Lecture "Experimental Physics I"
(Prof. Dr. R. Seidel)

Lecture 2

Motion in 1D: Velocity and acceleration

- Average & instantaneous velocity
- Acceleration
- Equations of motion
- Free fall

1. Displacement, velocity and acceleration

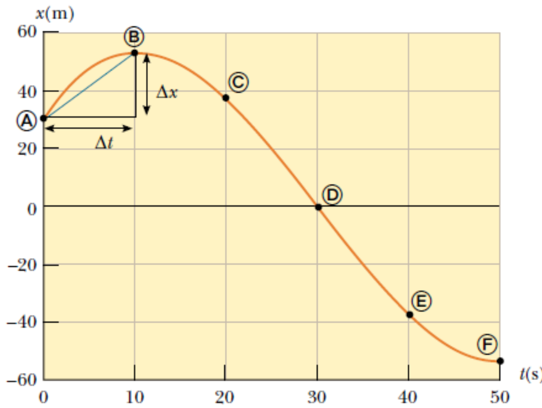
In our first venture into mechanics, we will describe mechanical objects just by their motion, which forms the field of **kinematics**: Motion is here described in terms of space and time while ignoring the agents that caused that motion. Our descriptors are **displacement, velocity and acceleration**. We will see in the following that we can convert them into each other.
We further introduce the **model of the point mass.** For simplicity, we neglect hereby the extension of an object but assume that all the **mass of the particle is concentrated in one-point mass having infinitesimal size**

<u>Examples of different types of motion</u>:

A car moving down a highway:	translational motion
Earth's orbit around the sun:	rotational motion
back-and-forth movement of a pendulum:	vibrational motion

A) Displacement and average velocity

Let us define the 3 mentioned quantities to describe motion of an object. For this we look at motion of a car that first goes in one direction, slows down and then turns accelerating in the other direction.



Position of the Car at Various Times

Position	t(s)	x(m)
Ⓐ	0	30
Ⓑ	10	52
Ⓒ	20	38
Ⓓ	30	0
Ⓔ	40	− 37
Ⓕ	50	− 53

For a moving object we now define the following quantities:

- **Displacement Δx:** being a change in its position from an initial position x_i to a final position x_f:

  $$\Delta x = x_f - x_i$$

  Δx is a vectorial quantity. In 1D it can just be positive or negative.

- **Distance travelled**: being the sum of the absolute values of the displacements, which is scalar quantity since it is always positive (Do not confuse with displacement)

  $$\Delta s = \sum_i |\Delta x|$$

- **Average velocity v_x :** being the **displacement Δx per time (interval) Δt** during which the given displacement occurred:

  $$\bar{v}_x = \frac{\Delta x}{\Delta t} \qquad \text{with unit} \qquad [\bar{v}_x] = \frac{m}{s}$$

It is again a vectorial quantity since it can be positive or negative (see plot).
For example, let us calculate the average velocity from A to F:

$$\overline{v}_x = \frac{-53\text{ m} - 30\text{ m}}{50\text{ s} - 0\text{ s}} = \frac{-83\text{ m}}{50\text{ s}} = \boxed{-1.7\text{ m/s}}$$

Experiment: We determine **the average velocity of the bullet of an air gun**.
We use two rotating paper disks at a distance of 1 m. First the angular velocity is determined by a stroboscobic illumination at 50 Hz (period of 0.02s). The first disks has large dots on its outer ring every 60°. If the disk is spinning with 60° per 0.02 s the motion appears to stall since each dot just occupies the position of the previous dot. If we turn faster, we get another stall at 120° per 0.02 s which is our desired angular velocity. Now we shoot through both disks and determine the angular difference Δφ ~30° between the bullet entries on the front and the rear. The flight time t_{flight} we get by expressing the known angular velocity with Δφ and t_{flight}:

$$\omega = \frac{120°}{0.02s} = \frac{\Delta\varphi}{t_{flight}}$$

$$t_{flight} = \frac{\Delta\varphi}{120°}0.02s \approx \frac{30°}{120°}0.02s = 0.005s$$

