

Math for Students of Faculty of Physics  
Mathematics 1  
Linear Algebra and Calculus of Functions  
of One Variable

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## References

The programm of the course “Mathematics 1. Linear Algebra and Calculus of Functions of One Variable” is standard and the related material can be found practically in any book devoted to the calculus of functions of one variable and the linear algebra.

Books recommended in the official syllabus:

<https://www.physgeo.uni-leipzig.de/en/studying/courses-of-study/>

- Serge Lang: Linear Algebra, Springer
- Serge Lang: A First Course in Calculus, Springer
- Kenneth A. Ross: Elementary Analysis, Springer
- Stephen Abbott: Understanding Calculus, Springer

Books I frequently use myself to prepare my lectures (to be completed):

### **Logic, sets, functions**

- Keith Devlin, Sets, Functions, and Logic. An Introduction to Abstract Mathematics, CRC.
- Steven Galovich, Introduction to Mathematical Structures, HBJ, Academic Press.

### **Calculus of functions of one real variable**

- Vladimir A. Zorich, Mathematical Analysis I, Springer.

### **Linear Algebra**

- Steven J. Leon, Linear Algebra with Applications, Prentice Hall.

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## 1. Notation

$\forall$  means “for all”

$\exists$  means “there exists”

$\exists!$  means “there exists a unique”

$:$  means “such that”

$:=$  means “denote by”

$\mathbb{N}$  is a set of all natural numbers,  $\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{Z}$  is a set of all integer numbers,  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$\mathbb{Q}$  is a set of all rational numbers,  $\mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N} \right\}$

$\mathbb{R}$  is a set of all real numbers

an interval  $(a, b) \subset \mathbb{R}$  is the set of all real  $x$  such that  $a < x < b$

an interval  $[a, b] \subset \mathbb{R}$  is the set of all real  $x$  such that  $a \leq x \leq b$

$\operatorname{tg} x := \tan x$  is the tangent of  $x$ ,  $\operatorname{ctg} x := \cot x$  is the cotangent of  $x$

$\operatorname{arctg} y := \arctan y$  is the arctangent of  $y$ ,  $\operatorname{arcctg} y := \operatorname{arccot} y$  is the arccotangent of  $y$

# 1 One-dimensional calculus

## 1.1 Basic concepts

### 1. Logic

- Logical operations: and, or, negation, implication, equivalence.
- Negation of “and” and “or”. Negation of an implication.
- Equivalence of the implication to the contrapositive. Necessary and sufficient condition.

### 2. Proof by contradiction

EXAMPLE.  $\sqrt{2}$  is not rational.

PROOF. By contradiction, assume  $\sqrt{2}$  is rational. Then there exists integers  $n, m \in \mathbb{Z}$ :

$$\sqrt{2} = \frac{n}{m}, \quad \text{greatest common divisor of } n \text{ and } m \text{ is } 1.$$

Taking a square of this relation we obtain

$$2 = \frac{n^2}{m^2} \quad \Longleftrightarrow \quad n^2 = 2m^2$$

$2m^2$  is even and hence  $n^2$  even. But then  $n$  itself even, i.e.

$$n = 2k \quad \text{for some } k \in \mathbb{Z}.$$

Then we obtain

$$n^2 = 2m^2 \quad \Longleftrightarrow \quad (2k)^2 = 2m^2 \quad \Longleftrightarrow \quad 4k^2 = 2m^2 \quad \Longleftrightarrow \quad 2k^2 = m^2$$

Then  $m^2$  is even and hence  $m$  itself is even. As both  $n$  and  $m$  are even, this contradicts to the assumption that the greatest common divisor of  $n$  and  $m$  is 1. Hence our assumption that  $\sqrt{2}$  is rational can not be true. Hence  $\sqrt{2}$  is irrational.  $\square$

### 3. Quantifiers

- sentences and predicates
- $\forall, \exists, \exists!$  (there exists a unique)
- Negation of the statement with quantifiers

EXAMPLE. Construct the explicit negation of the statement

$$\left( \forall n \in \mathbb{N} \quad \exists x \in (1, +\infty) \quad \forall y \in [0, 1] \quad x + y > n^2 \right)$$

Determine which of the statements (the original one or its negation) is true.

#### 4. Proof by induction

$$\left( P(1) \text{ is true } \right) \wedge \left( \forall n \in \mathbb{N} \quad P(n) \implies P(n+1) \right) \implies \forall n \in \mathbb{N} \quad P(n) \text{ is true}$$

EXAMPLE.

- 1) For any  $n \in \mathbb{N}$  the number  $5^{n+2} + 6^{2n+1}$  is divisible by 31
- 2) Prove by induction that for any  $n \in \mathbb{N}$   $1 + 2 + \dots + n = \frac{n(n+1)}{2}$

PROOF.

1) Base of the induction  $P(1)$ . Let us verify that for  $n = 1$  the statement is true. Indeed,

$$5^{1+2} + 6^{2 \cdot 1 + 1} = 5^3 + 6^3 = 125 + 216 = 341 \text{ — is divisible by 31 } (341 : 31 = 11)$$

Inductional step  $P(n) \Rightarrow P(n+1)$ . Let us take some arbitrary  $n \in \mathbb{N}$  and let us assume

$$5^{n+2} + 6^{2n+1} \text{ is divisible by 31} \quad \text{— we assume that this is true}$$

Basing on this information, let us try to derive that

$$5^{(n+1)+2} + 6^{2(n+1)+1} = 5^{n+3} + 6^{2n+3} \text{ is divisible by 31} \quad \text{— we want to prove this}$$

Indeed, we have

$$5^{n+3} + 6^{2n+3} = 5 \cdot 5^{n+2} + 36 \cdot 6^{2n+1} = \underbrace{5 \cdot 5^{n+2} + 6 \cdot 6^{2n+1}}_{\text{is divisible by 31}} + \underbrace{31 \cdot 6^{2n+1}}_{\text{is divisible by 31}}$$

So, we see that if the statement  $P(n)$  is true then the statement  $P(n+1)$  is also true. Hence by induction all statements  $P(n)$  are true for all  $n \in \mathbb{N}$ .  $\square$

2) Base of the induction  $P(1)$ . Let us verify that for  $n = 1$  the statement is true. Indeed,

$$1 = \frac{1 \cdot (1+1)}{2} \quad \text{— is true}$$

Inductional step  $P(n) \Rightarrow P(n+1)$ . Let us assume that for some  $n \in \mathbb{N}$

$$1 + 2 + \dots + n = \frac{n(n+1)}{2} \quad \text{— we assume that this is true}$$

and basing on this information let us try to derive that

$$1 + 2 + \dots + n + (n+1) = \frac{(n+1)((n+1)+1)}{2} \quad \text{— we want to prove this}$$

Indeed, using the inductional assumption we can transform

$$\underbrace{1 + 2 + \dots + n}_{= \frac{n(n+1)}{2}} + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1) \left( \frac{n}{2} + 1 \right) = \frac{(n+1)(n+2)}{2}$$

So, we see that if the statement  $P(n)$  is true then the statement  $P(n+1)$  is also true. Hence by induction all statements  $P(n)$  are true for all  $n \in \mathbb{N}$ .  $\square$

## 5. Sets

- $x \in A$  and  $B \subset A$  — elements and subsets
- Barber paradox, universal set  $X$
- Equality of sets:  $A = B \Leftrightarrow (A \subset B) \wedge (B \subset A)$ , proofs of set identities
- Operations with sets:  $A \cup B$ ,  $A \cap B$ ,  $A \setminus B$ ,  $A' = cA = X \setminus A$
- Infinite union and intersection
- Cartesian product

EXAMPLE. For any an indexed family of subsets  $\{A_i\}_{i \in I}$  of some universal set  $X$  the following equalities hold:

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \setminus A_i) \quad X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i)$$

## 6. Relations

- $a, b \in A$ ,  $a \mathrel{\mathcal{R}} b$
- Examples:  $a \leq b$  (real numbers),  $a : b$  (integers),  $a \perp b$  (lines on the plane) etc
- Reflexive, symmetric, transitive relations
- Equivalence relation is a relation which is reflexive, symmetric, transitive
- Formally: relation  $\mathcal{R}$  is some subset of  $A \times A$ . Then  $a \mathrel{\mathcal{R}} b \Leftrightarrow (a, b) \in \mathcal{R}$

## 7. Functions

DEFINITION. Let  $A$  and  $B$  be any non-empty sets.

- A *function* (or a *map*)  $f : A \rightarrow B$  is a rule which associate with each member of  $A$  a unique element of  $B$ .
- $A$  is called the *domain* of  $f$  (we denote the domain  $D(f)$ ), and  $B$  is called *codomain*.
- Functions  $f$  and  $g$  are called *equal* if  $D(f) = D(g)$  and  $\forall a \in D(f) f(a) = g(a)$ .
- $g$  is an *extension* of  $f$  if  $D(f) \subset D(g)$  and  $\forall a \in D(f) f(a) = g(a)$ .

## 8. Domain and range of a function

DEFINITION. If  $f : A \rightarrow B$  is a function then

$$D(f) := A \text{ is the domain of } f$$
$$R(f) := \{ b \in B \mid \exists a \in A : b = f(a) \} \text{ is the range of } f$$

If  $f(a) = b$  then we say  $b$  is the *image* of  $a$  and  $a$  is a *pre-image* of  $b$ .

## 9. Injection, surjection and bijection functions

DEFINITION. Let  $f : A \rightarrow B$  be a function.

- $f$  is *injection*  $\iff f$  is one-to-one  $\iff \forall a_1, a_2 \in A$  if  $a_1 \neq a_2$  then  $f(a_1) \neq f(a_2)$
- $f$  is *surjection*  $\iff f$  is onto  $\iff R(f) = B \iff \forall b \in B \exists a \in A: f(a) = b$
- $f$  is a *bijection*  $\iff f$  is both an injection and a surjection

## 10. Countable sets

- The set  $A$  is *countable*  $\iff \exists$  a bijection  $A \leftrightarrow \mathbb{N}$
- The set of all rational numbers  $\mathbb{Q}$  is countable

## 11. Inverse function

DEFINITION. A function  $f : A \rightarrow B$  is *invertible* on  $A$  if  $f$  is an injection on  $A$ . If  $f$  is invertible on  $D(f)$  then the *inverse function* to  $f$  is defined by

$$g := f^{-1}, \quad D(g) = R(f), \quad g(b) = a \iff b = f(a), \quad \forall a \in D(f), \quad \forall b \in R(f)$$

## 12. Composition of functions

DEFINITION. Assume functions  $f : A \rightarrow E$  and  $g : B \rightarrow C$  satisfy  $R(f) \subset B$ . A *composition* of  $f$  and  $g$  is the function  $h : A \rightarrow C$  defined by

$$h := g \circ f, \quad D(h) = D(f) = A, \quad h(a) = g(f(a)), \quad a \in A$$

## 13. Real numbers

- Common sense concept: infinite decimal fractions
- More formal: the real number can be associated with certain sequences of rational numbers  $\{r_n\}$  which “approximate” (in a certain sense) this real number.
- Axioms of  $\mathbb{R}$ : operations  $x + y$  and  $x \cdot y$  possess commutativity, associativity, existence on neutral element, existence of inverse, distributivity, relation  $x \leq y$
- Intervals  $(a, b)$ ,  $[a, b)$ ,  $[a, b]$ ,  $(-\infty, b)$ ,  $[a, +\infty)$  etc

## 14. Bounded sets, lower and upper bounds

DEFINITION.

- $M \in \mathbb{R}$  is an *upper bound* of a set  $A \subset \mathbb{R}$   $\iff \forall x \in A \quad x \leq M$
- $m \in \mathbb{R}$  is a *lower bound* of a set  $A \subset \mathbb{R}$   $\iff \forall x \in A \quad x \geq m$
- $A \subset \mathbb{R}$  is *bounded from above*  $\iff \exists$  at least one upper bound of  $A$
- $A \subset \mathbb{R}$  is *bounded from below*  $\iff \exists$  at least one lower bound of  $A$
- $A \subset \mathbb{R}$  is *bounded*  $\iff A$  is bounded both from above and from below



## 15. Supremum and infimum

DEFINITION.

$M \in \mathbb{R}$  is the *supremum* of a set  $A \subset \mathbb{R}$   $\iff$

- 1)  $M$  is an upper bound of  $A$
- 2) if  $M' \in \mathbb{R}$  is some other upper bound of  $A$  then  $M \leq M'$

$m \in \mathbb{R}$  is the *infimum* of a set  $A \subset \mathbb{R}$   $\iff$

- 1)  $m$  is a lower bound of  $A$
- 2) if  $m' \in \mathbb{R}$  is some other lower bound of  $A$  then  $m \geq m'$

NOTATION.

$\sup A$  is the supremum of  $A$ ,  $\sup A = +\infty \iff A$  is not bounded from above

$\inf A$  is the infimum of  $A$ ,  $\inf A = -\infty \iff A$  is not bounded from below

## 16. Characterization of supremum and infimum using quantifiers

THEOREM.

$M \in \mathbb{R}$  is the supremum of  $A \subset \mathbb{R}$   $\iff$

- 1)  $\forall x \in A \quad x \leq M$
- 2)  $\forall \varepsilon > 0 \quad \exists x_\varepsilon \in A: \quad M - \varepsilon < x_\varepsilon$

$m \in \mathbb{R}$  is the infimum of  $A \subset \mathbb{R}$   $\iff$

- 1)  $\forall x \in A \quad m \leq x$
- 2)  $\forall \varepsilon > 0 \quad \exists x_\varepsilon \in A: \quad x_\varepsilon < m + \varepsilon$

PROBLEM. Find the supremum and the infimum of the set  $\left\{ \frac{2nm}{n^2+m^2} \mid n, m \in \mathbb{N} \right\} \subset \mathbb{R}$

## 17. Least upper bound axiom

AXIOM OF THE SET  $\mathbb{R}$ .

