CS1231/CS1231S Assignment #1
AY2020/21 Semester 1
Deadline: Wednesday, 16 September 2020, 4:00pm

IMPORTANT: Please read the instructions below

This is a graded assignment worth 10% of your final grade. Please work on it by yourself, not in a group or in collaboration with anybody. Anyone found plagiarising (submitting other's work as your own), or sending your answers to others will be penalised with a straight zero for the assignment, and if found re-committing this offence, will be referred to the disciplinary board.

You are to submit your assignment to **LumiNUS Files**. A submission folder has been created for you at Files > Assignment #1 > Your tutorial group > Your personal folder.

Your answers may be typed or handwritten. Make sure that it is legible (for example, don't use very light pencil or ink, or very small font) or marks may be deducted.

You are to submit **a SINGLE pdf file**, where each page is A4 size. Do not submit multiple files or files in other format, or we will not accept your submission.

You may test out your submission before the deadline, but make sure you remove any test files you have submitted earlier.

Late submission will NOT be accepted, as the folder will automatically close on the dot. We will set the closing time to slightly later than 4pm to provide a grace period, but in your mind, you should treat **4pm** as the deadline. If you think you might be too busy on the day of the deadline, please submit earlier. Also, avoid submitting in the last minute; if everybody does that (and we have more than 1000 students in CS1231 and CS1231S) the system may get sluggish due to the overload, or worse, it may break down, and you will miss the deadline.

Note the following as well:

- Name your pdf file with your **Name**, preferably as spelled in your Student Card (for example: SantaClause.pdf). (You may use Upper Camel Case style of naming. Ref: https://whatis.techtarget.com/definition/UpperCamelCase)
- At the top of the first page of your submission, write your **Name**, **Student Number** and **Tutorial Group**.
- To keep the submitted document short, you may submit your answers without including the questions.
- As this is an assignment given well ahead of time, we expect you to work on it early. You should submit **polished work**, not answers that are untidy or appear to have been done in a hurry, for example, with scribbling and cancellation all over the places.

To combine all pages into a single pdf document for submission, you may find the following scanning apps helpful if you intend to scan your handwritten answers:

* for Android: https://fossbytes.com/best-android-scanner-apps/

* for iphone:
https://www.switchingtomac.com/tutorials/ios-tutorials/the-best-ios-scanner-apps-to-scan-documents-images/

If you need any clarification about this assignment, please post on the **LumiNUS > Assignments** forum.

Question 1. (1 mark)

You do not need to answer this question. Your tutor will check if you have done it correctly.

(a) Have you named your file with your name in it? [½ mark]

(b) Have you written your Name, Student Number and Tutorial group number (all three must be present) on the first page of your submission? [½ mark]

Note: It is a relief that almost all students followed the instructions on naming the file and writing the particulars in the file, unlike in the past when there were so many violations. It seems like awarding marks to following instructions works. However, I hope that students follow all instructions, whether marks are given for that or not.

Some mistakes students made in this assignment: not checking that the question file downloaded is complete; not checking that the file submitted is the correct file; remembering the folder closing time wrongly. Please do not make such mistakes again.

Question 2. (4 marks)

Simplify the proposition below using the laws given in **Theorem 2.1.1 (Epp)** and the **implication law**. Make sure you justify every step. Any step that is skipped, or law cited wrongly will be penalised. (Refer to tutorial #1 question 2a.) Use **true** and **false** instead of **t** and **c** for tautology and contradiction respectively.

$$\boldsymbol{p \wedge (\sim p \rightarrow r \wedge q) \wedge \sim(q \rightarrow \sim p)}$$

Answer:

$p \wedge (\sim p \rightarrow r \wedge q) \wedge \sim(q \rightarrow \sim p)$

$\equiv p \wedge \big(\sim(\sim p) \vee (r \wedge q)\big) \wedge \sim(q \rightarrow \sim p)$ (by implication law)

$\equiv p \wedge \big(p \vee (r \wedge q)\big) \wedge \sim(q \rightarrow \sim p)$ (by double negative law)