The bullet velocity we calculate from the known distance of 1 m between the two disks and t_{flight}:

$$v_{bullet} = \frac{\Delta x}{t_{flight}} = \frac{1m}{0.005s} = 200m/s$$

We further define the **average speed** from the distance travelled per time which is always positive:

$$\text{average speed} = \frac{\Delta s}{\Delta t}$$

An example is a marathon with the same start and end point, where the average velocity is zero but the average speed is 20 km/h for a total time of about 2h.
For the car example above we can calculate the average speed from the sections where the car is moving in the same direction:

$$\text{Average speed} = \frac{22\text{ m} + 52\text{ m} + 53\text{ m}}{50\text{ s}} = \boxed{2.5\text{ m/s}}$$

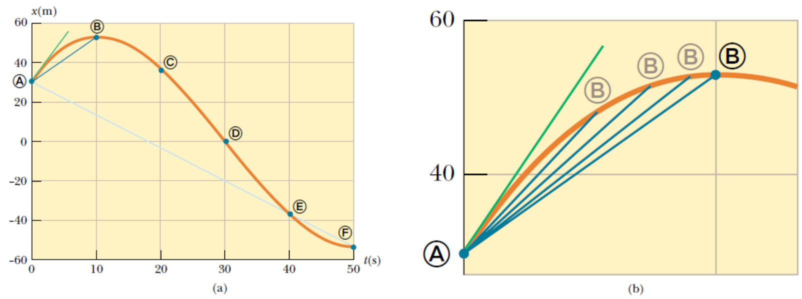
B) Instantaneous velocity

The police making a speed control is not really interested in average velocities but rather the velocity at a given moment, i.e. the velocity during a short time interval:

Experiment on the inclined plane: We obtain the average velocity for smaller and smaller distances. We see that for a small distances the velocity at the end of the inclined track is much higher than at the beginning or than the average velocity.
We now define the **instantaneous velocity** v_x equaling the limiting value of $\Delta x/\Delta t$ as Δt approaches zero:

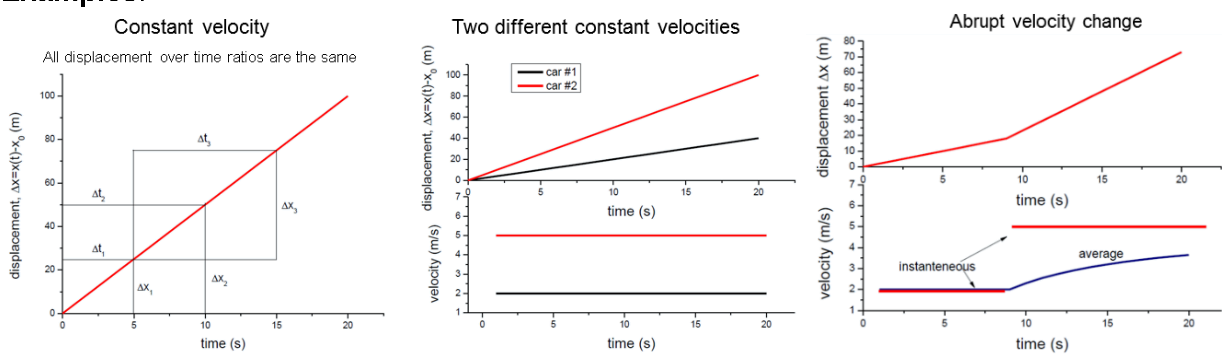
$$v_x \overset{\text{def}}{=} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

It is thus the **derivative of the position by time** and thus the **slope of the tangent in the position time plot** at time t:



The instantaneous velocity is of course also a **vectorial quantity** since it has a sign. A **constant velocity is a linear function in the displacement-time plot** since it has a constant derivative/slope.

