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IPSP 3720433

TP1 - HW9
Differential Equations

Problem 9.1

a) $\frac{dy}{dx} = \frac{\cos^2 y}{\sin^2 x}$

$$\int_{y(\frac{\pi}{4})=0}^{y} \frac{dy'}{\cos^2 y'} = \int_{\frac{\pi}{4}}^{x} \frac{dx'}{\sin^2 x'}$$

$$\tan y' \Big|_0^y = \cdot \cot x' \Big|_x^{\frac{\pi}{4}}$$

$$\tan y = 1 - \cot x$$

$$y = \arctan(1-\cot x)$$

b) $\frac{dy}{dx} = \frac{3x^2 y}{2y^2+1}$ $y(0)=1$

$$\int_1^y \frac{2y'^2+1}{y'} dy' = \int_0^x 3x'^2 dx'$$

$$\int_1^y 2y' + \frac{1}{y'} dy' = \frac{x'^3}{\cancel{3x}}\Big|_0^x$$

$$(y')^2 + \ln|y'| \Big|_1^y = x^3$$

$$y^2 + \ln|y| - 1 = x^3$$

$$G(x,y) = x^3 - y^2 - \ln|y| = -1$$

Solutions satisfy this contour line

c) $\frac{dy}{dx} = -\frac{1+y^3}{xy^2(1+x^2)}$

$$-\int_2^y \frac{y'^2 dy'}{1+y'^3} = +\int_1^x \frac{dx'}{x'(1+x'^2)}$$

$$\begin{bmatrix} 1+y'^3 = u \\ 3y'^2 dy' = du \end{bmatrix}$$
$u(2) = 9$
$u(y) = 1+y^3$

$$-\frac{1}{3}\int_9^{1+y^3} \frac{du}{u} = \int_1^x \frac{dx'}{x'(1+x'^2)}$$

$$\boxed{\frac{1}{x} - \frac{x}{1+x^2} = \frac{1+x^2-x^2}{x(1+x^2)} = \frac{1}{x(1+x^2)}}$$

$$-\frac{1}{3}\ln|u|\Big|_9^{1+y^3} = \int_1^x \frac{1}{x'} - \frac{x'}{1+x'^2} dx'$$

$$\frac{1}{3}\ln|u|\Big|_{1+y^3}^{9} = \ln|x'|\Big|_1^x - \frac{1}{2}\int_2^{1+x^2} \frac{dw}{w}$$

$$\frac{1}{3}[\ln 9 - \ln|1+y^3|] = \ln|x| - \frac{1}{2}[\ln(1+x^2) - \ln 2]$$

$$\ln|1+y^3|^{-\frac{1}{3}} = \ln|x| + \ln(1+x^2)^{-\frac{1}{2}} + \underbrace{\ln\sqrt{2} + \ln 9^{-\frac{1}{3}}}_{\ln A} \quad \Big| \exp \text{ all}$$

$$|1+y^3|^{-\frac{1}{3}} = \frac{|x|}{\sqrt{1+x^2}} \cdot A, \quad 1+y^3 = \left(\frac{|x|}{\sqrt{1+x^2}} A\right)^{-3}, \quad A = \pm\sqrt{2}\, 9^{-\frac{1}{3}}$$

$$y^3 = -1 + \left(\frac{x}{\sqrt{1+x^2}} A\right)^{-3}$$

Finally $y = \left(-1 + \left(\frac{xA}{\sqrt{1+x^2}}\right)^{-3}\right)^{\frac{1}{3}}$ where $A = \pm\sqrt{2}\, g^{-\frac{1}{3}}$. 2.1

Problem 9.2 Dimensionalization

$$\dot{v} = -g - Av^3$$

a) $[Av^3] = \frac{m}{s^2} = [A]\cdot\frac{m^3}{s^3}$, $[A] = \frac{s}{m^2}$ 2.2

b) friction opposes motion and has sign of $-v$,

$\text{sign}(f_r) = \text{sign}(-v) = \text{sign}(-v^3) \underset{\uparrow}{=} \text{sign}(-Av^3)$ 2.3

True iff $\underline{A > 0}$

c) $v_\infty = f(g, A)$

$$\frac{m}{s} = \left[\frac{m}{s^2}\right]^x \left[\frac{s}{m^2}\right]^y$$

$$\begin{matrix} m^1 = m^{x-2y} \\ s^{-1} = s^{-2x+y} \end{matrix} \Longrightarrow \begin{cases} 1 = x - 2y \\ -1 = -2x + y \end{cases} \quad \begin{cases} 2 = 2x - 4y \\ -1 = -2x + y \end{cases}$$

$1 = -3y$,

so $y = -\frac{1}{3}$, $x = \frac{1}{3}$

$v_\infty \approx g^{\frac{1}{3}} A^{-\frac{1}{3}} = \sqrt[3]{\frac{g}{A}}$, $\quad v_\infty = C\sqrt[3]{\frac{g}{A}}$. 2.4

d) v_∞ is reached $\longleftrightarrow \dot{v} = 0$, 2.5

$$0 = -g - Av_\infty^3$$

$$v_\infty = -\sqrt[3]{\frac{g}{A}}, \quad C = -1.$$

Same result, but emphasizing negative direction. In unit analysis only it is „hidden" inside constants.

e) $\hat{v} = \frac{v}{v_\infty}$ $[\hat{v}] = 1$

$[T] = 1$

$T = \frac{t - t_0}{s}$ ← something. Find s: (and hence T)

$$\frac{dv}{dt} = -g - Av^3 \xrightarrow[\text{must work}]{} \frac{d\hat{v}}{dT} = 1 - \hat{v}^3$$

$$\frac{d(v_\infty \hat{v})}{d(sT)} = -g - A(v_\infty \hat{v})^3$$

$$\frac{v_\infty \, d\hat{v}}{s \, dT} = -g - Av_\infty^3 \hat{v}^3 \quad \Big| \cdot \frac{1}{Av_\infty^3}$$

$$\frac{v_\infty}{s} \cdot \frac{1}{Av_\infty^3} \cdot \frac{d\hat{v}}{dT} = \underbrace{-\frac{g}{Av_\infty^3}}_{1} - \hat{v}^3$$

$$\underbrace{\frac{1}{Asv_\infty^2}}_{\text{must become } 1} \cdot \frac{d\hat{v}}{dT} = 1 - \hat{v}^3$$

must become 1 $\longrightarrow s = \frac{1}{Av_\infty^2} \longrightarrow T = \frac{t - t_0}{1/Av_\infty^2} = \underline{(Av_\infty^2)(t - t_0)}.$

Check $[T] = 1$: $\frac{s}{m^2} \cdot \frac{m^2}{s^2} \cdot s = 1$ yes.

