Mathematics 1. Selected proofs
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Derivative. Lagrange's and Cauchy's theorems

1. Statement of Lagrange's theorem

   THEOREM 3. Assume $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there is a point $c \in (a,b)$ such that

   $$f'(c) = \frac{f(b)-f(a)}{b-a}$$

2. PROOF. Define the auxiliary function $h(x)$:

   $$h(x) \ := \ f(x) \ - \ f(a) \ - \ \frac{f(b)-f(a)}{b-a} \cdot (x-a), \qquad x \in [a,b]$$

3. Verify the assumptions of Rolle's theorem for $h(x)$:

   $h(a) = f(a) - f(a) = 0, \quad h(a) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a} \cdot (b-a) = 0 \quad \Longrightarrow \quad h(a) = h(b) = 0$

   $h : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b).

4. Use Rolle's theorem for $h(x)$:

   $$\exists\, c \in (a,b) : \quad h'(c) = 0, \quad h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \quad \Longrightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

5. Statement of Cauchy's theorem:

   THEOREM 4. Assume functions $f,\ g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). Assume $g'(x) \neq 0$ for any $x \in (a,b)$. Then there is a point $c \in (a,b)$ such that

   $$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

6. Show that $g(b) \neq g(a)$ and hence $\frac{f(b)-f(a)}{g(b)-g(a)}$ is well-defined.

   By contradiction, assume $g(b) \neq g(a) \overset{\text{Rolle}}{\Longrightarrow} \ \exists\, c \in (a,b)$: $g'(c) = 0$ — contradicts to $g'(x) \neq 0$

7. Define the auxiliary function $h(x)$:

   $$h(x) \ := \ f(x) \ - \ f(a) \ - \ \frac{f(b)-f(a)}{g(b)-g(a)} \cdot \big(g(x) - g(a)\big), \qquad x \in [a,b]$$

8. Use Rolle's theorem for $h(x)$:

   $h(a) = f(a) - f(a) = 0, \ h(a) = f(b) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot \big(g(b) - g(a)\big) = 0 \implies h(a) = h(b) = 0$

   $h : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b)

   $$\exists\, c \in (a,b) : \quad h'(c) = 0, \quad h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g'(x) \quad \Longrightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a} \cdot \underbrace{g'(c)}_{\neq 0}$$