Matriculation Number:

A								

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2016-2017

MA1521 CALCULUS FOR COMPUTING

November 2016 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** Do not write your name anywhere in this booklet. This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **FIVE (5)** questions and comprises **TWENTY ONE (21)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question. The marks for each question are indicated at the beginning of the question. The maximum possible total score for this examination paper is 80 marks.
4. This is a **closed book (with authorized material)** examination. Students are only allowed to bring into the examination hall **ONE** piece A4 size help-sheet which must be handwritten and can be written on both sides.
5. Candidates may use any calculators that satisfy MOE A-Level examination guidelines. However, they should lay out systematically the various steps in the calculations.

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Question	1	2	3	4	5
(a)					
(b)					

Question 1 (a) [8 marks]

(i) (Multiple Choice Question)

Let $y = x^{2016}$. Find the **exact value** of $\frac{dy}{dx}$ at $x = 1$.

(A) 2016 (B) 2020 (C) 2001

(ii) Let $y = t^3 - t + 1$ and $x = t^2 + t + 2$. Find the **exact value** of $\frac{d^2y}{dx^2}$ when $t = 1$. Give your answer in the form of one single fraction in its simplest form.

Answer 1(a)(i)	A	Answer 1(a)(ii)	$\frac{14}{27}$

(Show your working below and on the next page.)

(i) $\frac{dy}{dx} = 2016x^{2015}$

$x=1 \Rightarrow \frac{dy}{dx} = \underline{\underline{2016}}$

(ii) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-1}{2t+1}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{6t(2t+1)-(3t^2-1)2}{(2t+1)^3}$

$t=1 \Rightarrow \frac{d^2y}{dx^2} = \frac{18-4}{27} = \underline{\underline{\frac{14}{27}}}$

Question 1 (b) [8 marks]

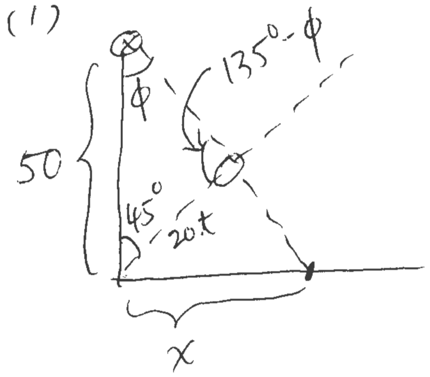
(i) A light is located at the top of a building which is 50 metres tall. At time $t = 0$, a bird flies out from the foot of the building at a constant velocity of 20 metre/second in a straight line flight path which makes a 45^o angle with the ground. Find the speed of the shadow of the bird on the ground at time $t = 2$ second. Give your answer correct to two decimal places.

(ii) Let a denote a positive constant with $a > 1$. It is known that $\int_1^{a^2} \frac{1}{2\sqrt{x}} e^{\sqrt{x}} dx = 10$. Find the value of a. Give your answer correct to two decimal places.
(Hint: You may want to use the substitution $u = \sqrt{x}$.)

Answer 1(b)(i)	74.97	Answer 1(b)(ii)	2.54

(Show your working below and on the next page.)

(i)



$$\frac{20t}{\sin\phi} = \frac{50}{\sin(135^\circ - \phi)} = \frac{50}{\frac{1}{\sqrt{2}}\cos\phi + \frac{1}{\sqrt{2}}\sin\phi}$$

$$20t\cos\phi + 20t\sin\phi = 50\sqrt{2}\sin\phi$$

$$2t + 2t\tan\phi = 5\sqrt{2}\tan\phi$$

$$\tan\phi = \frac{2t}{-2t + 5\sqrt{2}}$$

$$x = 50\tan\phi = \frac{100t}{5\sqrt{2} - 2t} \Rightarrow \frac{dx}{dt} = \frac{100(5\sqrt{2} - 2t) - 100t(-2)}{(5\sqrt{2} - 2t)^2}$$

$$\therefore t = 2 \Rightarrow \frac{dx}{dt} = \frac{500\sqrt{2}}{(5\sqrt{2} - 4)^2} \approx \underline{\underline{74.973\ldots}}$$

(ii) Let $u = \sqrt{x}$

$$\therefore du = \frac{1}{2\sqrt{x}} dx \Rightarrow \int_1^{a^2} \frac{1}{2\sqrt{x}} e^{\sqrt{x}} dx = \int_1^a e^u du = e^a - e$$

$$\therefore e^a - e = 10 \Rightarrow e^a = 10 + e \Rightarrow a = \ln(10 + e)$$

$$= \underline{\underline{2.543\ldots}}$$

Question 2 (a) [8 marks]

(i) It is known that the area of the region bounded between $y = 101\cos^2 x$, $y = -201\sin^2 x$, $x = 0$ and $x = \frac{3\pi}{4}$ is equal to $n + k\pi$, where n denotes a positive integer and k denotes a fraction. Find the **exact value** of n.