3.1

Problem 9.3 Anharmonic oscillator

$$F(x) = -f \tanh \frac{x}{L}$$

a) $[f] = N = \frac{kg \cdot m}{s^2}$ Length: $\tilde{x} = \frac{x}{L}$

3.2 $[L] = m$ Time: $\tau = \frac{t - t_0}{\left(\frac{f}{mL}\right)^{-\frac{1}{2}}} = \frac{t - t_0}{\left(\frac{mL}{f}\right)^{\frac{1}{2}}} = \sqrt{\frac{f}{mL}}(t - t_0)$

With units:

$$m\ddot{x} = -f \tanh \frac{x}{L}$$

Without

$$\frac{m \, d^2(L\tilde{x})}{d\left(\sqrt{\frac{mL}{f}}\tau\right)^2} = -f \tanh \tilde{x}$$

$$f \cdot \frac{\cancel{mL} \, d^2\tilde{x}}{\cancel{mL} \, d\tau^2} = -f \tanh \tilde{x} \quad \Big| \cdot \frac{1}{f}$$

$$\underline{\frac{d^2\tilde{x}}{d\tau^2} = -\tanh \tilde{x}}$$ (later use x)

b) $\ddot{x} = -\tanh(x)$ | $\ln(\cosh(x))' = \frac{\sinh x}{\cosh x} = \tanh x$

$-\tanh(x) = -\nabla\varphi$

Potential

$\varphi = \int \tanh(x)\,dx + C = \ln(\cosh(x)) + C,$

↑ any antiderivative. Put $C=0$, so $\varphi(0) = \ln 1 = 0.$

$\varphi = \ln(\cosh(x)).$

Kinetik

$T = \frac{\dot{x}^2}{2}$

Conservation

$$\frac{dE}{dt} = \frac{d}{dt}(\varphi + T) = \frac{d}{dt}\left(\frac{\dot{x}^2}{2}\right) + \underbrace{\nabla\varphi}_{=-\ddot{x}}\cdot\frac{dx}{dt} = \dot{x}\ddot{x} - \ddot{x}\dot{x} = \underline{0.}$$

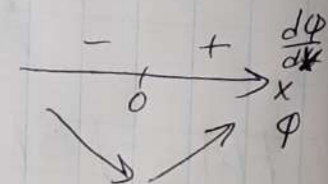
(in any cons. field)

(can also dif. φ directly)

c) $\varphi = \ln(\cosh(x))$

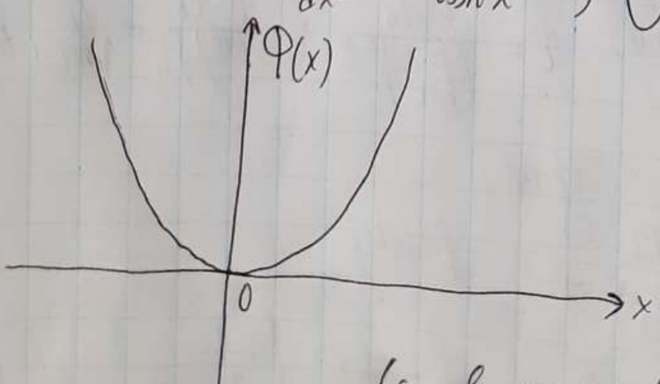
Analysis to build plot:

extrema $\frac{d\varphi}{dx} = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, $\tanh(x) = 0$ at 0, minima



concavity

$$\frac{d\tanh(x)}{dx} = \frac{d^2\varphi}{dx^2} = \frac{1}{\cosh^2 x} > 0, \ \cup$$



(can be compared with harmonic $\frac{x^2}{2}$ using Taylor)

Since $\underset{(t)}{E} = const$,

$$\ln(\cosh(x)) + \frac{\dot{x}^2}{2} = C_0$$ – contour lines of energy in $(x, \dot{x})$ phase space

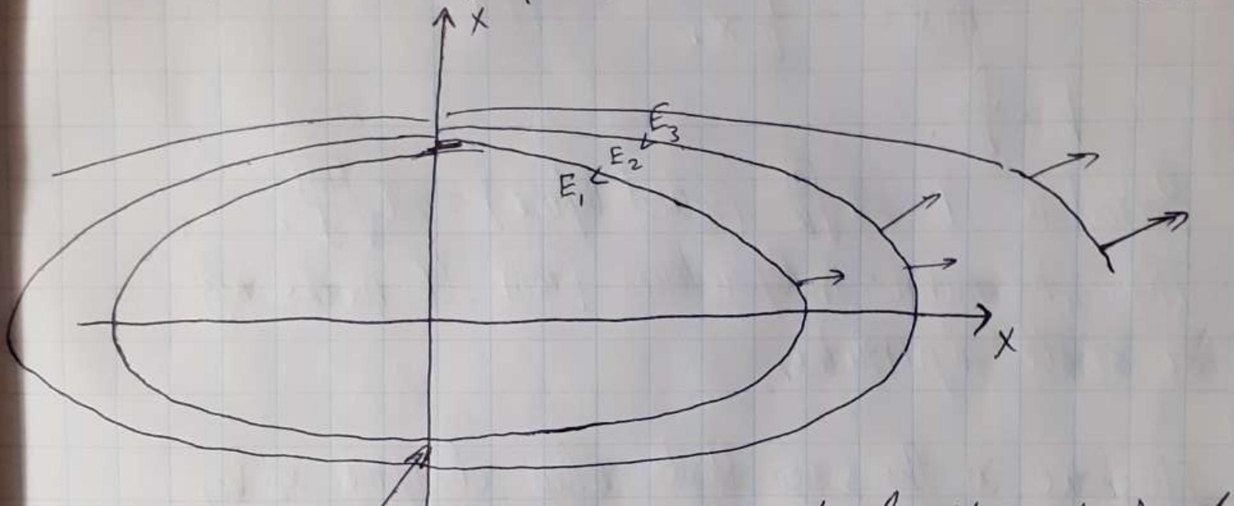
1) $\frac{\dot{x}^2}{2} + C = \ln(\cosh(x))$

$\cosh(x) \approx e^{\frac{\dot{x}^2}{2}}$

(approximate) $\frac{e^x + e^{-x}}{2} \approx e^x \approx e^{\frac{\dot{x}^2}{2}}$,

$x \approx$ parabolic relative to $\dot{x}$

2) Also, check $\nabla \varphi = \begin{pmatrix} \partial x \\ \partial \dot{x} \end{pmatrix} \left(\ln(\cosh x) + \frac{\dot{x}^2}{2}\right) = \begin{pmatrix} \tanh x \\ \dot{x} \end{pmatrix}$ $\tanh x \xrightarrow{\infty} 1$, $\dot{x}$ will increase



differences are smaller on vertical than horizontal, since $\dot{x}$ contributes to faster increase of E, $\left(\frac{\dot{x}^2}{2}\right)$ than x $(\approx \ln(\cosh(x)) \approx x)$.

5.1 d) Taylor:

$$\ddot{x} = -\tanh(x) = -\sum_{i=0}^{\infty} \frac{\tanh^{(i)}(0)\, x^i}{i!} \approx -\left(0 + \frac{x}{1!} + 0 - \frac{2x^3}{3!}\right) \approx -\left(x - \frac{x^3}{3}\right)$$

use here

$f^{(0)} = \tanh(x)$, $f^{(0)}(0) = 0$

$f^{(1)} = \frac{1}{\cosh^2(x)}$, $f^{(1)}(0) = 1$

$f^{(2)} = -2\cosh^{-3}\sinh$, $f^{(2)}(0) = 0$

$f^{(3)} = 6\cosh^{-4}\sinh - 2\cosh^{-2}$, $f^{(3)}(0) = -2$

When x is small, $\ddot{x} \approx -x$.

