List of theoretical questions for the final exam

Dr. Tim Shilkin

Leipzig University, WiSe 2023/24

Information on the final exam

- Form of the final attestation: written exam
- Date of the final exam: February 21, 2024 (Wednesday), from 12:00 till 14:00
- Date of the re-examination: March 27, 2024 (Wednesday), from 12:00 till 14:00
- For the final exam students will be asked to solve several computational problems similar to homework problems (estimated from 5 points each problem) and write exactly **one** theoretical proof of a result from the list below (the theoretical proof will be estimated from 8 points).

Contents

1 List of theoretical questions in the final exam

1.1 Limit of a function. Equivalence of definitions

THEOREM. Definitions of the limit of a function according to Cauchy and Heine are equivalent. We are given: $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$, $y_0\in\mathbb{R}$

PROOF of (Cauchy $\Longrightarrow$ Heine)

1. Use Cauchy's definition of the limit of a function:

   $$\forall\,\varepsilon>0\quad\exists\,\delta(\varepsilon)>0:\quad\forall\,x\in(a,b),\quad x\neq x_0\qquad|x-x_0|<\delta\quad\Longrightarrow\quad|f(x)-y_0|<\varepsilon\quad\text{(C)}$$

   Take $\{x_n\}_{n=1}^{\infty}\subset(a,b)$: $x_n\to x_0$, $x_n\neq x_0$. We want to show that $f(x_n)\to y_0$.

   Assume $\varepsilon>0$ is arbitrary. Applying (C) we obtain $\exists\,\delta=\delta(\varepsilon)>0$ such that (C) holds.

2. Use the definition of the limit of a sequence:

   $x_n\to x_0\quad\Longrightarrow\quad\exists\,N(\delta)\in\mathbb{N}$: $\forall\,n\geq N(\delta)\quad|x_n-x_0|<\delta$.

3. Use the assumption:

   (C) $\wedge\ |x_n-x_0|<\delta\quad\Longrightarrow\quad|f(x_n)-y_0|<\varepsilon$

4. Check the definition of $\lim\limits_{n\to\infty}f(x_n)=y_0$

   The following moments must be indicated (marked above with a blue color):

   $$\boxed{\forall\,\varepsilon>0}\qquad\boxed{\exists\,N(\delta(\varepsilon))\in\mathbb{N}:}\qquad\boxed{\forall\,n\geq N(\delta(\varepsilon))}\qquad\boxed{|f(x_n)-y_0|<\varepsilon}$$

PROOF of (Heine $\Longrightarrow$ Cauchy)

5. Use Heine's definition of the limit of a function:

   $$\forall\,\{x_n\}_{n=1}^{\infty}\subset(a,b):\quad x_n\neq x_0\qquad x_n\xrightarrow[n\to\infty]{}x_0\quad\Longrightarrow\quad f(x_n)\xrightarrow[n\to\infty]{}f(x_0)\qquad\text{(H)}$$

   We want to show $\forall\,\varepsilon>0\ \ \exists\,\delta(\varepsilon)>0$: $\quad|x-x_0|<\varepsilon\quad\Longrightarrow\quad|f(x)-y_0|<\varepsilon$

6. Proof by contradiction. Assume

   $\exists\,\varepsilon_0>0$: $\ \forall\,\delta>0\ \ \exists\,x_\delta\in(a,b)$: $\quad|x_\delta-x_0|<\delta\wedge|f(x_\delta)-y_0|\geq\varepsilon_0$

7. Choose $\delta>0$ in the specific way:

   $$\begin{array}{lcll}
   \text{Take }\delta=1 & \Longrightarrow & \exists\,x_1\in(a,b): & |x_1-x_0|<1\ \wedge\ |f(x_1)-y_0|\geq\varepsilon_0\\
   \text{Take }\delta=\frac{1}{2} & \Longrightarrow & \exists\,x_2\in(a,b): & |x_2-x_0|<\frac{1}{2}\ \wedge\ |f(x_2)-y_0|\geq\varepsilon_0\\
   \text{Take }\delta=\frac{1}{3} & \Longrightarrow & \exists\,x_3\in(a,b): & |x_3-x_0|<\frac{1}{3}\ \wedge\ |f(x_3)-y_0|\geq\varepsilon_0\\
   \dots & \dots & \dots\quad\dots & \dots\quad\dots\quad\dots\\
   \text{Take }\delta=\frac{1}{n} & \Longrightarrow & \exists\,x_n\in(a,b): & |x_n-x_0|<\frac{1}{n}\ \wedge\ |f(x_n)-y_0|\geq\varepsilon_0\\
   \dots & \dots & \dots\quad\dots & \dots\quad\dots\quad\dots
   \end{array}$$

   We obtain $\{x_n\}_{n=1}^{\infty}$: $\ \forall\,n\in\mathbb{N}\quad|x_n-x_0|<\frac{1}{n}\ \wedge\ |f(x_n)-y_0|\geq\varepsilon_0$.

8. Obtain the contradiction:

$$\left.\begin{array}{rcl} |x_n - x_0| < \frac{1}{n} & \Longrightarrow & x_n \longrightarrow x_0 \\ |f(x_n) - y_0| \geq \varepsilon_0 & \Longrightarrow & f(x_n) \not\longrightarrow y_0 \end{array}\right\} \Longrightarrow \quad \text{This contradicts to (H) !!!}$$

1.2 Continuity. Intermediate value theorems

THEOREM 1. f is continuous on $[a, b]$, $f(a) \leq 0$, $f(b) \geq 0 \quad \Longrightarrow \quad \exists\, c \in [a, b]\colon \; f(c) = 0$

PROOF. Denote $a_0 = a$, $b_0 = b$

1. Construct the sequence of nested intervals:

   Step 1.

