CSC373 – Problem Set 1

Remember to write your **full name(s)** and **student number(s)** prominently on your submission. To avoid suspicions of plagiarism: at the beginning of your submission, **clearly state any resources (people, print, electronic) outside of your group, the course notes, and the course staff, that you consulted**.

Remember that you are required to submit your problem sets as both LaTeX `.tex` source files and `.pdf` files. There is a 10% penalty on the assignment for failing to submit both the `.tex` and `.pdf`.

Due Oct 7, 2022, 22:00; required files: ps1.pdf, ps1.tex

Answer each question completely, always justifying your claims and reasoning. Your solution will be graded not only on correctness, but also on clarity. Answers that are technically correct that are hard to understand will not receive full marks. Mark values for each question are contained in the [square brackets].

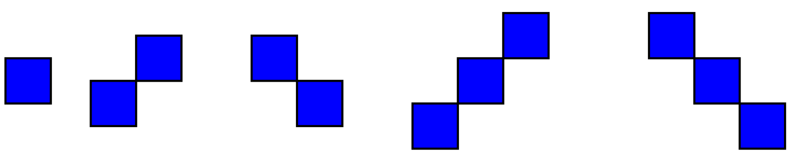
You may work in groups of up to THREE to complete these questions.

Please see the course information sheet for the late submission policy.

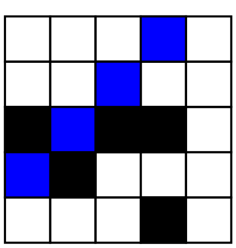
1. [**15 points**]

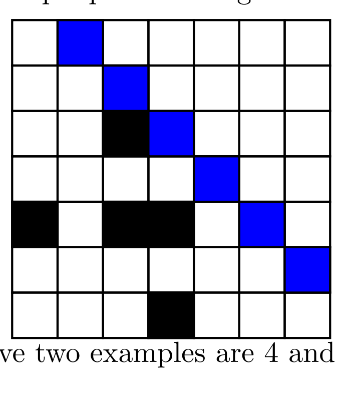
   Sushant finds crosswords fascinating. However, rather than attempt to solve the word clues, he enjoys finding patters in the crossword grid. So whenever he sees a crossword grid, he's looking to find the biggest diagonal he can fit in it.

   You are given an $n \times n$ array with entries in 0 or 1. 0 indicates that that location is empty, and 1 indicates that the location is filled. Your goal is to determine the largest diagonal that can be filled in the grid while utilizing only the 0 locations. The following figures below demonstrate diagonals with sizes 1, 2, 2, 3, and 3 respectively (a zero sized diagonal would have no blue squares). Note that the diagonal can go from top-left to bottom-right, or bottom-left to top-right.



   In the example grids drawn below, the squares with 1 in the input array are filled with black color, and all the squares colored blue or white are 0 locations. The blue squares visualize a diagonal with the biggest possible size that can be fit inside the grid using only non-black locations. (Note that there could be multiple possible diagonals with the biggest size)





   Thus the expected answers for the above two examples are 4 and 6.

Design an algorithm for the above problem with a worst case running-time complexity of $O(n^2)$ following the steps below.

(a) **[1 point]** Clearly and precisely specify in English the problem you wish to solve.

(b) **[4 points]** Give a recursive solution for the problem (including base cases) and justify it. (Hint: You may need to define more than one recursive function)

(c) **[1 point]** Specify all the subproblems that your algorithm needs to solve.

(d) **[3 point]** Specify the memoization data structure(s), clearly define what value will be stored in each location at the end of the algorithm, and give a good bottom-up evaluation order for filling the memoization datastructure(s).

(e) **[6 point]** Write down the final dynamic-programming algorithm (non-recursive), and analyze its time and space complexity.

2. **[15 points]**

The pandemic unleashed due to the SARS-CoV-2 virus (also referred to as Covid-19) has resulted in mandatory social distancing requirements.

As the University plans to reopen its classrooms to be used for in-person lectures again, it needs to ensure that the physical distancing guidelines are followed. As a result, the university needs to reconsider the layout of its classrooms.

Assume that each student seat is exactly 1 feet wide. As per the distancing requirement, every two seated students must have at least 6 feet of distance between them. You are intrigued, and want to count the resulting number of possible student seating arrangements.

For some $n \geqslant 0$, consider a row with exactly $n + 1$ seats. You are given an array of n non-negative integers `A[1...n]`, where `A[i]` gives the distance in feet between seat i and seat $i + 1$.

Your goal is to determine the number of possible valid student seating arrangements.

e.g. Given `A = [2, 3]`, specifying the distances between 3 seats, there are a total of 5 valid seating arrangements:

- 1 arrangement with no students `␣␣␣`,
- 3 possible arrangements with 1 student `X␣␣`, `␣X␣`, `␣␣X`,
- and one arrangement with 2 students `X␣X`.

Note that the last arrangement is valid since the empty seat in the middle is 1 feet wide, giving that the distance between the two students is exactly 6 feet.

Systematically design a Dynamic Programming based algorithm for the above problem with worst case running-time complexity of $O(n)$ by answering the following questions:

(a) **[1 point]** Clearly and precisely specify in English the problem you wish to solve recursively.

(b) **[3 point]** Give a clear and precise recursive formula / algorithm for solving the problem and justification for its correctness. Identify and specify the base cases.

(c) **[1 point]** Identify all the subproblems that your recursive algorithm needs to solve.

(d) **[1 point]** What is the memoization data structure you will use?

(e) **[2 point]** Find a good bottom-up evaluation order of the memoization data-structure (non-recursive) such that before solving a subproblem instance, all necessary subproblems have been solved.

(f) **[5 point]** Write down the final dynamic-programming algorithm (non-recursive).

(g) **[2 point]** Analyze its time and space complexity.

3. **[15 points]**

You are given an array S of n distinct numbers (not necessarily integers) in sorted order such that all numbers in S are non-negative, i.e. $a \geqslant 0$ for all $a \in S$. You are also given a target value V. Your goal is to determine the number of distinct pairs (a, b) such that $a, b \in S$ and $a \neq b$ such that

$$a^4 + b^4 + 4ab \leqslant V.$$

We will consider (a, b) to be the same pair as (b, a). e.g. Given $S = [4, 6, 7, 9, 11]$, and $V = 3865$, the answer is 3 (consider the pairs $(4, 6), (4, 7), (6, 7)$), and if $V = 1000$, the answer is 0.

Systematically design an algorithm for this problem that has a worst-case time complexity of $O(n)$. Note: you're not allowed to use a formula for solving the quartic equation.

(a) **(3 points)** Clearly describe all the subproblems that will be solved by your algorithm, and the values that will be stored in all the data-structure being used by your algorithm.

(b) **(5 points)** Write your algorithm in pseudo-code and analyze its complexity.

(c) **(7 points)** Give a convincing proof of correctness for your algorithm. Remember that if your algorithm is greedy, you will not get any marks for the previous parts without a proof of correctness.