1. Find the exact value of

$$\lim_{x\to(-1)} \frac{1+x^{1521}}{1+x}.$$

Answer 1521

$$\lim_{x\to(-1)} \frac{1+x^{1521}}{1+x} = \lim_{x\to(-1)} \frac{1521\, x^{1520}}{1}$$

$$= \underline{\underline{1521}}$$

2. Let a denote a positive constant. Let L denote the tangent line to the curve

$$y = \frac{a - \sqrt{x}}{a + \sqrt{x}}$$

at the point $(a^2, 0)$. If L passes through the point $(-1, \frac{2020}{1521})$, find the value of a. Give your answer correct to two decimal places.

Answer 0.48

$$\frac{dy}{dx} = \frac{-\frac{1}{2}x^{-\frac{1}{2}}(a+x^{\frac{1}{2}}) - (a-x^{\frac{1}{2}})(\frac{1}{2}x^{-\frac{1}{2}})}{(a+\sqrt{x})^2}$$

$$x = a^2 \Rightarrow \frac{dy}{dx} = -\frac{1}{4a^2}$$

$$L: \frac{y}{x-a^2} = -\frac{1}{4a^2} \Rightarrow y = -\frac{1}{4a^2}x + \frac{1}{4}$$

$$\therefore \frac{2020}{1521} = \frac{1}{4a^2} + \frac{1}{4} = \frac{1+a^2}{4a^2}$$

$$8080a^2 = 1521 + 1521a^2$$

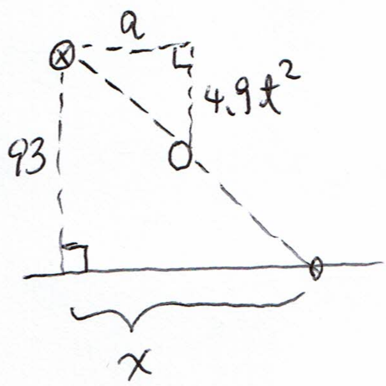
$$6559a^2 = 1521$$

$$a = \sqrt{\frac{1521}{6559}} = 0.4815\ldots$$

$$\approx \underline{\underline{0.48}}$$

3. Let a denote a positive constant. The torch on the Statue of Liberty is 93 metres above the ground. At time $t = 0$ a ball is dropped from the same height as the torch at a distance a metres from the torch. It is known that the ball falls a distance of $4.9t^2$ metres at time t seconds. If the speed (i.e. the absolute value of the velocity) of the shadow of the ball on the ground is $\sqrt{1521}$ metre per second at the moment just before the ball hits the ground, find the value of a. Give your answer correct to two decimal places.

Answer 84.95



$$\frac{x}{93} = \frac{a}{4.9t^2}$$

$$\therefore x = \frac{93a}{4.9t^2}$$

$$\frac{dx}{dt} = -\frac{2 \times 93a}{4.9t^3}$$

$$4.9t^2 = 93 \Rightarrow t = \sqrt{\frac{93}{4.9}}$$

$$\therefore \sqrt{1521} = \left| -\frac{2 \times 93a}{4.9\left(\frac{93}{4.9}\right)^{3/2}} \right|$$

$$\therefore a = \frac{1}{2}\sqrt{\frac{1521 \times 93}{4.9}} = 84.952\ldots$$

$$\approx \underline{\underline{84.95}}$$

4. Let a denote a positive constant. Let C denote the Cissoid which has equation $r = \frac{2a\sin^2\theta}{\cos\theta}$ in polar coordinates. Let L denote the tangent line to C at the point when $\theta = \frac{\pi}{3}$. If L passes through the point $(0, -2020)$ in Cartesian coordinates, find the value of a. Give your answer correct to two decimal places.

Answer 388.75

$$y = r\sin\theta = \frac{2a\sin^3\theta}{\cos\theta} = 2a\tan\theta\sin^2\theta$$

$$x = r\cos\theta = 2a\sin^2\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2a\sec^2\theta\sin^2\theta + 4a\tan\theta\sin\theta\cos\theta}{4a\sin\theta\cos\theta}$$

$$\text{at } \theta = \pi/3 \implies x = \frac{3}{2}a, \quad y = \frac{3\sqrt{3}}{2}a, \quad \frac{dy}{dx} = 3\sqrt{3}$$

$$\therefore L: \quad \frac{y - \frac{3\sqrt{3}}{2}a}{x - \frac{3}{2}a} = 3\sqrt{3}$$

