1). $\underline{\underline{48}}$

$$f(x) = ax^3 - bx^2$$

$$f(1) = 8 \Rightarrow 8 = a - b \text{ ---------- } ①$$

$$f'(x) = 3ax^2 - 2bx$$

$$f''(x) = 6ax - 2b = 2(3ax - b)$$

$$f''(1) = 0 \Rightarrow 3a - b = 0 \text{ ----- } ②$$

$$① + ② \Rightarrow 3a - b + 8 = a - b$$

$$\Rightarrow a = -4$$

$$② \Rightarrow b = 3a = -12$$

$$\therefore ab = (-4) \times (-12) = \underline{\underline{48}}$$

2). $\underline{\underline{6.28}}$

$$(9,7,4)-(1,1,2)=(8,6,2)$$

$$\therefore S:\ 8x+6y+2z+k=0$$

$$(2,6,9)\in S \Rightarrow 16+36+18+k=0$$

$$\Rightarrow k=-70$$

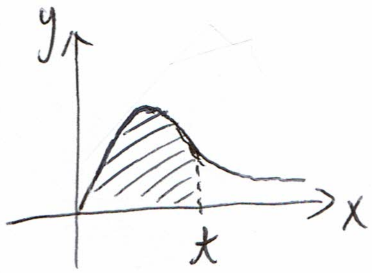
$$d=\frac{|120+12+2-70|}{\sqrt{8^2+6^2+2^2}}=\frac{64}{\sqrt{104}}$$

$$=\frac{64\times\sqrt{104}}{104}$$

$$=6.275716\ldots$$

$$\approx \underline{\underline{6.28}}$$

3). $\underline{\underline{1.75}}$



$$V(t) = \int_0^t \pi \left(\frac{\sqrt{x}}{1+x^2} \right)^2 dx$$

$$= \pi \int_0^t \frac{x}{(1+x^2)^2} dx$$

$$= \frac{\pi}{2} \int_0^t \frac{d(1+x^2)}{(1+x^2)^2}$$

$$= \frac{\pi}{2} \left[-\frac{1}{1+x^2} \right]_0^t$$

$$= \frac{\pi}{2} \left(1 - \frac{1}{1+t^2} \right)$$

$$\therefore \lim_{t \to \infty} V(t) = \frac{\pi}{2}$$

$$\frac{V(a)}{\lim_{t \to \infty} V(t)} = 1 - \frac{1}{1+a^2} = \frac{1521}{2020}$$

$$\frac{1}{1+a^2} = \frac{499}{2020}$$

$$a = \sqrt{\frac{1521}{499}} = 1.7458\ldots$$

$$\simeq \underline{\underline{1.75}}$$

4). $\underline{\underline{24.43}}$

$$\frac{dQ}{dt} = -kQ$$

$$\Rightarrow Q = Q_0 e^{-kt} = Q_0 e^{-\frac{\ln 2}{a} t}$$

$$2020 = Q_0 e^{-\frac{\ln 2}{a} \times 10}$$

$$1521 = Q_0 e^{-\frac{\ln 2}{a} \times 20}$$

$$\frac{2020}{1521} = e^{\frac{-10 \ln 2 + 20 \ln 2}{a}}$$

$$\ln 2020 - \ln 1521 = \frac{-10 \ln 2 + 20 \ln 2}{a}$$

$$a = \frac{-10 \ln 2 + 20 \ln 2}{\ln 2020 - \ln 1521} = \frac{6.931471\cdots}{7.610852\cdots - 7.32712\cdots}$$

