MA1 – HW 10

1)

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 5 \\ 3x_2 + x_3 - 2x_4 &= 1 \\ -x_3 + 2x_4 &= -1 \\ 4x_4 &= 4 \end{aligned} \iff \begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 5 \\ x_2 + \frac{x_3}{3} - \frac{2}{3}x_4 &= \frac{1}{3} \\ x_3 - 2x_4 &= 1 \\ x_4 &= 1 \end{aligned} \iff$$

$$\begin{aligned} x_1 &= 5 - 2x_2 - 2x_3 - x_4 \\ x_2 &= \tfrac{1}{3}(1 - x_3 + 2x_4) \\ x_3 &= 1 + 2x_4 = 3 \\ x_4 &= 1 \end{aligned} \iff \begin{aligned} x_1 &= 5 - 2\cdot 0 - 2\cdot 3 - 1 = -2 \\ x_2 &= \tfrac{1}{3}(1 - 3 + 2) = 0 \\ x_3 &= 3 \\ x_4 &= 1 \end{aligned}$$

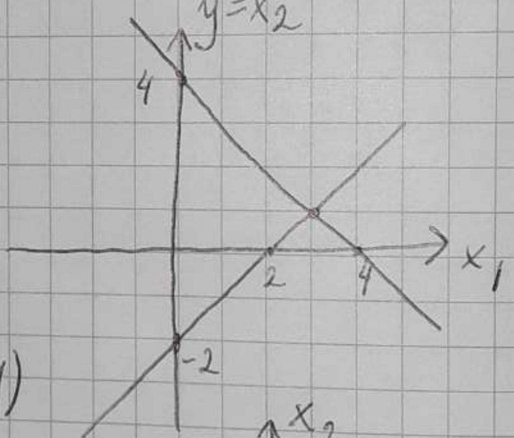
$\begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \end{pmatrix}$ is a solution.

2)

$$4x_1 - 3x_2 + x_3 + 2x_4 = 4$$
$$3x_1 + x_2 - 5x_3 + 6x_4 = 5$$

3)

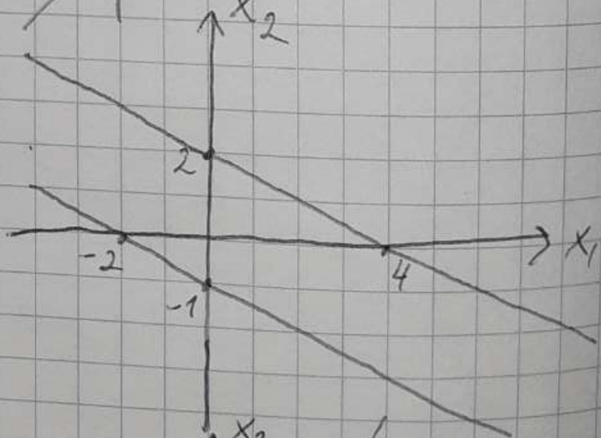
a) $\begin{cases} x_1 + x_2 = 4 \\ x_1 - x_2 = 2 \end{cases}$



One solution

$(x_1, x_2) = (3, 1)$

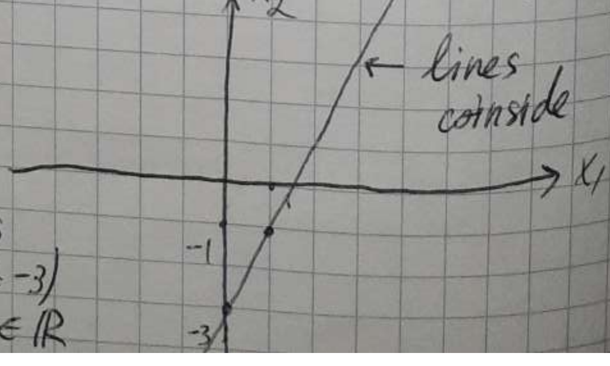
b) $\begin{cases} x_1 + 2x_2 = 4 \\ -2x_1 - 4x_2 = 4 \end{cases}$



