14. Exam Preparation

Your solution to the problems 14.1–14.3 should uploaded
to your Moodle account
as a PDF-file
by Wednesday, Jan 31, 10:30 (with a grace time till the start of the seminars).

The parts marked by ⋆ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

The self-test problems serve to check you understanding of the analysis of the motion of different systems.

Problems

Problem 1. Vectors, derivatives and phase-space portraits

We consider a particle with unit mass that resides at a dimensionless position $\big(x(t), y(t)\big)$, and moves under the influence of a dimensionless force

$$\mathbf{F} = \mathrm{e}^{x^2}\,\mathrm{e}^{-y^2}\,\begin{pmatrix} -a\,x \\ b\,y \end{pmatrix}$$

We will be interested in the work, W, performed by this force, when the particle is moved from the origin to a position $\mathbf{q}$.

a) Determine the work performed for motion along the path
$\gamma_1 = \{\mathbf{q}(t) = (t\,x_E,\; t\,y_E) \mid 0 \le t \le 1\}$
to the fixed end point $(x_E,\, y_E)$.

b) Determine the work for a motion parallel to the coordinate axis,
$\gamma_2 = \{\mathbf{q}(t) = (t\,x_E,\, 0) \mid 0 \le t \le 1\} \cup \{\mathbf{q}(t) = (x_E,\, t\,y_E) \mid 0 \le t \le 1\}$.

⋆ c) Is $\mathbf{F}$ a conservative force?
How would your reply depend on the choice of the parameters a and b?

⋆ d) Show that for $a = b$ the force is described by the potential

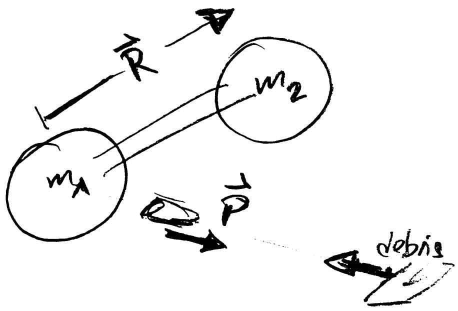
$$\Phi(x, y) = c \, \exp(x^2 - y^2)$$

How is c related to a and b?

e) Determine the expression $y(x)$ for the contour lines and the gradient of the work $W(x, y)$ provided in d).

Sketch the contour lines and indicate the gradient by arrows.

Problem 2. Space-station lost



We consider a cluster of two coupled space stations (masses m_1 and m_2) that are connected by a supply channel whose mass may be neglected as compared to m_1 and m_2. The supply channel takes the form of a stiff tube that cannot bend. However, it is attached to the stations in such a way that the distance R between the stations may change; with a restoring force $-k\,(R - \ell)$ towards the rest length ℓ.
We consider an emergency where one of the stations is approached by a heavy piece of space debris. To prevent damage the debris is intercepted by a space torpedo that leaves the endangered station with momentum $\mathbf{P}$. We choose a coordinate frame where the station is at rest initially. How will it move after launching the torpedo?
We approach the problem by first determining the equations of motion of the coupled space stations. Subsequently, we discuss the situation at hand.

a) Determine the center of mass $\mathbf{Q}$ of the coupled space stations, and its equation of motion.

Before launching the torpedo the center of mass is at rest at the origin of our coordinate system. How does it move after the launch?

b) The launch of the torpedo induces a torque that leads to a rotation of the coupled space stations. We will now discuss this rotation in their center of mass frame, i.e., we discuss the evolution of the vector $\mathbf{R}(t)$ that describes their orientation in space.

Show that the angular momentum $\mathbf{L} = \mathbf{R}(t) \times \mu\dot{\mathbf{R}}(t)$ is preserved.

⋆ c) Why does this imply that the coupled space station will rotate in the plane defined by the initial position of m_1 and m_2, and the approaching piece of debris?

d) Determine the Lagrange function for the motion of $\mathbf{R}$ in the plane selected by angular-momentum conservation, and determine the equation of motion for $\mathbf{R}(t)$.

e) Show that the conservation of energy E and the modulus of angular momentum imply that

$$E = a\dot{R}^2 + \frac{b}{R^2} + c\,R^2 - d\,R + e$$

and determine the positive real constants a, b, c, d, and e in terms of the masses m_1 and m_2, the rest length ℓ of the supply channel, strength k of the restoring force, and the angular momentum $|\mathbf{L}| = L$.

