Matriculation Number: MA1521

A								

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2015-2016

MA1521 CALCULUS FOR COMPUTING

May 2016 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** Do not write your name anywhere in this booklet. This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **FIVE (5)** questions and comprises **TWENTY THREE (23)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question. The marks for each question are indicated at the beginning of the question. The maximum possible total score for this examination paper is 80 marks.
4. This is a **closed book (with authorized material)** examination. Students are only allowed to bring into the examination hall **ONE** piece A4 size help-sheet which must be handwritten and can be written on both sides.
5. Candidates may use any calculators that satisfy MOE A-Level examination guidelines. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5
(a)					
(b)					

Question 1 (a) [8 marks]

(i) (Multiple Choice Question)

Let $y = (13x - 25)^{117}$. Find the **exact value** of $\frac{dy}{dx}$ at $x = 2$.

(A) 1521 (B) 1520 (C) 1522

(ii) It is known that the tangent line to the curve

$$e^y + \sin(x + y) - 2x = 1$$

at the origin passes through the point $(2016, k)$. Find the **exact value** of the constant k.

Answer 1(a)(i)	A	**Answer 1(a)(ii)**	1008

(Show your working below and on the next page.)

(i) $\frac{dy}{dx} = 117\,(13x-25)^{116}(13)$

$x=2 \Rightarrow \frac{dy}{dx} = (117)(13) = \underline{\underline{1521}}$

(ii) $e^y \frac{dy}{dx} + \left(1+\frac{dy}{dx}\right)\cos(x+y) - 2 = 0$

at $(0,0) \Rightarrow \frac{dy}{dx} + \left(1+\frac{dy}{dx}\right) - 2 = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{2}$

$k = \frac{1}{2}(2016) = \underline{\underline{1008}}$

Question 1 (b) [8 marks]

(i) Let a denote a given positive constant. Let $f(x)$ be the function defined on the interval $-a \le x \le 3a$ by

$$f(x) = \frac{1}{2}x^2 + 2ax + 1001a^2.$$

If $f(c)$ is the absolute minimum value of f on this interval, find the **exact value** of c. Give your answer in terms of a.

(ii) Find the **exact value** of the definite integral $\int_0^{\frac{91\pi}{6}} |\cos x|\, dx$.

Answer 1(b)(i)	$-a$	**Answer 1(b)(ii)**	$\frac{61}{2}$

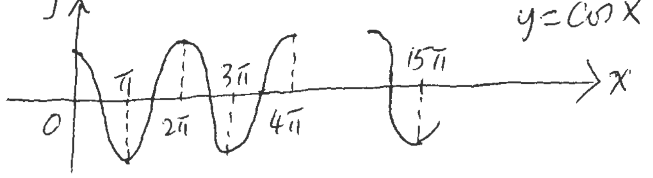
(Show your working below and on the next page.)

(i) $f'(x) = x + 2a \geq a > 0$ in $-a \le x \le 2a$

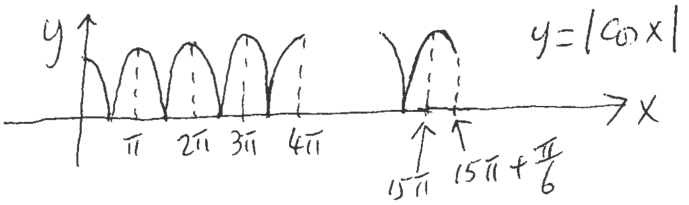
∴ absolute minimum occurs at $x = -a$

∴ $c = -a$

(ii)



$$\frac{91}{6}\pi = 15\pi + \frac{\pi}{6}$$



$$\int_0^{\pi} |\cos x|\, dx = 2\int_0^{\pi/2} \cos x\, dx = 2$$

$$\int_0^{\frac{91}{6}\pi} |\cos x|\, dx = 15 \times 2 + \int_0^{\pi/6} \cos x\, dx = 30 + \frac{1}{2} = \frac{61}{2}$$

Question 2 (a) [8 marks]

(i) Find the **exact value** of the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^{33} x \ dx.$$

Give your answer in the form of one single fraction in its simplest form.

