Revision
Past Years' Exams

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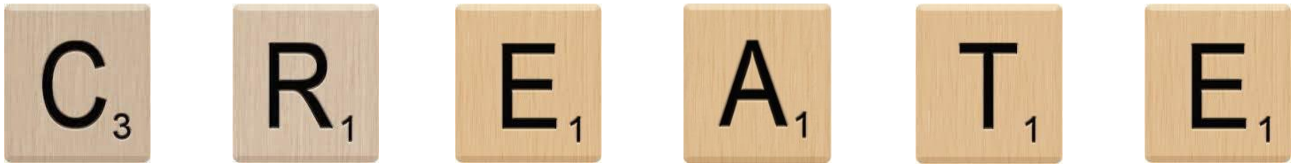
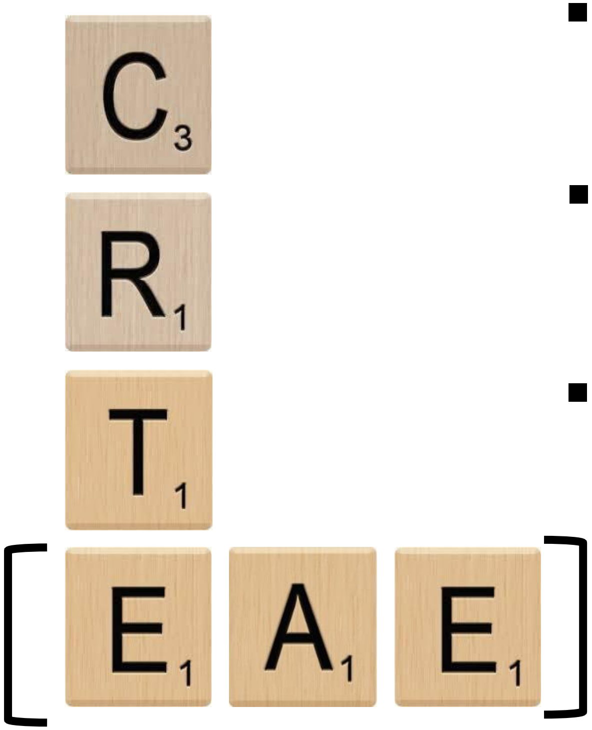
Contents

1. In how many ways can the letters of the word “CREATE” be arranged such that the vowels always stay together?





- Treating E,A,E as one group, there are 4! =24 ways to arrange CRT(EAE).
- There are 3!/2! = 3 ways to arrange the letter in the group (EAE).
- Therefore, total number of ways = $24 \times 3 =$ **72**.

2. Suppose a random sample of 2 lightbulbs is selected from a group of 6 bulbs in which 2 are defective, what is the expected value of the number of defective bulbs in the sample?

Let X represent the number of defective bulbs that occur on a given selection, where $X = 0,1,2$.

$$P(X = 0) = \frac{4}{6} \times \frac{3}{5} = \frac{2}{5}$$

$$P(X = 1) = \left(\frac{4}{6} \times \frac{2}{5}\right) + \left(\frac{2}{6} \times \frac{4}{5}\right) = \frac{8}{15}$$

$$P(X = 2) = \frac{2}{6} \times \frac{1}{5} = \frac{1}{15}$$

$$E[X]$$
$$= P(X = 0) \cdot 0 + P(X = 1) \cdot 1 + P(X = 2) \cdot 2$$
$$= \frac{8}{15} + \frac{2}{15} = \mathbf{\frac{2}{3}}$$

3. What is the coefficient of x^7 in the expansion of $(x+3)^9$?

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \boldsymbol{\binom{n}{r} a^{n-r} b^r} + \cdots + b^n$$

$$\binom{9}{2} x^{9-2} 3^2 = 36 \times 9x^7 = \mathbf{324}x^7$$

Answer: $\mathbf{324}$

4. How many non-negative integer solutions for a, b and c does the following equation have?

$$a + b + c = 100.$$

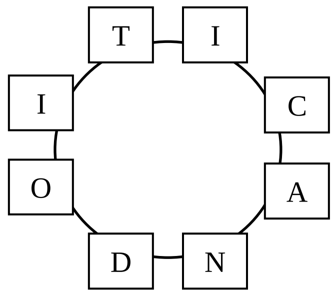
This is a multiset problem with $n = 3$ and $r = 100$.

