Lecture "Experimental Physics I"
(Prof. Dr. R. Seidel)

Lecture 3 + 4

Point mass: Velocity and acceleration in three dimensions

- Vectors and scalars
- Trajectory (path) of a point mass in 3D
- Superposition of motions along orthogonal directions
- Projectile motion
- Circular motion

Motivation-Experiment:

- City-Tunnel model in Leipzig: A train that goes first down into the tunnel and then up again is faster than a train that stays and moves down a small incline. One sees from this that one has to consider the full path to understand motion between two point!

NOTE: If you want to delve deeper, check: Brachistochrone problem (http://mathworld.wolfram.com/BrachistochroneProblem.html). This will be done in the lab course.

- Pulling handle of a trolley (Handwagen) in different directions shows that there is a best direction at which the trolley will follow.

1) Scalars and Vectors

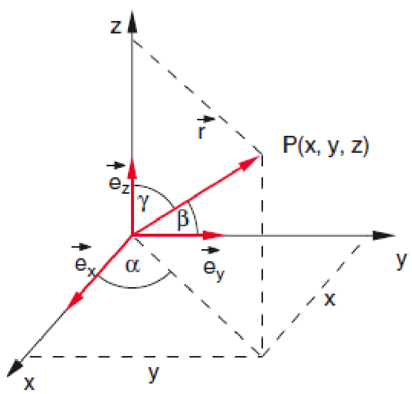
From the introductory experiments we see that a particular path and the direction of motion (or force) matters. To take directions into account we have to describe our kinematic quantities by **vectors**.

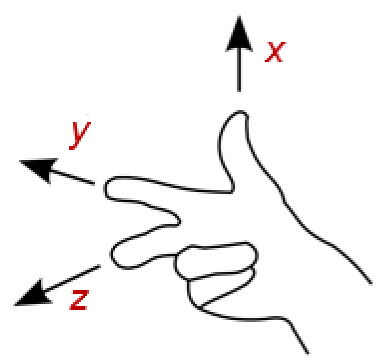
General definition (see also slide)**:**
A **scalar quantity** is specified by a **single value** with an appropriate unit and has **no direction**.
A **vector quantity** has both a **magnitude and direction**.

A) Representation of vectors

In 3D space each vector can be represented as a linear combination of 3 linearly independent base vectors.

Cartesian coordinate system
It is the most used coordinate system. It consists of **three mutually perpendicular axes, the** $\boldsymbol{x-, y-\ and\ z-axes}$. This ensures independence of these directions, i.e. movement along one coordinate axis does not change the position along the other axes. It forms a so-called **right-handed coordinate system** (when spanned by the thumb (x), the index (y) and the middle (z) finger of the right hand or circular permutations of it)





If a vector starts at zero then its endpoint is a point P with the coordinates (x,y,z), the **coordinates are the orthogonal projections** of the vector onto the coordinate axes and are called the **vector components**, which are written as:

$$\vec{r} = (x, y, z) \qquad or \qquad \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The vector components define the **absolute value** (length) of the vector (according to the Pythagoras' theorem):

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

The **unit vector** is a vector with a length of 1, i.e. it is unitless! It can be written as:

$$\hat{e} = \frac{\vec{r}}{|\vec{r}|} \quad \text{such that } |\hat{e}| = 1$$

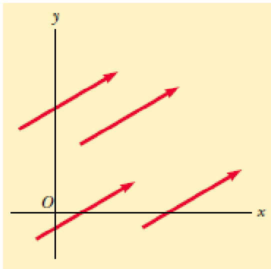
Special unit vectors point along the coordinate axes:

$$\hat{e}_x = (1,0,0), \qquad \hat{e}_y = (1,0,0), \hat{e}_z = (1,0,0)$$

such that:

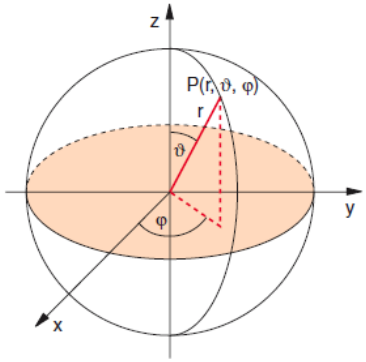
$$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z = (x, y, z)$$

Two vectors are defined to be equal if they have the same magnitude and point in the same direction.



Spherical coordinates

In spherical coordinates the vector between 0 and P is defined by its length (**radius**) $\boldsymbol{r} = |\boldsymbol{r}|$, the **polar angle** $\boldsymbol{\vartheta}$ and the **azimuthal angle** $\boldsymbol{\varphi}$. It is useful to describe problems with spherical symmetry, since all points on a sphere shall have the same coordinate *r*.



The transformation from spherical to Cartesian coordinates is given by (see slide):

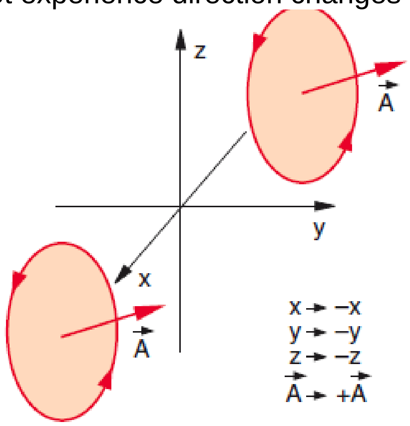
$$x = r \sin\vartheta \cos\varphi$$
$$y = r \sin\vartheta \sin\varphi$$
$$z = r \cos\vartheta$$

Polar & axial vectors

We generally distinguish:

Polar vectors: the vectors considered so far change their sign when the coordinate system is mirrored: $x \rightarrow -x$, $y \rightarrow -y$, $z \rightarrow -z$: $\vec{r} \rightarrow -\vec{r}$**.** They are therefore called polar vectors

Axial (pseudo) vectors: are vectors that define a rotation. These vectors define size and orientation of an area. The vector itself is normal (i.e. perpendicular) to the area. The length of the vector is equal to the area. The rotation direction is a right-handed screw. Examples are quantities like angular velocity, magnetic field, torque and angular momentum. Axial vectors do not experience direction changes upon the mirror operation.