   - Split the interval $[a_0, b_0]$ onto $\left[a_0, \frac{a_0+b_0}{2}\right]$ and $\left[\frac{a_0+b_0}{2}, b_0\right]$
   - If $f\left(\frac{a_0+b_0}{2}\right) \geq 0$ denote $[a_1, b_1] := \left[a_0, \frac{a_0+b_0}{2}\right]$
   - If $f\left(\frac{a_0+b_0}{2}\right) < 0$ denote $[a_1, b_1] := \left[\frac{a_0+b_0}{2}, b_0\right]$
   - In any case we obtain the interval $[a_1, b_1] \subset [a_0, b_0]$ such that $f(a_1) \leq 0$ and $f(b_1) \geq 0$

   Step 2.

   - Split the interval $[a_1, b_1]$ onto $\left[a_1, \frac{a_1+b_1}{2}\right]$ and $\left[\frac{a_1+b_1}{2}, b_1\right]$
   - If $f\left(\frac{a_1+b_1}{2}\right) \geq 0$ denote $[a_2, b_2] := \left[a_1, \frac{a_1+b_1}{2}\right]$
   - If $f\left(\frac{a_1+b_1}{2}\right) < 0$ denote $[a_2, b_2] := \left[\frac{a_1+b_1}{2}, b_1\right]$
   - In any case we obtain the interval $[a_2, b_2] \subset [a_1, b_1]$ such that $f(a_2) \leq 0$ and $f(b_2) \geq 0$

   ...

   Step n.

   - Split the interval $[a_{n-1}, b_{n-1}]$ onto $\left[a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}\right]$ and $\left[\frac{a_{n-1}+b_{n-1}}{2}, b_{n-1}\right]$
   - If $f\left(\frac{a_{n-1}+b_{n-1}}{2}\right) \geq 0$ denote $[a_n, b_n] := \left[a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}\right]$
   - If $f\left(\frac{a_{n-1}+b_{n-1}}{2}\right) < 0$ denote $[a_n, b_n] := \left[\frac{a_{n-1}+b_{n-1}}{2}, b_{n-1}\right]$
   - In any case we obtain $[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$ such that $f(a_n) \leq 0$ and $f(b_n) \geq 0$

   ...

2. Use the monotone sequence theorem:

   $[a_n, b_n] \subset [a_{n-1}, b_{n-1}] \quad \Longrightarrow \quad \forall\, n \in \mathbb{N} \quad a_{n-1} \leq a_n \leq b_n \leq b_{n-1}$

   $\{a_n\}_{n=1}^{\infty}$ — $\nearrow$ and bounded from above $(a_n \leq b_0) \quad \Longrightarrow \quad \exists\, c_1 \in [a, b]\colon \; c = \lim\limits_{n\to\infty} a_n$

   $\{b_n\}_{n=1}^{\infty}$ — $\searrow$ and bounded from below $(b_n \geq a_0) \quad \Longrightarrow \quad \exists\, c_2 \in [a, b]\colon \; c_2 = \lim\limits_{n\to\infty} b_n$

3. Show that $c_1 = c_2$:

   $$c_2 - c_1 \;=\; \lim_{n\to\infty}(b_n - a_n) \;=\; \lim_{n\to\infty} \frac{b_0 - a_0}{2^n} \;=\; 0 \quad \Longrightarrow \quad c := c_1 = c_2$$

4. Use continuity of f:

   f is continuous on $[a,b]$, $c \in [a,b] \quad \Longrightarrow \quad f$ is continuous at c

   f is continuous at c, $a_n \to c \quad \Longrightarrow \quad f(c) = \lim_{n\to\infty} f(a_n)$

   f is continuous at c, $b_n \to c \quad \Longrightarrow \quad f(c) = \lim_{n\to\infty} f(b_n)$

5. Show that $f(c) = 0$:

   If limits exist, one can pass to the limit in the inequality: $x_n \le y_n \implies \lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$

   $\forall\, n \in \mathbb{N} \quad f(a_n) \le 0 \quad \Longrightarrow \quad f(c) = \lim_{n\to\infty} f(a_n) \le 0$

   $\forall\, n \in \mathbb{N} \quad f(b_n) \ge 0 \quad \Longrightarrow \quad f(c) = \lim_{n\to\infty} f(b_n) \ge 0$

   $f(c) \le 0$ and $f(c) \ge 0 \quad \Longrightarrow \quad f(c) = 0$

□

THEOREM 2. f is continuous on $[a,b]$, $f(a) = y_1$, $f(b) = y_2$, $y_1 \le y_2 \quad \Longrightarrow \quad [y_1, y_2] \subset R(f)$

PROOF.

6. Use the definition of $R(f)$

   Take any $y_0 \in [y_1, y_2]$. We want to show: $\exists\, x_0 \in [a,b] \;\; f(x_0) = y_0$

7. Construct an auxiliary function $g : [a,b] \to \mathbb{R}$:

   Define $g(x) := f(x) - y_0$, $x \in [a,b] \quad \Longrightarrow \quad g$ is continuous on $[a,b]$

   As $y_0 \in [y_1, y_2]$ we obtain $y_1 \le y_0 \le y_2$.

   Hence $g(a) \le 0$ and $g(b) \ge 0$.