Restore units:

$\tilde{x} = \frac{x}{L}$

$\tilde{\tau} = (t - t_0)\sqrt{\frac{f}{mL}}$

$$\ddot{\tilde{x}} = -\tilde{x}$$

$$\frac{d^2\tilde{x}}{d\tau^2} = -\tilde{x}$$

$$\frac{d^2\left(\frac{x}{L}\right)}{\frac{f}{mL}d(t-t_d)^2} = -\frac{x}{L}$$

$$\frac{mL}{f} \cdot \frac{d^2x}{dt^2} = -x$$

$$m\ddot{x} = -\frac{f}{L}x.$$

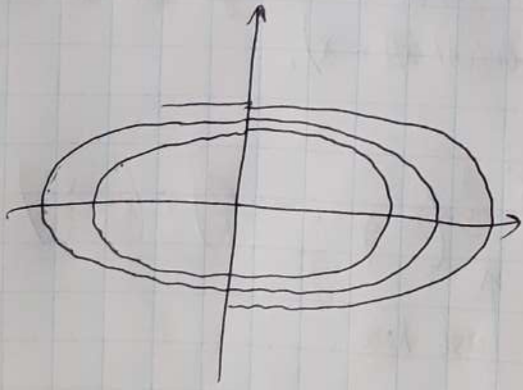
$$k \propto \frac{f}{L}.$$

Period is $T = 2\pi\sqrt{\frac{m}{k}}$ (solved before) $= 2\pi\sqrt{\frac{mL}{F}}$.

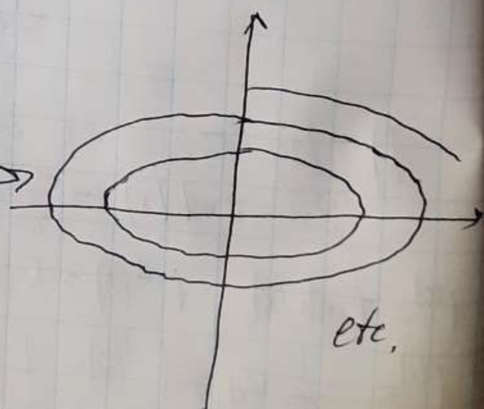
This „harmonic" independence from IC works only for small amplitudes.

* e) It shows in the sense that when changing x_0, v_0, on horizontal curves scale differently than on vertical.

This



and Not



etc.

* f) Inharmonic effect. + different when sinh.

g) The idea is that period becomes different for different IC.

both here

This is because $\tanh x \xrightarrow{\infty} 1$, and

$\ddot{x} = -\tanh x$ becomes constant force, and for large amplitudes time of motion (like in free fall) grows, so period grows when $x_0 \uparrow$, v_0 same.

2) For the case of $F = -f \sinh\left(\frac{x}{\tau}\right)$ inharmonicity acts in reverse way: $F \approx \sinh(x) = \frac{e^{x} - e^{-x}}{2} \approx e^{x}$ on larger x,

so force grows faster than distance, and acceleration/velocity also, ~~(when~~ when choosing other x_0 and the periods will decrease.

3) I checked all 3 situations: (writing in dimens. units)

* $\ddot{x} = -x$ (harmonic $T(x_1 > x_0) = T(x_0)$
* $\ddot{x} = -\tanh x$ (inharmonic $T(x_1 > x_0) > T(x_0)$)
* $\ddot{x} = -\sinh x$ (inharmonic $T(x_1 > x_0) < T(x_0)$)

(for cosh is same since e^{-x} does not play role)

By writing Python program and integrating numerically.

Algorithm: specify x_0, v_0 (input).

while (in given time range)

$x = x_{old} + v \cdot dt$

$a =$ (choose from above)

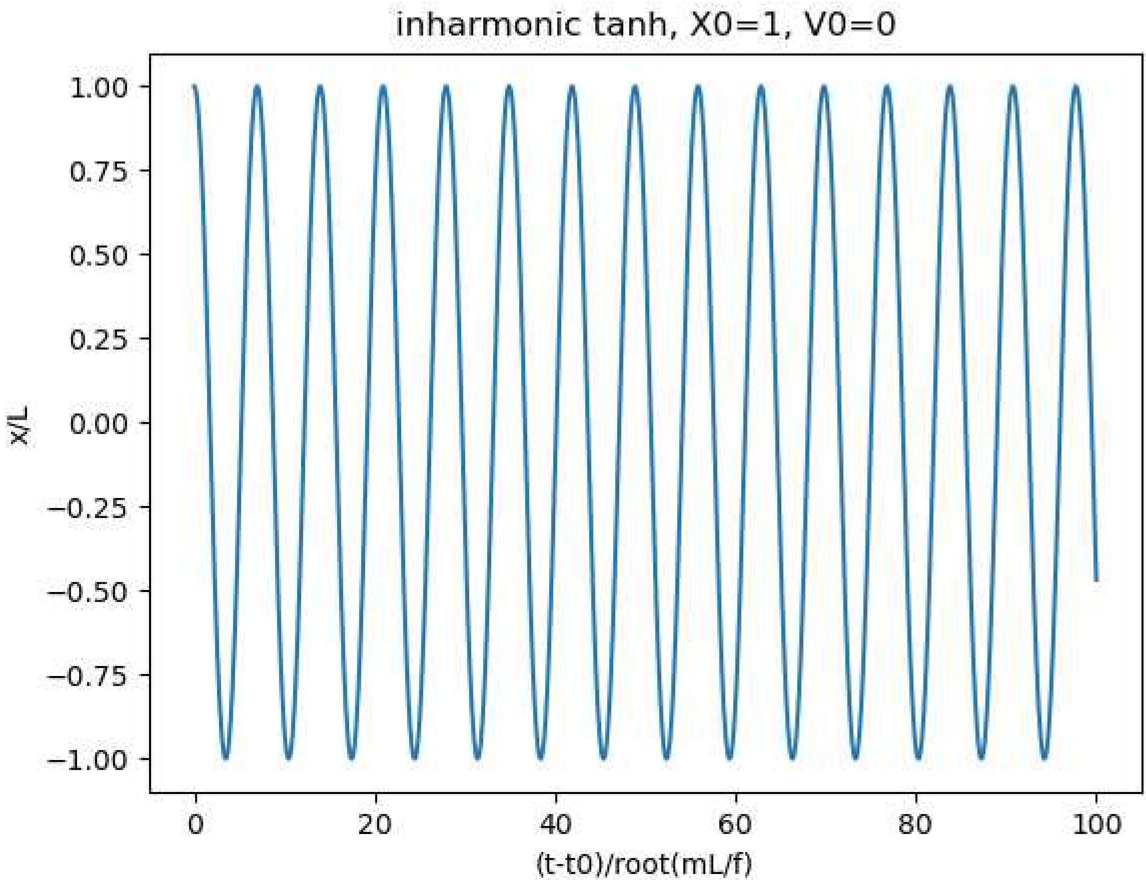
$v = v + a \cdot dt$

plot (t and x)

