Lecture "Experimental Physics I"
(Prof. Dr. R. Seidel)

Lecture 21

Hydrodynamics of ideal fluids

- Basic definitions
- Euler’s equation for ideal fluids
- Continuity equation
- Bernoulli’s equation & applications

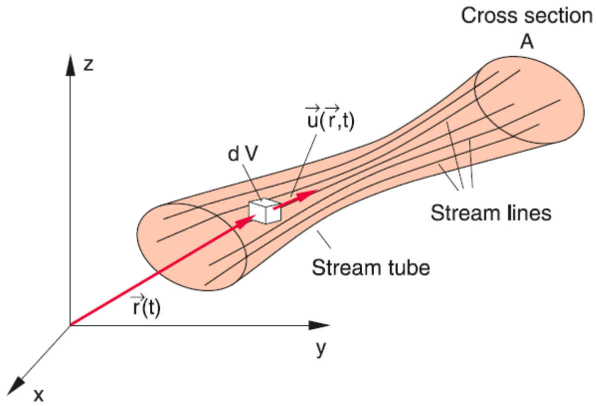
1) Describing fluid dynamics

So far, we considered only static fluids without flow. In the following we will describe moving fluids, i.e. the so-called fluid flow. To describe fluid flow, it is helpful to introduce and use the following quantities:

- **Fluid velocity $\vec{u}(\vec{r}, t)$:** A full description of a moving fluid is provided by the time-dependent vector field of the fluid velocity, since it provides the flow velocity at any point in time and space.
- **Flow field (velocity field) $\vec{u}(\vec{r}, t_0)$** is the **velocity field at a given time t_0.** It provides a snapshot of the spatial distribution of the fluid velocity. The flow field can change in time. The flow is called **stationary flow** if the flow field does not change in time, i.e. $\vec{u}(\vec{r}, t) = \vec{u}(\vec{r}, t_0)$ for all times t.
- **Streamline** is the trajectory $\vec{r}(t)$ of a given volume element dV in time. A comprehensive set of streamlines provides an intuitive visualization of the fluid flow in a liquid. A streamline starting from a given point can be obtained by integrating the displacements of the fluid flow over time:

$$\vec{r}(t) = \int_{t_0}^{t} \vec{u}(\vec{r}(t), t)dt + \vec{r}_0(t_0)$$

- **Stream tube** is a set of all the streamlines passing through a chosen area A. Since the liquid is always moving along the stream lines no liquid leaks out of the "virtual walls" of a stream tube.

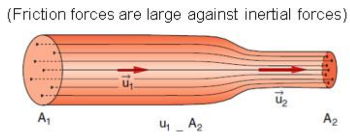


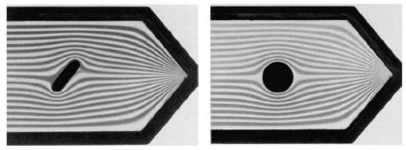
Fluid flow can be very complex (just imagine at a mountain stream), since the motion of a given fluid element is determined by different competing forces. We classify two extreme cases:

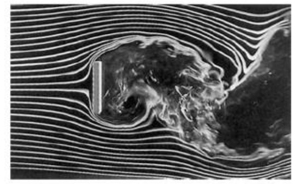
(i) The ideal liquid where we only have inertial forces from the acceleration of the fluid elements but no friction forces (e.g. liquid helium)

(ii) The viscous liquid where the friction forces are large compared to all other forces such that inertia can be neglected (e.g. honey)

If the frictional forces are dominant (viscous liquid) and the flow velocities are not too high, we have a so-called **laminar (non-turbulent) flow,** where the **stream lines do not cross each other** and **which is stationary.** If inertial forces cannot be neglected, we typically obtain together with the weaker friction forces **non-stationary turbulent flows** with very complex time variant patterns, such as vertices etc.







Experiment: Laminar flow around differently formed objects (circular disk, wing) illustrated by a string model on the OHP. A transition from "laminar" to "turbulent" flow occurs at high flow velocities.

