MATH-411 Numerical Analysis—Homework 6
Rochester Institute of Technology, Fall 2022

Due: Monday November 21, 2022 at 11.59pm EST.
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Remark:

- All assignments are uploaded on MyCourses as pdf.
- You can discuss ideas on how to tackle the problems on **Piazza** but do not post solutions.
- Please show all your work clearly. If the assignment involves MATLAB, please turn in your code and figures as well.
- Do you put your answers on your code comments, I strongly ask that you write down your answers for any assignment (homework and projects) on a different sheet (handwritten or typeset), include your figures and tables on this sheet and all your supporting calculations and narratives.

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1. Consider the following nonlinear autonomous initial value problem in $\mathbb{R}$ with $f \in C^2$:

$$y' = f(y), \qquad y(0) = y_0$$

   (a) As we discussed in class, one way you can derive numerical ODE methods is by re-writing the ODE through integrating and then applying a quadrature method. Write this ODE in integral form and use the mid-point quadrature rule to derive the *mid-point method* with uniform time-step h:
$$y_{n+1} = y_n + hf\left(\frac{y_{n+1}+y_n}{2}\right).$$

   (b) Prove that the truncation error, $y(t_{n+1}) - [y(t_n) + hf(\frac{y(t_{n+1})+y(t_n)}{2})]$, is $O(h^3)$ for this method. (Hint: expand first $y(t_{n+1})$ and $y(t_n)$ around $\tau_n = t_n + \frac{h}{2}$ and then expand $f(\frac{1}{2}(y(t_{n+1}) + y(t_n)))$.)

   (c) Based on your result from (b) and our results in class, what is the order of the method (that is, the global error)?

2. Give the order and stability type for each of the following multi-step methods.

   (a) $w_{j+1} = 3w_j - 2w_{j-1} + \frac{h}{12}(13f_{j+1} - 20f_j - 5f_{j-1})$

   (b) $w_{j+1} = \frac{4}{3}w_j - \frac{1}{3}w_{j-1} + \frac{2}{3}hf_{j+1}$

   (c) $w_{j+1} = \frac{4}{3}w_j - \frac{1}{3}w_{j-1} + \frac{h}{9}(4f_{j+1} + 4f_j - 2f_{j-1})$

   (d) $w_{j+1} = 2w_j - w_{j-1} + \frac{h}{2}(f_{j+1} - f_{j-1})$

3. Remember that Legendre polynomials were the set of polynomials that are orthogonal on $[-1,1]$ (meaning $\int_{-1}^{1} \phi_n(x)\phi_m(x)dx = 0$ if $m \neq n$. As we discussed in class, orthogonality can be

generalized to include a weight function $w(x)$ so that functions are orthogonal on $[a,b]$ with respect to $w(x)$ if $\int_a^b \phi_n(x)\phi_m(x)w(x)dx = 0$ if $m \neq n$.

The Gaussian quadratue method we derived in class is for weight function $w(x) = 1$, but the same process for deriving the Gaussian quadrature formula generalizes to any weight function. Keeping this in mind, find the three points x_j and the weights w_j for the three-point Gaussian quadrature formula with weighting function $w(x) = x^2$, that is find the x_j and w_j such that

$$\int_{-1}^{1} f(x)x^2 dx \approx \sum_{j=1}^{3} f(x_j)w_j.$$