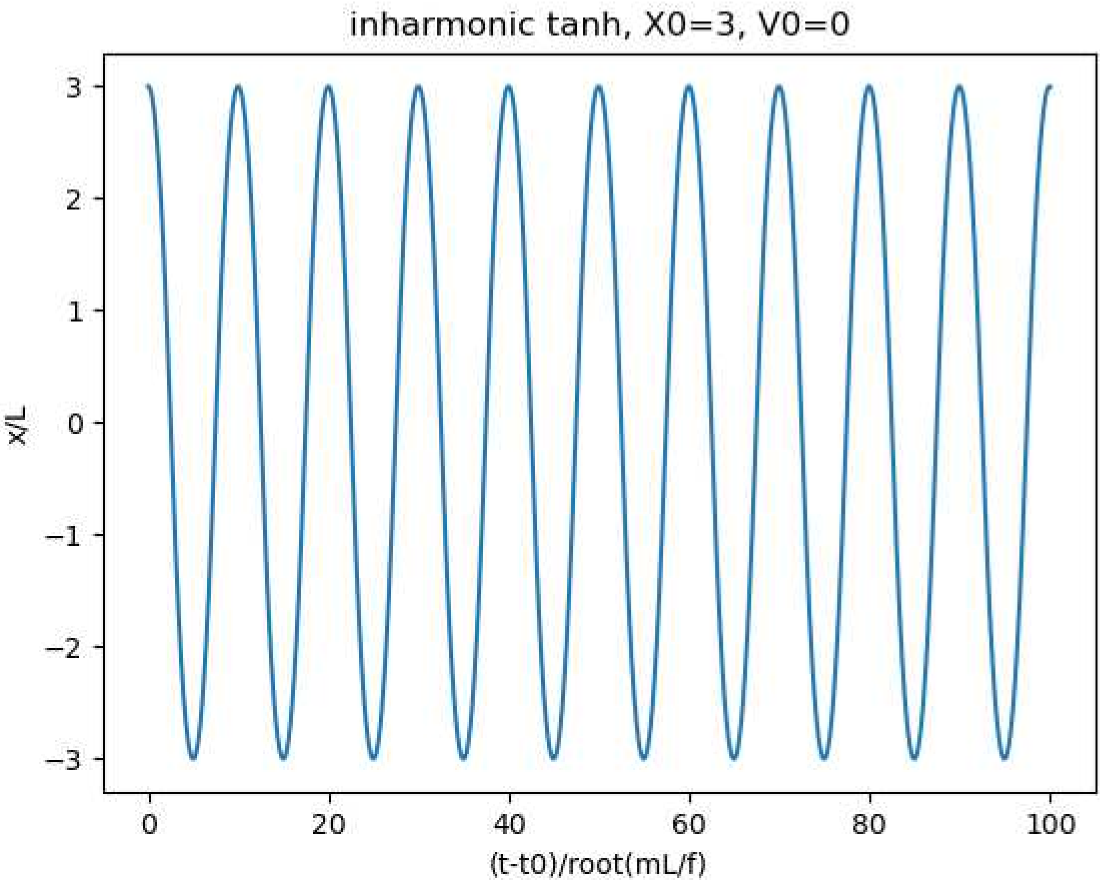
I run all for different initial conditions pairs. Here results:

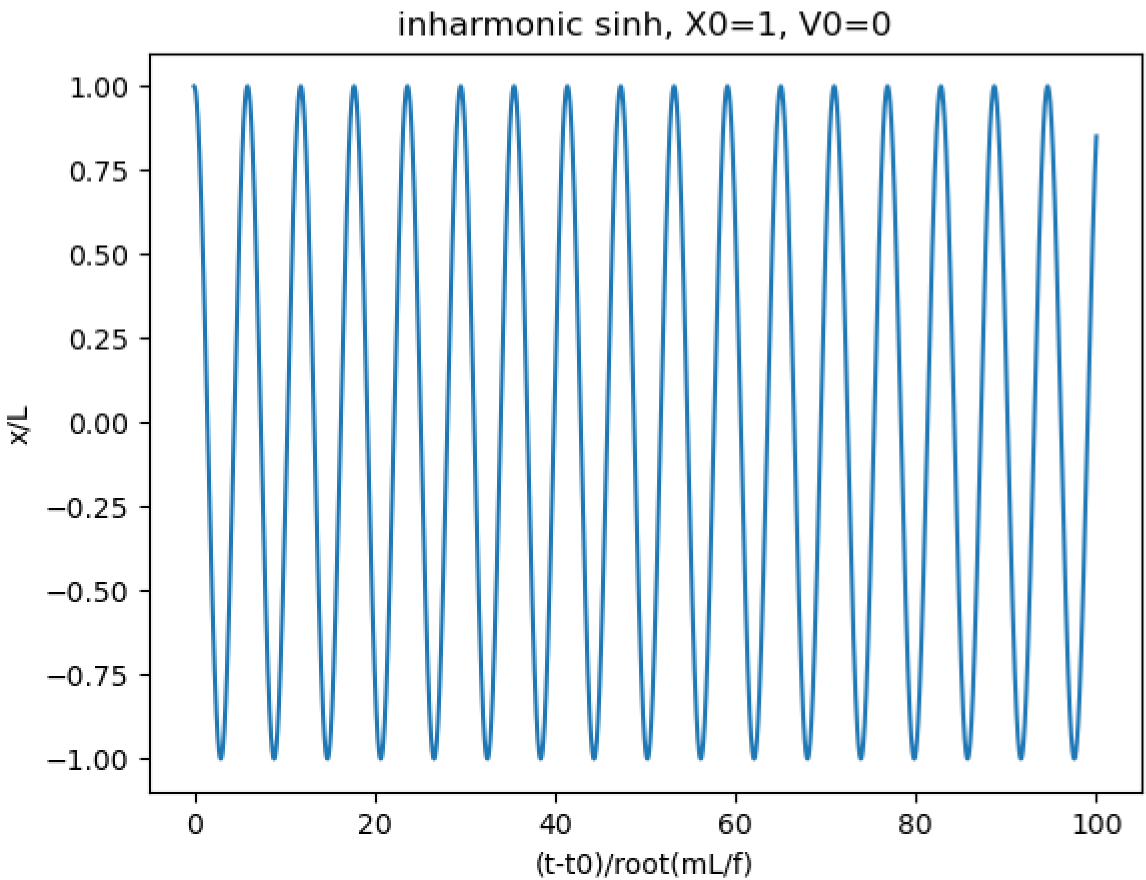
(it is clear how period changes)

