National University of Singapore

Semester 1, 2020/2021 MA1101R Practice Assignment 1 Solution

1. A certain linear system has the augmented matrix

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 0 & 2 & 2 & c \end{array}\right)$$

for some real numbers a, b, c.

(i) [3 marks] Reduce the augmented matrix to a row echelon form using <u>two</u> elementary row operations (show the two e.r.o. in your working.)

(ii) [3 marks] Write down the condition in terms of a, b, c (if possible) for the system to have (a) no solution; (b) only one solution; (c) infinitely many solutions.

(iii) [2 marks] If the above linear system is a homogeneous system in variables x, y, z (in that order), write down a general solution of this system.

Answer

(i)

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 0 & 2 & 2 & c \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & b \\ 0 & 1 & 1 & a \\ 0 & 2 & 2 & c \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & b \\ 0 & 1 & 1 & a \\ 0 & 0 & 0 & c-2a \end{array}\right)$$

(ii) (a) no solution: $c - 2a \neq 0$;
(b) only one solution: not possible;
(c) infinitely many solutions: $c - 2a = 0$.

(iii) For homogeneous system, $a = b = c = 0$.
Set $z = t$. Then $y = -t$ and $x = -t$.
So the general solution is given by:

$$\begin{cases} x &= -t \\ y &= -t \\ z &= t \end{cases}$$

2. [4 marks] Let

$\boldsymbol{A} = (a_{ij})_{2\times 3}$ with $a_{ij} = 2i - j$ and $\boldsymbol{B} = (b_{ij})_{3\times 2}$ with $b_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ 2 & \text{if } j = 2 \end{cases}$.

Write down $\boldsymbol{A}$ and $\boldsymbol{B}$ explicitly.

Answer

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix}.$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

3. Given that the following linear system is consistent:

$$\begin{cases} x & + & y & = & 1 \\ x & - & y & = & 1 \\ x & - & 3y & = & 1 \\ 3x & + & y & = & 3 \end{cases}$$

(i) [1 mark] Write the linear system in matrix equation form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

(ii) [2 marks] Compute $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}^T\boldsymbol{b}$ for $\boldsymbol{A}$ and $\boldsymbol{b}$ in part (i).

(iii) [2 marks] Pre-multiply $\boldsymbol{A}^T$ on both sides of the matrix equation in (i), derive the solution of the linear system without using Gaussian Elimination. Show your working.

Answer

(i)

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

(ii) $\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$ and

$\boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$

(iii) Pre-multiplying $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ by $\boldsymbol{A}^T$ on both sides gives: $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$. By (ii), this gives

$$\begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

Multiply out the matrices on the left hand side, we get

$$12x = 12 \text{ and } 12y = 0.$$

This gives the solution $x = 1$ and $y = 0$.

4. [3 marks] Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & a & b & a \\ 0 & a & b & a \\ 0 & 0 & b & a \end{array}\right)$$

for some real numbers a and b.
Suppose the linear system has only one solution. Find the reduced row echelon form of the above augmented matrix. Show how you derive your answer.

Answer

For the system to have only one solution, both a and b must not be 0.

The augmented matrix reduces to

$$\left(\begin{array}{ccc|c} 1 & a & b & a \\ 0 & a & b & a \\ 0 & 0 & b & a \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & a & b & a \\ 0 & 0 & b & a \end{array}\right) \xrightarrow{R_2-R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & a \end{array}\right)$$

$$\xrightarrow{\frac{1}{a}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & a \end{array}\right) \xrightarrow{\frac{1}{b}R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{a}{b} \end{array}\right)$$

where the last augmented matrix is the reduced row echelon form.