$\equiv p \wedge \sim(q \rightarrow \sim p)$ (by absorption law)

$\equiv p \wedge \sim(\sim q \vee \sim p)$ (by implication law)

$\equiv p \wedge \big(\sim(\sim q) \wedge \sim(\sim p)\big)$ (by De Morgan's law)

$\equiv p \wedge (q \wedge p)$ (by double negative law)

$\equiv p \wedge (p \wedge q)$ (by commutative law)

$\equiv (p \wedge p) \wedge q$ (by associative law)

$\equiv p \wedge q$ (by idempotent law)

Question 3. (2 marks)

(a) Given the following statement:

$$\forall x \in \mathbb{R} \left(\_\_\_\_\_\_ \leftrightarrow \forall y \in \mathbb{R}\ (x \neq y^2)\right)$$

Fill in the blank so that the statement is true.
(Do not fill in the blank with $\forall y \in \mathbb{R}\ (x \neq y^2)$ since that is trivial.) [1 mark]

(b) Given the following predicate:

$$P(x) = \left(x \neq 1 \land \forall y, z \in \mathbb{N}\ \left(x = yz \rightarrow (y = 1 \lor y = x)\right)\right), \forall x \in \mathbb{N}.$$

If $P(x)$ is true, what can you say about x? [1 mark]

Answers:

(a) $x < 0$.

We also accepted trivial but correct answers such as "$\forall y \in \mathbb{R}\ (\sqrt{x} \neq y)$" and "$\forall y \in \mathbb{R}\ (x - y^2 \neq 0)$" and the like as the question does not explicitly forbid that. This is not the intention of the question but unfortunately the preclusion was not stated explicitly in the question.

But make sure "$\forall y \in \mathbb{R}$" is added in front, otherwise it is wrong.

(b) x is a prime number. (This is actually in the lecture slide in Lecture #4!)

Question 4. (4 marks)

Determine **whether** $\left((\boldsymbol{p} \lor \boldsymbol{q} \lor \boldsymbol{r}) \land (\sim\boldsymbol{p} \rightarrow \boldsymbol{s}) \land (\sim\boldsymbol{q} \rightarrow \boldsymbol{s})\right) \rightarrow (\boldsymbol{r} \rightarrow \boldsymbol{s})$ is a tautology.

Show <u>all</u> your steps in your working (however, you do not need to justify your steps in this question, unlike question 2.) Do not use truth table.

Answer:

It is not a tautology. Counterexample: $p = q = r = true, s = false$.

Working:

$\left((p \lor q \lor r) \land (\sim p \rightarrow s) \land (\sim q \rightarrow s)\right) \rightarrow (r \rightarrow s)$

$\equiv \left((true \lor true \lor true) \land (\sim true \rightarrow false) \land (\sim true \rightarrow false)\right) \rightarrow (true \rightarrow false)$

$\equiv \left((true) \land (\sim true \rightarrow false) \land (\sim true \rightarrow false)\right) \rightarrow (true \rightarrow false)$ (def of disjunction)

$\equiv (true \land true \land true) \rightarrow (false)$ (definition of conditional statement)

$\equiv true \rightarrow false$ (definition of conjunction)

$\equiv false$ (definition of conditional statement)

Some students simplified the statement into $\sim p \lor \sim q \lor \sim r \lor s$ and said that it is false with the counterexample: $p = q = r = true, s = false$. However, they did not substitute the values into the simplified statement to show that it is indeed false. We accepted this.

If, however, the students simplified the statement into $\sim p \lor \sim q \lor \sim r \lor s$ and just said that it is not a tautology WITHOUT giving any counterexample, then only 2 marks are awarded.

Question 5. (2 marks)

Two sequences β and γ are said to span a space S over field F if and only if

"every sequence α in S can be expressed as $\alpha = b\beta + c\gamma$ for some $b, c \in F$."

Dueet wrote:

"ρ and σ span S because $\omega \in S, 0 \in F$ and $\omega = 0\rho + 0\sigma$."

