Lecture "Experimental Physics I"

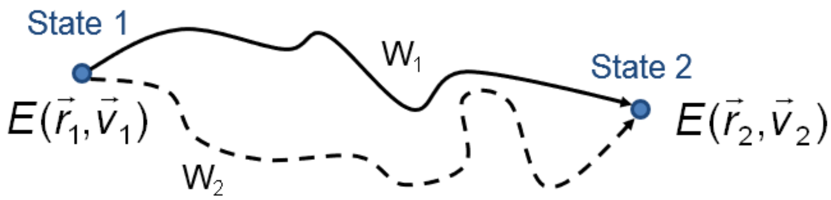
(Prof. Dr. R. Seidel)

Lecture 7

Energy conservation - Applications

- (Mechanical) equilibrium
- The potential energy landscape of a force field
- Oscillators (mass on a spring, pendulum)

Previously we have seen **that energy is a quantity defined for each state of the system.** In mechanics the system state of point masses is defined by the position and velocity vectors ($\vec{r}$ and $\vec{v}$):



The energy of a mechanical system is described by the two energy forms **potential and kinetic energy** that are calculated from the position and velocity vectors.
Energy is always the capacity of the system to do work. Specifically, the **system energy increases, when we do work on the system**. In contrary, the system energy decreases when the system is doing work on the object itself
For a conservative force field, we can formulate the **principle of energy conservation**, which gets the simple form: **The total energy of an isolated system is always constant.**

In this lecture we want to apply the principle of energy conservation to study a few important processes in mechanics, such as oscillations. **The principle of energy conservation often allows to address these problems in a much simpler manner** compared to solving the equations of motion from dynamics.
An illustrative example is the **City tunnel experiment**. Last lecture we have seen that at the tunnel end both cars are equally fast, even though they required different times to pass the system. Energy conservation readily explains this observation, since both cars exhibited the same height difference and thus the same decrease in potential energy that was powering their acceleration. Using the dynamics approach, i.e. forces, accelerations, velocities and positions the explanation would be very tedious. For the City tunnel one could get e.g. this via **numeric integration** (see slides).

1) Equilibrium

One can approach questions in mechanics likewise by looking at the forces and the resulting dynamics or alternatively by using conservation laws such as energy conservation. Each of the two approaches must provide the same result and can be used as a consistency check, since energy conservation was derived from equations from dynamics. To test this, we will approach different examples in the following using energy conservation.
Let us start with Newton's 1st law that defines the **equilibrium state** as the one of uniform motion incl. rest at which **no effective force is acting on the system**. Let us look at this from the energy perspective.

A) From potential to force

The potential energy function that we defined last lecture was:

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \underbrace{\vec{F}}_{\substack{field\\force}} \cdot d\vec{r} + U_0$$

where the increment inside the integral is given as

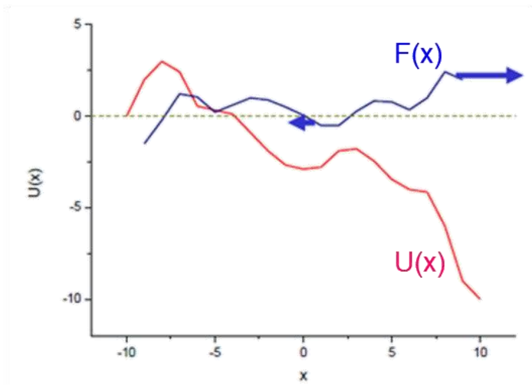
$$dU = -F \cdot dx \qquad \text{(in 1D) or}$$

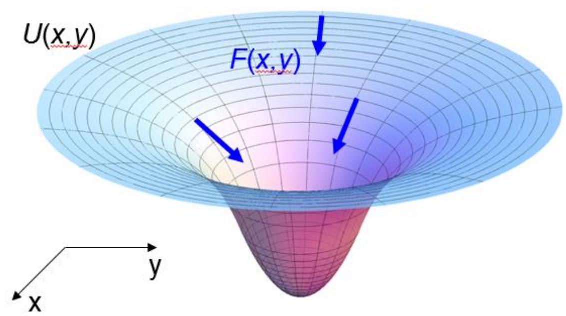
$$dU = -\vec{F} \cdot d\vec{r} = -F_x dx - F_y dy - F_z dz \qquad \text{(in 3D)}$$

Integration over the force-field thus provides the potential. Vice versa, we can extract from a known potential the force field by the inverse operation of integration, i.e. differentiation. In 1D we simply get for the acting forces:

$$F(x) = -\frac{dU}{dx}$$

We can thus get from any potential energy function (or potential) the locally acting force. A **steep slope/increase in the potential translates thus into a large force into the opposite direction. The force always acts in the direction of decreasing potential energy** (see left plot below).





