Problem 2: **Keppler laws** **1 + 2+1 Points**

A planet which is orbiting around the sun is discovered. It is observed that its orbital period is about 7 years.

a. What is the planets average distance to the sun?
b. Assume that the maximum and minimum distance of the planet to the sun are $r_{\max} = 4\mathrm{AE}$ and $r_{\min} = 3.2\ \mathrm{AE}$. Calculate the elliptic parameters ε and a.
c. Another celestial body with mass $m = 0.3 \cdot 10^{-10} M_{sun}$ is parabolically passing by the sun. What is its angular momentum if its minimal distance is the same as that of the newly discovered planet?

Solution:

a.)

Kepplers third law:

$$\left(\frac{T}{T_E}\right)^2 = \left(\frac{r}{r_E}\right)^3$$

$$r = r_E\left(\frac{T}{T_E}\right)^{\frac{2}{3}} = 1AE * \left(\frac{7a}{1a}\right)^{\frac{2}{3}} = 3.66\ AE$$

$$\underline{= 5.5 * 10^{11} m}$$

b.)

minimum distance: $r_{\min} = a(1-\varepsilon)$ maximum distance: $r_{\max} = a(1+\varepsilon)$

$$\frac{(1-\varepsilon)}{(1+\varepsilon)} = \frac{r_{\min}}{r_{\max}} = 0.8$$

$$\leftrightarrow$$

$$1 - \varepsilon = 0.8 + 0.8\varepsilon$$

$$\leftrightarrow$$

$$\frac{1-0.8}{1.8} = \varepsilon$$

$$\underline{= 0.111}$$

$$a(1-\varepsilon) = 3.2AE$$

$$\leftrightarrow$$

$$a = \frac{3.2AE}{1-\varepsilon} = 3.599\ AE$$

$$= 5.38 * 10^{11} m$$

c.)

for parabolic orbit we have $r_{\text{min}} = \frac{L^2}{2Gm^2 M_{sun}}$

$$L = \sqrt{r_{min} 2Gm^2 M_{sun}}$$

$$= 6.40 * 10^{35} \frac{kg\ m^2}{s}$$

Problem 2: Spinning earth 6 Points

Calculate the kinetic energy of the earth due to its spinning about its axis. Compare your answere with the kinetic energy of the orbital motion of Earth's center of mass about the sun. Assume Earth to be a homogeneous sphere of mass $6 \cdot 10^{24} kg$ and a radius $6.4 \cdot 10^6 m$. The radius of Earth's orbit is $1.54 \cdot 10^{11} m$.

66 •• Calculate the kinetic energy of Earth due to its spinning about its axis, and compare your answer with the kinetic energy of the orbital motion of Earth's center of mass about the Sun. Assume Earth to be a homogeneous sphere of mass 6.0×10^{24} kg and radius 6.4×10^{6} m. The radius of Earth's orbit is 1.5×10^{11} m.

Picture the Problem Earth's rotational kinetic energy is given by $K_{\text{rot}} = \frac{1}{2}I\omega^2$ where I is its moment of inertia with respect to its axis of rotation. The center of mass of the Earth-Sun system is so close to the center of the Sun and the Earth-Sun distance so large that we can use the Earth-Sun distance as the separation of their centers of mass and assume each to be a point mass.

Express the rotational kinetic energy of Earth:

$$K_{\text{rot}} = \tfrac{1}{2}I\omega^2 \qquad (1)$$

Find the angular speed of Earth's rotation using the definition of ω:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\,\text{rad}}{24\,\text{h}\times\dfrac{3600\,\text{s}}{\text{h}}} = 7.27\times10^{-5}\,\text{rad/s}$$

From Table 9-1, for the moment of inertia of a homogeneous sphere, we find:

$$I = \tfrac{2}{5}MR^2 = \tfrac{2}{5}\left(6.0\times10^{24}\,\text{kg}\right)\left(6.4\times10^{6}\,\text{m}\right)^2 = 9.83\times10^{37}\,\text{kg}\cdot\text{m}^2$$

Substitute numerical values in equation (1) to obtain:

$$K_{\text{rot}} = \tfrac{1}{2}\left(9.83\times10^{37}\,\text{kg}\cdot\text{m}^2\right)\times\left(7.27\times10^{-5}\,\text{rad/s}\right)^2 = 2.60\times10^{29}\,\text{J} = \boxed{2.6\times10^{29}\,\text{J}}$$

