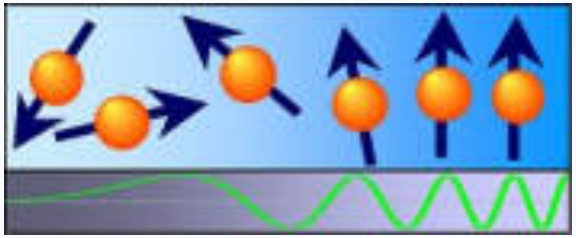
Experimental Physics
EP1 MECHANICS

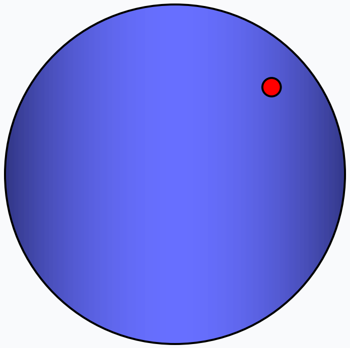
- Rotation. Basics -

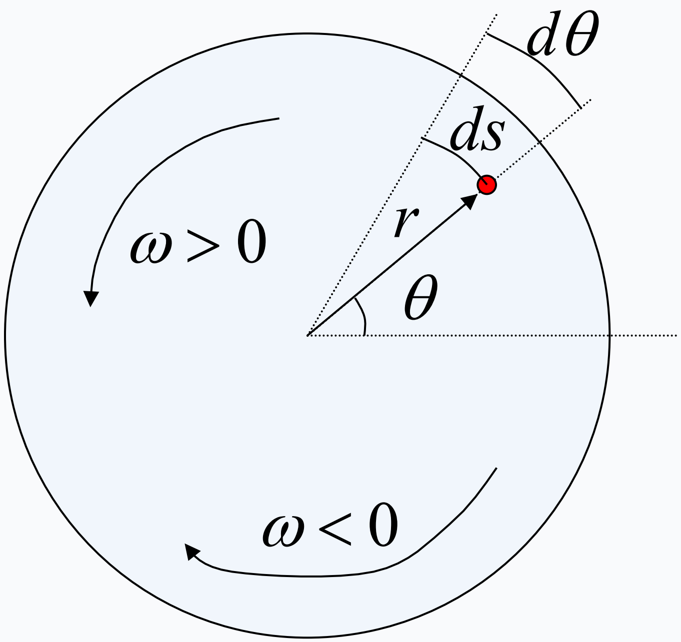


Rustem Valiullin

https://www.physgeo.uni-leipzig.de/en/fbi/applied-magnetic-resonance

Some basics





$$ds = v\,dt \qquad ds = r\,d\theta$$

$$\frac{d\theta}{dt} = \frac{v}{r} \equiv \omega$$

- angular velocity

$$\alpha = a/r$$

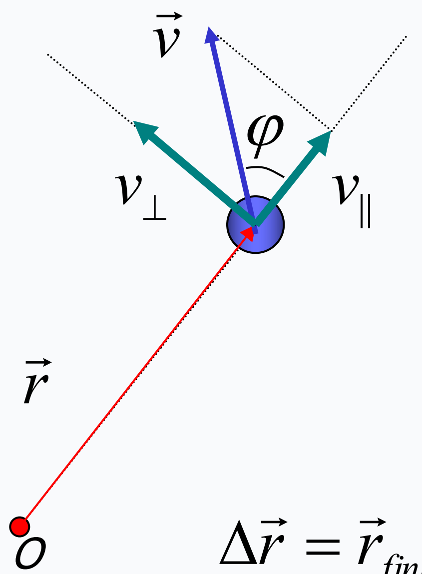
$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d\omega}{dt} \equiv \alpha$$

- angular acceleration

constant angular acceleration

#	Along line	Rotational
1	$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
2	$x = x_0 + v_0 t + at^2/2$	$\theta = \theta_0 + \omega_0 t + \alpha t^2/2$
3	$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
4	$x = x_0 + (v + v_0)t/2$	$\theta = \theta_0 + (\omega + \omega_0)t/2$
5	$x = x_0 + vt - at^2/2$	$\theta = \theta_0 + \omega t - \alpha t^2/2$

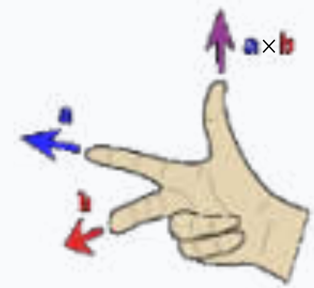
Some more basics



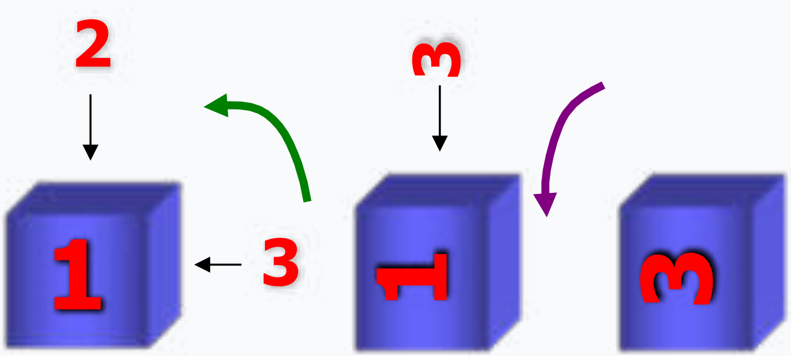
$$\frac{d\theta}{dt} = \frac{v}{r} \equiv \omega \qquad \omega = \frac{v\sin\varphi}{r} = \frac{rv\sin\varphi}{r^2}$$

