Math for Students of Faculty of Physics
Mathematics 1
Linear Algebra and Calculus of Functions of One Variable

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References

The programm of the course "Mathematics 1. Linear Algebra and Calculus of Functions of One Variable" is standard and the related material can be found practically in any book devoted to the calculus of functions of one variable and the linear algebra.

Books recommended in the official syllabus:

https://www.physgeo.uni-leipzig.de/en/studying/courses-of-study/

- Serge Lang: Linear Algebra, Springer
- Serge Lang: A First Course in Calculus, Springer
- Kenneth A. Ross: Elementary Analysis, Springer
- Stephen Abbott: Understanding Calculus, Springer

Books I frequently use myself to prepare my lectures (to be completed):

Logic, sets, functions

- Keith Devlin, Sets, Functions, and Logic. An Introduction to Abstract Mathematics, CRC.
- Steven Galovich, Introduction to Mathematical Structures, HBJ, Academic Press.

Calculus of functions of one real variable

- Vladimir A. Zorich, Mathematical Analysis I, Springer.

Linear Algebra

- Steven J. Leon, Linear Algebra with Applications, Prentice Hall.

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1. **Notation**

$\forall$ means “for all”

$\exists$ means “there exists”

$\exists!$ means “there exists a unique”

: means “such that”

:= means “denote by”

$\mathbb{N}$ is a set of all natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{Z}$ is a set of all integer numbers, $\mathbb{Z} = \{\, \dots, -3, -2, -1, 0, 1, 2, 3, \dots \,\}$

$\mathbb{Q}$ is a set of all rational numbers, $\mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z},\ n \in \mathbb{N} \right\}$

$\mathbb{R}$ is a set of all real numbers

an interval $(a, b) \subset \mathbb{R}$ is the set of all real x such that $a < x < b$

an interval $[a, b] \subset \mathbb{R}$ is the set of all real x such that $a \le x \le b$

$\operatorname{tg} x := \tan x$ is the tangent of x, $\quad \operatorname{ctg} x := \cot x$ is the cotangent of x

$\operatorname{arctg} y := \arctan y$ is the arctangent of y, $\quad \operatorname{arcctg} y := \operatorname{arccot} y$ is the arccotangent of y

1 One-dimensional calculus

1.1 Basic concepts

1. **Logic**

   - Logical operations: and, or, negation, implication, equivalence.
   - Negation of "and" and "or". Negation of an implication.
   - Equivalence of the implication to the contrapositive. Necessary and sufficient condition.

2. **Proof by contradiction**

   Example. $\sqrt{2}$ is not rational.

   Proof. By contradiction, assume $\sqrt{2}$ is rational. Then there exists integers n, $m \in \mathbb{Z}$:

   $$\sqrt{2} = \frac{n}{m}, \qquad \text{greatest common divisor of } n \text{ and } m \text{ is } 1.$$

   Taking a square of this relation we obtain

   $$2 = \frac{n^2}{m^2} \qquad \Longleftrightarrow \qquad n^2 = 2m^2$$

   $2m^2$ is even and hence n^2 even. But then n itself even, i.e.

   $$n = 2k \quad \text{for some} \quad k \in \mathbb{Z}.$$

   Then we obtain

   $$n^2 = 2m^2 \quad \Longleftrightarrow \quad (2k)^2 = 2m^2 \quad \Longleftrightarrow \quad 4k^2 = 2m^2 \quad \Longleftrightarrow \quad 2k^2 = m^2$$

   Then m^2 is even and hence m itself is even. As both n and m are even, this contradicts to the assumption that the greatest common divisor of n and m is 1. Hence our assumption that $\sqrt{2}$ is rational can not be true. Hence $\sqrt{2}$ is irrational. □

3. **Quantifiers**

   - sentences and predicates
   - $\forall$, $\exists$, $\exists!$ (there exists a unique)
   - Negation of the statement with quantifiers

   Example. Construct the explicit negation of the statement

   $$\Big(\forall\, n \in \mathbb{N} \;\; \exists\, x \in (1, +\infty) \;\; \forall\, y \in [0,1] \quad x + y > n^2\Big)$$

   Determine which of the statements (the original one or its negation) is true.

4. **Proof by induction**

$$\boxed{\Big(P(1)\text{ is true}\Big)\wedge\Big(\forall n\in\mathbb{N}\quad P(n)\implies P(n+1)\Big)\quad\implies\quad\forall n\in\mathbb{N}\quad P(n)\text{ is true}}$$

Example.

1) For any $n \in \mathbb{N}$ the number $5^{n+2} + 6^{2n+1}$ is divisible by 31
2) Prove by induction that for any $n \in \mathbb{N}$ $\quad 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

Proof.

1) Base of the induction $P(1)$. Let us verify that for $n = 1$ the statement is true. Indeed,

$$5^{1+2} + 6^{2\cdot 1+1} = 5^3 + 6^3 = 125 + 216 = 341 \text{ — is divisible by } 31 \quad (341 : 31 = 11)$$

Inductional step $P(n) \Rightarrow P(n+1)$. Let us take some arbitrary $n \in \mathbb{N}$ and let us assume

$$5^{n+2} + 6^{2n+1} \text{ is devisable by } 31 \qquad \text{— we **assume** that this is true}$$

Basing on this information, let us try to derive that

$$5^{(n+1)+2} + 6^{2(n+1)+1} = 5^{n+3} + 6^{2n+3} \text{ is divisible by } 31 \qquad \text{— we **want** to prove this}$$

Indeed, we have

$$5^{n+3} + 6^{2n+3} = 5 \cdot 5^{n+2} + 36 \cdot 6^{2n+1} = \underbrace{5 \cdot 5^{n+2} + 6 \cdot 6^{2n+1}}_{\text{is divisible by 31}} + \underbrace{31 \cdot 6^{2n+1}}_{\text{is divisible by 31}}$$

So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n \in \mathbb{N}$. $\square$

2) Base of the induction $P(1)$. Let us verify that for $n = 1$ the statement is true. Indeed,

$$1 = \frac{1 \cdot (1+1)}{2} \qquad \text{— is true}$$

Inductional step $P(n) \Rightarrow P(n+1)$. Let us assume that for some $n \in \mathbb{N}$

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \qquad \text{— we **assume** that this is true}$$

and basing on this information let us try to derive that

$$1 + 2 + \ldots + n + (n+1) = \frac{(n+1)\big((n+1)+1\big)}{2} \qquad \text{— we **want** to prove this}$$

Indeed, using the inductional assumption we can transform

$$\underbrace{1 + 2 + \ldots + n}_{=\frac{n(n+1)}{2}} + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)\Big(\frac{n}{2} + 1\Big) = \frac{(n+1)(n+2)}{2}$$

So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n \in \mathbb{N}$. $\square$

5. **Sets**

- $x \in A$ and $B \subset A$ — elements and subsets
- Barber paradox, universal set X
- Equality of sets: $A = B \;\Leftrightarrow\; (A \subset B) \wedge (B \subset A)$, proofs of set identities
- Operations with sets: $A \cup B$, $A \cap B$, $A \setminus B$, $A' = cA = X \setminus A$
- Infinite union and intersection
- Cartesian product

EXAMPLE. For any an indexed family of subsets $\{A_i\}_{i \in I}$ of some universal set X the following equalities hold:

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \setminus A_i) \qquad X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i)$$

6. **Relations**

- $a, b \in A$, $\; a \overset{\mathcal{R}}{\sim} b$
- Examples: $a \leq b$ (real numbers), $a \,\vdots\, b$ (integers), $a \perp b$ (lines on the plane) etc
- Reflexive, symmetric, transitive relations
- Equivalence relation is a relation which is reflexive, symmetric, transitive
- Formally: relation $\mathcal{R}$ is some subset of $A \times A$. Then $a \overset{\mathcal{R}}{\sim} b \;\Leftrightarrow\; (a, b) \in \mathcal{R}$

7. **Functions**

DEFINITION. Let A and B be any non-empty sets.

- A *function* (or a *map*) $f : A \to B$ is a rule which associate with each member of A a unique element of B.
- A is called the *domain* of f (we denote the domain $D(f)$), and B is called *codomain*.
- Functions f and g are called *equal* if $D(f) = D(g)$ and $\forall\, a \in D(f)\; f(a) = g(a)$.
- g is an *extension* of f if $D(f) \subset D(g)$ and $\forall\, a \in D(f)\; f(a) = g(a)$.

8. **Domain and range of a function**

DEFINITION. If $f : A \to B$ is a function then

$$\begin{aligned} D(f) &:= A \quad \text{is the } \textit{domain} \text{ of } f \\ R(f) &:= \big\{\, b \in B \;\big|\; \exists\, a \in A : \; b = f(a) \,\big\} \quad \text{is the } \textit{range} \text{ of } f \end{aligned}$$

If $f(a) = b$ then we say b is the *image* of a and a is a *pre-image* of b.

9. **Injection, surjection and bijection functions**

   DEFINITION. Let $f : A \to B$ be a function.

   - f is *injection* $\iff$ f is one-to-one $\iff$ $\forall\, a_1,\, a_2 \in A$ if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$
   - f is *surjection* $\iff$ f is onto $\iff$ $R(f) = B$ $\iff$ $\forall\, b \in B\ \ \exists\, a \in A$: $f(a) = b$
   - f is a *bijection* $\iff$ f is both an injection and a surjection

10. **Countable sets**

   - The set A is *countable* $\iff$ $\exists$ a bijection $A \leftrightarrow \mathbb{N}$
   - The set of all rational numbers $\mathbb{Q}$ is countable

11. **Inverse function**

   DEFINITION. A function $f : A \to B$ is *invertible* on A if f is an injection on A. If f is invertible on $D(f)$ then the *inverse function* to f is defined by

   $$g := f^{-1}, \quad D(g) = R(f), \quad g(b) = a \quad \iff \quad b = f(a), \quad \forall\, a \in D(f), \quad \forall\, b \in R(f)$$

12. **Composition of functions**

   DEFINITION. Assume functions $f : A \to E$ and $g : B \to C$ satisfy $R(f) \subset B$. A *composition* of f and g is the function $h : A \to C$ defined by

   $$h := g \circ f, \qquad D(h) = D(f) = A, \qquad h(a) \;=\; g(f(a)), \qquad a \in A$$

13. **Real numbers**

   - Common sense concept: infinite decimal fractions
   - More formal: the real number can be associated with certain sequences of rational numbers $\{r_n\}$ which "approximate" (in a certain sense) this real number.
   - Axioms of $\mathbb{R}$: operations $x + y$ and $x \cdot y$ possess commutativity, associativity, existence on neutral element, existence of inverse, distributivity, relation $x \leq y$
   - Intervals (a, b), $[a, b)$, $[a, b]$, $(-\infty, b)$, $[a, +\infty)$ etc

14. **Bounded sets, lower and upper bounds**

   DEFINITION.

   - $M \in \mathbb{R}$ is an *upper bound* of a set $A \subset \mathbb{R}$ $\iff$ $\forall\, x \in A \quad x \leq M$
   - $m \in \mathbb{R}$ is a *lower bound* of a set $A \subset \mathbb{R}$ $\iff$ $\forall\, x \in A \quad x \geq m$
   - $A \subset \mathbb{R}$ is *bounded from above* $\iff$ $\exists$ at least one upper bound of A
   - $A \subset \mathbb{R}$ is *bounded from below* $\iff$ $\exists$ at least one lower bound of A
   - $A \subset \mathbb{R}$ is *bounded* $\iff$ A is bounded both from above and from below

15. **Supremum and infimum**

DEFINITION.

$M \in \mathbb{R}$ is the *supremum* of a set $A \subset \mathbb{R} \iff$

1) M is an upper bound of A
2) if $M' \in \mathbb{R}$ is some other upper bound of A then $M \leq M'$

$m \in \mathbb{R}$ is the *infimum* of a set $A \subset \mathbb{R} \iff$

1) m is a lower bound of A
2) if $m' \in \mathbb{R}$ is some other lower bound of A then $m \geq m'$

NOTATION.

$\sup A$ is the supremum of A, $\quad \sup A = +\infty \iff A$ is not bounded from above

$\inf A$ is the infimum of A, $\quad \inf A = -\infty \iff A$ is not bounded from below

16. **Characterization of supremum and infimum using quantifiers**

THEOREM.

$M \in \mathbb{R}$ is the supremum of $A \subset \mathbb{R} \iff$

1) $\forall x \in A \quad x \leq M$
2) $\forall \varepsilon > 0 \quad \exists x_\varepsilon \in A\colon \quad M - \varepsilon < x_\varepsilon$

$m \in \mathbb{R}$ is the infimum of $A \subset \mathbb{R} \iff$

1) $\forall x \in A \quad m \leq x$
2) $\forall \varepsilon > 0 \quad \exists x_\varepsilon \in A\colon \quad x_\varepsilon < m + \varepsilon$

PROBLEM. Find the supremum and the infimum of the set $\left\{ \frac{2nm}{n^2+m^2} \;\middle|\; n, m \in \mathbb{N} \right\} \subset \mathbb{R}$

17. **Least upper bound axiom**

AXIOM OF THE SET $\mathbb{R}$.

$A \subset \mathbb{R}$ is bounded from above $\implies \exists$ the number $M \in \mathbb{R}$ which is the supremum of A

$A \subset \mathbb{R}$ is bounded from below $\implies \exists$ the number $m \in \mathbb{R}$ which is the infimum of A

REMARK. The least upper bound axiom does not hold for the set of rational numbers $\mathbb{Q}$:

$$A = \{ r \in \mathbb{Q} : \; r^2 < 2 \}, \quad A \text{ is bounded from above, BUT } \quad \nexists\, q \in \mathbb{Q} : \; q = \sup A$$

18. **Functions of a real variable**

DEFINITION. $f : D(f) \subset \mathbb{R} \to \mathbb{R}$

- Admissible set (maximal domain) of an algebraic expression

- Graph of a function of a real variable
- Range of a function of a real variable

19. **Monotone functions**

    DEFINITION. Assume $f : [a, b] \to \mathbb{R}$.

    - f — $\nearrow$ on $[a, b] \iff \forall x_1, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) \le f(x_2)$
    - f — $\uparrow$ on $[a, b] \iff \forall x_1, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) < f(x_2)$
    - f — $\searrow$ on $[a, b] \iff \forall x_1, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) \ge f(x_2)$
    - f — $\downarrow$ on $[a, b] \iff \forall x_1, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) > f(x_2)$

20. **Invertibility of strictly monotone functions**

    THEOREM. A strictly monotone function $f : D(f) \subset \mathbb{R} \to \mathbb{R}$ in invertible on the interval of its monotonicity and the inverse function f^{-1} is strictly monotone in the same sense to f, i.e.

    - f — $\uparrow$ on $D(f) \implies f^{-1}$ — $\uparrow$ on $R(f)$
    - f — $\downarrow$ on $D(f) \implies f^{-1}$ — $\downarrow$ on $R(f)$

1.2 Limit of a sequence

1. **Definition of a sequence**

   Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ is a map (a function) $f : \mathbb{N} \to \mathbb{R}$, $x_n := f(n)$

2. **Informal notion helping to understand quantifiers in the definition of a limit**

   We say an interval (α, β) is a "feeder" for $\{x_n\}_{n=1}^{\infty} \iff \forall N \in \mathbb{N}\ \exists n \geq N$: $x_n \in (\alpha, \beta)$
   (i.e. the sequence $\{x_n\}_{n=1}^{\infty}$ "returns" to the interval (α, β) infinitely many times).

   We say an interval (α, β) is a "catcher" for $\{x_n\}_{n=1}^{\infty} \iff \exists N \in \mathbb{N}$: $\forall n \geq N$: $x_n \in (\alpha, \beta)$
   (i.e. once entering the interval (α, β), the sequence $\{x_n\}_{n=1}^{\infty}$ can never get out of it).

   We say the sequence $\{x_n\}_{n=1}^{\infty}$ is *bounded* $\iff$ $\exists\ M \geq 0$: $|x_n| \leq M\ \ \forall n \in \mathbb{N}$.
   (i.e. there exists at least one "catcher" for a sequence $\{x_n\}_{n=1}^{\infty}$).

3. **Limit of a sequence**

   Definition. $a \in \mathbb{R}$ is the *limit* of a sequence $\{x_n\}_{n=1}^{\infty}$ (equivalently, the sequence $\{x_n\}_{n=1}^{\infty}$ *converges* to $a \in \mathbb{R}$) iff

   $$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) : \quad \forall n \geq N \qquad |x_n - a| < \varepsilon$$

   i.e. for any $\varepsilon > 0$ the interval $(a - \varepsilon, a + \varepsilon)$ is a "catcher" for a sequence $\{x_n\}_{n=1}^{\infty}$.

   Notation. $a = \lim\limits_{n\to\infty} x_n$ or $x_n \underset{n\to\infty}{\longrightarrow} a$.

4. **Infinitesimal sequences**

   Definition.
   A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is called *infinitesimal* $\iff$ $\lim\limits_{n\to\infty} \alpha_n = 0$
   A sequence $\{x_n\}_{n=1}^{\infty}$ is called *infinitely large* $\iff$ $\lim\limits_{n\to\infty} |x_n| = +\infty$ $\iff$

   $$\forall M > 0 \quad \exists N = N(M) : \quad \forall n \geq N \quad |x_n| \geq M$$

5. **Characterization of a limit in terms of infinitesimal**

   Theorem. Assume $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $a \in \mathbb{R}$. Then

   $$\exists \lim_{n\to\infty} x_n = a \quad \iff \quad \exists \text{ an infinitesimal } \{\alpha_n\}_{n=1}^{\infty} \text{ such that } \forall n \in \mathbb{N} \quad x_n = a + \alpha_n$$

6. **Properties of infinitesimal sequences**

1) $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are infinitesimal $\implies$ $\{\alpha_n + \beta_n\}_{n=1}^{\infty}$ is infinitesimal

2) $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal, $\{x_n\}_{n=1}^{\infty}$ is bounded $\implies$ $\{\alpha_n x_n\}_{n=1}^{\infty}$ is infinitesimal

3) $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal, $\alpha_n \neq 0$, $\forall\, n \in \mathbb{N}$ $\iff$ $\{\frac{1}{\alpha_n}\}_{n=1}^{\infty}$ is infinitely large

7. **Basic properties of limits**

Theorem 1. The limit of a sequence, if exists, is unique.

Theorem 2. $\exists \lim\limits_{n\to\infty} x_n = a$ (a is a finite real number) $\implies$ $\{x_n\}_{n=1}^{\infty}$ is bounded.

Theorem 3. $\exists \lim\limits_{n\to\infty} x_n = a$, $\exists \lim\limits_{n\to\infty} y_n = b$, $\forall n \in \mathbb{N}$ $x_n \leq y_n$ $\implies$ $a \leq b$

Theorem 4. $\forall n \in \mathbb{N}$ $x_n \leq y_n \leq z_n$, $\exists \lim\limits_{n\to\infty} x_n = a$, $\exists \lim\limits_{n\to\infty} z_n = a$ $\implies$ $\exists \lim\limits_{n\to\infty} y_n = a$

Corollary. $\{\beta_n\}_{n=1}^{\infty}$ is infinitesimal, $|\alpha_n| \leq |\beta_n|$ $\implies$ $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal.

8. **Arithmetic operations with limits**

Theorem 1. $\exists \lim\limits_{n\to\infty} x_n = a$ $\exists \lim\limits_{n\to\infty} y_n = b$ $\implies$ $\exists \lim\limits_{n\to\infty} (x_n \pm y_n) = a \pm b$

Theorem 2. $\exists \lim\limits_{n\to\infty} x_n = a$ $\exists \lim\limits_{n\to\infty} y_n = b$ $\implies$ $\exists \lim\limits_{n\to\infty} (x_n y_n) = ab$

Lemma. $\exists \lim\limits_{n\to\infty} y_n = b$ and $y_n \neq 0$, $b \neq 0$ $\implies$ $\{\frac{1}{y_n}\}_{n=1}^{\infty}$ is bounded.

Theorem 3. $\exists \lim\limits_{n\to\infty} x_n = a$ $\exists \lim\limits_{n\to\infty} y_n = b$ and $y_n \neq 0$, $b \neq 0 \implies \exists \lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{a}{b}$

9. **Monotone sequences**

Definition.

- $\{x_n\}_{n=1}^{\infty}$ — $\nearrow$ $\iff$ $\forall\, n \in \mathbb{N}$ $x_n \leq x_{n+1}$
- $\{x_n\}_{n=1}^{\infty}$ — $\searrow$ $\iff$ $\forall\, n \in \mathbb{N}$ $x_n \geq x_{n+1}$

Theorem.

