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Model Predictive Control

Linear Systems

NLTI CT D State Space

$$\begin{cases} \dot{x} = g(x,u) \\ y = h(x,u) \end{cases}$$

$x \in \mathbb{R}^n$ states

$u \in \mathbb{R}^m$ inputs

$y \in \mathbb{R}^p$ outputs

Mathematical Models → LTI DT (D)

↳ Most important part!

NL → L

$$\begin{cases} \delta x := x - x_s \\ \delta u := u - u_s \\ \delta y := y - y_s \end{cases}$$

where $\dot{x}_s = g(x_s, u_s) (= 0)$

$y_s = h(x_s, u_s)$

$$\delta\dot{x} = \dot{x} - \dot{x}_s = g(x,u) - g(x_s,u_s) \cong g(x_s,u_s) + \overbrace{\frac{\partial g}{\partial x}\bigg|_s}^{=:A_c} \delta x + \overbrace{\frac{\partial g}{\partial u}\bigg|_s}^{=:B_c} \delta u - g(x_s,u_s)$$

$$\delta y = y - y_s = h(x,u) - h(x_s,u_s) \cong \cancel{h(x_s,u_s)} + \underbrace{\frac{\partial h}{\partial x}\bigg|_s}_{=:C} \delta x + \underbrace{\frac{\partial h}{\partial u}\bigg|_s}_{=:D} \delta u - \cancel{h(x_s,u_s)}$$

$$\begin{cases} \delta\dot{x} = A_c\delta x + B_c \delta u \\ \delta y = C\delta x + D\delta u \end{cases}$$

Good approximation for small δx, δu

$\delta x(t) = e^{A_c(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau$

Ev → N

$$\begin{cases} \delta x = \overbrace{T_x}^{\mathrm{diag}(x_N)} \delta x_N \\ \delta u = \underbrace{T_u}_{\mathrm{diag}(u_N)} \delta u_N \\ \delta y = \underbrace{T_y}_{\mathrm{diag}(y_N)} \delta y_N \end{cases}$$

$$\begin{cases} \delta\dot{x}_N = \overbrace{T_x^{-1} A_c T_x}^{=:A_N} \delta x_N + \overbrace{T_x^{-1} B_c T_u}^{=:B_N} \delta u_N \\ \delta y_N = \underbrace{T_y^{-1} C_c T_x}_{=:C_N} \delta x_N + \underbrace{T_y^{-1} D T_u}_{=:D_N} \delta u_N \end{cases}$$

$$\begin{cases} \delta\dot{x}_N = A_N \delta x_N + B_N \delta u_N \\ \delta y_N = C_N \delta x_N + D_N \delta u_N \end{cases}$$

CT → DT $\quad t_{k+1} = t_k + T_s$ (sampling time) $\quad u(t) = u(k) \ \forall t \in [t_k, t_{k+1})$ implemented

neglect δ $\quad t_0 = t_k \quad x_0 = x(k) \quad t = t_{k+1}$

$$x(k+1) = \underbrace{e^{A_c T_s}}_{=:A} x(k) + \underbrace{\int_0^{T_s} e^{A_c(T_s-\tau)} B_c \, d\tau}_{=:B \,=\, A_c^{-1}(A-I)B_c \text{ if } \exists A_c^{-1}} u(k)$$

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Transformation $\quad [e_i, \lambda_i] \in \mathrm{eig}(A) \quad \tilde{x} = Tx \quad T := [e_1, \dots, e_n]^{-1}$ (assume lin. indep.)

$\tilde{A} = \Lambda = TAT^{-1} = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \Rightarrow x(k) = \Lambda^k x(0)$ otherwise Jordan

GAS of LTI DT sys (i.e. of trivial $x^* = 0$) $\hat{=} \lim_{k\to\infty} x(k) = 0 \quad \forall x(0) \in \mathbb{R}^n$

$$\boxed{x^* = 0 \text{ of } x(k+1) = Ax(k) \text{ GAS} \Leftrightarrow |\lambda_i| < 1 \quad \forall \lambda_i \in \mathrm{Spec}(A)}$$

Lyapunov Stability for NLTI DT $\begin{cases} x_{k+1} = g(x_k) & \text{where } g(0) = 0 \\ x^* = 0 \end{cases}$

$x^* = 0$ is $\begin{cases} \text{stable if } \forall \varepsilon > 0 \ \exists \delta > 0 \mid x_0 \in B_\delta(0) \Rightarrow x_k \in B_\varepsilon(0) \ \forall k \geq 0 \\ \text{unstable otherwise} \end{cases}$

G/LAS if $\forall / \exists \Omega \subseteq \mathbb{R}^n \mid \lim_{k\to\infty} x_k = 0 \quad \forall x_0 \in \Omega$

Lyapunov Function system theoretic generalization of energy.

$V: \mathbb{R}^n \to \mathbb{R}$ continuous @ $x = 0$, finite $\forall x \in \Omega \subseteq \mathbb{R}^n$ and s.t.

LPDF: $V(0) = 0 \quad V(x) > 0 \quad \forall x \in \Omega \setminus \{0\}$ $\quad$ PDF in Ω

$V(g(x_k)) - V(x_k) \leq -\alpha(x_k) \quad \forall x_k \in \Omega \setminus \{0\}$ $\quad$ where $\alpha: \mathbb{R}^n \to \mathbb{R}$ C PDF

is called Lyapunov Function.

Lyapunov Theorem

$\boxed{\begin{array}{llll} x^* = 0 & AS \text{ in } \Omega & \Leftarrow & x_{k+1} = g(x_k) \text{ admits Lyapunov Function } V(x) \\ & GAS & \Leftarrow & \text{'' admits Lyap Fcn s.t. } \|x\| \to \infty \Rightarrow V(x) \to \infty \end{array}}$

LTI $\quad x(k+1) = Ax(k)$

$$V(x) = x^T P x \quad P = P^T > 0 \qquad \alpha(x) = x^T Q x$$

$$\boxed{x^* = 0 \text{ GAS} \Leftrightarrow \forall \lambda_i(A) \ |\lambda_i| < 1 \Leftrightarrow \forall Q = Q^T > 0 \ \exists! P = P > 0 \text{ s.t. } A^T P A - P = -Q \quad \text{DTLE}}$$

$\Rightarrow$ use P to determine infinite horizon cost to go

$$\Psi(x(0)) = \sum_{k=0}^{\infty} x^T(k) Q x(k) = x^T(0) \sum_{k=0}^{\infty} \underbrace{(A^k)^T Q A^k}_{=: H_k} x(0) = x^T(0) P x(0)$$

$$H_0 = Q, \quad H_{k+1} = A^T H_k A$$

$$A^T P A = \sum_{k=0}^{\infty} A^T H_k A = \sum_{k=0}^{\infty} H_{k+1} = \sum_{k=1}^{\infty} H_k = \sum_{k=0}^{\infty} H_k - H_0 = P - Q \qquad \text{QED}$$

Controllability

$x(k+1) = Ax(k) + Bu(k)$ controllable $\Leftrightarrow$

$\forall (x(0), x^*) \ \exists N, \exists \{u(0), \dots, u(N-1)\}$ s.t. $x(N) = x^* \Leftrightarrow$

$= \text{range}(\mathcal{C}_n) = \mathbb{R}^n \Leftrightarrow \text{rank}(\mathcal{C}_n) = n \qquad \mathcal{C}_n := (B \ AB \ \dots \ A^{n-1}B)$

$\rightarrow$ if not ctrb in n steps then not in $\forall N > n$

$\Rightarrow$ stabilizable $\Leftrightarrow \forall x(0) \ \exists \{u(0), \dots\}$ s.t. $\lim_{k \to \infty} x(k) = 0 \Leftrightarrow$

all uncontrollable modes are astable $\Leftrightarrow \text{rank}([\lambda_i I - A \ \ B]) = n$
$\forall \lambda_i \in \Lambda_A^+$

Observability

$x(k+1) = Ax(k), \quad y(k) = Cx(k)$ observable $\Leftrightarrow$

$\exists N$, s.t. $\{y(0), \dots, y(N-1)\}$ uniquely distinguish $x(0) \quad \forall x(0) \Leftrightarrow$

$\text{range}(\mathcal{O}_n) = \mathbb{R}^n \Leftrightarrow \text{rank}(\mathcal{O}_n) = n \qquad \mathcal{O}_n := (C^T \ (CA)^T \dots (CA^{n-1})^T)^T$

$\hat{x}(k) \underset{\neq}{\rightarrow} x(k)$

$\Rightarrow$ detectable $\Leftrightarrow$ if can reconstruct $\lim_{k \to \infty} x(k) \Leftrightarrow$

all unobservable modes are astable $\Leftrightarrow \text{rank}\left(\begin{bmatrix} I\lambda_i - A \\ C \end{bmatrix}\right) = n \quad \forall \lambda_i \in \Lambda_A^+$

Optimal Control

Choose optimal input sequence $U^*_{0\to N} := \{u^*_0, \dots, u^*_{N-1}\}$ s.t.

Cost function $J_{0\to N}(x_0, U_{0\to N}) = p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k)$

where
$$\left.\begin{aligned} x_{k+1} &= g(x_k, u_k) \\ x_0 &= x(0) \end{aligned}\right\} \text{Dynamics}$$
$$\left.\begin{aligned} &h(x_k, u_k) \leq 0 \\ &x_N \in \mathcal{X}_f \quad \text{(finite horizon)} \end{aligned}\right\} \text{constraints}$$

Formulation

$$\boxed{\begin{aligned} J^*_{0\to N}(x(0)) &:= \min_{U_{0\to N}} J_{0\to N}(x(0), U_{0\to N}) \\ \text{s.t.}\quad & x_{k+1} = g(x_k, u_k) \quad k = 0, \dots, N-1 \\ & h(x_k, u_k) \leq 0 \quad k = 0, \dots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_0 = x(0) \end{aligned}}$$

LQOC for LTI DT - No constraints!

