CS1231S Chapter 5

Sets

5.1 Basics

Definition 5.1.1. (1) A *set* is an unordered collection of objects.

(2) These objects are called the *members* or *elements* of the set.

(3) Write $x \in A$ for x is an element of A;
$x \notin A$ for x is not an element of A;
$x, y \in A$ for x, y are elements of A;
$x, y \notin A$ for x, y are not elements of A; etc.

Symbol	Meaning	Examples	Non-examples
$\mathbb{N}$	the set of all natural numbers	$0, 1, 2, 3, 31 \in \mathbb{N}$	$-1, \frac{1}{2} \notin \mathbb{N}$
$\mathbb{Z}$	the set of all integers	$0, 1, -1, 2, -10 \in \mathbb{Z}$	$\frac{1}{2}, \sqrt{2} \notin \mathbb{Z}$
$\mathbb{Q}$	the set of all rational numbers	$-1, 10, \frac{1}{2}, -\frac{7}{5} \in \mathbb{Q}$	$\sqrt{2}, \pi, \sqrt{-1} \notin \mathbb{Q}$
$\mathbb{R}$	the set of all real numbers	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi \in \mathbb{R}$	$\sqrt{-1}, \sqrt{-10} \notin \mathbb{R}$
$\mathbb{C}$	the set of all complex numbers	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi, \sqrt{-1}, \sqrt{-10} \in \mathbb{C}$	
$\mathbb{Z}^+$	the set of all positive integers	$1, 2, 3, 31 \in \mathbb{Z}^+$	$0, -1, -12 \notin \mathbb{Z}^+$
$\mathbb{Z}^-$	the set of all negative integers	$-1, -2, -3, -31 \in \mathbb{Z}^-$	$0, 1, 12 \notin \mathbb{Z}^-$
$\mathbb{Z}_{\geqslant 0}$	the set of all non-negative integers	$0, 1, 2, 3, 31 \in \mathbb{Z}_{\geqslant 0}$	$-1, -12 \notin \mathbb{Z}_{\geqslant 0}$

$\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{Q}_{\geqslant m}$, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_{\geqslant m}$, etc. are defined similarly.

Table 5.1: Common sets

Note 5.1.2. Some define $0 \notin \mathbb{N}$.

Definition 5.1.3 (roster notation)**.** (1) The set whose only elements are $x_1, x_2, \ldots, x_n$ is denoted $\{x_1, x_2, \ldots, x_n\}$.

(2) The set whose only elements are $x_1, x_2, x_3, \ldots$ is denoted $\{x_1, x_2, x_3, \ldots\}$.

Example 5.1.4. (1) The only elements of $A = \{1, 5, 6, 3, 3, 3\}$ are 1, 5, 6 and 3. So $6 \in A$ but $7 \notin A$.

(2) The only elements of $B = \{0, 2, 4, 6, 8, \ldots\}$ are the non-negative even integers. So $4 \in B$ but $5 \notin B$.

To check whether an object z is an element of a set $S = \{x_1, x_2, \ldots, x_n\}$. If z is in the list $x_1, x_2, \ldots, x_n$, then $z \in S$, else $z \notin S$.

Definition 5.1.5 (set-builder notation)**.** Let U be a set and $P(x)$ is a predicate over U. Then the set of all elements $x \in U$ such that $P(x)$ is true is denoted

$$\{x \in U : P(x)\}.$$

This is read as "the set of all x in U such that $P(x)$".

Note 5.1.6. Some write $\{\ldots \mid \ldots\}$ for $\{\ldots : \ldots\}$.

Example 5.1.7. (1) The elements of $C = \{x \in \mathbb{Z}_{\geqslant 0} : x \text{ is even}\}$ are precisely the elements of $\mathbb{Z}_{\geqslant 0}$ that are even, i.e., the non-negative even integers. So $6 \in C$ but $7 \notin C$.

(2) The elements of $D = \{x \in \mathbb{Z} : x \text{ is a prime number}\}$ are precisely the elements of $\mathbb{Z}$ that are prime numbers, i.e., the prime integers. So $7 \in D$ but $9 \notin D$.

To check whether an object z is an element of $S = \{x \in U : P(x)\}$. If $z \in U$ and $P(z)$ is true, then $z \in S$, else $z \notin S$. Hence $z \notin U$ implies $z \notin S$, and $P(z)$ is false implies $z \notin S$.

