Matriculation Number: MA1521

A								

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2016-2017

MA1521 CALCULUS FOR COMPUTING

April/May 2017 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** Do not write your name anywhere in this booklet. This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **FIVE (5)** questions and comprises **TWENTY ONE (21)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question. The marks for each question are indicated at the beginning of the question. The maximum possible total score for this examination paper is 80 marks.
4. This is a **closed book (with authorized material)** examination. Students are only allowed to bring into the examination hall **ONE** piece A4 size help-sheet which can be used on both sides.
5. Candidates may use any calculators that satisfy MOE A-Level examination guidelines. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5
(a)					
(b)					

Question 1 (a) [8 marks]

(i) (Multiple Choice Question)

The line $y = mx + c$ is parallel to the tangent line of the curve $y = x^3$ at the point $(1, 1)$. Find the value of m.

(A) 3 (B) 5 (C) 15

(ii) The function $f(x) = \frac{2x^2-3x}{x^2-2x+3}$ has two critical points. One of them is at $x = c$ where $c > 1$. Find the **exact value** of c.

Answer 1(a)(i)	A	**Answer 1(a)(ii)**	$6+3\sqrt{3}$

(Show your working below and on the next page.)

(i) $\frac{dy}{dx} = 3x^2$

$x=1 \Rightarrow \frac{dy}{dx} = \underline{\underline{3}}$

(ii) $f'(x) = \frac{-x^2+12x-9}{(x^2-2x+3)^2}$

$f'(x) = 0$

$\Rightarrow x = 6 \pm 3\sqrt{3}$

$\because c > 1$

$\therefore c = \underline{\underline{6+3\sqrt{3}}}$

Question 1 (b) [8 marks]

(i) Let k denote a positive constant. It is known that

$$\int_0^k x\sqrt{1+x}dx = \frac{4}{15}.$$

Find the **exact value** of k. Give your answer in the form of a single fraction in its simplest form.

(ii) Let n denote a positive integer. The area of the finite region bounded by the curves $y = \frac{14}{x}$, $y = \frac{1}{x}$, and the vertical lines $x = \frac{1}{e}$ and $x = e^n$ is equal to 1521. Find the **exact value** of n.

Answer 1(b)(i)	$\frac{2}{3}$	**Answer 1(b)(ii)**	116

(Show your working below and on the next page.)

(i) $\int_0^k x\sqrt{1+x}\,dx = \int_0^k (1+x-1)(1+x)^{1/2}dx$

$= \int_0^k \{(1+x)^{3/2} - (1+x)^{1/2}\}\, d(1+x)$

$= \left[\frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2}\right]_0^k$

$= \frac{2}{5}(1+k)^{5/2} - \frac{2}{3}(1+k)^{3/2} + \frac{4}{15}$

$\therefore \frac{2}{5}(1+k)^{5/2} = \frac{2}{3}(1+k)^{3/2}$

$\because k>0 \quad \therefore 1+k = \frac{5}{3}$

$\therefore k = \frac{2}{3}$

(ii) $\int_{e^{-1}}^{e^n} \left(\frac{14}{x} - \frac{1}{x}\right)dx$

$= 13 \ln x \Big|_{e^{-1}}^{e^n}$

$= 13(n+1)$

$\therefore 13(n+1) = 1521$

$n+1 = 117$

$n = 116$

Question 2 (a) [8 marks]

(i) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{-n}{(56)^n}(7x+9)^n$.

(ii) Find the **exact value** of the sum $\sum_{n=0}^{\infty} \frac{2^n+3^{n+1}}{5^n}$. Give your answer in the form of a single fraction in its simplest form.

Answer 2(a)(i)	8	**Answer 2(a)(ii)**	$\frac{55}{6}$

(Show your working below and on the next page.)

