Linear Systems

NLTI CT: $\dot x = g(x,u)$, $y = h(x,u)$

LTI CT: $\dot x = A_c x + B_c u$, $y = Cx + Du$

LTI DT: $x_{k+1} = Ax_k + Bu_k$, $y_k = Cx_k + Du_k$

$A = e^{A_c T_s}$ $\quad B = \int_0^{T_s} e^{A_c(T_s-\tau)} B_c d\tau = A_c^{-1}(A-I)B_c$

GAS of LTIDT $x^*=0$ $\quad x_{k+1} = Ax_k \Leftrightarrow \underline{|\lambda_i|<1}$

Lyapunov Stability of NLTIDT $x^*=0$ $\quad x_{k+1} = g(x_k)$

$\begin{cases} \text{stable if } \forall \varepsilon>0\ \exists\delta>0: x_0 \in B_\delta(0) \Rightarrow x_k \in B_\varepsilon(0)\ \forall k\geq 0 \\ \text{unstable otherwise} \end{cases}$

G/LAS if $\forall/\exists\, \Omega \subseteq \mathbb{R}^n: \lim_{k\to\infty} x_k = 0\ \forall x_0 \in \Omega$

Lyapunov Function $V:\mathbb{R}^n \to \mathbb{R}$ c@$x=0$, finite $\forall x \in \Omega \subseteq \mathbb{R}^n$

and s.t. LPDF: $V(0)=0$ $\quad V(x)>0\ \forall x \in \Omega\setminus\{0\}$ PDF in Ω

$V(g(x_k)) - V(x_k) \leq -\alpha(x_k)\ \forall x_k \in \Omega\setminus\{0\}$ $\quad \alpha:\mathbb{R}^n\to\mathbb{R}$ cPDF

Lyapunov Thm

$x^*=0$ AS in $\Leftarrow$ / GAS $\Leftarrow$ $\quad x_{k+1} = g(x_k)$ admits Lyap fcn s.t. $\|x\|\to\infty \Rightarrow V(x)\to\infty$

LTIDT $x_{k+1} = Ax_k$ $\quad V(x) = x^TPx$ $\quad P^T=P\succ 0$ $\quad \alpha(x) = x^TQx$ $\quad Q^T=Q\succ 0$

$x^*=0$ GAS $\Leftrightarrow \forall\lambda_i(A)\ |\lambda_i|<1 \Leftrightarrow \forall Q=Q^T\succ 0\ \exists! P=P^T\succ 0$ s.t. $A^TPA - P = -Q$ **DTLE**

∞-horizon cost-to-go

$\Phi(x(0)) = \sum_{k=0}^{\infty} x_k^TQx_k = x_0^T \underbrace{\sum_{k=0}^{\infty}(A^k)^TQA^k}_{=:H_k,\ H_0=Q,\ H_{k+1}=A^TH_kA} x_0 = \underline{x_0^TPx_0}$

Controllability

$x_{k+1} = Ax_k + Bu_k$ ctrb $\Leftrightarrow \forall(x_0,x^*)\ \exists N, \exists\{u_0,\dots,u_{N-1}\}: x_N = x^*$ — if not ctrb in n steps then not $\forall N>n$

$\Leftrightarrow \text{range}(\mathcal{C}_n) = \mathbb{R}^n \Leftrightarrow \text{rank}(\mathcal{C}_n) = n \Leftrightarrow |\mathcal{C}_n| \neq 0$ $\quad \mathcal{C}_n := (B \dots A^{n-1}B)$

$\Rightarrow$ stabilizable $\Leftrightarrow \forall x_0\ \exists\{u_0,\dots\}: \lim_{k\to\infty} x_k = 0 \Leftrightarrow$

all uncontrollable modes astable $\Leftrightarrow \text{rank}([\lambda_i I - A\ \ B]) = n$ $\quad \forall\lambda_i \in \Lambda_A^+$

Observability

$x_{k+1} = Ax_k$, $y_k = Cx_k$ obsv $\Leftrightarrow \exists N: \{y_0,\dots,y_{N-1}\}$ uniq. distinguish $x_0\ \forall x_0$

$\Leftrightarrow \text{range}(\mathcal{O}_n) = \mathbb{R}^n \Leftrightarrow \text{rank}(\mathcal{O}_n) = n \Leftrightarrow |\mathcal{O}_n| \neq 0$ $\quad \mathcal{O}_n = (C^T \dots (CA^{n-1})^T)^T$

$\Rightarrow$ detectable $\Leftrightarrow$ can reconstruct $\lim_{k\to\infty} x_k \Leftrightarrow \hat x_k \xrightarrow[\infty]{k} x_k$

all unobs modes astable $\Leftrightarrow \text{rank}\left(\begin{bmatrix}\lambda_i I - A \\ C\end{bmatrix}\right) = n$ $\quad \forall\lambda_i \in \Lambda_A^+$

Model Predictive Control

Mauro Salazar

Optimal Control

$J^*_{0\to N}(x(0)) = \min_{U_{0\to N}} J_{0\to N}(x(0), U_{0\to N}) = p(x_N) + \sum_{k=0}^{N-1} q(x_k,u_k)$

Horizon

s.t. $x_{k+1} = g(x_k,u_k)$ Dynamics

constraints $\begin{cases} h(x_k,u_k) \leq 0 \\ x_N \in \mathcal{X}_f \quad x_0 = x(0) \end{cases}$

LQOC LTI DT

Unconstrained $\quad \underline{p(x_N) = x_N^T \overbrace{P}^{=P^T\succeq 0} x_N}$ $\quad \underline{q(x_k,u_k) = x_k^T \overbrace{Q}^{=Q^T\succeq 0} x_k + u_k^T \overbrace{R}^{=R^T\succ 0} u_k}$

$\Rightarrow$ Batch Approach

$$\underbrace{\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}}_{\mathcal{X}} = \underbrace{\begin{bmatrix} I \\ A \\ \vdots \\ A^N \end{bmatrix}}_{S^x} x_0 + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ \vdots & B & \cdots & \vdots \\ A^{N-1}B & \vdots & \cdots & B \end{bmatrix}}_{S^u} \cdot \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{U_0}$$

$\Rightarrow \underline{J(x_0,U_0)} = \mathcal{X}^T\bar Q\mathcal{X} + U_0^T\bar RU_0 =$

$\underline{= U_0^THU_0 + 2x_0^TFU_0 + x_0^TS^{x^T}\bar QS^xx_0}$

$\bar Q = \text{blckd}\{Q,\dots,Q,P\}$ $\quad \bar R := \text{blckd}\{R,\dots\}$

$\underline{0 \prec H := S^{u^T}\bar QS^u + \bar R}$ $\quad \underline{F := S^{x^T}\bar QS^u}$

