ST2334 PROBABILITY AND STATISTICS
SEMESTER I, AY 2022/2023

Example Questions for the Midterm Test: Solution

- The format of the exam is open book (hard copies or/and soft copies physically stored on the laptop for exam are allowed), onsite through **exam software: Exemplify**. A laptop with the software installed and solid battery that can last for at least 1.5 hours is needed.
- Calculators of any kind are allowed.
- Coverage:
  - Chapter 1: Basic Probability Concepts and Definitions;
  - Chapter 2: Random Variables.

1. (MCQ) $A \cup (B \cap C) =$

   (a) $(A \cup B) \cap (A \cup C)$
   (b) $(A \cup B) \cap C$
   (c) $A \cup B' \cup C'$
   (d) $(A \cap B) \cup (A \cap C)$

   SOLUTION
   (a)

2. (Fill in the blank) How many ways are there to choose an arbitrary number of students (including the possibility of choosing 0 student) from 6 students?

   Answer: ________

   SOLUTION
   For each student, there are 2 possibilities: "chosen" or "not chosen". So the total number of possibilities is $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$.

3. (MRQ) Which of the following can be used as the sample space for the problem: "choose two students from four students to complete a project"? Assume students are labeled as S_1, S_2, S_3, and S_4.

   (a) $\Big\{(S_1,S_2),(S_1,S_3),(S_1,S_4),(S_2,S_1),(S_2,S_3),(S_2,S_4),$
   $(S_3,S_1),(S_3,S_2),(S_3,S_4),(S_4,S_1),(S_4,S_2),(S_4,S_3)\Big\}.$
   (b) $\Big\{\{S_1,S_2\},\{S_1,S_3\},\{S_1,S_4\},\{S_2,S_3\},\{S_2,S_4\},\{S_3,S_4\}\Big\}.$
   (c) $\Big\{\{S_1,S_1\},\{S_1,S_2\},\{S_1,S_3\},\{S_1,S_4\},\{S_2,S_2\},\{S_2,S_3\}\Big\}.$
   (d) $\Big\{S_1,S_2,S_3,S_4,S_5,S_6\Big\}$

   SOLUTION
   (a), (b). (a) corresponds to the sample space that order is considered; (b) corresponds to the sample space that order is not considered.

4. (MCQ) Draw 4 balls randomly without replacement from a basket containing 4 blue balls, 4 green balls, and 2 red balls. What is the probability to get 3 blue balls and 1 green ball?

   (a) $7/105$
   (b) $8/105$
   (c) $8/35$
   (d) $9/35$

   SOLUTION

   - The total number of ways to draw 4 balls from 10 balls is $\binom{10}{4} = 210$.
   - The number of ways to get 3 blue balls and 1 green ball is $\binom{4}{3}\binom{4}{1} = 16$.
   - The probability is $16/210 = 8/105$.

   So the answer is (b).

5. (Fill in the blank) Suppose

$$P(A') = 1/2, \quad P(B) = 3/8, \quad \text{and,} \quad P(B'|A) = 3/4.$$

Find $P(B \cap A)$.

Answer: ________

SOLUTION

$$\begin{aligned} P(B' \cap A) &= P(B'|A)P(A) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}, \\ P(B \cap A) &= P(A) - P(B' \cap A) = \frac{1}{2} - \frac{3}{8} = \frac{1}{8} = 0.125. \end{aligned}$$

6. (Fill in the blank) A group of 8 friends A,B,C,D,E,F,G,H go to a restaurant. Due to safe-distancing measures, the group needs to split up into two groups of 4. How many ways are there to split the group such that A and B are together but away from C?

Answer: ________

SOLUTION

Except A, B, C, there are 5 people left. The group with A and B only has two more slots, the groups are set if and only if we select two more people out of 5 to fill in the slots of the group with A, B, and the rest 3 are with C. So the number of ways is $\binom{5}{2} = 10$.

7. (MCQ) A worker needs to drive to work from his home daily. There is only one route available, on which there are two speeding cameras working independently. The speeding cameras at each of these locations operates 50% and 75% of the time respectively. Based on the worker's driving habit, he will speed 40% of the time; and whether he will speed at different time points are independent. What is the probability that the worker will not receive a speeding ticket for each day?

