Problem 1

$$\vec{q}(t)=\begin{bmatrix}x(t)\\y(t)\end{bmatrix} \qquad \vec{F}=e^{x^2-y^2}\begin{bmatrix}-ax\\by\end{bmatrix}$$

a) $$W=\int_{\vec{q}(0)}^{\vec{q}(1)}\vec{F}\,d\vec{q}=\int_{t=0}^{t=1}\vec{F}(t)\,\dot{\vec{q}}(t)\,dt=\int_0^1 e^{t^2(x_E^2-y_E^2)}\begin{bmatrix}-atx_E\\bty_E\end{bmatrix}\cdot\begin{bmatrix}x_E\\y_E\end{bmatrix}dt=$$

$$\vec{q}(t)=\begin{bmatrix}tx_E\\ty_E\end{bmatrix} \qquad \dot{\vec{q}}(t)=\begin{bmatrix}x_E\\y_E\end{bmatrix}$$

$$=\int_0^1 e^{t^2(x_e^2-y_e^2)}\left[-atx_e^2+bty_e^2\right]dt=$$

$$=\int_0^1\left[by_e^2-ax_e^2\right]t\,e^{t^2(x_e^2-y_e^2)}dt=\begin{bmatrix}u=t^2(x_e^2-y_e^2)\\du=2t(x_e^2-y_e^2)dt\\u(0)=0\quad u(1)=x_e^2-y_e^2\end{bmatrix}=$$

$$=\int_0^{x_e^2-y_e^2}\frac{(by_e^2-ax_e^2)}{2\cancel{t}(x_e^2-y_e^2)}\cancel{t}\,e^u\,du=\frac{by_e^2-ax_e^2}{2(x_e^2-y_e^2)}\left[e^{x_e^2-y_e^2}-1\right].$$

b) $$\gamma_2=\left\{\vec{q}(t)=\begin{bmatrix}tx_e\\0\end{bmatrix}\right\}_{t=1}\cup\left\{\vec{q}(t)=\begin{bmatrix}x_e\\ty_e\end{bmatrix}\right\}$$

$$W=\int_{t=0}^{t=1}e^{t^2x_e^2}\begin{bmatrix}-atx_e\\0\end{bmatrix}\begin{bmatrix}x_e\\0\end{bmatrix}dt+\int_{t=0}^{t=1}e^{x_e^2-t^2y_e^2}\begin{bmatrix}-ax_e\\bty_e\end{bmatrix}\begin{bmatrix}0\\y_e\end{bmatrix}dt=$$

$$=\int_{t=0}^{t=1}e^{t^2x_e^2}(-atx_e^2)dt+\int_0^1 e^{x_e^2-t^2y_e^2}(bty_e^2)dt=$$

$$=\frac{-a}{2}e^{t^2x_e^2}\Big|_0^1-\frac{b}{2}e^{x_e^2-t^2y_e^2}\Big|_0^1=\overbrace{-\frac{a}{2}\left[e^{x_e^2}-1\right]}^{W_1}-\underbrace{\frac{b}{2}\left[e^{x_e^2-y_e^2}-e^{x_e^2}\right]}_{W_2}.$$

~~$+\frac{b}{2}e^{\sim(1+2ty_e^2)}$~~

*c) W_a must (at least) $=\left[W_b=W_1+W_2\right]$

So $\dfrac{by_e^2-ax_e^2}{2(x_e^2-y_e^2)}\left[e^{\overbrace{x_e^2-y_e^2}^{A}}-\cancel{1}\right]=-\dfrac{a}{2}\left[e^{x_e^2}-1\right]-\dfrac{b}{2}\left[e^{x_e^2-y_e^2}-e^{x_e^2}\right]$

For same endpoint $[x_e, y_e]$.

If $\underline{a=b}$, $-\frac{a}{2}[e^A-1]=-\frac{a}{2}[e^A-1]$.

Necessary (not sufficient): $\underline{a=b}$.

*d) but when $\boxed{a=b}$ =>

$$-\nabla\varphi(x,y)=\begin{bmatrix}-\frac{\partial}{\partial x}\varphi\\ -\frac{\partial}{\partial y}\varphi\end{bmatrix}=\begin{bmatrix}-2xce^{x^2-y^2}\\ 2yce^{x^2-y^2}\end{bmatrix}=$$

$$=e^{x^2-y^2}\begin{bmatrix}-2cx\\ 2cy\end{bmatrix},\ \boxed{c=\tfrac{1}{2}a.}$$

e) $\varphi(x,y(x))=C=ce^{x^2-y^2}$

$A=e^{x^2-y^2}$

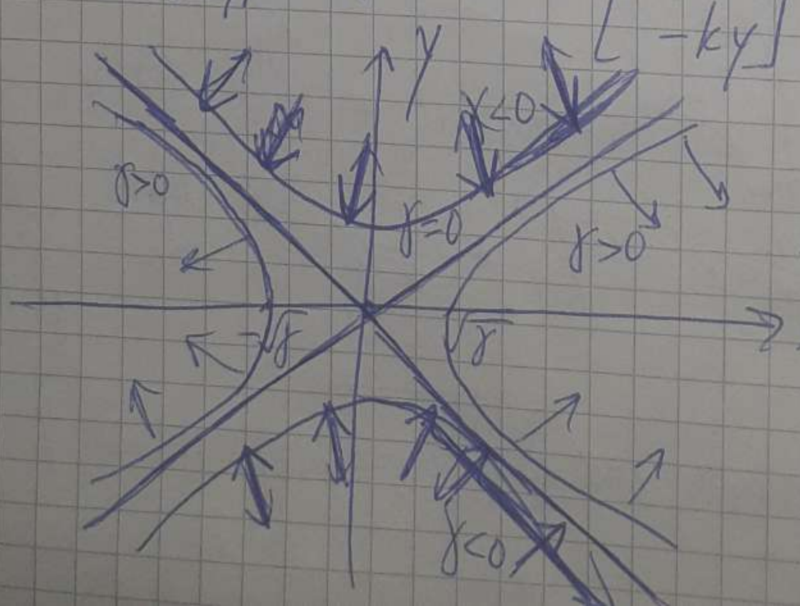
If

$e^{y^2}=e^{x^2}\cancel{B}e^{\gamma}$

$y^2=x^2+\gamma$, $y=\pm\sqrt{x^2+\gamma}$

$$\nabla\varphi(x,y)=-F=e^{x^2-y^2}\begin{bmatrix}\cancel{2}kx\\ -ky\end{bmatrix},\ \underline{k=2c}$$