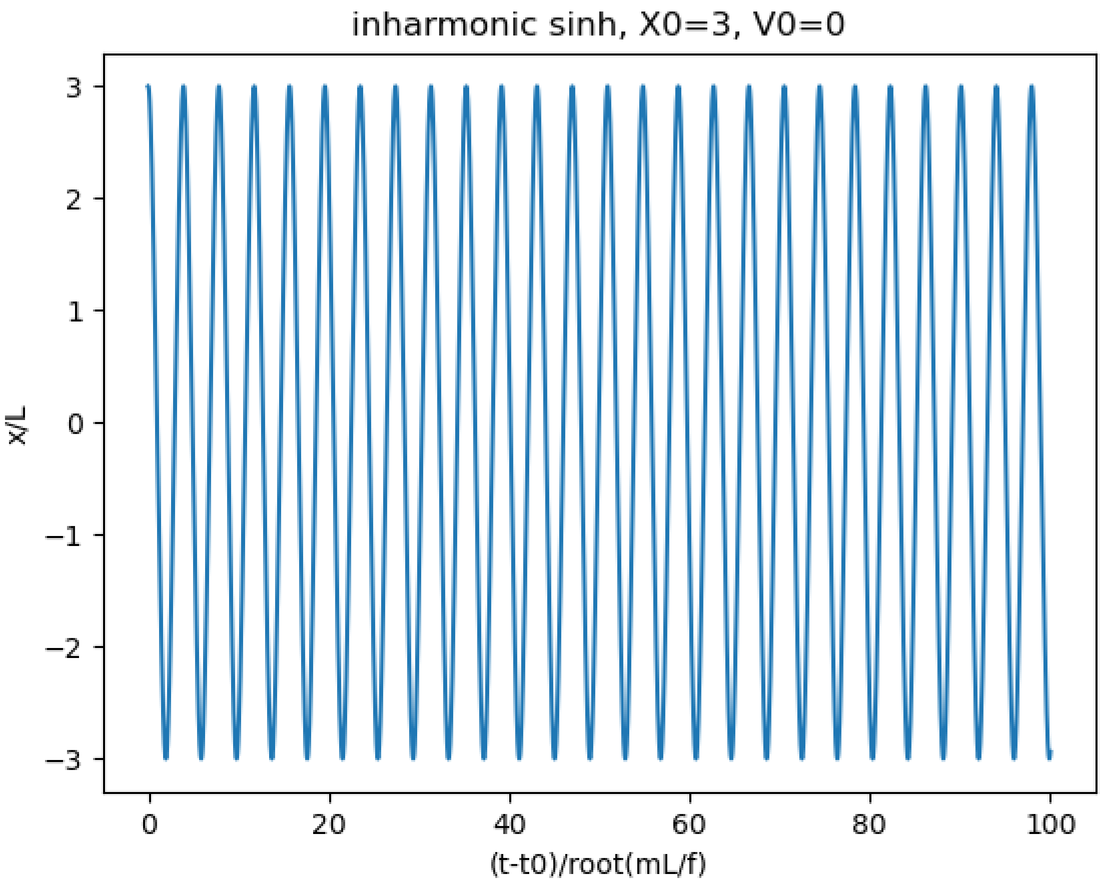
```python
#this is example for tanh, replace the line with a for sinh or harmonic case
# to use, launch
# python3 <this file> <x0> <v0>


import sys
import numpy as np
import matplotlib.pyplot as plt
import math

x0=sys.argv[1]
v0=sys.argv[2]

x=float(x0)
v=float(v0)
a=0
t=0


x_vals = []
t_vals = []

dt = 0.01
for t in np.arange(0, 100, dt):
    x_vals.append(x)
    t_vals.append(t)
    x=x+v*dt
    a=-math.tanh(x)
    # or a=-math.sinh(x)
    # or a=-x
    v=v+a*dt

plt.plot(t_vals, x_vals)
plt.xlabel('(t-t0)/root(mL/f)')
plt.ylabel('x/L')
plt.title("inharmonic tanh, X0=" + str(x0) + ", V0=" + str(v0))

plt.show()
```



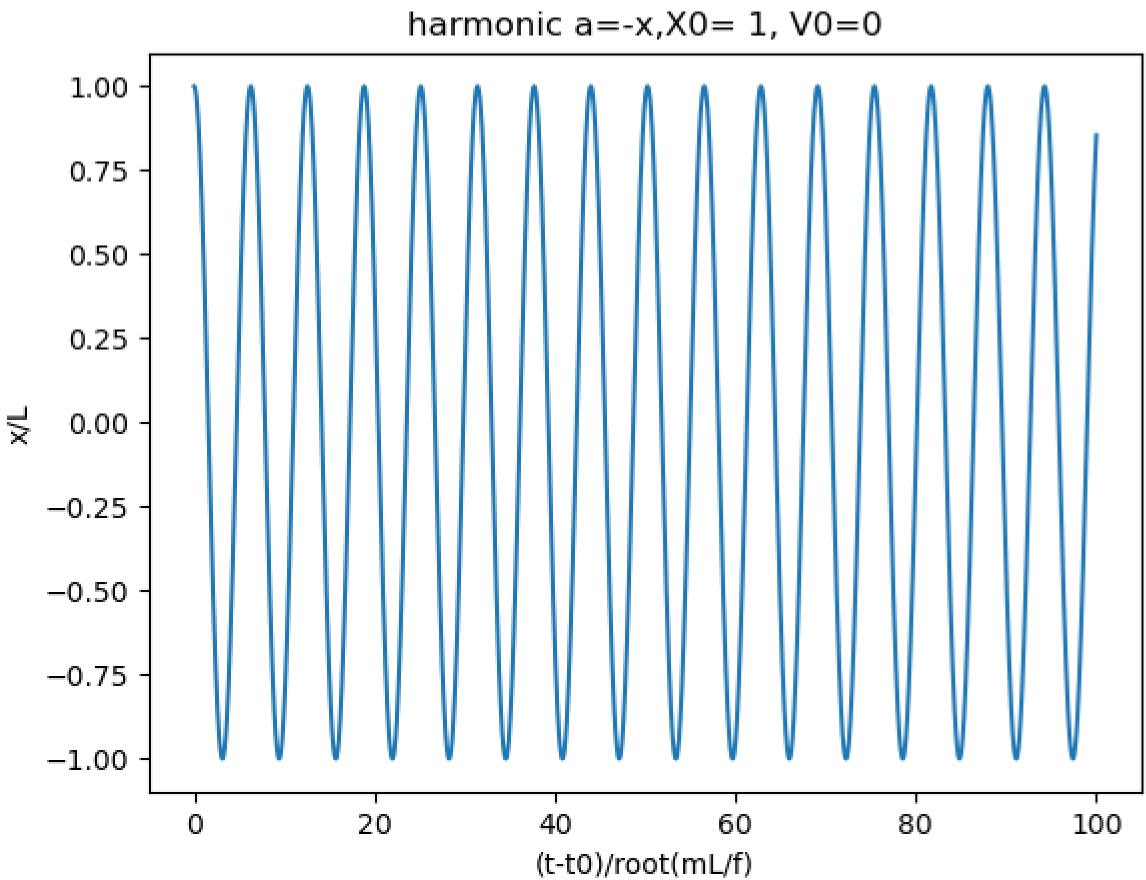
inharmonic tanh, X0=1, V0=0
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
x/L
0
20
40
60
80
100
(t-t0)/root(mL/f)


inharmonic tanh, X0=3, V0=0
x/L
(t-t0)/root(mL/f)
3
2
1
0
−1
−2
−3
0
20
40
60
80
100