B) Mathematical operations with vectors

Adding vectors

Vectors can be added geometrically by moving the tail of one vector to the tip of the other. Doing this for both possibilities, a parallelogram is produced where one of the diagonals stretches from 0 to the end of the vector sum:

This addition is equal to the addition of the corresponding components, since the x, y and z components of the two vectors add up independently (as can be seen from the geometric figure). In vector representation it can be written as:

$$\vec{c} = \vec{a} + \vec{b}, \qquad \vec{c} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} + \begin{pmatrix} b_x \\ b_z \\ b_y \end{pmatrix} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{pmatrix}$$

This rule for vector addition can also be shown using the vector representation of the linear combination of unit vectors (see slides):

$$\vec{c} = \left(a_x\hat{e}_x + a_y\hat{e}_y + a_z\hat{e}_z\right) + \left(b_x\hat{e}_x + b_y\hat{e}_y + b_z\hat{e}_z\right)$$
$$\vec{c} = (a_x + b_x)\hat{e}_x + (a_y + b_y)\hat{e}_y + (a_z + b_z)\hat{e}_z$$

Furthermore, we have the following mathematical rules that apply for vector addition:

Commutative law: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Associative law: $\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$

Vector substraction: $\vec{s} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

The latter is often used to substract vectors by adding the negative of the second vector to the first.

Multiplying a vector with a scalar

For the component-wise addition we used already the multiplication of a vector with a scalar, for which we can write:

$$c\vec{a} = c\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} ca_x \\ ca_y \\ ca_z \end{pmatrix}$$

i.e. each component is multiplied the same way with the scalar. For the absolute value holds then:

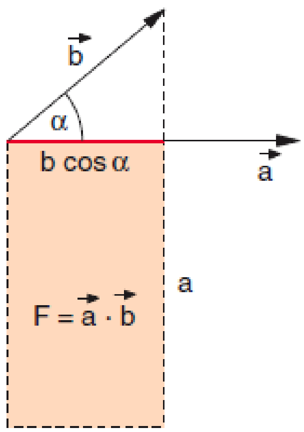
$$|c\vec{a}| = |c||\vec{a}|$$

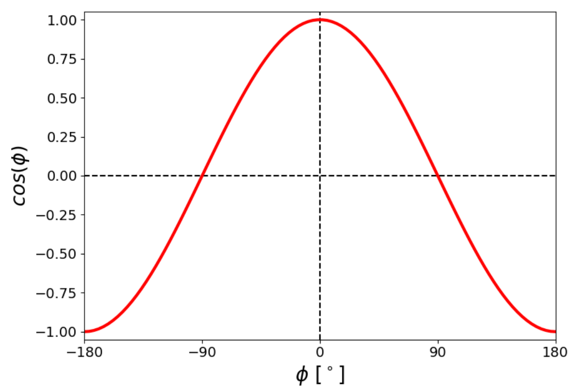
The scalar (dot) product

The scalar product is the scalar:

$$c = \vec{a} \cdot \vec{b} = |\vec{a}| \underbrace{|\vec{b}| \cos\alpha}_{\text{projection onto } \vec{a}}$$

Generally, c is the absolute value of the projection of $\vec{b}$ onto $\vec{a}$ multiplied by $|\vec{a}|$ (see eqn. above). This can be represented graphically as the area *F* in the figure below:





Using or knowledge about the cosine function (**see figure**), we see that the scalar product becomes ab if both vectors point in the same direction and 0 if the vectors are perpendicular to each other:

$$\vec{a} \cdot \vec{b} = 0 \quad \text{if} \quad \vec{a} \perp \vec{b}$$

$$\vec{a} \cdot \vec{b} = ab \quad \text{if} \quad \vec{a} \,||\, \vec{b}$$

This finding provides for the unit vectors: $\hat{e}_i \cdot \hat{e}_k = \delta_{ik}$ with $i, k = x, y, z$, where δ_{ik} is the **Kronecker-delta** symbol for which holds:

$$\delta_{ik} = \begin{cases} 1, & i = k, \qquad \text{i.e. } \hat{e}_x \cdot \hat{e}_x;\ \hat{e}_y \cdot \hat{e}_y;\ \hat{e}_z \cdot \hat{e}_z; \\ 0, & i \neq k, \qquad \text{i.e. all unit vectors for for different axes} \end{cases}$$

Using the scalar products of the unit vectors, we can express the scalar product by its components and obtain:

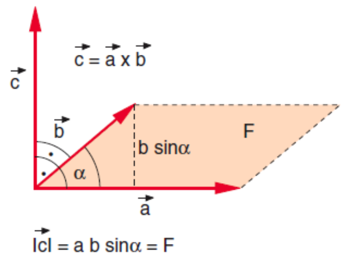
$$\vec{a} \cdot \vec{b} = (a_x\hat{e}_x + a_y\hat{e}_y + a_z\hat{e}_z) \cdot (b_x\hat{e}_x + b_y\hat{e}_y + b_z\hat{e}_z) = a_xb_x + a_yb_y + a_zb_z$$

To better understand the sense of calculating with Kronecker symbol and later in theory courses with the Levi-Cevita symbol check an instructive and fun-to-read blog (http://aryaprasetya.com/einstein-notation) by Arya Prasetya an Higher semester IPSP student about the so called Einstein notation.

