Lecture "Experimental Physics I"

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Lecture 6

Work, Energy & Power

- Work & Power
- Kinetic energy
- Conservative and non-conservative force fields
- Potential energy
- Energy conservation

Many forms of energy are known to everyone (**see slides**). The SI energy unit of energy is Joule J = kg m^2 s^{-2}. However, what is energy? Here we will try to understand this by first introducing work and certain forms of energy (kinetic, potential energy …).
Overall, we will see that **energy is a quantity/function that describes the state of the system** independent of how this state was reached.
In a complex situation **description of the dynamics of a system through the "energy approach" can often be much simpler than the direct application of Newton's second law**.

1) Work

A) Definition of work

We first define work. Motivation to this is that we understand **energy as the capacity of a system to do work** or vice versa **work as a quantity that changes the energy of a system.**
Work is the product of a displacement $d\vec{r}$ and the force $\vec{F}$ acting along it:

$$dW = \vec{F} \cdot d\vec{r} = F \cos\varphi \; dr; \qquad [W] = Nm = \frac{kg\,m^2}{s^2}$$

It is a scalar quantity! From the scalar product in the equation, we see that work is only produced if there is a force component acting along the displacement!

Examples:

- no work for uniform circular motion since force acts perpendicular to displacement
- no work of the normal force done during uniform linear motion (draw!)

The factor ccos φ in the definition of work automatically defines the sign of the work (see slide):

$$dW = \begin{cases} Fdr, & \varphi = 0 \\ 0, & \varphi = \frac{\pi}{2} \\ -Fdr, & \varphi = \pi \end{cases}$$

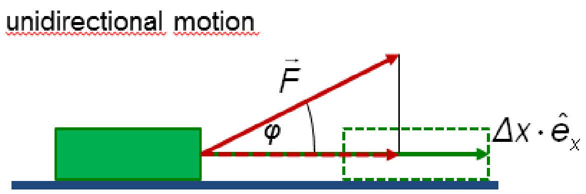
Generally, **work is an energy transfer** for which we define the following signs:

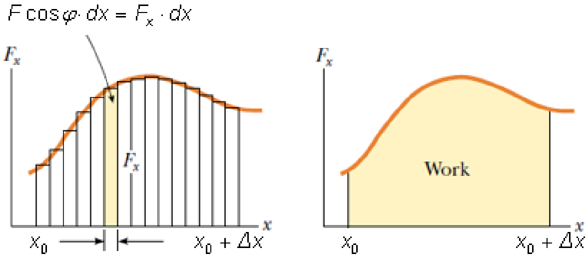
- **W > 0 as work done on a system** (movement with force) i.e. energy transfer to the system
- **W < 0 as work done by the system** (movement against force) as energy transfer from the system (loss of energy)

When plotting a force-position diagram for unidirectional motion under a non-constant force F_x, we see that the work done during a larger displacement corresponds to the area below the force-position curve, such that:

$$W = \int F_x \, dx$$

with F_x denoting the force component along x.





For an arbitrary three-dimensional path, we get the expression:

$$W = \int \vec{F} \cdot d\vec{r}$$

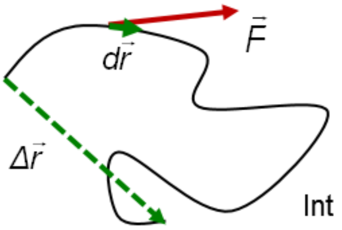
Using $d\vec{r} = dx\,\hat{e}_x + dy\,\hat{e}_y + dz\,\hat{e}_z$, the scalar product can in the vector component representation be written as

$$W = \int (F_x \, dx + F_y \, dy + F_z \, dz)$$

i.e. the work integral is split into three integrals for the respective components and equals the sum of the work done in each of the three orthogonal directions:

$$W = \int F_x \, dx + \int F_y \, dy + \int F_z \, dz$$

Note, the calculation of work requires **integration along the actual trajectory**, which is also called **line or path integral**