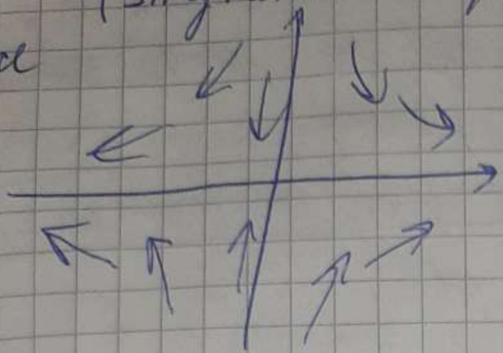
(for correct signs).



$x\to\infty$, $y\to$ straight line

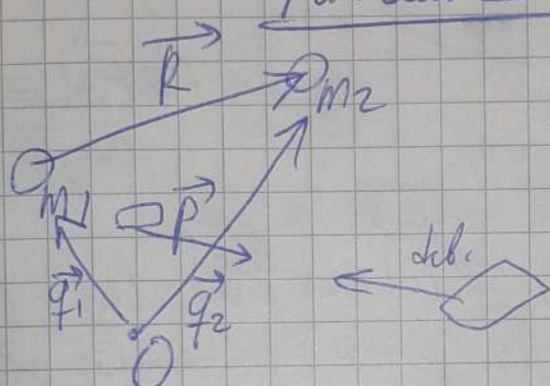
(Imag. continuous potential & gradient)

Since



(for $c<0$ ~~situation~~ directions are inverted).

Problem 2



$F_R = -k(R-l)$

a) $\vec Q = \dfrac{m_1\vec q_1(t)+m_2\vec q_2(t)}{M}$, but $\ddot{\vec Q}(t) = \dfrac{m_1\ddot{\vec q}_1+m_2\ddot{\vec q}_2}{M} = \dfrac{\vec F-\vec F}{M} = \vec 0$.

So $\underline{\vec Q(t)} = \vec Q_0 + \int_0^t \dot{\vec Q}\,dt =$

$= \vec Q_0 + \int_0^t \dot{\vec Q}_0\,dt = \vec Q_0 + \underbrace{\dot{\vec Q}_0}_{\vec 0} t = \underline{\vec Q_0} = \underline{\vec 0}$.

CM does not move.

EOM $\underline{\ddot{\vec Q}(t) = \vec 0}$ — enough to describe only relative motion

~~(after describ~~

Moreover:

1) $m_1\vec q_1 + m_2\vec q_2 = \vec 0$

2) $\vec R = \vec q_2 - \vec q_1 \Longrightarrow$

knowing $\vec R(t)$ describes everything.

b) $\vec R(t) = \vec q_2(t) - \vec q_1(t)$

$\Downarrow$

Our further task is just to describe $R(t)$ — not even $\vec R(t)$, because motion will be in the

~~E~~ plane spanned by $\vec{R}(0)$ and $\dot{\vec{R}}(0)$ and $\dot{\vec{L}}=\vec{0} \Longrightarrow$ it will be in the same plane (we choose that plane).

<u>Full conclusion</u> — 1DOF problem for $R(t)$ (after showing below →) (due to $\dot{\vec{L}}=\vec{0}$, see below).

b) $\vec{L} = \vec{R}(t)\times\mu\dot{\vec{R}}(t)$, $\frac{d\vec{L}}{dt} = \underbrace{\dot{\vec{R}}\times\mu\dot{\vec{R}}}_{\vec{0}} + \vec{R}\times\underbrace{\mu\ddot{\vec{R}}}_{\vec{F}_1\,(=-\vec{F}_2)} = \vec{R}\times(K\vec{R}) = \vec{0}$, for some K) □

[easily shown $\mu\ddot{\vec{R}} = \frac{m_1 m_2}{m_1+m_2}(\ddot{\vec{q}}_2-\ddot{\vec{q}}_1) \equiv$

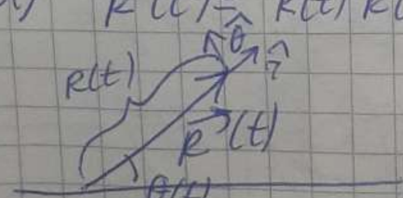
$\equiv$ $m_2\ddot{\vec{q}}_2 = -m_1\ddot{\vec{q}}_1$ because since ($\ddot{\vec{Q}}_{CM}=\vec{0}$),

$$\frac{m_1 m_2\ddot{\vec{q}}_2 - m_1 m_2\ddot{\vec{q}}_1}{m_1+m_2} = \frac{m_1 m_2\ddot{\vec{q}}_2 + m_2^2\ddot{\vec{q}}_2}{m_1+m_2} = m_2\ddot{\vec{q}}_2\Big].$$

$= \vec{F} = -k(R-\ell)\hat{R}$ where $\hat{R} = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R}$.

*c) (1) $\dot{\vec{q}}_1$ and $\dot{\vec{q}}_2$ (since $\dot{\vec{Q}}_{CM}=\vec{0}$) lie in a plane, and (2) $\dot{\vec{q}}_1$ and $\vec{p}$ lie in plane (by ~~definition~~ momentum conserv.) and (3) $\vec{p}$ and debris lie in plane (by definition of task) and (4) $\dot{\vec{q}}_1$ and $\dot{\vec{q}}_2$ determine $\vec{R}$ $\xRightarrow{1,2,3,4}$ motion <u>must</u> be in the same plane with m_1, m_2, debris <u>initially</u> and (5) it <u>will</u> stay there because $\dot{\vec{L}}=\vec{0}$.

d) $\vec{R}(t) = R(t)\hat{R}(t)$, $\dot{\vec{R}}(t) = \dot{R}\hat{R} + R\dot{\theta}\hat{\theta}$.



