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Signature

Angular Momentum Theory and The Atiyah Conjecture

Abstract

The objective of this paper is two-fold. Firstly, to attempt to showcase the intriguing relationship of $SU(2)$ representations used to explain express rotation and angular momentum operators in quantum mechanics. Secondly, to elucidate on the Atiyah conjecture giving enough background information required to understand the concepts involved in the construction of the conjecture.

Contents

Part I. Introduction

The concept of a Representation was invented in order to be able to define the action of groups on vector spaces. It is important to study such representations as such actions arise in different parts of physics and mathematics alike. One group whose action on spaces is of primary importance is the Rotation group $SO(3)$. The representations of $SO(3)$ help define the rotations on a vector space and at times helps reveal some features regarding the rotational symmetry of a system. However, in some cases, the rotation group chooses to act through another, the special unitary group of 2 dimensions $SU(2)$.

We start with the most basic definition of $SU(2)$ and try to see this relationship that $SU(2)$ with $SO(3)$. This interesting relationship is showcased by a representation of $SU(2)$ called the orthogonal representation.

Representations of $SU(2)$ are often appear in physics. Infact, Angular momentum theory uses the irreducible represention of $SU(2)$ in order to describe the rotations in quantum mechanical state space.

The goal of this thesis is two fold. Firstly, we shall independently develop the first part of this dissertation involves the independent derivations and development of the representations of $SU(2)$ and its Lie algebra $\mathfrak{su}(2)$ and representations of angular momentum and rotations in quantum state space (Usually given by a Hilbert space). Through these independent ventures, we hope to the beauty in witnessing the use of abstract mathematical concepts in a physical system.

In the second part of the thesis, we shall look at the necessary background material required to understand and appreciate the famous Atiyah conjecture. The conjecture was motivated by a problem of finding a map which helps bridging the gap between classical physical states and quantum physical states.

After looking at the conjecture, we shall move forward and see the different properties that this map holds.

Part II. Prelimnaries

1 Stereographic Projection

Stereographic Projection is a method used to interpret complex numbers as points on a sphere, rather than imagining them to be points on a plane. Inorder to do that, we need to establish a one-one correspondence between the points on the unit sphere and the points on the plane.

Let S be a unit sphere centered at the origin of $\mathbb{C}$ and placed in such a way that the "equator" coincides with the unit circle of the plane $\mathbb{C}$. Let N be the north pole of the sphere S and let p_s be a point on the sphere. Drawing a line through N joining p and extending it such that it meets $\mathbb{C}$ at a point p. The point p_s is the stereographic image of the point p on the sphere S. We know that there exists a unique line joining N to any point on the sphere S and these unique lines intersect $\mathbb{C}$ at unique points. This gives us a one-one corresspondence between the points in $\mathbb{C}$ and the points on S.

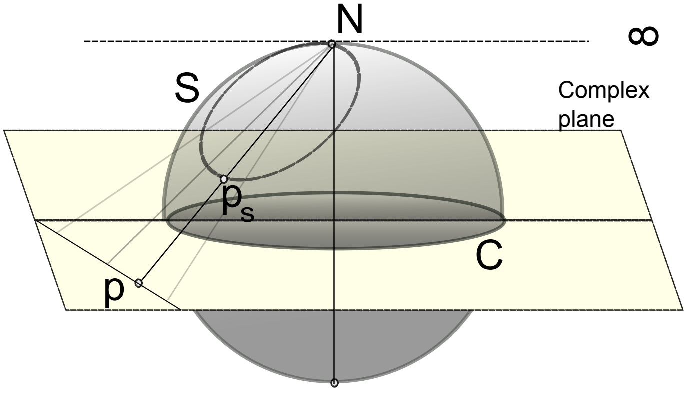

Fig. 1.1: Stereographic projection of $\mathbb{C}$

Note :

1. Having used the northern hemisphere to map the complex plane except the unit circle, we map the interior of the unit circle to the southern hemisphere of S and in particular, we note that the line through the north pole joining the south pole meets the complex plane at the origin. Therefore, 0 is mapped to the south pole.

2. Each point on the unit circle is mapped to itself. Therefore, we can say that it lies on the equator of the sphere S.

3. The tangent to the sphere S passing through N lies parallel to $\mathbb{C}$, hence, not intersecting it at any finite point. So, we can say that N is the stereographic image of the complex no. ∞.

4. The stereographic image of a line in $\mathbb{C}$ is a circle on S passing through N. From this fact, one could say that stereographic projections preserve angles.

Let us describe $z \in \mathbb{C}$ with cartesian coordinates (x, y) on the plane. Similarly, we can define the corressponding point on the sphere$\hat{z}$ as (X, Y, Z). We project the points on S to points on $\mathbb{C}$ by

$$(X, Y, Z) \to (\frac{X}{1-Z}, \frac{Y}{1-Z}) = (x, y)$$

The reasons for making such a transformation to represent the stereographic projections is quite intutitive. We need to map the northpole $N(0,0,1)$ to the complex no. $z = \infty$ which we assume to be given by (∞, ∞). The best way to do this was to make the denominators $(1 - Z)$, so that when $Z = 1$,$\frac{X}{1-Z} = \frac{Y}{1-Z} = \infty$. As N is only point on the $Z = 1$ plane, we map $(0, 0, 1) \to \infty$.

Also, the unit circle lies on the $Z = 0$ plane and since, the stereographic projection maps the points on the unit circle to itself, we can see that this transformation maps $(X, Y, 0) \to (X, Y)$.

2 Complex Projective Line

Our first task here is to try understand how we map vectors in $\mathbb{C}^2$ to the complex projective line CP^1. Since, it is quite hard (virtually impossible) to visualize vectors in $\mathbb{C}^2$, we shall try to deduce this projection intuitively by working with a vector space which is relatively easier to visualize.

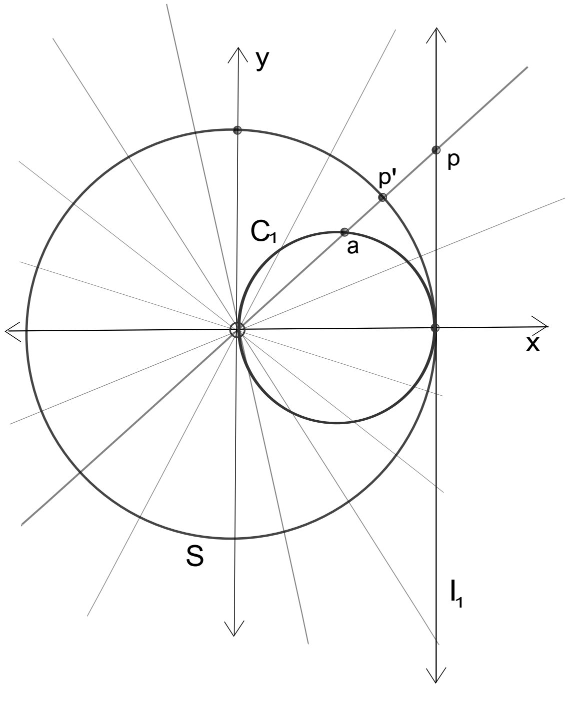

Fig. 2.1:

Let us consider the space $\mathbb{R}^2$ where S^1 is the unit circle and l_1 is the line given by $x = 1$. Also, let C_1 be a circle through the origin as shown in the figure. Consider the set of all lines passing through the origin in $\mathbb{R}^2$. Every line through the origin intersects S^1 at diametrically opposite points (antipodal points) which is nothing but the $S^0(0 - Sphere)$. Notice that every line through the origin also intersects the line $l_1$at an unique point. (we consider that the $x = 0$ line interesects $l_1$at ∞.) Hence, this shows us that every line passing through the origin can be uniquely represented by a point on the line l_1. Taking any point $(x, y) \in \mathbb{R}^2$, we know that we can draw a line through the origin that passes through $p' = (x, y)$ and so we can map this point to a point $p = (1, \frac{y}{x})$ on the line l_1. Therefore, we can project every point in $\mathbb{R}^2$ on to $l_1$and we call $l_1$the real projective line $RP^1$and a point $(0, y)$ is mapped to ∞.

Further, every line through the origin also interescts the circle C_1at two points, namely, the origin and an unique point on the circle.(We can assume that the y- axis interesects C_1at the origin twice) This tells us that every point on $l_1$can be mapped to $C_1$where the origin is mapped to ∞. Therefore, the real projective line can be thought of as a circle.

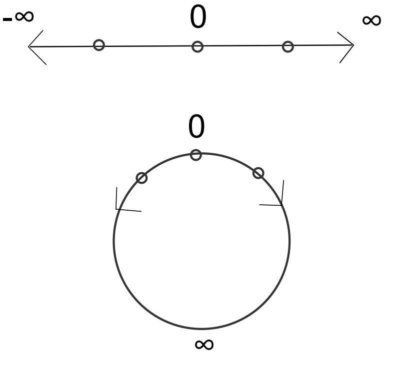

Fig. 2.2: Real line to S^1

From $\mathbb{C}^2$ to CP^1 Imagine a similar situation in $\mathbb{C}^2$. Consider all the complex line that pass through the origin where the set of all unit vectors are now represented by S^3 and now every complex line in $\mathbb{C}^2$intersects S^3 at a unit circle (a copy of S^1). And l_1 is now represented by a complex line (a copy of $\mathbb{C}$). Drawing conclusions from the case of $\mathbb{R}^2$, we can say that the set of all complex lines in $\mathbb{C}^2$can be projected onto the l_1. We can call $l_1$as the complex projective line $\mathbb{C}P^1$.

Consider a point $z = \big(z_1, z_2\big) \in \mathbb{C}^2$. If we consider the set of all the lines in $\mathbb{C}^2$ through the origin then all points on the line passing through the point (z_1, z_2) is now mapped to a point $\frac{z_1}{z_2} \in \mathbb{C}P^1$. This enables us to map the the z_1 axis to the point ∞. These coordinates (z_1, z_2) which map to $\frac{z_1}{z_2}$ are called homogeneous coordinates and this technique of using homogeneous coordinates is prevalent in the field of projective geometry.

Part III. Representations of $SU(2)$

3 The Special unitary Group $SU(2)$

The special unitary group $SU(2)$ which represents 2×2 unitary matrices with determinant 1. A general element $U \in SU(2)$ can be represented in the form

$$U = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$

Since, a and b are complex numbers, we can write them out in terms of their real and imaginary parts and so

$$U = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix}$$

where x_1, y_1 and x_2, y_2 are the real and imaginary parts of a and b respectively and

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \tag{3.1}$$

The equation 3.1 defines a $3 - sphere$ in $\mathbb{R}^4$.

3.1 Latitudes and longitudes in S^3

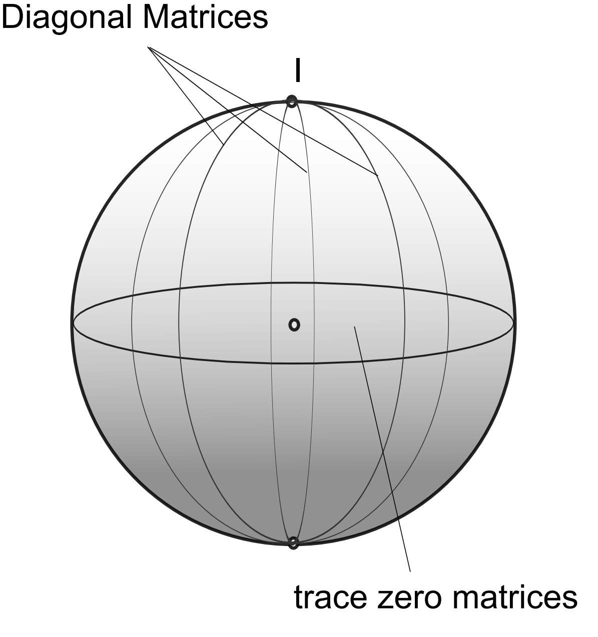

Fig. 3.1: Equatorial Latitude is S^2 embedded in S^3

Although it is quite impossible to visualize S^3, there are several ways to imagine how they look locally. If $x_4 = 0$, then (3.1) becomes,

$$x_1^2 + x_2^2 + x_3^2 = 1$$

This is the equation of a S^2. So, we know that there is a S^2embedded in S^3 at the origin. If we assume the poles of the S^3 to lie on the points $(\pm 1, 0, 0, 0)$ (similar to the conventional idea of poles on the S^2), then we see that the latitues are just surfaces which are given by keeping a coordinate constant (say $x_1 = c$). Then the cartesian equation of the 3-sphere becomes,

$$x_2^2 + x_3^2 + x_4^2 = (1 - c^2),\ -1 < c < 1 \tag{3.2}$$

The conjugacy class of $U \in SU(2)$ is given by

$$Cl(U) = VUV^\dagger$$

where V is unitary.

Now, if we assume that $V \in SU(2)$ (even if this is not true, it is quite rudimentary to show that there exists V_1 having determinant 1 which can conjugate U), then all the elements in the conjugacy class of U_1 will have the same eigenvalues as U_1.

In order to find the eigenvalues of the general element $U \in SU(2)$ by writing down the characteristic equation of U.

$$det(U - \lambda I) = det(\begin{bmatrix} a - \lambda & b \\ -\bar{b} & \bar{a} - \lambda \end{bmatrix})$$

$$\lambda^2 - 2(a + \bar{a})\lambda + 1 = 0$$

$$\lambda^2 - 2x_1\lambda + 1 = 0 \tag{3.3}$$

The eigenvalues of $U \in SU(2)$ depends only on x_1. As in the case of latitudes, we fix the value of $x_1 = c$, so the latitutes of S^3 represent the non-trivial conjugacy classes of $SU(2)$. However, we shall have to consider the trivial conjugacy classes as well, $\{I\}$ and $\{-I\}$. These corresspond to the north and south poles of the 3-sphere. We claim that these latitudes of S^3 define the conjugacy classes in $SU(2)$ and each of these non-trivial conjugacy classes of $SU(2)$ are represented by a 2-sphere.

