MA1101R

LIVE LECTURE 2

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Topics for week 2

1.4 Gaussian Elimination

1.5 Homogeneous Linear System

2.1 Introduction to Matrices

2.2 Matrix Operations

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Let's revise

- The solutions of a LS can be easily obtained from the $\boxed{\text{REF}}$ of its augmented matrix
- An augmented matrix has $\boxed{\text{many}}$ REF but $\boxed{\text{only one}}$ RREF
- A LS has no solution if and only if the last column of its REF is a $\boxed{\text{pivot column}}$
- In a REF,

  number of non-zero rows = number of $\boxed{\text{leading entries}}$

  = number of $\boxed{\text{pivot columns}}$
- In the REF of a consistent LS,

  if number of variables in LS = number of non-zero rows in REF,

  then the LS has $\boxed{\text{exactly one}}$ solution

  if number of variables in LS > number of non-zero rows in REF,

  then the LS has $\boxed{\text{infinitely many}}$ solutions

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Merging two augmented matrices

$$\begin{cases} x + 2y - 3z = 1 \\ 2x + 6y - 11z = 1 \\ x - 2y + 7z = 1 \end{cases} \qquad \begin{cases} x + 2y - 3z = 1 \\ 2x + 6y - 11z = 2 \\ x - 2y + 7z = 1 \end{cases}$$

no solution

infinitely many solutions

Same coefficients

You can perform G.E. (G.J.E.) on the two systems "simultaneously"

$$\left(\begin{array}{ccc|c|c} 1 & 2 & -3 & 1 & 1 \\ 2 & 6 & -11 & 1 & 2 \\ 1 & -2 & 7 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c|c} 1 & 2 & -3 & 1 & 1 \\ 0 & 2 & -5 & -1 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right)$$

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Linear System with 3 variables

REF		**Solutions**	**Geometrical interpretation**
3 leading entries	$\left(\begin{array}{ccc\|c} \otimes & & & \\ & \otimes & & \\ & & \otimes & \\ 0 & 0 & 0 & \end{array}\right)$	0 parameter	Intersect at 1 point
2 leading entries	$\left(\begin{array}{ccc\|c} \otimes & & & \\ & \otimes & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{array}\right)$	1 parameter	Intersect at a line
1 leading entry	$\left(\begin{array}{ccc\|c} \otimes & & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{array}\right)$	2 parameters	Intersect at a plane
0 leading entry	$\left(\begin{array}{ccc\|c} 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{array}\right)$	3 parameters	NA

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Linear Systems with "unknown" terms

$$\left(\begin{array}{cc|c} 1 & a & b \\ 0 & (a+1)(b-2) & (a+2)(b-1) \end{array}\right)$$

Determine the values of a and b so that the system has

i. no solution

ii. only 1 solution

iii. infinitely many solutions

Row 1 has no effect on the # solutions.

Only need to analyse row 2.

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Linear Systems with “unknown” terms

$$\left(\begin{array}{cc|c} 1 & a & b \\ 0 & (a+1)(b-2) & (a+2)(b-1) \end{array}\right)$$

- $\neq 0$ → Unique solution
- $= 0$ → No solution or infinite solutions
  - $(a + 2)(b - 1) \neq 0$ → No solution
  - $(a + 2)(b - 1) = 0$ → Infinite solutions

$$\left(\begin{array}{cc|c} 1 & a & b \\ 0 & (a+1)(b-2) & (a+2)(b-1) \end{array}\right)$$

