Theoretical Mechanics IPSP

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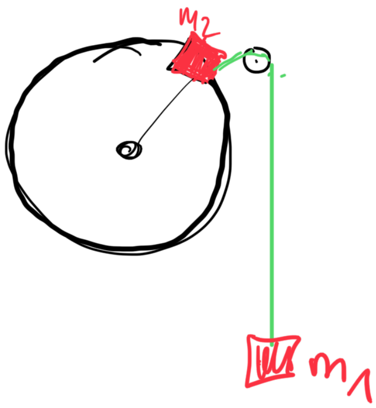
13.3. Particles on a wheel and pulley

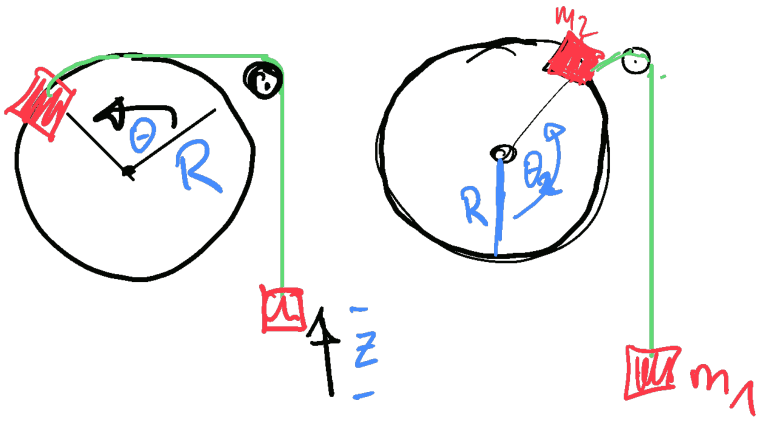
We consider two particles of masses m_1 and m_2 that are connected by a rope of fixed length. The rope is led over a pulley such that mass 1 is hanging straight down at a height z. Mass 2 is attached to a wheel such that the chord wraps along the wheel circumference when the wheel is turned. The sketch to the left shows the setting when mass 2 is attached at the closest point on the wheel. When the wheel is then turned (in either direction!) the chord wraps around the wheel, and mass 1 moves up, as shown in the following sketch:



a) Sketch and notations

Let the wheel have radius R and let it be turned by an angle θ, i.e. $\theta = 0$ in the sketch to the left. Mark these notations in the sketches.

Solution:



b) Height of the countermass

For $|\theta| \gtrsim \pi/4$ the function $z(\theta)$ is linear. Determine its slope for positive and negative θ. How are the slopes related?

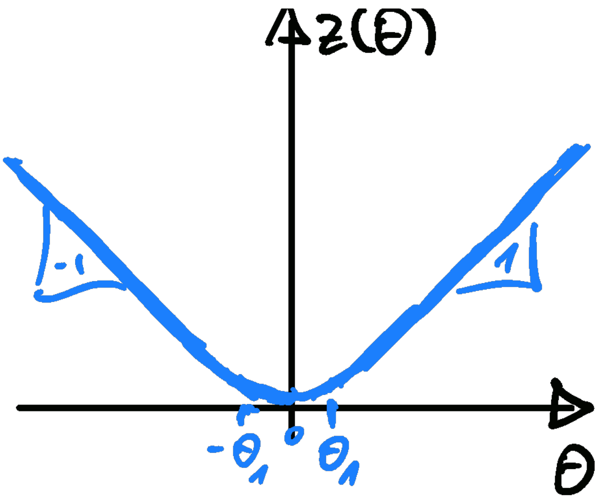
Solution:

The height change Δz must match the length of the rope wrapped around the wheel $R\,\Delta\theta$. Hence,

$$\frac{\mathrm{d}z(\theta)}{\mathrm{d}\theta} = \frac{\Delta z}{\Delta \theta} = R$$

Sketch $z(\theta)$.

Solution:



c) Kinetic energy and potential energy

Provide the kinetic energy and the potential energy of particle 1 and 2.

Solution:

kinetic energy and the potential energy of particle 1 and 2

$$T_1 = \frac{m_1}{2}\dot{z}^2 = \frac{m_1 R^2}{2}\dot{\theta}^2$$
$$T_2 = \frac{m_2 R^2}{2}\dot{\theta}^2$$

potential energy of particle 1 and particle 2

$$V_1 = m_1 g\, z = m_1\, g\, R\,\big(|\theta| - \theta_1\big) \quad \text{for} \quad |\theta| \gtrsim \frac{\pi}{4}$$
$$V_2 = -m_2 g\, R\, \cos(\theta + \theta_2)$$

d) Lagrange function

Determine the Lagrange function and express it in terms of $z(\theta(t))$, $\theta(t)$ and their time derivatives.