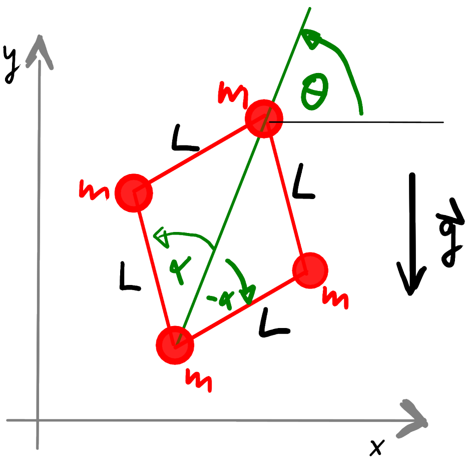
f) Determing the effective potential $\Phi_{\text{eff}}(R)$ for the evolution of the distance $R(t)$, and sketch the solutions of its equation of motion in phase space.

⋆ g) Mark the initial condition of $R(t)$ in the phase-space plot, i.e. its value and speed immediately after launch of the torpedo, and discuss the entailed evolution of $\mathbf{R}(t)$.

⋆ h) How does the minimum *energy* required to stop its rotation depend on the direction of approach of the debris?

Problem 3. Sliding diamonds

In geometry a quadrilateral where all sides have the same length, L, is referred to as a diamond. Its geometry and position in a plane can uniquely be characterized by selecting one corner, and specifying the angle 2α between its two neighboring sides, and the orientation θ of the diagonal proceeding through the corner (cf. the sketch to the right). We attach four identical masses m to the corners of such a structure, and explore how it moves in the sketched 2D plane. There is gravity acting vertically downwards, $\mathbf{g} = -g\,\hat{\boldsymbol{y}}$.



a) Let $\mathbf{q}_i$, $i \in \{1, 2, 3, 4\}$ be the position of corner i of the diamond, and $\mathbf{Q}$ its center of mass. Express the positions $\mathbf{q}_i$ as

$$\mathbf{q}_i(t) = \mathbf{Q}(t) + L_i(\alpha(t))\,\hat{\boldsymbol{r}}(\theta_i(t))$$

Determine $L_i(\alpha(t))$ and $\theta_i(t)$.

Hint: You may use that the diagonals always intersect vertically. Why does this imply that $\theta_i(t)$ does not depend on $\alpha(t)$, and that the positions of opposite corners differ only by the sign of the respective L_i.

Bonus: Show that the diagonals always intersect vertically.

b) Determine the potential energy V of the diamond in the gravitational field.

c) Determine the velocities $\mathbf{q}_i$ and the kinetic energy of the diamond.

d) The Langrangian can be expressed in the form

$$\mathcal{L} = a\,\dot{\boldsymbol{Q}}^2 + b\,\dot{\alpha}^2 + c\,\dot{\theta}^2 + d\,\hat{\boldsymbol{y}}\cdot\mathbf{Q}$$

Adopt dimensional analysis to determine how a, b, c, and d depend on the system parameters m, L, and g.

Compare your results to the values that result from the calculations performed in b) and c).

e) Determine the EOM for $\mathbf{Q}$, $\theta(t)$ and $\alpha(t)$.

f) Let the diamond start at the origin with $\theta(t_0) = 0$ and $\alpha(t_0) = \pi/4$, and velocities $\dot{\boldsymbol{Q}}(t_0) = \mathbf{V}$, $\dot{\theta}(t_0) = \Omega$ and $\dot{\alpha}(t_0) = \omega$. Provide the positions of the corners at time t.

⋆ g) Provide a physical interpretation of the motion.

What is fishy about the solution for $\alpha(t)$?

How does the motion change when the disks at the corners have a diameter $R < L/2$ and they undergo elastic hard-core collisions?

Sketch the evolution of α in phase space.

Challenge: How does the phase-space plot change when the particles undergo elastic soft-core collisions?