$\nabla_{U_0} J_0 \overset{!}{=} 0 \Leftrightarrow \underline{U_0^*(x_0) = -H^{-1}F^Tx_0}$

$\Rightarrow$ Recursive Approach

j-step cost-to-go $\quad J_j^*(x_j) := \min_{u_j,\dots,u_{N-1}} x_N^TPx_N + \sum_{k=j}^{N-1} x_k^TQx_k + u_k^TRu_k$

s.t. $x_{k+1} = Ax_k + Bu_k$ $\quad x_j = x(j)$

Bellman: Pieces of optimal traj. are optimal

$J_j^*(x_j) = J_{j+1}^*(Ax_j + Bu_j^*) + x_j^TQx_j + u_j^{*T}Ru_j^* \Rightarrow \underline{u_k^* = F_kx_k}$

RDE $P_k = A^TP_{k+1}A + Q - A^TP_{k+1}B(B^TP_{k+1}B+R)^{-1}B^TP_{k+1}A$, $P = P_N$

$F_k := -(B^TP_{k+1}B + R)^{-1}B^TP_{k+1}A$ $\quad J_k^*(x_k) = x_k^TP_kx_k$

Infinite Horizon Unconstrained $\to$ LQR $\quad p(x_N) = 0$ $\quad N\to\infty$

Assume $\lim_{k\to\infty} P_k = \lim_{k\to\infty} P_{k+1} = P_\infty$ $\Rightarrow \underline{u_k^* := F_\infty x_k}$

RDE$\to$<u>ARE</u> $\quad P_\infty = A^TP_\infty A + Q - A^TP_\infty B(B^TP_\infty B+R)^{-1}B^TP_\infty A$

$F_\infty := -(B^TP_\infty B+R)^{-1}B^TP_\infty A$ $\quad J_\infty^*(x_k) = x_k^TP_\infty x_k$

If (A,B) stabilizable & $(Q^{\frac12},A)$ detectable $\quad$ RDE $\xrightarrow[\infty]{k}$ ARE & $A+BF_\infty$ astab

Terminal Weight in Finite Horizon

1. Solution = ∞-horizon $\to P = \text{sol}(\text{ARE})$ $\to$ No constraints $k\geq N$ assumed
2. No control action for $k\geq N$ $\to P = \text{sol}(\text{DTLE})$ $\quad x_{k+1} = Ax_k$ $\quad k\geq N$ — only if astab!
3. $u_k = 0$ $\quad x_k = 0$ $\quad k\geq N$ $\to$ extra constraint $x_{k+N} = 0$

State Estimation

$\to$ Crucial for MPC!

Estimate $x(k+i)$ given $\{u(j), y(j)\}_{j=0,\dots,k}$ $\quad \hat x_{k+i|k}$ $\quad i$ $\begin{cases} <0 \text{ smooth} \\ =0 \text{ estimate} \\ >0 \text{ predict} \end{cases}$

LTI DT $\quad E[\varepsilon(i)\varepsilon(j)^T] = \begin{cases} \begin{pmatrix} R_1 & \cancel{R_{12}} \\ \cancel{R_{12}^T} & R_2 \end{pmatrix} & i=j \\ 0 & \text{else} \end{cases}$ $\quad$ white noise $\quad$ 1 process, 2 meas

$D=0$ - strictly proper $\quad$ indep of $\hat x_{0|0}$

Structure

Predict $\hat x_{k|k-1} = A\hat x_{k-1|k-1} + Bu(k-1)$

Update $\hat x_{k|k} = \hat x_{k|k-1} + K_k(y(k) - C\hat x_{k|k-1})$ $\quad$ $\hat y(k|k-1)$ lin. obsv $\quad K_k$: filter gain

Error $\hat x^e_{i|j} := x(i) - \hat x_{i|j} \Rightarrow \hat x^e_{k|k} = (A - K_kCA)\hat x^e_{k-1|k-1} + (I - K_kC)\varepsilon_1(k-1) - K_k\varepsilon_2(k)$

State Observer

$K: |\lambda(A-KCA)|<1$ $\quad$ Obs stable: $\lim_{k\to\infty} \hat x^e_{k|k} = 0\ \forall \hat x^e_{0|0}$ $\quad \varepsilon_1 = \varepsilon_2 = 0$

Pole placement $\Leftrightarrow (CA,A)$ obsv $\overset{|A|\neq 0}{\Leftrightarrow} (C,A)$ obsv $\quad$ choose λ_i not v_i ⚠

Kalman Filter assume $R_{1,2} = 0$

$P_{i|j} := \text{cov}\{\hat x^e_{i|j}\}$ $\quad$ Optimality criterion $\min_{K_k} P_{k|k}$

Algorithm

- Initialize $\hat x_{0|0}$ $\quad P_{0|0}$
- Online or in advance $\quad P_{k|k-1} = AP_{k-1|k-1}A^T + R_1$
  $K_k = P_{k|k-1}C^T(CP_{k|k-1}C^T + R_2)^{-1}$
  $P_{k|k} = (I - K_kC)P_{k|k-1}$
- @ time k
  Predict $\hat x_{k|k-1} = A\hat x_{k-1|k-1} + Bu(k-1)$
  Measure $y(k)$
  Update $\hat x_{k|k} = \hat x_{k|k-1} + K_k(y(k) - C\hat x_{k|k-1})$
- $k \to k+1$

Stability $\to K_k \xrightarrow[\infty]{k} K_\infty$

$P_{k|k-1} \xrightarrow[\infty]{k} P_\infty$, $\quad \underline{|\lambda(A - K_\infty CA)|<1} \Rightarrow \underline{\lim_{k\to\infty} E[\hat x^e_{k|k}] = \prod_{i=1}^{\infty}(A - K_iCA)E[\hat x^e_{0|0}] = 0}$ — unbiasedness

$\hookrightarrow$ if bounded cov $\Rightarrow$ KF stable

$\hookrightarrow$ if (C,A) det $\Rightarrow \forall P_{0|0} \to$ If $(A,R_1^{\frac12})$ stab, $P_{0|0}\succeq 0$

$\Rightarrow P_\infty = AP_\infty A^T - AP_\infty C^T(CP_\infty C^T + R_2)^{-1}CP_\infty A^T + R_1$

Uncertainty Model

Convex Optimization

Optimization Problem

$\min_{x \in \mathcal{X}} f_0(x) =: J^* = p^* = \min_{x \in \mathcal{X} \cap \mathcal{X}} f_0(x)$ — obj fcn, domain

s.t. $f_i(x) \le 0$ $i = 1, \dots, m$ ineq. constr.

$h_i(x) = 0$ $i = 1, \dots, p$ eq.