Examples:



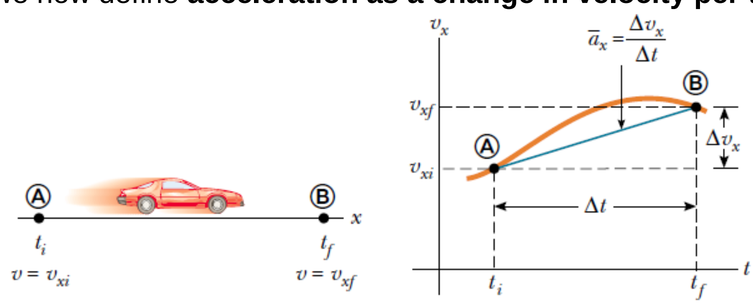
Experiment: Instantaneous velocity as derivative of the position with respect to time:
We measure the position of a toy car with position sensor moving over different surfaces. Its velocity is obtained from the derivative of the position

We additionally define the absolute value of the instantaneous velocity as:

Instantaneous speed $= |v_x|$

C) Acceleration

Similar to velocity, we now define **acceleration as a change in velocity per time**



- We define the **average acceleration** as a change of the velocity per time interval Δt:

$$\bar{a}_x \stackrel{\text{def}}{=} \frac{\Delta v_x}{\Delta t} = \frac{v_{xf} - v_{xi}}{t_f - t_i}$$

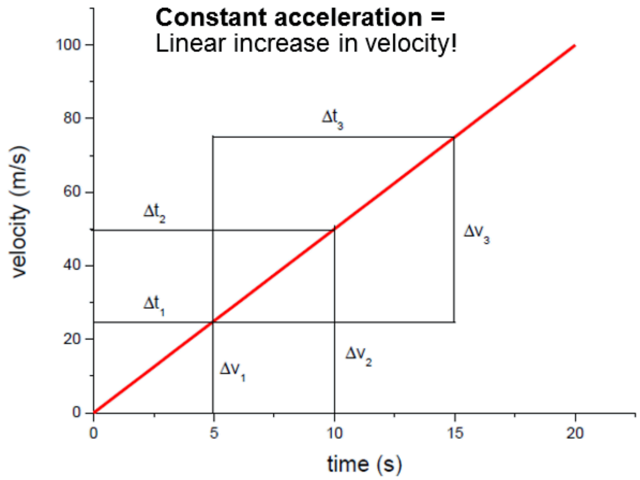
- The **instantaneous acceleration** is obtained for Δt becoming infinitesimally small:

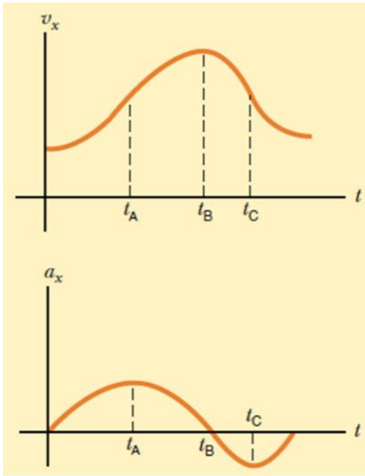
$$a_x \stackrel{\text{def}}{=} \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt}$$

The instantaneous acceleration is thus the **derivative of the velocity by the time, i.e. the local slope in the velocity-time plot.** We have for:

- zero acceleration - a constant velocity
- constant acceleration - a linear velocity change
- positive acceleration - a velocity increase,
- negative acceleration - a velocity decrease but not necessarily a direction change!

as can be seen in the following examples:

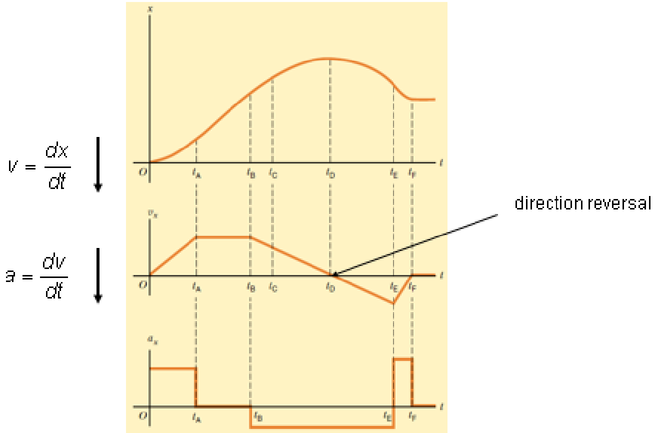




With the previous definition of the instantaneous velocity we can also say that the **acceleration is the second time derivative of the position x**:

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

As can be seen in the following example:



Experiment: Instantaneous acceleration as derivative of the velocity with respect to time

We obtain the **acceleration** in the toy car experiment from the **time derivative of the velocity** that we previously calculated..