Self Test

Problem 4. Geometry of a Tetrahedron

The four corners of a tetrahedron, $\mathcal{E}$, form a subsect of the eight corners of a cube

$$\mathcal{E} = \left\{ P_0 = \frac{R}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \; P_1 = \frac{R}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \; P_2 = \frac{R}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \; P_3 = \frac{R}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

a) What is the length of the edges of the cube?
What is the distance of the corners of the tetrahedron from the center of the cube?
What is the length L of the edges of the tetrahedron?

b) Determine the cosine of the angle between two edges of the tetrahedron. (I am looking for an algebraic expression, not for the numerical value!)

c) Adopt spherical coordinates where the origin is placed at the center of the cube, the reference axis $\theta = 0$ proceeds through the point P_0, and point P_2 resides at a position $P_2 = D\,\hat{\boldsymbol{r}}(\theta, 0)$. Determine $\cos\theta$.

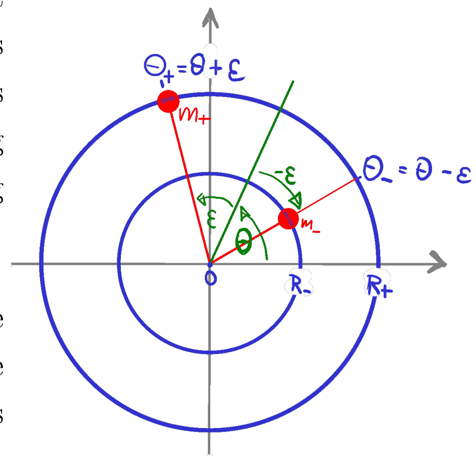
⋆ d) Provide symmetry arguments to show that

$$P_{2\pm1} = D\,\hat{\boldsymbol{r}}(\theta, \pm\phi)$$

Determine $\cos\phi$.

Problem 5. Two coupled masses on concentric circles

We consider two pearls of mass m_+ and m_- that move without friction on concentric circles of radius R_+ and R_-, respectively. We denote their positions as $\theta_\pm(t) = \theta(t) \pm \epsilon(t)$, and we consider a setting where they are connected by a harmonic spring with rest length ℓ and spring constant k.



a) Specify the positions and the velocities of the two pearls in terms of polar coordinates where R is the distance from the center of the circles (cf. the sketch to the right).

b) Determine the kinetic energy of the pearls in terms of the generalized coordinates $\theta(t)$ and $\epsilon(t)$.

⋆ c) Show that the potential energy of the system can be expressed as

$$V = \frac{k\,L^2}{2}\,\left[\sqrt{1 + \rho\,\sin^2\epsilon} - \rho\,\lambda\right]^2$$

where, $L = R_+ - R_-$.
How do the dimensionless parameters ρ and λ depend on $R_\pm$ and ℓ?
Which values do ρ and λ take in the limits $R_- \ll R_+$ and $L \ll R_+$?
Hint: Note that $\cos(2\epsilon) = 1 - 2\,\sin^2\epsilon$.

d) The coordinate θ is a cyclic variable. Determine the associated conservation law.

e) Verify that the EOM for $\epsilon(t)$ can be written in the form

$$\ddot{\epsilon} = \omega^2 \, \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left[\cos^2 \epsilon + 2\,\lambda \, \sqrt{1 + \rho \, \sin^2 \epsilon} \right]$$

and provide the dependence of ω on the system parameters.

$\star$ f) Determine the fixed points of the motion, and show that

for $\rho\,\lambda < 1$ there are stable fixed point at $\epsilon = \pi\,\mathbb{Z}$, and unstable fixed points right in the middle between the stable fixed points,

for $\rho\,\lambda > \sqrt{1+\rho}$ there are unstable fixed points at $\epsilon = \pi\,\mathbb{Z}$, and stable fixed points right in the middle between the stable fixed points,

otherwise there are unstable fixed points at $\epsilon = (\pi/2)\,\mathbb{Z}$, and stable fixed points somewhere in the middle between each neighboring pair of stable fixed points.

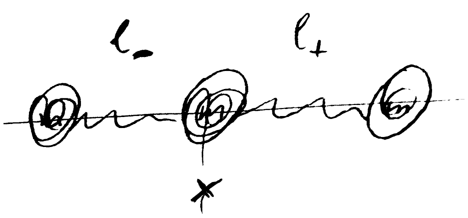
Provide a *physical* interpretation for the positions of the stable and unstable fixed points in the different regimes!

g) Sketch the potentials and phase-space portraits for the three different parameter regimes identified in f).