1) $\{x_n\}_{n=1}^{\infty}$ — $\nearrow$ $\exists\, M \in \mathbb{R}$: $\forall\, n \in \mathbb{N}$ $x_n \leq M$ $\implies$ $\exists \lim\limits_{n\to\infty} x_n \leq M$
2) $\{x_n\}_{n=1}^{\infty}$ — $\searrow$ $\exists\, m \in \mathbb{R}$: $\forall\, n \in \mathbb{N}$ $x_n \geq m$ $\implies$ $\exists \lim\limits_{n\to\infty} x_n \geq m$

10. **Important examples**

1) $q > 1 \implies \lim\limits_{n\to\infty} \frac{n}{q^n} = 0$

2) $q \in \mathbb{R} \implies \lim\limits_{n\to\infty} \frac{q^n}{n!} = 0$

3) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

4) $a > 0 \implies \lim\limits_{n\to\infty} \sqrt[n]{a} = 1$

11. **Euler's number e**

LEMMA 1. For all non-negative $a_1,\, a_2,\, a_3,\, \ldots a_n \in \mathbb{R}$

$$\underbrace{\sqrt[n]{a_1 a_2 a_3 \cdot \ldots \cdot a_n}}_{\text{Geometric mean}} \;\le\; \underbrace{\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}}_{\text{Arithmetic mean}}$$

LEMMA 2. Assume $a > 0$, $b > 0$. Then

$$\sqrt[n+1]{ab^n} \;\le\; \frac{a + nb}{n+1}$$

LEMMA 3. Denote $x_n := \left(1 + \frac{1}{n}\right)^n, \quad z_n := \left(1 - \frac{1}{n}\right)^n, \quad y_n := \left(1 + \frac{1}{n}\right)^{n+1}.$ Then

$$\forall\, n \in \mathbb{N} \qquad x_n < x_{n+1}, \qquad z_n < z_{n+1}, \qquad y_{n+1} < y_n$$

LEMMA 4. The following limits exist and coincide

$$\lim_{n\to\infty} x_n \;=\; \lim_{n\to\infty} y_n$$

DEFINITION. Euler's number e is defined by

$$e \;:=\; \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n, \qquad (e \approx 2{,}718281828459045\ldots)$$

1.3 Subsequences

1. **Definition of a subsequence**

   DEFINITION. Assume $f : \mathbb{N} \to \mathbb{R}$ is a sequence and assume $\varphi : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Denote by $h := f \circ \varphi$ the comosition $h(k) = f(\varphi(k))$. Then the sequence $h : \mathbb{N} \to \mathbb{R}$ is called a *subsequence* of the sequence $f : \mathbb{N} \to \mathbb{R}$.

   $$x_n := f(n), \qquad x_{n_k} = h(k), \qquad n_k := \varphi(k), \qquad n_1 < n_2 < \ldots < n_k < \ldots$$

   NOTATION. $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty} \iff \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$

2. **Any subsequence of a convergent sequence converges to the same limit**

   THEOREM. $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, $\exists \lim\limits_{n\to\infty} x_n = a \implies \exists \lim\limits_{k\to\infty} x_{n_k} = a$

3. **Nested interval theorem**

   THEOREM. Assume a sequence of closed intervals $[a_n, b_n] \subset \mathbb{R}$ satisfies the properties

   1) $\forall\, n \in \mathbb{N} \quad [a_{n+1}, b_{n+1}] \subset [a_n, b_n]$
   2) $b_n - a_n \to 0 \quad \text{as} \quad n \to \infty$

   Then $\exists !\; c \in \mathbb{R}$: $c \in \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$

4. **Bolzano–Weierstrass theorem**

   THEOREM. Each bounded sequence has a convergent subsequence:

   $$\{x_n\}_{n=1}^{\infty} \text{ is bounded} \implies \exists\, \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty} \text{ such taht } \exists\, a \in \mathbb{R}: \quad a = \lim_{k\to\infty} x_{n_k}$$

5. **Cauchy sequence**

   DEFINITION. $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is a *Cauchy sequence* (or a *fundamental sequence*) if

   $$\forall\, \varepsilon > 0 \quad \exists\, N(\varepsilon) \in \mathbb{N}: \quad \forall\, n, m \geq N(\varepsilon) \qquad |x_n - x_m| < \varepsilon$$

   THEOREM. Every fundamental sequence is convergent:

   $$\{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \text{ is a Cauchy sequence} \iff \exists\, a \in \mathbb{R}: \quad x_n \underset{n\to\infty}{\longrightarrow} a$$

1.4 Limit of a function

1. **Definition of the limit of a function according to Cauchy**

   DEFINITION (C). Assume $f : (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$, $y_0 \in \mathbb{R}$. We write

   $$\lim_{x\to x_0} f(x) = y_0 \qquad \text{or} \qquad f(x) \underset{x\to x_0}{\to} y_0$$

   and say y_0 is the *limit* of $f(x)$ as x tends to x_0 iff

   $$\forall\, \varepsilon > 0 \;\; \exists\, \delta = \delta(\varepsilon) > 0 : \;\forall\, x \in (a,b) \quad (x \neq x_0) \;\wedge\; |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - y_0| < \varepsilon$$

   The set $\overset{\circ}{U}_\delta(x_0) := (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ is called a *deleted neighbourhood* of x_0.

2. **Definition of the limit of a function according to Heine**

   DEFINITION (H). Assume $f : (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$, $y_0 \in \mathbb{R}$. We write

   $$\lim_{x\to x_0} f(x) = y_0 \qquad \text{or} \qquad f(x) \underset{x\to x_0}{\to} y_0$$

   and say y_0 is the *limit* of $f(x)$ as x tends to x_0 iff

   $$\forall\, \{x_n\}_{n=1}^{\infty} \subset (a,b) \setminus \{x_0\} : \qquad \lim_{n\to\infty} x_n = x_0 \quad \Longrightarrow \quad \lim_{n\to\infty} f(x_n) = y_0$$

3. **Equivalence of definitions**

   THEOREM. Definition (C) $\Longleftrightarrow$ Definition (H)

4. **Arithmetic operations with limits of functions**

   THEOREM 1. $\exists \lim\limits_{x\to x_0} f(x), \exists \lim\limits_{x\to x_0} g(x) \quad \Longrightarrow \quad \exists \lim\limits_{x\to x_0} (f(x) + g(x)) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x)$

   THEOREM 2. $\exists \lim\limits_{x\to x_0} f(x), \exists \lim\limits_{x\to x_0} g(x) \quad \Longrightarrow \quad \exists \lim\limits_{x\to x_0} (f(x) \cdot g(x)) = \lim\limits_{x\to x_0} f(x) \cdot \lim\limits_{x\to x_0} g(x)$

   LEMMA. $\lim\limits_{x\to x_0} g(x) > 0 \quad \Longrightarrow \quad \exists\, \delta > 0\text{:} \quad |x - x_0| < \delta \;\Rightarrow\; g(x) > 0$

   THEOREM 3. $\exists \lim\limits_{x\to x_0} f(x), \exists \lim\limits_{x\to x_0} g(x) \neq 0 \quad \Longrightarrow \quad \exists \lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$

5. **One-sided limits**

   DEFINITION. Assume $f : [a,b] \to \mathbb{R}$, $x_0 \in [a,b]$.

   $\exists \lim\limits_{x\to x_0 - 0} f(x) = y_0 \quad \Longleftrightarrow \quad \forall\, \varepsilon > 0 \;\; \exists\, \delta > 0\text{: } x \in (x_0 - \delta, x_0) \;\Rightarrow\; |f(x) - y_0| < \varepsilon$

   $\exists \lim\limits_{x\to x_0 + 0} f(x) = y_0 \quad \Longleftrightarrow \quad \forall\, \varepsilon > 0 \;\; \exists\, \delta > 0\text{: } x \in (x_0, x_0 + \delta) \;\Rightarrow\; |f(x) - y_0| < \varepsilon$

   THEOREM. $\exists \lim\limits_{x\to x_0} f(x) = y_0 \quad \Longleftrightarrow \quad \exists \lim\limits_{x\to x_0 - 0} f(x) = \lim\limits_{x\to x_0 + 0} f(x) = y_0$

1.5 Indeterminate forms and important examples

1. **Types of indeterminate forms**

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty \quad 1^{\infty}, \quad \infty^0, \quad 0^0$$

2. **Auxiliary inequalities**

THEOREM.

1) $0 < x < \frac{\pi}{2} \implies \sin x < x < \operatorname{tg} x, \quad x \in \mathbb{R} \implies |\sin x| \le |x|$
2) $x_1,\, x_2 \in \mathbb{R} \implies |\sin x_1 - \sin x_2| \le |x_1 - x_2|, \quad |\cos x_1 - \cos x_2| \le |x_1 - x_2|$
3) $\lim\limits_{x\to 0} \sin x = 0, \quad \lim\limits_{x\to 0} \cos x = 1$

3. **Properties of a^x and $\log_a x$ which are postulated for now**

We postulate the properties of the exponential and the logarithmic functions which actually come together with the definition of the exponential function and its inverse. We will return to discussion of these properties in the section "Continuity". These properties also allowed to use without further explanation when you solve homework problems.

THEOREM. Assume $a > 0$ and $a \neq 1$. Then

1) $\forall \{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \quad x_n \to x_0 \quad \implies \quad a^{x_n} \to a^{x_0}$
2) $\forall \{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \quad x_n \to x_0, \quad x_0 > 0 \quad \implies \quad \log_a x_n \to \log_a x_0$

4. **Important examples**

1) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

2) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

3) $\lim\limits_{x\to +\infty} (1 + \frac{1}{x})^x = e \qquad \lim\limits_{x\to -\infty} (1 + \frac{1}{x})^x = e \qquad \lim\limits_{x\to 0} (1 + x)^{\frac{1}{x}} = e$

4) $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$

5) $\lim\limits_{x\to 0} \frac{a^x-1}{x} = \ln a$

6) $\lim\limits_{x\to 0} \frac{(1+x)^{\mu}-1}{x} = \mu$

1.6 Infinite limits and asymptotes

1. **Infinite limits, vertical asymptotes**

   Definition.

$$\lim_{x\to x_0-0} f(x) = +\infty \iff \forall\, M>0\ \exists\,\delta>0\colon\ x\in(x_0-\delta, x_0)\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to x_0+0} f(x) = +\infty \iff \forall\, M>0\ \exists\,\delta>0\colon\ x\in(x_0, x_0+\delta)\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to x_0-0} f(x) = -\infty \iff \forall\, M<0\ \exists\,\delta>0\colon\ x\in(x_0-\delta, x_0)\ \Rightarrow\ f(x)<M$$
$$\lim_{x\to x_0+0} f(x) = -\infty \iff \forall\, M<0\ \exists\,\delta>0\colon\ x\in(x_0, x_0+\delta)\ \Rightarrow\ f(x)<M$$

   Definition. The line $x = x_0$ is a *vertical asymptote* of the function $f:\mathbb{R}\to\mathbb{R}$ if at least one of the four statements is true:

$$\lim_{x\to x_0+0} f(x) = +\infty, \quad \lim_{x\to x_0-0} f(x) = +\infty, \quad \lim_{x\to x_0-0} f(x) = +\infty, \quad \lim_{x\to x_0-0} f(x) = -\infty$$

2. **Behavior of functions at infinity**

   Definition.

$$\lim_{x\to+\infty} f(x) = +\infty \iff \forall\, M>0\ \exists\, R>0\colon\ x>R\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to-\infty} f(x) = +\infty \iff \forall\, M>0\ \exists\, R<0\colon\ x<R\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to+\infty} f(x) = -\infty \iff \forall\, M<0\ \exists\, R>0\colon\ x>R\ \Rightarrow\ f(x)<M$$
$$\lim_{x\to-\infty} f(x) = -\infty \iff \forall\, M<0\ \exists\, R<0\colon\ x<R\ \Rightarrow\ f(x)<M$$

3. **Limits at infinity, horizontal asymptotes**

   Definition.

$$\lim_{x\to+\infty} f(x) = y_0 \iff \forall\,\varepsilon>0\ \exists\, R>0\colon\ x>R\ \Rightarrow\ |f(x)-y_0|<\varepsilon$$
$$\lim_{x\to-\infty} f(x) = y_0 \iff \forall\,\varepsilon>0\ \exists\, R<0\colon\ x<R\ \Rightarrow\ |f(x)-y_0|<\varepsilon$$

   Definition. The line $y = y_0$ is a *horizontal asymptote* of the function $f:\mathbb{R}\to\mathbb{R}$ if at least one of the two statements is true:

$$\lim_{x\to+\infty} f(x) = y_0 \quad \text{or} \quad \lim_{x\to-\infty} f(x) = y_0$$

4. **How to remember definitions of finite/infinite limits?**

$y_0 = \lim\limits_{x \to x_0} f(x)$

x_0 is finite or infinite
y_0 is finite or infinite

	Values of $f(x)$	Values of x	When is used?
Small number	$\forall\, \varepsilon > 0$	$\exists\, \delta > 0$	For a finite value
Large number	$\forall\, M > 0$	$\exists\, R > 0$	For an infinite value

5. **Oblique asymptotes**

Definition. The line $y = kx + b$ is an *oblique asymptote* of the function $f : \mathbb{R} \to \mathbb{R}$ if at least one of the two statements is true:

$$\lim_{x \to +\infty} \Big(f(x) - kx - b \Big) = 0 \quad \text{or} \quad \lim_{x \to -\infty} \Big(f(x) - kx - b \Big) = 0$$

Remark. A horizontal asymptote is a particular case of an oblique asymptote with $k = 0$.

Theorem. $y = kx + b$ is a horizontal or an oblique asymptote of $f : \mathbb{R} \to \mathbb{R}$ as $x \to +\infty$ iff the following two limits exist:

1) $\exists \lim\limits_{x \to +\infty} \frac{f(x)}{x} = k$
2) $\exists \lim\limits_{x \to +\infty} (f(x) - kx) = b$

The similar statements hold as $x \to -\infty$.

1.7 Continuity

1. **Continuity of a function at a point**

   DEFINITION. $f:(a,b)\to\mathbb{R}$ is *continuous* at a point $x_0 \iff \lim\limits_{x\to x_0} f(x)=f(x_0)$

   REMARK. The following statements are equivalent:

   1) $f:(a,b)\to\mathbb{R}$ is continuous at x_0
   2) $\forall\,\varepsilon>0\ \ \exists\,\delta(\varepsilon)>0$: $\quad |x-x_0|<\delta \quad\Longrightarrow\quad |f(x)-f(x_0)|<\varepsilon$
   3) $\forall\ \{x_n\}_{n=1}^{\infty}\subset(a,b)$: $\quad x_n\to x_0 \quad\Longrightarrow\quad f(x_n)\to f(x_0)$

   DEFINITION. $f:(a,b)\to\mathbb{R}$ is continuous *on* $(a,b) \iff \forall\,x_0\in(a,b)\ \ f$ is continuous at x_0

2. **Stability of sign theorem**

   THEOREM. f is continuous at x_0, $f(x_0)>0 \implies \exists\,\delta>0$: $\forall\,x\in(x_0-\delta,x_0+\delta)\ \ f(x)>0$

3. **Arithmetic operations with continuous functions**

   THEOREM.

   1) f, g are continuous at $x_0 \implies f\pm g$ are continuous at x_0
   2) f, g are continuous at $x_0 \implies f\cdot g$ is continuous at x_0
   3) f, g are continuous at x_0, $\ g(x_0)\neq 0 \implies f/g$ is continuous at x_0

4. **Left and right continuity**

   DEFINITION.

   $f:[a,b]\to\mathbb{R}$ is *left continuous* at $x_0 \iff \lim\limits_{x\to x_0-0} f(x)=f(x_0)$

   $f:[a,b]\to\mathbb{R}$ is *right continuous* at $x_0 \iff \lim\limits_{x\to x_0+0} f(x)=f(x_0)$

   $$f:[a,b]\to\mathbb{R} \text{ is continuous } \textit{on } [a,b] \iff \begin{cases} f \text{ is continuous on } (a,b)\\ f \text{ is left continuous at } a\\ f \text{ is right continuous at } b\end{cases}$$

   THEOREM. f is continuous at $x_0\in(a,b) \iff f$ is left and right continuous at x_0.

5. **Types of discontinuity**

   DEFINITION. Assume $x_0\in(a,b)$, $\ f:(a,b)\setminus\{x_0\}\to\mathbb{R}$

   - x_0 is a *removable discontinuity* $\iff \exists$ finite $\lim\limits_{x\to x_0\pm 0} f(x)$, $\ \lim\limits_{x\to x_0-0} f(x)=\lim\limits_{x\to x_0+0} f(x)$

- x_0 is a *jump discontinuity* $\iff$ $\exists$ both finite $\lim\limits_{x\to x_0\pm 0} f(x)$, $\lim\limits_{x\to x_0-0} f(x) \neq \lim\limits_{x\to x_0+0} f(x)$
- $x = x_0$ is a *vertical asymptote* $\iff$ $\exists \lim\limits_{x\to x_0\pm 0} f(x) = \pm\infty$ (at least one of limits)
- x_0 may be a discontinuity of no above type $\iff$ $\nexists \lim\limits_{x\to x_0\pm 0} f(x)$ $\quad (f(x) = \sin\frac{1}{x})$

6. **Continuity of some elementary functions**

- $f(x) = x^n$ is continuous on $\mathbb{R}$, $n \in \mathbb{N}$
- $f(x) = \frac{1}{x^n}$ is continuous on $\mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$
- $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is continuous on $\mathbb{R}$
- $f(x) = \sin x$, $g(x) = \cos x$ are continuous on $\mathbb{R}$
- $f(x) = \operatorname{tg} x$ is continuous at $x \neq \frac{\pi}{2} + \pi k$, $g(x) = \operatorname{ctg} x$ is continuous at $x \neq \pi k$, $k \in \mathbb{Z}$
- $f(x) = x^p$, $p \in (0, +\infty)$ is continuous on $[0, +\infty)$
- $f(x) = a^x$, $a > 0$, is continuous on $\mathbb{R}$

7. **Continuity of a composition of functions**

Theorem. $f : (a,b) \to \mathbb{R}$, $g : (c,d) \to \mathbb{R}$, $x_0 \in (a,b)$, $f(x_0) \in (c,d)$

$$f \text{ is continuous at } x_0, \quad g \text{ is continuous at } f(x_0) \quad \Longrightarrow \quad g \circ f \text{ is continuous at } x_0$$

8. **Two intermediate value theorems (Bolzano–Cauchy theorems)**

Theorem 1. f is continuous on $[a,b]$, $f(a) \le 0$, $f(b) \ge 0$ $\Longrightarrow$ $\exists\, c \in [a,b]$: $f(c) = 0$

Theorem 2. f is continuous on $[a,b]$, $f(a) = y_1$, $f(b) = y_2$, $y_1 \le y_2$ $\Longrightarrow$ $[y_1, y_2] \subset R(f)$

9. **Continuity of the inverse function**

Theorem. Assume $f : (a,b) \to \mathbb{R}$ is continuous and strictly monotone on (a,b). Then f is invertible on (a,b) and f^{-1} is continuous on $R(f)$.

10. **Continuity of some elementary inverse functions**

- $f(x) = \sqrt[n]{x}$ is continuous on $D(f)$, $n \in \mathbb{N}$
- $f(x) = \log_a x$, $a > 0$, $a \neq 1$, is continuous on $(0, +\infty)$
- $f(x) = \arcsin x$, $g(x) = \arccos x$ are continuous on $[-1, 1]$
- $f(x) = \operatorname{arctg} x$, $g(x) = \operatorname{arcctg} x$ are continuous on $\mathbb{R}$

11. **Two extreme value theorems (Weierstrass theorems)**

THEOREM 1. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\Longrightarrow$ f is bounded on $[a,b]$, i.e.

$$\exists\, M > 0 : \qquad \forall\, x \in [a,b] \qquad |f(x)| \le M$$

THEOREM 2. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\Longrightarrow$ f achieves on $[a,b]$ its maximum and minimal values, i.e. $\exists\ c_1,\, c_2 \in [a,b]$ such that

$$f(c_1) = \inf_{x\in[a,b]} f(x) \qquad \text{and} \qquad f(c_2) = \sup_{x\in[a,b]} f(x)$$

12. **Uniform continuity**

DEFINITION. A function $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* on $D(f)$ iff

$$\forall\, \varepsilon > 0 \quad \exists\, \delta(\varepsilon) > 0 : \quad \forall\, x_1, x_2 \in D(f) \qquad |x_1 - x_2| < \delta \quad \Longrightarrow \quad |f(x_1) - f(x_2)| < \varepsilon$$

13. **Heine–Cantor theorem**

THEOREM. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\Longrightarrow$ f is uniformly continuous on $[a,b]$.

1.8 Differentiability and derivatives

1. **Main definition**

   Definition. Assume $f : (a, b) \to \mathbb{R}$, $x_0 \in (a, b)$. A function f is *differentiable at* a point x_0 iff there exists a finite limit

   $$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

   The number $f'(x_0)$ is called the *derivative* of a function f at x_0. Another notation

   $$\frac{df}{dx}(x_0) \; := \; f'(x_0)$$

   - We say f is *differentiable on* (a, b) if for any $x_0 \in (a, b)$ the function f is differentiable at x_0.
   - We say f is *continuously differentiable* on (a, b) if f is differentiable on (a, b) and the function f' is continuous on (a, b).

2. **Physical and geometric meaning of the derivative**

   - From a physical point of view the derivative of a function is the instantaneous speed (or the rate of change) of the function.
   - From a geometrical point of view the derivative of a function f at a point x_0 is equal to the slope of the tangent line to the graph of the function at the point x_0.

3. **Differentiable function is continuous**

   Theorem. $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b) \quad \Longrightarrow \quad f$ is continuous at x_0

   Remark. The inverse is false: $f(x) = |x|$ is continuous but non-differentiable at 0.

4. **Basic properties of derivatives**

   Theorem . Assume $f,\ g : (a, b) \to \mathbb{R}$ are differentiable at $x \in (a, b)$. Then $f \pm g$ and fg are differentiable at x. If additionally $g(x) \neq 0$ then f/g is also differentiable at x. Besides,

   1) $(f \pm g)'(x) = f'(x) \pm g'(x)$

   2) $(cf)'(x) = cf'(x)$

   3) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

   4) $\left(\frac{f}{g}\right)'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

5. Derivatives of elementary functions

1. $(C)' = 0$

2. $(x)' = 1$

3. $(x^m)' = mx^{m-1}, \quad m \in \mathbb{Z}, \quad x \in \mathbb{R} \quad (x \neq 0 \text{ if } m < 0)$

4. $(x^r)' = rx^{r-1}, \quad r \in \mathbb{R}, \quad x > 0$

5. $(e^x)' = e^x, \qquad (a^x)' = a^x \ln a, \quad a > 0, \quad x \in \mathbb{R}$

6. $(\ln|x|)' = \dfrac{1}{x}, \quad x \neq 0, \qquad (\log_a |x|)' = \dfrac{1}{x \ln a}, \qquad a > 0, \quad a \neq 1$

7. $(\sin x)' = \cos x, \quad (\cos x)' = -\sin x$

8. $(\operatorname{tg} x)' = \dfrac{1}{\cos^2 x}, \quad (\operatorname{ctg} x)' = -\dfrac{1}{\sin^2 x}$

1.9 Chain rule

1. **Differential of a function**

   Definition. Assume $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$. A linear map $l_{x_0} : \mathbb{R} \to \mathbb{R}$,

   $$l_{x_0}(h) \ = \ k_{x_0}h, \qquad \forall\, h \in \mathbb{R}, \qquad (k_{x_0} \in \mathbb{R})$$

   is called the *differential* of f at a point x_0 iff there exists a number $h_0 > 0$ and a function $\alpha_{x_0} : (-h_0, h_0) \to \mathbb{R}$ such that

   $$\lim_{h\to 0} \alpha_{x_0}(h) = 0,$$

   $$f(x_0 + h) = f(x_0) + k_{x_0}h + \alpha_{x_0}(h)h, \qquad \forall\, |h| < h_0$$

   Notation. $\quad l_{x_0} := df(x_0) \quad \Longleftrightarrow \quad df(x_0)(h) \ := \ l_{x_0}(h) \ = \ k_{x_0}h, \quad \forall\, h \in \mathbb{R}$

2. **When the differential of a function is well-defined?**

   Theorem. Assume $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$. The differential of f at x_0 exists if and only if f is differentiable at x_0. Moreover, if $df(x_0)(h) = k_{x_0}h$ is the differential of f at x_0 then

   $$k_{x_0} = f'(x_0) \qquad \Longleftrightarrow \qquad df(x_0)(h) = f'(x_0)h, \qquad \forall\, h \in \mathbb{R}$$

   where $f'(x_0)$ is the derivative of f at x_0.