(Note that $\omega \in S, 0 \in F$ and $\omega = 0\rho + 0\sigma$ are all correct.)

(a) Explain why Dueet's argument might be false/wrong. [1 mark]

(b) Explain why Dueet's argument might be true/correct. [1 mark]

Answers:

(a) If $S \neq \{\omega\}$, then Dueet must also check cases where $\alpha \in S$ and $\alpha \neq \omega$.

(b) If $S = \{\omega\}$, then Dueet's argument is correct

Note: The essence of this question is on the "for all sequences in S", so the correctness of Dueet's answer would depend on whether ω is the sole element in S.

Question 6. (4 marks)

Let the domain of discourse be the set of natural number $\mathbb{N}$ and you may omit this in your answers. Assuming that the following predicates are given and you may use them:

- $Prime(x)$ = "x is prime".
- $Even(x)$ = "x is even".

You may assume that if a natural number is not odd, then it is even; if it is not even, then it is odd.

Write the logical statements for the following sentences. You are not to create new predicates (such as $Odd(x)$).

(a) For every odd natural number there is a different natural number such that their sum is even. [2 marks]

(b) The sum of any two prime numbers except the prime number 2 is even. [2 marks]

Answers: (Note: We may accept $\forall x, y$ for $\forall x\ \forall y$.)

(a) $\forall x\ \big({\sim}Even(x) \rightarrow \exists y\ (y \neq x \land Even(x + y))\big)$.

(b) $\forall x\ \forall y\ \big((Prime(x) \land Prime(y) \land x \neq 2 \land y \neq 2) \rightarrow Even(x + y)\big)$.

Question 7. (4 marks)

Let a be a rational number and b an irrational number. Prove the following:

$$\boldsymbol{a \neq 0 \rightarrow ab} \textbf{ is irrational.}$$

(Remember to use numbering in your proof.)

Answer:

Proof by contradiction:

1. Suppose $a \neq 0$ and ab is rational.
2. Since a is rational and is not zero, $a = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $\underline{p, q \neq 0}$.
3. Since ab is rational, $ab = \frac{r}{s}$ where $r, s \in \mathbb{Z}$ and $\underline{s \neq 0}$.
4. Now, $b = \frac{ab}{a} = \frac{rq}{sp}$ where $rq \in \mathbb{Z}$ and $sp \in \mathbb{Z}$ (closure of integers under $\times$) and $\underline{sp \neq 0}$ (as $s \neq 0$ and $p \neq 0$).
5. This means that b is rational, contradicting the assumption and b is irrational.
6. Therefore, $a \neq 0 \rightarrow ab$ is irrational.

Note that the three underlined parts ($q \neq 0, s \neq 0, sp \neq 0$) are important. If omitted, marks will be deducted.

Some students quoted "closure of rational numbers under division". This is incorrect since there might be division by zero.

Students who cited this are likely to use proof by contraposition, i.e. ab is rational $\rightarrow a = 0$. Since a is rational and b is irrational, some students said that for ab to be rational, a must be 0 (or they may express ab as a fraction and claim that for it to be rational the denominator must be 0). All these are not proven so they are not acceptable.

Question 8. (4 marks)

Prove or disprove this statement:

$$\boldsymbol{\forall n \in \mathbb{Z}\; n^2 + n} \textbf{ is even.}$$

(Remember to use numbering in your proof.)

Answer:

Proof (by division into cases):

1. Case 1: n is even
   1.1 Then $n = 2k$ for some $k \in \mathbb{Z}$ (by definition of even numbers)
   1.2 $n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$ (by basic algebra)
   1.3 Since $(2k^2 + k)$ is an integer (closure of integers under $\times$ and $+$), $n^2 + n$ is even (by definition of even numbers)
2. Case 2: n is odd
   2.1 Then $n = 2k + 1$ for some $k \in \mathbb{Z}$ (by definition of odd numbers)
   2.2 $n^2 + n = (2k + 1)^2 + 2k + 1 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$ (by basic algebra)
   2.3 Since $(2k^2 + 3k + 1)$ is an integer (closure of integers under $\times$ and $+$), $n^2 + n$ is even (by definition of even numbers)
3. In all cases, $n^2 + n$ is even, therefore the statement $\forall n \in \mathbb{Z}\; n^2 + n$ is even is true.

