Lecture "Experimental Physics I"

(Prof. Dr. R. Seidel)

Lecture 10 + 11

Planetary motion and Gravity

- Kepler's laws
- Newton's law of gravity
- Planetary motion
- Gravitational force of extended object

1) Kepler's laws

Precise measurements of planetary motion by Tycho Brahe (1546–1601) even without any telescope provided a basis for Johannes Kepler to define the laws for planetary motion. Johannes Kepler (1571–1630) provided with his heliocentric (sun centered) model of planetary motion a much better description than the complicated geocentric (earth centered) model that was the dominating model at his time also due to the religious dogma.
The laws formulated by Kepler state:

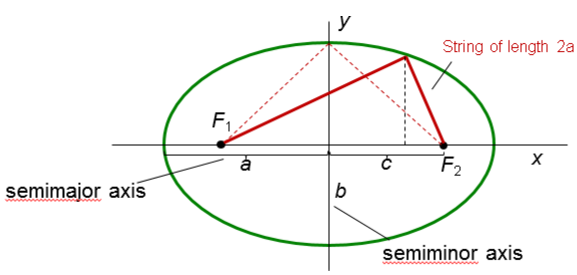
> **Law 1:** All planets move in elliptical orbits with the sun at one focal point.
>
> **Law 2:** The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
>
> **Law 3:** The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

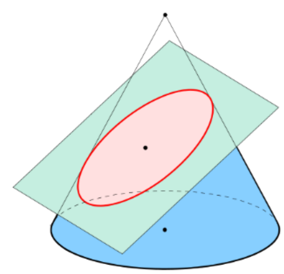
Sidenote: ellipse basics
Let us briefly repeat some essentials about the ellipse mentioned in the 1st law. One possible **definition for an ellipse** is:

"An ellipse is composed by all points of a plane for which the sum of the distances to two focal points F_1 and F_2 is constant."

Experiment: Drawing an ellipse at the blackboard using the Gardener's method, i.e. attaching both ends of a cord to the two focal points and using a pen that is guided by the maximum lateral displacement of the cord.

This construction provides the typical ellipse form, an oval that is elongated along the axis that connects the focal points (see figure below). If one puts the focal points at one spot one would obtain a circle.
If we set the **string length to 2*a***, then **half of the maximum ellipse extension equals to *a***, which is called the **semimajor axis**. Perpendicular to the semimajor axis, i.e. the long symmetry axis, we have the **shorter semiminor axis *b***.





Let the distance between the two focal points be 2c: $\overline{F_1F_2} = 2c$. We then can write a simple relation between a, b and c when looking at the point (0,b):

$$a^2 = b^2 + c^2$$

We can also easily derive the ellipse equation using the definition by summing up the distances at a given ellipse point to the focal points using the Pythagorean theorem in the two triangles that are formed by the normal of the point onto the x axis (**not shown**):

$$2a = \sqrt{(c+x)^2 + y^2} + \sqrt{(c-x)^2 + y^2}$$

Bring one square root to the other equation side, squaring and simplifying (**not shown**) yields:

$$4a^2 - 4a\sqrt{(c+x)^2 + y^2} + (c+x)^2 + y^2 = (c-x)^2 + y^2$$

$$4a^2 - 4a\sqrt{(c+x)^2 + y^2} + 4cx = 0$$

Isolating the root on one equation side and squaring provides:

$$(a^2 + cx)^2 = a^2[(c + x)^2 + y^2]$$
$$a^4 + 2a^2cx + c^2x^2 = a^2c^2 + 2a^2cx + a^2x^2 + a^2y^2$$

and we arrive at:

$$a^2(\underbrace{a^2 - c^2}_{b^2}) = x^2(\underbrace{a^2 - c^2}_{b^2}) + a^2y^2$$

Replacing $a^2 - c^2 = b^2$ and devision by a^2b^2 finally provides the **ellipse equation**:

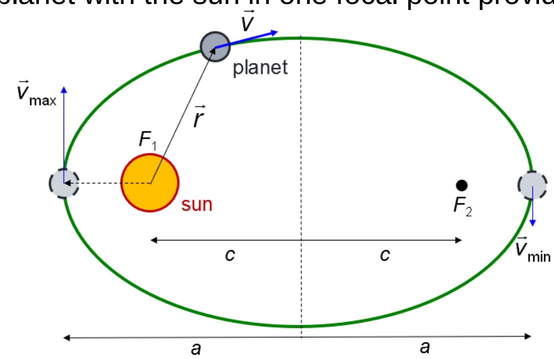
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Thus, knowing a and b fully describes the ellipse shape. An ellipse can also be obtained by making a plane cut through a cone making the ellipse to one of the well-known conical sections.

Let us now have a look at the meaning and the consequences of the individual laws of Kepler.

1st law: Elliptic trajectory of planets

The elliptic trajectory of a planet with the sun in one focal point provides the following picture:



The **smallest distance** of the planet to the sun is called **perihelion** located at the closer intersection of the ellipse with the semimajor axis. According to the figure it is given as:

$$r_{min} = a - c$$

The **largest distance** is called **aphelion** located at the farther intersection of the ellipse with the semimajor axis. It is given by

$$r_{\max} = a + c$$

We will see later that as closer the planet is to the sun as lower is its potential energy. Due to energy conservation it has therefore the **maximal velocity at the shortest distance** and the **minimal velocity at the largest distance**. Only at these **extreme points the velocity is perpendicular to the radial vector**. At all other points there is a non-90° angle between the two vectors. When looking at the characteristic orbit data for the planets of our solar system in particular the numerical eccentricity defined as $\varepsilon = c/a$, we see that with the exception of mercury all orbits are practically circular.