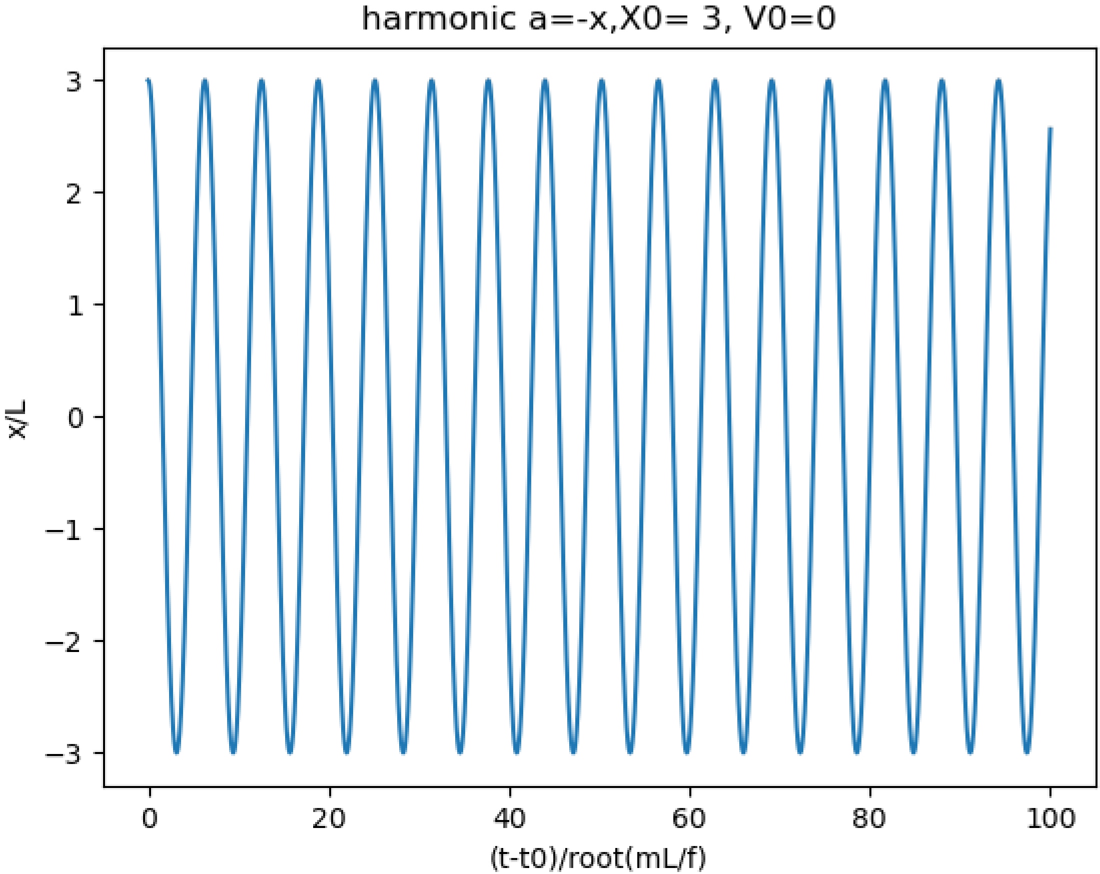

inharmonic sinh, X0=1, V0=0
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
x/L
0
20
40
60
80
100
(t-t0)/root(mL/f)



inharmonic sinh, X0=3, V0=0
3
2
1
0
−1
−2
−3
x/L
0
20
40
60
80
100
(t-t0)/root(mL/f)



harmonic a=-x,X0= 1, V0=0
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
x/L
0
20
40
60
80
100
(t-t0)/root(mL/f)


harmonic a=-x,X0= 3, V0=0
x/L
(t-t0)/root(mL/f)
3
2
1
0
−1
−2
−3
0
20
40
60
80
100

Problem 9.4 Solutions of lde

$$I(t) = \sum_{k=0}^{N} c_k(t)\, q^{(k)}(t)$$

$I: \mathbb{R} \to \mathbb{C}$, $c_k(t) \in \mathbb{C}$, $q^{(k)}: \mathbb{R} \to \mathbb{C}$

$\{q(t)\}$ – all solutions we want.

15.1

a) $I(t_0) = c_0(t_0) q^{(0)}(t_0) + \dots + c_{N-1}(t_0) q^{(N-1)}(t_0) + c_N(t_0) q^{(N)}(t_0)$,

then $$q^{(N)}(t_0) = \frac{I(t_0) - \sum_{i=0}^{n-1} c_i(t) q^{(i)}(t)}{\underbrace{c_N(t_0)}_{\text{assume } \neq 0,\text{ otherwise reduce } N}}$$

b) $$0 = \sum_{k=0}^{N} c_k(t)\, q^{(k)}(t)$$

$V = \{q(t)\}$ Show: all linear combinations $\in V$.

$h_1, h_2 \in V$, $$\sum_{k=0}^{N} c_k(t) \underbrace{\left(a_1 h_1^{(k)}(t) + a_2 h_2^{(k)}(t)\right)}_{\text{any combination}} = a_1 \underbrace{\sum_{k=0}^{N} \left(c_k(t) h_1^{(k)}(t)\right)}_{0} + a_2 \underbrace{\sum_{k=0}^{N} c_k(t) h_2^{(k)}(t)}_{0} =$$

$= 0$. So $a_1 h_1(t) + a_2 h_2(t)$ also in V.

15.2

This is enough to prove VS, since other properties like $f_1 + (f_2 + f_3) = (f_1 + f_2) + f_3$ are completely trivial properties of $\mathbb{C}$-field, not related to ODE or giving anything new.

c) $s(t)$ is solution $\longleftrightarrow$ $I(t) = \sum_{i=0}^{N} c_i(t) \cdot s^{(i)}(t)$

arbitrary sol. $q(t) \longleftrightarrow I(t) = \sum_{i=0}^{N} c_i(t)\, q^{(i)}(t)$

$-\downarrow$

$$0=\sum_{i=0}^{N} c_i(t)\big(\underbrace{s^{(i)}(t)-q^{(i)}(t)}_{-h^{(i)}(t)}\big)$$

$h(t)$ – solution of homogenous, $h(t)=-s(t)+q(t)$.

In other words, $q(t) = h(t)+s(t)$

any solution — all homogenous — any particular

16.1

d) $\sum_{i=0}^{N} c_i q^{(i)}(t)=0$

$$q(t)=e^{\lambda t}\in V \longleftrightarrow \sum_{i=0}^{N} c_i (e^{\lambda t})^{(i)} = \sum_{i=0}^{N} c_i e^{\lambda t}\lambda^i =$$

$$=\underset{\neq 0}{e^{\lambda t}} \sum_{i=0}^{N} c_i\lambda^i \longleftrightarrow \sum_{i=0}^{N} c_i\lambda^i = 0 \longleftrightarrow \lambda \text{ is root.}$$

If not constant, well get $\sum_{i=0}^{N} c_i(t)\lambda^i=0$, e.g. for $N=2$

$$c_0(t)+c_1(t)\lambda+c_2(t)\lambda^2=0.$$

Does not give any clue, and λ is not necessary root.