(ii) Let R denote the finite region in the first quadrant bounded by the curve $y = \sqrt{|x \ln x|}$, the x-axis, the line $x = e^{-30}$ and the line $x = 1$. It is known that the volume of the solid formed by revolving R one complete round about the x-axis is equal to $\frac{\pi}{4}\left(A - Be^{-C}\right)$ where A, B, C denote three positive integers. Find the **exact value** of B.

Answer 2(a)(i)	25	**Answer 2(a)(ii)**	61

(Show your working below and on the next page.)

(i) $\int_0^{\frac{3\pi}{4}} \left[101\cos^2 x - (-201\sin^2 x)\right] dx$

$= \int_0^{\frac{3\pi}{4}} (101 + 100\sin^2 x)\, dx = \frac{303\pi}{4} + \int_0^{\frac{3\pi}{4}} 50(1-\cos 2x)\, dx$

$= \frac{303\pi}{4} + \frac{150\pi}{4} - \left[25\sin 2x\right]_0^{\frac{3\pi}{4}} = \underline{\underline{25}} + \frac{453}{4}\pi$

(ii) $\int_{e^{-30}}^{1} \pi y^2\, dx = \pi \int_{e^{-30}}^{1} |x \ln x|\, dx = -\pi \int_{e^{-30}}^{1} x \ln x\, dx$

$= \frac{1}{4}\pi - \frac{61}{4}\pi e^{-60} = \frac{\pi}{4}(1 - 61e^{-60})$

$\therefore B = \underline{\underline{61}}$

Question 2 (b) [8 marks]

(i) (Multiple Choice Question)

Find the **exact value** of the sum of the geometric series

$$3 - \frac{1}{2} + \ldots.$$

(A) $\frac{18}{7}$ (B) $\frac{17}{8}$ (C) $\frac{16}{5}$

(ii) Let $f(x) = \int_0^x t^{20} \sin t dt$. Find the **exact value** of $f^{(86)}(0)$. Give your answer in the form $\frac{n!}{m!}$ where n and m denote two positive integers.

Answer 2(b)(i)	A	**Answer 2(b)(ii)**	$\frac{85!}{65!}$

(Show your working below and on the next page.)

(i) $r = \frac{-1/2}{3} = -\frac{1}{6}$

$S = \frac{3}{1-(-\frac{1}{6})} = \frac{18}{7}$

(ii) $f(x) = \int_0^x t^{20} \sin t dt = \int_0^x t^{20} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} dt$

$= \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n+21}}{(2n+1)!} dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+22}}{(2n+22)[(2n+1)!]}$

$2n+22 = 86 \Rightarrow n = 32$

$\therefore \frac{f^{(86)}(0)}{86!} = \frac{(-1)^{32}}{(86)(65!)} \Rightarrow f^{(86)}(0) = \frac{85!}{65!}$

Question 3 (a) [8 marks]

(i) Let $\sum_{n=0}^{\infty} c_n x^n$ denote the Taylor series for $\frac{x^2}{1-2x}$ at $x = 0$.
Find the **exact value** of c_5.

(ii)It is known that

$$\sum_{n=1}^{\infty} \frac{1}{n!\,(n+3)} = Ae - B,$$

where A denotes a positive integer and B denotes a positive fraction in its simplest form. Find the **exact value** of B.
(Hint: Integrate the Taylor series of $x^2 e^x$. Note that the sum starts from $n = 1$ and not from $n = 0$.)

Answer 3(a)(i)	8	Answer 3(a)(ii)	$\frac{7}{3}$

(Show your working below and on the next page.)

(i) $\frac{x^2}{1-2x} = \sum_{n=0}^{\infty} x^2 (2x)^n = \sum_{n=0}^{\infty} 2^n x^{n+2}$

$C_5 = 2^3 = \underline{\underline{8}}$

(ii) $x^2 e^x = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$

$\int_0^1 x^2 e^x \, dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^{n+2}}{n!} dx = \sum_{n=0}^{\infty} \frac{1}{n!(n+3)} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{n!(n+3)}$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n!(n+3)} = \int_0^1 x^2 e^x \, dx - \frac{1}{3} = e - 2 - \frac{1}{3} = e - \underline{\underline{\frac{7}{3}}}$

Question 3 (b) [8 marks]

(i) An elliptic cylinder has semi-major axis x metres, semi-minor axis y metres and height z metres. Its volume is given by the formula $V = \pi xyz$ cubic metres. At time t_0 we have $x = 5$, $y = 3$ and $z = 2$ and we know that x is increasing at a rate of 3 metres per second, y is increasing at a rate of 2 metres per second and z is increasing at a rate of 4 metres per second. Its volume at time t_0 is found to be increasing at a rate of $n\pi$ cubic metres per second, where n is a positive integer. Find the **exact value** of n.