Remark 5.1.8. Sometimes people write $\{y^2 : y \text{ is an odd integer}\}$, for example, to mean "the set of all objects of the form y^2 such that y is an odd integer". More generally, if $t(y_1, y_2, \ldots, y_n)$ is an expression involving $y_1, y_2, \ldots, y_n$, and $P(y_1, y_2, \ldots, y_n)$ is a predicate in $y_1, y_2, \ldots, y_n$, then one may use

$$\{t(y_1, y_2, \ldots, y_n) : P(y_1, y_2, \ldots, y_n)\}$$

to denote

$$\big\{x : \exists y_1, y_2, \ldots, y_n \ \big(P(y_1, y_2, \ldots, y_n) \wedge x = t(y_1, y_2, \ldots, y_n)\big)\big\}.$$

Definition 5.1.9. Two sets are *equal* if they have the same elements, i.e., for all sets A, B,

$$A = B \quad \Leftrightarrow \quad \forall z\ (z \in A \Leftrightarrow z \in B).$$

Example 5.1.10. $\{1, 5, 6, 3, 3, 3\} = \{1, 5, 6, 3\} = \{1, 3, 5, 6\}$.

Slogan 5.1.11. Order and repetition do not matter.

Example 5.1.12. If E denotes the set considered in the first sentence of Remark 5.1.8, then

$$\begin{aligned} E &= \{y^2 : y \text{ is an odd integer}\} \\ &= \{x \in \mathbb{Z} : x = y^2 \text{ for some odd integer } y\} \\ &= \{1^2, 3^2, 5^2, \ldots\}. \end{aligned}$$

Example 5.1.13. $\{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}$.

Proof.
1. $(\Rightarrow)$
   1.1. Take any $z \in \{x \in \mathbb{Z} : x^2 = 1\}$.
   1.2. Then $z \in \mathbb{Z}$ and $z^2 = 1$.
   1.3. So $\quad z^2 - 1 = (z-1)(z+1) = 0$
   1.4. $\therefore \quad z - 1 = 0 \quad \text{or} \quad z + 1 = 0$
   1.5. $\therefore \quad z = 1 \quad \text{or} \quad z = -1.$
   1.6. This means $z \in \{1, -1\}$.
2. $(\Leftarrow)$
   2.1. Take any $z \in \{1, -1\}$.
   2.2. Then $z = 1$ or $z = -1$.
   2.3. In either case, we have $z \in \mathbb{Z}$ and $z^2 = 1$.
   2.4. So $z \in \{x \in \mathbb{Z} : x^2 = 1\}$. □

Theorem 5.1.14. There exists a unique set with no element, i.e.,

- there is a set with no element; and (existence part)
- for all sets A, B, if both A and B have no element, then $A = B$. (uniqueness part)

Proof. 1. (existence part) The set $\{\}$ has no element.
2. (uniqueness part)
   2.1. Let A, B be sets with no element.
   2.2. Then trivially,
   $$\forall z\ (z \in A \Rightarrow z \in B) \quad \text{and} \quad \forall z\ (z \in B \Rightarrow z \in A)$$
   because the antecedents are never true.
   2.3. So $A = B$. $\square$

Definition 5.1.15. The set with no element is called the *empty set*. It is denoted by $\varnothing$.

Definition 5.1.16. Let A, B be sets. Call A a *subset* of B, and write $A \subseteq B$, if

$$\forall z\ (z \in A \Rightarrow z \in B).$$

Alternatively, we may say that B *includes* A, and write $B \supseteq A$ in this case.

Example 5.1.17. (1) $\{1, 5, 2\} \subseteq \{5, 2, 1, 4\}$ but $\{1, 5, 2\} \not\subseteq \{2, 1, 4\}$.

(2) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Remark 5.1.18. Let A, B be sets.

(1) $$A \not\subseteq B \quad \Leftrightarrow \quad \exists z\ (z \in A \text{ and } z \notin B);$$

(2) $$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A;$$

(3) $$\varnothing \subseteq A \quad \text{and} \quad A \subseteq A.$$

Definition 5.1.19. Let A, B be sets. Call A a *proper subset* of B, and write $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$. In this case, we may say that the inclusion of A in B is *proper* or *strict*.

