AY2020/2021 Semester I
MA1101R Take-home assignment declaration form

Read A, B and C. Sign and submit this declaration form together with your answers.

A. Academic, Professional and Personal Integrity

1. *The University is committed to nurturing an environment conducive for the exchange of ideas, advancement of knowledge and intellectual development. Academic honesty and integrity are essential conditions for the pursuit and acquisition of knowledge, and the University expects each student to maintain and uphold the highest standards of integrity and academic honesty at all times.*
2. *The University takes a strict view of cheating in any form, deceptive fabrication, plagiarism and violation of intellectual property and copyright laws. Any student who is found to have engaged in such misconduct will be subject to disciplinary action by the University.*
3. *It is important to note that all students share the responsibility of protecting the academic standards and reputation of the University. This responsibility can extend beyond each student's own conduct, and can include reporting incidents of suspected academic dishonesty through the appropriate channels. Students who have reasonable grounds to suspect academic dishonesty should raise their concerns directly to the relevant Head of Department, Dean of Faculty, Registrar, Vice Provost or Provost.*

B. I have read and understood the rules of the assessments stated below.

a. *Students should attempt the assessments on their own. (For homework assignments, students may discuss with or consult their friends and instructors, but they have to work on the assignment independently.)*

b. *Students should not reproduce of any assessment materials, e.g. by photography, videography, screenshots, or copying down of questions, etc.*

C. I understand that by breaching any of the rules above, I would have committed offences under clause 3(l) of the NUS Statute 6, Discipline with Respect to Students which is punishable with disciplinary action under clause 10 or clause 11 of the said statute.

3) *Any student who is alleged to have committed or attempted to commit, or caused or attempted to cause any other person to commit any of the following offences, may be subject to disciplinary proceedings:*
   (l) plagiarism, giving or receiving unauthorised assistance in academic work, or other forms of academic dishonesty.

I have read and will abide by the NUS Code of Student Conduct (in particular, (A) Academic, Professional and Personal Integrity), B and C when attempting this assessment.

Signature: ______________________

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Semester 1, 2020/2021 **MA1101R** **Homework Assignment 2**

- This assignment consists of 5 pages and 12 questions.
- Some hints are given on the last page.
- Each question is 8 marks, except question 9 which is 12 marks. Total 100 marks.
- Deadline for submission is 6 November, 2020 by 11.59pm.

Please read the following instructions carefully. Answer scripts that do not follow the instructions will be subject to mark deduction.

(a) Use A4 size paper and pen (blue or black ink) to write your answers. (Students may also type out the answers or write the answers electronically using their devices.)

(b) Write down your student number and full name clearly on the top left of every page of the answer scripts.

(c) Write the page number on the top right corner of each page of answer scripts.

(d) To submit your answer scripts, do the following:

 (i) Scan or take pictures of your work (make sure the images can be read clearly).

 (ii) Merge the declaration form and all your answers into one pdf file. Arrange them in order of the page with the declaration form as the first page.

 (iii) Name the pdf file by **StudentNo HW2** (e.g. **A123456R HW2**).

 (iv) Upload your pdf into the LumiNUS folder Homework 2 submission.

(e) Late submission will not be accepted.

1. (a) Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 \end{pmatrix}$.
Use the reduced row echelon form of $\boldsymbol{A}$ (you may use MATLAB command) to find a basis for the row space of $\boldsymbol{A}$.

   (b) Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$ where $\boldsymbol{u}_1 = (1, 1, 0, -1)$, $\boldsymbol{u}_2 = (-1, -1, 0, 1)$, $\boldsymbol{u}_3 = (0, 1, 1, 1)$, $\boldsymbol{u}_4 = (2, 3, 1, -1)$, $\boldsymbol{u}_5 = (1, 1, 1, -1)$ and let $V = \text{span}(S)$.
Find a basis for V consisting of vectors from S.

   (c) Extend the bases in (a) and (b) to bases for $\mathbb{R}^5$ and $\mathbb{R}^4$ respectively using the row space method.

2. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 2 & 3 & -1 & 2 \\ 1 & 3 & -1 & 0 \\ 4 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   (a) Find a basis for the column space of $\boldsymbol{A}$. Hence find the rank and nullity of $\boldsymbol{A}$. (MATLAB command can be used to find rref.)