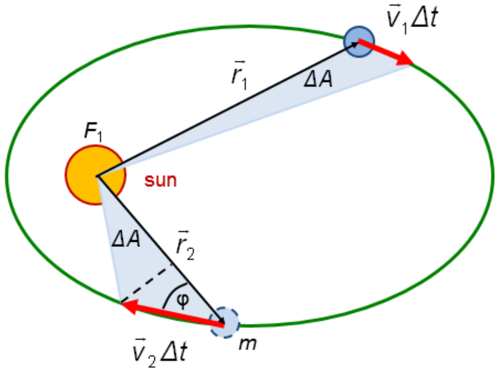
Name	Symbol	Large semi axis a of orbit			Revolution period T	Mean velocity	Numerical excentricity	Inclination of orbit	Distance from earth	
		In AU	In 10^6 km	In light travel time t		In km s^{-1}			Minimum in AU	Maximum in AU
Mercury	☿	0.39	57.9	3.2 min	88 d	47.9	0.206	7.0°	0.53	1.47
Venus	♀	0.72	108.2	6.0 min	225 d	35.0	0.007	3.4°	0.27	1.73
Earth	♁	1.00	149.6	8.3 min	1.00 a	29.8	0.017	–	–	–
Mars	♂	1.52	227.9	12.7 min	1.9 a	24.1	0.093	1.8°	0.38	2.67
Jupiter	♃	5.20	778.3	43.2 min	11.9 a	13.1	0.048	1.3°	3.93	6.46
Saturn	♄	9.54	1427	1.3 h	29.46 a	9.6	0.056	2.5°	7.97	11.08
Uranus	♅	19.18	2870	2.7 h	84 a	6.8	0.047	0.8°	17.31	21.12
Neptun	♆	30.06	4496	4.2 h	165 a	5.4	0.009	1.8°	28.80	31.33
Earth moon	☾	0.00257	0.384	1.3 s	27.32 d	1.02	0.055	5.1°	356410 km	406740 km

Interestingly, the orbit planes of the planets almost coincide with the exception of some smaller angular deviations. This indicates similar but not identical initial conditions at the birth of our solar system.

Most comets originated from our solar system that's why they move also on elliptical tracks around the sun.

2^{nd} law: Conservation of angular momentum

The 2^{nd} law says that the area ΔA from sweeps of the radius vector in a short time Δt is always the same.



To look what this means we calculate the **area of such a triangular sweep** over which we assume the velocity vector to be approximately constant. The area is then one **half of the distance to the sun multiplied by the height of the formed triangle**, which is given by $v_2\Delta t\, sin\,\varphi$. This product can also be expressed by the vector product between radial vector and velocity:

$$\Delta A = \frac{1}{2} r \cdot \underbrace{v_2 \Delta t\, sin\,\varphi}_{\text{triangle height}} = \frac{1}{2}|\vec{r} \times \vec{v}|\Delta t$$

Transformation provides the area per time, which is a constant:

$$\frac{2\Delta A}{\Delta t} = |\vec{r} \times \vec{v}| = const$$

Using the definition of the angular velocity we can write.

$$\vec{\omega} = \frac{1}{r^2}\vec{r} \times \vec{v}$$

With this we get:

$$|\vec{r} \times \vec{v}| = |r^2\vec{\omega}| = const$$

Since the plane of the orbit remains constant, the angular velocity that is perpendicular to the plane points always in the same direction such that even the vector in the equation above is a constant:

$$\vec{r} \times \vec{v} = r^2\vec{\omega} = const$$

Multiplying this equation by the mass of the planet *m* provides the so-called **angular momentu**m which is thus conserved:

$$\vec{L} = m \cdot \vec{r} \times \vec{v} = mr^2\vec{\omega} = const$$

The **angular momentum** is a **fundamental quantity and angular momentum conservation is a fundamental conservation law** (as energy and linear momentum conservation). We will discuss this in more detail later. Here it is important that angular momentum conservation provides a relation between angular velocity and radius ($1/r^2$ dependence) that together with energy conservation helps to calculate the planetary trajectories:

$$\omega = \frac{L}{mr^2} \propto \frac{1}{r^2}$$

For the maximal and minimal distance of the planet to the sun the velocity is perpendicular to the radial vector, such that we can write:

$$v_{max,min} = r\,\omega_{max,min} = \frac{L}{mr} \propto \frac{1}{r}$$

i.e. the velocities at maximum and minimum distance scale with the reciprocal of the radius.
We will see later that another consequence of angular momentum conservation is that the gravitational force must act along the radial vector (due to $0 = d\vec{L}/dt = \vec{\tau} = \vec{r} \times \vec{F}$), i.e. it is only dependent on the radius:

$$\vec{F}_G = -f(r) \cdot \hat{e}_r$$

The gravitational force that attracts sun, planets etc. is a central force!

2) Newton's law of universal gravitation

A) Formulation

Newton formulated the famous law of universal gravitation based on the **3rd law of Kepler** and on the following reasoning:

1) Due to angular momentum conservation, the gravitational force must be a central force.
2) Actio = Reactio: A planet (apple falling on his head) is as much attracted by the sun (earth) as it attracts itself the sun (earth).
3) The weight of an object, i.e. the gravity on earth, is proportional to its mass *m*. For symmetry reasons the gravity should also be proportional to the mass *M* of the earth

Together this reasoning provides the following proportionality for the gravitational force:

$$\vec{F}_G = const \cdot mM \cdot f(R) \cdot (-\hat{e}_r)$$

4) The **3rd Kepler's law** provides finally the dependence on the distance *r*
   It states that
   $$\frac{R^3}{T^2} = const$$
   Replacing the period with the angular velocity ($\omega = 2\pi/T$), we get:
   $$R^3 \left(\frac{\omega}{2\pi}\right)^2 = const$$
   or
   $$\omega^2 = \frac{const}{R^3} \propto \frac{1}{R^3}$$

If we assume that the planets have a circular orbit, then the gravitational force must be equal to the centripetal force in order to keep them on the circular track:

$$-const \cdot mM \cdot f(R) = -m\omega^2 R \propto \frac{1}{R^2}$$

where we got the right side by inserting the relation for ω^2. Since the masses are constant for a given system we can conclude that

$$f(r) = \frac{1}{R^2}$$

Thus, the **gravitational force decays with a 1/R² dependence**. Inserting into the initial relation from above provides then the **law of gravity**:

$$\vec{F}_G = -G\frac{mM}{R^2}\hat{e}_r$$

Here we combined all constants in the **universal gravity constant G**:

$G = 6.67 \cdot 10^{-11}\ N \cdot m^2 \cdot kg^{-2}$

Newton's finding published in 1687: Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

B) Determination of the gravitation constant according to Cavendish:

