12. Lagrange Formasism

The Chapters 6.1–6.4 of my lecture notes provides the background to solve the following exercises. Your solution to the problems 12.1–12.3 should uploaded

to your Moodle account

as a PDF-file

by Wednesday, Dec 17, 10:30 (with a grace time till the start of the seminars).

The parts marked by $\star$ are bonus problems suggestions for further exploration. They should not be submitted and they will not be graded. However, they can be followed up in the seminars.

The self-test problems serve to check you understanding of vectors.

The bonus problem provides the solution to a problem that we introduced in class. It might take some extra effort to solve.

Problems

Problem 1. Evolution of a particle in a Mexican-hat potential

We explore the motion of a particle of mass m in a rotation-symmetric potential

$$\Phi(R) = \frac{m\,A}{4}\;R^2\;(R^2 - 2\,R_0^2)$$

The particle evolves in a plane where its position is specified by the polar coordinates (R, θ).

a) Sketch the potential. Where are its maxima and minima?

b) Determine the Lagrange function for this problem, and determine the equations of motion for $\theta(t)$ and $R(t)$.

Bonus. The angular momentum and the energy of the particle are conserved. How do you see this without calculation based on the Lagrange function?

c) Determine a frequency ω, a length scale ℓ and a constant K, such that the dimensionless length r/R as function of the dimensionless time $\tau = \omega t$ obeys the following equation of motion,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}(\omega t)^2} = r - r^3 + \frac{K}{r^3}$$

where r denotes the dimensionless (scalar) distance

$$\text{with} \;\; r(t) = \frac{r(t)}{\ell} \, .$$

In the following we discuss the dimensionless equations, where we absorb ω into the time scale and drop the hat to avoid clutter in the equations.

d) Show that the effective energy E_{eff} is conserved,

$$E = \frac{\dot{r}^2}{2} + V_{\text{eff}}(r) \qquad \text{with} \qquad V_{\text{eff}}(r) = \frac{r^4}{4} - \frac{r^2}{2} + \frac{K}{2\,r^2} \, .$$

e) Sketch the effective potential $V_{\text{eff}}(r)$ and the phase portrait of the motion for $K > 0$.

Bonus. Why is it necessary to give a separate discussion of $K = 0$?

Problem 2. The driven pendulum

We consider a (mathematical) pendulum of mass m with a pendulum arm of length ℓ. The deflection of the arm with respect to its rest position is denoted as $\theta(t)$. The fulcrum of the pendulum is moving vertically, residing at the position $z(t)$ at time t. Here, $z(t)$ should be considered as a time-dependent parameter, with a time dependence that must be specified upon discussing the solutions of $\theta(t)$.

a) Sketch the setup of for this problem, and mark the relevant physical parameters and coordinates.

b) We assume that the mass of the pendulum arm does not play a significant role for the motion. Determine the kinetic energy T and the potential energy V of the pendulum mass.

c) Use the Lagrange formalism to determine the equation of motion for θ.

Can you imagine why $\ddot{\theta}(t) = 0$ for $\ddot{z} = -g$?

d) Assume now that $\ddot{z}$ =const. Discuss the stability of the fixed points of the motion for $\ddot{z} > -g$ and $\ddot{z} < -g$. Sketch corresponding phase-space plots.

⋆ e) Consider now the T-periodic motion with

$$z(t) = \begin{cases} a\,t\,(t - T/2) & \text{for} \quad 0 < t < T/2 \\ a\,(T - t)\,(t - T/2) & \text{for} \quad T/2 < t < T \end{cases}$$

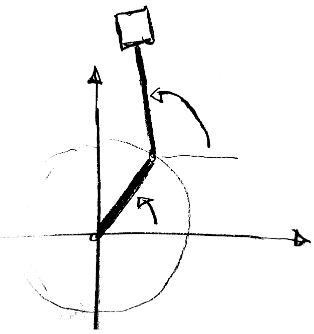
What does this imply for $\ddot{z}$? How would the trajectories close to $\theta = 0$ and $\theta = \pi$ look like for $a > g$?

f) It turns out that for vigorous, high-frequency driving the fixed point where the pendulum stands upside down, will become stable. You might wish to explore this transition numerically.

Problem 3. Horizontal driven double pendulum

A double pendulum is a mathematical pendulum with another pendulum attached to its loose end (see sketch). We will consider here a horizontal double pendulum where the arms run in a horizontal plane perpendicular to gravity. Moreover, we will control the position of the outer pendulum by a motor such that the angle between the x-axis and the outer arms revolves with with a *fixed, constant* frequency Ω.

a) Label the quantities in the sketch:
The pendulum is moving in the (x, y) plane. How is gravity oriented?
The inner arm has length R. Its fixed end lies in the origin of the (x, y)-plane, and it is pointing in direction $\theta(t)$ with respect to the positive x axis.
The outer arm has a length L. It connects the loose end of the inner arm and a weight of mass M. The angle between this arm and the x-axis is Ωt.



