Lab Exercise 1

1) $Y|\pi \sim$ binomial $(n = 4, \pi)$ and suppose $Y = 3$ was observed

a) Find the likelihood of $Y = 3$

First we know the pmf: $f(y|\pi) = \binom{n}{y} \pi^y (1-\pi)^{n-y}$, $y = 0, 1, 2, \dots$

For $\pi = 0.4$, the lkhd of $Y = 3$ is:

$$L(\pi|3) = \binom{4}{3} \cdot (0.4)^3 (0.6)^1 = \underline{0.1536}$$

For $\pi = 0.5$, the Lkhd of $Y = 3$ is:

$$L(\pi|3) = \binom{4}{3} \cdot (0.5)^3 (0.5)^1 = \underline{0.25}$$

For $\pi = 0.6$, the Lkhd of $Y = 3$ is:

$$L(\pi|3) = \binom{4}{3} \cdot (0.6)^3 (0.4)^1 = \underline{0.3456}$$

b) Obtain the posterior probability of π given $Y = 3$

π	Prior	Lkhd	Lkhd X Prior	Posterior
0.4	1/3	0.1536	0.0512	0.0512 / 0.2497 = 0.205
0.5	1/3	0.2500	0.0833	0.0833 / 0.2497 = 0.334
0.6	1/3	0.3456	0.1152	0.1152 / 0.2497 = 0.461
Sum			0.2497	1.00

c) Using the Bernoulli trials instead of the binomial distribution, determ
the likelihood of $Y = 3$ and obtain the posterior probability of π
given $Y = 3$

	π	Prior	Lkhd	Lkhd x Prior	Posterior
	.4	1/3	$.4^3 \times .6^1 = .0384$	.0128	.0128 / .0624 = .205
	.5	1/3	$.5^3 \times .5^1 = .0625$	.0208	.334
	.6	1/3	$.6^3 \times .4^1 = .0864$	.0288	.461
sum				.0624	1.00

2) a)

$$X|\theta \sim \text{Poisson}(\theta)$$

$$\Rightarrow f(x|\theta) \propto \theta^x / x! \text{ (kernel)}$$

$$\sum_{x=0}^{\infty} f(x|\theta) = \sum_{x=0}^{\infty} \theta^x e^{-\theta} / x! = e^{-\theta} \sum_{x=0}^{\infty} \theta^x / x! = 1$$

b) $Y|\theta, \beta \sim \text{Beta}(\beta\theta, \beta)$

$$\Rightarrow f(y|\theta,\beta) = \frac{\Gamma(\beta\theta+\beta)}{\Gamma(\beta\theta)\Gamma(\beta)} y^{\beta\theta-1}(1-y)^{\beta-1} \propto y^{\beta\theta-1}(1-y)^{\beta-1}$$

c) $\theta|\alpha, \beta, x \sim \text{Gamma}(\alpha + x + 1, 1/(\beta - 3x))$

$$\Rightarrow f(\theta|\alpha,\beta,x) = \frac{1}{\Gamma(\alpha+x+1)\left(\frac{1}{\beta-3x}\right)^{\alpha+x+1}} \theta^{\alpha+x} \exp(-\theta(\beta-3x))$$

$$\propto \theta^{\alpha+x} \exp(-\theta(\beta-3x))$$

d) $\phi \mid \mu, \bar{x}, T \sim \text{Normal}\left(T\mu + (1-T)\bar{x}, \bar{x}^2 T^{-2}\right)$

$$\Rightarrow f(\phi \mid \mu, \bar{x}, T) = \frac{1}{\sqrt{2\pi \bar{x} T^{-1}}} \exp\left(-\frac{\{\phi - T\mu - (1-T)\bar{x}\}^2}{2\bar{x}^2 T^{-2}}\right)$$

$$\propto \exp\left(-\frac{T^2}{2\bar{x}^2}\{\phi - T\mu - (1-T)\bar{x}\}^2\right)$$

$$\propto \exp\left(-\frac{T^2}{2\bar{x}^2}\{\phi^2 - 2(T\mu + (1-T)\bar{x})\phi\}\right)$$

(3) In each of the following state the distribution of the corresponding random quantity.

