Re-take Exam Solutions. 30 March

Problem 1. Prove that for any $n \geq 2$, $n \in \mathbb{N}$, one has

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

Solution: Let us prove this by induction in n. Base case: $n = 2$,

$$\left(1-\frac{1}{4}\right)=\frac{2+1}{2\cdot 2}=\frac{3}{4},$$

holds. Assume that the statement holds for n, and let is prove it for $(n+1)$. Using the induction hypothesis, one has

$$\begin{aligned}\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)&=\frac{n+1}{2n}\left(1-\frac{1}{(n+1)^2}\right)\\&=\frac{(n+1)}{2n}\frac{(n^2+2n)}{(n+1)^2}=\frac{n+2}{2(n+1)},\end{aligned}$$

which concludes the proof.

Problem 2. Determine the supremum of the following set

$$\{\sqrt{n+1}-\sqrt{n} : n \in \mathbb{N}\}.$$

Solution: Let us prove that $a_n := \sqrt{n+1}-\sqrt{n}$ is a monotonically decreasing sequence. In order to show this, one needs to prove that

$$\sqrt{n+1}-\sqrt{n} > \sqrt{n+2}-\sqrt{n+1}, \qquad \text{for all } n \in \mathbb{N}. \tag{1}$$

This follows from the definition of the concave function and the fact that $\sqrt{x}$ is a concave function, since its second derivative is negative. Let us give also a direct proof. Squaring both sides of the inequality $2\sqrt{n+1} > \sqrt{n+2}+\sqrt{n}$, one gets

$$4(n+1) > 2n+2+2\sqrt{n(n+2)},$$

cancelling terms and squaring again, one gets

$$4(n+1)^2 > 4n(n+2),$$

which is correct. Going via these inequalities from bottom to top, one obtains the proof of (1).

Therefore, a_1 is the largest element of the sequence (a_n), and the supremum is equal to $a_1 = \sqrt{2}-1$.

Problem 3. Provide an example of a positive sequence (a_n) such that $\sqrt[n]{a_n} \to 1$ as $n \to \infty$, but a_{n+1}/a_n does not tend to 1, as $n \to \infty$.

Solution: For example, let $a_1 = a_3 = a_5 = a_7 = \cdots = 1$, and $a_2 = a_4 = a_6 = a_8 = \cdots = 2$. Then $\sqrt[n]{a_n} \to 1$, since $\sqrt[n]{1} \to 1$ and $\sqrt[n]{2} \to 1$. The sequence a_{n+1}/a_n has a form $2, 0.5, 2, 0.5, 2, 0.5, 2, 0.5, \ldots$, thus, it does not have a limit.

Problem 4. Compute the following limit

$$\lim_{n\to\infty} \frac{2+n+5n\sqrt{n}}{3+6\sqrt{n}+2n\sqrt{n}}.$$

Solution: Divide both the numerator and the denominator by $n\sqrt{n}$, which is the fastest growing term in the expression. One obtains

$$\lim_{n\to\infty} \frac{2n^{-3/2}+n^{-1/2}+5}{3n^{-3/2}+6n^{-1}+2}$$

We know that all non-constant summands converge to 0. Applying arithmetics of limits, we obtain that this limit equals

$$\frac{0+0+5}{0+0+2} = \frac{5}{2}.$$

Problem 5. Check the following series for convergence

$$\sum_{n=1}^{\infty} \frac{n^{100}}{2^n + n^{101}}.$$

Solution: There exists N such that for all $n > N$ one has $1.1^n > n^{100}$. Therefore, for all $n > N$ one has

$$\frac{n^{100}}{2^n + n^{101}} < \frac{1.1^n}{2^n} = (\frac{1.1}{2})^n.$$

Since $\sum_{n=1}^{\infty} \left(\frac{1.1}{2}\right)^n$ converges (this is a geometric series), the series in question also converges by the Comparison Test.

Problem 6. Provide an example of a continuous function $f : (1;5) \to \mathbb{R}$ which is not differentiable at points $x = 2$ and $x = 4$, and is differentiable at all other points of $(1;5)$.