8. Apply Theorem 1:

   $g : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, $g(a) \le 0$, $g(b) \ge 0 \quad \Longrightarrow \quad \exists\, x_0 \in [a,b]: \quad g(x_0) = 0$

   $g(x_0) = 0 \quad \Longleftrightarrow \quad f(x_0) = y_0$

□

1.3 Continuity. Extreme value theorems

THEOREM 1. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b] \implies f$ is bounded on $[a,b]$, i.e.

$$\exists\, M > 0 : \qquad \forall\, x \in [a,b] \qquad |f(x)| \le M$$

PROOF.

1. Proof by contradiction:

   Assume $\forall\, M > 0 \quad \exists\, x_M \in [a,b]$: $\quad |f(x_M)| > M$

2. Construct a sequence $\{x_n\}_{n=1}^{\infty} \subset [a,b]$

   Take $M = 1 \quad \implies \quad \exists\, x_1 \in [a,b]$: $\quad |f(x_1)| > 1$

   Take $M = 2 \quad \implies \quad \exists\, x_2 \in [a,b]$: $\quad |f(x_2)| > 2$

   Take $M = 3 \quad \implies \quad \exists\, x_3 \in [a,b]$: $\quad |f(x_3)| > 3$

   ...

   Take $M = n \quad \implies \quad \exists\, x_n \in [a,b]$: $\quad |f(x_n)| > n$

   ...

   So, we obtain $\{x_n\}_{n=1}^{\infty} \subset [a,b]$: $\quad \forall\, n \in \mathbb{N} \quad |f(x_n)| > n$

3. Use Bolzano–Weierstrass theorem:

   $\{x_n\}_{n=1}^{\infty}$ is bounded $\quad \implies \quad \exists$ a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$, $\exists\, c \in \mathbb{R}$: $\quad x_{n_k} \to c$

   One can pass to the limit in the inequality: $a \le x_{n_k} \le b \quad \implies \quad a \le c \le b \quad \implies \quad c \in [a,b]$

4. Use continuity to obtain a contradiction:

   $x_{n_k} \to c$, f is continuous on $[a,b] \quad \implies \quad f(x_{n_k}) \to f(c)$

   Convergent sequence is bounded $\quad \implies \quad \exists\, L > 0$: $\quad \forall\, k \in \mathbb{N} \quad |f(x_{n_k})| \le L$

   $\forall\, k \in \mathbb{N} \quad |f(x_{n_k})| > n_k \to \infty$ — this contradicts to the boundedness of $\{f(x_{n_k})\}_{k=1}^{\infty}$!!!

□

THEOREM 2. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b] \implies f$ achieves on $[a,b]$ its maximum and minimal values, i.e. $\exists\, c_1, c_2 \in [a,b]$ such that

$$f(c_1) = \inf_{x \in [a,b]} f(x) \qquad \text{and} \qquad f(c_2) = \sup_{x \in [a,b]} f(x)$$

PROOF. Let us prove that f achieves its maximum. The proof for the minimum is analogous.

5. Function which is continuous on a closed bounded interval is bounded:

   f is continuous on $[a,b] \implies f$ is bounded on $[a,b] \implies \exists\, M \in \mathbb{R}: \quad M = \sup_{x\in[a,b]} f(x)$

6. Use the characterization of supremum using the quantifiers:

   $\forall\, \varepsilon > 0 \;\; \exists\, x_\varepsilon \in [a,b]: \quad M - \varepsilon < f(x_\varepsilon) \le M$

   Take $\varepsilon = 1 \quad \exists\, x_1 \in [a,b]: \quad M - 1 < f(x_1) \le M$

   Take $\varepsilon = \frac{1}{2} \quad \exists\, x_2 \in [a,b]: \quad M - \frac{1}{2} < f(x_2) \le M$

   Take $\varepsilon = \frac{1}{3} \quad \exists\, x_3 \in [a,b]: \quad M - \frac{1}{3} < f(x_3) \le M$

   ...

   Take $\varepsilon = \frac{1}{n} \quad \exists\, x_n \in [a,b]: \quad M - \frac{1}{n} < f(x_n) \le M$

   ...

   So, we obtain $\{x_n\}_{n=1}^{\infty} \subset [a,b]: \quad \forall\, n \in \mathbb{N} \quad M - \frac{1}{n} < f(x_n) \le M$

   Two policemen theorem $\implies f(x_n) \to M$

7. Use Bolzano–Weierstrass theorem:

   $\{x_n\}_{n=1}^{\infty}$ is bounded $\implies \exists$ a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}, \;\; \exists\, c \in [a,b]: \quad x_{n_k} \to c$

8. Use continuity of f and uniqueness of the limit:

   f is continuous at $c \in [a,b], \;\; x_{n_k} \to c \implies f(x_{n_k}) \to f(c)$

   $f(x_{n_k}) \to M, \;\; f(x_{n_k}) \to f(c) \implies f(c) = M$

$\square$

1.4 Fermat's and Rolle's theorems

1. Statement of Fermat's theorem:

   THEOREM 1. Assume $f:(a,b)\to\mathbb{R}$ has a local extremum (maximum or minimum) on the interval (a,b) at some internal point $c\in(a,b)$, i.e.

   $$\exists\, c\in(a,b):\quad \forall\, x\in(a,b)\quad f(x)\le f(c)\qquad \Big(\text{ or }\quad \forall\, x\in(a,b)\quad f(x)\le f(c)\Big)$$

   If f is differentiable at c then $f'(c)=0$.

   PROOF. Assume $\forall\, x\in(a,b)\ \ f(x)\le f(c)$. The case of minimum is similar.