(i) $\left| \dfrac{\frac{-(n+1)}{56^{n+1}}(7x+9)^{n+1}}{\frac{-n}{56^n}(7x+9)^n} \right|$

$= \frac{n+1}{n}\left(\frac{1}{56}\right)|7x+9|$

$\rightarrow \frac{1}{8}|x+\frac{9}{7}|$

$\therefore \frac{1}{8}|x+\frac{9}{7}| < 1$

$\Rightarrow |x+\frac{9}{7}| < \underline{\underline{8}}$

(ii) $\sum_{n=0}^{\infty} \frac{2^n+3^{n+1}}{5^n}$

$= \sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n + 3\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n$

$= \frac{1}{1-\frac{2}{5}} + \frac{3}{1-\frac{3}{5}}$

$= \frac{5}{3} + \frac{15}{2} = \underline{\underline{\frac{55}{6}}}$

Question 2 (b) [8 marks]

(i) The equation $e^{-x} = \frac{x^3}{8}$ has a solution between 1 and 2. To find an approximate value of this solution, you replace e^{-x} by its first order Taylor polynomial centred at 1, and replace $\frac{x^3}{8}$ by its first order Taylor polynomial centred at 1 to obtain a linear equation. Solving this linear equation, you find an answer which you use as an approximate value of the solution to your original equation. What is your answer after you solve the linear equation? Give your answer correct to two decimal places.

(ii) Let $f(x) = \int_{x^2}^{0} \cos\sqrt{t}\,dt$. It is known that $f^{(100)}(0)$ is a positive integer less than 1000. Find the **exact value** of $f^{(100)}(0)$.

Answer 2(b)(i)	1.33	**Answer 2(b)(ii)**	198

(Show your working below and on the next page.)

(i) $e^{-x} = e^{-1} + (-e^{-1})(x-1) + \cdots$

$\frac{x^3}{8} = \frac{1}{8} + \frac{3}{8}(x-1) + \cdots$

$\therefore e^{-1} - e^{-1}(x-1) = \frac{1}{8} + \frac{3}{8}(x-1)$

$e^{-1} - \frac{1}{8} = (x-1)(e^{-1} + \frac{3}{8})$

$x = 1 + \dfrac{\frac{1}{e} - \frac{1}{8}}{\frac{1}{e} + \frac{3}{8}}$

$= 1.3269\cdots$

$\approx \underline{\underline{1.33}}$

(ii) $f(x) = -\int_0^{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{t})^{2n}}{(2n)!} dt$

$= \sum_{n=0}^{\infty} (-1)^{n+1} \int_0^{x^2} \frac{t^n}{(2n)!} dt$

$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+2}}{(n+1)(2n)!}$

$\therefore \frac{f^{(100)}(0)}{100!} = \frac{(-1)^{50}}{50(98!)}$

$f^{(100)}(0) = \underline{\underline{198}}$

Question 3 (a) [8 marks]

(i) Let $f(x,y) = \left(x^2+y^2+1\right)\ln\left(x^2+y^2+1\right)$. Find the value of $f_x(3,5)$. Give your answer correct to two decimal places.

(ii)Let $f(x,y) = x^3 + xy + y^2 + 2017$. Find the largest possible value of $D_{\mathbf{u}}f(2,3)$. Give **exact value** for your answer.

Answer 3(a)(i)	27.33	Answer 3(a)(ii)	17

(Show your working below and on the next page.)

(i) $f_x = 2x\ln(x^2+y^2+1) + 2x$

$f_x(3,5) = 6\ln 35 + 6$

$= 27.332\cdots$

$\approx \underline{\underline{27.33}}$

(ii) $\nabla f = (3x^2+y,\ x+2y)$

$\therefore \nabla f(2,3) = (15, 8)$

$\max D_u f(2,3)$

$= \|\nabla f(2,3)\|$

$= \sqrt{15^2+8^2}$

$= \sqrt{289}$

$= \underline{\underline{17}}$

Question 3 (b) [8 marks]

(i) Let $f(x,y,z)$ be a differentiable function of three variables. It is known that $f(1,2,3) = 0$, $f_x(1,2,3) = 4$, $f_y(1,2,3) = 5$, $f_z(1,2,3) = 6$. It was found that if the point Q moved from $(1,2,3)$ a distance 0.1 units towards the point $(7,8,k)$, the value of f became $\frac{1}{5}$. Estimate the value of k. Give your answer correct to two decimal places.

