Mathematics 1. Selected proofs
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Continuity. Extreme value theorems

THEOREM 1. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b] \implies f$ is bounded on $[a,b]$, i.e.

$$\exists\, M > 0 : \qquad \forall\, x \in [a,b] \qquad |f(x)| \leq M$$

PROOF.

1. Proof by contradiction:

   Assume $\forall\, M > 0 \quad \exists\, x_M \in [a,b]$: $\quad |f(x_M)| \geq M$

2. Construct a sequence $\{x_n\}_{n=1}^{\infty} \subset [a,b]$

   Take $M = 1 \quad \implies \quad \exists\, x_1 \in [a,b]$: $\quad |f(x_1)| \geq 1$

   Take $M = 2 \quad \implies \quad \exists\, x_2 \in [a,b]$: $\quad |f(x_2)| \geq 2$

   Take $M = 3 \quad \implies \quad \exists\, x_3 \in [a,b]$: $\quad |f(x_3)| \geq 3$

   ...

   Take $M = n \quad \implies \quad \exists\, x_n \in [a,b]$: $\quad |f(x_n)| \geq n$

   ...

   So, we obtain $\{x_n\}_{n=1}^{\infty} \subset [a,b]$: $\quad \forall\, n \in \mathbb{N} \quad |f(x_n)| \geq n$

3. Use Bolzano–Weierstrass theorem:

   $\{x_n\}_{n=1}^{\infty}$ is bounded $\quad \implies \quad \exists$ a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$, $\exists\, c \in \mathbb{R}$: $\quad x_{n_k} \to c$

   One can pass to the limit in the inequality: $a \leq x_{n_k} \leq b \quad \implies \quad a \leq c \leq b \quad \implies \quad c \in [a,b]$

4. Use continuity to obtain a contradiction:

   $x_{n_k} \to c$, $\ f$ is continuous on $[a,b] \quad \implies \quad f(x_{n_k}) \to c$

   Convergent sequence is bounded $\quad \implies \quad \exists\, L > 0$: $\quad \forall\, k \in \mathbb{N} \quad |f(n_k)| \leq L$

   $\forall\, k \in \mathbb{N} \quad |f(n_k)| \geq n_k \to \infty$ — this contradicts to the boundedness of $\{f(x_{n_k})\}_{k=1}^{\infty}$!!!

$\square$

THEOREM 2. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b] \implies f$ achieves on $[a,b]$ its maximum and minimal values, i.e. $\exists\ c_1,\ c_2 \in [a,b]$ such that

$$f(c_1) = \inf_{x \in [a,b]} f(x) \qquad \text{and} \qquad f(c_2) = \sup_{x \in [a,b]} f(x)$$

PROOF. Let us prove that f achieves its maximum. The proof for the minimum is analogous.

5. Function which is continuous on a closed bounded interval is bounded:

   f is continuous on $[a, b] \implies f$ is bounded on $[a, b] \implies \exists\, M \in \mathbb{R}: \quad M = \sup\limits_{x \in [a,b]} f(x)$

6. Use the characterization of supremum using the quantifiers:

   $\forall\, \varepsilon > 0 \;\; \exists\, x_\varepsilon \in [a, b]: \quad M - \varepsilon < f(x_\varepsilon) \leq M$

   Take $\varepsilon = 1 \quad \exists\, x_1 \in [a, b]: \quad M - 1 < f(x_1) \leq M$

   Take $\varepsilon = \frac{1}{2} \quad \exists\, x_2 \in [a, b]: \quad M - \frac{1}{2} < f(x_2) \leq M$

   Take $\varepsilon = \frac{1}{3} \quad \exists\, x_3 \in [a, b]: \quad M - \frac{1}{3} < f(x_3) \leq M$

   ...

   Take $\varepsilon = \frac{1}{n} \quad \exists\, x_n \in [a, b]: \quad M - \frac{1}{n} < f(x_n) \leq M$

   ...

   So, we obtain $\{x_n\}_{n=1}^{\infty} \subset [a, b]: \quad \forall\, n \in \mathbb{N} \quad M - \frac{1}{n} < f(x_n) \leq M$

   Two policemen theorem $\implies f(x_n) \to M$

7. Use Bolzano–Weierstrass theorem:

   $\{x_n\}_{n=1}^{\infty}$ is bounded $\implies \exists$ a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}, \;\exists\, c \in [a, b]: \quad x_{n_k} \to c$

8. Use continuity of f and uniqueness of the limit:

   f is continuous at $c \in [a, b], \; x_{n_k} \to c \implies f(x_{n_k}) \to f(c)$

   $f(x_{n_k}) \to M, \; f(x_{n_k}) \to f(c) \implies f(c) = M$

$\square$