Self Test

Your Mathematics background, as well as Chapter 1 and 2.1–2.3 of my lecture notes provide the background to solve the following exercises. This self test serves to check your background and understanding of intervals, geometry, trigonometric functions, complex numbers, differentiation, and integration. I will publish the solutions at the end of next week, and I will henceforth assume that you can deal with these topics *unless* you indicate where additional explanation and help is needed.
Bonus problems are illuminating fun problems for those participants who might be bored otherwise. Everybody else can safely ignore them. They are marked here by .

Problems

Problem 1. An astronomical unit

An astronomical unit (1 AU) amounts to the mean distance between Earth and Sun. Light traverses this distance in 8.3 min. Provide the distance $L = 1\,\mathrm{AU}$ is SI units.
Hint: The speed of light in vacuum is $c = 299\,792\,458\,\mathrm{m\,s^{-1}}$.

Problem 2. Printing the output of Phantom cameras

With a set of three phantom cameras one can simultaneously follow the motion of 100 particles in a violent 3d turbulent flow. Data analysis of the images provides particle positions with a resolution of 25 000 frames per second. You follow the evolution for 20 min, print it double paged with 8 coordinates per line and 70 lines per page. A bookbinder makes 12 cm thick books from every 1000 pages. You put these books into bookshelves with seven boards in each shelf. How many meters of bookshelves will you need to store your data on paper?

Problem 3. Intervals of real numbers

Sketch the following subsets of $\mathbb{R}$, and describe them as unions of disjoint intervals

a) $[-1, 4] \setminus [1, 2[$ b) $[2, 4[\cup ([3, 10] \setminus (]3, 4[\cup [6, 7]))$

Problem 4. Euler's equation and trigonometric relations

Euler's equation $e^{ix} = \cos x + i \sin x$ relates complex values exponential functions and trigonometric functions.

a) Sketch the position of e^{ix} in the complex plain, and indicate how Euler's equation is related to the Theorem of Pythagoras.

b) Complex valued exponential functions obey the same rules as their real-valued cousins. In particular one has $e^{i(x+y)} = e^{ix} e^{iy}$. Compare the real and complex parts of the expressions on both sides of this relation. What does this imply about the relation of $\sin(x+y)$ and $\cos(x+y)$ to $\sin x$, $\cos x$, $\sin y$, and $\cos y$, respectively.

c) Simplify the general expressions for the special cases $\sin(2x)$ and $\cos(2x)$.

d) Determine also the trigonometric relations for the triple angles, i.e. express $\sin(3x)$ and $\cos(3x)$ in terms of $\sin x$ and $\cos x$.
Hint: Take into account that $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$. Look up the rules for Pascal's triangle and its relation to the expansion of $(a+b)^n$ with $n \in \mathbb{N}$ if you do not know about them.

Problem 5. Real and imaginary parts of complex numbers

Determine the real parts, imaginary parts and complex conjugate of the following complex numbers

a) $z_1 = \dfrac{1}{i}$ b) $z_2 = \dfrac{1}{1 - i\sqrt{3}}$ c) $z_3 = \dfrac{1}{2} e^{i\pi/3}$

Problem 6. Properties of right-angled triangles

a) Fill in the gaps for the values of the angle θ in radians, and employ the symmetry of the trigonometric sine and cosine functions to determine the values in the right columns

θ		$\sin\theta$	$\cos\theta$
$\circ$	rad		
0	--------	0	--------
30	--------	$\frac{1}{2}$	--------
45	--------	$\frac{\sqrt{2}}{2}$	--------
60	--------	$\frac{\sqrt{3}}{2}$	--------
90	--------	1	--------
120	--------	--------	--------
135	--------	--------	--------
150	--------	--------	--------
180	--------	--------	--------

b) Consider a right triangle where one of the angles is θ. How are the length of its sides related to $\sin\theta$ and $\cos\theta$? Check that the Theorem of Pythagoras holds! Do you see a systematics for the values provided for $\sin\theta$?

c) Use the symmetries of the trigonometric functions to determine the values provided for $\theta = \pi/4$.

d) Use the symmetries of the trigonometric functions and the trigonometric relation $\sin(2\theta) = 2\,\sin\theta\,\cos\theta$ to determine the values provided for $\theta = \pi/6$ and $\theta = \pi/3$.

e) The values for $\pi/10$, $\pi/8$, and $\pi/5$ can also be stated explicitly in elementary form. Determine the expressions for these values!