Solution: For example, let $f(x) = |x-2| + |x-4|$. One can equivalently write it as

$$f(x) = \begin{cases} 6-2x, & \text{for } x \in (1;2] \\ 2, & \text{for } x \in [2;4] \\ 2x-6, & \text{for } x \in [4;5) \end{cases}$$

$f(x)$ is everywhere continuous since it is the sum of two continuous functions. It is also differentiable at all points of $(1;5)$ except of 2 and 4 since $f(x)$ is linear in the neighborhood of these points. At $x_0 = 2$ and $x_0 = 4$ the expression $\frac{f(x)-f(x_0)}{x-x_0}$ does not have a limit, since it has constant and different values depending on whether $x > x_0$ or $x < x_0$, therefore, the function is not differentiable there.

Problem 7. Compute the following limit

$$\lim_{x\to 1} \frac{x+x^2+x^3+x^4+x^5-5}{x-1}.$$

Solution: Note that the limit of both numerator and denominator is 0. Applying L'Hospital's rule, one gets that this limit is equal to $1+2+3+4+5 = 15$.

Problem 8. Compute the following series

$$\sum_{n=1}^{\infty}(-1)^n \frac{2^{2n}}{(2n)!}$$

Solution: Recall that

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}, \qquad \text{for any } x \in \mathbb{R}.$$

Therefore, the series in question equals $\cos(2) - 1$.

Problem 9. Find all local and global extrema of the following function

$$f(x) = x^3 e^x, \qquad f : [-10; 1] \to \mathbb{R}.$$

Solution: We have

$$f'(x) = x^3 e^x + 3x^2 e^x = x^2(x+3)e^x.$$

Therefore, one has $f'(x) = 0$ if and only if $x = 0, -3$, and the function $f'(x)$ is negative in the interval $[-10; -3)$, positive in the interval $(-3; 0)$, and again positive in the interval $(0; 1]$. Therefore, $x = -3$ is a point of a local minimum, and $x = 0$ is not an extremum.

In order to determine the global minimum and maximum, notice that $f(x)$ is decreasing in $[-10; -3)$ and increasing in $(-3; 1)$ due to the analysis of the signs of the derivative above. Therefore, (-3) is a point of the global minimum with the value $f(-3) = \frac{-27}{e^3}$. In order to find the global maximum, we need to compare the boundary values: $f(-10) < 0$, $f(1) > 0$, therefore $f(1) > f(-10)$ and we conclude that $f(1) = e$ is the global maximum attained at $x = 1$.

Problem 10. Compute the area of a region bounded by curves

$$y = x^2, \qquad y = x^3.$$

Solution: Note that the region (of finite area) bounded by these curves appears for $x \in [0; 1]$, and the curves intersect at points $(0, 0)$ and $(1, 1)$. The are of the region can be computed as the area below the curve $y = x^2$ minus the area below the curve $y = x^3$, for $x \in [0; 1]$. Therefore,

$$A = \int_0^1 x^2 - \int_0^1 x^3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Problem 11. Determine all possible values of $\alpha \in \mathbb{R}$, $\alpha > 0$, such that the following improper integral is convergent

$$\int_0^1 \frac{3 + \frac{1}{\sqrt{x}}}{x^\alpha} dx$$

Solution: Recall that $\int_0^1 \frac{1}{x^\beta} dx$ is convergent if and only if $\beta < 1$. Therefore, the integral

$$\int_0^1 \left(\frac{3}{x^\alpha} + \frac{1}{x^{\frac{1}{2}+\alpha}} \right) dx$$

converges if and only if $\frac{1}{2} + \alpha < 1$, which means that $\alpha < \frac{1}{2}$.

Problem 12. Let $u = (1,1,1)$ be a vector from $\mathbb{R}^3$. Provide an example of different vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ such that (u, v_1, v_2) and (u, v_3, v_4) are both linear bases of $\mathbb{R}^3$. In other words, provide two different pairs of vectors which together with $(1,1,1)$ form a linear basis of $\mathbb{R}^3$.

Solution: For example, let $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$, and $v_4 = (-1,0,0)$. Then (u, v_1, v_2) is a linear basis, because the equation

$$\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(1,1,1) = (0,0,0)$$

implies that $\alpha_3 = 0$ and $\alpha_1 = -\alpha_3$, $\alpha_2 = -\alpha_3$, so all coefficients must be 0. Similarly, (u, v_3, v_4) is a linear basis, because the equation

$$\alpha_1(0,0,1) + \alpha_2(-1,0,0) + \alpha_3(1,1,1) = (0,0,0)$$

implies that $\alpha_3 = 0$ and $\alpha_1 = -\alpha_3$, $\alpha_2 = \alpha_3$, so all coefficients must be 0 as well.