Question 9. (3 marks)

Let $A = \{-2, -1, 0, 1, 2\}, B = \{0, 1, 4\}$ and $C = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

Determine whether the following is true or false. Justify your answer. (Merely writing true or false without explanation will not get you full mark even if your answer is correct.)

$$\boldsymbol{\forall x \in C\; \big((x \in A) \leftrightarrow (x^2 \in B)\big)}.$$

Answer:

True.

Case $x = -4, -3, 3, 4$: $x \in A$ is false and $x^2 \in B$ is false ($(-4)^2 = 4^2 = 16$; $(-3)^2 = 3^2 = 9$),
so $(x \in A) \leftrightarrow (x^2 \in B)$ is true.

Case $x = -2, -1, 0, 1, 2$: $x \in A$ is true and $x^2 \in B$ is true ($(-2)^2 = 2^2 = 4$; $(-1)^2 = 1^2 = 1$; $0^2 = 0$),
so $(x \in A) \leftrightarrow (x^2 \in B)$ is true.

Alternatively, as most students have done, is to prove the "only if" and "if" directions.

Question 10. (6 marks)

For each $k \in \mathbb{Z}_{\geqslant 0}$, let $D_k = \{n \in \mathbb{Z}_{\geqslant 0} : k = mn \text{ for some } m \in \mathbb{Z}_{\geqslant 0}\}$. Write down each of the following sets in roster notation:

(a) $D_1, D_2, D_3, D_4, D_5, D_6, D_7$; [2 marks]

(b) $\bigcup_{k=1}^{7} D_k$ and $\bigcap_{k=1}^{7} D_k$; [2 marks]

(c) $\{n \in \mathbb{Z}_{\geqslant 0} : n \in D_k \text{ for some } k \in \mathbb{Z}_{\geqslant 0}\}$ and $\{n \in \mathbb{Z}_{\geqslant 0} : n \in D_k \text{ for all } k \in \mathbb{Z}_{\geqslant 0}\}$. [2 marks]

Answer:

(a) $D_1 = \{1\}$, $D_2 = \{1,2\}$, $D_3 = \{1,3\}$, $D_4 = \{1,2,4\}$, $D_5 = \{1,5\}$, $D_6 = \{1,2,3,6\}$, $D_7 = \{1,7\}$.
(b) $\bigcup_{k=1}^{7} D_k = \{1,2,3,4,5,6,7\}$ and $\bigcap_{k=1}^{7} D_k = \{1\}$.
(c) $\{n \in \mathbb{Z}_{\geqslant 0} : n \in D_k \text{ for some } k \in \mathbb{Z}_{\geqslant 0}\} = \{0,1,2,3,\dots\}$ and

$\{n \in \mathbb{Z}_{\geqslant 0} : n \in D_k \text{ for all } k \in \mathbb{Z}_{\geqslant 0}\} = \{1\}$.

Note that the question asks for roster notation. Answers such as $\mathbb{Z}_{\geqslant 0}$ or $\{n \in \mathbb{Z} : n \geq 0\}$ are not accepted.

Question 11. (6 marks)

(a) Let A, B be sets. Show that if $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$, then either $A \subseteq B$ or $B \subseteq A$. [4 marks]

(b) Find $A, B \subseteq \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$ such that $\mathcal{P}(A \cup B) \nsubseteq \mathcal{P}(A) \cup \mathcal{P}(B)$. Explain why your choice of A and B satisfies the required condition. [2 marks]

Answer:

(a) We prove by contraposition.