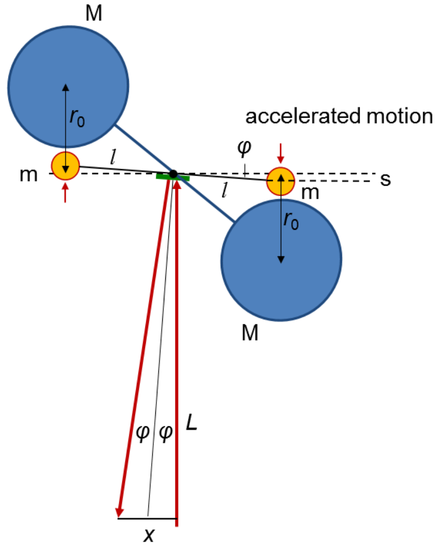
Newton's formula was at that time a great achievement, but the gravity constant *G* still remained unknown for a long time.
The problem is that one cannot determine *G* from the gravity on earth, if one does not know the mass of the earth, since both parameters determine the value of the free fall acceleration *g*:

$$F_g = -m \underbrace{\frac{G M_E}{r_E^2}}_{g}$$

In **1798**, **Henry Cavendish** carried out an experiment where he measured the attraction between two smaller masses in the laboratory, from which he determined *G*:

His **idea** was hereby:

1) Bring a large mass in vicinity of a small mass that undergoes slow accelerated motion due to the mutual gravitation attraction.
2) Detect the accelerated motion of the small mass with a long "light lever" (show slides). The Gravity force is the determined from force required for the observed acceleration



In Cavendish's words the apparatus is described as:
*"The apparatus is very simple it consists of a wooden arm, 6 feet long, made so, as to unite great strength with little weight. This arm is suspended in a horizontal position, by' a slender wire 40 inches long, and to each extremity is hung a leaden ball, about 2 inches in diameter; and the whole is in- closed in a narrow wooden case, to defend it from the wind***"**
(**Henry Cavendish,** Philosophical Transactions of the Royal Society of London, Vol. 88 (1798), pp. 469-526)

Amplification of the displacement *s* using the "light lever"

Using the apparatus geometry, we can write for the displacement of the small mass considering that the light beam undergoes twice the angular displacement as the rod with the two masses:

$$s = l\,\varphi = l\frac{x/2}{L} = \frac{x}{2}\frac{l}{L} = x\frac{0.1m}{2\cdot 19.8m} \approx \frac{x}{400}$$

Using the real parameters (l = 10 cm and L = 19.8 m), we see that the displacement s of the mass m is 400-fold amplified when detecting the light beam displacement x!

Calculating the universal gravitation constant from the acceleration

Measuring *x* as function of time provides thus the displacement *s* of the small mass *m* as function of time. In first approximation we consider a constant acceleration, such that:

$$s \approx \frac{a}{2} t^2$$

which can be transformed to:

$$\frac{a}{2} \approx \frac{s}{t^2}$$

i.e. the slope in a $s - t^2$ plot provides half the acceleration. The acceleration is induced by the gravitational force between small and large mass, thus:

$$F_G = ma = G\frac{mM}{(r_0 - s)^2} \approx G\frac{mM}{{r_0}^2}$$

and we arrive at the simple formula for G:

$$G \approx \frac{a{r_0}^2}{M}$$

Inserting the measured slope in the $s - t^2$ plot of $\approx 5 \cdot 10^{-8} ms^{-2}$ provides for the gravitation constant:

$$G = \frac{2 \cdot 5 \times 10^{-8} m\, s^{-2} (0.08m)^2}{10kg} = 6.4 \cdot 10^{-11} \quad \frac{m^2}{\underbrace{kgs^2}_{Nm^2kg^{-2}}}$$

which is a bit lower than the real value. For curiosity we can also calculate the actual gravitational force that acted on the small lead sphere of 2 cm diameter. Its mass is

$$m = \rho \frac{\pi}{6} d^3 = 11.34 \frac{g}{cm^3} \frac{\pi}{6} 8cm^3 = 48g \approx 0.050kg$$

and the force is:

$$F_G \approx ma = 0.05kg \cdot 1 \times 10^{-7} m\, s^{-2} = 5 \cdot 10^{-9} N = 5\, nN$$

This is a spectacularly low force. Nano-newton is the force scale to break a single covalent bond, e.g. a C-C bond. It is a remarkable achievement of Cavendish to measure such low forces at that time.

For a more precise experiment we have to consider all other acting forces in particular the friction of the accelerating sphere with the air and the back-driving torsion of the cord at which the rod with the two small masses is attached. A more correct equation of motion including back-driving torsion spring and friction is given by (see slide):

$$G\frac{mM}{{r_0}^2} + \underbrace{k_{cord}}_{\substack{\text{torsion} \\ \text{spring} \\ \text{const.}}} \frac{s}{l^2} + \underbrace{\gamma \dot{s}}_{\text{friction in air}} = m\ddot{s}$$

Mass and density of earth

By knowing the gravity constant and gravitational acceleration at the earth surface we can calculate the mass of the earth, which is quite spectacular. On the surface of the earth we have:

$$g = \frac{GM_E}{r_E^2}$$

From this we can calculate the weight of the earth! Using G = 6.67 · 10^{-11} N· m^2·kg^{-2} and r_E = 6370 km we arrive at:

$$M_E = \frac{g{r_E}^2}{G} = 5.96 \times 10^{24}\ \text{kg}$$

with $M = \rho_E \frac{4}{3}\pi {R_E}^3 \Rightarrow 5.5{\times}10^3\ \mathrm{kg/m^3}$ one obtains an estimate for the density of the earth.

C) Potential energy in the gravitational field

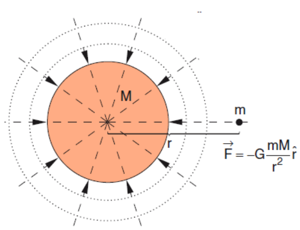
The gravitational law defines a corresponding gravitational force field around any mass M:

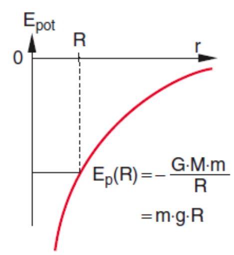
$$\vec{F}_g = -G\frac{Mm}{r^2}\hat{e}_r$$

From this, we can calculate the potential energy (as already previously done) according to:

$$U(\vec{r}) = -\int_\infty^{\vec{r}} -\frac{GMm}{r^2}\hat{e}_r \cdot d\vec{r} = \int_\infty^{\vec{r}} \frac{GMm}{r^2}dr = -\frac{GMm}{r}\bigg|_\infty^r$$

To do so we use infinity distance as reference point, since the force becomes zero at infinity. Integration provides a $-1/r$ dependence of the potential energy. The negative sign ensures that we lower our potential energy (fall down the potential) when we approach the attracting mass. **$-E_{pot}$ provides the minimum work (kinetic energy) required for escaping the field towards infinity, e.g. our solar system.**





3) Planetary motion

To derive the trajectory of a planet around the sun we can again either use the **equation of motion** or an approach via **energy & angular momentum conservation.**

A) Planetary motion via forces

Let us place the sun into the origin of the coordinate system. The planet has then the position

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

in its plane of motion.