b) Consider the vectors $\hat{\mathbf{r}}(\phi) = \hat{\mathbf{x}}\,\cos\phi + \hat{\mathbf{y}}\,\sin\phi$ and $\hat{\phi}(\phi) = \frac{\mathrm{d}}{\mathrm{d}\phi}\hat{\mathbf{r}}(\phi)$. Show that for each value of ϕ they form an orthonormal pair of unit vectors in the (x, y)-plane.

c) Express the position $\mathbf{q}$ of the weight with respect to the origin as a linear combination of $\hat{\boldsymbol{r}}(\theta)$ and $\hat{\boldsymbol{r}}(\Omega t)$; i.e. determine a and b such that

$$\mathbf{q}(t) = a\,\hat{\boldsymbol{r}}(\theta) + b\,\hat{\boldsymbol{r}}(\Omega t)$$

d) Determine $\dot{\boldsymbol{q}}(t)$ and the kinetic energy of the horizontal driven double pendulum.

e) Determine (i) the potential energy,

(ii) the Lagrange function, and

(iii) the equation of motion for $\theta(t)$.

f) Show that – with an appropriate choice of the time scale – the dimensionless equation of motion for $\alpha(t) = \theta(t) - \Omega\, t$ takes the form

$$\ddot{\alpha}(t) = -\sin\alpha(t)$$

Which time scale was chosen to obtain this result?

g) Show that $E = \dfrac{\dot{\alpha}^2}{2} - \cos\alpha$ is a constant of motion.

⋆ h) Determine the stable and unstable fixed points of the equation of motion.

Sketch its solutions in phase space.

⋆ i) The results of f) and g) imply that $\alpha(t)$ behaves like a mathematical pendulum. Interpret its solutions:

- Which motion of the double pendulum is described by the stable fixed points?
- Which track does the double pendulum follow for closed trajectories around a stable fixed point?
- Which motion is described the the unstable fixed points?
- How do trajectories look like where $\dot{\alpha}$ never changes its sign?
- The systems has a homoclinic trajectory. How does the system's homoclinic trajectory look like in terms of $\theta(t)$?

Self Test

Problem 4. Pendulum on rails

We set a mathematical pendulum on a cart that can move without friction horizontally in the direction of motion of the pendulum. The cart has an overall mass M. This mass will also include the pendulum frame which is considered to be a part of the cart. The pendulum comprises of a mass m that moves on an arm of length ℓ. We choose the coordinates such that the fulcrum of the pendulum is located at $(x(t), 0)$, and the mass of the pendulum at $(x + \ell \sin\theta, -\ell\cos\theta)$.

a) Sketch the setup, indicating the parameters and coordinates of the problem.

b) Determine the kinetic energy of the cart, the kinetic energy of the pendulum weight, and the potential energy of the pendulum weight.
Provide the resulting Lagrange function for the pendulum on the cart.

c) Identify the equation of motion for x, and show that it leads to a conserved quantity of the form

$$p_x = \alpha\,\dot{x} + \beta\,\dot{\theta}\,\cos\theta$$

How do α and β depend on the parameter ℓ, m, M, and g?

d) Determine the x-component of the center of mass of the pendulum weight and the cart. Which interpretation does this provide for the result of part (c)?

e) In the following we work in the center of mass frame, where $p_x = 0$. Show that the equation of motion for θ will then take the form

$$\ddot{\theta} = -\sin\theta\,\frac{1 + \mu\,\dot{\theta}^2\,\cos\theta}{1 - \mu\,\cos^2\theta}$$

This form of the equation involves a particular form of the dimensionless time scale! Which time scale has been chosen?
How does μ depend on the masses m and M?

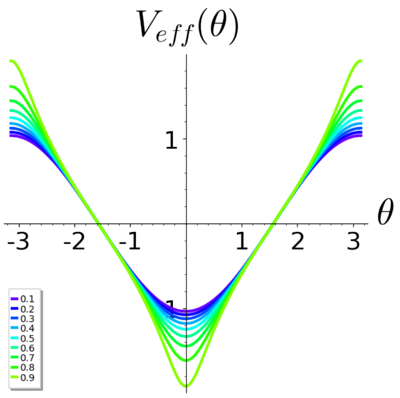
* f) Why is it clear a priori that μ can only depend on m and M and not on the other parameters of the problem, g and ℓ?

* g) In the limit $\mu \to 0$ one recovers the equation of motion of the mathematical pendulum. Why would one expect this result?

* h) We consider the case of small oscillations where terms of quadratic and higher order in θ and $\dot{\theta}$ may be dropped. Verify that the equation of motion for θ amounts in that case to the motion in an effective potential

$$V_{\text{eff}} = \frac{1}{2\sqrt{\mu}} \ln \frac{1 - \sqrt{\mu}\,\cos\theta}{1 + \sqrt{\mu}\,\cos\theta}$$

* i)



The plot to the left shows the effective potential for different values of μ. For small μ it amounts to $V_{\text{eff}} \simeq \cos\theta$. For $\mu \to 1$ one obtains a very narrow, very deep potential. What does this tell about the motion of the pendulum in the two limiting cases?

