NATIONAL UNIVERSITY OF SINGAPORE

CS1231 - DISCRETE STRUCTURES

(AY2018/19 Semester 1)

Midterm Assessment

Time Allowed: 2 Hours

SOLUTIONS

Part I
(30 marks) Multiple choice questions. Answer on the OCR form.

Use **2B pencil** for this section. **Shade** and **write** your student number completely and correctly (please check!) on the OCR form. You do not need to fill in any other particulars on the OCR form.

For each multiple choice question, choose the best answer and **shade** the corresponding choice on the OCR form. Each multiple choice question is worth 2 marks. No mark is deducted for wrong answers.

Q1. Assuming that i, can, do and it are statement variables, simplify the following statement:

$$((i \wedge \sim can) \rightarrow ((can \wedge do) \rightarrow (it \vee \sim it))) \rightarrow can$$

A. true
B. false
C. can
D. $\sim can$
E. $i \wedge can \wedge do \wedge it$

Solution:

$((i \wedge \sim can) \rightarrow ((can \wedge do) \rightarrow (it \vee \sim it))) \rightarrow can$

$\equiv ((i \wedge \sim can) \rightarrow ((can \wedge do) \rightarrow true)) \rightarrow can$ [by negation law: $p \vee \sim p \equiv true$]

$\equiv ((i \wedge \sim can) \rightarrow true) \rightarrow can$ [because $p \rightarrow true \equiv true$]

$\equiv true \rightarrow can$ [because $p \rightarrow true \equiv true$]

$\equiv can$ [because $true \rightarrow p \equiv p$]

Q2. On the fabled Island of Knights, Knaves and Spies, you meet three natives Aiken, Dueet, and Kenyu. One of them is a knight, one a knave, and one a spy. The knight always tells the truth, the knave always lies, and the spy can either lie or tell the truth. Given the following conversation, who is the knight, who the knave, and who the spy?

You: Who is the spy?
Aiken: Dueet is the spy.
Dueet: No, Kenyu is the spy.
Kenyu: No, actually Dueet is the spy.

A. Aiken is the knight, Dueet the knave, and Kenyu the spy.
B. Aiken is the knight, Dueet the spy, and Kenyu the knave.
C. Aiken is the knave, Dueet the knight, and Kenyu the spy.
D. Aiken is the spy, Dueet the knave, and Kenyu the knight.
E. Aiken is the spy, Dueet the knight, and Kenyu the knave.

Q3. Given the following four statements:

$p \to p \vee q$

$p \wedge q \to p$

$(p \to p) \vee q$

$p \wedge (q \to p)$

Pick the odd one out.

A. $p \to p \vee q$

B. $p \wedge q \to p$

C. $(p \to p) \vee q$

D. $p \wedge (q \to p)$

E. None of the above

> **Solution:** $(p \to p \vee q)$, $(p \wedge q \to p)$ and $((p \to p) \vee q)$ are tautology, but $p \wedge (q \to p) \equiv p$.

Q4. Study the following statements, assuming that the domain is the set of real numbers $\mathbb{R}$.

(I) $\forall x \exists y \, (x + y = 0)$

(II) $\exists x \forall y \, (x + y = 0)$

(III) $\forall x \forall y \exists z \, (z = 3/(x - y))$

(IV) $\forall z \exists y \exists x \, (x^2 + y^2 = z^2)$

Which of the above statements are TRUE?

A. (I) only.

B. (II) and (III) only.

C. (I) and (IV) only.

D. (III) and (IV) only

E. (I), (III) and (IV) only.

Q5. Given the sentence "Everyone loves everyone except himself" (in other words, "Everyone loves everyone else"), and the predicate $love(x, y)$ means "x loves y", which one of the following is the correct statement for the above sentence?