(ii) Let R denote the finite region in the first quadrant bounded by the curve $y = \sqrt{x^3+1}$, the x-axis, the y-axis and the line $x = 3$. Find the volume of the solid formed by revolving R one complete round about the x-axis. Give your answer correct to the nearest integer.

Answer 2(a)(i)	$\frac{1}{612}$	**Answer 2(a)(ii)**	73

(Show your working below and on the next page.)

(i) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^{33} x dx = -\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{33} x \, d(\cos x)$

$= \int_0^{\frac{\pi}{2}} (-\cos^{33} x + \cos^{35} x) d(\cos x) = \left[-\frac{1}{34}\cos^{34} x + \frac{1}{36}\cos^{36} x \right]_0^{\frac{\pi}{2}}$

$= \frac{1}{34} - \frac{1}{36} = \underline{\underline{\frac{1}{612}}}$

(ii) $Vol = \int_0^3 \pi(x^3+1) dx = \pi\left[\frac{x^4}{4} + x\right]_0^3 = \frac{93}{4}\pi$

$\approx \underline{\underline{73.04}}$

Question 2 (b) [8 marks]

(i) Find the **exact value** of the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(3x-111)^n}{n+1}$.

(ii) Let

$$\sum_{n=0}^{\infty} c_n (x+1)^n$$

denote the Taylor Series of $\frac{-1}{3x+1}$ at $x = -1$. Find the **exact value** of c_6. Give your answer in the form of one single fraction in its simplest form.

Answer 2(b)(i)	$\frac{1}{3}$	**Answer 2(b)(ii)**	$\frac{729}{128}$

(Show your working below and on the next page.)

(i) $\left| \frac{\frac{(3x-111)^{n+1}}{n+2}}{\frac{(3x-111)^n}{n+1}} \right| = \frac{n+1}{n+2}|3x-111| \to 3|x-37|$

$3|x-37| < 1 \Rightarrow |x-37| < \underline{\underline{\frac{1}{3}}}$

(ii) $\frac{-1}{3x+1} = \frac{-1}{3(x+1)-2} = \frac{1}{2}\frac{1}{1-\frac{3}{2}(x+1)} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{3^n}{2^n}(x+1)^n$

$= \sum_{n=0}^{\infty} \frac{3^n}{2^{n+1}}(x+1)^n$

$n=6 \Rightarrow c_n = \frac{3^6}{2^7} = \underline{\underline{\frac{729}{128}}}$

Question 3 (a) [8 marks]

(i) To approximate the equation $e^{-x} = \cos x$, you replace e^{-x} by its Taylor polynomial of order 3 at $x = 0$, replace $\cos x$ by its Taylor polynomial of order 2 at $x = 0$. After simplifying the resulting expression, you arrive at a cubic equation of the form $x^3 + Ax^2 + Bx + C = 0$, where A, B, C denote three constants. Find the **exact value** of the constant A.

(ii)Find the **exact value** of $\sum_{n=1}^{\infty} \frac{n}{5^n}$. Give your answer in the form of one single fraction in its simplest form.

(Hint: Differentiate the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, where $-1 < x < 1$.)

Answer 3(a)(i)	-6	Answer 3(a)(ii)	$\frac{5}{16}$

(Show your working below and on the next page.)

(i) $1 - x + \frac{x^2}{2} - \frac{x^3}{6} = 1 - \frac{x^2}{2} \Rightarrow x^3 - 6x^2 + 6x = 0$

$\therefore A = \underline{\underline{-6}}$

(ii) $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \Rightarrow \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$

$x = \frac{1}{5} \Rightarrow \sum_{n=1}^{\infty} \frac{n}{5^n} = \frac{1/5}{(4/5)^2} = \underline{\underline{\frac{5}{16}}}$

Question 3 (b) [8 marks]

(i) Let $f(x,y) = e^x + e^{3y}$. Find the directional derivative of f at the point $(1, \frac{1}{3})$ in the direction of the vector $\mathbf{i}+\mathbf{j}$. Give your answer correct to one decimal place.