Solution:

$$\mathcal{L} = \frac{(m_1 + m_2)\, R^2}{2}\dot{\theta}^2 - m_1 g\, R\,\big(|\theta| - \theta_1\big) + m_2 g\, R\, \cos(\theta + \theta_2)$$

e) EOM

Show that the equation of motion for $\theta(t)$ takes the form

$$\ddot{\theta} = K\,\big(\kappa\, s(\theta) - \sin(\theta + \theta_2)\big) \tag{1}$$

where K is a constant that can be absorbed into the time scale and κ is a dimensionless parameter of the system. Moreover, $s(\theta)$ is a rescaled version of $z'(\theta)$ that takes the values ± 1 for large arguments and it crosses over from -1 to $+1$ in a narrow range of angles close to $\theta = 0$.
Determine K and κ.

Solution:

$$(m_1 + m_2)\, R^2\, \ddot{\theta} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} = -m_1 g\, R\, \mathrm{sgn}(\theta - \theta_\pm) - m_2 g\, R\, \sin(\theta - \theta_2)$$

Now we devide by $m_2 g\, R$ and we choose θ_2 as origin of the angle θ,

$$\frac{m_1 + m_2}{m_2}\,\frac{R}{g}\,\ddot{\theta} = -\frac{m_1}{m_2}\,\mathrm{sgn}(\theta - \theta_\pm + \theta_2) - \sin(\theta - \theta_2)$$

This agrees with Eq. (\ref{rope-eq:EOM}) for

$$K^{-1} = \frac{m_1 + m_2}{m_2}\,\frac{R}{g} \qquad \text{and} \qquad \kappa = -\frac{m_1}{m_2}$$

⋆ f) Phase space for $\kappa \gg 1$

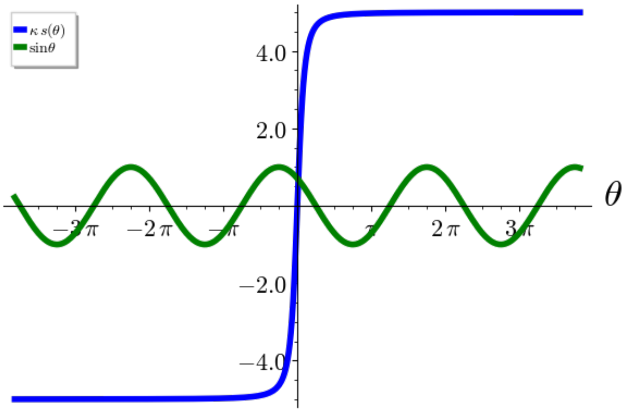
Show that the system has a single fixed point when $\kappa \gg 1$.
Hint: The fixed points are the roots of Eq. (1).

Solution:

The system has fixed points at angles where the force vanishes, i.e., for

$$\kappa\, s(\theta) = \sin\theta$$

For very large κ the function $\kappa\, s(\theta)$ takes values of modulus larger than one everywhere except for a steep crossover from negative to positive values for small angles θ. This implies that there is a single root:



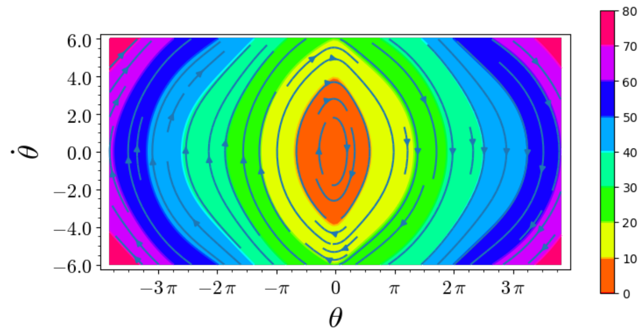
What does this mean physically?

Solution:

The mass m_1 is larger than m_2 such that it keeps mass m_1 in a unique stable equilibrium position close to the positions sketched in the left plot.

How does the phase-space plot look like in that case?