(ii) Let $f(x, y, z) = x^2 + y^2 + z^2$. Find the largest possible value of $D_{\mathbf{u}} f(1, 2, 3)$. Give your answer correct to two decimal places.

Answer 3(b)(i)	98	Answer 3(b)(ii)	7.48

(Show your working below and on the next page.)

(i) $V = \pi x y z \Rightarrow \frac{dV}{dt} = \pi y z \frac{dx}{dt} + \pi x z \frac{dy}{dt} + \pi x y \frac{dz}{dt}$

$\therefore n\pi = \pi(3)(2)(3) + \pi(5)(2)(2) + \pi(5)(3)(4)$

$= 18\pi + 20\pi + 60\pi = 98\pi$

$\therefore n = \underline{\underline{98}}$

(ii) $\nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\nabla f(1,2,3) = 2\vec{i} + 4\vec{j} + 6\vec{k}$

$\therefore \max D_u f(1,2,3) = \|2\vec{i} + 4\vec{j} + 6\vec{k}\| = \sqrt{56} \approx \underline{\underline{7.483\ldots}}$

Question 4 (a) [8 marks]

(i) Let $f(x,y,z) = 2\sqrt{1+e^{xyz}}$. Use the method of directional derivative to estimate how much the value of f will change if a point Q moves 0.15 unit from the point $(1,1,1)$ towards $(2,4,1+\sqrt{15})$. Give your answer correct to two decimal places.
(Note: ZERO mark if you do not use the method of directional derivative.)

(ii) (Multiple Choice Question)

It is known that the function $f(x,y) = x^3+y^3-3xy+1521$ has two critical points. Classify the two critical points.

(A) One saddle point and one Local Maximum (B) One saddle point and one Local Minimum (C) Both are Saddle Points

Answer 4(a)(i)	0.33	Answer 4(a)(ii)	B

(Show your working below and on the next page.)

(i) $\nabla f = \dfrac{yze^{xyz}}{\sqrt{1+e^{xyz}}}\vec{i} + \dfrac{xze^{xyz}}{\sqrt{1+e^{xyz}}}\vec{j} + \dfrac{xye^{xyz}}{\sqrt{1+e^{xyz}}}\vec{k}$

$\nabla f(1,1,1) = \dfrac{e}{\sqrt{1+e}}(\vec{i}+\vec{j}+\vec{k})$

$\vec{u} = \dfrac{\vec{i}+3\vec{j}+\sqrt{15}\vec{k}}{\|\vec{i}+3\vec{j}+\sqrt{15}\vec{k}\|} = \frac{1}{5}(\vec{i}+3\vec{j}+\sqrt{15}\vec{k})$

$\Delta f \approx (D_{\vec{u}}f(1,1,1))(\Delta t) = \dfrac{e}{5\sqrt{1+e}}(1+3+\sqrt{15})(0.15) \approx \underline{\underline{0.332}}$

(ii) $f_x = 3x^2-3y = 0 \Rightarrow x^2 = y$
$f_y = 3y^2-3x = 0 \Rightarrow y^2 = x$
$\Rightarrow (0,0), (1,1)$ are the critical points.

$f_{xx} = 6x, \ f_{xy} = -3, \ f_{yy} = 6y$

$f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9$

$\therefore (0,0)$ = saddle
$(1,1)$ = local minimum //

Question 4 (b) [8 marks]

(i) Let y denote the solution of the differential equation $\frac{dy}{dx} = e^{x-2y}$ such that $y = 0$ when $x = 0$. Find the value of y when $x = 2$. Give your answer correct to two decimal places.

(ii) A body was found at UTown. You are a member of the CSI team and you arrived at the crime scene at 8:15am. Immediately upon arrival, you took the temperature of the victim and found that it was 32°C. At 9:05am you took the temperature of the victim again and found that it was 30°C. You estimated that the victim's temperature was 37°C just before death and that the temperature of UTown stayed approximately constant at 25°C. What is your estimate on the time of death? Give your answer correct to the nearest minute in the form X:YZam.