$$= 24.429\cdots$$

$$\simeq \underline{\underline{24.43}}$$

5). $\underline{\underline{2183.93}}$

$$\frac{d}{dx}\int_{\sqrt{x}}^{1521} f(t)\,dt$$

$$= \frac{d}{dx}\int_{1521}^{\sqrt{x}} -f(t)\,dt$$

$$= \left(-f(\sqrt{x})\right)\frac{1}{2\sqrt{x}}$$

$$\therefore \; -\frac{f(\sqrt{x})}{2\sqrt{x}} = -\sqrt{x}\,e^{x}$$

$$f(\sqrt{x}) = 2xe^{x}$$

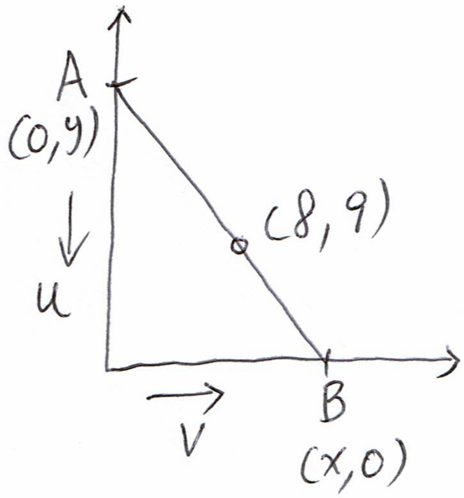
$$f(x) = 2x^2e^{x^2}$$

$$f'(x) = 4xe^{x^2} + 4x^3e^{x^2}$$

$$f'(2) = 40e^4 = 2183.9260\cdots$$

$$\simeq \underline{\underline{2183.93}}$$

6).



$$\frac{-y}{x-0} = \frac{-9}{x-8}, \quad x>8$$

$$y = \frac{9x}{x-8}, \quad x>8$$

$$T = \frac{y}{u} + \frac{x}{v} = \frac{9x}{u(x-8)} + \frac{x}{v}, \quad x>8$$

$$\frac{dT}{dx} = \frac{9}{u}\left(\frac{x-8-x}{(x-8)^2}\right) + \frac{1}{v} = \frac{-72v + u(x-8)^2}{uv(x-8)^2}$$

$$= \frac{(x-8)^2 - 72v/u}{v(x-8)^2} = \frac{\left(x-8-\sqrt{\frac{72v}{u}}\right)\left(x-8+\sqrt{\frac{72v}{u}}\right)}{v(x-8)^2}$$

$\because x>8$

$\therefore T$ attains global min when $x = 8+\sqrt{\frac{72v}{u}}$

$\frac{dT}{dx}$	$+$		$-$		$+$
		$8-\sqrt{\frac{72v}{u}}$	8	$8+\sqrt{\frac{72v}{u}}$	
		↑ out ($\because x>8$)		↑ min	

$$\min T = \frac{72 + 9\sqrt{\frac{72v}{u}}}{2\left(\sqrt{\frac{72v}{u}}\right)} + \frac{8+\sqrt{\frac{72v}{u}}}{v}$$

$$u=2,\ v=3 \Rightarrow \min T = \frac{72+9\sqrt{108}}{2\sqrt{108}} + \frac{8+\sqrt{108}}{3}$$

$$= \frac{432\sqrt{108} + 4644}{648} = 14.0948\ldots$$

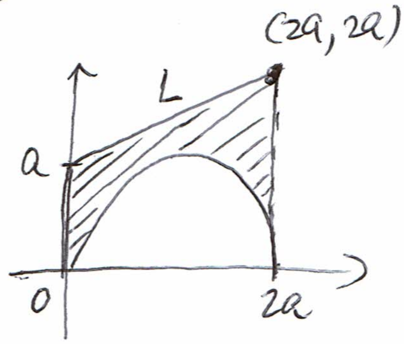
$$\approx \underline{\underline{14.09}}$$

7).

$$y^2 = 2ax \Rightarrow 2yy' = 2a \Rightarrow 4ay' = 2a \text{ at } (2a, 2a)$$

$$\Rightarrow y' = \frac{1}{2} \text{ at } (2a, 2a)$$

$$\therefore L: \frac{y-2a}{x-2a} = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + a$$



$$x^2 + y^2 = 2ax$$

$$\Rightarrow x^2 - 2ax + a^2 + y^2 = a^2$$

$$\Rightarrow (x-a)^2 + y^2 = a^2$$

$$\iint_R y\,dA = \int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\frac{1}{2}x+a} y\,dy\,dx$$

