Exercise 1

Consider a (univariate) lognormal random variable: $X \sim LogN\left(\mu, \sigma^2\right)$.

a) Write a Python function that determines the value of the parameters μ and σ^2 from the expectation $\mathbb{E}\{X\}$ and the variance $\mathbb{V}\{X\}$.

b) Write a Python script in which you:

- Use the function created in a) to determine the parameters μ and σ^2 such that $\mathbb{E}\{X\} = 3$ and $\mathbb{V}\{X\} = 5$.
- Generate a large sample from the distribution of X with such parameters;
- Plot the sample (do not join the observations, show them as dots);
- Plot the histogram (suitably normalized) and superimpose the exact pdf for comparison;
- Plot the empirical cdf and superimpose the exact cdf for comparison.

Note: display title, labels and legend in each plot.

Exercise 2

a) Given a sample $\{\boldsymbol{\epsilon}_t\}_{t=1,\ldots,\bar{t}}$ and a vector of probabilities $\{p_t\}_{t=1,\ldots,\bar{t}}$ expressing the relative weights of the observations, the recursive routine *fit_locdisp_mlfp* outputs the maximum likelihood estimates of the location and dispersion parameters under the assumption of a multivariate Student t distribution with ν degrees of freedom.

Write a Python function that implements the routine, here written in pseudo-code

$$(\hat{\boldsymbol{\mu}}_{\boldsymbol{\varepsilon}}^{MLFP}, \hat{\boldsymbol{\sigma}}_{\boldsymbol{\varepsilon}}^{2MLFP}) \leftarrow \textit{fit\_locdisp\_mlfp}(\{\boldsymbol{\epsilon}_t, p_t\}_{t=1,\ldots,\bar{t}}, \nu)$$

0. Initialize	$\begin{cases} \boldsymbol{\mu} \leftarrow \hat{\boldsymbol{\mu}}_{\boldsymbol{\varepsilon}}^{HFP} & (1) \\ \boldsymbol{\sigma}^2 \leftarrow \begin{cases} \frac{\nu-2}{\nu}\hat{\boldsymbol{\sigma}}_{\boldsymbol{\varepsilon}}^{2HFP} & \text{if } \nu > 2 \\ \hat{\boldsymbol{\sigma}}_{\boldsymbol{\varepsilon}}^{2HFP} & \text{otherwise} \end{cases} & (2) \end{cases}$	
1. Update weights and FP	$\begin{cases} w_t \leftarrow \frac{\nu+\bar{\imath}}{\nu+(\boldsymbol{\epsilon}_t-\boldsymbol{\mu})'(\boldsymbol{\sigma}^2)^{-1}(\boldsymbol{\epsilon}_t-\boldsymbol{\mu})} \\ q_t \leftarrow \frac{p_t w_t}{\sum_{s=1}^{\bar{t}} p_s w_s} \end{cases}$	$, \quad t = 1, \ldots, \bar{t}$
2. Update output	$\begin{cases} \boldsymbol{\mu} \leftarrow \sum_{t=1}^{\bar{t}} q_t \boldsymbol{\epsilon}_t \\ \boldsymbol{\sigma}^2 \leftarrow \sum_{t=1}^{\bar{t}} q_t(\boldsymbol{\epsilon}_t - \boldsymbol{\mu})(\boldsymbol{\epsilon}_t - \boldsymbol{\mu})' \end{cases}$	
3. If convergence, output $(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$; else go to 1		

where

- each $\boldsymbol{\epsilon}_t$ is a $\bar{\imath} \times 1$ vector, each p_t is a scalar, ν is a positive integer;
- the mean and covariance for the initialization (Step 0) are defined as

$$\hat{\boldsymbol{\mu}}_{\boldsymbol{\varepsilon}}^{HFP} \equiv \sum_{t=1}^{\bar{t}} p_t \boldsymbol{\epsilon}_t \tag{1}$$

$$\hat{\boldsymbol{\sigma}}_{\boldsymbol{\varepsilon}}^{2HFP} \equiv \sum_{t=1}^{\bar{t}} p_t(\boldsymbol{\epsilon}_t - \hat{\boldsymbol{\mu}}_{\boldsymbol{\varepsilon}}^{HFP})(\boldsymbol{\epsilon}_t - \hat{\boldsymbol{\mu}}_{\boldsymbol{\varepsilon}}^{HFP})'; \tag{2}$$

- convergence in the above routine occurs when the relative Euclidean norm $\|\tilde{\boldsymbol{\mu}} - \boldsymbol{\mu}\|/\|\boldsymbol{\mu}\|$ and the relative Frobenius norm $\|\tilde{\boldsymbol{\sigma}}^2 - \boldsymbol{\sigma}^2\|_F/\|\boldsymbol{\sigma}^2\|_F$ between two subsequent updates are smaller than a given threshold (required as an additional input).

b) To test the function, write a script that calls it with the following inputs:

- $\{\boldsymbol{\epsilon}_t\}_{t=1,\ldots,\bar{t}}$ is a sample of $\bar{t}$ =1000 simulations from a standard bivariate normal distribution;
- $p_t = 1/\bar{t}$ for every t (observations are equally weighted);
- $\nu = 100$;
- convergence threshold $= 10^{-9}$.