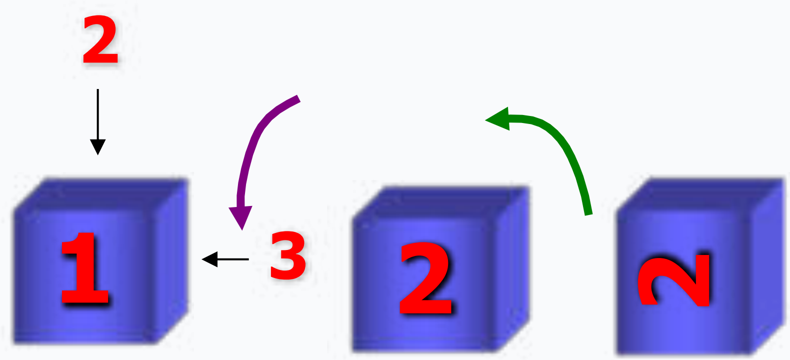
$$\boxed{\vec{\omega} = \frac{\vec{r}\times\vec{v}}{r^2}}$$

- it is a vector! (right-hand rule)



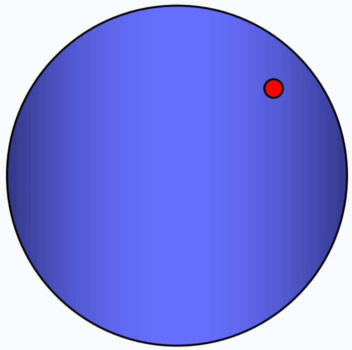
$$\Delta\vec{r} = \vec{r}_{final} - \vec{r}_{initial} \qquad \Delta\theta = \theta_{final} - \theta_{initial} - ?$$



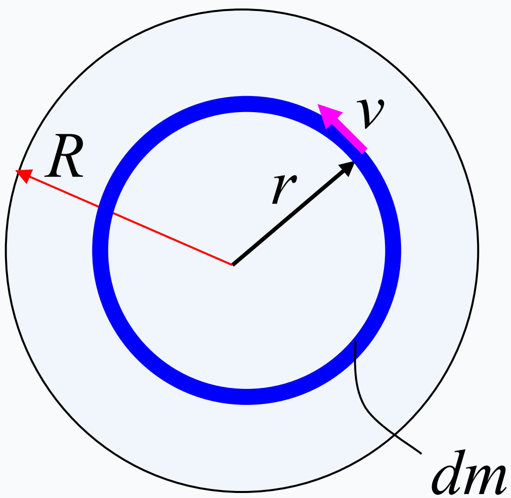


One of the properties of vectors $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ **is not reproduced.**

Kinetic energy of rotation



$$I = \tfrac{1}{2} MR^2$$



$$dE_k = \tfrac{1}{2} v^2 dm \Rightarrow E_k = \tfrac{1}{2}\omega^2 \int_{body} r^2 dm$$

$$I \equiv \int_{body} r^2 dm \left(\equiv \sum_i m_i r_i^2 \right)$$

moment of inertia
rotational inertia

$$E_{k,rot} = \tfrac{1}{2} I\omega^2 \quad \Leftrightarrow \quad E_{k,tr} = \tfrac{1}{2} mv^2$$

$$E_k = \tfrac{1}{2}\sum_i m_i v_i^2 = \tfrac{M}{2}\sum_i \tfrac{m_i}{M} v_i^2 = \tfrac{1}{2} M\left\langle v^2 \right\rangle$$

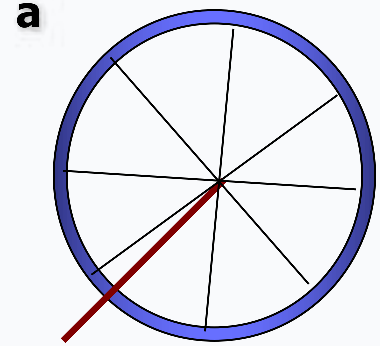
$$\left\langle v_{rot}^2 \right\rangle = \frac{1}{M}\int_0^R v^2 dm = \frac{\omega^2}{M}\int_0^R r^2 (2\pi r\rho_0) dr = \frac{\pi\rho_0\omega^2 R^4}{2M}$$

$$E_k = \tfrac{1}{4} M\omega^2 R^2$$

- the same result!

Moment of inertia

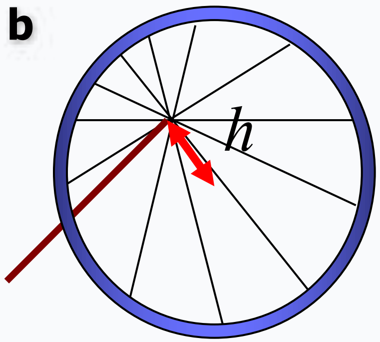
a



Depends on the rotation axis!

$$I = \int r^2 dm = M \int r^2 \frac{dm}{M} = M\left\langle r^2 \right\rangle \qquad I_a = MR^2$$

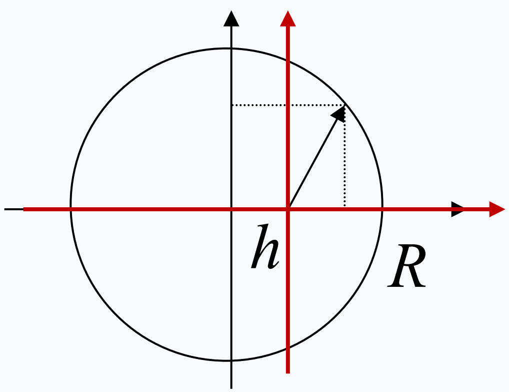
b

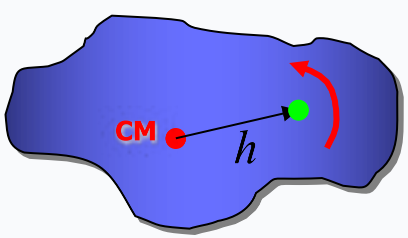


$$y^2 + (x+h)^2 = R^2 \Rightarrow \qquad r^2 = R^2 - h^2 - 2xh$$

$$\left\langle r^2 \right\rangle = \frac{\int_{-R-h}^{R-h} r^2 dx}{\int_{-R-h}^{R-h} dx} = \frac{1}{2R} \int_{-R-h}^{R-h} \left(R^2 - h^2 - 2hx\right) dx = R^2 + h^2$$