Earth's orbital kinetic energy is:

$$K_{\text{orb}} = \tfrac{1}{2}I\omega_{\text{orb}}^2 \qquad (2)$$

Find the angular speed of the center of mass of the Earth-Sun system:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\,\text{rad}}{365.24\,\text{days}\times24\dfrac{\text{h}}{\text{day}}\times\dfrac{3600\,\text{s}}{\text{h}}} = 1.99\times10^{-7}\,\text{rad/s}$$

The orbital moment of inertia of Earth is:

$$I = M_{\text{E}}R_{\text{orb}}^2 = \left(6.0\times10^{24}\,\text{kg}\right)\left(1.50\times10^{11}\,\text{m}\right)^2 = 1.35\times10^{47}\,\text{kg}\cdot\text{m}^2$$

Substitute for I in equation (2) and evaluate K_{orb}:

$$K_{\text{orb}} = \tfrac{1}{2}\left(1.35\times10^{47}\,\text{kg}\cdot\text{m}^2\right)\times\left(1.99\times10^{-7}\,\text{rad/s}\right)^2 = 2.68\times10^{33}\,\text{J}$$

Evaluate the ratio $\dfrac{K_{\text{orb}}}{K_{\text{rot}}}$:

$$\frac{K_{\text{orb}}}{K_{\text{rot}}} = \frac{2.68\times10^{33}\,\text{J}}{2.60\times10^{29}\,\text{J}} \approx \boxed{10^4}$$

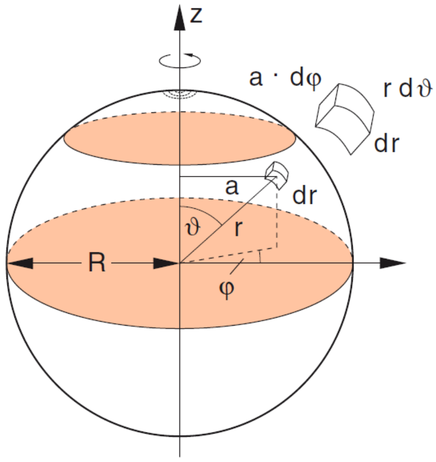
Momentum of Inertia $\mathbf{4 + 2}$ **Points**

The moment of inertia of the sphere depicted in **A** (mass M, radius R) is:

$$I = \int a^2 \, \rho(\vec{r}) \, \mathrm{d}V = \int a^2 \, \mathrm{d}m$$

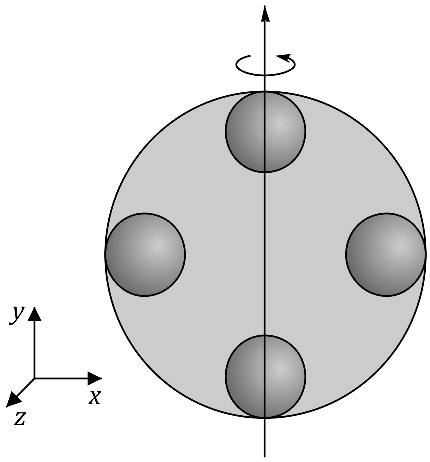
For a sphere with a constant density ρ this can be expressed in spherical coordinates:

$$I = \rho \int_0^R \int_0^\pi \int_0^{2\pi} a^3 r \cdot \mathrm{d}\varphi \cdot \mathrm{d}\theta \cdot \mathrm{d}r$$



A

B



a. Show that the Moment of Inertia I of an isotropic sphere is:

$$I = \frac{2}{5} M R^2$$

Hint: Show all steps of the calculation!

b. In the image B, a sphere (Radius R, density ρ) with 4 spherical holes (radius $r = R/4$, the holes are displaced from the centre of the sphere in x and y direction) is shown. Calculate the moment of inertia for a rotation around the y axis for this object.