2) Euler equation of ideal fluids

A) Equation of motion in dynamic fluids

To derive an equation of motion for fluid flow we look at a small volume element ΔV within the flow. From static fluids we know the force on a fluid element due to a pressure gradient inside the fluid $\vec{F}_p = -\vec{\nabla} p\, \Delta V$ as well as the external (gravity) force on a fluid element $\vec{F}_g = \rho \vec{g}\, \Delta V$. In the **dynamic case** we have additionally to consider the **internal friction force $\vec{F}_R$ on the element** due to the motion of the neighboring fluid elements. If the velocity vector of the moving fluid element is changing in time, it gets accelerated, such that we have furthermore to consider the **inertia of the fluid element** that equals the sum of all the externally acting forces. This provides the following **equation of motion for the element ΔV:**

Inertia of fluid element

$$\vec{F} = \vec{F}_p + \vec{F}_g + \vec{F}_R = \Delta m \frac{d^2 \vec{r}}{dt^2} = \rho\, \Delta V \frac{d\vec{u}}{dt}$$

$\vec{F}_p = -\vec{\nabla} p\, \Delta V$

velocity of fluid element

$\vec{F}_g = \rho \vec{g}\, \Delta V$

friction within fluid

Dividing the derived equation of motion by ΔV provides:

$$\rho \frac{d\vec{u}}{dt} = \underbrace{-\vec{\nabla} p}_{\substack{\text{pressure} \\ \text{gradient}}} + \underbrace{\rho\, \vec{g}}_{\text{gravity}} + \underbrace{\frac{f_R}{\Delta V}}_{\text{friction}}$$

For **ideal fluids** we can omit the friction term since no friction shall occur by definition (inviscid fluid). To derive our first hydrodynamic equation, we have now practically everything, except a good description of the acceleration as well as the friction force inside the fluid. Corresponding descriptions we will derive in the following lecture parts.

B) The Euler equation

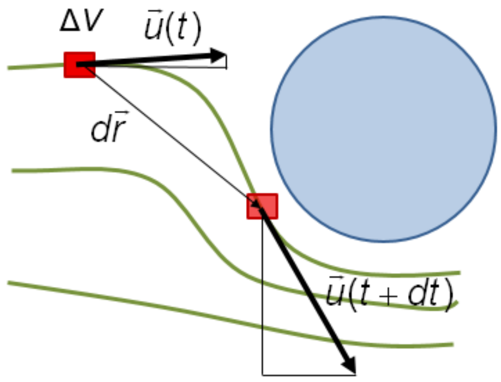
The first hydrodynamic equation we will derive is the **Euler equation**. We assume that we can neglect friction, i.e. an **ideal (inviscid) fluid.** The flow is then dominated by inertial forces, i.e. the momentum change of all the volume elements dV in time. The Euler equation is therefore also called **momentum equation**.

From our previous considerations, we know already all acting forces. We have, however, to look closer on the acceleration $d\vec{u}/dt$ in our initial equation. The velocity field of the fluid is given by $\vec{u} = \vec{u}(\vec{r}, t)$. Let us now look at the volume element dV and its change in velocity between t and $t + dt$. During this time, the element travels with the fluid approximately by $d\vec{r} = \vec{u}\,dt$. After time dt its velocity is thus given as:

$$\vec{u}_{(dV)} = \vec{u}(\vec{r} + \underbrace{\vec{u}dt}_{d\vec{r}}, t + dt)$$

The change in the total velocity can therefore be due to:

1) A change of the local velocity in time $\partial\vec{u}/\partial t$ in case of a time dependent flow field.
2) A change of the velocity vector due to the traveling within the flow to a new place. Such a change can even occur inside a stationary flow:



The change in velocity due to the position change is for each component of the velocity vector given by a total differential. For example, the x-component of the velocity vector would for a displacement of $d\vec{r}$ change by:

$$du_{x,d\vec{r}} = \left(\frac{\partial u_x}{\partial x}\right)_{\vec{r}} dx + \left(\frac{\partial u_x}{\partial y}\right)_{\vec{r}} dy + \left(\frac{\partial u_x}{\partial z}\right)_{\vec{r}} dz$$

Each term of the sum describes the change of u_x for a small displacement dx, dy or dz along the corresponding coordinate axis.