Linear Systems with “unknown” terms

<u>One solution</u>: $(a + 1)(b - 2) \neq 0$

$(a + 1) \neq 0$ AND $(b - 2) \neq 0$

$a \neq -1$ AND $b \neq 2$

<u>Infinite solutions</u>: $(a + 1)(b - 2) = 0$, $(a + 2)(b - 1) = 0$

$(a + 1) = 0$ OR $(b - 2) = 0$ $(a + 2) = 0$ OR $(b - 1) = 0$

$a = -1$ OR $b = 2$ AND $a = -2$ OR $b = 1$

Simplify as

$a = -1$ AND $b = 1$ OR $b = 2$ AND $a = -2$

<u>No solution</u>: $(a + 1)(b - 2) = 0, (a + 2)(b - 1) \neq 0$

Linear Systems with "unknown" terms

<u>No solution</u>: $(a + 1)(b - 2) = 0, (a + 2)(b - 1) \neq 0$

$(a + 1) = 0$ OR $(b - 2) = 0$ $\quad$ $(a + 2) \neq 0$ AND $(b - 1) \neq 0$

$a = -1$ OR $b = 2$ $\quad$ AND $\quad$ $a \neq -2$ AND $b \neq 1$

Simplify as

$a = -1$ AND $b \neq 1$ $\quad$ OR $\quad$ $b = 2$ AND $a \neq -2$

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Linear Systems with "unknown" terms

Exercise 1 Q24

$$\left(\begin{array}{ccc|c} a & a & a & c \\ 0 & b & b & a \\ 0 & 0 & c & b \end{array}\right)$$

Determine the values of a, b, c so that the system has

i. no solution
ii. only 1 solution
iii. infinitely many solutions

All three rows have effect on the # solutions

Depending on whether a, b, c are 0 or not.

There are 8 cases

$$\left(\begin{array}{ccc|c} a & a & a & c \\ 0 & b & b & a \\ 0 & 0 & c & b \end{array}\right)$$

Linear Systems with “unknown” terms

i. All are not 0

only 1 solution

ii. Exactly one 0

no solution

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & c \\ 0 & b & b & 0 \\ 0 & 0 & c & b \end{array}\right) \quad \left(\begin{array}{ccc|c} a & a & a & c \\ 0 & 0 & 0 & a \\ 0 & 0 & c & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} a & a & a & 0 \\ 0 & b & b & a \\ 0 & 0 & 0 & b \end{array}\right)$$

iii. Exactly two 0

no solution

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} a & a & a & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & b & b & 0 \\ 0 & 0 & 0 & b \end{array}\right)$$

iv. All are 0

infinite solutions

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases}$$

Homogeneous system

Linear Systems

Homogeneous system

(always consistent)

Non-homogeneous system

(may be inconsistent)

Trivial Solution (zero solution)

(always exists)

Non-trivial Solutions (with parameters)

(may not exist)

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Summary

- A linear system that has the zero solution is called a $\boxed{\text{homogeneous}}$ system.
- A homogeneous system is always consistent, as it always has the $\boxed{\text{trivial}}$ solution.
- If a homogeneous system has a $\boxed{\text{non-trivial}}$ solution, then it has infinitely many solutions.
- A homogeneous system with more variables than equations has $\boxed{\text{infinitely many}}$ solutions.
- A homogeneous system with more equations than variables has $\boxed{\text{one or many}}$ solutions.

True or False

F (i) The unique solution of a linear system is called the trivial solution

F (ii) The trivial solution of a linear system is the unique solution

Trivial solution $\neq$ Unique solution

We do not refer to solutions for a non-homogeneous system as trivial or non-trivial.

Matrices

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Terms associated to matrix

- rows (m)
- columns (n)
- size (m x n)
- entries (a_{ij})

Special Matrices

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Identity matrix

Zero matrix

Diagonal matrix

Triangular matrix

Symmetric matrix

Scalar matrix

Matrix Operations

Matrix Operations	Operation	Requirement
	Matrix addition/subtraction $\mathbf{A} \pm \mathbf{B}$	same sizes
	Scalar multiplication $c\mathbf{A}$	any
	Matrix multiplication $\mathbf{AB}$	compatible sizes
	Matrix transpose $\mathbf{A}^t$	any
	Matrix power $\mathbf{A}^n$	square matrix

Matrix Multiplication (row x column)

$$\begin{array}{cc} & A \\ \begin{array}{c} a_1 \\[4pt] a_2 \\[4pt] a_3 \end{array} & \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \end{array} \begin{array}{c} B \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\ \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \end{array} = \begin{array}{c} AB \\ \begin{pmatrix} \underset{a_1b_1}{5} & \underset{a_1b_2}{7} & \underset{a_1b_3}{9} \\ \underset{a_2b_1}{14} & \underset{a_2b_2}{19} & \underset{a_2b_3}{24} \\ \underset{a_3b_1}{-9} & \underset{a_3b_2}{-12} & \underset{a_3b_3}{-15} \end{pmatrix} \end{array}$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} \qquad \mathbf{B} = (\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n) \Rightarrow \mathbf{AB} = \begin{pmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \cdots & \mathbf{a}_1\mathbf{b}_n \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \cdots & \mathbf{a}_2\mathbf{b}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_m\mathbf{b}_1 & \mathbf{a}_m\mathbf{b}_2 & \cdots & \mathbf{a}_m\mathbf{b}_n \end{pmatrix}$$