$x(k+1) = Ax(k) + Bu(k)$

Unconstrained Finite Horizon $\underline{J^*_0(x(0)) = \min_{U_0} \left\{ x_N^T P x_N + \sum_{k=0}^{N-1} [x_k^T Q x_k + u_k^T R u_k] \right\}}$

$$\left.\begin{aligned} P^T = P \geq 0 &\quad \text{terminal} \\ Q^T = Q \geq 0 &\quad \text{state} \\ R^T = R > 0 &\quad \text{input} \end{aligned}\right\} \text{weight}$$

N horizon

s.t. $x_{k+1} = Ax_k + Bu_k \quad k = 0, \dots, N-1$

$x_0 = x(0)$

Batch Approach

$$\underbrace{\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix}}_{\mathcal{X}} = \underbrace{\begin{pmatrix} I \\ A \\ \vdots \\ A^N \end{pmatrix}}_{S^x} x(0) + \underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ A^{N-1}B & \cdots & \cdots & B \end{pmatrix}}_{S^u} \cdot \underbrace{\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}}_{U_0}$$

$\bar{Q} := \text{blckdiag}\{Q, \dots, Q, P\}$

$\bar{R} := \text{blckdiag}\{R, \dots, R\}$

$$\Rightarrow \overbrace{J_0(x(0), U_0)}^{\text{PDF}} = \mathcal{X}^T \bar{Q} \mathcal{X} + U_0^T \bar{R} U_0 =$$
$$= U_0^T H U_0 + 2x^T(0) F U_0 + x^T(0) S^{xT} \bar{Q} S^x x(0)$$

$0 < H := S^{uT} \bar{Q} S^u + \bar{R} \qquad F := S^{xT} \bar{Q} S^u$

$\nabla_{U_0} J_0 \overset{!}{=} 0$

$$\underline{U_0^*(x(0)) = -H^{-1} F^T x(0)}$$

$$\underline{J_0^*(x(0)) = x^T(0) \left(S^{xT} \bar{Q} S^x - S^{xT} \bar{Q} S^u (S^{uT} \bar{Q} S^u + \bar{R})^{-1} S^{uT} \bar{Q} S^x \right) x(0)}$$

→ Recursive Approach

j-step cost-to-go $J_j^*(x(j)) := \min_{u_j,\dots,u_{N-1}} \; x_N^T P x_N + \sum_{k=j}^{N-1} [x_k^T Q x_k + u_k^T R u_k]$

s.t. $\begin{cases} x_{k+1} = A x_k + B u_k & k = j, \dots, N-1 \\ x_j = x(j) \end{cases}$

Bellman's Principle of Optimality: pieces of optimal trajectories are optimal

→ $u^*(k) = F_k x(k)$ $\quad F_k := -(B^T P_{k+1} B + R)^{-1} B^T P_{k+1} A$

RDE: $P_k = A^T P_{k+1} A + Q - A^T P_{k+1} B (B^T P_{k+1} B + R)^{-1} B^T P_{k+1} A$ $\quad P_N = P$

$J_k^*(x(k)) = x^T(k) P_k x(k)$

→ Same as Batch if no model errors / disturbances / ...

→ Infinite Horizon → LQR

$J_\infty^*(x(0)) = \min_{u(\cdot)} \left\{ \sum_{k=0}^{\infty} [x_k^T Q x_k + u_k^T R u_k] \right\}$

s.t. $x_{k+1} = A x_k + B u_k \quad k \in \mathbb{N}$

$x_0 = x(0)$

Assume: $\lim_{k\to\infty} P_k = \lim_{k\to\infty} P_{k+1} = P_\infty$

$u^*(k) = F_\infty x(k)$ $\quad F_\infty := -(B^T P_\infty B + R)^{-1} B^T P_\infty A$ → LQR

RDE → ARE $\quad P_\infty = A^T P_\infty A + Q - A^T P_\infty B (B^T P_\infty B + R)^{-1} B^T P_\infty A$

$J_\infty^*(x(k)) = x^T(k) P_\infty x(k)$

If (A, B) stabilizable and $(Q^{\frac{1}{2}}, A)$ detectable $\quad$ RDE $\xrightarrow[\;]{k\to\infty}$ ARE and $A + BF_\infty$ astab

→ Terminal Weight P in Finite Horizon

1) Solution = infinite horizon $\quad P = A^T P A + Q - A^T P B (B^T P B + R)^{-1} B^T P A$ ARE

↳ Assume no constraints for $k \geq N$

2) No control action for $k \geq N$ $\quad x(k+1) = A x(k) \quad k \geq N$ $\quad A^T P A + Q = P$ DTLE

↳ only if astab! (otherwise $\nexists P \succeq 0$)

3) $u_k = 0 \quad x_k = 0 \quad k \geq N$ $\quad$ extra constraint $x_{k+N} = 0$

Uncertainty Model

Stochastic Process

$\{w(0), w(1), \dots\} \Rightarrow$ Model with joint pdf $P(w(0), w(1), \dots)$

Normal Stochastic Process

$$\mu_w(k) := E\{w(k)\} = \mu_w$$
$$R_w(k,\tau) := E\{w(k+\tau)\,w^T(k)\} - \mu_w(k+\tau)\mu_w(k) = R_w(\tau)$$

if stationary

Normal white noise stochastic process $\varepsilon(k)$

$$\begin{cases} \mu_\varepsilon = 0 \\ R_\varepsilon(\tau) = \begin{cases} R_\varepsilon & \tau = 0 \\ 0 & \text{else} \end{cases} \end{cases}$$

uncorrelated $\Rightarrow$ time indep!

Stationary State Space Model

stochastic process driven by white noise

$$\begin{cases} x_w(k+1) = A_w x_w(k) + B_w \varepsilon(k) \\ w(k) = C_w x_w(k) + \varepsilon(k) \end{cases}$$

$|\lambda_i| < 1$

$$E\{x_w(k)\} = A_w^k\, E\{x_w(0)\}$$
$$E\{x_w(k)x_w^T(k)\} = A_w\, E\{x_w(k-1)x_w^T(k-1)\}\, A_w^T + B_w R_\varepsilon B_w^T$$
$$E\{x_w(k+\tau)x_w^T(k)\} = A_w^\tau\, E\{x_w(k)x_w^T(k)\}$$

$$\bar{w} = \lim_{k\to\infty} E\{w(k)\} = 0 \qquad 0 \le \bar{P}_w^T = \bar{P}_w = A_w \bar{P}_w A_w^T + B_w R_\varepsilon B_w^T \quad \text{DTLE}$$
$$R_w(\tau) = \lim_{k\to\infty} E\{w(k+\tau)w^T(k)\} = C_w A_w^\tau \bar{P}_w C_w^T + C_w A_w^{\tau-1} B_w R_\varepsilon \Rightarrow A_w, B_w, C_w$$

Nonstationary State Space Model $\rightarrow$ if offset present

Integrated white noise $\varepsilon_{int}(k+1) = \varepsilon_{int}(k) + \varepsilon(k)$

$$\begin{cases} x_w(k+1) = A_w x_w(k) + B_w \varepsilon_{int}(k) \\ w(k) = C_w x_w(k) + \varepsilon_{int}(k) \end{cases}$$

eq. to $\varepsilon(k-1)$

$$\Delta w(k) := w(k) - w(k-1) \Rightarrow \begin{cases} \Delta x_w(k+1) = A_w \Delta x_w(k) + B_w \varepsilon(k) \\ \Delta w(k) = C_w \Delta x_w(k) + \varepsilon(k) \end{cases}$$

(zero mean $\bar{w} = 0$ stationary process!)

$\rightarrow$ same Considerations!

Models from First Principles

$$\dot{x}_p = A_p^c x_p + B_p^c u_p + F_p^c w \quad \Rightarrow \quad x_p(k+1) = A_p x_p(k) + B_p u_p(k) + \underbrace{F_p}_{=0} w(k)$$

p: process

$$y = C_p x_p + \underbrace{G_p}_{=I} w$$

if overall effect in output

Stationary case

$$\begin{cases} x(k+1) = A\,x(k) + B\,u(k) + F\,\varepsilon(k) \\ y(k) = C\,x(k) + G\,\varepsilon(k) \end{cases}$$

$A = \begin{pmatrix} A_p & F_p C_w \\ 0 & A_w \end{pmatrix}$ $B = \begin{pmatrix} B_p \\ 0 \end{pmatrix}$ $F = \begin{pmatrix} F_p \\ B_w \end{pmatrix}$

$C = \begin{pmatrix} C_p & G_p C_w \end{pmatrix}$ $G = G_p$ $x = \begin{pmatrix} x_p \\ x_w \end{pmatrix}$

Nonstationary case

$$\begin{cases} \bar{x}(k+1) = \bar{A}\,\bar{x}(k) + \bar{B}\,\Delta u(k) + \bar{F}\,\varepsilon(k) \\ y(k) = \bar{C}\,\bar{x}(k) \end{cases}$$

$\bar{A} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ $\bar{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$ $\bar{x} = \begin{pmatrix} \Delta x \\ y \end{pmatrix}$

$\bar{C} = \begin{pmatrix} 0 & I \end{pmatrix}$ $\bar{F} = \begin{pmatrix} F \\ G \end{pmatrix}$

Models from System Identification

$$y(z) = \underbrace{H_1(z)}_{r \geq 1} u(z) + H_2(z) \cdot \varepsilon_{int}(z)$$

if nonstationary

Stationary
$$\begin{cases} x(k+1) = A\,x(k) + B\,u(k) + F\,\varepsilon(k) \\ y(k) = C\,x(k) + D\,u(k) + G\,\varepsilon(k) \end{cases}$$

can assume w/o loss of gen.: $D = 0$, $H_2(0) = I$, $G = I$

Nonstationary $(1 - z^{-1})\,y(z) = (\Delta y)(z)$
$$\Rightarrow \begin{cases} x(k+1) = A\,x(k) + B\,\Delta u(k) + F\,\varepsilon(k) \\ \Delta y(k) = C\,x(k) + \varepsilon(k) \end{cases}$$

State Estimation

General Problem
$$x(k+1) = g(x(k), u(k), w_1(k), k)$$
$$y(k) = h(x(k), u(k), w_2(k), k)$$

Estimate $x(k+i)$ given $\{y(j), u(j)\}_{j=0,\dots,k} : \hat{x}_{k+i|k}$, i < 0 smoothing, $= 0$ filtering, > 0 prediction

↳ Crucial part of MPC! Disturbance Model important!

LTI DT
$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + \varepsilon_1(k) \\ y(k) = Cx(k) + \varepsilon_2(k) \end{cases}$$
white noise indep of $\hat{x}_{0|0}$
$$E\{\varepsilon(i)\varepsilon^T(j)\} = \begin{cases} \begin{pmatrix} R_1 & R_{1,2} \\ R_{1,2}^T & R_2 \end{pmatrix} & i = j \\ 0 & \text{else} \end{cases}$$

Structure

Prediction $\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu(k-1)$

Update $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y(k) - C\hat{x}_{k|k-1})$ K_k filter gain

Error $x^e_{i|j} := x(i) - \hat{x}_{i|j}$ $\Rightarrow x^e_{k|k} = (A - K_k CA)x^e_{k-1|k-1} + (I - K_k C)\varepsilon_1(k-1) - K_k\varepsilon_2(k)$

↳ min in meaningful way!