$A \subset \mathbb{R}$  is bounded from above  $\implies \exists$  the number  $M \in \mathbb{R}$  which is the supremum of  $A$

$A \subset \mathbb{R}$  is bounded from below  $\implies \exists$  the number  $m \in \mathbb{R}$  which is the infimum of  $A$

REMARK. The least upper bound axiom does not hold for the set of rational numbers  $\mathbb{Q}$ :

$$A = \{ r \in \mathbb{Q} : r^2 < 2 \}, \quad A \text{ is bounded from above, BUT } \nexists q \in \mathbb{Q} : q = \sup A$$

## 18. Functions of a real variable

DEFINITION.  $f : D(f) \subset \mathbb{R} \rightarrow \mathbb{R}$

- Admissible set (maximal domain) of an algebraic expression

- Graph of a function of a real variable
- Range of a function of a real variable

## 19. Monotone functions

DEFINITION. Assume  $f : [a, b] \rightarrow \mathbb{R}$ .

- $f$  —  $\nearrow$  on  $[a, b] \iff \forall x_1, x_2 \in [a, b] \ x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
- $f$  —  $\uparrow$  on  $[a, b] \iff \forall x_1, x_2 \in [a, b] \ x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- $f$  —  $\searrow$  on  $[a, b] \iff \forall x_1, x_2 \in [a, b] \ x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$
- $f$  —  $\downarrow$  on  $[a, b] \iff \forall x_1, x_2 \in [a, b] \ x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

## 20. Invertibility of strictly monotone functions

THEOREM. A strictly monotone function  $f : D(f) \subset \mathbb{R} \rightarrow \mathbb{R}$  is invertible on the interval of its monotonicity and the inverse function  $f^{-1}$  is strictly monotone in the same sense to  $f$ , i.e.

- $f$  —  $\uparrow$  on  $D(f) \implies f^{-1}$  —  $\uparrow$  on  $R(f)$
- $f$  —  $\downarrow$  on  $D(f) \implies f^{-1}$  —  $\downarrow$  on  $R(f)$

## 1.2 Limit of a sequence

### 1. Definition of a sequence

DEFINITION. A sequence  $\{x_n\}_{n=1}^{\infty}$  is a map (a function)  $f : \mathbb{N} \rightarrow \mathbb{R}$ ,  $x_n := f(n)$

### 2. Informal notion helping to understand quantifiers in the definition of a limit

We say an interval  $(\alpha, \beta)$  is a “feeder” for  $\{x_n\}_{n=1}^{\infty} \iff \forall N \in \mathbb{N} \exists n \geq N: x_n \in (\alpha, \beta)$   
(i.e. the sequence  $\{x_n\}_{n=1}^{\infty}$  “returns” to the interval  $(\alpha, \beta)$  infinitely many times).

We say an interval  $(\alpha, \beta)$  is a “catcher” for  $\{x_n\}_{n=1}^{\infty} \iff \exists N \in \mathbb{N}: \forall n \geq N: x_n \in (\alpha, \beta)$   
(i.e. once entering the interval  $(\alpha, \beta)$ , the sequence  $\{x_n\}_{n=1}^{\infty}$  can never get out of it).

We say the sequence  $\{x_n\}_{n=1}^{\infty}$  is *bounded*  $\iff \exists M \geq 0: |x_n| \leq M \quad \forall n \in \mathbb{N}$ .  
(i.e. there exists at least one “catcher” for a sequence  $\{x_n\}_{n=1}^{\infty}$ ).

### 3. Limit of a sequence

DEFINITION.  $a \in \mathbb{R}$  is the *limit* of a sequence  $\{x_n\}_{n=1}^{\infty}$  (equivalently, the sequence  $\{x_n\}_{n=1}^{\infty}$  *converges* to  $a \in \mathbb{R}$ ) iff

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) : \quad \forall n \geq N \quad |x_n - a| < \varepsilon$$

i.e. for any  $\varepsilon > 0$  the interval  $(a - \varepsilon, a + \varepsilon)$  is a “catcher” for a sequence  $\{x_n\}_{n=1}^{\infty}$ .

NOTATION.  $a = \lim_{n \rightarrow \infty} x_n$  or  $x_n \xrightarrow{n \rightarrow \infty} a$ .

### 4. Infinitesimal sequences

DEFINITION.

A sequence  $\{\alpha_n\}_{n=1}^{\infty}$  is called *infinitesimal*  $\iff \lim_{n \rightarrow \infty} \alpha_n = 0$

A sequence  $\{x_n\}_{n=1}^{\infty}$  is called *infinitely large*  $\iff \lim_{n \rightarrow \infty} |x_n| = +\infty \iff$

$$\forall M > 0 \quad \exists N = N(M) : \quad \forall n \geq N \quad |x_n| \geq M$$

### 5. Characterization of a limit in terms of infinitesimal

THEOREM. Assume  $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ ,  $a \in \mathbb{R}$ . Then

$$\exists \lim_{n \rightarrow \infty} x_n = a \iff \exists \text{ an infinitesimal } \{\alpha_n\}_{n=1}^{\infty} \text{ such that } \forall n \in \mathbb{N} \quad x_n = a + \alpha_n$$

## 6. Properties of infinitesimal sequences

- 1)  $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$  are infinitesimal  $\implies \{\alpha_n + \beta_n\}_{n=1}^{\infty}$  is infinitesimal
- 2)  $\{\alpha_n\}_{n=1}^{\infty}$  is infinitesimal,  $\{x_n\}_{n=1}^{\infty}$  is bounded  $\implies \{\alpha_n x_n\}_{n=1}^{\infty}$  is infinitesimal
- 3)  $\{\alpha_n\}_{n=1}^{\infty}$  is infinitesimal,  $\alpha_n \neq 0, \forall n \in \mathbb{N} \iff \{\frac{1}{\alpha_n}\}_{n=1}^{\infty}$  is infinitely large

## 7. Basic properties of limits

THEOREM 1. The limit of a sequence, if exists, is unique.

THEOREM 2.  $\exists \lim_{n \rightarrow \infty} x_n = a$  ( $a$  is a finite real number)  $\implies \{x_n\}_{n=1}^{\infty}$  is bounded.

THEOREM 3.  $\exists \lim_{n \rightarrow \infty} x_n = a, \exists \lim_{n \rightarrow \infty} y_n = b, \forall n \in \mathbb{N} \ x_n \leq y_n \implies a \leq b$

THEOREM 4.  $\forall n \in \mathbb{N} \ x_n \leq y_n \leq z_n, \exists \lim_{n \rightarrow \infty} x_n = a, \exists \lim_{n \rightarrow \infty} z_n = a \implies \exists \lim_{n \rightarrow \infty} y_n = a$

COROLLARY.  $\{\beta_n\}_{n=1}^{\infty}$  is infinitesimal,  $|\alpha_n| \leq |\beta_n| \implies \{\alpha_n\}_{n=1}^{\infty}$  is infinitesimal.

## 8. Arithmetic operations with limits

THEOREM 1.  $\exists \lim_{n \rightarrow \infty} x_n = a \exists \lim_{n \rightarrow \infty} y_n = b \implies \exists \lim_{n \rightarrow \infty} (x_n \pm y_n) = a \pm b$

THEOREM 2.  $\exists \lim_{n \rightarrow \infty} x_n = a \exists \lim_{n \rightarrow \infty} y_n = b \implies \exists \lim_{n \rightarrow \infty} (x_n y_n) = ab$

LEMMA.  $\exists \lim_{n \rightarrow \infty} y_n = b$  and  $y_n \neq 0, b \neq 0 \implies \{\frac{1}{y_n}\}_{n=1}^{\infty}$  is bounded.

THEOREM 3.  $\exists \lim_{n \rightarrow \infty} x_n = a \exists \lim_{n \rightarrow \infty} y_n = b$  and  $y_n \neq 0, b \neq 0 \implies \exists \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{a}{b}$

## 9. Monotone sequences

DEFINITION.

- $\{x_n\}_{n=1}^{\infty} \nearrow \iff \forall n \in \mathbb{N} \ x_n \leq x_{n+1}$
- $\{x_n\}_{n=1}^{\infty} \searrow \iff \forall n \in \mathbb{N} \ x_n \geq x_{n+1}$

THEOREM.

- 1)  $\{x_n\}_{n=1}^{\infty} \nearrow \exists M \in \mathbb{R}: \forall n \in \mathbb{N} \ x_n \leq M \implies \exists \lim_{n \rightarrow \infty} x_n \leq M$
- 2)  $\{x_n\}_{n=1}^{\infty} \searrow \exists m \in \mathbb{R}: \forall n \in \mathbb{N} \ x_n \geq m \implies \exists \lim_{n \rightarrow \infty} x_n \geq m$

## 10. Important examples

$$1) \quad q > 1 \implies \lim_{n \rightarrow \infty} \frac{n}{q^n} = 0$$

$$2) \quad q \in \mathbb{R} \implies \lim_{n \rightarrow \infty} \frac{q^n}{n!} = 0$$

$$3) \quad \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$4) \quad a > 0 \implies \lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$$

## 11. Euler's number $e$

LEMMA 1. For all non-negative  $a_1, a_2, a_3, \dots, a_n \in \mathbb{R}$

$$\underbrace{\sqrt[n]{a_1 a_2 a_3 \cdot \dots \cdot a_n}}_{\text{Geometric mean}} \leq \underbrace{\frac{a_1 + a_2 + a_3 + \dots + a_n}{n}}_{\text{Arithmetic mean}}$$

LEMMA 2. Assume  $a > 0, b > 0$ . Then

$$\sqrt[n+1]{ab^n} \leq \frac{a + nb}{n+1}$$

LEMMA 3. Denote  $x_n := \left(1 + \frac{1}{n}\right)^n$ ,  $z_n := \left(1 - \frac{1}{n}\right)^n$ ,  $y_n := \left(1 + \frac{1}{n}\right)^{n+1}$ . Then

$$\forall n \in \mathbb{N} \quad x_n < x_{n+1}, \quad z_n < z_{n+1}, \quad y_{n+1} < y_n$$

LEMMA 4. The following limits exist and coincide

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$$

DEFINITION. Euler's number  $e$  is defined by

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n, \quad (e \approx 2,718281828459045\dots)$$

## 1.3 Subsequences

### 1. Definition of a subsequence

DEFINITION. Assume  $f : \mathbb{N} \rightarrow \mathbb{R}$  is a sequence and assume  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  is strictly increasing. Denote by  $h := f \circ \varphi$  the composition  $h(k) = f(\varphi(k))$ . Then the sequence  $h : \mathbb{N} \rightarrow \mathbb{R}$  is called a *subsequence* of the sequence  $f : \mathbb{N} \rightarrow \mathbb{R}$ .

$$x_n := f(n), \quad x_{n_k} = h(k), \quad n_k := \varphi(k), \quad n_1 < n_2 < \dots < n_k < \dots$$

NOTATION.  $\{x_{n_k}\}_{k=1}^{\infty}$  is a subsequence of  $\{x_n\}_{n=1}^{\infty} \iff \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$

### 2. Any subsequence of a convergent sequence converges to the same limit

THEOREM.  $\{x_{n_k}\}_{k=1}^{\infty}$  is a subsequence of  $\{x_n\}_{n=1}^{\infty}$ ,  $\exists \lim_{n \rightarrow \infty} x_n = a \implies \exists \lim_{k \rightarrow \infty} x_{n_k} = a$

### 3. Nested interval theorem

THEOREM. Assume a sequence of closed intervals  $[a_n, b_n] \subset \mathbb{R}$  satisfies the properties

- 1)  $\forall n \in \mathbb{N} \quad [a_{n+1}, b_{n+1}] \subset [a_n, b_n]$
- 2)  $b_n - a_n \rightarrow 0 \quad \text{as } n \rightarrow \infty$

Then  $\exists! c \in \mathbb{R} : c \in \bigcap_{n=1}^{\infty} [a_n, b_n]$

### 4. Bolzano–Weierstrass theorem

THEOREM. Each bounded sequence has a convergent subsequence:

$$\{x_n\}_{n=1}^{\infty} \text{ is bounded } \implies \exists \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty} \text{ such that } \exists a \in \mathbb{R} : a = \lim_{k \rightarrow \infty} x_{n_k}$$

### 5. Cauchy sequence

DEFINITION.  $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$  is a *Cauchy sequence* (or a *fundamental sequence*) if

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in \mathbb{N} : \quad \forall n, m \geq N(\varepsilon) \quad |x_n - x_m| < \varepsilon$$

THEOREM. Every fundamental sequence is convergent:

$$\{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \text{ is a Cauchy sequence } \iff \exists a \in \mathbb{R} : x_n \xrightarrow[n \rightarrow \infty]{} a$$

## 1.4 Limit of a function

### 1. Definition of the limit of a function according to Cauchy

DEFINITION (C). Assume  $f : (a, b) \rightarrow \mathbb{R}$ ,  $x_0 \in (a, b)$ ,  $y_0 \in \mathbb{R}$ . We write

$$\lim_{x \rightarrow x_0} f(x) = y_0 \quad \text{or} \quad f(x) \xrightarrow{x \rightarrow x_0} y_0$$

and say  $y_0$  is the *limit* of  $f(x)$  as  $x$  tends to  $x_0$  iff

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 : \forall x \in (a, b) \quad (x \neq x_0) \wedge |x - x_0| < \delta \implies |f(x) - y_0| < \varepsilon$$

The set  $\overset{\circ}{U}_\delta(x_0) := (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$  is called a *deleted neighbourhood* of  $x_0$ .