A. $\forall x \forall y \, (x \neq y \to love(x, y))$

B. $\forall x \forall y \, (x = y \to \sim love(x, y))$

C. $\forall x \forall y \, (x \neq y \leftrightarrow love(x, y))$

D. $\exists x \exists y \sim (x = y \leftrightarrow \sim love(x, y))$

E. None of the above.

Q6 to Q7 refer to Figure 1, which shows six readers and four book genres (Children's

Fiction, Fantasy, Mystery and Classic Literary Fiction) with selected book titles of each genre. A line is drawn between a reader and a book title if and only if that reader reads that book. For example, Ms Aiken reads "The Little Prince", but Mr Dueet does not read "Black Beauty".

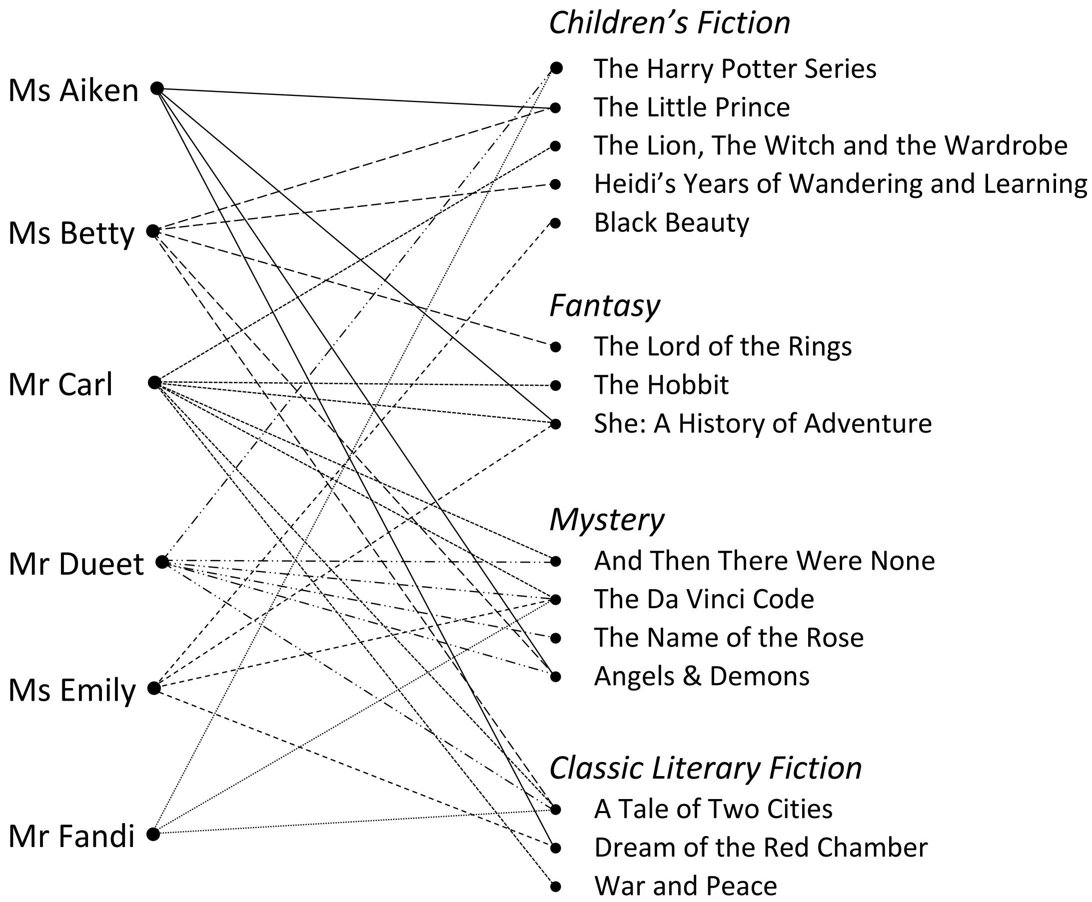

Figure 1: Readers and book titles.

Q6. Which of the following statements are TRUE?

(I) Some title is read by all the three female readers.
(II) Some title that is read by two or more readers is read by readers of different genders.
(III) Every reader reads some title in every genre.
(IV) Some reader reads all titles of some genre.