Unbounded below $J^* = -\infty$

Infeasible $\mathcal{X} = \emptyset \Rightarrow J^* = \infty$

Unconstrained $\mathcal{X} = \mathbb{R}^n$ $m = p = 0$

Terminology

Feasible point $x \in \mathcal{X}$: $f_i(x) \le 0$ $h_i(x) = 0$ — strictly: only $<$

Optimal Value $f_0(x^*) = J^* = p^*$ → infimum → Minimizer

Local Optimality $\exists R > 0: z = x$ optimal for $\min_{z \in \mathcal{X}} f_0(z) \mid \|x - z\| \le R$

Optimal Solution $\forall x^* \in \mathcal{X}: f_0(x^*) \le f_0(x)$ $\forall x \in \mathcal{X}$

Constraint active / inactive @ $\bar{x}$ if $f_i(\bar{x}) = 0$ / < 0 — or $h_i(x)$ always active!

redundant doesn't change $\mathcal{X}$

Satisfiability find $x \in \mathcal{X}$ special case of $\min_{x \in \mathcal{X}} 0$ or $0x$

LP $\min_x c^T x$ s.t. $Gx \le h$, $Ax = b$

QP $\min_x \frac{1}{2} x^T P x + q^T x$ s.t. $Gx \le h$, $Ax = b$ — convex if $P \succeq 0$

MILP $x \in \{0,1\}^n$ or $\mathbb{Z}^n$

Convex Sets

Convex Set $\mathcal{X} \Leftrightarrow \forall x, y \in \mathcal{X}$ $\lambda x + (1-\lambda) y \in \mathcal{X}$ $\forall \lambda \in (0,1)$

Affine Set $\mathcal{X} = \{x \mid Ax = b\}$ subspace if $b = 0$ convex!

Hyperplane / closed Halfspace / Open Halfspace $\mathcal{X} = \{x \mid a^T x \stackrel{=}{\underset{<}{\le}} b\}$ $\forall a \in \mathbb{R}^n \setminus \{0\}$ convex! — everything on one side of a hyperplane

Cone $\mathcal{X} \Leftrightarrow \forall x \in \mathcal{X}$ $\theta \cdot x \in \mathcal{X}$ $\forall \theta > 0$ — pointed if $0 \in \mathcal{X}$

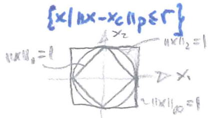
Polyhedron $\mathcal{X} = \{x \mid a_1^T x \le b_1, \dots, a_m^T x \le b_m\} = \{x \mid Ax \le b\}$ — finite number, $\cap$ of closed Halfspaces

Polytope $\triangleq$ Bounded Polyhedron

Norm $f: \mathbb{R}^n \to \mathbb{R}$ 1) $f(x) \ge 0$ $f(x) = 0 \Rightarrow x = 0$ 2) $f(t \cdot x) = |t| f(x)$ $\forall t \in \mathbb{R}$ 3) $f(x+y) \le f(x) + f(y)$

ℓ_p $\|x\|_p := \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}}$ $p \ge 1$

Norm ball $\{x \mid \|x - x_c\|_p \le r\}$



Ellipsoid $\mathcal{X} = \{x \mid (x - x_c)^T A^{-1} (x - x_c)\}$ $A \succ 0$

Thm $\mathcal{X}, \mathcal{Y}$ convex $\Rightarrow$ $\mathcal{X} \cap \mathcal{Y}$ convex $\mathcal{X} \cup \mathcal{Y}$ not!

Convex Hull $Co(\mathcal{X}) = \{x \mid x = \lambda a + (1-\lambda) b$ $\forall \lambda \in (0,1)$ $\forall a, b \in \mathcal{X}\}$

smallest convex set containing $\mathcal{X}$

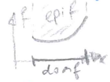
Convex Functions

Convex Function $f: \mathrm{dom}(f) \to \mathbb{R} \Leftrightarrow$ $\mathrm{dom}(f)$ convex $\forall x, y \in \mathrm{dom} f$ $\forall \lambda \in (0,1)$ $f(\lambda x + (1-\lambda) y) \stackrel{<}{\le} \lambda f(x) + (1-\lambda) f(y)$ — strictly

concave $\Leftrightarrow -f$ convex $\Leftrightarrow g(t) = f(x + tv)$ convex → eval along line convex

$\Leftrightarrow f(y) \ge f(x) + \nabla f(x)^T (y - x)$ $\forall y \in \mathrm{dom}(f) \Leftrightarrow \nabla^2 f(x) \stackrel{\succ}{\succeq} 0$ $\forall x \in \mathrm{dom}(f)$

Epigraph $\mathrm{epi}(f) = \{\binom{x}{t} \mid x \in \mathrm{dom}(f), f(x) \le t\} \subseteq \mathrm{dom}(f) \times \mathbb{R}$ — convex $\Leftrightarrow f$ convex



Sublevel Set $C_\alpha = \{x \in \mathrm{dom}(f) \mid f(x) \stackrel{\le}{=} \alpha\}$ f convex $\not\Leftarrow\Rightarrow C_\alpha$ convex $\forall \alpha$

f convex $\Leftrightarrow \tilde{f}$ convex $\mathrm{dom} f$ convex $\Rightarrow f$ quasiconvex

Extended Value function $\tilde{f}(x) = \begin{cases} f(x) & x \in \mathrm{dom}(f) \\ \infty & \text{else} \end{cases} \Rightarrow \mathrm{epi}(f) = \mathrm{epi}(\tilde{f})$

Convexity preserving

f_i convex $\Rightarrow \sum \alpha_i f_i$ convex $\forall \alpha_i \ge 0$, $\max\{f_1(x), \dots, f_m(x)\}$

f convex $\Rightarrow f(Ax + b)$ convex

$f(x,y)$ convex in x $\forall y \in \mathcal{Y}$ / (x,y), $\mathcal{C}$ convex $\Rightarrow$ $q(x) = \sup_{y \in \mathcal{Y}} f(x,y)$, $g(x) = \min_{y \in \mathcal{C}} f(x,y)$ convex

$f(x) = h(g(x))$ convex $\Leftarrow$ g_i / $-g_i$, h convex, $\tilde{h}$ non decr / incr in each arg

Convex Optimization Problems

$\min_{x \in \mathcal{X}} f_0(x)$ s.t. $f_i(x) \le 0$ $i \in [m]$, $Ax = b$ $A \in \mathbb{R}^{p \times n}$ — $\mathcal{X}$ convex set, f_i convex fcn $\Rightarrow$ local opt = global opt

Optimality Criterion $x \in \mathcal{X}$ opt $\Leftrightarrow \nabla f_0(x)^T (y - x) \ge 0$ $\forall y \in \mathcal{X}$