2. Use the characterization of the limit in terms of one-sided limits:

   $$\exists\, f'(c) = \lim_{x\to c}\tfrac{f(x)-f(c)}{x-c} = \lim_{x\to c-0}\tfrac{f(x)-f(c)}{x-c} = \lim_{x\to c+0}\tfrac{f(x)-f(c)}{x-c}$$

3. Compute the limit from the left:
   $$\forall\, x\in(a,c),\quad f(x)\le f(c)\quad\Rightarrow\quad \tfrac{f(x)-f(c)}{x-c}\ge 0\quad\Rightarrow\quad \lim_{x\to c-0}\tfrac{f(x)-f(c)}{x-c}\ge 0\quad\Rightarrow\quad f'(c)\ge 0$$
   Compute the limit from the right:
   $$\forall\, x\in(c,b),\quad f(x)\le f(c)\quad\Rightarrow\quad \tfrac{f(x)-f(c)}{x-c}\le 0\quad\Rightarrow\quad \lim_{x\to c+0}\tfrac{f(x)-f(c)}{x-c}\le 0\quad\Rightarrow\quad f'(c)\le 0$$
   Compare limits from the left and from the right:
   $$f'(c)\ge 0\quad\text{and}\quad f'(c)\le 0\quad\Longrightarrow\quad f'(c)=0$$

4. Statement of Rolle's theorem:

   THEOREM 2. Assume $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Assume that $f(a)=f(b)$. Then there exists $c\in(a,b)$ such that $f'(c)=0$.

5. PROOF. Use the extreme value theorem:
   $$\exists\, c_1,\, c_2\in[a,b]:\qquad f(c_1)=\inf_{x\in[a,b]} f(x),\qquad f(c_2)=\sup_{x\in[a,b]} f(x)$$

6. Consider the case $f(x)=\mathit{const}$:
   $$f(c_1)=f(c_2)\quad\Longrightarrow\quad \forall\, x\in[a,b]\quad f(x)=f(a)=f(b)\quad\Longrightarrow\quad \forall\, x\in(a,b)\quad f'(x)=0$$

7. Consider the case $f(x)\ne\mathit{const}$:
   $$f(c_1)\ne f(c_2)\quad\Longrightarrow\quad\text{at least one of the points } c_1 \text{ and } c_2 \text{ is different from } a \text{ and } b$$
   $$\text{denote by } c \text{ those of } c_1 \text{ and } c_2 \text{ for which } c\ne a \text{ and } c\ne b\quad\Longrightarrow\quad c\in(a,b)$$

8. Use Fermat's theorem:
   $$\text{Assume}\quad c\in(a,b),\quad f(c)=\sup_{x\in[a,b]} f(x)\quad\Rightarrow\quad \forall\, x\in(a,b)\quad f(x)\le f(c)\quad\overset{\text{Fermat}}{\Longrightarrow}\quad f'(c)=0$$
   $$\text{The case}\quad c\in(a,b),\quad f(c)=\inf_{x\in[a,b]} f(x)\quad\text{is similar.}$$

1.5 Lagrange's and Cauchy's theorem

1. Statement of Lagrange's theorem

   THEOREM 3. Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

2. PROOF. Define the auxiliary function $h(x)$:

$$h(x) \;:=\; f(x) \;-\; f(a) \;-\; \frac{f(b) - f(a)}{b - a} \cdot (x - a), \qquad x \in [a, b]$$

3. Verify the assumptions of Rolle's theorem for $h(x)$:

$$h(a) = f(a) - f(a) = 0, \quad h(a) = f(b) - f(a) - \tfrac{f(b)-f(a)}{b-a} \cdot (b - a) = 0 \quad \Longrightarrow \quad h(a) = h(b) = 0$$

   $h : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b).

4. Use Rolle's theorem for $h(x)$:

$$\exists\, c \in (a, b) : \quad h'(c) = 0, \quad h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \quad \Longrightarrow \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$

5. Statement of Cauchy's theorem:

   THEOREM 4. Assume functions $f,\, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Assume $g'(x) \neq 0$ for any $x \in (a, b)$. Then there is a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

6. Show that $g(b) \neq g(a)$ and hence $\frac{f(b)-f(a)}{g(b)-g(a)}$ is well-defined.

   By contradiction, assume $g(b) = g(a) \overset{\text{Rolle}}{\Longrightarrow} \exists\, c \in (a, b)$: $g'(c) = 0$ — contradicts to $g'(x) \neq 0$

7. Define the auxiliary function $h(x)$:

$$h(x) \;:=\; f(x) \;-\; f(a) \;-\; \frac{f(b) - f(a)}{g(b) - g(a)} \cdot \big(g(x) - g(a)\big), \qquad x \in [a, b]$$

8. Use Rolle's theorem for $h(x)$:

$$h(a) = f(a) - f(a) = 0, \; h(a) = f(b) - f(a) - \tfrac{f(b)-f(a)}{g(b)-g(a)} \cdot \big(g(b) - g(a)\big) = 0 \;\Longrightarrow\; h(a) = h(b) = 0$$

   $h : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b)

$$\exists\, c \in (a, b) : \quad h'(c) = 0, \quad h'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(x) \quad \Longrightarrow \quad f'(c) = \frac{f(b) - f(a)}{b - a} \cdot \underbrace{g'(c)}_{\neq 0}$$

1.6 Investigation of functions using derivatives. Convexity.

THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is twice differentiable on (a, b). Then

$$f \text{ is convex on } (a, b) \qquad \Longleftrightarrow \qquad f'' \geq 0 \text{ on } (a, b)$$

