MA - HW8 Stanislav

① $\lim\limits_{x\to0}\dfrac{tgx-x}{x-\sin x}=\left(\dfrac{0}{0}\right)=\left[\begin{array}{l}\text{if } \lim \frac{f'}{g'}\\ \text{exists}\end{array}\right]=\lim\limits_{x\to0}\dfrac{(tgx-x)'}{(x-\sin x)'}=\lim\limits_{x\to0}\dfrac{\frac{1}{\cos^2x}-1}{1-\cos x}=\left(\dfrac{0}{0}\right)=$

$=\lim\limits_{x\to0}\dfrac{(\cos^{-2}x-1)'}{(1-\cos x)'}=\lim\limits_{x\to0}\dfrac{+2\cos^{-3}x\,(+\sin x)}{\sin x}=2\underbrace{\lim\limits_{x\to0}\dfrac{1}{\cos^3x}}_{1}=2.$

② $\lim\limits_{x\to6}\dfrac{6^x-x^6}{x-6}=\left(\dfrac{0}{0}\right)=\lim\limits_{x\to6}\dfrac{6^x\ln6-6x^5}{1}=6^6\ln6-6^6=6^6(\ln6-1)=6^6\ln\dfrac{6}{e}.$

③ $\sqrt[3]{\sin(x^3)}$ — expand as $x\to0$ up to x^{13}.

$x\to0\Longrightarrow \sin(x^3)=(x^3)-\dfrac{(x^3)^3}{3!}+\dfrac{(x^3)^5}{5!}+\dots+(-1)^{n-1}\dfrac{(x^3)^{2n-1}}{(2n-1)!}+o((x^3)^{2n})\Longrightarrow$

$\sqrt[3]{\sin(x^3)}=\sqrt[3]{(x^3)\cdot\left[1-\dfrac{(x^3)^2}{3!}+\dfrac{(x^3)^4}{5!}+\dots+(-1)^{n-1}\dfrac{(x^3)^{2n-2}}{(2n-1)!}+o((x^3)^{2n-1})\right]}=$

$=x\cdot\sqrt[3]{1+\underbrace{\left(-\dfrac{(x^3)^2}{3!}+\dfrac{(x^3)^4}{5!}+\dots+(-1)^{n-1}\dfrac{(x^3)^{2n-2}}{(2n-1)!}+o((x^3)^{2n-1})\right)}_{A}}=x\cdot(1+A)^{\frac13}\overset{\text{as } A\to0}{=}$

$\overset{\text{as } A\to0}{=}x\cdot\left(1+\dfrac13A+\dfrac{\frac13\left(-\frac23\right)}{2!}A^2+\underbrace{\dfrac{\frac13\left(-\frac23\right)\left(-\frac53\right)}{3!}A^3}_{o(x^{18-1})}+o(A^3)\right)=\left[\begin{array}{l}\text{only } A, A^2\\ \text{have not } o(x^{13})\\ \text{terms}\end{array}\right]=$

$=x\cdot\left(1+\dfrac13\left[-\dfrac{(x^3)^2}{3!}+\dfrac{(x^3)^4}{5!}+o(x^{12})\right]+\dfrac13\left(-\dfrac13\right)\left[-\dfrac{(x^3)^2}{3!}+o(x^6)\right]^2+o(x^{12})\right)=$

$=x\left(1-\dfrac{x^6}{18}+\dfrac{x^{12}}{3\cdot5!}+\dfrac{(-1)}{9}\cdot\dfrac{x^{12}}{36}\right)+o(x^{13})=x-\dfrac{x^7}{18}+x^{13}\left[\underbrace{\dfrac{1}{3\cdot5!}}_{360}-\dfrac{1}{324}\right]+$

$+o(x^{13})=x-\dfrac{x^7}{18}+x^{13}\cdot\dfrac{-36}{360\cdot324}+o(x^{13})=\underline{x-\dfrac{x^7}{18}-\dfrac{x^{13}}{3240}+o(x^{13})}$ as $x\to0$.