Bad situation.

16.2

* e) $\sum_{k=0}^{N} c_k x^k = \prod_{k=1}^{M}(x-\lambda_k)$

If $N\neq M$, $\prod_{n=1}^{M}(x-\lambda_k) = x^M+\dots$ which in general $\neq c_N x^N$

1) Show all λ_m are roots

$$\sum_{k=0}^{N} c_k\lambda_m^k = \prod_{k=1}^{N}(\lambda_m-\lambda_k) = \Big(\prod_{\substack{k=1,\\ k\neq m}}^{N}(\lambda_m-\lambda_k)\Big)\cdot\underbrace{(\lambda_m-\lambda_m)}_{0}=0.$$

2) Show no other. Suppose λ is another sol.

Then $\sum_{k=0}^{N} c_k\lambda^k = \prod_{k=1}^{N}(\lambda-\lambda_k)$ never zero. Fail.

16.3

f) $e^{\lambda_k t} \in V$ since if $\lambda_1, ..., \lambda_N$ are roots I proved before.

* Their linear combinations $C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + ... + C_N e^{\lambda_N t}$ are also in V as proven before.
* I am not sure how to prove there cannot be roots of other form at all.
* If assumed, last check is linear independence:

$$0 = C_1 e^{\lambda_1 t} + ... + C_N e^{\lambda_N t} \text{ has only one set } \{C_1, ..., C_N\} = \{0\}$$

since $e^{\lambda_k t}$ are different and $\neq 0$.

* If allow pairwise, then:
  - $0 = \underbrace{C_1 e^{\lambda_1 t} + C_2 e^{\lambda_1 t}}_{} + ...$

    can take $C_1 = -C_2$, for any $C_1 \neq 0 \longrightarrow$ not LI.
  - does not span all V since dimension is smaller.

Problem 9.5 Separation

a) $\dot y = \frac{x}{y}$ (next I do not change integral variables; $\dot y$ means $\frac{dy}{dx}$)

$\dot y y = x$

$$\int_{y_0}^{y} y\,dy = \int_{x_0}^{x} x\,dx$$

$\dot y y = x \downarrow$ phase sp. variables

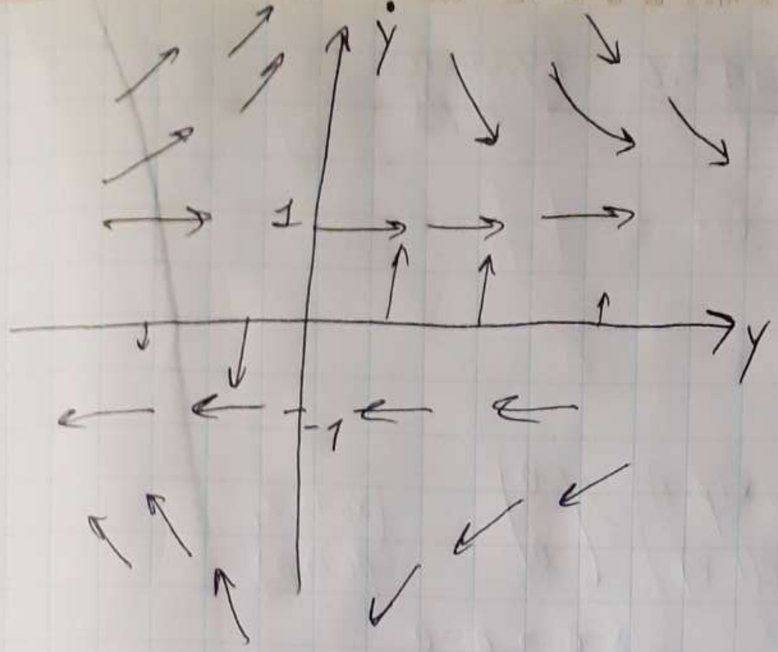
$$y^2 - x^2 = y_0^2 - x_0^2$$

$$y = \pm\sqrt{x^2 + (y_0^2 - x_0^2)}$$

---

b) $v = \begin{pmatrix} \dot y \\ \ddot y \end{pmatrix} = \begin{pmatrix} \dot y \\ \left(\frac{x}{y}\right)^{\cdot} \end{pmatrix} = \left[\begin{matrix} \text{using} \\ \dot x = 1 \\ x = y\dot y \end{matrix}\right] = \begin{pmatrix} \dot y \\ \frac{y - y(\dot y)^2}{y^2} \end{pmatrix} =$

$= \begin{pmatrix} \dot y \\ \frac{1 - (\dot y)^2}{y} \end{pmatrix}$ dir field.



If $\dot y = 1$ $\begin{pmatrix}1\\0\end{pmatrix}$

$\dot y = -1$ $\begin{pmatrix}-1\\0\end{pmatrix}$

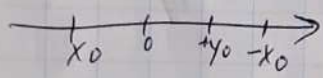
$1-(\dot y)^2$ drops quickly.

$\dot y$-component is odd

If $\dot y = 0$, $\dot y$-comp $= \frac{1}{y}$

c) e) $y = \pm\sqrt{x^2+(y_0^2-x_0^2)}$

Case 1)



$|x_0| > |y_0|$

$x_0^2 > y_0^2$

$y = \pm\sqrt{x^2 + C}$

<0, will be real y when $x^2 \geq (x_0^2 - y_0^2)$

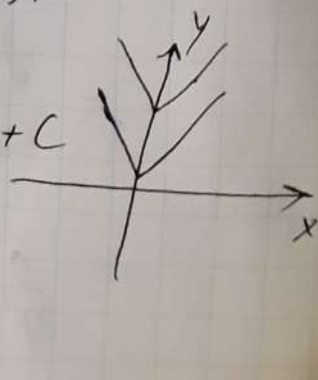
Case 2) } Similar (generic already done)
Case 3) }

Case 4) $y = \pm\sqrt{x^2} = \pm|x|$ is a solution

$y \neq 0$ from start, in all other $\dot{|x|}$ exists.