The vector (cross) product

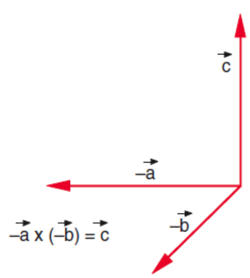
The vector product $\vec{c} = \vec{a} \times \vec{b}$ is a vector that:

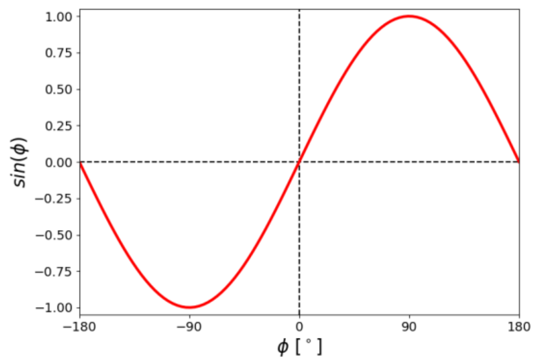
1) is perpendicular to $\vec{a}$ and $\vec{b}$**.**
2) has a length of $|\vec{c}| = |\vec{a}||\vec{b}| \sin\alpha$, where α is the angel between vector $\vec{a}$ and $\vec{b}$.
3) forms a right-handed screw, if $\vec{a}$ is turned on the shortest way to $\vec{b}$**.**



Consequences:

The length of $\vec{c}$ equals the **area of the parallelogram** that is spanned by $\vec{a}$ and $\vec{b}$**.** The vector product provides an **axial vector** (invariance against point mirror operation of coordinate system):





Using or knowledge about the sine function (**see figure**), we see that the vector product becomes 0 if both vectors point in the same direction and has an absolute value of ab if the vectors are perpendicular to each other:

$$\vec{a} \times \vec{b} = 0 \quad \text{if} \quad \vec{a} \,||\, \vec{b}$$

$$|\vec{a} \times \vec{b}| = ab \quad \text{if} \quad \vec{a} \perp \vec{b}$$

For the **unit vectors** we have thus:

$$\hat{e}_x \times \hat{e}_y = -(\hat{e}_y \times \hat{e}_x) = \hat{e}_z$$
$$\hat{e}_y \times \hat{e}_z = -\left(\hat{e}_z \times \hat{e}_y\right) = \hat{e}_x$$
$$\hat{e}_z \times \hat{e}_x = -(\hat{e}_x \times \hat{e}_z) = \hat{e}_y$$
$$\hat{e}_i \times \hat{e}_i = 0 \quad \text{with} \quad i = x.y.z$$

Note that we have to keep a circular permutation of the indices in these equations! When exchanging the order of the two multiplied factors, the vector product changes sign!

We have for the vector product the following rules (see slides):

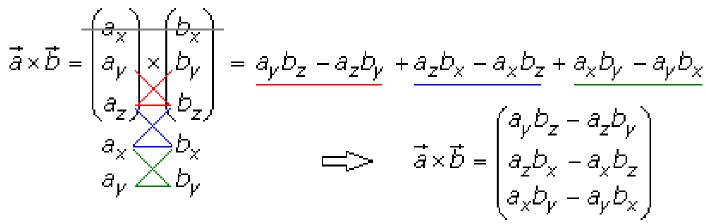
Anti-commutative: $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

Distributive over addition: $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Using the vector product for the unit vectors we can derive the **vector product by components** (see slides):

$$\vec{a} \times \vec{b} = \left(a_x\hat{e}_x + a_y\hat{e}_y + a_z\hat{e}_z\right) \times \left(b_x\hat{e}_x + b_y\hat{e}_y + b_z\hat{e}_z\right)$$
$$= (a_y b_z - a_z b_y)\hat{e}_x + (a_z b_x - a_x b_z)\hat{e}_y + (a_x b_y - a_y b_x)\hat{e}_z$$

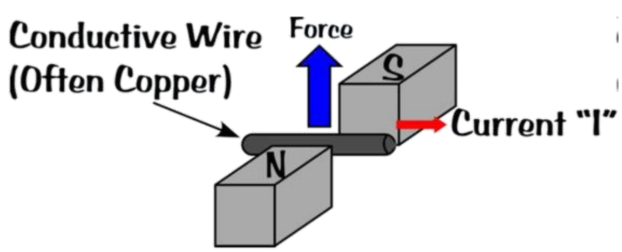
A simple scheme for memorizing the vector product is illustrated in the following:



One extends each vector at its bottom by the x and y components. A given component of the vector product is then provided by the “over-cross” multiplication of the following two lines of the components in the extended vectors (see red, blue and green lines for x, y and z components of the vector product, respectively)
One can also us the so-called **determinant method** by forming a square matrix of the input vectors with the unit vectors und calculate the determinant of the matrix (look up in the internet).

An example for the application of the vector product is the formula for the force on a current-carrying wire in a magnetic field *B*:

$$\vec{F} = I \cdot \underset{\substack{\downarrow \\ \text{wire length}}}{\vec{L}} \times \vec{B}$$



2) Trajectories in three dimensions

A) The trajectory (path) of motion in 3D

So far, we had for motion in 1D a scalar position as function of time *x* = *x*(t). For **motion in 3D our position becomes a vectorial position:**

$$\vec{r} = \vec{r}(t)$$

In Cartesian coordinates with the orthogonal unit vectors $\hat{e}_x, \hat{e}_y, \hat{e}_z$, we can express $\vec{r}(t)$ by its components x, y, z:

$$\vec{r}(t) = x(t)\hat{e}_x + y(t)\hat{e}_y + z(t)\hat{e}_z = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

$\vec{r}(t)$ forms a path in space, which is also called a **trajectory.** Any movement of a mass point is called **translation**, since it has no orientation.