In 3D we can write the same relation for each component using the **Nabla operator**, which is a vector that contains in each component the corresponding partial derivative:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}\right)$$

The forces are then obtained by the negative scalar product between Nabla operator and force vector. The components of the force are thus the negative partial derivatives of the potential energy along the given coordinate:

$$\vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r}) = \begin{pmatrix} -\frac{\partial U}{\partial x} \\ -\frac{\partial U}{\partial y} \\ -\frac{\partial U}{\partial z} \end{pmatrix} = -\mathrm{grad}\, U(\vec{r})$$

$\vec{\nabla}U(\vec{r})$ is the **gradient of the potential energy** (see alternative representation by '$grad$'). It points in the direction of the strongest potential increase! Due to the negative sign, the force points towards the strongest potential decrease (see right plot above). One can imagine this by the path that a ball would follow when placed in this energy landscape.

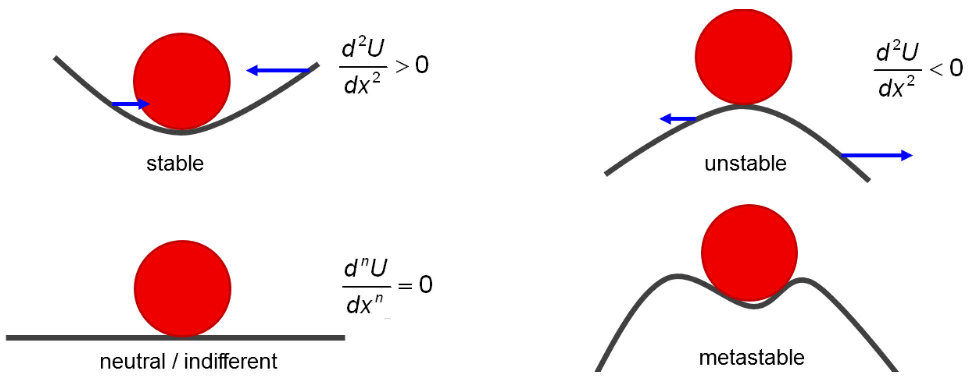
B) Equilibrium

Using the finding that force is the negative gradient (derivative in 1D) of the potential we can reformulate the equilibrium condition. We have **equilibrium** if

$$0 = F(x) = -\frac{dU(x)}{dx}$$

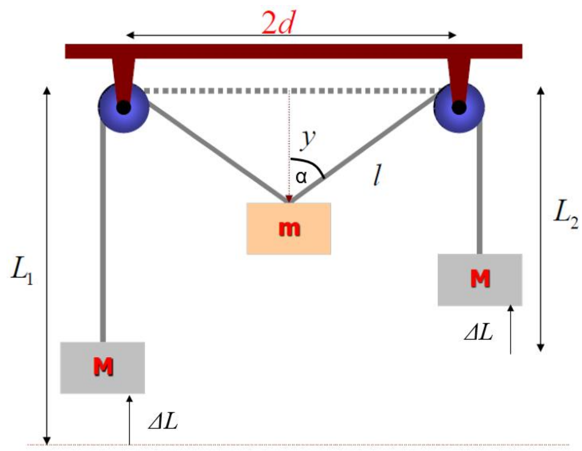
, i.e. **if the potential energy has an extreme value**. We distinguish different types of equilibrium:

- **Stable equilibrium** is reached **when the potential energy has a global minimum**.
  Using the potential of a spring $U = (k/2)\,x^2$ we get $F = -dU/dx = -kx$, i.e. any displacement from the equilibrium position $x = 0$ results in a *backdriving force* towards the equilibrium position.
  Remember, a (local) minimum of a function can be identified from the second derivative (see slide) by the condition:
  $$\frac{d^2U}{dx^2} > 0$$
- **Unstable equilibrium** is reached **when the potential energy has a local maximum.**
  Using a potential $U = -(k/2)x^2$ we get $F = +dU/dx = +kx$, i.e. any displacement from the equilibrium position $x = 0$ results in an driving *force away from the equilibrium position.*
  Remember a (local) maximum of a function can be identified by the following condition:
  $$\frac{d^2U}{dx^2} < 0$$
- **Neutral or indifferent equilibrium** is reached if $U(\vec{r}) = const$. In this case all higher order derivatives must be zero:
  $$\frac{d^nU}{dx^n} = 0$$
- **Metastable equilibrium:** if $U(\vec{r})$ has only a local minimum, such that a large enough displacement can make the particle to leave the local energy minimum towards the global minimum.



C) Mapping a potential landscape (1D)

Let us apply the principle of potential energy minimization to find the equilibrium position of a previously studied system of 3 masses and 2 pulleys:



To this end we assume that the central mass *m* is symmetrically located between the pulleys, since the other two masses are equal and the forces along the horizontal direction need to cancel each other. We now calculate the **potential energy of the whole system as function of the position *y* of *m* from the pulley level**.