(i, j)-entry of matrix multiplication

$\boldsymbol{A} = (a_{ij})_{m\times p}$ and $\boldsymbol{B} = (b_{ij})_{p\times n}$

(i, j)-entry of $\boldsymbol{AB} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$

$(1, 2)$-entry of $\boldsymbol{AB} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1p}b_{p2}$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

Matrix Multiplication

$$\boldsymbol{A} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

row matrix

$$\boldsymbol{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

column matrix

What is ***BA***? n x n matrix

1. $\begin{pmatrix} b_1a_1 & + & b_2a_2 & + & \cdots & + & b_na_n \end{pmatrix}$ 1 x 1 matrix

2. $\begin{pmatrix} b_1a_1 & b_2a_2 & \cdots & b_na_n \end{pmatrix}$ 1 x n matrix

3. $\boxed{\begin{pmatrix} b_1a_1 & b_1a_2 & \dots & b_1a_n \\ b_2a_1 & b_2a_2 & \dots & b_2a_n \\ \vdots & \vdots & & \vdots \\ b_na_1 & b_na_2 & \dots & b_na_n \end{pmatrix}}$

4. $\begin{pmatrix} b_1a_1 & b_2a_1 & \dots & b_na_1 \\ b_1a_2 & b_2a_2 & \dots & b_na_2 \\ \vdots & \vdots & & \vdots \\ b_1a_n & b_2a_n & \dots & b_na_n \end{pmatrix}$

n x n matrix

Matrix Multiplication (matrix x column)

$$\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{B} & & \boldsymbol{AB} \\ \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} & = & \begin{pmatrix} 5 & 7 & 9 \\ 14 & 19 & 24 \\ -9 & -12 & -15 \end{pmatrix} \\ & \boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \boldsymbol{b}_3 & & \boldsymbol{Ab}_1 \quad \boldsymbol{Ab}_2 \quad \boldsymbol{Ab}_3 \end{array}$$

$\boldsymbol{A}$(j th column of $\boldsymbol{B}$) = j th column of $\boldsymbol{AB}$

$$\boxed{\mathbf{AB} = (\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \cdots \quad \mathbf{Ab}_n)}$$

Matrix Multiplication (row x matrix)

$$\begin{array}{c} \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{array} \overset{\boldsymbol{A}}{\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}} \overset{\boldsymbol{B}}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} = \overset{\boldsymbol{AB}}{\begin{pmatrix} 5 & 7 & 9 \\ 14 & 19 & 24 \\ -9 & -12 & -15 \end{pmatrix}} \begin{array}{c} \boldsymbol{a}_1\boldsymbol{B} \\ \boldsymbol{a}_2\boldsymbol{B} \\ \boldsymbol{a}_3\boldsymbol{B} \end{array}$$

(*i* th row of ***A***) ***B*** = *i* th row of ***AB***

$$\boxed{\mathbf{AB} = \begin{pmatrix} \mathbf{a}_1\mathbf{B} \\ \mathbf{a}_2\mathbf{B} \\ \vdots \\ \mathbf{a}_m\mathbf{B} \end{pmatrix}}$$

What is **A**?

$$\mathbf{A}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}9\\3\\8\end{pmatrix} \qquad \mathbf{A}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}5\\7\\2\end{pmatrix} \qquad \mathbf{A}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\2\\4\end{pmatrix}$$

$\mathbf{Ab}_1$ $\mathbf{Ab}_2$ $\mathbf{Ab}_3$

stacking

$$\mathbf{AB} = (\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \cdots \quad \mathbf{Ab}_n)$$

splitting

$$\mathbf{A}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}9 & 5 & 0\\3 & 7 & 2\\8 & 2 & 4\end{pmatrix}$$