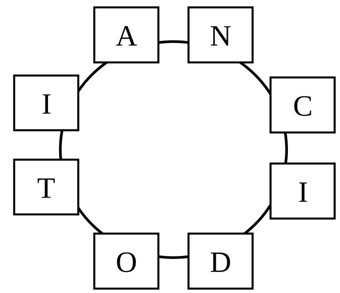
$$\binom{r+n-1}{r} = \binom{100+3-1}{100} = \binom{102}{100} = \mathbf{5151}.$$

5. Four children are to be selected from 6 boys and 4 girls to participate in a training camp. There must be at least one boy among the selected children. In how many ways can the children be selected?

N(4 children) – N(4 girls) = $\binom{10}{4} - \binom{4}{4} = 210 - 1 = \mathbf{209}$.

6. You want to lay the letter tiles of these four words “I”, “CAN”, “DO”, “IT” in a circular arrangement. The letters in the groups “CAN”, ”DO” and “IT” must be kept together in each group, but the letters within each group may be arranged in any order within that group. Also, no two similar letters should be placed next to each other. In how many ways can this be done? The diagram below shows two possible arrangements.





Case 1: “I” and “IT” are opposite of each other.

$$2 \times 3! \times 2! \times 2! = 48$$

Swapping “I” and “IT”

Permuting “CAN”

Permuting “DO”

Permuting “IT”

Case 2: “I” and “IT” are next to each other, i.e. “ITI”

$$2 \times 3! \times 2! = 24$$

Swapping “CAN” and “DO”

Permuting “CAN”

Permuting “DO”

Total: $48 + 24 = \mathbf{72}$

7. Each of the 9 cells in the 3×3 grid below is to be filled with the number -1, 0 or 1. Prove that among the 3 row-sums, 3 column-sums and 2 diagonal-sums, there are two sums that are equal in value.

$$1+1+1=3;$$
$$1+1+0=2;$$
$$1+0+0=1;$$
$$0+0+0=0;$$
$$(-1)+0+0=-1;$$
$$(-1)+(-1)+0=-2;$$
$$(-1)+(-1)+(-1)=-3.$$

There are 7 possible values (pigeonholes): -3, -2, -1, 0, 1, 2, 3.

There are 8 pigeons: 3 row-sums, 3 column-sums and 2 diagonal-sums.

By PHP, there must be two sums with the same value.

8. 0.3% of the population are sufferers of a certain disease. The probability of a sufferer tested positive is 98%, while the probability of a non-sufferer tested negative by the test is 95%.

Assume that the test is administered to a person randomly.

(i) What is the probability that the test result of the person will be positive?

Let T = "tested positive", S = "sufferer".

$P(S) = 0.003; P(T|S) = 0.98; P(T|\bar{S}) = 0.05.$

$$P(T) = P(T|S) \cdot P(S) + P(T|\bar{S}) \cdot P(\bar{S})$$
$$= (0.98 \times 0.003) + (0.05 \times 0.997) = 0.05279 = \mathbf{5.28\%}$$

(ii) If the test result is positive, what is the probability that the person is a sufferer?

$$P(S|T) = \frac{P(T|S) \cdot P(S)}{P(T)} = \frac{0.98 \times 0.003}{0.05279} = 0.05569 = \mathbf{5.57\%}$$

(iii) What is the probability that the person will be misclassified?

$$P(misclassified) = P(T \cap \bar{S}) + P(\bar{T} \cap S)$$
$$= P(T|\bar{S}) \cdot P(\bar{S}) + P(\bar{T}|S) \cdot P(S)$$
$$= (0.05 \times 0.997) + (0.02 \times 0.003) = 0.04991 = \mathbf{4.99\%}$$

9. Given a fair coin, what is the expected number of coin tosses to get 5 consecutive heads?

Solution 1:

1. Let E be the expected number of tosses.
2. If we get a T (with probability ½), then the expected value is $E + 1$.
3. If we get HT (with probability ¼), then the expected value is $E + 2$.
4. If we get HHT (probability 1/8), then the expected value is $E + 3$.
5. If we get HHHT (probability 1/16), then the expected value is $E + 4$.
6. If we get HHHHT (probability 1/32), then the expected value is $E + 5$.
7. If we get HHHHH (probability 1/32), then the expected value is 5.
8. Putting these together, we have

$$E = \frac{1}{2}(E+1) + \frac{1}{4}(E+2) + \frac{1}{8}(E+3) + \frac{1}{16}(E+4) + \frac{1}{32}(E+5) + \frac{1}{32}(5)$$

9. Solving the above equation, we get $E = \mathbf{62}$.

Generalizing, we may get $E_n = 2(2^n - 1)$, where n is the number of consecutive heads.