PROOF $\boxed{\Longrightarrow}$

1. Assume f is convex on (a, b). Use the definition of convexity:

$$\forall\, x_1,\, x_2 \in (a, b), \quad x_1 < x_2, \quad \forall\, \lambda \in [0, 1] \qquad f\big((1-\lambda)x_1 + \lambda x_2\big) \leq (1-\lambda)f(x_1) + \lambda f(x_2)$$

2. Take arbitrary $x_1 < x < x_2$ and choose $\lambda \in (0, 1)$ in a specific way:

$$\lambda := \frac{x - x_1}{x_2 - x_1} \qquad \Longrightarrow \qquad 1 - \lambda = \frac{x_2 - x}{x_2 - x_1}, \qquad (1-\lambda)x_1 + \lambda x_2 = x$$

3. Proof the inequality for slopes of the secants:

$$x_1 < x < x_2 \qquad \Longrightarrow \qquad \frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}$$

Indeed,

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \leq (1-\lambda)f(x_1) + \lambda f(x_2) \quad \Longleftrightarrow$$
$$f(x) \leq \tfrac{x_2 - x}{x_2 - x_1}\, f(x_1) + \tfrac{x - x_1}{x_2 - x_1}\, f(x_2) \quad \Longleftrightarrow$$
$$\underbrace{\tfrac{x - x_1}{x_2 - x_1}\, f(x) + \tfrac{x_2 - x}{x_2 - x_1}\, f(x)}_{=\, f(x)} \leq \tfrac{x_2 - x}{x_2 - x_1}\, f(x_1) + \tfrac{x - x_1}{x_2 - x_1}\, f(x_2) \quad \Longleftrightarrow$$
$$\tfrac{x_2 - x}{x_2 - x_1}\Big(f(x) - f(x_1)\Big) \leq \tfrac{x - x_1}{x_2 - x_1}\Big(f(x_2) - f(x)\Big) \quad \Longleftrightarrow$$
$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}$$

(You can omit everything in blue color if you find it routine).

4. Proof f' — $\nearrow$ on (a, b). Take any $x_1,\, x_2 \in (a, b)$ such that $x_1 < x_2$.

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}, \quad x \to x_1 + 0 \quad \Longrightarrow \quad f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}, \quad x \to x_2 - 0 \quad \Longrightarrow \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2)$$

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2), \;\; \forall\, x_1 < x_2 \;\; \Longrightarrow \;\; f' \text{ — } \nearrow \text{ on } (a, b) \;\; \Longrightarrow \;\; f'' \geq 0 \text{ on } (a, b)$$

Proof $\boxed{\Longleftarrow}$

5. Assume $f'' \geq 0$ on (a,b) $\quad \Longrightarrow \quad$ f' — $\nearrow$ on (a,b).
6. Proof the inequality for slopes of the secants:

$$x_1 < x < x_2 \qquad \Longrightarrow \qquad \frac{f(x)-f(x_1)}{x-x_1} \leq \frac{f(x_2)-f(x)}{x_2-x}$$

Indeed, from the Lagrange theorem we obtain

$$\exists\, c_1 \in (x_1, x): \qquad \frac{f(x)-f(x_1)}{x-x_1} = f'(c_1)$$

$$\exists\, c_2 \in (x, x_2): \qquad \frac{f(x_2)-f(x)}{x_2-x} = f'(c_2)$$

$$c_1 \in (x_1, x), \quad c_2 \in (x, x_2) \quad \Longrightarrow \quad c_1 < c_2$$

$$f \text{ — } \nearrow \text{ on } (a,b) \quad \Longrightarrow \quad \frac{f(x)-f(x_1)}{x-x_1} = f'(c_1) \leq f'(c_2) = \frac{f(x_2)-f(x)}{x_2-x}$$

7. Take arbitrary $x_1 < x_2$ and $\lambda \in (0,1)$ and choose $x \in (x_1, x_2)$ in a specific way:

$$x = (1-\lambda)x_1 + \lambda x_2 \qquad \Longrightarrow \qquad \lambda = \frac{x-x_1}{x_2-x_1}, \qquad 1-\lambda = \frac{x_2-x}{x_2-x_1}$$

8. Verify the definition of convexity for f:

$$\frac{f(x)-f(x_1)}{x-x_1} \leq \frac{f(x_2)-f(x)}{x_2-x} \quad \Longleftrightarrow$$

$$\tfrac{x_2-x}{x_2-x_1}\Big(f(x)-f(x_1)\Big) \leq \tfrac{x-x_1}{x_2-x_1}\Big(f(x_2)-f(x)\Big) \quad \Longleftrightarrow$$

$$\underbrace{\tfrac{x-x_1}{x_2-x_1} f(x) + \tfrac{x_2-x}{x_2-x_1} f(x)}_{=\, f(x)} \leq \tfrac{x_2-x}{x_2-x_1} f(x_1) + \tfrac{x-x_1}{x_2-x_1} f(x_2) \quad \Longleftrightarrow$$

$$f(x) \leq \tfrac{x_2-x}{x_2-x_1} f(x_1) + \tfrac{x-x_1}{x_2-x_1} f(x_2) \quad \Longleftrightarrow$$

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \leq (1-\lambda) f(x_1) + \lambda f(x_2)$$

(You can omit everything in blue color if you find it routine). $\square$

1.7 Riemann integral. Basic condition for integrability

THEOREM. Assume f is bounded on $[a, b]$. Then f is integrable on $[a, b]$ if and only if $\forall\, \varepsilon > 0$ $\exists\, \delta(\varepsilon) > 0$ such that for any partition T of $[a, b]$ the following implication holds

$$\lambda(T) < \delta \quad \Longrightarrow \quad S(T) - s(T) < \varepsilon,$$

where $\lambda(T)$ is the mesh of the partition T.