State Observer

K s.t. $\lambda(A - KCA) \in B_1(0) \setminus \partial B_1(0)$

Observer-stable $\lim_{k\to\infty} x^e_{k|k} = 0 \quad \forall x^e_{0|0}, \quad \varepsilon_1 = \varepsilon_2 = 0$

Pole placement $\Leftrightarrow$ (CA, A) observable $\overset{\text{rank}(A)=n}{\Leftrightarrow}$ (C, A) observable

↳ can freely choose λ_i, but not e_i → obs could be slow

↳ no unique K!

→ Kalman Filter

$P_{i|j} := \mathrm{cov}\{x^e_{i|j}\}$ Optimality Criterion: $\min_{K_k} P_{k|k}$

Assume $R_{1,2} = 0$

Unbiasedness $E\{x^e_{k|k}\} = \left[\prod_{i=1}^{k} (A - K_i C A)\right] \cdot E\{x^e_{0|0}\} \xrightarrow[\infty]{k} 0$ If $K_k \xrightarrow[\infty]{k} K_\infty$ & $(A - K_\infty C A)$ asstabl

Algorithm

- Initialize $\hat{x}_{0|0}$, $P_{0|0}$
- Online or in advance
$$P_{k|k-1} = A P_{k-1|k-1} A^T + R_1$$
$$K_k = P_{k|k-1} C^T (C P_{k|k-1} C^T + R_2)^{-1}$$
$$P_{k|k} = (I - K_k C) P_{k|k-1}$$
- @ time k

Prediction: $\hat{x}_{k|k-1} = A \hat{x}_{k-1|k-1} + B u(k-1)$

Measure: $y(k)$

Update: $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y(k) - C \hat{x}_{k|k-1})$

- $k \to k+1$

Stability

If $P_{k|k-1} \xrightarrow[\infty]{k} P_\infty$ and $\lambda(A - K_\infty C A) \in B_1(0) \setminus \partial B_1(0) \Rightarrow \lim_{k \to \infty} E\{x^e_{k|k}\} = 0$

Moreover, if bounded covariances ⇒ KF is stable

↳ If (C, A) detectable then $\forall P_{0|0}$

↳ if also $(A, R_1^{1/2})$ stabilizable and $P_{0|0} \geq 0$

$$\Rightarrow P_\infty = A P_\infty A^T - A P_\infty C^T (C P_\infty C^T + R_2)^{-1} C P_\infty A^T + R_1$$

ARE

Convex Optimization

Optimization Problem NLP $\min_x f(x)$ — objective function

s.t. $x \in X \subseteq \mathcal{X}$ ~ domain; X — constraint set; $= \min_{x \in X \subseteq \mathcal{X}} f(x)$

→ More common format $\min_{x \in \mathcal{X}} f_0(x)$

s.t. $f_i(x) \leq 0 \quad i = 1, \dots, m$ ineq. constr. } explicit

domain $h_i(x) = 0 \quad i = 1, \dots, p$ eq. "

implicit: $\mathrm{dom}(f_0) \cap \mathrm{dom}(f_i) \cap \mathrm{dom}(h_i)$

Properties of NLP $J^* = \min_{x \in \mathcal{X}} f(x)$

Unbounded below	$J^* = -\infty$
Infeasible	$\mathcal{X} = \emptyset \Rightarrow J^* = \infty$
Unconstrained	$\mathcal{X} = \mathbb{R}^n$ — $m = p = 0$

Set of solutions $\operatorname{argmin}_{x \in \mathcal{X}} f(x) := \{x \in \mathcal{X} \mid f(x) = J^*\}$

Terminology

Feasible point $x \in \mathcal{X}$: $f_i(x) \leq 0 \quad i = 1, \dots, m$, $h_i(x) = 0 \quad i = 1, \dots, p$

Strictly feasible point $x \in \mathcal{X}$: $f_i(x) < 0 \quad i = 1, \dots, m$ ⚠ $\inf_{x \geq 0} e^{-x}$ ↳ $J^* = 0$ but $\nexists x^*$

Optimal value $f_0(x^*) = J^* = p^*$ → infimum → Minimizer

Local optimality $\exists R > 0$: $z = x$ optimal for $\min_{z \in \mathcal{X}} f_0(x)$ s.t. $f_i(x) \leq 0$, $h_i(x) = 0$, $\|x - z\|_2 < R$

Optimal Solution $\forall x^* \in \mathcal{X}_{feas}$: $f_0(x^*) \leq f_0(x) \ \forall x \in \mathcal{X}_{feas}$

Constraint $f_i(x) \leq 0$ active @ $\bar{x}$ if $f_i(\bar{x}) = 0$

inactive otherwise

$h_i(x) = 0$ always active!

redundant - doesn't change $X \subseteq \mathcal{X}$

Constraint Satisfiability $\operatorname{find}_{x \in \mathcal{X}} x$ special case of $\min_{x \in \mathcal{X}} 0$ → $p^* = 0$ constr ✓ → $p^* = \infty$ ✗

s.t. $f_i(x) \leq 0$ $h_i(x) = 0$

LP $\min_x c^T x$ QP $\min_x \frac{1}{2} x^T P x + q^T x$ convex if $P \succeq 0$

s.t. $Gx \leq h$ =

$Ax = b$

MILP $x \in \{0,1\}^n$ or $\mathbb{Z}^n$

Convex Sets

$$\boxed{\text{Convex set } \mathcal{X} \Leftrightarrow \forall x, y \in \mathcal{X} \quad \lambda x + (1-\lambda) y \in \mathcal{X} \quad \forall \lambda \in (0,1)}$$

Affine set $\mathcal{X} = \{x \in \mathbb{R}^n \mid Ax = b\}$ subspace if $b = 0$

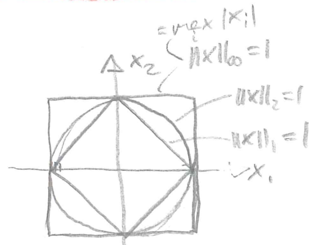
Hyperplane
Closed Halfspace $\mathcal{X} = \{x \in \mathbb{R}^n \mid a^T x \overset{=}{\underset{<}{\leq}} b\} \quad \forall a \in \mathbb{R}^n \setminus \{0\} \Rightarrow$ convex!
Open Halfspace — everything on one side of a hyperplane

Cone $\mathcal{X} \Leftrightarrow \forall x \in \mathcal{X} \quad \theta x \in \mathcal{X} \quad \forall \theta > 0$ pointed if $0 \in \mathcal{X}$

Polyhedron $\hat{=}$ intersection of finite number of halfspaces
$$\mathcal{X} = \{x \in \mathbb{R}^n \mid a_1^T x \leq b_1, \dots, a_m^T x \leq b_m\} = \{x \in \mathbb{R}^n \mid \underbrace{A}_{[a_1, \dots, a_m]^T} x \leq \underbrace{b}_{[b_1, \dots, b_m]^T}\}$$
Polytope $\hat{=}$ bounded polyhedron

Norm $f: \mathbb{R}^n \to \mathbb{R}$
1) $f(x) \geq 0 \quad f(x) = 0 \Rightarrow x = 0$
2) $f(tx) = |t| f(x) \quad \forall t \in \mathbb{R}$
3) $f(x+y) \leq f(x) + f(y)$

$\ell_p \quad \|x\|_p := \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} \quad \forall p \geq 1$



Norm ball $\{x \mid \|x - x_c\|_p \leq r\}$

Ellipsoid $\mathcal{X} = \{x \in \mathbb{R}^n \mid (x - x_c)^T A^{-1} (x - x_c)\} \quad A \succ 0$

Euclidean ball $\hat{=}$ Ellipsoid with $A = r^2 I$

Thm: $\mathcal{X}, \mathcal{Y}$ convex $\Rightarrow \mathcal{X} \cap \mathcal{Y}$ convex

⚠ $\mathcal{X} \cup \mathcal{Y}$ not convex in general!

Convex Hull $Co(\mathcal{X}) = \{x \in \mathbb{R}^n \mid x = \lambda a + (1-\lambda) b, \ \forall \lambda \in [0,1], \ \forall a, b \in \mathcal{X}\}$
↳ smallest convex set containing $\mathcal{X}$

Convex Functions

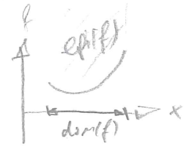
Convex Function $f: dom(f) \to \mathbb{R} \iff \begin{cases} dom(f) \text{ convex} \\ f(\lambda x+(1-\lambda)y) \overset{<}{\leq} \lambda f(x)+(1-\lambda) f(y) \quad \forall \lambda \in (0,1) \end{cases}$ strictly! $\forall x,y \in dom(f)$

concave if $-f$ convex

$\Updownarrow$ 1st diff

$\nabla^2 f(x) \overset{\succ}{\succeq} 0 \overset{2^{nd}\,diff}{\iff} f(y) \geq f(x) + \nabla f(x)^T (y-x) \quad \forall x,y \in dom(f)$

Epigraph $epi(f) = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \middle| x \in dom(f),\ f(x) \leq t \right\} \subseteq dom(f) \times \mathbb{R}$



$epi(f)$ convex $\iff$ f convex

Level set $L_\alpha = \{x \mid x \in dom(f),\ f(x) = \alpha\}$

$dom(f)$ convex $\Rightarrow$ f quasi-convex

$\Uparrow$

Sublevel set $C_\alpha = \{x \mid x \in dom(f),\ f(x) \leq \alpha\}$ $\quad f$ convex $\overset{\not\Leftarrow}{\Rightarrow}$ C_α convex $\forall \alpha$

Function f convex $\iff$ evaluation along any line in its domain is convex

$g(t) = f(x+tv) \quad dom(g) = \{t \mid x+tv \in dom(f)\}$ $\quad \Updownarrow$ g convex in t $\forall x \in dom(f),\ v \in \mathbb{R}^n$

Extended value function $\tilde{f}(x) := \begin{cases} f(x) & x \in dom(f) \\ \infty & \text{else} \end{cases} \Rightarrow epi(f) = epi(\tilde{f})$

$dom(f)$ convex, f convex $\iff$ $\tilde{f}$ convex

Convexity preserving operations

f_i convex $\Rightarrow$ $\sum_i f_i \cdot \alpha_i$ convex $\forall \alpha_i \geq 0$, $\max\{f_1(x), \dots, f_m(x)\}$ convex

f convex $\Rightarrow$ $f(Ax+b)$ convex

$f(x,y)$ convex in x $\forall y \in \mathcal{Y}$ $\Rightarrow$ $g(x) = \sup_{y \in \mathcal{Y}} f(x,y)$ convex

$f(x,y)$ convex in (x,y), C convex $\Rightarrow$ $g(x) = \min_{y \in C} f(x,y)$ convex

$g: \mathbb{R}^n \to \mathbb{R}^k \quad h: \mathbb{R}^k \to \mathbb{R} \quad f(x) = h(g(x)) = h(g_1(x), \dots, g_k(x))$ convex if

- g_i, h convex, $\tilde{h}$ non decreasing in each argument
- $-g_i$, h convex, $\tilde{h}$ non increasing in each argument

Convex Optimization Problems

Convex Optimization Problem

$$\min_{x \in \mathcal{X}} f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0 \quad i = 1, \dots, m; \qquad Ax = b \quad A \in \mathbb{R}^{p \times n}$$

→ If $\mathcal{X}$ convex set, f_i convex fcn $j = 0, \dots, m$

→ ! Every locally optimal solution is globally optimal!