Note: whether the camera is working at any time is also independent with whether a driver is speeding when s/he drives through that camera.

   (a) 0.56
   (b) 0.48
   (c) 0.36
   (d) 0.72

SOLUTION

Derivation: $A_i = \{\text{speed at carmera } i\}, B_i = \{\text{camera } i \text{ is on}\}, C_i = \{\text{recorded speeding at camera } i\}$. Then $C_i = A_i B_i$. Then $C_1' C_2'$ is the event that he will not receive a speeding ticket. Using independence leads to

$$\begin{aligned} Pr(C_1' C_2') &= Pr(C_1')Pr(C_2') = \{1 - Pr(A_1 B_1)\}\{1 - Pr(A_2 B_2) \\ &= \{1 - Pr(A_1)Pr(B_1)\}\{1 - Pr(A_2)Pr(B_2)\} \\ &= (1 - 0.5 \times 0.4)(1 - 0.75 \times 0.4) = 0.56. \end{aligned}$$

So, the answer is (a).

8. (MCQ) A new Covid test kit detects the virus 90% of the time if a patient is infected. However, it also detects the virus 5% of the time if a patient is uninfected. Given that the overall Covid infection rate is 1%, what is the probability of being infected if your test kit detects the virus?

   (a) 0.114
   (b) 0.215
   (c) 0.154
   (d) 0.322

   SOLUTION
   Let T = Tested Covid; D = Diseased, then

$$P(T|D) = 0.9, \quad P(T|D') = 0.05, \quad P(D) = 0.01.$$

   Therefore

$$\begin{aligned} P(D|T) &= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D')P(D')} \\ &= \frac{0.9 \times 0.01}{0.9 \times 0.01 + 0.05 \times 0.99} = 0.154. \end{aligned}$$

   Thus, the answer is (c).

9. (MCQ) Suppose $P(F) = P(G) = 0.4$. Which of the following statements must be true?

   (a) $P(F \cup G) = 0.8$
   (b) $P(F \cup G) = 0.4$
   (c) $P(F \cup G) > 0.4$
   (d) $P(F \cup G) \leq 0.8$

   SOLUTION
   Answer: (d).

10. (MCQ) There are 10 women and 20 men in a class. Find the number of samples of three that can be formed with two women and one man.

   (a) $\binom{30}{3}$
   (b) $\binom{30}{1} \cdot \binom{10}{2}$
   (c) $\binom{10}{2} \cdot \binom{20}{1}$
   (d) $\binom{30}{2} \cdot \binom{20}{1}$

   SOLUTION
   Answer: (c).

11. (MRQ) Let A and B be two events. Which of the following statements is/are true?

   (a) If $A \neq B$, then $P(A) \neq P(B)$.
   (b) If A and B are independent, then we must have $Pr(A \cup B) = 1 - \{1 - Pr(A)\}\{1 - Pr(B)\}$.
   (c) If $Pr(A) = 1 - Pr(B')$, then $Pr(A) = Pr(B)$.

(d) $(A \cap B') \cup (A' \cap B) = \emptyset$, then $A = B$.

SOLUTION
Answer: (b), (c), (d).

12. (MCQ) Consider the following statements about Peter whom you have not met before.

(A): He is not married.

(B): He is not married and smokes.

(C): He is married.

(D): He is married and does not smoke.

You are to assign probabilities to these statements. Which answer below is consistent with the laws of probability?

(a) $P(A) = 0.45, \quad P(B) = 0.5, \quad P(C) = 0.55, \quad P(D) = 0.4$

(b) $P(A) = 0.45, \quad P(B) = 0.1, \quad P(C) = 0.6, \quad P(D) = 0.3$

(c) $P(A) = 0.45, \quad P(B) = 0.2, \quad P(C) = 0.55, \quad P(D) = 0.5$

(d) $P(A) = 0.45, \quad P(B) = 0.4, \quad P(C) = 0.55, \quad P(D) = 0.6$

SOLUTION
Answer: (c).