9. Given a fair coin, what is the expected number of coin tosses to get 5 consecutive heads?

Solution 2:

Find recurrence relation for E_n, the expected number of tosses to get n consecutive heads. After getting $n-1$ consecutive heads, there are two scenarios:

- We get a head (with probability ½), in which case we have got n consecutive heads and the expected value is $E_{n-1}+1$.
- We get a tail (with probability ½), in which case we have to start all over, and hence the expected value is $E_{n-1}+1+E_n$.

$$E_0 = 0$$

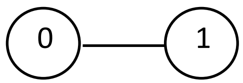
$$E_n = \frac{1}{2}(E_{n-1}+1) + \frac{1}{2}(E_{n-1}+1+E_n)$$

or $$E_n = 2\,E_{n-1} + 2$$

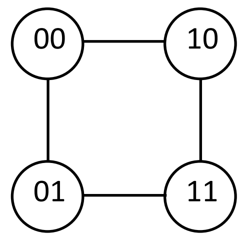
Therefore, $E_5 = \mathbf{62}$.

Solving the recurrence relation, we get the closed-form formula $E_n = 2(2^n - 1)$.

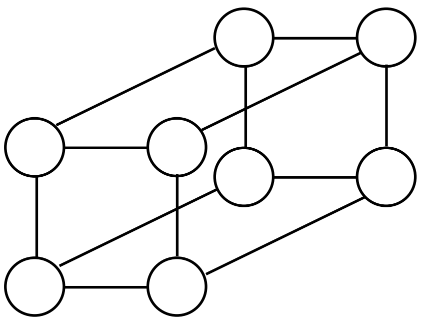
10. A hypercube graph Q_n may be constructed by creating a vertex for each n-digit binary number (leading zeroes are added if necessary to form n digits), with two vertices adjacent when their binary representations differ in a single digit. $N_v(G)$ and $N_e(G)$ denote the number of vertices and number of edges, respectively, in graph G.



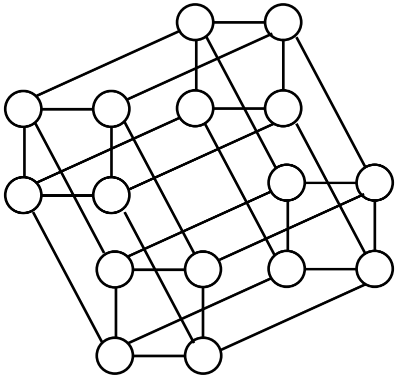
Graph Q_1
$N_v(Q_1) = 2$
$N_e(Q_1) = 1$



Graph Q_2
$N_v(Q_2) = 4$
$N_e(Q_2) = 4$



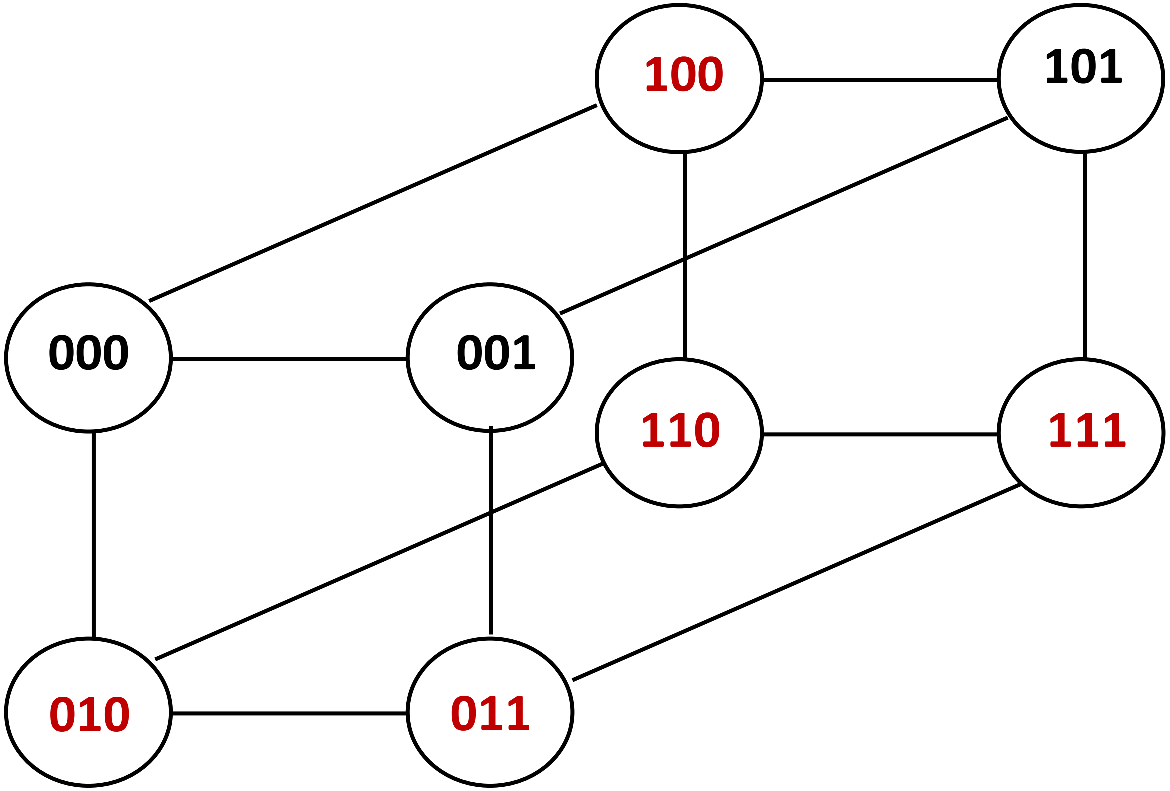
Graph Q_3
$N_v(Q_3) = 8$
$N_e(Q_3) = 12$



