MA1101R Summary Notes

<table>
<tr><td></td><td>Week 1</td></tr>
<tr><td>scope</td><td>• 1.1 Linear Systems and their solutions
• 1.2 Elementary Row Operations
• 1.3 Row-Echelon Forms
• 1.4 Gaussian Elimination</td></tr>
<tr><td>objective</td><td>• What is a linear equation and a linear system?
• What is a general solution of a linear equation/system?
• What is the geometrical interpretation of a linear equation/syste and its solutions?
• How to find a general solution of a linear equation?
• What are the three elementary row operations (ERO)?
• How to perform ERO?
• What is meant by row equivalence?
• How to identify a row-echelon form (REF) and a reduced row-echelon form (RREF)?
• How to use REF / RREF to get solutions of linear system?
• What are Gaussian elimination (GE) and Gauss-Jordan elimination (GJE)?
• How to use GE / GJE to reduce an augmented matrix to a REF / RREF ?</td></tr>
<tr><td>summary</td><td>• A linear equation with two or more variables has infinitely many solutions.
• A linear system has either no solution, exactly one solution, or infinitely many solutions.
• Elementary row operations do not change the solution set of a linear system.
• Two linear systems have the same solution set if their augmented matrices are row equivalent.
• The solutions of a LS can be obtained from its REF
• An augmented matrix has many REF but only one RREF
• Given any matrix, we can always apply GE (resp. GJE) to reduce the matrix into REF (resp. RREF)</td></tr>
<tr><td>Ex</td><td>• Exercise 1: 1 - 21</td></tr>
</table>

	Week 2
scope	• 1.4 Gaussian Elimination • 1.5 Homogeneous Linear System • 2.1 Introduction to Matrices • 2.2 Matrix Operations
objective	• How to tell the number of solutions of linear system from REF? • How to use GE / GJE to solve indirect linear system problems? • What is a **homogeneous** system? • What is a **trivial / non-trivial** solution of a homogeneous system? • What are the **size**, **entries**, **order** of a matrix? • What are **diagonal**, **identity**, **symmetric**, **triangular** matrices? • How to perform **matrix addition**, **matrix multiplication**, **scalar multiplication** and **transpose**? • How to express certain matrices and operations using (i, j)-entries? • What are some properties of matrix operations? • What are some different ways to express matrix multiplication? • How to express linear system in matrix equation form?
summary	• A LS has no solution if and only if the last column of its REF is a pivot column • In a REF, # non-zero rows = # leading entries = # pivot columns • In a consistent linear system, • if # variables = # non-zero rows, then the system has exactly one soln • if # variables > # non-zero rows, then the system has infinite solns • A homogeneous system is always consistent, as it always has the trivial solution. • If a homogeneous system has a non-trivial solution, then it has infinitely many solutions. • A homogeneous system with more variables than equations has infinitely many solutions. • We do not refer to solutions for a non-homogeneous system as trivial or non-trivial. • (i, j)-entry of $\boldsymbol{AB} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$ • Matrix multiplication: $\boldsymbol{AB} \neq \boldsymbol{BA}$ (in general) • Matrix multiplication: $\boldsymbol{AB} = \mathbf{0}$ does not imply $\boldsymbol{A} = \mathbf{0}$ or $\boldsymbol{B} = \mathbf{0}$ • Linear system can be expressed in *matrix equation form* and *column form* • Matrix transpose: $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$ • $\boldsymbol{A}$ is a symmetric matrix if and only if $\boldsymbol{A} = \boldsymbol{A}^T$
Ex	• **Exercise 1**: 22 - 30 • **Exercise 2**: 1 - 24