A. (II) only.
B. (III) only.
C. (I) and (IV) only.
D. (II) and (IV) only.
E. (I), (II) and (IV) only.

Solution:

(I) False: None of the titles is read by all female readers.

(II) True: "She: A History of Adventure" is read by Ms Aiken and Mr Carl.

(III) False: Mr Dueet doesn't read any Fantasy title.

(IV) True: Mr Dueet reads all Mystery titles.

Q7. Which of the following statements are TRUE?

(I) There is a genre in which every title of that genre that is read by a female reader is also read by a male reader.

(II) There is some genre for which some reader does not read any of its titles.

(III) There is a unique genre with the highest readership. Readership of a genre is defined as the ratio of the number of unique readers who read titles of that genre to the number of titles of that genre.

A. (I) only.

B. (II) only.

C. (I) and (II) only.

D. (II) and (III) only.

E. All of (I), (II) and (III).

Solution:

(I) True: Every Mystery title that is read by a female reader is also read by a male reader.

(II) True: None of the Fantasy titles is read by Dueet (and Fandi).

(III) True: Readership of CF=6/5=1.2; F=4/3=1.33; M=6/4=1.5; CLF=6/3=2. Hence Classic Literary Fiction has the highest readership.

Q8. Calculate $\gcd(5!, 10!)$.

A. 5!

B. 10!

C. $\frac{10!}{5!}$

D. $10! \times 5!$

E. None of the above.

Q9. Let a, b be any two integers, not both 0. Which of the following statements are TRUE?

(I) Given any prime p, $\sum_{x=1}^{p} \gcd(x, p) = p$.

(II) $\gcd(a, b) = \gcd(a + b, \ -b)$.

(III) $\gcd(\frac{a}{\gcd(a,b)}, \ \frac{b}{\gcd(a,b)}) = 1$.

(IV) If an integer d is such that $ax + by = d$ for some $x, y \in \mathbb{Z}$, then $d = \gcd(a, b)$.

A. (I) and (II) only.

B. (II) and (III) only.

C. (III) and (IV) only.

D. (II), (III) and (IV) only.

E. All of (I), (II), (III) and (IV).

Solution:

(I) False. All x between $1 \le x < p$ are coprime with p, ie. $\gcd(x,p) = 1$. But $\gcd(p,p) = p$. Thus the sum is $2p - 1$.

(II) True. Write $a = (a+b) + (-b)$. Thus a is a linear combination of $(a+b)$ and $-b$. Thus $\gcd(a+b,\ -b)$ divides a, and of course divides b. Also, $(a+b)$ is clearly a linear combination of a and b, so that $\gcd(a,b)$ divides $(a+b)$ and $-b$. Hence the common divisors of a, b are the same as those of $(a+b), -b$, which in turn means their gcd are equal.

(III) True. This is the proposition in Slide 39 of `Week4_NumberTheory2.pdf`.

(IV) False. This is the converse of Bézout's Identity.

Q10. The $x \in \mathbb{Z}$ such that $x^2 \equiv 1 \pmod 7$ could be:

A. 6

B. 10

C. 19

D. -2

E. There is no solution.

Q11. $(207)_{10} =$

A. $(207)_{11}$

B. $(10000111)_2$

C. $(11001111)_2$

D. $(3112)_5$

E. $(B3)_{16}$

Q12. Dueet hastily wrote the following proof that $\sqrt{2}$ is irrational. Unfortunately, the lines are not in the correct order. Reorder them to make a logically valid proof.

Proof.