Graph Q_4
$N_v(Q_4) = 16$
$N_e(Q_4) = 32$

10. A hypercube graph Q_n may be constructed by creating a vertex for each n-digit binary number (leading zeroes are added if necessary to form n digits), with two vertices adjacent when their binary representations differ in a single digit. $N_v(G)$ and $N_e(G)$ denote the number of vertices and number of edges, respectively, in graph G.

(i) Fill in the vertex labels for Q_3. Note that 3 of the vertices have been labelled 000, 001 and 101 for you.

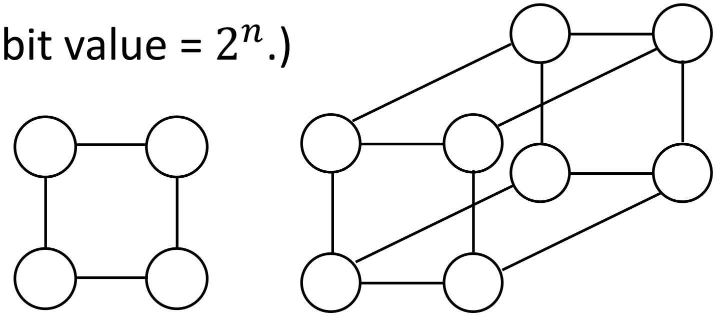


10. A hypercube graph Q_n may be constructed by creating a vertex for each n-digit binary number (leading zeroes are added if necessary to form n digits), with two vertices adjacent when their binary representations differ in a single digit. $N_v(G)$ and $N_e(G)$ denote the number of vertices and number of edges, respectively, in graph G.

(Number of n-bit value = 2^n.)

(ii) What is $N_v(Q_n)$? $N_v\ (Q_n) = 2^n$.

(iii) What is $N_e(Q_n)$? $N_e\ (Q_n) = \frac{n2^n}{2}$.



Graph Q_2
$N_v(Q_2) = 4$
$N_e(Q_2) = 4$

Graph Q_3
$N_v(Q_3) = 8$
$N_e(Q_3) = 12$

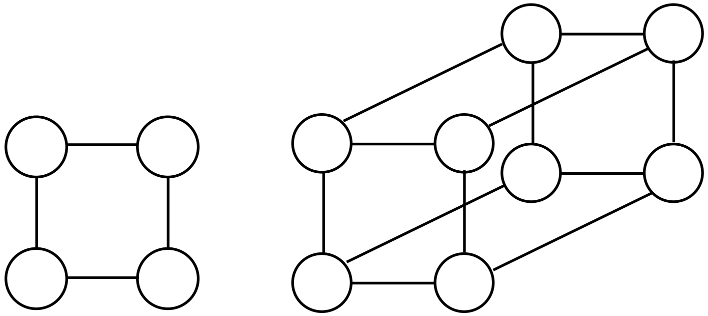
Proof

1. An n-bit number can differ by 1 bit in n ways (eg: 1011 → 0011, 1111, 1001 or 1010.)
2. Therefore, each vertex has degree n.
3. There are 2^n vertices (from part (ii)).
4. Total degree of $Q_n = n2^n$.
5. By Handshake Theorem, total degree of graph = 2 × number of edges.
6. Therefore, $N_e\ (Q_n) = \frac{n2^n}{2}$.