We look at the longitudes of the 3-sphere which can be naturally thought of as extrapolation of the idea of a longitudes in 2−sphere. A longitude on S^3 will the intersection of S^3 with 2 dimensional flat subspace of R^4 containing the poles. Except for the poles , there exists a unique longitude passing through every point on S^3 lies. Algebrically, $SU(2) \cap V$ represents a longitude where V is subspace of $\mathbb{R}^4$ containing the poles. The intersection of S^3 with the subspace V given by $x_3 = x_4 = 0$ is represented by a great circle S^1 which contains the poles. This is represented by the set of diagonal matrices in $SU(2)$.

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} \quad | \, |a|^2 = 1 \right.$$

4 The Spinor Representation

As every element in $SU(2)$ is a 2×2 complex matrix, it has a natural representation on $\mathbb{C}^2 i.e$ the representation of $SU(2)$ on $\mathbb{C}^2$ is just the identity map : For $U \in SU(2)$,

$$\rho(U) = U$$

In physics, this trivial representation of $SU(2)$ is referred to as the Spinor representation.

5 The Orthogonal Representation

It is quite convinient to identify the $SU(2)$ with the $3 - sphere$ and the conjugacy classes of $SU(2)$ with 2-spheres. The assertion is that every element $U \in SU(2)$ acts as a rotation on these 2-sphere and we know that SO(3) is the group of all rotations of S^2. Therefore, we can define a surjective map

$$\phi : SU(2) \to SO(3) \tag{5.1}$$

As the trivial conjugacy classes $\{\pm I\}$ of $SU(2)$ are just points on the 3−sphere, the action of any U on the them is trivial. So, $\phi(\pm I) \to e$, where e is the identity element of $SO(3)$. As the kernel of ϕ is $\{\pm I\}$, its cosets are the sets $\{\pm P\}$, there exists 2 elements in $SU(2)$ that map to the same element of $SO(3)$. (If $\phi(U) = R$, where $R \in SO(3)$ then $\phi(-U) = R$). So, we call the group $SU(2)$, a double cover of the group $SO(3)$.

Inorder to help gain some geometric intuition, we shall consider another example of a double covering. Lets consider a map from the group of rotations on the unit circle S^1, $SO(2)$ to itself

$$R : SO(2) \to SO(2)$$

such that rotation by an angle θ is mapped to rotation by an angle 2θ $i.e$ $R(\theta) \to R(2\theta)$.

Every rotation by an angle θ will be the same as the rotation by an angle $\pi+\theta$ ($R(\theta+\pi) \to R(2\pi + 2\theta)$). This is an example of a double covering.

Every element in $SO(3)$ corresponds to 2 elements in $SU(2)$, namely, $\{\pm P\}$. Now, identifying these unitary matrices as points on S^3, it is straightforward to see that P and $-P$ are antipodes. Therefore, we could say that $SO(3)$ idenfies antipodal points on the $SU(2)$ 3-sphere.

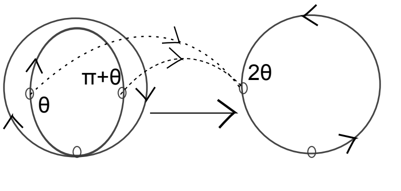

Fig. 5.1: Double cover of S^1

The $SO(3)$ group is group of all rotations on the S^2. Every element of the group represents a rotation by an angle about some axis. So, every element of the group (except $\{I\}$) can be represented the pair (v, θ) where v corresponds to the unit vector about the axis of rotation and θ denotes the angle of rotation about the axis. By this notation, the elements (v, θ) and $(-v, -\theta)$ correspond to the same rotation.

Therefore, either of these elements of $SO(3)$ can be used to represent a rotation. In quantum physics, this choice is called the choice of *spin*.

Now, we have a map from $S^2 \times \angle$ which represents all the pairs (v, θ) (where $\angle = \{\theta | 0 < \theta < 2\pi\}$ to the Rotation group.

$$\psi : S^2 \times \angle \to SO(3) - \{I\} \tag{5.2}$$

As we have two ordered pairs in (v, θ) and $(-v, -\theta)$ that map to the same element in the rotation group, it gives another double cover of $SO(3)$.

From 5.1 and 5.2 , inituitively, we could say that there exists a map between the two double covers of $SO(3)$ such that $(SU(2) - \{\pm I\})$ and $S^2 \times \angle$ are equivalent.

Now, consider

$$\xi : (SU(2) - \{\pm I\}) \to S^2 \times \angle \tag{5.3}$$

We claim that ξ is a homomorphism which is compatible with the rotation group $SO(3)$ *i.e* $\psi \circ \xi = \phi$.

By removing the poles of the 3-sphere, we can think of it as a space spanned by latitudes which are 2-spehres , like the "equator". Any point now lies on a S^2

As for the compatibility with $SO(3)$, a point on the S^3 lies on a latitude (which is a S^2) and if we let $SO(3)$ act on the point, it is rotated about some axis and by some angle but it still lies on the same latitude.

Therefore, given a choice of *spin* ,every element of $SU(2) - \{\pm I\}$ can be descriced as a rotation in $\mathbb{R}^3$. This result is quite important in physics as the representations of $SU(2)$ are often used to express different abstract concepts. For this reason, $SU(2)$ is often referred to as the *Spin group* in physics.

6 Lie group and Lie Algebra Representation of $SU(2)$

Lie groups are tools designed to study different aspects of continuous symmetry. They can, roughly, be thought of as differentiable manifolds which have group structure. Associated with Lie group, we often have a mathematical structure called a Lie algebra. It can be thought of as the tangent space of a Lie group at the identity. Here, we are only interested in looking at the Lie group and Lie algebra structure of $SU(2)$.

6.1 Brief Introduction

A Lie group G is a group that is also a finite-dimensional differentiable manifold in which the group operations of multiplication and inversion are smooth maps. *i.e*

$$\mu : G \times G \to G \quad \mu(x, y) = xy$$

$$\delta : G \to G \quad \delta(g) = g^{-1}$$

whereμ, δ are differentiable maps.

Example. $SO(3)$ and $SU(2)$

The group of rotations in 3 dimensions , $SO(3)$ is an example of a Lie group. It is probably the best example because it describes the abstract mathematical structure and properties of a Lie group in a geometric point of view. We have already seen how $SO(3)$ can be mapped to the S^2 which is a smooth manifold and for a given choice of spin, it is easy to see that the group operation of rotations (which is composition of rotations in this case) and the inverse rotation map are both smooth. The space of rotations is continuous and each rotation has a neighborhood of rotations which are almost the same, and this neighborhood becomes flat as we towards the poles on the S^2.

Similarly, it is also easy to see that $SU(2)$ is a lie group as we already know about its manifold structure from its relationship with the 3-sphere. Considering that Lie groups as smooth manifolds, we can 'move' along the manifold from one element to another element which is infinitesimally close to the other. However, the idea behind this transformation which connect these infinitesimally close elements is completely captured by another mathematical structure called a Lie algebra.

Definition 1. Lie Algebra

A finite-dimensional complex Lie Algebra is a finite dimensional complex vector space together with a operation $[] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

1. $[]$ is bilinear
2. $[A, B] = -[B, A] \, \forall A, B \in \mathfrak{g}$
3. $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \, \forall A, B, C \in \mathfrak{g}$

The vector space of invertible complex $n \times n$ matrices $GL(n, \mathbb{C})$ corresponds to a Lie algebra $\mathfrak{gl}(\mathfrak{n}, \mathfrak{C})$ with respect to the commutator operation $[A, B] = AB - BA$.

Lie Algebras were developed in order to study these infinitesimal transformations. In general, we can think of the Lie algebra associated with any Lie group as the vector space of all possible tangents to smooth paths that pass through the identity element of the group. For example, in the case of the group of all rotations $SO(3)$ acting continuously on S^2over time, then the tangent to the path at the identity can be thought of the angular velocity of the sphere at that instant.

6.2 Lie algebra of $SU(2)$

Although, we have the formal definiton of a Lie algebra, we need to able to associate a Lie algebra to a Lie group.

Let G be a Lie group where elements can be represented by matrices then the Lie algebra of G denoted $\mathfrak{g}$, is the set of all matrices X such that $e^{tX} \in G \ \forall t \in \mathbb{R}$. Now, the tangent at the identity would be given by

$$\frac{d}{dt}e^{tX}|_{t=0} = X$$

Therefore, the tangent at the identity of the Lie group element gives the corresponding Lie algebra element.

We choose the exponential function to help us map the elements of a Lie algebra to the Lie group because it gives us a natural path through the identity matrix I from its definition

$$exp(tA) = I + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + ...$$

And also note that at the identity, the tangent to such a path is given by

$$\frac{d}{dt}(exp(tA))|_{t=0} = A$$

which is the Lie algebra element corresponding to the Lie group element $exp(tA)$.

Now, if $U \in SU(2)$ then

$$U = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

A general element of $U \in SU(2)$ can be conjugated to be represented in the following form :

$$U = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} = exp(X_3)$$

where $X_3 = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Therefore, the element of $\mathfrak{su}(2)$ corresponding to U is given by

$$\frac{d}{dt}exp(X_3)|_{t=0} = X_3$$

Elements of $SU(2)$ are unitary by definition, $U^\dagger = U^{-1}$. We have resolved to representing elements of $SU(2)$ in the form of e^{tX} in order to help with the identification of $\mathfrak{su}(2)$ and so for e^{tX} to be unitary, we need the follwing condition to hold.

$$(e^{tX})^\dagger = (e^{tX})^{-1} = e^{-tX}$$

But $(e^{tX})^\dagger = e^{tX^\dagger}$ as $t \in \mathbb{R}$. So, EQ becomes,

$$e^{tX^\dagger} = e^{-tX} = e^{t(-X)}$$

Therefore, for e^{tX} to unitary, we need the condition

$$X^{\dagger} = -X$$

The Lie algebra of unitary matrices is space of all complex matrices X such that $X^{\dagger} = -X$. However, for special untary matrices we need the extra condition that the $det(e^{tX}) = 1$. Since, $X \in SU(2)$ is represented in a diagonal form, the $det(e^{tX}) = e^{t(trace(X))} = 1$.

Therefore, the Lie algebras of $SU(2)$, denoted by $\mathfrak{su}(2)$, is given the space of all complex matrices X such that $X^{\dagger} = -X$ and $trace(X) = 0$. A suitable basis for the space of $\mathfrak{su}(2)$ is given by,

$$X_1 = \frac{i}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X_2 = -\frac{i}{2}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, X_3 = \frac{i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.1}$$

It is quite easy to check that the commutation relations hold.

$$[X_1, X_2] = X_3 \;\; [X_2, X_3] = X_1 \;\; [X_3, X_1] = X_2 \tag{6.2}$$

Definition 2. Lie Group Representation

Let G be a Lie group, the representation of G is given by a Lie group homomorphism ρ,

$$\rho : G \to GL(V)$$

This a finite-dimensional representation if V is finite dimensional vector space.

Definition 3. Irreducible Representation

Let G be a Lie group acting on a vector space V. A subspace $W \subset V$ is called invariant if

$$\rho(g)w \in W \;\; \forall w \in W \; and \, g \in G$$

A representation with no non-trivial invariant subspaces is called an irreducible representation of V.

Lemma 4. *Lie group and Lie algebra representation*

Let G be Lie group with Lie algebra $\mathfrak{g}$ and let ρ be a representation of G on V , there exists a unique representation $\rho_1$of $\mathfrak{g}$ acting on V such that

$$\rho(e^X) = e^{\rho_1(X)} \;\; \forall X \in \mathfrak{g}.$$

This representation $\rho_1$is given

$$\rho(X) = \frac{d}{d\epsilon}|_{\epsilon=0}\rho(e^{\epsilon X})$$

and satifies

$$\rho_1(gXg^{-1}) = \rho(g)\rho_1(X)\rho(g)^{-1} \;\; \forall X \in \mathfrak{g} \; and \, g \in G$$

Definition 5. Complexification

The complexification of a real vector space V is a vector space V^c obtained by extending scalar multiplication to include multiplication of complex numbers. V^c is the space of linear combinations of $v_1, v_2 \in V$ in the form $v_1 + iv_2$. This can be thought of as a real vector space in the obvious way. It can be thought of a complex vector space if we define

$$i(v_1 + iv_2) = -v_2 + iv_1$$

Lemma 6. $\mathfrak{su}(2) \to \mathfrak{sl}(2, \mathfrak{c})$

Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{g}_C$ be its complexification. Then there exists a unique representation of $\mathfrak{g}_{\mathfrak{C}}$ given by ρ_1

$$\rho_1(X + iY) = \rho_1(X) + i\rho_1(Y) \;\; \forall X, Y \in \mathfrak{g} \tag{6.3}$$

It will be useful to check the above statement for the case of the Lie algebra $\mathfrak{su}(2)$ and its complexification $\mathfrak{sl}(2,\mathbb{C})$.

