Tutorial 6

Exercise 3

32. Determine which of the following sets are bases for $\mathbb{R}^3$.

    (a) $S_1 = \{(1, 0, -1), (-1, 2, 3)\}$.

    (b) $S_2 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 0)\}$.

    (c) $S_3 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 3)\}$.

    (d) $S_4 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 0), (1, -1, 1)\}$.

33. Find a basis for the solution space of each of the following homogeneous systems.

$$\text{(c)} \quad \begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 0 \\ -3x_2 + x_3 = 0 \\ x_1 - x_4 = 0. \end{cases}$$

40. Let $\boldsymbol{u_1} = (1, 0, 1, 1)$, $\boldsymbol{u_2} = (-3, 3, 7, 1)$, $\boldsymbol{u_3} = (-1, 3, 9, 3)$, $\boldsymbol{u_4} = (-5, 3, 5, -1)$ and let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ and $V = \text{span}(S)$.

    (a) Find a non-trivial solution to the equation
    $$a\boldsymbol{u_1} + b\boldsymbol{u_2} + c\boldsymbol{u_3} + d\boldsymbol{u_4} = \boldsymbol{0}.$$

    (b) Express $\boldsymbol{u_3}$ and $\boldsymbol{u_4}$ (separately) as linear combinations of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

    (c) Find a basis for and determine the dimension of V.

    (d) Find a subspace W of $\mathbb{R}^4$ such that $\dim(W) = 3$ and $\dim(W \cap V) = 2$. Justify your answer.

44. Let $U = \text{span}\{u_1, u_2, u_3\}$ and $V = \text{span}\{v_1, v_2, v_3\}$ be subspaces of $\mathbb{R}^5$ such that $\dim(U \cap V) = 2$. Suppose W is the smallest subspace of $\mathbb{R}^5$ that contains both U and V. Determine all possible dimensions of W. Justify your answers.

46. (a) Let $\boldsymbol{u_1} = (1, 2, -1)$, $\boldsymbol{u_2} = (0, 2, 1)$, $\boldsymbol{u_3} = (0, -1, 3)$. Show that $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ forms a basis for $\mathbb{R}^3$.

    (b) Suppose $\boldsymbol{w} = (1, 1, 1)$. Find the coordinate vector of $\boldsymbol{w}$ relative to S.

    (c) Let $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ be another basis for $\mathbb{R}^3$ where $\boldsymbol{v_1} = (1, 5, 4)$, $\boldsymbol{v_2} = (-1, 3, 7)$, $\boldsymbol{v_3} = (2, 2, 4)$. Find the transition matrix from T to S.

    (d) Find the transition matrix from S to T.

    (e) Use the vector $\boldsymbol{w}$ in Part (b). Find the coordinate vector of $\boldsymbol{w}$ relative to T.

49. Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ be a basis for $\mathbb{R}^3$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ where
    $$\boldsymbol{v_1} = \boldsymbol{u_1} + \boldsymbol{u_2} + \boldsymbol{u_3}, \quad \boldsymbol{v_2} = \boldsymbol{u_2} + \boldsymbol{u_3} \quad \text{and} \quad \boldsymbol{v_3} = \boldsymbol{u_2} - \boldsymbol{u_3}.$$

    (a) Show that T is a basis for $\mathbb{R}^3$.

    (b) Find the transition matrix from S to T.