10. A hypercube graph Q_n may be constructed by creating a vertex for each n-digit binary number (leading zeroes are added if necessary to form n digits), with two vertices adjacent when their binary representations differ in a single digit. $N_v(G)$ and $N_e(G)$ denote the number of vertices and number of edges, respectively, in graph G.

(ii) What is $N_v(Q_n)$? $N_v\ (Q_n) = 2^n$.

(iii) What is $N_e(Q_n)$? $N_e\ (Q_n) = \frac{n2^n}{2}$.



Graph Q_2
$N_v(Q_2) = 4$
$N_e(Q_2) = 4$

Graph Q_3
$N_v(Q_3) = 8$
$N_e(Q_3) = 12$

Prove $N_e(Q_n)$ by M.I.

1. Let $P(n)$ be $N_e\ (Q_n) = \frac{n2^n}{2}$ for $n \in \mathbb{Z}^+$.
2. Basis: $N_e\ (Q_1) = 1 = \frac{1\times 2^1}{2}$. Hence, $P(1)$ is true.
3. Assume $P(k)$ is true, i.e. $N_e\ (Q_k) = \frac{k2^k}{2}$.
4. Then $N_e\ (Q_{k+1}) = 2N_e\ (Q_k) + N_v\ (Q_k) = k2^k + 2^k = (k+1)2^k$ $= \frac{(k+1)2^{k+1}}{2}$.
5. Hence, $P(n)$ is true for $n \in \mathbb{Z}^+$.

11. The lazy caterer's sequence describes the number of maximum pieces of a pancake (or pizza) that can be made with a given number of straight cuts.

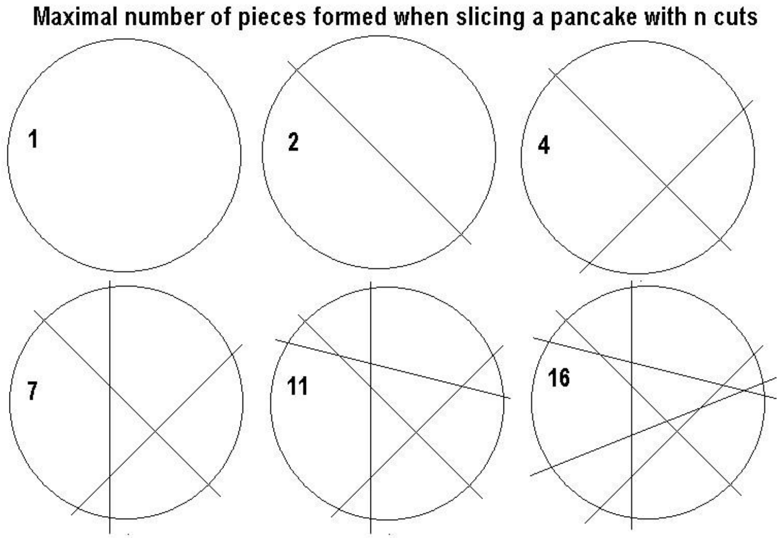
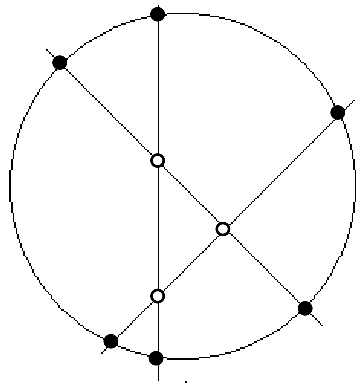


Figure 1: Lazy Caterer's Sequence.



$P(3) = 7;$
$V(3) = 9;$
$V_3(3) = 6;$
$V_4(3) = 3;$
$E(3) = 15.$

Figure 2: Graph representation.

$P(n)$: number of pieces of pancakes with n cuts;

$V(n)$: number of vertices of a graph corresponding to a pancake with n cuts;

- $V_3(n)$: number of vertices with degree 3;
- $V_4(n)$: number of vertices with degree 4;
- $V(n) = V_3(n)+V_4(n)$.

