MA1101R

LIVE LECTURE 1

Topics for week 1

1.1 Linear Systems and their solutions

1.2 Elementary Row Operations

1.3 Row-Echelon Forms

1.4 Gaussian Elimination

Linear Systems – Different Expressions

$$\begin{array}{rcrcrcrcr} x & + & y & + & z & + & w & = & 1 \\ 2x & - & y & + & 3z & + & 5w & = & 2 \\ x & + & 2y & + & 7z & + & 0w & = & 5 \\ 0x & - & 6y & + & 2z & + & 9w & = & 0 \\ 5x & + & 2y & - & 4z & + & 7w & = & 8 \end{array}$$

standard form

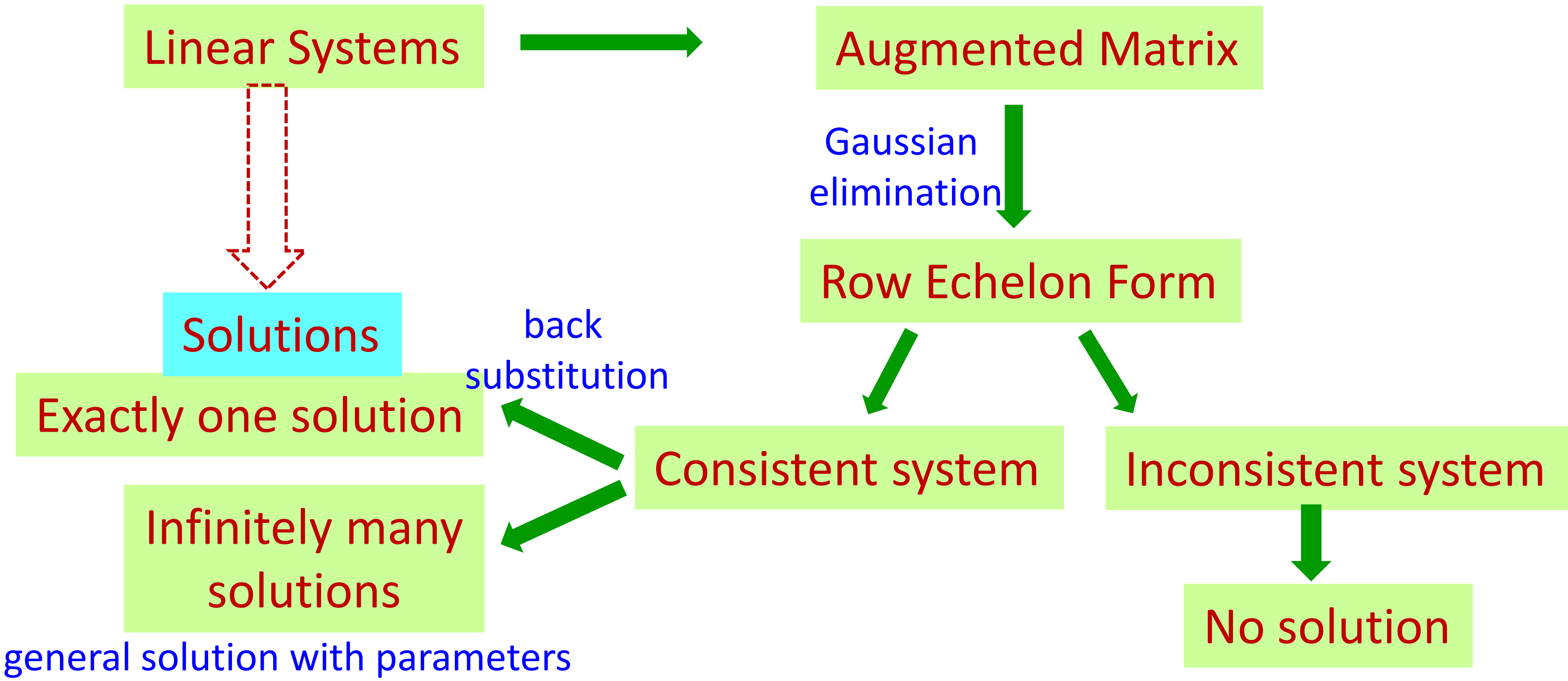
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 5 \\ 1 & 2 & 7 & 0 \\ 0 & -6 & 2 & 9 \\ 5 & 2 & -4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 0 \\ 8 \end{pmatrix}$$