B) Power

Power determines how much work is (can be) done per time, i.e. how fast a car can drive up a mountain if its engine allows a certain power (kW) or how fast it can accelerate to a certain velocity (i.e. kinetic energy)

$$P = \frac{dW}{dt}$$

$$dW = \vec{F} \cdot d\vec{r} = \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \vec{F} \cdot \vec{v} dt$$

$$P = \vec{F} \cdot \vec{v}$$

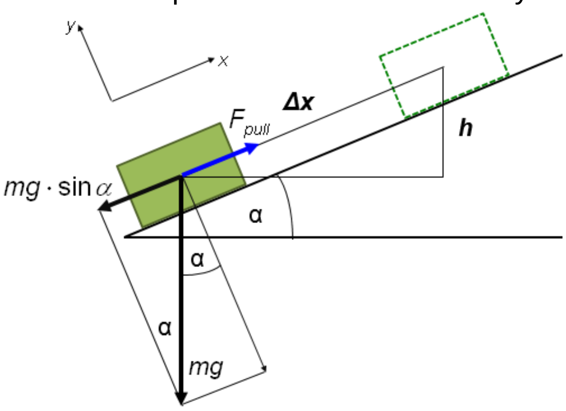
The unit of power is Watt:

$$[P] = W = \frac{J}{s} = \frac{Nm}{s} = \frac{kg \cdot m^2}{s^3}$$

Some useful conversions (see slide)
1 Watt = 1 J/s
1 Horsepower = 1 hp = 746 W
1 kilowatt-hour = 1 kWh = 3.6×10^6 J

C) Work done against gravity

Let us get familiar with the concept of work by calculating the work when pulling an object upwards an inclined plane with constant velocity



Since there is no acceleration we can write:

$$\sum F_x = F_{pull} - mg\, sin\, \alpha = 0$$

such that

$$F_{pull} = mg\, sin\, \alpha$$

The work for a displacement Δx is given by:

$$W = \int_{x_0}^{x_0+\Delta x} F_{pull}\, dx$$

which results in:

$$W = F_{pull}\Delta x = mg\, \Delta x \sin \alpha$$

$\Delta x \sin \alpha$ corresponds to the height difference, such that the work simplifies to:

$$W = mg\Delta h$$

Thus, **the work under the influence of a constant gravity force is only dependent on the height difference but not on the angle or the path along the plane**. An inclined plane can be helpful in practical terms since less force is required to pull object upwards, e.g. when trying to lift a cupboard into lorry, but finally the same work is required.

Experiment: To check the derived relation, we measure the work done when pulling a wagon up an inclined plane. The wagon has built-in force and position sensors und thus provides all parameters to integrate the work. We find:

- ➔ The integrated **work is independent on the pulling speed** or any additional forward & backward motion during the pulling.
- ➔ The work done in the experiment is: W ≈ 1 Nm (1.1 Nm) for m = 750 g and h = 0.15 m

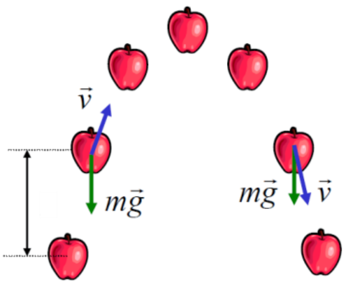
We get the same work for any path including some partial backward motion due to the fact that "we do negative work" when moving backward. Gravity does in this case due work on the wagon which has a negative sign. This depletes the initial work that we did. More elegantly we can show that **work against gravity is independent of the particular path also in three dimensions** by:

$$W = \int_{\vec{r}_0}^{\vec{r}_0+\Delta\vec{r}} m\, \vec{g} \cdot d\vec{r} = \int_{\vec{r}_0}^{\vec{r}_0+\Delta\vec{r}} mg\, \hat{e}_z \cdot d\vec{r}$$

$$W = mg \int_{z_0}^{z} dz = mg(z - z_0) = mg\Delta z$$

The scalar product of $\hat{e}_z$ and $d\vec{r}$ gives directly dz. The path independence is due to the fact that we finally integrate only over the vertical but not the lateral displacements.

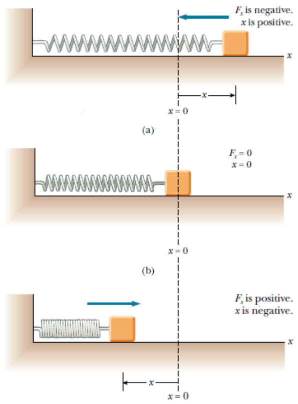
Watch out that **work done by gravity has a negative sign**, since gravity acts in the negative direction (or it does a "real" positive work on an object, when it can drag it downwards, i.e. for negative displacements):



$$W = \int_{\vec{r}_0}^{\vec{r}_0+\Delta\vec{r}} m(-\vec{g}) \cdot d\vec{r} = -mg\Delta z$$

D) Work done by a spring

We next explore further the concept of work and will calculate the work that an extended spring does on an object.