$$T = \frac{m(\dot{\vec{R}})^2}{2} = \frac{m}{2}\left[\dot{R}^2 + R^2\dot{\theta}^2\right]$$

$$V = \frac{k}{2}(R-\ell)^2,$$

$\Longrightarrow$ ~~L~~

$$\Longrightarrow \mathcal{L}(R,\theta) = \frac{m}{2}\left[\dot R^2 + R^2\dot\theta^2\right] - \frac{k}{2}(R-\ell)^2$$

for θ: (θ is cyclic)

(**) $\frac{\partial \mathcal{L}}{\partial \theta} = 0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = (mR^2\dot\theta)^{\cdot}$ $\quad \vec R = R(t)\,\hat r(t) = \vec R(R,\theta)$

$\dot{\vec L} = \vec 0$

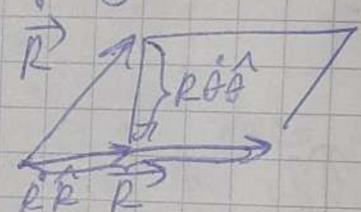
for R:

$$\frac{\partial \mathcal{L}}{\partial R} = mR\dot\theta^2 - k(R-\ell) =$$
$$= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot R}\right) = \frac{d}{dt}\left[m\dot R\right] = m\ddot R \quad (*)$$

~~Or alternatively for~~ $\cancel{\vec R(t)}$: Lagrange is not used in this task anymore.

$\cancel{T = \frac{m}{2}\dot{\vec R}^2 \quad V = \frac{k}{2}(R}$

e) But $\dot{\vec L} = \vec 0$, $\dot E = 0 \Longrightarrow$



$$L = |\vec L| = \mu \cdot R \cdot R\dot\theta = \mu R^2\dot\theta$$

$$\Longrightarrow \dot\theta = \frac{L}{\mu R^2}, \text{ (same information as in (**))}$$

~~Now from (*)~~

$$\cancel{mR\dot\theta^2 - k(R-\ell) = m\ddot R} \Longrightarrow$$

$$\cancel{mR\frac{L^2}{\mu R^4}} =$$

Now from $\dot E = 0$,

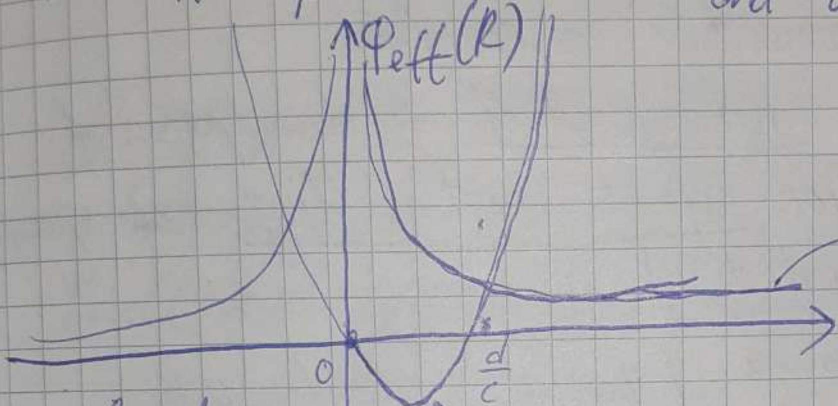
$$\frac{d}{dt}E = \frac{d}{dt}\left(\frac{\mu}{2}\left[\dot R^2 + R^2\dot\theta^2\right] + \frac{k}{2}(R-\ell)^2\right) =$$
$$= \frac{d}{dt}\left(\frac{\mu}{2}\left[\dot R^2 + R^2\frac{L^2}{\mu^2R^4}\right] + \frac{k}{2}(R-\ell)^2\right) =$$
$$= \frac{d}{dt}\underbrace{\left(\frac{\mu}{2}\left[\dot R^2 + \frac{L^2}{\mu^2R^2}\right] + \frac{k}{2}(R-\ell)^2\right)}_{E}$$

So $E = \frac{\mu}{2}\dot{R}^2 + \frac{L^2}{2\mu R^2} + \frac{k}{2}R^2 - k\ell R + \frac{k\ell^2}{2}$

So $a = \frac{\mu}{2}$ $\quad c = \frac{k}{2}$ $\quad e = \frac{k\ell^2}{2}$

$b = \frac{L^2}{2\mu}$ $\quad d = k\ell$, but easier to look at.

f) $\varphi_{eff} = \frac{b}{R^2} + cR^2 - dR$ [e is just constant shift, and a-part depends on $\dot{R}$ and not on R]



Consider just $R>0$ $(R = |\vec{R}|)$

$cR^2 - dR =$
$= R(cR-d) = 0$

Check minima:

$\varphi' = -\frac{2b}{R^3} + 2cR - d \; | = 0$

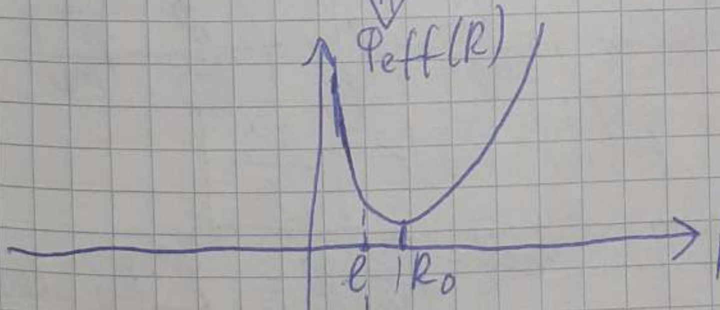
$-2b + 2cR^4 - dR^3 = 0$

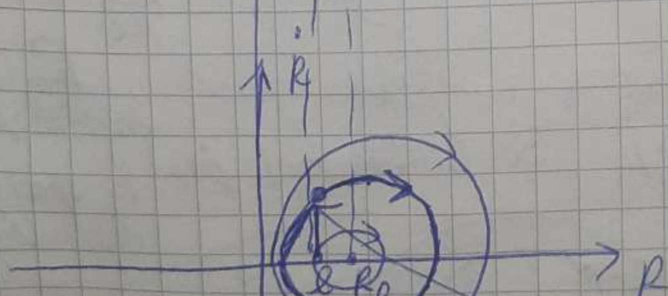
How related to $R = \ell$?