- Unconstrained $\nabla f_0(x) = 0$
- Eq. constrained $x \in \mathrm{dom} f_0$ $Ax = b$ $\exists v: \nabla f_0(x) + A^T v = 0$
- Min over $x \ge 0$ $0 \le x \perp \nabla f_0(x) \ge 0$

Equivalent Problems

$\min_x f_0(A_0 x + b_0)$ s.t. $f_i(A_i x + b_i) \ge 0$ $\Leftrightarrow$ $\min_{x,y} f_0(y_0)$ s.t. $f_i(y_i) \ge 0$, $y_i = A_i x + b_i$ $\Leftrightarrow$ $\min_{x,y} f_0(y_0)$ s.t. $f_i(y_i) + s_i = 0$, $y_i = A_i x + b_i$ $s_i \le 0$ — slack variables

LP $\min_x c^T x + d$ s.t. $Gx \le h$, $Ax = b$

QP $\min_x \frac{1}{2} x^T P x + q^T x + r$ s.t. $Gx \le h$, $Ax = b$

PWA $\min_x [\max\{c_i^T x + d_i\}]$ s.t. $Gx \le h$ $\Leftrightarrow$ $\min_{t,x} t$ s.t. $c_i^T x + d_i \le t$, $Gx \le h$

ℓ_∞ $\min_x \|x\|_\infty$ s.t. $Gx \le h$ $\Leftrightarrow$ $\min_{x,t} t$ s.t. $x_i \le t$, $-x_i \le t$, $Gx \le h$

ℓ_1 $\min_x \|x\|_1$ s.t. $Gx \le h$ $\Leftrightarrow$ $\min_{x,t} \sum_i t_i$ s.t. $x_i \le t_i$, $-x_i \le t_i$, $Gx \le h$

Duality

(P) $\min_{x \in \mathcal{X}} f_0(x) = p^*$ s.t. $f_i(x) \le 0$, $h_j(x) = 0$ $\Rightarrow$ Lagrangian Function $L: \mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$L(x, \lambda, v) := f_0(x) + \sum_i \lambda_i f_i(x) + \sum_j v_j h_j(x)$

(D) $\max_{\lambda, v} g(\lambda, v) = d^* \le p^*$ s.t. $\lambda \ge 0$ — convex even if P is not $\Leftarrow$ Dual Function $g: \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$g(\lambda, v) := \inf_{x \in \mathcal{X}} (L(x, \lambda, v)) \le p^*$ concave $\forall \lambda \ge 0, v \in \mathbb{R}^p$

LP (P) $\min_x c^T x$ s.t. $Ax = b$, $Cx \le d$

(D) $\max_{\lambda, v} -b^T v - d^T \lambda$ s.t. $A^T v + C^T \lambda + c = 0$, $\lambda \ge 0$

QP (P) $\min_x \frac{1}{2} x^T Q x + c^T x$ s.t. $Cx \le d$ $Q \succ 0$

(D) $\max_\lambda -\frac{1}{2} \lambda^T C Q^{-1} C^T \lambda - (C Q^{-1} c + d)^T \lambda + -\frac{1}{2} c^T Q^{-1} c$ s.t. $\lambda \ge 0$

MILP (P) $\min_x c^T x$ s.t. $Ax \le b$ $x \in \{-1, 1\}^n$

(D) $\max_\lambda -\|A^T \lambda + c\|_1 - b^T \lambda$ s.t. $\lambda \ge 0$

Week Duality $d^* \le p^*$ Duality Gap $p^* - d^*$ Strong Duality $d^* = p^*$

Slater Condition $\{x \mid Ax = b, f_i(x) < 0$ $\forall i\} \neq \emptyset$ $\stackrel{\text{suff}}{\Rightarrow}$ $p^* = d^*$ — convex P

KKT Conditions: necessary for optimality

- Primal Feasibility $f_i(x^*) \le 0$ $h_i(x^*) = 0$
- Dual Feasibility $\lambda^* \ge 0$ — $0 \le \lambda^* \perp -f(x^*) \ge 0$
- Complementary Slackness $\lambda_i^* \cdot f_i(x^*) = 0$
- Stationarity $\nabla f_0(x^*) + \sum_i \lambda_i^* \nabla f_i(x^*) + \sum_j v_j^* \nabla h_j(x^*) = 0$

Convex: KKT ✓ $\Rightarrow$ $p^* = d^* = L(x^*, \lambda^*, v^*)$

Slater ✓ $\Rightarrow$ x^* opt $\Leftrightarrow \exists (\lambda^*, v^*)$: KKT ✓

Sensitivity Analysis

P $\min_x f_0(x) = p^*(u, v)$ s.t. $f_i(x) \le 0 + u_i$, $h_i(x) = 0 + v_i$ — perturb

D $\max_{\lambda, v} g(\lambda, v) - u^T \lambda - v^T v$ s.t. $\lambda \ge 0$

Strong Duality $p^*(0,0) = g^*(\lambda^*, v^*) \Rightarrow p^*(u, v) \ge \underbrace{g^*(\lambda^*, v^*)}_{= p^*(0,0)} - u^T \lambda^* - v^T v^*$

Global Sensitivity

$\lambda_i \gg 0$ $u_i < 0$ $\Rightarrow$ $p^* \uparrow\uparrow$ $\Leftarrow$ $|v^*| \gg 0$ $v^* v_i \le 0$

$\lambda_i \ge 0$ $u_i > 0$ $\Rightarrow$ $p^* \downarrow$ $\Leftarrow$ $|v^*| \ge 0$ $v^* v_i > 0$

Local Sensitivity

$\lambda_i^* = -\frac{\partial p^*}{\partial u_i}(0,0)$, $v_i^* = -\frac{\partial p^*}{\partial v_i}(0,0)$ sensitivity w.r.t. i-th inequality / equality

MPC

Concept: Solve Opt → Get $u_0, \dots, u_{N-1}$ → Apply u_0 → Repeat
↳ t-domain with control and process constraints

Mathematical Formulation Need Model & Observer!

$$U_t^*(x(t)) := \arg\min_{U_t} \sum_{k=0}^{N-1} q(x_{t+k}, u_{t+k}) + p(x_{t+N}) \quad N: \text{Horizon}$$

implement

$$\text{s.t. } x_t = x(t), \quad x_{t+k+1} = A x_{t+k} + B u_{t+k} \quad \text{Dyn.}$$

$$U_t = \{u_t, \dots, u_{t+N-1}\} \qquad x_{t+k} \in \mathcal{X} \quad x_{t+N} \in \mathcal{X}_f \quad u_{t+k} \in \mathcal{U} \quad \text{constr.}$$