Example 5.1.20. All the inclusions in Example 5.1.17 are strict.

Note 5.1.21. Sets can be elements of sets.

Example 5.1.22. (1) The set $A = \{\varnothing\}$ has exactly 1 element, namely the empty set. So A is not empty.

(2) The set $B = \{\{1\}, \{2\}, \{3\}\}$ has exactly 3 elements, namely $\{1\}$, $\{2\}$, and $\{3\}$. So $\{3\} \in B$, but $3 \notin B$.

Note 5.1.23. Membership and inclusion can be different.

Question 5.1.24. Let $C = \{\{1\}, 2, \{3\}, 3, \{\{4\}\}\}$. Which of the following are true? ✎ 5a

- $\{1\} \in C$.
- $\{2\} \in C$.
- $\{3\} \in C$.
- $\{4\} \in C$.
- $\{1\} \subseteq C$.
- $\{2\} \subseteq C$.
- $\{3\} \subseteq C$.
- $\{4\} \subseteq C$.

5.2 Powers and products

Definition 5.2.1. Let A be a set. The set of all subsets of A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example 5.2.2. (1) $\mathcal{P}(\varnothing) = \{\varnothing\}$.

(2) $\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$.

(3) $\mathcal{P}(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$.

Definition 5.2.3. (1) A set is *finite* if it has finitely many (distinct) elements. It is *infinite* if it is not finite.

(2) Let A be a finite set. The *cardinality* of A, or the *size* of A, is the number of (distinct) elements in A. It is denoted by $|A|$.

(3) Sets of size 1 are called *singletons*.

Theorem 5.2.4. Let A be a finite set. Then $|\mathcal{P}(A)| = 2^{|A|}$.

Example 5.2.5. (1) $|\varnothing| = 0$ and $|\mathcal{P}(\varnothing)| = 1 = 2^0$.

(2) $|\{1\}| = 1$ and $|\mathcal{P}(\{1\})| = 2 = 2^1$.

(3) $|\{1,2\}| = 2$ and $|\mathcal{P}(\{1,2\})| = 4 = 2^2$.

Definition 5.2.6. An *ordered pair* is an expression of the form

$$(x, y).$$

Let (x, y) and (x', y') be ordered pairs. Then

$$(x, y) = (x', y') \qquad \Leftrightarrow \qquad x = x' \quad \text{and} \quad y = y'.$$

Example 5.2.7. (1) $(1,2) \neq (2,1)$, although $\{1,2\} = \{2,1\}$.

(2) $(3, 0.5) = (\sqrt{9}, \frac{1}{2})$.

Definition 5.2.8. Let A, B be sets. The *Cartesian product* of A and B, denoted $A \times B$, is defined to be

$$\{(x, y) : x \in A \text{ and } y \in B\}.$$

Read $A \times B$ as "A cross B".

Example 5.2.9. $\{a, b\} \times \{1,2,3\} = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$.

Note 5.2.10. $|\{a, b\} \times \{1,2,3\}| = 6 = 2 \times 3 = |\{a, b\}| \times |\{1,2,3\}|$.

Definition 5.2.11. Let $n \in \{x \in \mathbb{Z} : x \geqslant 2\}$. An *ordered n-tuple* is an expression of the form

$$(x_1, x_2, \ldots, x_n).$$

Let $(x_1, x_2, \ldots, x_n)$ and $(x_1', x_2', \ldots, x_n')$ be ordered n-tuples. Then

$$(x_1, x_2, \ldots, x_n) = (x_1', x_2', \ldots, x_n') \quad \Leftrightarrow \quad x_1 = x_1' \text{ and } x_2 = x_2' \text{ and } \ldots \text{ and } x_n = x_n'.$$

Example 5.2.12. (1) $(1,2,5) \neq (2,1,5)$, although $\{1,2,5\} = \{2,1,5\}$.