matrix equation form

$$\mathrm{x} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 5 \end{pmatrix} + \mathrm{y} \begin{pmatrix} 1 \\ -1 \\ 2 \\ -6 \\ 2 \end{pmatrix} + \mathrm{z} \begin{pmatrix} 1 \\ 3 \\ 7 \\ 2 \\ -4 \end{pmatrix} + \mathrm{w} \begin{pmatrix} 1 \\ 5 \\ 0 \\ 9 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 0 \\ 8 \end{pmatrix}$$

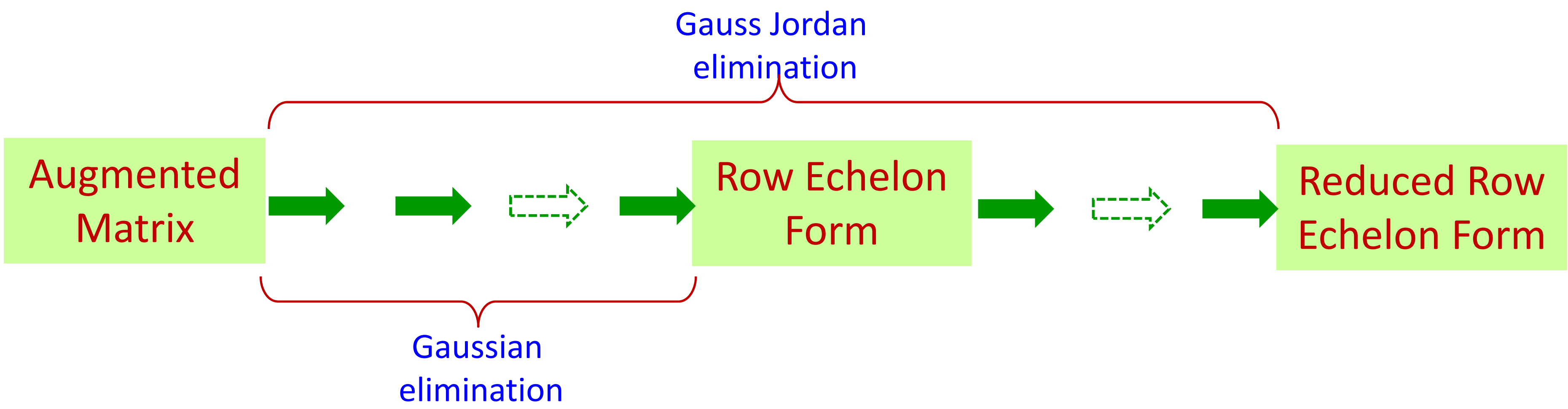
vector equation form

Overview


Linear Systems
Augmented Matrix
Gaussian elimination
Row Echelon Form
Solutions
back substitution
Exactly one solution
Consistent system
Inconsistent system
Infinitely many solutions
general solution with parameters
No solution

Elementary Row Operations

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.



Elementary Row Operations

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ \boxed{2 \quad 3 \quad 0 \quad -2} \\ 3 & 3 & 6 & -9 \end{pmatrix} \xrightarrow{\text{Interchange row 1 and 3}} \mathbf{B} = \begin{pmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Multiply row 3 by 1/3