1. Suppose $A \nsubseteq B$ and $B \nsubseteq A$.
2. Use the definition of $\subseteq$ to find an element x of A that is not in B, and an element y of B that is not in A.
3. Define $S = \{x, y\}$.
4. Note $x \in S$ but $x \notin B$ — by the choice of x.
5. So $S \nsubseteq B$ — by the definition of $\subseteq$.
6. Thus $S \notin \mathcal{P}(B)$ — by the definition of $\mathcal{P}$.
7. Note $y \in S$ but $y \notin A$ — by the choice of y.
8. So $S \nsubseteq A$ — by the definition of $\subseteq$.
9. Thus $S \notin \mathcal{P}(A)$ — by the definition of $\mathcal{P}$.
10. Hence $S \notin \mathcal{P}(A) \cup \mathcal{P}(B)$ — by lines 6 and 9.
11. Note $x \in A$ — by the choice of x.
12. So $x \in A$ or $x \in B$ — by the definition of "or".
13. Thus $x \in A \cup B$ — by the definition of $\cup$.
14. Note $y \in B$ — by the choice of y.
15. So $y \in A$ or $y \in B$ — by the definition of "or".

16. Thus $y \in A \cup B$ by the definition of $\cup$.

17. Every element of S is an element of $A \cup B$ by lines 13 and 16, as $S = \{x, y\}$.

18. So $S \subseteq A \cup B$ by the definition of $\subseteq$.

19. Hence $S \in \mathcal{P}(A \cup B)$ by the definition of $\mathcal{P}$.

20. Therefore, we conclude that $\mathcal{P}(A \cup B) \nsubseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Alternative proof:

1. Suppose $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.
2. Note $A \cup B \subseteq A \cup B$ by Remark 5.1.18(3).
3. So $A \cup B \in \mathcal{P}(A \cup B)$ by the definition of $\mathcal{P}$.
4. This implies $A \cup B \in \mathcal{P}(A) \cup \mathcal{P}(B)$ by line 1 and the definition of $\subseteq$.
5. Thus $A \cup B \in \mathcal{P}(A)$ or $A \cup B \in \mathcal{P}(B)$ by the definition of $\cup$.
6. Suppose $A \cup B \in \mathcal{P}(A)$.
   6.1. Then $A \cup B \subseteq A$ by the definition of $\mathcal{P}$.
   6.2. Now if $x \in B$, then
   6.2.1. $x \in A$ or $x \in B$ by the definition of "or";
   6.2.2. $\therefore\ x \in A \cup B$ by the definition of $\cup$;
   6.2.3. $\therefore\ x \in A$ by line 6.1 and the definition of $\subseteq$.
   6.3. This shows $B \subseteq A$ by the definition of $\subseteq$.
   6.4. Hence $A \subseteq B$ or $B \subseteq A$ by the definition of "or".
7. Suppose $A \cup B \in \mathcal{P}(B)$.
   7.1. Then $A \cup B \subseteq B$ by the definition of $\mathcal{P}$.
   7.2. Now if $x \in A$, then
   7.2.1. $x \in A$ or $x \in B$ by the definition of "or";
   7.2.2. $\therefore\ x \in A \cup B$ by the definition of $\cup$;
   7.2.3. $\therefore\ x \in B$ by line 7.1 and the definition of $\subseteq$.
   7.3. This shows $A \subseteq B$ by the definition of $\subseteq$.
   7.4. Hence $A \subseteq B$ or $B \subseteq A$ by the definition of "or".
8. It follows from lines 6 and 7 that $A \subseteq B$ or $B \subseteq A$ in all cases.

Note: It is fine to omit half of lines 4–9 by appealing to symmetry. Similarly, it is fine to omit half of lines 11–16 by appealing to symmetry. Lines 12, 15 and 17 are less important and may thus be omitted without losing marks.

Note: Some students didn't use numbering. It is advisable to use numbering especially when your proof is long.

(b) In view of (a), one can take any $A, B \subseteq \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$ such that $A \nsubseteq B$ and $B \nsubseteq A$. For instance, one can set $A = \{\diamondsuit\}$ and $B = \{\clubsuit\}$. These are the only possibilities because the converse to the statement in (a) is also true.