MA1-HW6

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1. $\int\left(\frac{1-x}{x}\right)^2 dx = \int(x^{-1}-1)^2 dx = \int x^{-2}+1-2x^{-1}dx = \frac{x^{-1}}{-1}+x-2\ln|x|+C =$
$= -\frac{1}{x}+x-2\ln|x|+C.$

2. $\int\left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx = \int(1-x^{-2})x^{3/4}\,dx = \int x^{3/4}-x^{-5/4}dx = \frac{4}{7}x^{7/4}+4x^{-1/4}+C$

3. $\int\frac{dx}{e^x+e^{-x}} = \int\frac{e^x\,dx}{e^{2x}+1} = \begin{bmatrix} e^x=u \\ du=e^x dx\end{bmatrix} = \int\frac{du}{u^2+1} = \operatorname{arctg}(u)+C = \operatorname{arctg} e^x + C$

4. $\int \underbrace{\cos^5(x)}_{\cos x\cdot\underbrace{(1-\sin^2 x)^2}_{\cos^4 x}}\sqrt{\sin x}\,dx = \begin{bmatrix}\sin x=u \\ du=\cos x\,dx\end{bmatrix} = \int(1-u^2)^2u^{1/2}du = \int(u^4+1-2u^2)u^{1/2}du =$
$= \int u^{9/2}+u^{1/2}-2u^{5/2}du = \frac{u^{11/2}}{11/2}+\frac{u^{3/2}}{3/2}-2\frac{u^{7/2}}{7/2}+C = \frac{2}{11}(\sin x)^{\frac{11}{2}}+$
$+\frac{2}{3}(\sin x)^{3/2}-\frac{4}{7}(\sin x)^{7/2}+C.$

6. $\int\frac{x}{\cos^2 x}dx = \begin{bmatrix} t=\operatorname{tg}x,\ x=\operatorname{arctg}t \\ dt=(\operatorname{tg}x)'dx=\frac{dx}{\cos^2 x}\end{bmatrix} = \int(\operatorname{arctg}t)dt = [\text{by parts}] =$
$= \int(t')\operatorname{arctg}t\,dt = t\cdot\operatorname{arctg}t-\int t(\operatorname{arctg}t)'dt = (t)\operatorname{arctg}(t)-\frac{1}{2}\int\frac{2t}{1+t^2}dt =$
$= (t)\operatorname{arctg}t-\frac{1}{2}\int\frac{d(1+t^2)}{(1+t^2)} = (t)\operatorname{arctg}t-\frac{1}{2}\ln(1+t^2)+C = [t=\operatorname{tg}x]$
$= \operatorname{tg}x\cdot\operatorname{arctg}(\operatorname{tg}x)-\frac{1}{2}\ln(1+(\operatorname{tg}x)^2)+C = x\cdot\operatorname{tg}(x)-\frac{1}{2}\ln(1+\operatorname{tg}^2(x))+C.$

5. $\int x^2\sqrt[3]{1-x}\,dx = \begin{bmatrix}1-x=u \\ du=-dx \\ x=1-u\end{bmatrix} = \int(1-u)^2u^{1/3}(-du) = -\int(1+u^2-2u)u^{1/3}du =$
$= \int -u^{1/3}-u^{7/3}+2u^{4/3}du = -\frac{u^{4/3}}{4/3}-\frac{u^{10/3}}{10/3}+2\frac{u^{7/3}}{7/3}+C = -\frac{3}{4}(1-x)^{4/3}-\frac{3}{10}(1-x)^{10/3}+$
$+\frac{6}{7}(1-x)^{7/3}+C.$