$\varphi'(\ell) = -2b + 2c\ell^4 - d\ell^3 =$
$= -\frac{L^2}{\mu} + \cancel{k\ell^4} - \cancel{k\ell^4} < 0$

$\Longrightarrow$ therefore $\underset{R_0}{R(\varphi_{min})} > \ell$,

due to rotational energy impact.





Stable minima is not at ℓ, but at $R_0 > \ell$.

~~*g) Initial speed determined from $\vec{p}$, let it~~ R_0

~~In phase space it is point $(\ell, \dot{R}_0)$~~

and then oscillates near minimum R_0.

$\longrightarrow$ see after

~~*h)~~

Intro to *g,h) Let us investigate situation better for given $\vec{p}$.

Debris direction is $\vec{d}$. $\vec{p} = -c\vec{d}$ for some c.

And by momentum conservation

(note, this is not important for description of further motion up to just setting one specific initial condition)

$m_1 \dot{\vec{q}}_1 + \vec{p} = \vec{0}$ (happens instantaneously before changes in $R(t)$)

Then $\dot{\vec{q}}_1 = -\frac{\vec{p}}{m_1}$

$m_1\dot{\vec{q}}_1 + m_2\dot{\vec{q}}_2 = \vec{0} \Longrightarrow \dot{\vec{q}}_2 = -\frac{m_1}{m_2}\cdot\left(-\frac{\vec{p}}{m_1}\right) = \frac{\vec{p}}{m_2}$.

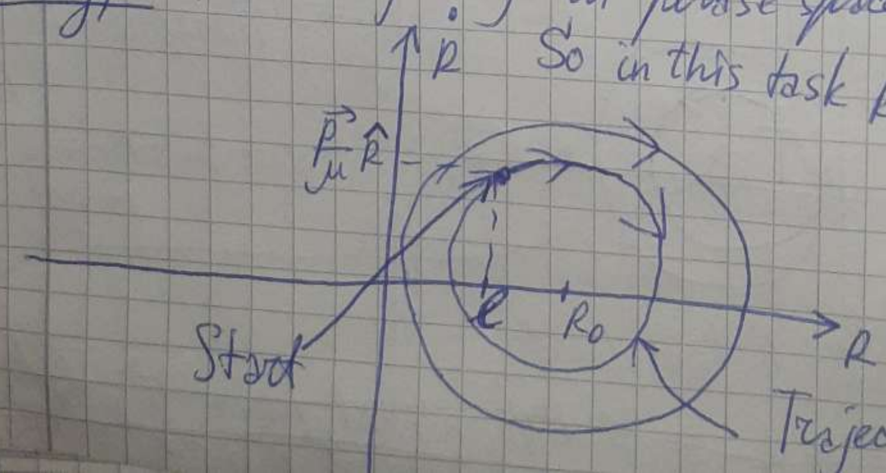
So $\dot{\vec{R}}$ initially is $-(\dot{\vec{q}}_1 - \dot{\vec{q}}_2) = -\left(-\frac{\vec{p}}{m_1} - \frac{\vec{p}}{m_2}\right) =$

$= \vec{p}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \boxed{\frac{\vec{p}}{\mu}}$.

$\Longrightarrow \dot{\vec{R}}(0) = \frac{\vec{p}}{\mu}$. Then $\dot{R}(0) = \frac{\vec{p}}{\mu}\cdot\hat{R} = \frac{\vec{p}}{\mu}\vec{R}_0$ where $\vec{R}_0$ is initial direction

part *g) Full trajectory in phase space:

So in this task $\dot{R}(0)$ is specified, depending on $\vec{p}$.



part *h) $\frac{b}{R^2}$ is energy to stop rotation.

So $E_{stop} = \frac{L^2}{2\mu R^2} = \frac{L_0^2}{2\mu R^2}$ (due to $\dot{L}=0$, $L=L_0$)

but $L_0 = \mu R_0^2 \dot{\theta}_0 = \mu \ell^2 \dot{\theta}_0$, So $E_{\text{to stop rotation}} =$

$$= \frac{\mu^2 \ell^4 \dot{\theta}_0^2}{2\mu R^2} = \frac{\mu \ell^4}{2R^2}\dot{\theta}_0^2.$$

At any given R, $E(R) = \frac{\mu \ell^4}{2R^2}\dot{\theta}_0^2$, where $\dot{\theta}_0$ stores dependency on $\vec{p}$.

Recall that $\dot{\vec{R}}(0) = \frac{\vec{p}}{\mu} = \dot{R}(0)\hat{R} + R_{(0)}\hat{\theta}\dot{\theta}(0) =$

$$= \dot{R}(0)\hat{R} + \ell\hat{\theta}\dot{\theta}(0)$$

set up by task situation

set up by coordinate system

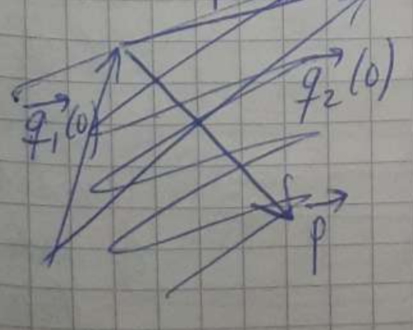
See above in phase space $\Longrightarrow$

$$\frac{p^2}{\mu^2} = \dot{R}_0^2 + \ell^2\dot{\theta}_0^2, \quad \dot{\theta}_0^2 = \frac{\left(\frac{p}{\mu}\right)^2 - \dot{R}_0^2}{\ell^2}$$

$$\Longrightarrow E(R) = \frac{\mu\ell^4}{2R^2}\left[\frac{\left(\frac{p}{\mu}\right)^2 - \dot{R}_0^2}{\ell^2}\right] = \frac{\mu\ell^2}{2R^2}\left[\left(\frac{p}{\mu}\right)^2 - \dot{R}_0^2\right]$$

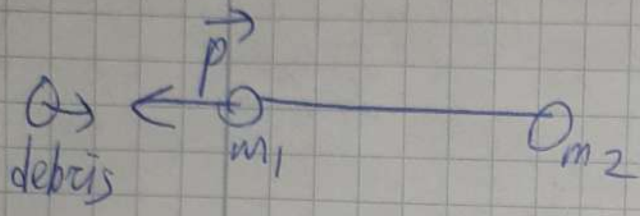
$= \cancel{\frac{\mu\ell^2}{2}}$

Now,



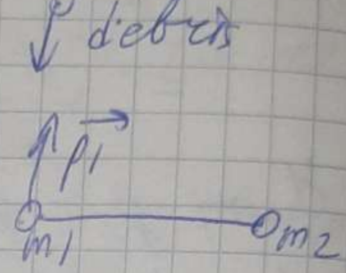
For given R E is minima when $\dot{R}_0^2$ initially was the maxima, so $\frac{\vec{p}}{\mu}\hat{R}$ is maxima, So $\vec{p}$ is along $\vec{R}(0)$.