1. Suppose $\sqrt{2}$ is rational.
2. Then there exist positive integers m, n such that $\sqrt{2} = \frac{m}{n}$.
3. Then $2n^2 = m^2$, by basic algebra.
4. The left hand side is odd, while the right hand side is even. Contradiction.
5. Then equating powers of 2: $2e_1 + 1 = 2t_1$.
6. Then $n \neq 1$, since otherwise $\sqrt{2} = m$, which contradicts the fact that $\sqrt{2}$ is not an integer.
7. Without loss of generality, we may let $p_1 = q_1 = 2$.
8. Then $m \neq 1$, since otherwise $\sqrt{2} = \frac{1}{n} < 1$, which contradicts the fact that $\sqrt{2} > 1$.
9. Then $2(p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k})^2 = (q_1^{t_1} q_2^{t_2} \ldots q_r^{t_r})^2$, by the unique prime factorization of m, n, where each p_i and q_j are primes, each e_i and t_j are natural numbers.
10. Therefore $\sqrt{2}$ is irrational. ■

A. 1,2,3,4,6,9,7,5,8,10

B. 1,2,6,8,3,9,7,5,4,10

C. 1,2,3,9,5,4,7,8,6,10

D. 1,2,3,8,6,5,9,7,4,10

E. None of the above.

Q13 to Q15 refer to the following definition:

> A natural number is said to be *wobbly* if, and only if, its decimal representation is an alternating sequence of zero and non-zero digits, with the rightmost (ie. units) digit being non-zero.
>
> Examples: $9, 203, 80104$ are wobbly numbers; but not $53, 700201, 5090$.

Q13. Each of these primes can be a factor of a wobbly number, EXCEPT:

A. 2

B. 3

C. 7

D. 11

E. None of the above.

Solution: $2, 3, 7$ are clearly wobbly. $11 \times 19 = 209$ is wobbly.

Q14. Each of these composites can be a factor of a wobbly number, EXCEPT:

A. 4

B. 10

C. 12

D. 28

E. None of the above.

Solution: 4 is clearly wobbly. 10 cannot be a factor, since any multiple of 10 must have its rightmost digit = 0, which is not wobbly. $12 \times 17 = 204$ and $28 \times 11 = 308$ are wobbly numbers.

Q15. Which of the following statements are FALSE?

(I) No wobbly number > 9 is prime.

(II) A wobbly number cannot be a multiple of 13.

(III) If a prime p divides a wobbly number w, then $p \mid (w+1)(w-1)$.

(IV) If w is a wobbly number, then $w \; mod \; 10^2 = w \; mod \; 10$.

A. (I) only.

B. (IV) only.

C. (II) and (III) only.

D. (I), (II) and (III) only.

E. None of (I), (II), (III) and (IV).

Solution:

(I) False. $101, 103, 107, 109$ are wobbly primes.

(II) False. $13 \times 16 = 308$ is wobbly.

(III) False. If $p \mid w$ and $p \mid (w^2 - 1)$, then $p \mid w \cdot w + (w^2 - 1) \cdot (-1) = 1$, by Theorem 4.1.1. That is, $p \mid 1$, which is a contradiction since p is prime.

(IV) True. The remainder when w is divided by 100 or 10 is the same.

Part II

(20 marks) Structured questions. Write your answer in the space provided in the Answer sheet.

Q16. Quantified Statements (9 marks)

Sudoku is a number-placement puzzle. The objective is to fill a 9×9 grid of cells with digits so that each column, each row, and each of the nine 3×3 subgrids that compose the grid contains all of the digits from 1 through 9. The rows and columns are numbered from 1 through 9. (Credit: wikipedia.)

The puzzle setter provides a partially completed grid, as shown in the left figure below. The solution is shown in the right figure below.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

The following definitions are given:

- $in(d, r, c)$, for $1 \leq d, r, c \leq 9$: digit d appears in the cell at row r and column c. Rows are numbered from top to bottom; while columns are numbered from left to right.
- $\bigwedge_{i=1}^{n} P(i) = P(1) \wedge P(2) \wedge \cdots \wedge P(n)$
- $\bigvee_{i=1}^{n} P(i) = P(1) \vee P(2) \vee \cdots \vee P(n)$

Hence, the following logical statement means "every row contains all digits from 1 through 9":

$$\bigwedge_{r=1}^{9} (\bigwedge_{d=1}^{9} (\bigvee_{c=1}^{9} in(d, r, c)))$$

(a) (2 marks) Write a logical statement to mean "every column contains all digits from 1 through 9".