PROOF $\boxed{\Longrightarrow}$

1. Use the definition of the integrable function:

$$\exists\, I \in \mathbb{R}: \quad \forall\, \varepsilon > 0 \;\; \exists\, \delta\left(\tfrac{\varepsilon}{3}\right) > 0 : \;\; \forall \text{ tagged partition } (T, \xi) : \;\; T = \{x_j\}, \;\; \xi = \{\xi_j\}, \;\; \xi_j \in [x_{j-1}, x_j]$$
$$\lambda(T) < \delta \quad \Longrightarrow \quad I - \tfrac{\varepsilon}{3} < \sigma(T, \xi) < I + \tfrac{\varepsilon}{3}$$

2. Use the property of the Darboux sums:

$$s(T) = \inf_{\xi} \sigma(T; \xi), \quad I - \tfrac{\varepsilon}{3} < \sigma(T, \xi) < I + \tfrac{\varepsilon}{3} \quad \Longrightarrow \quad I - \tfrac{\varepsilon}{3} \le s(T) < I + \tfrac{\varepsilon}{3}$$
$$S(T) = \sup_{\xi} \sigma(T; \xi), \quad I - \tfrac{\varepsilon}{3} < \sigma(T, \xi) < I + \tfrac{\varepsilon}{3} \quad \Longrightarrow \quad I - \tfrac{\varepsilon}{3} < S(T) \le I + \tfrac{\varepsilon}{3}$$

3. Use the fact that both numbers $s(T)$ and $S(T)$ lay in the interval $[I - \frac{\varepsilon}{3}, I + \frac{\varepsilon}{3}]$:

$$\lambda(T) < \delta\left(\tfrac{\varepsilon}{3}\right) \quad \Longrightarrow \quad I - \tfrac{\varepsilon}{3} \le s(T) \le S(T) \le I + \tfrac{\varepsilon}{3} \quad \Longrightarrow \quad S(T) - s(T) \le \tfrac{2\varepsilon}{3} < \varepsilon.$$

PROOF $\boxed{\Longleftarrow}$

4. Use the property if the Darboux sums:

$$\forall \text{ partitions } T \text{ and } T_0 \text{ of } [a, b] \qquad s(T) \le S(T_0) \quad \Longrightarrow \quad \begin{array}{c}\text{the set } \{s(T)\}_T \\ \text{is bounded from above}\end{array}$$

5. Define the number $I \in \mathbb{R}$. Denote

$$\text{Least upper bound axiom} \quad \Longrightarrow \quad \exists\, I := \sup \left\{ s(T) \;\middle|\; T \text{ is a partition of } [a, b] \right\}$$

   where the supremum is taken over all possible partitions T of the interval $[a, b]$

6. Use the property of the sup (the least upper bound less or equal than some upper bound):

$$\forall \text{ partitions } T, T_0 \qquad s(T) \le S(T_0) \quad \Longrightarrow \quad \underbrace{\sup_T s(T)}_{\text{the least u.b.}} \le \underbrace{S(T_0)}_{\text{some u.b.}} \quad \Longrightarrow \quad I \le S(T_0)$$

7. Use the fact that the partition T_0 is also arbitrary:

$$\forall \text{ partition } T_0 \qquad I \le S(T_0) \quad \Longrightarrow \quad \forall \text{ partition } T \quad s(T) \le I \le S(T)$$

8. Use assumption and the definition of the Riemann integral:

$$\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 : \quad \forall \text{ partition } T = \{x_j\}_{j=1}^N \quad \lambda(T) < \delta \quad \Longrightarrow \quad S(T) - s(T) < \varepsilon$$

Let $\xi = \{\xi_j\}_{j=1}^N$, $\xi_j \in [x_{j-1}, x_j]$. Then $\forall$ tagged partition (T, ξ) if $\lambda(T) < \delta(\varepsilon)$ then

$$\Longrightarrow \qquad \left.\begin{array}{c} s(T) \le \sigma(T,\xi) \le S(T) \\ s(T) \le I \le S(T) \\ S(T) - s(T) < \varepsilon \end{array}\right\} \quad \Longrightarrow \quad |I - \sigma(T,\xi)| \le \varepsilon$$

Use the definition of the Riemann integral: the lined colored in blue imply

$$\exists\, I \; = \int_a^b f(x)\, dx$$

1.8 Mean value formula and fundamental theorem of calculus

THEOREM. $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b] \implies \exists\, c \in [a, b]$:

$$f(c) = \frac{1}{b-a} \int\limits_a^b f(x)\, dx$$

PROOF.

