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EXERCISE 4321

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Problem 1: Derivatives **6 · 2 Points**

Solution

Range for Numbers:

a. 123

(a)

$$f'(x) = \frac{1}{\sqrt{1-[(ax)^3\exp(-bx)]}}\left(-\frac{1}{2}\right)[(ax)^3\exp(-bx)]^{-3/2}\cdot$$
$$\cdot\,[3a^3x^2\exp(-bx)-b(ax)^3\exp(-bx)]$$
$$=\frac{-1/2}{\sqrt{1-[(ax)^3\exp(-bx)]}}\frac{1}{[(ax)^3\exp(-bx)]^{1/2}}\cdot$$
$$\cdot\,\frac{3a^3x^2\exp(-bx)-b(ax)^3\exp(-bx)}{(ax)^3\exp(-bx)}$$
$$=\frac{b/2-3/2x}{\sqrt{(ax)^3\exp(-bx)-1}}$$

(b)

$$(1),\qquad \frac{\partial g}{\partial y} = 3y^2+6yz+3z^2$$
$$(2),\qquad \frac{\partial g}{\partial y} = \exp[y^2\arctan(yz)]\left\{2y\arctan(yz)+\frac{y^2z}{1+y^2z^2}\right\}$$

(c)

$$(1),\qquad \int f(x)\,\mathrm{d}x = \int\frac{\mathrm{d}x}{x\ln(ax)}$$
$$z=\ln(ax)\,,\ \mathrm{d}z=\frac{\mathrm{d}x}{x}$$
$$\int f(x)\,\mathrm{d}x = \int\frac{\mathrm{d}z}{z} = \ln(z)+\text{const.} = \ln(\ln(ax))+\text{const.}$$
$$(2),\qquad \int f(x)\,\mathrm{d}x = \int\frac{\mathrm{d}x}{a^2+bx^2}$$
$$(i),\qquad a=0,\qquad b\neq 0:$$
$$\int\frac{\mathrm{d}x}{bx^2} = -\frac{1}{bx}+\text{const.}$$
$$(ii),\qquad b=0,\qquad a\neq 0:$$

$$\int \frac{\mathrm{d}x}{a^2} = \frac{x}{a^2} + \text{const.}$$

(iii), $\quad a \neq 0, \quad b \neq 0$:

$$\int \frac{\mathrm{d}x}{a^2 + bx} = \frac{1}{a^2} \int \frac{\mathrm{d}x}{1 + bx^2/a^2}$$

$(iiia)$, $\quad b > 0$:

$$z = 0\sqrt{\frac{b}{a^2}}x, \ \mathrm{d}z = \sqrt{\frac{b}{a^2}}\mathrm{d}x$$

$$\int \frac{\mathrm{d}x}{a^2 + bx^2} = \frac{1}{a^2}\sqrt{\frac{a^2}{b}} \int \frac{\mathrm{d}z}{1 + z^2}$$

$(iiib)$, $\quad b < 0$:

$$z = \sqrt{\frac{|b|}{a^2}}x, \ \mathrm{d}z = \sqrt{\frac{|b|}{a^2}}\mathrm{d}x$$

$$\int \frac{\mathrm{d}x}{a^2 + bx^2} = \frac{1}{a^2}\sqrt{\frac{a^2}{b}} \int \frac{\mathrm{d}z}{1 - z^2}$$

$$= \frac{1}{\sqrt{a^2|b|}} \begin{cases} \operatorname{arctanh}\left(\sqrt{\frac{|b|}{a}}x\right) + \text{const.} \\ \qquad \text{for } x < \sqrt{\frac{a^2}{|b|}} \\ \operatorname{arccoth}\left(\sqrt{\frac{|b|}{a}}x\right) + \text{const.} \\ \qquad \text{for } x > \sqrt{\frac{a^2}{|b|}} \end{cases}$$

(d)