No solutions

$(x_1, x_2) \in \emptyset$

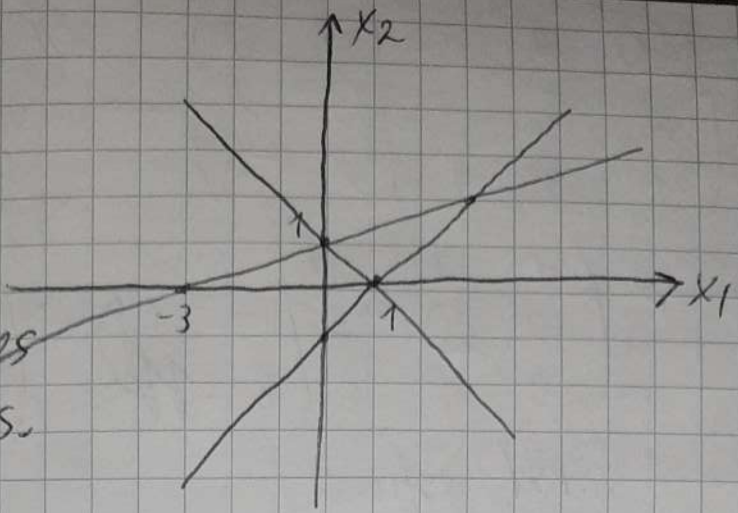
c) $\begin{cases} 2x_1 - x_2 = 3 \\ -4x_1 + 2x_2 = -6 \end{cases}$



Infinite set of solutions

$(x_1, x_2) = (t, 2t - 3) \quad \forall t \in \mathbb{R}$

d) $\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 1 \\ -x_1 + 3x_2 = 3 \end{cases}$



No intersection off all 3 lines — no solutions.

$(x_1, x_2, x_3) \in \emptyset$

4

$\begin{cases} 3x_1 + 2x_2 + x_3 = 0 \\ -2x_1 + x_2 - x_3 = 2 \\ 2x_1 - x_2 + 2x_3 = -1 \end{cases}$

Constr. augm. matrix:

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ -2 & 1 & -1 & 2 \\ 2 & -1 & 2 & -1 \end{array}\right) \xrightarrow{\cdot\frac{1}{3}\ 1st\ r.} \begin{array}{cccc} 2 & \frac{4}{3} & \frac{2}{3} & 0 \end{array} \left(\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ -2 & 1 & -1 & 2 \\ 2 & -1 & 2 & -1 \end{array}\right)$$

$$\longrightarrow \begin{array}{cccc} -2 & -\frac{4}{3} & -\frac{2}{3} & \end{array} \left(\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{7}{3} & -\frac{1}{3} & 2 \\ 2 & -1 & 2 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 7/3 & -1/3 & 2 \\ 0 & -7/3 & 4/3 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 7/3 & -1/3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{\cdot 2nd\ by\ \frac{3}{7}} \left(\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{7} & 6/7 \\ 0 & 0 & 1 & 1 \end{array}\right) \Longleftrightarrow \begin{cases} x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 = 0 \\ x_2 - \frac{1}{7}x_3 = \frac{6}{7} \\ x_3 = 1 \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} x_1 = -\frac{1}{3}\cdot 1 - \frac{2}{3}\cdot 1 = -1 \\ x_2 = \frac{6}{7} + \frac{1}{7} = 1 \\ x_3 = 1 \end{cases}$$

$(-1, 1, 1)$ is the solution.

5

$\begin{cases} x_1 + 2x_2 - 2x_3 = 1 \\ 2x_1 + 5x_2 + x_3 = 9 \\ x_1 + 3x_2 + 4x_3 = 9 \end{cases}$ $\quad$ $\begin{cases} x_1 + 2x_2 - 2x_3 = 9 \\ 2x_1 + 5x_2 + x_3 = 9 \\ x_1 + 3x_2 + 4x_3 = -2 \end{cases}$