According to (Hooke's law) we can write for the force that the spring exerts on the object:

$$F = -kx$$

According to our definition, the work then becomes:

$$W = \int_{x_0}^{x_1} -kx\,dx = -\frac{k}{2}x^2\Big|_{x_0}^{x_1}$$

with this, one gets:

$$W = -\frac{k}{2}({x_1}^2 - {x_0}^2)$$

We can distinguish different cases:

$x_1 = \pm x_0$ $\quad W = 0;$

$x_1 = 0$ $\quad W = \frac{k}{2}{x_0}^2$; work done by spring

$x_0 = 0$ $\quad W = -\frac{k}{2}{x_1}^2$; work done on spring, thus spring "does negative work"

Generally, we see that the **work done by the spring is independent of the "path"** since it depends only on the difference of the squared displacements. In contrast to friction (see below) it does not matter how we reached the final position (e.g. by multiple forward-backward excursions).

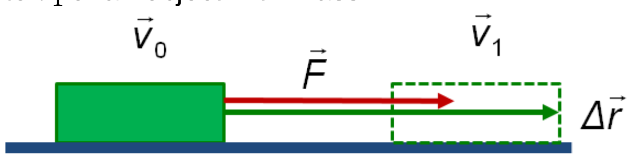
2) Kinetic energy & work

A) Kinetic energy

Let us derive a relation to describe the energy associated a mass that travels with a speed v. From everyday experience we know that it has an energy called kinetic energy. It is typically known as:

$E_k = \frac{m}{2}v^2$. $\quad [W] = \frac{kg\cdot m^2}{s^2}$

As kinetic energy we define the work that is needed to accelerate a particle to a given velocity. In the following we calculate the work by an external force $\vec{F}$ that causes a change in velocity from v_0 to v_1 of an object with mass m.



We use the previously introduced relations:

$$\vec{F} = m\frac{d\vec{v}}{dt}; \qquad d\vec{r} = \vec{v}\,dt; \qquad d\vec{v} = \vec{a}\,dt$$

Starting with:

$$W = \int_{\vec{r}_0}^{\vec{r}_1} \vec{F} \cdot d\vec{r}$$

We then get:

$$W = \int_{\vec{r}_0}^{\vec{r}_1} m\frac{d\vec{v}}{dt} \cdot d\vec{r} = \int_{t_0}^{t_1} m\underbrace{\frac{d\vec{v}}{dt}}_{\vec{a}} \cdot \vec{v}\,dt$$

$$W = m\int_{\vec{v}_0}^{\vec{v}_1} \vec{v} \cdot d\vec{v}$$

Thus, we arrive at:

$$W = m\left(\frac{\vec{v}^2}{2}\right)\Bigg|_{\vec{v}_0}^{\vec{v}_1} = m\left(\frac{v^2}{2}\right)\Bigg|_{v_0}^{v_1}$$

Inserting the integration boundaries provides the desired relationship:

$$W = \frac{m}{2}(v_1^2 - v_0^2)$$

Thus, the work is not dependent on how the velocity was reached, just how large the difference of the squared velocities is!

We now define **kinetic energy** as, which is known already to you:

$$E_k = \frac{m}{2}v^2$$

With this, the **work required to accelerate an object is equal to the change in kinetic energy** (**show on slides**):

$$W = E_{k1} - E_{k0}$$
$$W = \Delta E_k$$

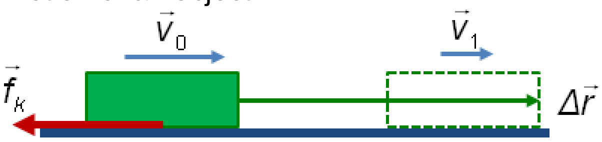
This is called the **work-kinetic energy theorem.**

Our derivation is valid in 3D and does not make any assumptions over the time course of the force we applied. **Thus, the kinetic energy does not depend on how the velocity was reached, just how large the velocity it is!** With this we can define (kinetic) **energy as a property that defines the state of a system**. In this context **energy becomes the capacity of a system to do work**.