Which makes sense, because then there is no rotation;



and

maximum rotation when



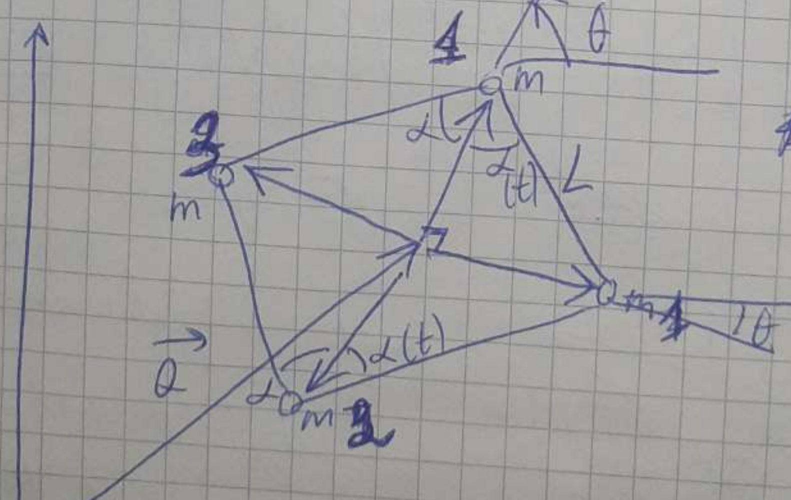
Problem 3

a) $\cancel{\vec Q(t) = \frac{\sum_{i=1}^{4} m_i \vec q_i}{4m} \Longrightarrow 4m\vec Q = \sum_i m_i \vec q_i}$

$\cancel{\Longrightarrow \vec q_i(t) =}$

From geometry it is clear that

$\vec q_i(t) = \vec Q(t) + L_i(\alpha(t))\,\hat r(\theta_i(t)) = \cancel{\vec Q(t) + L\cos}$ $(\delta_i \frac{\pi}{2})\,\hat r(\theta_i(t))$



(independently of α

$\theta_i(t=0)$ are determined by initial orientation θ)

$$\vec{q}_i = \begin{cases} \vec{Q}(t) + L\sin\delta\,\hat{r}(\theta_1) & (m1) \\ \vec{Q}(t) + L\sin\delta\,\hat{r}(\theta_3) & (m3) \\ \vec{Q}(t) + L\cos\delta\,\hat{r}(\theta_2) & (m2) \\ \vec{Q}(t) + L\cos\delta\,\hat{r}(\theta_4) & (m4) \end{cases}$$

Can be further simplified, but that is unnecessary for $\mathcal{L}$

b) $V = -Mg\vec{\ }\vec{Q}$

c) $$\dot{\vec{q}}_i = \begin{cases} \dot{\vec{Q}} + L\cos\delta\,\dot{\delta}\,\hat{r}(\theta_1) + L\sin\delta\,\hat{\theta}(\theta_1)\,\dot{\theta} \\ \dot{\vec{Q}} + L\cos\delta\,\dot{\delta}\,\hat{r}(\theta_3) + L\sin\delta\,\hat{\theta}(\theta_3)\,\dot{\theta} \\ \dot{\vec{Q}} - L\sin\delta\,\dot{\delta}\,\hat{r}(\theta_2) + L\cos\delta\,\hat{\theta}(\theta_2)\,\dot{\theta} \\ \dot{\vec{Q}} - L\sin\delta\,\dot{\delta}\,\hat{r}(\theta_4) + L\cos\delta\,\hat{r}(\theta_4)\,\dot{\theta} \end{cases}$$

because all masses have constant shifts from $\theta(t)$

$$T = \underbrace{\frac{4m\dot{\vec{Q}}^2}{2}}_{CM} + \underbrace{\frac{m}{2}\left[L^2\cos^2\delta\,\dot{\delta}^2 + L^2\sin^2\delta\,\dot{\theta}^2\right]\cdot 2}_{1,3} + \underbrace{\frac{m}{2}\left[L^2\sin^2\delta\,\dot{\delta}^2 + L^2\cos^2\delta\,\dot{\theta}^2\right]\cdot 2}_{2,4} =$$

$$= 2m\dot{\vec{Q}}^2 + m\left[L^2\dot{\delta}^2 + L^2\dot{\theta}^2\right] = 2m\dot{\vec{Q}}^2 + mL^2\left[\dot{\delta}^2 + \dot{\theta}^2\right]$$

d) $$\mathcal{L} = T - V = \underbrace{2m}_{a}\dot{\vec{Q}}^2 + \underbrace{mL^2}_{b}\dot{\delta}^2 + \underbrace{mL^2}_{c}\dot{\theta}^2 + \underbrace{Mg\vec{\ }\vec{Q}}_{-\underbrace{Mg}_{d}\hat{y}\vec{Q}}.$$

e) for $\theta(t)$: $\frac{\partial\mathcal{L}}{\partial\theta} = 0 = \frac{d}{dt}\left[\underbrace{2c\dot{\theta}}_{const}\right] = 2c\ddot{\theta}$

$\underline{const,\ \dot{\theta} = const}$

Similarly for $\delta(t)$: $\frac{\partial\mathcal{L}}{\partial\delta} = 0 = \frac{d}{dt}\left[2b\dot{\delta}\right]$