3. **What is the difference between the derivative $f'(x_0)$ and the differential $df(x_0)$?**

   It is well-known that any linear map $l : \mathbb{R} \to \mathbb{R}$ has the form

   $$l(x) = kx, \qquad x \in \mathbb{R}$$

   i.e. any linear map from $\mathbb{R}$ to $\mathbb{R}$ can be characterized in the unique way by the coefficient $k \in \mathbb{R}$ which is called a *slope* of the linear map $l(x)$. The theorem above shows that the slope of the differential $df(x_0)$ is equal to $f'(x_0)$. So, the differential and the derivative of a function are related in the same way as a linear transformation $l(x) = kx$ is related to its slope $k \in \mathbb{R}$.

   Notation. Very often people write $df(x_0) = f'(x_0)dx$ meaning that $f'(x_0)$ is a slope of a linear function $df(x_0)$ and dx stands for the identity linear map, i.e. $dx(h) = h$.

4. **Chain rule**

   Theorem. Assume $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$, $g : (c, d) \to \mathbb{R}$, $R(f) \subset (c, d)$, $x_0 \in (a, b)$. If f is differentiable at x_0 and g is differentiable at $y_0 = f(x_0)$ then the composition $\varphi := g \circ f$ is differentiable at x_0 and

   $$\varphi'(x_0) = g'(y)\Big|_{y=f(x_0)} \cdot f'(x_0)$$

5. **Inverse function rule**

THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is strictly monotone and continuous (so the inverse function f^{-1} exists, f^{-1} is strictly monotone in the same sense to f and continuous on $R(f)$). Assume additionally f is differentiable at $x_0 \in (a, b)$ and $f'(x_0) \neq 0$. Then the function $g := f^{-1}$ is differentiable at $y_0 := f(x_0)$ and

$$g'(y_0) = \frac{1}{f'(x_0)}$$

6. **Derivatives of inverse trigonometric functions**

9. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}, \qquad (\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

10. $(\operatorname{arctg} x)' = \dfrac{1}{1+x^2}, \qquad (\operatorname{arcctg} x)' = -\dfrac{1}{1+x^2}$

1.10 The basic theorems of differential calculus

1. **Fermat's theorem (or Fermat's lemma)**

   THEOREM. Assume $f : (a,b) \to \mathbb{R}$ has a local extremum (maximum or minimum) on the interval (a,b) at some internal point $c \in (a,b)$, i.e.

   $$\begin{array}{rlcl} \text{either} & f(c) = \max\limits_{x\in(a,b)} f(x) & \iff & \forall\, x \in (a,b) \quad f(x) \le f(c) \\ \text{or} & f(c) = \min\limits_{x\in(a,b)} f(x) & \iff & \forall\, x \in (a,b) \quad f(x) \ge f(c) \end{array}$$

   If f is differentiable at c then $f'(c) = 0$.

2. **Rolle's theorem**

   THEOREM. Assume $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Assume that $f(a) = f(b)$. Then there exists $c \in (a,b)$ such that $f'(c) = 0$.

3. **Lagrange's theorem**

   THEOREM. Assume $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there is a point $c \in (a,b)$ such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

4. **Cauchy's theorem**

   THEOREM. Assume functions f, $g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). Assume $g'(x) \neq 0$ for any $x \in (a,b)$. Then there is a point $c \in (a,b)$ such that

   $$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

5. **Functions with identically zero derivatives are constants**

   THEOREM. Assume $f : (a,b) \to \mathbb{R}$ is differentiable on (a,b) and $f'(x) = 0$ for any $x \in (a,b)$. Then f is equal to a constant, i.e.

   $$\exists\, y_0 \in \mathbb{R} : \qquad f(x) = y_0 \qquad \forall\, x \in (a,b)$$

1.11 Investigation of functions using derivatives

1. **Investigation of the monotonicity of functions**

   THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Then

   1) $f'(x) \geq 0 \;\; \forall x \in (a,b) \iff f - \nearrow$ (is non-decreasing) on (a, b)
   2) $f'(x) \leq 0 \;\; \forall x \in (a,b) \iff f - \searrow$ (is non-increasing) on (a, b)
   3) $f'(x) > 0 \;\; \forall x \in (a,b) \implies f - \uparrow$ (is strictly increasing) on (a, b)
   4) $f'(x) < 0 \;\; \forall x \in (a,b) \implies f - \downarrow$ (is strictly decreasing) on (a, b)

2. **Conditions for extreme in terms of the first derivative**

   THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Then and assume $c \in (a, b)$ satisfies $f'(c) = 0$. Then

   1) if $f'(x) \leq 0 \; \forall \, x \in (a, c)$ and $f'(x) \geq 0 \; \forall \, x \in (c, b)$ then c is the point of minimum of f
   2) if $f'(x) \geq 0 \; \forall \, x \in (a, c)$ and $f'(x) \leq 0 \; \forall \, x \in (c, b)$ then c is the point of maximum of f

3. **Higher order derivatives**

   DEFINITION. Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). If the function $g = f' : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ then the function f is called *twice differentiable* at x_0. In this case the derivative of the function $g = f'$ at x_0 is called the *second derivative* of f at x_0.

   $$f''(x_0) \;:=\; g'(x_0) \;=\; \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

   The higher-order derivatives are defined similarly:

   $$f''' = (f'')', \quad \ldots, \quad f^{(k)} = (f^{(k-1)})', \quad \ldots$$

   ALTERNATIVE NOTATION.

   $$\frac{d^2 f}{dx^2}(x) = f''(x), \qquad \frac{d^3 f}{dx^3}(x) = f'''(x), \quad \ldots \quad \frac{d^k f}{dx^k}(x) = f^{(k)}(x)$$

4. **Conditions for extreme in terms of the second derivative**

   THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Let f be twice differentiable at $c \in (a, b)$ and $f'(c) = 0$. Then

   1) $f''(c) > 0 \implies c$ is a point of a local min of $f \;\Leftrightarrow\; \exists \, \delta > 0\colon\; f(c) = \min\limits_{x \in (x_0 - \delta, x_0 + \delta)} f(x)$
   2) $f''(c) < 0 \implies c$ is a point of a local max of $f \;\Leftrightarrow\; \exists \, \delta > 0\colon\; f(c) = \max\limits_{x \in (x_0 - \delta, x_0 + \delta)} f(x)$

5. **Convex functions**

DEFINITION. $f : [a,b] \to \mathbb{R}$ is *convex* on (a,b) $\iff$ $\forall\ x_1,\ x_2 \in (a,b),\ \forall\ \lambda \in [0,1]$

$$f\big((1-\lambda)x_1 + \lambda x_2\big)\ \le\ (1-\lambda)f(x_1) + \lambda f(x_2)$$

$f : [a,b] \to \mathbb{R}$ is *strictly convex* on (a,b) $\iff$ $\forall\ x_1 \ne x_2 \in (a,b),\ \forall\ \lambda \in (0,1)$

$$f\big((1-\lambda)x_1 + \lambda x_2\big)\ <\ (1-\lambda)f(x_1) + \lambda f(x_2)$$

$f : [a,b] \to \mathbb{R}$ is *concave* on (a,b) $\iff$ $\forall\ x_1,\ x_2 \in (a,b),\ \forall\ \lambda \in [0,1]$

$$f\big((1-\lambda)x_1 + \lambda x_2\big)\ \ge\ (1-\lambda)f(x_1) + \lambda f(x_2)$$

$f : [a,b] \to \mathbb{R}$ is *strictly concave* on (a,b) $\iff$ $\forall\ x_1 \ne x_2 \in (a,b),\ \forall\ \lambda \in (0,1)$

$$f\big((1-\lambda)x_1 + \lambda x_2\big)\ >\ (1-\lambda)f(x_1) + \lambda f(x_2)$$

6. **Inequalities for slopes of secants**

THEOREM.

1) f is convex on (a,b) $\iff$ $\forall\ x_1,\ x_2 \in (a,b),\ \ x_1 < x_2,\ \ \forall\ x \in (x_1, x_2)$

$$\frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x_2) - f(x)}{x_2 - x}$$

2) f is strictly convex on (a,b) $\iff$ $\forall\ x_1,\ x_2 \in (a,b),\ \ x_1 < x_2,\ \ \forall\ x \in (x_1, x_2)$

$$\frac{f(x) - f(x_1)}{x - x_1} < \frac{f(x_2) - f(x)}{x_2 - x}$$

3) Let f be differentiable on (a,b). Then f is convex on (a,b) $\iff$ $\forall\ x_1,\ x_2 \in (a,b)$

$$x_1 < x_2 \quad \Longrightarrow \qquad f'(x_1)\ \le\ \frac{f(x_2) - f(x_1)}{x_2 - x_1}\ \le\ f'(x_2)$$

7. **Investigation of convexity using the first derivative**

THEOREM. Let $f : (a,b) \to \mathbb{R}$ be differentiable on (a,b).

1) f is convex on (a,b) $\iff$ f' — $\nearrow$ on (a,b)
2) f is concave on (a,b) $\iff$ f' — $\searrow$ on (a,b)
3) f is strictly convex on (a,b) $\iff$ f' — $\uparrow$ on (a,b)
4) f is strictly concave on (a,b) $\iff$ f' — $\downarrow$ on (a,b)

8. **Investigation of convexity using the second derivative**

THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is twice differentiable on (a, b).

1) $\forall\, x \in (a, b) \quad f''(x) \geq 0 \iff f$ is convex on (a, b)
2) $\forall\, x \in (a, b) \quad f''(x) \leq 0 \iff f$ is concave on (a, b)
3) $\forall\, x \in (a, b) \quad f''(x) > 0 \implies f$ is strictly convex on (a, b)
4) $\forall\, x \in (a, b) \quad f''(x) < 0 \implies f$ is strictly concave on (a, b)

9. **Tangent line to the graph of a function**

DEFINITION. Assume $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$. The straight line given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is called the *tangent* to the function $y = f(x)$ (or to the graph of f) at the point x_0.

10. **Convexity and tangent lines**

THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b).

1) f is convex on $(a, b) \iff$ for any $x_0 \in (a, b)$ the graph of the function f lies above the tangent to f at the point x_0, i.e.

$$\forall x \in (a, b) \quad f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

2) f is concave on $(a, b) \iff$ for any $x_0 \in (a, b)$ the graph of the function f lies below the tangent to f at the point x_0, i.e.

$$\forall x \in (a, b) \quad f(x) \leq f(x_0) + f'(x_0)(x - x_0)$$

3) f is strictly convex on $(a, b) \iff \forall\, x_0 \in (a, b),\ \forall\, x \in (a, b)$,

$$x \neq x_0 \quad \implies \quad f(x) > f(x_0) + f'(x_0)(x - x_0)$$

4) f is strictly concave on $(a, b) \iff \forall\, x_0 \in (a, b),\ \forall\, x \in (a, b)$,

$$x \neq x_0 \quad \implies \quad f(x) < f(x_0) + f'(x_0)(x - x_0)$$

11. **Inflection points**

DEFINITION. Assume $f : (a, b) \to \mathbb{R}$. The point $c \in (a, b)$ is called the *inflection point* of the function f if $\exists\, \delta > 0$, such that on the intervals $(c - \delta, c)$ and $(c, c + \delta)$ the function f changes convexity (was convex, became concave, or vice versa).

12. **Necessary condition for an inflection point**

THEOREM. Let $f : (a, b) \to \mathbb{R}$ be twice differentiable on (a, b) and let $c \in (a, b)$ be the inflection point of f. Then

$$f''(c) = 0.$$

1.12 Primitive function and indefinite integral

1. **Primitive function (antiderivative)**

   Definition. $F : (a, b) \to \mathbb{R}$ is called a *primitive function* (or an *antiderivative*) of the function $f : (a, b) \to \mathbb{R}$ on the interval (a, b) if F is differentiable on (a, b) and

   $$F'(x) = f(x), \qquad \forall\, x \in (a, b).$$

2. **Which functions possess antiderivatives?**

   This is a non-trivial question. In the section "Riemann integral" we will show that at least any *continuous* function possesses antiderivatives on the interval of its continuity.

   Discussing antiderivatives below we will always assume that all functions involved in the formulas possess the corresponding antiderivatives by assumption and we will focus only on the derivation of relations between these antiderivatives.

3. **Properties of primitives**

   Theorem. Assume $f : (a, b) \to \mathbb{R}$ has a primitive on (a, b). Note that in this case f has infinitely many other primitives on (a, b), but any two primitives of f can differ only by a constant:

   1) if $F(x)$ is a primitive of $f(x)$ on (a, b) then for any $c \in \mathbb{R}$ the function $F(x) + c$ is also a primitive of $f(x)$ on (a, b)
   2) if F_1 and F_2 are two primitives of f on (a, b) then there exists $c \in \mathbb{R}$ such that

   $$\forall\, x \in (a, b) \qquad F_1(x) - F_2(x) = c$$

4. **Indefinite integral**

   Definition. The set $\{F(x) + C\}$ of all primitives of the function $f : (a, b) \to \mathbb{R}$, defined on the interval (a, b), is called the indefinite integral of f on (a, b) and is denoted by

   $$\int f(x)\ dx$$

5. **Properties of the indefinite integral**

   Theorem.

   $$\int \Big(\alpha f(x) + \beta g(x)\Big)\, dx \;=\; \alpha \int f(x)\, dx \,+\, \beta \int g(x)\, dx, \qquad \forall\, \alpha, \beta \in \mathbb{R}$$

   $$\frac{d}{dx} \int f(x)\ dx \;=\; f(x), \qquad \int g'(x)\, dx \;=\; g(x) + C$$

6. **List of simplest antiderivatives**

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$
2. $\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int a^x \, dx = \frac{a^x}{\ln a} + C \quad (a > 0,\, a \neq 1)$
5. $\int \sin x \, dx = -\cos x + C, \quad \int \cos x \, dx = \sin x + C$
6. $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C, \quad \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$
7. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C \quad (a \neq 0)$
8. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C \quad (a > 0)$

7. **Antiderivatives computed by standard methods (see the next section)**

1. $\int \operatorname{tg} x \, dx = -\ln|\cos x| + C, \quad \int \operatorname{ctg} x \, dx = \ln|\sin x| + C$
2. $\int \frac{x\,dx}{a^2 \pm x^2} = \pm\frac{1}{2} \ln|a^2 \pm x^2| + C$
3. $\int \frac{x\,dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2} + C \quad (a > 0)$
4. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C \quad (a > 0)$
5. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C \quad (a \neq 0)$
6. $\int \ln x \, dx = x(\ln x - 1) + C$
7. $\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$
8. $\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C, \quad \int \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$
9. $\int \frac{dx}{\sin x} = \ln\left|\operatorname{tg} \frac{x}{2}\right| + C$

8. **Non-elementary integrals (can not be computed in terms of elementary functions)**

1. $\mathrm{Ei}(x) = \int \frac{e^x}{x} \, dx$ (exponential integral)
2. $\mathrm{Si}(x) = \int \frac{\sin x}{x} \, dx, \quad \mathrm{Ci}(x) = \int \frac{\cos x}{x} \, dx$ (sine and cosine integrals)
3. $\mathrm{S}(x) = \int \sin x^2 \, dx, \quad \mathrm{C}(x) = \int \cos x^2 \, dx$ (Fresnel integral)
4. $\Phi(x) = \int e^{-x^2} \, dx$ (Gaussian integral)
5. $\mathrm{li}(x) = \int \frac{dx}{\ln x}$ (logarithmic integral)

1.13 Basic methods for finding primitives

1. **Notation for the differential of a function**

$$df(x) \ = \ f'(x)\,dx$$

2. **Change of variables**

THEOREM. If $f : (a, b) \to \mathbb{R}$ and $\varphi : (\alpha, \beta) \to (a, b)$ is differentiable on (α, β) then

$$\int f(\underbrace{\varphi(x)}_{y})\, \underbrace{\varphi'(x)\,dx}_{dy} \ = \ \left[\begin{array}{c} y = \varphi(x) \\ dy = \varphi'(x)\,dx \end{array}\right] \ = \ \int f(y)\,dy\,\Big|_{y=\varphi(x)}$$

ABBREVIATED NOTATION.

$$\int f(\varphi(x))\,d\varphi(x) \ = \ \int f(y)\,dy\,\Big|_{y=\varphi(x)}$$

EXAMPLES.

1) $\int \frac{dx}{x+a} \ = \ \ln|x+a| + C$
2) $\int \cos(ax)\,dx \ = \ \frac{1}{a}\,\sin(ax) + C, \qquad a \neq 0$
3) $\int \operatorname{tg} x \ dx = -\ln|\cos x| + C, \qquad \int \operatorname{ctg} x \ dx = \ln|\sin x| + C$
4) $\int \frac{x}{x^2+a}\,dx = \frac{1}{2}\ln|x^2+a| + C, \qquad a \in \mathbb{R}$
5) $\int \frac{x}{\sqrt{x^2+a}}\,dx = \sqrt{x^2+a} + C, \qquad a \in \mathbb{R}$
6) $\int \frac{dx}{\sqrt{x^2+a}} = \ln\left|x + \sqrt{x^2+a}\right| + C, \qquad a \in \mathbb{R}$

PROOFS.

1.

$$\int \frac{dx}{x+a} \ = \ \int \frac{d(x+a)}{x+a} \ = \ \left[\begin{array}{c} y = x+a \\ dy = dx \end{array}\right] \ = \ \int \frac{dy}{y} \ = \ \ln|y| + C \ = \\ = \ \left[\ y = x+a\ \right] \ = \ \ln|x+a| + C$$

2.

$$\int \cos(ax)\,dx \ = \ \frac{1}{a}\int \cos(ax)\,d(ax) \ = \ \left[\begin{array}{c} y = ax \\ dy = a\,dx \end{array}\right] \ = \ \frac{1}{a}\int \cos y\,dy \ = \ \frac{1}{a}\,\sin y + C \ = \\ = \ \left[\ y = ax\ \right] \ = \ \frac{1}{a}\,\sin(ax) + C$$

3.

$$\int \operatorname{tg} x\, dx \;=\; \int \frac{\sin x}{\cos x}\, dx \;=\; -\int \frac{d(\cos x)}{\cos x}\, dx \;=\; \left[\begin{array}{c} y = \cos x \\ dy = -\sin x\, dx \end{array}\right] \;=$$

$$= -\int \frac{dy}{y} \;=\; -\ln|y| + C \;=\; [\; y = \cos x \;] \;=\; -\ln|\cos x| + C$$

4.

$$\int \frac{x}{x^2+a}\, dx \;=\; \frac{1}{2}\int \frac{d(x^2+a)}{x^2+a} \;=\; \left[\begin{array}{c} y = x^2+a \\ dy = 2x\, dx \end{array}\right] \;=\; \frac{1}{2}\int \frac{dy}{y} \;=\; \frac{1}{2}\ln|y| + C \;=$$

$$= [\; y = x^2+a \;] \;=\; \frac{1}{2}\ln|x^2+a| + C$$

5.

$$\int \frac{x}{\sqrt{x^2+a}}\, dx \;=\; \frac{1}{2}\int \frac{d(x^2+a)}{\sqrt{x^2+a}} \;=\; \left[\begin{array}{c} y = x^2+a \\ dy = 2x\, dx \end{array}\right] \;=\; \frac{1}{2}\int \frac{dy}{\sqrt{y}} \;=\; \sqrt{y} + C \;=$$

$$= [\; y = x^2+a \;] \;=\; \sqrt{x^2+a} + C$$

6.

$$\int \frac{dx}{\sqrt{x^2+a}} \;=\; \int \frac{1}{x+\sqrt{x^2+a}}\,\frac{x+\sqrt{x^2+a}}{\sqrt{x^2+a}}\, dx \;=\; \int \frac{1}{x+\sqrt{x^2+a}}\left(1+\frac{x}{\sqrt{x^2+a}}\right) dx \;=$$

$$= \left[\begin{array}{c} y = x+\sqrt{x^2+a} \\ dy = \left(1+\frac{x}{\sqrt{x^2+a}}\right) dx \end{array}\right] \;=\; \int \frac{dy}{y} \;=\; \ln|y| + C \;=$$

$$= [\; y = x+\sqrt{x^2+a} \;] \;=\; \ln(x+\sqrt{x^2+a}) + C$$

3. Decomposition of an integrand into a sum of elementary functions

THEOREM. If $f = f_1 + f_2$ on (a,b) then

$$\int f(x)\; dx \;=\; \int f_1(x)\; dx \;+\; \int f_2(x)\; dx$$

EXAMPLES.

1) $\int \frac{x+a}{x+b}\, dx \;=\; x + (a-b)\ln|x+b| + C$

2) $\int \frac{dx}{a^2-x^2} \;=\; \frac{1}{2a}\,\ln\left|\frac{a+x}{a-x}\right| + C, \qquad a \neq 0$

PROOFS.

1.

$$\frac{x+a}{x+b} \;=\; \frac{x+b-b+a}{x+b} \;=\; 1 + \frac{a-b}{x+b}$$

$$\int \frac{x+a}{x+b}\,dx \;=\; \int dx \;+\; (a-b)\int \frac{dx}{x+b} \;=\; x+(a-b)\ln|x+b|+C$$

2.

$$\frac{1}{a^2-x^2} \;=\; \frac{1}{2a}\cdot\frac{(a-x)+(a+x)}{a^2-x^2} \;=\; \frac{1}{2a}\left(\frac{1}{a+x}+\frac{1}{a-x}\right)$$

$$\int \frac{dx}{a^2-x^2} \;=\; \frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx \;=\; \frac{1}{2a}\big(\ln|a+x|-\ln|a-x|\big)+C \;=$$

$$=\; \frac{1}{2a}\,\ln\left|\frac{a+x}{a-x}\right|+C$$

4. **Integrations by parts**

Theorem.

$$\int f(x)g'(x)\,dx \;=\; f(x)g(x) \;-\; \int f'(x)g(x)\,dx$$

Abbreviated notation.

$$\int f\,dg \;=\; fg \;-\; \int g\,df$$

Examples.

1) $\int \ln x\;dx = x(\ln x - 1)+C$
2) $\int x\,\cos x\;dx = x\sin x+\cos x+C$
3) $\int x\,e^x\;dx = xe^x - e^x + C$
4) $\int x\,\ln x\;dx = \frac{x^2}{2}\,\ln x \;-\; \frac{x^2}{4}+C$
5) $\int e^x\sin x\,dx = \frac{1}{2}(\sin x-\cos x)\,e^x+C, \qquad \int e^x\cos x\,dx = \frac{1}{2}(\sin x+\cos x)\,e^x+C,$

Proofs.