### 2. Definition of the limit of a function according to Heine

DEFINITION (H). Assume  $f : (a, b) \rightarrow \mathbb{R}$ ,  $x_0 \in (a, b)$ ,  $y_0 \in \mathbb{R}$ . We write

$$\lim_{x \rightarrow x_0} f(x) = y_0 \quad \text{or} \quad f(x) \xrightarrow{x \rightarrow x_0} y_0$$

and say  $y_0$  is the *limit* of  $f(x)$  as  $x$  tends to  $x_0$  iff

$$\forall \{x_n\}_{n=1}^\infty \subset (a, b) \setminus \{x_0\} : \quad \lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} f(x_n) = y_0$$

### 3. Equivalence of definitions

THEOREM. Definition (C)  $\iff$  Definition (H)

### 4. Arithmetic operations with limits of functions

$$\text{THEOREM 1.} \quad \exists \lim_{x \rightarrow x_0} f(x), \exists \lim_{x \rightarrow x_0} g(x) \implies \exists \lim_{x \rightarrow x_0} (f(x) + g(x)) = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$$

$$\text{THEOREM 2.} \quad \exists \lim_{x \rightarrow x_0} f(x), \exists \lim_{x \rightarrow x_0} g(x) \implies \exists \lim_{x \rightarrow x_0} (f(x) \cdot g(x)) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x)$$

$$\text{LEMMA.} \quad \lim_{x \rightarrow x_0} g(x) > 0 \implies \exists \delta > 0 : |x - x_0| < \delta \implies g(x) > 0$$

$$\text{THEOREM 3.} \quad \exists \lim_{x \rightarrow x_0} f(x), \exists \lim_{x \rightarrow x_0} g(x) \neq 0 \implies \exists \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)}$$

### 5. One-sided limits

DEFINITION. Assume  $f : [a, b] \rightarrow \mathbb{R}$ ,  $x_0 \in [a, b]$ .

$$\exists \lim_{x \rightarrow x_0 - 0} f(x) = y_0 \iff \forall \varepsilon > 0 \quad \exists \delta > 0 : x \in (x_0 - \delta, x_0) \implies |f(x) - y_0| < \varepsilon$$

$$\exists \lim_{x \rightarrow x_0 + 0} f(x) = y_0 \iff \forall \varepsilon > 0 \quad \exists \delta > 0 : x \in (x_0, x_0 + \delta) \implies |f(x) - y_0| < \varepsilon$$

$$\text{THEOREM.} \quad \exists \lim_{x \rightarrow x_0} f(x) = y_0 \iff \exists \lim_{x \rightarrow x_0 - 0} f(x) = \lim_{x \rightarrow x_0 + 0} f(x) = y_0$$

## 1.5 Indeterminate forms and important examples

### 1. Types of indeterminate forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad 1^\infty, \quad \infty^0, \quad 0^0$$

### 2. Auxiliary inequalities

THEOREM.

- 1)  $0 < x < \frac{\pi}{2} \implies \sin x < x < \operatorname{tg} x, \quad x \in \mathbb{R} \implies |\sin x| \leq |x|$
- 2)  $x_1, x_2 \in \mathbb{R} \implies |\sin x_1 - \sin x_2| \leq |x_1 - x_2|, \quad |\cos x_1 - \cos x_2| \leq |x_1 - x_2|$
- 3)  $\lim_{x \rightarrow 0} \sin x = 0, \quad \lim_{x \rightarrow 0} \cos x = 1$

### 3. Properties of $a^x$ and $\log_a x$ which are postulated for now

We postulate the properties of the exponential and the logarithmic functions which actually come together with the definition of the exponential function and its inverse. We will return to discussion of these properties in the section “Continuity”. These properties also allowed to use without further explanation when you solve homework problems.

THEOREM. Assume  $a > 0$  and  $a \neq 1$ . Then

- 1)  $\forall \{x_n\}_{n=1}^\infty \subset \mathbb{R} \quad x_n \rightarrow x_0 \implies a^{x_n} \rightarrow a^{x_0}$
- 2)  $\forall \{x_n\}_{n=1}^\infty \subset \mathbb{R} \quad x_n \rightarrow x_0, \quad x_0 > 0 \implies \log_a x_n \rightarrow \log_a x_0$

### 4. Important examples

- 1)  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
- 2)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$
- 3)  $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e \quad \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = e \quad \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$
- 4)  $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$
- 5)  $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$
- 6)  $\lim_{x \rightarrow 0} \frac{(1+x)^\mu - 1}{x} = \mu$



## 1.6 Infinite limits and asymptotes

### 1. Infinite limits, vertical asymptotes

DEFINITION.

$$\begin{aligned}
 \lim_{x \rightarrow x_0-0} f(x) = +\infty &\iff \forall M > 0 \exists \delta > 0: x \in (x_0 - \delta, x_0) \Rightarrow f(x) > M \\
 \lim_{x \rightarrow x_0+0} f(x) = +\infty &\iff \forall M > 0 \exists \delta > 0: x \in (x_0, x_0 + \delta) \Rightarrow f(x) > M \\
 \lim_{x \rightarrow x_0-0} f(x) = -\infty &\iff \forall M < 0 \exists \delta > 0: x \in (x_0 - \delta, x_0) \Rightarrow f(x) < M \\
 \lim_{x \rightarrow x_0+0} f(x) = -\infty &\iff \forall M < 0 \exists \delta > 0: x \in (x_0, x_0 + \delta) \Rightarrow f(x) < M
 \end{aligned}$$

DEFINITION. The line  $x = x_0$  is a *vertical asymptote* of the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  if at least one of the four statements is true:

$$\lim_{x \rightarrow x_0+0} f(x) = +\infty, \quad \lim_{x \rightarrow x_0-0} f(x) = +\infty, \quad \lim_{x \rightarrow x_0-0} f(x) = +\infty, \quad \lim_{x \rightarrow x_0-0} f(x) = -\infty$$

### 2. Behavior of functions at infinity

DEFINITION.

$$\begin{aligned}
 \lim_{x \rightarrow +\infty} f(x) = +\infty &\iff \forall M > 0 \exists R > 0: x > R \Rightarrow f(x) > M \\
 \lim_{x \rightarrow -\infty} f(x) = +\infty &\iff \forall M > 0 \exists R < 0: x < R \Rightarrow f(x) > M \\
 \lim_{x \rightarrow +\infty} f(x) = -\infty &\iff \forall M < 0 \exists R > 0: x > R \Rightarrow f(x) < M \\
 \lim_{x \rightarrow -\infty} f(x) = -\infty &\iff \forall M < 0 \exists R < 0: x < R \Rightarrow f(x) < M
 \end{aligned}$$

### 3. Limits at infinity, horizontal asymptotes

DEFINITION.

$$\begin{aligned}
 \lim_{x \rightarrow +\infty} f(x) = y_0 &\iff \forall \varepsilon > 0 \exists R > 0: x > R \Rightarrow |f(x) - y_0| < \varepsilon \\
 \lim_{x \rightarrow -\infty} f(x) = y_0 &\iff \forall \varepsilon > 0 \exists R < 0: x < R \Rightarrow |f(x) - y_0| < \varepsilon
 \end{aligned}$$

DEFINITION. The line  $y = y_0$  is a *horizontal asymptote* of the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  if at least one of the two statements is true:

$$\lim_{x \rightarrow +\infty} f(x) = y_0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = y_0$$

#### 4. How to remember definitions of finite/infinite limits?

$$y_0 = \lim_{x \rightarrow x_0} f(x)$$

$x_0$  is finite or infinite

$y_0$  is finite or infinite

	Values of $f(x)$	Values of $x$	When is used?
Small number	$\forall \varepsilon > 0$	$\exists \delta > 0$	For a finite value
Large number	$\forall M > 0$	$\exists R > 0$	For an infinite value

#### 5. Oblique asymptotes

DEFINITION. The line  $y = kx + b$  is an *oblique asymptote* of the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  if at least one of the two statements is true:

$$\lim_{x \rightarrow +\infty} (f(x) - kx - b) = 0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} (f(x) - kx - b) = 0$$

REMARK. A horizontal asymptote is a particular case of an oblique asymptote with  $k = 0$ .

THEOREM.  $y = kx + b$  is a horizontal or an oblique asymptote of  $f : \mathbb{R} \rightarrow \mathbb{R}$  as  $x \rightarrow +\infty$  iff the following two limits exist:

$$1) \quad \exists \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = k$$

$$2) \quad \exists \lim_{x \rightarrow +\infty} (f(x) - kx) = b$$

The similar statements hold as  $x \rightarrow -\infty$ .

## 1.7 Continuity

### 1. Continuity of a function at a point

DEFINITION.  $f : (a, b) \rightarrow \mathbb{R}$  is *continuous* at a point  $x_0 \iff \lim_{x \rightarrow x_0} f(x) = f(x_0)$

REMARK. The following statements are equivalent:

- 1)  $f : (a, b) \rightarrow \mathbb{R}$  is continuous at  $x_0$
- 2)  $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0: |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$
- 3)  $\forall \{x_n\}_{n=1}^\infty \subset (a, b): x_n \rightarrow x_0 \implies f(x_n) \rightarrow f(x_0)$

DEFINITION.  $f : (a, b) \rightarrow \mathbb{R}$  is continuous *on*  $(a, b) \iff \forall x_0 \in (a, b)$   $f$  is continuous at  $x_0$

### 2. Stability of sign theorem

THEOREM.  $f$  is continuous at  $x_0$ ,  $f(x_0) > 0 \implies \exists \delta > 0: \forall x \in (x_0 - \delta, x_0 + \delta) f(x) > 0$

### 3. Arithmetic operations with continuous functions

THEOREM.

- 1)  $f, g$  are continuous at  $x_0 \implies f \pm g$  are continuous at  $x_0$
- 2)  $f, g$  are continuous at  $x_0 \implies f \cdot g$  is continuous at  $x_0$
- 3)  $f, g$  are continuous at  $x_0$ ,  $g(x_0) \neq 0 \implies f/g$  is continuous at  $x_0$

### 4. Left and right continuity

DEFINITION.

$f : [a, b] \rightarrow \mathbb{R}$  is *left continuous* at  $x_0 \iff \lim_{x \rightarrow x_0-0} f(x) = f(x_0)$

$f : [a, b] \rightarrow \mathbb{R}$  is *right continuous* at  $x_0 \iff \lim_{x \rightarrow x_0+0} f(x) = f(x_0)$

$f : [a, b] \rightarrow \mathbb{R}$  is continuous *on*  $[a, b] \iff \begin{cases} f \text{ is continuous on } (a, b) \\ f \text{ is left continuous at } a \\ f \text{ is right continuous at } b \end{cases}$

THEOREM.  $f$  is continuous at  $x_0 \in (a, b) \iff f$  is left and right continuous at  $x_0$ .

### 5. Types of discontinuity

DEFINITION. Assume  $x_0 \in (a, b)$ ,  $f : (a, b) \setminus \{x_0\} \rightarrow \mathbb{R}$

- $x_0$  is a *removable discontinuity*  $\iff \exists$  finite  $\lim_{x \rightarrow x_0 \pm 0} f(x)$ ,  $\lim_{x \rightarrow x_0-0} f(x) = \lim_{x \rightarrow x_0+0} f(x)$

- $x_0$  is a *jump discontinuity*  $\iff \exists$  both finite  $\lim_{x \rightarrow x_0 \pm 0} f(x)$ ,  $\lim_{x \rightarrow x_0 - 0} f(x) \neq \lim_{x \rightarrow x_0 + 0} f(x)$
- $x = x_0$  is a *vertical asymptote*  $\iff \exists \lim_{x \rightarrow x_0 \pm 0} f(x) = \pm\infty$  (at least one of limits)
- $x_0$  may be a discontinuity of no above type  $\iff \nexists \lim_{x \rightarrow x_0 \pm 0} f(x)$  ( $f(x) = \sin \frac{1}{x}$ )

## 6. Continuity of some elementary functions

- $f(x) = x^n$  is continuous on  $\mathbb{R}$ ,  $n \in \mathbb{N}$
- $f(x) = \frac{1}{x^n}$  is continuous on  $\mathbb{R} \setminus \{0\}$ ,  $n \in \mathbb{N}$
- $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$  is continuous on  $\mathbb{R}$
- $f(x) = \sin x$ ,  $g(x) = \cos x$  are continuous on  $\mathbb{R}$
- $f(x) = \operatorname{tg} x$  is continuous at  $x \neq \frac{\pi}{2} + \pi k$ ,  $g(x) = \operatorname{ctg} x$  is continuous at  $x \neq \pi k$ ,  $k \in \mathbb{Z}$
- $f(x) = x^p$ ,  $p \in (0, +\infty)$  is continuous on  $[0, +\infty)$
- $f(x) = a^x$ ,  $a > 0$ , is continuous on  $\mathbb{R}$

## 7. Continuity of a composition of functions

THEOREM.  $f : (a, b) \rightarrow \mathbb{R}$ ,  $g : (c, d) \rightarrow \mathbb{R}$ ,  $x_0 \in (a, b)$ ,  $f(x_0) \in (c, d)$

$f$  is continuous at  $x_0$ ,  $g$  is continuous at  $f(x_0) \implies g \circ f$  is continuous at  $x_0$

## 8. Two intermediate value theorems (Bolzano–Cauchy theorems)

THEOREM 1.  $f$  is continuous on  $[a, b]$ ,  $f(a) \leq 0$ ,  $f(b) \geq 0 \implies \exists c \in [a, b]: f(c) = 0$

THEOREM 2.  $f$  is continuous on  $[a, b]$ ,  $f(a) = y_1$ ,  $f(b) = y_2$ ,  $y_1 \leq y_2 \implies [y_1, y_2] \subset R(f)$

## 9. Continuity of the inverse function

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  is continuous and strictly monotone on  $(a, b)$ . Then  $f$  is invertible on  $(a, b)$  and  $f^{-1}$  is continuous on  $R(f)$ .