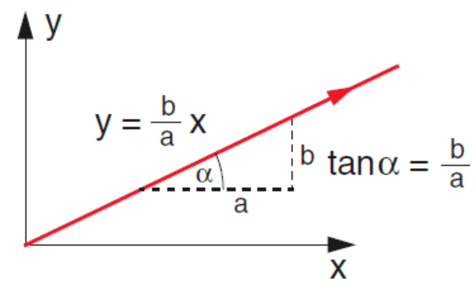
An **example of a linear path** (i.e. on a straight line) in 3D is given by:

$$x = at, \qquad y = bt, \qquad z = 0$$

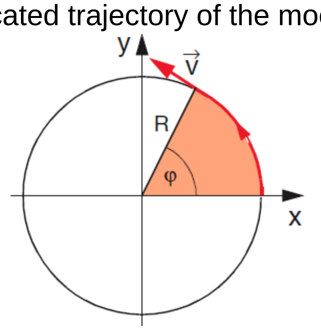
Replacement of *t* by one of the equations provides a linear function for the trajectory:

$$y = \left(\frac{b}{a}\right) x$$

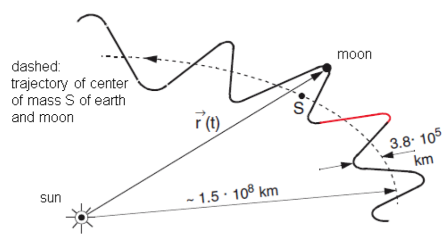
Since $z = 0 = const.$, this motion occurs in the x,y plane



Another simple example is a circular path with constant distance R from a fixed point or the rather complicated trajectory of the moon orbiting around the earth, which itself orbits around the sun:



circular path



trajectory of the moon

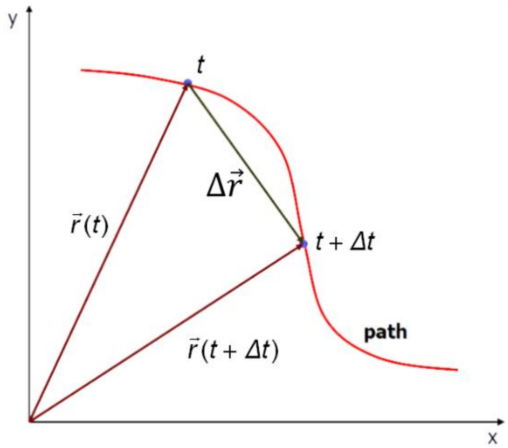
B) Velocity and acceleration for motion in 3D

For the trajectory $\vec{r}(t)$ we define displacement, velocity and acceleration as vectorial quantities in a similar fashion as we did in 1D:

The **displacement** is defined as:

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t) = \begin{pmatrix} x(t+\Delta t) - x(t) \\ y(t+\Delta t) - y(t) \\ z(t+\Delta t) - z(t) \end{pmatrix} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Hint: The vector difference $\vec{r}_2 - \vec{r}_1$ can be written as $\vec{r}_2 + (-\vec{r}_1)$. It is thus the sum of of the inverted vecor $-\vec{r}_1$ (with swapped direction) and $\vec{r}_2$. It thus points from the tip of $\vec{r}_1$ to the tip of $\vec{r}_2$:



We continue to define velocity and acceleration. Here the calculations of difference and of the differentiation are done individually for each component, since the whole vector is a sum of components of unit vectors. We thus get for **average and instantaneous velocity**:

Average quantity

$$\langle\vec{v}\rangle = \frac{\Delta\vec{r}}{\Delta t} = \begin{pmatrix}\Delta x/\Delta t\\ \Delta y/\Delta t\\ \Delta z/\Delta t\end{pmatrix} = \begin{pmatrix}\langle v_x\rangle\\ \langle v_y\rangle\\ \langle v_z\rangle\end{pmatrix}$$

Instantaneous quantity

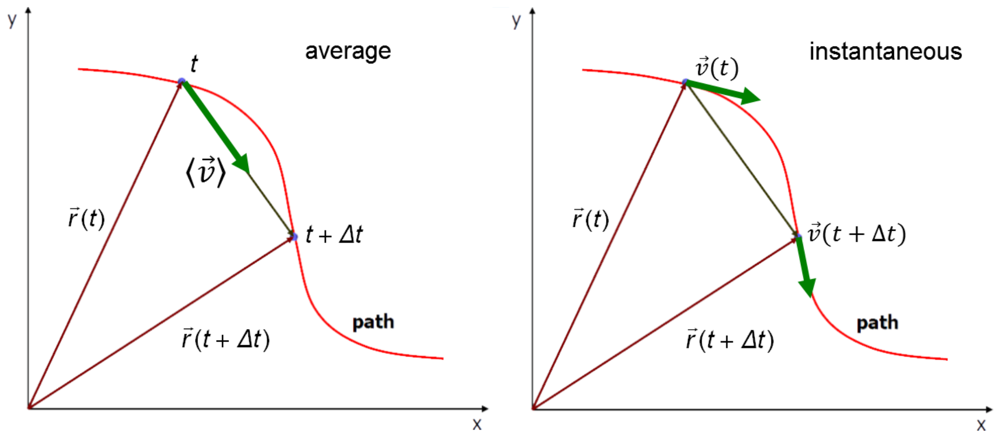
$$\vec{v} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \begin{pmatrix}dx/dt\\ dy/dt\\ dz/dt\end{pmatrix} = \begin{pmatrix}v_x\\ v_y\\ v_z\end{pmatrix} = \dot{\vec{r}}$$

For simplification we introduced the following conventions for time and position derivatives:

$\frac{df}{dt} = \dot{f}$ (with respect to time)

$\frac{df}{dx} = f'$ (with respect to position)

Average and instantaneous velocity have the following directions in our graphical representation of the path:



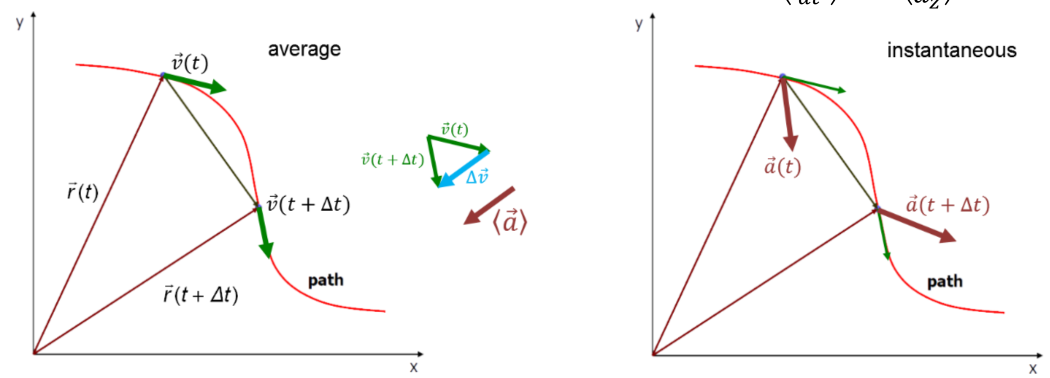
The average velocity points along the macroscopic displacement. The **instantaneous velocity points along the tangent on the path**.

The absolute value of the instantaneous velocity is then given by its components as (slide):

$$v = |\vec{v}| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

We similarly define **average and instantaneous acceleration**:

$$\langle \vec{a} \rangle = \frac{\Delta \vec{v}}{\Delta t} = \begin{pmatrix} \Delta v_x/\Delta t \\ \Delta v_y/\Delta t \\ \Delta v_z/\Delta t \end{pmatrix} = \begin{pmatrix} \langle a_x \rangle \\ \langle a_y \rangle \\ \langle a_z \rangle \end{pmatrix} \qquad \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \begin{pmatrix} \frac{dv_x}{dt} \\ \frac{dv_y}{dt} \\ \frac{dv_z}{dt} \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = \dot{\vec{v}}$$



In the graphical vector representation, acceleration is hereby defined by the triangle of initial minus the final velocity vector. As can be seen (from the graphical representation of the average acceleration), an **acceleration in 3D can occur upon**:

1) **a change of the magnitude of the velocity as well as**
2) **a change of the direction of the velocity!!!**

C) Kinematic equations in 3D

So far, we went from position to velocity and acceleration by differentiation. To go the way back we can derive kinematic equations that are similarly to 1D obtained by integration. Again, this is done by components (each component is part of a sum) since integration reflects a simple vector addition (shown only for one component, rest on slides):

$$\vec{v} = \vec{v}_0 + \int_{\vec{v}_0}^{\vec{v}} d\vec{v} = \vec{v}_0 + \int_0^t \vec{a}\, dt = \begin{pmatrix} v_{0,x} + \int_0^t a_x\, dt \\ v_{0,y} + \int_0^t a_y\, dt \\ v_{0,z} + \int_0^t a_z\, dt \end{pmatrix} = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix}$$

$$\vec{r} = \vec{r}_0 + \int_{\vec{r}_0}^{\vec{r}} d\vec{r} = \vec{r}_0 + \int_0^t \vec{v}\, dt = \begin{pmatrix} x_0 + \int_0^t v_x\, dt \\ y_0 + \int_0^t v_y\, dt \\ z_0 + \int_0^t v_z\, dt \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

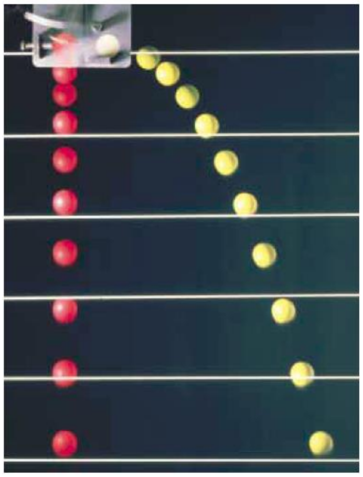
Each component (direction) is integrated separately (independently), i.e. the **trajectory is a superposition of the individual (and independent) motions in each direction**

Experiments: Does the superposition principle occur in reality? The consequence of it would be that every falling object with the same initial velocity along z reaches the ground equally fast independent of an initial velocity in the lateral direction.

We can confirm this superposition principle by looking at:

- Two falling spheres, one with lateral and one with no lateral velocity
- A toy “monkey hunt” where a dart arrow that initially flies in the direction of a falling “monkey” still catches it during the fall

Thus, objects fall equally fast independent of a lateral velocity component as illustrated in the following figure with stroboscopic snapshots of two falling spheres:



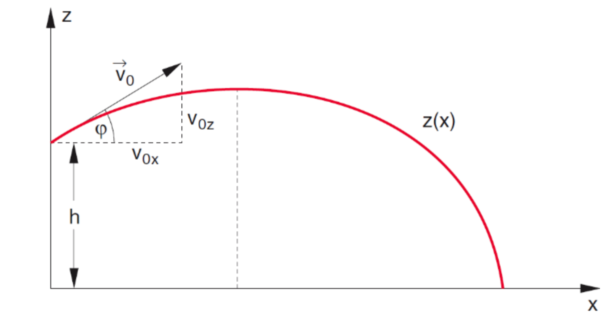
We therefore can use the kinematic equations that we derived for 1D trajectories to describe the **motion of a particle in 3D in case of constant acceleration** ($\vec{a} = const$):