a) $f(x) \propto e^{-2x}, \quad x > 0 \quad \Rightarrow X \sim \text{Exponential}\left(\frac{1}{2}\right)$ or $\text{Gamma}\left(1, \frac{1}{2}\right)$

b) $f(x) \propto 1, \quad 0 \leq x \leq 1, \quad \Rightarrow X \sim \text{Uniform}(0,1)$

c) $f(y \mid \mu) \propto \mu^y / y! \quad y = 0, 1, 2, \dots, \mu > 0 \quad \Rightarrow Y \sim \text{Poisson}(\mu)$

d) $f(y \mid \alpha, \beta) \propto y^{-\alpha} e^{-1/(2\beta y)}, \quad y > 0, \alpha > 1, \beta > 0$

$$\propto y^{-\alpha+1-1} e^{-1/(2\beta y)} \propto y^{-(\alpha+1-1)} e^{-1/(2\beta y)}$$

$$\Rightarrow Y \sim \text{Inverse Gamma}(\alpha - 1, 2\beta)$$

cf. $Y_i \sim \text{Inverse Gamma}(\alpha, \beta) \iff f(y_i \mid \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} y_i^{-(\alpha+1)} \exp(1/(-y_i\beta))$

e) $f(x \mid m) \propto (1-x)^{(m-1)/2} \quad 0 \leq x \leq 1, \quad m > -1/2$

$$= (1-x)^{(m/2 - 1/2)} \times x^0 = x^{1-1}(1-x)^{(m/2 - 1/2 + 1 - 1)}$$

$$\Rightarrow X \sim \text{Beta}\left(1, \frac{m}{2} + \frac{1}{2}\right)$$

f) $f(z|\theta,\phi) \propto \exp\left(-\frac{\theta}{\phi}z^2 + (\phi+1)z\right), \quad \theta > 0, \phi > 0$

$$= \exp\left[-\frac{\theta}{\phi}\left(z^2 - \frac{(\phi+1)z}{\frac{\theta}{\phi}}\right)\right] = \exp\left[-\frac{\theta}{\phi}\left(z - \frac{(\phi+1)}{2\frac{\theta}{\phi}}\right)\right]$$

$$= \exp\left[\frac{-\left(z - \frac{(\phi+1)}{2\frac{\theta}{\phi}}\right)^2}{\frac{\phi}{\theta} \times 2\frac{1}{2}}\right] \therefore \text{Kernel of Normal}\left(\frac{(\phi+1)}{2\frac{\theta}{\phi}}, \frac{\phi}{2\theta}\right)$$

$$\Rightarrow Z \sim \text{Normal}\left(\frac{\phi(\phi+1)}{2\theta}, \frac{\phi}{2\theta}\right)$$

④ a)

$$L(\theta|x_1,\dots,x_n) = L(\theta|x) = f(x|\theta)$$

$$f(x|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(x_i-\mu)^2}{2\theta}\right)$$

$$= \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n \exp\left[\frac{-\sum_{i=1}^{n}(x_i-\mu)^2}{2\theta}\right] = (2\pi\theta)^{-n/2} \exp\left[\frac{-\sum_{i=1}^{n}(x_i-\mu)^2}{2\theta}\right]$$

b)

$$L(\theta|x) = \prod_{i=1}^{18} f(x_i|\theta) = \prod_{i=1}^{18} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-\theta}\cdots e^{-\theta} \times \theta^{x_1}\cdots\theta^{x_{18}}}{x_1!\cdots x_{18}!}$$

$$= \frac{e^{18\theta}\,\theta^{\sum_{i=1}^{18} x_i}}{\prod_{i=1}^{18} x_i!}$$

c) $L(\theta|x) = \prod_{i=1}^{r} P(x_i|\theta) \prod_{j=1}^{n-r} P(x_j|\theta) = \prod_{i=1}^{r} \alpha \prod_{j=1}^{n-r} (1-\alpha)\beta(1-\beta)^{x_i-1}$

$= \alpha^r (1-\alpha)^{n-r} \beta^{n-r} (1-\beta) \sum_{i=1}^{n-r} x_i - (n-r)$

(5)

a) $\sum_{y=0}^{\infty} \mu^y / y!, \quad \mu > 0$

$\mu^y / y!$ is a likelihood (pmf) for poisson (μ)

$\sum_{y=0}^{\infty} \frac{\mu^y}{y!} = \frac{1}{n.c} = \frac{1}{e^{-\mu}} = e^{\mu}$ or $\sum_{y=0}^{\infty} \frac{\mu^y e^{-\mu}}{y!} = e^{-\mu} \sum_{y=0}^{\infty} \frac{\mu^y}{y!} = 1$

$\Rightarrow \sum_{y=0}^{\infty} \frac{\mu^y}{y!} = e^{\mu}$

b) $\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx, \quad \alpha > 0, \beta > 0$

Note that $x^{\alpha-1}(1-x)^{\beta-1}$ is a kernel of Beta (α, β) and $\int_0^1 f(x|\alpha,\beta)\, dx = 1$