(2) $(3, (-2)^2, 0.5, 0) = (\sqrt{9}, 4, \frac{1}{2}, 0)$

Definition 5.2.13. Let $n \in \{x \in \mathbb{Z} : x \geqslant 2\}$ and $A_1, A_2, \ldots, A_n$ be sets. The *Cartesian product* of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$, is defined to be

$$\{(x_1, x_2, \ldots, x_n) : x_1 \in A_1 \text{ and } x_2 \in A_2 \text{ and } \ldots \text{ and } x_n \in A_n\}.$$

If A is a set, then $A^n = \underbrace{A \times A \times \cdots \times A}_{n\text{-many } A\text{'s}}$.

Example 5.2.14. $\{0,1\} \times \{0,1\} \times \{x, y\} = \{(0,0,x), (0,0,y), (0,1,x), (0,1,y), (1,0,x), (1,0,y), (1,1,x), (1,1,y)\}$.

5.3 Boolean operations

Definition 5.3.1. Let A, B be sets.

(1) The *union* of A and B, denoted $A \cup B$, is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Read $A \cup B$ as "A union B".

(2) The *intersection* of A and B, denoted $A \cap B$, is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Read $A \cap B$ as "A intersect B".

(3) The *complement* of B in A, denoted $A - B$ or $A \setminus B$, is defined by

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Read $A \setminus B$ as "A minus B".

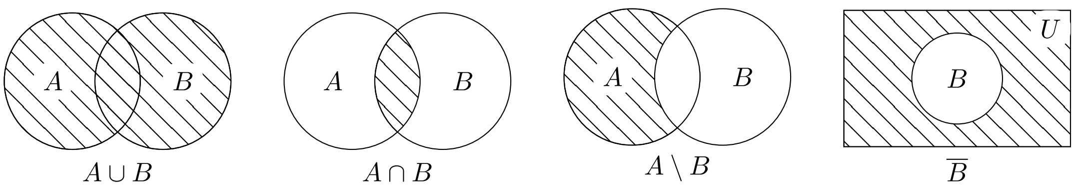


Figure 5.2: Boolean operations on sets

Convention and terminology 5.3.2. When working in a particular context, one usually works within a set that contains all the objects one may talk about. Such a set is called a *universal set.*

Definition 5.3.3. Let B be a set. In a context where U is the universal set (so that implicitly $U \supseteq B$), the *complement* of B, denoted $\overline{B}$ or B^{c}, is defined by

$$\overline{B} = U \setminus B.$$

Example 5.3.4. Let $A = \{x \in \mathbb{Z} : x \leqslant 10\}$ and $B = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 15\}$. Then

$$\begin{aligned}
A \cup B &= \{x \in \mathbb{Z} : (x \leqslant 10) \vee (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x \leqslant 15\};\\
A \cap B &= \{x \in \mathbb{Z} : (x \leqslant 10) \wedge (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 10\};\\
A - B &= \{x \in \mathbb{Z} : (x \leqslant 10) \wedge {\sim}(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x < 5\};\\
\overline{B} &= \{x \in \mathbb{Z} : {\sim}(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : (x < 5) \vee (x > 15)\},
\end{aligned}$$

in a context where $\mathbb{Z}$ is the universal set. To show the first equation, one shows

$$\forall x \in \mathbb{Z} \ \big((x \leqslant 10) \vee (5 \leqslant x \leqslant 15) \Leftrightarrow (x \leqslant 15)\big), \qquad \text{etc.}$$

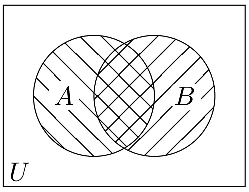
Theorem 5.3.5 (Set Identities)**.** For all set A, B, C in a context where U is the universal set, the following hold.

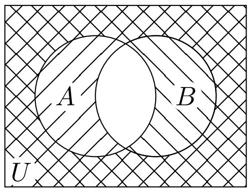
Identity Laws	$A \cup \varnothing = A$	$A \cap U = A$
Universal Bound Laws	$A \cup U = U$	$A \cap \varnothing = \varnothing$
Idempotent Laws	$A \cup A = A$	$A \cap A = A$
Double Complement Law	$\overline{(\overline{A})} = A$	
Commutative Laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative Laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive Laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption Laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Complement Laws	$A \cup \overline{A} = U$	$A \cap \overline{A} = \varnothing$
Set Difference Law	$A \setminus B = A \cap \overline{B}$	
	$\overline{\varnothing} = U$	$\overline{U} = \varnothing$

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Venn Diagrams. In the left diagram below, hatch the regions representing A and B with and respectively. In the right diagram below, hatch the regions representing $\overline{A}$ and $\overline{B}$ with and respectively.