1.

$$\int \ln x\,dx \;=\; [\text{ by parts }] \;=\; x\ln x \;-\; \int x\,d(\ln x) \;=\; x\ln x \;-\; \int x\,\frac{1}{x}\,dx \;=\; x\ln x - x + C$$

2.

$$\int x\,\cos x\;dx \;=\; \int x\,d(\sin x) \;=\; [\text{ by parts }] \;=\; x\sin x - \int \sin x\,dx \;=\; x\sin x+\cos x+C$$

3.

$$\int x\,e^x\;dx \;=\; \int x\,d(e^x) \;=\; [\text{ by parts }] \;=\; xe^x - \int e^x\,dx \;=\; xe^x - e^x + C$$

4.

$$\int x\,\ln x\;dx = \int \ln x\,d\left(\tfrac{x^2}{2}\right) = [\text{ by parts }] = \frac{x^2}{2}\,\ln x - \int \frac{x^2}{2}\,\underbrace{d(\ln x)}_{\frac{1}{x}\,dx} =$$

$$= \frac{x^2}{2}\,\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\,\ln x - \frac{x^2}{4} + C$$

5.

$$\int e^x \sin x\,dx = \int \sin x\,d(e^x) = [\text{ by parts }] = \sin x\,e^x - \int e^x\,\underbrace{d(\sin x)}_{\cos x\,dx} =$$

$$= \sin x\,e^x - \int e^x \cos x\,dx = \sin x\,e^x - \int \cos x\,d(e^x) = [\text{ again by parts }] =$$

$$= \sin x\,e^x - \cos x\,e^x + \int e^x\,\underbrace{d(\cos x)}_{-\sin x\,dx} = \sin x\,e^x - \cos x\,e^x - \int e^x \sin x\,dx$$

So, we obtain

$$\int e^x \sin x\,dx = \frac{\sin x - \cos x}{2} + C$$

5. **Method of substitution**

THEOREM. Assume $f:(a,b)\to\mathbb{R}$ and $\varphi:(\alpha,\beta)\to(a,b)$ is differentiable and invertible on (α,β). Then

$$\int f(x)\;dx = \begin{bmatrix} x = \varphi(t) \\ dx = \varphi'(t)\,dt \end{bmatrix} = \int f(\varphi(t))\varphi'(t)\;dt\;\Big|_{t=\varphi^{-1}(x)}$$

EXAMPLES.

1) $\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$

PROOFS.

1.

$$\int \frac{dx}{(1-x^2)^{3/2}} = \begin{bmatrix} x = \sin t \\ dx = \cos t\,dt \\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{bmatrix} = \int \frac{\cos t}{(1-\sin^2 t)^{3/2}}\,dt = \int \frac{dt}{\cos^2 t} =$$

$$= \operatorname{tg} t + C = [\; t = \arcsin x \;] = \operatorname{tg}(\arcsin x) + C =$$

$$= \begin{bmatrix} \operatorname{tg} t = \frac{\sin t}{\sqrt{1-\sin^2 t}} \\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{bmatrix} = \frac{x}{\sqrt{1-x^2}} + C$$

6. **Using trigonometrical formulas**

Power-reduction formulas:

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Double angle formulas:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

EXAMPLE.

1) $\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C, \qquad \int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$

2) $\int \frac{dx}{\sin x} = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C, \qquad \int \frac{dx}{\cos x} = -\ln\left|\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C$

PROOFS.

1.

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

2.

$$\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{2}\int \frac{dx}{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\cos^2\frac{x}{2}} = \frac{1}{2}\int \frac{dx}{\operatorname{tg}\frac{x}{2}\cdot\cos^2\frac{x}{2}} =$$

$$= \left[\begin{array}{c} y = \operatorname{tg}\frac{x}{2} \\ dy = \frac{dx}{2\cos^2\frac{x}{2}} \end{array}\right] = \int \frac{dy}{y} = \ln|y| + C = \left[\ y = \operatorname{tg}\frac{x}{2}\ \right] = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(\frac{\pi}{2}-x\right)} = \left[\begin{array}{c} y = \frac{\pi}{2} - x \\ dy = -dx \end{array}\right] = -\int \frac{dy}{\sin y} = -\ln\left|\operatorname{tg}\frac{y}{2}\right| + C =$$

$$= \left[\ y = \frac{\pi}{2} - x\ \right] = -\ln\left|\operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + C$$

1.14 Definite integral

1. **A partition of an interval**

   A *partition* of an interval $[a, b]$ is a finite sequence of numbers $T = \{x_i\}_{i=0}^N$ such that

   $$a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b$$

   The *mesh* $\lambda(T)$ of a partition T is the length of the longest interval in this partition:

   $$\lambda(T) := \max_{i=1,\ldots,N} \Delta x_i, \qquad \Delta x_i := x_i - x_{i-1}$$

   Let $T = \{x_i\}_{i=0}^N$ be a partition of $[a, b]$. Choose a finite set of points $\xi = \{\xi_i\}_{i=1}^N$ so that

   $$\xi_i \in [x_{i-1}, x_i], \qquad i = 1, \ldots, N$$

   The pair (T, ξ) is called a *partition with distinguished points* or a *tagged partition* of $[a, b]$.

2. **Riemann sum**

   DEFINITION. Assume $f : [a, b] \to \mathbb{R}$ and let (T, ξ) be a tagged partition of $[a, b]$, where

   $$T = \{x_i\}_{i=0}^N, \qquad \xi = \{\xi_i\}_{i=1}^N, \qquad \xi_i \in [x_{i-1}, x_i].$$

   The *Riemann sum* of the function f corresponding to the tagged partition (T, ξ) is the sum

   $$\sigma(T; \xi) := \sum_{i=1}^N f(\xi_i)\, \Delta x_i, \qquad \Delta x_i := x_i - x_{i-1}$$

3. **Riemann integral**

   DEFINITION. A function $f : [a, b] \to \mathbb{R}$ is called *Riemann-integrable* on $[a, b]$ if there exists a finite limit

   $$\exists\, I \in \mathbb{R} : \qquad I = \lim_{\lambda(T) \to 0} \sum_{i=1}^N f(\xi_i)\, \Delta x_i,$$

   as the mesh $\lambda(T)$ of the partition tends to zero, and this limit does not depend neither on the method of partitioning the interval $[a, b]$ by points $T = \{x_i\}_{i=0}^N$, nor on the choice of points ξ_i in the intervals $[x_{i-1}, x_i]$. If such limit exists, then the number I is called the *definite integral* (or the *Riemann integral*) of the function f over the interbal $[a, b]$ and is denoted by

   $$\int_a^b f(x)\, dx := I$$

4. **Refinement of the definition of the Riemann integral**

The definition of the integral above is an informal one: we know only the definition of a function depending on some variable. But the Riemann sum is not a function of a *mesh* of the partition, since it also depends on the choice of points x_j and ξ_j. The rigorous definition of the integral can be done only using ε–δ formalism: the function f is called *integrable* on $[a, b]$ if there exists a number $I \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any partition (T, ξ) of the interval $[a, b]$ with $T = \{x_i\}_{i=0}^{N}$ and $\xi = \{\xi_i\}_{i=1}^{N}$, $\xi_i \in [x_{i-1}, x_i]$, the following implication holds:

$$\lambda(T) < \delta \qquad \Longrightarrow \qquad \left| \, I - \sum_{i=1}^{N} f(\xi_i)\Delta x_i \, \right| \; < \; \varepsilon$$

5. **The geometric meaning of the definite integral**

   - For a non-negative function f, the Riemann sum is the sum of the areas of rectangles with bases $[x_{i-1}, x_i]$ and heights $f(\xi_i)$.
   - For a non-negative function f, the definite integral is equal to the area of the figure bounded by the graph of the function f, the axis OX, and the parallel lines $x = a$, $x = b$.
   - For an sign-indefinite function f, the definite integral is equal to the algebraic sum of the areas of the figures formed by the parts of the graph f lying above the axis OX (taken with a “plus” sign) and the ones lying below the axis OX (taken with a “minus” sign).

6. **Example of a non-integrable function**

In the next section we will show that any continuous or piecewise-continuous function is integrable. A typical example of a non-integrable function is the following function (which is discontinuous at every point):

$$f(x) \;=\; \begin{cases} 1, & x \in [a, b] \cap \mathbb{Q} \\ 0, & x \in [a, b] \setminus \mathbb{Q} \end{cases}$$

7. **Integral with “upside down” limits**

$$a < b \qquad \Longrightarrow \qquad \int\limits_b^a f(x)\; dx \;\stackrel{\text{def}}{=}\; -\int\limits_a^b f(x)\; dx$$

8. **Integrable functions are bounded**

THEOREM. f is integrable on $[a, b]$ $\Longrightarrow$ f is bounded on $[a, b]$

9. **Darboux sums**

Definition. Assume $f : [a,b] \to \mathbb{R}$ is bounded on $[a,b]$ and $T = \{x_j\}_{j=1}^{N}$ is a partition of the interval $[a,b]$, i.e.

$$a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b$$

Denote

$$m_j = \inf_{x\in[x_{j-1},x_j]} f(x), \qquad M_j = \sup_{x\in[x_{j-1},x_j]} f(x), \qquad j = 1,\ldots,N.$$

The *lower* and the *upper Darboux sums* corresponding to the partition T are

$$s(T) := \sum_{j=1}^{N} m_j\,\Delta x_j, \qquad S(T) := \sum_{j=1}^{N} M_j\,\Delta x_j, \qquad \Delta x_j := x_j - x_{j-1}.$$

10. **Refinement and common refinement of partitions**

Definition. A partition $\tilde{T} = \{y_j\}_{j=1}^{N}$ is the *refinement* of a partition $T = \{x_i\}_{i=1}^{m}$ if all points included in T are also included in $\tilde{T}$, i.e. $x_i = y_{n_i}$ for some $n_1 < n_2 < \ldots < n_m$.

The partition $T_1 \cdot T_2$ is the *common refinement* of partitions T_1 and T_2 if it is formed by all points included in T_1 or T_2.

The common refinement of two partitions is a refinement of each of the original partitions.

11. **Properties of the Darboux sums**

1) for any tagged partition the Riemann sum $\sigma(T,\xi)$ is in between the corresponding Darboux sums $s(T)$ and $S(T)$: $\forall\, T = \{x_j\}_{j=0}^{N}$, $\forall\, \xi = \{\xi_j\}_{j=1}^{N}$ $\quad s(T) \le \sigma(T,\xi) \le S(T)$
2) if T_1 is a refinement of T then $\quad s(T_1) \ge s(T), \quad S(T_1) \le S(T)$
3) $\forall$ partitions T_1 and T_2 $\quad\Longrightarrow\quad s(T_1) \le S(T_2)$
4) for any fixed partition $T = \{x_j\}_{j=1}^{N}$ the infimum and supremum of all possible values of Riemanns' sums $\sigma(T;\xi)$ are equal respectively to the lower and upper Darboux sums $s(T)$ and $S(T)$

$$s(T) = \inf_{\xi} \sigma(T;\xi), \qquad S(T) = \sup_{\xi} \sigma(T;\xi)$$

where the infimum is taken over all sets of points $\xi = \{\xi_j\}_{j=1}^{N}$ such that $\xi_j \in [x_{j-1}, x_j]$.

12. **The basic condition for Riemann integrability**

Theorem. Assume f is bounded on $[a,b]$. Then f is integrable on $[a,b]$ if and only if $\forall\, \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0$ such that for any partition T of $[a,b]$ the following implication holds

$$\lambda(T) < \delta \quad\Longrightarrow\quad S(T) - s(T) < \varepsilon,$$

where $\lambda(T)$ is the mesh of the partition T.

13. Oscillation and the Darboux sums

NOTATION. For any partition T of the interval $[a, b]$ we have

$$S(T) - s(T) = \sum_{j=1}^{N} \omega_i(f)\,\Delta x_j, \qquad \omega_j(f) := M_j - m_j, \qquad \Delta x_j := x_j - x_{j-1}$$

The number $\omega_j(f)$ is called the *oscillation* of f over the interval $[x_{j-1}, x_j]$.

14. Continuous functions are integrable

THEOREM. Assume f is continuous on $[a, b]$. Then f is integrable on $[a, b]$.

1.15 Properties of definite integrals

1. **Basic properties**

    1. f, g are integrable on $[a, b]$, $\alpha,\ \beta \in \mathbb{R} \implies \alpha f + \beta g$ is integrable on $[a, b]$,

    $$\int_a^b (\alpha f(x) + \beta g(x))\,dx \;=\; \alpha \int_a^b f(x)\,dx \;+\; \beta \int_a^b g(x)\,dx$$

    2. $a < c < b \implies f$ is integrable on $[a, b] \Leftrightarrow f$ is integrable on $[a, c]$ and $[c, b]$,

    $$\int_a^b f(x)\;dx \;=\; \int_a^c f(x)\,dx \;+\; \int_c^b f(x)\,dx$$

    3. f, g are integrable on $[a, b]$, $f(x) \le g(x)$, $\forall x \in [a, b] \implies$

    $$\int_a^b f(x)\,dx \;\le\; \int_a^b g(x)\,dx$$

    4. f is integrable on $[a, b] \implies |f|$ is integrable on $[a, b]$ and

    $$\Big|\int_a^b f(x)\,dx\Big| \;\le\; \int_a^b |f(x)|\,dx$$

    5. f is integrable on $[a, b]$, $m \le f(x) \le M$, $\forall\, x \in [a, b] \implies$

    $$m\,(b-a) \;\le\; \int_a^b f(x)\,dx \;\le\; M\,(b-a)$$

    6. f is continuous on $[a, b] \implies \exists\, c \in [a, b]$:

    $$f(c) \;=\; \frac{1}{b-a}\int_a^b f(x)\,dx \qquad \text{(Mean value theorem)}$$

    7. f is continuous on $[a, b]$, g is integrable and non-negative on $[a, b] \implies \exists\, c \in [a, b]$:

    $$\int_a^b f(x)g(x)\,dx \;=\; f(c)\int_a^b g(x)\,dx$$

2. **Fundamental theorem of calculus**

THEOREM. Assume $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$. Define $\Phi : [a, b] \to \mathbb{R}$,

$$\Phi(x) := \int_a^x f(t)\ dt, \quad x \in [a, b] \qquad \text{— the } \textit{integral with variable upper limit.}$$

If f is continuous on $[a, b]$ then Φ is differentiable on $[a, b]$ and

$$\forall\, x_0 \in (a, b) \qquad \Phi'(x_0) = f(x_0).$$

COROLLARY. $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b] \implies f$ has a primitive on (a, b).

3. **Newton–Leibniz formula**

THEOREM. Let f be continuous on $[a, b]$ and let F be any primitive of f on (a, b). Then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

4. **Integration by parts in a definite integral**

THEOREM. Let f, g be differentiable on $[a, b]$ and f', g' be continuous on $[a, b]$. Then

$$\int_a^b f(x)g'(x)\, dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\, dx.$$

ABBREVIATED NOTATION.

$$\int_a^b fg'\ dx = fg\Big|_a^b - \int_a^b f'g\ dx$$

5. **Change of variables in the Riemann integral**

THEOREM. Assume $\varphi : [\alpha, \beta] \to [a, b]$ is differentiable bijection of $[\alpha, \beta]$ onto $[a, b]$, φ' is continuous and strictly positive on $[\alpha, \beta]$. Assume f is continuous on $[a, b]$. Then the function $(f \circ \varphi) \cdot \varphi'$ is continuous on $[\alpha, \beta]$ and

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)\ dt = \left[\begin{array}{c} x = \varphi(t) \\ dx = \varphi'(t)\ dt \end{array} \right] = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\ dx$$

1.16 Applications of definite integrals

1. **Area of a plane figure**

   Definition. A subset of the plane F is called *Jordan measurable* if for any $\varepsilon > 0$ there exist two polygons F_1 and F_2 such that
$$F_1 \subset F \subset F_2 \qquad \text{and} \qquad S(F_2 \setminus F_1) < \varepsilon$$
   (here for every polygon P on the plane we denote by $S(P)$ its area). For the Jordan measurable figure F its area is defined
$$S(F) \;=\; \sup_{F_1 \subset F} S(F_1) \;=\; \inf_{F \subset F_2} S(F_2)$$
   where the supremum is taken over all polygons F_1 contained in F, and the infimum is taken over all polygons F_2 containing F. (These supremum and infimum are equal, since the figure is Jordan measurable).

2. **Area of a curved trapezoid**

   Theorem.

   1) $f : [a,b] \to \mathbb{R},\;\; f(x) \geq 0,\;\; \forall\, x \in [a,b],$ and assume $\mathcal{F} \subset \mathbb{R}^2$ is given by
$$\mathcal{F} \;=\; \{\, (x,y) \in \mathbb{R}^2 : \, x \in [a,b],\; 0 \leq y \leq f(x) \,\}$$
   Then
$$S_{\mathcal{F}} \;=\; \int\limits_a^b f(x)\, dx$$

   2) $f, g : [a,b] \to \mathbb{R},\;\; f(x) \geq g(x),\;\; \forall x \in [a,b],$ and assume $\mathcal{F} \subset \mathbb{R}^2$ is given by
$$\mathcal{F} \;=\; \{\, (x,y) \in \mathbb{R}^2 : \, x \in [a,b],\; g(x) \leq y \leq f(x) \,\}$$
   Then
$$S_{\mathcal{F}} \;=\; \int\limits_a^b \big(f(x) - g(x)\big)\, dx$$

3. **Polar coordinate on the plane**

   Converting from the polar coordinates r and φ to the Cartesian coordinates x and y:
$$\begin{cases} x \;=\; r\,\cos\varphi & r \in [0, +\infty) \\ y \;=\; r\,\sin\varphi & \varphi \in [0, 2\pi) \end{cases}$$
   Converting from the Cartesian coordinates x and y to the polar coordinates r and φ:
$$\begin{cases} r \;=\; \sqrt{x^2 + y^2} & \varphi \in [0, \pi], \quad \text{if} \quad y \geq 0 \\ \operatorname{tg}\varphi \;=\; \dfrac{y}{x} & \varphi \in (\pi, 2\pi), \quad \text{if} \quad y < 0 \end{cases}$$

4. **Area of a curved sector**

THEOREM. $f : [\alpha, \beta] \to [0, +\infty)$, $0 \le \alpha < \beta \le 2\pi$, and assume $\mathcal{F} \subset \mathbb{R}^2$ is given by

$$\mathcal{F} := \{ (r, \varphi) : \varphi \in [\alpha, \beta], \ 0 \le r \le f(\varphi) \}$$

Then

$$S_{\mathcal{F}} = \frac{1}{2} \int_a^b (f(\varphi))^2 \, d\varphi$$

5. **Curves on the plane**

DEFINITION. Let $\gamma : [a, b] \to \mathbb{R}^2$, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be continuous on $[a, b]$

- The mapping $\gamma : [a, b] \to \mathbb{R}^2$ is called a *parameterized plane curve.*
- γ is *regular* if γ is differentiable on $[a, b]$, γ' is continuous on $[a, b]$ and

$$|\gamma'(t)| := \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2} \ne 0, \qquad \forall\, t \in [a, b].$$

- If $T = \{t_j\}_{j=0}^N$ is some partition of $[a, b]$, i.e. $a = t_0 < t_1 < t_2 < \ldots < t_N = b$, then a broken line consisting of segments $[\gamma(t_{j-1}), \gamma(t_j)]$ is said to be *inscribed* in the curve γ. The *length* of a broken line corresponding to the partition T is

$$\sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})|$$

where

$$|\gamma(t_j) - \gamma(t_{j-1})| := \sqrt{(\gamma_1(t_j) - \gamma_1(t_{j-1}))^2 + (\gamma_2(t_j) - \gamma_2(t_{j-1}))^2}$$

- A curve γ is called *rectifiable* if there is a limit on the lengths of all broken lines inscribed in this curve as the mesh of the partition tends to zero. In this case, the *length* is

$$l_\gamma = \lim_{\lambda(T) \to 0} \sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})|$$

- More precisely, a curve is called rectifiable if there is a number $l_\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that for any partition $T = \{t_j\}_{j=1}^N$ of the segment $[a, b]$ the implication is true

$$\lambda(T) < \delta \qquad \Longrightarrow \qquad \left| \, l_\gamma - \sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})| \, \right| < \varepsilon$$

Here $\lambda(T) = \max\limits_{j=1,\ldots,N} (t_j - t_{j-1})$ is the mesh of the partition T.

6. **Length of a curve**

THEOREM.

1) Length of a parameterized curve

Any regular parameterized curve $\gamma : [a,b] \to \mathbb{R}^2$ is rectifiable and its length is

$$l_\gamma \;=\; \int_a^b |\gamma'(t)|\, dt \;:=\; \int_a^b \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}\, dt$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t))$.

2) Length of the curve that is the graph of the function

$f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ and f' is continuous on $[a,b]$,

$$\Gamma \;:=\; \{\, (x,y) \in \mathbb{R}^2 : \; x \in [a,b], \; y = f(x) \,\}$$

Then

$$l_\Gamma \;=\; \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

3) Length of the curve, specified in polar coordinates

Let (r,φ) be polar coordinates on the plane $\mathbb{R}^2$, let $f : [\alpha,\beta] \to [0,+\infty)$ be continuously differentiable on $[\alpha,\beta]$, $0 \le \alpha < \beta \le 2\pi$, and

$$\Gamma \;:=\; \{\, (r,\varphi) : \; \varphi \in [\alpha,\beta], \; r = f(\varphi) \,\}$$

Then

$$l_\Gamma \;=\; \int_\alpha^\beta \sqrt{(f(\varphi))^2 + (f'(\varphi))^2}\, d\varphi$$

7. **Generalized Cavalieri's principle (1635)**

For any solid of revolution (such as a barrel)

$$\text{Volume of the barrel} \;\approx\; \sum_j \Big(\text{Area of a crossection of a barrel}\Big)_j \cdot \Big(\text{Vertical height}\Big)_j$$

$$\text{Area of a side surface} \;\approx\; \sum_j \Big(\text{Length of a crossection of a barrel}\Big)_j \cdot \Big(\text{Vertical } \underline{\text{length}}\Big)_j$$

8. **The volume of a solid of revolution**

THEOREM. Assume $f : [a,b] \to \mathbb{R}$ is non-negative and continuous on $[a,b]$ and

$$\Omega \;:=\; \Big\{\,(x,y,z) \in \mathbb{R}^3 :\; x \in [a,b],\;\; y^2 + z^2 \le f^2(x)\,\Big\}$$

Then the volume $V(\Omega)$ of a solid of revolution Ω is equal to

$$V(\Omega) \;=\; \pi \int\limits_a^b f^2(x)\,dx$$

9. **The area of the side surface of a circular cone**

THEOREM. The side surface of a circular cone whose base radius is R and height H:

$$S_{R,H} \;=\; \pi\, R\,\sqrt{R^2 + H^2}$$

For a truncated cone:

$$S_{R,H} - S_{r,h} \;=\; \pi\,\sqrt{1 + \overset{2}{\operatorname{tg}}\,\alpha} \cdot (RH - rh) \;=\; 2\pi\,\sqrt{1 + \overset{2}{\operatorname{tg}}\,\alpha} \cdot \frac{R + r}{2} \cdot (H - h),$$

where $\operatorname{tg}\alpha \;=\; \frac{r}{h} \;=\; \frac{R}{H}$.