Answer 4(b)(i)	1.31	Answer 4(b)(ii)	6:55

(Show your working below and on the next page.)

(i) $\frac{dy}{dx} = \frac{e^x}{e^{2y}} \Rightarrow e^{2y}dy = e^x dx \Rightarrow \frac{1}{2}e^{2y} = e^x + C$

$y=0$ when $x=0 \Rightarrow \frac{1}{2} = 1 + C \Rightarrow C = -\frac{1}{2}$

$\therefore \frac{1}{2}e^{2y} = e^x - \frac{1}{2}$

$x = 2 \Rightarrow \frac{1}{2}e^{2y} = e^2 - \frac{1}{2} \Rightarrow y = \frac{1}{2}\ln(2e^2 - 1) \approx \underline{1.31}1\cdots$

(ii) Set $t=0$ at 8:15 am, t measured in minutes.

$\frac{dT}{dt} = k(T-25) \Rightarrow T - 25 = Ae^{kt}$; $T(0) = 32 \Rightarrow A = 7 \Rightarrow T = 25 + 7e^{kt}$

$T(50) = 30 \Rightarrow 5 = 7e^{50k} \Rightarrow k = \frac{\ln 5 - \ln 7}{50}$

$37 = 25 + 7e^{kt} \Rightarrow t = \frac{50(\ln 12 - \ln 7)}{\ln 5 - \ln 7} \approx -80.09$

$\therefore$ Answer is $\underline{\underline{6:55 \text{ am}}}$

Question 5 (a) [8 marks]

~~(i) A certain species of birds were protected by law and they settled down to a logistic equilibrium population of 182000 with a birth rate per capita of 8.6% per year. Eventually the hunting ban was lifted and the hunters were allowed to shoot E birds per year. Find the theoretical maximum value of E which will not make the bird population go to extinction. Give your answer correct to the nearest integer.~~

(ii) Suppose that y is a function of x that satisfies the differential equation

$$x\frac{dy}{dx} + 2y = 8x^2, \quad x > 0, \quad y(1) = 3.$$

Find the value of $y(3)$. Give your answer correct to two decimal places.

Answer 5(a)(i)	3913	**Answer 5(a)(ii)**	18.11

(Show your working below and on the next page.)

(i) $\frac{B}{s} = 182000 \Rightarrow s = \frac{0.086}{182000}$

$$\max E = \frac{B^2}{4s} = \frac{0.086^2 \times 182000}{4 \times 0.086} = \underline{\underline{3913}}$$

(ii) $\frac{dy}{dx} + \frac{2}{x}y = 8x$

$$R = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$y = \frac{1}{x^2}\int 8x^3 dx = \frac{1}{x^2}(2x^4 + c)$$

$$y(1) = 3 \Rightarrow 3 = (2 + c) \Rightarrow c = 1$$

$$\therefore y = 2x^2 + \frac{1}{x^2}$$

$$y(3) = 18 + \frac{1}{9} = \frac{163}{9} \approx \underline{\underline{18.111\ldots}}$$

Question 5 (b) [8 marks]

(i) (Multiple Choice Question)
Find the general solution of the differential equation

$$y'' - 8y' + 15y = 0.$$

(A) $y = C_1e^{2x} + C_2e^{5x}$
(B) $y = C_1e^{3x} + C_2e^{6x}$
(C) $y = C_1e^{3x} + C_2e^{5x}$

(ii) Let a and b denote two positive constants. The population of a certain species of fish is described by the equation

$$\frac{dN}{dt} = -10N^4 + aN^3 - bN^2 + 504N,$$

where N is the population measured in millions. After a long time the population settled down to a stable equilibrium at around 5 million. Then there was a terrible storm which killed a large number of fish. Another long time after the storm, the population settled down to a new stable equilibrium at around 2.8 million. Find the exact value of the constant a.

Answer 5(b)(i)	C	**Answer 5(b)(ii)**	114

(Show your working below and on the next three pages.)

(i) $\lambda^2 - 8\lambda + 15 = 0 \Rightarrow \lambda = 3, 5 \Rightarrow \underline{y = C_1e^{3x} + C_2e^{5x}}$

(ii) We have $-10N^4 + aN^3 - bN^2 + 504N = -10N(N-5)(N-2.8)(N-c)$

$= -10N^4 + N^3(10c + 78) - N^2(78c + 140) + 140cN$

$\therefore \; c = \frac{504}{140} = 3.6$

$\therefore \; a = 10c + 78 = 36 + 78 = \underline{\underline{114}}$