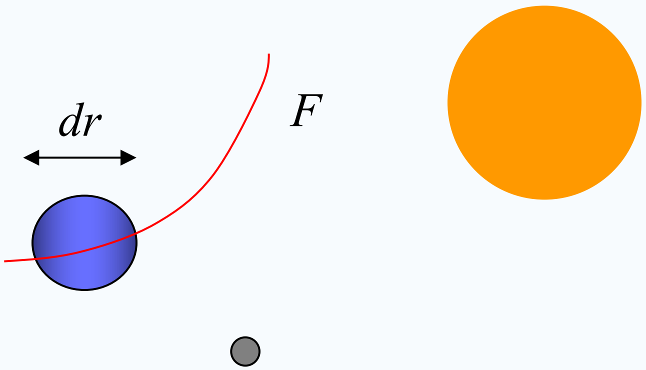
$$F_S = \frac{GmM_S}{r_S^2} \qquad F_M = \frac{GmM_M}{r_M^2}$$

$$\frac{F_S}{F_M} = \frac{M_S r_M^2}{M_M r_S^2}$$

$$M_S = 1.98\times10^{30}\ \text{kg} \qquad M_M = 7.35\times10^{22}\ \text{kg}$$

$$r_S = 1.49\times10^{8}\ \text{km} \qquad r_M = 3.84\times10^{5}\ \text{km}$$

$$F_S / F_M \approx 200$$

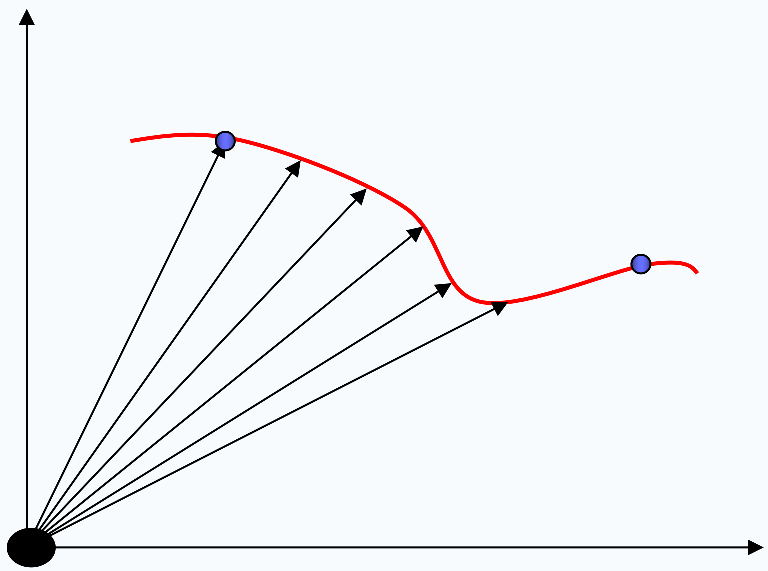


$$dF = \frac{dF(r)}{dr}dr \qquad dF = \frac{2Gm_1m_2}{r^3}dr$$

$$\frac{dF}{F} = -\frac{2dr}{r} \qquad = \frac{4R_{Earth}}{r}$$

$$\frac{\Delta F_S}{\Delta F_M} = \frac{F_S}{F_M}\frac{R_M}{R_S} \approx 0.4$$

Gravitational potential energy



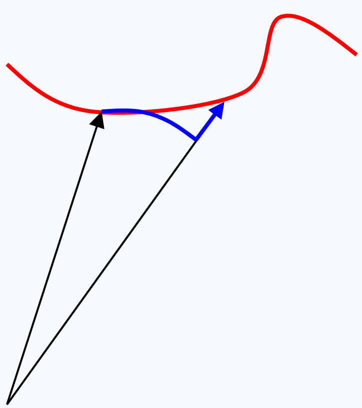
$$dW = \vec{F} \cdot d\vec{s} \qquad dW_{arc} = 0$$

$$dW = \vec{F} \cdot d\vec{r} = F(r)dr$$

$$\Delta U = -W = -\int_{r_1}^{r_2} \left(-\frac{GmM}{r^2} \right) dr = -GmM \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

Let us choose r_1 such that $U(r_1)=0$

$$\boxed{U = -\frac{GmM}{r}}$$



Bound systems; m << M

$$E = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

$$\frac{GmM}{r^2} = \frac{mv^2}{r}$$

$$\frac{GmM}{r} = mv^2$$

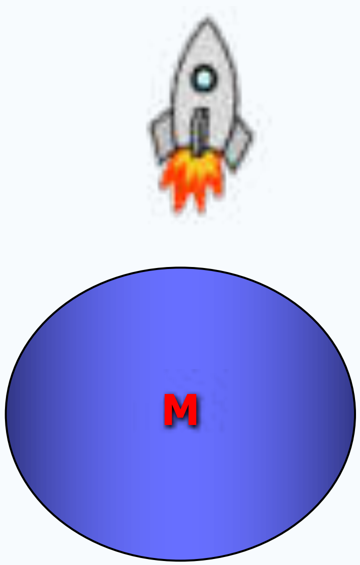
$$\boxed{E = -\frac{1}{2}mv^2}$$

Escape velocity

$$\frac{1}{2}mv^2 - \frac{GmM}{R} = -\frac{GmM}{r}$$

$$v = \sqrt{\frac{2GM}{R} - \frac{2GM}{r}}$$

$$\boxed{v_{esc} = \sqrt{\frac{2GM}{R}}}$$



1 a.m.u. = 1.66 10^{-27} kg
k_B = 1.38 10^{-23} J/K

Hydrogen ~ 1
Oxygen ~ 16

Sun - 617.7 km/s

Mercury - 4.25 km/s

Venus - 10.46 km/s

Earth - 11.186 km/s

Moon - 2.38 km/s

Mars - 5.027 km/s

Jupiter - 59.5 km/s

Saturn - 35.5 km/s

Uranus - 21.3 km/s

Neptune - 23.5 km/s

Pluto - 1.27 km/s

To remember!

- **Three empirical Kepler' laws:**

- Planets are moving in elliptical orbits.

- Line joining the Sun and a planet sweeps out equal areas in equal times.

- The square of the period of a planet is proportional to the cube of the planet's distance from the Sun.

- **The Newton's law of gravitation:**

Every particle attracts every other particle with a force directly proportional to their masses and inversely proportional to the square of the distance between them .