	Week 3
scope	• 2.3 Inverses of Square Matrices • 2.4 Elementary Matrices • 2.4 Elementary Matrices • 2.5 Determinants
objective	• What is an **invertible** matrix? • What is the **inverse** of a matrix? • What are the **powers** of a matrix? • What are **elementary** matrices? • How are elementary matrices related to elementary row operations? • How to find inverse of an elementary matrix? • What are some different ways to show a matrix is invertible? • How to find the inverse of an invertible matrix? • What is the **determinant** of a matrix? • What is **cofactor expansion** of a matrix?
summary	• If $\boldsymbol{A}$ is invertible, then $\boldsymbol{AB_1} = \boldsymbol{AB_2} \Rightarrow \boldsymbol{B_1} = \boldsymbol{B_2}$ • If $\boldsymbol{A}$ is invertible, then $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$ • If $\boldsymbol{A}$, $\boldsymbol{B}$ are invertible, then $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ • There are three types of elementary matrices • All elementary matrices are invertible • $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent if $\boldsymbol{A} = \boldsymbol{E_n} \ldots \boldsymbol{E_2 E_1 B}$ where all $\boldsymbol{E_i}$ are elementary matrices. • $\boldsymbol{A}$ is invertible $\leftrightarrow$ RREF of $\boldsymbol{A}$ is the identity matrix $\boldsymbol{I}$ • $\boldsymbol{A}$ is invertible $\leftrightarrow$ $\boldsymbol{Ax} = \boldsymbol{0}$ has only trivial solution • Cofactor expansion of a matrix along any row (column) gives the determinant • Determinant of a triangular (diagonal) matrix is the product of its diagonal entries
Ex	• **Exercise 2**: 25 -47

	Week 4
scope	• 2.5 Determinants • 3.1 Euclidean n-spaces
objective	• How do matrix operations affect determinants? • What is the relation between invertibility and determinant? • What is the **adjoint** of a matrix? • What is **Cramer's rule**? • What is an **n-vector**? • What are some operations on n-vectors? • What is a **Euclidean n-space** $\mathbf{R}^n$? • How to express subsets of $\mathbf{R}^n$?
summary	• $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$ • If $\boldsymbol{A}$ has two identical rows (columns), then $\det(\boldsymbol{A}) = 0$ • Interchanging two rows/columns will change determinant by a negative sign • Adding a multiple of a row (column) to another will not change determinant. • $\boldsymbol{A}$ is invertible $\leftrightarrow \det(\boldsymbol{A}) \neq 0$ • If $\boldsymbol{A}$ is $n \times n$ and c is a scalar, then $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$ • $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$ • $\det(\boldsymbol{A}^{-1}) = 1/\det(\boldsymbol{A})$ • $\boldsymbol{A}^{-1} = [1/\det(\boldsymbol{A})]\ \mathrm{adj}(\boldsymbol{A})$ • Cramer's rule is a method to solve $\boldsymbol{Ax} = \boldsymbol{b}$ when $\boldsymbol{A}$ is invertible • 2-vectors and 3-vectors can be expressed geometrically and algebraically • n-vectors ($n > 3$) can only be expressed algebraically • Subsets of $\mathbf{R}^n$ can be expressed in implicit and explicit forms • Lines/planes are subsets of $\mathbf{R}^2$ and $\mathbf{R}^3$ • Solution set of n variable LS is a subset of $\mathbf{R}^n$
Ex	• **Exercise 2**: 48 - 61 • **Exercise 3**: 1 - 7

	Week 5
scope	• 3.2 Linear Combinations and Linear Spans • 3.3 Subspaces
objective	• What is a **linear combination**? • How to express a vector as a linear combination? • What is a **linear span**? • What is a **subspace**? • What are some examples of subspaces of $\mathbf{R}^n$? • What is a **solution space** of a linear system? • How to show a linear span is contained in another?
summary	• Linear span (of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$) = the set of all linear combinations (of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$) • A subset of $\mathbf{R}^n$ is a subspace if it is a linear span of some fixed n-vectors • A subspace of $\mathbf{R}^n$ always contains the zero vector • Any linear combination of vectors in a subspace *V* is again a vector in *V*. • $\{\mathbf{0}\}$ and $\mathbf{R}^n$ are subspaces of $\mathbf{R}^n$ • In $\mathbf{R}^2$ and $\mathbf{R}^3$, span$\{\boldsymbol{u}\}$ is a line if $\boldsymbol{u} \neq \mathbf{0}$; span$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a plane if $\boldsymbol{u}$ not parallel to $\boldsymbol{v}$. • The solution set of a homogeneous system with n variables is a subspace of $\mathbf{R}^n$. • To show span$(S_1) \subseteq$ span(S_2), just need to show every vector in S_1 is a linear combination of vectors in S_2. • If $\boldsymbol{u} \in$ span(S) , then span(S) = span($S \cap \boldsymbol{u}$)
Ex	• **Exercise 3**: 8 - 24

