Mathematics 1, Homework 11
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Due Date: 28.01.24 until 23:59 on-line
or 29.01.24 until 9:15 am in person

Each problem is estimated by one point. Explain your answers.

1. Given the matrices

$$A = \begin{pmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{pmatrix}$$

find the matrix

$$(2A)^T - (3B)^T$$

2. For each of the pairs of matrices that follow, determine whether it is possible to multiply the first matrix times the second. If it is possible, perform the multiplication:

(a) $\begin{pmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & -2 \\ 6 & -4 \\ 8 & -6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \\ 4 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 4 & 6 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{pmatrix}$

(e) $\begin{pmatrix} 4 & 6 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{pmatrix}$

(f) $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 4 & 5 \end{pmatrix}$

3. Two matrices $n \times n$ A and B are said to commute if

$$AB = BA$$

Determine if there exist the values of a parameter $\alpha \in \mathbb{R}$ such that the 2×2 matrices

$$A = \begin{pmatrix} \alpha & 1 \\ 2 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 5 & 2 \\ \alpha & 1 \end{pmatrix}$$

commute and find all such values of α.

4. Given the 2×2 matrix

$$A = \begin{pmatrix} 1 & 2024 \\ 0 & 1 \end{pmatrix}$$

compute the matrix A^{10}.

5. For two column vectors a, $b \in \mathbb{R}^3$ let us define their vector product $a \times b$ as a column vector

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \Longrightarrow \qquad a \times b = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Given a column vector

$$\omega = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

determine the components a_{ij} of the 3×3-matrix A such that

$$Ax = \omega \times x \qquad \text{for any column vector} \quad x \in \mathbb{R}^3,$$

and verify that the matix A possess the property $A^T = -A$.

6. Which of the matrices that follow are elementary matrices?

$$\text{(a)} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(b)} \quad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \qquad \text{(c)} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix} \qquad \text{(d)} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7. For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$

$$\text{(a)} \quad A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix}$$

$$\text{(b)} \quad A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{pmatrix}$$

$$\text{(c)} \quad A = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{pmatrix}$$

8. Given the matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

find elementary matrices E_1, E_2, E_3 such that the matrix $E_3 E_2 E_1 A$ is upper triangular.

9. Find the inverse of the following matrix using the Gauss elimination method:

$$\begin{pmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{pmatrix}$$

10. Compute the LU factorization of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{pmatrix}$$