We know that $\mathfrak{sl}(2,\mathbb{C})$ is the space of all 2 x 2 complex matrices with trace zero. Let $X \in \mathfrak{sl}(2, \mathbb{C})$

$$X = \frac{X - \bar{X}}{2} + \frac{X + \bar{X}}{2}$$

This can be written as,

$$X = \frac{X - \bar{X}}{2} + i\frac{(X + \bar{X})}{2i}$$

It is quite easy to show that $\frac{X-\bar{X}}{2}, \frac{(X+\bar{X})}{2i} \in \mathfrak{su}(2)$. This decomposition is unique as every X has a unique$\bar{X}$, Therefore, $X \in \mathfrak{sl}(2,\mathbb{C})$ can be represented as $X = W + iZ$ where $W, Z \in \mathfrak{su}(2)$. From 6.3 we know that for $\mathfrak{su}(2)$ to be isomorphic to $\mathfrak{sl}(2, \mathfrak{C})$, we require that,

$$[W_1 + iZ_1, W_2 + iZ_2] = [W_1, W_2] - [Z_1, Z_2] + i([W_1, Z_2] + [W_2, Z_1])$$

$$\forall W_1, W_2, Z_1, Z_2 \in \mathfrak{su}(2) \quad W_1 + iZ_1, W_2 + iZ_2 \in \mathfrak{sl}(2, \mathfrak{C})$$

It is quite easy to check. So, any representation ρ of $\mathfrak{su}(2)$ extends to a representation of $\mathfrak{sl}(2,\mathbb{C})$ and can al so be denoted byρ.

6.3 Irreducible Representation of $SU(2)$

If a representation ρ of a group G on a nonzero vector space V has no proper G-invariant subspace, it is called an *irreducible* representation. If there is a proper invariant subsapce then ρ is said to be reducible.

Further, every reducible representation of a group G on V can be represented as a direct sum of irreducible representations. In short, the importance of irreducible representations arise from the fact that they are the fundamental components from which all the representations are constructed.

In this section, we shall see what these Irreducible representations of $SU(2)$ are.

We know that $SU(2)$ is a Lie group and it acts on the vector space $\mathbb{C}^2$ and so it has a representation ρ on the space of all functions on $\mathbb{C}^2$ and we can get this representation ρ of any function f on $\mathbb{C}^2$ by defining it as

$$\rho(U)f(a) = f(U^{-1}a) \tag{6.4}$$

Every element in $SU(2)$ is a 2×2 complex matrix, it has natural representation on $\mathbb{C}^2$ given by the identity map. (Spinor representation)

This gives the obvious action of $SU(2)$ on $\mathbb{C}^2$. So, we can look to get a representation on a space of functions on $\mathbb{C}^2$ and because of its inherent complex structure, the best choice is the space of polynomials. Although, the space of polynomials in a complex variable in $\mathbb{C}^2$ is infinite dimensional, we divide it into finite dimensional subspaces by considering homogeneous polynomials.

Let V_n be a linear complex vector space of dimension $n+1$, of polynomials of homogeneous degree n in the complex variable $z = \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} \in \mathbb{C}^2$ of the form

$$p(z) = p(t_0, t_1) = c_n t_0^n + c_{n-1} t_0^{n-1} t_1 + ... + c_1 t_0 t_1^{n-1} + c_0 t_1^n$$

Now, using the action we defined in 6.4 of $SU(2)$ on the space of functions, let us investigate the action of the representation of U on the polynomial $p(z)$.

As $U \in SU(2)$,

$$U = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \; and \; U^{-1} = \begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix}$$

From 6.4, we get,

$$\rho(U)p(z) = p(U^{-1}\begin{pmatrix} t_0 \\ t_1 \end{pmatrix})$$

$$\rho(U)p(z) = p(\bar{a}t_0 - bt_1, \bar{b}t_0 + at_1)$$

$$\rho(U)p(z) = \sum_{k=0}^{n} c_k (\bar{a}t_0 - bt_1)^{n-k} (\bar{b}t_0 + at_1)^k \tag{6.5}$$

Looking at the right-hand side of this equation, we can see that $\rho(U)p$ is also a homogeneous polynomial of degree $(n-1)$. So, $\rho(U)$ maps V_n to V_n . As there are no invariant subspaces in V_n with respect to $\rho(U)$, we could say that $\rho(U)$ is an irreducible representation of $SU(2)$. And so, any finite-dimensional representation of $SU(2)$ can be constructed with $\rho(U)$.

However, in order to understand the irreducible representations of $SU(2)$ at an infinitesimal stage, we should study the representation of the lie algebra $\mathfrak{su}(2)$. Let ρ_1 be the lie algebra representation of $SU(2)$ on V_n. We know that

$$\rho_1(X) = \frac{d}{dt}\rho(e^{tX})|_{t=0} \tag{6.6}$$

This is the Lie algebra representation of $SU(2)$ where we have taken $U = e^{tX}$, where $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in \mathfrak{su}(2)$. Now, considering the action on space of polynomials V_n from 6.5,

$$\frac{d}{dt}\rho(e^{tX})p|_{t=0} = \frac{d}{dt}p(e^{-tX}z)$$

Using the equation 6.6,

$$(\rho_1(X))p)(z) = \frac{d}{dt}p(e^{-tX}\begin{pmatrix} t_0 \\ t_1 \end{pmatrix})|_{t=0}$$

Using the Chain Rule, we get,

$$\begin{aligned} \rho_1(X)p(z) &= (\frac{\partial p}{\partial t_0}, \frac{\partial p}{\partial t_1})(\frac{d}{dt}e^{-tX}\begin{pmatrix} t_0 \\ t_1 \end{pmatrix})|_{t=0} \\ &= (\frac{\partial p}{\partial t_0}, \frac{\partial p}{\partial t_1})(-X.\begin{pmatrix} t_0 \\ t_1 \end{pmatrix}) \end{aligned}$$

And so, we get,

$$\rho_1(X)p(z) = -(\frac{\partial p}{\partial t_0}(x_1t_0 + x_2t_1) + \frac{\partial p}{\partial t_1}(x_3t_0 + x_4t_1)) \tag{6.7}$$

We know that $U \in SU(2)$ can be diagonalised to be brought to the form [1]

$$U = \begin{bmatrix} e^{-i\frac{t}{2}} & 0 \\ 0 & e^{i\frac{t}{2}} \end{bmatrix} = exp(-iX_3)$$

Let $H = -iX_3$ and so $H \in \mathfrak{su}(2)$,

$$H = -iX_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Plugging in $X = H$ in [eq], we get,

$$\rho_1(H)p(z) = \rho_1(\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})p(z) = \frac{1}{2}(t_1\frac{\partial p}{\partial t_1} - t_0\frac{\partial p}{\partial t_0})$$

$$\rho_1(H) = \frac{1}{2}(t_1\frac{\partial}{\partial t_1} - t_0\frac{\partial}{\partial t_0}) \tag{6.8}$$

[1] We have $\frac{t}{2}$ instead of as it help us see clearly the similarities presented with th Schwinger model.

Notice that the Lie algebra representation ρ_1 of $SU(2)$ for general diagonalized element U is independent of the polynomial.

Looking at this expression, we get the idea that the representation of H does not alter the degree of the monomial vector $t_0^{n-k}t_1^k \in V_n$. Applying this to a basis vector of V_n,

$$\rho_1(H)t_0^{n-k}t_1^k = \frac{1}{2}(n-2k)t_0^{n-k}t_1^k \tag{6.9}$$

It is quite clear that all the basis vectors of V_n are eigenvectors for $\rho_1(H)$ with corressponding eigenvalues $\frac{1}{2}(n-2k)$. As this representation of H gives us the basis $\{t_0^{n-k}t_1^k\}$ of V_n as the eigenvectors, it becomes of great significance.

6.4 Raising and Lowering Operators

The function of raising and lowering operators are help us explore the spectrum of eigenvectors and eigenvalues of the space and also figure out the bounds on eigenvalues. (if we have a finite dimensional space) Here, we need to find such operators, which help us to increase or decrease the eigenvalues of our representation of the Lie algebra $\mathfrak{su}(2)$ on the space of polynomials V_n.

We define the raising and lowering operator as follows,

$$E_+ = -i(X_1 - iX_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E_- = -i(X_1 + iX_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

where $X_1, X_2 \in \mathfrak{su}(2)$ which are elements in the basis of $\mathfrak{su}(2)$ (from 6.1). To understand completely these operators, let us take a look at the action of their representations on $p \in V_n$ using 6.7,

$$\rho_1(E_+)p(z) = \rho_1(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})p(z) \qquad \rho_1(E_-)p(z) = \rho_1(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})p(z)$$

$$\rho_1(E_+)p(z) = -t_1\frac{\partial P}{\partial t_0} \qquad \rho_1(E_-)p(z) = -t_0\frac{\partial p}{\partial t_1}$$

$$\rho_1(E_+) = -t_1\frac{\partial}{\partial t_0} \qquad \rho_1(E_-) = -t_0\frac{\partial}{\partial t_1} \tag{6.10}$$

Now, applying the raising and lowering operators on the an eigenvector $t_0^{n-k}t_1^k$, our we hope to see the spectrum of eigenvalues which correspond to the eigenvectors.

$$\rho_1(E_+)t_0^{n-k}t_1^k = -(n-k)t_0^{n-k-1}t_1^{k+1}$$

We can see that $\rho_1(E_+)$ has taken us to the next eigenvector $t_0^{n-k-1}t_1^{k+1}$. We can check the eigenvalue of this eigenvector by applying $\rho_1(H)$.

By a very similar action, $\rho_1(E_-)$ takes the eigenvector $t_0^{n-k}t_1^k$ to the another eigenvector $t_0^{n-k+1}t_1^{k-1}$.

We can see that the raising and lowering operators increase and decrease the eigenvalues of $\rho_1(H)$, respectively. Since, we are dealing with a finitie dimensional Vector space there are only a finite number of eigenvectors and so there has to be upper and lower limit for the eigenvalues.

$$\rho_1(H)t_1^n = \frac{-n}{2}t_1^n \tag{6.11}$$

Therefore, $\rho_1(E_+)$ cannot raise the degree of t_1 as t_1^{n+1}does not belong to vector space V_n. So, we set

$$\rho_1(E_+)t_1^n = 0 \tag{6.12}$$

Similarly, we can set $\rho_1(E_-)t_0^n = 0$ where the eigenvalue $\rho_1(H)$ corresponding to t_0^n will be $\frac{n}{2}$. And so we have our bounds on the eigenvalues.

7 Haar Measure and Inner Product on V_n

Let G be a compact Lie group, then there exists an unique volume element in G such that $\int_G dg = 1$ where $g \in G$. This invariant measure is called the Haar Measure. By invariant , we mean that for a map ρ acting on G, and $a \in G$

$$\int_G \rho(ag)dg = \int_G \rho(ga)dg = \int_G \rho(g)dg$$

As explained in (), $SU(2)$ can be thought of as the $3-$sphere. We shall use this idea inorder to help us derive an $SU(2)$ invariant measure.If we have a represntation ρ of G on a vector space V, we can define an $G - invariant$ inner product on V such that $< gv_1, gv_2 >=< v_1, v_2 > \forall v_1, v_2 \in V$ and $g \in G$.

To normalize our eigenvectors of our representation of H, we need to define a inner product on the space of polynomials V_n. How do we do this?

We are working on the space $\mathbb{C}^2$. It is natural for us to identify $\mathbb{C}^2$ with space $\mathbb{R}^4$, splitting each complex vector into real and imaginary parts. Moreover, we want to get orthonormal vectors so it only natural that we want to map the vectors in $\mathbb{C}^2$ to orthogonal unit vectors in $\mathbb{R}^4$. So, the answer to our problem lies with S^3. Now, let us consider the coordinates $(t_0, t_1) \in \mathbb{C}^2$ such that the monomial $t_0^{n_1}t_1^{k_1} \in V_n$. To define a suitable inner product we embed our vector space V_n in S^3 by parametrising

$$t_0 = \cos(\frac{\theta}{2})e^{\frac{i}{2}(\phi-\xi)}\, t_1 = \sin(\frac{\theta}{2})e^{\frac{i}{2}(\phi+\xi)}$$

It is easy to see that $|t_0^2| + |t_1|^2 = 1$.

We define our hermitian inner-product in $\mathbb{C}^2$by

$$< w, z >= \bar{w_1} z_1 + \bar{w_2} z_2 \;\; w, z \in \mathbb{C}^2$$

Using this, lets define our inner product on V_n,

$$< t_0^{n_1} t_1^{k_1}, t_0^{n_2} t_1^{k_2} >= \int \bar{t_0}^{n_1} \bar{t_1}^{k_1} t_0^{n_2} t_1^{k_2} d\mu$$

Here, $d\mu = \frac{1}{2\pi^2} \sin^2 \phi \sin \theta d\theta d\phi d\xi$ is a infinitisimal volume strip of the 3-sphere, is the *haar* measure. It is defined in such a way so that $\int_{S^3} d\mu$=1. It is easy to see that this if we know how to solve $\int_0^{2\pi} \sin^2 \phi d\phi = \pi$. It is also possible to see that over the course of this calculation.