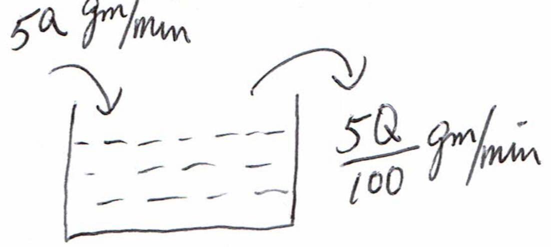
$$= \int_0^{2a} \frac{1}{2}\left\{\left(\frac{1}{2}x+a\right)^2 - (2ax - x^2)\right\} dx$$

$$= \frac{1}{8}\left[4a^2x - 2ax^2 + \frac{5}{3}x^3\right]_0^{2a} = \frac{5}{3}a^3$$

$$\therefore \frac{5}{3}a^3 = 898 \Rightarrow a \simeq 8.1372\cdots$$

$$\simeq \underline{\underline{8.14}}$$

8)



$$\frac{dQ}{dt} = 5a - \frac{5Q}{100}$$

$$= -\frac{5}{100}(Q - 100a)$$

$$\frac{dQ}{Q-100a} = -\frac{1}{20}\,dt$$

$t = 0 \Rightarrow Q = 10 \times 100 = 1000\,\text{gm}$

$t = 32 \Rightarrow Q = 15.21 \times 100 = 1521\,\text{gm}$

$$\therefore \int_{1000}^{1521} \frac{dQ}{Q-100a} = -\frac{1}{20}\int_0^{32} dt = -\frac{8}{5}$$

$$\therefore \ln \frac{1521 - 100a}{1000 - 100a} = -\frac{8}{5}$$

$$1521 - 100a = e^{-\frac{8}{5}}(1000 - 100a)$$

$$1521 - 1000e^{-\frac{8}{5}} = 100(1 - e^{-8/5})a$$

$$a = \frac{15.21 - 10e^{-8/5}}{1 - e^{-8/5}}$$

$$\approx 16.5280\ldots \approx \underline{\underline{16.53}}$$

9). $F(x, y, z) = f(x^2+y^2+z^2)$

$\nabla F(x, y, z) = (2f'(x^2+y^2+z^2))\,(x,\ y,\ z)$

$\nabla F(a, b, c) = (2f'(a^2+b^2+c^2))\,(a,\ b,\ c)$

$$\vec{v} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{(1, 5, 21) - (20, 2, 0)}{\|(1, 5, 21) - (20, 2, 0)\|} = \frac{(-19, 3, 21)}{\sqrt{811}}$$

$$D_{\vec{v}} F(a, b, c) = \nabla F(a, b, c) \cdot \vec{v}$$

$$= \frac{2f'(a^2+b^2+c^2)}{\sqrt{811}}(-19a+3b+21c)$$

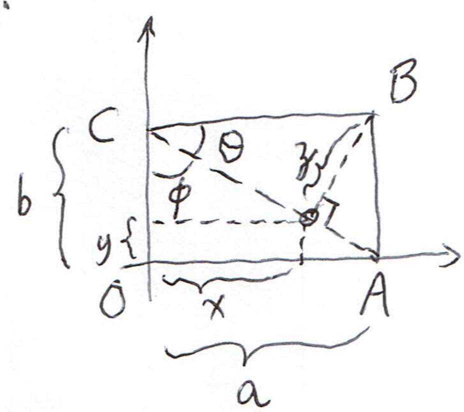
$a=6,\ b=5,\ c=6$

$$D_{\vec{v}} F(6, 5, 6) = \frac{2f'(97)}{\sqrt{811}}(-19\times 6+3\times 5+21\times 6)$$

$$= 88$$

$$\Rightarrow f'(97) \approx \underline{\underline{46.41}}$$

10).



Let $c = AC = \sqrt{a^2+b^2}$

$$z = a\sin\theta = \frac{ab}{\sqrt{a^2+b^2}}$$

$$x = a\cos\theta\sin\phi$$

$$= a\frac{a}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}} = \frac{a^3}{a^2+b^2}$$

$$y = b - a\cos\theta\cos\phi = b - a\frac{a}{\sqrt{a^2+b^2}}\frac{b}{\sqrt{a^2+b^2}}$$

$$= b - \frac{a^2 b}{a^2+b^2} = \frac{b^3}{a^2+b^2}$$

$$\therefore B' = \left(\frac{a^3}{a^2+b^2}, \frac{b^3}{a^2+b^2}, \frac{ab}{\sqrt{a^2+b^2}}\right)$$