Optimality Criterion

x optimal ⟺ feasible and $\nabla f_0(x)^T (y - x) \ge 0 \;\; \forall y$ feas.

can be $\neq 0$ as far as other feas points not "downhill"

- Unconstrained: $x \in dom(f_0)$ $\quad \nabla f_0(x) = 0$
- Eq. constrained: $x \in dom(f_0)$ $\quad Ax = b$ $\quad \exists \nu: \nabla f_0(x) + A^T \nu = 0$
- Min over $x \ge 0$: $x \in dom(f_0)$ $\quad x \ge 0$ $\quad 0 \le x \perp \nabla f_0(x) \ge 0$

Equivalent Optimization Problems

solution from one can be inferred to sol. of other one

$$\min_x f_0(A_0 x + b_0) \;\; \text{s.t.} \;\; f_i(A_i x + b_i) \ge 0 \;\; 1 \to m \quad \Longleftrightarrow \quad \min_{x, y_i} f_0(y_0) \;\; \text{s.t.} \;\; f_i(y_i) \ge 0 \;\; 1 \to m, \quad A_i x + b_i = y_i \;\; 0 \to m$$

$$\min_x f_0(x) \;\; A_i x \le b_i \quad \Longleftrightarrow \quad \min_{x, s_i} f_0(x) \;\; A_i x + s_i = b_i \;\; 1 \to n$$

slack variables: $s_i \ge 0$ $\;1 \to n$

LP

$$\min_x c^T x + d \quad \text{s.t.} \quad Gx \le h, \quad Ax = b$$

PWA

$$\min_x \left[\max_{i=1,\dots,m} \{ c_i^T x + d_i \} \right] \;\; \text{s.t.} \;\; Gx \le h \quad \Longleftrightarrow \quad \text{LP} \;\; \min_{x,t} t \;\; \text{s.t.} \;\; c_i^T x + d_i \le t \;\; 1 \to m, \quad Gx \le h$$

ℓ_∞

$$\min_{x \in \mathbb{R}^n} \|x\|_\infty \;\; Fx \le g \quad \Longleftrightarrow \quad \text{LP} \;\; \min_{x,t} t \;\; x_i \le t, \;\; -x_i \le t, \;\; Fx \le g$$

ℓ_1

$$\min_{x \in \mathbb{R}^n} \|x\|_1 \;\; Fx \le g \quad \Longleftrightarrow \quad \text{LP} \;\; \min_{x,t} \sum_{i=1}^n t_i \;\; x_i \le t_i, \;\; -x_i \le t_i, \;\; Fx \le g$$

QP

$$\min_x \tfrac{1}{2} x^T P x + q^T x + \cancel{r} \quad Gx \le h, \quad Ax = b$$

not necessary!

Duality

Primal (P)

$$\min_{x\in\mathcal{X}} f_0(x) = p^* \quad \text{s.t.} \quad f_i(x) \le 0 \quad i\in[1,m], \qquad h_j(x) = 0 \quad j\in[1,p]$$

$\Rightarrow$ Lagrangian Function $L: \mathcal{X}\times\mathbb{R}^m\times\mathbb{R}^p \to \mathbb{R}$

$$L(x,\lambda,\nu) := f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x)$$

ineq. eq. Lagrange multiplier

$\Downarrow$

Dual (D)

convex even if P is not!

$$\max_{\lambda,\nu} g(\lambda,\nu) = d^* \le p^* \quad \text{s.t.} \quad \lambda \ge 0$$

Dual Function $g: \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ PW infimum of affine fcts

$$g(\lambda,\nu) := \inf_{x\in\mathcal{X}} L(x,\lambda,\nu) \le p^* \quad \text{concave}$$

$\forall \lambda \ge 0,\ \nu\in\mathbb{R}^p$ $\quad \mathrm{dom}(g) = \{(\lambda,\nu) \mid g(\lambda,\nu) > -\infty\}$

LP

(P) $\min_{x\in\mathbb{R}^n} c^T x$ s.t. $Ax = b$, $Cx \le d$

(D) $\max_{\lambda,\nu} -b^T\nu - d^T\lambda$ s.t. $A^T\nu + C^T\lambda + c = 0$, $\lambda \ge 0$ — LP!

QP

(P) $\min_{x\in\mathbb{R}^n} \frac{1}{2}x^TQx + c^Tx$ s.t. $Cx \le d$, $Q \succ 0$

(D) $\max_{\lambda} -\frac{1}{2}\lambda^T C^T Q^{-1} C\lambda - (CQ^{-1}c + d)^T\lambda - \frac{1}{2} c^T Q^{-1} c$ s.t. $\lambda \ge 0$ — QP!

MILP

(P) $\min_{x\in\mathcal{X}} c^T x$ s.t. $Ax \le b$, $\mathcal{X} = \{-1,1\}^n$

(D) $\max_{\lambda} -\|A^T\lambda + c\|_1 - b^T\lambda$ s.t. $\lambda \ge 0$ — LP!

$\rightarrow$ easier!

Week Duality $d^* \le p^*$ Duality Gap $p^* - d^*$

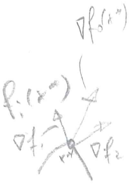
Strong Duality $d^* = p^*$ usually holds / doesn't for convex / non-convex $\rightarrow$ Can impose conditions!

For convex problems: Slater Condition $\{x \mid Ax = b,\ f_i(x) < 0\ \forall i\} \ne \emptyset \Rightarrow p^* = d^*$

KKT Conditions: necessary conditions for optimality

- Primal Feasibility $f_i(x^*) \le 0$, $h_i(x^*) = 0$ $\qquad 0 \le \lambda^* \perp -f(x^*) \ge 0$
- Dual Feasibility $\lambda^* \ge 0$
- Complementary Slackness $\lambda_i^* f_i(x^*) = 0 \quad \forall i$
- Stationarity $\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0$

if only ineq.: $-\nabla f_0(x^*) = \sum_{i=1}^{m} \lambda_i^* f_i(x^*)$



For convex:
1) If KKT ✓ $\Rightarrow p^* = d^* = L(x^*, \lambda^*, \nu^*)$
2) If Slater ✓ x^* opt $\Leftrightarrow \exists (\lambda^*, \nu^*) \rightarrow$ KKT ✓

Sensitivity Analysis

$$P \left| \begin{array}{ll} \min_x & f_0(x) = p^*(u,v) \\ \text{s.t.} & f_i(x) \le 0 + u_i \\ & h_i(x) = 0 + v_i \end{array} \right. \qquad D \left| \begin{array}{ll} \max_{\nu,\lambda} & g(\nu,\lambda) - u^T\lambda - v^T\nu \\ \text{s.t.} & \lambda \ge 0 \end{array} \right.$$

perturbed

Strong duality for unperturbed with (ν^*, λ^*) dual optimal

Weak duality for perturbed $p^*(u,v) \ge \underbrace{g^*(\nu^*, \lambda^*)}_{=p^*(0,0)} - u^T\lambda^* - v^T\nu^*$

Global Sensitivity

- $\lambda_i^* \gg 0 \quad u_i < 0 \Rightarrow p^*(u,v) \uparrow \Leftarrow |\nu^*| \gg 0 \quad \nu^* v_i < 0$
- $\lambda_i^* > 0 \quad u_i > 0 \Rightarrow p^*(u,v) \downarrow \Leftarrow |\nu^*| > 0 \quad \nu^* \cdot v_i > 0$

↳ Not symmetrical bcs only lower bound on $p^*(u,v)$

Local Sensitivity

assume $p^*(u,v)$ differentiable @ $(0,0)$

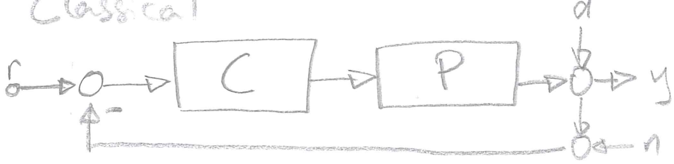
$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial u_i}$ sensitivity of p^* relative to i-th inequality

$\nu_i^* = -\frac{\partial p^*(0,0)}{\partial v_i}$ " " " " " " equality

MPC

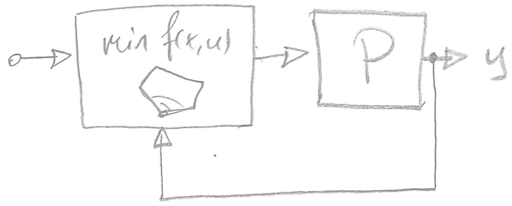
Concept
1) Solve Opt. Problem
2) Obtain Series of Control Actions
3) Apply First Control Action
4) Repeat

Classical



VS

MPC



- Disturbance rejection
- Noise insensitivity
- Model Uncertainty

ω-Domain

→ Ad hoc constraint management
→ Set point far from constraint
→ Suboptimal Plant Operation

- Control Constraints
- Process Constraints

t-Domain

→ Constraints in Design
→ Set point optimal
→ Optimal Plant Operation

Mathematical Formulation

Need Model & State Observer!

$$U_t^*(x(t)) := \underset{U_t}{\operatorname{argmin}} \sum_{k=0}^{N-1} q(x_{t+k}, u_{t+k}) \qquad N: \text{horizon}$$

$$\text{s.t.} \quad x_t = x(t) \qquad \text{measurement}$$

$$x_{t+k+1} = A x_{t+k} + B u_{t+k} \qquad \text{system model}$$

$$x_{t+k} \in \mathcal{X} \qquad \text{state constraints}$$

$$u_{t+k} \in \mathcal{U} \qquad \text{input constraints}$$

$$U_t = \{u_t, u_{t+1}, \dots, u_{t+N-1}\} \qquad \text{opt. variables}$$

At each step: Measure/Estimate $x(t)$
Find Optimal U_t^* for entire planning window N
Implement only first control action u_t^*

→ Verify! ⚠ Implementation (solve in $< T_s$!) ↳ Opt. software! ⚠ Stability, Robustness & Feasibility Not guaranteed!