Problem 6. Springs in line

We consider three particles with identical masses m that reside at the positions $x_1(t)$, $x_2(t)$, and $x_3(t)$ on a one-dimensional track. The particles $(1,2)$ and $(2,3)$ are connected by identical Hookian springs with a spring constant k and rest length ℓ. No further forces are acting.

a) We adopt a parameterization in terms of x, l_- and l_+, where $x_1 = x - l_-$, $x_2 = x$, and $x_3 = x + l_+$ (see the sketch to the right).
Determine the kinetic energy and the potential energy.



b) Observe that x is a cyclic observable. Determine the related conservation law, and verify that it entails

$$\ddot{x} = c\,\left(\ddot{l}_+ - \ddot{l}_-\right)$$

Determine the value of c.

Bonus: What does the conservation law tell about the center-of-mass motion of the three particles?

c) Show that l_+ and l_- follow the EOMs

$$\begin{aligned} \ddot{l}_+ + c\,\left(\ddot{l}_+ - \ddot{l}_-\right) &= -\omega^2\,(l_+ - \ell) \\ \ddot{l}_- - c\,\left(\ddot{l}_+ - \ddot{l}_-\right) &= -\omega^2\,(l_- - \ell) \end{aligned}$$

How does ω depend on the system parameters k, m, ℓ.

Bonus: Determine the dependence of ω on k, m, ℓ also by dimensional analysis.

d) Introduce the dimensionless variables $\Delta = (l_+ - l_-)/\ell$ and $\Sigma = (l_+ + l_- - 2\,\ell)/\ell$, and the dimensionless time $\omega\,(t - t_0)$. Show that they follow the dimensionless EOM

$$\ddot{\Delta}(\tau) = -3\,\Delta \qquad \text{and} \qquad \ddot{\Sigma}(\tau) = -\Sigma$$

where the dots denote here derivatives with respect to τ.

e) Determine the solution of $\Delta(\tau)$ and $\Sigma(\tau)$ for the initial condition $\dot{l}_-(t_0) = \dot{l}_+(t_0) = 0$ and $l_-(t_0) = l_+(t_0) = \ell + L$.

How would x evolve in this case?

f) Determine the solution of $\Delta(\tau)$ and $\Sigma(\tau)$ for the initial condition $\dot{l}_-(t_0) = \dot{l}_+(t_0) = 0$, $l_-(t_0) = \ell$, and $l_+(t_0) = \ell + L$.

Bonus: How does $x(t)$ evolve in this case.

⋆ g) Verify that for the latter initial condition

$$l_\pm = \ell + \frac{L}{2}\,\Big(\cos\big((\Omega + \epsilon)\,\tau\big) \pm \cos\big((\Omega - \epsilon)\,\tau\big)\Big)$$

Determine the real numbers Ω and ϵ.

Expand the cosine function by trigonometric relations to show that

$$l_+(t) = \ell + L\ \cos(\Omega\,\tau)\ \cos(\epsilon\,\tau)$$
$$l_-(t) = \ell - L\ \sin(\Omega\,\tau)\ \sin(\epsilon\,\tau)$$

The energy of a Hookian spring amounts to k times the vibration amplitude squared. Assume that ϵ is much smaller than Ω, such that it can be interpreted as a modulation of the vibration amplitude. How is the vibrational energy distributed then among the two springs, and how does the distribution evolve in time?

Bonus Problem

Problem 7. Line integral for a velocity-dependent force

We consider a particle with unit mass that resides at a dimensionless position $\mathbf{q}(t)$ and moves under the influence of a dimensionless force

$$\mathbf{F} = \mathbf{A} \times \dot{\boldsymbol{q}} + \mathbf{B}$$

Note that this force depends on the velocity $\dot{\boldsymbol{q}}$ of the particle, but not on its position! We will be interested in the work, W, performed by this force, when the particle is moved from the origin to a position $\mathbf{x}$.

a) Determine the work performed for motion along the path $\gamma = \{\mathbf{q}(t) = t\,\mathbf{x} \mid 0 \le t \le 1\}$.

⋆ b) Show that

$$W(\mathbf{x}) = -\frac{1}{2}\,\mathbf{B} \cdot \mathbf{x}$$

irrespective of the path taken from the origin to $\mathbf{x}$.

c) Sketch the contour lines and the gradient of $W(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^2$.

⋆ d) Is this $\mathbf{F}$ a conservative force?
If yes: Provide an argument to support your conclusion.
If no: Can you think of a special case where it would be conservative?