$\Rightarrow \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx = 1$

$\Rightarrow \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

c) $\int_0^1 \frac{1}{2}(\beta+3)(\beta+2)(\beta+1)x(1-x)^\beta dx, \quad \beta > -1$

Since $x(1-x)^\beta$ is the kernel of Beta $(2, \beta+1)$

$\frac{\Gamma(2+\beta+1)}{\Gamma(2)\Gamma(\beta+1)} x(1-x)^\beta$ is the pdf of Beta $(2, \beta+1)$ and

$$\int_0^1 \frac{\Gamma(2+\beta+1)}{\Gamma(2)\Gamma(\beta+1)} x(1-x)^\beta dx = 1$$

Gamma function : $\Gamma(a) = \int_0^\infty x^{a-1} \exp(-x) dx$

$$\Gamma(n) = (n-1)! \text{ for } n \in \mathbb{Z}$$

$$\Gamma(a+1) = a\Gamma(a)$$

$$\frac{\Gamma(2+\beta+1)}{\Gamma(2)\Gamma(\beta+1)} = \frac{\Gamma(\beta+3)}{\Gamma(2)\Gamma(\beta+1)} = \frac{(\beta+2)\Gamma(\beta+2)}{1!\Gamma(\beta+1)} = \frac{(\beta+2)(\beta+1)\Gamma(\beta+1)}{1!\Gamma(\beta+1)}$$

$$= (\beta+2)(\beta+1)$$

$$\int_0^1 \frac{1}{2}(\beta+3)(\beta+2)(\beta+1)x(1-x)^\beta dx = \frac{1}{2}(\beta+3)\int_0^1 (\beta+2)(\beta+1)x(1-x)^\beta dx$$

$$= \frac{1}{2}(\beta+3)$$

d)

$$\int_{-\infty}^{\infty} 4\mu^{(\alpha-1)} \left(\frac{s^2+T^2}{sT}\right)^{1/2} \exp\left(-\left(\frac{s^2+T^2}{sT}\right)(\theta-\mu)^2\right) d\theta, \quad s>0,\ T>0$$

Note that $\exp\left(-\left(\frac{s^2+T^2}{sT}\right)(\theta-\mu)^2\right)$ is kernel of Normal distribution

$$\exp\left[-\left(\frac{s^2+T^2}{sT}\right)(\theta-\mu)^2\right] = \exp\left[\frac{-(\theta-\mu)^2}{\frac{1}{\left(\frac{r^2+T^2}{sT}\right)}}\right] = \exp\left[\frac{-(\theta-\mu)^2}{\frac{1}{\left(\frac{r^2+T^2}{rT}\right)} \times 2\frac{1}{2}}\right]$$

$\therefore$ Kernel of $N\left(\mu, \frac{1}{2\left(\frac{r^2+T^2}{rT}\right)}\right)$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{1}{2\left(\frac{r^2+T^2}{rT}\right)}}} \exp\left[\frac{-(\theta-\mu)^2}{\frac{1}{2\left(\frac{r^2+T^2}{rT}\right)} \times 2}\right] d\theta = \int_{-\infty}^{\infty} \frac{\sqrt{\left(\frac{r^2+T^2}{rT}\right)}}{\sqrt{\pi}} \exp\left[\frac{-(\theta-\mu)^2}{\frac{1}{2\left(\frac{r^2+T^2}{rT}\right)} \times 2}\right] d\theta = 1$$

$$\int_{-\infty}^{\infty} 4\mu^{(a-1)} \left(\frac{r^2+T^2}{rT}\right)^{1/2} \exp\left(-\left(\frac{r^2+T^2}{rT}\right)(\theta-\mu)^2\right) d\theta$$

$$= 4\mu^{(a-1)} \int_{-\infty}^{\infty} \left(\frac{r^2+T^2}{rT}\right)^{1/2} \exp\left(-\left(\frac{r^2+T^2}{rT}\right)(\theta-\mu)^2\right) d\theta$$

$$= 4\mu^{(a-1)} \int_{-\infty}^{\infty} \left(\frac{r^2+T^2}{rT}\right)^{1/2} \frac{1}{\sqrt{\pi}} \sqrt{\pi} \exp\left(-\left(\frac{r^2+T^2}{rT}\right)(\theta-\mu)^2\right) d\theta$$

$$= 4\mu^{(a-1)} \sqrt{\pi} \int_{-\infty}^{\infty} \left(\frac{r^2+T^2}{rT}\right)^{1/2} \frac{1}{\sqrt{\pi}} \exp\left(-\left(\frac{r^2+T^2}{rT}\right)(\theta-\mu)^2\right) d\theta$$

$$= 4\mu^{(a-1)} \sqrt{\pi}$$