## 10. Continuity of some elementary inverse functions

- $f(x) = \sqrt[n]{x}$  is continuous on  $D(f)$ ,  $n \in \mathbb{N}$
- $f(x) = \log_a x$ ,  $a > 0$ ,  $a \neq 1$ , is continuous on  $(0, +\infty)$
- $f(x) = \arcsin x$ ,  $g(x) = \arccos x$  are continuous on  $[-1, 1]$
- $f(x) = \operatorname{arctg} x$ ,  $g(x) = \operatorname{arcctg} x$  are continuous on  $\mathbb{R}$

### 11. Two extreme value theorems (Weierstrass theorems)

THEOREM 1.  $[a, b] \subset \mathbb{R}$  is a closed bounded interval,  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b] \implies f$  is bounded on  $[a, b]$ , i.e.

$$\exists M > 0 : \quad \forall x \in [a, b] \quad |f(x)| \leq M$$

THEOREM 2.  $[a, b] \subset \mathbb{R}$  is a closed bounded interval,  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b] \implies f$  achieves on  $[a, b]$  its maximum and minimal values, i.e.  $\exists c_1, c_2 \in [a, b]$  such that

$$f(c_1) = \inf_{x \in [a, b]} f(x) \quad \text{and} \quad f(c_2) = \sup_{x \in [a, b]} f(x)$$

### 12. Uniform continuity

DEFINITION. A function  $f : D(f) \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is *uniformly continuous* on  $D(f)$  iff

$$\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 : \quad \forall x_1, x_2 \in D(f) \quad |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$$

### 13. Heine–Cantor theorem

THEOREM.  $[a, b] \subset \mathbb{R}$  is a closed bounded interval,  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b] \implies f$  is uniformly continuous on  $[a, b]$ .

## 1.8 Differentiability and derivatives

### 1. Main definition

DEFINITION. Assume  $f : (a, b) \rightarrow \mathbb{R}$ ,  $x_0 \in (a, b)$ . A function  $f$  is *differentiable at a point*  $x_0$  iff there exists a finite limit

$$f'(x_0) := \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The number  $f'(x_0)$  is called the *derivative* of a function  $f$  at  $x_0$ . Another notation

$$\frac{df}{dx}(x_0) := f'(x_0)$$

- We say  $f$  is *differentiable on*  $(a, b)$  if for any  $x_0 \in (a, b)$  the function  $f$  is differentiable at  $x_0$ .
- We say  $f$  is *continuously differentiable* on  $(a, b)$  if  $f$  is differentiable on  $(a, b)$  and the function  $f'$  is continuous on  $(a, b)$ .

### 2. Physical and geometric meaning of the derivative

- From a physical point of view the derivative of a function is the instantaneous speed (or the rate of change) of the function.
- From a geometrical point of view the derivative of a function  $f$  at a point  $x_0$  is equal to the slope of the tangent line to the graph of the function at the point  $x_0$ .

### 3. Differentiable function is continuous

THEOREM.  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable at  $x_0 \in (a, b) \implies f$  is continuous at  $x_0$

REMARK. The inverse is false:  $f(x) = |x|$  is continuous but non-differentiable at 0.

### 4. Basic properties of derivatives

THEOREM . Assume  $f, g : (a, b) \rightarrow \mathbb{R}$  are differentiable at  $x \in (a, b)$ . Then  $f \pm g$  and  $fg$  are differentiable at  $x$ . If additionally  $g(x) \neq 0$  then  $f/g$  is also differentiable at  $x$ . Besides,

- 1)  $(f \pm g)'(x) = f'(x) \pm g'(x)$
- 2)  $(cf)'(x) = cf'(x)$
- 3)  $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- 4)  $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

## 5. Derivatives of elementary functions

1.  $(C)' = 0$

2.  $(x)' = 1$

3.  $(x^m)' = mx^{m-1}, \quad m \in \mathbb{Z}, \quad x \in \mathbb{R} \quad (x \neq 0 \text{ if } m < 0)$

4.  $(x^r)' = rx^{r-1}, \quad r \in \mathbb{R}, \quad x > 0$

5.  $(e^x)' = e^x, \quad (a^x)' = a^x \ln a, \quad a > 0, \quad x \in \mathbb{R}$

6.  $(\ln |x|)' = \frac{1}{x}, \quad x \neq 0, \quad (\log_a |x|)' = \frac{1}{x \ln a}, \quad a > 0, \quad a \neq 1$

7.  $(\sin x)' = \cos x, \quad (\cos x)' = -\sin x$

8.  $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}, \quad (\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}$

## 1.9 Chain rule

### 1. Differential of a function

DEFINITION. Assume  $f : (a, b) \rightarrow \mathbb{R}$  and  $x_0 \in (a, b)$ . A linear map  $l_{x_0} : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$l_{x_0}(h) = k_{x_0}h, \quad \forall h \in \mathbb{R}, \quad (k_{x_0} \in \mathbb{R})$$

is called the *differential* of  $f$  at a point  $x_0$  iff there exists a number  $h_0 > 0$  and a function  $\alpha_{x_0} : (-h_0, h_0) \rightarrow \mathbb{R}$  such that

$$\lim_{h \rightarrow 0} \alpha_{x_0}(h) = 0,$$

$$f(x_0 + h) = f(x_0) + k_{x_0}h + \alpha_{x_0}(h)h, \quad \forall |h| < h_0$$

NOTATION.  $l_{x_0} := df(x_0) \iff df(x_0)(h) := l_{x_0}(h) = k_{x_0}h, \quad \forall h \in \mathbb{R}$

### 2. When the differential of a function is well-defined?

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  and  $x_0 \in (a, b)$ . The differential of  $f$  at  $x_0$  exists if and only if  $f$  is differentiable at  $x_0$ . Moreover, if  $df(x_0)(h) = k_{x_0}h$  is the differential of  $f$  at  $x_0$  then

$$k_{x_0} = f'(x_0) \iff df(x_0)(h) = f'(x_0)h, \quad \forall h \in \mathbb{R}$$

where  $f'(x_0)$  is the derivative of  $f$  at  $x_0$ .

### 3. What is the difference between the derivative $f'(x_0)$ and the differential $df(x_0)$ ?

It is well-known that any linear map  $l : \mathbb{R} \rightarrow \mathbb{R}$  has the form

$$l(x) = kx, \quad x \in \mathbb{R}$$

i.e. any linear map from  $\mathbb{R}$  to  $\mathbb{R}$  can be characterized in the unique way by the coefficient  $k \in \mathbb{R}$  which is called a *slope* of the linear map  $l(x)$ . The theorem above shows that the slope of the differential  $df(x_0)$  is equal to  $f'(x_0)$ . So, the differential and the derivative of a function are related in the same way as a linear transformation  $l(x) = kx$  is related to its slope  $k \in \mathbb{R}$ .

NOTATION. Very often people write  $df(x_0) = f'(x_0)dx$  meaning that  $f'(x_0)$  is a slope of a linear function  $df(x_0)$  and  $dx$  stands for the identity linear map, i.e.  $dx(h) = h$ .

### 4. Chain rule

THEOREM. Assume  $f : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : (c, d) \rightarrow \mathbb{R}$ ,  $R(f) \subset (c, d)$ ,  $x_0 \in (a, b)$ . If  $f$  is differentiable at  $x_0$  and  $g$  is differentiable at  $y_0 = f(x_0)$  then the composition  $\varphi := g \circ f$  is differentiable at  $x_0$  and

$$\varphi'(x_0) = g'(y) \Big|_{y=f(x_0)} \cdot f'(x_0)$$



## 5. Inverse function rule

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  is strictly monotone and continuous (so the inverse function  $f^{-1}$  exists,  $f^{-1}$  is strictly monotone in the same sense to  $f$  and continuous on  $R(f)$ ). Assume additionally  $f$  is differentiable at  $x_0 \in (a, b)$  and  $f'(x_0) \neq 0$ . Then the function  $g := f^{-1}$  is differentiable at  $y_0 := f(x_0)$  and

$$g'(y_0) = \frac{1}{f'(x_0)}$$

## 6. Derivatives of inverse trigonometric functions

$$9. (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$10. (\operatorname{arctg} x)' = \frac{1}{1+x^2}, \quad (\operatorname{arcctg} x)' = -\frac{1}{1+x^2}$$

## 1.10 The basic theorems of differential calculus

### 1. Fermat's theorem (or Fermat's lemma)

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  has a local extremum (maximum or minimum) on the interval  $(a, b)$  at some internal point  $c \in (a, b)$ , i.e.

$$\begin{aligned} \text{either} \quad f(c) &= \max_{x \in (a, b)} f(x) && \iff \quad \forall x \in (a, b) \quad f(x) \leq f(c) \\ \text{or} \quad f(c) &= \min_{x \in (a, b)} f(x) && \iff \quad \forall x \in (a, b) \quad f(x) \geq f(c) \end{aligned}$$

If  $f$  is differentiable at  $c$  then  $f'(c) = 0$ .

### 2. Rolle's theorem

THEOREM. Assume  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Assume that  $f(a) = f(b)$ . Then there exists  $c \in (a, b)$  such that  $f'(c) = 0$ .

### 3. Lagrange's theorem

THEOREM. Assume  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there is a point  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

### 4. Cauchy's theorem

THEOREM. Assume functions  $f, g : [a, b] \rightarrow \mathbb{R}$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Assume  $g'(x) \neq 0$  for any  $x \in (a, b)$ . Then there is a point  $c \in (a, b)$  such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

### 5. Functions with identically zero derivatives are constants

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable on  $(a, b)$  and  $f'(x) = 0$  for any  $x \in (a, b)$ . Then  $f$  is equal to a constant, i.e.

$$\exists y_0 \in \mathbb{R} : \quad f(x) = y_0 \quad \forall x \in (a, b)$$

## 1.11 Investigation of functions using derivatives

### 1. Investigation of the monotonicity of functions

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable on  $(a, b)$ . Then

- 1)  $f'(x) \geq 0 \quad \forall x \in (a, b) \iff f \text{ — } \nearrow$  (is non-decreasing) on  $(a, b)$
- 2)  $f'(x) \leq 0 \quad \forall x \in (a, b) \iff f \text{ — } \searrow$  (is non-increasing) on  $(a, b)$
- 3)  $f'(x) > 0 \quad \forall x \in (a, b) \implies f \text{ — } \uparrow$  (is strictly increasing) on  $(a, b)$
- 4)  $f'(x) < 0 \quad \forall x \in (a, b) \implies f \text{ — } \downarrow$  (is strictly decreasing) on  $(a, b)$

### 2. Conditions for extreme in terms of the first derivative

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable on  $(a, b)$ . Then and assume  $c \in (a, b)$  satisfies  $f'(c) = 0$ . Then

- 1) if  $f'(x) \leq 0 \quad \forall x \in (a, c)$  and  $f'(x) \geq 0 \quad \forall x \in (c, b)$  then  $c$  is the point of minimum of  $f$
- 2) if  $f'(x) \geq 0 \quad \forall x \in (a, c)$  and  $f'(x) \leq 0 \quad \forall x \in (c, b)$  then  $c$  is the point of maximum of  $f$

### 3. Higher order derivatives

DEFINITION. Let  $f : (a, b) \rightarrow \mathbb{R}$  be differentiable on  $(a, b)$ . If the function  $g = f' : (a, b) \rightarrow \mathbb{R}$  is differentiable at  $x_0 \in (a, b)$  then the function  $f$  is called *twice differentiable* at  $x_0$ . In this case the derivative of the function  $g = f'$  at  $x_0$  is called the *second derivative* of  $f$  at  $x_0$ .

$$f''(x_0) := g'(x_0) = \lim_{x \rightarrow x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

The higher-order derivatives are defined similarly:

$$f''' = (f'')', \quad \dots, \quad f^{(k)} = (f^{(k-1)})', \quad \dots$$

ALTERNATIVE NOTATION.

$$\frac{d^2 f}{dx^2}(x) = f''(x), \quad \frac{d^3 f}{dx^3}(x) = f'''(x), \quad \dots \quad \frac{d^k f}{dx^k}(x) = f^{(k)}(x)$$

### 4. Conditions for extreme in terms of the second derivative

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable on  $(a, b)$ . Let  $f$  be twice differentiable at  $c \in (a, b)$  and  $f'(c) = 0$ . Then

- 1)  $f''(c) > 0 \implies c$  is a point of a local min of  $f \iff \exists \delta > 0: f(c) = \min_{x \in (x_0 - \delta, x_0 + \delta)} f(x)$
- 2)  $f''(c) < 0 \implies c$  is a point of a local max of  $f \iff \exists \delta > 0: f(c) = \max_{x \in (x_0 - \delta, x_0 + \delta)} f(x)$

## 5. Convex functions

DEFINITION.  $f : [a, b] \rightarrow \mathbb{R}$  is *convex* on  $(a, b) \iff \forall x_1, x_2 \in (a, b), \forall \lambda \in [0, 1]$

$$f((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)f(x_1) + \lambda f(x_2)$$

$f : [a, b] \rightarrow \mathbb{R}$  is *strictly convex* on  $(a, b) \iff \forall x_1 \neq x_2 \in (a, b), \forall \lambda \in (0, 1)$

$$f((1 - \lambda)x_1 + \lambda x_2) < (1 - \lambda)f(x_1) + \lambda f(x_2)$$

$f : [a, b] \rightarrow \mathbb{R}$  is *concave* on  $(a, b) \iff \forall x_1, x_2 \in (a, b), \forall \lambda \in [0, 1]$

$$f((1 - \lambda)x_1 + \lambda x_2) \geq (1 - \lambda)f(x_1) + \lambda f(x_2)$$

$f : [a, b] \rightarrow \mathbb{R}$  is *strictly concave* on  $(a, b) \iff \forall x_1 \neq x_2 \in (a, b), \forall \lambda \in (0, 1)$

$$f((1 - \lambda)x_1 + \lambda x_2) > (1 - \lambda)f(x_1) + \lambda f(x_2)$$

## 6. Inequalities for slopes of secants

THEOREM.