Including the time dependence of the velocity field, the **total (substantial) acceleration** of the fluid element along *x* thus becomes:

$$\frac{du_x}{dt} = \frac{\partial u_x}{\partial t} + \left(\frac{\partial u_x}{\partial x}\right)_{\vec{r}} \underbrace{\frac{dx}{dt}}_{u_x} + \left(\frac{\partial u_x}{\partial y}\right)_{\vec{r}} \underbrace{\frac{dy}{dt}}_{u_y} + \left(\frac{\partial u_x}{\partial z}\right)_{\vec{r}} \underbrace{\frac{dz}{dt}}_{u_z} = \frac{\partial u_x}{\partial t} + (\vec{u} \cdot \vec{\nabla})u_x$$

with the position derivatives being the velocity components u_x, u_y, u_z. One can rewrite this equation using the **scalar product between velocity and Nabla operator** (see right hand side). We can derive similar expressions for the velocity components u_y and u_z, such that the total acceleration for the fluid element dV becomes:

$$\frac{d\vec{u}}{dt} = \frac{\partial\vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u}$$

In case of stationary flow, the partial time derivative (1st term) vanishes. The **2nd term is the so-called convective acceleration** that originates from the position change within the flow as just discussed.

Inserting the expression for the total acceleration of the fluid element dV into the **equation of motion** from above and setting the friction to zero, we get the **Euler equation** (from Euler in 1755):

$$\rho\frac{d\vec{u}}{dt} = \rho\left[\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla}p + \rho g$$

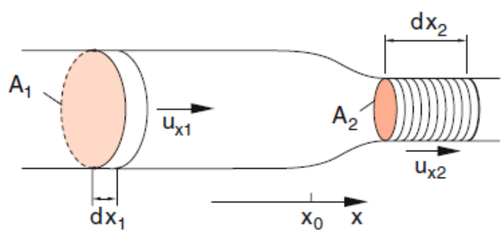
It states that a velocity change of a fluid element is driven either by a pressure difference and/or an external force (such as gravity, …).

3) Continuity equation

To describe fluid dynamics, we need further tools in addition to the Euler equation. An important equation is the **continuity equation** that ensures **mass conservation** in a fluid. Since we consider individual fluid elements that move around and potentially bump into each other, we must make sure that no mass gets lost, which is provided by this relation.

A) Simplified derivation

To understand the concept of the continuity equation, we first look at a fluid moving through at a tube with two different cross-sectional areas. In both cross-sectional areas the fluid flow is parallel to the tube axis. The average fluid velocity for the cross-sectional area A_1 shall be u_{x1} and for A_2 it shall be u_{x2}.



The mass transport per time, i.e. the **mass current**, through the tube cross section at a given position is obtained from the volume that is passing the cross section:

$$\frac{dM}{dt} = \rho\frac{\overbrace{A_1 dx_1}^{dV_1}}{dt} = \underbrace{\rho\, u_{x1}}_{j_{x1}}\, A_1$$

According to the right side of the equation the **mass current** can be expressed as

$$\frac{dM}{dt} = \vec{j}\cdot\vec{A}$$

where

$$\vec{j} = \rho\,\vec{u}$$

is the **mass flux**, being defined as the **mass current per cross-sectional area**. It points in the direction of the velocity vector, while $\vec{A}$ is normal to the considered cross sectional area. For an **incompressible fluid** we must have the **same mass current at each position**, since the mass cannot leave the tube and can also not accumulate in between the two positions:

$$\frac{dM_1}{dt} = \frac{dM_2}{dt}$$

Inserting provides

$$\rho\, u_{x1}A_1 = \rho\, u_{x2}A_2$$

And transformation gives:

$$\frac{u_{x1}}{u_{x2}} = \frac{A_2}{A_1}$$

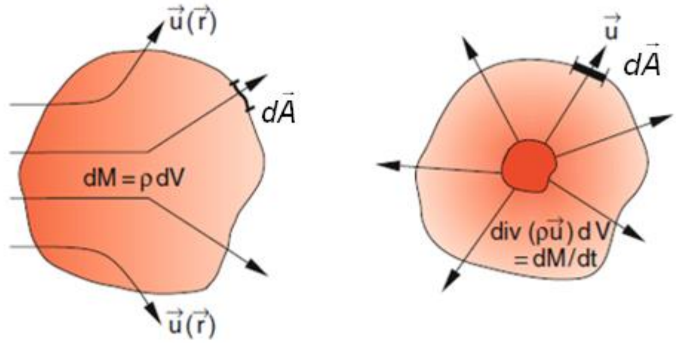
Thus, the **fluid is faster in narrower tube part, since per time the same mass must be transported at any position along the tube independent of the cross-sectional area (incompressible fluid)**. This reflects intuitively the principle of the continuity equation.

B) General derivation

We now want to derive a more general form of the continuity equation independent of a particular geometry. To this end we look at the fluid mass M in a given volume V for which we can write:

$$M = \int_V \rho \, dV$$

Now we must consider that a mass change in V can only be achieved by a non-zero mass current through the surface area A of the volume. In simple words, a mass reduction in V is only obtained if per time more mass leaves than enters through the total surface:



We get the total mass decrease per time by integrating the mass flux over the whole surface:

$$\frac{\partial M}{\partial t} = -\int_A \vec{j} \cdot d\vec{A} = -\underbrace{\int_A \overbrace{\rho \, \vec{u} \cdot d\vec{A}}^{\text{mass current through dA}}}_{\substack{\text{net mass loss per dt} \\ \text{through whole surf.}}}$$

where $d\vec{A}$ is an outside-directed vector normal to the surface at the given point with length dA. Thus, $\vec{j} \cdot d\vec{A}$ describes the local mass current through dA out of the volume. The negative sign of the mass current ensures that a positive net mass current out of V leads to a mass loss in V. Now, we apply the divergence theorem, which transforms the surface integral into a volume integral using the divergence of the vector field:

$$\underbrace{\int_S \overbrace{\rho \, \vec{u} \, d\vec{A}}^{\text{mass current through dA}}}_{\substack{\text{net mass loss per dt} \\ \text{through whole surf.}}} = \underbrace{\int_V \overbrace{\mathrm{div}(\underbrace{\rho \, \vec{u}}_{flux}) \, dV}^{\text{mass loss per dt of dV}}}_{\substack{\text{sum of all mass losses} \\ \text{per dt of entire volume}}}$$

with $\mathrm{div}(\vec{r}) = \vec{\nabla} \cdot \vec{r}$. The term $\mathrm{div}(\rho \, \vec{u}) \, dV$ represents the mass loss per time of the infinitesimal volume dV at a given position. The sum over all mass losses/gains in each of the volume elements equals the net mass current through the total surface.

This sounds trivial but is an important ingredient for many vector field calculations. You will encounter the application of the divergence theorem many times in future, e.g. in electrodynamics for the electric and the magnetic field where we also look more into the mathematics of it. For a fixed volume we can write using the first equation for the mass change per time:

$$-\frac{\partial}{\partial t} \underbrace{\int_V \rho \, dV}_{\partial M} = -\int_V \frac{\partial \rho}{\partial t} \, dV = \int_V \mathrm{div}(\rho \, \vec{u}) \, dV$$

Since this equation must hold for any selected volume, the integrands on each side must be equal:

$$-\frac{\partial \rho}{\partial t} \, dV = \mathrm{div}(\rho \, \vec{u}) \, dV$$

Transformation provides us the **general form of the continuity equation**:

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\vec{u}) = 0$$

It states that an **effective mass loss per time within a volume element dV** (divergence term) **can only be achieved by a density reduction in time.** For an **incompressible fluid** the **density is a constant** and the continuity equation becomes:

$$\mathrm{div}(\vec{u}) = 0$$

This is equivalent to:

$$(\vec{\nabla}\cdot\vec{u}) = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$

This expression states that **the same amount of fluid that enters a volume element must leave it.** (Imagine little cube in which a change of u_x must get compensated by a change of u_y).