④ $\ln\left(\dfrac{\sin x}{x}\right)$ up to x^6 as $x\to0$.

$\ln\left(\dfrac{\sin x}{x}\right)=\ln\left(\dfrac1x\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\dots+(-1)^{n-1}\dfrac{x^{2n-1}}{(2n-1)!}+o(x^{2n})\right)\right)=$

$=\ln\left(1+\underbrace{\left(-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}-\dfrac{x^6}{7!}+\dots+o(x^{2n-1})\right)}_{A\xrightarrow[x\to0]{}0}\right)=\ln(1+A)=A-\dfrac{A^2}{2}+\dfrac{A^3}{3}+\dots$

$\dots+(-1)^{n-1}\dfrac{A^n}{n}+o(A^n)=\underbrace{-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}-\dfrac{x^6}{7!}+o(x^6)}_{A}\underbrace{-\dfrac12\left(-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}\right)^2+o(x^6)}_{-A^2/2}+$

$+\underbrace{\dfrac13\left(-\dfrac{x^2}{3!}\right)^3+o(x^6)}_{A^3}=-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}-\dfrac{x^6}{7!}-\dfrac12\left(\dfrac{x^4}{36}+\cancel{\dfrac{x^8}{120^2}}-\dfrac{2x^6}{3!\,5!}\right)-\dfrac{x^6}{3\cdot6^3}+o(x^6)=$

$=-\dfrac{x^2}{6}+x^4\left(\dfrac{1}{120}-\dfrac{1}{72}\right)+x^6\left(-\dfrac{1}{7!}+\dfrac{1}{3!\,5!}-\dfrac{1}{3\cdot6^3}\right)+o(x^6)=-\dfrac{x^2}{6}+x^4\dfrac{(-48)}{120\cdot72}+$

$+x^6\cdot\left(\underbrace{\dfrac{-3!\cdot3\cdot6^3+6\cdot7\cdot3\cdot6^3-3!\,7!}{3!\,7!\,3\cdot6^3}}_{B}\right)=\left[B=\dfrac{-3\cdot6^2+7\cdot3\cdot6^2-5!\,7}{7!\,3\cdot6^2}=\dfrac{-840+756-108}{120\cdot42\cdot108}=\dfrac{-192}{120\cdot42\cdot109}=\right.$

$\left.=\dfrac{-\cancel{2}^6\cdot3}{\cancel{2}^3\cdot15\cdot\cancel{2}\cdot21\cdot\cancel{2}^2\cdot27}=-\dfrac{1}{9\cdot21\cdot15}=-\dfrac{1}{315\cdot9}=-\dfrac{1}{2835}\right]$

$$= -\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} + o(x^6).$$

(5) $$\lim_{x\to0}\frac{e^{-\frac{x^2}{2}}-\cos x}{x^4} = \lim_{x\to0}\frac{1+\left(\frac{x^2}{-2}\right)+\frac{\left(-\frac{x^2}{2}\right)^2}{2!}+o(x^4)-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+o(x^4)\right)}{x^4} =$$

$$= \lim_{x\to0}\frac{\frac{x^4}{8}-\frac{x^4}{24}}{x^4} = \frac{1}{8}-\frac{1}{24} = \frac{2}{24} = \frac{1}{12}.$$

(6) $$\lim_{x\to0}\frac{e^x\sin x - x(1+x)}{x^3} = \lim_{x\to0}\frac{\left(1+x+\frac{x^2}{2}\right)\left(x-\frac{x^3}{3!}\right)-x(1+x)+o(x^3)}{x^3} =$$

$$= \lim_{x\to0}\frac{x-\frac{x^3}{3!}+x^2-\cancel{\frac{x^4}{3!}}+\frac{x^3}{2}-\cancel{\frac{x^5}{2\cdot3!}}-x-x^2+o(x^3)}{x^3} =$$

$$= \lim_{x\to0}\frac{-\frac{x^3}{3!}+\frac{x^3}{2}}{x^3} = -\frac{1}{6}+\frac{1}{2} = \frac{1}{3}.$$