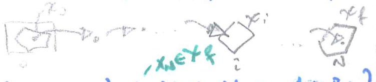
⚠ Stability, Robustness and Feasibility! Solve in $<T_s$!
↳ TI: $t=0$

Constrained Linear Optimal Control

ℓ_2 $p(x_N) = x_N^T P x_N$, $q(x_k, u_k) = x_k^T Q x_k + u_k^T R u_k$

ℓ_1, ℓ_∞ $p(x_N) = \|P x_N\|_{1,\infty}$, $q(x_k,u_k) = \|Q x_k\|_{1,\infty} + \|R u_k\|_{1,\infty}$

Polyhedral $\mathcal{X}, \mathcal{X}_f, \mathcal{U}$



Feasible Sets $\mathcal{X}_i := \{x \in \mathbb{R}^n \mid \exists (u_i, \dots, u_{N-1}): x_k \in \mathcal{X}, u_k \in \mathcal{U}, x_N \in \mathcal{X}_f, x_{k+1} = A x_k + B u_k\}$

Constrained Optimal Control: 2-norm

Quadratic Cost Function with $P \succeq 0$ $Q \succeq 0$ $R \succ 0$
↳ Unconstrained = LQR $\mathcal{X}_f = \{x \mid A_f x \le b_f\}$ $\mathcal{U} := \{u \mid A_u u \le b_u\}$

With substitution

1) Cost $J_0(x(0), U_0) = (U_0^T \; x(0)^T) \begin{pmatrix} H & F^T \\ F & Y \end{pmatrix} \begin{pmatrix} U_0 \\ x(0) \end{pmatrix}$ ← Batch!
⇒ QP

2) Constraints $G_0 U_0 \le w_0 + E_0 x(0)$

$$G_0 = \begin{bmatrix} A_u & 0 & \cdots & 0 \\ 0 & A_u & & \vdots \\ \vdots & \vdots & & A_u \\ 0 \; \text{size}(A_u,1) & 0 & & 0 \\ A_x B & & & \vdots \\ A_x A B & A_x B & & 0 \\ \vdots & & & \\ A_f A^{N-1} B & A_f A^{N-2} B & & A_f B \end{bmatrix} \quad E_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^N \end{bmatrix} \quad w_0 = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

$N \cdot m$, n
$N \cdot \text{length}(b_u) + N \cdot \text{length}(b_x) + \text{length}(b_f)$ (rows)

Without substitution

1) Cost $J_0(x(0), z) = (z^T \; x(0)^T) \begin{pmatrix} \bar{H} & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} z \\ x(0) \end{pmatrix}$ | $\bar{H} = \text{blckd}(\underbrace{Q, \dots, Q}_{N-1}, P, \underbrace{R, \dots, R}_{N})$
$z^T = (x_1^T \cdots x_N^T \; u_0^T \cdots, u_{N-1}^T)$

2) Constraints $G_{0,in} z \le w_{0,in} + E_{0,in} x(0)$ $G_{0,eq} z \le E_{eq} x(0)$

$$G_{0,eq} = \left[\begin{array}{cccc|cccc} I & & & 0 & -B & & & 0 \\ -A & I & \cdots & & & -B & & \\ & -A & & & 0 & & \ddots & \\ 0 & & -A & I & & & & -B \end{array}\right] \quad E_{0,eq} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$n \cdot N + m \cdot N$ (columns), $n \cdot N$ (rows) ⇐ $x_{k+1} = A x_k + B u_k$

$$G_{0,in} = \begin{bmatrix} 0 & & & & \\ & A_x & & 0 & \\ & & \ddots & & \\ \text{sparse!} & & A_f & & \\ & & & A_u & \\ 0 & & & & \ddots \; A_u \end{bmatrix} \quad w_{0,in} = \begin{bmatrix} b_x \\ b_x \\ \vdots \\ b_f \\ b_u \\ \vdots \\ b_u \end{bmatrix} \quad E_{0,in} = \begin{bmatrix} -A_x \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix}$$

Lower better!

Constrained Optimal Control: ℓ_1, ∞-norm

PWL cost function with P, Q, R with full rank

∞-Norm LP $\min_{z_0} c_0^T z_0$ $\bar{G}_0 = \text{blckd}(G_\varepsilon, G_0)$ $\bar{S}_0 = \begin{pmatrix} S_\varepsilon \\ S_0 \end{pmatrix}$
↳ 1-Norm similar! $\text{s.t. } \bar{G}_0 z_0 \le \bar{w}_0 + \bar{S}_0 z_0$ $\bar{w}_0 = \begin{pmatrix} w_\varepsilon \\ w_0 \end{pmatrix}$ $z_0 = (\varepsilon_0^x \dots \varepsilon_N^x \; \varepsilon_0^u \dots \varepsilon_{N-1}^u, U^T)^T$

Feasibility and Stability

Infinite Horizon: feasible ⇒ feasible CL + stable
Finite Horizon: finite cost ⇒ OL ≠ CL?

↳ Mimic $N = \infty$ with $p(\cdot)$, $\mathcal{X}_f$: $\begin{cases} \mathcal{X}_N \subseteq \mathcal{O}_\infty \subseteq \mathcal{X}_0 \\ \text{stable} \end{cases}$

1) $\mathcal{X}_f = \{0\}$ ⇒ recursive feas + stable ✓ But smaller $\mathcal{X}_0$!

2) Finite $\mathcal{X}_f$ invariant $\exists u = v(x_k)$, constraints ✓, $q(\cdot,\cdot)$, $p(\cdot)$ PDF
$p(x_{k+1}) - p(x_k) \le -q(x_k, v(x_k)) \; \forall x_k \in \mathcal{X}_f$ ⇒ CL under $u_0^*(x)$ stable, $\mathcal{X}_f$ inv

Invariant set $\mathcal{O}$ pos invariant if $x(0) \in \mathcal{O} \Rightarrow x(t) \in \mathcal{O} \; \forall t \ge 0$
— $\mathcal{O}_\infty$ contains all $\mathcal{O}$

With $N = \infty$ LQR $p(x_N) = x_N^T P_\infty x_N$ $\mathcal{X}_f = \left\{x \,\middle|\, \begin{bmatrix} A_x \\ A_u(A + B F_\infty) \\ A_u F_\infty \end{bmatrix} x \le \begin{bmatrix} b_x \\ b_x \\ b_u \end{bmatrix}\right\}$

Remark $\mathcal{X}_f \to x_0 \notin \mathcal{O}_\infty$ often $N\uparrow$ → check stability by sampling
↳ $\lim_{N \to \infty} \mathcal{X}_0 = \mathcal{O}_\infty$

Nonlinear MPC

not relied on L!