Constrained Linear Optimal Control

Cost function $J_0(x(0), U_0) = p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k)$

Squared Euclidean Norm $p(x_N) = x_N^T P x_N$ $\quad q(x_k, u_k) = x_k^T Q x_k + u_k^T R u_k$

$p=1$ or $p=\infty$ $\quad p(x_N) = \|P x_N\|_p$ $\quad q(x_k, u_k) = \|Q x_k\|_p + \|R u_k\|_p$

Constrained Finite Time Optimal Control problem (CFTOC)

$$\boxed{\begin{aligned} J_0^*(x(0)) = \min_{U_0} \quad & J_0(x(0), U_0) \\ \text{s.t.} \quad & x_{k+1} = A x_k + B u_k \\ & x_k \in \mathcal{X} \quad u_k \in \mathcal{U} \\ & x_N \in \mathcal{X}_f \quad x_0 = x(0) \end{aligned} \quad \Big\} \; k = 0, \dots, N-1}$$

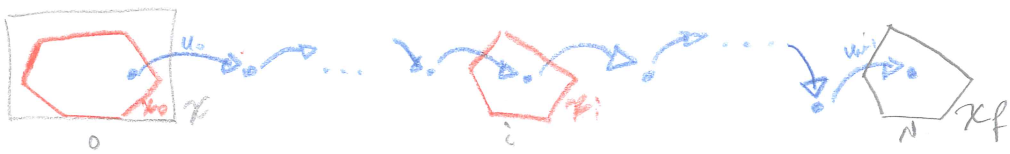
time horizon

polyhedral regions

Feasible Sets

Set of states at time i for which CFTOC is feasible

$\mathcal{X}_i = \{x_i \in \mathbb{R}^n \mid \exists (u_i, \dots, u_{N-1}) : x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ x_N \in \mathcal{X}_f,\ x_{k+1} = A x_k + B u_k \ \forall k\}$



Unconstrained Solution for quadratic cost $\mathcal{X} = \mathcal{X}_f = \mathbb{R}^n$ $\quad \mathcal{U} = \mathbb{R}^m$

→ TV linear control law $u^*(k) = F_k x(k)$

→ TI linear control law $u^*(k) = F_\infty x(k)$ $\quad N \to \infty$

} LQR

Constrained Optimal Control: 2-norm

Quadratic Cost Function $J_0(x(0), U_0) = x_N^T P x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k$

$P \succeq 0 \quad Q \succeq 0 \quad R \succ 0$

QP with substitution

1) Cost $\underline{J_0(x(0), U_0)} = U_0^T H U_0 + 2x^T(0) F U_0 + x^T(0) Y x(0) = \underline{(U_0^T \; x^T(0)) \begin{pmatrix} H & F^T \\ F & Y \end{pmatrix} \begin{pmatrix} U_0 \\ x(0) \end{pmatrix}}$

↳ Batch Approach!

2) Constraints $\underline{G_0 U_0 \le W_0 + E_0 x(0)}$

If $\mathcal{X} = \{x \mid A_x x \le b_x\}$ $\mathcal{U} = \{u \mid A_u u \le b_u\}$ $\mathcal{X}_f = \{x \mid A_f x \le b_f\}$

$$G_0 = \begin{bmatrix} A_u & 0 & \cdots & 0 \\ 0 & A_u & & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & A_u \\ 0 \;]\, \text{size}(A_x,1) & 0 & & 0 \\ A_x B & 0 & & 0 \\ A_x A B & A_x B & \cdots & 0 \\ \vdots & & \ddots & \\ A_f A^{N-1} B & A_f A^{N-2} B & \cdots & A_f B \end{bmatrix} \quad E_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^N \end{bmatrix} \quad W_0 = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

$N \cdot m$ n m

$N \cdot \text{length}(b_u)$ + $N \cdot \text{length}(b_x)$ + $\text{length}(b_f)$

3) $J_0^*(x(0)) = \min\limits_{U_0} \; [U_0^T \; x^T(0)] \begin{bmatrix} H & F^T \\ F & \cancel{Y} \end{bmatrix} \begin{bmatrix} U_0 \\ x(0) \end{bmatrix}$ ← useless!

$\underbrace{G_0 U_0}_{A_{ineq}} \le \underbrace{W_0 + E_0 x(0)}_{b_{ineq}}$

QP

QP without substitution

1) Cost $\underline{J_0(x(0), z) = [z^T \; x^T(0)] \begin{bmatrix} \bar{H} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} z \\ x(0) \end{bmatrix}}$ ← useless!

$z^T = (x_1^T \; \cdots \; x_N^T \; U_0^T \; \cdots \; u_{N-1}^T)$

$$\bar{H} = \begin{bmatrix} Q & & & & 0 \\ & \ddots & & & \\ & & Q & & \\ & & & P & \\ & & & & R \\ 0 & & & & \ddots \\ & & & & & R \end{bmatrix}$$

2) Constraints $\underline{G_{0,in} z \le W_{0,in} + E_{0,in} x(0)}$ $\quad \underline{G_{0,eq} z = E_{0,eq} x(0)}$

$$G_{0,eq} = \left[\begin{array}{cccc|cccc} I & & & 0 & -B & & & 0 \\ -A & I & & & & -B & & \\ & -A & \ddots & & & & \ddots & \\ 0 & & -A & I & 0 & & & -B \end{array}\right] \quad E_{0,eq} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Leftarrow x_{k+1} = A x_k + B u_k$$

$N \cdot N$ + $m \cdot N$ $n \cdot N$ $n \cdot N$

$$G_{0,in} = \begin{bmatrix} 0 & A_x & & & & \\ & & \ddots & & 0 & \\ & & & A_x A_f & & \\ & & & & A_u & \\ & 0 & & & & \ddots \\ & & & & & A_u \end{bmatrix} \quad W_{0,in} = \begin{bmatrix} b_x \\ b_x \\ \vdots \\ b_x \\ b_f \\ b_u \\ \vdots \\ b_u \end{bmatrix} \quad E_{0,in} = \begin{bmatrix} -A_x \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

sparse → QP better!

Constrained Optimal Control: 1- and ∞-Norm

Piecewise Linear Cost Function $J_0(x(0), U_0) := \|Px_N\|_p + \sum_{k=0}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p$

$p = 1$ or ∞ P, Q, R full rank matrices

∞-Norm Problem

$$\min_{z_0} \quad \varepsilon_0^x + \dots + \varepsilon_N^x + \varepsilon_0^u + \dots + \varepsilon_{N-1}^u$$

$$\text{s.t.} \quad \begin{aligned} & -1_n \varepsilon_k^x \leq \pm Q \left[A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j}\right] \\ & -1_r \varepsilon_N^x \leq \pm P \left[A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j}\right] \\ & -1_m \varepsilon_k^u \leq \pm R u_k \\ & A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-j-1} \in \begin{matrix} \mathcal{X} & k \in [0, N-1] \\ \mathcal{X}_f & k = N \end{matrix} \\ & u_k \in \mathcal{U} \\ & x_0 = x(0) \end{aligned}$$

LP

$$\boxed{\begin{aligned} \min_{z_0} \quad & c_0^T z_0 \\ \text{s.t.} \quad & \bar{G}_0 z_0 \leq \bar{W}_0 + \bar{S}_0 x(0) \end{aligned}}$$

where: $z_0 := [\varepsilon_0^x, \dots, \varepsilon_N^x, \varepsilon_0^u, \dots, \varepsilon_{N-1}^u, u_0^T, \dots, u_{N-1}^T]^T \in \mathbb{R}^{s}$, $s = (n+1)N + N + 1$

$$\bar{G}_0 = \begin{bmatrix} G_\varepsilon & 0 \\ 0 & G_0 \end{bmatrix} \quad \bar{S}_0 = \begin{bmatrix} S_\varepsilon \\ S_0 \end{bmatrix} \quad \bar{W}_0 = \begin{bmatrix} W_\varepsilon \\ W_0 \end{bmatrix}$$

→ 1-Norm similarly

Open Loop Optimal Control Function → mp-LP

$u_0^* = [0 \; \dots \; 0 \; I_m \; 0 \dots 0] \, z_0^* = f_0(x(0)) \quad \forall x(0) \in \mathcal{X}_0$

PWA on polyhedra $f_0(x) = F_0^i x + g_0^i \quad x \in CR_0^i \quad i = 1, \dots, N_0^r$

Polyhedral sets $CR_0^i := \{x \in \mathbb{R}^n \mid H_0^i x \leq K_0^i\}$ $i = 1, \dots, N_0^r$ partition of $\mathcal{X}_0$ polyhedron!

Feasibility and Stability

Would like to solve for $N=\infty$ but can NOT $\to U_0 \in \mathbb{R}^{m\cdot\infty}$

$\to$ Receding Horizon Control (RHC) $\triangleq$ MPC

$p(x_{t+N})$, $\mathcal{X}_f$ to approximate "tail" of costs and constraints for $k: N \to \infty$

Notation $\dots|t \triangleq$ given t, based on $x(t)$ $\Rightarrow$ $u_{2|1} \neq u_{3|2}$ (in general!) $u(t) = u^*_{t|t}$

$\hookrightarrow$ Drop it bcs system, constraints and cost TI $\Rightarrow U_0$, J_0 TI

Features

Pros: NL/L, $n \in \mathbb{N}$, Tdelay, constraints ✓ — Any objective ✓

Cons: τ computation, stable?, feasible?

Parameters have complex effect on CL trajectory!

Infinite horizon $\to$ feasible + finite cost $\Rightarrow$ feasible CL + stable trajectory

Finite horizon $\to$ Deviation between OL prediction and CL system $\to$ stable? $\to$ feasible?

$\hookrightarrow$ Design it s.t approximates infinite horizon $\begin{cases} \mathcal{X}_0 \triangleq \mathcal{O}_\infty \subseteq \mathcal{X}_0 \\ \text{Stable} \end{cases}$

$\hookrightarrow$ Mimic with $p(\cdot)$, $\mathcal{X}_f$

Terminal constraint $\mathcal{X}_f = \{0\}$ $\Rightarrow$ recursive feasibility + stability ✓

$\hookrightarrow$ but smaller $\mathcal{X}_0$!