Solution:

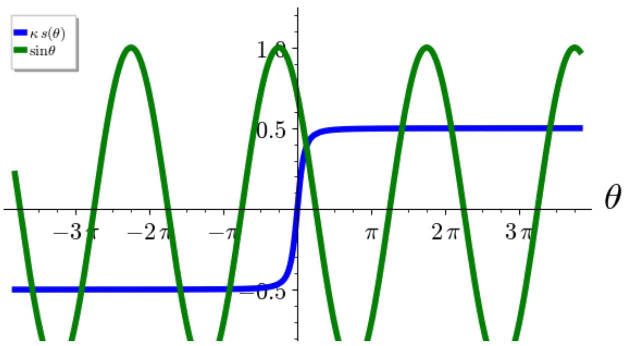


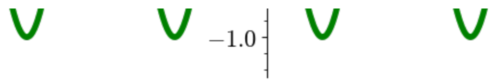
⋆ g) Fixed points for $\kappa \ll 1$

Show that the equation of motion has infinitely many fixed points for $\kappa \ll 1$.

Solution:

For small κ the function $\kappa\, s(\theta)$ takes values of modulus smaller than one everywhere except for a steep crossover from negative to positive values for small angles θ. This implies that there are infinitely many intersections with the sine function





What does this mean physically?

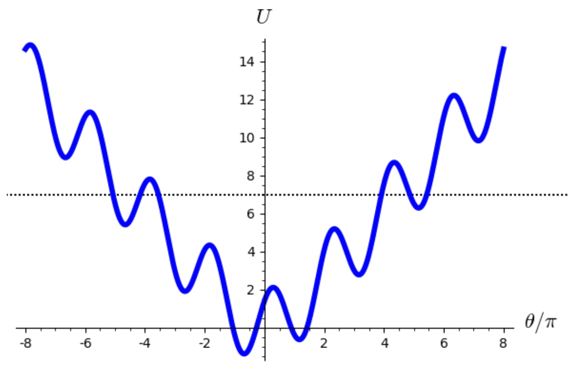
Solution:

The mass m_2 is heavier than m_2 and the system has a stable fixed point whenever this mass is hanging at the bottom side of the wheel. A 2π turn of the wheel has no effect on the force balance, except when moving between positive and negative values of θ.

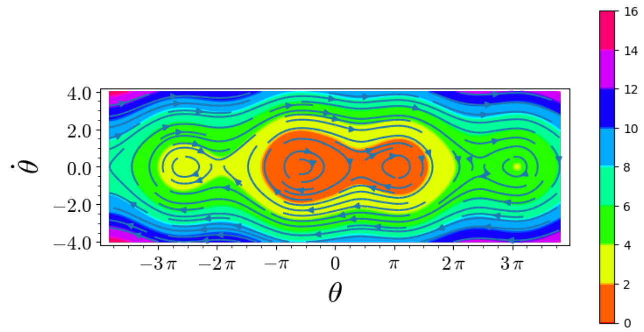
What does this mean physically?

h) Phase space for $\kappa \ll 1$

The plot to the left shows the effective potential of the dynamics for $\kappa \ll 1$. Provide the corresponding phase-space plot.



Solution:



⋆ i) Limitations of the model

Let the rope have a length L. For which range of angles θ would you trust the predictions of the model?

Solution:

When $\theta\,R$ approaches L the mass m_1 will approach the roll, and subsequently tip over the roll. At that point the EOM does no longer apply.

Discussion

Eni Hoxhalli, 2022/01/23 12:39, 2022/01/23 17:46

Hello.

If we consider the position of the pulley very close to the perimeter of the wheel and also very far to its perimeter the angle at which the rope becomes tangent to the wheel changes to interval $(0, \pi/2)$. So is it correct $\theta \geq \pi/4$ or is there another assumption missing? Or we just can take $\theta > \theta(0)$?

Jürgen Vollmer, 2022/01/23 17:53

What I want to say here is that there always are two angles $\theta_l, \theta_r \in [-\pi/2, \pi/2]$ such that the rope is wrapped around the wheel for $\theta \notin [\theta_l, \theta_r]$.

In my sketch these angles are roughly at $\pm\pi/4$, such that the rope is wrapped around the wheel for $|\theta| \geq \pi/4$, and it is not part of the task to determine the angles and the exact form of the resulting potential in the range $[\theta_l, \theta_r]$.

(Of course you may certainly do so, and discuss your results in the seminars, or come back to me for further discussion.)