Experiment: We **confirm the work-kinetic energy theorem.** We let a trolley move down an inclined plane over a height of $h = 0.18\,m$ and measure the position over time. We get the final speed from the derivative of the position-time curve and then calculate the work done by the gravity and the gained kinetic energy using $W = m(-g)(-h)$ and $E_k = (m/2)\,v^2$.

B) Work done by kinetic friction

With the definition of work, we can also calculate the work done by kinetic friction, e.g. when friction slows down the motion of an object:



The friction force acts always opposite of the motion direction, i.e. opposite of the displacement $d\vec{r}$

$$W_{fr} = \int_{\vec{r}_0}^{\vec{r}_1} \vec{f}_k \cdot d\vec{r} = \int_{\vec{r}_0}^{\vec{r}_1} -f_k dr = -f_k \int_{\vec{r}_0}^{\vec{r}_1} dr = -f_k \Delta s$$

with Δs being the length of the actual path. **Work done by kinetic friction is therefore dependent on the path. It is negative since the system itself does work based on its energy**

According to the work-kinetic energy theorem, a change (reduction) in kinetic energy powers the work done by the kinetic friction:

$$W_{fr} = \Delta E_k$$

$$-f_k \Delta s = E_{k1} - E_{k0}$$

which provides:

$$E_{k1} = E_{k0} - f_k \Delta s$$

Thus, **friction always reduces the kinetic energy of an object**. The energy is converted into heat at both contact surfaces.

3) Energy conservation

A) Conservative and nonconservative forces

Experiment: Again, we look at the **model of the Leipzig city tunnel**, where the car through the tunnel was faster than the car on the shallow incline. We now additionally **measure the velocity at the end of the track**

Finding: Both cars have about the **same velocity at the end of the track**! (The fast car through the tunnel is slightly slower due to friction on longer path).

To understand this, the kinematics approach would require a careful integration of the acceleration and the velocity over the whole path. There seems however to be a simpler principle to understand the end velocity, which is the principle of **energy conservation**!

To work out a proper definition of this principle, it is worthwhile to look at the work done by the different force types we considered so far (→ **see slides**). For some of the forces, the work does but for other does not depend on the path! Dependent on this we make a distinction between conservative and non-conservative force fields:

__**Conservative force field**__

Let $\vec{F}(\vec{r})$ be a force field, i.e. a defined position-dependent force, that is time independent. **$\vec{F}(\vec{r})$ is called conservative if the work between two points P_1 and P_2 is path-independent, i.e.**

$$W_a = W_b$$

with

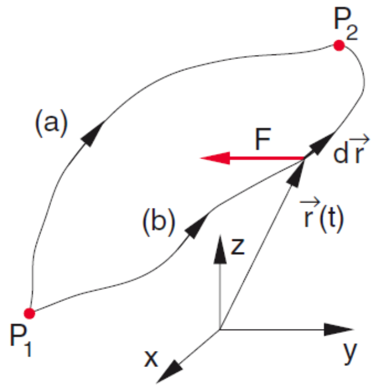
$$W_a = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}_a, \qquad W_b = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}_b$$

for any two paths a and b between P_1 and P_2

A consequence of this definition is that the **line integral of the work over any closed curve/loop is zero:**

$$0 = W_a - W_b = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}_a - \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}_b = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}_a + \int_{P_2}^{P_1} \vec{F} \cdot d\vec{r}_b = \oint \vec{F} \cdot d\vec{r}$$

since the second expression from the right side represents an integral from P_1 to P_2 and back to P_1. This means**, in a conservative force field we cannot gain or lose energy when moving along a closed loop. The energy within the field is conserved!**



For advanced students (not part of lecture): An alternative formulation to the line integral being zero for any closed loop can be obtained using (differential) vector operators. For this situation (according to the Stokes theorem) the curl of the force field must be zero at any spot. Mathematically this is expressed as:

$$\mathrm{rot}\,\vec{F} = \vec{\nabla} \times \vec{F} = 0$$

with

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z}\right)$$

being the Nabla operator which is a vector formed by partial differentials

Non-conservative force field

A non-conservative force field is then defined by:

$$\oint \vec{F} \cdot d\vec{r} \neq 0$$

e.g. any time-dependent field such as kinetic friction but also certain types of time-independent fields can form non-conservative force fields.