Aug. matrix:

$$\begin{array}{c} \begin{matrix} -2 & -4 & 4 & -2 & -18 \end{matrix} \\ \left(\begin{array}{ccc|cc} 1 & 2 & -2 & 1 & 9 \\ 2 & 5 & 1 & 9 & 9 \\ 1 & 3 & 4 & 9 & -2 \end{array}\right) \end{array} \to \begin{array}{c} \begin{matrix} -1 & -2 & 2 & -1 & -9 \end{matrix} \\ \left(\begin{array}{ccc|cc} 1 & 2 & -2 & 1 & 9 \\ 0 & 1 & 5 & 7 & -9 \\ 1 & 3 & 4 & 9 & -2 \end{array}\right) \end{array} \to \left(\begin{array}{ccc|cc} 1 & 2 & -2 & 1 & 9 \\ 0 & 1 & 5 & 7 & -9 \\ 0 & 1 & 6 & 8 & -11 \end{array}\right) \to$$

$$\begin{array}{ccccc} -1 & -5 & -7 & 9 \end{array}$$

$$\to \left(\begin{array}{ccc|cc} 1 & 2 & -2 & 1 & 9 \\ 0 & 1 & 5 & 7 & -9 \\ 0 & 0 & 1 & 1 & -2 \end{array}\right) \Longleftrightarrow$$

First system:

$$\begin{cases} x_1 + 2x_2 - 2x_3 = 1 \\ x_2 + 5x_3 = 7 \\ x_3 = 1 \end{cases} \qquad \begin{cases} x_1 = 1 - 2x_2 + 2x_3 = 1 - 4 + 2 = -1 \\ x_2 = 7 - 5x_3 = 7 - 5 = 2 \\ x_3 = 1 \end{cases}$$

Second system:

$$\begin{cases} x_1 + 2x_2 - 2x_3 = 9 \\ x_2 + 5x_3 = -9 \\ x_3 = -2 \end{cases} \qquad \begin{cases} x_1 = 9 + 2x_3 - 2x_2 = 9 + 2(-2) - 2 = 3 \\ x_2 = -9 - 5(-2) = 1 \\ x_3 = -2 \end{cases}$$

Answer: $(-1, 2, 1)$ and $(3, 1, -2)$ are solutions to 1st and 2nd systems.

(6)

a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ Row echelon form, not reduced.

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Not row echelon → not reduced.

c) $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ Reduced row echelon → row echelon.

d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ Reduced row echelon → row echelon.

e) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ Row echelon, but not reduced.

f) $\begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ Non row echelon → not reduced.

g) $\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{pmatrix}$ Reduced row echelon → row echelon

h) $\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Row echelon, ~~but~~ not reduced.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
REF	✓	✗	✓	✓	✓	✗	✓	✓
RREF	✗	✗	✓	✓	✗	✗	✓	✗

(7)

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 5 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Lead variables: x_1, x_4

Free variables: x_2, x_3.

Let $x_2 = \alpha, x_3 = \beta, \alpha, \beta \in \mathbb{R}$.

Then $x_4 = 6$, $x_2 = 3 - 5x_2 + 2x_3 = 3 - 5\alpha + 2\beta$.

$\forall \alpha, \beta \in \mathbb{R}$ $\begin{pmatrix} 3-5\alpha+2\beta \\ \alpha \\ \beta \\ 6 \end{pmatrix}$ are solutions of system.

(8)

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \\ 3x_1 + 4x_2 + 2x_3 = 4 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right) \xrightarrow[1\&2]{\text{int. rows}} \begin{array}{c} \begin{array}{cccc} -2 & -2 & -2 & -6 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & 4 \end{array}\right) \end{array} \rightarrow \begin{array}{c} \begin{array}{cccc} -3 & -3 & -3 & -9 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 4 & 2 & 4 \end{array}\right) \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -1 & -5 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The aug. m. is in row echelon, zero rows do not have non-zero in RHS $\Longrightarrow$ consistent.

After removing zero-rows:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \end{array}\right)$$ The system is not strict triangular $\Longrightarrow$ infinitely many solutions.

$\Big\downarrow$ G-J reduction

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -5 \end{array}\right)$$

Lead: x_1, x_2.
Free: $x_3 \to f$, $x_2 = -5 + f$
$x_1 = 8 - 2f$

Then $\begin{pmatrix} 8-2f \\ -5+f \\ f \end{pmatrix}$ are solutions $\forall f \in \mathbb{R}$.