10. **The area of the side surface of a solid of revolution**

THEOREM. Assume $f : [a,b] \to \mathbb{R}$ is non-negative and differentiable on (a,b), f' is continuous on $[a,b]$ and

$$\Sigma \;:=\; \Big\{\,(x,y,z) \in \mathbb{R}^3 :\; x \in [a,b],\;\; y^2 + z^2 = f^2(x)\,\Big\}$$

Then the area of the side surface $S(\Sigma)$ the corresponding solid of revolution is equal to

$$S(\Sigma) \;=\; 2\pi \int\limits_a^b f(x)\,\sqrt{1 + (f'(x))^2}\,dx$$

1.17 Comparison of infinitesimals and Taylor's formula

1. **L'Hôpital's (actually Bernoulli's) rule for an indeterminate form $\frac{0}{0}$**

   THEOREM. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable on (a, b) and $g'(x) \neq 0$, $\forall\, x \in (a, b)$. Assume

   1) $\exists\, c \in (a, b)$: $\quad f(c) = g(c) = 0$

   2) $\exists\, k \in \mathbb{R}$: $\quad \exists \lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = k$

   Then there exists the limit
$$\exists \lim_{x \to c} \frac{f(x)}{g(x)} = k.$$

2. **Other versions of the L'Hôpital rule**

   There are also versions of the L'Hôpital rule which suit to the following cases:

   - Limit at infinity: $\quad f,\, g : [a, +\infty) \to \mathbb{R}, \quad \lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} g(x) = 0,$
$$\exists\, k \in \mathbb{R} : \qquad \exists \lim_{x \to +\infty} \tfrac{f'(x)}{g'(x)} = k \qquad \Longrightarrow \qquad \exists \lim_{x \to +\infty} \tfrac{f(x)}{g(x)} = k$$
   - Indeterminate form $\frac{\infty}{\infty}$: $\quad f,\, g : (a, b) \to \mathbb{R}, \quad c \in (a, b), \quad \lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to c} g(x) = +\infty,$
$$\exists\, k \in \mathbb{R} : \qquad \exists \lim_{x \to c} \tfrac{f'(x)}{g'(x)} = k \qquad \Longrightarrow \qquad \exists \lim_{x \to c} \tfrac{f(x)}{g(x)} = k$$
   - Version of the above statements with $k = +\infty$ or $k = -\infty$
   - Version of the above statements with $x \to -\infty$ instead of $x \to +\infty$

   All these statements can be proved similarly to the proof of the first statement and we omit their proofs in our course. Informally we can summarize the L'Hôpital rules as follows:

The limit of a ratio of functions is equal to the limit of the ratio of their derivatives if the latter exists.

3. **Taylor's formula**

   THEOREM. $f : [a, b] \to \mathbb{R}$ is differentiable $(n + 1)$ times on $(a, b) \quad \Longrightarrow \quad \forall\, x, x_0 \in (a, b)$
$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1}(x),$$

where

$$R_{n+1}(x) := \frac{1}{n!}\int\limits_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt \quad \text{— the integral form of the remainder}$$

If $f^{(n+1)}$ is continuous on $[a,b]$ then there exists $\xi \in [x_0, x]$ such that

$$R_{n+1}(x) := \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1} \quad \text{— the Lagrange form of the remainder}$$

Corollary. $[a,b]$ is bounded, $f^{(n+1)}$ is continuous on $[a,b] \implies R_{n+1}(x) = O(|x-x_0|^{n+1})$

4. **$O-$ asymptotic notation**

Assume $f,\, g : (a,b) \to \mathbb{R}$ (we allow here a and b be infinite), $x_0 \in (a,b)$. Then

- $g = O(f)$ as $x \in (a,b) \iff \exists\, M > 0$: $\forall\, x \in (a,b)$: $|g(x)| \le M\,|f(x)|$
- $g = O(f)$ as $x \to x_0 \iff \exists\, M > 0, \delta > 0$: $\forall\, x \in (x_0-\delta, x_0+\delta)$: $|g(x)| \le M\,|f(x)|$
- $g = O(f)$ as $x \to +\infty \iff \exists\, M > 0,\, R > 0$: $\forall\, x \ge R$: $|g(x)| \le M\,|f(x)|$
- $g = O(f)$ as $x \to -\infty \iff \exists\, M > 0,\, R < 0$: $\forall\, x \le R$: $|g(x)| \le M\,|f(x)|$

5. **$o-$ asymptotic notation**

Assume $f,\, g : (a,b) \to \mathbb{R}$, $-\infty \le a \le x_0 \le b \le +\infty$.

- $g = o(f)$ as $x \to x_0 \iff \exists\, \alpha : (a,b) \to \mathbb{R}$: $\lim\limits_{x\to x_0} \alpha(x) = 0$ and $g(x) = \alpha(x) f(x)$

6. **Properties of $O-$ and $o-$ symbols**

Theorem. Assume $f,\, g : (a,b) \to \mathbb{R}$, $-\infty \le a \le x_0 \le b \le +\infty$.

1) for $x \in (a,b)$ we have
   - $O(f) + O(f) = O(f)$
   - $O(f) \cdot O(g) = O(fg)$
2) for $x \to x_0$ we have
   - $g = o(f) \implies g = O(f)$
   - $o(f) + o(f) = o(f)$
   - $o(f) \cdot O(g) = o(fg)$
3) in particular, for $x \to 0$ we have
   - $O(x^n) \cdot O(x^m) = O(x^{n+m})$
   - $o(x^n) \cdot O(x^m) = o(x^{n+m})$
   - $O(x^n) = o(x^m)$, if $n > m$

7. **Asymptotics of the reminder in Taylor's expansion**

THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is differentiable $(n+1)$ times on (a, b) and $f^{(n+1)}$ is continuous on (a, b). Then for any $x_0 \in (a, b)$

$$R_{n+1}(x) \;=\; O\left((x - x_0)^{n+1}\right) \qquad \text{as} \quad x \to x_0,$$

where we denote by $R_{n+1}(x)$ the reminder in the Taylor formula for f.

8. **Taylor expansions of some elementary functions**

THEOREM. The following asymptotics holds as $x \to 0$:

1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + o(x^n)$
2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + o(x^{2n})$
3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$
4) $(1+x)^r = 1 + \frac{r}{1!}\, x + \frac{r(r-1)}{2!}\, x^2 + \frac{r(r-1)(r-2)}{3!}\, x^3 + \ldots + \frac{r(r-1)\ldots(r-n+1)}{n!}\, x^n + o(x^n)$
5) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$

1.18 Improper integrals

1. **Improper integral of an unbounded function**

   Definition. Assume $f : [a, b) \to \mathbb{R}$ is unbounded near $x = b$ and integrable over $[a, c]$ for any $c \in [a, b)$. We say the *improper integral* of f over $[a, b)$ is *convergent* if there exists a finite limit

   $$\int_a^b f(x)\ dx = \lim_{c \to b-0} \int_a^c f(x)\ dx,$$

   Otherwise we say the corresponding improper integral is *divergent*. For $f : (a, b] \to \mathbb{R}$ which is unbounded near $x = a$ the improper integral is defined in a similar way:

   $$\int_a^b f(x)\ dx = \lim_{c \to a+0} \int_c^b f(x)\ dx,$$

2. **Improper integral over an unbounded interval**

   Definition. Assume $f : [a, +\infty) \to \mathbb{R}$ is integrable over $[a, c]$ for any $c \in [a, +\infty)$. We say the *improper integral* of f over $[a, +\infty)$ is *convergent* if there exists a finite limit

   $$\int_a^{+\infty} f(x)\ dx = \lim_{c \to +\infty} \int_a^c f(x)\ dx,$$

   Otherwise we say the corresponding improper integral is *divergent*. For $f : (-\infty, b] \to \mathbb{R}$ the improper integral is defined in a similar way:

   $$\int_{-\infty}^b f(x)\ dx = \lim_{c \to -\infty} \int_c^b f(x)\ dx,$$

3. **Cauchy's criterion for convergence of improper integrals**

   Theorem.

   1) Assume $F : [a, b) \to \mathbb{R}$. Then

   $$\exists \lim_{c \to b-0} F(c) \quad \Longleftrightarrow \quad \forall \varepsilon > 0\ \ \exists \delta > 0 : \ \ \forall c_1, c_2 \in (b - \delta, b) \quad |F(c_1) - F(c_2)| < \varepsilon.$$

   2) In particular, for $f : [a, b) \to \mathbb{R}$

   $$\int_a^b f(x)\, dx \text{ is convergent} \iff \forall \varepsilon > 0\ \exists \delta > 0 : \ \forall c_1, c_2 \in (b - \delta, b) \qquad \Bigl| \int_{c_1}^{c_2} f(x)\, dx \Bigr| < \varepsilon.$$

3) Assume $F : [a, +\infty) \to \mathbb{R}$. Then

$$\exists \lim_{c \to +\infty} F(c) \quad \Longleftrightarrow \quad \forall \varepsilon > 0 \ \exists R > 0 : \ \forall c_1, c_2 \in [R, +\infty) \quad |F(c_1) - F(c_2)| < \varepsilon.$$

4) In particular, for $f : [a, b) \to \mathbb{R}$

$$\int_a^{+\infty} f(x)\,dx \text{ is convergent } \Longleftrightarrow \forall \varepsilon > 0 \ \exists R > 0 : \ \forall c_1, c_2 \in [R, +\infty) \qquad \Big| \int_{c_1}^{c_2} f(x)\,dx \Big| < \varepsilon.$$

4. **Absolute convergence of improper integrals**

Definition. Assume $f : [a, b) \to \mathbb{R}$ is integrable over $[a, c]$ for any $c \in [a, b)$ and either the interval $[a, b)$ is finite and f is unbounded near $x = b$ or $b = +\infty$. We say the improper integral of f over the interval $[a, b)$ *converges absolutely* if the improper integral of $|f|$ over $[a, b)$ is convergent:

$$\int_a^b |f(x)|\,dx \ < \ +\infty.$$

5. **Absolutely convergent improper integral is convergent**

Theorem. If the improper integral of f over $[a, b)$, $b \le +\infty$, converges absolutely then it converges:

$$\int_a^b |f(x)|\,dx \ \text{ is convergent} \qquad \Longrightarrow \qquad \int_a^b f(x)\,dx \ \text{ is convergent.}$$

6. **Necessary and sufficient condition for absolute convergence of improper integrals**

Theorem. The improper integral of f over $[a, b)$, $b \le +\infty$, converges absolutely if and only if the primitive of $|f|$ defined by

$$F(c) := \int_a^c |f(x)|\,dx, \qquad c \in [a, b)$$

is bounded on the interval $[a, b)$.

7. **Comparison test for absolute convergence of improper integrals**

Theorem. Assume $f,\ g : [a, b) \to \mathbb{R}$ are integrable over $[a, c]$ for any $c \in [a, b)$ and either the interval $[a, b)$ is finite and f is unbounded near $x = b$ or $b = +\infty$. Assume

$$0 \le f(x) \le g(x) \qquad \forall\, x \in [a, b)$$

If the improper integral of g over $[a, b)$ is convergent then the improper integral of f over $[a, b)$ is also convergent:

$$\int_a^b g(x)\, dx \;\text{ is convergent} \qquad \Longrightarrow \qquad \int_a^b f(x)\, dx \;\text{ is convergent}$$

$$\int_a^b f(x)\, dx \;\text{ is divergent} \qquad \Longrightarrow \qquad \int_a^b g(x)\, dx \;\text{ is divergent}$$

8. **Dirichlet–Abel test for non-absolute convergence of improper integrals**

   Theorem. Assume $f,\, g : [a, +\infty) \to \mathbb{R}$ satisfy the conditions

   1) f is continuous on $[a, +\infty)$ and its primitive F is bounded on $[a, +\infty)$
   2) g is non-negative, non-increasing, differentiable on $[a, +\infty)$, g' is continuous on $[a, +\infty)$
   3) $\lim\limits_{x \to +\infty} g(x) = 0$

   Then the improper integral of fg over $[a, +\infty)$ is convergent:

$$\int_a^{+\infty} f(x) g(x)\, dx \;\text{ is convergent.}$$

9. **There are improper integrals that are convergent, but not absolutely convergent**

$$\int_1^{+\infty} \frac{\sin x}{x}\, dx, \qquad \int_1^{+\infty} \frac{\cos x}{x}\, dx \qquad \text{are convergent non-absolutely,}$$

$$\int_1^{+\infty} \frac{|\sin x|}{x}\, dx, \qquad \int_1^{+\infty} \frac{|\cos x|}{x}\, dx \qquad \text{are divergent}$$

10. **Improper integrals with more than one singularity**

   - If $f : (a, b) \to \mathbb{R}$ is unbounded near both ends of the interval (a, b) then we split the interval (a, b) by some point $c \in (a, b)$ so that f possess only one singularity on each of the intervals $(a, c]$ and $[c, b)$ and define the improper integral of f over (a, b) as the sum of improper integrals of f over $(a, c]$ and $[c, b)$.
   - If $f : (a, +\infty) \to \mathbb{R}$ is unbounded near $x = a$ then we split the interval $(a, +\infty)$ by some point $c \in (a, +\infty)$ and define the improper integral of f over $(a, +\infty)$ as the sum of improper integrals of f over $(a, c]$ and $[c, +\infty)$.
   - If $f : (a, b) \to \mathbb{R}$ is unbounded near an internal point $c \in (a, b)$ then we define the improper integral of f over (a, b) as the sum of improper integrals of f over (a, c) and (c, b).

1.19 Infinite series

1. **Convergent and divergent series**

   DEFINITION. Assume $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is a sequence. We say

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \iff \exists\, S \in \mathbb{R}: \quad S = \lim_{n\to\infty} \sum_{j=1}^{n} a_j$$

$$\sum_{n=1}^{\infty} a_n \text{ is divergent} \iff \text{there is no finite limit of } \sum_{j=1}^{n} a_j \quad \text{as } n \to \infty$$

2. **Cauchy's criterion for convergence**

   THEOREM. The following conditions are equivalent:

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \iff \forall \varepsilon > 0 \;\; \exists\, N = N(\varepsilon): \;\; \forall\, n \geq m \geq N(\varepsilon) \quad \sum_{j=m}^{n} a_j < \varepsilon$$

3. **A necessary condition for convergence of a series**

   THEOREM. $\sum\limits_{n=1}^{\infty} a_n$ is convergent $\implies \lim\limits_{n\to\infty} a_n = 0$

4. **Absolute convergence of series**

   DEFINITION. $\sum\limits_{n=1}^{\infty} a_n$ is *absolutely convergent* $\iff \sum\limits_{n=1}^{\infty} |a_n|$ is convergent.

5. **Absolutely convergent series is convergent**

   THEOREM. If the series converges absolutely then it converges:

$$\sum_{n=1}^{\infty} |a_n| \text{ is convergent} \implies \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

6. **Comparison test for series with non-negative terms**

   THEOREM. Assume $0 \leq a_n \leq b_n \quad \forall\, n \in \mathbb{N}$. Then

   1) $\sum\limits_{n=1}^{\infty} b_n$ is convergent $\implies \sum\limits_{n=1}^{\infty} a_n$ is convergent

   2) $\sum\limits_{n=1}^{\infty} a_n$ is divergent $\implies \sum\limits_{n=1}^{\infty} b_n$ is divergent

7. **M-test (majorant test) of convergence**

THEOREM. Assume $\{b_n\}_{n=1}^{\infty}$ is non-negative and $\{a_n\}_{n=1}^{\infty}$ satisfies

$$|a_n| \le b_n, \qquad \forall\, n \in \mathbb{N}$$

Then

$$\sum_{n=1}^{\infty} b_n \text{ is convergent} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent.}$$

8. **Cauchy's test for the absolute convergence**

THEOREM. Assume $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and assume there exists a finite limit

$$\alpha := \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then

1) $\alpha < 1 \quad \Longrightarrow \quad \sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent;
2) $\alpha > 1 \quad \Longrightarrow \quad \sum\limits_{n=1}^{\infty} a_n$ is divergent;
3) there exist both absolutely convergent and divergent series with $\alpha = 1$.

9. **D'Alembert's test for the absolute convergence**

THEOREM. Assume $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and assume there exists a finite limit

$$\beta := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then

1) $\beta < 1 \quad \Longrightarrow \quad \sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent;
2) $\beta > 1 \quad \Longrightarrow \quad \sum\limits_{n=1}^{\infty} a_n$ is divergent;
3) there exist both absolutely convergent and divergent series with $\beta = 1$.

10. **Leibniz's test for alternating series**

THEOREM. Assume $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ satisfies

1) $0 \le a_{n+1} \le a_n, \quad \forall\, n \in \mathbb{N}$
2) $\lim\limits_{n\to\infty} a_n = 0$

Then

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{is convergent.}$$

11. **Integral test for the absolute convergence**

Theorem. Assume $f : [1, +\infty) \to \mathbb{R}$ is continuous on $[1, +\infty)$ and

1) $f \geq 0$ on $[1, +\infty)$
2) f — $\searrow$ on $[1, +\infty)$

Then

$$\sum_{n=1}^{\infty} f(n) \quad \text{and} \quad \int_1^{+\infty} f(x)\, dx \quad \text{converges or diverges simultaniously.}$$

12. **There exist infinite series that converge but do not converge absolutely**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \quad \text{is convergent,} \qquad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{is divergent}$$

2 Matrices and determinants

2.1 Gaussian elimination method

1. **System of linear equations**

   We consider the $m \times n$ system of linear algebraic equations of the following form:

$$\left\{\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n & = & b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n & = & b_3 \\ \ldots \quad \ldots \quad \ldots & & \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n & = & b_m \end{array}\right. \qquad (*)$$

   Here x_1, x_2, x_3, $\ldots$, x_n are unknowns, a_{jk}, $j = 1, \ldots m$, $k = 1, \ldots, n$ and b_1, b_2, b_3, $\ldots$, b_m are given (real or complex) numbers.

   By a solution of an $m \times n$ system, we mean an ordered n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ that satisfies all the m equations of the system.

   We say the system $(*)$ is *consistent* if there exists at least one solution to this system. We say the system $(*)$ is *inconsistent* if there is no solutions to this system.

2. **Geometrical interpretation of a 2×2 system**

   Consistent, inconsistent and uniquely solvable systems:

$$\left\{\begin{array}{l} x_1 + x_2 = 3 \\ x_1 - x_2 = 1 \end{array}\right. \qquad \left\{\begin{array}{c} x_1 + x_2 = 2 \\ 2x_1 + 2x_2 = 6 \end{array}\right. \qquad \left\{\begin{array}{c} x_1 + x_2 = 2 \\ 2x_1 + 2x_2 = 4 \end{array}\right.$$

3. **Overdetermined and underdetermined linear systems**

   A linear system $(*)$ is said to be overdetermined if there are more equations than unknowns (i.e. $m > n$). Overdetermined systems are usually (but not always) inconsistent.

   A linear system $(*)$ is said to be underdetermined if there are fewer equations than unknowns (i.e. $m < n$). Although it is possible for underdetermined systems to be inconsistent, they are usually consistent with infinitely many solutions.

4. **Equivalent systems**

   DEFINITION. Two systems of equations involving the same variables are said to be *equivalent* if they have the same solution set.

5. **Elementary operations with linear systems**

THEOREM. Applying any of the following operations with the equation of a linear system we obtain the new linear system which is equivalent to the original one:

I. Interchange equations in the system.

II. Multiply both sides of some equation by a nonzero number.

III. Replace some equation by its sum with a multiple of another equation.

6. **The systems in the strict triangular form are easy to solve**

We say an $n \times n$ systems is in *strict triangular form* if in the k-th equation the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero ($k = 1, \dots, n$).

The systems in the strict triangular form are easy to solve by the back substitution:

$$\left\{\begin{array}{rcl} x_1 - 3x_2 + 2x_3 & = & 7 \\ 2x_2 - 5x_3 & = & 1 \\ x_3 & = & 3 \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} x_1 - 3x_2 + 2x_3 & = & 6 \\ x_2 \quad\quad & = & \frac{1}{2}(1 + 5 \cdot 3) = 8 \\ x_3 & = & 3 \end{array}\right.$$

$$\Longrightarrow \left\{\begin{array}{lllcl} x_1 & & & = & 7 + 3 \cdot 8 - 2 \cdot 3 = 25 \\ & x_2 & & = & 8 \\ & & x_3 & = & 3 \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} x_1 & = & 25 \\ x_2 & = & 8 \\ x_3 & = & 3 \end{array}\right.$$

So, our strategy is: applying the elementary operations with the equations I, II and III to try to reduce our system to the equivalent system which has the strict triangular form.

7. **Coefficient matrix, augmented matrix**

To avoid extra writing we organize the coefficient of the linear system $(*)$ as an $m \times n$ array

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

We will call this array the *coefficient matrix* of the system $(*)$.

To include the right-hand side of the system $(*)$ into consideration we can build an *augmented matrix* corresponding to the system $(*)$:

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{array}\right)$$

8. **Elementary row operations with matrices**

Now the elementary operations I, II and III with the equations in the linear system $(*)$ corresponds to the following elementary operations with the rows of the augmented matrix of the system $(*)$:

I. Interchange rows.
II. Multiply a row by a nonzero real number.
III. Replace a row by its sum with a multiple of another row.

9. **Gaussian elimination**

We want to try to the linear system $(*)$ to an equivalent system in a triangular form.

1) Assume the first column of the coefficient matrix contain at least one non-zero entry. We can fix this element and refer to this element as a *pivot* element of the first column of the coefficient matrix. The row of an augmented matrix containing the pivot element is referred as the *pivotal row*.

   Interchanging (if necessary) the rows of the augmented matrix (i.e. applying to the augmented matrix the elementary row operation I) we can move the pivotal row as the new first row.