(1), $$\int_{\pi/6}^{\pi/4} f(x)\,\mathrm{d}x = \int_{\pi/6}^{\pi/4} \cos^3 x\,\mathrm{d}x$$

$$= \sin x \cos^2 x|_{\pi/6}^{\pi/4} + 2\int_{\pi/6}^{\pi/4} \sin^2 x \cos x\,\mathrm{d}x$$

$$= \sin x \cos^2 x|_{\pi/6}^{\pi/4} + \frac{2}{3}\sin^3 x\Big|_{\pi/6}^{\pi/4}$$

$$= \sin x\left(1 - \frac{1}{3}\sin^2 x\right)_a^b$$

$$= \frac{5\sqrt{2}}{12} - \frac{11}{24} \approx 0.131\,.$$

(2), $$\int_{-1}^{1} f(x)\,\mathrm{d}x = \int_{-1}^{1} x\exp(-x^2)\,\mathrm{d}x = 0\,.$$

(e)

Implicit differentiation:

The expression

$$y = \operatorname{arctanh} x$$

is equivalent to

$$x = \tanh y\,.$$

The derivative of the latter with respect to x yields

$$1 = \frac{1}{\cosh^2 y}\,y'.$$

From the identity

$$\cosh^2 x - \sinh^2 x = 1$$

we obtain

$$1 - \tan^2 x = \frac{1}{\cosh^2 x}\,,$$

such that

$$y' = \cosh^2 y = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x}\,.$$

(f)

The trivial solution would be a function that is zero everywhere. A non-trivial solution would be the function

$$f(x) = \frac{(x-5)(x+5)}{x^2+1}$$

Problem 2: Forces, Skateboards **2 + 2 Points**

Solution

Assuming they uphold a constant force, where do they meet?

(Prof. X starts at Position 0, Mr. X at $x = L$)

Only internal Forces, total Force is 10 N

$$a_1 = \frac{F}{m_1} \text{ and } a_2 = -\frac{F}{m_2}$$

Equation of motions:

$$x_1 = \frac{a_1 t^2}{2} \text{ and } x_2 = \frac{a_2 t^2}{2} + L$$

When they meet (at t_1): $x = x_1 = x_2$ so:

$$\frac{a_1 t_1^2}{2} = \frac{a_2 t_1^2}{2} + L$$

$$t_1^2 = \frac{2L}{a_1 - a_2}$$
$$a_1 - a_2 = \frac{F}{m_1} + \frac{F}{m_2} = \frac{Fm_2 + Fm_1}{m_1 \cdot m_2} = F\frac{m_1 + m_2}{m_1 \cdot m_2}$$
$$t_1^2 = \frac{2L}{F}\frac{m_1 \cdot m_2}{m_1 + m_2}$$

Using this in the equation of motion:

$$x = \frac{a_1 L}{F}\frac{m_1 \cdot m_2}{m_1 + m_2} = L\frac{m_2}{m_1 + m_2} = 12.73\ \mathrm{m}$$

Note: Since only internal forces are acting the center of mass of the whole system cannot move. This trivialises this task considerably since only the center of mass needs to be calculated (However the time needs to be calculated for part b) anyway):

$$x_{CoM} = \frac{x_{0,1} \cdot m_1 + x_{0,2} \cdot m_2}{m_1 + m_2} = \frac{0 \cdot m_1 + L \cdot m_2}{m_1 + m_2} = L\frac{m_2}{m_1 + m_2}$$

<u>What is their speed when they meet?</u>

Using the equations above:

$$t = \sqrt{\frac{2L}{F}\frac{m_1 \cdot m_2}{m_1 + m_2}} = 12.36\ \mathrm{s}$$
$$v_1 = a_1 \cdot t = \frac{F}{m_1}\sqrt{\frac{2L}{F}\frac{m_1 \cdot m_2}{m_1 + m_2}} = \sqrt{\frac{2L \cdot F}{m_1}\frac{m_2}{m_1 + m_2}} = 2.06\,\frac{\mathrm{m}}{\mathrm{s}}$$

Similarly, for v2:

$$v_2 = \sqrt{\frac{2L \cdot F}{m_2}\frac{m_1}{m_1 + m_2}} = 1.17\,\frac{\mathrm{m}}{\mathrm{s}}$$