Invariant set $\mathcal{O}$ positively invariant if $x(0) \in \mathcal{O} \Rightarrow x(t) \in \mathcal{O}\ \forall t \in \mathbb{N}_+$

Set that contains every $\mathcal{O}$ is $\mathcal{O}_\infty$

Theorem

$q(\cdot,\cdot)$ PDF, $\mathcal{X}_f$ invariant under $u = v(x_k)$, all constraints ✓,

$p(\cdot)$ PDF $\forall x_k \in \mathcal{X}_f$: $p(x_{k+1}) - p(x_k) \leq -q(x_k, v(x_k))\ \forall x_k \in \mathcal{X}_f$

$\Rightarrow$ CL under MPC $u_0^*(x)$ stable and set $\mathcal{X}_f$ positive invariant under $u = u_0^*(x_k)$

Choice of Terminal Set and Cost - Linear System, Quadratic Cost

$p(x_N) = x_N^T P x_N \quad x_N \in \mathcal{X}_f$

Unconstrained LQR
$$F_\infty = -(B^T P_\infty B + R)^{-1} B^T P_\infty$$
$$P_\infty = A^T P_\infty A + Q - A^T P_\infty B (B^T P_\infty B + R)^{-1} B^T P_\infty A$$

$P = P_\infty$

$x_{k+1} = Ax_k + BF_\infty x_k \in \mathcal{X}_f \subseteq \mathcal{X}, \quad F_\infty x_k \in \mathcal{U} \quad \forall x_k \in \mathcal{X}_f$

$\mathcal{X}_f$ = max invariant set for $x_{k+1} = (A + BF_\infty) x_k \in \mathcal{X}_{cl} = \left\{ x \,\middle|\, \begin{bmatrix} A_x \\ A_u F_\infty \end{bmatrix} x \leq \begin{bmatrix} b_x \\ b_u \end{bmatrix} \right\}$

$\hookrightarrow \mathcal{X}_f = \left\{ x \,\middle|\, \begin{bmatrix} A_x \\ A_x(A+BF_\infty) \\ A_u F_\infty \end{bmatrix} x \leq \begin{bmatrix} b_x \\ b_x \\ b_u \end{bmatrix} \right\}$

Remarks

$\mathcal{X}_f$, $p(x_N)$ ensure recursive feasibility and stability of CL,

but $\mathcal{X}_f \rightarrow \mathcal{X}_0 \downarrow \subseteq \mathcal{O}_\infty$ → real controller must provide input ∀ circumstance

$\hookrightarrow$ Extend horizon $N \uparrow \rightarrow \mathcal{X}_0 \uparrow \subseteq \mathcal{O}_\infty$

→ often unnecessary

$\hookrightarrow$ $N\uparrow$ and check stability by sampling

$\lim_{N \to \infty} \mathcal{X}_0 = \mathcal{O}_\infty \rightarrow \mathcal{X}_f$ irrelevant!

RHC may be unstable and unfeasible VS ∞-Horizon

$\hookrightarrow$ Fake ∞-Horizon

→ Can extend to Nonlinear, but sets difficult to compute

Nonlinear MPC

don't rely on Linearity!

$$\boxed{\begin{aligned} J_0^*(x(0)) = \min_{U_0} \quad & p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \\ \text{s.t.} \quad & x_{k+1} = g(x_k, u_k) \\ & x_k \in \mathcal{X} \quad u_k \in \mathcal{U} \\ & x_N \in \mathcal{X}_f \\ & x_0 = x(0) \end{aligned}}$$

→ Results can be extended!

But, compute $p(\cdot)$, $\mathcal{X}_f$ very difficult!

Reference Tracking

Steady-State Problem for $D = 0$ $\quad \lim_{k \to \infty} y_k = r$

$$\Rightarrow \begin{cases} x_s = A x_s + B u_s \\ C x_s = r \end{cases}$$

If feasible unfeasible

$$\min \; u_s^T R_s u_s \quad (C x_s - r)^T Q_s (C x_s - r)$$

cheapest input
closest state

$$\text{s.t.} \begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \Rightarrow (x_s, u_s)$$

$$x_s \in \mathcal{X} \quad u_s \in \mathcal{U}$$

MPC to get there

$$\min_{u_0} \; \| y_N - C x_s \|_P^2 + \sum_{k=0}^{N-1} \| y_k - C x_s \|_Q^2 + \| u_k - u_s \|_R^2$$

$$\text{s.t.} \quad x_{k+1} = A x_k + B u_k$$

$$y_k = C x_k$$

$$x_k \in \mathcal{X} \quad u_k \in \mathcal{U} \quad x_N \in \mathcal{X}_f \quad x_0 = x(t)$$

↳ ⚠ offset if model error, d

Offset Free Reference Tracking

Augmented Model

$$x_{k+1} = A x_k + B u_k + B_d d_k$$

$$d_{k+1} = d_k$$

$$y_k = C x_k + C_d d_k$$

constant $d_k \in \mathbb{R}^{n_d}$

State Observer $\begin{pmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{pmatrix} = \begin{pmatrix} A & B_d \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{x}_k \\ \hat{d}_k \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u_k + \begin{pmatrix} L_x \\ L_d \end{pmatrix} (-y_{m,k} + C \hat{x}_k + C_d \hat{d}_k)$

If $n_d = p$, $C \hat{x}_\infty + C_d \hat{d}_\infty = y_{m,\infty}$ without offset!

MPC

$$\min_{u_0} \; \| x_N - \bar{x}_t \|_P^2 + \sum_{k=0}^{N-1} \| x_k - \bar{x}_t \|_Q^2 + \| u_k - \bar{u}_t \|_R^2$$

$$\text{s.t.} \quad x_{k+1} = A x_k + B u_k + B_d d_k$$

$$d_{k+1} = d_k$$

$$x_k \in \mathcal{X} \quad u_k \in \mathcal{U} \quad x_N \in \mathcal{X}_f \quad x_0 = \hat{x}(t) \quad d_0 = \hat{d}(t)$$

where targets: $\begin{pmatrix} A - I & B \\ C & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_t \\ \bar{u}_t \end{pmatrix} = \begin{pmatrix} -B_d \hat{d}(t) \\ r(t) - C_d \hat{d}(t) \end{pmatrix}$

$n_d = p$, MPC rec. feasible

↳ convergence!

$C_d = I$

sys observable $\Leftrightarrow$ (C, A) obs & $\det(A - I - B_d C) \neq 0$

if no $B_d = 0$

Soft Constraints

State/output constraints rarely hard $\longrightarrow$ complications

Softened Problem

$$\min_{z,\varepsilon} \; f(z) + \ell(\varepsilon) \longrightarrow \text{Requirement } z^*_{hard} = z^*_{soft}, \; \varepsilon^* = 0$$
$$\text{s.t.} \quad g(z) \leq \varepsilon$$
$$\varepsilon \geq 0$$

Exact Penalty Function $\quad \ell(\varepsilon) = u \cdot \varepsilon + v\varepsilon^2 \quad$ (smooth) $\quad$ where $u > u^* := -\lim_{z \to z^*} f'(z) \geq 0$

Multiple Constraints $\quad \ell(\varepsilon) = \sum_j u_j \varepsilon_j + v_j \varepsilon_j^2$

Generalizing Problem

Modify MPC as

$$\min_{U_0} \; \|x_{N_y}\|_P^2 + \sum_{k=0}^{N_y-1} \|x_k\|_Q^2 + \|u_k\|_R^2$$

s.t.

$$\begin{cases} y_{min}(k) \leq y_k \leq y_{max}(k) & k = 1, \dots, N_y - 1 \\ u_{min}(k) \leq u_k \leq u_{max}(k) & k = 0, \dots, N_u \end{cases}$$

TV constr!

to reduce complexity

$$x_0 = x(t)$$
$$x_{k+1} = Ax_k + Bu_k \qquad k \geq 0$$
$$y_k = Cx_k \qquad k \geq 0$$
$$u_k = Kx_k \qquad N_u \leq k \leq N_y$$

feas + stab guarantees lost!

Explicit Model Predictive Control

Optimization Problem parametrized by state

↳ pre-compute PWA $u_0^*(x)$ for L systems/constraints $\tau_{online,comp} \downarrow\downarrow$

mpQP

$$J^*(x) = \min_z \; z^T H z \qquad H \succ 0$$
$$\text{s.t.} \quad Gz \le W + Sx \qquad z \in \mathbb{R}^s \quad x \in \mathbb{R}^n \quad G \in \mathbb{R}^{m \times s}$$

$x \in \mathcal{K} \subset \mathbb{R}^n$ closed and bounded polyhedral set

Feasible set $\mathcal{K}^* := \{x \in \mathcal{K} \mid \exists z : Gz \le W + Sx\}$ is polyhedron

$$\boxed{\begin{array}{ll} \text{Active set} & A(x) := \{i \in I \mid G_i z^*(x) - S_i x = W_i\} \\ \text{its complement} & NA(x) := \{i \in I \mid G_i z^*(x) - S_i x < W_i\} \end{array}}$$

G_i = i-th row of G

$I := \{1, \dots, m\}$

↳ set of constraint indices

$$\boxed{\text{Critical Region } CR_A := \{x \in \mathcal{K}^* \mid A(x) = A\}}$$

set of parameters $x \in \mathcal{K}^*$, for which same set of constraints $A \subseteq I$ active @ *

$z^*(x) : \mathcal{K}^* \to \mathbb{R}^m$ is continuous, polyhedral PWA over $\mathcal{K}^* = \bigcup CR_i$

Affine in each CR_i

$J^*(x) : \mathcal{K}^* \to \mathbb{R}$ is continuous, convex, polyhedral PWA over $\mathcal{K}^*$, quadratic in each CR_i

mpLP

$$J^*(x) = \min_z \; c^T z$$
$$\text{s.t.} \quad Gz \le W + Sx$$

If unique, $z^*(x)$ continuous, polyhedral PWA over $\mathcal{K}^*$, affine in each CR_i

$J^*(x)$ is continuous, convex PPWA over $\mathcal{K}^*$, affine in each CR_i

2-Norm State Feedback Solution

$$J_0^*(x(0)) = \min_{U_0} [U_0^T \; x(0)^T] \begin{bmatrix} H & F^T \\ F & Y \end{bmatrix} \begin{bmatrix} U_0 \\ x(0) \end{bmatrix} \quad \text{s.t.} \quad G_0 U_0 \le W_0 + E_0 x(0)$$

$$\xrightarrow[S_0 := E_0 + G_0 H^{-1} F^T]{z := U_0 + H^{-1} F^T x(0)}$$

$$\hat{J}^*(x(0)) = \min_z \; z^T H z \quad \text{s.t.} \quad G_0 z \le W_0 + S_0 x(0)$$

$= \hat{J}^*(x(0)) - x(0)^T (Y - F H^{-1} F^T) x(0)$

Open Loop Optimal Control $U_0^*(x(0)) = z^*(x(0)) - H^{-1} F^T x(0)$ solve mpQP $\forall x(0) \in \mathcal{X}_0$

→ First Component $u^*(0) = f_0(x(0)) \quad \forall x(0) \in \mathcal{X}_0$ $\quad f_0(x) = F_0^i x + g_0^i$ if $x \in CR_0^i = \{x \mid H_0^i x \le K_0^i\}$

const.