Solution

(a)

Since moment of inertia for the constant density can be written as

$$I = \rho \int_0^R \int_0^\pi \int_0^{2\pi} a^3 r \cdot \mathrm{d}\varphi \cdot \mathrm{d}\theta \cdot \mathrm{d}r$$

Taking in to account that $a = r \cdot \sin\theta$ (1 point for that)

$$I_{sphere} = \rho \int_0^R \int_0^\pi \int_0^{2\pi} r^4 \cdot \sin^3\theta \cdot \mathrm{d}\varphi \cdot \mathrm{d}\theta \cdot \mathrm{d}r$$

Since the coordinates are independent variables the integrals can be split:

$$I_{sphere} = \rho \int_o^R r^4 \cdot \mathrm{d}r \cdot \int_o^{2\pi} \mathrm{d}\varphi \cdot \int_o^\pi \sin^3\theta \cdot \mathrm{d}\theta = \rho \cdot \frac{R^5}{5} \cdot 2\pi \cdot \int_o^\pi \sin^3\theta \cdot \mathrm{d}\theta$$

Since $M_{sphere} = \rho \cdot V = \rho \cdot \frac{4}{3}\pi \cdot R^3$ one can transform $\rho \cdot \frac{R^5}{5} \cdot 2\pi\ to\ M_{sphere} \cdot R^2 \cdot \frac{3}{10}$ *(1 point for this transformation to the total mass of the sphere)*

Let's take $\int_o^\pi \sin^3\theta \ \cdot \mathrm{d}\theta$ here: *(2 points for this integral)*

$$\int_o^\pi \sin^3\theta \ \cdot \mathrm{d}\theta = \int_o^\pi \sin\theta \ \cdot (1 - \cos^2\theta) \cdot \mathrm{d}\theta = \int_o^\pi \sin\theta \ \cdot \mathrm{d}\theta - \int_o^\pi \cos^2\theta \cdot \sin\theta \ \cdot \mathrm{d}\theta$$

$$= -\cos(\pi) + \cos(0) + \int_1^{-1} \cos^2\theta \ \cdot \mathrm{d}(\cos(\theta)) = 1 + 1 + \int_1^{-1} y^2 \ \cdot \mathrm{d}y$$

$$= 2 + \frac{-1^3}{3} - \frac{1^3}{3} = \frac{4}{3}$$

Thus,

$I_{sphere} = \rho \cdot \frac{R^5}{5} \cdot 2\pi \cdot \frac{4}{3} = M_{sphere} \cdot R^2 \cdot \frac{3}{10}\frac{4}{3} = \frac{2}{5} M \cdot R^2$

Students really should derive all the integrals and formulas. It's is not necessary to split the integrals in order to take them, so if they didn't split, no points should be removed as long as the calculations are right. If they take $\int_o^\pi \sin^3\theta \ \cdot d\theta$and the rest is wrong 2-3 points can be assigned to them since this integral is most difficult calculation part.

P.s. here ϑ is changed for θ but it's the same angle.

(b)

So, this object can be considered as a full sphere with 4 spheres of the negative mass with a density value equal to $-\rho$. (It's physically reasonable since the density was not specified to have positive value form the math point of view on the integral of the moment of inertia. And all the masses are here have a the linear additive property $(M(total\ in\ the\ dV\ volume) = \sum_{all\ masses\ in\ this\ volume} M_i(dV)$.

For simplification the holes in the spheres will be called "bottom", "top", "left" and "right" spheres.

Thus:

$$I_{bigsphere}\,(around\ the\ central\ y\ axis) = \frac{2}{5}M \cdot R^2$$

Moment of inertia of the top and bottom spheres since their centres are on the central y axis.

$$I_{bottom}\,(around\ the\ central\ y\ axis) = I_{top}\ = -\frac{2}{5}\frac{M}{64}\cdot\frac{R^2}{16} = -\frac{2}{5}M \cdot R^2\frac{1}{1024}$$

For "left" and "right" spherical holes the moments on inertia will be the same along their central axis. But since their central vertical axis is actually on the distance $\frac{3}{4}R$ from the central y axis, we will need to use a formula provided on the lecture for the moment of inertia for the CM and displaced axis:

$$I_{displaced} = I_{CM} + M \cdot a^2$$

where "a" is the distance of displacement. (1 point for right using of the displacement formula for only left and right holes)

Then (minuses are from the "negative" masses):

$$I_{left}(around\ the\ central\ y\ axis) = I_{right} = -\frac{2}{5}M \cdot R^2\frac{1}{1024} + \frac{-M}{64}\cdot\frac{9}{16}R^2$$

