3. The Logic of Quantified Statements (aka Predicate Logic) Summary

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Summary

3.1 Predicates and Quantified Statements I

- Predicate; domain; truth set
- Universal quantifier $\forall$, existential quantifier $\exists$
- Universal conditional statements; Implicit quantification

3.2 Predicates and Quantified Statements II

- Negation of quantified statements; negation of universal conditional statements
- Vacuous truth of universal statements
- Variants of universal conditional statements (contrapositive, converse, inverse)
- Necessary and sufficient conditions, only if

3.3 Statements with Multiple Quantifiers

- Negations of multiply-quantified statements; order of quantifiers
- Prolog

3.4 Arguments with Quantified Statements

- Universal instantiation; universal modus ponens; universal modus tollens

Summary

Definition 3.1.1 (Predicate)

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Definition 3.1.2 (Truth set)

If $P(x)$ is a predicate and x has domain D, the **truth set** is the set of all elements of D that make $P(x)$ true when they are substituted for x.
The truth set of $P(x)$ is denoted $\{x \in D \mid P(x)\}$.

Definition 3.1.3 (Universal Statement)

Let $Q(x)$ be a predicate and D the domain of x.
A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$".

- It is defined to be true iff $Q(x)$ is true for every x in D.
- It is defined to be false iff $Q(x)$ is false for at least one x in D.

A value for x for which $Q(x)$ is false is called a **counterexample.**

Definition 3.1.4 (Existential Statement)

Let $Q(x)$ be a predicate and D the domain of x.
An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$".

- It is defined to be true iff $Q(x)$ is true for at least one x in D.
- It is defined to be false iff $Q(x)$ is false for all x in D.

$\exists!$ is the **uniqueness quantifier symbol**. It means "there exists a unique" or "there is one and only one".

Notation

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $\boldsymbol{P(x) \Rightarrow Q(x)}$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x, P(x) \rightarrow Q(x)$.
- The notation $\boldsymbol{P(x) \Leftrightarrow Q(x)}$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x, P(x) \leftrightarrow Q(x)$.

Summary

Theorem 3.2.1 Negation of a Universal Statement

The **negation** of a statement of the form

$$\forall x \in D,\ P(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D \text{ such that } {\sim}P(x)$$

Symbolically,

$$\sim(\forall x \in D,\ P(x)) \equiv \exists x \in D \text{ such that } {\sim}P(x)$$

Theorem 3.2.2 Negation of an Existential Statement

The **negation** of a statement of the form

$$\exists x \in D \text{ such that } P(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D,\ {\sim}P(x)$$

Symbolically,

$$\sim(\exists x \in D \text{ such that } P(x)) \equiv \forall x \in D,\ {\sim}P(x)$$

Summary

Definition 3.2.1 (Contrapositive, converse, inverse)

Consider a statement of the form: $\forall x \in D, P(x) \to Q(x)$.

1. Its **contrapositive** is: $\forall x \in D, \sim Q(x) \to \sim P(x)$.
2. Its **converse** is: $\forall x \in D, Q(x) \to P(x)$.
3. Its **inverse** is: $\forall x \in D, \sim P(x) \to \sim Q(x)$.

Definition 3.2.2 (Necessary and Sufficient conditions, Only if)

- "$\forall x, r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x, r(x) \to s(x)$".
- "$\forall x, r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x, \sim r(x) \to \sim s(x)$" or, equivalently, "$\forall x, s(x) \to r(x)$".
- "$\forall x, r(x)$ **only if** $s(x)$" means "$\forall x, \sim s(x) \to \sim r(x)$" or, equivalently, "$\forall x, r(x) \to s(x)$" .

Universal Modus Ponens

Formal version	*Informal version*
$\forall x, P(x) \rightarrow Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
• $Q(a)$.	• a makes $Q(x)$ true.

Universal Modus Tollens

Formal version	*Informal version*
$\forall x, P(x) \rightarrow Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$ for a particular a.	a does not make $Q(x)$ true.
• $\sim P(a)$.	• a does not makes $P(x)$ true.

Definition 3.4.1 (Valid Argument Form)

To say that **an argument form is valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

An **argument is called valid** if, and only if, its form is valid.

Summary

Converse Error (Quantified Form)

Formal version	*Informal version*
$\forall x, P(x) \rightarrow Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
• $P(a)$.	• a makes $P(x)$ true.

Inverse Error (Quantified Form)

Formal version	*Informal version*
$\forall x, P(x) \rightarrow Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$ for a particular a.	a does not make $P(x)$ true.
• $\sim Q(a)$.	• a does not make $Q(x)$ true.

Universal Transitivity

Formal version	*Informal version*
$\forall x, P(x) \rightarrow Q(x)$.	Any x that makes $P(x)$ true makes $Q(x)$ true.
$\forall x, Q(x) \rightarrow R(x)$.	Any x that makes $Q(x)$ true makes $R(x)$ true.
• $\forall x, P(x) \rightarrow R(x)$.	• Any x that makes $P(x)$ true makes $R(x)$ true.

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