1)  $f$  is convex on  $(a, b) \iff \forall x_1, x_2 \in (a, b), x_1 < x_2, \forall x \in (x_1, x_2)$

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}$$

2)  $f$  is strictly convex on  $(a, b) \iff \forall x_1, x_2 \in (a, b), x_1 < x_2, \forall x \in (x_1, x_2)$

$$\frac{f(x) - f(x_1)}{x - x_1} < \frac{f(x_2) - f(x)}{x_2 - x}$$

3) Let  $f$  be differentiable on  $(a, b)$ . Then  $f$  is convex on  $(a, b) \iff \forall x_1, x_2 \in (a, b)$

$$x_1 < x_2 \implies f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2)$$

## 7. Investigation of convexity using the first derivative

THEOREM. Let  $f : (a, b) \rightarrow \mathbb{R}$  be differentiable on  $(a, b)$ .

1)  $f$  is convex on  $(a, b) \iff f' \text{ — } \nearrow \text{ on } (a, b)$

2)  $f$  is concave on  $(a, b) \iff f' \text{ — } \searrow \text{ on } (a, b)$

3)  $f$  is strictly convex on  $(a, b) \iff f' \text{ — } \uparrow \text{ on } (a, b)$

4)  $f$  is strictly concave on  $(a, b) \iff f' \text{ — } \downarrow \text{ on } (a, b)$

## 8. Investigation of convexity using the second derivative

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  is twice differentiable on  $(a, b)$ .

- 1)  $\forall x \in (a, b) \quad f''(x) \geq 0 \iff f$  is convex on  $(a, b)$
- 2)  $\forall x \in (a, b) \quad f''(x) \leq 0 \iff f$  is concave on  $(a, b)$
- 3)  $\forall x \in (a, b) \quad f''(x) > 0 \implies f$  is strictly convex on  $(a, b)$
- 4)  $\forall x \in (a, b) \quad f''(x) < 0 \implies f$  is strictly concave on  $(a, b)$

## 9. Tangent line to the graph of a function

DEFINITION. Assume  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable at  $x_0 \in (a, b)$ . The straight line given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is called the *tangent* to the function  $y = f(x)$  (or to the graph of  $f$ ) at the point  $x_0$ .

## 10. Convexity and tangent lines

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable on  $(a, b)$ .

- 1)  $f$  is convex on  $(a, b) \iff$  for any  $x_0 \in (a, b)$  the graph of the function  $f$  lies above the tangent to  $f$  at the point  $x_0$ , i.e.

$$\forall x \in (a, b) \quad f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

- 2)  $f$  is concave on  $(a, b) \iff$  for any  $x_0 \in (a, b)$  the graph of the function  $f$  lies below the tangent to  $f$  at the point  $x_0$ , i.e.

$$\forall x \in (a, b) \quad f(x) \leq f(x_0) + f'(x_0)(x - x_0)$$

- 3)  $f$  is strictly convex on  $(a, b) \iff \forall x_0 \in (a, b), \forall x \in (a, b),$

$$x \neq x_0 \implies f(x) > f(x_0) + f'(x_0)(x - x_0)$$

- 4)  $f$  is strictly concave on  $(a, b) \iff \forall x_0 \in (a, b), \forall x \in (a, b),$

$$x \neq x_0 \implies f(x) < f(x_0) + f'(x_0)(x - x_0)$$

## 11. Inflection points

DEFINITION. Assume  $f : (a, b) \rightarrow \mathbb{R}$ . The point  $c \in (a, b)$  is called the *inflection point* of the function  $f$  if  $\exists \delta > 0$ , such that on the intervals  $(c - \delta, c)$  and  $(c, c + \delta)$  the function  $f$  changes convexity (was convex, became concave, or vice versa).

## 12. Necessary condition for an inflection point

THEOREM. Let  $f : (a, b) \rightarrow \mathbb{R}$  be twice differentiable on  $(a, b)$  and let  $c \in (a, b)$  be the inflection point of  $f$ . Then

$$f''(c) = 0.$$

## 1.12 Primitive function and indefinite integral

### 1. Primitive function (antiderivative)

DEFINITION.  $F : (a, b) \rightarrow \mathbb{R}$  is called a *primitive function* (or an *antiderivative*) of the function  $f : (a, b) \rightarrow \mathbb{R}$  on the interval  $(a, b)$  if  $F$  is differentiable on  $(a, b)$  and

$$F'(x) = f(x), \quad \forall x \in (a, b).$$

### 2. Which functions possess antiderivatives?

This is a non-trivial question. In the section “Riemann integral” we will show that at least any *continuous* function possesses antiderivatives on the interval of its continuity.

Discussing antiderivatives below we will always assume that all functions involved in the formulas possess the corresponding antiderivatives by assumption and we will focus only on the derivation of relations between these antiderivatives.

### 3. Properties of primitives

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  has a primitive on  $(a, b)$ . Note that in this case  $f$  has infinitely many other primitives on  $(a, b)$ , but any two primitives of  $f$  can differ only by a constant:

- 1) if  $F(x)$  is a primitive of  $f(x)$  on  $(a, b)$  then for any  $c \in \mathbb{R}$  the function  $F(x) + c$  is also a primitive of  $f(x)$  on  $(a, b)$
- 2) if  $F_1$  and  $F_2$  are two primitives of  $f$  on  $(a, b)$  then there exists  $c \in \mathbb{R}$  such that

$$\forall x \in (a, b) \quad F_1(x) - F_2(x) = c$$

### 4. Indefinite integral

DEFINITION. The set  $\{F(x) + C\}$  of all primitives of the function  $f : (a, b) \rightarrow \mathbb{R}$ , defined on the interval  $(a, b)$ , is called the indefinite integral of  $f$  on  $(a, b)$  and is denoted by

$$\int f(x) dx$$

### 5. Properties of the indefinite integral

THEOREM.

$$\int (\alpha f(x) + \beta g(x)) dx = \alpha \int f(x) dx + \beta \int g(x) dx, \quad \forall \alpha, \beta \in \mathbb{R}$$

$$\frac{d}{dx} \int f(x) dx = f(x), \quad \int g'(x) dx = g(x) + C$$

## 6. List of simplest antiderivatives

1.  $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
2.  $\int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3.  $\int \frac{dx}{x} = \ln|x| + C$
4.  $\int a^x dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
5.  $\int \sin x dx = -\cos x + C, \quad \int \cos x dx = \sin x + C$
6.  $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C, \quad \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$
7.  $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C \quad (a \neq 0)$
8.  $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C \quad (a > 0)$

## 7. Antiderivatives computed by standard methods (see the next section)

1.  $\int \operatorname{tg} x dx = -\ln|\cos x| + C, \quad \int \operatorname{ctg} x dx = \ln|\sin x| + C$
2.  $\int \frac{x dx}{a^2 \pm x^2} = \pm \frac{1}{2} \ln|a^2 \pm x^2| + C$
3.  $\int \frac{x dx}{\sqrt{a^2 \pm x^2}} = \pm \sqrt{a^2 \pm x^2} + C \quad (a > 0)$
4.  $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C \quad (a > 0)$
5.  $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C \quad (a \neq 0)$
6.  $\int \ln x dx = x(\ln x - 1) + C$
7.  $\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$
8.  $\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} + C, \quad \int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$
9.  $\int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C$

## 8. Non-elementary integrals (can not be computed in terms of elementary functions)

1.  $\operatorname{Ei}(x) = \int \frac{e^x}{x} dx$  (exponential integral)
2.  $\operatorname{Si}(x) = \int \frac{\sin x}{x} dx, \quad \operatorname{Ci}(x) = \int \frac{\cos x}{x} dx$  (sine and cosine integrals)
3.  $\operatorname{S}(x) = \int \sin x^2 dx, \quad \operatorname{C}(x) = \int \cos x^2 dx$  (Fresnel integral)
4.  $\Phi(x) = \int e^{-x^2} dx$  (Gaussian integral)
5.  $\operatorname{li}(x) = \int \frac{dx}{\ln x}$  (logarithmic integral)



## 1.13 Basic methods for finding primitives

### 1. Notation for the differential of a function

$$df(x) = f'(x) dx$$

### 2. Change of variables

THEOREM. If  $f : (a, b) \rightarrow \mathbb{R}$  and  $\varphi : (\alpha, \beta) \rightarrow (a, b)$  is differentiable on  $(\alpha, \beta)$  then

$$\int \underbrace{f(\varphi(x))}_y \underbrace{\varphi'(x) dx}_{dy} = \left[ \begin{array}{l} y = \varphi(x) \\ dy = \varphi'(x) dx \end{array} \right] = \int f(y) dy \Big|_{y=\varphi(x)}$$

ABBREVIATED NOTATION.

$$\int f(\varphi(x)) d\varphi(x) = \int f(y) dy \Big|_{y=\varphi(x)}$$

EXAMPLES.

- 1)  $\int \frac{dx}{x+a} = \ln|x+a| + C$
- 2)  $\int \cos(ax) dx = \frac{1}{a} \sin(ax) + C, \quad a \neq 0$
- 3)  $\int \operatorname{tg} x dx = -\ln|\cos x| + C, \quad \int \operatorname{ctg} x dx = \ln|\sin x| + C$
- 4)  $\int \frac{x}{x^2+a} dx = \frac{1}{2} \ln|x^2+a| + C, \quad a \in \mathbb{R}$
- 5)  $\int \frac{x}{\sqrt{x^2+a}} dx = \sqrt{x^2+a} + C, \quad a \in \mathbb{R}$
- 6)  $\int \frac{dx}{\sqrt{x^2+a}} = \ln|x + \sqrt{x^2+a}| + C, \quad a \in \mathbb{R}$

PROOFS.

1.

$$\begin{aligned} \int \frac{dx}{x+a} &= \int \frac{d(x+a)}{x+a} = \left[ \begin{array}{l} y = x+a \\ dy = dx \end{array} \right] = \int \frac{dy}{y} = \ln|y| + C = \\ &= \left[ \begin{array}{l} y = x+a \end{array} \right] = \ln|x+a| + C \end{aligned}$$

2.

$$\begin{aligned} \int \cos(ax) dx &= \frac{1}{a} \int \cos(ax) d(ax) = \left[ \begin{array}{l} y = ax \\ dy = a dx \end{array} \right] = \frac{1}{a} \int \cos y dy = \frac{1}{a} \sin y + C = \\ &= \left[ \begin{array}{l} y = ax \end{array} \right] = \frac{1}{a} \sin(ax) + C \end{aligned}$$

3.

$$\begin{aligned}\int \operatorname{tg} x \, dx &= \int \frac{\sin x}{\cos x} \, dx = - \int \frac{d(\cos x)}{\cos x} \, dx = \left[ \begin{array}{l} y = \cos x \\ dy = -\sin x \, dx \end{array} \right] = \\ &= - \int \frac{dy}{y} = -\ln |y| + C = \left[ y = \cos x \right] = -\ln |\cos x| + C\end{aligned}$$

4.

$$\begin{aligned}\int \frac{x}{x^2 + a} \, dx &= \frac{1}{2} \int \frac{d(x^2 + a)}{x^2 + a} = \left[ \begin{array}{l} y = x^2 + a \\ dy = 2x \, dx \end{array} \right] = \frac{1}{2} \int \frac{dy}{y} = \frac{1}{2} \ln |y| + C = \\ &= \left[ y = x^2 + a \right] = \frac{1}{2} \ln |x^2 + a| + C\end{aligned}$$

5.

$$\begin{aligned}\int \frac{x}{\sqrt{x^2 + a}} \, dx &= \frac{1}{2} \int \frac{d(x^2 + a)}{\sqrt{x^2 + a}} = \left[ \begin{array}{l} y = x^2 + a \\ dy = 2x \, dx \end{array} \right] = \frac{1}{2} \int \frac{dy}{\sqrt{y}} = \sqrt{y} + C = \\ &= \left[ y = x^2 + a \right] = \sqrt{x^2 + a} + C\end{aligned}$$

6.

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + a}} &= \int \frac{1}{x + \sqrt{x^2 + a}} \frac{x + \sqrt{x^2 + a}}{\sqrt{x^2 + a}} \, dx = \int \frac{1}{x + \sqrt{x^2 + a}} \left( 1 + \frac{x}{\sqrt{x^2 + a}} \right) \, dx = \\ &= \left[ \begin{array}{l} y = x + \sqrt{x^2 + a} \\ dy = \left( 1 + \frac{x}{\sqrt{x^2 + a}} \right) \, dx \end{array} \right] = \int \frac{dy}{y} = \ln |y| + C = \\ &= \left[ y = x + \sqrt{x^2 + a} \right] = \ln(x + \sqrt{x^2 + a}) + C\end{aligned}$$

### 3. Decomposition of an integrand into a sum of elementary functions

THEOREM. If  $f = f_1 + f_2$  on  $(a, b)$  then

$$\int f(x) \, dx = \int f_1(x) \, dx + \int f_2(x) \, dx$$

EXAMPLES.

$$1) \int \frac{x+a}{x+b} \, dx = x + (a-b) \ln |x+b| + C$$

$$2) \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C, \quad a \neq 0$$

PROOFS.