For a **tube with constant diameter** we have only lateral velocities in case of non-turbulent flow. Thus,

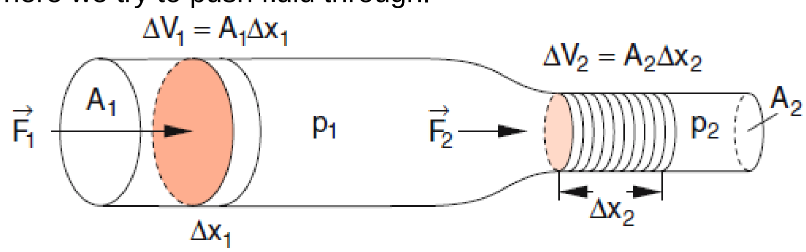
$$div\,\vec{u} = \frac{\partial u_x}{\partial x} + 0 + 0 = 0$$

Thus, the velocity inside the tube must be constant at all points.

4) Bernoulli equation

A) Derivation of the Bernoulli equation

In the following we want to derive an important relation for the pressure in a flowing fluid. For this, we consider an **incompressible ideal fluid** (no friction) for which we must have **energy conservation** (no work from compression nor friction). We again look at a tube with two different cross sections, where we try to push fluid through.



Since the fluid becomes faster in the thinner section, one needs to do work at the entry of the tube in order to accelerate the fluid. The **work done to displace a fluid volume ΔV** at the tube entry (or somewhere else) against the acting pressure is provided by:

$$\Delta W = F\,\Delta x = p\,A\,\Delta x = p\,\Delta V$$

The kinetic energy of the fluid volume is given by:

$$E_{kin} = \frac{1}{2}\Delta m u^2 = \frac{1}{2}\rho u^2 \Delta V$$

If one does work at position 1 then this work is used at position 2 to change the velocity of the volume ΔV_2 from u_1 to u_2 and to allow the fluid to do work against the pressure p_2 at this spot. Thus,

$$\underbrace{p_1\Delta V_1}_{\Delta W_1} = \frac{1}{2}\rho {u_2}^2\Delta V_2 - \frac{1}{2}\rho {u_1}^2\Delta V_1 + \underbrace{p_2\Delta V_2}_{\Delta W_2}$$

Incompressibility (from the continuity equation) demands that

$$\Delta V_1 = \Delta V_2 = \Delta V$$

With this, we can convert the upper equation to

$$p_1 + \frac{1}{2}\rho {u_1}^2 = p_2 + \frac{1}{2}\rho {u_2}^2\Delta V_2 = const.$$

It describes nothing else than the **conservation of total mechanical energy** per volume ΔV. Thus, the total mechanical energy is conserved, since we did not specify a concrete point along the tube. The derived equation is called **Bernoulli equation**, which in its simple form is given as (**see slide**):

$$\underbrace{p}_{\substack{\text{static}\\\text{pressure}}} + \underbrace{\frac{1}{2}\rho u^2}_{\substack{\text{ram}\\\text{pressure}}} = \underbrace{p_0}_{\substack{\text{total}\\\text{pressure}}} = const.$$

The individual pressure terms are named according to the equation and are explained in more detail below. When considering a height change within the tube we have to take it additionally into account (similar as for the extension of Pascal's principle to gravity), such that we arrive at:

$$p + \rho g h + \frac{1}{2}\rho u^2 = p_0 = const.$$

The pressure terms are:

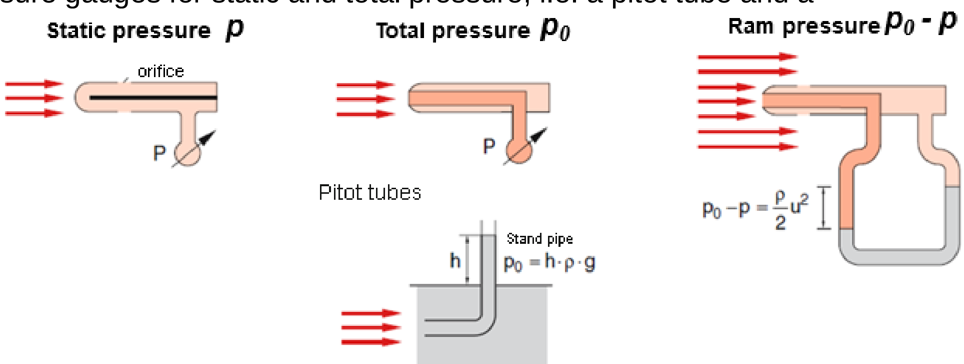
p_0 **total or stagnation pressure**

p **static pressure**, which is the pressure component that acts in all directions, but is best measured perpendicular to the flow at which no dynamic pressure acts

$p_0 - p = \frac{1}{2}\rho u^2$ **dynamic pressure/ram pressure**, which is directional and reflects the pressure increase when flow hits a perpendicularly oriented surface

Using different geometries of pressure meters one can directly measure the different types of pressure:

- The static pressure measured through an orifice perpendicular to the flow
- The total pressure is measured with a pitot tube with the tube opening oriented towards the flow at which the flow velocity becomes zero, such that one has the total pressure.
- The dynamic pressure is provided by the pressure difference between two counteracting pressure gauges for static and total pressure, i.e. a pitot tube and a



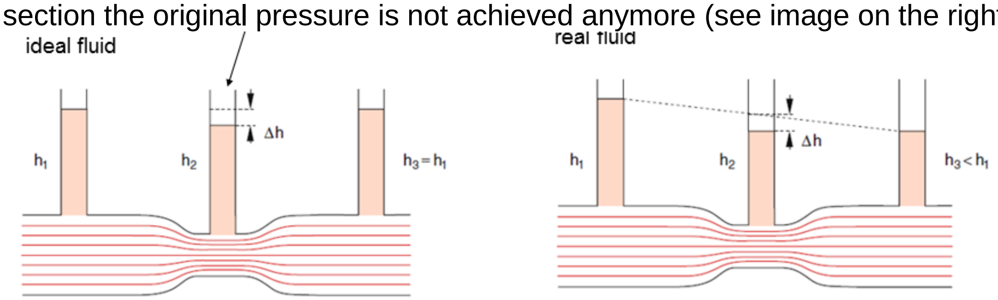
The measurement of the dynamic pressure is used in aircrafts to determine the airspeed of an aircraft (speed relative to the air), which is given according to Bernoullis law as (see slides):

$$u = \sqrt{\frac{2(p_0 - p)}{\rho}}$$

The Bernoulli equation states that the total pressure is the sum of static and dynamic pressure and remains constant. This is remarkable, since in the flow the static pressure becomes velocity dependent. Particularly, it predicts a reduced static pressure perpendicular to a rapidly flowing fluid. There is a larger number of observations that are somewhat counterintuitive and that can only be understood by considering the reduced static pressure for rapidly streaming liquids/gases:

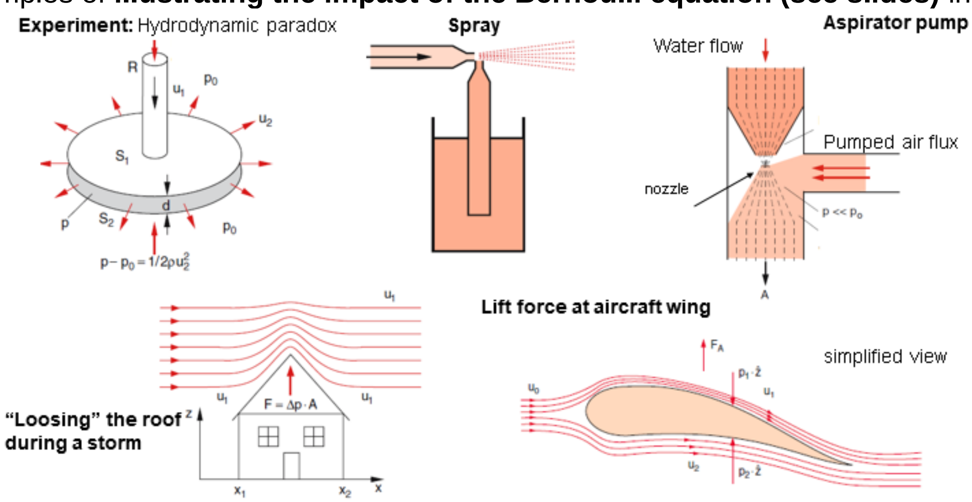
Experiments:

- **Venturi tube**: a tube with different cross sections that probes the static pressure for these sections using thin liquid columns perpendicular to the tube axis. These liquid columns act as pressure sensors, since height differences correspond to the difference in hydrostatic pressure.
  The experiment shows nicely that for the small cross section where the liquid is faster (higher dynamic pressure) a low hydrostatic pressure is obtained.
  For a real fluid there is a net pressure loss over distance due to friction, such that for the large cross section the original pressure is not achieved anymore (see image on the right).



- **Hydrodynamic paradox:** Objects with flow in between get pressed together instead of getting pushed apart
  - Spoons with air flow in between
  - Disks with flow in between (small and large scale)

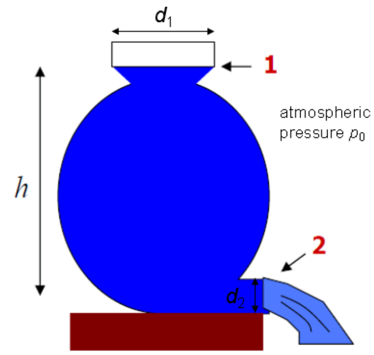
Other examples of **illustrating the impact of the Bernoulli equation (see slides)** include:



B) Torricelli's law

The Bernoulli equation is helpful to explain a number of phenomena but also to estimate fluid velocities assuming that friction does not play a role. For example, for our water rocket we can calculate the exhaust velocity, if we know the internal pressure in the rocket (which is approximately the total pressure by neglecting the slower movement of the liquid in the bottle). On the exit we have only the atmospheric pressure and the ram pressure, which gives us the velocity.

A similar problem is given by the **fluid velocity at the exit of a vessel** which is provided by the so-called **Torricelli's law**:



To derive Torricelli's law, we write down the total pressure at the fluid surface (point 1) and just outside the vessel exit (point 2). The vessel diameter shall be much larger than the diameter of the vessel exit. In this case, the velocity at point1 $u_2 \approx 0$. If p is the atmospheric pressure that provides the static pressure at point 1 as well as point 2 we get:

$$\underbrace{p + \rho g h}_{\text{point 1}} \approx \underbrace{p + \frac{1}{2}\rho u_2^2}_{\text{point 2}}$$

Transformation then provides for the fluid velocity at the outlet, which is known as Torricelli's law:

$$u_2 = \sqrt{2gh}$$

Lecture 21: Experiments

1) String model (OHP) for laminar flow around differently formed objects (circular disk, wing) and the transition from laminar to turbulent flow
2) Bernoulli equation: Static pressure reduction in a water flow within the confined region of a tube (Venturi tube)
3) Bernoulli equation / hydrodynamic paradox: airflow between two hanging spoons pushes the sppons together
4) Bernoulli equation / hydrodynamic paradox: airflow between two discs pulls the disks towards each other (small disks and/ large disks that can hold the body weight of a person)
5) Aspiration pump
6) Model of an ‚atomizer', i.e. liquid spraying based on Bernoulli's principle