$$y - \frac{3\sqrt{3}}{2}a = 3\sqrt{3}x - \frac{9\sqrt{3}}{2}a$$

$$y = 3\sqrt{3}x - 3\sqrt{3}a$$

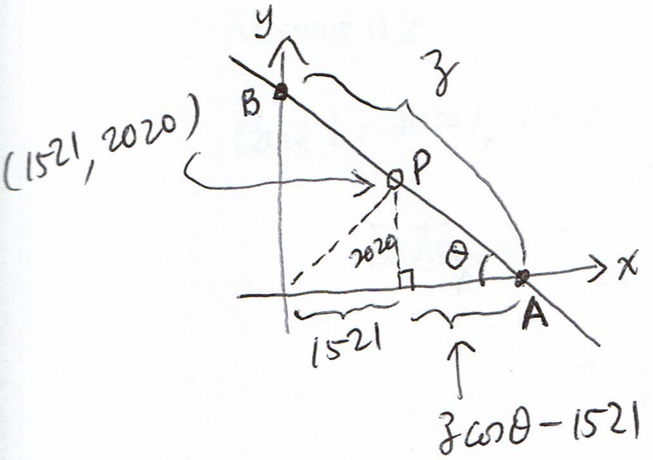
$$x = 0, \ y = -2020 \implies a = \frac{2020}{3\sqrt{3}} = \frac{2020\sqrt{3}}{9}$$

$$= 388.7491\ldots$$

$$\approx \underline{\underline{388.75}}$$

5. Let P denote the point $(1521, 2020)$. Let L denote a straight line that passes through P. It is known that L intersects the positive x-axis at A and L intersects the positive y-axis at B. Find the smallest possible length of the line segment AB. Give your answer correct to the nearest integer.

Answer 4991



$$\tan\theta = \frac{2020}{z\cos\theta - 1521}$$

$$z\sin\theta - 1521\tan\theta = 2020$$

$$z = 2020\operatorname{cosec}\theta + 1521\sec\theta$$

$$\frac{dz}{d\theta} = -2020\operatorname{cosec}\theta\cot\theta + 1521\sec\theta\tan\theta$$

$$= \frac{-2020\cos^3\theta + 1521\sin^3\theta}{\sin^2\theta\cos^2\theta} = \frac{1521\cos\theta}{\sin^2\theta}\left\{\tan^3\theta - \frac{2020}{1521}\right\}$$

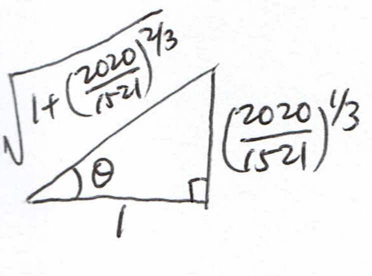
From the picture we have $0<\theta<\frac{\pi}{2}$, $\therefore \cos\theta, \sin\theta, \tan\theta > 0$

$\therefore \frac{dz}{d\theta} = 0 \Rightarrow \tan\theta = \left(\frac{2020}{1521}\right)^{1/3}$

$\because 0<\theta<\frac{\pi}{2}$, $\therefore \tan^3\theta$ is increasing $\Rightarrow$

$$\begin{array}{c|ccccc} \frac{dz}{d\theta} & 0 & - & \left(\frac{2020}{1521}\right)^{1/3} & + & \frac{\pi}{2} \end{array}$$

absolute minimum



$\therefore$ smallest possible z

$$= 2020 \times \frac{\sqrt{1+\left(\frac{2020}{1521}\right)^{2/3}}}{\left(\frac{2020}{1521}\right)^{1/3}} + 1521 \times \frac{\sqrt{1+\left(\frac{2020}{1521}\right)^{2/3}}}{1}$$

$$= 4991.078\ldots \approx \underline{\underline{4991}}$$

6. Let m and n denote two positive integers with $m+n=3$. Find the smallest possible value of the integral

$$\int_0^{\frac{\pi}{3}} \cos^m x \sin^n x dx.$$

Give your answer correct to two decimal places.

Answer 0.22

Case 1 : $m=1$, $n=2$

$$\text{Integral} = \int_0^{\pi/3} \cos x \sin^2 x \, dx = \frac{1}{3} \sin^3 x \Big|_0^{\pi/3}$$

$$= \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 = 0.2165\cdots$$

Case 2 : $m=2$, $n=1$

$$\text{Integral} = \int_0^{\pi/3} \cos^2 x \sin x \, dx = -\frac{1}{3}\cos^3 x \Big|_0^{\pi/3}$$

$$= \frac{1}{3}\left(1-\frac{1}{8}\right) = 0.2916\cdots$$

$\therefore$ smallest $= 0.2165\cdots \approx \underline{\underline{0.22}}$

7. Let $g(t)$ denote a continuous function which satisfies

$g(2) = 2$, $g(4) = 3$, $g(6) = 4$, $\int_0^2 g(t)dt = 4$, $\int_0^6 g(t)dt = 18$ and $\int_4^6 g(t)dt = 6$. Let

$$f(x) = \int_0^{x^2} x^3 g(t)dt.$$

Find the exact value of $f'(2)$.