Production Cost Matrix

Three products A, B, C

produce: 3000 of A
2000 of B
5000 of C

Production cost per item

	A	B	C
Raw materials	0.10	0.30	0.15
Labor	0.30	0.40	0.25
Overhead & Misc.	0.10	0.20	0.13

$$\begin{pmatrix} 0.10 & 0.30 & 0.15 \\ 0.30 & 0.40 & 0.25 \\ 0.10 & 0.20 & 0.13 \end{pmatrix} \begin{pmatrix} 3000 \\ 2000 \\ 5000 \end{pmatrix} = \begin{pmatrix} 1650 \\ 2950 \\ 1350 \end{pmatrix} \begin{array}{l} \text{total raw material cost} \\ \text{total labor cost} \\ \text{total overhead cost} \end{array}$$

Easier to manipulate the data

True or False

Suppose **A, B, C** are square matrices of the same size.

1. $\boldsymbol{AB} = \boldsymbol{BA}$
2. If $\boldsymbol{AB} = \mathbf{0}$, then $\boldsymbol{A} = \mathbf{0}$ or $\boldsymbol{B} = \mathbf{0}$
3. If $\boldsymbol{A}^2 = \mathbf{0}$, then $\boldsymbol{A} = \mathbf{0}$
4. If $\boldsymbol{A} = \boldsymbol{B}$, then $\boldsymbol{CA} = \boldsymbol{BC}$
5. If $\boldsymbol{AC} = \boldsymbol{BC}$, then $\boldsymbol{A} = \boldsymbol{B}$
6. $(\boldsymbol{AB})^n = \boldsymbol{A}^n \boldsymbol{B}^n$

All are false

Matrix Equation Form of Linear System

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$$

a solution

$x_1 = 1,\ x_2 = 0,\ x_3 = -1,\ x_4 = 0$

$\boldsymbol{Ax = b}$ linear system

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

the trivial solution

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0$

$\boldsymbol{Au = b}$ linear system substituted with solution

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

True or False

1. $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$
2. $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{AB}_1 + \boldsymbol{AB}_2$

   $(\boldsymbol{C}_1 + \boldsymbol{C}_2)\,\boldsymbol{A} = \boldsymbol{C}_1\boldsymbol{A} + \boldsymbol{C}_2\boldsymbol{A}$
3. $c(\boldsymbol{AB}) = (c\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(c\boldsymbol{B})$

Suppose $\boldsymbol{u}$ is a solution of the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$. Then

a. $2\boldsymbol{u}$ is a solution of $\boldsymbol{Ax} = \boldsymbol{0}$.

b. $\boldsymbol{u}$ is a solution of $\boldsymbol{BAx} = \boldsymbol{0}$.
(**B** is any matrix compatible with **A**)

(a) is true: $\mathbf{A}(2\mathbf{u}) = 2(\mathbf{Au}) = 2(\mathbf{0}) = \mathbf{0}$ using property 3

(b) is true: $\mathbf{BA}(\mathbf{u}) = \mathbf{B}(\mathbf{Au}) = \mathbf{B}(\mathbf{0}) = \mathbf{0}$ using property 1

Transpose

Given $\boldsymbol{I}$ is n x n identity matrix; $\boldsymbol{A}$ and $\boldsymbol{B}$ are m x n matrices.

Is the following true?

$$\boxed{(3\boldsymbol{I} + \boldsymbol{A}^T\boldsymbol{B})^T = 3\boldsymbol{I} + \boldsymbol{B}^T\boldsymbol{A}}$$

$$(3\boldsymbol{I})^T + (\boldsymbol{A}^T\boldsymbol{B})^T = 3(\boldsymbol{I})^T + (\boldsymbol{B})^T(\boldsymbol{A}^T)^T = 3\boldsymbol{I} + \boldsymbol{B}^T\boldsymbol{A}$$

1. $(\boldsymbol{A}^T)^T = \boldsymbol{A}$
2. If $\boldsymbol{B}$ is an m x n matrix, then $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$.
3. If a is a scalar, then $(a\boldsymbol{A})^T = a\boldsymbol{A}^T$.
4. If $\boldsymbol{B}$ is an n x p matrix, then $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$.