Then the region represents $\overline{A \cup B}$ in the left diagram, and the region represents $\overline{A} \cap \overline{B}$ in the right diagram. Since these regions occupy the same region in the respective diagrams, we infer that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Note 5.3.6. This argument depends on the fact that each possibility for membership in A and B is represented by a region in the diagram.

Proof using a truth table. The rows in the following table list all the possibilities for an element $x \in U$:

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in \overline{A \cup B}$	$x \in \overline{A}$	$x \in \overline{B}$	$x \in \overline{A} \cap \overline{B}$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since the columns under "$x \in \overline{A \cup B}$" and "$x \in \overline{A} \cap \overline{B}$" are the same, for any $x \in U$,

$$x \in \overline{A \cup B} \quad \Leftrightarrow \quad x \in \overline{A} \cap \overline{B}$$

no matter in which case we are. So $\overline{A \cup B} = \overline{A} \cap \overline{B}$. □

Direct proof. 1. Let $z \in U$.

2. 2.1.	Then	$z \in \overline{A \cup B}$	
2.2.	$\Leftrightarrow$	$z \notin A \cup B$	by the definition of $\overline{\cdot}$;
2.3.	$\Leftrightarrow$	$\sim\big((z \in A) \vee (z \in B)\big)$	by the definition of $\cup$;
2.4.	$\Leftrightarrow$	$(z \notin A) \wedge (z \notin B)$	by De Morgan's Laws for propositions;
2.5.	$\Leftrightarrow$	$(z \in \overline{A}) \wedge (z \in \overline{B})$	by the definition of $\overline{\cdot}$;
2.6.	$\Leftrightarrow$	$z \in \overline{A} \cap \overline{B}$	by the definition of $\cap$. $\square$

Example 5.3.7. Fix a universal set U. Show that $(A \cap B) \cup (A \setminus B) = A$ for all sets A, B.

Proof.

1.	$(A \cap B) \cup (A \setminus B) = (A \cap B) \cup (A \cap \overline{B})$	by the Set Difference Law;
2.	$= A \cap (B \cup \overline{B})$	by the Distributive Law;
3.	$= A \cap U$	by the Complement Law;
4.	$= A$	by the Identity Law. $\square$

Question 5.3.8. Is the following true? 5b

$$(A \setminus B) \cup (B \setminus C) = A \setminus C \quad \text{for all sets } A, B, C.$$

Definition 5.3.9. (1) Two sets A, B are *disjoint* if $A \cap B = \varnothing$.

(2) Sets $A_1, A_2, \ldots, A_n$ are *pairwise disjoint* or *mutually disjoint* if $A_i \cap A_j = \varnothing$ for all distinct $i, j \in \{1, 2, \ldots, n\}$.

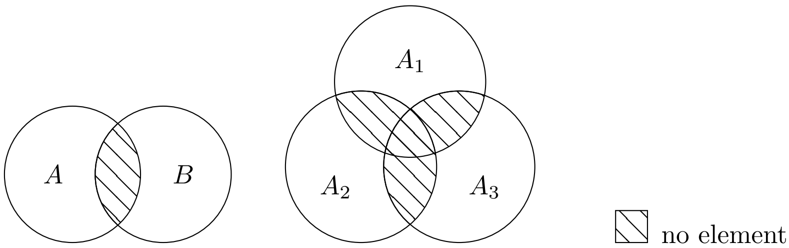


Figure 5.3: (Pairwise) disjoint sets

Example 5.3.10. The sets $A = \{1, 3, 5\}$ and $B = \{2, 4\}$ are (pairwise) disjoint. Note

$$|A \cup B| = |\{1, 2, 3, 4, 5\}| = 5 = 3 + 2 = |A| + |B|.$$

Theorem 5.3.11. (1) Let A, B be disjoint finite sets. Then $|A \cup B| = |A| + |B|$.

(2) Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Proof. Count the elements set by set. Every element in the union is counted exactly once because the sets are (pairwise) disjoint. $\square$

Theorem 5.3.12 (Inclusion–Exclusion Principle)**.** For all finite sets A, B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$