Answer 240

$$f'(x) = \frac{d}{dx}\left\{ x^3 \int_0^{x^2} g(t)dt \right\}$$

$$= 3x^2 \int_0^{x^2} g(t)dt + x^3\left\{ (g(x^2))(2x) \right\}$$

$$f'(2) = 12\int_0^4 g(t)dt + 8\left\{ 4g(4) \right\}$$

$$= 12\left\{ \int_0^6 g(t)dt - \int_4^6 g(t)dt \right\} + 8 \times 4 \times 3$$

$$= 12\{18 - 6\} + 8 \times 12$$

$$= (12 + 8) \times 12 = \underline{\underline{240}}$$

8. Let a denote a positive constant with $a > 1$. If

$$\tan\left(\int_{\ln a}^{\ln(8a)} \frac{e^x}{e^{2x}+1}dx\right) = 0.1521$$

find the value of a. Give your answer correct to two decimal places.

Answer 5.73

$$\tan\left(\int_{\ln a}^{\ln(8a)} \frac{e^x}{e^{2x}+1}dx\right) = \tan\left(\int_{\ln a}^{\ln(8a)} \frac{d(e^x)}{(e^x)^2+1}\right)$$

$$= \tan\left(\left[\tan^{-1} e^x\right]_{\ln a}^{\ln 8a}\right)$$

$$= \tan\left(\tan^{-1} 8a - \tan^{-1} a\right)$$

$$= \frac{\tan(\tan^{-1} 8a) - \tan(\tan^{-1} a)}{1 + (\tan \tan^{-1} 8a)(\tan \tan^{-1} a)} = \frac{7a}{1+8a^2}$$

$$\therefore \frac{7a}{1+8a^2} = 0.1521$$

$$\therefore 1.2168a^2 - 7a + 0.1521 = 0$$

$$\therefore a = \frac{7 \pm \sqrt{49 - 4 \times 1.2168 \times 0.1521}}{2 \times 1.2168} = 0.0218\cdots \text{ or } 5.7309\cdots$$

$$\because a > 1$$

$$\therefore a \simeq \underline{\underline{5.73}}$$

9. Let a denote a positive constants. Let R denote the finite region in the first quadrant bounded between the x-axis, the y-axis, the line $x = a$ and the curve $y = \frac{1}{4a^2 - x^2}$. If the area of R is equal to 1.521, find the value of a. Give your answer correct to two decimal places.

Answer 0.18

$$\int_0^a \frac{1}{4a^2 - x^2}\,dx = \frac{1}{4a}\int_0^a \left(\frac{1}{x+2a} - \frac{1}{x-2a}\right) dx$$

$$= \frac{1}{4a}\Big[\ln|(x+2a)| - \ln|(x-2a)|\Big]_0^a$$

$$= \frac{1}{4a}\ln 3$$

$$\frac{1}{4a}\ln 3 = 1.521$$

$$\Rightarrow \quad a = \frac{\ln 3}{4 \times 1.521} = 0.18057\ldots$$

$$\approx \underline{\underline{0.18}}$$

10. Let a denote a positive constant. Let C denote the curve

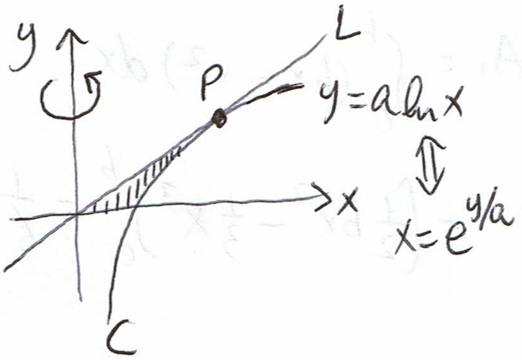
$$y = a \ln x.$$

Let L denote the tangent line at a point on C in the first quadrant such that L passes through the origin. Let R denote the finite region in the first quadrant bounded by the x-axis, the curve C and the line L. If the volume of the solid of revolution obtained by rotatating R one complete round about the y-axis is equal to 1521, find the value of a. Give your answer correct to two decimal places.

Answer 661.85

Let $P = (k, a\ln k)$

$$\frac{dy}{dx} = \frac{a}{x} = \frac{a}{k} \text{ at } P$$



$$\therefore \frac{a\ln k}{k} = \frac{a}{k}$$

$$\Rightarrow \ln k = 1 \Rightarrow k = e$$

$$\therefore P = (e, a\ln e) = (e, a), \quad L: y = \frac{a}{e}x$$

For C: $$\int_0^a \pi x^2 dy = \int_0^a \pi e^{\frac{2y}{a}} dy$$

$$= \frac{a}{2}\pi e^{\frac{2y}{a}} \Big|_0^a = \frac{a}{2}\pi(e^2-1)$$

For L: $$\int_0^a \pi x^2 dy = \int_0^a \pi \frac{e^2}{a^2} y^2 dy = \frac{1}{3}\pi e^2 a$$

$$\therefore \frac{a}{2}\pi(e^2-1) - \frac{1}{3}\pi e^2 a = 1521$$

$$a = \frac{1521 \times 6}{\pi(e^2-3)} = 661.849\ldots \approx \underline{\underline{661.85}}$$