1. Use the extreme value theorem:

$$f \text{ is continuous on } [a, b] \quad \implies \quad f \text{ is bounded on } [a, b]$$

   Denote

$$m := \inf_{x\in[a,b]} f(x), \qquad M := \sup_{x\in[a,b]} f(x) \qquad \implies \qquad R(f) = [m, M]$$

2. Use the property of the definite integral:

$$\forall\, x \in [a, b] \qquad m \le f(x) \le M \qquad \implies \qquad m\,(b-a) \le \int\limits_a^b f(x)\, dx \le M\,(b-a)$$

3. Use the intermediate value theorem:

$$y_0 := \frac{1}{b-a} \int\limits_a^b f(x)\, dx \qquad \implies \qquad m \le y_0 \le M \qquad \implies \qquad y_0 \in R(f)$$

$$f \text{ is continuous on } [a, b] \quad \implies \quad \exists\, c \in [a, b] : \quad f(c) = y_0 \quad \implies \quad f(c) = \frac{1}{b-a} \int\limits_a^b f(x)\, dx$$

THEOREM. Assume $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$. Define $\Phi : [a, b] \to \mathbb{R}$,

$$\Phi(x) := \int\limits_a^x f(t)\ dt, \quad x \in [a, b] \qquad \text{is the } \textit{integral with variable upper limit.}$$

If f is continuous on $[a, b]$ then Φ is differentiable on $[a, b]$ and

$$\forall\, x_0 \in (a, b) \qquad \Phi'(x_0) = f(x_0).$$

Proof.

4. Use the property of the definite integral:

$$\forall\, x \in [a,b], \quad x > x_0 \qquad \Longrightarrow \qquad \frac{\Phi(x)-\Phi(x_0)}{x-x_0} \;=\; \frac{1}{x-x_0}\int\limits_{x_0}^{x} f(t)\,dt$$

5. Use the mean value formula:

$$f \text{ is continuous on } [x_0,x] \quad \Longrightarrow \quad \exists\, c_x \in [x_0,x] : \qquad \frac{1}{x-x_0}\int\limits_{x_0}^{x} f(t)\,dt \;=\; f(c_x)$$

6. Use the two policemen theorem and continuity of f:

$$x_0 \le c_x \le x \qquad \Longrightarrow \qquad c_x \underset{x\to x_0}{\longrightarrow} x_0 \qquad \overset{f \text{ is continuous}}{\Longrightarrow} \qquad f(c_x) \underset{x\to x_0}{\longrightarrow} f(x_0)$$

7. Use the definition of the derivative

$$\exists\, \Phi'(x_0) \;=\; \lim_{x\to x_0}\frac{\Phi(x)-\Phi(x_0)}{x-x_0} \;=\; \lim_{x\to x_0} f(c_x) \;=\; f(x_0)$$

8. Consider the case $x < x_0$:

$$x < x_0 \quad \Longrightarrow \quad \frac{\Phi(x)-\Phi(x_0)}{x-x_0} \;=\; \frac{1}{x-x_0}\int\limits_{x_0}^{x} f(t)\,dt \;=\; \frac{1}{x_0-x}\int\limits_{x}^{x_0} f(t)\,dt \;\overset{c_x\in[x,x_0]}{=}\; f(c_x) \;\to\; f(x_0)$$

1.9 Dirichlet–Abel test for convergence of improper integrals

PROBLEM. Proof the Dirichlet–Abel test for convergence of improper integrals.

1. Recall the statement of the Dirichlet–Abel test:

THEOREM. Assume $f,\, g : [a,+\infty) \to \mathbb{R}$ satisfy the conditions

1) f is continuous on $[a,+\infty)$ and its primitive F is bounded on $[a,+\infty)$
2) g is non-negative, non-increasing, differentiable on $[a,+\infty)$ and g' is continuous on $[a,+\infty)$
3) $\lim\limits_{x\to+\infty} g(x) = 0$

Then the improper integral of fg over $[a,+\infty)$ is convergent:

$$\int_a^{+\infty} f(x)g(x)\,dx \quad \text{is convergent.}$$

PROOF.

2. Recall the definition of a primitive:

$$F : [a,+\infty) \to \mathbb{R} \text{ is differentiable on } [a,+\infty), \quad F'(x) = f(x), \quad |F(x)| \le M, \ \forall\, x \in [a,+\infty)$$

3. Recall the property of a non-increasing differentiable function:

$$\forall\, x \in [a,+\infty) \qquad g(x) \ge 0, \qquad g'(x) \le 0 \qquad \Longrightarrow \qquad |g'(x)| = -g'(x)$$

4. Use integration by parts formula:

$$\int_a^c f(x)g(x)\,dx = \int_a^c g(x)\,dF(x) = [\text{ by parts }] = g(x)F(x)\Big|_{x=a}^{x=c} - \int_a^c f(x)dg(x) =$$

$$= g(c)F(c) - g(a)F(a) - \int_a^c F(x)g'(x)\,dx$$

5. Use the Newton–Leibniz formula:

$$\int_a^c |g'(x)|\,dx = -\int_a^c g'(x)\,dx = g(a) - g(c) \le g(a) \quad \Longrightarrow \quad \int_a^{+\infty} |g'(x)|\,dx \quad \text{is convergent}$$

6. Use the comparison test of absolute convergence of an improper integral:

$$|F(x)g'(x)| \le M\,|g'(x)|, \quad \forall\, x \in [a,+\infty) \quad \Longrightarrow \quad \int_a^{+\infty} F(x)g'(x)\,dx \quad \text{is absolutely convergent}$$

7. Use the fact that $g(x)$ vanishes as $x \to +\infty$

$$\lim_{c\to+\infty} g(c) = 0, \quad |F(c)| \le M, \quad \forall\, c \in [a, +\infty) \quad \Longrightarrow \quad \lim_{c\to+\infty} \Big(g(c)F(c) \Big) = 0$$

8. Use the definition of an improper integral:

$$\exists \lim_{c\to+\infty} \int\limits_a^c f(x)g(x)\,dx = \underbrace{\lim_{c\to+\infty} \Big(g(c)F(c) \Big)}_{=0} - g(a)F(a) - \underbrace{\lim_{c\to+\infty} \int\limits_a^c F(x)g'(x)\,dx}_{\text{is convergent}} =$$

$$= -g(a)F(a) - \int\limits_0^{+\infty} F(x)g'(x)\,dx$$

1.10 Cauchy's and D'Alambert's tests for convergence of infinite series

Theorem. Assume $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and assume there exists a finite limit

$$\alpha := \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then

1) $\alpha < 1 \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n$ is absolutely convergent;

2) $\alpha > 1 \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n$ is divergent;

Proof.