↳ PWA on P

↳ partition $\mathcal{X}_0$

$J_0^*(x(0))$ convex and PWQ on polyhedra

1-/∞-Norm State Feedback Solution

$$J_0^*(x(0)) = \min_{z_0} \; c_0^T z_0 \quad \text{s.t. } \bar{G}_0 z_0 \le \bar{w}_0 + \bar{S}_0 x(0) \quad \rightarrow u_0^* = [0 \dots 0 \; I_m \; 0 \dots 0] z_0^*(x(0)) = f_0(x(0))$$

→ Same as 2-Norm! $J^*(x(0))$ convex and PWA on polyhedra

Online Evaluation: Point Location → Evaluation of Affine Function

- Sequential search: $\forall i \quad A_i x + b_i \le 0 \Rightarrow x \in CR^i$ ~lin in # regions
- Logarithmic search: Find hyperplane that seperates regions in two
  ↳  ↳ Repeat for region where point is ~ not logarithmic

→ MPT 3.0

Summary

Linear MPC + Quadratic or linear-norm cost ⇒ PWA $u^*(x(0))$

↳ precompute offline → Only for small systems $n \in [1,6]$

↳ Online evaluation fast (ns-μs)!

Modeling of Hybrid Systems

Hybrid Systems: State Evolution depends on interaction between CT dynamics and discrete events

Interaction thru events and mode switches

PWA systems (eg. lin. of NL @ different OP, CL MPC with liner constrains)

- Affine dynamics and output $\left\{ \begin{array}{l} x(t+1) = A_i x(t) + B_i u(t) + f_i \\ y(t) = C_i x(t) + D_i u(t) + g_i \end{array} \right.$ if $(x(t), u(t)) \in \mathcal{X}_{i(t)}$
- Polyhedral partition of (x,u)-space $\{\mathcal{X}_i\}_{i=1}^{s} := \{x, u \mid H_i x + J_i u \leq K_i\}$

> Well-Posedness PWA system P with $\mathcal{X} = \bigcup_{i=1}^{s} \mathcal{X}_i \subseteq \mathbb{R}^{n+m}$
> well posed $\Leftrightarrow$ $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset \quad \forall i \neq j$?

Binary States, Inputs and Outputs $x = \begin{pmatrix} x_c \\ x_\ell \end{pmatrix} \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ → same for u & y

- → Number
- → Boolean variable $\delta \in \{0,1\}$ $\quad \delta = 0 \,\hat{=}\,$ false $\quad \delta = 1 \,\hat{=}\,$ true
  - Expression $\neg$ = not $\quad \vee$ = or $\quad \wedge$ = and $\quad \rightarrow$ implies $\quad \leftrightarrow$ iff
  - Function $f: \{0,1\}^{n-1} \rightarrow \{0,1\}$ $\quad \delta_n = f(\delta_1, \dots, \delta_{n-1})$

Mixed Logical Dynamical System

Associate Boolean p_i to binary integer δ_i $\quad$ $p_i \Leftrightarrow \{\delta_i = 1\}$, $\neg p_i \Leftrightarrow \{\delta_i = 0\}$

1) Boolean formula as Linear Integer Inequality

$F(p_1, \dots, p_n)$ "TRUE" $\Leftrightarrow$ $A\delta \leq B$ $\quad \delta \in \{0,1\}^n$ $\quad$ where: $\{\delta_i = 1\} \Leftrightarrow p_i =$ TRUE

Analytic Approach

conjunctive normal form \ CNF $F(p_1, p_2, \dots) = \bigwedge_j \left[\bigvee_i p_i \right]$

CNF into algebraic inequalities

Relation	CNF	AI
AND	$\delta_1 \wedge \delta_2$	$\delta_1 + \delta_2 \geq 2$
OR	$\delta_1 \vee \delta_2$	$\delta_1 + \delta_2 \geq 1$
NOT	$\neg \delta_1$	$1 - \delta_1 \geq 1$
XOR	$\delta_1 \oplus \delta_2$	$\delta_1 + \delta_2 = 1$
IMPLY	$\delta_1 \rightarrow \delta_2$	$\delta_1 - \delta_2 \leq 0$
IFF	$\delta_1 \leftrightarrow \delta_2$	$\delta_1 - \delta_2 = 0$
ASSIGN	$\delta_3 \leftrightarrow \delta_1 \wedge \delta_2$	$\left\{ \begin{array}{l} \delta_1 + (1-\delta_3) \geq 1 \\ \delta_2 + (1-\delta_3) \geq 1 \\ 2 - \delta_1 - \delta_2 + \delta_3 \geq 1 \end{array} \right.$

2) Linear Inequality as Logic Condition

$$\boxed{\text{Event Generator } f_{EG}: \mathcal{X} \times \mathcal{U} \times \mathbb{N}_0 \Rightarrow D: \ \delta_e(t) = f_{EG}(x_c(t), u_c(t), t)}$$

Boolean expression $p \Leftrightarrow a^T x \le b$ where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ $x \in \mathcal{X} := \{x \mid a^T x - b \in [m, M]\}$

Translated to LI $m\delta \le a^T x - b \le M(1-\delta)$

Switched Affine Dynamics

$$z_j(t) = \begin{cases} A_j x_c(t) + B_j u_c(t) + f_j & \text{if } i(t) = j \\ 0 & \text{otherwise} \end{cases} \Rightarrow x_c(t+1) = \sum_{j=1}^{s} z_j(t)$$

In general, use "IF-THEN-ELSE" relations

$$\left| \begin{array}{l} \text{IF } p \text{ THEN} \\ \quad z_t = a_1^T x_t + b_1 \\ \text{ELSE} \\ \quad z_t = a_2^T x_t + b_2 \end{array} \right. \Leftrightarrow \begin{cases} (m_2 - M_1)\delta + z_t \le a_2^T x_t + b_2 \\ (m_1 - M_2)\delta - z_t \le -(a_2^T x_t + b_2) \\ (m_1 - M_2)(1-\delta) + z_t \le a_1^T x_t + b_1 \\ (m_2 - M_1)(1-\delta) - z_t \le -(a_1^T x_t + b_1) \end{cases}$$

where: $x \in \mathcal{X}$, $M_i \ge \sup_{x \in \mathcal{X}} a_i^T x + b_i$ $m_i \le \inf_{x \in \mathcal{X}} a_i^T x + b_i$

MLD Hybrid Model from DHA

$$\begin{aligned} x_{t+1} &= A x_t + B_1 u_t + B_2 \delta_t + B_3 z_t \\ y_t &= C x_t + D_1 u_t + D_2 \delta_t + D_3 z_t \\ E_2 \delta_t + E_3 z_t &\le E_4 x_t + E_1 u_t + E_5 \end{aligned}$$

$x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_e}$, $u \in \dots$, $y \in \dots$, $\delta \in \{0,1\}^{r_e}$, $z \in \mathbb{R}^{r_c}$

Phys. Constr. $\mathcal{C} := \left\{ \begin{bmatrix} x_c \\ u_c \end{bmatrix} \in \mathbb{R}^{n_c + m_c} \mid F x_c + G u_c \le H \right\}$

Complete Well-Posedness: given $\begin{pmatrix} x_t \\ u_t \end{pmatrix} \Rightarrow$ unique $x_{t+1}, y_t, \delta_t, z_t$

↳ Sufficient for computation of state & output

↳ allows transformation into eq. hybrid models → /~hybrid/hysdel/

HYSDEL: based on DHA → automatically generates MLD models for MATLAB

CFTOC Problem for Hybrid Systems

$$\begin{aligned} J^*(x(t)) = \min_{U_0} \ & p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k, \delta_k, z_k) \\ \text{s.t. } & x_{k+1} = A x_k + B_1 u_k + B_2 \delta_k + B_3 z_k \\ & E_2 \delta_k + E_3 z_k \le E_4 x_k + E_1 u_k + E_5 \\ & x_N \in \mathcal{X}_f \quad x_0 = x(t) \end{aligned}$$

MILP $\to$ in general nonconvex

$$\inf_{z_c, z_b} \; c_c^T z_c + c_b^T z_b + d$$
s.t. $G_c z_c + G_b z_b \leq W$
$z_c \in \mathbb{R}^{s_c}$ $z_b \in \{0,1\}^{s_b}$

fix $\bar{z}_b \to$ LP
$\hookrightarrow$ Bruteforce: solve LP for all 2^{s_b} integer values of $z_b \to$ get lowest

MIQP $\to$ in general nonconvex

$$\inf_{z_c, z_b} \; \frac{1}{2} z^T \overset{H \succeq 0}{H} z + q^T z + r$$
s.t. $G_c z_c + G_b z_b \leq W$
$z = [z_c; z_b]$, $z_c \in \mathbb{R}^{s_c}$ $z_b \in \{0,1\}^{s_b}$

fix $\bar{z}_b \to$ QP $\to$ Bruteforce

Branch & Bound method based on relaxation $\{0,1\} \to [0,1]$
$\hookrightarrow$ LBound on optimal cost
$\to$ Feasible solution to original $\to$ UBound
$\to$ rule out parts of B&B trees

MPC needs solution of MI$\frac{L}{Q}$P online (speed!)

Explicit MPC

Quadratic Case $\to$ $\overset{\text{TVPWA}}{u_t^*(x_t) = F_t^i x_t + G_t^i}$ if $x_t \in R_t^i$ where $\mathcal{X}_t^* = \bigcup_i R_t^i$

closure $\bar{R}_k^i := \{x \mid x_k^T L(j)_k^i x_k + M(j)_k^i x_k \leq N(j)_k^i \quad j = 1, \dots, n_i\}$
$\hookrightarrow$ Quad and Lin boundaries, not polyhedral!

$\{v_i\}_{i=1}^{s^N}$ set of all possible switching sequences $\to$ fix it and constrain state acc.
$\hookrightarrow$ $u^i(x(0)) = \tilde{F}^{i,j} x(0) + \tilde{g}^{i,j}$ $\forall x(0) \in \mathcal{T}^{i,j}$ $j = 1, \dots, N^{r_i}$ $D^i = \bigcup_j \mathcal{T}^{i,j}$
$\mathcal{X}_0 = \bigcup_i D^i$ in general not convex
$\hookrightarrow$ can overlap!
$\hookrightarrow$ x_0 feas for diff. switchings

1-, ∞-Norm Case $\to$ $u_t^*(x_t) = F_t^i x_t + G_t^i$ if $x_t \in R_t^i$ $\to$ polyh!
$\hookrightarrow$ partition $\mathcal{X}_t^*$

Summary

Mixed-Integer Programming HARD $\to$ solvers CPLEX, XPRESS-MP
B&B, B&Cut
$\hookrightarrow$ Online: $T_{comp} \approx$ mins, expensive hardware & software
$\hookrightarrow$ small problems $\to$ explicit MPC

$\to$ Many systems! $\to$ Many tricks
$\hookrightarrow$ Optimization Problem $\to$ MI$\frac{L}{Q}$P, NP Hard to solve
$\hookrightarrow$ advanced solvers!

