Bayesian Analysis I, Fall 2023

Lab Exercise 1: on **Thursday Oct. 26, 14-16**

1. Let $Y|\pi$ be binomial $(n = 4, \pi)$. Suppose that we consider there are only three possible (equally likely) values for π, .4,.5, and .6. The prior distribution of π : $P(\pi = .4) = P(\pi = .5) = P(\pi = .6) = 1/3$. Suppose $Y = 3$ was observed.

   a) Find the likelihood of $Y = 3$ (i.e. binomial probabilities).

   b) Obtain the posterior probability of π given $Y = 3$.

   c) Using the Bernoulli trials instead of the binomial distribution, determine the likelihood of $Y = 3$ and obtain the posterior probability of π given $Y = 3$.

2. For each of the following distributions, write down the probability density function and find a corresponding kernel. Your kernel can be as simple as possible.

   a) $X|\theta \sim$ Poisson (θ).

   b) $Y|\theta, \beta \sim \text{Beta}(\beta\theta, \beta)$.

   c) $\theta|\alpha, \beta, x \sim \text{Gamma}(\alpha + x + 1, 1/(\beta - 3x))$.

   d) $\phi|\mu, \overline{x}, \tau \sim \text{Normal}(\tau\mu + (1 - \tau)\overline{x}, \overline{x}^2\tau^{-2})$.

3. In each of the following, state the distribution of the corresponding random quantity.

   a) $f(x) \propto e^{-2x}, \quad x > 0$.

   b) $f(x) \propto 1, \quad 0 \leq x \leq 1$.

   c) $f(y|\mu) \propto \mu^y/y! \ \ y = 0, 1, 2, ..., \ \mu > 0$.

   d) $f(y|\alpha, \beta) \propto y^{-\alpha}e^{-1/(2\beta y)}, \ \ y > 0, \alpha > 1, \beta > 0$.

   e) $f(x|m) \propto (1 - x)^{(m-1)/2}, \ \ 0 \leq x \leq 1, m > -1/2$.

   f) $f(z|\theta, \phi) \propto \exp\left(-\frac{\theta}{\phi}z^2 + (\phi + 1)z\right), \quad \theta > 0, \phi > 0$.

4. Form the likelihood function of the model in each case.

a) Let $X_1, ..., X_n$ be conditionally independent given θ. Each $X_i|\theta \sim \text{Normal}(\mu, \theta)$ where μ is known, and $\theta > 0$.

b) It is believed that the number of accidents in a new factory will follow a Poisson distribution with mean θ per month. There are 18 accidents in the first six months.

c) The number of offspring X in a certain population has probability function

$$p(x|\alpha,\beta) = \begin{cases} \alpha & x = 0 \\ (1-\alpha)\beta(1-\beta)^{x-1} & x = 1, 2, ... \end{cases}$$

where α and β are unknown parameters lying in the unit interval. Obtain the likelihood function when r zeroes and $n-r$ non-zero values $X_1, ..., X_{n-r}$ are obtained from n independent observations on X.

5. Find the following sums and integrals by identifying a kernel of a probability density function and using properties of probability density functions.

a) $\sum_{y=0}^{\infty} \mu^y/y!, \quad \mu > 0.$

b) $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx, \quad \alpha > 0, \beta > 0.$

c) $\int_0^1 \frac{1}{2}(\beta+3)(\beta+2)(\beta+1)x(1-x)^{\beta}dx, \quad \beta > -1.$

d) $\int_{-\infty}^{\infty} 4\mu^{(a-1)} \left(\frac{\sigma^2+\tau^2}{\sigma\tau}\right)^{1/2} \exp\left(-\left(\frac{\sigma^2+\tau^2}{\sigma\tau}\right)(\theta-\mu)^2\right) d\theta, \quad \sigma > 0, \tau > 0.$