9

$$\begin{cases} x_1 - x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - x_3 = 1 \\ 7x_1 + 3x_2 + 4x_3 = 7 \end{cases}$$

$$\begin{array}{c} -2 \quad 2 \quad -4 \quad -8 \\ \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & 3 & -1 & 1 \\ 7 & 3 & 4 & 7 \end{array}\right) \end{array} \to \begin{array}{c} -7 \quad 7 \quad -14 \quad -28 \\ \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & -7 \\ 7 & 3 & 4 & 7 \end{array}\right) \end{array} \to$$

$$\to \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & -7 \\ 0 & 10 & -10 & -21 \end{array}\right) \xrightarrow{\cdot\frac{1}{5}\text{ 2nd}} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -\frac{7}{5} \\ 0 & 10 & -10 & -21 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -\frac{7}{5} \\ 0 & 0 & 0 & -7 \end{array}\right)$$

In row echelon form, with zero row in coefficient m. & correspond. non-zero entry in RHS $\Longrightarrow$ inconsistent.

10

$$\begin{cases} x_1 + 3x_2 + x_3 + x_4 = 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 = 8 \\ 3x_1 + x_2 + 2x_3 - x_4 = -1 \end{cases}$$

$$\begin{array}{c} -2 \quad -6 \quad -2 \quad -2 \quad -6 \\ \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 2 & -2 & 1 & 2 & 8 \\ 3 & 1 & 2 & -1 & -1 \end{array}\right) \end{array} \to$$

$-3 \quad -9 \quad -3 \quad -3 \quad -9$

(10)

$$\begin{cases} x_1+3x_2+x_3+x_4=3\\ 2x_1-2x_2+x_3+2x_4=8\\ 3x_1+x_2+2x_3-x_4=-1\end{cases}$$

Construct augmented matrix:

$$\begin{array}{c} \begin{array}{ccccc} -2 & -6 & -2 & -2 & -6 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3\\ 2 & -2 & 1 & 2 & 8\\ 3 & 1 & 2 & -1 & -1\end{array}\right)\end{array} \longrightarrow \begin{array}{c} \begin{array}{ccccc} -3 & -9 & -3 & -3 & -9 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3\\ 0 & -8 & -1 & 0 & 2\\ 3 & 1 & 2 & -1 & -1\end{array}\right)\end{array} \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3\\ 0 & -8 & -1 & 0 & 2\\ 0 & -8 & -1 & -4 & -10\end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3\\ 0 & -8 & -1 & 0 & 2\\ 0 & 0 & 0 & -4 & -12\end{array}\right) \xrightarrow[\bullet\ \text{2nd by } -\frac{1}{8}]{\bullet\ \text{3rd by } -\frac{1}{4}} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3\\ 0 & 1 & \frac{1}{8} & 0 & -\frac{1}{4}\\ 0 & 0 & 0 & 1 & 3\end{array}\right) \longrightarrow$$

[Row echelon matrix & consistent system.
(no non-zero RHS for zero coef. row)]

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3\\ 0 & 1 & \frac{1}{8} & 0 & -\frac{1}{4}\\ 0 & 0 & 0 & 1 & 3\end{array}\right) \xrightarrow{G-J} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0\\ 0 & 1 & \frac{1}{8} & 0 & -\frac{1}{4}\\ 0 & 0 & 0 & 1 & 3\end{array}\right) \xrightarrow{G-J}$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & \frac{5}{8} & 0 & \frac{3}{4}\\ 0 & 1 & \frac{1}{8} & 0 & -\frac{1}{4}\\ 0 & 0 & 0 & 1 & 3\end{array}\right)$$

Lead: x_1, x_2, x_4.
Free: $x_3 \longrightarrow$ assign s.

$x_1 \quad x_2 \quad x_3 \quad x_4$

$x_1 = \frac{3}{4} - \frac{5}{8}x_3 = \frac{3}{4} - \frac{5}{8}s$

$x_2 = -\frac{1}{4} - \frac{1}{8}x_3 = -\frac{1}{4} - \frac{1}{8}s$

$x_3 = s$

$x_4 = 3$

$$\Longrightarrow \forall s \in \mathbb{R} \begin{pmatrix} -\frac{5}{8}s + \frac{3}{4} \\ -\frac{1}{4} - \frac{1}{8}s \\ s \\ 3 \end{pmatrix} \text{ is a solution.}$$