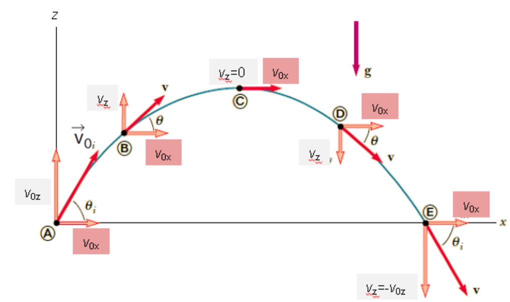
$$\vec{v}(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} = \begin{pmatrix} v_{0,x} + a_x t \\ v_{0,y} + a_y t \\ v_{0,z} + a_z t \end{pmatrix} = \vec{v}_0 + \vec{a}t$$

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 + v_{0,x}t + \frac{1}{2}a_x t^2 \\ y_0 + v_{0,y}t + \frac{1}{2}a_y t^2 \\ z_0 + v_{0,z}t + \frac{1}{2}a_x t^2 \end{pmatrix} = \vec{r}_0 + \vec{v}_0 t + \frac{\vec{a}}{2}t^2$$

3) Projectile motion

Let us better understand the concept of the superposition of the motions along the different independent directions of the coordinate system. For this we look at the motion of an projectile under the action of gravity. The projectile is shot with velocity v_0 under angle φ with respect to the horizontal line upwards. We have to consider that the free fall acceleration acts only along the vertical direction z. The projectile undergoes therefore a uniform translation (with const. velocity) along the lateral direction x and a free fall along the vertical direction. Thus, for the lateral velocity $v_{0x} = const.$, while v_z is subjected to $a = -g$.





The starting conditions are:

$$\vec{r}_0 = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}, \vec{v}_0 = \begin{pmatrix} v_{0x} \\ 0 \\ v_{0z} \end{pmatrix} = \begin{pmatrix} v_0 \cos\varphi \\ 0 \\ v_0 \sin\varphi \end{pmatrix}, \vec{a} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

With this and the above solutions for the kinematic equations in case of constant acceleration we get:

$$x(t) = v_{0,x}t$$
$$y(t) = 0$$
$$z(t) = h + v_{0,z}t - \frac{1}{2}gt^2$$

$v_{0,x} = 0$ is a vertical shooting, while $v_{0,z} = 0$ corresponds to a horizontal shooting. By **eliminating** $t = x/v_{0,x}$, which we get from the first equation, we get the projectile path in the $z - x$ plane:

$$z(x) = -\frac{1}{2}\frac{g}{v_{0,x}^2}x^2 + \frac{v_{0,z}}{v_{0,x}}x + h$$

This is a parabola (as for the time dependence of the free fall *x*(t)). Its maximum height h_{max} is reached when

$$0 = \frac{dz}{dx} = -\frac{g}{{v_{0,x}}^2}x_{max} + \frac{v_{0,z}}{v_{0,x}}$$

This provides for the position of the maximum x_{max} in the lateral direction:

$$x_{max} = \frac{v_{0,z}v_{0,x}}{g} = \frac{{v_0}^2 \sin\varphi\cos\varphi}{g} = \frac{{v_0}^2 \sin 2\varphi}{2g}$$

for the second transformation we inserted the initial velocity components $v_{0,z}$ and $v_{0,x}$ (see above). For the last transformation we used the double angle identity (see slide):

$$\sin 2\varphi = 2\sin\varphi\ \cos\varphi$$

Inserting x_{max} into $z(x)$ provides the maximum height:

$$h_{max} = z(x_{max}) = h - \frac{1}{2}\frac{g}{{v_{0,x}}^2}\frac{{v_{0,z}}^2{v_{0,x}}^2}{g^2} + \frac{{v_{0,z}}^2 v_{0,x}}{g v_{0,x}}$$

$$h_{max} = h + \frac{1}{2}\frac{{v_{0,z}}^2}{g}$$

which only depends on the *z*-component of the initial velocity in agreement with the superposition principle for the two motion directions.

The distance x_{ground} at which the projectile hits the ground can be calculated by solving:

$$0 = h - \frac{1}{2}\frac{g}{{v_{0,x}}^2}x_{ground}^2 + \frac{v_{0,z}}{v_{0,x}}x_{ground}$$

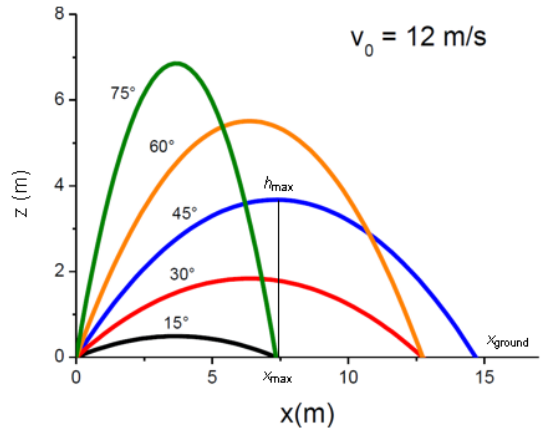
If we start shooting from the ground (i.e. $h = 0$) then we can transform this equation to:

$$x_{ground} = \frac{2v_{0,z}v_{0,x}}{g} = \frac{{v_0}^2 \sin 2\varphi}{g}$$

Here we used again the trick with the double angle identity. x_{ground} is twice x_{max} due to the symmetry of the parabolic curve around x_{max}. x_{ground} gets maximal for φ = 45°, since then $\sin 2\varphi = 1$. The maximal distance then becomes

$$x_{ground}^{max} = x_{ground}(\varphi = 45°) = \frac{{v_0}^2}{g}$$

The maximum height is reached for vertical shooting since in this case we have the maximum initial velocity in the vertical direction $v_{0,z} = v_0$. As shallower the start angle is as smaller are the initial heights. We thus get the following set of parabolas for the different start angles:



Notably, for a given starting angle φ we get the same maximum distance for an initial angle of 90°-φ (e.g. 15° and 75°). This can be explained by the symmetry of the $\sin 2\varphi$ function with respect to 45°.