Same with $x_{k+1} = g(x_k, u_k)$ → Can extend, but difficult → $p(\cdot)$, $\mathcal{X}_f$

Reference Tracking

S.S. $D = 0$ $\lim_{k \to \infty} y_k = r$ ⇒ $\begin{cases} x_s = A x_s + B u_s \\ C x_s = r \end{cases}$

feasible / unfeasible $\min \; u_s^T R u_s$ cheapest u / $(Cx_s - r)^T Q (Cx_s - r)$ closest x } Min combo in MPC
⚠ Model error, Disturbance → offset

$$\text{s.t. } \begin{pmatrix} I - A & -B \\ C & 0 \end{pmatrix} \begin{pmatrix} x_s \\ u_s \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix} \Rightarrow \underline{(x_s, u_s)}$$

$x_s \in \mathcal{X}$ $u_s \in \mathcal{U}$

Offset Free → Augment Model $d_{k+1} = d_k$ and add $B_d d_k$ (x_{k+1}), $C_d d_k$ (y_k)

observer $\begin{pmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{pmatrix} = \begin{pmatrix} A & B_d \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{x}_k \\ \hat{d}_k \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u_k - \begin{pmatrix} L_x \\ L_d \end{pmatrix} (y_{m,k} \underbrace{- C\hat{x}_k - C_d \hat{d}_k}_{-\hat{y}_k})$ $n_d = p$ ↳ conv!

MPC $\min_{U_0} \|x_N - \bar{x}_t\|_P^2 + \sum_{k=0}^{N-1} \|x_k - \bar{x}_t\|_Q^2 + \|u_k - \bar{u}_t\|_R^2$
s.t. $x_{k+1} = A x_k + B u_k + B_d d_k$
$d_{k+1} = d_k$ $x_0 = \hat{x}(t)$ $d_0 = \hat{d}(t)$
$x_k \in \mathcal{X}$ $x_N \in \mathcal{X}_f$ $u_k \in \mathcal{U}$

each t: $\begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ \bar{u}_t \end{bmatrix} = \begin{bmatrix} -B_d \hat{d} \\ r - C_d \hat{d} \end{bmatrix}$
↳ get target $\bar{x}_t$ $\bar{u}_t$

(C, A) obs & $|A - I - B_d C| \ne 0$ & $n_d = p$ & $C_d = I$ (if no ∫ $B_d = 0$) ⇒ ✓

Soft Constraints

! $z^*_{hard} = z^*_{soft}$, $\varepsilon^* = 0$

Softened Problem $\min_{z,\varepsilon} f(z) + \ell(\varepsilon)$ s.t. $g(z) \le \varepsilon$, $\varepsilon \ge 0$ $\ell(\varepsilon) = u\varepsilon + \underline{v\varepsilon^2}$ — smooth
$u > u^* = -\lim_{z \to z^*} f'(z) \ge 0$

Generalizing Problem

MPC → $\min_{U_0} \|x_{N_y}\|_P^2 + \sum_{k=0}^{N_y - 1} \|x_k\|_Q^2 + \|u_k\|_R^2$
s.t. $y_{min}(k) \le y_k \le y_{max}(k)$ $k = 1, \dots, N_y - 1$ → reduce complexity
TV $u_{min}(k) \le u_k \le u_{max}(k)$ $k = 0, \dots, N_u - 1$
Constr. $x_0 = x(t)$ $x_{k+1} = A x_k + B u_k$ $y_k = C x_k$
$u_k = K x_k$ $N_u \le k \le N_y$ ↳ Stab + Feas Guarantees lost!

Explicit MPC

Precompute PWA $u_0^*(x)$ → Tonlinecomp ↓   Only small systems! $n \le 6$ → linear, LP & QP

mpLQP | $J^*(x) = \min_z c^T z \; z^T H z$ $H \succ 0$
s.t. $Gz \le W + Sx$, $x \in \mathcal{K} \subset \mathbb{R}^n$ polytope

Feasible set $\mathcal{K}^* = \{x \in \mathcal{K} \mid \exists z: Gz \le W + Sx\}$ polytope

Active set $A(x) := \{i \in I \mid G_i z^*(x) - S_i x = W\}$ $I := \{1, \dots, m\}$ set of constraint indices
Inactive set $NA(x) := \{i \in I \mid G_i z^*(x) - S_i x < W\} = I \setminus A(x)$

Critical Region $CR_A := \{x \in \mathcal{K}^* \mid A(x) = A\}$ set of x: const A active

$z^*(x): \mathcal{K}^* \to \mathbb{R}^s$ c, polyh PWA over $\mathcal{K}^* = \bigcup CR_i$, affine in each CR_i
$J^*(x): \mathcal{K}^* \to \mathbb{R}$ c, convex, polyh PWA over $\mathcal{K}^*$, affine / quadratic in each CR_i

→ 2-Norm with substitution

$z := U_0 + H^{-1} F^T x(0)$, $S_0 := E_0 + G_0 H^{-1} F^T$ → | $\hat{J}^*(x(0)) = \min_z z^T H z = J^*(x(0)) - x(0)^T (Y - F H^{-1} F^T) x(0)$
s.t. $G_0 z \le w_0 + S_0 x(0)$ Solve mpQP $\forall x_0 \in \mathcal{X}_0$

$U_0^*(x(0)) = z^*(x(0)) - H^{-1} F^T x(0)$ → $u^*(0) = f_0(x(0)) \; \forall x(0) \in \mathcal{X}_0$

$f_0(x) = F_0^i x + g_0^i$ if $x \in CR_0^i = \{x \mid H_0^i x \le K_0^i\}$ partition of $\mathcal{X}_0$

→ 1- / ∞-Norm → Same as 2-Norm!