$$I_b = MR^2 + Mh^2$$

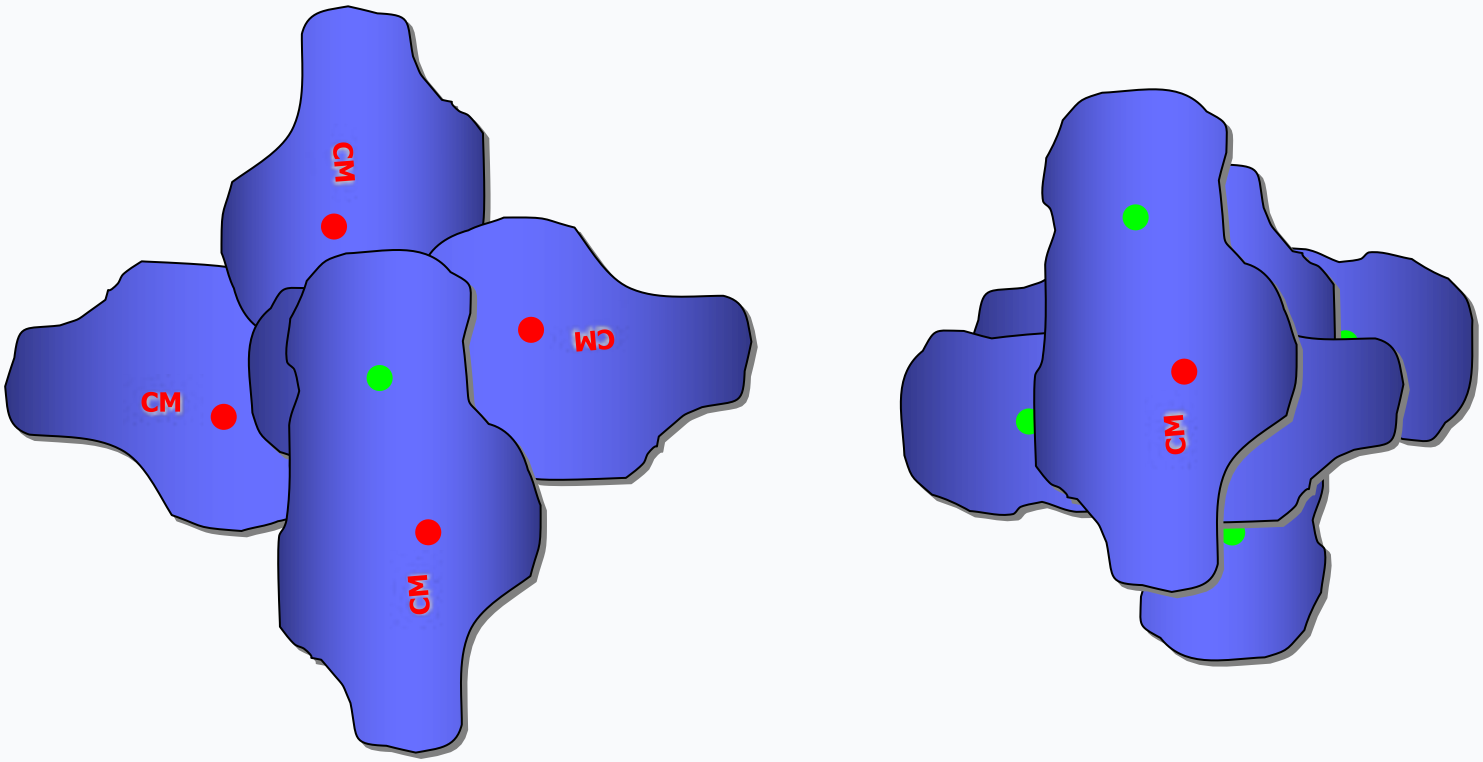




$$\boxed{I = MR_{CM}^2 + Mh^2}$$

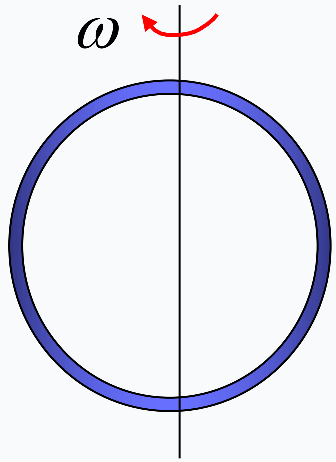
The parallel-axis theorem.

Rotation seen from two reference systems



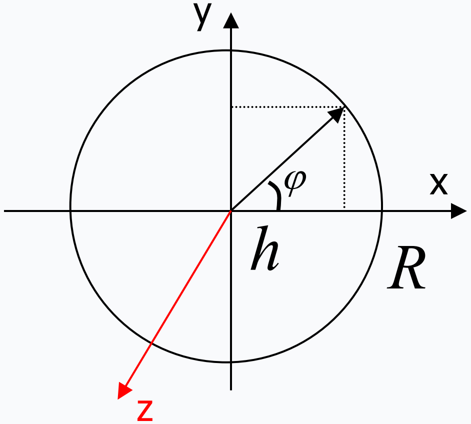
Rotating of an object around an arbitrary point is accompanied by the respective rotation around its center of mass.

Perpendicular-axis theorem



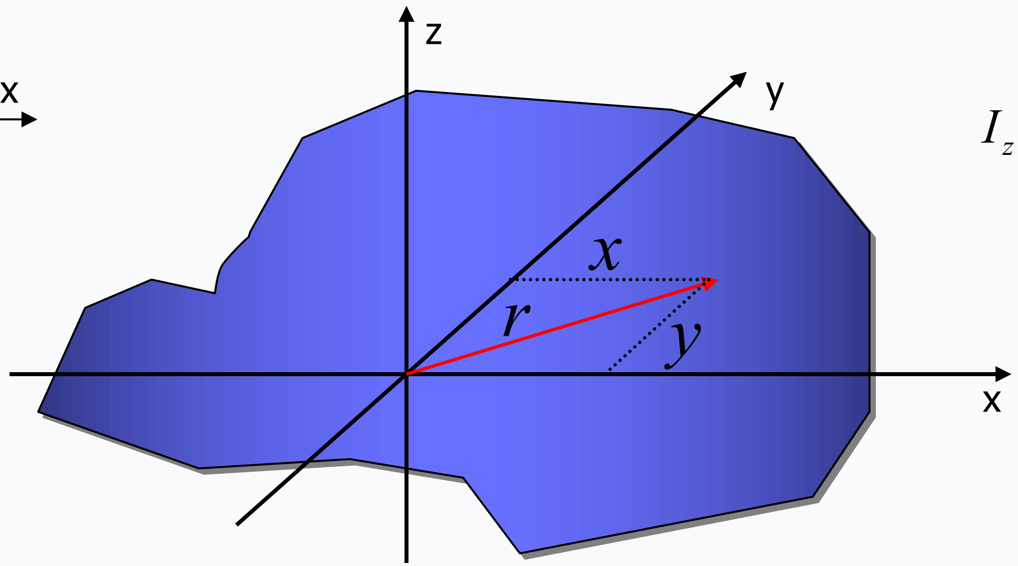
$$I = \int r^2 dm = M\left\langle r^2 \right\rangle \qquad \int \cos^2\varphi \, d\varphi = \tfrac{1}{2}\left(\varphi + \cos\varphi \sin\varphi\right) + c$$