	Week 6
scope	• 3.4 Linear Independence • 3.5 Bases
objective	• What is a **linearly independent/dependent** set? • How to show that a set is linearly (in)dependent? • What are some conditions on linearly (in)dependent sets? • What is a **basis** for a vector space? • How to show that a set is a basis? • How to find a basis for a vector space? • What are **coordinate vectors**?
summary	• $\boldsymbol{u}$ and $\boldsymbol{v}$ are scalar multiples of each other $\leftrightarrow$ $\{\boldsymbol{u}, \boldsymbol{v}\}$ is linearly dependent. • If S contains **0**, then S is linearly dependent. • S is linearly dependent $\leftrightarrow$ at least one vector in S is a linear combination of the other vectors in S • $\{\boldsymbol{u}, \boldsymbol{v}\} \subseteq \mathbf{R}^2$ are linearly independent $\leftrightarrow$ span$\{\boldsymbol{u}, \boldsymbol{v}\} = \mathbf{R}^2$ • $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \subseteq \mathbf{R}^3$ are linearly independent $\leftrightarrow$ span$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = \mathbf{R}^3$ • If $S \subseteq \mathbf{R}^n$ and S has more than n elements, then S is linearly dependent. • $\mathbf{R}^n$ has the standard basis (for all n) • Every non-zero vector space has infinitely many different bases • All bases for the same vector space V has the same number of vectors (called the dimension of V). • Every vector in a vector space can be expressed as linear combination of a basis in a unique way • S is a basis for span(S) $\leftrightarrow$ S is linearly indep
Ex	• **Exercise 3**: 25 - 35

	Week 7
scope	• 3.6 Dimensions • 3.7 Transition Matrices
objective	• What is the **dimension** of a vector space? • How to compute dimension for a vector space? • What are some conditions for a set to be a basis for a vector space? • What is a **transition matrix**? • How to compute transition matrices? • What is the relation between same coordinate vectors w.r.t. different bases?
summary	• If S has more than dim(V) of vectors, then S is linearly dependent • If S has less than dim(V) of vectors, then S cannot span V • dim(solution space) = # parameters in gen. soln. • $W \subseteq V \Rightarrow \dim(W) \leq \dim(V)$ • $W \subseteq V$ and $\dim(W) = \dim(V) \Rightarrow W = V$ • S is linearly independent and $\lvert S\rvert = \dim V \Rightarrow S$ is a basis for V • S spans V and $\lvert S\rvert = \dim V \Rightarrow S$ is a basis for V • $\boldsymbol{A}$ is an invertible n×n matrix $\leftrightarrow$ rows (columns) of $\boldsymbol{A}$ form a basis for $\mathbf{R}^n$ • Suppose **P** is the transition matrix from S to T. Then ➢ $[\boldsymbol{w}]_T = \mathbf{P}\,[\boldsymbol{w}]_S$ for any vector $\boldsymbol{w}$ in V ➢ $\boldsymbol{P}$ is invertible ➢ $\boldsymbol{P}^{-1}$ is the transition matrix from T to S
Ex	• **Exercise 3**: 36 - 49

	Week 8
scope	• 4.1 Row spaces and Column spaces • 4.2 Ranks • 4.3 Nullspaces and Nullities
objective	• What are **row space** and **column space** of a matrix? • How to find bases for row /column spaces? • How to use row /column spaces to find bases for vector spaces? • How to **extend a basis**? • What is the relation between column space and consistency of linear system? • What is the **rank** of a matrix? • What is the relation between rank and invertibility of a matrix? • What is the relation between rank and consistency of linear system? • What is the **nullspace** and **nullity** of a matrix? • What is the **Dimension Theorem**? • What is the relation between nullspace and solution set of a linear system?
summary	• The row space (resp. column space) of an m x n matrix is a subspace of $\mathbf{R}^n$ (resp. $\mathbf{R}^m$) • Row operations preserve row space but do not preserve column space • If ***R*** is an REF of ***A***, then the non-zero rows of ***R*** form a basis for row space of ***A*** • Those columns of ***A*** that correspond to the pivot columns in an REF form a basis for the column space of ***A*** • A basis for span(S) can be found using row space or column space methods • We can extend a basis using row space method • Row space and column space of a matrix have the same dimension (rank) • Largest possible rank of an m×n matrix is min{m,n} • An n×n matrix ***A*** is invertible ↔ rank(***A***) = n ↔ nullity(***A***) = 0 • Dimension Theorem: rank(***A***) + nullity(***A***) = # of columns of ***A*** • For the linear system ***Ax*** = ***b*** • if ***b*** belongs to column space of ***A***, system is consistent • solution set of the system= (Nullspace of ***A***) + (fix solution of ***Ax* = *b***)
Ex	• **Exercise 4**: 1 - 26