2. Kinematic equations

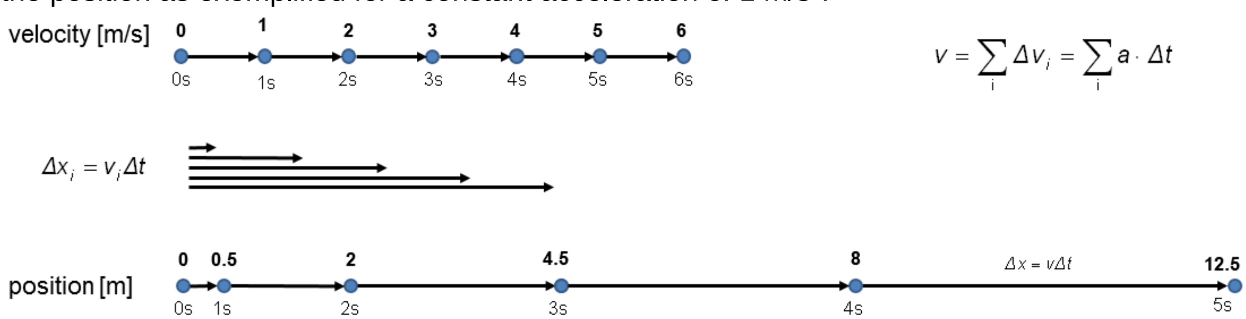
A) From acceleration to velocity to position

So far, we went from **displacement per time to velocity** and **velocity change per time to acceleration**. Often on knows however the acceleration or the velocity (e.g. from the mobile phone sensors) and wants to calculate the velocity change or the displacement after some time. For this we have to revert the differentiation by separation of the variables. We get new equations:

$$a_x = \frac{dv_x}{dt} \quad \rightarrow \quad a_x\, dt = dv_x$$

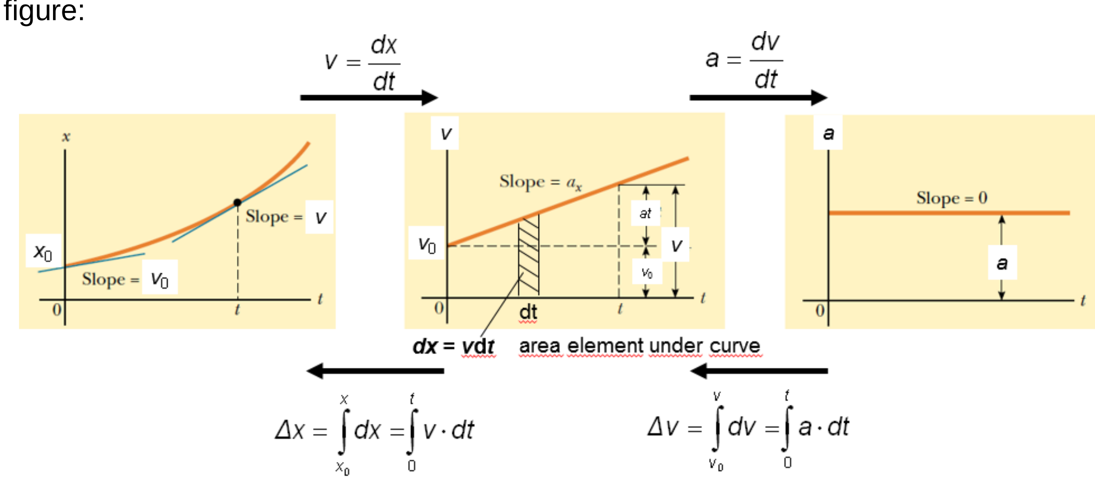
$$v_x = \frac{dx}{dt} \quad \rightarrow \quad v_x\, dt = dx$$

that define infinitesimal velocity or displacement changes within dt. Summing over the velocity changes gives the velocity as function of time and summing over the displacements gives the position as exemplified for a constant acceleration of 1 m/s^2:



In this case we have constant velocity increments per time interval Δt, such that the velocity increases linearly with time. The displacements per Δt increase thus also linearly with time (arrows in the center of the time intervals, since we took the average velocity of the time interval). Adding these displacements provides then the position over time, which is not liner in time anymore.