   Multiplying the pivotal row of the augmented matrix by an appropriate nonzero number and subtracting the results from each of the remaining $m-1$ rows (i.e. applying to the augmented matrix the elementary row operation III) we can obtain zeros in the first entries of all rows from 2 to m.

2) At the second step, a pivot element is chosen from the nonzero entries in column 2, rows 2 through m (if there are no non-zero entries in among them then we just switch to the next column an look for a pivotal element there).

   The row containing the pivot is then interchanged with the second row of the matrix and is used as the new pivotal row.

   Multiples of the pivotal row are then subtracted from the remaining $m-2$ rows so as to eliminate all entries below the pivot in the second column.

etc) The same procedure is repeated for all the rest columns of the coefficient matrix until

   either there is no more rows under the next pivotal row or

   all rows of the coefficient matrix below the next pivotal row consist on zero entries.

   Note that at the second step row 1 and column 1 remain unchanged, at the third step the first two rows and first two columns remain unchanged, and so on.

What we obtain at the end is we reduce the original augmented matrix to some matrix which has the *row echelon form* (see the definition below). The algorithm described above is called *Gaussian elimination*.

10. **A matrix in the row echelon form**

DEFINITION. A matrix is said to be in the *row echelon form* if

- The first nonzero entry in each row is 1.
- If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.
- If there are rows whose entries are all zero, they are below the rows having nonzero entries.

A matrix is said to be in the *reduced row echelon form* if additionally

- The first nonzero entry in each row is the only nonzero entry in its column.

11. **Solving of the linear systems by the Gaussian elimination method**

    Theorem. With the help of row operations I, II, and III any linear system $(*)$ can be to transformed to an equivalent linear system whose augmented matrix is in the row echelon form.

12. **Analysis of consistency and the unique solvability of a linear system from the row echelon form**

    Theorem.

    1) The system $(*)$ is consistent if and only if after the Gaussian reduction of the augmented matrix of the system $(*)$ to a row echelon form the non-pivotal rows of the coefficient matrix (i.e. those rows of the coefficient matrix whose all entries are zeros) have also zero entries in the right-hand side column.
    2) The system $(*)$ is uniquely solvable if and only if after the Gaussian reduction of the augmented matrix of the system $(*)$ to a row echelon form and elimination of those rows of the augmented matrix which are identically zeros (both in the coefficient matrix and in the right-hand side column) the coefficient matrix has a strictly triangular form.

13. **Lead variables, free variables**

    Definitoin. If the matrix of a linear system is in the row echelon form then the variables corresponding to the columns containing the pivotal elements are called the *lead variables* and the variables corresponding to all other columns are called *free variables.*

14. **Gauss-Jordan reduction**

    If the system in the row echelon form is consistent and contains free variables, it is convenient to continue the elimination process and simplify the form of the reduced matrix even further.

    1) First we take the last pivotal row and using the elementary row operation III transform into zeros all entries above in the pivotal element in this row.
    2) Then we take the previous pivotal row and do all the same with the entries of the column of the pivotal element in this row.

etc) The same procedure is repeated for all the rest pivotal rows until the first one.

What we obtain at the end is the *reduced row echelon form* of the augmented matrix of the system. The process of using of elementary row operations I, II, and III to transform a linear system into reduced row echelon form is called *Gauss-Jordan reduction*.

15. **Solving of the underdetermined consistent linear systems**

THEOREM.

1) If the linear system in the reduced row echelon form is consistent and contains free variables then we can assign arbitrary values to free variables and represent lead variables in the unique way via right and sides and free variables.
2) Underdetermined consistent linear systems have infinitely many solutions.
3) Any solution of such system can be specified by the particular choice of the values of parameters corresponding to the free variables.

16. **Example**

PROBLEM. Solve the following underdetermined system:

$$\begin{aligned} x_1 + 7x_2 + 9x_3 + 4x_4 &= 2 \\ 2x_1 + 2x_2 + 3x_3 + 5x_4 &= 4 \\ 5x_1 + 3x_2 + 5x_3 + 12x_4 &= 10 \end{aligned}$$

SOLUTION.

$$\left(\begin{array}{cccc|c} 1 & 7 & 9 & 4 & 2 \\ 2 & 2 & 3 & 5 & 4 \\ 5 & 3 & 5 & 12 & 10 \end{array}\right) \quad \left[\begin{array}{c} \text{use 1 as a pivotal entry} \\ \text{multiply the 1st row by } -2 \\ \text{and add to the 2nd row} \end{array}\right]$$

$$\left(\begin{array}{cccc|c} 1 & 7 & 9 & 4 & 2 \\ 0 & -12 & -15 & -3 & 0 \\ 5 & 3 & 5 & 12 & 10 \end{array}\right) \quad \left[\begin{array}{c} \text{use 1 as a pivotal entry} \\ \text{multiply the 1st row by } -5 \\ \text{and add to the 3rd row} \end{array}\right]$$

$$\left(\begin{array}{cccc|c} 1 & 7 & 9 & 4 & 2 \\ 0 & -12 & -15 & -3 & 0 \\ 0 & -32 & -40 & -8 & 0 \end{array}\right) \quad \left[\begin{array}{c} \text{intermidiate step:} \\ \text{multiply the 2nd row by } -\frac{1}{3} \\ \text{multiply the 3rd row by } -\frac{1}{8} \end{array}\right]$$

$$\left(\begin{array}{cccc|c} 1 & 7 & 9 & 4 & 2 \\ 0 & 4 & 5 & 1 & 0 \\ 0 & 4 & 5 & 1 & 0 \end{array}\right) \quad \left[\begin{array}{c} \text{use 4 as a pivotal entry} \\ \text{multiply the 2nd row by } -1 \\ \text{and add to the 3rd row} \end{array}\right]$$

$$\left(\begin{array}{cccc|c} 1 & 7 & 9 & 4 & 2 \\ 0 & 4 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \left[\begin{array}{c} \text{intermidiate step:} \\ \text{eliminate the identically zero row} \end{array}\right]$$

$$\left(\begin{array}{cccc|c} 1 & 7 & 9 & 4 & 2 \\ 0 & 4 & 5 & 1 & 0 \end{array}\right) \quad \left[\begin{array}{c} \text{intermidiate step:} \\ \text{multiply the 2nd row by } \frac{1}{4} \\ \text{to obtain the row echelon form} \end{array}\right]$$

$$\left(\begin{array}{cccc|c} 1 & 7 & 9 & 4 & 2 \\ 0 & 1 & \frac{5}{4} & \frac{1}{4} & 0 \end{array}\right) \quad \left[\begin{array}{c} \text{Gauss–Jordan reduction} \\ \text{multiply the 2nd row by } -7 \\ \text{and add to the 1st row} \end{array}\right]$$

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & \frac{9}{4} & 2 \\ 0 & 1 & \frac{5}{4} & \frac{1}{4} & 0 \end{array}\right) \quad \left[\begin{array}{c} \text{this is the reduced} \\ \text{row echelon form} \end{array}\right]$$

$$\text{Lead variables:} \quad x_1,\, x_2 \quad \left[\begin{array}{c} \text{correspond to 1st and 2nd columns} \\ \text{containing pivotal elements} \end{array}\right]$$

$$\text{Free variables:} \quad x_3,\, x_4 \quad \left[\begin{array}{c} \text{correspond to 3rd and 4th columns} \\ \text{which do not contain pivotal elements} \end{array}\right]$$

$$\begin{array}{rcl} x_1 & = & 2 - \frac{1}{4}x_3 - \frac{9}{4}x_4 \\ x_2 & = & -\frac{5}{4}x_3 - \frac{1}{4}x_4 \end{array} \quad \left[\begin{array}{c} \text{represent lead variables} \\ \text{as functions of free variables} \end{array}\right]$$

$$\begin{array}{rcl} x_3 & = & \alpha \\ x_4 & = & \beta \end{array} \quad \left[\begin{array}{c} \text{assign arbitrary values (parameters)} \\ \text{to the free variables} \end{array}\right]$$

$$\begin{array}{rcl} x_1 & = & 2 - \frac{1}{4}\alpha - \frac{9}{4}\beta \\ x_2 & = & -\frac{5}{4}\alpha - \frac{1}{4}\beta \end{array} \quad \left[\begin{array}{c} \text{represent lead variables} \\ \text{as functions of arbitrary parameters} \end{array}\right]$$

ANSWER. Any solution of the system is given by

$$\begin{array}{rcl} x_1 & = & 2 - \frac{1}{4}\alpha - \frac{9}{4}\beta \\ x_2 & = & -\frac{5}{4}\alpha - \frac{1}{4}\beta \\ x_3 & = & \alpha \\ x_4 & = & \beta \end{array} \qquad \text{where } \alpha,\, \beta \in \mathbb{R} \text{ are arbitrary}$$

17. **Homogeneous systems**

DEFINITION. The system $(*)$ is said to be *homogeneous* if all right-hand sides are zeros, i.e.

$$b_1 = b_2 = \ldots = b_m = 0$$

THEOREM.

1) Homogeneous system are always consistent.
2) Underdetermined homogeneous system of always has a nontrivial solution.

2.2 Matrix algebra

1. **Matrix, augmented matrix**

   In general, a $m \times n$ *matrix* is the $m \times n$ array whose entries are numbers (real or complex). The entry denoted by a_{jk} stands in the j-th row and k-th column of the array:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

   Given a $m \times n$ matrix $A = (a_{jk})$ and a $m \times r$ matrix $B = (b_{jl})$ we can always build a $m \times (n + r)$ *augmented* matrix $(A|B)$ which is

$$\left(\begin{array}{ccccc|cccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_{11} & b_{12} & \dots & b_{1r} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_{21} & b_{22} & \dots & b_{2r} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_{31} & b_{32} & \dots & b_{3r} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_{m1} & b_{m2} & \dots & b_{mr} \end{array}\right)$$

   Two $m \times n$ matrices A and B are said to be equal iff $a_{ij} = b_{ij}$ for each i and j.

2. **Row vectors and column vectors**

   A *scalar* is a 1×1 matrix. We identify this matrix with a number which is the only entry of this matrix (so, essentially a 'scalar' means a 'number'):

$$(a) \;=\; a, \qquad a \in \mathbb{R}$$

   A *row vector* of order n is a $1 \times n$ matrix:

$$\mathbf{a} \;=\; (a_1, a_2, a_3, \dots, a_n), \qquad a_j \in \mathbb{R}$$

   A *column vector* (or just a *vector*) of order m is a $m \times 1$ matrix:

$$\mathbf{b} \;=\; \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ b_m \end{pmatrix}, \qquad b_j \in \mathbb{R}$$

   We denote by $\mathbb{R}^m$ the set of all column vectors of order m with real entries.

3. **Linear operations with matrices**

DEFINITION. Assume $A = (a_{jk})$ and $B = (b_{jk})$ are $m \times n$ matrices and $\alpha \in \mathbb{R}$ is a scalar.

- The sum $C = A + B$ is the $m \times n$ matrix $C = (c_{jk})$ whose entries are

$$c_{jk} := a_{jk} + b_{jk}, \qquad j = 1, \ldots, m, \qquad k = 1, \ldots, n.$$

- The product $C = \alpha A$ is the $m \times n$ matrix $C = (c_{jk})$ whose entries are

$$c_{jk} := \alpha\, a_{jk}, \qquad j = 1, \ldots, m, \qquad k = 1, \ldots, n.$$

- The zero $m \times n$ matrix $\mathbb{O}$ is the matrix whose all entries are zeros.
- We denote by $-A := (-1)A$ the *additive inverse* of A and define $A - B := A + (-B)$

4. **Properties of linear operations with matrices**

THEOREM.

1) $A + B = B + A$
2) $(A + B) + C = A + (B + C)$
3) $A + \mathbb{O} = A$
4) $A + (-A) = \mathbb{O}$
5) $(\alpha\beta)A = \alpha(\beta A)$
6) $1 \cdot A = A$
7) $(\alpha + \beta)A = \alpha A + \beta A$
8) $\alpha(A + B) = \alpha A + \alpha B$

Further properties:

9) $0 \cdot A = \mathbb{O}$
10) $\alpha \cdot \mathbb{O} = \mathbb{O}$

5. **Matrix product**

DEFINITION.

- If $\mathbf{a}$ is a row vector of order n and $\mathbf{x}$ is a column vector of the same order n then the *product* of a row vector by a column vector is a scalar

$$\mathbf{a} \cdot \mathbf{x} = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{pmatrix} = a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n$$

- Assume A is an $m \times n$ matrix and $\mathbf{x}$ is a column vector of the order n then the *product* of a matrix by a column vector is a column vector of order m

$$A\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}$$

  In other words, the product of a matrix $A = (a_{jk})$ by a column vector $\mathbf{x} = (x_k)$ is the column vector $\mathbf{b} = (b_j)$ whose entries are the products of rows of the matrix A by the column $\mathbf{x}$

$$b_i \ = \ \sum_{k=1}^{n} a_{ik} x_k \ = \ (\ i\text{-th row of } A\ ) \cdot \mathbf{x}$$

- Assume A is an $m \times n$ matrix, B is an $n \times l$ matrix. The the matrix product of $A = (a_{ik})$ by $B = (b_{kj})$ is the $m \times l$ matrix $C = (c_{ij})$ whose entries are

$$c_{ij} \ := \ \sum_{k=1}^{n} a_{ik} b_{kj} \ = \ (\ i\text{-th row of } A\ ) \cdot (\ j\text{-th column of } B\ )$$

- Just as in ordinary algebra, if an expression involves both multiplication and addition and there are no parentheses to indicate the order of the operations, multiplications are carried out before additions. This is true for both scalar and matrix multiplications.

6. **Properties of matrix product**

   Theorem.

   1) $\alpha(AB) = (\alpha A)B$
   2) $(AB)C = A(BC)$
   3) $A(B + C) = AB + AC$
   4) $(A + B)C = AC + BC$

7. **Transpose of a matrix**

   Definition. An $n \times m$ matrix B is called a *transpose* of an $m \times n$ matrix A if $b_{ij} = a_{ji}$ for any $i = 1, \dots, n$ and $j = 1, \dots, m$. A transpose of A is denoted by A^T.

   Theorem.

   1) $(A^T)^T = A$
   2) $(\alpha A)^T = \alpha A^T$
   3) $(A + B)^T = A^T + B^T$
   4) $(AB)^T = B^T A^T$

2.3 Square matrices

1. **Algebra of matrices**

   The product AB of two $n \times n$ matrices A and B is again a $n \times n$ matrix. So, the matrix product of two square matrix returns a matrix of the same type. Attention! To the contrast to scalars, the matrix product in general is not commutative:

   $$\text{In general:} \qquad \boxed{AB \neq BA}$$

2. **Commuting matrices**

   DEFINITION. The *commutator* of two $n \times n$ matrices A and B is the matrix

   $$[A, B] := AB - BA$$

   Matrices A and B *commute* if $AB = BA$ or, equivalently, $[A, B] = \mathbb{O}$.

3. **Power of a matrix**

   DEFINITION. For a $n \times n$ matrix A we can define the k-th power of A:

   $$\boxed{A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k}}$$

   THEOREM. For any $n \times n$ matrix A and for any k, $m \in \mathbb{N}$

   $$A^m A^k = A^k A^m = A^{k+m}$$

4. **Identity matrix**

   DEFINITION. The $n \times n$ identity matrix I is a $n \times n$ matrix whose diagonal entries are equal to 1's and all other elements are zeros.

   $$I = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

   Note that for any $n \times n$ matrix A

   $$\boxed{AI = IA = A}$$

5. **Inverse matrix**

DEFINITION. An $n \times n$ matrix A is called *nonsingular* (or *invertible*) if there exists an $n \times n$ matrix $B \in \mathbb{M}^{n\times n}$ such that

$$AB = BA = I$$

The matrix B is called an *multiplicative inverse* (or simply *inverse*) of A and is denoted A^{-1}. An $n \times n$ matrix A is called *singular* if it does not have a multiplicative inverse.

THEOREM. If $B_1A = I$ and $AB_2 = I$ then $B_1 = B_2$.

6. **Properties of inverse matrices**

THEOREM. A and B are invertible $\implies$ AB and A^{-1} are invertible and

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}} \qquad \boxed{(A^{-1})^{-1} = A}$$

7. **Symmetric and skew-symmetric matrices**

DEFINITION. An $n \times n$ matrix A is called *symmetric* if

$$\boxed{A^T = A}$$

An $n \times n$ matrix A is called *skew symmetric* if

$$\boxed{A^T = -A}$$

8. **Upper triangular, lower triangular and diagonal matrices**

DEFINITION. Assume A is $n \times n$ matrix whose elements are $A = (a_{ij})$

- A is called *upper triangular* if $a_{ij} = 0$ for any $i < j$
- A is called *lower triangular* if $a_{ij} = 0$ for any $i > j$
- A is called *diagonal* if $a_{ij} = 0$ for any $i \neq j$

THEOREM. Assume A and B are $n \times n$ matrices. Then

- A and B are upper triangular $\implies$ AB is upper triangular
- A and B are lower triangular $\implies$ AB is lower triangular
- A and B are diagonal $\implies$ AB is diagonal

9. ***LU* factorization of a matrix**

DEFINITION. The factorization of a square matrix A into a product

$$A = LU$$

of a lower triangular matrix L and an upper triangular matrix U is called an *LU factorization.*

2.4 Matrices and linear systems

1. **Linear system in the matrix form**

   Using the notion of matrix multiplication we can rewrite the linear system $(*)$ in the form

$$\left\{\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &=& b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &=& b_2 \\ \ldots \quad \ldots \quad \ldots && \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &=& b_m \end{array}\right. \quad \Longleftrightarrow \quad A\mathbf{x} = \mathbf{b} \qquad (*)$$

   where A is a given $m \times n$ coefficient matrix, $\mathbf{b} \in \mathbb{R}^m$ is a given vector and $\mathbf{x} \in \mathbb{R}^n$ is the unknown vector.

2. **Linear combination of the vectors**

   Definition. If $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are (row or column) vectors of order m and $c_1, c_2, \ldots, c_n$ are scalars, then the vector

$$c_1\mathbf{a_1} + c_2\mathbf{a_2} + \ldots + c_n\mathbf{a_n}$$

   is said to be a *linear combination* of the vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$.

3. **Consistency for linear systems**

   Theorem. Let A be $m \times n$ matrix whose columns are $\mathbf{a_1}$, $\mathbf{a_2}$, $\ldots$, $\mathbf{a_n}$:

$$A = \left(\mathbf{a_1} \,\middle|\, \mathbf{a_2} \,\middle|\, \ldots \,\middle|\, \mathbf{a_n} \right)$$

   Then for any $\mathbf{x} \in \mathbb{R}^n$ the product $A\mathbf{x}$ is the linear combination of columns of the matrix A:

$$A\mathbf{x} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \ldots + x_n\mathbf{a_n}$$

   Corollary. A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be written as a liner combination of the column vectors of A.

4. **Equivalent systems**

   Theorem. A is a $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$ and M is a $m \times m$ non-singular matrix. Then the systems

$$A\mathbf{x} = \mathbf{b} \qquad \text{and} \qquad MA\mathbf{x} = M\mathbf{b}$$

   are equivalent, i.e. if $\mathbf{x} \in \mathbb{R}^n$ is a solution of one of these systems then it is also a solution of the another one. In the matrix form:

$$(A \,|\, \mathbf{b}) \qquad \Longleftrightarrow \qquad (MA \,|\, M\mathbf{b}) \qquad \text{if } M \text{ is non-singular}$$

5. **Elementary matrices**

Definition. We define elementary $m \times m$ matrices of the following types

1) Type I elementary matrix is a $m \times m$ matrix obtained from the $m \times m$ identity matrix by interchanging of some its two rows;

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad m = 3$$

2) Type II elementary matrix is a $m \times m$ matrix obtained by multiplying some row of the $m \times m$ identity matrix by a nonzero constant.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha \neq 0$$

3) Type III elementary matrix is a $m \times m$ matrix obtained from the $m \times m$ identity matrix by adding a multiple of one row to another row.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{pmatrix}, \qquad \alpha \in \mathbb{R}$$

6. **Multiplication by an elementary matrix**

Theorem. Assume A is an $m \times n$ matrix and E is an $m \times m$ elementary matrix. Then multiplication of A by E from the left has the effect of performing the same operation with rows of A, i.e.

1) if E is the elementary matrix of type I obtained from the $m \times m$ identity matrix by interchanging its i-th and j-th rows then the matrix EA is obtained from A by interchanging its i-th and j-th rows;
2) if E is the elementary matrix of type II obtained from the $m \times m$ identity matrix by multiplication of its i-th row by α then the matrix EA is obtained from A by multiplication of its i-th row by α;
3) if E is the elementary matrix of type I obtained from the $m \times m$ identity matrix by adding a multiple of j-th row to i-th row then the matrix EA is obtained from A by adding a multiple of j-th row of A to i-th row of A.

Moreover, if E is an $n \times n$ elementary matrix then multiplication of A by E from the right has the effect of performing the same operation with columns of A.

7. **Inverse of an elementary matrix**

THEOREM. If E is an elementary matrix then E is nonsingular and E^{-1} is an elementary matrix of the same type. So, the product of finite number of elementary matrices is invertible.

Type	EA	Algebraic form of E	E^{-1}
Type I	Interchange two rows of A	Obtained from I by interchanging of it's two rows	$E^{-1} = E$
Type II	Multiply a row of A by a scalar α	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Type III	Replace a row of A by it's sum with a multiple of another row	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha & 1 \end{pmatrix}$

8. **Row equivalent matrices**

DEFINITION. Let A and B be $m \times n$ matrices. We say the matrix B is *row equivalent* to A if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of $m \times m$ elementary matrices such that

$$B = E_k E_{k-1} \ldots E_1 A$$

REMARK. Row equivalence is an equivalence relation on the set of $m \times n$ matrices.

9. **Matrix form of the Gaussian–Jordan elimination**

THEOREM. Every $m \times n$ matrix A is row equivalent to some $m \times n$ matrix U in the reduced row echelon form:

$$\exists \text{ elementary } m \times m \text{ matrices } E_1, E_2, \ldots, E_k : \qquad E_k E_{k-1} \ldots E_1 A = U,$$
$$A\mathbf{x} = \mathbf{b} \qquad \Longleftrightarrow \qquad U\mathbf{x} = E_k E_{k-1} \ldots E_1 \mathbf{b}$$

10. **Equivalent conditions for nonsingularity**

THEOREM. Assume A is a square matrix. Then the following conditions are equivalent:

1) A is nonsingular.
2) $A\mathbf{x} = \mathbf{0}$ has only trivial solution $\mathbf{0}$.
3) A is row equivalent to I.