$a=20,\ b=21 \Rightarrow a^2+b^2 = 841 \Rightarrow \sqrt{a^2+b^2} = 29$

$$\therefore B' = \left(\frac{8000}{841}, \frac{9261}{841}, \frac{420}{29}\right)$$

$$d = \frac{\left|6\times\frac{8000}{841} + 17\times\frac{9261}{841} + 6\times\frac{420}{29}\right|}{\sqrt{6^2+17^2+6^2}}$$

$$= 17.4301\ldots$$

$$\approx \underline{\underline{17.43}}$$

11).

$$\begin{array}{cccccc} & a+d & a+2d & a+3d & a+4d & a+5d \\ & \| & \| & \| & \| & \| \\ a & u & v & b & w & c \\ & & & \| & & \| \\ & & & ar & & ar^2 \end{array}$$

$$a+3d=ar \Rightarrow 3d=a(r-1)$$
$$a+5d=ar^2 \Rightarrow 5d=a(r-1)(r+1)$$

$\because$ S converges, $\therefore r \neq \pm 1$

$$\therefore \frac{3}{5}=\frac{1}{r+1} \Rightarrow 3r+3=5$$
$$\Rightarrow r=\frac{2}{3}$$

$$\therefore S=\frac{a}{1-r}=\frac{a}{1-2/3}=\frac{a}{1/3}$$

$$\Rightarrow S=3a.$$

$\because a, u, v, b, w, c$ is an A.P.

$$\therefore a+b+c+u+v+w=3(a+c)$$
$$=3(a+ar^2)=3a(1+r^2)=\frac{13S}{9}$$
$$=\frac{13\times 639}{9}$$
$$=\underline{\underline{923}}$$

12).

$$(\|\overrightarrow{OQ}\|)(\|\overrightarrow{OP}\|)=a^2$$

$$\Rightarrow \sqrt{u^2+v^2+w^2}\sqrt{x^2+y^2+z^2}=a^2$$

$$\overrightarrow{OQ}=t\overrightarrow{OP}\Rightarrow(u,v,w)=(tx,ty,tz)$$

$$\therefore t(x^2+y^2+z^2)=a^2$$

$$\therefore t=\frac{a^2}{x^2+y^2+z^2}$$

$$f(x,y,z)=u+v+w=\frac{a^2(x+y+z)}{x^2+y^2+z^2}$$

$$\nabla f=\frac{a^2}{(x^2+y^2+z^2)^2}\Big(-x^2+y^2+z^2-2xy-2xz,\; x^2-y^2+z^2-2xy-2yz,\; x^2+y^2-z^2-2xz-2yz\Big)$$

$$\nabla f(1,2,3)=\frac{a^2}{196}(2,-10,-22)$$

$$\|\nabla f(1,2,3)\|=\frac{a^2}{196}\sqrt{2^2+10^2+22^2}=\frac{a^2\sqrt{588}}{196}$$

$$\frac{a^2\sqrt{588}}{196}=689\Rightarrow a=\sqrt{\frac{689\times196\times\sqrt{588}}{588}}$$

$$=74.6265\cdots$$

$$\approx \underline{\underline{74.63}}$$

13).

$$z = f(x, y, w) \qquad w = g(x, y, z)$$

$$z_x = f_x + f_w w_x \text{ --- ①} \qquad w_x = g_x + g_z z_x \text{ --- ②}$$

$$g_z ① + ② \Rightarrow g_z z_x + w_x = g_z f_x + g_z f_w w_x + g_x + g_z z_x$$

$$w_x (1 - g_z f_w) = g_z f_x + g_x$$

$$\therefore w_x = \frac{g_z f_x + g_x}{1 - g_z f_w}$$

$$① + f_w ② \Rightarrow z_x + f_w w_x = f_x + f_w w_x + f_w g_x + f_w g_z z_x$$

$$z_x (1 - g_z f_w) = g_x f_w + f_x$$

$$\therefore z_x = \frac{g_x f_w + f_x}{1 - g_z f_w}$$

$$\therefore \frac{\partial z}{\partial x} + \frac{\partial w}{\partial x} = \frac{g_x f_w + f_x + g_z f_x + g_x}{1 - g_z f_w}$$

$$= \frac{g_x (f_w + 1) + f_x (g_z + 1)}{1 - g_z f_w}$$

$$= \frac{2 \times (\frac{1}{2} + 1) + 5 \times (\frac{5}{3} + 1)}{1 - \frac{5}{3} \times \frac{1}{2}} = \underline{\underline{98}}$$