$E(n)$: number of edges of a graph corresponding to a pancake with n cuts.

11. The lazy caterer's sequence describes the number of maximum pieces of a pancake (or pizza) that can be made with a given number of straight cuts.

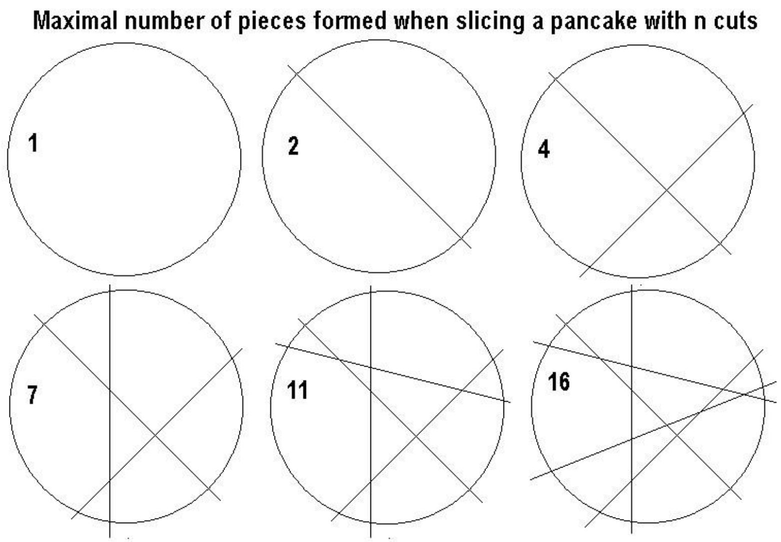


Figure 1: Lazy Caterer's Sequence.

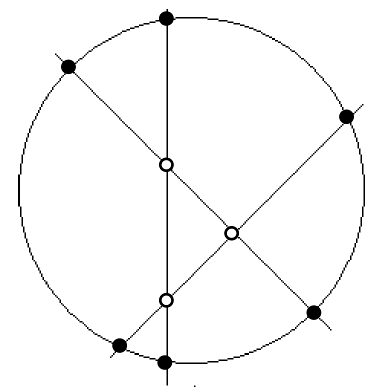


Figure 2: Graph representation.

(a) Express $E(n)$ in terms of $V_3(n)$ and $V_4(n)$.

(b) Write the recurrence relation for $V(n)$. The base case is $V(0) = 0$.

(c) Write the closed form formula for $V(n)$.

(d) Write the recurrence relation for $E(n)$. The base case is $E(0) = 0$.

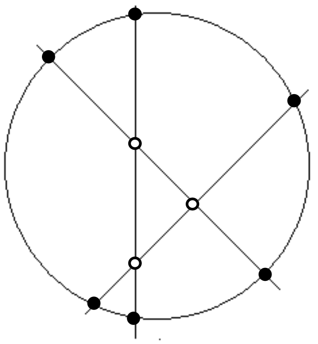
(e) Write the closed form formula for $E(n)$.

(f) Euler's formula is given as $v - e + f = 2$. Relate v, e and f with the functions defined in this question.

(g) From part (f), or otherwise, derive the closed form forum for $P(n)$.

11. The lazy caterer's sequence describes the number of maximum pieces of a pancake (or pizza) that can be made with a given number of straight cuts.

n	0	1	2	3	4	5	6	7
$P(n)$	1	2	4	7	11	16	22	29
$E(n)$	0	3	8	15	24	35	48	63
$V(n)$	0	2	5	9	14	20	27	35
$V_3(n)$	0	2	4	6	8	10	12	14
$V_4(n)$	0	0	1	3	6	10	15	21



$V_3(n) = 2n$ (every cut creates 2 new vertices on the boundary)

$V_4(n) = V_4(n-1) + n - 1$ (the n^{th} cut intersects the existing $n-1$ cuts)

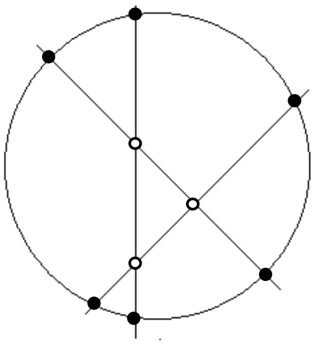
$V_4(n) = n(n-1)/2$ for $n > 0$.

$V(n) = V(n-1) + n + 1$ (the n^{th} cut cuts through the existing $n-1$ cuts and it also cuts the boundary at two places)

$V(n) = V_3(n) + V_4(n) = 2n + \frac{n(n-1)}{2} = \frac{n^2+3n}{2}$.

