Exam

Instructions for working on this exam

1. There are 120 min to execute the exam. In total there are 60 points + 30 bonus points. You pass the exam with 25 points. For the best grade, 1.0, you need 45 points. The points for the different sub-tasks are provided on the next page. If you want to go through the full exam, there will be about 1 min/point.

2. In the present exam there are two exercises that address skills that we need to solve physical problems, and subsequently you will discuss in some depth two physical problems.

3. The best strategy to attack the exam is to strive to collect as many points as possible. Do not try to solve the full exam. Work on things that you can solve — ignore anything where you get stuck.

4. Each exercise should be solved on a separate set of sheets. Please write neatly and leave space on the margin for remarks and indicating credits.

5. Bonus problems are marked by (*). Often they involve a tricky argument that might not be immediately obvious. I recommend that you first work on the other exercises. Only attack the *-problems in the end when there is still time, or when you immediately see a fast and straightforward solution.

6. Explain carefully what you are doing. What do you assume? What do you intend to show? We will only give points when we understand what you intend to do.

7. You may also use my lecture notes and one sheet of A4 paper with hand-written notes that you prepared to take the exam. You must not use other resources; in particular no calculators and algebra programs.

Declaration of independence (written examinations)

For the exam of the course:	Theoretical Physics I. Theoretical Mechanics
Name of the examiner:	Prof. Dr. Jürgen Vollmer
Date:	29 March 2022

Note: If unauthorized aids are used and/or the exam is not written independently or in the case of obvious cooperation with fellow students or obvious copying, this will be treated as an attempt to cheat!

I confirm with my signature that I have prepared the answers/solutions of the examination task(s) independently and have not used any unauthorized aids. I did not ask any other person for help in preparing the answers/solutions and I did not copy from the answers/solutions of other students.

name, given name   matriculation number

Point overview

1 (a)	(b)	(c)	*[d]	2 (a)	2 (b)	*[c]	*[d]	(e)
3	2	2	4	3	4	3	3	5

3 (a)	(b)	*[c]	(d)	(e)	*[f]	(g)	4 (a)	(b)	(c)	(d)	(e)	(f)	*[g]
3	4	6	2	5	6	5	4_{+2}	2	4	5	3	4	6

total points

grade

Problem 1. Geometry of a Tetrahedron

The four corners of a tetrahedron, $\mathcal{E}$, form a subsect of the eight corners of a cube

$$\mathcal{E} = \left\{ P_0 = \frac{R}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} , \; P_1 = \frac{R}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} , \; P_2 = \frac{R}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} , \; P_3 = \frac{R}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

a) What is the length of the edges of the cube?
What is the distance of the corners of the tetrahedron from the center of the cube?
What is the length L of the edges of the tetrahedron?

b) How does the cosine of the angle between two edges of the tetrahedron

c) Adopt spherical coordinates where the origin is placed at the center of the cube, the reference axis $\theta = 0$ proceeds through the point P_0, and point P_2 resides at a position $P_2 = D\,\hat{\boldsymbol{r}}(\theta, 0)$. Determine $\cos\theta$.

⋆ d) Provide symmetry arguments to show that

$$P_{2\pm 1} = D\,\hat{\boldsymbol{r}}(\theta, \pm\phi)$$

Determine $\cos\phi$.

Problem 2. Conservative forces, potentials, and contour lines

We consider a particle with unit mass that resides at a dimensionless position $\big(x(t), y(t)\big)$, and moves under the influence of a dimensionless force

$$\boldsymbol{F} = \mathrm{e}^{x^2}\,\mathrm{e}^{-y^2} \begin{pmatrix} -a\,x \\ b\,y \end{pmatrix}$$

We will be interested in the work, W, performed by this force, when the particle is moved from the origin to a position $\boldsymbol{q}$.

a) Determine the work performed for motion along the path
$\gamma = \{\boldsymbol{q}(t) = (t\,x_E,\; t\,y_E) \mid 0 \le t \le 1\}$
to the fixed end point $(x_E,\; y_E)$.

b) Determine the work for a motion parallel to the coordinate axis,
$\gamma = \{\boldsymbol{q}(t) = (t\,x_E,\; 0) \mid 0 \le t \le 1\} \cup \{\boldsymbol{q}(t) = (x_E,\; t\,y_E) \mid 0 \le t \le 1\}$.

⋆ c) Is $\boldsymbol{F}$ a conservative force?
How would your reply depend on the choice of the parameters a and b?

⋆ d) Show that for $a = b$

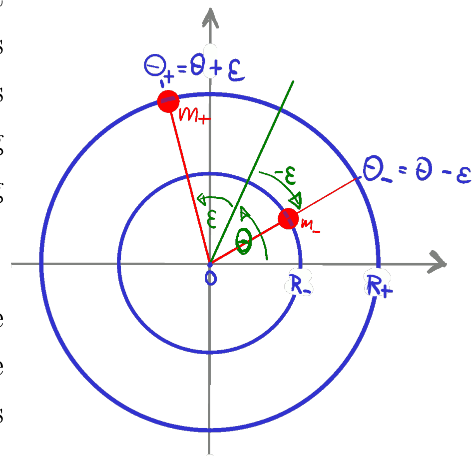
$$W(x, y) = c \, \exp(x^2 - y^2)$$

irrespective of the path taken from the origin to (x, y).
How is c related to a and b?

e) Determine the expression $y(x)$ for the contour lines and the gradient of the work $W(x, y)$ provided in d).
Sketch the contour lines and indicate the gradient by arrows.