<table>
<tr><td></td><td>Week 9</td></tr>
<tr><td>scope</td><td>• 5.1 Inner Products in $\mathbf{R}^n$
• 5.2 Orthogonal and Orthonormal Bases</td></tr>
<tr><td>objective</td><td>• What are the algebraic representation of length, distance and angles in $\mathbf{R}^n$?
• What is the dot product of vectors?
• What is an orthogonal/orthonormal set?
• How to normalize a vector?
• What are the properties of orthogonal sets?
• What is the projection of a vector onto a subspace?
• What is Gram-Schmidt Process?</td></tr>
<tr><td>summary</td><td>• $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{u}\boldsymbol{v}^T$ (RHS is matrix multiplication, $\boldsymbol{u}, \boldsymbol{v}$ as rows)
• $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{u}^T\boldsymbol{v}$ (RHS is matrix multiplication, $\boldsymbol{u}, \boldsymbol{v}$ as columns)
• $\boldsymbol{u}\cdot\boldsymbol{u} = 0 \leftrightarrow ||\boldsymbol{u}|| = 0 \leftrightarrow \boldsymbol{u} = \mathbf{0}$
• If $\boldsymbol{u}$ is non-zero, then $\frac{1}{||\mathbf{u}||}\mathbf{u}$ is a unit vector
• If S is an orthogonal set of nonzero vectors, then S is linearly independent
• If $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is an orthogonal basis for V and $\boldsymbol{w} \in V$, then
$$\mathbf{w} = \frac{\mathbf{w}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\mathbf{w}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 + \cdots + \frac{\mathbf{w}\cdot\mathbf{u}_k}{\|\mathbf{u}_k\|^2}\mathbf{u}_k$$
• If $\boldsymbol{p}$ is the projection of $\boldsymbol{w}$ onto a subspace V, then
    • $\boldsymbol{w} - \boldsymbol{p}$ is orthogonal to V
    • $d(\boldsymbol{w}, \boldsymbol{p}) \le d(\boldsymbol{w}, \boldsymbol{v})$ for any vector $\boldsymbol{v}$ in V
• If $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is an orthogonal basis for V, then the projection of $\boldsymbol{w}$ onto V is
$$\mathbf{p} = \frac{\mathbf{w}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\mathbf{w}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 + \cdots + \frac{\mathbf{w}\cdot\mathbf{u}_k}{\|\mathbf{u}_k\|^2}\mathbf{u}_k$$
• Gram-Schmidt Process converts any basis for a vector space to an orthogonal (orthonormal) basis</td></tr>
<tr><td>Ex</td><td>• Exercise 5: 1 - 20</td></tr>
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	Week 10
scope	• 5.3 Best Approximation • 5.4 Orthogonal Matrices • 6.1 Eigenvalues and Eigenvectors
objective	• What is a **Least Squares solution**? • How to find the best approximate solution to inconsistent system? • What is an **orthogonal matrix**? • How is orthogonal matrix related to orthonormal basis? • How is transition matrix related to orthogonal matrix? • What are **eigenvalue**, **eigenvectors** and **eigenspace**? • How to find eigenvalues and eigenvectors of a matrix? • How to find basis for eigenspace of a matrix? • How is eigenvalue related to invertibility of matrix?
summary	• The least squares solutions to $\boldsymbol{Ax} = \boldsymbol{b}$ is given by • the solutions of $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$ • the solutions of $\boldsymbol{Ax} = \boldsymbol{p}$ • The inverse of an orthogonal matrix is its transpose • Rows/columns of n x n orthogonal matrix form orthonormal basis for $\mathbf{R}^n$ • Transition matrix between two orthonormal bases is an orthogonal matrix • λ is an eigenvalue of $\boldsymbol{A} \Leftrightarrow \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ • 0 is an eigenvalue of $\boldsymbol{A} \Leftrightarrow \det(\boldsymbol{A}) = 0$ • $\boldsymbol{A}$ is invertible $\Leftrightarrow$ 0 is not an eigenvalue of $\boldsymbol{A}$. • The eigenvalues of a triangular matrix are the diagonal entries. • The eigenvectors of $\boldsymbol{A}$ are the solutions of $(\lambda \boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{x} = \mathbf{0}$ • If $\boldsymbol{u}$ and $\boldsymbol{v}$ are eigenvectors of $\boldsymbol{A}$ associated with the same eigenvalue λ, then $\boldsymbol{u} + \boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$.
Ex	• **Exercise 5**: 21 - 34 • **Exercise 6**: 1 - 8