Summing up infinitesimal elements is nothing else than integration, which can be seen on the next figure:



$dx = v_x \cdot dt$ defines the infinitesimal area element of width dt below the velocity curve. The full displacement between t_0 and t from integration corresponds thus to the total area below the velocity curve. The same we can derive for the integration of the velocity. This provides in general the relations:

$$v(t) = v_0 + \int_{v_0}^{v} dv = v_0 + \int_0^t a(t)\, dt \qquad with \quad v(0) = v_0$$

$$x(t) = x_0 + \int_{x_0}^{x} dx = x_0 + \int_0^t v(t)\, dt \qquad with \quad x(0) = x_0$$

B) 1D motion with constant acceleration

Let us now study the case of **constant acceleration,** i.e. $\boldsymbol{a(t)} = \boldsymbol{const}$, which we have for example for a free fall of an object. We derive the relevant velocity and position equations. For the **velocity** we can write:

$$v(t) = \int_0^t a\, dt = v_0 + \int_0^t a\ \ t^0 dt = v_0 + \frac{1}{1} a\, t^1 \Big|_0^t = v_0 + a(t^1 - 0^1)$$

$$v(t) = v_0 + at$$

where we applied the integration rule for a power law. This equation can intuitively be understood by calculating the area increase under the acceleration curve of $\Delta v = at$.

For the **position** we then get:

$$x(t) = x_0 + \int_0^t v(t) \cdot dt = x_0 + \int_0^t (v_0 + at) \cdot dt = x_0 + \int_0^t v_0 \cdot t^0 \cdot dt + \int_0^t at^1 \cdot dt$$

$$= x_0 + v_0 t + \frac{1}{2} at^2 \Big|_0^t$$

$$x(t) = x_0 + \underbrace{v_0 t}_{\substack{\text{uniform motion} \\ \text{with } v_0}} + \underbrace{\tfrac{1}{2} at^2}_{\text{accelertion only}}$$

We have here a **superposition of a uniform motion with $\boldsymbol{v_0}$ and an accelerated motion starting with zero velocity both for the velocity and the position!** We can now describe the position of our object at any time! Also, if we can use these relations to calculate the time that an object needs to reach a position x by solving a quadratic equation.

Let us verify the equation by obtaining the velocity from differentiating the position formula:

$$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left(x_0 + v_0 t + \frac{1}{2} at^2\right) = v_0 + at$$

Let us further derive another kinematic equation by substituting the time *t* with the instantaneous velocity by transforming the latter equation:

$$t = (v - v_0)/a$$

Inserting into the position equation gives:

$$x - x_0 = v_0 \left(\frac{v - v_0}{a}\right) + \frac{1}{2} a \left(\frac{v - v_0}{a}\right)^2 = \left(\frac{vv_0 - {v_0}^2}{a} + \frac{1}{2} \frac{v^2 - 2vv_0 + {v_0}^2}{a} = \frac{v^2 - {v_0}^2}{2a}\right)$$

$$v^2 = {v_0}^2 + 2a(x - x_0)$$

Two additional kinematic equations are (see slides):

$$x = x_0 + (v + v_0)\frac{t}{2}$$

$$x = x_0 + vt - \frac{1}{2}at^2$$

We thus can calculate any desired relationship for the motion with constant acceleration.

3. The Free Fall

The free fall is the most important motion with constant acceleration. It is a classical example of physics. The Greeks thought that heavier objects fall faster, while Galileo stated the opposite.

Definition: A freely falling object is any object moving freely under the influence of gravity alone, regardless of its initial motion.