Problem 7. Derivatives of Elementary Functions

Determine the derivatives of the following functions.

a) $\sin x$

b) $\cos x$

c) $\tan x$

d) x^n

e) $\sinh x$

f) $\cosh x$

g) $\tanh x$

h) e^x

i) $\ln x$

$\log_{10} x$

n^x

x^x

Problem 8. Derivatives of Common Composite Expressions

Evaluate the following derivatives.

a) $\frac{\mathrm{d}}{\mathrm{d}x}(a+x)^b$

b) $\frac{\partial}{\partial x}(x+b\,y)^2$

c) $\frac{\mathrm{d}}{\mathrm{d}x}(x+y(x))^2$

d) $\frac{\mathrm{d}}{\mathrm{d}t}\sin\theta(t)$

e) $\frac{\mathrm{d}}{\mathrm{d}t}\big(\sin\theta(t)\ \cos\theta(t)\big)$

f) $\frac{\mathrm{d}}{\mathrm{d}t}\sin\big(2\theta(t)\big)$

g) $\frac{\mathrm{d}}{\mathrm{d}z}\sqrt{a+b\,z^2}$

h) $\frac{\partial}{\partial x_3}\left[\sum_{j=1}^{6} x_j^2\right]^{-1/2}$

i) $\frac{\partial}{\partial y_1}\ln(\mathbf{x}\cdot\mathbf{y})$

In these expressions a and b are real constants, and $\mathbf{x}$ and $\mathbf{y}$ are 6-dimensional vectors.

Problem 9. Integrals of Elementary Functions

Evaluate the following integrals.

a) $\displaystyle\int_{-1}^{1} \mathrm{d}x\,(a+x)^2$

b) $\displaystyle\int_{-5}^{5} \mathrm{d}q\,(a+b\,q^3)$

$\displaystyle\int_{0}^{B} \mathrm{d}k\ \tanh^2(kx)$

c) $\displaystyle\int_{0}^{\infty} \mathrm{d}x\,\mathrm{e}^{-x/L}$

d) $\displaystyle\int_{-L}^{L} \mathrm{d}y\,\mathrm{e}^{-y/\xi}$

e) $\displaystyle\int_{0}^{L} \mathrm{d}z\,\frac{z}{a+b\,z^2}$

f) $\displaystyle\int_{0}^{\infty} \mathrm{d}x\,x\,\mathrm{e}^{-x^2/(2Dt)}$

g) $\displaystyle\int_{-\sqrt{Dt}}^{\sqrt{Dt}} \mathrm{d}\ell\,\ell\,\mathrm{e}^{-\ell^2/(2Dt)}$

$\displaystyle\int_{-\sqrt{Dt}}^{\sqrt{Dt}} \mathrm{d}z\,x\,\mathrm{e}^{-z\,x^2}$

Except for the integration variable all quantities are considered to be constant.

Hint: Sometimes symmetries can substantially reduce the work needed to evaluate an integral.

Problem 10. Solving Integrals by Partial Integration

Evaluate the following integrals by partial integration

$$\int \mathrm{d}x\, f(x)\, g'(x) = f(x)\, g(x) - \int \mathrm{d}x\, f'(x)\, g(x)$$

a) $\displaystyle\int_a^b \mathrm{d}x\, x\, \mathrm{e}^{kx}$ b) $\displaystyle\int_a^b \mathrm{d}x\, x^2\, \mathrm{e}^{kx}$ $\displaystyle\int_a^b \mathrm{d}x\, x^n\, \mathrm{e}^{kx}$, $n \in \mathbb{N}$

The integral c) can only be given as a sum over $j = 0, \ldots, n$.