(ii) Let $f(x,y)$ be a differentiable function of two variables which satisfies $f(1,2) = 1510$, $\frac{\partial f}{\partial y}(1,2) = -104$ and $f(1.1, 1.8) = 1521$. Estimate the value of $\frac{\partial f}{\partial x}(1,2)$.

Answer 3(b)(i)	7.7	**Answer 3(b)(ii)**	-98

(Show your working below and on the next page.)

(i) $\vec{u} = \frac{\vec{i}+\vec{j}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$, $\nabla f = e^x\vec{i} + 3e^{3y}\vec{j}$

$D_{\vec{u}} f(1, \frac{1}{3}) = \nabla f(1, \frac{1}{3}) \cdot \vec{u} = (e\vec{i} + 3e\vec{j}) \cdot (\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j})$

$= \frac{4e}{\sqrt{2}} \approx 7.688 \approx \underline{\underline{7.7}}$

(ii) $(1.1, 1.8) - (1, 2) = (0.1, -0.2)$

$\Delta t = \sqrt{0.1^2 + 0.2^2} = \sqrt{0.05}$

$\Delta f = 1521 - 1510 \approx \left[\left(\frac{\partial f}{\partial x}(1,2)\vec{i} - 104\vec{j}\right) \cdot \frac{0.1\vec{i} - 0.2\vec{j}}{\sqrt{0.05}}\right] \Delta t$

$\therefore 11 \approx (0.1)\frac{\partial f}{\partial x}(1,2) + 0.2 \times 104$

$-9.8 \approx (0.1)\frac{\partial f}{\partial x}(1,2) \Rightarrow \frac{\partial f}{\partial x}(1,2) \approx \underline{\underline{-98}}$

Question 4 (a) [8 marks]

(i) Let k denote a constant. If

$$\nabla f(x,y) = (3x^2y^2 + 10xy + 6y^3)\mathbf{i} + (2x^3y + 5x^2 + kxy^2)\mathbf{j},$$

find the **exact value** of the constant k.

(ii) (Multiple Choice Question)

It is known that $(5,5)$ is the only critical point of the function

$$f(x,y) = xy + (x+y)(15 - x - y).$$

Classify this critical point.

(A) Local Maximum (B) Local Minimum (C) Saddle Point

Answer 4(a)(i)	18	**Answer 4(a)(ii)**	A

(Show your working below and on the next page.)

(i) $\frac{\partial}{\partial y}(3x^2y^2 + 10xy + 6y^3) = \frac{\partial}{\partial x}(2x^3y + 5x^2 + kxy^2)$

$\Rightarrow 6x^2y + 10x + 18y^2 = 6x^2y + 10x + ky^2$

$\Rightarrow k = \underline{\underline{18}}$

(ii) $f_x = y + (15 - x - y) - (x+y) = 15 - 2x - y$

$f_y = x + (15 - x - y) - (x+y) = 15 - x - 2y$

$f_{xx} = -2, \quad f_{xy} = -1, \quad f_{yx} = -1, \quad f_{yy} = -2$

$\begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = 3 > 0, \quad f_{xx} = -2 < 0 \Rightarrow$ $\underline{\underline{\text{local maximum}}}$

Question 4 (b) [8 marks]

(i) You found a fossil at NUS. You took it to the chemistry lab and determined that it contained 70% of the original amount of Carbon-14. The half-life of Carbon-14 is known to be 5730 years. Find the age of this fossil. Give your answer in years correct to the nearest integer.

(ii) Let $y(x)$ denote the solution of the differential equation

$$\frac{1}{x}y' - y = \frac{1}{x}e^{(\frac{1}{2}x^2)},$$

with $x > 0$ and $y(1) = 7\sqrt{e}$. Find the value of $y(2)$. Give your answer correct to one decimal place.

Answer 4(b)(i)	2949	**Answer 4(b)(ii)**	59.1

(Show your working below and on the next page.)