   (b) Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be column vectors in $\mathbb{R}^4$ such that $\boldsymbol{Au}, \boldsymbol{Av}, \boldsymbol{Aw}$ are linearly independent. Show that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent.

   (c) Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be linearly independent column vectors in $\mathbb{R}^4$. Are $\boldsymbol{Au}, \boldsymbol{Av}, \boldsymbol{Aw}$ linearly independent? Justify your answer.

3. (a) Let $\boldsymbol{A}$ be a square matrix such that $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}^2)$.

      (i) Show that the nullspace of $\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{A}^2$.

      (ii) Show that
(the nullspace of $\boldsymbol{A}$) $\cap$ (the column space of $\boldsymbol{A}$) $= \{\boldsymbol{0}\}$.

   (b) Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ be an $n \times m$ matrix with $n < m$. Show that $\boldsymbol{AB}$ is singular.

4. Let $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ -1 \end{pmatrix}$

   (i) Show that $\boldsymbol{b}$ does not belong to the column space of $\boldsymbol{A}$.

   (ii) Find the least squares solutions to the linear system $\boldsymbol{Ax} = \boldsymbol{b}$. Show your working.

   (iii) Find the projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto the subspace $V = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

   (iv) Use $\boldsymbol{p}$ in (iii) to find a vector that is orthogonal to V.

5. Let $V = \{(a+2b-c, \ -a+3c, \ a+c, \ a-c, \ -a+2b+c) \in \mathbb{R}^5 \mid a,b,c \in \mathbb{R}\}$.

   (i) Show that $S = \{(1,1,1,0,1),(0,2,0,-1,2),(2,-2,0,1,0)\}$ is a basis for V.

   (ii) Perform Gram Schmidt process on S to obtain an orthogonal basis for V. (Do not use MATLAB command. Show your steps.)

   (iii) Find all vectors that are orthogonal to V.

6. Let $S = \{(1,1,2,1),(3,3,-1,-4),(-2,3,-1,1),(1,0,-1,1)\}$ and $T = \{(1,0,1,0),(1,2,-1,2),(-1,3,1,-2),(-1,0,1,1)\}$ be two subsets of $\mathbb{R}^4$.

   (i) Show that both S and T are orthogonal bases for $\mathbb{R}^4$.

   (ii) Normalise S and T to get orthonormal bases S' and T' respectively.

   (iii) Find the transition matrix from S' to T'.

   (For (ii) and (iii), give the exact values for the entries in terms of square roots.)

7. (a) Suppose $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k\}$ is an orthogonal basis for a subspace V of $\mathbb{R}^n$. Let $S' = \left\{\dfrac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|}, \dfrac{\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|}, ..., \dfrac{\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|}\right\}$ be an orthonormal basis of V obtained from normalizing S. If the coordinate vector of v with respect to S is given by

   $$(\boldsymbol{v})_S = \begin{pmatrix} c_1 & c_2 & \dots & c_k \end{pmatrix},$$

   what is $(\boldsymbol{v})_{S'}$, the coordinate vector of $\boldsymbol{v}$ with respect to S'?

   (b) Let $T = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k\}$ be a basis for a subspace $V \subseteq \mathbb{R}^n$ and $T' = \{\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_k\}$ be an orthonormal basis of V obtained from T by applying the Gram-Schmidt process (Theorem 5.2.19) with normalisation so that each $\boldsymbol{w}_i$ is a unit vector.

   Find the transition matrix $\boldsymbol{P}$ from T to T'. Give your answer in terms of $\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k$ and $\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_k$.

8. Let V be a subspace of $\mathbb{R}^n$. Recall (from exercise 5.7) that $V^\perp$ is also a subspace of $\mathbb{R}^n$.

   (i) Suppose $\boldsymbol{A}$ is an $n \times n$ matrix such that $\boldsymbol{Av} \in V$ for any $\boldsymbol{v} \in V$. Show that $\boldsymbol{A}^T\boldsymbol{w} \in V^\perp$ for any $\boldsymbol{w} \in V^\perp$.