Experiment: Using a water stream out of a tube one can visualize the parabola shape of the "projectile" trajectory. One can also demonstration the influence of different magnitudes and directions of the initial velocity. Particularly, one can see that the maximum distance is bridged when $\varphi = 45°$.

4) Circular motion

A) Uniform Circular motion

Now let us have a look at uniform circular motion, where a point mass is moving along a circular path with radius R with constant velocity v. This is an interesting trajectory since only the direction but not the absolute value of the velocity vector $\vec{v} = v \cdot \hat{e}_t$ changes, which points always in direction of the tangent vector. Let's first define an **angular velocity** starting from the linear velocity. We use that the travelled distance along the arc Δs is related to a change in the angular position $\Delta\varphi$ (**when given in radians**) by:

$$\Delta s = R\,\Delta\varphi$$

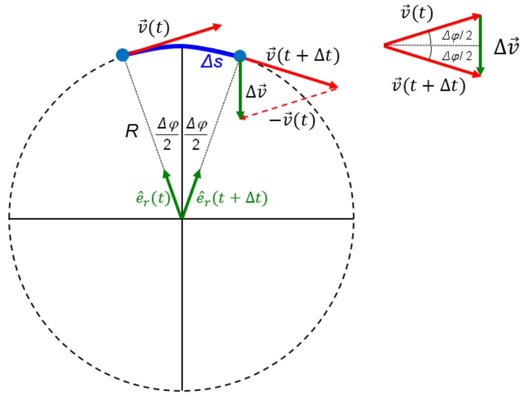
With this we can write for the magnitude of the velocity:

$$v = \frac{\Delta s}{\Delta t} = R\frac{\Delta\varphi}{\Delta t} = R\omega$$

where the angular displacement per time provides the **(average) angular velocity $\boldsymbol{\omega}$**. In the limit of small Δt we get the **instantaneous angular velocity**:

$$\omega = \frac{d\varphi}{dt} = \frac{v}{R}, \qquad [\omega] = rad/s$$

Let us look at the acceleration during the circular motion. For this, we distribute the angular displacement from t to $t + \Delta t$ symmetrically around the $y - axis$, such that the two radial vectors form an angle $\Delta\varphi$ with the $y - axis$:



Geometric subtraction of the velocity vectors provides now a vector that points downwards, i.e. into the direction of the circle center. For the acceleration we can write (see also sketch on the top right):

$$|\vec{a}| = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{2v\,sin(\Delta\varphi/2)}{\Delta t} \approx \frac{2v\Delta\varphi/2}{\Delta t} = v\omega$$

with $\sin(\Delta\varphi/2) \approx \Delta\varphi/2$ in the limits of small angular displacements. According to the drawing $\vec{a}$ points always to the center of the circle and thus perpendicular to the velocity. It is therefore called **centripetal (center-seeking) acceleration**. Replacing either velocity or angular velocity using $v = R\omega$, we can therefore write:

$$\vec{a} = \frac{v^2}{R}(-\hat{e}_r) = \omega^2 R(-\hat{e}_r)$$

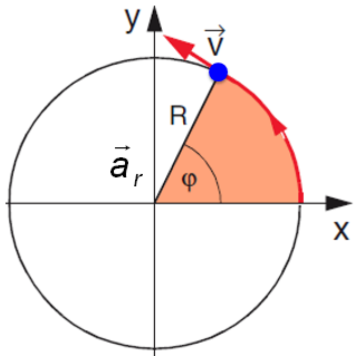
where $\hat{e}_r$ is the rotating radial unit vector that points from the circle center to the rotating object.

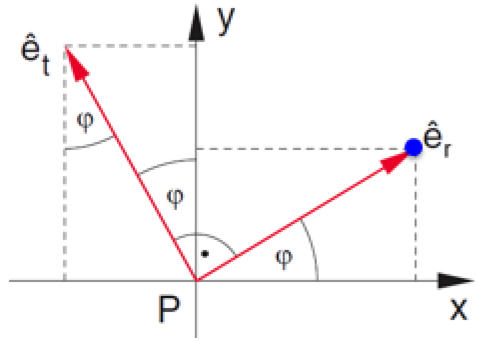
Experiment: We measure the centripetal acceleration using an acceleration sensor on a lever that rotates around a fixed axis. We record in real time the angular velocity and the acceleration along the radial vector. A plot of the centripetal acceleration a over ω^2 provides a linear relationship. The slope in this plot provides the rotation radius R.

Experiment: We look at sparks escaping from a grinding wheel. The escape occurs in tangential direction and follows initially on a straight path, since there is no acceleration anymore that forces the spark particles on a circular track.

B) Generalized circular motion

In the following we want to derive the acceleration during circular motion in a more elegant and complete form using vector notation.





We can represent our position vector $\vec{r}$ of the circulating object using cosine and sine of the respective angle. With this we can also express the radial unit vector:

$$\vec{r} = \begin{pmatrix} R\cos\varphi\,(t) \\ R\sin\varphi\,(t) \end{pmatrix} = R\begin{pmatrix} \cos\varphi\,(t) \\ \sin\varphi\,(t) \end{pmatrix} = R\hat{e}_r$$

where we would have for circular motion with constant angular velocity $\varphi(t) = \omega t$.