$\downarrow$ $\underline{\dot{\delta}\ const}$

$$\boxed{\begin{aligned} \theta(t) &= \theta_0 + \dot{\theta}t \\ \delta(t) &= \delta_0 + \dot{\delta}t \end{aligned}}$$

f) Full description then:

Initial Conditions

$\theta(t_0)=0,\ \alpha(t_0)=\frac{\pi}{4}$

$\dot{\vec{Q}}(t_0)=\vec{v},\quad \dot{\theta}(t_0)=\Omega,$

$\dot{\alpha}(t_0)=\omega,$

CM motion:

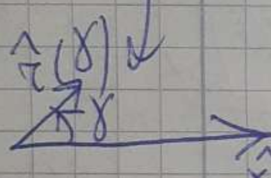
$$\vec{Q}(t)=\vec{Q}(t_0)+\dot{\vec{Q}}(t_0)(t-t_0)+\frac{\vec{g}(t-t_0)^2}{2},$$

$$\alpha(t)=\cancel{\alpha}\,\frac{\pi}{4}+\omega(t-t_0)$$

$$\theta(t)=\Omega(t-t_0)$$

$$\Longrightarrow\ \vec{q}_1(t)=\underbrace{\vec{Q}(t_0)+\vec{V}(t-t_0)+\frac{\vec{g}(t-t_0)^2}{2}}_{\vec{Q}(t)}+L\sin\left(\frac{\pi}{4}+\omega(t-t_0)\right)\cancel{\hat{r}(\Omega(t-t_0))}\ \hat{r}\left(-\frac{\pi}{2}+\Omega(t-t_0)\right)$$

Using convention like

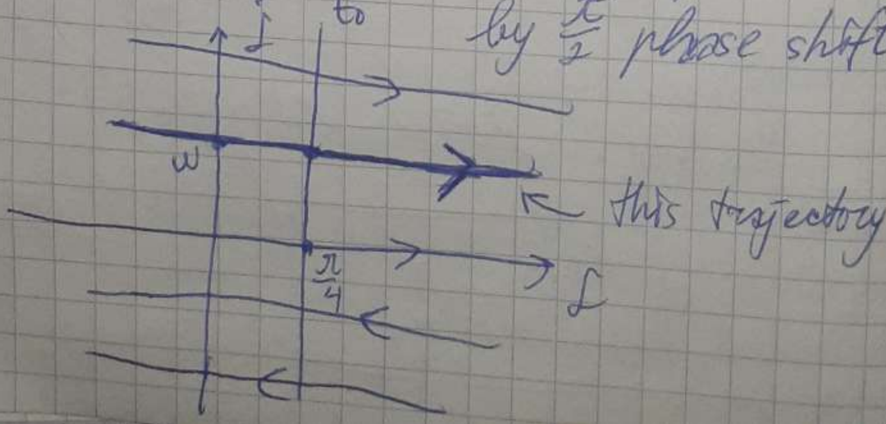


$$\vec{q}_2(t)=\cancel{\vec{Q}(t_0)}\ \vec{Q}(t)+L\cos\left(\frac{\pi}{4}+\omega(t-t_0)\right)\cdot\hat{r}(\pi+\Omega(t-t_0))$$

$$\vec{q}_3(t)=\vec{Q}(t)+L\sin\left(\frac{\pi}{4}+\omega(t-t_0)\right)\cdot\hat{r}\left(+\frac{\pi}{2}+\Omega(t-t_0)\right)$$

$$\vec{q}_4(t)=\vec{Q}(t)+L\cos\left(\frac{\pi}{4}+\omega(t-t_0)\right)\hat{r}(\Omega(t-t_0)).$$

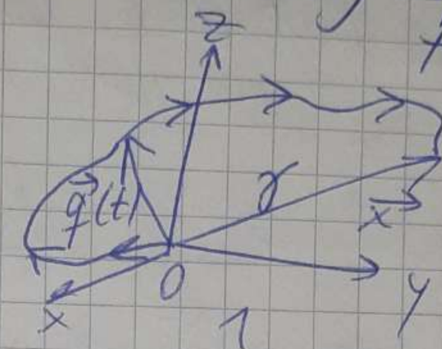
~~*~~ g) CM moves in free fall, and corners rotate around it separated by $\frac{\pi}{2}$ phase shift.



$\gamma(t)$ in this description changes linearly as in rigid body, without any additional potential.

Problem 7 – Velocity-dependent force

$$\vec{F} = \vec{A} \times \dot{\vec{q}} + \vec{B}$$



a) $\gamma = \{\vec{q}(t) = t\vec{x} \mid 0 \le t \le 1\}$

$$W = \int_\gamma [\vec{A} \times \dot{\vec{q}} + \vec{B}]\, d\vec{q} = \int_0^1 [\vec{A} \times \vec{x} + \vec{B}] \cdot \vec{x}\, dt =$$

$\dot{\vec{q}}(t) = \vec{x}$

$$= \int_0^1 \vec{x} \cdot (\vec{A} \times \vec{x}) + \vec{B}\vec{x}\, dt =$$

$$= \int_0^1 \vec{A} \cdot \underbrace{(\vec{x} \times \vec{x})}_{\vec{0}} + \vec{B}\vec{x}\, dt = \vec{B}\vec{x} \cdot 1.$$