$\to$ MPC theory applies
$\hookrightarrow$ invariant sets + opt. solution difficult!
$\hookrightarrow$ subopt ok

Numerical Optimization Methods

Optimization Problem $\quad x^* \in \arg\min f(x)$ s.t. $x \in Q$

Iterative optimization

Given initial guess x_0, produce iterates

$$x_{i+1} = \Psi(x_i, f, Q) \quad i = 0, 1, \dots, m-1$$

s.t. $|f(x_m) - f(x^*)| \leq \varepsilon \quad \mathrm{dist}(x_m, Q) \leq \delta$ tolerances

Important aspects

Convergence, Convergence speed, Feasibility, Numerical Robustness, Warm-starting (advantage x_0 close to x^*), Preconditioning (transform → faster?)

Unconstrained Optimization

Unconstrained Problem $\quad x^* \in \arg\min f(x)$

$$\boxed{\text{Gradient Method} \quad x_{i+1} = x_i - h_i \nabla f(x_i)}$$

L-Smoothness $\quad \|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\| \quad \forall x, y \in \mathbb{R}^n$

$\Leftrightarrow \nabla f$ Lipschitz continuous $\Leftrightarrow f(x) \leq f(y) + \nabla f(y)^T (x-y) + \frac{L}{2} \|x-y\|^2 \quad \forall x, y$

Constant step size $\quad h_i = \frac{1}{L} \Rightarrow$ Convergence

if too small $(L < \frac{L_{real}}{2})$ none!

Fast Gradient Method

Set x_0, $y_0 = x_0 \quad \alpha_0 = (\sqrt{5} - 1) \cdot \frac{1}{2}$

Repeat $x_{i+1} = y_i - \frac{1}{L} \nabla f(y_i)$

$\alpha_{i+1} = \alpha_i \cdot \left(\sqrt{\alpha_i^2 + 4} - \alpha_i\right) \cdot \frac{1}{2}$

$\beta_i = \dfrac{\alpha_i (1 - \alpha_i)}{\alpha_i^2 + \alpha_{i+1}}$

$y_{i+1} = x_{i+1} + \beta_i (x_{i+1} - x_i) \qquad i = 0, \dots, m-1$

Until $|f(x_m) - f(x^*)| < \varepsilon_1 \;\|\; \|x_m - x_{m-1}\| \leq \varepsilon_2$

Strong Convexity $f(x) \lessgtr f(y) + \nabla f(y)^T (x-y) + \frac{L/\mu}{2} \|x-y\|^2 \quad \forall x,y \in \mathbb{R}^n$

Condition number $\kappa := \frac{L}{\mu} \geq 1$ $\qquad f(x) = \frac{1}{2} x^T H x = \frac{1}{2} x^T V \Lambda V^T x$, $\quad H^T \succeq 0$, $\quad \mu = \min \lambda(H)$, $L = \max \lambda(H)$

Fast Gradient Method $M \sim \mathcal{O}\left(\sqrt{\kappa \|x_0 - x^*\|^2} \ln\left(\frac{1}{\varepsilon}\right)\right)$

$\hookrightarrow$ Convergence speed $\sim \kappa$!

Preconditioning $\triangleq$ Variable transformation to improve κ

$x = Py$ s.t. $\kappa_h < \kappa_f$ where $\min_y h(y) = \min_y f(Py)$ vs $\min_x f(x)$

$$\boxed{\text{Newton's Method } x_{i+1} = x_i - h_i \left(\nabla^2 f(x_i)\right)^{-1} \nabla f(x_i)}$$

$\Delta x_{nt} := -\left(\nabla^2 f(x_i)\right)^{-1} \nabla f(x_i)$ $\qquad \hookrightarrow$ Find $h_i > 0 : f(x_i + h_i \Delta x_{nt}) < f(x_i)$

Line search

- Exact $h_i^* = \arg\min_{h>0} f(x_i + h \Delta x_{nt})$
- Inexact $h_i = 1$; while $f(x_i + h_i \Delta x_{nt}) > f(x_i) + \alpha h_i \nabla f(x_i)^T \Delta x_{nt}$; $h_i = \beta h_i$ end $\quad \alpha \in (0, 0.05)$, $\beta \in (0,1)$

Two phases

- Damped Newton $\|\nabla f(x)\|_2 \geq \eta \in \left(0, \frac{\mu^2}{M}\right)$ $\quad$ f strong convex: $\nabla^2 f(x) \succeq \mu I$, $\quad$ Lipschitz c. for $\nabla^2 f$
- Quadratically Convergent $\|\nabla f(x)\|_2 < \eta$ $\quad h_i = 1$, $\|\nabla f(x)\|_2 \longrightarrow 0$ quadratically

Aspects	Gradient	Newton
Convergence	$M \in \mathbb{N} \ \forall \varepsilon, \delta$ ✓	✓
Convergence Speed	$\varepsilon_i \leftrightarrow M$	✓
Robustness	✓ ⚠ Fast	✓
Warm-starting	✓	✓
Preconditioning	✓	No
Computational Cost	cheap	Expensive

Constrained Optimization

Constrained Problem
$x^* = \arg\min f(x)$ — L-smooth
s.t. $x \in Q$ → convex

$\boxed{\text{Gradient Method} \quad x_{i+1} = \Pi_Q(x_i - h_i \nabla f(x_i))}$ where: $\Pi_Q(y) = \arg\min_x \|x-y\|_2^2$ s.t. $x \in Q$ — Projection, $\mathrm{prox}_Q(y)$

$\boxed{\text{Fast Gradient Method} \quad x_{i+1} = \Pi_Q(y_i - h_i \nabla f(x_i))}$ — Rest same

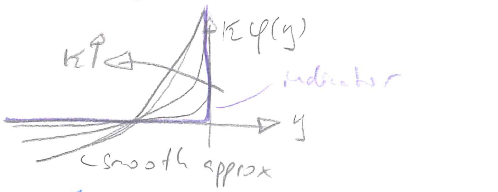
Projection → If easy directly, else solve dual
↳ See tables 3-27,28

MPC Projection on intersection $\mathcal{K} := \mathcal{X} \times \mathcal{U}$ not easy → dual

Interior Point Method
$\min f(x)$ — convex, 2x diff
$g_i(x) \le 0 \quad i = 1, \dots, m$

Assume $f(x^*) < \infty$, $\exists \tilde{x} \in \mathrm{dom} f$ $g_i(\tilde{x}) < 0$ → strictly feasible

$\boxed{\text{Barrier Method} \quad \min f(x) + \kappa \varphi(x)}$



Indicator Function $\varphi(x) := \sum_{i=1}^{m} I_-(g_i(x)) \approx -\sum_{i=1}^{m} \ln(-g_i(x))$ $\quad I_-(y) = \begin{cases} 0 & y \le 0 \\ \infty & y > 0 \end{cases}$

Analytic Center $\arg\min_x \varphi(x)$
Central Path $\{x^*(\kappa) \mid \kappa > 0\}$ $\quad \lim_{\kappa \to 0} x^*(\kappa) = x^*$

Path Following
1) Centering Step $x^*(\kappa_i) = \arg\min f(x) + \kappa_i \varphi(x)$ $\quad x_{i-1}$ to start
Require: x_0 strictly feas, κ_0, $\mu > 1$, $\varepsilon > 0$
2) Update $x_i := x^*(\kappa_i)$
3) Stop if $\kappa_i < \varepsilon$
4) Decrease $\kappa_{i+1} = \frac{\kappa_i}{\mu}$

centering step

Newton $\Delta x_{nt} = -\left(\nabla^2 f(x) + \kappa \nabla^2 \varphi(x)\right)^{-1} \left(\nabla f(x) + \kappa \nabla \varphi(x)\right)$
1) Retain feasibility $h_{max} = \arg\max_{h > 0} \{h \mid g_i(x + h\Delta x_{nt}) < 0 \ \forall i = 1, \dots, m\}$
2) Find $h^* = \arg\min_{h \in [0, h_{max}]} \{f(x + h\Delta x_{nt}) + \kappa \varphi(x + h\Delta x_{nt})\}$ — exactly or backtracking

Centering step with Equality Contraints

$$x^*(\kappa_i) = \arg\min \; f(x) + \kappa_i \varphi(x) \quad \text{s.t.} \quad Cx = d$$

$\Rightarrow$ Newton

$$\begin{bmatrix} \nabla^2 f(x) + \kappa \nabla^2 \varphi(x) & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{nt} \\ \nu \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + \kappa \nabla \varphi(x) \\ 0 \end{bmatrix}$$

KKT for Barrier IPM

$$Cx^* = d$$

$$g_i(x^*) \leq 0$$

$$\nabla f(x^*) + \kappa \sum_{i=1}^{m} \frac{1}{-g_i(x^*)} \nabla g_i(x^*) + C^T \nu^* = 0$$

$\lambda_i^* := \kappa \dfrac{1}{-g_i(x^*)} \geq 0 \Rightarrow$ Complementerity Slackness replaced by relaxed condition $\lambda_i^* g_i(x^*) = -\kappa$

$\Rightarrow$ Relaxed KKT

$$Cx^* = d$$

$$g_i(x^*) + s_i^* = 0$$

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + C^T \nu^* = 0$$

$$\lambda_i^* g_i(x^*) = -\kappa$$

$$\lambda_i^*, s_i^* \geq 0$$

Primal-Dual Search Direction Computation $\Rightarrow$ Most efficient, infeasible to v

$$\begin{bmatrix} H(x,\lambda) & C^T & G(x)^T & 0 \\ C & 0 & 0 & 0 \\ G(x) & 0 & 0 & I \\ 0 & 0 & S & \Lambda \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta \lambda \\ \Delta s \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + C^T y + G(x)^T \lambda \\ Cx - d \\ g(x) + s \\ S\lambda - \nu \end{bmatrix}$$

where: $S := \mathrm{diag}(s_1, \ldots, s_m)$ $\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ $H(x,\lambda) := \nabla^2 f(x) + \sum_{i=1}^{m} \lambda_i \nabla^2 g_i(x)$

$\Delta[\ldots](\nu) \rightarrow$ Standard Newton $\nu = 0$
$\quad \rightarrow \kappa 1$ centering direction
$\quad\quad$ next iterate on central path

$\left.\right\}$ $\sigma \cdot \kappa 1$ $\sigma \in (0,1)$
$\quad \rightarrow$ linear combo for fast convergence

Software

Modeling: Matlab Embedded, Desktop PCs: IPM, Embedded Platforms: solvers Code-gen inc $\rightarrow$ MPC: FORCES