(7) $$\lim_{x\to1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \lim_{x\to1}\left(\frac{x-1-\ln x}{\ln x(x-1)}\right) = \left(\frac{0}{0}\right) = [\text{L'Hôpitals}] =$$

$$= \lim_{x\to1}\frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x} = \lim_{x\to1}\frac{1-\frac{1}{x}}{1-\frac{1}{x}+\ln x} = \left(\frac{0}{0}\right) = [\text{L'Hôpital}] =$$

$$= \lim_{x\to1}\frac{\frac{1}{x^2}}{\frac{1}{x^2}+\frac{1}{x}} = \frac{1}{1+1} = \frac{1}{2}.$$

(8) $$\lim_{x\to0}\frac{\arcsin(2x)-2\arcsin(x)}{x^3} = \left(\frac{0}{0}\right) = [\text{L'Hôpitals}] =$$

$$= \lim_{x\to0}\frac{\frac{-2}{\sqrt{1-4x^2}}-\frac{2}{\sqrt{1-x^2}}}{3x^2} = \frac{2}{3}\lim_{x\to0}\frac{(1-4x^2)^{\left(-\frac{1}{2}\right)}-(1-x^2)^{\left(-\frac{1}{2}\right)}}{x^2} =$$

$$= \left(\frac{0}{0}\right) = [\text{L'Hôpitals}] = \frac{2}{3}\lim_{x\to0}\frac{\left(-\frac{1}{2}\right)(1-4x^2)^{-\frac{3}{2}}(-8x)-\left(-\frac{1}{2}\right)\cdot(1-x^2)^{-\frac{3}{2}}(-2x)}{2x} =$$

$$= \frac{2}{3}\lim_{x\to0}\frac{4\cancel{x}(1-4x^2)^{-\frac{3}{2}}-\cancel{x}(1-x^2)^{-\frac{3}{2}}}{2\cancel{x}} = \frac{2}{3}\cdot\frac{4\cdot(1-0)-1}{2} = 1.$$

(9) $$\lim_{x\to0}\frac{x\sqrt[3]{\sin(x^3)}+\ln\left(\frac{\sin x}{x}\right)}{x^2} = \left[\begin{matrix}\text{Taylor, use the results}\\ \text{of tasks 3,4}\end{matrix}\right] =$$

$$= \lim_{x\to0}\frac{x\left(x-\cancel{\frac{x^7}{18}}\right)+\left(-\frac{x^2}{6}\right)+o(x^2)}{x^2} = \lim_{x\to0}\frac{x^2-\frac{x^2}{6}}{x^2} = 1-\frac{1}{6} = \frac{5}{6}.$$

(10) $$\lim_{x\to+\infty}\left(x-x^2\ln\left(1+\frac{1}{x}\right)\right) = \left[\begin{matrix}\mu=\frac{1}{x}\\ \mu\xrightarrow{x\to+\infty}0\end{matrix}\right] = \lim_{\mu\to0}\left(\frac{1}{\mu}-\frac{\ln(1+\mu)}{\mu^2}\right) =$$

= [Note – we cannot use table of limits and set $\frac{\ln(1+\mu)}{\mu}\to1$ because it is difference of expressions, and $\frac{1}{\mu}\to+\infty$, $\frac{\ln(1+\mu)}{\mu^2}\to+\infty$, so $(\infty-\infty)$] =

= [instead Taylor] =

$$= \lim_{\mu \to 0} \frac{\mu - \left[\mu - \frac{\mu^2}{2} + \frac{\mu^3}{3} + o(\mu^3)\right]}{\mu^2} = \lim_{\mu \to 0} \frac{\mu^2/2 + o(\mu^2)}{\mu^2} = \frac{1}{2}.$$