Solution: $\bigwedge_{c=1}^{9} (\bigwedge_{d=1}^{9} (\bigvee_{r=1}^{9} in(d, r, c)))$

(b) (3 marks) Write a logical statement to mean "every 3×3 subgrid contains all digits from 1 through 9". To help you, part of the statement has been filled for you on the Answer Sheet.

> **Solution:** For any $0 \le k, h \le 2$, $\bigwedge_{d=1}^{9} (\bigvee_{r=1}^{3} (\bigvee_{c=1}^{3} in(d, 3k+r, 3h+c)))$

(c) (4 marks) Write a logical statement to mean "a cell cannot contain two digits".

> **Solution:** For any $1 \le d, e, c, r \le 9$, $(d \ne e) \to (in(d, r, c) \to \sim in(e, r, c))$
> or
> For any $1 \le d, e, c, r \le 9$, $in(d, r, c) \wedge in(e, r, c) \to (d = e)$
> or any equivalant statement.

Q17. Modulo Arithmetic (7 marks)
(No working needed. Just write the final answer in the Answer Sheet.)

(a) (2 marks) All integers k such that $(-3)^k \equiv 4 \pmod{13}$ may be written in the form: $k = aq + b, \forall q \in \mathbb{Z}$, for some nonnegative integers a and b. Determine the values of a and b.

> **Solution:** Trying different powers $k = 1, 2\ldots$ gives these residues: $10, 9, 12, 3, 4, 1$. This is a cycle of 6, hence:
>
> $k = 6q + 5, \forall q \in \mathbb{Z}$, which means $\underline{a = 6,\ b = 5}$.

(b) (5 marks) Determine the smallest natural number n with both properties:
(i) In decimal representation, its rightmost digit is 6.
(ii) When this 6 is removed and placed on the left of the remaining digits (eg. $abc6$ becomes $6abc$), the new number is 4 times n.

> **Solution:** Let $n = (a_k a_{k-1} \ldots a_1 6)_{10}$ be the desired number. Re-write: $n = 10N + 6$, where $N = (a_k a_{k-1} \ldots a_1)_{10}$. Then property (ii) becomes:
>
> $$\begin{aligned} 4(10N+6) &= 6 \cdot 10^k + N \\ \Rightarrow \qquad 39N + 24 &= 6 \cdot 10^k \\ \Rightarrow \qquad 13N + 8 &= 2 \cdot 10^k \\ \text{Taking modulo 13:} \quad 8 &\equiv 2 \cdot 10^k \pmod{13} \\ \Rightarrow \qquad 10^k &\equiv 4 \pmod{13}, \text{ since 2 and 13 are coprime} \end{aligned}$$
>
> Since $10 \equiv -3 \pmod{13}$, we get $(-3)^k \equiv 4 \pmod{13}$
>
> From part (a), $k = 6q + 5$.
>
> $$\begin{aligned} &\text{The least positive } k \text{ is therefore } k = 5 \\ \text{Hence,} \qquad 13N + 8 &= 2 \cdot 10^5 \\ \Rightarrow \qquad N &= 15384 \\ &\text{which means } \underline{n = 153846}. \end{aligned}$$

Q18. Proof (4 marks)

Fill in the two boxes **in the ANSWER SHEET** to complete the proof that there is no integer between 0 and 1.

Proof (by Contradiction).

1. Suppose there is some integer between 0 and 1.
   - 1.1. Let $S = \{ n \in \mathbb{Z} \mid 0 < n < 1 \}$.
   - 1.2. From Line 1., S is not empty.
   - 1.3. Clearly, 0 is a lower bound for S.
   - 1.4. Then by the Well Ordering Principle, S has a least element. Call this x.
   - 1.5. Let $y =$ _____ x^2 _____.
   - 1.6. Then $y \in \mathbb{Z}$ and $0 < y < 1$, and thus $y \in S$.
   - 1.7. Also, _____ $y < x$ _____.
   - 1.8. Contradiction, since x is the least element.
2. Hence, there is no integer between 0 and 1. ■

END OF PAPER