$$I = \frac{M}{\int d\varphi} \int (R\cos\varphi)^2 d\varphi = MR^2 \frac{2}{\pi} \int_0^{\pi/2} \cos^2\varphi \, d\varphi = \frac{MR^2}{2}$$



$$\boxed{I_z = I_x + I_y}$$

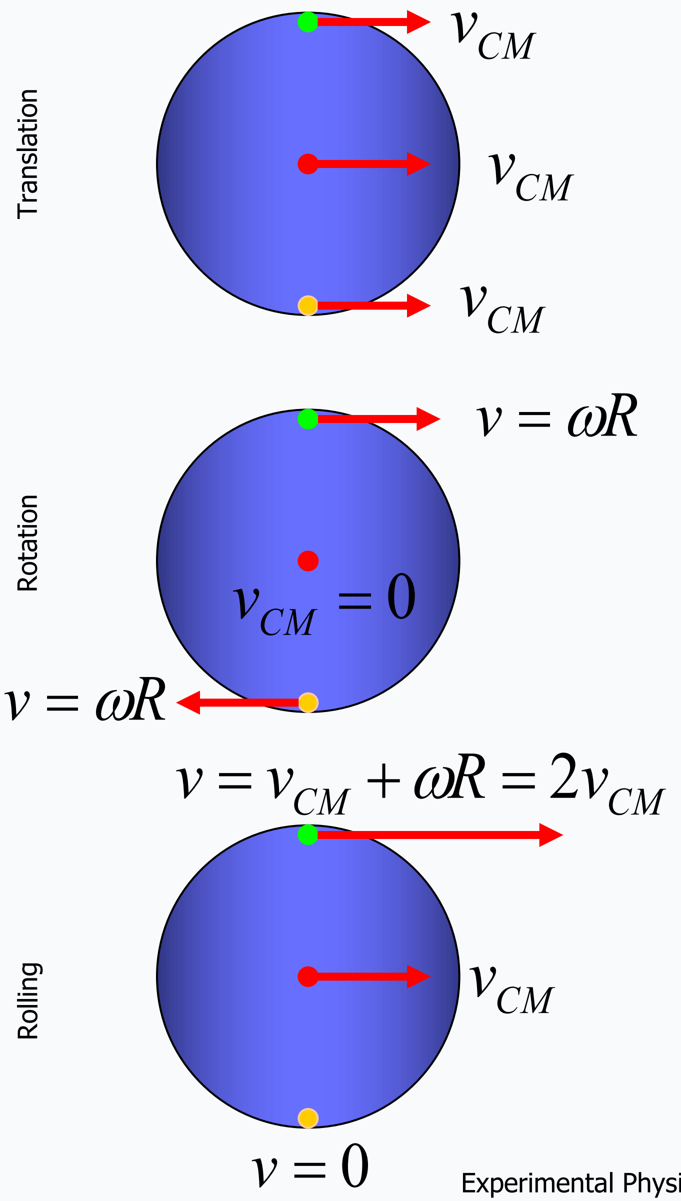
The perpendicular-axis theorem.



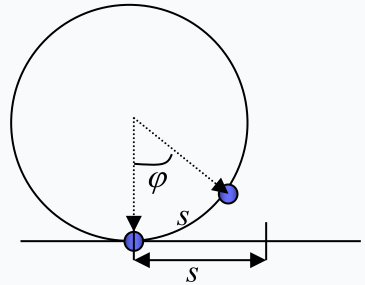
$$I_z = \int r^2 dm$$

$$I_z = \int (x^2 + y^2) dm$$

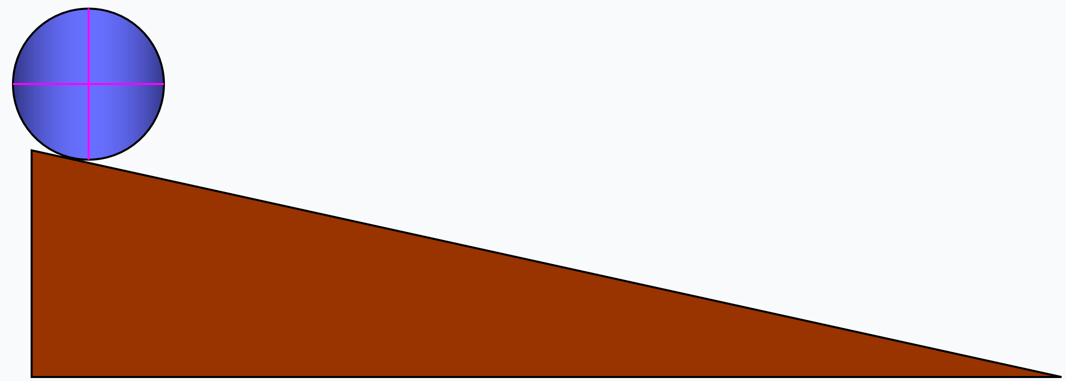
Rolling motion



$$v_{CM} = \frac{ds}{dt} = R\frac{d\varphi}{dt} = R\omega$$



$$E_k = \frac{1}{2} I_y \omega^2 = \frac{1}{2}\left(I_{CM} + MR^2\right)\omega^2 = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v^2$$

