Tutorial 11

Exercise 7

1. Determine whether the following are linear transformations. Write down the standard matrix for each of the linear transformations.

   (a) $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   (b) $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_2\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2^x \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   (c) $T_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   (d) $T_4 : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_4\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 \\ y-x \\ y-z \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

   (e) $T_5 : \mathbb{R}^n \to \mathbb{R}$ such that $T_5(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{y}$ for $\boldsymbol{x} \in \mathbb{R}^n$ where $\boldsymbol{y} = (y_1, y_2, \dots, y_n)^{\mathrm{T}}$ is a fixed vector in $\mathbb{R}^n$.

   (f) $T_6 : \mathbb{R}^n \to \mathbb{R}$ such that $T_6(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^n$.

   (In Parts (e) and (f), $\mathbb{R}$ is regarded as $\mathbb{R}^1$.)

2. For each of the following linear transformations, (i) determine whether there is enough information for us to find the formula of T; and (ii) find the formula and the standard matrix for T if possible.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that

   $$T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \quad T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 4 \end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 6 \end{pmatrix}.$$

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

   $$T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

   (f) $T : \mathbb{R}^3 \to \mathbb{R}$ such that

   $$T\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right) = -1, \quad T\left(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) = 1 \quad \text{and} \quad T\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right) = 0.$$

7. Let $\boldsymbol{n}$ be a unit vector in $\mathbb{R}^n$. Define $P : \mathbb{R}^n \to \mathbb{R}^n$ such that

   $$P(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n.$$

   (a) Show that P is a linear transformation and find the standard matrix for P.

   (b) Prove that $P \circ P = P$.

Tutorial 11 (cont.)

10. A linear operator T on $\mathbb{R}^n$ is called an *isometry* if $||T(\boldsymbol{u})|| = ||\boldsymbol{u}||$ for all $\boldsymbol{u} \in \mathbb{R}^n$.

   (a) If T is an isometry on $\mathbb{R}^n$, show that $T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. (Hint: Compute $T(\boldsymbol{u}+\boldsymbol{v}) \cdot T(\boldsymbol{u}+\boldsymbol{v})$ in two different ways.)

   (b) Let $\boldsymbol{A}$ be the standard matrix for a linear operator T. Show that T is an isometry if and only if $\boldsymbol{A}$ is an orthogonal matrix. (See also Question 5.32.)

   (c) Find all isometries on $\mathbb{R}^2$. (Hint: See Question 2.57.)

13. In each of the following parts, use the given information to find the nullity of the linearly transformation T.

   (a) $T : \mathbb{R}^4 \to \mathbb{R}^6$ has rank 2.

   (b) The range of $T : \mathbb{R}^6 \to \mathbb{R}^4$ is $\mathbb{R}^4$.

   (c) The reduced row-echelon form of the standard matrix for $T : \mathbb{R}^6 \to \mathbb{R}^6$ has 4 nonzero rows.