So, the total moment of inertia will be the moment of inertia of the large sphere minus the momenta of the 4 hollowed spheres:

$$I_{total} = \frac{2}{5}M \cdot R^2 - \frac{2}{5}M \cdot R^2\frac{1}{1024} - \frac{2}{5}M \cdot R^2\frac{1}{1024} - \frac{2}{5}M \cdot R^2\frac{1}{1024} - \frac{M}{64}\cdot\frac{9}{16}R^2 - -\frac{2}{5}M \cdot R^2\frac{1}{1024} - \frac{M}{64}\cdot\frac{9}{16}R^2 =$$

$$= \frac{2}{5}M \cdot R^2 - \frac{2}{5}M \cdot R^2\frac{4}{1024} - \frac{2}{5}M \cdot R^2\frac{9 \cdot 5 \cdot 2}{64 \cdot 16 \cdot 2}$$

$$= \frac{2}{5}M \cdot R^2 \cdot \left(1 - \frac{4}{1024} - \frac{45}{1024}\right) =$$

$$= \frac{2}{5}M \cdot R^2 \cdot \frac{975}{1024} \quad \text{or}$$

$$= \frac{195}{512}\,MR^2 \quad \text{or}$$

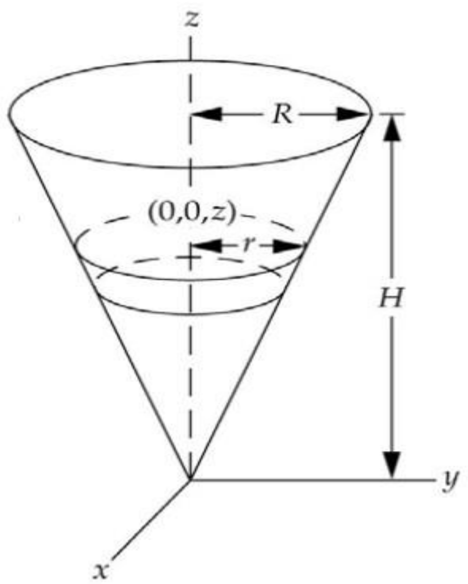
$$= \frac{65}{128}\cdot\ \pi \cdot \rho \cdot R^5\,.$$

(1 point for right answer in any of those forms)

Any of those answers should be taken as a correct one. No points for writing the formula for the sphere as it is $\frac{2}{5}M \cdot R^2$ *(since it was derived just above). Only correct use of the displacement theorem and correct answer are graded.*

Problem 3: **Inertia of cone** **7** Points

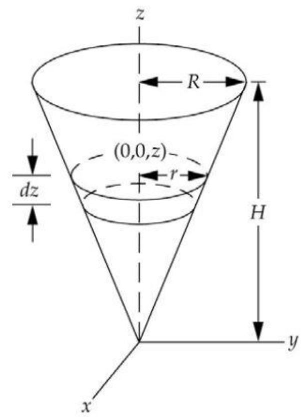
Use integration to determine the moment of inertia about its axis of a uniform right circular cone of height *H*, base radius *R*, and mass *M.*



Hint: use the transition $\int dm = \int \rho dV = \int \rho \pi r^2 dz$

55 ••• Use integration to determine the moment of inertia about its axis of a uniform right circular cone of height H, base radius R, and mass M.

Picture the Problem Let the origin be at the apex of the cone, with the z axis along the cone's symmetry axis. Then the radius of the elemental ring, at a distance z from the apex, can be obtained from the proportion $\frac{r}{z}=\frac{R}{H}$. The mass dm of the elemental disk is $\rho dV = \rho\pi r^2 dz$. We'll integrate $r^2 dm$ to find the moment of inertia of the disk in terms of R and H and then integrate dm to obtain a second equation in R and H that we can use to eliminate H in our expression for I.



Express the moment of inertia of the cone in terms of the moment of inertia of the elemental disk:

$$\begin{aligned} I &= \tfrac{1}{2}\int r^2 dm \\ &= \tfrac{1}{2}\int_0^H \frac{R^2}{H^2}z^2\left(\rho\pi\frac{R^2}{H^2}z^2\right)dz \\ &= \frac{\pi\rho R^4}{2H^4}\int_0^H z^4 dz = \frac{\pi\rho R^4 H}{10} \end{aligned}$$

Express the total mass of the cone in terms of the mass of the elemental disk:

$$\begin{aligned} M &= \pi\rho\int_0^H r^2 dz = \pi\rho\int_0^H \frac{R^2}{H^2}z^2 dz \\ &= \tfrac{1}{3}\pi\rho R^2 H \end{aligned}$$

Divide I by M, simplify, and solve for I to obtain:

$$I = \boxed{\tfrac{3}{10}MR^2}$$