It is easy to see that this if we know how to solve $\int_0^{2\pi} \sin^2 \phi d\phi = \pi$. It is also possible to see that over the course of this calculation.

$$= \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{2\pi} \int_0^{2\pi} \cos(\frac{\theta}{2})^{n_1+n_2} e^{\frac{i}{2}(n_2-n_1)(\phi-\xi)} \sin(\frac{\theta}{2})^{k_2+k_1} e^{\frac{i}{2}(k_2-k_1)(\phi+\xi)} \sin^2 \phi \sin \theta d\theta d\phi d\xi$$

As we that these monomials are orthogonal, the expression is simplified with the help of the *kronecker* delta functions.

$$= \frac{1}{2\pi^2} \delta_{n_1,n_2} \delta_{k_1,k_2} \int_0^{2\pi} d\xi \int_0^{2\pi} \sin^2 \phi d\phi \int_0^{\pi} \cos(\frac{\theta}{2})^{2n_1} \sin(\frac{\theta}{2})^{2k_1} \sin \theta d\theta$$

Its is quite easy to show that $\int_0^{2\pi} \sin^2 \phi d\phi = \pi$ and $\int_0^{2\pi} d\xi = 2\pi$. Using this, we get,

$$= \delta_{n_1,n_2} \delta_{k_1,k_2} \int_0^{\pi} (\cos \frac{\theta}{2})^{2n_1} (\sin \frac{\theta}{2})^{2k_1} \sin \theta d\theta$$

Let $r = \cos \theta$ and so $dr = -\sin \theta d\theta$. Also, using the trignometric indenties: $\cos(\theta) = 2\cos^2(\frac{\theta}{2}) - 1$ and $\cos(\theta) = 2\sin^2(\frac{\theta}{2}) + 1$, we can simplify the above expression,

$$= \delta_{n_1,k_1} \delta_{n_2,k_2} \int_1^{-1} (\frac{1+r}{2})^{n_1} (\frac{1-r}{2})^{n_2} (-dr)$$

$$= \delta_{n_1,k_1} \delta_{n_2,k_2} \int_{-1}^{1} (\frac{1+r}{2})^{n_1} (\frac{1-r}{2})^{n_2} dr$$

Inorder to simply things, we make a substitution $s = \frac{1+r}{2}$ and $ds = \frac{dr}{2}$,

$$< t_0^{n_1} t_1^{k_1}, t_0^{n_1} t_1^{k_1} >= \int_0^1 s^{n_1} (1-s)^{k_1} ds$$

Using the gamma functions, we know that $\int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and so we can solve the integral,

$$||t_0^{n_1}t_1^{k_1}||^2 = \frac{n_1!k_1!}{(n_1+k_1+1)!} \tag{7.1}$$

This gives us an inner product on the space of polynomials V_n. We can write this result in different ways in order to suit our need. One way is to express it as follows,

$$||t_0^{n+k}t_1^{n-k}|| = (\frac{(n-k)!(n+k)!}{(2n+1)!})^{1/2} \tag{7.2}$$

An orthonormal basis of eigenvectors for the space of V_n is given by

$$\{e^k(z)\} = \{\frac{t_0^{n+k}t_1^{n-k}}{\sqrt{(n-k)!(n+k)!}}\}\, where\, k = -n, -n+1, ..., n \tag{7.3}$$

where all the elements of the basis have unit length with respect to the unique (upto a scale) $SU(2)$ invariant scalar product. We have omitted the scalar factor $(2n+1)!$ as for a given n, it will just be a common factor of the elements of the basis.

Part IV. Essential Quantum Mechanics

8 Angular Momentum Theory

8.1 Rotation Operators

In $\mathbb{R}^3$, when you rotate a point or a vector by 30 degrees and then about 60 degrees about the x-axis, the result is the same as when you rotate the vector by 60 degrees and then by 30 degrees about the x-axis. However, when we perform rotations about different axes, do our rotations still commute?

Since, a rotational transformation is linear, we can represent a rotation in 3 dimensions using a 3 x 3 orthogonal matrix , say R, acting on $\mathbb{R}^3$with real entries. We can see that R acts on a vector y to give another vector x in $\mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = R \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

We also know from the orthogonality condition that $RR^T = R^TR$ and $||x|| = ||y||$. Lets consider a rotation of an angle ϕ in anti-clockwise direction about the z-axis. Its not difficult to see that the rotation matrix looks like,

$$R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We are going to look at rotation by an infinetisimally small angle ϵ. So, approximating the cosine and sine functions for a really small value, we get,

$$R_z(\epsilon) = \begin{bmatrix} 1-\epsilon^2 & -\epsilon & 0 \\ \epsilon & 1-\epsilon^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, we can represent rotation by an infinitesimal small angle about the x and y axes as rotation matrices.

$$R_x(\epsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\epsilon^2 & -\epsilon \\ 0 & \epsilon & 1-\epsilon^2 \end{bmatrix}, \; R_y(\epsilon) = \begin{bmatrix} 1-\epsilon^2 & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\epsilon^2 \end{bmatrix}$$

We can see that infinitesimal rotations about the x and y axes are not commutative.

$$R_x(\epsilon)R_y(\epsilon) - R_y(\epsilon)R_x(\epsilon) = \begin{bmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R_z(\epsilon^2) - I$$

$$R_x(\epsilon)R_y(\epsilon) - R_y(\epsilon)R_x(\epsilon) = R_z(\epsilon^2) - R(0) \tag{8.1}$$

The identity matrix I can be thought of as rotation of 0 degrees about any axis and so we write it as $R(0)$. We can how rotations affect physical systems especially in $\mathbb{R}^3$. However, we know that quantum mechanical systems are however represented in some Hilbert Space. 8.1 is a fundamental commutation relation between rotation operations about different axes and its easy to see that infinitesimal small rotations about different axes do commute if we omit terms of order higher that ϵ^2.

So far, we have discussed rotations in $\mathbb{R}^3$. However, we need find an operator which performs rotations in the state space of the quantum mechanical system and lets call this space V. Let us represent a rotational operator in the space V as $D(R)$ where R performs an analogous rotation in $\mathbb{R}^3$.

$$\psi_R = D(R)\psi$$

where $\psi, \psi_R \in V$ are initial and rotated vectors in V.

Now, we need figure out how this rotational operator $D(R)$ actually looks like. For that, we can use the same ideas employed in developing the time evolution operator. The time evolution operator U for an infinitesimally small time lapse dt was given by

$$U = 1 - i\frac{H}{\hbar}dt$$

where H is the Hamiltonian of the system. We know that the Hamiltonian is a hermitian operator responsible for the generation of time translations. Similarly, we can construct the $D(R)$ using Angular momentum, which we know from Classical physics , generates rotation (from the famous Noether's theorem.)

$$D(R) = D_n(R_{d\phi}) = 1 - i\frac{J_n}{\hbar}d\phi$$

where the Hermitian operatorJ_n is the angular momentum operator about the n-axis i.e $J_n = J.n$ where n is the unit vector along the n-axis.

So, we have defined the rotation operator for an infinitesimally small rotation about an axis. However, one might be more interested in developing the operator for a finite rotation. Inorder to do that let us consider a rotation of angle ϕ about the n-axis,

$$D_n(R) = \lim_{N\to\infty}[1 - i\frac{J_n}{\hbar}\frac{\phi}{N}]^N$$

We have split up the finite angle into infinitesimally small angles and performed a large number of a small rotations which essentially adds up to a finite rotation. To simplify the above expression we use two keys ideas, the following identity and series expansion of the exponential function.

$$\lim_{k\to\infty}[1 + \frac{a}{k}]^k = e^a \;\; exp(x) = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + ...$$

Using these, we simplify the expression for the Rotation operator to

$$D_n(R) = exp(-i\frac{J_n\phi}{\hbar}) = 1 - i\frac{J_n\phi}{\hbar} - \frac{J_n^2\phi^2}{2\hbar^2} + ...$$

Having developed an expression to associate with the rotation operator, we can now try to see the fundamental commutation relation looks like. Recall 8.1 the commutation relation we had developed for the infinitesimal rotation operator in R^3

$$D_x(R)D_y(R)(\epsilon) - D_y(R)D_z(R)(\epsilon) = D_z(R)(\epsilon^2) - D(R)(0)$$

$$(1-\frac{iJ_x\epsilon}{\hbar}-\frac{J_x^2\epsilon^2}{2\hbar^2}+...)(1-\frac{iJ_y\epsilon}{\hbar}-\frac{J_y^2\epsilon^2}{2\hbar^2}+...)-(1-\frac{iJ_y\epsilon}{\hbar}-\frac{J_y^2\epsilon^2}{2\hbar^2}+...)(1-\frac{iJ_x\epsilon}{\hbar}-\frac{J_x^2\epsilon^2}{2\hbar^2}+...) = 1-\frac{iJ_z\epsilon^2}{\hbar}-1$$

Simplifying this, we get the commutation relation $[J_x, J_y] = i\hbar J_z$. It is also easy to infer that $[J_y, J_x] = -i\hbar J_z$ and $[J_x, J_x] = 0$. We could generalize this result for other axes by,

$$[J_i, J_j] = i\hbar\xi_{ijk}J_k \;\; where\, \xi_{ijk} = \begin{cases} 1 & if\, i,j,k\, are\, a\, cyclic\, permutation \\ -1 & if\, i,j,k\, are\, an\, anti-cyclic\, permutation \\ 0 & if\, otherwise \end{cases} \tag{8.2}$$

This result is called the Fundamental Commutation relations of angular momentum and from this we understand that rotations about different axes don't commute. Its is quite instructive to see that due its relationship (by definition) to the Euclidean rotation group $SO(3)$, the quantum mechanical rotation operator $D(R)$ for infitesimal rotations forms a Lie group (much like $SO(3)$). Consequently, the relationship between angular momentum operators and rotation operators is the same as the relationship between Lie algebras and Lie groups. We shall provide some evidence of this in the following sections.

8.2 Ladder operators and Eigenstates of Angular Momentum

The importance of commutating observables in quantum mechanics cannot be over-emphasized. The fact that obervables commute, means they share no uncertainty relation between them. Although we have a commutative relation from the result 8.2 for the angular momentum about different axes, we know that they don't commute and therefore we can never measure the angular momentum about any two of the axes with certainty. So, it is of paramount importance for us now to develop a new operator which commutes with our angular momentum operator about all axes. Let us consider the operator J^2,

$$J^2 = J_x J_x + J_y J_y + J_z J_z$$

where J_i is the angular momentum about the i-axis. First, we can check the commutation relation $[J^2, J_z]$

$$[J^2, J_z] = [J_x J_x + J_y J_y + J_z J_z, J_z] = [J_x J_x, J_z] + [J_y J_y, J_x] + [J_z J_z, J_z]$$

Exapnding and simplifying, we get,

$$[J^2, J_z] = J_y[J_y, J_x] + [J_y, J_x]J_y + J_x[J_x, J_z] + [J_x, J_z]J_x$$

Using our result from **8.2**,

$$[J^2, J_z] = J_y(-i\hbar J_x) + (-i\hbar J_x)J_y + J_x(i\hbar J_y) + (i\hbar J_y)J_x = 0$$

We find that J^2 commutes with J_z.With similar arguments,we can deduce that that

$$[J^2, J_x] = [J^2, J_y] = 0. \tag{8.3}$$

So, J^2 operator commutes with the angular momentum about each axis. However, we can choose only one of J_i to be the observable which can be measured simultaneously with J^2 in order to measure them without any uncertainty. As we started working with J_z, we shall continue to work with $J^2$and J_z and find their eigenstate and the eigenvalues but keeping in mind that these following arguments can be used similarly for $J_x$or J_y. Let the eigenvalues of J^2 and J_z be λ_1 and λ_2 respectively and both the these eigenvalues correspond to the same eigenvector v.(asJ^2 and J_i commute).

$$J^2.v = \lambda_1 v \;\; J_z.v = \lambda_2 v \tag{8.4}$$

In Quantum physics, we deal with discrete phenomenon. Eigenstates of Angular momenta of atoms is always discrete. We are looking for operators that help us some more generate possible quantum states from given states with which we can have broader view of how the system looks like. Our job is now to construct the eigenstates of the angular momentum operators and determine the spectrum of eigenvalues for these eigenstates. Inorder find the whole spectrum of eigenvalues of the angular momentum operators, we use operators of the form

$$J_\pm = J_x \pm iJ_y \tag{8.5}$$

The $J_+$and J_- operators are called the Ladder operators in quantum physics. Before we go into what these operators are and their properties, it might be useful to check their commututative relation with $J^2$and J_z.

$[J^2, J_\pm] = J^2 J_\pm - J_\pm J^2 = (J_xJ_x + J_yJ_y + J_zJ_z)(J_x \pm iJ_y) - (J_x \pm iJ_y)(J_xJ_x + J_yJ_y + J_zJ_z)$

Simplifying, we get,

$$[J^2, J_\pm] = [J^2, J_x] \pm i[J^2, J_y] = 0$$

$$[J_z, J_\pm] = J_zJ_\pm - J_\pm J_z = J_z(J_x \pm iJ_y) - (J_x \pm iJ_y)J_z$$

After some simplification and rearraging, we get,

$$[J_z, J_\pm] = \pm\hbar J_\pm$$

So, J^2 and $J_\pm$ commute and hence can conclude that $J_\pm$ shares the same eigenstates as J^2 and also J_i. Therefore, we can now apply these ladder operators on the angular momentum eigenvectors and see hopw they affect the state.

$$J_z(J_\pm.v) = ([J_z, J_\pm] + J_\pm J_z).v = (\pm\hbar J_\pm).v + J_\pm(\lambda_2 v)$$

$$J_zJ_\pm v = (\lambda_2 \pm \hbar)J_\pm v \tag{8.6}$$

This is result gives us enough knowledge to define this $J_\pm$ operator that we have been using. 8.6 shows that $J_\pm$ acts on the eigenstate v of J_z and gives us the same eigenstate v but its eigenvalue is increased or decreased by $\hbar$. So, our $J_\pm$ operator increases or decreases the eigenvalue of J_z by one unit($\hbar$). This is the reason for calling $J_\pm$ as the *ladder operators* of J_z.