Experiment: The upper relation can also be visualized in a simulation where the radius (and later on the velocity and the acceleration) vectors are shown for circular motion: https://www.leifiphysik.de/mechanik/kreisbewegung/versuche#Kreisbewegung%20mit%20konstanter%20Winkelgeschwindigkeit%20(Simulation)

We can check that our position vector describes a circular trajectory by verifying its constant length:

$$x^2 + y^2 = R^2[\cos^2\varphi\,(t) + \sin^2\varphi\,(t)] = R^2$$

For the velocity we know that it acts along the tangent vector that is perpendicular to the radial unit vector, i.e.

$$\vec{v} = v\,\hat{e}_t$$

From the definition of the velocity we can derive the velocity explicitly:

$$\vec{v} = \frac{d\vec{r}}{dt} = R\frac{d}{dt}\begin{pmatrix} \cos\varphi\,(t) \\ \sin\varphi\,(t) \end{pmatrix} = R\underbrace{\frac{d\varphi}{dt}}_{\omega}\underbrace{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}}_{\hat{e}_t} = R\frac{d\varphi}{dt}\hat{e}_t$$

Here we had to apply the chain rule for the differentiation, since $\varphi = \varphi(t)$ is time-dependent. One can show with the scalar product that $\vec{r}$ and $\vec{v}$ are indeed perpendicular to each other, such that $\hat{e}_t \perp \hat{e}_r$ as demanded (see Figure above).

From the velocity we can then calculate the acceleration by differentiation is *R* is constant. Since φ and $\hat{e}_t$ are time-dependent, we have to apply the product rule in this case:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(R\frac{d\varphi}{dt}\hat{e}_t\right) = R\frac{d^2\varphi}{dt^2}\hat{e}_t + R\frac{d\varphi}{dt}\frac{d\hat{e}_t}{dt} = R\frac{d^2\varphi}{dt^2}\hat{e}_t + R\frac{d\varphi}{dt}\frac{d\varphi}{dt}\begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix}$$

$$= R\frac{d^2\varphi}{dt^2}\hat{e}_t + R\left(\frac{d\varphi}{dt}\right)^2(-\hat{e}_r)$$

With $\omega = d\varphi/dt$, we get the following form for the total acceleration of the object:

$$\vec{a} = \underbrace{R\frac{d\omega}{dt}\hat{e}_t}_{\substack{\text{tangential}\\ \text{acceleration}}} + \underbrace{R\omega^2(-\hat{e}_r)}_{\substack{\text{centripetal}\\ \text{acceleration}}}$$

The first part of the sum is **the tangential acceleration that causes the change in the angular velocity of the particle**. The second part of the sum is the previously derived **centripetal (also**

radial) acceleration that causes exclusively a change in the direction of the mass point (see upper eqn.). We can thus write for the two different components of the acceleration:

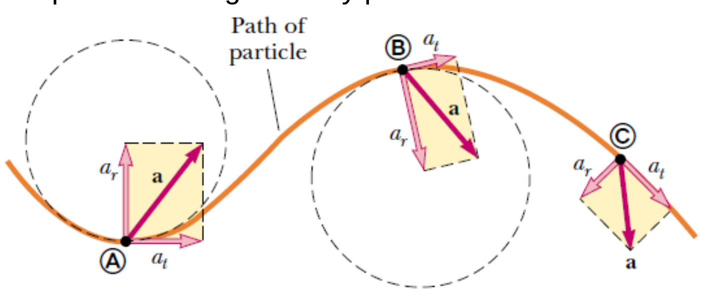
$$\vec{a} = \vec{a}_t + \vec{a}_r$$

Using $v = R\,\omega$ we can rewrite the acceleration formula from above:

$$\vec{a} = \frac{dv}{dt}\hat{e}_t + \frac{v^2}{R}(-\hat{e}_r)$$

The centripetal acceleration is thus also proportional to the square of the velocity and inversely proportional to the radius of the path's curvature. If the absolute value of the velocity does not change we have a uniform circular motion and only the centripetal component.

The splitting of the local acceleration into a tangential and a radial (centripetal) component is used to describe the motion of particles along arbitrary paths



One defines local curvatures with radii $R(s)$ to the path and can calculate locally the corresponding vectors $a_t(s)$ and $a_r(s)$. Since both vectors are perpendicular it follows for the total acceleration:

$$|\vec{a}| = \sqrt{|\vec{a}_t|^2 + |\vec{a}_r|^2}$$

5) Angular velocity in 3D (Not done in lecture)

ω becomes vectorial quantity in order to specify plane in which the circular motion occurs

ω becomes a normal vector of the plane (i.e. perpendicular to the plane)

One defines:

$$\frac{d\vec{r}}{dt} = \vec{v} = \vec{w} \times \vec{R}$$

Which can be transformed to:

$$\vec{\omega} = \frac{1}{R^2}\vec{R} \times \vec{v}$$

$\boldsymbol{R}, \boldsymbol{v}$ and $\boldsymbol{\omega}$ are thus perpendicular to each other. In 2D $\boldsymbol{\omega}$ would be the vector that comes out perpendicularly of the plane. The vector product defines a right-handed screw, such that $\boldsymbol{\omega}$ even defines the direction of the rotation! (Example on slides)

One uses the right hand with the thumb pointing along $\boldsymbol{\omega}$ to reveal the rotation direction

Lecture 3: Experiments

1. City tunnel Leipzig: Train on curved track through the tunnel is faster than on straight incline track.

2. Pulling handle of a trolley (Handwagen) in different directions shows that there is a best direction at which the trolley will follow.

3. Free fall - Superposition of velocities and forces:
- "falling monkey": Falling arrow hits falling disk.
- "Schnipser": same falling time for free fall and projectile motion

4. Using a water stream out of a tube one can visualize the parabola shape of the "projectile" trajectory. One can also demonstration the influence of different magnitudes and directions of the initial velocity. Particularly, one can see that the maximum distance is bridged when $\varphi = 45°$.
5. Centrifugal acceleration: Wii remote at a rotating lever, $F \propto \omega^2$, $F \propto R$.

6. Grinding wheel: sparks from grinding escape tangentially, since there is no acceleration that forces the grinded particles on a circular track.

7. Simulation where the radius, the velocity and the acceleration vectors are shown for circular motion

https://www.leifiphysik.de/mechanik/kreisbewegung/versuche#Kreisbewegung%20mit%20konstanter%20Winkelgeschwindigkeit%20(Simulation)