(i) $\frac{dQ}{dt} = -kQ \Rightarrow Q = Ce^{-kt}$, $k = \frac{\ln 2}{5730}$

$Q = 0.7C \Rightarrow 0.7 = e^{-kt} \Rightarrow t = \frac{\ln(0.7)}{-k} = -\frac{\ln(0.7)}{\ln 2} \times 5730$

$\approx 2948.50 \approx \underline{\underline{2949}}$

(ii) $y' - xy = e^{\frac{1}{2}x^2}$

$R = e^{\int -x\,dx} = e^{-\frac{1}{2}x^2} \Rightarrow y = e^{\frac{1}{2}x^2}\int dx = e^{\frac{1}{2}x^2}(x + C)$

$y(1) = 7\sqrt{e} \Rightarrow 7\sqrt{e} = \sqrt{e}(1 + C) \Rightarrow C = 6$

$\therefore y(2) = 8e^2 \approx 59.11 \approx \underline{\underline{59.1}}$

Question 5 (a) [8 marks]

~~(i) Find the general solution of the differential equation~~

$$\cancel{y'' - 3y' + 2y = 0.}$$

(ii) A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at time $t = 0$, water that contains 0.5 pound of salt per gallon is poured into the tank at a rate of 4 gallons per minute and the well-stirred solution is drained from the tank at the same rate. At time $t = k$ minutes, the amount of salt in the tank is found to be 150 pounds. Find the value of k. Give your answer correct to one decimal place.

Answer 5(a)(i)	$y=C_1e^{x}+C_2e^{2x}$	**Answer 5(a)(ii)**	82.5

(Show your working below and on the next page.)

(i) $\lambda^2-3\lambda+2=0 \Rightarrow \lambda=1,2 \Rightarrow \underline{\underline{y=C_1e^{x}+C_2e^{2x}}}$

(ii) $\frac{dQ}{dt}=2-\frac{Q}{600}\times 4 \Rightarrow \frac{dQ}{dt}+\frac{1}{150}Q=2$

$R=e^{\int\frac{1}{150}dt}=e^{\frac{1}{150}t}$

$Q=e^{-\frac{t}{150}}\int 2e^{\frac{t}{150}}dt=e^{-\frac{t}{150}}\left(300e^{\frac{t}{150}}+C\right)$

$Q(0)=40 \Rightarrow 40=300+C \Rightarrow C=-260$

$\therefore Q=300-260e^{-\frac{t}{150}}$

$Q(k)=150 \Rightarrow e^{-\frac{k}{150}}=\frac{150}{260} \Rightarrow k=-150\ln\frac{150}{260}\approx\underline{\underline{82.5}}$

Question 5 (b) [8 marks]

(i) The growth of deer in a certain national forest follows a logistic model given by the equation

$$\frac{dN}{dt} = 10N - 0.02N^2,$$

where N denotes the number of deer at time t (measured in year). Initially at time $t = 0$, there were 200 deer in that forest. What is the carrying capacity of the deer population in that national forest? Give **exact value** for your answer.

(ii) You had 1000 bugs in a bottle. It is known that this bug population is given by a logistic model with a birth rate per capita of 20% per week. After a very long time, you found that the bug population had attained an equilibrium value of 5000. You then started an experiment with an antibiotic which killed E bugs per week, where E denotes a constant. After another very long time, you found that the bug population had attained another equilibrium value of 3000. Find the value of E. Give your answer correct to the nearest integer.

Answer 5(b)(i)	500	**Answer 5(b)(ii)**	240

(Show your working below and on the next three pages.)

(i) $N_\infty = \frac{B}{S} = \frac{10}{0.02} = \underline{\underline{500}}$

(More working space for Question 5(b))

(ii) $\beta = 0.2$, $\frac{\beta}{S} = 5000 \Rightarrow S = \frac{0.2}{5000}$

$$\frac{dN}{dt} = 0.2N - \frac{0.2}{5000}N^2 - E$$

$$2000 \times 3000 = \frac{-E}{-\frac{0.2}{5000}} = 25000E$$

$$\therefore E = \frac{2000 \times 3000}{25000} = \underline{\underline{240}}$$