13. (TRUE/FALSE) Let A and B be mutually exclusive events. If $P(A) = 0.1$, $P(B) = 0.01$, then A and B are not independent.

- TRUE
- FALSE

SOLUTION
TRUE; If otherwise, $P(A \cap B) = P(A)P(B) > 0$, which contradicts that A and B are mutually exclusive.

14. (MCQ) Suppose that A and B are any two events where $P(A) = 0.4$ and $P(A \cap B) = 0.2$. Then $P(A|B) =?$

(a) 0.4

(b) 0.5

(c) Not enough information to determine

(d) None of the above

SOLUTION
Answer: (c).

15. (TRUE/FALSE) Probability density function can not take on values greater than 1.

- TRUE
- FALSE

SOLUTION
Answer: FALSE.

16. (Fill in the blank) Suppose that random variable X has the cumulative distribution function given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{100}, & 0 \le x \le 10 \\ 1, & x > 10 \end{cases}$$

Compute $P(X \ge 4)$.

Answer: ________

SOLUTION
$P(X \ge 4) = 1 - P(X < 4) = 1 - F(4) = 1 - 4^2/100 = 0.84.$

17. (MCQ) The continuous random variable X has the following probability density function

$$f_X(x) = \begin{cases} \frac{1}{8}(1+3x), & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

The median of a continuous random variable Y, denoted by m_Y, is a real number satisfying $P(Y \le m_Y) = 0.5$. What is the median of X?

(a) 4/3

(b) 2/3

(c) 1

(d) 5/3

SOLUTION
For any $x \in [0,2]$,

$$P(X \le x) = \int_0^x \frac{1}{8}(1+3t)dt = \frac{1}{8}\left(x + \frac{3}{2}x^2\right).$$

Set $P(X \le m_X) = 0.5$,

$$\frac{1}{8}\left(m_X + \frac{3}{2}m_X^2\right) = 0.5,$$

which leads to $m_X = 4/3$ or $m_X = -2$ (removed because $m_X \in [0,2]$).

The answer is (a).

18. (Fill in the blank) Let X have probability mass function given by the following table.

x	0	2	5	6
$f(x)$	0.3	0.5	0.1	0.1

Compute $E(X)$.

Answer: ________

SOLUTION
$E(X) = 0(0.3) + 2(0.5) + 5(0.1) + 6(0.1) = 2.1.$

19. (Fill in the blank) The probability function for random variable X is given by

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 0.5, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Compute $E(X)$.

Answer: ________ (Write your answer in form x.xxx, rounded to the third decimal place.)

SOLUTION

$$E(X) = \int_0^1 x \cdot x dx + \int_2^3 x \cdot 0.5 dx = {}^1\!/_3 + {}^5\!/_4 = 1.583.$$

20. (MCQ) Let X be a random variable. Which of the following statement is **INCORRECT**?

(a) If $P(X = 1) = 0.1$ and $E(X)$ exists, then we must have $E(X^2) > (E(X))^2$.

(b) If $V(X) > 0$, then for any x, $P(X = x) < 1$.

(c) By the definition of the random variable, the range of X is a subset of $\mathbb{R}$; therefore, it is impossible that $P(X = x) = 0$ for any $x \in \mathbb{R}$.

(d) There are cases that $E(X)$ does not exist.

SOLUTION

Answer: (c)

- (a) is correct because $V(X) = E(X^2) - [E(X)]^2$ and $P(X = 1) = 0.1$ implies $V(X) > 0$ (since otherwise $P(X = E(X)) = 1$).
- (b) is correct with similar reason to (a).
- For any continuous RV, we must have $P(X = x) = 0$ for any $x \in \mathbb{R}$.
- For example, X is a random variable with p.d.f.

$$f(x) = \begin{cases} \frac{1}{2x^2}, & \text{for } |x| \ge 1, \\ 0, \text{elsewhere} \end{cases}$$

Then

$$E(X) = \int_{-\infty}^{1} x \cdot \frac{1}{2x^2} dx + \int_1^{\infty} x \cdot \frac{1}{2x^2} dx = -\infty + \infty,$$

which implies $E(X)$ does not exist.