11. **Equivalent conditions for the unique solvability of linear systems**

THEOREM. Assume A is an $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$. The system of n linear equations in n unknowns $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A is nonsingular.

12. **Finding the inverse matrix by the Gaussian method**

THEOREM. Assume A is nonsingular and $E_1, E_2, \ldots, E_k$ are elementary matrices. Then

$$\text{if} \quad E_k E_{k-1} \ldots E_1 A = I \quad \text{then} \quad E_k E_{k-1} \ldots E_1 I = A^{-1},$$

i.e. the same series of elementary row operations that transform nonsingular A into I will transform also I into A^{-1}:

$$(A \,|\, I) \quad \longrightarrow \quad (I \,|\, A^{-1})$$

13. **Example**

PROBLEM. Find the matrix A^{-1} inverse to the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

SOLUTION.

$$\left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \left[\begin{array}{c} \text{use } \color{red}{1} \text{ as a pivotal entry} \\ \text{multiply the 1st row by } -1 \\ \text{and add to the 2nd row} \end{array}\right]$$

$$\left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \left[\begin{array}{c} \text{use } \color{red}{1} \text{ as a pivotal entry} \\ \text{multiply the 1st row by } -2 \\ \text{and add to the 3rd row} \end{array}\right]$$

$$\left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & \color{blue}{1} & -2 & 0 & 1 \end{array}\right) \quad \left[\begin{array}{c} \text{use } \color{blue}{1} \text{ as a pivotal entry} \\ \text{multiply the 3rd row by } -3 \\ \text{and add to the 2nd row} \end{array}\right]$$

$$\left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & -3 \\ 0 & 0 & \color{blue}{1} & -2 & 0 & 1 \end{array}\right) \quad \Longrightarrow \quad A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{pmatrix}$$

Verification:

$$A^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{— OK}$$

ANSWER. $\quad A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{pmatrix}$

14. **Finding LU factorization of a square matrix**

Theorem. The $n \times n$ matrix A can be transformed into the matrix U in the row echelon form with the help of lower triangular elementary matrices E_1, E_2, ..., E_k of Type III only (i.e. without interchanging rows and multiplication of rows by non-zero numbers):

$$E_k E_{k-1} \dots E_1 A = U$$

Define the matrix

$$L = E_1^{-1} E_2^{-1} \dots E_k^{-1}$$

Then the matrix L is lower triangular with all diagonal entries equal to 1 and the matrices L and U provide an LU–factorization of the matrix A:

$$A = LU,$$

where L is lower triangular and U is upper triangular.

15. **Solving of consistent underdetermined systems**

Theorem. Assume the linear system $A\mathbf{x} = \mathbf{b}$ is consistent and row equivalent to the undetermined system

$$U\mathbf{x} = \mathbf{c} \qquad (**)$$

where U is $m \times n$ matrix ($m < n$) in the reduced row echelon form. Assume the system $(**)$ contains $k = n - m$ free variables and denote by $\mathbf{u^{(1)}}$, $\mathbf{u^{(2)}}$, ..., $\mathbf{u^{(k)}}$ the columns of the matrix U corresponding to free variables (i.e. each $\mathbf{u^{(i)}} \in \mathbb{R}^m$). Then any solution of the equivalent systems $(*)$ and $(**)$ can be represented in the form

$$\mathbf{x} = \hat{\mathbf{c}} \underbrace{- \alpha_1 \hat{\mathbf{u}}^{\mathbf{(1)}} - \alpha_2 \hat{\mathbf{u}}^{\mathbf{(2)}} - \dots - \alpha_k \hat{\mathbf{u}}^{\mathbf{(k)}}}_{\substack{\text{general solution} \\ \text{of homogeneous system}}}$$

where

- $\hat{\mathbf{c}} \in \mathbb{R}^n$ is the *extended column* corresponding to $\mathbf{c} \in \mathbb{R}^m$, i.e. $\hat{\mathbf{c}}$ is a column vector of order n whose entries matching to free variables are all zeros and entries matching to lead variables are equal to the corresponding entries of $\mathbf{c}$
- $\hat{\mathbf{u}}^{\mathbf{(i)}} \in \mathbb{R}^n$ is the *extended column* corresponding to $\mathbf{u^{(i)}} \in \mathbb{R}^m$, i.e. $\hat{\mathbf{u}}^{\mathbf{(i)}}$ is a column vector of order n whose entry matching to i-th free variable is equal to -1, entries matching to other free variables are zeroes, and entries matching to free variables are all zeros and entries matching to lead variables are equal to the corresponding entries of $\mathbf{u^{(i)}}$
- α_1, α_2, ..., α_k are arbitrary scalars.

Example. Solve the underdetermined linear system

$$(\, U \,|\, \mathbf{c} \,) = \left(\begin{array}{cccccc|c} 1 & 2 & 3 & 0 & 4 & 0 & 10 \\ 0 & 0 & 0 & 1 & 5 & 0 & 11 \\ 0 & 0 & 0 & 0 & 0 & 1 & 12 \end{array}\right)$$

Solution.

$$x_1,\ x_4,\ x_6 \quad \text{— lead variables}$$
$$x_2,\ x_3,\ x_5 \quad \text{— free variables}$$

Represent lead variables via free variables:

$$\begin{cases} x_1 = 10 - 2x_2 - 3x_3 - 4x_5 \\ x_4 = 11 - 5x_4 \\ x_6 = 12 \end{cases}$$

Solution in the algebraic form: $\alpha,\ \beta,\ \gamma \in \mathbb{R}$ are arbitrary parameters.

$$\begin{cases} x_2 = \alpha \\ x_3 = \beta \\ x_5 = \gamma \end{cases} \quad \Longrightarrow \quad \begin{cases} x_1 = 10 - 2\alpha - 3\beta - 4\gamma \\ x_2 = \alpha \\ x_3 = \beta \\ x_4 = 11 - 5\gamma \\ x_5 = \gamma \\ x_6 = 12 \end{cases}$$

Solution in matrix form: column vectors $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}, \mathbf{c}$ in the augmented matrix $(U \,|\, \mathbf{c})$ corresponding to the right hand side and free variables:

$$\mathbf{u}^{(1)} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{u}^{(2)} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{u}^{(3)} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}$$

Extended columns are column vectors of order 6 (number of unknowns in the system). The positions, corresponding to the lead variables, are black (we put the corresponding entries of $\mathbf{u}^{(\mathbf{i})}$ in this positions), and positions, corresponding to the free variables, are marked in red (the entry matching to i-th free variable is equal to -1, entries matching to other free variables are zeroes).

$$\hat{\mathbf{u}}^{(1)} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{u}}^{(2)} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{u}}^{(3)} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 5 \\ -1 \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{c}} = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 11 \\ 0 \\ 12 \end{pmatrix}$$

The solution in the matrix form (we represent the vector od unknowns as a linear combination of given vector with arbitrary parameters $\alpha,\ \beta,\ \gamma$:

$$\mathbf{x} = \hat{\mathbf{c}} - \alpha\hat{\mathbf{u}}^{(1)} - \beta\hat{\mathbf{u}}^{(2)} - \gamma\hat{\mathbf{u}}^{(3)} \quad \Longleftrightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 11 \\ 0 \\ 12 \end{pmatrix} - \alpha \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \beta \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \gamma \begin{pmatrix} 4 \\ 0 \\ 0 \\ 5 \\ -1 \\ 0 \end{pmatrix}$$

2.5 Determinants

1. **Determinant of 2×2- matrix**

   DEFINITION. For any 2×2 matrix A define its *determinant* by

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

   For a 1×1 matrix $A = (a)$ we formally define $\det A = a$.

2. **Motivation for the determinant of 2×2- matrix**

   Assume $a_{11} \neq 0$. Then we can use a_{11} as a pivot element and reduce the matrix A to the row echelon form:

$$\begin{pmatrix} {\color{red}a_{11}} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad {\color{green}\left[\begin{array}{c} \text{use } {\color{red}a_{11}} \text{ as a pivotal entry} \\ \text{multiply the 1st row by } -\frac{a_{21}}{a_{11}} \\ \text{and add to the 2nd row} \end{array}\right]}$$

$$\begin{pmatrix} {\color{red}a_{11}} & a_{12} \\ 0 & a_{22} - \frac{a_{21}a_{12}}{a_{11}} \end{pmatrix} \quad {\color{green}\left[\text{ row echelon form }\right]}$$

   We see that

$$a_{22} - \frac{a_{21}a_{12}}{a_{11}} \neq 0 \qquad \Longleftrightarrow \qquad \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} \neq 0 \qquad \Longleftrightarrow \qquad \det A \neq 0$$

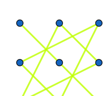
   Conclusion:

$$\boxed{\begin{array}{c} \text{The row echelon form of a } 2 \times 2 \text{ matrix is strictly triangular} \\ \text{if and only if the determinant of the matrix is non-zero.} \end{array}}$$

3. **Determinant of 3×3- matrix**

   DEFINITION. For any 3×3 matrix A define its *determinant* by

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{c} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \end{array}$$



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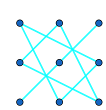

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4. **Motivation for the determinant of 3×3- matrix**

   Assume $a_{11} \neq 0$. Then we can use a_{11} as a pivot element and reduce the matrix A to the row echelon form:

$$\begin{pmatrix} {\color{red}a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad {\color{green}\left[\begin{array}{c} \text{use } {\color{red}a_{11}} \text{ as a pivotal entry} \\ \text{multiply the 1st row by } -\frac{a_{21}}{a_{11}} \\ \text{and add to the 2nd row} \end{array}\right]}$$

$$\begin{pmatrix} {\color{red}a_{11}} & a_{12} & a_{13} \\ 0 & \dfrac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \left[\begin{array}{c} \text{use } {\color{red}a_{11}} \text{ as a pivotal entry} \\ \text{multiply the 1st row by } -\frac{a_{31}}{a_{11}} \\ \text{and add to the 3rd row} \end{array} \right]$$

$$\begin{pmatrix} {\color{red}a_{11}} & a_{12} & a_{13} \\ 0 & \dfrac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}} \\ 0 & \dfrac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{13}a_{31}}{a_{11}} \end{pmatrix}$$

We see that

$$\begin{vmatrix} \dfrac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}} \\ \dfrac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{13}a_{31}}{a_{11}} \end{vmatrix} \neq 0 \qquad \Longleftrightarrow \qquad \det A \neq 0$$

Conclusion:

$$\boxed{\begin{array}{c} \text{The row echelon form of a } 3 \times 3 \text{ matrix is strictly triangular} \\ \text{if and only if the determinant of the matrix is non-zero.} \end{array}}$$

5. **Permutations of n elements**

DEFINITION. A *permutation* of n elements is a bijection of the set $\{1, 2, \ldots, n\}$ onto itself:

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}, \qquad f \text{ is a bijection.}$$

Any permutation of set elements can be characterized by the array of by the row vector

$$f = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ i_1 & i_2 & i_3 & \ldots & i_n \end{pmatrix} \qquad \longleftrightarrow \qquad f = (\, i_1,\, i_2,\, i_3 \, \ldots \, i_n \,)$$

6. **Group of permutations**

DEFINITION. We denote by S_n the set of all permutations of n elements.

THEOREM. The set S_n consists of $n!$ permutations.

EXAMPLE. The set S_3 of permutations of three elements $\{1, 2, 3\}$ consists of 6 permutations:

$$\begin{array}{lll} f_1 = (1, 2, 3) & f_3 = (2, 1, 3) & f_5 = (2, 3, 1) \\ f_2 = (1, 3, 2) & f_4 = (3, 1, 2) & f_6 = (3, 2, 1) \end{array}$$

7. **Inversions in a permutation**

Assume $f = (i_1, i_2, i_3, \ldots, i_n)$ is some permutation of n elements. Consider the set consisting of ordered pairs

$$
\begin{array}{ccccccc}
(i_1,i_2) & (i_1,i_3) & (i_1,i_4) & \ldots & (i_1,i_{n-2}) & (i_1,i_{n-1}) & (i_1,i_n) \\
 & (i_2,i_3) & (i_2,i_4) & \ldots & (i_2,i_{n-2}) & (i_2,i_{n-1}) & (i_2,i_n) \\
 & & (i_3,i_4) & \ldots & (i_1,i_{n-2}) & (i_1,i_{n-1}) & (i_1,i_n) \\
 & & & \ldots & \ldots & \ldots & \ldots \\
 & & & & (i_{n-3},i_{n-2}) & (i_{n-3},i_{n-1}) & (i_{n-3},i_n) \\
 & & & & & (i_{n-2},i_{n-1}) & (i_{n-2},i_n) \\
 & & & & & & (i_{n-1},i_n)
\end{array}
$$

The pair (i_k, i_l) in this table is called the *inversion* in the permutation f if the first element in this pair is greater than the second one. The total number of such pairs in the table above is called the *number of inversions in the permutation f*.

Example. Let $f = (2, 6, 4, 1, 3, 5)$ be a permutation of 6 elements. Then the table above is

$$
\begin{array}{ccccc}
(2,6) & (2,4) & {\color{red}(2,1)} & (2,3) & (2,5) \\
 & {\color{red}(6,4)} & {\color{red}(6,1)} & {\color{red}(6,3)} & {\color{red}(6,5)} \\
 & & {\color{red}(4,1)} & {\color{red}(4,3)} & (4,5) \\
 & & & (1,3) & (1,5) \\
 & & & & (3,5)
\end{array}
$$

The inversions in f are marked in red. The number of inversions in the permutation f is 7.

8. **Signature of a permutation**

Definition. *Sign* or *signature* of a permutation $f \in S_n$ is the number

$$\operatorname{sign} f \;=\; (-1)^k, \qquad \text{where } k \text{ is the number of inversions in } f$$

Assume $g = (2, {\color{blue}4}, {\color{blue}6}, 1, 3, 5)$ is obtained from $f = (2, {\color{magenta}6}, {\color{magenta}4}, 1, 3, 5)$ by transposition of elements 6 and 4. Then

$$
f: \begin{array}{ccccc}
{\color{green}(2,6)} & {\color{green}(2,4)} & (2,1) & (2,3) & (2,5) \\
 & {\color{magenta}(6,4)} & {\color{green}(6,1)} & {\color{green}(6,3)} & {\color{green}(6,5)} \\
 & & {\color{green}(4,1)} & {\color{green}(4,3)} & {\color{green}(4,5)} \\
 & & & (1,3) & (1,5) \\
 & & & & (3,5)
\end{array}
\qquad
g: \begin{array}{ccccc}
{\color{green}(2,4)} & {\color{green}(2,6)} & (2,1) & (2,3) & (2,5) \\
 & {\color{blue}(4,6)} & {\color{green}(4,1)} & {\color{green}(4,3)} & {\color{green}(4,5)} \\
 & & {\color{green}(6,1)} & {\color{green}(6,3)} & {\color{green}(6,5)} \\
 & & & (1,3) & (1,5) \\
 & & & & (3,5)
\end{array}
$$

Hence $\operatorname{sign} g = -\operatorname{sign} f$.

Theorem. If the permutation $g \in S_n$ is obtained from the permutation $f \in S_n$ by the transposition of two elements then

$$\operatorname{sign} g \;=\; -\operatorname{sign} f$$

9. **Group of permutations S_3**

Let us compute signatures of the elements of S_3:

$f_1 = (1,2,3)$	$(1,2)$	$(1,3)$ $(2,3)$	$\operatorname{sign} f_1 = (-1)^0 = 1$
$f_2 = (1,3,2)$	$(1,3)$	$(1,2)$ $(3,2)$	$\operatorname{sign} f_2 = (-1)^1 = -1$
$f_3 = (2,1,3)$	$(2,1)$	$(2,3)$ $(1,3)$	$\operatorname{sign} f_3 = (-1)^1 = -1$
$f_4 = (3,1,2)$	$(3,1)$	$(3,2)$ $(1,2)$	$\operatorname{sign} f_4 = (-1)^2 = 1$
$f_5 = (2,3,1)$	$(2,3)$	$(2,1)$ $(3,1)$	$\operatorname{sign} f_5 = (-1)^2 = 1$
$f_6 = (3,2,1)$	$(3,2)$	$(3,1)$ $(2,1)$	$\operatorname{sign} f_6 = (-1)^3 = -1$

10. **Combinatoric interpretation of the determinant of 3×3 matrix**

The expression for the determinant of a 3×3 matrix A

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underset{f_1=(1,2,3)}{a_{11}a_{22}a_{33}} + \underset{f_3=(2,3,1)}{a_{12}a_{23}a_{31}} + \underset{f_4=(3,1,2)}{a_{13}a_{21}a_{32}} - \underset{f_6=(3,2,1)}{a_{13}a_{22}a_{31}} - \underset{f_3=(2,1,3)}{a_{12}a_{21}a_{33}} - \underset{f_2=(1,3,2)}{a_{11}a_{23}a_{32}}$$

is the linear combination of the products $a_{k_1j_1}a_{k_2j_2}a_{k_3j_3}$ of elements of the matrix A, taken one from each row and each column. By rearranging elements in each product we can re-write each product in the form

$$a_{1i_1}a_{2i_2}a_{3i_3}$$

Then the 3-tuple (i_1, i_2, i_3) form some permutation of three elements. Note that the sign in front of the product coincide with the signature of the corresponding permutation. So, we can interpret the expression of the determinant of a 3×3 matrix as a sum taken over all possible permutations of 3 elements:

$$\det A = \sum_{f \in S_3} \operatorname{sign} f \cdot a_{1f(1)}a_{2f(2)}a_{3f(3)}$$

This definition can be easily extended in for arbitrary $n \times n$ matrices.

11. **Combinatoric definition of the determinant of $n \times n$ matrix**

Definition. The determinant of an $n \times n$ matrix is the scalar

$$\det A = \sum_{f \in S_n} \operatorname{sign} f \cdot a_{1f(1)}a_{2f(2)} \dots a_{nf(n)}$$

where S_n denotes the set of all possible permutations of n elements (so, the sum contains $n!$ terms) and $\operatorname{sign} f = \pm 1$ is the signature of the permutation f (equal to the -1 power to number of inversions in the permutation f).

12. **Minors and cofactors**

Definition. Let $A = (a_{ij})$ be an $n \times n$ matrix, and let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The determinant of M_{ij} is called the *minor* of a_{ij}. We define the *cofactor* A_{ij} of a_{ij} by

$$A_{ij} \;=\; (-1)^{i+j} \det M_{ij}$$

13. **Cofactor expansion of the determinants with respect to the first row**

Note that for a determinant of 2×2 and 3×3 matrices the following formulas hold:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \;=\; a_{11}\, a_{22} \;-\; a_{12}\, a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \;=\; a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \;-\; a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \;+\; a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

These formulas are called the *cofactor expansion of the determinant with respect to the first row*. It turns out that similar expansions hold with respect to any row or a column of any $n \times n$ matrices (we will prove it in the next sections). The interesting feature of these formulas is that they allow us to reduce the computation of the determinant of $n \times n$ matrix to the computation of n determinants of some $(n-1) \times (n-1)$ matrices. This makes it possible to consider another approach to the definition of the determinant of $n \times n$ matrices for arbitrary n which is alternative to the combinatoric approach.

14. **Recursive approach to the definition of determinants**

The combinatoric definition of the determinant is a bit complicated (it presumes that readers are familiar with the concept of permutations). The alternative approach to define the determinant of the $n \times n$ matrix for arbitrary n is the recursive one. The readers who wants to avoid studying of permutations can use the definition below as the basic one. In the next sections we will prove that the recursive definition leads to the same value of the determinant as the value given by the combinatoric definition of the determinant.

Definition. The determinant of an $n \times n$ matrix A, denoted $\det A$, is a scalar associated with the matrix A that is defined inductively as

$$\det A \;=\; \begin{cases} a_{11} & \text{if} \quad n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}, & \text{if} \quad n > 1 \end{cases}$$

where

$$A_{1j} \;=\; (-1)^{1+j} \det M_{1j}, \qquad j = 1, \ldots, n$$

are the cofactors associated with the entries in the first row of A.

2.6 Properties of determinants

1. **Determinant of the transposed matrix**

THEOREM. For any $n \times n$ matrix A

$$\boxed{\det(A^T) \;=\; \det A}$$

PROOF.

1. $\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$

$$\begin{array}{ll} a_{13}a_{21}a_{32} & f = (3,1,2) \\ a_{21}a_{32}a_{13} & g = (2,3,1) \end{array} \quad \Longrightarrow \quad a_{1f(1)}a_{2f(2)}\dots a_{3f(3)} = a_{g(1)1}a_{g(2)2}\dots a_{g(n)n}$$

2.

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

$$g \circ f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = id$$

$$f \in S_n \quad \Longrightarrow \quad g = f^{-1}$$

3.

$$\begin{array}{ll} f_1 = (1,2,3) & f_1^{-1} = (1,2,3) \\ f_2 = (1,3,2) & f_2^{-1} = (1,3,2) \\ f_3 = (2,1,3) & f_3^{-1} = (2,1,3) \\ f_4 = (3,1,2) & f_4^{-1} = (2,3,1) \\ f_5 = (2,3,1) & f_5^{-1} = (3,1,2) \\ f_6 = (3,2,1) & f_6^{-1} = (3,2,1) \end{array} \quad \Longrightarrow \quad \{\, f \mid f \in S_3 \,\} = \{\, f^{-1} \mid f \in S_3 \,\}$$

4.

$$\begin{array}{l} \text{sign}\, f_1^{-1} = \text{sign}\, f_1 = 1 \\ \text{sign}\, f_2^{-1} = \text{sign}\, f_2 = -1 \\ \text{sign}\, f_3^{-1} = \text{sign}\, f_3 = -1 \\ \text{sign}\, f_4^{-1} = \text{sign}\, f_5 = 1 \\ \text{sign}\, f_5^{-1} = \text{sign}\, f_4 = 1 \\ \text{sign}\, f_6^{-1} = \text{sign}\, f_6 = -1 \end{array} \quad \Longrightarrow \quad \text{sign}\, f^{-1} = \text{sign}\, f$$

5.

$$\det A = \sum_{f \in S_3} \text{sign}\, f \cdot a_{1f(1)}a_{2f(2)}a_{3f(3)} \overset{g=f^{-1}}{=} \sum_{f \in S_3} \underbrace{\text{sign}\, f}_{=\,\text{sign}\, g} \cdot a_{g(1)1}a_{g(2)2}a_{g(3)3} =$$

$$= \sum_{g \in S_3} \text{sign}\, g \cdot a_{g(1)1}a_{g(2)2}a_{g(3)3} = \sum_{g \in S_3} \text{sign}\, g \cdot (A^T)_{1g(1)}(A^T)_{2g(2)}(A^T)_{3g(3)} = \det(A^T)$$

2. **Interchanging any pair of rows or columns changes the sign of the determinant**

THEOREM. If the matrix B is obtained from the $n \times n$ matrix A by the interchange of its two rows or two columns then

$$\boxed{\det B \;=\; -\det A}$$

PROOF.