7. $\int e^{\sqrt{x}}dx = [\text{by parts}] = \int(x')e^{\sqrt{x}}dx = xe^{\sqrt{x}}-\int x(e^{\sqrt{x}})'dx = xe^{\sqrt{x}}-$
$-\int xe^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}dx = xe^{\sqrt{x}}-\frac{1}{2}\int\sqrt{x}e^{\sqrt{x}}dx = \begin{bmatrix}\sqrt{x}=u \\ du=\frac{dx}{2\sqrt{x}}\end{bmatrix} = xe^{\sqrt{x}}-\frac{1}{2}\int ue^u(2u)du =$
$= xe^{\sqrt{x}}-\int u^2e^u du = [\text{by parts}] = xe^{\sqrt{x}}-\int u^2 d(e^u) = xe^{\sqrt{x}}-\left[u^2e^u-\int e^u d(u^2)\right] =$
$= xe^{\sqrt{x}}-u^2e^u+\int e^u 2u\,du = xe^{\sqrt{x}}-u^2e^u+2\int ue^u du = [\text{by parts}] =$
$= xe^{\sqrt{x}}-u^2e^u+2\left[ue^u-\int e^u du\right] = xe^{\sqrt{x}}-u^2e^u+2ue^u-2e^u+C = [u=\sqrt{x}] =$
$= \cancel{xe^{\sqrt{x}}}-\cancel{xe^{\sqrt{x}}}+2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C = 2e^{\sqrt{x}}[\sqrt{x}-1]+C.$

Quick check: $\frac{d}{dx}\left[2e^{\sqrt{x}}(\sqrt{x}-1)\right] = 2e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} + \cancel{2}(\sqrt{x}-1)\cdot\frac{1}{\cancel{2}\sqrt{x}}e^{\sqrt{x}} =$

$$= \cancel{\frac{e^{\sqrt{x}}}{\sqrt{x}}} + e^{\sqrt{x}} - \cancel{\frac{e^{\sqrt{x}}}{\sqrt{x}}} = \underline{e^{\sqrt{x}}}.$$

8. $$\int\sqrt{1-x^2}\,dx = \begin{bmatrix} x=\sin t,\ t=\arcsin x \\ dx=\cos t\,dt \end{bmatrix} = \int\sqrt{1-\sin^2 t}\,\cos t\,dt = \int|\cos t|\cdot\cos t\,dt =$$

$$= \begin{bmatrix} \cos t \geq 0 \text{ since } t=\arcsin x, \\ t\in[-\frac{\pi}{2}, +\frac{\pi}{2}] \end{bmatrix} = \int\cos^2 t\,dt = \int\frac{1+\cos 2t}{2}\,dt = \frac{t}{2}+\frac{\sin 2t}{4}+C =$$

$$= \begin{bmatrix} t=\arcsin x,\ \cos t\geq 0 \\ \sin 2t = \sin(2\arcsin x) = 2\sin(\arcsin x)\cos(\arcsin x) = 2x\sqrt{1-x^2} \end{bmatrix} =$$

$$= \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C.$$

9. $$\int\sin x\cos(x+1)\,dx = \int\sin x\left[\cos x\cos 1 - \sin x\sin 1\right]dx = \int\sin x\cos x(\cos 1)\,dx -$$

$$-\int\sin^2 x(\sin 1)\,dx = \begin{bmatrix} a=\cos 1 \\ b=\sin 1 \end{bmatrix} = a\int\sin x\cos x\,dx - b\int\sin^2 x\,dx =$$

$$= \frac{a}{2}\int\sin(2x)\,dx - \frac{b}{2}\int 1-\cos(2x)\,dx = -\frac{a}{4}\cos(2x) - \frac{b}{2}\left[x-\frac{\sin(2x)}{2}\right] + C =$$

$$= -\frac{a}{4}\cos(2x) + \frac{b}{4}\sin(2x) - \frac{b}{2}x + C = \frac{1}{4}\left[\sin 2x\cdot\sin 1 - \cos 2x\cdot\cos 1\right] - \frac{\sin 1}{2}x + C =$$

$$= -\frac{1}{4}\cos(2x+1) - \frac{(\sin 1)}{2}x + C.$$

10. $$\int\frac{2x+3}{(x-2)(x+5)}\,dx = \int\frac{A}{x-2}+\frac{B}{x+5}\,dx = \begin{bmatrix} A(x+5)+B(x-2) = 2x+3 \\ A+B=2 \\ 5A-2B=3 \Rightarrow A=B=1 \end{bmatrix} =$$

$$= \int\frac{1}{x-2}+\frac{1}{x+5}\,dx = \ln|x-2| + \ln|x+5| + C = \ln|(x-2)(x+5)| + C.$$