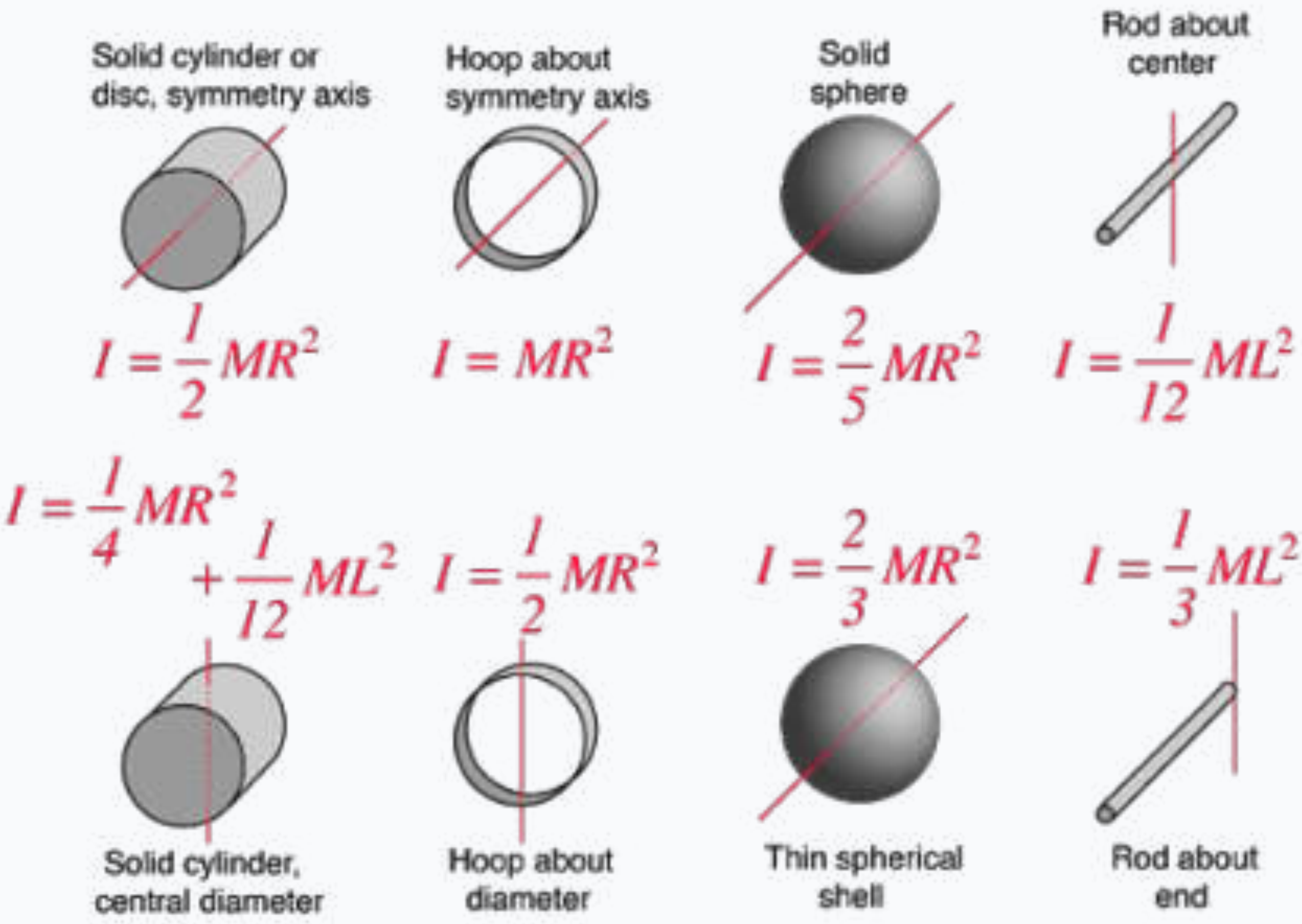


$$Mgh = \frac{1}{2} I_{CM} \frac{v^2}{R^2} + \frac{1}{2} M v^2 \qquad v^2 = \frac{2gh}{\left(1 + I_{CM} / MR^2\right)}$$

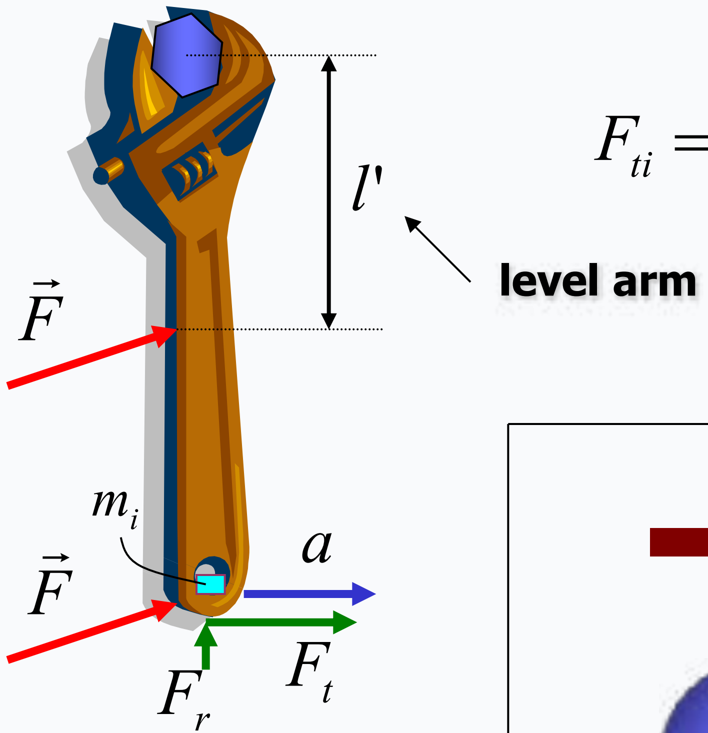
$$v_{sphere} = \sqrt{\tfrac{10}{7} gh}$$

To remember!