11. The lazy caterer's sequence describes the number of maximum pieces of a pancake (or pizza) that can be made with a given number of straight cuts.

n	0	1	2	3	4	5	6	7
$P(n)$	1	2	4	7	11	16	22	29
$E(n)$	0	3	8	15	24	35	48	63
$V(n)$	0	2	5	9	14	20	27	35
$V_3(n)$	0	2	4	6	8	10	12	14
$V_4(n)$	0	0	1	3	6	10	15	21



$E(n) = E(n-1) + 2n + 1$ (the n^{th} cut adds $n+1$ vertices as it cuts through the existing $n-1$ cuts and opposite sides of the circle, splitting each of the $n+1$ edges it cuts through into 2, and also introducing n new edges).

$E(n) = \frac{3V_3(n)+4V_4(n)}{2}$ (Handshake Theorem: total degree of a graph is twice the number of edges)

$$= \frac{3(2n)+2n(n-1)}{2} = n^2 + 2n.$$

$P(n) = E(n) - V(n) + 1$ (using Euler's formula $v - e + f = 2$, where $f = P(n) + 1$ since the region outside the circle constitutes one face)

$P(n) = P(n-1) + n.$

All the above give the closed form formula $P(n) = \frac{n^2+n+2}{2}$.

11. The lazy caterer's sequence describes the number of maximum pieces of a pancake (or pizza) that can be made with a given number of straight cuts.

(a) Express $E(n)$ in terms of $V_3(n)$ and $V_4(n)$.

$$\boldsymbol{E(n) = \frac{3V_3(n) + 4V_4(n)}{2}.}$$

(b) Write the recurrence relation for $V(n)$. The base case is $V(0) = 0$.

$\boldsymbol{V(n) = V(n-1) + n + 1}$**, for** $\boldsymbol{n > 0.}$

(c) Write the closed form formula for $V(n)$.

$\boldsymbol{V(n) = \frac{n^2+3n}{2}}$**, for** $\boldsymbol{n \geq 0.}$

(d) Write the recurrence relation for $E(n)$. The base case is $E(0) = 0$.

$\boldsymbol{E(n) = E(n-1) + 2n + 1}$**, for** $\boldsymbol{n > 0.}$

(e) Write the closed form formula for $E(n)$.

$\boldsymbol{E(n) = n^2 + 2n}$**, for** $\boldsymbol{n \geq 0.}$

(f) Euler's formula is given as $v - e + f = 2$. Relate v, e and f with the functions defined in this question.

$\boldsymbol{v = V(n), e = E(n)}$**, and** $\boldsymbol{f = P(n) + 1.}$

(g) From part (f), or otherwise, derive the closed form forum for $P(n)$.

$\boldsymbol{P(n) = \frac{n^2+n+2}{2}}$**, for** $\boldsymbol{n \geq 0.}$

12. License plate format: $S\alpha_1\alpha_2x_1x_2x_3x_4c$, where each $\alpha_1$and α_2 is a single letter in the English alphabet <u>excluding</u> I and O, and each $x_1, \cdots, x_4$ is a single digit. The last letter c is a checksum letter.

Let $\mathcal{L}$ denote the set of all possible strings of the form: $\alpha_1\alpha_2x_1x_2x_3x_4$. Also, let $\kappa = \{A, Z, Y, X, U, T, S, R, P, M, L, K, J, H, G, E, D, C, B\}$. Then the checksum function may be defined as $f: \mathcal{L} \to \kappa$, where $f(\alpha_1\alpha_2x_1x_2x_3x_4)$ is calculated in 3 steps:

F1. Let n_1 be the positional value of α_1 in the English alphabet, i.e. $A = 1, B = 2, \cdots, Z = 26$. And let n_2 be the positional value of α_2.

F2. Compute $t = 9n_1 + 4n_2 + 5x_1 + 4x_2 + 3x_3 + 2x_4$, and $r = t \underline{\text{mod}}\ 19$.

F3. The checksum letter $c =$ the letter in κ indexed by r, where $0 = A, 1 = Z, 2 = Y, \cdots, 18 = B$.

Using the example in the figure above:

F1. $\alpha_1 = D, \alpha_2 = N$, so $n_1 = 4, n_2 = 14, x_1 = 7, x_2 = 4, x_3 = 8,\ x_2 = 4$.