	Week 11
scope	• 6.2 Diagonalization • 6.3 Orthogonal Diagonalization
objective	• What is a **diagonalizable** matrix? • How to determine if a matrix is diagonalizable? • How to diagonalize a matrix? • How to compute powers of matrix using diagonalization? • How to solve linear recurrence relation using diagonalization? • What is **orthogonal diagonalization**? • What is the characterization of a matrix that is orthogonally diagonalizable? • How to orthogonally diagonalize a symmetric matrix?
summary	• An $n \times n$ matrix $\boldsymbol{A}$ is diagonalizable $\leftrightarrow$ $\boldsymbol{A}$ has n linearly independent eigenvectors • A set of eigenvectors associated with different eigenvalues are linearly independent • If geometric multiplicity < algebraic multiplicity for some eigenvalue, the matrix is not diagonalizable. • If an $n \times n$ matrix $\boldsymbol{A}$ has n distinct eigenvalues, then $\boldsymbol{A}$ is diagonalizable. • If $\boldsymbol{A}$ is diagonalizable, then $$\mathrm{A}^m = \mathrm{P}\begin{pmatrix} \lambda_1^m & & & \\ & \lambda_2^m & & \\ & & \ddots & \\ & & & \lambda_n^m \end{pmatrix}\mathrm{P}^{-1}$$ where $\boldsymbol{P}$ is the matrix of eigenvectors and λ_i are the eigenvalues • A matrix is orthogonally diagonalizable if and only if it is symmetric. • If $\boldsymbol{A}$ is a symmetric matrix, and $\boldsymbol{u}$, $\boldsymbol{v}$ are two eigenvectors of $\boldsymbol{A}$ associated with distinct eigenvalues, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal
Ex	• **Exercise 6**: 9 - 30

	Week 12
scope	• 7.1 Linear Transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$ • 7.2 Ranges and Kernel
objective	• What is a **linear transformation**? • How are linear transformations related to matrices? • What are the conditions of a linear transformation? • How to use basis to determine linear transformation? • What is the **composition** of linear transformations? • What are the **range** and **kernel** of a linear transformation? • What are the **rank** and **nullity** of a linear transformation? • What is the **Dimension Theorem of linear transformation**?

summary	• A linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$
	    • is a mapping between two vector spaces
	    • is defined by matrix multiplication
	    • maps zero vector to zero vector
	    • preserves linear combinations
	    • If $\boldsymbol{A}$ is the standard matrix of T, then $T(\boldsymbol{u}) = \boldsymbol{Au}$ for all $\boldsymbol{u} \in \mathbf{R}^n$
	    • If $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ is a basis for $\mathbf{R}^n$, then T is completely determined by the images $T(\boldsymbol{u_1}), T(\boldsymbol{u_2}), \ldots, T(\boldsymbol{u_n})$
	    • The standard matrix of the composition $T \circ S$ is the product of the standard matrices of T and S.
	• T: $\mathbf{R}^n \to \mathbf{R}^m$ linear transformation with standard matrix $\boldsymbol{A}$
	    • Range of T = column space of $\boldsymbol{A}$ (subspace of $\mathbf{R}^m$)
	    • Kernel of T = nullspace of $\boldsymbol{A}$ (subspace of $\mathbf{R}^n$)
	    • rank(T) = rank($\boldsymbol{A}$)
	    • nullity(T) = nullity($\boldsymbol{A}$)
	    • rank(T) + nullity(T) = n

Ex	• **Exercise 7**: 1 - 17