- **Rotational motion** is similar to one-dimensional translational motion.
- **Moment of inertia** is an analogue of mass, which might be considered as resistance to rotation.
- For moment of inertia of planar objects the parallel-axis and perpendicular-axis theorems can be applied.
- For **rolling motion** velocity of a selected point is the sum of that of the center of mass and of the point in the center of mass frame.

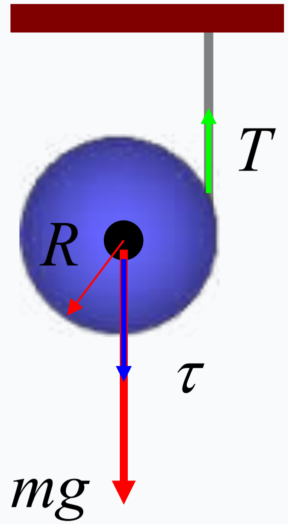

Solid cylinder or disc, symmetry axis
$I = \frac{1}{2}MR^2$
Hoop about symmetry axis
$I = MR^2$
Solid sphere
$I = \frac{2}{5}MR^2$
Rod about center
$I = \frac{1}{12}ML^2$
$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
Solid cylinder, central diameter
$I = \frac{1}{2}MR^2$
Hoop about diameter
$I = \frac{2}{3}MR^2$
Thin spherical shell
$I = \frac{1}{3}ML^2$
Rod about end

Torque



$$F_{ti} = m_i a_i = m_i \alpha l_i \qquad \sum l_i F_{ti} = \alpha \sum m_i l_i^2$$

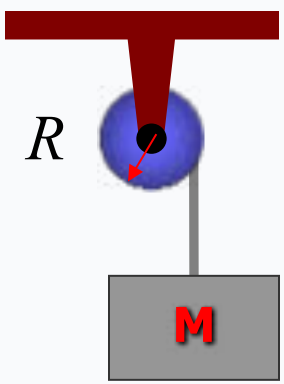
Torque $\boxed{\tau_{net} = \alpha I}$



$$\tau = mgR = \alpha I = \frac{a}{R} I$$

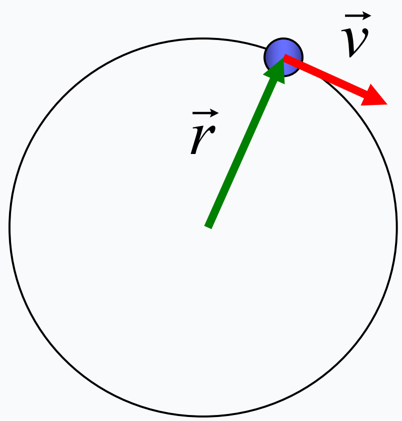
$$I = I_{CM} + MR^2 = \tfrac{3}{2} MR^2$$

$$T - mg = -ma = -\frac{2}{3} mg$$



$$a = {}^{2g}\!/_{3} \qquad T = \frac{1}{3} mg$$

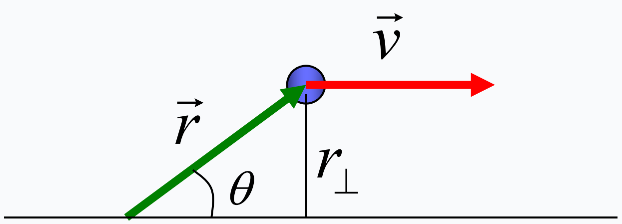
Angular momentum



$$L = I\omega = mr^2 \frac{v}{r} = mvr = pr$$

$$\boxed{L = I\omega}$$

$$L = mvr\sin\theta = m(\vec{r}\times\vec{v}) = \vec{r}\times\vec{p}$$



$$L = \sum L_i = \omega \sum m_i r_i^2 = I\omega$$

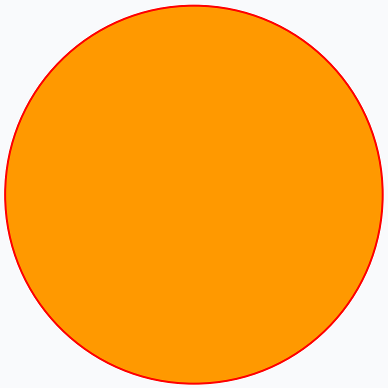
$$\tau_{net} = \alpha I = I\frac{d\omega}{dt} = \frac{d(I\omega)}{dt} = \frac{dL}{dt}$$

The net external torque = the rate of change of *L*.

Conservation of angular momentum

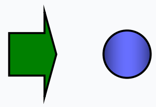
$\boxed{\textbf{If} \quad \tau_{net} = 0 \quad \textbf{then} \quad L = \text{const}}$

Conservation of angular momentum



$$R_{Sun} = 6.96\times 10^{8}\ \text{m}$$

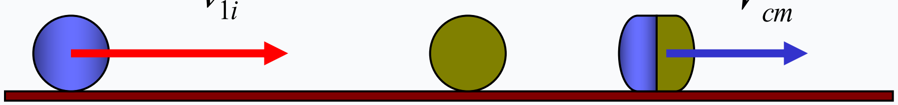
$$T_{Sun} = 25.3\ \text{d}$$



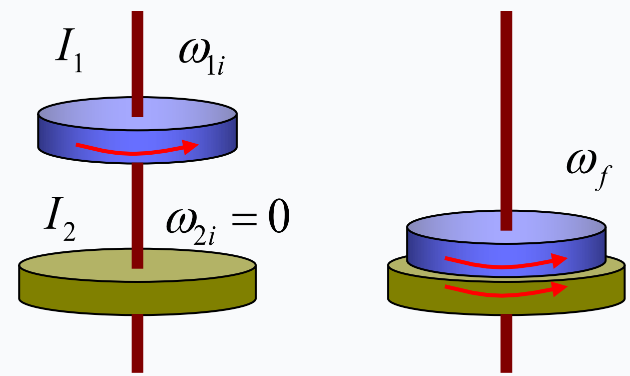
$$R_{ns} = 5.0\times 10^{3}\ \text{m}$$

$$I = KmR^2 \qquad L = KmR^2\frac{2\pi}{T}$$

$$\frac{R_{Sun}^2}{T_{Sun}} = \frac{R_{ns}^2}{T_{ns}} \Rightarrow T_{ns} = T_{Sun}\left(\frac{R_{ns}}{R_{Sun}}\right)^2 \approx 1.12\times 10^{-4}\ \text{s}$$