$x>0 \longrightarrow \dot y = \frac{x}{|x|} = 1$, $y = x + C$

$x<0$, $\dot y = \frac{x}{|x|} = -1$, $y = -x + C$



Problem 9.6 Dynamics in PS

$$\dot{x}(t) = \alpha x(t)(p(t)-1)$$
$$\dot{p}(t) = p(t)(1-x(t))$$

a) Fixed point

$$\begin{pmatrix} x_0 \\ p_0 \end{pmatrix}, \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} \alpha x(p-1) \\ p(1-x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(x_0 is x, p_0 is p)

$$\begin{cases} \alpha x(p-1) = 0 \\ p(1-x) = 0 \end{cases} \quad \begin{matrix} \nearrow x=0 \to p=0 \\ \searrow x=1, p=1 \end{matrix}$$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are f.p.

b) $$\frac{dI}{dt} = \dot{x} + \alpha\dot{p} - \frac{\dot{x}p^\alpha + x\alpha p^{\alpha-1}\dot{p}}{xp^\alpha} =$$

(x, p mean $x(t)$, $p(t)$)

$$= \alpha x(p-1) + \alpha(1-x)p - \frac{\alpha x(p-1)p^\alpha + x\alpha p^{\alpha-1}p(1-x)}{xp^\alpha} =$$

$$= \frac{\alpha x^2(p-1)p^\alpha + \alpha(1-x)xp^{\alpha+1} - \alpha x(p-1)p^\alpha - x\alpha p^\alpha(1-x)}{xp^\alpha} =$$

$$= \frac{\cancel{\alpha x^2 p^{\alpha+1}} - \cancel{\alpha x^2 p^\alpha} + \alpha x p^{\alpha+1} - \cancel{\alpha x^2 p^{\alpha+1}} - \alpha x p^{\alpha+1} + \alpha x p^\alpha - \cancel{\alpha x p^\alpha} + \cancel{\alpha x^2 p^\alpha}}{xp^\alpha}$$

$= \frac{0}{xp^\alpha} = 0$. (and if x/p 0, already showed f.p.)

c) Let's check! Should be like with energy in before.

$$\downarrow \begin{cases} \dot{x} = \alpha x(p-1) \\ \dot{p} = p(1-x) \end{cases} \quad \int_{p_0}^{p} \frac{dx}{dp'}\,dp' = \int_{p_0}^{p} \frac{\alpha x(p'-1)}{p'(1-x)}\,dp'$$

$$\int_{x_0}^{x} \frac{1-x'}{x'}\,dx' = \alpha \int_{p_0}^{p} \frac{p'-1}{p'}\,dp'$$

$$\ln|x'| - x' \Big|_{x_0}^{x} = \alpha \, |p' - \ln|p'|| \Big|_{p_0}^{p}$$

$$\ln x - x - (\ln x_0 - x_0) = \alpha(p - \ln p) - \alpha(p_0 - \ln p_0)$$

$$\ln x + \alpha \ln p = \alpha(p - p_0) + x - x_0 + \ln x_0 + \alpha \ln p_0$$

$$\underbrace{\ln x + \alpha \ln p} = \alpha p + x \underbrace{- \alpha p_0 - x_0 + \ln x_0 + \alpha \ln p_0}_{C}$$

$$\ln x p^{\alpha} = \alpha p + x - C$$

Then $\alpha p + x - \ln(xp^{\alpha}) = \mathbf{I}$ – all curves satisfying this equation

Each contour line $\longleftrightarrow$ some I.

d) $\alpha p + x - \ln(xp^{\alpha}) = \alpha p + x - \ln x - \alpha \ln p = \alpha(p - \ln p) + (x - \ln x) =$

$$= \alpha\left[\underbrace{(p-1)}_{\varepsilon} + 1 - \ln(\underbrace{(p-1)}_{\varepsilon} + 1)\right] + \left[(x-1) + 1 - \ln((x-1)+1)\right] \approx$$

only near origin

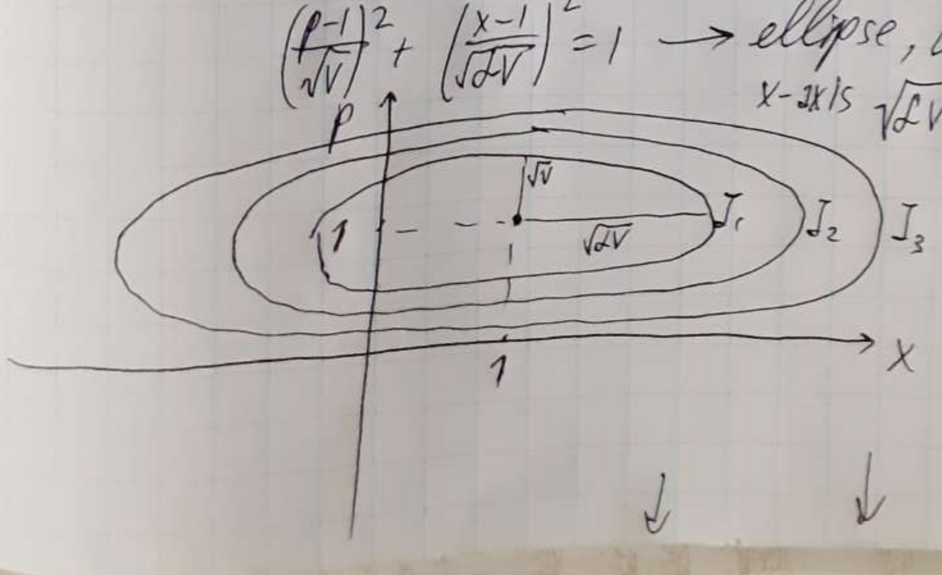
$$\approx \alpha\left[1 + \frac{(p-1)^2}{2}\right] + \left[1 + \frac{(x-1)^2}{2}\right] = I \quad \text{(drop constants)}$$

$$\alpha(p-1)^2 + (x-1)^2 = K$$

$$(p-1)^2 + \frac{(x-1)^2}{\alpha} = V$$

$$\left(\frac{p-1}{\sqrt{V}}\right)^2 + \left(\frac{x-1}{\sqrt{\alpha V}}\right)^2 = 1 \longrightarrow \text{ellipse, center } (1,1),$$

x-axis $\sqrt{\alpha V}$, p-axis $\sqrt{V}$ ($\sqrt{I}$ smaller)



does not work here

* e) For example, in $\begin{pmatrix}1\\1\end{pmatrix}$ $\dot{p} = \dot{x} = 0$, while $p = 1$,

if p is momentum $p = \dot{x} = 0$

(Arrive at contradiction for generic case?)

$$\delta p - \delta \ln p + x - \ln x = C \,\Big|\, \frac{d}{dt}$$

$$\delta\left(\dot{p} - \frac{\dot{p}}{p}\right) + \left(\dot{x} - \frac{\dot{x}}{x}\right) = 0$$

$$\delta\left(\ddot{x} - \frac{\ddot{x}}{\dot{x}}\right) + \left(\dot{x} - \frac{\dot{x}}{x}\right) = 0$$ (expand further?)

—— ask Dr. Vollmer on this also.

Index of comments

1.1 2/3 solve for x

1.2 3/3

2.1 3/4

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2.3 1/1

2.4 2/2

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3.1 3/3

3.2 5/5

4.1 4/4

4.2 4/4

5.1 4/6

missing taylor exp. for potential

15.1 1. Do not divide by something you know for sure is not 0. Good that you have mentioned it and provided an alternative.
2. this does not fully reason uniqueness. it usually comes from the mapping between two entities being unique. 1.5/2

15.2 yes. 4/4

16.1 3/3

16.2 True. Good identification! This only works if c and lambda are time indep among other subtleties. 2/2

16.3 Good!

17.1 its okay, you can go forward with the assumptions of them being constant.

17.2 very good!!! 2/2