1.

$$\frac{x+a}{x+b} = \frac{x+b-b+a}{x+b} = 1 + \frac{a-b}{x+b}$$

$$\int \frac{x+a}{x+b} dx = \int dx + (a-b) \int \frac{dx}{x+b} = x + (a-b) \ln|x+b| + C$$

2.

$$\begin{aligned} \frac{1}{a^2-x^2} &= \frac{1}{2a} \cdot \frac{(a-x) + (a+x)}{a^2-x^2} = \frac{1}{2a} \left( \frac{1}{a+x} + \frac{1}{a-x} \right) \\ \int \frac{dx}{a^2-x^2} &= \frac{1}{2a} \int \left( \frac{1}{a+x} + \frac{1}{a-x} \right) dx = \frac{1}{2a} (\ln|a+x| - \ln|a-x|) + C = \\ &= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C \end{aligned}$$

#### 4. Integrations by parts

THEOREM.

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

ABBREVIATED NOTATION.

$$\int f dg = fg - \int g df$$

EXAMPLES.

- 1)  $\int \ln x dx = x(\ln x - 1) + C$
- 2)  $\int x \cos x dx = x \sin x + \cos x + C$
- 3)  $\int x e^x dx = x e^x - e^x + C$
- 4)  $\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$
- 5)  $\int e^x \sin x dx = \frac{1}{2}(\sin x - \cos x) e^x + C, \quad \int e^x \cos x dx = \frac{1}{2}(\sin x + \cos x) e^x + C,$

PROOFS.

1.

$$\int \ln x dx = [\text{by parts}] = x \ln x - \int x d(\ln x) = x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C$$

2.

$$\int x \cos x dx = \int x d(\sin x) = [\text{by parts}] = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

3.

$$\int x e^x dx = \int x d(e^x) = [\text{by parts}] = x e^x - \int e^x dx = x e^x - e^x + C$$

4.

$$\begin{aligned}\int x \ln x \, dx &= \int \ln x \, d\left(\frac{x^2}{2}\right) = [\text{by parts}] = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \underbrace{d(\ln x)}_{\frac{1}{x} dx} = \\ &= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C\end{aligned}$$

5.

$$\begin{aligned}\int e^x \sin x \, dx &= \int \sin x \, d(e^x) = [\text{by parts}] = \sin x \, e^x - \int e^x \underbrace{d(\sin x)}_{\cos x \, dx} = \\ &= \sin x \, e^x - \int e^x \cos x \, dx = \sin x \, e^x - \int \cos x \, d(e^x) = [\text{again by parts}] = \\ &= \sin x \, e^x - \cos x \, e^x + \int e^x \underbrace{d(\cos x)}_{-\sin x \, dx} = \sin x \, e^x - \cos x \, e^x - \int e^x \sin x \, dx\end{aligned}$$

So, we obtain

$$\int e^x \sin x \, dx = \frac{\sin x - \cos x}{2} + C$$

## 5. Method of substitution

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  and  $\varphi : (\alpha, \beta) \rightarrow (a, b)$  is differentiable and invertible on  $(\alpha, \beta)$ . Then

$$\int f(x) \, dx = \left[ \begin{array}{l} x = \varphi(t) \\ dx = \varphi'(t) \, dt \end{array} \right] = \int f(\varphi(t)) \varphi'(t) \, dt \Big|_{t=\varphi^{-1}(x)}$$

EXAMPLES.

$$1) \int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$$

PROOFS.

1.

$$\begin{aligned}\int \frac{dx}{(1-x^2)^{3/2}} &= \left[ \begin{array}{l} x = \sin t \\ dx = \cos t \, dt \\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{array} \right] = \int \frac{\cos t}{(1-\sin^2 t)^{3/2}} \, dt = \int \frac{dt}{\cos^2 t} = \\ &= \operatorname{tg} t + C = [t = \arcsin x] = \operatorname{tg}(\arcsin x) + C = \\ &= \left[ \begin{array}{l} \operatorname{tg} t = \frac{\sin t}{\sqrt{1-\sin^2 t}} \\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{array} \right] = \frac{x}{\sqrt{1-x^2}} + C\end{aligned}$$

## 6. Using trigonometrical formulas

Power-reduction formulas:

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Double angle formulas:

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

EXAMPLE.

$$\begin{aligned} 1) \int \sin^2 x \, dx &= \frac{x}{2} - \frac{\sin 2x}{4} + C, & \int \cos^2 x \, dx &= \frac{x}{2} + \frac{\sin 2x}{4} + C \\ 2) \int \frac{dx}{\sin x} &= \ln \left| \operatorname{tg} \frac{x}{2} \right| + C, & \int \frac{dx}{\cos x} &= -\ln \left| \operatorname{tg} \left( \frac{\pi}{4} - \frac{x}{2} \right) \right| + C \end{aligned}$$

PROOFS.

1.

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx = \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x \, dx = \frac{x}{2} - \frac{1}{4} \sin 2x + C$$

2.

$$\begin{aligned} \int \frac{dx}{\sin x} &= \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \frac{1}{2} \int \frac{dx}{\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \cdot \cos^2 \frac{x}{2}} = \frac{1}{2} \int \frac{dx}{\operatorname{tg} \frac{x}{2} \cdot \cos^2 \frac{x}{2}} = \\ &= \left[ \begin{array}{l} y = \operatorname{tg} \frac{x}{2} \\ dy = \frac{dx}{2 \cos^2 \frac{x}{2}} \end{array} \right] = \int \frac{dy}{y} = \ln |y| + C = \left[ y = \operatorname{tg} \frac{x}{2} \right] = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C \end{aligned}$$

$$\cos x = \sin \left( \frac{\pi}{2} - x \right)$$

$$\begin{aligned} \int \frac{dx}{\cos x} &= \int \frac{dx}{\sin \left( \frac{\pi}{2} - x \right)} = \left[ \begin{array}{l} y = \frac{\pi}{2} - x \\ dy = -dx \end{array} \right] = - \int \frac{dy}{\sin y} = -\ln \left| \operatorname{tg} \frac{y}{2} \right| + C = \\ &= \left[ y = \frac{\pi}{2} - x \right] = -\ln \left| \operatorname{tg} \left( \frac{\pi}{4} - \frac{x}{2} \right) \right| + C \end{aligned}$$

## 1.14 Definite integral

### 1. A partition of an interval

A *partition* of an interval  $[a, b]$  is a finite sequence of numbers  $T = \{x_i\}_{i=0}^N$  such that

$$a = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = b$$

The *mesh*  $\lambda(T)$  of a partition  $T$  is the length of the longest interval in this partition:

$$\lambda(T) := \max_{i=1, \dots, N} \Delta x_i, \quad \Delta x_i := x_i - x_{i-1}$$

Let  $T = \{x_i\}_{i=0}^N$  be a partition of  $[a, b]$ . Choose a finite set of points  $\xi = \{\xi_i\}_{i=1}^N$  so that

$$\xi_i \in [x_{i-1}, x_i], \quad i = 1, \dots, N$$

The pair  $(T, \xi)$  is called a *partition with distinguished points* or a *tagged partition* of  $[a, b]$ .

### 2. Riemann sum

DEFINITION. Assume  $f : [a, b] \rightarrow \mathbb{R}$  and let  $(T, \xi)$  be a tagged partition of  $[a, b]$ , where

$$T = \{x_i\}_{i=0}^N, \quad \xi = \{\xi_i\}_{i=1}^N, \quad \xi_i \in [x_{i-1}, x_i].$$

The *Riemann sum* of the function  $f$  corresponding to the tagged partition  $(T, \xi)$  is the sum

$$\sigma(T; \xi) := \sum_{i=1}^N f(\xi_i) \Delta x_i, \quad \Delta x_i := x_i - x_{i-1}$$

### 3. Riemann integral

DEFINITION. A function  $f : [a, b] \rightarrow \mathbb{R}$  is called *Riemann-integrable* on  $[a, b]$  if there exists a finite limit

$$\exists I \in \mathbb{R} : \quad I = \lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^N f(\xi_i) \Delta x_i,$$

as the mesh  $\lambda(T)$  of the partition tends to zero, and this limit does not depend neither on the method of partitioning the interval  $[a, b]$  by points  $T = \{x_i\}_{i=0}^N$ , nor on the choice of points  $\xi_i$  in the intervals  $[x_{i-1}, x_i]$ . If such limit exists, then the number  $I$  is called the *definite integral* (or the *Riemann integral*) of the function  $f$  over the interval  $[a, b]$  and is denoted by

$$\int_a^b f(x) dx := I$$

#### 4. Refinement of the definition of the Riemann integral

The definition of the integral above is an informal one: we know only the definition of a function depending on some variable. But the Riemann sum is not a function of a *mesh* of the partition, since it also depends on the choice of points  $x_j$  and  $\xi_j$ . The rigorous definition of the integral can be done only using  $\varepsilon$ - $\delta$  formalism: the function  $f$  is called *integrable* on  $[a, b]$  if there exists a number  $I \in \mathbb{R}$  such that for any  $\varepsilon > 0$  there exists  $\delta(\varepsilon) > 0$  such that for any partition  $(T, \xi)$  of the interval  $[a, b]$  with  $T = \{x_i\}_{i=0}^N$  and  $\xi = \{\xi_i\}_{i=1}^N$ ,  $\xi_i \in [x_{i-1}, x_i]$ , the following implication holds:

$$\lambda(T) < \delta \quad \implies \quad \left| I - \sum_{i=1}^N f(\xi_i) \Delta x_i \right| < \varepsilon$$

#### 5. The geometric meaning of the definite integral

- For a non-negative function  $f$ , the Riemann sum is the sum of the areas of rectangles with bases  $[x_{i-1}, x_i]$  and heights  $f(\xi_i)$ .
- For a non-negative function  $f$ , the definite integral is equal to the area of the figure bounded by the graph of the function  $f$ , the axis  $OX$ , and the parallel lines  $x = a$ ,  $x = b$ .
- For an sign-indefinite function  $f$ , the definite integral is equal to the algebraic sum of the areas of the figures formed by the parts of the graph  $f$  lying above the axis  $OX$  (taken with a “plus” sign) and the ones lying below the axis  $OX$  (taken with a “minus” sign).

#### 6. Example of a non-integrable function

In the next section we will show that any continuous or piecewise-continuous function is integrable. A typical example of a non-integrable function is the following function (which is discontinuous at every point):

$$f(x) = \begin{cases} 1, & x \in [a, b] \cap \mathbb{Q} \\ 0, & x \in [a, b] \setminus \mathbb{Q} \end{cases}$$

#### 7. Integral with “upside down” limits

$$a < b \quad \implies \quad \int_b^a f(x) \, dx \stackrel{\text{def}}{=} - \int_a^b f(x) \, dx$$

#### 8. Integrable functions are bounded

THEOREM.  $f$  is integrable on  $[a, b] \implies f$  is bounded on  $[a, b]$

## 9. Darboux sums

DEFINITION. Assume  $f : [a, b] \rightarrow \mathbb{R}$  is bounded on  $[a, b]$  and  $T = \{x_j\}_{j=1}^N$  is a partition of the interval  $[a, b]$ , i.e.

$$a = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = b$$

Denote

$$m_j = \inf_{x \in [x_{j-1}, x_j]} f(x), \quad M_j = \sup_{x \in [x_{j-1}, x_j]} f(x), \quad j = 1, \dots, N.$$

The *lower* and the *upper Darboux sums* corresponding to the partition  $T$  are

$$s(T) := \sum_{j=1}^N m_j \Delta x_j, \quad S(T) := \sum_{j=1}^N M_j \Delta x_j, \quad \Delta x_j := x_j - x_{j-1}.$$

## 10. Refinement and common refinement of partitions

DEFINITION. A partition  $\tilde{T} = \{y_j\}_{j=1}^N$  is the *refinement* of a partition  $T = \{x_i\}_{i=1}^m$  if all points included in  $T$  are also included in  $\tilde{T}$ , i.e.  $x_i = y_{n_i}$  for some  $n_1 < n_2 < \dots < n_m$ .

The partition  $T_1 \cdot T_2$  is the *common refinement* of partitions  $T_1$  and  $T_2$  if it is formed by all points included in  $T_1$  or  $T_2$ .

The common refinement of two partitions is a refinement of each of the original partitions.

## 11. Properties of the Darboux sums

- 1) for any tagged partition the Riemann sum  $\sigma(T, \xi)$  is in between the corresponding Darboux sums  $s(T)$  and  $S(T)$ :  $\forall T = \{x_j\}_{j=1}^N, \forall \xi = \{\xi_j\}_{j=1}^N \quad s(T) \leq \sigma(T, \xi) \leq S(T)$
- 2) if  $T_1$  is a refinement of  $T$  then  $s(T_1) \geq s(T), \quad S(T_1) \leq S(T)$
- 3)  $\forall$  partitions  $T_1$  and  $T_2 \implies s(T_1) \leq S(T_2)$
- 4) for any fixed partition  $T = \{x_j\}_{j=1}^N$  the infimum and supremum of all possible values of Riemanns' sums  $\sigma(T; \xi)$  are equal respectively to the lower and upper Darboux sums  $s(T)$  and  $S(T)$

$$s(T) = \inf_{\xi} \sigma(T; \xi), \quad S(T) = \sup_{\xi} \sigma(T; \xi)$$

where the infimum is taken over all sets of points  $\xi = \{\xi_j\}_{j=1}^N$  such that  $\xi_j \in [x_{j-1}, x_j]$ .

## 12. The basic condition for Riemann integrability

THEOREM. Assume  $f$  is bounded on  $[a, b]$ . Then  $f$  is integrable on  $[a, b]$  if and only if  $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$  such that for any partition  $T$  of  $[a, b]$  the following implication holds

$$\lambda(T) < \delta \implies S(T) - s(T) < \varepsilon,$$

where  $\lambda(T)$  is the mesh of the partition  $T$ .