(ii) Let $f(x,y) = 5xy - \frac{5}{2}x^2 - y^5$. How many critical points does f have?

Answer 3(b)(i)	-2.61	Answer 3(b)(ii)	2

(Show your working below and on the next page.)

(i) $\vec{u} = \dfrac{(7,8,k)-(1,2,3)}{\|(7,8,k)-(1,2,3)\|}$

$= \dfrac{(6,6,k-3)}{\sqrt{72+(k-3)^2}}$

$\nabla f(1,2,3) = (4,5,6)$

$D_{\vec{u}} f(1,2,3) = \dfrac{24+30+6(k-3)}{\sqrt{72+(k-3)^2}}$

$0.1 \times \dfrac{36+6k}{\sqrt{k^2-6k+81}} = 0.2$

$18+3k = \sqrt{k^2-6k+81}$ --- ①

$8k^2+114k+243=0$

$k = -11.64\ldots$ or $-2.609\ldots$

$k = -11.64\ldots \Rightarrow$ L.H.S. of ① is $-$ve

$\therefore k \approx \underline{\underline{-2.61}}$

(ii) $f_x = 5y - 5x = 0$

$f_y = 5x - 5y^4 = 0$

$\therefore 5y(1-y^3) = 0$

$\therefore y = 0$ or 1

$\therefore (0,0), (1,1)$

$\therefore$ <u>2 critical points</u>

Question 4 (a) [8 marks]

(i) Let y denote the solution of the differential equation $(\tan x)\frac{dy}{dx} = y$ with $0 < x < \frac{\pi}{2}$ and $y(\frac{\pi}{6}) = 1$. Find the value of $y(\frac{\pi}{3})$. Give your answer correct to two decimal places.

(ii) A piece of wood is known to be 15500 years old. You took it to the lab and found that it contained 15% of the original amount of Carbon-14. Find the half-life of Carbon-14. Give your answer correct to the nearest integer.

Answer 4(a)(i)	1.73	**Answer 4(a)(ii)**	5663

(Show your working below and on the next page.)

(i) $\frac{dy}{y} = \frac{\cos x}{\sin x} dx$

$\ln|y| = \ln|\sin x| + C$

$y = A \sin x$

$y(\frac{\pi}{6}) = 1 \Rightarrow A = 2$

$y(\frac{\pi}{3}) = 2 \times \frac{\sqrt{3}}{2}$

$= \sqrt{3}$

$\approx \underline{\underline{1.73}}$

(ii) $y = Ae^{-kt}$

$0.15A = Ae^{-k(15500)}$

$k = \frac{-\ln 0.15}{15500}$

half-life $= \frac{\ln 2}{k}$

$= \frac{15500 \ln 2}{-\ln 0.15}$

$\approx \underline{\underline{5663}}$

Question 4 (b) [8 marks]

(i) Let y denote the solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x^3 e^{x^2}}{y}$$

with $x > 0$, $y > 0$ and $y(1) = 2\sqrt{e}$. Find the value of $y(\sqrt{2})$. Give your answer correct to two decimal places.

(ii) Let y denote the solution of the differential equation $x^2 y' - xy = 1$, with $x > 0$ and $y(1) = 1$. Find the **exact value** of $y(2)$. Give your answer in the form of a single fraction in its simplest form.

Answer 4(b)(i)	5.58	Answer 4(b)(ii)	$\frac{11}{4}$

(Show your working below and on the next page.)