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{pmatrix}$$

-2 x row 2 and add to row 3

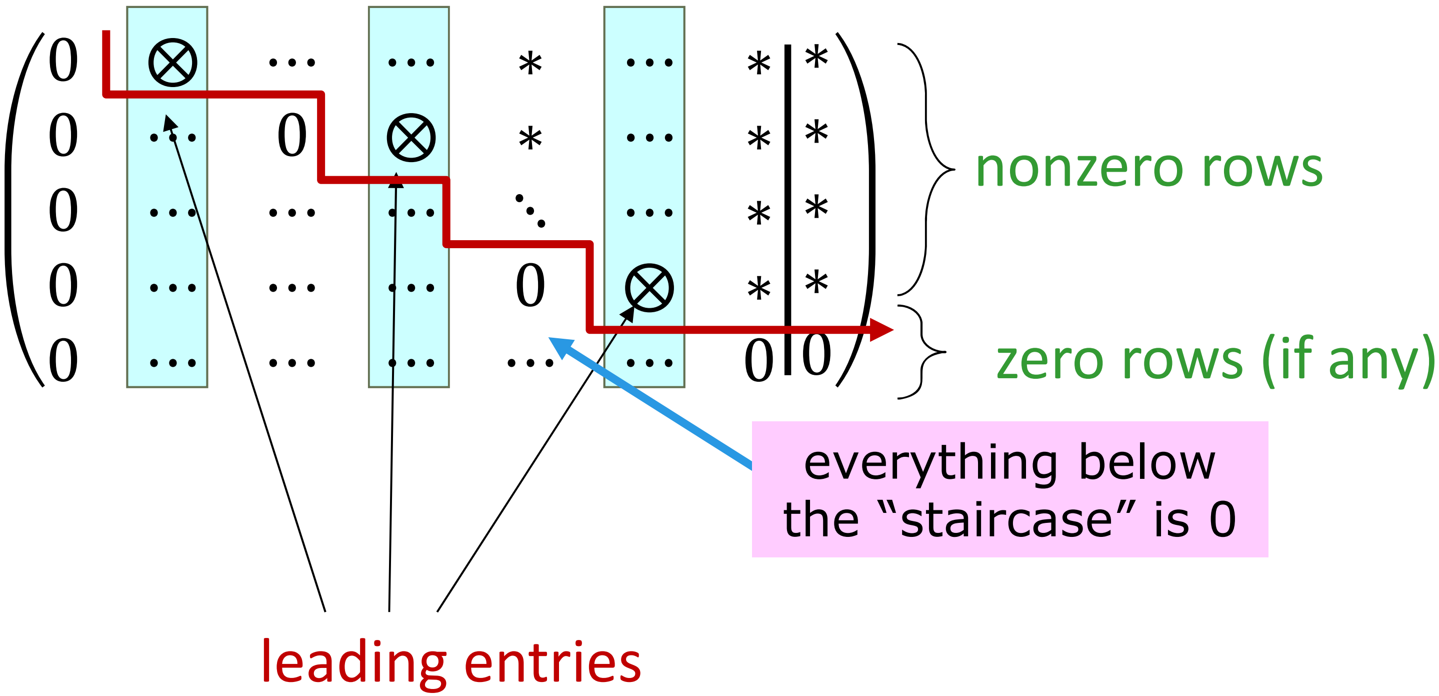
$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{pmatrix}$$