There is only one force acting on the planet, which is the gravitational force acting in the direction of the planet position, that leads to an acceleration of the planet according to Newton's 2nd law:

$$\vec{F}_g(\vec{r}) = -G\frac{Mm}{r^2}\hat{e}_r = -G\frac{Mm}{r^3}\vec{r} = m\frac{d^2\vec{r}}{dt^2}$$

Inserting $\vec{r}$ in components provides two differential equations, one for each component:

$$\frac{d^2\vec{r}}{dt^2} = \begin{pmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \end{pmatrix} = -G\frac{M}{(x^2+y^2)^{3/2}}\begin{pmatrix} x \\ y \end{pmatrix}$$

Both **equations are coupled** since they both contain x AND y and are thus apparently difficult to solve. However, one can use numeric integration and simultaneously sum up the velocities and the displacements for the x and y directions (**see slide**):

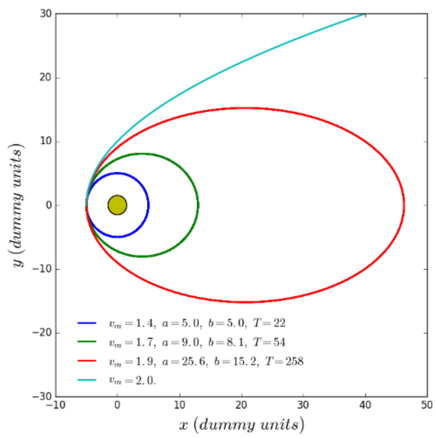
$$v_x(t+\Delta t) = v_x(t) + a_x(x,y)\cdot\Delta t$$

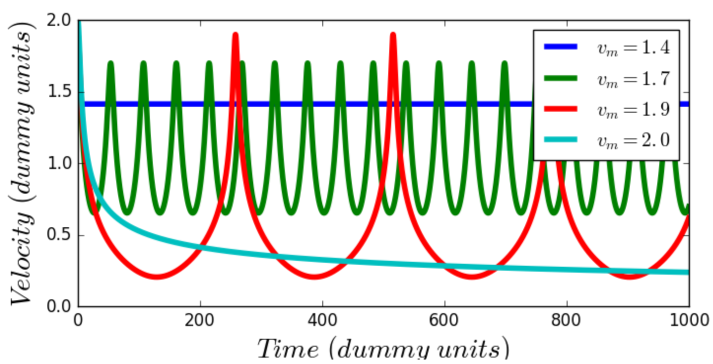
$$x(t+\Delta t) = x(t) + v_x(t)\Delta t$$
$$v_y(t+\Delta t) = v_y(t) + a_y(x,y)\cdot\Delta t$$
$$y(t+\Delta t) = y(t) + v_y(t)\Delta t$$

Covenient starting conditions are that the planet is at *t* = 0 at minimum or maximum distance from the sun, i.e. it has only a tangential velocity component:

$$\vec{r}(0) = \begin{pmatrix} -r_{min,max} \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}(0) = \begin{pmatrix} 0 \\ v_{max,min} \end{pmatrix}$$

The numeric solution for the problem provides elliptic orbits in agreement with Kepler's 1st law in which the sun is in a focal point (**show animation**). Increasing the initial velocities at the minimum distance provide more and more elongated ellipses with increased eccentricity. At some velocity threshold the planet is not returning to its initial position but completely leaves the gravitational field:





When looking at the velocities, one sees that there is a **maximum velocity** (initial velocity) **at the minimum distance** (with minimum potential energy) and a minimum velocity at the largest distance (with maximum potential energy), in agreement with energy conservation. Also, the **period of the orbit is increasing** with the semimajor axis of the orbit.

A special case is a **circular orbit**, for which we can provide a simple solution, if we know as starting condition either its radius or the velocity on the orbit. As used before for this orbit, the gravity force directly equals the centripetal force:

$$-\frac{GmM}{{r_c}^2}\hat{e}_r = -m\omega^2 r_c \hat{e}_r$$

which provides a relation between orbit radius and (angular velocity). The **orbit period is obtained** by transforming the angular frequency:

$$\frac{GM}{{r_c}^2} = \left(\frac{2\pi}{T}\right)^2 r_c$$

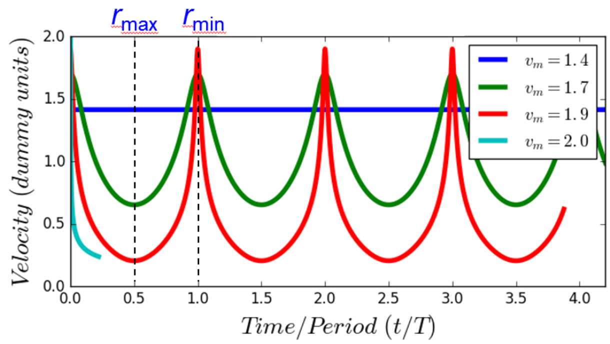
Such that we get for the period:

$$T^2 = \frac{(2\pi)^2}{GM}{r_c}^3$$

which agrees with the 3rd Kepler's law since for a circle the semimajor axis equals the circle radius ($a = r_c$). Generally we can thus write for the period on any elliptic trajectory:

$$T^2 = \frac{(2\pi)^2}{GM}a^3$$

Dividing the time of the simulation by the period calculated from this formula proofs the validity of the formula and thus also Kepler's 3rd law:



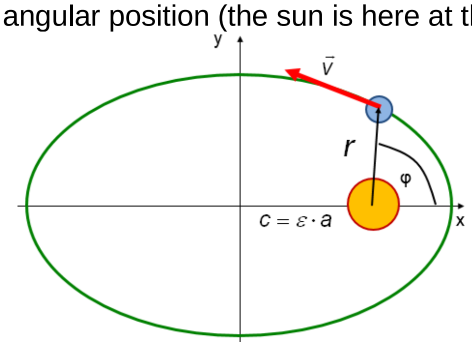
Overall understanding planetary motion is very central in physics, since **other force fields, such as the electric field also decay with 1/r²**, generalizing the results of these findings.

