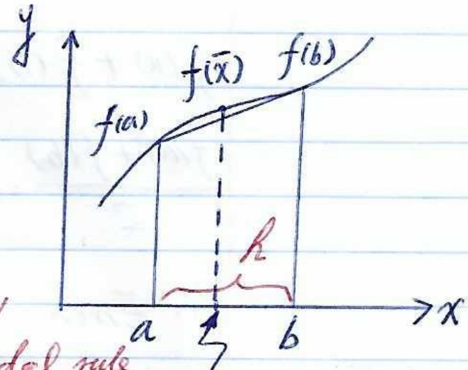
8.1.3 Error Analysis for Trapezoidal Rule



Consider one stripe for simplicity:

$$\text{Error } E = \underbrace{\int_a^b f(x)\,dx}_{\text{True Integration}} - \underbrace{\frac{f(a)+f(b)}{2}(b-a)}_{\text{Approxi by Trapezoidal rule}}$$

$$E = \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\,h$$

next: use T.S.E. to express $\int f(x)dx$, f(a), f(b) as function of h

Introduce variable substitution: $p = x - \bar{x}$ $\quad \begin{cases} @\ x = a,\ p = \frac{a-b}{2} = -\frac{h}{2} \\ @\ x = b,\ p = \frac{b-a}{2} = \frac{h}{2} \end{cases}$

recall calculus $\quad x = \bar{x} + p$

True Integration $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(\bar{x}+p)\,d(\bar{x}+p) = \int_{-\frac{h}{2}}^{\frac{h}{2}} f(\bar{x}+p)\,dp$

treat $\bar{x} = \frac{a+b}{2}$ as constant!

Taylor's Series Expansion for $f(\bar{x}+p)$ @ $\bar{x}$:

$$f(\bar{x}+p) = f(\bar{x}) + \frac{f'(\bar{x})}{1!}p + \frac{f''(\bar{x})}{2!}p^2 + \frac{f'''(\bar{x})}{3!}p^3 + \cdots \quad (1)$$

$$\int_a^b f(x)\,dx = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[f(\bar{x}) + f'(\bar{x})p + \frac{f''(\bar{x})}{2!}p^2 + \cdots \right] dp$$

$$= f(\bar{x})\,p\Big|_{-\frac{h}{2}}^{\frac{h}{2}} + f'(\bar{x})\frac{p^2}{2}\Big|_{-\frac{h}{2}}^{\frac{h}{2}} + f''(\bar{x})\frac{p^3}{3 \times 2!}\Big|_{-\frac{h}{2}}^{\frac{h}{2}} + \cdots$$

$$= h f(\bar{x}) + 0 + \frac{h^3}{24} f''(\bar{x}) + \cdots$$

Now work on $f(a)$ and $f(b)$:

substitute $p = -\frac{h}{2}$ to eqn. (1) $\Rightarrow$

$$f(a) = f\left(\bar{x} - \frac{h}{2}\right) = f(\bar{x}) - f'(\bar{x})\frac{h}{2} + \frac{f''(\bar{x})}{2!}\cdot\frac{h^2}{4} - \cdots$$

$p = \frac{h}{2} \Rightarrow f(b) = f\left(\bar{x} + \frac{h}{2}\right) = f(\bar{x}) + f'(\bar{x})\frac{h}{2} + \frac{f''(\bar{x})}{2!}\frac{h^2}{4} + \cdots$

$$f(a)+f(b) = 2f(\bar{x}) + f''(\bar{x})\cdot\frac{h^2}{4} + \cdots$$

$$\frac{f(a)+f(b)}{2}\cdot h = f(\bar{x})h + f''(\bar{x})\frac{h^3}{8} + \cdots$$

$$\therefore \text{Error } E = \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\cdot h$$

$$= hf(\bar{x}) + \frac{h^3}{24}f''(\bar{x}) + \cdots - \left[f(\bar{x})h + \frac{h^3}{8}f''(\bar{x}) + \cdots\right]$$

$$\approx -\frac{h^3}{12}f''(\bar{x})$$ neglect higher order terms.

now consider domain $[a,b]$ is divided to n intervals. $h = \frac{b-a}{n}$.

$$E = \sum_{i=1}^{n} E_i = \sum_{i=1}^{n} -\frac{h^3}{12}f''(\bar{x}_i) = -\frac{h^3}{12}\sum_{i=1}^{n} f''(\bar{x}_i)$$

where $\bar{x}_i = \frac{x_i + x_{i+1}}{2}$

Average of all $f''(\bar{x}_i)$

Assume $\sum_{i=1}^{n} f''(\bar{x}_i) = n\cdot\overline{f''}$

Then Error $E = \sum_{i=1}^{n} E_i = -\frac{h^2}{12}\left(\frac{b-a}{n}\right)\cdot n\overline{f''}$

$$= -\frac{h^2}{12}(b-a)\overline{f''} \sim O(h^2)$$

for given function $f(x)$ and domain $(b-a)$, only way to improve accuracy is to reduce h. or increase n!

8.2 Simpson's Rule.