8.2.2 Error Analysis for Simpson's 1/3 Rule.

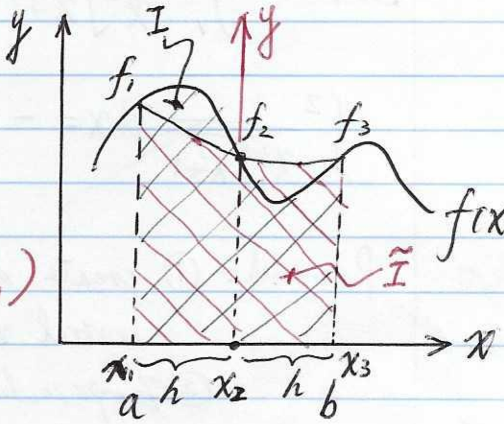
Start from simple case: consider 2 intervals in $[a, b]$

$$I = \int_a^b f(x)\,dx$$

$$\tilde{I} = \frac{h}{3}(f_1 + 4f_2 + f_3)$$

Simpson 1/3 Rule: $\tilde{I} = \frac{h}{3}\left(f_1 + 4\sum_{i=2,4,6\ldots}^{n} f_i + 2\sum_{i=3,5,7\ldots}^{n-1} + f_{n+1}\right)$

HERE $n=2$ $\quad h = (a-b)/2$



$$\text{Error: } E = \int_a^b f(x)\,dx - \frac{h}{3}(f_1 + 4f_2 + f_3)$$

As I and $\tilde{I}$ are independent with relative position of y Axis.
so for easy calculation, move y axis going through x_2.

Perform variable change: $p = x - x_2$

$$\therefore x = p + x_2 \qquad dx = dp$$

$$\begin{cases} x = x_1 = a & p = -h \\ x = x_2 = b & p = h \end{cases}$$

$$f(x) = f(x_2 + p) \xlongequal{T.S.E.} f_2 + \frac{f_2'}{1!}p + \frac{f_2''}{2!}p^2 + \frac{f_2'''}{3!}p^3 + \frac{f_2^{(4)}}{4!}p^4 + \cdots$$

$$\int_a^b f(x)\,dx = \int_a^b f(x_2+p)\,d(x_2+p) = \int_{-h}^{h}\left[f_2 + f_2'p + \frac{f_2''}{2!}p^2 + \frac{f_2'''}{3!}p^3 + \frac{f_2^{(4)}}{4!}p^4 + \cdots\right]dp$$

$$= f_2 p\Big|_{-h}^{h} + f_2'\frac{p^2}{2}\Big|_{-h}^{h} + \frac{f_2''}{6}p^3\Big|_{-h}^{h} + \frac{f_2'''}{24}p^4\Big|_{-h}^{h} + \frac{f_2^{(4)}}{120}p^5\Big|_{-h}^{h} + \cdots$$

$$= 2hf_2 + 0 + \frac{h^3}{3}f_2'' + 0 + \frac{h^5}{60}f_2^{(4)} + \cdots$$

$$f_1 = f(x_2 - h) \xlongequal{T.S.E.} f_2 - \frac{f_2'}{1!}h + \frac{f_2''}{2!}h^2 - \frac{f_2'''}{3!}h^3 + \frac{f_2^{(4)}}{4!}h^4 - \cdots$$

$$f_3 = f(x_2+h) = f_2 + hf_2' + \frac{h^2}{2!}f_2'' + \frac{f_2'''}{3!}h^3 + \frac{f_2^{(4)}}{4!}h^4 + \cdots$$

$$\tilde{I} = \frac{h}{3}(f_1+4f_2+f_3) = \frac{h}{3}\Big[\overset{f_2+f_2+4f_2}{6f_2} + h^2 f_2'' + \frac{h^4}{12}f_2^{(4)} + \cdots\Big]$$

$$= 2hf_2 + \frac{h^3}{3}f_2'' + \frac{h^5}{36}f_2^{(4)} + \cdots$$

$$\therefore E = I - \tilde{I} = \left(2hf_2 + \frac{h^3}{3}f_2'' + \frac{h^5}{60}f_2^{(4)} + \cdots\right)$$

neglect higher order terms

$$-\left(2hf_2 + \frac{h^3}{3}f_2'' + \frac{h^5}{36}f_2^{(4)} + \cdots\right) \approx -\frac{h^5}{90}f_2^{(4)}$$

If $[a,b]$ is divided into n intervals (n is even number)

quantity of $f_i^{(4)}$ in Σ

$$E = -\frac{h^5}{90}\sum_{i=2,4,6,\cdots}^{n} f_i^{(4)} \qquad \text{Assume } \sum_{i=2,4,6\cdots}^{n} f_i^{(4)} = \frac{n}{2}\overline{f_i^{(4)}}$$

$$E = -\frac{h^5}{90}\cdot\frac{n}{2}\overline{f_i^{(4)}} = -\frac{h^4}{90}\cdot\left(\frac{b-a}{n}\right)\frac{n}{2}\overline{f_i^{(4)}}$$

$$= -\frac{(b-a)}{180}\overline{f_i^{(4)}}\,h^4 \sim O(h^4)$$

Compare: Rectangular Rule LS & RS $\sim O(h)$

Rectangular Rule Avg & Trapezoidal Rule $\sim O(h^2)$

Simpson's 1/3 Rule : $O \sim (h^4)$

8.2.3 Simpson's Three Eighth's Rule

Approximate $f(x)$ by 3rd degree polynomial

$$p(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3$$

need at least 4 points in 3 intervals. Final Formulation

$$I \approx \frac{3h}{8}\Big[f_1 + 3\sum_{i=2,5,8,\cdots}^{n-1}(f_i + f_{i+1}) + 2\sum_{i=4,7,10,\cdots}^{n-2} f_i + f_{n+1}\Big] \sim O(h^4)$$

same order with 1/3 rule