Note: Similarly, we can construct ladder operators for angular momentum operator about other axes as well. Also, it is important for one note that these ladder operators are non-Hermitian. However, that does not seem to affect our purposes for using them so far. We now know the effect that the ladder operators have on our J_z operator. Lets check what kind of an effect it has on our J^2operator.

$$J^2J_\pm v = ([J^2, J_\pm] + J_\pm J^2)v = \lambda_1 J_\pm v$$

As we know $J^2\, and\, J_\pm$ commute with each other, so $[J^2, J_\pm] = 0$. Also, since, $\lambda_1$is the eigenvalue of J^2. Now, $J_+ = J_2 + iJ_3$ then $J_+^\dagger = J_2 - iJ_3 = J_-$.

$$J^2 - J_1^2 = \frac{1}{2}(J_+J_+^\dagger + J_-J_-^\dagger)$$

Assuming that our eigenvector v is normalized.

$$v^\dagger(J^2 - J_1^2)v \geq 0$$

$$\lambda_1 - \lambda_2^2 \geq 0 \implies -\sqrt{\lambda_1} \leq \lambda_2 \leq \sqrt{\lambda_1}$$

Hence, we have a bound on the eigenvalue of J_z operator. This result tells us that there is a maximum and a minimum eigenvalue which the angular momentum operator can take. Therefore, we can start with J_z at the minimum eigenvalue and repeatedly apply our up-ladder operator (J_+) and we should reach the maximum eigenvalue, after a finite number of iterations. We shall not indulge in finding the physical significance of these min/max eigenvalues. Let v_{max} represent the eigenvector corresponding to the maximum eigenvalue of J_z.

$$J_+ v_{max} = 0$$

By this argument it follows that,

$$J_- J_+ v_{max} = 0$$

We know that,

$$J_- J_+ = J_x^2 + J_y^2 - i(J_y J_x - J_x J_y) = J^2 - J_z^2 - i[J_y, J_x]$$

From $P.4$, $[J_y, J_x] = -i\hbar J_z$

$$J_- J_+ = J^2 - J_z^2 - \hbar J_z$$

$$J_- J_+ v_{max} = (J^2 - J_z^2 - \hbar J_z) v_{max} = (\lambda_1 - \lambda_{2,max}^2 - \hbar \lambda_{2,max}) v_{max} = 0 = 0$$

$$\implies \lambda_1 - \lambda_{2,max}^2 - \hbar \lambda_{2,max} = 0$$

Hence, we can write $\lambda_1$in terms of $\lambda_{2,max}$

$$\lambda_1 = \lambda_{2,max}(\lambda_{2,max} + \hbar) \tag{8.7}$$

In quite similar fashion, we could consider,

$$J_- v_{min} = 0$$

where $v_{min}$was the eigenvector corresponding to the minimum eigenvalue of J_z. The above statement is true because there exists a minimum eigenvalue for J_z,say $\lambda_{2,min}$and if we apply the down-ladder operator J_-then the result should be zero as there is no eigenvalue to 'climb down' to. Similar to 8.7, we can write $\lambda_1$in terms of $\lambda_{2,min}$

$$\lambda_1 = \lambda_{2,min}(\lambda_{2,min} - \hbar) \tag{8.8}$$

From 8.7 and 8.8, we can see that

$$\lambda_{2,max}^2 + \hbar \lambda_{2,max} = \lambda_{2,min}^2 - \hbar \lambda_{2,min}$$

we perform rotations about different axes, do our rotations still commute?

[Picture(s)]Comparing the coefficients, we get that$-\lambda_{2,max} = \lambda_{2,min}$. This give us the bounds of λ_2

$$-\lambda_{2,max} \leq \lambda_2 \leq \lambda_{2,max} \tag{8.9}$$

Since, both v_{max} and v_{min} are eigenstate of J_z,we should able to reach v_{max} from v_{min} by applying J_+ n times.

$$\lambda_{2,max} = \lambda_{2,min} + n\hbar = \lambda_{2,max} + n\hbar$$

$$\implies \lambda_{2,max} = \frac{n\hbar}{2} \tag{8.10}$$

To make it easier to the eye, lets take $j = \frac{n}{2}$,

$$\lambda_{2,max} = j\hbar$$

We can see that j can be an integer or a half-integer. Using 8.7 and 8.10, we can find an expression for λ_1,

$$\lambda_1 = \frac{n\hbar}{2}(\frac{n\hbar}{2} + \hbar) = \hbar^2\frac{n}{2}(\frac{n}{2} + 1) = \hbar^2 j(j+1) \tag{8.11}$$

Lets define $\lambda_2 = m\hbar$.

$$-j\hbar \leq m\hbar \leq j\hbar \implies -j \leq m \leq j$$

where $m = -j, -j+1, ..., j-1, +j$.

$$\implies \lambda_2 = -j\hbar, -(j-1)\hbar, ..., (j-1)\hbar, j\hbar \tag{8.12}$$

This means that there $2j+1$ values that m can take and by extension, $2j+1$ values that λ_2 can take.

Results 8.11 and 8.12 have given us the eigenvalues and eigen vectors of $J^2$and $J_z$operators.

$$J^2 v = j(j+1)\hbar^2 v_{j,m},\ J_z v = m\hbar v_{j,m} \tag{8.13}$$

9 The Irreducible Representation of $D(R)$

Our priorities lie with finding a suitable way to describe rotations in a quantum mechanical system. The operator $D(R)$ is representation of the rotation group, which acts on the entire state space V and carries out a rotation. If $v \in V$, then $D(R)$ acts on v and rotates it.

$$v \to D(R)v$$

If the representation $D(R)$ acts on V such that it leaves any subspace invariant under the action of $D(R)$, then $D(R)$ is called a reducible representation. So, if we have a reducible representation $D(R)$, then we can spilt up the space V into subspaces and see what happens in each subspace.

Typically, the state space V be be decomposed into orthogonal subspaces V_i so that every $v \in V$ can be written as a sum

$$v = \sum_i v_i$$

where $v_i \in V_i$ such that $D(R)v_i \in V_i$.

A representation $D(R)$ acting V_i is called irrdeucible if there are no linear subsapces in V_i that is invariant under the action of $D(R)$. Hence, inorder to know about the rotations in state space, we must find an irreducible representation.

Let us now consider the operator J^2. Now we know that J^2 is a hermitian operator and so we could say that they are diagonalizable. (Proof)

And so we can find eigenvalues λ and corresponding eigenvectors v_λ such that

$$J^2 v_\lambda = \lambda v_\lambda$$

Here we assume v_λ is the normalized eigenvector corresponding to the eigenvalue λ. However, it is straight forward to see that there is a linear subspace $V_\lambda$of eigenvectors corresponding to the eigenvalue λ where $v_\lambda \in V_\lambda$. If J^2 has different eigenvalues, then the corresponding V_λ's must be orthogonal to each other. Consider the operator J_i which is the angular momentum operator about the $i-th$ axis $(i = 1, 2, 3)$ and we know that J_i commutes with J^2 (from)

$$J^2 J_i v_\lambda = J_i J^2 v_\lambda = \lambda J_i v_\lambda$$

Therefore, we can that $J_i v_\lambda$ is also an eigenvector of J^2 corresponding to the eigenvector λ.

$$J_i v_\lambda \in V_\lambda$$

If there were different λ values , then the corresponding subspaces V_λ must be orthogonal. These subspaces V_λ corresponding to different eigenvalues are invariant under the operation of J_i. However, as we have already seen that the eigen values of J^2 are of the form $\lambda_1 = j(j+1)\hbar^2$. So, for a given j there is only one eigenvalue of J^2 and so V_λ is only trivially invariant under the action of J_i. Therefore, we can say that J_i is an irreducible representation on the space of eigenvectors V_λ.

We know that J^2 commutes with $J_i$and so it easy to show that it commutes with any function of J_i. And so we can say that

$$[J^2, exp(\frac{-iJ_i\phi}{\hbar})] = 0$$

And from (eq), we get,

$$[J^2, D(R)] = 0$$

This tells us that the quantum mechanical rotation operator $D(R)$ also commutes with $J^2 i.e$ if $v_{j,m}$ is an eigenvector of J^2 then $D(R)v_{j,m}$ is also an eigenvector of J^2.

$$J^2 D(R) v_{j,m} = D(R) J^2 v_{j,m} = j(j+1)\hbar^2 D(R) v_{j,m}$$

From this we can infer that $D(R)$ (much like J_i) is an irreducible representation on the space V_λ for a fixed value of j. To have a better idea of the represention $D(R)$, we try find how the matrix $D(R)$ looks like.

From (8.13), we can see that,

$$J_z v_{j,m} = m\hbar v_{j,m}$$

We can use this result to get the matrix elements of J_3 by,

$$v^\dagger_{j',m'} J_z v_{j,m} = m\hbar \delta_{j,j'} \delta_{m,m'} \tag{9.1}$$

Here, we have considered the angular momentum about the z axis and so we use J_z. However, we can generalize this to any angular momentum operator J_i about the ith axis.

Using this 9.1, we can construct the matrix elements of $D(R)$

$$\mathfrak{D}^j_{m,m'} = v^\dagger_{j,m'} exp(\frac{-iJ_i\phi}{\hbar}) v_{j,m} \tag{9.2}$$

This gives us the elements of the rotation operator for a given value of j and for a fixed value of j, we have $2j+1$ values of m. And $\mathfrak{D}^j_{m,m'}$ gives us a $(2j+1) \times (2j+1)$ matrix. As we have seen that $D(R)$ is an irreducible representation on the space eigenvectors for a given value of j, any rotation has a representation $D(R)$ will have block diagonal form where these blocks are given by the $\mathfrak{D}^j_{m,m'}$.

We have now got an idea of the representation of rotations in the quantum mechanical state space looks like. However, inorder to get accurate description of the matrix we need an expression for the eigenvectors $v_{j,m}$ and the angular momentum operator J_i.

10 Schwinger's Oscillator Model of Angular Momentum

This model was developed with the idea that *spin* $\frac{1}{2}$ system can be represented using 2 quantum mechanical harmonic oscillators where the spin up ($m = \frac{1}{2}$) could be associated to an oscillator and spin down ($m = -\frac{1}{2}$) particles could be associated with another oscillator.

Let us consider the system comprising of 2 uncoupled quantum harmonic oscillators [2] Say Oscillators 1 and 2. We have creation and annhialation operators $A_1^\dagger, A_2^\dagger$ and A_1, A_2 respectively for the oscillators. Each oscillator is associated with a number operator which is defined by

$$N_1 = A_1^\dagger A_1,\ N_2 = A_2^\dagger A_2 \tag{10.1}$$

From the previous section, we have the commuation relations for the operators of the harmonic oscillators,

$$[A_1, A_1^\dagger] = 1,\ [A_2, A_2^\dagger] = 1\ [N_1, A_1] = -A_1\ [N_2, A_2] = -A_2\ [N_1, A_1^\dagger] = A_1^\dagger\ [N_2, A_2] = A_2^\dagger \tag{10.2}$$

[2] A detailed deductions of quantum mechanical harmonic oscillators has been covered in the appendix.

Since, we have assumed the oscillators are uncoupled, we can assume that any pair of operators between different oscialltors commute.

$$[A_1, A_2^\dagger] = [A_2, A_1^\dagger] = 0$$

So, we can say that these operators between the different oscillators share the same eigenstates. From this result, we can find the commutation relation between the number operators of the oscillators $N_1$and N_2. Using (10.1) and (10.2)

$$[N_1, N_2] = [A_1^\dagger A_1, A_2^\dagger A_2] = 0$$

And now that we know that $N_1$and N_2 commute we can construct simultaneuous eigenstates $v_{n_1 n_2}$ for both the operators with eigenvalues $n_1$and n_2 respectively.

$$N_1 v_{n_1,n_2} = n_1 v_{n_1,n_2} \quad N_2 v_{n_1,n_2} = n_2 v_{n_1,n_2}$$

Using the relations we derived for a single harmonic oscillator in the previous section, we can wrtie the following results.

$$A_1^\dagger v_{n_1,n_2} = \sqrt{n_1+1} v_{n_1+1,n_2} \; A_2^\dagger v_{n_1,n_2} = \sqrt{n_2+1} v_{n_1,n_2+1}$$

$$A_1 v_{n_1,n_2} = \sqrt{n_1} v_{n_1-1,n_2} \; A_2 v_{n_1,n_2} = \sqrt{n_2} v_{n_1,n_2-1}$$

We also know that the lowest value that $n_1$and $n_2$can take is 0. So, the ground state of our system of 2 oscillators is given by $v_{0,0}$. All the eigenstates of $N_1$and N_2 can be constructed by applying $A_1^\dagger$and $A_2^\dagger$. By this process we obtain,

$$v_{n_1,n_2} = \frac{(A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}} v_{0,0}$$

The idea behind the schwinger model is that we should think of each of these oscillators represent a certain type of particle in a system and the eigenvalues give the number of partices of that type in the system. Keeping this in mind, we can develop the angular momentum operators in this model.

$$J_+ = \hbar A_1^\dagger A_2 \quad J_- = \hbar A_2^\dagger A_1$$

$$J_z = \frac{\hbar}{2}(A_1^\dagger A_1 - A_2^\dagger A_2) = \frac{\hbar}{2}(N_1 - N_2)$$

$$J^2 = J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+) = \frac{\hbar^2}{2} N(\frac{N}{2} + 1)$$

where we have used a total number operator $N = N_1 + N_2$. We can check if the Angular momentum commutation relations hold,

$$[J_z, J_\pm] = \pm\hbar J_\pm \; [J_+, J_-] = 2\hbar J_z$$

Note that the ladder operators $J_+$and J_-make use of the creation and annhialation operators of the oscillators to produce increase and decrease the number of particles in a given state, respectively. Using our definitions of the angular momentum operators, we can see how act on the states,

$$J_+ v_{n_1,n_2} = \hbar A_1^\dagger A_2 v_{n_1,n_2} = \sqrt{n_2(n_1+1)}\hbar v_{n_1+1,n_2-1}$$

$$J_- v_{n_1,n_2} = \hbar A_2^\dagger A_1 v_{n_1,n_2} = \sqrt{n_1(n_2+1)}\hbar v_{n_1-1,n_2+1}$$

$$J_z v_{n_1,n_2} = \frac{\hbar}{2}(N_1 - N_2) v_{n_1,n_2} = \frac{1}{2}(n_1 - n_2)\hbar v_{n_1,n_2}$$

Note: In all these operations, we can see that the total number of spin $\frac{1}{2}$ particles in the system is conserved. Recall our expression for $J_\pm$and $J_z$operators dervied in ()(). Comparing that to the above equations we can infer that,

$$n_1 = j + m \;\; n_2 = j - m$$
$$j = \frac{n_1 + n_2}{2} \;\; m = \frac{n_1 - n_2}{2}$$

Using these relations, we can see that the expression for eigenvalues of J_z and J^2 operators are the same as the ones we developed in () ().