Experiment: Demonstrate the problem on a real system of masses and pulleys

We assume that initially the left mass is at height 0, the central mass at pulley-level height L_1 and the right mass at height $L_1 - L_2$. The initial potential energy when mass *m* is at the pulley level is then given as:

$$U_0 = Mg0 + mgL_1 + Mg(L_1 - L_2)$$

If *m* moves down by distance y then the rope length from *m* to the pulley is given by:

$$l^2 = d^2 + y^2$$

such that each outer mass is lifted by

$$\Delta L = l - d = \sqrt{d^2 + y^2} - d$$

With 2*d* being the distance between the pulleys. The initial potential energy U_0 gets now corrected due to the lifting of both masses M by ΔL and the height decrease of *m* by *y*:

$$U(y) = U_0 \underbrace{-mgy}_{m} + 2\underbrace{Mg\Delta L}_{M} = U_0 - mgy + 2Mg(\sqrt{d^2 + y^2} - d)$$

(Stable) equilibrium is reached if

$$0 = \frac{dU(y)}{dy} = -mg + Mg\frac{2y_{eq}}{\sqrt{d^2 + {y_{eq}}^2}}$$

From which we get by transformation:

$$m^2(d^2 + {y_{eq}}^2) = 4M^2{y_{eq}}^2$$

and finally:

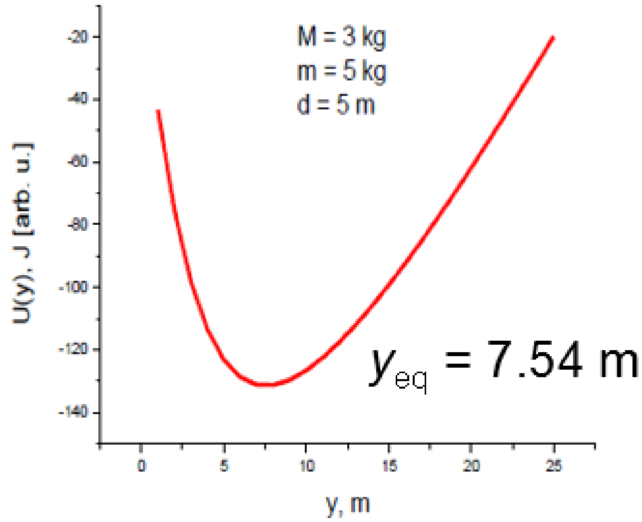
$$y_{eq} = d\frac{m/2M}{\sqrt{1 - \frac{m^2}{4M^2}}}; \quad l_{eq} = \sqrt{d^2 + y_{eq}^2} = \sqrt{d^2\frac{1}{1 - \frac{m^2}{4M^2}}} = d\frac{1}{\sqrt{1 - \frac{m^2}{4M^2}}}$$

The angle α at which the rope from m to the pully is inclined is given by the relationship:

$$\cos\alpha = \frac{y_{eq}}{l} = \frac{m}{2M}$$

It equals the expression, which we got for $\cos\alpha$ by considering the force equilibrium.

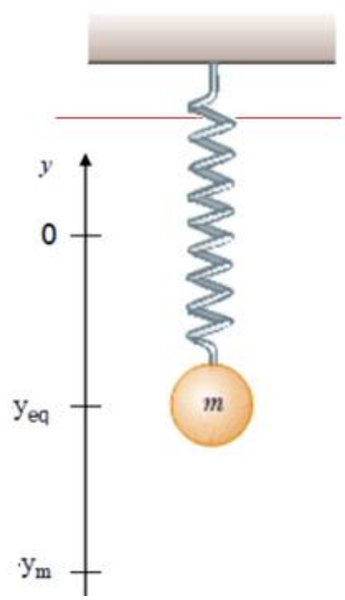
Thus, energy minimization is an alternative approach to force balancing when finding an equilibrium position.
An example of the calculated potential energy function (landscape) is given in the following:



2) Mass on a spring

We will now further practice the force and the energy approach when considering a problem in dynamics. In particular we will look at a mass on a spring under gravity.

A) Equilibrium and force for mass on a spring under gravity



The spring shell be not extended if the mass is at position zero. When not supported the mass extends the spring to the equilibrium position y_{eg}. We can then write for the **total force** (y-scale is directed upwards):

$$F = \sum F_i = -ky - mg = -k\left(y + \frac{mg}{k}\right) = 0$$

if we assume equilibrium where the total force is zero. With this we get for the equilibrium position:

$$y_{eq} = -\frac{mg}{k}$$

This allows rewriting the upper equation by subtracting the equilibrium position from *y*:

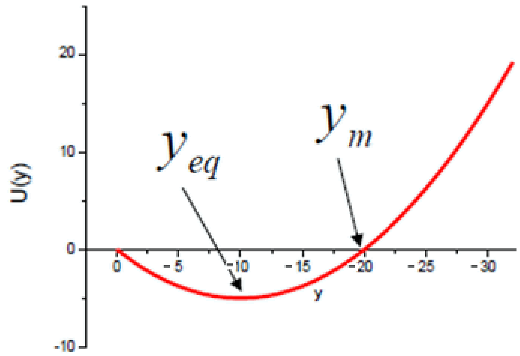
$$F = -k(\underbrace{y - y_{eq}}_{y*}) = -ky*$$

The new coordinate *y** is zero at y_{eg}. Thus, **for the force on the mass we can just use Hooke's law (i.e. ignore gravity), if the equilibrium position y_{eq} is taken as reference point.**