Remark: Our approach with a fixed central sun is a simplification, since the center of sun is not directly in the focal point. Rather both sun and planet(s) move around their common center of mass.

Experiment: One can mimic a planetary orbit using a small ferromagnet in an inhomogeneous magnetic field

B) Planetary motion via energy and angular momentum conservation

As for oscillations, an easier approach to planetary motion comes from energy conservation combined with angular momentum conservation. For this approach we use for practical reasons so called **plane polar coordinates $\{r, \varphi\}$** where we describe the planet position in terms of distance from the sun and its angular position (the sun is here at the coordinate system origin).



The position in polar coordinates can be transformed into Cartesian coordinates by taking the projections of *r* onto the *x* and *y* axes and considering the shift of the coordinate system by $c = \varepsilon\, a$. We thus get:

$$x = r\cos\varphi + \varepsilon a$$
$$y = r\sin\varphi$$

Inserting these equations into the ellipse equation for Cartesian coordinates allows to derive an ellipse equation in polar coordinates:

Sidenote: Deriving **the ellipse equation on polar coordinates** (not shown in lecture)

We start with the ellipse equation, which can in Cartesian coordinates being transformed to:

$$\frac{b^2}{a^2}x^2 + y^2 = b^2$$

Inserting polar coordinates and transformations yield:

$$\frac{b^2}{a^2}(\varepsilon\, a + r\cos\varphi)^2 + r^2\sin^2\varphi = b^2,$$

$$b^2\varepsilon^2 + 2\frac{b^2}{a}\varepsilon\, r\cos\varphi + \frac{b^2}{a^2}r^2\cos^2\varphi + r^2(1-\cos^2\varphi) = b^2,$$

$$-b^2\underbrace{(1-\varepsilon^2)}_{\frac{b^2}{a^2}} + 2\frac{b^2}{a}\varepsilon\cdot r\,cos\,\varphi - \underbrace{\left(1-\frac{b^2}{a^2}\right)}_{\frac{c^2}{a^2}=\varepsilon^2} r^2\,cos^2\,\varphi + r^2 = 0,$$

$$\frac{b^4}{a^2} - 2\frac{b^2}{a}\varepsilon\, r\;\cos\varphi + \varepsilon^2 r^2\cos^2\varphi = r^2$$

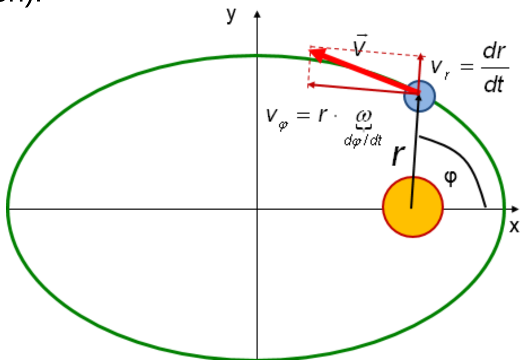
Simplifying provides then:

$$\left(\frac{b^2}{a} - \varepsilon\, r\cos\varphi\right)^2 = r^2$$

Taking the square root and replacing $b^2 = a^2(1-\varepsilon^2)$ provides the final form of the ellipse equation in polar coordinates:

$$r(\varphi) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\varphi}$$

which describes the distance to the focal point as function of the angle. We now split the velocity vector of the planet into its radial component and its tangential component (the latter only changing the angular position).



These components are given by:

$$v_r = \frac{dr}{dt} \quad \text{and} \quad v_\varphi = r\underbrace{\omega}_{d\varphi/dt}$$

Using the two components we can write for the **total conserved energy:**

$$const = E = \overbrace{-G\frac{mM}{r}}^{E_{pot}} + \overbrace{\underbrace{\frac{m}{2}\dot{r}^2}_{\text{radial}} + \underbrace{\frac{m}{2}(r\omega)^2}_{\text{angular}}}^{E_{kin}}$$

Angular momentum conservation (Kepler's 3rd law) demands:

$$const = \left|\vec{L}\right| = |m\vec{r}\times\vec{v}| = mr^2\omega$$

We can transform this equation towards the angular velocity ω:

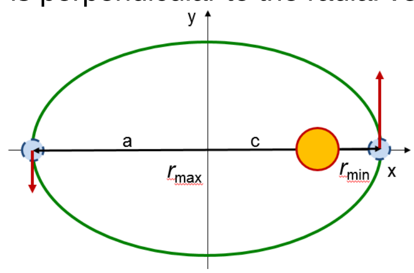
$$\omega = \frac{d\varphi}{dt} = \frac{L}{mr^2}$$

Inserting this into the energy equation provides:

$$const = E = -G\frac{mM}{r} + \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2}$$

C) Evaluating the energy equation with conserved angular momentum

Let us now evaluate the last equation in order to obtain the **orbit parameters** of our planet **from the conserved values of energy and angular momentum.**
To this end we consider only the two points on our orbit of maximal and minimal distance from the sun. Here the velocity vector is perpendicular to the radial vector, such that dr/dt is zero.



We can thus write:

$$E = -G\frac{mM}{r_{min,max}} + \frac{L^2}{2mr_{min,max}^2}$$

Transformation provides a quadratic equation

$$r_{min,max}^2 + G\frac{mM}{E}r_{min,max} - \frac{L^2}{E\ 2m} = 0$$

which can be easily solved:

$$r_{min,max} = \underbrace{-G\frac{mM}{2E}}_{a} \pm \underbrace{\sqrt{\left(G\frac{mM}{2E}\right)^2 + \frac{L^2}{2mE}}}_{c}$$

This is an important finding, since from the minimal and maximal distance we can calculate the two ellipse parameters a and c, since:

$$r_{min} = a - c \quad \text{and} \quad r_{max} = a + c$$

With this we obtain for the **semimajor axis**:

$$a = \frac{r_{max} + r_{min}}{2} = -G\frac{mM}{2E}$$

as well as for the **half focal-point distance**:

$$c = \frac{r_{max} - r_{min}}{2} = \sqrt{\left(G\frac{mM}{2E}\right)^2 + \frac{L^2}{2mE}} = a\underbrace{\sqrt{1 + \frac{2EL^2}{G^2m^3M^2}}}_{\varepsilon}$$