B) Potential energy

For a conservative force field, the work ***W* is thus only dependent on $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$** but not the path. If we choose a fixed reference point P_0 at $\vec{r}_0$ then the work from P_0 to any point $P(\vec{r})$ is a true function of $\vec{r}$**.** We now define **the potential energy** with respect to $\vec{r}_0$ of an object within a force field as:

$$E_{pot}(\vec{r}) = U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \underbrace{\vec{F}(\vec{r})}_{\substack{field \\ force}} \cdot d\vec{r} + U(\vec{r}_0)$$

This has the opposite sign as the definition of work, such that

$$U = (U(\vec{r}) - U(\vec{r}_0)) = -W_{field}$$

Please note:

1) The sign of U opposite to the work definition was chosen since we look at a field force like gravity. The sign choice provides that
   - if **we do work against the field then the potential energy of the object gets increased** (e.g. lifting a stone)
   - if the **field does work on the object** then **the potential energy gets reduced**, such that we **gain energy** from the field (e.g. a stone that falls done gains kinetic energy from the gravity field)
2) Typically, the reference point is chosen to be at $U_0 = 0$. The choice of the reference (zero point) is arbitrary and follows convenience (e.g. $h = 0$ at earth surface or at infinite distance for the gravity field of planets or coulomb fields).

3) The work between any other points P_1 and P_2 is independent of the choice of the reference point, since all paths from P_1 to P_2 through any selected references point require the same work.
4) The potential energy is a scalar

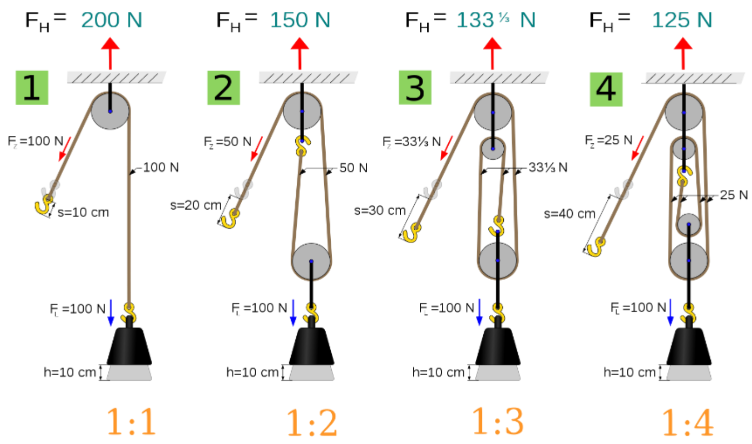
Potential energy for different force fields (see slide)

In the following can we calculate the potential energy functions of different force fields:

1) Constant gravitational field (close to planet surface):

$$U = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r} = -\int_{\vec{r}_0}^{\vec{r}} (-mg\,\hat{e}_z) \cdot d\vec{r} = mg \int_0^h dh = mgh$$

Experiment: Pulley system. In a pulley system the force to hold the attached weight gets distributed equally over 1,2,3 or more rope sections, such that one needs only 1x, 1/2x, 1/3x … of the weight force to lift it.



The work done at the rope end should equal to the potential energy change the weight experiences when lifted:

$$W_{on} = \Delta U$$
$$F_{pull}\Delta s = F_g \Delta h$$

where Δs is the length of the rope that we pulled in. Transformation provides:

$$\Delta s = \frac{F_g}{F_{pull}} \Delta h$$

i.e. one has to pull over a distance of 1x, 2x, 3x of the actually lifted height to compensate for the lower pulling force. This can be confirmed in the experiment.

2) Spring with reference at zero force position

To calculate the potential energy of a spring we start from Hooke's law:

$$F(x) = -kx$$

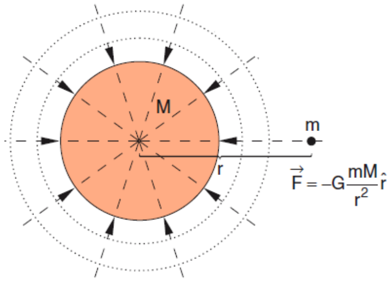
We then can write for the potential energy with respect to $x_0 = 0$:

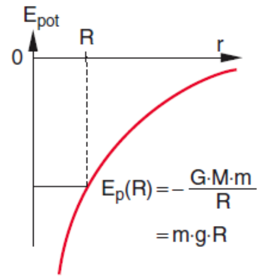
$$U(x) = -\int_0^x (-kx)\,dx = \frac{k}{2}x^2\Big|_0^x = \frac{k}{2}x^2$$

3) Gravitational field of a planet with mass *M* for large distance changes

At far distance r from its center, the gravitational force that a celestial body with mass *M* exerts on an object with mass *m* (see image below) is given as:

$$\vec{F}_g = -G\frac{Mm}{r^2}\hat{e}_r$$
where G is the universal constant of gravity.





As reference point for the potential energy we choose the infinite distance, since the force vanishes at infinity while it would be infinite at zero distance. With this, the potential energy in the gravity field of a celestial body becomes:

$$U(\vec{r}) = -\int_{\infty}^{\vec{r}} -G\frac{Mm}{r^2}\hat{e}_r \cdot d\vec{r} = \int_{\infty}^{\vec{r}} G\frac{Mm}{r^2}dr = -G\frac{Mm}{r^2}\bigg|_{\infty}^{r} = -G\frac{Mm}{r}$$

It becomes $-\infty$ at zero distance and increase towards zero at increasing distance. Note that any time-independent center-symmetric force field is conservative.

C) Energy conservation

Now we combine our knowledge about kinetic energy and potential energy to formulate energy conservation:

Conservative force field:
Let us consider an object that moves under the influence of an external conservative force field. The work done on an object by the field is provided by:

$$W = -\Delta U$$

The kinetic energy change resulting from this work W on the object is:

$$W = \Delta E_k$$

Combining both equations by subtraction provides then:

$$0 = \Delta E_k + \Delta U$$

i.e. the total change of kinetic and potential energy together is zero. When looking at the **total mechanical energy $\boldsymbol{E}$** being the sum of the kinetic energy and potential energy, we can thus say that it is constant:

$$E = E_k + U = const.$$

We can formulate energy conservation for a conservative force field as (**see slide**):

> **The total mechanical energy of a system remains constant in any isolated system of objects that interact only through conservative forces**.

Non-conservative forces:
In presence of additional non-conservative forces, we treat both force types separately:

$$\vec{F} = \vec{F}_{nc} + \vec{F}_c$$

We can thus write for the work done by this field:

$$W = \int (\vec{F}_{nc} + \vec{F}_c)d\vec{r} = \int \vec{F}_{nc}d\vec{r} + \int \vec{F}_c d\vec{r}$$

i.e. it is the sum of the work from conservative and non-conservative forces. Replacing the work by conservative forces with the corresponding potential energy change and equating the work done by the field with the kinetic energy provides:

$$W = W_{nc} + W_c = W_{nc} - \Delta U = \Delta E_k$$

The change in total mechanical energy as sum of the changes of kinetic and potential energy is then given by:

$$W_{nc} = \Delta E_k + \Delta U = \Delta E$$

Thus, energy conservation states in this case that **the work done by non-conservative forces is equal to the change of mechanical energy.** For example, the work done by friction is negative and thus always reduces the mechanical energy.

This concept of **energy conservation** can be generally formulated as:

> **Energy can never be created or destroyed. Energy may be transformed from one form to another, but the total energy of an isolated system is always constant.**

This is **one of most fundamental laws** without exceptions. Even the energy of the universe is constant.

Note that the system in case of a friction force is not isolated, since energy can escape through heat, which is nothing else than kinetic energy of the involved atoms. Only when taking this energy loss into account the energy balance is correctly described.

Lecture 6: Experiments

1. Inclined plane: Measuring work to pull up a wagon that measures in real time is position and the applied force: Work independent of the pulling speed and of the path (forward & backward movements), $W = mgh$
2. Inclined plane: Work – kinetic energy relation, measure with the wagon the final speed at the end of the inclined plane. Show that $mgh = m/2\ v^2$
3. Model of the Leipzig city tunnel, where the car through the tunnel was faster. Now measure velocity at the end of the track.
4. Pulley system. In a pulley system the force to hold the attached weight gets distributed equally over 1,2,3,… rope sections, such that one needs only 1x, 1/2x, 1/3x … of the weight force to lift it. Compare work done during pulling and the potential energy increase.