### 13. Oscillation and the Darboux sums

NOTATION. For any partition  $T$  of the interval  $[a, b]$  we have

$$S(T) - s(T) = \sum_{j=1}^N \omega_j(f) \Delta x_j, \quad \omega_j(f) := M_j - m_j, \quad \Delta x_j := x_j - x_{j-1}$$

The number  $\omega_j(f)$  is called the *oscillation* of  $f$  over the interval  $[x_{j-1}, x_j]$ .

### 14. Continuous functions are integrable

THEOREM. Assume  $f$  is continuous on  $[a, b]$ . Then  $f$  is integrable on  $[a, b]$ .

## 1.15 Properties of definite integrals

### 1. Basic properties

1.  $f, g$  are integrable on  $[a, b]$ ,  $\alpha, \beta \in \mathbb{R} \implies \alpha f + \beta g$  is integrable on  $[a, b]$ ,

$$\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

2.  $a < c < b \implies f$  is integrable on  $[a, b] \Leftrightarrow f$  is integrable on  $[a, c]$  and  $[c, b]$ ,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

3.  $f, g$  are integrable on  $[a, b]$ ,  $f(x) \leq g(x)$ ,  $\forall x \in [a, b] \implies$

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$

4.  $f$  is integrable on  $[a, b] \implies |f|$  is integrable on  $[a, b]$  and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

5.  $f$  is integrable on  $[a, b]$ ,  $m \leq f(x) \leq M$ ,  $\forall x \in [a, b] \implies$

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

6.  $f$  is continuous on  $[a, b] \implies \exists c \in [a, b]$ :

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx \quad (\text{Mean value theorem})$$

7.  $f$  is continuous on  $[a, b]$ ,  $g$  is integrable and non-negative on  $[a, b] \implies \exists c \in [a, b]$ :

$$\int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx$$

## 2. Fundamental theorem of calculus

THEOREM. Assume  $f : [a, b] \rightarrow \mathbb{R}$  is integrable on  $[a, b]$ . Define  $\Phi : [a, b] \rightarrow \mathbb{R}$ ,

$$\Phi(x) := \int_a^x f(t) dt, \quad x \in [a, b] \quad \text{— the integral with variable upper limit.}$$

If  $f$  is continuous on  $[a, b]$  then  $\Phi$  is differentiable on  $[a, b]$  and

$$\forall x_0 \in (a, b) \quad \Phi'(x_0) = f(x_0).$$

COROLLARY.  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b] \implies f$  has a primitive on  $(a, b)$ .

## 3. Newton–Leibniz formula

THEOREM. Let  $f$  be continuous on  $[a, b]$  and let  $F$  be any primitive of  $f$  on  $(a, b)$ . Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

## 4. Integration by parts in a definite integral

THEOREM. Let  $f, g$  be differentiable on  $[a, b]$  and  $f', g'$  be continuous on  $[a, b]$ . Then

$$\int_a^b f(x)g'(x) dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x) dx.$$

ABBREVIATED NOTATION.

$$\int_a^b fg' dx = fg \Big|_a^b - \int_a^b f'g dx$$

## 5. Change of variables in the Riemann integral

THEOREM. Assume  $\varphi : [\alpha, \beta] \rightarrow [a, b]$  is differentiable bijection of  $[\alpha, \beta]$  onto  $[a, b]$ ,  $\varphi'$  is continuous and strictly positive on  $[\alpha, \beta]$ . Assume  $f$  is continuous on  $[a, b]$ . Then the function  $(f \circ \varphi) \cdot \varphi'$  is continuous on  $[\alpha, \beta]$  and

$$\int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t) dt = \left[ \begin{array}{c} x = \varphi(t) \\ dx = \varphi'(t) dt \end{array} \right] = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x) dx$$

## 1.16 Applications of definite integrals

### 1. Area of a plane figure

DEFINITION. A subset of the plane  $F$  is called *Jordan measurable* if for any  $\varepsilon > 0$  there exist two polygons  $F_1$  and  $F_2$  such that

$$F_1 \subset F \subset F_2 \quad \text{and} \quad S(F_2 \setminus F_1) < \varepsilon$$

(here for every polygon  $P$  on the plane we denote by  $S(P)$  its area). For the Jordan measurable figure  $F$  its area is defined

$$S(F) = \sup_{F_1 \subset F} S(F_1) = \inf_{F \subset F_2} S(F_2)$$

where the supremum is taken over all polygons  $F_1$  contained in  $F$ , and the infimum is taken over all polygons  $F_2$  containing  $F$ . (These supremum and infimum are equal, since the figure is Jordan measurable).

### 2. Area of a curved trapezoid

THEOREM.

1)  $f : [a, b] \rightarrow \mathbb{R}$ ,  $f(x) \geq 0$ ,  $\forall x \in [a, b]$ , and assume  $\mathcal{F} \subset \mathbb{R}^2$  is given by

$$\mathcal{F} = \{ (x, y) \in \mathbb{R}^2 : x \in [a, b], 0 \leq y \leq f(x) \}$$

Then

$$S_{\mathcal{F}} = \int_a^b f(x) dx$$

2)  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $f(x) \geq g(x)$ ,  $\forall x \in [a, b]$ , and assume  $\mathcal{F} \subset \mathbb{R}^2$  is given by

$$\mathcal{F} = \{ (x, y) \in \mathbb{R}^2 : x \in [a, b], g(x) \leq y \leq f(x) \}$$

Then

$$S_{\mathcal{F}} = \int_a^b (f(x) - g(x)) dx$$

### 3. Polar coordinate on the plane

Converting from the polar coordinates  $r$  and  $\varphi$  to the Cartesian coordinates  $x$  and  $y$ :

$$\begin{cases} x = r \cos \varphi & r \in [0, +\infty) \\ y = r \sin \varphi & \varphi \in [0, 2\pi) \end{cases}$$

Converting from the Cartesian coordinates  $x$  and  $y$  to the polar coordinates  $r$  and  $\varphi$ :

$$\begin{cases} r = \sqrt{x^2 + y^2} & \varphi \in [0, \pi], \quad \text{if } y \geq 0 \\ \operatorname{tg} \varphi = \frac{y}{x} & \varphi \in (\pi, 2\pi), \quad \text{if } y < 0 \end{cases}$$

#### 4. Area of a curved sector

THEOREM.  $f : [\alpha, \beta] \rightarrow [0, +\infty)$ ,  $0 \leq \alpha < \beta \leq 2\pi$ , and assume  $\mathcal{F} \subset \mathbb{R}^2$  is given by

$$\mathcal{F} := \{ (r, \varphi) : \varphi \in [\alpha, \beta], 0 \leq r \leq f(\varphi) \}$$

Then

$$S_{\mathcal{F}} = \frac{1}{2} \int_{\alpha}^{\beta} (f(\varphi))^2 d\varphi$$

#### 5. Curves on the plane

DEFINITION. Let  $\gamma : [a, b] \rightarrow \mathbb{R}^2$ ,  $\gamma(t) = (\gamma_1(t), \gamma_2(t))$  be continuous on  $[a, b]$

- The mapping  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  is called a *parameterized plane curve*.
- $\gamma$  is *regular* if  $\gamma$  is differentiable on  $[a, b]$ ,  $\gamma'$  is continuous on  $[a, b]$  and

$$|\gamma'(t)| := \sqrt{(\gamma'_1(t))^2 + (\gamma'_2(t))^2} \neq 0, \quad \forall t \in [a, b].$$

- If  $T = \{t_j\}_{j=0}^N$  is some partition of  $[a, b]$ , i.e.  $a = t_0 < t_1 < t_2 < \dots < t_N = b$ , then a broken line consisting of segments  $[\gamma(t_{j-1}), \gamma(t_j)]$  is said to be *inscribed* in the curve  $\gamma$ . The *length* of a broken line corresponding to the partition  $T$  is

$$\sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})|$$

where

$$|\gamma(t_j) - \gamma(t_{j-1})| := \sqrt{(\gamma_1(t_j) - \gamma_1(t_{j-1}))^2 + (\gamma_2(t_j) - \gamma_2(t_{j-1}))^2}$$

- A curve  $\gamma$  is called *rectifiable* if there is a limit on the lengths of all broken lines inscribed in this curve as the mesh of the partition tends to zero. In this case, the *length* is

$$l_{\gamma} = \lim_{\lambda(T) \rightarrow 0} \sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})|$$

- More precisely, a curve is called rectifiable if there is a number  $l_{\gamma} \in \mathbb{R}$  such that for any  $\varepsilon > 0$  there is  $\delta > 0$  such that for any partition  $T = \{t_j\}_{j=1}^N$  of the segment  $[a, b]$  the implication is true

$$\lambda(T) < \delta \quad \implies \quad \left| l_{\gamma} - \sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})| \right| < \varepsilon$$

Here  $\lambda(T) = \max_{j=1, \dots, N} (t_j - t_{j-1})$  is the mesh of the partition  $T$ .

## 6. Length of a curve

THEOREM.

### 1) Length of a parameterized curve

Any regular parameterized curve  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  is rectifiable and its length is

$$l_\gamma = \int_a^b |\gamma'(t)| dt := \int_a^b \sqrt{(\gamma'_1(t))^2 + (\gamma'_2(t))^2} dt$$

where  $\gamma(t) = (\gamma_1(t), \gamma_2(t))$  and  $\gamma'(t) = (\gamma'_1(t), \gamma'_2(t))$ .

### 2) Length of the curve that is the graph of the function

$f : [a, b] \rightarrow \mathbb{R}$  is differentiable on  $[a, b]$  and  $f'$  is continuous on  $[a, b]$ ,

$$\Gamma := \{ (x, y) \in \mathbb{R}^2 : x \in [a, b], y = f(x) \}$$

Then

$$l_\Gamma = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

### 3) Length of the curve, specified in polar coordinates

Let  $(r, \varphi)$  be polar coordinates on the plane  $\mathbb{R}^2$ , let  $f : [\alpha, \beta] \rightarrow [0, +\infty)$  be continuously differentiable on  $[\alpha, \beta]$ ,  $0 \leq \alpha < \beta \leq 2\pi$ , and

$$\Gamma := \{ (r, \varphi) : \varphi \in [\alpha, \beta], r = f(\varphi) \}$$

Then

$$l_\Gamma = \int_\alpha^\beta \sqrt{(f(\varphi))^2 + (f'(\varphi))^2} d\varphi$$

## 7. Generalized Cavalieri's principle (1635)

For any solid of revolution (such as a barrel)

$$\text{Volume of the barrel} \approx \sum_j \left( \text{Area of a crossection of a barrel} \right)_j \cdot \left( \text{Vertical height} \right)_j$$

$$\text{Area of a side surface} \approx \sum_j \left( \text{Length of a crossection of a barrel} \right)_j \cdot \left( \text{Vertical \underline{length}} \right)_j$$

## 8. The volume of a solid of revolution

THEOREM. Assume  $f : [a, b] \rightarrow \mathbb{R}$  is non-negative and continuous on  $[a, b]$  and

$$\Omega := \left\{ (x, y, z) \in \mathbb{R}^3 : x \in [a, b], y^2 + z^2 \leq f^2(x) \right\}$$

Then the volume  $V(\Omega)$  of a solid of revolution  $\Omega$  is equal to

$$V(\Omega) = \pi \int_a^b f^2(x) dx$$

## 9. The area of the side surface of a circular cone

THEOREM. The side surface of a circular cone whose base radius is  $R$  and height  $H$ :

$$S_{R,H} = \pi R \sqrt{R^2 + H^2}$$

For a truncated cone:

$$S_{R,H} - S_{r,h} = \pi \sqrt{1 + \operatorname{tg}^2 \alpha} \cdot (RH - rh) = 2\pi \sqrt{1 + \operatorname{tg}^2 \alpha} \cdot \frac{R+r}{2} \cdot (H-h),$$

where  $\operatorname{tg} \alpha = \frac{r}{h} = \frac{R}{H}$ .

## 10. The area of the side surface of a solid of revolution

THEOREM. Assume  $f : [a, b] \rightarrow \mathbb{R}$  is non-negative and differentiable on  $(a, b)$ ,  $f'$  is continuous on  $[a, b]$  and

$$\Sigma := \left\{ (x, y, z) \in \mathbb{R}^3 : x \in [a, b], y^2 + z^2 = f^2(x) \right\}$$

Then the area of the side surface  $S(\Sigma)$  the corresponding solid of revolution is equal to

$$S(\Sigma) = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx$$

## 1.17 Comparison of infinitesimals and Taylor's formula

### 1. L'Hôpital's (actually Bernoulli's) rule for an indeterminate form $\frac{0}{0}$

THEOREM. Let  $f, g : (a, b) \rightarrow \mathbb{R}$  be differentiable on  $(a, b)$  and  $g'(x) \neq 0, \forall x \in (a, b)$ . Assume

$$1) \quad \exists c \in (a, b): \quad f(c) = g(c) = 0$$

$$2) \quad \exists k \in \mathbb{R}: \quad \exists \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = k$$

Then there exists the limit

$$\exists \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = k.$$

### 2. Other versions of the L'Hôpital rule

There are also versions of the L'Hôpital rule which suit to the following cases:

- Limit at infinity:  $f, g : [a, +\infty) \rightarrow \mathbb{R}, \quad \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} g(x) = 0,$

$$\exists k \in \mathbb{R} : \quad \exists \lim_{x \rightarrow +\infty} \frac{f'(x)}{g'(x)} = k \quad \implies \quad \exists \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = k$$

- Indeterminate form  $\frac{\infty}{\infty}$ :  $f, g : (a, b) \rightarrow \mathbb{R}, \quad c \in (a, b), \quad \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow c} g(x) = +\infty,$

$$\exists k \in \mathbb{R} : \quad \exists \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = k \quad \implies \quad \exists \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = k$$

- Version of the above statements with  $k = +\infty$  or  $k = -\infty$
- Version of the above statements with  $x \rightarrow -\infty$  instead of  $x \rightarrow +\infty$

All these statements can be proved similarly to the proof of the first statement and we omit their proofs in our course. Informally we can summarize the L'Hôpital rules as follows:

The limit of a ratio of functions is equal to the limit of the ratio of their derivatives if the latter exists.