only row 3 is changed

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Row Echelon Form

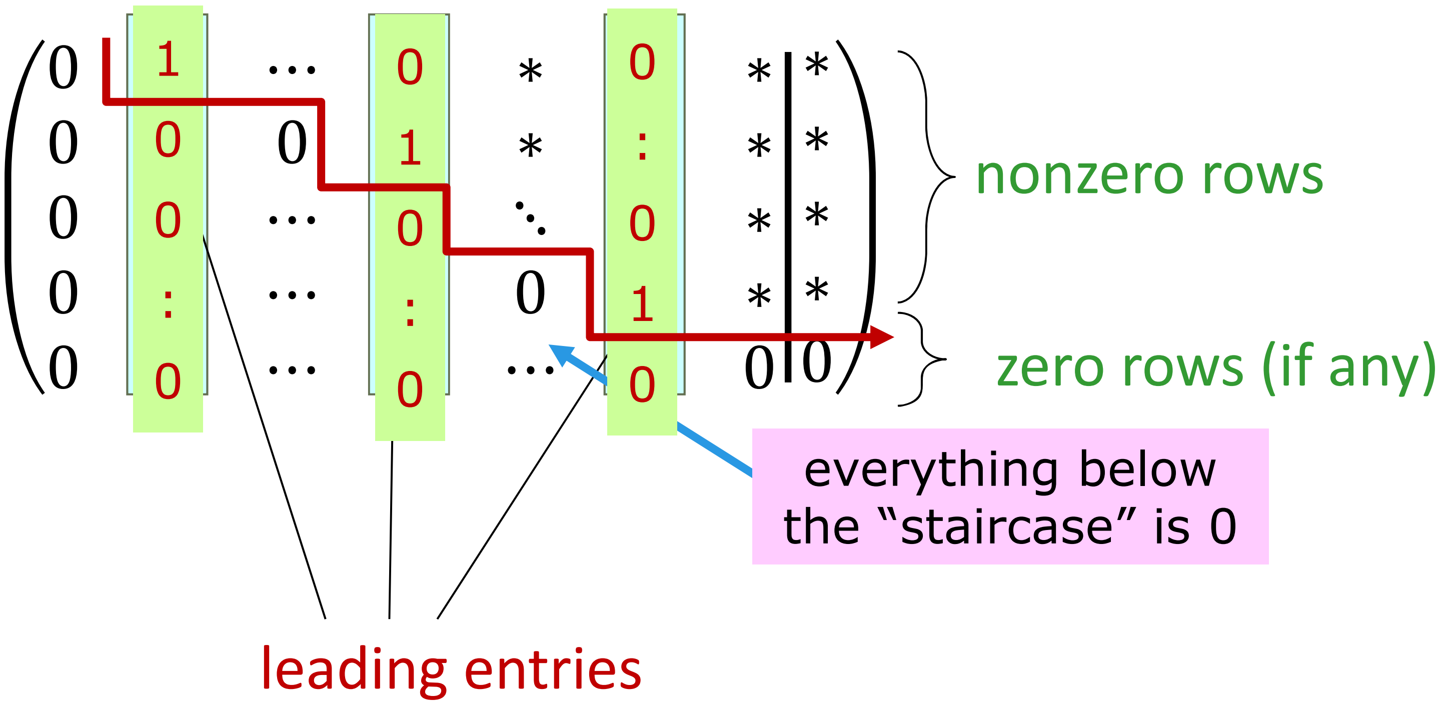
columns that contain leading entries called pivot columns



$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced Row Echelon Form

columns that contain leading entries called pivot columns



True or False

In a row echelon form,

1. The number of pivot columns is the same as the number of leading entries
2. The number of pivot columns is the same as the number of non-zero rows

$$\left(\begin{array}{ccccccc|c}
0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\
0 & \cdots & 0 & \otimes & * & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0
\end{array}\right)$$