For the **potential energy** we can write in an alternative approach:

$$U(y) = mgy + \frac{1}{2}ky^2$$

This provides the following quadratic potential energy function (for a suitable m/k ratio):



At equilibrium we get:

$$0 = \frac{dU}{dy} = mg + ky_{eq}$$

from which we get as before:

$$y_{eq} = -\frac{mg}{k}$$

Experiment: From determining y_{eq} of a real mass-spring system we can determine the spring constant of the system:

$$k = \frac{mg}{-y_{eq}} = \frac{0.1kg \cdot 9.81m/s^2}{0.05m} = 19.6\frac{N}{m}$$

B) Dynamics of a mass on a spring

Now let us do a dynamic rather than a static experiment. Particularly, we **let the mass fall from position $\boldsymbol{y = 0}$**. We actually see that the **mass is undergoing oscillations**, where it periodically changes between downward and upward motion.

To understand this, let us look at its energy. At y = 0 it has zero potential energy but also zero kinetic energy, i.e. the total energy is zero for the chosen coordinates. When we let the mass move freely downwards we can write using **energy conservation**:

$$E(y) = \frac{1}{2}mv^2 + U(y) = 0$$

The potential energy gets initially negative, such that we have a corresponding positive kinetic energy, i.e. the mass moves downwards with velocity $v(y)$. At y_{eq} the mass has the lowest potential energy and this reaches its maximal velocity At y_m the potential energy becomes zero again such that no kinetic energy is left i.e. the downward motion stops and the mass that is below the equilibrium point returns upwards. We thus get an **oscillation of the mass between 0 and $\boldsymbol{y_m}$.** The turning point(s) y_m can be obtained by letting the potential energy become the total mechanical energy of the system (which is zero):

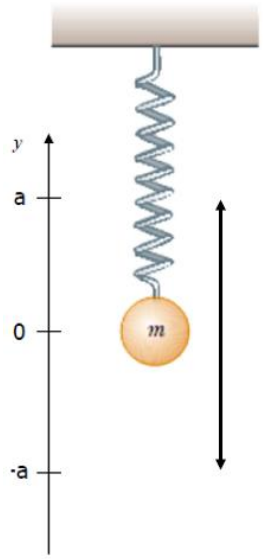
$$0 = mgy_m + \frac{1}{2}ky_m{}^2 = \left(mg + \frac{1}{2}ky_m\right)y_m$$

This equation becomes zero at $y_m = 0$, corresponding to the upper turning point of the oscillation, or at:

$$y_m = -\frac{2mg}{k} = 2y_{eq}$$

Corresponding to the lower turning point.

Let us describe these oscillations quantitatively. For simplicity we put $y_{eq} = 0$. At time t = 0 we start at $y(0) = -a$ and zero velocity $v(0) = 0$, i.e. $-a$ and a are the turning points of the oscillation for symmetry reasons (see drawing):



Force at mass-spring system

Using the **force approach** we can write according to Newton's 2nd law**:**

$$-ky = m\frac{d^2y}{dt^2}$$

Transformation provides:

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

This is a **differential equation that is difficult to solve** (see also below) since our acceleration depends on the position, such that we cannot use a simple integration over time.

Not part of lecture: Numerically this would be doable by splitting the time into small intervals and simultaneously integrating velocity and time using the **Euler method** (as described for the City tunnel):

$$\frac{dv}{dt} = -\frac{k}{m}y \Rightarrow dv = -\frac{k}{m}ydt \qquad \Rightarrow v(t+\Delta t) = v(t) - \frac{k}{m}y\Delta t$$

and $\frac{dx}{dt} = v \Rightarrow dx = vdt \qquad \Rightarrow x(t+\Delta t) = x(t) + v(t)\Delta t$

Energy at mass-spring system

As an alternative we can use **energy conservation** to describe the oscillatory motion:

$$E = \frac{1}{2}mv^2 + U(y) = \text{const.}$$

with $U(y) = (k/2)y^2$ and $E = U(|a|) = (k/2)a^2$ since we initially start at $-a$ with zero kinetic energy. Transformation of this equation towards the velocity and inserting the energies gives then the velocity as function of the position:

$$v = \frac{dy}{dt} = \sqrt{\frac{2[E - U(y)]}{m}} = \sqrt{\frac{2[k/2a^2 - k/2y^2]}{m}} = \sqrt{\frac{k}{m}}\sqrt{a^2 - y^2}$$

Now we can apply separation of the variables and arrive at:

$$\frac{dy}{\sqrt{a^2 - y^2]}} = \sqrt{\frac{k}{m}}dt$$

Integrating now the position from $-a$ to $y(t)$ and the time from zero to t provides:

$$\int_{-a}^{y(t)} \frac{dy}{\sqrt{a^2 - y^2]}} = \int_0^t \sqrt{\frac{k}{m}}dt = \sqrt{\frac{k}{m}}t$$

The integral at the left-hand side can be solved by substituting $y = a\sin\varphi$ (**see slides**)

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with this we have:

$\frac{dy}{d\varphi} = a\cos\varphi$ and thus $dy = a\cos\varphi\, d\varphi$

We furthermore have to change the integration borders using:

$\varphi = \arcsin(y/a)$

Replacing y, dy and the integration border in the integral then provides:

$$\int_{-a}^{y(t)} \frac{dy}{\sqrt{a^2 - y^2]}} = \int_{\arcsin(-a/a)}^{\arcsin(y(t)/a)} \frac{a\cos\varphi\ d\varphi}{\sqrt{a^2 - a^2\sin^2\varphi]}} = \int_{-\pi/2}^{\arcsin(y(t)/a)} \frac{a\cos\varphi\ d\varphi}{a\underbrace{\sqrt{1 - \sin^2\varphi]}}_{\cos\varphi}}$$

$$= \int_{-\pi/2}^{\arcsin(y(t)/a)} d\varphi = \arcsin\frac{y(t)}{a} - \left(-\frac{\pi}{2}\right)$$

The integral over $d\varphi$ on the right hand side has a trivial solution, such that we arrive after inserting the integration boundaries at:

$$\arcsin\left(\frac{y(t)}{a}\right) - \left(-\frac{\pi}{2}\right) = \sqrt{\frac{k}{m}}t$$

Transformation then provides the time dependence of the position of the mass (**Draw**):

$$y(t) = a\sin\left(\sqrt{\frac{k}{m}}t - \frac{\pi}{2}\right)$$

This is a sinusoidal oscillation, with a single defined period/frequency (pure tone). It is therefore called a single harmonics and **such an oscillator is called harmonic oscillator**.

The amplitude reaches periodically the same value if the argument of the sine functions is a multiple of 2π, i.e. at times t of

$$\sqrt{\frac{k}{m}}t = 2\pi n$$

The **period**, i.e. the **time of a single oscillation** is obtained by division through *n*

$$T = \frac{t}{n} = 2\pi\sqrt{\frac{m}{k}}$$

Important to note is that **it is independent of the initial displacement.** It **increases for a more heavy mass** (large inertia) but decreases for a very stiff system.

The **frequency**, i.e. how many oscillations occur per time is provided by:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The so-called **angular frequency** is given by

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

It tells us how the phase angle (in rad) of the phase of the sine function changes per time (with 2π corresponding to the phase change of one period). One can view it also as a angular velocity given that the sine function und thus the spring motion can be seen as the projection of a circular motion onto the y-coordinate.

Experiment: Spring constant from a dynamic oscillation measurement. We now determine the spring constant from the period of an oscillating mass on a spring. To obtain a precise value for the period, we record the time for a larger number of oscillations. To this end, we monitor the force at the anchor point.

$$k = \frac{(2\pi)^2 m}{T^2} = \frac{(2\pi)^2}{(\frac{1}{2.19}s)^2} 0.1kg \approx \frac{36}{1/5} 0.1\frac{N}{m} = 5 \cdot 3.6\frac{N}{m} = 18\frac{N}{m}$$

We also show that the period is **independent of the initial displacement** and that it **doubles when having a 4 times heavier mass**. In agreement with the derived formula.
With the expressions for *T* and ω we can **rewrite our solution** to:

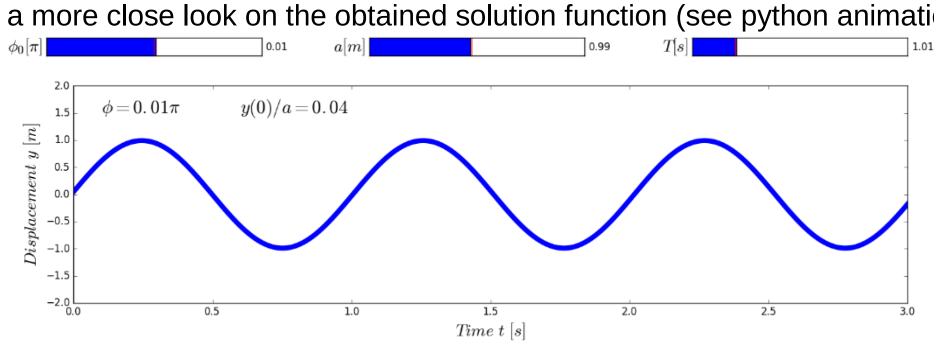
$$y(t) = a\sin(\omega t + \varphi_0) = a\ \sin\left(\frac{2\pi}{T}t + \varphi_0\right)$$

This solution solves also our initial differential equation, which one can show by inserting.
Rewriting **the initial differential equation** gives:

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

Always when we see in future a differential equation of that type we can directly write the found solution *y*(t) as the corresponding solution!