   (ii) Suppose $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k\}$. Form a $k \times n$ matrix $\boldsymbol{B}$ with $\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k$ as its rows.
   Show that nullspace of $\boldsymbol{B}$ is equal to $V^\perp$.

   (iii) Show that $\dim V + \dim V^\perp = n$.

9. (a) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

   (i) Without using MATLAB, find the characteristic polynomial and eigenvalues of $\boldsymbol{A}$. Explain briefly how your answers are derived.

   (ii) Find a basis for the eigenspace associated to each eigenvalue of $\boldsymbol{A}$.

   (iii) Without finding $\boldsymbol{A}^5$, write down the characteristic polynomial, eigenvalues, and a basis for the eigenspace associated to each eigenvalue of $\boldsymbol{A}^5$. Explain briefly how your answers are derived.

   (iv) Is $\boldsymbol{A}$ diagonalizable? Why?

   (b) Let $\boldsymbol{C} = \begin{pmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ 2 & 2 & 3 \end{pmatrix}$.

   (i) Find the eigenvalues and the corresponding eigenvectors for $\boldsymbol{C}$. Show your working clearly. (Do not use MATLAB command.)

   (ii) Diagonalize $\boldsymbol{C}$ in three different ways so that the resulting diagonal matrices $\boldsymbol{D}_1, \boldsymbol{D}_2$ and $\boldsymbol{D}_3$ are all different.

10. A sequence $\{a_n\}$ is defined by $a_0 = a_1 = a_2 = 1$ and

$$a_n = 7a_{n-2} + 6a_{n-3} \qquad n \geq 3.$$

Find a general formula for a_n (in terms of n).

(You may use MATLAB to find eigenvalues and eigenvectors.)

11. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ be an orthogonal basis for $\mathbb{R}^n$ and

$$\boldsymbol{A} = \boldsymbol{u}_1\boldsymbol{u}_1^T + \boldsymbol{u}_2\boldsymbol{u}_2^T + \cdots \boldsymbol{u}_n\boldsymbol{u}_n^T,$$

where the vectors are written as columns vectors.

   (i) Show that $\boldsymbol{A}$ is a symmetric matrix.

   (ii) Show that each of $\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n$ is an eigenvector of $\boldsymbol{A}$ and find the corresponding eigenvalues.

   (iii) Find an orthogonal matrix $\boldsymbol{P}$ and diagonal matrix $\boldsymbol{D}$ such that

$$\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}.$$

12. Let $\boldsymbol{A}$ be a square matrix of order n such that $\boldsymbol{A}^2 = \boldsymbol{I}$. Show that

   (i) the only eigenvalues of $\boldsymbol{A}$ are ± 1;

   (ii) $\text{nullity}(\boldsymbol{I} + \boldsymbol{A}) + \text{nullity}(\boldsymbol{I} - \boldsymbol{A}) = n$;

   (iii) $\boldsymbol{A}$ is diagonalizable.

Hints

- **2(c)** Look at the nullity of $\boldsymbol{A}$.
- **3(a(ii))** Show that, if $\boldsymbol{x} \in$ nullspace of $\boldsymbol{A}$ and $\boldsymbol{x} \in$ column space of $\boldsymbol{A}$, then $\boldsymbol{x} = \boldsymbol{0}$. You may need Theorem 4.1.16.
- **3(b)** Consider the ranks of the matrices involved.
- **7(a)** Express $\boldsymbol{v}$ as a linear combination of the vectors in S.
- **7(b)** Use Theorem 5.2.8.
- **8(a)** Express dot product as column matrix multiplication.
- **8(b)** Consider $\boldsymbol{Bx}$ for $\boldsymbol{x} \in V^{\perp}$.
- **8(c)** Use Dimension Theorem.
- **10** Refer to Example 6.2.11.2. Here you need a 3×3 recurrent matrix.
- **11(ii)** Use the matrix multiplication interpretation of dot product.
- **12(ii)** Rewrite $\boldsymbol{A}^2 = \boldsymbol{I}$ in terms of $\boldsymbol{I} + \boldsymbol{A}$ and $\boldsymbol{I} - \boldsymbol{A}$. You may need to use Exercise 5.23 and Dimension Theorem.)