The square-root term on the right side gives us the eccentricity defined by $c = \varepsilon a$. The semi-minor axis is then given by:

$$b = \sqrt{a^2 - c^2} = a\sqrt{1 - \varepsilon^2}$$

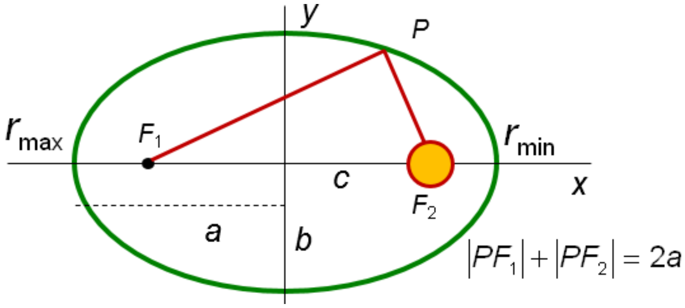
These relations tell us that **both energy and angular momentum are required to fully determine the orbit**. The length of the semimajor axis is determined only by the energy of the orbiting object, while the eccentricity is determined by energy and angular momentum.

D) Orbit shapes

The total energy *E* of an orbiting object can be positive or negative, which correspondingly flips the sign of a and makes the excentricity smaller or larger than 1. Let us look at the different cases and the consequences on the orbit form:

- **Negative total energy ($E < 0$) provides elliptic orbit**

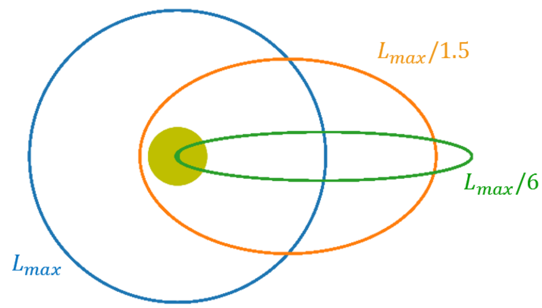
For $E < 0$, we get for our ellipse parameters $\boldsymbol{a} > \mathbf{0}$ **and** $\boldsymbol{\varepsilon} < \mathbf{1}$, which provides **the typical elliptic orbit** we discussed so far for all trajectories with conserved total energy. We have also positive solutions for the minimal and the maximal distance of the planet from the sun ($r_{min} = a - \varepsilon a$ and $r_{max} = a + \varepsilon a$).



A negative total energy means that even if we can convert all kinetic energy into potential energy, we are still left with a negative value for the maximum potential energy, i.e. the object is still trapped inside the gravitational potential and we cannot escape from the central mass M (see figure below).

From $b = a\sqrt{1-\varepsilon^2}$, we see that we get a **circular orbit for** $\boldsymbol{\varepsilon} = \mathbf{0}$ and a **highly stretched ellipse for** $\boldsymbol{\varepsilon}$ **approaching 1**.

Looking into the expression for ε found above, we see that $\boldsymbol{\varepsilon}$ **approaches 1 when** $\boldsymbol{L}$ **approaches zero** (vanishes). In this case the trajectory approaches a linear oscillation in radial direction (see black trajectory on the left).



A circular trajectory is obtained if a maximum value for the angular momentum is obtained, such that ε vanishes. Transforming the relation for ε provides for the maximum angular momentum:

$$L_{max} = \sqrt{\frac{G^2 m^3 M^2}{-2E}}$$

The period of the orbit is however only dependent on the energy, which exclusively determined the semi-major axis according to:

$$T^2 = \frac{(2\pi)^2}{GM} a^3$$

The depicted orbits in the figure above have thus the same periods.

- **Positive total energy ($\boldsymbol{E} > \mathbf{0}$) provides escape on a hyperbolic trajectory**

For $E > 0$, we get $\boldsymbol{a} < \mathbf{0}$ and $\boldsymbol{\varepsilon} > \mathbf{1}$, i.e. the focal points lie "outside" of the curve. Using the formula for the semi-minor axis $b^2 = a^2(1-\varepsilon^2)$, we obtain a negative value for b^2, such that we have the curve **equation of a hyperbola**:

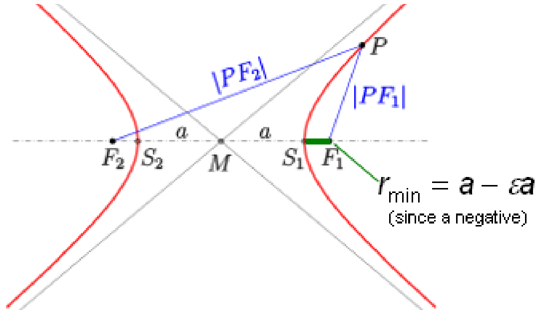
$$\frac{x^2}{a^2} - \frac{y^2}{|b^2|} = 1$$

Like an ellipse, it can be constructed using two focal points. For a hyperbola, is however the **distance difference** to the two focal points a constant and equals 2a ($\big||PF_1| - |PF_2|\big| = 2a$). This construction provides 2 oppositely oriented hyperbolas at distance 2a (see figure below).

In this case **only a minimal distance** of the planet from the sun is obtained since the second solution of the quadratic eq. is negative, such that we get:

$$r_{min} = a - \varepsilon a$$

The hyperbola corresponds to objects that enter our solar system from outside (infinity) and that leave it again or to a rocket that has been brought to a sufficient velocity to leave our solar system. Energetically this makes sense, because a positive energy means that there is kinetic energy left even at infinite distance from the sun where there is no attractive force anymore.