F2. $t = 9 \cdot 4 + 4 \cdot 14 + 5 \cdot 7 + 4 \cdot 4 + 3 \cdot 8 + 2 \cdot 4 = 175$,
and $r = 175\ \underline{\text{mod}}\ 19 = 4$.

F3. Hence, $c = U$.

12. Let $\mathcal{L}$ denote the set of all possible strings of the form: $\alpha_1\alpha_2x_1x_2x_3x_4$. Also, let $\kappa = \{A, Z, Y, X, U, T, S, R, P, M, L, K, J, H, G, E, D, C, B\}$. Then the checksum function may be defined as $f: \mathcal{L} \to \kappa$, where $f(\alpha_1\alpha_2x_1x_2x_3x_4)$ is calculated in 3 steps:

F1. Let n_1 be the positional value of α_1 in the English alphabet, i.e. $A = 1, B = 2, \cdots, Z = 26$. And let n_2 be the positional value of α_2.

F2. Compute $t = 9n_1 + 4n_2 + 5x_1 + 4x_2 + 3x_3 + 2x_4$, and $r = t \underline{\text{mod}}\ 19$.

F3. The checksum letter $c =$ the letter in κ indexed by r, where $0 = A, 1 = Z, 2 = Y, \cdots, 18 = B$.

(a) Determine the checksum letter for "CS1231".

F1. $\alpha_1 = C$, $\alpha_2 = S$, so $n_1 = 3$, $n_2 = 19$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_2 = 1$.

F2. $t = 9 \cdot 3 + 4 \cdot 19 + 5 \cdot 1 + 4 \cdot 2 + 3 \cdot 3 + 2 \cdot 1 = 127$, and $r = 127 \underline{\text{mod}}\ 19 = 13$.

F3. Hence, $c = \boldsymbol{H}$.

(b) Show that f is not one-to-one by finding another $y \in \mathcal{L}$ such that $f(y)$ is the same checksum letter as in this figure.

Many possible answers. One easy way is to note that the weights of n_2 and x_2 are the same, which allows us to increase n_2 by the same amount as we decrease x_2, yielding the same value for t. Thus, $y = DP7284$ is one possible answer.

12. (c) Is f onto? Prove or disprove.

f is onto.

Construct table which shows the values of a, b such that $3a + 2b \equiv r \pmod{19}$, for $r = 0,1,2,\ldots,18$. Note that this table is a bijection between a, b and r.

a\b	0	1	2	3	4
0	0	2	4		
1	3	5	7		
2	6	8	10	12	
3	9	11	13	15	17
4		14	16	18	1

Proof (by construction)

1. Take any $w \in \kappa$.
2. From step F3, there is a unique r which is the positional index of w. Note that $0 \leq r < 19$.
3. (We will construct a string: $y = BE00ab$, where a, b will be derived from r, such that $f(y) = w$.)
4. Given the r from line 2, look up the table for the unique pair a, b.
5. Let $y = BE00ab$. Clearly, $y \in \mathcal{L}$.
6. Now, $f(y)$ may be calculated as follows:
   6.1 F2: $t = 9 \cdot 2 + 4 \cdot 5 + 5 \cdot 0 + 4 \cdot 0 + 3a + 2b = 38 + 3a + 2b$.
   6.2 Since $38 + 3a + 2b \equiv 3a + 2b \equiv r \pmod{19}$, we have $f(y) = w$.
7. Thus $\exists y \in \mathcal{L}$ such that $f(y) = w$.
8. Hence, f is onto.

MCQ answers for past-years' papers.

- If you want the answers for MCQs for past-years' papers (only for semester 1 paper from AY2015/16 onwards), post your own MCQ answers on LumiNUS > Forum > Exams and we will verify.

Mass consultation for next week

- 4:00pm – 5:30pm
- Please go to this doodle page to pick one of the dates (Monday – Friday):
- https://www.doodle.com/poll/8qbnf3wrzxpvbeqg?utm_source=poll&utm_medium=link

This Semester's Exam Format

- Section A (MCQs), Section B (MRQs), Section C (long questions)
- Q0 (admin): 3 marks
- Q1-12 (MCQs): 12×2 marks = 24 marks
- Q13-17 (MRQs): 5×3 marks = 15 marks
- Aaron's part:
  - Counting and Probability: 20 marks
  - Graphs and Trees: 18 marks
- Lawrence's part:
  - 3 or 4 questions: 20 marks
- Total: 100 marks

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