The operator J_+ changes n_1 to $n_1 + 1$ and n_2 to $n_2 - 1$ which means that j does not change but m becomes $m + 1$. Similarly, the J_-operators does not change j but reduces m to $m - 1$. So, in essence the change in m-value after the action of the ladder operators characterizes the change in both n_1 and n_2 values. Hence, we write the eigenvectors of N_1 and N_2 in the form,

$$v_{j,m} = \frac{(A_1^\dagger)^{j+m}(A_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}} v_{0,0} \tag{10.3}$$

This expression gives us all we need to know about the Angular momentum states of the system. As far as rotational transformations go, we can say that any object with angular momentum j can be thought of a system of $2j$ spin $\frac{1}{2}$particles such that each state is given by 10.3. To help visualize this system, lets consider to cases

Case 1: $j = m$, the state is given by

$$v_{j,j} = \frac{(A_1^\dagger)^{2j}}{\sqrt{(2j)!}} v_{0,0}$$

This can be thought of as the state where $2j$ spin $\frac{1}{2}$ particles with all their spin pointing in the positive z direction.

Case 2: $j = -m$

$$v_{j,-j} = \frac{(A_2^\dagger)^{2j}}{\sqrt{(2j)!}} v_{0,0}$$

This can be thought of as the state where $2j$ spin $\frac{1}{2}$ particles with all their spin pointing in the negative z direction.

The Schwinger model enables us to think about a quantum mechanical system using harmonic oscillators to describe the different types of particles. Most importantly, it gives an expression for the angular momentum operators their eigenvectors $v_{j,m}$ which will help us find out the exact nature of the rotation operator $D(R)$.

Using the expressions (),() which can substitute them in () and get the matrices $\mathfrak{D}^j_{m,m'}$ and see that they cannot be broken down into smaller blocks. Further, we can construct a matrix for $D(R)$ with all $\mathfrak{D}^j_{m,m'}$ appearing on the diagonal as blocks. We shall go into getting this explicit form as it requires basic, yet extensive computation. However, we have learnt that rotations have an irreducible representation on the quantum mechanical state space.

11 $SU(2)$ representations in Angular Momentum theory

From our detailed excursions into Representations of $SU(2)$ and Angular momentum theory, it has been quite evident that there is a natural link between these 2 distinct areas of Mathematics and Physics. Nevertheless, I shall try to point out these similarities and possibly give a reason for these apparently 'coincidental' connections.

Knowing that rotations have an irreducible representation $D(R)$ on the state space and the eigenvectors of $D(R)$ take the form in 8.1 shows us a lot similarity with the irreducible representation of $SU(2)$.

Lets consider 8.1 and EQ ,

$$e^k(z) = \frac{t_0^{n+k}t_1^{n-k}}{\sqrt{(n-k)!(n+k)!}} \, where \, k = -n, -n+1, ..., n$$

$$v_{j,m} = \frac{(A_1^\dagger)^{j+m}(A_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}} v_{0,0} \, where \, m = -j, -j+1...j$$

We witness the fact that angular momentum eigenvectors and exactly the same as the eigenvectors of the irreducible representation of Lie algebra $\mathfrak{su}(2)$.

This is not just by accident as we can see the similarities in their construction quite clearly. For example, the ladder-up, ladder-down operators and raising, lowering operators execute similar actions on the respective eigenvectors.

$$J_+ = \hbar A_1^\dagger A_2 \;\; \rho_1(E_+) = -t_1 \frac{\partial}{\partial t_0}$$

$$J_- = \hbar A_2^\dagger A_1 \;\; \rho_1(E_-) = -t_0 \frac{\partial}{\partial t_1}$$

Also, we compare the action of the angular momentum operator J_z and the operator H which is the Lie algebra representation of the general diagonalized element of $SU(2)$.

$$\frac{J_z}{\hbar} = \frac{1}{2}(A_1^\dagger A_1 - A_2^\dagger A_2) \ \ \rho_1(H) = \frac{1}{2}(t_1\frac{\partial}{\partial t_1} - t_0\frac{\partial}{\partial t_0})$$

It is also instructive to note the bounds of eigenvalues of both representations are also the same .(from 6.12, 6.11 and 8.12) .It is easy infer from these distinct similarities that the angular momentum operators J_i act on the state space through a Irreducible Lie algebra representation of $SU(2)$.

We have seen that

$$D(R)v_{j,m} = exp(\frac{iJ_i\phi}{\hbar})v_{j,m}$$

and we know that J_i has diagonal elements given by $m\hbar$. If we consider the case of rotating by $\phi = 2\pi$, a system where j takes half-integer values, (then consequently m takes half-integer values), we'd expect to get back thse vector. But, we get

$$D(R)v_{j,m} = -v_{j,m}$$

However, if we rotate the system by an angle $\phi = 4\pi$, then we get

$$D(R)v_{j,m} = v_{j,m}$$

Thus, $D(R)$ is a double covering of the group of rotations on the space of quantum states. In other words, there are 2 elements in the rotation group $SO(3)$ which map to $D(R)$.

We know that $SU(2)$ offers a double cover to $SO(3)$ (from Section 5) and so there exists a isomorphism between $D(R)$ and $SU(2)$.

From its definition along with the commutation relations, we can say that $D(R)$ and J_i share a Lie group -Lie algebra relationship. However, we need to show that $D(R)$ is a Lie group.

This is not a ground breaking development to our idea of rotations in quantum mechanical state space as the idea behind the conception of the $D(R)$ and J_i operators was to make them a representation of $SU(2)$ for various purposes. However, we have developed both ,theories of $SU(2)$ representations and angular momentum theory , independently and consequently seen the connection between these diverse areas of mathematics and physics .

Part V. Configuration of points

12 Euclidean Conjecture

Physicists and mathematicians, since the evolution of quantum theories, have tried to understand and invent methods to visualize the quantum states of a system. The problem proposed by M.V. Berry and J.M. Robbins[ref] was to find a map which, if at all it exists, bridges the gap between ideas of states in classical physics (positions in $\mathbb{R}^3$) to the quantum states (vectors in $\mathbb{C}^n$).

To materialize this idea, the problem connects two famous spaces:

The configuration Space $C_n(\mathbb{R}^3)$ When we consider a physical system, the parameters that define the system are represented by coordinates and these coordinates define a vector space which is often refered to as the Configuration space of the system. The position of a single particle moving in 3- dimensional space is given by a vector $(x, y, z) \in \mathbb{R}^3$. So, the confirguration space of a single particle is $\mathbb{R}^3$.

Now, if we consider a system of n distinct particles in 3- dimensional space, we have $3n$ coordinates which define the configuration of the system. Hence, the configuration space of the system will be a subset of $\mathbb{R}^{3n}$. It will only be a subset because we define our system to have n distinct particles. Hence, we have to remove the subspaces of $\mathbb{R}^{3n}$ where the coordinates of any 2 points coincide. This represents the configuration space of n distinct points in $\mathbb{R}^3$.

The Flag manifold $U(n)/T^n$ The Flag manifold, for all practical purposes, can be thought of as the space of n orthonormal vectors in $\mathbb{C}^n$. (defined Up to a phase)

The flag manifold is given by the space $\mathbb{F}_n$ which contains all the sequences

$$V_1 \subset V_2 \subset V_3 \subset ... \subset V_n = \mathbb{C}^n$$

where V_i is a linear subspace of $\mathbb{C}^n$ such that $dim(V_i) = i \ \forall i = 1, 2.., n$. Such a sequence is called a flag in $\mathbb{C}^n$. Now, each of these V_i are spanned by i orthonormal vectors $(e_1, e_2, ..e_i)$ which form the standard basis for V_i. Using these standard basis, we can define a standard flag ,$\{V_i^*\}$

$$Span(e_1) \subset Span(e_1, e_2) \subset ... \subset Span(e_1, e_2, ...e_n) = \mathbb{C}^n$$

We claim that for every flag $\{V_i\}$, there exists a $A \in GL(n, \mathbb{C})$ such that $A\{V_i^*\} = \{V_i\}$, i.e.

$$A(Span(e_1, e_2, .., e_i)) = V_i$$

Now, if we choose our matrix A in such way that $\{Ae_1, Ae_2, .., Ae_n\}$ is orthonormal basis of $\mathbb{C}^n$ with respect to the standard Hermitian inner product on $\mathbb{C}^n$, we see that A satifies

$$< Ac, Ad >=< c, d >$$

for any $c, d \in \mathbb{C}^n$.As we can write them as

$$c = \sum_{j=1}^{n} c_j e_j \, and \, d = \sum_{j=1}^{n} d_j e_j$$

$$< Ac, Ad >=< \sum_{j=1}^{n} c_j(Ae_j), \sum_{j=1}^{n} d_j(Ae_j) >=< \sum_{j=1}^{n} c_j e_j, \sum_{j=1}^{n} d_j e_j >=< c, d >$$

From the above statement, we can deduce that $A \in U(n)$. Therefore, flags can be identified with the elements of $U(n)$ if we remove the subgroup which leaves the standard flag fixed $i.e\, X = \{A \in U(n),\, Ae = e\}$ where e denotes the standard flag.

For the action of $U(n)$, this subgroup X is given the group of all diagonal matrices in $U(n)$, more specifically, X consists of all matrices of the form,

$$A = \begin{bmatrix} z_1 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & z_n \end{bmatrix}$$

where $z_j \in \mathbb{C}$, $|z_j| = 1$, $j = 1, 2, ..., n$. Each of the z_j can be written of the form $e^{i\theta_j}$and so,

$$A = \begin{bmatrix} e^{i\theta_1} & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & e^{i\theta_n} \end{bmatrix}$$

Now, X can be thought of the product space of n unit circles, $(S^1)^n$. This space is called the $n - torus$ and is denoted by T^n. Therefore, the flag manifold is represented by the group $U(n)/T^n$ and so any element in $U(n)/T^n$ represents a unique flag, and thus a unique, ordered set of n orthonormal vectors. (ambiguous up to a phase)

12.1 Action of Symmetry Group

Definition G.1 : Free Action

Let$G$ be a group and X be a non-empty set. If $g, h \in G$, $x \in X$ and $g.x = h.x \implies g = h$ then G is said to act freely on X.

If we have a collection of n distinct ordered points in $\mathbb{R}^3$say $(x_1, x_2, ..., x_n)$, this corresponds to a particular configuration $i.e$ a point in $C_n(\mathbb{R}^3)$. If we permute these points in some way, say exchange the points x_1 and x_2 $(\sigma_{21} \in \sum_n)$, although we still have the same collection of points, we have a different configuration $(x_2, x_1, ...x_n)$.

From the definition of $\sum_n$, we know that there is only one element, σ_{21} in the group which permutes $(x_1, x_2, ..., x_n)$ to $(x_2, x_1, ...x_n).i.e$ no two elements of the symmetry group act on an element $x \in C_n(\mathbb{R}^3)$ to give the same result. So, we can say that the Symmetry group $(\sum_2)$ acts freely on the configuration space $C_n(\mathbb{R}^3)$.

Similiarly, we can deduce that $\sum_n$acts freely on the ordered set of n orthonormal vectors. If we permute these n orthonormal vectors, then by definition, the newly formed set of orthonormal vectors correspond to an unique flag, and by extension correspond to another element of $U(n)/T^n$.

Clearly, the symmetry group acts freely on each of these manifolds by permuting points and vectors , respectively.

12.2 The Berry-Robins Problem

The inception of the Atiyah conjecture is attributed to the question posed by Berry-Robins in their paper [8]. The question was about the existence of a map, for each n, from $C_n(\mathbb{R}^3)$ to $U(n)/T^n$,

$$f_n : C_n(\mathbb{R}^3) \to U(n)/T^n \tag{12.1}$$

such that f_n was compatible with the action of the symmetry group.

For a fixed n, let us consider any $\sigma \in \sum_n$ and $x \in C_n(\mathbb{R}^3)$. We say that f_n is compatible with the action of symmetry group if

$$f_n(\sigma.x) = \sigma.f_n(x)$$

In other words, it means that σ permutes $f_n(x)$ the in the 'same manner' that it permutes x.

The case of $n = 1$ is trivial. The first non-trivial case of the Berry-Robins problem is for $n = 2$. Let $(x_1, x_2) \in C_2(\mathbb{R}^3)$ where $x_i \in \mathbb{R}^3$ and $x_1 \neq x_2$.