Problem 3. Two coupled masses on concentrical circles

We consider two pearls of mass m_+ and m_- that move without friction on concentric circles of radius R_+ and R_-, respectively. We denote their positions as $\theta_\pm(t) = \theta(t) \pm \epsilon(t)$, and we consider a setting where they are connected by a harmonic spring with rest length ℓ and spring konstant k.



a) Specify the positions and the velocities of the two pearls in terms of polar coordinates where R is the distance from the center of the circles (cf. the sketch to the right).

b) Determine the kinetic energy of the pearls in terms of the generalized coordinates $\theta(t)$ and $\epsilon(t)$.

⋆ c) Show that the potential energy of the system can be expressed as

$$V = \frac{k\,L^2}{2} \left[\sqrt{1 + \rho \, \sin^2 \epsilon} - \rho \, \lambda \right]^2$$

where, $L = R_+ - R_-$.
How do the dimensionless parameters ρ and λ depend on $R_\pm$ and ℓ?
Which values do ρ and λ take in the limits $R_- \ll R_+$ and $L \ll R_+$?
Hint: Note that $\cos(2\epsilon) = 1 - 2 \sin^2 \epsilon$.

d) The coordinate θ is a cyclic variable. Determine the associated conservation law.

e) Verify that the EOM for $\epsilon(t)$ can be written in the form

$$\ddot{\epsilon} = \omega^2 \, \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left[\cos^2 \epsilon + 2\,\lambda \, \sqrt{1 + \rho \, \sin^2 \epsilon} \right]$$

and provide the dependence of ω on the system parameters.

$\star$ f) Determine the fixed points of the motion, and show that

for $\rho\,\lambda < 1$ there are stable fixed point at $\epsilon = \pi\,\mathbb{Z}$, and unstable fixed points right in the middle between the stable fixed points,

for $\rho\,\lambda > \sqrt{1+\rho}$ there are unstable fixed points at $\epsilon = \pi\,\mathbb{Z}$, and stable fixed points right in the middle between the stable fixed points,

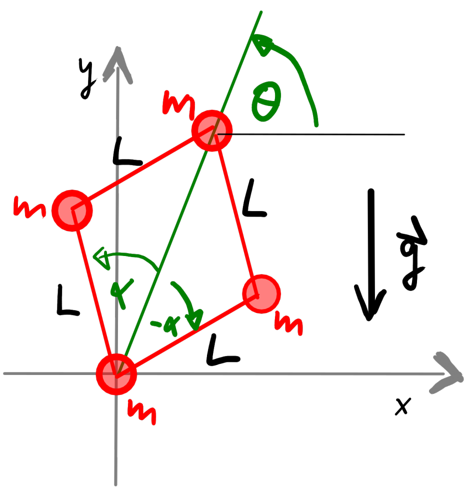
otherwise there are unstable fixed points at $\epsilon = (\pi/2)\,\mathbb{Z}$, and stable fixed points somewhere in the middle between each neighboring pair of stable fixed points.

Provide a *physical* interpretation for the positons of the stable and unstable fixed points in the different regimes!

g) Sketch the potentials and phase-space portraits for the three different parameter regimes identified in f).

Problem 4. Sliding diamonds

In geometry a quadrilateral where all sides have the same length, L, is referred to as a diamond. Its geometry and position in a plane can uniquly be characterized by selecting one corner, and specifying the angle 2α between its two neighboring sides, and the orientation θ of the diagonal proceeding through the corner (cf. the sketch to the right). We attach four identical masses m to the corners of such a structure, and explore how it moves in the sketched 2D plane. There is gravity acting vertically downwards, $\boldsymbol{g} = -g\,\hat{\boldsymbol{y}}$.



a) Let $\boldsymbol{q}_i$, $i \in \{1,2,3,4\}$ be the position of corner i of the diamond, and $\boldsymbol{Q}$ its center of mass. Express the positions $\boldsymbol{q}_i$ as

$$\boldsymbol{q}_i(t) = \boldsymbol{Q}(t) + L_i(\alpha(t))\,\hat{\boldsymbol{r}}(\theta_i(t))$$

Hint: Argue that the positions of opposite corners differ only by the sign of the respective L_i.

Bonus: Show that the diagonals always intersect vertically. Why does this imply that $\theta_i(t)$ does not depend on $\alpha(t)$?

b) Determine the potential energy of the diamond in the gravitational field.

c) Determine the velocities $\boldsymbol{q}_i$ and the kinetic energy of the diamond.

d) The Langrangian can be expressed in the form

$$\mathcal{L} = a\,\boldsymbol{Q}^2 + b\,\dot{\alpha}^2 + c\,\dot{\theta}^2 + d\,\hat{\boldsymbol{y}} \cdot \boldsymbol{Q}$$

Adopt dimensional analysis to determine how a, b, c, and d depend on the system parameters m, L, and g.

Compare your results to the values that result from the calculations preformed in b) and c).

e) Determine the EOM for $\boldsymbol{Q}$, $\theta(t)$ and $\alpha(t)$.

f) Let the diamond start at the origin with $\theta(t_0) = 0$ and $\alpha(t_0) = \pi/4$, and velocities $\dot{\boldsymbol{Q}}(t_0) = \boldsymbol{V}$, $\dot{\theta}(t_0) = \Omega$ and $\dot{\alpha}(t_0) = \omega$. Provide the positions of the corners at time t.

$\star$ g) Provide a physical interpretation of the motion.

What is fishy about the solution for $\alpha(t)$?

How does the motion change when the disks at the corners have a diameter $R < L/2$ and they undergo elastic hard-core collisions?

Sketch the evolution of α in phase space.

Challenge: How does the phase-space plot change when the particles undergo elastic soft-core collisions?