Do heavier objects actually fall faster? This can be seen in experiments where we exclude the drag forces from the motion within air:
Experiment: A feather and a cork fall at the same speed in a vacuum chamber
Movie: Apollo 15, hammer and feather

From these experiments we can conclude that all objects experience the same free fall acceleration *g* regardless their mass. *g* is approximately constant over small enough distances from the surface of the earth.

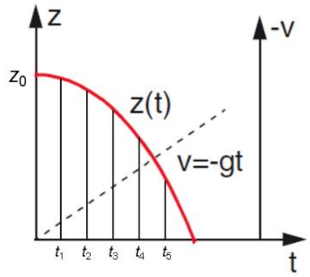
A) Kinematic equation for free fall:

To prove that *g* is approximately a constant we can use the kinematic equations from above by inserting $a = -g$. The negative sign indicates that the acceleration goes towards decreasing heights. *g* is hereby a positive value:

$$v = v_0 - gt$$

$$z = z_0 + v_0 t - \frac{1}{2}gt^2$$

For $v_0 = 0$, we have a parabolic curve with a maximum at $t = 0$ for the height. This can be qualitatively seen by making pictures of a falling sphere using stroboscopic illumination. Larger and larger distances are obtained since the velocity in the negative direction is linearly increasing:



Experiment with beads on a string: We further test a constant gravitational acceleration using a chain of beads on a string. When the extended string is dropped, we can hear the arrival times of the beads.

The beads start at $t = 0$ at heights z_n. The arrival time t_n of beads n at zero height can thus be calculated using the previous kinematic equation:

$$0 = z_n - \frac{1}{2}gt_n^2$$

Thus,

$$z_n = \frac{1}{2} g t_n^2$$

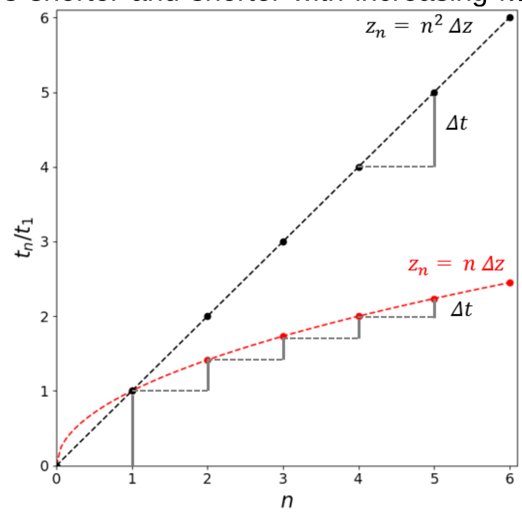
and

$$t_n = \sqrt{\frac{2z_n}{g}} = \sqrt{\frac{2\Delta z\, n}{g}}$$

In case of **equal distance between the beads,** $z_n = n\,\Delta z$, such that:

$$t_n = \sqrt{\frac{2\Delta z\, n}{g}} \propto \sqrt{n}$$

Due to the square-root dependence, the arrival times decrease less than linearly with bead number, i.e. the distance between two arrivals becomes shorter and shorter with increasing n. (see figure). This is due to the fact that the beads arrive with increasing velocity at the bottom according to $v_n = g t_n$. The decreasing times between the arrivals we can hear in the experiment.



Arrival times that increase linearly with the bead number n would lead to **equal time differences between two arrivals**. This can be achieved by **bead distances that increase quadratically** with bead number $z_n = n^2\,\Delta z$ such that

$$t_n = \sqrt{\frac{2\Delta z\, n^2}{g}} \propto n$$

which we can hear in the experiment.

Experiment: We now determine the **value of the gravitational acceleration** at the surface of the earth. By measuring the fall times of a sphere from different start heights z_0. Transforming the equation from above provides:

$$z_0 = \frac{1}{2} g {t_{fall}}^2 \rightarrow \frac{z_0}{{t_{fall}}^2} = \frac{1}{2} g$$

Thus, plotting $z_0$over $t2$ provides a linear function with slope $g/2$. This means that g corresponds to twice the slope in such a plot

Precise measurements of g provide:

Germany: 9.81 m/s^2

Pole: 9.83 m/s^2

Equator: 9.78 m/s^2

The reduced g value at the equator is due to the elliptic profile of the earth (0. 18% larger earth radius) and due to the centrifugal acceleration (0.35%)

B) Trajectory of an object that is thrown with v_0 upwards:

Let us now look at the trajectory of an object that is thrown with v_0 upwards from $z_0 = 0$

The equations of motion become:

$$z(t) = v_0 t - \frac{1}{2} g t^2$$

$$v = v_0 - gt$$

It is again a **superposition** of an upwards-directed translation with constant velocity v_0 and a motion with constant acceleration downwards. Initially the upward motion is faster but at some point, the downward motion wins.