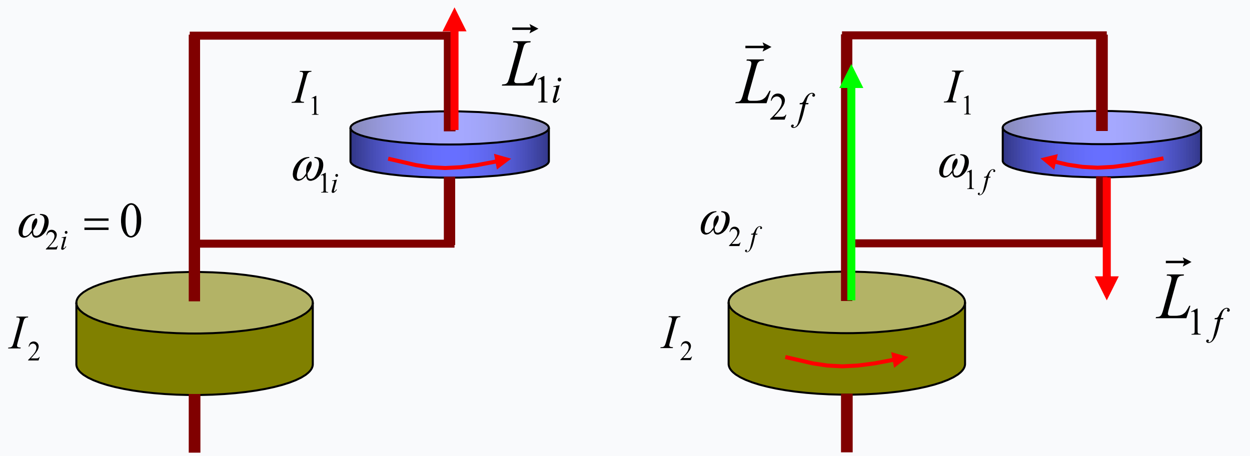


$$E_{k,f} = \frac{2m_1 E_{k,i}}{2(m_1+m_2)} = \frac{m_1}{m_1+m_2}\frac{m_1 v_{1i}^2}{2}$$



$$E_{k,1i} = \frac{1}{2}I_1\omega_{1i}^2 = \frac{L_{1i}^2}{2I_1} \qquad E_{k,f} = \frac{1}{2}(I_1+I_2)\omega_f^2 = \frac{1}{2}\frac{L_f^2}{(I_1+I_2)} = \left(\frac{I_1}{I_1+I_2}\right)E_{k,i}$$

Conservation of angular momentum



$$m_2 / m_1 \approx 10$$
$$T_1 \approx 0.1\,\mathrm{s}$$
$$R_1 \approx 0.5\,\mathrm{m}$$
$$R_2 \approx 1\,\mathrm{m}\,(!)$$

$$\vec{L}_{ti} = \vec{L}_{1i} \qquad \vec{L}_{tf} = \vec{L}_{1f} + \vec{L}_{2f} = -\vec{L}_{1i} + \vec{L}_{2f} \quad \Rightarrow \quad \vec{L}_{2f} = 2\vec{L}_{1i}$$

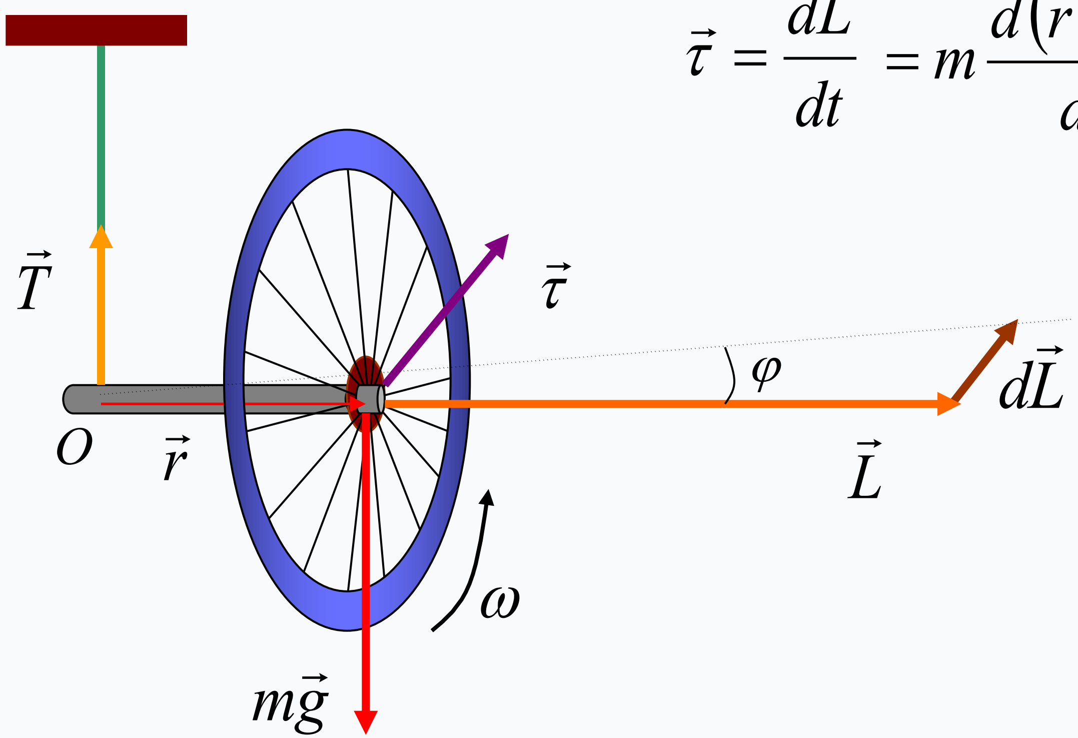
$$\frac{(m_1 + m_2)R_2^2}{2T_2} = 2\frac{m_1 R_1^2}{T_1} \qquad T_2 = T_1 \frac{(m_1 + m_2)R_2^2}{4m_1 R_1^2} \approx 1\,\mathrm{s}$$

There are only internal forces acting within the system. To turn the wheel around we have to apply force, i.e., torque. This will be balanced by the reaction torque.