14).

$$f(x) = 8 + ax + bx^2 + cx^3 + kx^4 + \cdots$$

$$\Rightarrow 8 = f(0),\ a = \frac{f'(0)}{1!},\ b = \frac{f''(0)}{2!},\ c = \frac{f'''(0)}{3!},\ k = \frac{f^{(4)}(0)}{4!}$$

Given $f''(x) = (f(x))^2 \Rightarrow f''(0) = (f(0))^2 = 8^2 = 64$

$$f'''(x) = 2f(x)f'(x)$$

$$f^{(4)}(x) = 2(f'(x))^2 + 2f(x)f''(x)$$

$$\therefore k = \frac{f^{(4)}(0)}{4!} = \frac{2(f'(0))^2 + 2f(0)f''(0)}{24} = \frac{a^2 + 512}{12}$$

$$\therefore a = \sqrt{12k - 512}$$

$$\lim_{x\to\infty}\left(\frac{f(\frac{1}{x})}{f(0)}\right)^x = \lim_{t\to 0}\left(\frac{f(t)}{f(0)}\right)^{1/t} \quad (\text{let } t = \tfrac{1}{x})$$

$$= \lim_{t\to 0} e^{\ln\left(\frac{f(t)}{f(0)}\right)^{1/t}} = e^{\lim_{t\to 0}\frac{\ln f(t) - \ln f(0)}{t}}$$

$$= e^{\lim_{t\to 0}\frac{\frac{1}{f(t)}f'(t)}{1}} = e^{\frac{f'(0)}{f(0)}} = e^{\frac{a}{8}}$$

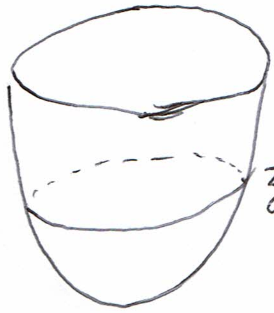
$$= e^{\sqrt{12k-512}/8}$$

$$k = 129 \Rightarrow \lim_{x\to\infty}\left(\frac{f(\frac{1}{x})}{f(0)}\right)^x = e^{\sqrt{12\times 129 - 512}/8}$$

$$= 55.8890\ldots$$

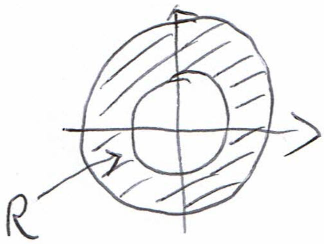
$$\approx \underline{\underline{55.89}}$$

15).



$z=a \Rightarrow x^2+y^2=2a=(\sqrt{2a})^2$

$z=1 \Rightarrow x^2+y^2=2=(\sqrt{2})^2$



$$z=\frac{1}{2}(x^2+y^2)$$

$$z_x=x,\quad z_y=y$$

$$\sqrt{1+{z_x}^2+{z_y}^2}=\sqrt{1+x^2+y^2}$$

$$A=\iint_R \sqrt{1+x^2+y^2}\,dA=\int_0^{2\pi}\int_{\sqrt{2}}^{\sqrt{2a}}\sqrt{1+r^2}\,r\,dr\,d\theta$$

$$=\pi\int_{\sqrt{2}}^{\sqrt{2a}}\sqrt{1+r^2}\,d(1+r^2)$$

$$=\frac{2}{3}\pi\left[(1+r^2)^{3/2}\right]_{\sqrt{2}}^{\sqrt{2a}}=\frac{2}{3}\pi\left((1+2a)^{3/2}-3^{3/2}\right)$$

$$\therefore a=\frac{1}{2}\left\{\left(\frac{3A}{2\pi}+3^{3/2}\right)^{2/3}-1\right\}$$

$$A=828 \Rightarrow a=\frac{1}{2}\left\{\left(\frac{3\times 828}{2\pi}+3^{3/2}\right)^{2/3}-1\right\}$$

$$=26.6684\cdots$$

$$\approx \underline{\underline{26.67}}$$