### 3. Taylor's formula

THEOREM.  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable  $(n+1)$  times on  $(a, b) \implies \forall x, x_0 \in (a, b)$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1}(x),$$



where

$$R_{n+1}(x) := \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t)(x-t)^n dt \quad \text{--- the integral form of the remainder}$$

If  $f^{(n+1)}$  is continuous on  $[a, b]$  then there exists  $\xi \in [x_0, x]$  such that

$$R_{n+1}(x) := \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1} \quad \text{--- the Lagrange form of the remainder}$$

COROLLARY.  $[a, b]$  is bounded,  $f^{(n+1)}$  is continuous on  $[a, b] \implies R_{n+1}(x) = O(|x-x_0|^{n+1})$

#### 4. $O$ - asymptotic notation

Assume  $f, g : (a, b) \rightarrow \mathbb{R}$  (we allow here  $a$  and  $b$  be infinite),  $x_0 \in (a, b)$ . Then

- $g = O(f)$  as  $x \in (a, b) \iff \exists M > 0: \forall x \in (a, b): |g(x)| \leq M |f(x)|$
- $g = O(f)$  as  $x \rightarrow x_0 \iff \exists M > 0, \delta > 0: \forall x \in (x_0 - \delta, x_0 + \delta): |g(x)| \leq M |f(x)|$
- $g = O(f)$  as  $x \rightarrow +\infty \iff \exists M > 0, R > 0: \forall x \geq R: |g(x)| \leq M |f(x)|$
- $g = O(f)$  as  $x \rightarrow -\infty \iff \exists M > 0, R < 0: \forall x \leq R: |g(x)| \leq M |f(x)|$

#### 5. $o$ - asymptotic notation

Assume  $f, g : (a, b) \rightarrow \mathbb{R}$ ,  $-\infty \leq a \leq x_0 \leq b \leq +\infty$ .

- $g = o(f)$  as  $x \rightarrow x_0 \iff \exists \alpha : (a, b) \rightarrow \mathbb{R}: \lim_{x \rightarrow x_0} \alpha(x) = 0 \text{ and } g(x) = \alpha(x)f(x)$

#### 6. Properties of $O$ - and $o$ - symbols

THEOREM. Assume  $f, g : (a, b) \rightarrow \mathbb{R}$ ,  $-\infty \leq a \leq x_0 \leq b \leq +\infty$ .

1) for  $x \in (a, b)$  we have

- $O(f) + O(f) = O(f)$
- $O(f) \cdot O(g) = O(fg)$

2) for  $x \rightarrow x_0$  we have

- $g = o(f) \implies g = O(f)$
- $o(f) + o(f) = o(f)$
- $o(f) \cdot O(g) = o(fg)$

3) in particular, for  $x \rightarrow 0$  we have

- $O(x^n) \cdot O(x^m) = O(x^{n+m})$
- $o(x^n) \cdot O(x^m) = o(x^{n+m})$
- $O(x^n) = o(x^m)$ , if  $n > m$

## 7. Asymptotics of the reminder in Taylor's expansion

THEOREM. Assume  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable  $(n + 1)$  times on  $(a, b)$  and  $f^{(n+1)}$  is continuous on  $(a, b)$ . Then for any  $x_0 \in (a, b)$

$$R_{n+1}(x) = O((x - x_0)^{n+1}) \quad \text{as } x \rightarrow x_0,$$

where we denote by  $R_{n+1}(x)$  the reminder in the Taylor formula for  $f$ .

## 8. Taylor expansions of some elementary functions

THEOREM. The following asymptotics holds as  $x \rightarrow 0$ :

- 1)  $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$
- 2)  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + o(x^{2n})$
- 3)  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$
- 4)  $(1 + x)^r = 1 + \frac{r}{1!} x + \frac{r(r-1)}{2!} x^2 + \frac{r(r-1)(r-2)}{3!} x^3 + \dots + \frac{r(r-1)\dots(r-n+1)}{n!} x^n + o(x^n)$
- 5)  $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$

## 1.18 Improper integrals

### 1. Improper integral of an unbounded function

DEFINITION. Assume  $f : [a, b) \rightarrow \mathbb{R}$  is unbounded near  $x = b$  and integrable over  $[a, c]$  for any  $c \in [a, b)$ . We say the *improper integral* of  $f$  over  $[a, b)$  is *convergent* if there exists a finite limit

$$\int_a^b f(x) dx = \lim_{c \rightarrow b-0} \int_a^c f(x) dx,$$

Otherwise we say the corresponding improper integral is *divergent*. For  $f : (a, b] \rightarrow \mathbb{R}$  which is unbounded near  $x = a$  the improper integral is defined in a similar way:

$$\int_a^b f(x) dx = \lim_{c \rightarrow a+0} \int_c^b f(x) dx,$$

### 2. Improper integral over an unbounded interval

DEFINITION. Assume  $f : [a, +\infty) \rightarrow \mathbb{R}$  is integrable over  $[a, c]$  for any  $c \in [a, +\infty)$ . We say the *improper integral* of  $f$  over  $[a, +\infty)$  is *convergent* if there exists a finite limit

$$\int_a^{+\infty} f(x) dx = \lim_{c \rightarrow +\infty} \int_a^c f(x) dx,$$

Otherwise we say the corresponding improper integral is *divergent*. For  $f : (-\infty, b] \rightarrow \mathbb{R}$  the improper integral is defined in a similar way:

$$\int_{-\infty}^b f(x) dx = \lim_{c \rightarrow -\infty} \int_c^b f(x) dx,$$

### 3. Cauchy's criterion for convergence of improper integrals

THEOREM.

1) Assume  $F : [a, b) \rightarrow \mathbb{R}$ . Then

$$\exists \lim_{c \rightarrow b-0} F(c) \iff \forall \varepsilon > 0 \exists \delta > 0 : \forall c_1, c_2 \in (b - \delta, b) \quad |F(c_1) - F(c_2)| < \varepsilon.$$

2) In particular, for  $f : [a, b) \rightarrow \mathbb{R}$

$$\int_a^b f(x) dx \text{ is convergent} \iff \forall \varepsilon > 0 \exists \delta > 0 : \forall c_1, c_2 \in (b - \delta, b) \quad \left| \int_{c_1}^{c_2} f(x) dx \right| < \varepsilon.$$

3) Assume  $F : [a, +\infty) \rightarrow \mathbb{R}$ . Then

$$\exists \lim_{c \rightarrow +\infty} F(c) \iff \forall \varepsilon > 0 \exists R > 0 : \forall c_1, c_2 \in [R, +\infty) \quad |F(c_1) - F(c_2)| < \varepsilon.$$

4) In particular, for  $f : [a, b) \rightarrow \mathbb{R}$

$$\int_a^{+\infty} f(x) dx \text{ is convergent} \iff \forall \varepsilon > 0 \exists R > 0 : \forall c_1, c_2 \in [R, +\infty) \quad \left| \int_{c_1}^{c_2} f(x) dx \right| < \varepsilon.$$

#### 4. Absolute convergence of improper integrals

DEFINITION. Assume  $f : [a, b) \rightarrow \mathbb{R}$  is integrable over  $[a, c]$  for any  $c \in [a, b)$  and either the interval  $[a, b)$  is finite and  $f$  is unbounded near  $x = b$  or  $b = +\infty$ . We say the improper integral of  $f$  over the interval  $[a, b)$  *converges absolutely* if the improper integral of  $|f|$  over  $[a, b)$  is convergent:

$$\int_a^b |f(x)| dx < +\infty.$$

#### 5. Absolutely convergent improper integral is convergent

THEOREM. If the improper integral of  $f$  over  $[a, b)$ ,  $b \leq +\infty$ , converges absolutely then it converges:

$$\int_a^b |f(x)| dx \text{ is convergent} \implies \int_a^b f(x) dx \text{ is convergent}.$$

#### 6. Necessary and sufficient condition for absolute convergence of improper integrals

THEOREM. The improper integral of  $f$  over  $[a, b)$ ,  $b \leq +\infty$ , converges absolutely if and only if the primitive of  $|f|$  defined by

$$F(c) := \int_a^c |f(x)| dx, \quad c \in [a, b)$$

is bounded on the interval  $[a, b)$ .

#### 7. Comparison test for absolute convergence of improper integrals

THEOREM. Assume  $f, g : [a, b) \rightarrow \mathbb{R}$  are integrable over  $[a, c]$  for any  $c \in [a, b)$  and either the interval  $[a, b)$  is finite and  $f$  is unbounded near  $x = b$  or  $b = +\infty$ . Assume

$$0 \leq f(x) \leq g(x) \quad \forall x \in [a, b)$$

If the improper integral of  $g$  over  $[a, b)$  is convergent then the improper integral of  $f$  over  $[a, b)$  is also convergent:

$$\begin{aligned} \int_a^b g(x) \, dx \text{ is convergent} &\implies \int_a^b f(x) \, dx \text{ is convergent} \\ \int_a^b f(x) \, dx \text{ is divergent} &\implies \int_a^b g(x) \, dx \text{ is divergent} \end{aligned}$$

## 8. Dirichlet–Abel test for non-absolute convergence of improper integrals

THEOREM. Assume  $f, g : [a, +\infty) \rightarrow \mathbb{R}$  satisfy the conditions

- 1)  $f$  is continuous on  $[a, +\infty)$  and its primitive  $F$  is bounded on  $[a, +\infty)$
- 2)  $g$  is non-negative, non-increasing, differentiable on  $[a, +\infty)$ ,  $g'$  is continuous on  $[a, +\infty)$
- 3)  $\lim_{x \rightarrow +\infty} g(x) = 0$

Then the improper integral of  $fg$  over  $[a, +\infty)$  is convergent:

$$\int_a^{+\infty} f(x)g(x) \, dx \text{ is convergent.}$$

## 9. There are improper integrals that are convergent, but not absolutely convergent

$$\begin{aligned} \int_1^{+\infty} \frac{\sin x}{x} \, dx, \quad \int_1^{+\infty} \frac{\cos x}{x} \, dx &\text{ are convergent non-absolutely,} \\ \int_1^{+\infty} \frac{|\sin x|}{x} \, dx, \quad \int_1^{+\infty} \frac{|\cos x|}{x} \, dx &\text{ are divergent} \end{aligned}$$

## 10. Improper integrals with more than one singularity

- If  $f : (a, b) \rightarrow \mathbb{R}$  is unbounded near both ends of the interval  $(a, b)$  then we split the interval  $(a, b)$  by some point  $c \in (a, b)$  so that  $f$  possess only one singularity on each of the intervals  $(a, c]$  and  $[c, b)$  and define the improper integral of  $f$  over  $(a, b)$  as the sum of improper integrals of  $f$  over  $(a, c]$  and  $[c, b)$ .
- If  $f : (a, +\infty) \rightarrow \mathbb{R}$  is unbounded near  $x = a$  then we split the interval  $(a, +\infty)$  by some point  $c \in (a, +\infty)$  and define the improper integral of  $f$  over  $(a, +\infty)$  as the sum of improper integrals of  $f$  over  $(a, c]$  and  $[c, +\infty)$ .
- If  $f : (a, b) \rightarrow \mathbb{R}$  is unbounded near an internal point  $c \in (a, b)$  then we define the improper integral of  $f$  over  $(a, b)$  as the sum of improper integrals of  $f$  over  $(a, c)$  and  $(c, b)$ .

## 1.19 Infinite series

### 1. Convergent and divergent series

DEFINITION. Assume  $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$  is a sequence. We say

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \text{ is convergent} &\iff \exists S \in \mathbb{R} : S = \lim_{n \rightarrow \infty} \sum_{j=1}^n a_j \\ \sum_{n=1}^{\infty} a_n \text{ is divergent} &\iff \text{there is no finite limit of } \sum_{j=1}^n a_j \text{ as } n \rightarrow \infty \end{aligned}$$

### 2. Cauchy's criterion for convergence

THEOREM. The following conditions are equivalent:

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \iff \forall \varepsilon > 0 \exists N = N(\varepsilon) : \forall n \geq m \geq N(\varepsilon) \sum_{j=m}^n a_j < \varepsilon$$

### 3. A necessary condition for convergence of a series

$$\text{THEOREM. } \sum_{n=1}^{\infty} a_n \text{ is convergent} \implies \lim_{n \rightarrow \infty} a_n = 0$$

### 4. Absolute convergence of series

$$\text{DEFINITION. } \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent} \iff \sum_{n=1}^{\infty} |a_n| \text{ is convergent.}$$

### 5. Absolutely convergent series is convergent

THEOREM. If the series converges absolutely then it converges:

$$\sum_{n=1}^{\infty} |a_n| \text{ is convergent} \implies \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

### 6. Comparison test for series with non-negative terms

THEOREM. Assume  $0 \leq a_n \leq b_n \quad \forall n \in \mathbb{N}$ . Then

$$\begin{aligned} 1) \quad \sum_{n=1}^{\infty} b_n \text{ is convergent} &\implies \sum_{n=1}^{\infty} a_n \text{ is convergent} \\ 2) \quad \sum_{n=1}^{\infty} a_n \text{ is divergent} &\implies \sum_{n=1}^{\infty} b_n \text{ is divergent} \end{aligned}$$

## 7. M-test (majorant test) of convergence

THEOREM. Assume  $\{b_n\}_{n=1}^{\infty}$  is non-negative and  $\{a_n\}_{n=1}^{\infty}$  satisfies

$$|a_n| \leq b_n, \quad \forall n \in \mathbb{N}$$

Then

$$\sum_{n=1}^{\infty} b_n \text{ is convergent} \implies \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent.}$$