True or False

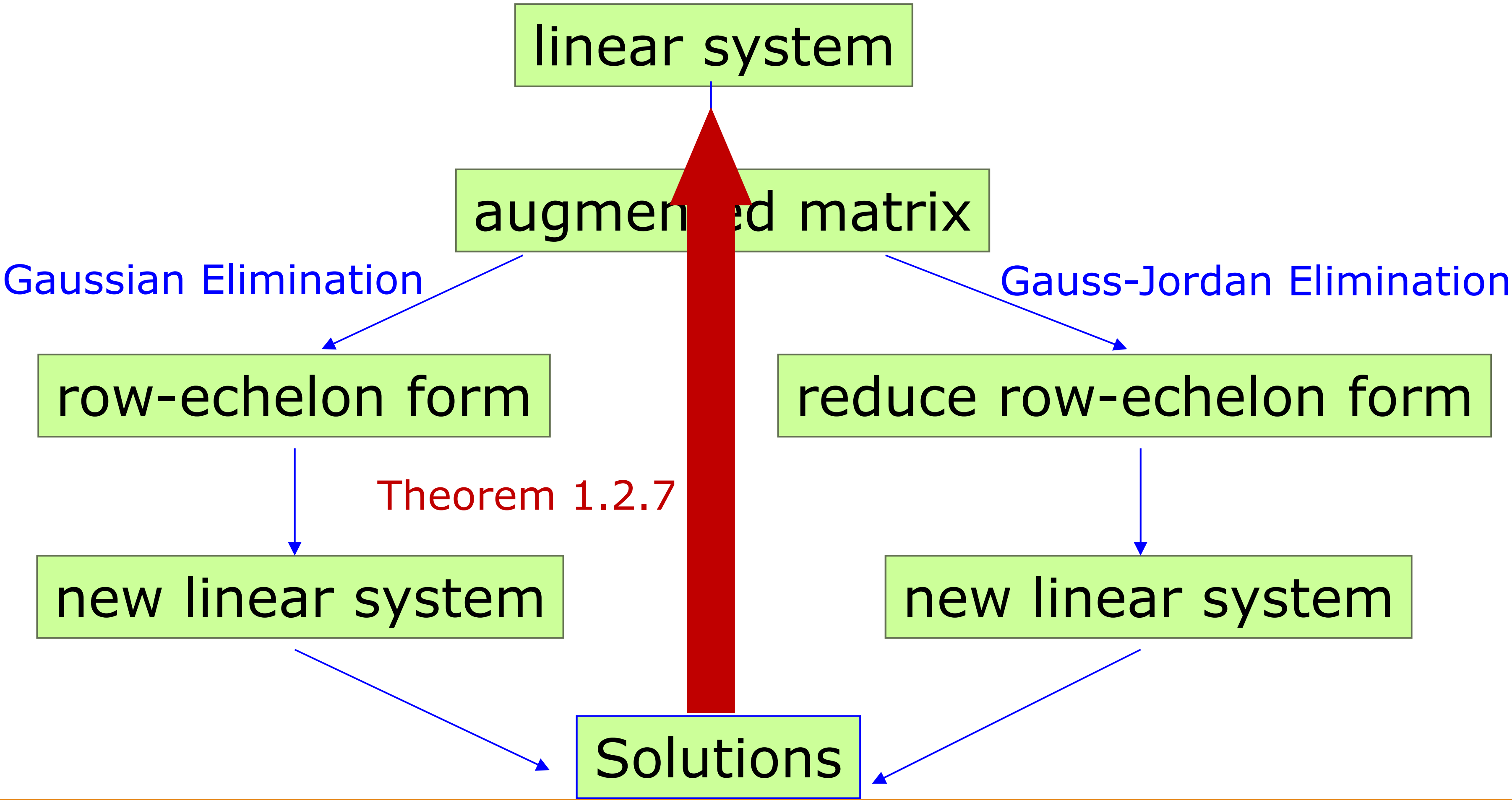
If a linear system has more equations than variables, then the system has at most one solution.

e.g. 5 equations 3 variables

→ 5 rows 3 columns

$$\left(\begin{array}{ccc|c} \boxed{\cdot} & \boxed{\cdot} & \boxed{\cdot} & \boxed{\cdot} \\ \boxed{\cdot} & \boxed{\cdot} & \boxed{\cdot} & \boxed{\cdot} \\ \boxed{\cdot} & \boxed{\cdot} & \boxed{\cdot} & \boxed{\cdot} \\ \boxed{\cdot} & \boxed{\cdot} & \boxed{\cdot} & \boxed{\cdot} \\ \boxed{\cdot} & \boxed{\cdot} & \boxed{\cdot} & \boxed{\cdot} \end{array}\right)$$

How to use GE/GJE to find solutions ?



Row Equivalence

Two augmented matrices are row equivalent (to each other)

if one can be obtained from the other
by a series of elementary row operations.