The velocity decreases linearly with time. When the velocity is zero the object reaches the maximum height since afterwards its direction (sign) changes . Thus:

$$0 = v_0 - g t_{max}$$

The time after which the maximum is reached is then obtained by transformation as:

$$t_{max} = \frac{v_0}{g}$$

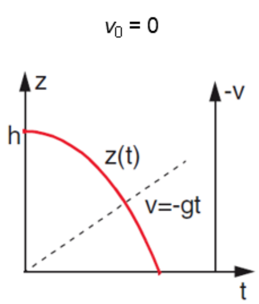
Inserting into the position equation provides the maximum height:

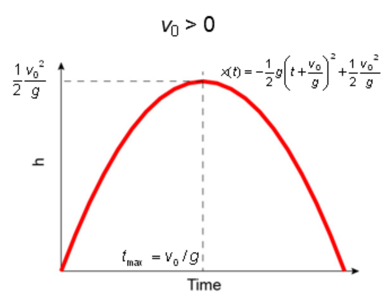
$$z_{max} = \frac{1}{2} \frac{{v_0}^2}{g}$$

Our initial equation of motion is a quadratic equation with a constant, a linear and a quadratic term. It can be transformed to have only a constant and a quadratic term. In this case, the constant term is the “turnig point” of the parabola at the maximum height z_{max}. Since z_{max} is reached at t_{max} at which a free fall with zero initial velocity starts, we transform the time time in our kinematic equation by subtracting t_{max}:

$$z(t) = -\frac{1}{2} g \left(t - \frac{v_0}{g}\right)^2 + \underbrace{\frac{1}{2} \frac{{v_0}^2}{g}}_{zmax}$$

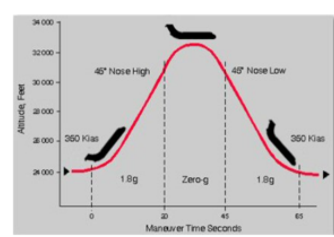
Inserting t_{max} we see that we get z_{max} as desired. Simplification of this equations provides the identical result as our initial equation of motion. The obtained equation describes a parabola that is centered around t_{max} with the given maximum height. The object reaches the ground at $2_{t_{max}}$ for symmetry reasons **(see slide)**





How does the velocity time plot look?

Flight trajectory of reduced gravity aircraft



Such a parabola trajectory with curvature -g/2 is taken by “reduced-gravity” aircrafts to simulate experimental conditions for weightlessness, where the aircraft engine compensates the drag of the air. One experiences weightlessness in free fall since the all gravitational force is taken to accelerate the object, such that “no force is left to generate a weight”. Weight thus depends on the frame of reference.

Experiment with Wii remote control on bungee cord: (see slide)

The Wii does not measure any acceleration in free fall due to the weightlessness. The sensor is not a real acceleration sensor but a “weight sensor”

Lecture 2: Experiments

1. Average velocity of the bullet of an air gun
2. Mean velocity at the inclined plane for different distances of the average, extrapolation to short distance -> transition from average velocity to instantaneous velocity
3. Displacement, velocity and acceleration of a toy car that reports its motion
4. Free fall in vacuum chamber (feather, cork) to exclude friction
5. Movie: hammer and feather on moon
6. Fallschnüre: dropping rope with equidistant spheres leads to faster hitting of the spheres onto the ground. Spheres at distances $z \propto n^2$ reach the ground at equal time intervals
7. Direct determination of the gravitational acceleration by measuring the time a sphere spends falling from different heights
8. Wii Remote control at Bungee-rope