Let us have a more close look on the obtained solution function (see python animation)



- ***a*** is called the **amplitude** it is the maximal displacement that is reached by the oscillation, since the sin-function becomes maximally 1.
- ***T*** and **ω** just determine how many oscillations we get per time
- ***φ₀*** is the **initial phase** and reflects the starting condition. The sine function starts at *t* = 0 with zero. φ_0 shifts the function to the left to achieve the corresponding starting condition. We can

simply see this by inserting into our solution *t* = 0: where $a \sin \varphi_0$ marks the starting point of the oscillation

The role of the initial phase becomes even clearer when looking at the velocity of the oscillating mass. It is given by the time derivative of the position:

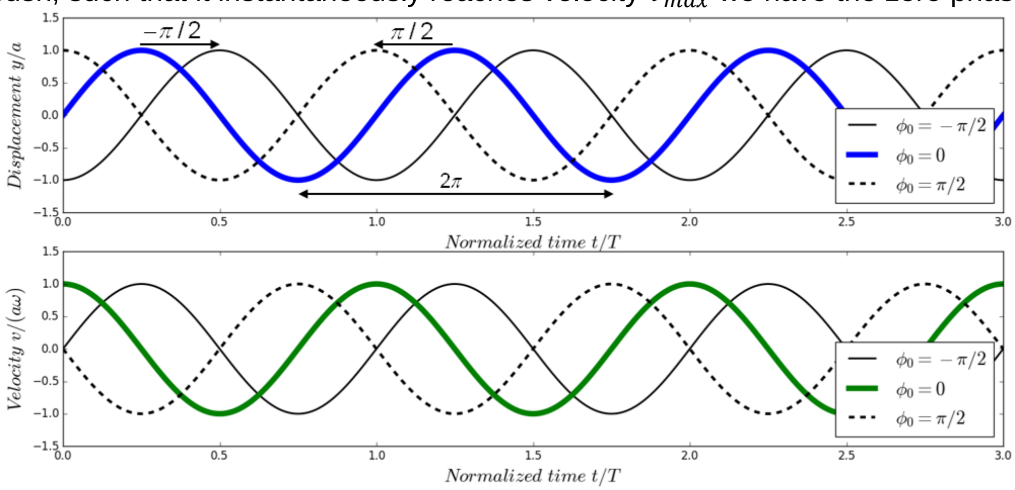
$$v(t) = \frac{dy(t)}{dt} = \underbrace{\omega\, a}_{v_{max}} \cdot \cos(\omega t + \varphi_0) = \frac{2\pi}{T} a \cos\left(\frac{2\pi}{T} t + \varphi_0\right)$$

The velocity is given by the cosine function. The factor before the cosine is the maximum speed. The cosine is phase shifted by π/2 compared to the sine function, i.e. when the displacement is maximal the velocity is zero and when the displacement is zero the velocity is maximal (**see function slider**).

This is also true for the initial phase, d.h. our starting condition for which we get (**see slider**):

$$y(t) = a \sin(\omega t + \varphi_0) \begin{cases} \varphi_0 = -\pi/2 & \rightarrow y(0) = -a, \quad v(0) = 0 \\ \varphi_0 = 0 & \rightarrow y(0) = 0, \quad v(0) = v_{\max} \\ \varphi_0 = \pi/2 & \rightarrow y(0) = +a, \quad v(0) = 0 \end{cases}$$

If we pull the spring to –a and then let it go, we have a phase of –π/2. If we push the phase to +a and then let go we have a phase of +π/2 and if we give the mass on the unextended spring a sudden push, such that it instantaneously reaches velocity v_{max} we have the zero phase.



Any other initial phase that is not a multiple of π/2 has as starting condition a partially extended spring AND a speed that is smaller than the maximal velocity.

Total energy of the mass spring system

In order to verify energy conservation of our solution we can calculate the potential and the kinetic energy at any time point:

$$U = \frac{k}{2} y^2 = \frac{k}{2} a^2 \cdot \sin^2(\omega t + \varphi_0)$$

$$E_k = \frac{m}{2} v^2 = \frac{m}{2} \omega^2 a^2 \cdot \cos^2(\omega t + \varphi_0)$$

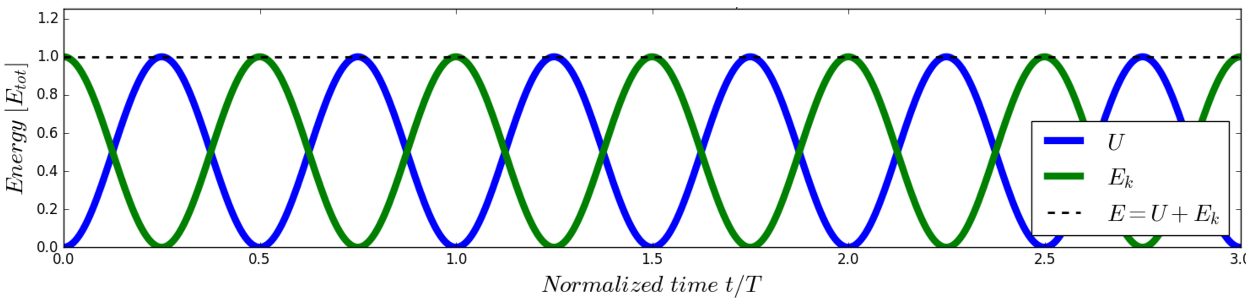
The total energy is then given by:

$$E_{tot} = U + E_k = \frac{1}{2} k a^2 \sin^2(\omega t + \varphi_0) + \frac{1}{2} \underbrace{m\omega^2}_{k} a^2 \cos^2(\omega t + \varphi_0)$$