Problem 3: Mass falling on a Coil (Newton) **3 + 2 Points**

<u>Solution</u>

<u>(a)</u>

The motion be along the z-axis. Let further the rest position define $z = 0$. Then the

equation of motion for the mass m is

$$m\vec{a} = \vec{G} + \vec{F_s} = m\vec{g} - D\vec{z} = -(mg + Dz)\hat{z}$$

since the motion is one-dimensional, this equation can be written in scalar form:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -(mg + Dz)\,.$$

Multiplication with v yields

$$mv\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}v^2 = -(mg + Dz)\frac{\mathrm{d}z}{\mathrm{d}t}\,.$$

This can be integrated over time from the time the ball hits the spring ($t = 0$, $v = v_0$, $z = 0$) to the time the ball comes to rest (t, $v = 0$, z):

$$\int_0^t mv\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = \int_{v_0}^0 mv\,\mathrm{d}v = -\frac{m}{2}v_0^2 = -\int_0^t (mg + Dz)\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t$$

$$= -\int_0^z (mg + Dz)\,\mathrm{d}z = -\left(mgz + \frac{1}{2}Dz^2\right)$$

$$\frac{1}{2}mv_0^2 = mgz + \frac{1}{2}Dz^2$$

$$v_0 = \sqrt{2gz + \frac{D}{m}z^2}$$

With a mass $m = 300\,g$, a spring constant $D = 2.5\,N/cm$ and an extension $z = -20\,cm$ this yields

$$v_0 = 5.42\,\frac{\mathrm{m}}{\mathrm{s}}\,.$$

(b)

If the initial velocity v_0 is known, the extension can be calculated:

$$z^2 + \frac{2mg}{D}z - \frac{mv_0^2}{D} = 0$$

$$z = -\frac{mg}{D}\left[1 + \sqrt{1 + \frac{Dv_0^2}{mg^2}}\right]$$

For a doubled mass $\mathrm{m} = 600\,\mathrm{g}$ one obtains $z = -29.0\,\mathrm{cm}$.

Problem 4: Stone out of a Window **$2 + 2 + 2$ Points**

Solution

For solving this task, one has to make use of the superposition principle and separate the movement $\vec{s}(t)$ in independent $x(t)$, $y(t)$ and $z(t)$. x and y are linear motions and z is a free fall motion without initial velocity. As the next step, the time t_0 needed for the stone to hit the ground ($z(t_0) = z_0$) has to be calculated. The elapsed time t_0 can now be plugged in $\dot{x}(t) = v_x(t)$ and $\dot{y}(t) = v_y(t)$ to gain the velocities needed.

$$z(t) = \frac{g}{2}t^2$$

$$\rightarrow z_0 = \frac{g}{2} t_0^2$$

$$\rightarrow \boldsymbol{t_0} = \sqrt{\frac{\boldsymbol{2z_0}}{\boldsymbol{g}}}$$

$$x(t) = v_x t\,, \qquad y(t) = v_y t\,, \qquad z(t) = \frac{g}{2} t^2$$

$$\rightarrow \boldsymbol{v_{x,0}} = \frac{\boldsymbol{x_0}}{\boldsymbol{t_0}}\,, \qquad \rightarrow \boldsymbol{v_{y,0}} = \frac{\boldsymbol{y_0}}{\boldsymbol{t_0}}\,, \qquad \rightarrow \boldsymbol{v_{z,0}} = \boldsymbol{g t_0}$$

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$\rightarrow \vec{\boldsymbol{v}}_{\mathbf{fin}} = \begin{pmatrix} \boldsymbol{v_{x,0}} \\ \boldsymbol{v_{y,0}} \\ \boldsymbol{v_{z,0}} \end{pmatrix}, \qquad \rightarrow |\vec{\boldsymbol{v}}_{\mathbf{fin}}| = \sqrt{\boldsymbol{v_{x,0}^2 + v_{y,0}^2 + v_{z,0}^2}}$$

Plugging in the given numbers results in the searched quantities of (a) and (c). The initial speed of the stone in (b) is given when setting $t = 0$.