MA 1521

Tutorial 3 Solutions

1. (a)

$$\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^2}\,ds = \int_1^{\sqrt{2}} (1+s^{-3/2})\,ds = (\sqrt{2}-1) - 2s^{-1/2}\Big|_1^{\sqrt{2}}$$
$$= (\sqrt{2}-1) - \frac{2}{\sqrt[4]{2}} + 2 = 1+\sqrt{2}-2^{3/4}.$$

(b)

$$\int_{-4}^{4} |x|\,dx = \int_0^4 x\,dx + \int_{-4}^{0}(-x)\,dx = \frac{1}{2}4^2 + \frac{1}{2}4^2 = 16.$$

(c)

$$\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx = \int_0^{\pi/2} \frac{1}{2}(\cos x + |\cos x|)\,dx + \int_{\pi/2}^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx$$
$$= \int_0^{\pi/2} \cos x\,dx + 0 = \sin x\Big|_0^{\pi/2} = 1.$$

(d)

$$\int_0^{\pi} \sin^2\Big(1+\frac{\theta}{2}\Big)\,d\theta = \int_0^{\pi} \frac{1}{2}\big[1-\cos(2+\theta)\big]\,d\theta = \frac{1}{2}\pi - \frac{1}{2}\sin(2+\theta)\Big|_0^{\pi}$$
$$= \frac{1}{2}\pi - \frac{1}{2}\big[\sin(2+\pi)-\sin 2\big] = \frac{1}{2}\pi + \sin 2.$$

2. The Fundamental Theorem of Calculus (I) says that

$$\frac{d}{du}\int_a^u f(t)\,dt = f(u)$$

for a continuous function f. Here a is a fixed number. It is a sort of *chain rule* to find

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt\,.$$

To see this, let

$$F(u) = \int_a^u f(t)\,dt \qquad \text{and} \quad u = g(x)\,.$$

It follows that

$$\frac{dF}{du} = \frac{d}{du}\int_a^u f(t)\,dt = f(u)\,.$$

Furthermore,

$$F\circ g(x) = F(g(x)) = \int_a^{g(x)} f(t)\,dt\,.$$

By the chain rule, we have

$$\frac{dF(g(x))}{dx} = \frac{dF}{du}\frac{dg(x)}{dx} = f(u)\,g'(x) = f(g(x))\,g'(x)\,.$$

(a) $y = \displaystyle\int_0^{\sqrt{x}} \cos t\,dt;\quad \cos\sqrt{x}\cdot\frac{d}{dx}\sqrt{x} = \frac{\cos\sqrt{x}}{2\sqrt{x}}.$

(b) $y = \displaystyle\int_0^{x^2} \cos\sqrt{t}\,dt;\quad \cos\sqrt{x^2}\cdot 2x = 2x\cos|x| = 2x\cos x.$

(c) $y = \displaystyle\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}},\quad |x| < \frac{\pi}{2}.\quad \frac{1}{\sqrt{1-\sin^2 x}}\cdot\frac{d}{dx}\sin x = \frac{1}{\cos x}\cos x = 1.$

3. (a) $\displaystyle\int x^{1/2}\sin(x^{3/2}+1)\,dx = \int \sin(x^{3/2}+1)\cdot\frac{2}{3}\,d(x^{3/2}+1) = -\frac{2}{3}\cos(x^{3/2}+1) + C.$

(b) $\displaystyle\int \csc^2 2t\,\cot 2t\,dt = \int \cot 2t\cdot\Big(-\frac{1}{2}\Big)d(\cot 2t) = -\frac{1}{4}\cot^2 2t + C.$

(c) $\displaystyle\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta = \int \sin\frac{1}{\theta}\cos\frac{1}{\theta}\cdot(-1)\,d\Big(\frac{1}{\theta}\Big) = -\int \sin\frac{1}{\theta}\,d\Big(\sin\frac{1}{\theta}\Big) = -\frac{1}{2}\sin^2\frac{1}{\theta} + C.$