$u^*(0) = [0 \dots 0 \; I_m \; 0 \dots 0] \, z_0^*(x(0)) = f_0(x(0))$

Online Evaluation: Point Location → Evaluation
- Sequential search: $\forall i$: $A_i x + b_i \le 0 \Rightarrow x \in CR^i$ → lin in # CR's
- Logarithmic search: Seperate in two → Repeat in region with point → pot. logarithmic

Modeling of Hybrid Systems

PWA systems

- Affine dynamics $\begin{cases} x(t+1) = A_i x(t) + B_i u(t) + f_i \\ y(t) = C_i x(t) + D_i u(t) + g_i \end{cases}$ if $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{X}_i(t)$ polyhedral partition

$\mathcal{X}_i := \{(x, u) \mid H_i x + J_i u \le K_i\}$

Well-Posedness PWA system P with $\mathcal{X} = \bigcup_{i=1}^s \mathcal{X}_i \subseteq \mathbb{R}^{n+m}$
well posed ⇔ $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset \; \forall i \ne j$

Binary states $x = \begin{pmatrix} x_c \\ x_\ell \end{pmatrix} \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ false / true
→ Number (math), Boolean $\delta \in \{0, 1\}$

Mixed Logical Dynamical System

Boolean p_i to binary δ_i: $\begin{cases} p_i \Leftrightarrow \{\delta_i = 1\} \\ \neg p_i \Leftrightarrow \{\delta_i = 0\} \end{cases}$ | $F(p)$ true ⇔ $A\delta \le B$

1) Boolean Formula as Linear Integer Inequality

CNF into algebraic inequalities

Relation	CNF	AI
AND	$\delta_1 \wedge \delta_2$	$\delta_1 + \delta_2 \ge 2$
OR	$\delta_1 \vee \delta_2$	$\delta_1 + \delta_2 \ge 1$
NOT	$\neg \delta_1$	$1 - \delta_1 \ge 1$
XOR	$\delta_1 \oplus \delta_2$	$\delta_1 + \delta_2 = 1$
IMPLY	$\delta_1 \to \delta_2$	$\delta_1 - \delta_2 \le 0$
IFF	$\delta_1 \leftrightarrow \delta_2$	$\delta_1 - \delta_2 = 0$
ASSIGN	$\delta_3 \Leftrightarrow \delta_1 \wedge \delta_2$	$\delta_1 + (1 - \delta_3) \ge 1$ $\delta_2 + (1 - \delta_3) \ge 1$ $2 - \delta_1 - \delta_2 + \delta_3 \ge 1$

2) Linear Inequality as Logic

Event Generator $\delta_e(t) = f_{EG}(x_c(t), u_c(t))$

Boolean $p \Leftrightarrow a^T x \le b$
↳ $m\delta \le a^T x - b \le M(1 - \delta)$
$x \in \mathcal{X} := \{x \mid a^T x - b \in [m, M]\}$

Switched Affine Dynamics

$z_j(t) = \begin{cases} A_j x_c(t) + B_j u_c(t) + f_i & \text{if } i(t) = j \\ 0 & \text{else} \end{cases} \Rightarrow x_c(t+1) = \sum_{j=1}^{s} z_j(t)$

"If-then-else" relations

If p then $z_t = a_1^T x_t + b_1$ else $z_t = a_2^T x_t + b_2$ $\Leftrightarrow$ $\begin{cases} (m_2 - M_1)\delta + z_t \le a_2^T x_t + b_2 \\ (m_1 - M_2)\delta - z_t \le -(a_2^T x_t + b_2) \\ (m_1 - M_2)(1-\delta) + z_t \le a_1^T x_t + b_1 \\ (m_2 - M_1)(1-\delta) - z_t \le -(a_1^T x_t + b_1) \end{cases}$ where: $x \in \mathcal{X}$, $M_i \ge \sup_{x \in \mathcal{X}} a_i^T x + b_i$, $m_i \le \inf_{x \in \mathcal{X}} a_i^T x + b_i$

MLD Hybrid Model from DHA

$x_{t+1} = A x_t + B_1 u_t + B_2 \delta_t + B_3 z_t$
$y_t = C x_t + D_1 u_t + D_2 \delta_t + D_3 z_t$
$E_2 \delta_t + E_3 z_t \le E_4 x_t + E_1 u_t + E_5$

Phys constr. $\mathcal{C} := \{ [\begin{smallmatrix} x \\ u \end{smallmatrix}] \in \mathbb{R}^{n+m} \mid F_x x + G_u u \le H \}$
Complete Well-Posedness: $\rightarrow$ allows trafo
Given $(x_t, u_t) \rightarrow \exists!\ x_{t+1}, y_t, \delta_t, z_t$

CFTOC Problem for Hybrid Systems

$J^*(x(t)) = \min_{U_0} p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k, \delta_k, z_k)$
s.t. $x_{k+1} = A x_k + B_1 u_k + B_2 \delta_k + B_3 z_k$
$E_2 \delta_k + E_3 z_k \le E_4 x_k + E_1 u_k + E_5$
$x_0 = x(t) \quad x_N \in \mathcal{X}_f$

MILQP

$\inf_{z_c, z_b} \ c_c^T z_c + c_b^T z_b + d \quad \frac{1}{2} z^T H z + q^T z + r$
s.t. $G_c z_c + G_b z_b \le W$
$z_c \in \mathbb{R}^{s_c} \quad z_b \in \{0,1\}^{s_b}$

fix $\bar{z}_b \rightarrow$ LQP, $\# = 2^{s_b}$
$\rightarrow$ BF: solve $\forall z_b \in \{0,1\}^{s_b}$
$\hookrightarrow$ get minimizer

Explicit for 2-Norm^2 1-, ∞-Norm

$u_t^*(x) = F_t^i x_t + G_t^i$ if $x_t \in R_t^i$ $\rightarrow$ polyh partitions
$\rightarrow$ Quad and Lin boundaries

Summary Mixed Integer Programming HARD $\rightarrow$ Solvers (tricks, ...)
$\rightarrow$ MPC theory applies, but difficult! invariant sets, opt, ...

Numerical Optimization Methods

Optimization Problem $x^* \in \arg\min f(x)$ s.t. $x \in \mathbb{Q}$

Iterative Optimization
x_0 (guess) $\rightarrow x_{i+1} = \Phi(x_i, f, \mathbb{Q})$ $i \in [0, M-1]$ (iterates) s.t. $|f(x_M) - f(x^*)| \le \varepsilon$ $\text{dist}(x_M, \mathbb{Q}) \le \delta$ (tolerances)

Unconstrained Optimization

Gradient Method $x_{i+1} = x_i - h_i \nabla f_i(x_i)$ $h_i = \frac{1}{L} \Rightarrow$ conv $\quad L < \frac{L_{real}}{2}$ ✗ cheap!