On the other hand, for $n = 2$, the space $U(2)/T^2$ represents the ordered pair of orthonormal vectors in $\mathbb{C}^2$. We also know that the set of all unit vectors in $\mathbb{C}^2$ is given by $S^3(3 - sphere)$. Using the hopf fibration, we can map all these unit vectors in $\mathbb{C}^2$ to the $S^2(2 - sphere)$.

$$U(2)/T^2 = S^2$$

The ordered pair (x_1, x_2) has to be mapped to a point on the sphere S^2. So, for the ease of explanation, the following transformation is applied.

$$(x_1, x_2) \to (\frac{1}{2}(x_2 + x_1), \frac{1}{2}(x_2 - x_1))$$

This helps us identify the space as

$$C_2(\mathbb{R}^3) = \mathbb{R}^3 \times (\mathbb{R}^3 - \{0\})$$

Observing that $\sum_2$permutes $x_1$and x_2 , thus changing the sign of $\frac{1}{2}(x_2 - x_1)$. So, if we have a $v \in C_2(\mathbb{R}^3)$, then $\sum_2$ acts on it to give $-v$. The map which connects vectors v and $-v$ on S^2 is called the antipodal map. We can say that $\sum_2$ acts on S^2 as the antipodal map. Hence, we have a natural map f_2 mapping $(x_1, x_2) \in C_2(\mathbb{R}^3)$ to points on the unit sphere S^2.

$$f_2(x_1, x_2) = \frac{x_2 - x_1}{|x_2 - x_1|} \tag{12.2}$$

12.2.1 Properties of f_2

In addition to the compatibility with symmetry group, the map f_2,

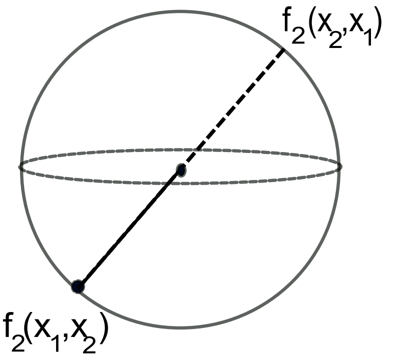

Fig. 12.1: Antipodal nature of f_2

1. Translation invariance If the vector $(x_1, x_2) \in C_n(\mathbb{R}^3)$ is translated by k units, then each of the points $x_1$and x_2 are translated by k units in $\mathbb{R}^3$.

$$k + (x_1, x_2) = (x_1 + k, x_2 + k)$$

It is quite obvious to see that such a translation will be invariant with respect to the map f_2.

$$f_2(x_1 + k, x_2 + k) = \frac{(x_2 + k) - (x_1 + k)}{|(x_2 + k) - (x_1 + k)|} = \frac{x_2 - x_1}{|x_2 - x_1|} = f_2(x_1, x_2)$$

2. Scalar invariance It is also quite obvious to see that the map f_2 is scalar invariant. If the vector $(x_1, x_2) \in C_n(\mathbb{R}^3)$ is multiplied by a scalar$k$, then the points $x_1$and x_2 becomes kx_1 and kx_2.

$$f_2(kx_1, kx_2) = \frac{k(x_2 - x_1)}{|k||x_2 - x_1|} = \frac{x_2 - x_1}{|x_2 - x_1|} = f_2(x_1, x_2)$$

3. Compatibility with rotation group $SO(3)$ As $x_1, x_2 \in \mathbb{R}^3$, there is natural action of $SO(3)$ on the both sides of the (12.2). Let $R \in SO(3)$, then

$$f_2(Rx_1, Rx_2) = \frac{Rx_2 - Rx_1}{|Rx_2 - Rx_1|} = \frac{R(x_2 - x_1)}{|x_2 - x_1|}$$

Now, the points $Rx_1$and Rx_2 map to the unit vector in the direction$R(x_2 - x_1)$. Therefore, it is pretty straight-forward that the map f_2 is compatible with the action of the rotation group $SO(3)$.

13 The Stellar Representation

The stellar repsentation helps us in constrcuting a practical way of thinking about vectors in $\mathbb{C}^n$. It was invented as a way to think about the spin of paritcles of quantum mechanical system as points on a sphere. We have seen how one could project the vectors in $\mathbb{C}^2$ on $\mathbb{C}P^1$. Extending this idea to $\mathbb{C}^n$, we can say that it is possible to project vectors in $\mathbb{C}^n$ to the complex projective space$\mathbb{C}P^{n-1}$. A complex polynomial P of degree $n-1$ in a complex variable z is given by

$$P(z) = a_0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + ... + a_{n-1} z^{n-1}$$

The polynomial can be factorized by their roots,
[3]

$$P(z) = a_0(z - z_1)(z - z_2)..(z - z_{n-1})$$

If we let $a_0 = 1$ and rescale the other $a_i's$, we can associate the set of these $(n-1)$ roots (given by $z_j's$) to a vector in $\mathbb{C}P^{n-1}$. If we let $a_0 = 0$, then all the roots $z_{ij} = \infty$.

So, we can establish an injective map with the unordered sets of $(n-1)$ complex numbers to $\mathbb{C}P^{n-1}$. Here we say unordered sets of complex numbers as we must remember that even if we change order of these $(n-1)$ roots , they still define only one polynomial. Therefore, we can say that every vector in $\mathbb{C}P^{n-1}(\mathbb{C}^n \amalg \infty)$ can be represented by a complex polynomial of degree $(n-1)$.

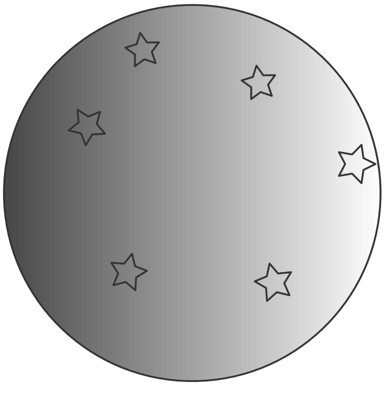

Fig. 13.1: Stellar Representation of $\mathbb{C}P^6$

As every root of the polynomial P is given by a point in $\mathbb{C}$, we can use the stereographic projection.[4] to map any point on the complex plane to the $2-sphere.$

$$\mathbb{C}P^1 \equiv S^2$$

Consequently, we can write

$$\mathbb{C}P^{n-1} \equiv (S^2 \times S^2 \times ... \times S^2)/\sum{}_{n-1}$$

[3] From the fundamental theoram of algebra, we know that every non-zero,single variable polynomial of degree n with complex coeffients, counted upto their multiplicity, has exactly n roots. So, one need not worry about the existence of roots for such polynomials.

[4] Dealt with in detail in section 1

where $\sum_{n-1}$ is the symmetric group which permutes the $(n-1)$ points. These points on S^2 which represent the roots are called stars.

14 Constructing a general map f_n

14.1 Polar Decomposition

If we have an invertible complex matrix $A \in GL(n, \mathbb{C})$, there always exists a unique decompostion of A to the form

$$A = UP$$

where $U \in U(n)$ and P is a positive definite Hermitian Matrix. This decomposition is called polar decomposition.

Conversely, we say that every $U \in U(n)$ can be uniquely represented in the form

$$U = AP^{-1}$$

$$g : GL(n, \mathbb{C}) \to U(n)$$

It is easy to see that from our definition of the map g that it maps T^n to itself. So, we can factor out the action of the $n - torus$ we get,

$$g : GL(n, \mathbb{C})/T^n \to U(n)/T^n$$

We should note that this map is also compatible with the $\sum_n$ as its action on $GL(n, \mathbb{C})/T^n$ is similar to that of $U(n)/T^n$.

The problem boils down to the construction of a map

$$F_n : C_n(\mathbb{R}^3) \to GL(n, \mathbb{C})/T^n$$

such that F_n is compatible with the action of the symmetry group $\sum_n$. Then, we can get our map

$$f_n : C_n(\mathbb{R}^3) \to U(n)/T^n$$

by the composition of the maps $g \circ F_n = f_n$.

14.2 From points to polynomials

Using the stellar representation, we can represent a vector in $\mathbb{C}^n$ as a complex polynomial of degree $n-1$. However, to represent an orthonormal vector in $\mathbb{C}^n$ by a polynomial, there must be certain restrictions on a polynomial. It might be difficult to find these restrictions on a polynomial so we consider a way around this problem. Lets try to construct a general map

$$F_n : C_n(\mathbb{R}^3) \to GL(n, \mathbb{C})/T^n \tag{14.1}$$

Note that $GL(n, \mathbb{C})$ represents $n \times n$ invertible matrics whose columns represent n linearly independent vectors. We have relaxed the unitary condition for the time being $i.e$ instead of considering n orthonormal vectors in $\mathbb{C}^n$, we consider shall consider n linearly independent vectors in $\mathbb{C}^n$ and T^n helps us factor out the ambiguous phases. These n linearly independent vectors must now correspond to a set of n complex polynomials of degree $n-1$. Once, we get these polynomials, we must figure out a way to normalize these polynomials such that $||p_i|| = 1$ in $\mathbb{C}^n$ such that the polynomial is determined upto a phase factor.

Assuming that this normalization procedure of polynomials is possible, we can then think of these normalized polynomials as the orthnormal vectors in $\mathbb{C}^n$ that we require.

In the $n = 2$ case, we mapped a point (x_1, x_2) to a unit vector lying on S^2.

$$x = (x_1, x_2) \to \frac{x_2 - x_1}{|x_2 - x_1|}$$

Consider a configuration $x = (x_1, x_2, ..., x_n) \in C_n(\mathbb{R}^3)$. Taking two of the components at a time, we have,

$$z_{ij} = \frac{x_j - x_i}{|x_j - x_i|}$$

If we fix i and let $j \neq i$, we get $(n-1)$ unit vectors each of which can be represented as a point on S^2. Let p_i be a complex polynomial on a complex variable z of degree $n-1$ having these $z_{ij}'s$ as its roots.

$$p_i = \prod_{j \neq i} (z - z_{ij}) \tag{14.2}$$

As any $x \in C_n(\mathbb{R}^3)$ has n components, we shall obtain a set of n polynomials $(p_1, p_2, .., p_n)$ which correspond to the point x.

$$x = (x_1, x_2, ..., x_n) \equiv (p_1, p_2, ..., p_n)$$

To summarize, from the stellar representation, we know that each of these polynomials p_i (which are given by an unordered set of roots, $\{z_{ij}\}$ upto a complex multiple) can be thought of as points on the $\mathbb{C}P^{n-1}$ and every root z_{ij}can be thought of as a point on the $2-sphere$.

$$p_i \in \mathbb{C}P^{n-1} = (S^2 \times S^2 \times ... \times S^2)/\sum{}_{n-1} \tag{14.3}$$

And a particular configuration $x \in C_n(\mathbb{R}^3)$ can be mapped to the set of these polynomials.

$$x = (x_1, x_2, ..., x_n) \equiv (p_1, p_2, ..., p_n)$$

It is also quite straight forward to see that $\sum_n$ acts on $x_i's$ and $p_i's$ in a similar manner. Hence, this map is compatible with $\sum_n$.

15 Euclidean Conjecture:

For any $(x_1, x_2, ..., x_n) \in C_n(\mathbb{R}^3)$, ***the polynomials*** $p_1, p_2 ..., p_n$ ***defined above, are linearly independent.***

If this conjecture is proven, then we get a map which fits our requirements for the map

$$F_n^i(x_1, x_2, ..., x_n) = p_i \;\; for\, i = 1, ..., n$$

and by extension the requirements for the map f_n (12.1) map.

Also, compatibility with the symmetry group $\sum_n$ is in-built for the polynomials p_i. However, we also need to prove that the map f_n exhibit the same properties as f_2 .

Although properties of translation and scalar invariance can be proven trivially, the property of compatability with $SO(3)$ is not very easy to prove. Discussions in section the orthogonal representation of $SU(2)$ (section 5) might be very useful here.

15.1 Compatibility with the Rotation group $SO(3)$

Inorder to show the compatibility of f_n with $SO(3)$, we need to show that the representations of $SO(3)$ adhere to the intertwining map with f_n. If $\pi$is a representation of $SO(3)$ on the $C_n(\mathbb{R}^3)$ and σ is the representation of $SO(3)$ on the space of complex polynomials of degree $n - 1$.

$$f_n \circ \pi(R) = \sigma(R) \circ f_n \tag{15.1}$$

However, we know that $SU(2)$ is the double cover of $SO(3)$ (From the section on Orthogonal representation) and from our previous sections, we have already seen $SU(2)$ representations on the space of polynomials. Hence, we can change (ref) to the following

$$f_n \circ (\pi \circ \phi) = \rho(U) \circ f_n \tag{15.2}$$

where ϕ is the orthogonal representation of $SU(2)$ (5.1) and ρ is the irreducible representation on the space of polynomials given by 6.5. Lets try to see this clearly. Given a configuration $x \in C_n(\mathbb{R}^3)$, $\phi(U)$ acts on x and gives an element in $SO(3)$ and π takes this rotation and applies it on the configuration x and produces a new configuration Rx and f_n takes Rx to a polynomial p_i'.