(d) $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)}\,dx = \int \frac{18\tan^2 x\,d(\tan x)}{(2+\tan^3 x)} = \int \frac{6\,d(\tan^3 x+2)}{(2+\tan^3 x)} = 6\,\ln|\tan^3 x + 2| + C.$

(e) $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}\,d\theta = -2\int (\cos\sqrt{\theta})^{-3}\,d(\cos\sqrt{\theta}) = (\cos\sqrt{\theta})^{-2} + C = \sec^2\sqrt{\theta} + C.$

4. (a)

$$\begin{aligned}\int x\sin\Big(\frac{x}{2}\Big)\,dx &= -2\int x\,d\Big[\cos\Big(\frac{x}{2}\Big)\Big] = -2\left[x\cos\Big(\frac{x}{2}\Big) - \int \cos\Big(\frac{x}{2}\Big)\,dx\right] + C\\ &= -2\left[x\cos\Big(\frac{x}{2}\Big) - 2\int \cos\Big(\frac{x}{2}\Big)\,d\Big(\frac{x}{2}\Big)\right] + C\\ &= -2\left[x\cos\Big(\frac{x}{2}\Big) - 2\sin\Big(\frac{x}{2}\Big)\right] + C\,.\end{aligned}$$

(b)

$$\begin{aligned}\int t^2 e^{4t}\,dt &= \frac{1}{4}\int t^2\,d(e^{4t}) = \frac{1}{4}\left[t^2 e^{4t} - 2\int t\,e^{4t}\,dt\right] + C = \frac{1}{4}\left[t^2 e^{4t} - \frac{1}{2}\int t\,d(e^{4t})\right] + C\\ &= \frac{1}{4}\left[t^2 e^{4t} - \frac{1}{2}\left(t\,e^{4t} - \int e^{4t}\,dt\right)\right] + C\\ &= \frac{1}{4}\left[t^2 e^{4t} - \frac{1}{2}\left(t\,e^{4t} - \frac{e^{4t}}{4}\right)\right] + C \quad \text{(continue to simplify)}\,.\end{aligned}$$

(c)

$$\begin{aligned}\int e^{-y}\cos y\,dy &= \int e^{-y}\,d(\sin y) = e^{-y}\sin y + \int e^{-y}\sin y\,dy + C\\ &= e^{-y}\sin y - \int e^{-y}\,d(\cos y) + C = e^{-y}\sin y - e^{-y}\cos y - \int e^{-y}\cos y\,dy\\ \Rightarrow\quad \int e^{-y}\cos y\,dy &= \frac{e^{-y}}{2}(\sin y - \cos y) + C\,.\end{aligned}$$

(There is no harm to rename $C/2$ as C .)

(d)

$$\begin{aligned}
\int \theta^2 \sin(2\theta)\, d\theta &= -\frac{1}{2}\int \theta^2 d[\cos(2\theta)] = -\frac{1}{2}\left[\theta^2\cos(2\theta) - 2\int \theta\, \cos(2\theta)\, d\theta\right] + C \\
&= -\frac{1}{2}\left[\theta^2\cos(2\theta) - \int \theta\ d[\sin(2\theta)]\right] + C \\
&= -\frac{1}{2}\left[\theta^2\cos(2\theta) - \theta\sin(2\theta) + \int \sin(2\theta)\, d\theta\right] + C \\
&= -\frac{1}{2}\left[\theta^2\cos(2\theta) - \theta\sin(2\theta) - \frac{1}{2}\cos(2\theta)\right] + C\,.
\end{aligned}$$

(e)

$$\begin{aligned}
\int z(\ln z)^2\, dz &= \frac{1}{2}\int (\ln z)^2\, d(z^2) = \frac{1}{2}\left[z^2(\ln z)^2 - 2\int z(\ln z)\, dz\right] + C \\
&= \frac{1}{2}\left[z^2(\ln z)^2 - \int (\ln z)\, d(z^2)\right] + C \\
&= \frac{1}{2}\left[z^2(\ln z)^2 - z^2(\ln z) + \int z\, dz\right] + C \\
&= \frac{1}{2}\left[z^2(\ln z)^2 - z^2(\ln z) + \frac{z^2}{2}\right] + C\,.
\end{aligned}$$