L-smoothness $\|\nabla f(x) - \nabla f(y)\| \le L \|x - y\| \ \forall x, y \in \mathbb{R}^n$
$\Leftrightarrow \nabla f$ Lipschitz c $\Leftrightarrow f(x) \le f(y) + \nabla f(y)^T (x-y) + \frac{L}{2} \|x-y\|^2 \ \forall x, y \in \mathbb{R}^n$

Fast Gradient Method $x_0, \ y_0 = x_0 \quad \alpha_0 = \frac{1}{2}(\sqrt{5} - 1)$ ΔRobustness $\rightarrow$ Numerics!
$x_{i+1} = y_i - \frac{1}{L} \nabla f(y_i) \quad \alpha_{i+1} = \frac{\alpha_i}{2} \left(\sqrt{\alpha_i^2 + 4} - \alpha_i \right) \quad \beta_i = \frac{\alpha_i (1 - \alpha_i)}{\alpha_i^2 + \alpha_{i+1}}$
$y_{i+1} = x_{i+1} + \beta_i (x_{i+1} - x_i) \quad i \in [0, n-1] \quad |f(x_M) - f(x^*)| \le \varepsilon_1 \quad \|x_n - x_{n-1}\| \le \varepsilon_2$

Strong Convexity $f(x) \lessgtr f(y) + \nabla f(y)^T (x-y) + \frac{L \mu}{2} \|x-y\|^2 \ \forall x, y \in \mathbb{R}^n$

$\rightarrow$ Condition Number $\kappa := \frac{L}{\mu} \ge 1$ $\quad f(x) = \frac{1}{2} x^T H x$ ($H \succeq 0$) $\rightarrow \begin{cases} \mu = \min(\lambda(H)) \\ L = \max(\lambda(H)) \end{cases}$
FGM: $m\,\mathcal{O}\left(\sqrt{\kappa} \ln \|x_0 - x^*\|^2 \frac{L}{\varepsilon}\right)$ $\rightarrow$ conv. speed acc

Preconditioning $\hat{=}$ Variable transformation to $\kappa \downarrow$
$x = P y$ s.t. $\kappa_h < \kappa_f$ where $\min_y h(y) := \min_y f(Py)$ vs $\min_y f(y)$

Newton's Method $x_{i+1} = x_i + h_i \Delta x_{nt} = -(\nabla^2 f(x_i))^{-1} \nabla f(x_i)$

$\rightarrow$ Find $h_i > 0 : f(x_i + h_i \Delta x_{nt}) < f(x_i)$ $\hookrightarrow$ No Preconditioning! Expensive!

Line search
- Exact $h_i^* = \arg\min_{h \ge 0} f(x_i + h \Delta x_{nt})$
- Inexact $h_i = 1$ while $(f(x_i + h_i \Delta x_{nt}) > f(x_i) + h_i \nabla f(x_i)^T \Delta x_{nt} \cdot \alpha)$: $h_i = h_i \beta$

Two phases
- Damped Newton $\|\nabla f(x)\|_2 \ge \eta \in (0, \frac{\mu^2}{M})$ — f strong convex: $\nabla^2 f(x^*) \succeq \mu I$, Lipschitz for ∇f
- Quadratically Convergent $\|\nabla f(x)\|_2 < \eta \rightarrow h_i = 1, \|\nabla f(x)\|_2 \rightarrow 0$

Constrained Optimization

Fast Gradient Method $x_{i+1} = \Pi_{\mathbb{Q}} (x_i y_i - h_i \nabla f(x_i))$ ($\text{prox}_{\mathbb{Q}}(\cdot)$)
Projection $\rightarrow$ If easy directly, else solve dual (convex)
$\hookrightarrow$ MPC Projection on intersection $\mathcal{X} \times \mathcal{U}$ not easy! $\rightarrow$ dual

Interior Point Methods $\min f(x)$ s.t. $g_i(x) \le 0 \quad i \in [1, M]$
Assume $f(x^*) < \infty \quad \exists \tilde{x} \in \text{dom} f : g(\tilde{x}) < 0$

Barrier Method $\min f(x) + \kappa \varphi(x)$

Indicator function $\varphi(x) := \sum_{i=1}^{m} I_-(g_i(x)) \approx -\sum_{i=1}^{m} \ln(-g_i(x))$
Analytic center $\arg\min \varphi(x)$ $\quad I_-(y) = \begin{cases} 0 & y \le 0 \\ \infty & y > 0 \end{cases}$
Central path $\{x^*(\kappa) \mid \kappa \ge 0\}$ $\quad \lim_{\kappa \to 0} x^*(\kappa) = x^*$

Path Following

x_0 strictly feas, κ_0, $\mu > 1$, $\varepsilon > 0$
1) Centering step $x^*(\kappa_i) = \arg\min f(x) + \kappa \varphi(x) \leftarrow x_0 = x_{i-1}$
2) Update step $x_i := x^*(\kappa_i)$
3) Stop if $\kappa_i < \varepsilon$
4) Decrease $\kappa_{i+1} = \frac{\kappa_i}{\mu}$

Newton $\Delta x_{nt} = -(\nabla^2 f(x) + \kappa \nabla^2 \varphi(x))^{-1} (\nabla f(x) + \kappa \nabla \varphi(x))$
1) Retain Feasibility $h_{max} = \arg\max_{h > 0} \{ h \mid g_i(x + h \Delta x_{nt}) < 0 \ \forall i \in [1, m] \}$
2) Find $h^* = \arg\min_{h \in [0, h_{max}]} \{ f(x + h \Delta x_{nt}) + \kappa \varphi(x + h \Delta x_{nt}) \}$ ← exact, ← backtrack

Centering with equality constraints $Cx = d$
$\rightarrow \begin{bmatrix} \nabla^2 f(x) + \kappa \nabla^2 \varphi(x) & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{nt} \\ \nu \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + \kappa \nabla \varphi(x) \\ 0 \end{bmatrix}$

$\hookrightarrow$ Relaxed KKT
$C x^* = d$
$g_i(x^*) + s_i^* = 0 \quad \kappa \frac{\partial \varphi(s)}{\partial s} = -\frac{1}{s} \cdot \kappa$
$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + C^T \nu^* = 0$
$\lambda_i^* g_i(x^*) = -\kappa \quad$ from $\lambda_i^* = \kappa \frac{1}{-g_i(x^*)}$
$\lambda_i^*, s_i^* \ge 0$ $\hookrightarrow$ replaces complementary slackness

Primal-Dual Search Direction Computation

$$\begin{bmatrix} H(x,\lambda) & C^T & G^T(x) & 0 \\ C & 0 & 0 & 0 \\ G(x) & 0 & 0 & I \\ 0 & 0 & S & \Lambda \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta \lambda \\ \Delta s \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + C^T y + G(x)^T \lambda \\ Cx - d \\ g(x) + s \\ S\lambda - \nu \end{bmatrix}$$

where: $S := \text{diag}(s_1, \ldots, s_m) \quad \Lambda := \text{diag}(\lambda_1, \ldots, \lambda_m)$
$H(x,\lambda) = \nabla^2 f(x) + \sum_{i=1}^{m} \lambda_i \nabla^2 g_i(x)$
$\Delta[\ldots](\nu) \rightarrow$ standard Newton $\nu = 0$ $\}$ $\nu = \sigma \kappa \mathbb{1}$ lin. comb. for fast conv. $\sigma \in (0,1)$
$\rightarrow \kappa \mathbb{1}$ centering direction
$\hookrightarrow$ next iterate on central path