Underpin your conclusion for $\mu \to 1$ by showing that in this limit the dimensionless height y of the pendulum weight obeys the EOM

$$\ddot{y} = -1$$

What will be its position x?

Problem 5. Stability of soap films

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When a soap film is suspended between two rings, it takes a cylinder-symmetric shape of minimal surface area. We discuss here the form of the film for rings of radius R_0 and R_1 positioned at the height x_0 and x_1, respectively. At the Mathematikum in Gießen there is a nice demonstration experiment: x_0 is the surface height of soap solution in a vessel around the platform where the children are standing, and x_1 is the height of the ring pulled upwards by the children.

`http://mathematikum.df-kunde.de/Wanderausstellung/index.php?m=2&la=de&id=314`

a) Let $w(x)$ be the radius of the cylinder-symmetric soap films at the vertical position x. Sketch the setup and mark the relevant notations for the problem.

b) Show that the surface area A of the soap film takes the form

$$A = \int_{x_0}^{x_1} \mathrm{d}x \, w(x) \, f(w'(x)) \, ,$$

Here, the factor $f(w'(x))$ takes into account that the area is larger when the derivative $w'(x) = \mathrm{d}x/\mathrm{d}x$ increases. Determine the function $f(w'(x))$ in this expression.

c) Show that A is extremal for shapes $w(x)$ that obey the differential equation

$$w''(x) = \frac{1 + (w'(x))^2}{w(x)} \, .$$

d) Determine the solutions of the differential equation.
Hint: Rewrite the equation into the form

$$\frac{w'(x) \, w''(x)}{1 + (w'(x))^2} = \frac{w'(x)}{w(x)} \, .$$

e) Consider now solutions with $-x_0 = x_1 = a$ and $R_0 = R_1 = R$, and denote the radius at the thinnest point of the soap film as w_0. Show that w_0 is the solution of

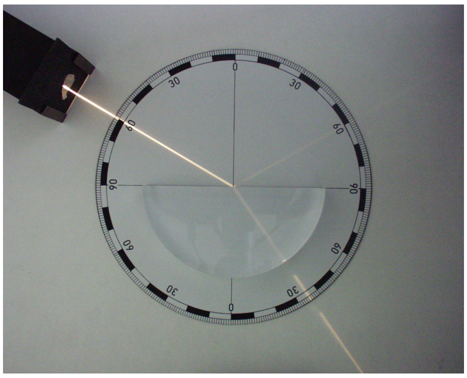
$$\frac{R}{a} = \frac{w_0}{a} \cosh \frac{a}{w_0} \, .$$

f) Sketch R/a as function of a/w_0. For given R and a you can then find w_0. For small separation of the rings you should find two solutions. What happens when one slowly rises the ring? Will an adult ever manage to pull up the ring to head height before the film ruptures?

Bonus Problem

Problem 6. Fermat's principle

Fermat's principle states that a light beam propagates along a path minimizing the flight time. When passing from air into glass it changes direction according to Sellius' refraction law.



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Here, we consider a setting where the beam starts in air at the position, $(x, y) = (0, 0)$, to the top left in the figure, with coordinates where $\hat{x}$ points downwards and $\hat{y}$ to the right. The path of the light is described by a function $y(x)$. We require that beam passes from air into the glass at the position (a, u) such that it will eventually proceed through the prescribed position (b, w) in the glass. The speed of light in air and in glass will be denoted as c_A and c_G, respectively.

a) Show that the time of flight T for a (hypothetical) trajectory $y(x)$ with derivative $y'(x)$ can be determined as follows

$$T = c_A^{-1} \int_0^a \mathrm{d}x \, \sqrt{1 + (y'(x))^2} + c_G^{-1} \int_a^b \mathrm{d}x \, \sqrt{1 + (y'(x))^2} \, .$$

b) In the following we consider a glas body with a planar surface, and align the coordinates such that glass surface is aligned parallel to the y-axis. Hence, we know that the light passes from air to glass at the fixed position a, but we still have to determine u. Determine δT for a variation $y(x) + \delta y(x)$ of the trajectory. What does this imply for $\delta y(x)\big|_{x=0}$, $\delta y(x)\big|_{x=a}$ and $\delta y(x)\big|_{x=b}$? What does it imply for the boundary terms that arise from the integration by parts, when determining δT?
Hint: Watch out! The position where the light beam passes from glass to air is not fixed in this variation. What does this imply for the variation of the time?

c) Show that the beam must go in a straight line in air and in glass.

d) Show that this implies that

$$T(u) = \frac{1}{c_A}\,\sqrt{u^2 + a^2} + \frac{1}{c_G}\,\sqrt{(w-u)^2 + (b-a)^2}\,.$$

Derive Snellius' law from the condition that $0 = \mathrm{d}T(u)/\mathrm{d}u$.

e) Snellius' Law can also be directly obtained from Fermat's principle. How?