(i) $y = ux \Rightarrow y' = u'x + u$

$\therefore u'x + u = u + \frac{x^3 e^{x^2}}{ux}$

$uu' = xe^{x^2}$

$\frac{1}{2}u^2 = \frac{1}{2}e^{x^2} + \frac{1}{2}C$

$y^2 = x^2(e^{x^2} + C)$

$4e = e + C \Rightarrow C = 3e$

$\therefore y = x\sqrt{e^{x^2} + 3e}$

$y(\sqrt{2}) = \sqrt{2}\sqrt{e^2 + 3e}$

$= 5.575\cdots$

$\approx \underline{\underline{5.58}}$

(ii) $y' - \frac{1}{x}y = \frac{1}{x^2}$

$R = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$y = x\int \frac{1}{x}\left(\frac{1}{x^2}\right)dx$

$= x\int \frac{1}{x^3}dx$

$= x\left(-\frac{1}{2}x^{-2} + C\right)$

$= -\frac{1}{2x} + Cx$

$y(1) = 1 \Rightarrow 1 = -\frac{1}{2} + C$

$\Rightarrow C = \frac{3}{2}$

$\therefore y(2) = -\frac{1}{4} + 3 = \underline{\underline{\frac{11}{4}}}$

Question 5 (a) [8 marks]

~~(i) Let k and n denote two positive integers. It is known that $y = 2017xe^{kx}$ is a solution of the differential equation~~

$$y'' - ny' + 1521y = 0.$$

~~Find the **exact value** of k.~~

(ii) The Klingons are a humanoid warrior species that live on the planet Qo'noS. It is known that their population follows a Malthus model with a birth rate per capita of 10% per year and a death rate per capita of D% per year. If the Klingon population doubles every 20 years, find the value of D. Give your answer correct to two decimal places.

Answer 5(a)(i)	39	Answer 5(a)(ii)	6.53

(Show your working below and on the next page.)

(i) k is a double root of

$\lambda^2 - n\lambda + 1521 = 0$

$\therefore k^2 = 1521$

$\therefore k = \underline{\underline{39}}$

(ii) $\frac{dy}{dt} = (0.1 - D\%)y$

$y = Ae^{(0.1 - D\%)t}$

$2A = Ae^{(0.1 - D\%)(20)}$

$\ln 2 = 2 - 20D\%$

$20D\% = 2 - \ln 2$

$D\% = \frac{2 - \ln 2}{20}$

$= 6.534\ldots\%$

$\therefore D \approx \underline{\underline{6.53}}$

Question 5 (b) [8 marks]

(i) The monkey population at the MacRitchie Reservoir Park follows a logistic model with a birth rate per capita of 8% per year. Initially at time $t = 0$ there were 1000 monkeys at the park. After a very long time, the population settled down to an equilibrium value of 465. What was the population when time $t = 5$ year? Give your answer correct to the nearest integer.

(ii) The lion population at the Tarangire National Park in Tanzania is governed by the equation

$$\frac{dN}{dt} = \frac{100}{111}N\left(1-\frac{N}{123}\right)\left(1-\frac{N}{234}\right)\left(1-\frac{N}{345}\right)^2\left(1-\frac{N}{456}\right).$$

Initially there were 250 lions. What will that lion population eventually be after a very long time? Give your answer correct to the nearest integer.

Answer 5(b)(i)	725	**Answer 5(b)(ii)**	345

(Show your working below and on the next three pages.)

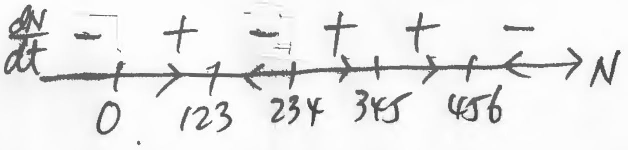
(i) $N_\infty = 465$, $\hat{N} = 1000$, $t = 5$

$$\Rightarrow N = \frac{465}{1+\left(\frac{465}{1000}-1\right)e^{-0.08\times 5}}$$

$= 725.00\ldots$

$\approx \underline{\underline{725}}$

(ii)



$t = 0$, $N = 250$

$\Rightarrow N \to \underline{\underline{345}}$ when $t \to \infty$