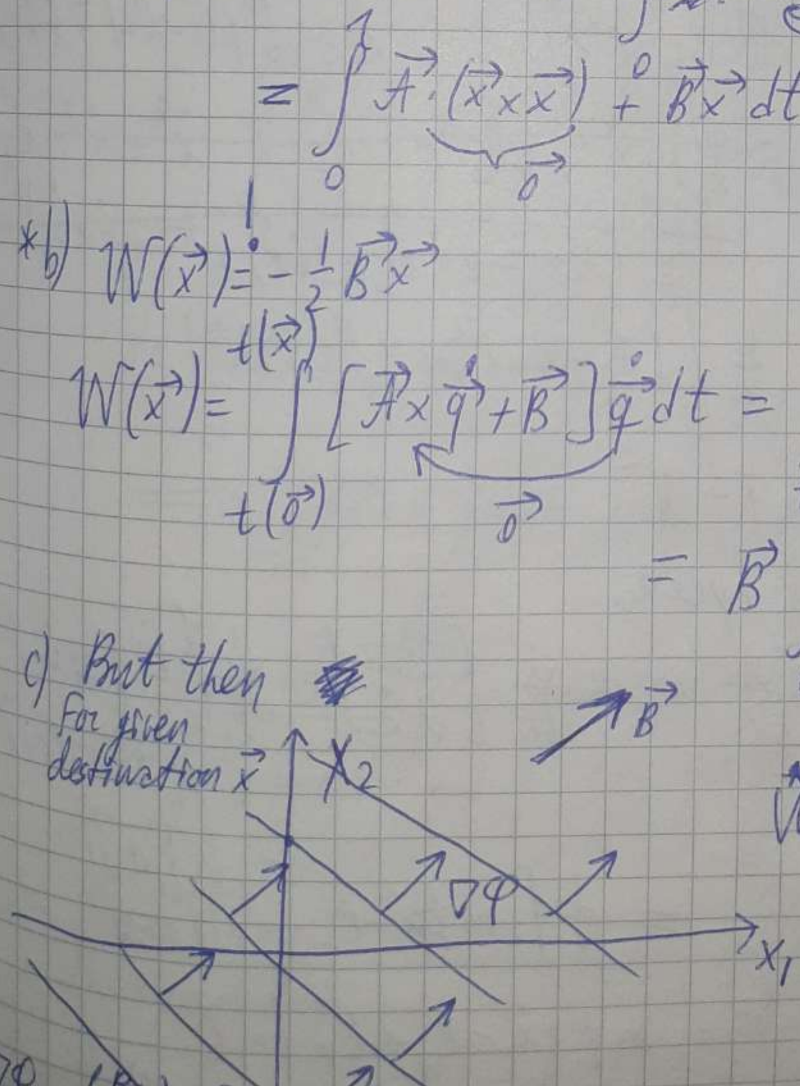
*b) $W(\vec{x}) \overset{!}{=} -\frac{1}{2}\vec{B}\vec{x}$

$$W(\vec{x}) = \int_{t(\vec{0})}^{t(\vec{x})} [\underbrace{\vec{A} \times \dot{\vec{q}}}_{\vec{0}} + \vec{B}]\, \dot{\vec{q}}\, dt = \int_{t(\vec{0})}^{t(\vec{x})} \vec{B}\dot{\vec{q}}\, dt =$$

$$= \vec{B} \int_{\vec{0}}^{\vec{x}} d\vec{q} = \vec{B}\vec{x}, \text{ always}$$

c) But then

For given destination $\vec{x}$



$W(\vec{x}) = \vec{B}\vec{x} = \text{const}$

$B_1 x_1 + B_2 x_2 = C$

$\to$ straight lines ($\vec{B}$ is constant)

$x_2 = \frac{C}{B_2} - \frac{B_1}{B_2} x_1$

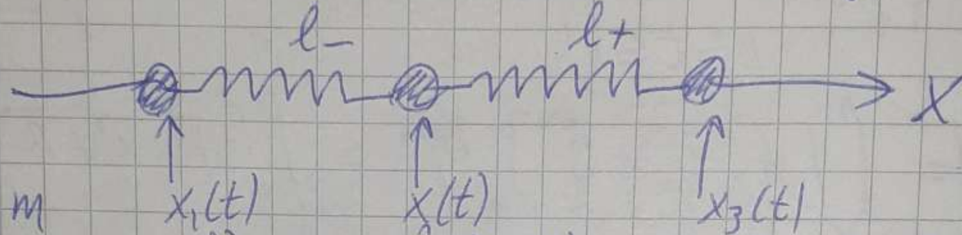
$(B_2, B_1 > 0)$

$\nabla\varphi = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \vec{B}$

*d) It is strictly speaking <u>not</u> conservative because $\nabla\varphi = \vec{B} \neq \vec{F}$.

But it <u>is</u> conservative when $\vec{A} = \vec{0}$ and there is no velocity dependence.

Problems Springs



l, m

$x_1(t) = x(t) - l_-(t)$ $\quad x_2(t) = x(t)$ $\quad x_3(t) = x(t) + l_+(t)$

a) $T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) = \frac{m}{2}\left((\dot{x} - \dot{l}_-)^2 + \dot{x}^2 + (\dot{x} + \dot{l}_+)^2\right)$

$\cancel{= \frac{m}{2}\left(\dot{x}^2 + \dot{l}_-^2 - 2\dot{x}\dot{l}_- + \dot{x}^2 + \dot{x}^2 + \dot{l}_+^2 + \right.}$

$\cancel{= \frac{m}{2}\left(3\dot{x}^2 - 2\dot{x}\dot{l}_- + 2\dot{x}\dot{l}_+ + + 2\dot{x}\dot{l}_+\right) =}$

$V = \frac{k}{2}\left[(l - l_-)^2 + (l - l_+)^2\right]$

b) $\mathcal{L} = T - V$,

x is cyclic.

Then $\frac{\partial \mathcal{L}}{\partial x} = 0 = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] = \frac{d}{dt}\left(\frac{m}{2}\left[2(\dot{x} - \dot{l}_-) + 2\dot{x} + 2(\dot{x} + \dot{l}_+)\right] - \frac{k}{2}[0]\right)$

$= \frac{d}{dt}\left[m\left[3\dot{x} - \dot{l}_- + \dot{l}_+\right]\right] =$

So $\quad m\left[3\dot{x} - \dot{l}_- + \dot{l}_+\right] = C$

$\dot{x} = \frac{1}{3}\left(\frac{C}{m} + \dot{l}_- - \dot{l}_+\right)$

So $\ddot{x} = \underbrace{-\frac{1}{3}}_{c}(\ddot{l}_+ + \ddot{l}_-)$

Center of mass moves with constant velocity.