- **Zero total energy ($E = 0$) provides escape on a parabolic trajectory**

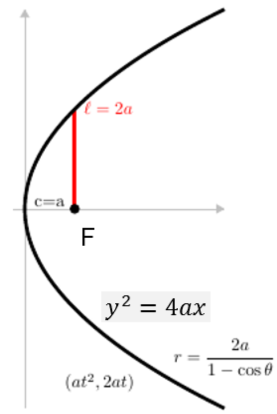
For $E = 0$, we get with $\boldsymbol{\varepsilon} = \mathbf{1}$ ($a = c$) a **parabola** described by $y^2 = 4ax$. The minimum distance $r_{min} = a$ from the focal point is obtained by setting the energy in the previous equation to zero:

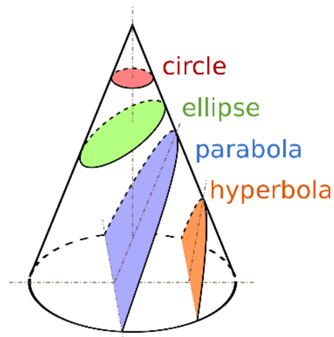
$$0 = E = -G\frac{mM}{r_{min}} + \frac{L^2}{2mr_{min}^2}$$

such that we get directly

$$r_{min} = a = c = \frac{L^2}{2Gm^2M}$$

The three obtained orbit forms are obtained by 3 different types of sections that can be obtained on cones:





$E = 0$ represents the lowest total energy at which an escape from the solar system would be possible (see slide). **Only with a negative total energy a planet, comet, asteroid, etc. remains on an elliptic or circular orbit around the sun !**

E) Deriving the orbit formula (not part of the lecture, maybe show on slides?)

The angular momentum conservation provided an equation for the angular velocity:

$$\omega = \frac{d\varphi}{dt} = \frac{L}{mr^2}$$

The energy equation incl. the angular momentum included only the radial velocity

$$const = E = E_{pot} + \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2}$$

Transformation towards dr/dt gives the radial component of the velocity vector:

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - E_{pot} - \frac{L^2}{2mr^2}\right)}$$

Knowing angular velocity and radial velocity at each distance would allow a successive numeric integration of the trajectory. An even simpler approach can be followed by combining the equations for $d\varphi/dt$ and dr/dt (by "division"), which provides the angular change per change of the radial distance:

$$\frac{d\varphi}{dr} = \frac{L}{mr^2}\left[\frac{2}{m}\left(E - E_{pot} - \frac{L^2}{2mr^2}\right)\right]^{-1/2}$$

Separation of the variables and integration provides:

$$\int_{\varphi_0}^{\varphi} d\varphi = \varphi - \varphi_0 = \frac{L}{m}\int \frac{dr}{r^2\sqrt{2/m\left(E - E_{pot} - L^2/(2mr^2)\right)}}$$

Inserting the potential energy and further transformation gives:

$$\varphi - \varphi_0 = \frac{L}{m}\int \frac{dr}{r\sqrt{(2mEr^2 + 2Gm^2Mr - L^2)}}$$

The integral is an elliptic integral whose general solution can be looked up in a table:

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\arccos\left(\frac{a}{x}\right)$$

If we set $\varphi_0 = 0$, then we get a polar representation of the trajectory:

$$\varphi = \arccos\left(\frac{L^2/r - Gm^2M}{\sqrt{(Gm^2M)^2 + 2mE\,L^2}}\right)$$

Abbreviating:

$$a = -\frac{GmM}{2E} \quad \text{and} \quad \varepsilon = \sqrt{1 + \frac{2EL^2}{G^2m^3M^2}}$$

provides:

$$\varphi = \arccos\left(\frac{a(1-\varepsilon^2) - r}{\varepsilon\, r}\right)$$

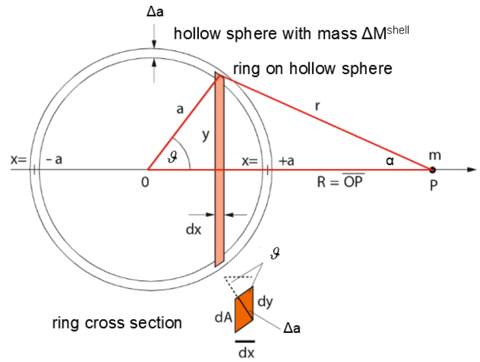
Solving for r gives:

$$r(\varphi) = \frac{a(1-\varepsilon^2)}{1 + \varepsilon\,\cos\varphi}$$

which is the ellipse/hyperbola equation in polar coordinates.

4) Gravitation field of extended bodies

So far, we treated all objects (planets, sun, earth) as point masses. Let's look at the **gravitation force of an extended object in particular a homogeneous large sphere that it exerts on a small point mass m at distance R from the sphere center**. This is a similar situation that we experience on the surface of our planet.



Starting point: We look first at the gravity of a **hollow sphere with radius a and a thin wall of thickness Δa**. From the hollow sphere we consider a **ring of height dx equidistant to m**:
By integrating the gravity force over all rings, we get then the gravity force that a hollow sphere causes. Subsequent integration over all hollow spheres gives the gravity force of the whole sphere. The mass of a **single ring at position x forming an angle ϑ** with respect to the connection between both masses is given by:

$$dM_{ring} = \rho \underbrace{L}_{\substack{ring \\ length}} \underbrace{dA}_{\substack{cross \\ section}} = \rho \underbrace{2\pi y}_{L} \underbrace{\Delta y dx}_{dA} = \rho \underbrace{2\pi a \sin\vartheta}_{L} \frac{\Delta a}{\sin\vartheta} dx = \rho\, 2\pi a\, \Delta a\, dx$$

The y-components of the gravitational force from all ring segments cancel each other out at point *P*. Therefore, we have only to consider the x-component of the gravitational force of each ring segment to get the total force from the ring:

$$dF_x^{ring} = -\frac{Gm\, dM_{ring}}{r^2}\cos\alpha = -\frac{Gm}{r^2} dM_{ring} \frac{R-x}{r} = -Gm\, dM_{ring} \frac{R-x}{r^3}$$

Integrating the force from all rings provides the gravity force of a single hollow shell:

$$\Delta F^{shell} = \int dF_x^{ring} = -Gm \cdot \underbrace{\rho \cdot 2\pi a \cdot \Delta a}_{\text{from } dM_{ring}} \int_{-a}^{a} \frac{R-x}{r^3} dx$$

Using geometric considerations (using both right triangles), we get for r:

$$r^2 = y^2 + (R-x)^2 = \underbrace{a^2 - x^2}_{y^2} + (R-x)^2 = R^2 + a^2 - 2Rx$$

Thus,

$$x = \frac{R^2 + a^2 - r^2}{2R} \quad \text{and} \quad \frac{dx}{dr} = -\frac{r}{R}$$

With this we can transform the integration over dx into an integration over dr. To this end, we distinguish 2 different cases:

A) $R \geq a$: *m* is outside of the hollow sphere

We can substitute x, dx and the integration boundaries. For the latter, we see that the integration would be from $r_{min} = R + a$ to $r_{max} = R - a$ such that we get:

$$\int_{-a}^{a} \frac{R-x}{r^3} dx = -\int_{R+a}^{R-a} \left(R - \frac{R^2 + a^2 - r^2}{2R}\right) \frac{1}{r^3} \frac{r}{R} dr = -\int_{R+a}^{R-a} \frac{R^2 - a^2 + r^2}{2R} \frac{1}{r^2} \frac{1}{R} dr$$

$$= -\frac{1}{2R^2} \int_{R+a}^{R-a} \left(\frac{R^2 - a^2}{2r^2} + 1\right) dr$$

$$= -\frac{1}{2R^2} \underbrace{\left(-\frac{R^2 - a^2}{r} + r \Big|_{R+a}^{R-a}\right)}_{-4a}$$

$$= 2a/R^2$$

Inserting the integration result provides:

$$\Delta F_x^{shell} = \Delta F_x^{shell} = -\frac{Gm}{R^2} \cdot \underbrace{\rho \cdot 4\pi a^2 \cdot \Delta a}_{\Delta M_{shell}} = -\frac{Gm\Delta M_{shell}}{R^2}$$

With this we can conclude:

The gravitation force and thus potential energy in the gravity field outside a thin homogeneous sphere are the same as if the same mass would be concentrated in the sphere center!

B) $R \leq a$: *m* is inside of the hollow sphere

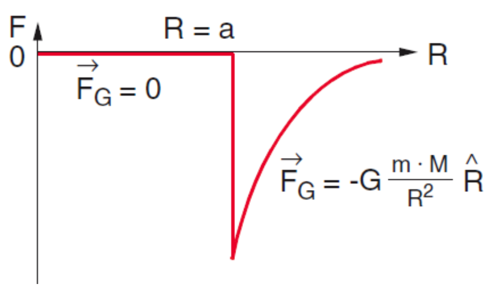
When the mass is inside the sphere we get the same integral as above but we have to change the integration borders from from $r_{min} = R + a$ to $r_{max} = a - R$:

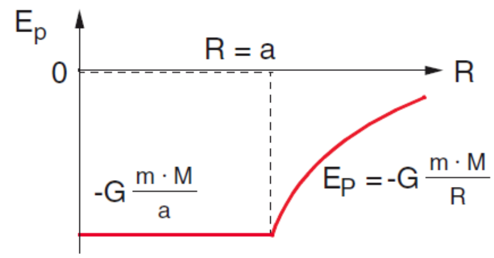
$$\int_{-a}^{a} \frac{R-x}{r^3} dx = -\frac{1}{2R^2} \underbrace{\left(-\frac{R^2 - a^2}{r} + r \Big|_{R+a}^{a-R} \right)}_{0} = 0$$

$$\Delta F^{shell} = \Delta F_x^{shell} = 0$$

Inside a thin homogeneous sphere there is no net gravitational field. The gravitation forces from the individual surface elements cancel each other exactly out!

We get therefore the depicted force profile when approaching the sphere from infinity. Integration of the force along this path provides the potential energy, that shows a $-1/R$ dependence at the outside as seen for a point mass and then remains constant within the sphere.





C) Gravity force of filled sphere (with radial symmetry) if standing outside:

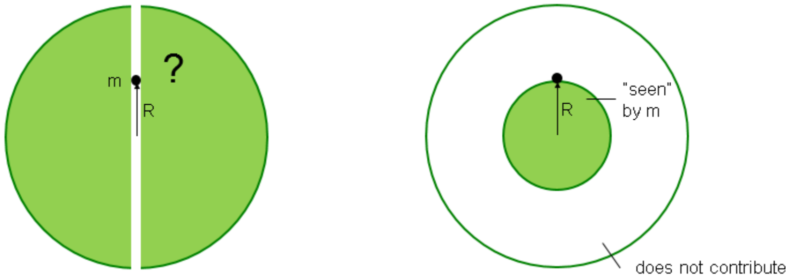
The gravity force outside a sphere with radially symmetric mass distribution is then simply obtained by integrating all shell contributions:

$$F^{spherel} = \int_0^a dF_x^{shell} = \int_0^a -Gm \frac{1}{R^2} dM^{shell} = -Gm \frac{1}{R^2} \int_0^a dM^{shell} = -GmM \frac{1}{R^2}$$

Since each shell generates a force as its mass would be concentrated in the center, also the total gravity force of the full sphere is the same as if the same mass would be in the sphere center**!**

D) Gravity force of within homogeneous filled sphere

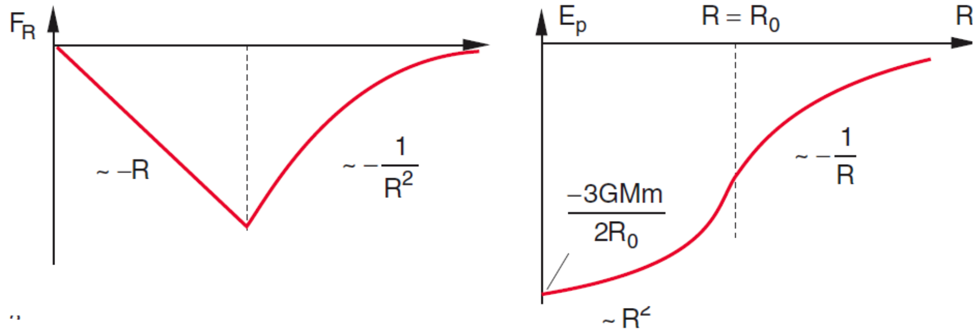
To the gravity force inside a homogeneously filled sphere with radius a and a small mass m being located at $R < a$ from center, only shell layers with radii $r < R$ are contributing while shell layers with $r > R$ do not contribute.



Thus, the mass *m* sees only the mass located within in a sphere of radius R! We can thus write for the force at radial distance *R*:

$$F(R) = -G\frac{m\,M(R)}{R^2} = -G\frac{m\rho 4/3\pi R^3}{R^2} = -Gm\rho\frac{4}{3}\pi R$$

The obtained force is proportional to the radial distance *R*, which is similar to Hooke's law!



The potential shows therefore a quadratic dependence in the inside.

Lecture 10: Experiments

1) Drawing an ellipse at the blackboard using the Gardener’s method, i.e. attaching both ends of a cord to the two focal points and using a pen that is guided by the maximum lateral displacement of the cord.
2) Determination of gravitation constant according to Cavendish
3) Orbit of a small ferromagnet in an inhomogeneous magnetic field