1. Assume, for example, that we interchange the 1st and the 2nd rows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

2.

$$\det A = \sum_{f \in S_3} \operatorname{sign} f \cdot a_{1f(1)} a_{2f(2)} a_{3f(3)}$$

$$\det B = \sum_{f \in S_3} \operatorname{sign} f \cdot a_{2f(1)} a_{1f(2)} a_{3f(3)} = \sum_{f \in S_3} \operatorname{sign} f \cdot a_{1f(2)} a_{2f(1)} a_{3f(3)}$$

Denote by g the permutation obtained from f by interchanging its 1st and 2nd elements.

$$\det B = \sum_{f \in S_3} \operatorname{sign} f \cdot a_{1g(1)} a_{2g(2)} a_{3g(3)}$$

3.

$$\operatorname{sign} g = -\operatorname{sign} f$$

4.

$$\begin{array}{ll} f_1 = (1,2,3) & g_1 = (2,1,3) \\ f_2 = (1,3,2) & g_2 = (3,1,2) \\ f_3 = (2,1,3) & g_3 = (1,2,3) \\ f_4 = (3,1,2) & g_4 = (1,3,2) \\ f_5 = (2,3,1) & g_5 = (3,2,1) \\ f_6 = (3,2,1) & g_6 = (2,3,1) \end{array} \qquad \Longrightarrow \qquad \{\ f \mid f \in S_3\ \} = \{\ g \mid f \in S_3\ \}$$

5.

$$\det B = \sum_{f \in S_3} \underbrace{\operatorname{sign} f}_{= -\operatorname{sign} g} \cdot a_{1g(1)} a_{2g(2)} a_{3g(3)} = -\sum_{g \in S_3} \operatorname{sign} g \cdot a_{1g(1)} a_{2g(2)} a_{3g(3)} = -\det A$$

3. **Determinant is a linear function of elements of a certain row**

THEOREM. Assume $\mathbf{x} \in \mathbb{R}^n$ is a row vector and $l : \mathbb{R}^n \to \mathbb{R}$ is the function

$$l(\mathbf{x}) \;=\; \det \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{pmatrix}, \qquad \mathbf{x} = (x_1, x_2, \dots, x_n)$$

where the variable vector $\mathbf{x}$ forms the i-th row of the matrix and the elements in all other rows are considered to be fixed coefficients. Then there are coefficients A_1, A_2, $\dots$, $A_n \in \mathbb{R}$ which are independent of x_1, x_2, $\dots$, x_n such that

$$l(\mathbf{x}) \;=\; A_1x_1 + A_2x_2 + \ldots + A_nx_n, \qquad \forall\, \mathbf{x} \in \mathbb{R}^n$$

PROOF.

$$\begin{aligned} \det A \;&=\; \sum_{f \in S_n} \operatorname{sign} f \cdot a_{1f(1)}a_{2f(2)} \dots a_{nf(n)} \;= \\ &=\; \sum_{f \in S_n} \operatorname{sign} f \cdot a_{f(1)}b_{f(2)} \dots \boxed{x_{f(i)}} \dots w_{f(n)} \;= \\ &=\; x_1 \underbrace{\left(\;\dots\; \right)}_{=:\, A_1} + x_2 \underbrace{\left(\;\dots\; \right)}_{=:\, A_2} + \;\dots\; + x_n \underbrace{\left(\;\dots\; \right)}_{=:\, A_n} \end{aligned}$$

4. **Properties of determinants**

THEOREM.

1) If A has a row/column consisting entirely of zeros, then $\det A = 0$

$$\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{vmatrix} \;=\; 0$$

2) A has two identical rows/columns then $\det A = 0$

$$\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{vmatrix} \;=\; 0$$

3) If some row/column of A is the sum of two vectors $\mathbf{x}$ and $\mathbf{y}$ then $\det A$ is the sum of two corresponding determinants:

$$\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ x_1+y_1 & x_2+y_2 & \dots & x_n+y_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ y_1 & y_2 & \dots & y_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{vmatrix}$$

4) Scaling a row/column of A by $\alpha \in \mathbb{R}$ multiplies the $\det A$ by α

$$\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ \alpha\, x_1 & \alpha\, x_2 & \dots & \alpha\, x_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{vmatrix} = \alpha \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{vmatrix}$$

5) Adding a scalar multiple of one row/column to another row/column does not change the value of the determinant.

$$\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ y_1+\alpha\, x_1 & y_2+\alpha\, x_2 & \dots & y_n+\alpha\, x_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ y_1 & y_2 & \dots & y_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{vmatrix}$$

6) The determinant of a diagonal matrix is the product of its diagonal elements:

$$\begin{vmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n \end{vmatrix} = d_1 \cdot d_2 \cdot \ldots \cdot d_n$$

In particular, the determinant of the identity matrix I is equal to 1.

5. Determinant of an elementary matrices

Theorem. If E is an elementary matrix then

$$\boxed{\det(EA) = \det(AE) = \det E \cdot \det A}$$

where

1) If E is of Type I then $\det E = -1$.
2) If E is of Type II then $\det E = \alpha$.
3) If E is of Type III then $\det E = 1$.

6. **Criterion of non-singularity**

THEOREM. An $n \times n$ matrix A is singular if and only if $\det A = 0$:

$$\boxed{A \text{ is invertible} \quad \Longleftrightarrow \quad \det A \neq 0}$$

PROOF. For any A there exist elementary matrices E_1, E_2, $\ldots$, E_k such that

$$E_k E_{k-1} \ldots E_1 A \;=\; I$$

where U is the $n \times n$ matrix in the reduced row echelon form. As $\det(EA) = \det E \cdot \det A$ we obtain

$$\underbrace{\det E_k \cdot \det E_{k-1} \cdot \ldots \cdot \det E_1}_{\neq 0} \cdot \det A \;=\; \det U$$

The matrix A is non-singular if and only if $U = I$. As $\det I = 1$ we obtain the result.

7. **Multiplicativity property**

THEOREM. If A and B are $n \times n$ matrices then

$$\boxed{\det(AB) \;=\; \det A \cdot \det B}$$

PROOF.

1. Assume B is singular, i.e. $\det B = 0$. Let us prove that AB is also singular. Indeed, from the equivalent condition of non-singularity we conclude that there exists a non-trivial solution to the linear system $B\mathbf{x} = \mathbf{0}$. Then $\mathbf{x} \neq \mathbf{0}$ is also a non-trivial solution to the system $(AB)\mathbf{x} = \mathbf{0}$. Hence the matrix AB is singular and hence $\det(AB) = 0$. So, we conclude

$$\underbrace{\det(AB)}_{=0} \;=\; \det A \cdot \underbrace{\det B}_{=0}$$

2. Assume now that B is non-singular. Then it is row equivalent to the identity matrix. Hence there exist elementary matrices E_1, E_2, $\ldots$, E_k such that

$$E_k E_{k-1} \ldots E_1 B \;=\; I \qquad \Longrightarrow \qquad B = E_1^{-1} E_2^{-1} \ldots E_k^{-1}$$

As $\det(AE) = \det A \cdot \det E$ for any elementary matrix E and the matrices, inverse to elementary matrices, are elementary matrices themselves, we obtain

$$\begin{aligned}
\det(AB) \;&=\; \det(A E_1^{-1} E_2^{-1} \ldots E_k^{-1}) \;= \\
&=\; \det A \cdot \det(E_1^{-1}) \cdot \det(E_2^{-1}) \ldots \det(E_k^{-1}) \;= \\
&=\; \det A \cdot \det(\underbrace{E_1^{-1} E_2^{-1} \ldots E_k^{-1}}_{=B}) \;=\; \det A \cdot \det B
\end{aligned}$$

2.7 Cofactor expansion of the determinant

1. **Cofactor matrix**

   Definition. Remind that for a $n\times n$ matrix $A=(a_{ij})$ we denote by M_{ij} the $(n-1)\times(n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The determinant of M_{ij} is called the *minor* of an element a_{ij} and the *cofactor* A_{ij} of a_{ij} is defined by

$$A_{ij} \;=\; (-1)^{i+j}\det M_{ij}$$

   The *cofactor matrix* of a matrix $A=(a_{ij})$ is the $n\times n$ matrix whose entries are (A_{ij})

$$\mathrm{Cof}\,A \;=\; \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix} \qquad \begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

2. **Example**

$$A \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathrm{Cof}\,A \;=\; \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{2,3} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

3. **Cofactor expansion of the 2×2 and 3×3 determinants**

   We have already came across the cofactor expansions of 2×2 and 3×3 determinants with respect to the first row:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{21} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

   Using the formulas for 2×2 and 3×3 determinants it is not difficult to verify that the similar expansions hold also with respect to any row or any column of the matrix. Namely, the row expansions of the 2×2 and 3×3 determinants are

   $n=2$

$$\det A \;=\; \begin{cases} a_{11}A_{11} + a_{12}A_{12} \\ a_{21}A_{21} + a_{22}A_{22} \end{cases}$$

   $n=3$

$$\det A \;=\; \begin{cases} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{cases}$$

The corresponding column expansions are

$$n = 2 \qquad\qquad\qquad\qquad n = 3$$

$$\det A \;=\; \begin{cases} a_{11}A_{11} + a_{21}A_{21} \\ a_{12}A_{12} + a_{22}A_{22} \end{cases} \qquad \det A \;=\; \begin{cases} a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \\ a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \\ a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \end{cases}$$

It turns out that the similar relations hold also in the case of arbitrary n.

4. **Cofactor expansion of the determinant with respect to arbitrary rows and columns**

THEOREM. Assume A is an $n \times n$ matrix whose elements are (a_{ij}) and denote by A_{ij} the cofactor associated with the element a_{ij}. Then for any i, $j = 1, \ldots, n$

$$\det A \;=\; \sum_{k=1}^{n} a_{ik}A_{ik} \quad \text{— the cofactor expansion with respect to } i\text{-th row}$$

$$\det A \;=\; \sum_{k=1}^{n} a_{kj}A_{kj} \quad \text{— the cofactor expansion with respect to } j\text{-th column}$$

PROOF.

1. We already know that the determinant of the $n \times n$ matrix is the linear function od the elements of some row. This means that for the i-th row there exists coefficients $x_1^{(i)}$, $x_2^{(i)}$, $\ldots$, $x_n^{(i)} \in \mathbb{R}$ which are independent of the elements a_{i1}, a_{i2}, $\ldots$, a_{in} of the i-th row of A and such that

$$\det A \;=\; x_1^{(i)}a_{i1} + x_2^{(i)}a_{i2} + \ldots + x_n^{(i)}a_{in}$$

Our goal is to show that

$$x_j^{(i)} \;=\; A_{ij}, \qquad i, j = 1, \ldots, n.$$

2. Let us show first that $x_n^{(n)} = A_{nn}$. Remind that in the combinatoric expression for $\det A$ all terms containing a_{nn} have the form

$$a_{1i_1}a_{2i_2} \ldots a_{n-1i_{n-1}}a_{nn},$$

where $(i_1, i_2, \ldots, i_{n-1})$ is some permutation of numbers $(1, 2, \ldots, n-1)$. In other words,

$$x_n^{(n)} \;=\; \sum_{f \in G} \operatorname{sign} f \cdot a_{1f(1)}a_{2f(2)} \ldots a_{n-1f(n-1)}$$

where $G := \{f \in S_n \mid f(n) = n\}$ is the set of all permutations of n elements which do not permute the last element.

3. Obviously, the set G is in one-to-one correspondence with the group S_{n-1} of all permutations of $n - 1$ elements. Hence we obtain

$$x_n^{(n)} \;=\; \sum_{g \in S_{n-1}} \operatorname{sign} f_g \cdot a_{1g(1)}a_{2g(2)} \ldots a_{n-1g(n-1)},$$

where $f_g \in S_n$ is a permutation of the form

$$f_g = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ g(1) & g(2) & \dots & g(n-1) & n \end{pmatrix}$$

4. Since the number of inversions in f_g and in g is the same, we obtain

$$\operatorname{sign} f_g \;=\; \operatorname{sign} g$$

5. So we obtain

$$x_n^{(n)} \;=\; \sum_{g \in S_{n-1}} \operatorname{sign} g \cdot a_{1g(1)} a_{2g(2)} \dots a_{n-1g(n-1)}$$

The last expression if nothing but the combinatoric definition of the determinant of the $(n-1) \times (n-1)$ matrix obtaiend from A by the eliminating of its last row and column. Hence we arrive at

$$x_n^{(n)} \;=\; \det M_{nn} \;=\; A_{nn}$$

6. Consider now the element a_{ij}. Let us introduce the matrix B, which is obtained from the matrix A by the following operations: first, we interchange the rows sequentially so that the i-th row of the matrix A goes to the last place (these are $n-i$ interchanges of rows), and then interchange the columns of the resulting matrix so that the j-th column of this matrix goes to the last column in the matrix B (this is another $n-j$ column rearrangement operations).

$$A \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \qquad \longrightarrow \qquad B \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{14} & a_{15} & a_{13} \\ a_{31} & a_{32} & a_{34} & a_{35} & a_{33} \\ a_{41} & a_{42} & a_{44} & a_{45} & a_{43} \\ a_{51} & a_{52} & a_{54} & a_{55} & a_{53} \\ a_{21} & a_{22} & a_{24} & a_{25} & a_{23} \end{pmatrix}$$

Note that

$$M_{nn}^B = M_{ij}^A$$

7. As B is obtained from A by $(n-i)+(n-j) = 2n-(i+j)$ interchange of rows and columns from the properties of determinants we conclude

$$\det B = (-1)^{2n-i-j} \det A = (-1)^{i+j} \det A.$$

8. We know that $\det A$ is a linear function of the elements of its i-th row and $\det B$ is the linear function of elements of its last row. Hence there exists coefficients $x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}$ (independent of the elements of i-th row of a matrix A) and coefficients $y_1^{(i)}, y_2^{(i)}, \dots, y_n^{(i)}$ (independent of the elements of the last row of a matrix B) such that

$$\begin{aligned} \det A \;&=\; x_1^{(i)} a_{i1} + x_2^{(i)} a_{i2} + \dots + x_j^{(i)} a_{ij} + \dots + x_n^{(i)} a_{in} \\ \det B \;&=\; y_1^{(n)} a_{i1} + y_2^{(n)} a_{i2} + \;\dots\; + y_{n-1}^{(n)} a_{n-1,n} + y_n^{(n)} a_{ij} \end{aligned}$$

9. We know that the coefficients $x_1^{(i)}$, $x_2^{(i)}$, ..., $x_n^{(i)}$ and $y_1^{(n)}$, $y_2^{(n)}$, ..., $y_n^{(n)}$ do not depend on the elements of the corresponding rows of the matrices A and B. Consider a matrix $\tilde{A}$ whose entries in all rows, except the i-th row, coincide with the elements of the matrix A, and in the i-th row all entries are zeros, except for the element $a_{ij} = 1$. Then for this matrix we obtain

$$\det \tilde{A} \;=\; x_1^{(i)} \cdot 0 + x_2^{(i)} \cdot 0 + \ldots + x_j^{(i)} \cdot 1 + \ldots + x_n^{(i)} \cdot 0 \;=\; x_j^{(i)}$$

Now transform that matrix $\tilde{A}$ by interchanging its rows and columns into the matrix $\tilde{B}$ in the same way as we transformed before A into B. Then

$$\det \tilde{B} \;=\; y_1^{(n)} \cdot 0 + y_2^{(n)} \cdot 0 + \;\ldots\; + y_{n-1}^{(n)} \cdot 0 + y_n^{(n)} \cdot 1 \;=\; y_n^{(n)}$$

As the matrix $\tilde{B}$ is obtained from $\tilde{A}$ by $2n - (i + j)$ interchange of rows and columns from properties of determinants we conclude

$$\det \tilde{B} \;=\; (-1)^{i+j} \det \tilde{A} \qquad \Longrightarrow \qquad x_j^{(i)} \;=\; (-1)^{i+j} y_n^{(n)}$$

10. We already know that

$$y_n^{(n)} \;=\; \det M_{nn}^B \;=\; \det M_{ij}^A$$

Hence we conclude that

$$x_j^{(i)} \;=\; (-1)^{i+j} \det M_{ij}^A \;=\; A_{ij}$$

Finally we arrive at

$$\det A \;=\; A_{i1} a_{i1} + A_{i2} a_{i2} + \ldots + A_{in} a_{in}$$

Theorem is proved. $\square$

5. **Determinants of upper and lower triangular matrices**

Theorem. If an $n \times n$ matrix A is either upper or lower triangular then determinant of A equals the product of the diagonal elements of A.

Proof. For a lower triangular matrix (for example) at each step we are doing the cofactor expansion of the corresponding determinant with respect to the first row:

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} = a_{11} \cdot \begin{vmatrix} a_{22} & 0 & 0 & 0 \\ a_{32} & a_{33} & 0 & 0 \\ a_{42} & a_{43} & a_{44} & 0 \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} = a_{11} \cdot a_{22} \cdot \begin{vmatrix} a_{33} & 0 & 0 \\ a_{43} & a_{44} & 0 \\ a_{53} & a_{54} & a_{55} \end{vmatrix} =$$

$$= a_{11} \cdot a_{22} \cdot a_{33} \cdot \begin{vmatrix} a_{44} & 0 \\ a_{54} & a_{55} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} \cdot a_{44} \cdot a_{55}$$

2.8 Inverse matrix and cofactor matrix. Cramer's rule

1. **Basic property of the cofactor matrix**

THEOREM. For any $n \times n$ matrix A the product of the transpose of A by the cofactor matrix Cof A equals to the $n \times n$ identity matrix $\mathbb{I}$ multiplied by the scalar det A:

$$\boxed{A^T \operatorname{Cof} A \;=\; \det A \; I}$$

In other words, for any i, $j = 1, \ldots, n$

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn} \;=\; \begin{cases} \det A, & i = j \\ 0, & i \neq j \end{cases}$$

PROOF.

1. In the case $i = j$ this formula is just a cofactor expansion of the determinant with respect to the i-th row that was already proved in the previous section.

2. Now consider the case $i \neq j$. Consider the matrix $\tilde{A}$ which is obtained from matrix A by replacing the elements in j-th row by the elements of i-th row (in the example $i = 4$ and $j = 2$):

$$A \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \qquad \longrightarrow \qquad \tilde{A} \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$$

3. As the matrix $\tilde{A}$ contains two identical rows we conclude

$$\det \tilde{A} \;=\; 0$$

4. Now we can write the cofactor expansion of det $\tilde{A}$ with respect to j-th row:

$$0 \;=\; \det \tilde{A} \;=\; a_{i1}\tilde{A}_{j1} + a_{i2}\tilde{A}_{j2} + \ldots + a_{in}\tilde{A}_{jn}$$

5. Note that the cofactors of all elements in j-th rows of matrices A and $\tilde{A}$ are the same:

$$A_{j1} = \tilde{A}_{j1}, \quad A_{j2} = \tilde{A}_{j2}, \quad \ldots \quad A_{jn} = \tilde{A}_{jn}$$

Hence we obtain

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn} \;=\; 0$$

2. **Computing the inverse matrix using determinants**

THEOREM. Assume an $n \times n$ matrix A is non-singular. Then

$$\boxed{A^{-1} \;=\; \frac{1}{\det A}\big(\,\mathrm{Cof}\,A\,\big)^T}$$

PROOF. We obtain

$$\big(\,\mathrm{Cof}\,A\,\big)^T A \;=\; \big(A^T\,\mathrm{Cof}\,A\big)^T \;=\; \big(\,\det A\;I\,\big)^T \qquad \Longrightarrow \qquad \big(\,\mathrm{Cof}\,A\,\big)^T A \;=\; \det A\; I$$

As A is non-singular we can multiply the last relation by the matrix A^{-1}. So we obtain

$$\big(\,\mathrm{Cof}\,A\,\big)^T \;=\; \det A\; A^{-1}$$

3. **Adjugate matrix**

DEFINITION. For the $n \times n$ matrix A its *adjugate* matrix $\mathrm{Adj}\,A$ is defined by

$$\mathrm{Adj}\,A \;:=\; \big(\,\mathrm{Cof}\,A\,\big)^T$$

Hence for a non-singular matrix A we obtain the formula

$$A^{-1} \;=\; \frac{1}{\det A}\;\mathrm{Adj}\,A$$

4. **Cramer's rule**

THEOREM. Let A be an $n \times n$ non-singular coefficient matrix, and let $\mathbf{b} \in \mathbb{R}^n$ be the given column vector

$$A \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathbf{b} \;=\; \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Denote by $A^{(i)}$ the matrix obtained from A by replacing the i-th column of A by $\mathbf{b}$:

$$A^{(1)} = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}, \qquad A^{(2)} = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}, \qquad A^{(3)} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}$$

Then the unique solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ to the linear system

$$A\mathbf{x} \;=\; \mathbf{b}$$

is given by the formulas

$$x_i \;=\; \frac{\det A^{(i)}}{\det A}, \qquad i = 1, 2, \ldots, n.$$

Proof. Using the formula $A^{-1} = \frac{1}{\det A} (\text{Cof } A)^T$ we obtain

$$\mathbf{x} \; = \; A^{-1}\mathbf{b} \; = \; \frac{1}{\det A} \, (\text{Cof } A)^T \, \mathbf{b}$$

Note that $(\text{Cof } A)^T$ has the entries

$$(\text{Cof } A)^T \;\; = \;\; \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

where A_{ij} is the cofactor of the element a_{ij} of the matrix A. Hence for the i-th entry of $\mathbf{x}$ we obtain the formula

$$x_i \;\; = \;\; \frac{1}{\det A} \Big(A_{1i}b_1 + A_{2i}b_2 + \ldots + A_{ni}b_n \Big)$$

The expression in the brackets is nothing but the cofactor expansion of $\det A^{(i)}$ with respect to the i-th column of the matrix $A^{(i)}$:

$$\det A^{(i)} \;\; = \;\; A_{1i}b_1 + A_{2i}b_2 + \ldots + A_{ni}b_n$$

Hence we arrive at

$$x_i \;\; = \;\; \frac{\det A^{(i)}}{\det A}$$

$\boxed{\text{END OF SEMESTER 1}}$