Using $\omega^2 = k/m$ one can replace $k = m\omega^2$ in the amplitude of the cosine function. This then gives:

$$E_{tot} = \frac{k}{2}a^2[sin^2(\omega t + \varphi_0) + cos^2(\omega t + \varphi_0)] = \frac{k}{2}a^2 = \frac{m}{2}\underbrace{(\omega a)}_{v_{max}}{}^2 = const$$

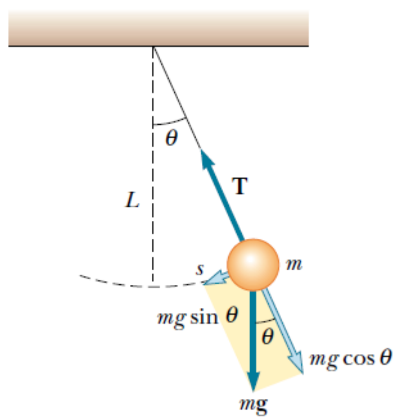
For the latter term we replaced the spring constant with $k = m\omega^2$. Thus, as we should have the total energy remains constant and we have a periodic exchange between potential and kinetic energy as can also be seen when plotting these functions:



3) The (mathematical) pendulum

A) Force at pendulum

Now we want to look at the oscillations of a pendulum. We will describe the pendulum motion in analogy to **circular motion**:



We look at the **forces in the tangential direction** that cause a change of the inertia of the system**.** Rewriting the velocity with an angular velocity provides:

$$-mg\,sin\,\theta = ma = m\frac{d}{dt}\underbrace{v}_{L\omega} = m\frac{d}{dt}\left(L\frac{d\theta}{dt}\right) = mL\frac{d^2\theta}{dt^2}$$

and we arrive at the differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}sin\,\theta = 0$$

This is difficult to solve. Therefore, we apply the small angle approximation $\sin\theta \approx 0$ (if θ inserted in rad and θ < 5°). With this we arrive at:

$$\frac{d^2\theta}{dt^2} + \underbrace{\frac{g}{L}}_{\omega^2}\theta = 0$$

This is the same type of differential equation as derived for the mass-spring system just that we have here an angle as coordinate instead of a position. We therefore should have **again a harmonic oscillation** as solution where the factor before θ corresponds to ω^2. Thus:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{L}} \quad \text{and} \quad T = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}}$$

We get then for the time dependence of the angle

$$\theta(t) = \theta_0 \sin\left(\sqrt{\frac{g}{L}}t + \varphi_0\right) \begin{cases} \varphi_0 = -\pi/2 & \rightarrow \theta(0) = -\theta_0 \\ \varphi_0 = 0 & \rightarrow \theta(0) = 0 \\ \varphi_0 = \pi/2 & \rightarrow \theta(0) = +\theta_0 \end{cases}$$

, where φ_0 defines the starting point of the oscillation.

Sidenote: Solving differential equations (see scheme)

In general a differential equation is an **equation that contains derivatives of a function (1)**, e.g. of the function x(t) of the oscillator position. The **first task is to solve the equation (2)** . This is often not straight forward and there are no general rules for it. One can do this by deriving a solution (e.g. separation of variables & subs. integration), by guessing or by knowing the solution of a similar equation (as done for the pendulum). Importantly one has to find all possible solutions that fullfill the equation. These solution sets can be simply **checked by inserting (3)** them into the differential equation. Finally one has to **consider the start conditions typically called boundary conditions (4)** to extract the correct solution out of the whole solution set (i.e. which concrete amplitude and phase do I have).

1) Differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

find *all* functions $x(t)$ that solve the equation

(by knowing, guessing, deriving…)

2) Solutions

$$x = x_0 \sin(\omega t + \varphi_0)$$

define infinite number of possible solutions

3) Check solution

$$\underbrace{-\omega^2 x_0 \sin(\omega t + \varphi_0)}_{\frac{d^2x}{dt^2}} + \omega^2 x_0 \sin(\omega t + \varphi_0) = 0$$

4) Start/boundary conditions

determine the particular solution to problem

x_0 … maximal displacement (amplitude)

φ_0 … sets $x(0)/x_0$ and $v(0)/v_{max}$

By knowing the solution for the oscillator differential equation, one can easily write down the solutions for the mass-spring oscillator and the pendulum, since they are described by the same type of equation (**see slide**)

The **remarkable finding** of the oscillation of a pendulum is that the **oscillation period *T* is independent of the mass** and **of the initial displacement** (if small enough). This has been used in the past to build very precise time keepers (in old precision clocks) as well as for some time to define the second based on a pendulum length

We will test our findings about the pendulum oscillations in a number of experiments:

Experiments: Period of a pendulum and energy conservation

- ***T* is independent of the mass**: Two parallel pendulums with an iron and a wooden sphere have the same period. Also we demonstrate **the independence of the period of the initial displacement**
- Determine **g from the period of the pendulum oscillation**

$$g = \frac{(2\pi)^2 L}{T^2} = \frac{(2\pi)^2}{(2s)^2} 1m \approx 3.14^2 \frac{m}{s^2} = 9.86 \frac{m}{s^2}$$

  We furthermore test the $T \propto \sqrt{L}$ scaling of the period with pendulum length
- Increasing "g" by additional attractive magnetic force results in a higher frequency
- **Energy conservation - "Nailpendulum"**: Obstacle for the pendulum string does not change maximum height
- **Energy conservation by numbers**: We measure the instantaneous velocity at the equilibrium position and calculate its expected value from the potential energy at the start height. Energy conservation demands that the maximum kinetic energy (at zero displacement) equals the maximum potential energy at initial displacement h:

$$\frac{1}{2} m v^2 = mgh$$

  such that we get:

$$v_{max} = \sqrt{2gh} = \sqrt{2\, 9.81 \frac{m}{s^2}\; 0.1\, m} \approx \sqrt{2} \frac{m}{s} \approx 1.4 \frac{m}{s}$$

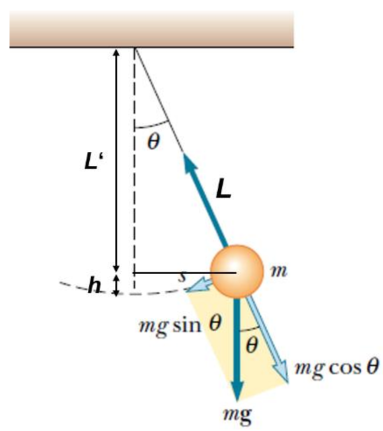
  We determine $v_{max}$from the sphere diameter and the time it blocks a photodetector at the equilibrium position of the pendulum:

$$v_{max} = \frac{23mm}{17ms} \approx 1.4 \frac{m}{s}$$

  Doubling the height increases the velocity (decreases the passage time) about 1.4-fold demonstrating square root scaling of v_{max}.

B) Deriving the pendulum period for large displacements using energy conservation

Not part of lecture



We can use **energy conservation** to derive an equation for the pendulum at **large displacements**:

$$E = \frac{1}{2} m \underbrace{v}_{L\dot{\theta}}{}^2 + U(\theta) = U(\theta_0)$$

where the right-hand side corresponds to potential energy at the maximum angle and zero angular velocity. The potential energy terms are given by:

$$U(\theta) = mgh = mgL(1 - \cos\theta)$$
$$U(\theta_0) = mgL(1 - \cos\theta_0)$$

The energy equation thus transforms to:

$$\frac{1}{2} m L^2 \left(\frac{d\theta}{dt}\right)^2 = mgL(\cos\theta - \cos\theta_0)$$

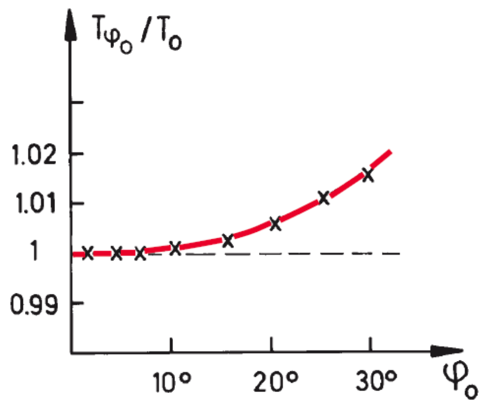
In analogy to the spring oscillator, transformation towards $d\theta/dt$ and separation of the variables provides the following integral:

$$\sqrt{\frac{L}{2g}} \int_{\theta_0}^{0} \frac{d\theta}{\sqrt{cos\,\theta - cos\theta_0}} = \int_0^{T/4} dt = \frac{T}{4}$$

To get the period of the integral we just integrate over the quarter of one oscillation. The integral can either be solved numerically or transformed into an elliptical integral and solved using Taylo rexpansion of the cosine function to provide:

$$T(\theta_0) = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{16}{\theta_0}^2 + \ldots\right)$$

For small angles the square term can be neglected and we get our original expression for the angular velocity. Plotting the angle dependent period normalized by the small angle period provides the relative change:



The small angle approximation is thus valid up to 5-10°

Experiment: The increase in the period can also be measured using a pendulum in the lecture hall.

Lecture 7: Experiments

1. System of 2 pulley and 3 masses to demonstrate the calculation of the potential energy function
2. Mass on spring: Determination of the spring constant from static and dynamic measurement
3. Period of pendulum is independnet of mass: Two parallel pendulums with an iron and a wooden sphere have the same period. With this one can also demonstrate the independence of the initial displacement
4. Determine g from the period of the pendulum oscillation
5. Increase "g" by additional attractive magnetic
6. Check energy conservation at the pendulum: Measure the instantaneous velocity at the equilibrium position as function of the start height. (0.1 m and 0.2 m height)
7. "**Nailpendulum**": Additional obstacle does not change maximum height
8. The increase in the period of a pendulum for large displacements