Enough to describe 2DOF — l_+ and l_-.

c) From $\mathcal{L}$:

1) for l_-

$$\frac{\partial \mathcal{L}}{\partial l_-} = \cancel{\frac{m}{2}[-2(\dot{x} - \dot{l}_-)] + \frac{k}{2}\cdot 2[l - l_-]} =$$

$$- \frac{k}{2}\cdot 2(l - l_-)(-1) = k(l - l_-) =$$

$$= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{l}_-} = \frac{d}{dt}\left[\frac{m}{2}\cdot 2(\dot{x} - \dot{l}_-)(-1)\right] =$$

$$= \frac{d}{dt}[-m(\dot{x} - \dot{l}_-)] =$$

$$= m\ddot{l}_- - m\ddot{x}.$$

So $k(l - l_-) = m\ddot{l}_- - m\ddot{x}$

$c(\ddot{l}_+ - \ddot{l}_-)$

$$\Longrightarrow \frac{k}{m}(l - l_-) = \ddot{l}_- - c(\ddot{l}_+ - \ddot{l}_-) \Longrightarrow$$

$$\ddot{l}_-[c+1] - c\ddot{l}_+ = -\frac{k}{m}(l_- + l)$$

$$\Longrightarrow \boxed{\ddot{l}_- - c(\ddot{l}_+ + \ddot{l}_-) = -\omega^2(l_- - l), \quad c = -\frac{1}{3}, \quad \omega = \sqrt{\frac{k}{m}}}$$

By symmetry ...

$$\ddot{l}_+ + c(\ddot{l}_+ - \ddot{l}_-) = -\omega^2(l_+ - l)$$

□

d) $\Delta = \dfrac{(l_+ - l_-)}{l} \qquad \Sigma = \dfrac{(l_+ + l_- - 2l)}{l}$

$$\tau = \omega(t - t_0)$$

$$\ddot{\Delta} = \frac{l_+'' - l_-''}{l} \qquad \ddot{\Sigma} = \frac{l_+'' + l_-''}{l}$$

$$\Downarrow$$

for $\Delta(\tau)$:

$$-\begin{cases} l_+'' + c\,l\,\Delta'' = -\omega^2(l_+ - l) & (*) \\ l_-'' - c\,l\,\Delta'' = -\omega^2(l_- - l) & (**) \end{cases}$$

$$\cancel{l}\Delta'' + 2c\cancel{l}\Delta'' = -\omega^2\cancel{l}\Delta$$

$$\ddot{\Delta}(1+2c) = -\omega^2\Delta$$

In new units

$\cancel{\ddot{\Delta}} \longrightarrow \cancel{\ddot{\Delta} = -\frac{1}{\underbrace{1+2c}_{\frac{5}{3}}}\Delta = -\frac{3}{5}\Delta}$

then $c = -\frac{1}{3}$, not $\frac{1}{3}$ (changed before)

~~(not -3 — then due to arithmetic errors I should got $c = -\frac{1}{3}$ before)~~

$$\ddot{\Delta}\cdot\frac{1}{3} = -\Delta \Longrightarrow \underset{(\tau)}{\ddot{\Delta}} = -3\underset{(\tau)}{\Delta}$$

for $\Sigma(\tau)$

(now adding (*) and (**))

$$\ddot{\Sigma} = \frac{1}{l}\left[-\omega^2(l_+ - l) - \omega^2(l_- - l)\right] =$$

$$= \frac{1}{\cancel{l}}[\omega^2][2\cancel{l} - \underbrace{(l_+ + l_-)}_{\cancel{l}\Sigma + 2\cancel{l}}] =$$

$$= -\omega^2\Sigma$$

in τ units $\ddot{\Sigma}(\tau) = -\Sigma(\tau)$

e) But then $\Delta(\tau) = A\cos(\sqrt{3}\tau + \varphi_0)$
f) $\Sigma(\tau) = B\cos(\tau + \varphi_1)$ } EOM

$\Delta(0) = A\cos\varphi_0$
$\dot{\Delta}(0) = -A\sqrt{3}\sin\varphi_0$
$\Sigma(0) = B\cos\varphi_1$
$\dot{\Sigma}(0) = -B\sin\varphi_1$

	Case e)	Case f)
$\Delta(0) = \frac{\ell_+(0) - \ell_-(0)}{\ell}$	0	$\frac{\ell + L - \ell}{\ell} = L/\ell$
$\dot{\Delta}(0) = \frac{\dot{\ell}_+(0) - \dot{\ell}_-(0)}{\ell}$	0	0
$\Sigma(0) = \frac{\ell_+(0) + \ell_-(0) - 2\ell}{\ell}$	$\frac{2(\ell + L) - 2\ell}{\ell} = \frac{2L}{\ell}$	$\frac{\cancel{\ell} + L + \cancel{\ell} - 2\cancel{\ell}}{\ell} = \frac{L}{\ell}$
$\dot{\Sigma}(0) = \frac{\dot{\ell}_+(0) + \dot{\ell}_-(0)}{\ell}$	0	0

$\Delta(\tau) = A\cos(\sqrt{3}\tau + \varphi_0)$

~~$\dot{\Delta}(\tau) = -As$~~

$\Sigma(\tau) = B\cos(\tau + \varphi_1)$

Case e): $\boxed{\Delta(\tau) = 0}$

$B\cos\varphi_1 = \frac{2L}{\ell}$
$-B\sin\varphi_1 = 0$, $\varphi_1 = 0$
$\Downarrow$
$\boxed{\Sigma(\tau) = \frac{2L}{\ell}\cos(\tau)}$

Case f): ~~$\Delta(\tau)$~~ $A\cos\varphi_0 = L/\ell$
$-A\sqrt{3}\sin\varphi_0 = 0$
$\varphi_0 = 0$

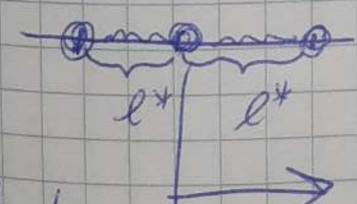
$\boxed{\Delta(\tau) = L/\ell \cos(\sqrt{3}\tau)}$

~~$B\cos\varphi_1 = L/\ell$~~

$\boxed{\Sigma(\tau) = \frac{L}{\ell}\cos(\tau)}$

Conclusion for (e)

S_0



System CM moves uniformly and ~~edge~~ left/right masses oscillate with same displacements.

Conclusion for (f)

~~$\Delta(\tau)$ $\Sigma(\tau)$~~

System CM moves uniformly, differences & sums of lengths oscillate.

With different frequencies / periods

→ Check if possible yes! ✓