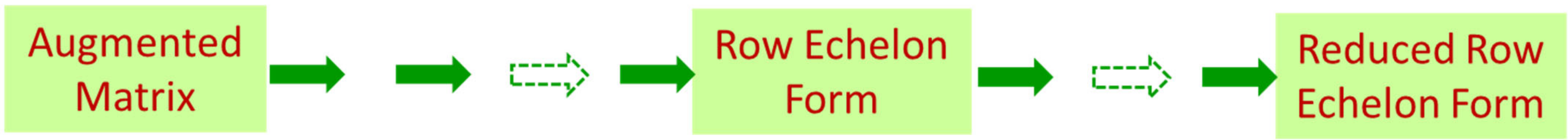
$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

Any 2 of the 4 augmented matrices are row equivalent

All 4 linear systems have the same solutions

True or False

Every augmented matrix is row equivalent to its RREF .



Linear System and Row echelon form

The following is a row echelon form of the augmented matrix of a linear system with a, b, c, d, e, f, g, h, j, k, l, m representing some real numbers.

$$M = \left(\begin{array}{cccc|c} a & b & c & d & e \\ 0 & f & g & h & j \\ 0 & 0 & k & l & m \end{array}\right)$$

Determine whether the following statements are true or false. Give brief justification for you answers.

Every column of REF is a pivot column, except the last column.

Linear System and Row echelon form

$$M = \left(\begin{array}{cccc|c} a & b & c & d & e \\ 0 & f & g & h & j \\ 0 & 0 & k & l & m \end{array}\right)$$

a, b, c, d, e, f, g, h, j, k, l, m representing some real numbers

(i) This linear system cannot have exactly one solution.

Ans: True.

There are four columns (on the left of the separator) but only three rows.

So there is at least one non-pivot column.

If the system is consistent, it will have infinitely many solutions.

There is a non-pivot column in the REF, other than the last column.

Linear System and Row echelon form

$$M = \left(\begin{array}{cccc|c} a & b & c & d & e \\ 0 & f & g & h & j \\ 0 & 0 & k & l & m \end{array}\right)$$

a, b, c, d, e, f, g, h, j, k, l, m representing some real numbers

(ii) If e, j, m are not all 0, then the linear system must have infinitely many solutions.

Ans: False.

If k and l are 0, but m is not 0, then the system will have no solution.

The last column of REF is a pivot column.

parameter in the solution is equal to the # of non-pivot columns in the REF.

Linear System and Row echelon form

$$M = \left(\begin{array}{cccc|c} a & b & c & d & e \\ 0 & f & g & h & j \\ 0 & 0 & k & l & m \end{array}\right)$$

a, b, c, d, e, f, g, h, j, k, l, m representing some real numbers

(iii) If the general solution of the linear system has $\boxed{\text{two parameters}}$, then the last row of M is a zero row.

Ans: True.

M has two non-pivot columns on the left of separator.

So M has two pivot columns and hence two leading entries.

This means M has exactly two non-zero rows.

Linear System and Row echelon form

$$M = \left(\begin{array}{cccc|c} a & b & c & d & e \\ 0 & f & g & h & j \\ 0 & 0 & k & l & m \end{array}\right)$$

a, b, c, d, e, f, g, h, j, k, l, m representing some real numbers

(iv) If $a = 0$, then f and k will be 0.

Ans: True.

M is a row echelon form and has a “staircase” of zeros.

If a = 0 in the top row, then there are at least two 0’s in the second row on the left of the leading entry. So f = 0.

This in turn implies there are at least three 0’s in the third row on the left of the leading entry. So k = 0.

The last column of REF is a pivot column.

Linear System and Row echelon form

$$M = \left(\begin{array}{cccc|c} a & b & c & d & e \\ 0 & f & g & h & j \\ 0 & 0 & k & l & m \end{array}\right)$$

a, b, c, d, e, f, g, h, j, k, l, m representing some real numbers

(v) If the system is $\boxed{\text{inconsistent,}}$ then $m \neq 0$.

Ans: False.

Let $f = g = h = 0$ but $j \neq 0$, and $k = l = m = 0$.

Then the system is inconsistent.