1. Assume $0 \le \alpha < 1$. Use the definition of a limit:

$$\alpha := \lim_{n\to\infty} \sqrt[n]{|a_n|} \quad \Longleftrightarrow \quad \forall\, \varepsilon > 0 : \quad \exists\, N(\varepsilon) : \quad \forall\, n \ge N(\varepsilon) \qquad \left| \sqrt[n]{|a_n|} - \alpha \right| < \varepsilon$$

2. Choose $\varepsilon > 0$ in a clever way:

Take any $q \in (\alpha, 1)$ and take $\varepsilon = q - \alpha \quad \Longrightarrow \quad \forall\, n \ge N(\varepsilon) \qquad \sqrt[n]{|a_n|} \; < \; \alpha + \varepsilon = q \; < \; 1$

3. Use majorant test (or comparison test) for absolute convergence:

$$\sqrt[n]{|a_n|} < q < 1 \quad \Longrightarrow \quad |a_n| < q^n, \qquad \forall\, n \ge N$$

$$q < 1 \quad \Longrightarrow \quad \sum_{n=N}^{\infty} q^n \quad \text{is convergent} \quad \Longrightarrow \quad \sum_{n=N}^{\infty} |a_n| \quad \text{is convergent}$$

$$\Longrightarrow \quad \sum_{n=1}^{\infty} |a_n| \; = \; \sum_{n=1}^{N-1} |a_n| \; + \; \sum_{n=N}^{\infty} |a_n| \quad \text{is convergent}$$

4. Assume $\alpha > 1$. Choose $\varepsilon > 0$ in a clever way:

Take any $q \in (1, \alpha)$ and take $\varepsilon = \alpha - q \quad \Longrightarrow \quad \forall\, n \ge N(\varepsilon) \qquad \sqrt[n]{|a_n|} \; > \; \alpha - \varepsilon = q \; > \; 1$

$$|a_n| > 1 \quad \Longrightarrow \quad \text{the necessary condition of convergence} \;\; \lim_{n\to\infty} a_n = 0 \;\; \text{is violated.}$$

THEOREM. Assume $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $a_n \neq 0$, and assume there exists a finite limit

$$\beta := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

Then

1) $\beta < 1 \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n$ is absolutely convergent;

2) $\beta > 1 \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n$ is divergent;

PROOF.

5. Assume $0 \le \beta < 1$. Use the definition of a limit and choose $\varepsilon > 0$ in a clever way:

$$\beta := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \Longleftrightarrow \quad \forall\, \varepsilon > 0 : \quad \exists\, N(\varepsilon) : \quad \forall\, n \ge N(\varepsilon) \qquad \left| \frac{|a_{n+1}|}{|a_n|} - \beta \right| < \varepsilon$$

Take any $q \in (\beta, 1)$ and take $\varepsilon = q - \beta \quad \Longrightarrow \quad \forall\, n \ge N(\varepsilon) \qquad \frac{|a_{n+1}|}{|a_n|} < \beta + \varepsilon = q < 1$

6. For any $n > N$ we have

$$\frac{|\,a_n\,|}{|a_N|} = \underbrace{\underbrace{\frac{|\,a_n\,|}{|a_{n-1}|}}_{<q} \cdot \underbrace{\frac{|a_{n-1}|}{|a_{n-2}|}}_{<q} \cdot \underbrace{\frac{|a_{n-2}|}{|a_{n-3}|}}_{<q} \cdot \ldots \cdot \underbrace{\frac{|a_{N+1}|}{|a_N|}}_{<q}}_{n-N \text{ factors}} < q^{n-N} \quad \Longrightarrow \quad |a_n| < |a_N|\, q^{n-N}$$

7. Use majorant test (or comparison test) for absolute convergence:

$$q < 1 \quad \Longrightarrow \quad |a_N| \sum_{n=N+1}^{\infty} q^{n-N} \ \text{ is convergent } \quad \Longrightarrow \quad \sum_{n=N+1}^{\infty} |a_n| \ \text{ is convergent}$$

$$\Longrightarrow \quad \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n| \ \text{ is convergent}$$

8. Assume $\beta > 1$. Choose $\varepsilon > 0$ in a clever way:

Take any $q \in (1, \beta)$ and take $\varepsilon = \beta - q \quad \Longrightarrow \quad \forall\, n \ge N(\varepsilon) \qquad \frac{|a_{n+1}|}{|a_n|} > \beta - \varepsilon = q > 1$

$$\forall\, n \ge N \qquad \frac{|a_{n+1}|}{|a_n|} > 1 \quad \Longrightarrow \quad \{a_n\}_{n=N}^{\infty} - \nearrow \quad \Longrightarrow \quad \forall\, n \ge N \quad a_n \ge a_N$$

$|a_n| > |a_N| \quad \Longrightarrow \quad$ the necessary condition of convergence $\lim_{n\to\infty} a_n = 0$ is violated.