On the right hand side, we have irreducible representation of $SU(2)$ which acts on the space of polynomials and f_n acts on the given configuration $x \in C_n(\mathbb{R}^3)$ to give a polynomial p_i. $\rho(U)$ acts on p_i as follows, 6.5

$$\rho(U)p_i(z) = p_i(\bar{a}t_0 - bt_1, \bar{b}t_0 + at_1) \tag{15.3}$$

Inorder to prove the compatibility condition, we just need to show that $p_i* = p_i'$ which in turn tells us that 15.2 holds,Therefore,the action of R on a configuration $(x_1, x_2, ..., x_n)$ corresponds to the action of U on the roots $(z_{i,1}, ..., z_{i,n-1})$.

$$(x_1, x_2 ..., x_n) \rightarrow (Rx_1, Rx_2, ..., Rx_n)$$

$$(z_{i,1}, ..., z_{i,n-1}) \to (Uz_{i,1}, ..., Uz_{i,n-1}) \tag{15.4}$$

We know that $U \in SU(2)$ can be represented as follows,

$$U = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

As $z_{ij} \in \mathbb{C}^2$,there exists a natural action of U on z_{ij}. Therefore, 15.4 becomes

$$z_{ij} \to \frac{az_{i,j} + b}{-\bar{b}z_{i,j} + \bar{a}}$$

We also know that the polynomial p_i is given by (14.2)

$$p_i = \prod_{j \neq i} (z - z_{ij})$$

where $z_{ij} = \frac{x_j - x_i}{|x_j - x_i|}$.

$$p_i'(z) = \prod_{j \neq i} (z - \frac{az_{i,j} + b}{-\bar{b}z_{i,j} + \bar{a}}) = \prod_{j \neq i} (\frac{-\bar{b}z_{ij}z + \bar{a}z - az_{ij} - b}{-\bar{b}z_{i,j} + \bar{a}}) \tag{15.5}$$

On the other hand, we have 15.3 and convert the homogeneous coordinates into inhomogeneous coordinates ,$z = \frac{t_0}{t_1}$,

$$\rho(U)p_i(z) = \sum_{k=0}^{n} c_k (\bar{a}t_0 - bt_1)^{n-k} (\bar{b}t_0 + at_1)^k$$

Multiplying and dividing the right hand side by $(\bar{b}t_0 + at_1)^{n-1}$, we get,

$$\rho(U)p_i(z) = (\bar{b}t_0 + at_1)^{n-1} p_i(\frac{\bar{a}t_0 - bt_1}{\bar{b}t_0 + at_1}) = (\bar{b}z + a)^{n-1} p_i(\frac{\bar{a}z - b}{\bar{b}z + a})$$

From our definition of the polynomials p_i (14.2),

$$p_i(\frac{\bar{a}z - b}{\bar{b}z + a}) = \prod_{j \neq i} (\frac{\bar{a}z - b}{\bar{b}z + a} - z_{ij}) = \prod_{j \neq i} (\frac{-\bar{b}z_{ij}z + \bar{a}z - az_{ij} - b}{\bar{b}z + a})$$

Therefore, our representation $\rho(U)$ acts on $p_i(z)$ to give

$$p_i * (z) = \rho(U)p_i(z) = (\bar{b}z + a)^{n-1} \prod_{j \neq i} (\frac{-\bar{b}z_{ij}z + \bar{a}z - az_{ij} - b}{\bar{b}z + a}) = \prod_{j \neq i} (-\bar{b}z_{ij}z + \bar{a}z - az_{ij} - b)$$

Comparing this polynomial with, 15.5, we can see that they are the same upto a phase factor determined by the denominator of 15.5. And so, the polynomials we get $p_i* = p_i'$. This proves that the map f_n is compatible with the group of rotations $SO(3)$.

15.2 Normalizing p_i

The approach to verify this conjecture taken by Sir micheal Atiyah in his paper **??**, is to define a determinant function of these polynomials p_i. We can think about these polynomials as vectors in $\mathbb{C}P^1 \cong \mathbb{C}^n$. Therefore $(p_1, p_2 ... p_n)$ can be visualized to form a $n \times n$ complex matrix.

We know that if the determinant has a non-zero value then, the matrix is invertible and hence, the polynomials $(p_1, p_2, ..., p_n)$ are linearly independent. Therefore, we need to find a method to normalize the polynomials $p_i's$.

We have shown using the stellar representation that vetors in $\mathbb{C}^n$ can be projected onto $\mathbb{C}P^{n-1}$ by using complex homogeneous polynomials of degree $(n-1)$ on a variable $z = (t_0, t_1) \in \mathbb{C}^2$.

In section 7 , inorder to find an $SU(2)$ invariant inner product on space of homogeneous polynomials, we had worked out the a normalization procedure on complex homogenous polynomials of degree n on a complex variable $z = (t_0, t_1)$. This could help us find a norm $||p_i||$ of the polynomials defined in 14.2. Once, we find an expression for the $||p_i||$, then we conviniently set $||p_i| = 1$ in order to get orthonormal polynomials.

This has been done in 7.1 where we have an expression for a normalized monomial $t_0^{n+k} t_1^{n-k}$. Extending this normalization process to a polynomial of degree $(n-1)$ in homogeneous coordinates (t_0, t_1)

$$p(t_0, t_1) = a_0 t_0^{n-1} + a_1 t_0^{n-2} t_1 + ... + a_{n-2} t_0 t_1^{n-2} + a_{n-1} t_1^{n-1}$$

As we are normalizing the whole polynomial, we have to take into account the coefficients of this polynomial. This can be done by

$$||p||^2 = \sum_{k=0}^{n-1} ||t_0^{n-k-1} t_1^k|| \, |a_k|^2$$

where $||t_0^{n-k-1} t_1^k||$ is given by (7.1).

$$||p||^2 = \sum_{k=0}^{n-1} \frac{(n-k-1)!k!}{n!} |a_k|^2 \tag{15.6}$$

As this is an $SU(2)$ invariant inner product (by construction) and we know that $SU(2)$ is the double cover of $SO(3)$, hence, this inner product on the space of polynomials preserves $SO(3)$ invariant. The Irreducible representation of $SU(2)$ acts on a polynomial p_i without changing its norm.

As we are dealing with complex polynomials, the process of making $||p||^2 = 1$ in 15.6 will normalize the polynomials only upto a phase factor. If we need to completely normalize the polynomials, we have to eliminate this problem of phases.

Conclusion

Having seen the explicit relationship the Lie group and Lie Algebra representation of $SU(2)$ with the Rotation and angular momentum operators, we can began to appreciate the fact that several constructions in physics use some abstract mathematical structures to fulfill their requirements. $SU(2)$ representations appear in many places in physics. Apart from Angular momentum theory, they also appear in the electroweak interactions which describe the interactions between electromagnetism and weak force of interaction.

In our investigations on the Atiyah conjecture, we have proven that if normalization procedure for the phases is determined, then we can prove that the properties of translation, scalar and compatibility with $SO(3)$ hold. we have have seen that the way to go inorder to solve the problem was to find a way to define a normalized determinant for these polynomials p_i, which even normalizes the arbitrary phase factor. We have seen how the euclidean conjecture works for the first non-trivial case of $n = 2$. However, there are proofs given for the case $n = 3$ by Micheal Atiyah[2] himself. The case $n = 4$ of the euclidean conjecture has also been solved.

There is also a similar conjecture which deals with a similar problem but with configurations in Hyperbolic 3- space.

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Appendix: Simple Harmonic Oscillators

In Quantum Mechanics, we refer to the Hamiltonian as the opertaor which represents the total energy in the system. When we attempt to measure the energy of the system, the set of all outcomes of measurement are given by the eigenvalues of the Hamiltonian operator. The hamiltonian operator for a simple harmonic operator can write in the form,

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \frac{p^2 + (m\omega x)^2}{2m}$$

where $\omega^2 = \frac{k}{m}$ is the angular frequency of the operator. We shall now introduce operators A and $A^\dagger$

$$A = \frac{m\omega x + ip}{\sqrt{2m\hbar\omega}} \;\&\; A^\dagger = \frac{m\omega x - ip}{\sqrt{2m\hbar\omega}}$$

Although the reason for the introduction of the these operators will be more apparent as we proceed, for now, we can just think of these operators as factors of the Hamiltonian. We are just trying to factorize the Hamiltonian operator.

$$A^\dagger A = \frac{p^2 + (m\omega x)^2}{2\hbar m\omega} + \frac{m\omega ixp - m\omega ipx}{2\hbar m\omega}$$

$$A^\dagger A = \frac{p^2 + (m\omega x)^2}{2\hbar m\omega} + \frac{m\omega i[x,p]}{2\hbar m\omega}$$

From the Heisenberg uncertainty relation between position and momentum (Dealt with in the Appendix), we get,

$$[x, p] = i\hbar$$

$$A^\dagger A = \frac{H}{\hbar\omega} - \frac{1}{2}$$

Now, we can write H in the form,

$$H = \hbar\omega(A^\dagger A + \frac{1}{2}) \tag{.7}$$

When we defined A and $A^\dagger$, we did not see if they were hermitian. However, for our analysis so far we have not needed them to Hermitian. Now, we have to deal with $A^\dagger A$. So, lets check if it is Hermitian.

$$(A^\dagger A)^\dagger = A^\dagger A^{\dagger\dagger} = A^\dagger A$$

Lets call this Hermitian operator $N = A^\dagger A$. Now, $P. - 1$ becomes,

$$H = \hbar\omega(N + \frac{1}{2})$$

Let n be the eigenvalue of N operator corresponding to the eigenstate v_n.

$$Nv_n = nv_n$$

Since, H is just a linear function of N, we can write,

$$Hv_n = (N + \frac{1}{2})\hbar\omega v_n = (n + \frac{1}{2})\hbar\omega v_n$$

This expression gives us the eigenvalues of H which are of the the form $(n + \frac{1}{2})\hbar\omega$. Before moving on to find the range of values that n can take, we shall look at some commutation relations of the operators $A, A^\dagger \, and \, N$.

$$[A^\dagger, A] = \frac{1}{2\hbar}(-i[x,p] + i[p,x]) = -1$$

Similarly,

$$[A, A^\dagger] = \frac{1}{2\hbar}(i[x,p] - i[p,x]) = 1$$

Using these commutation relations between the operators, we get,

$$[N, A] = [A^\dagger A, A] = A^\dagger AA - AA^\dagger A = [A^\dagger, A]A = -A$$
$$[N, A^\dagger] = [A^\dagger A, A^\dagger] = A^\dagger AA^\dagger - A^\dagger A^\dagger A = A^\dagger[A, A^\dagger] = A^\dagger$$

If we apply the operators $A, A^\dagger$on the eigenvectors of N,

$$NA^\dagger v_n = (NA^\dagger - A^\dagger N + A^\dagger N)v_n = ([N, A^\dagger] + A^\dagger N)v_n = A^\dagger v_n + A^\dagger(nv_n)$$
$$\implies NA^\dagger v_n = (n+1)A^\dagger v_n \tag{.8}$$

Similarly, we can get

$$NAv_n = (n-1)Av_n \tag{.9}$$

Expressions tell us

a.) Av_n and $A^\dagger v_n$are also eigenvectors of N with eigenvalues$(n-1)$ and $(n+1)$ respectively.

b.) Since we know that H is just a linear function of N, we can see that $A^\dagger$increasing the eigenvalue of N to $n+1$ is equivalent to increasing the eigenvalues of H to $((n+1) + \frac{1}{2})\hbar\omega$ (i.e. increasing the eigenvalue by $\hbar\omega$ which is thought of as one quantum unit of energy.) This can interpreted as creating a quantum of energy in the oscillator.

As $A^\dagger$increases the eigenvalue of the Hamiltonian which we can interpret as increasing the energy of the system by one quantum unit of enrgy, we call it the $creation\, operator$ of the quantum mechanical harmonic oscillator. Using a similar argument, we can call A as the $annhialation\, operator$ of the oscillator.

We have seen that n is the eigenvalue corresponding to the eigenvector v_n and so we denote the eigenvector corresponding to the eigenvalue $(n-1)$ by v_{n-1}. Using (.9) we can say that, $Av_n$and v_{n-1}differ only by a multiplicative constant.

$$Av_n = cv_{n-1}$$

With the assumption that both v_n and $v_{n-1}$are both orthonormal eigenvectors, we can find c by

$$< Av_n, Av_n >= |c|^2$$

$$< Av_n, Av_n >=< A^\dagger Av_n, v_n >=< Nv_n, v_n >= n < v_n, v_n >$$

$$< Av_n, Av_n >= n = |c|^2$$

$$c = \sqrt{n}$$

Note: We have assumed c to be positive and real.

$$Av_n = \sqrt{n}v_{n-1}$$

Using a similar argument, we get,

$$A^\dagger v_n = \sqrt{n+1}v_{n+1}$$

If we keep applying the lowering operator to $(P..)$,

$$A^2 v_n = \sqrt{n(n-1)}v_{n-2}$$

$$A^3 v_n = \sqrt{n(n-1)(n-2)}v_{n-3}$$

$$.$$

$$.$$

$$A^n v_n = \sqrt{n!}v_0$$

The sequence can continue if n can take negative values. However, we can show that n can take only non-negative integer values.

$$n =< Nv_n, v_n >=< A^\dagger Av_n, v_n >=< Av_n, Av_n >$$

By the non-negativity property of the hermitian inner product, this is always positive. However, if n was not an integer, then the sequence would terminate again with a negative value for n, which is not allowed. So, we conclude by saying that n can only take non-negative integral values $(n = 0, 1, 2, 3...)$ so that the sequence ends when we reach 0. Hence, the lowest possible energy state (eigenvalue of the hamiltonian) of the system is given by $H_0 = \frac{1}{2}\hbar\omega$.

By applying the raising operator $A^{\dagger}$continuously to v_0,

$$v_1 = A^{\dagger} v_0$$

$$v_2 = \frac{A^{\dagger}}{\sqrt{2!}} v_1 = \frac{(A^{\dagger})^2}{\sqrt{2!}} v_0$$

$$v_3 = \frac{(A^{\dagger})}{\sqrt{3!}} v_2 = \frac{(A^{\dagger})^3}{\sqrt{3}} v_0$$

$$.$$

$$.$$

$$v_n = \frac{(A^{\dagger})^n}{\sqrt{n!}} v_0$$

Thus, we have the expression for the eigenstates for the operator N and by extension, for the hamiltonian H.