Motion of a wheel

$$\vec{\tau} = \frac{d\vec{L}}{dt} = m\frac{d(\vec{r}\times\vec{v})}{dt} = m\vec{r}\times\frac{d\vec{v}}{dt} + m\frac{d\vec{r}}{dt}\times\vec{v}$$

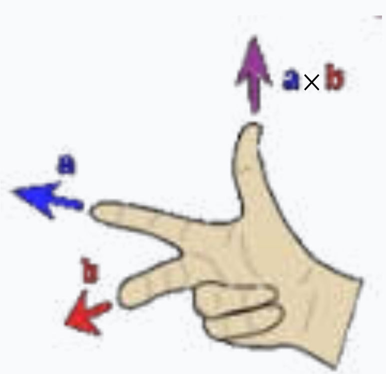
$$\vec{\tau} = m\vec{r}\times\vec{a} = \vec{r}\times\frac{d\vec{p}}{dt}$$

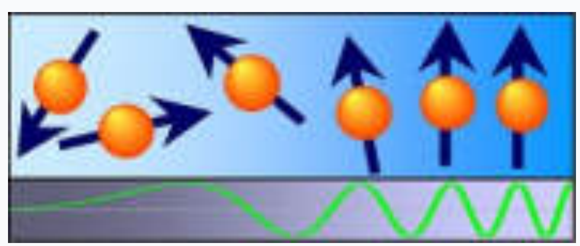


precession

$$dL = \tau dt = rmg\,dt = L\,d\varphi$$

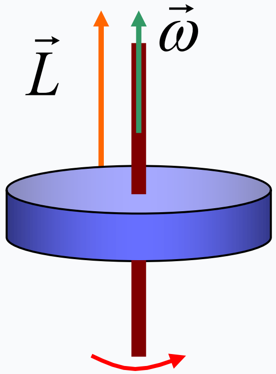
$$\omega_p = \frac{d\varphi}{dt} = \frac{rmg}{L}$$

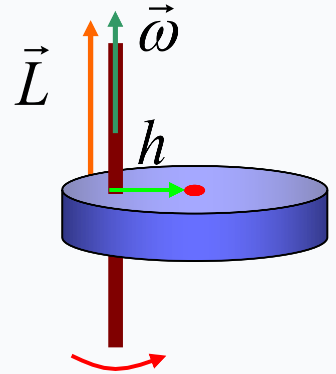




Nuclear Magnetic Resonance

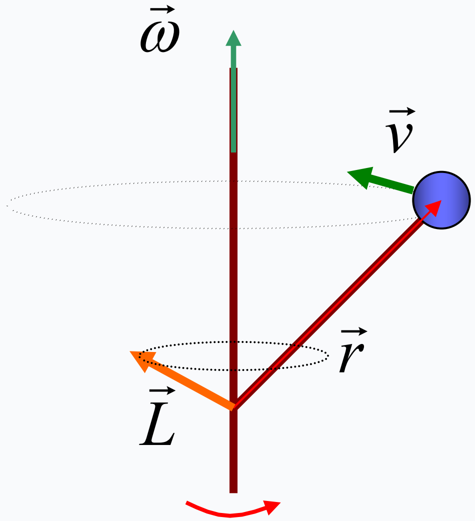
Static and dynamic imbalance

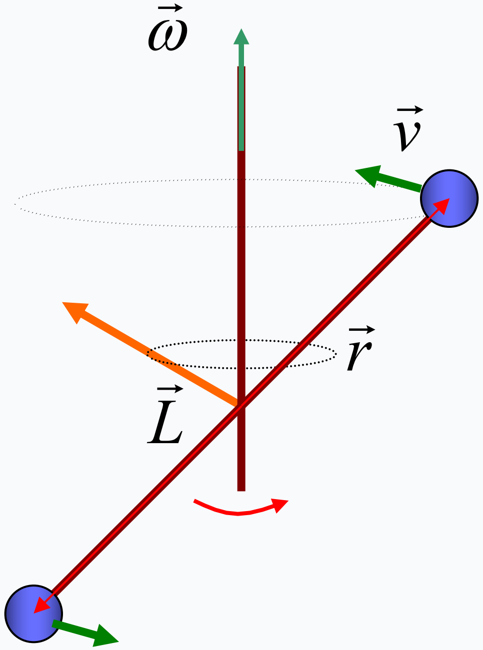


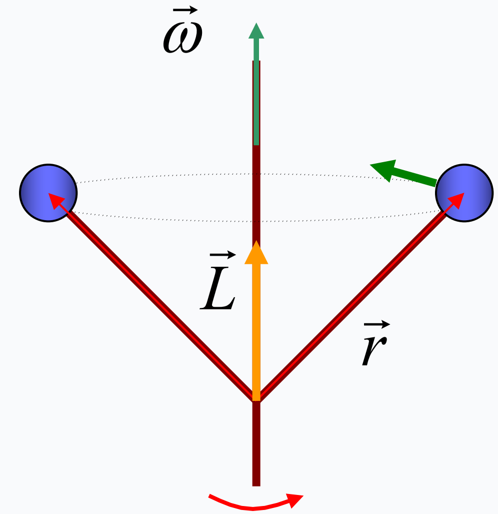


$$\vec{L} = m(\vec{r} \times \vec{v}) \qquad \vec{\tau} = \frac{d\vec{L}}{dt}$$

$$F = \frac{mv^2}{h} = m\omega^2 h$$







To remember!

- **<u>Torque</u> is the product of the tangential component of the force and the level arm.**

- **The <u>angular momentum</u> is the cross-product of the radius vector and the linear momentum.**

- **It is <u>fundamental</u> property that the angular momentum is always conserved.**

- **<u>Precession</u> is a reaction of the angular momentum to a net torque applied perpendicularly.**

- **A body is <u>dynamically imbalanced</u> when the angular velocity and momentum are not parallel to each other.**