Section 7.3: Recursion

CS1231S Discrete Structures

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- LumiNUS > Forum > Suggestions for Tin Lok (anonymous)
- open until 23:59 on Sunday 27 September, 2020

Exercise 7.3.3

Let $a_1 = 1$ and $a_{n+1} = a_n + (n+1)$ for all $n \in \mathbb{Z}_{\geqslant 1}$. Find a general formula for a_n in terms of n that does not involve $a_0, a_1, \ldots, a_{n-1}$.

Tell me your answer at `https://pollev.com/wtl`.

Solution

$$\begin{aligned}
a_n &= a_{n-1} + n && \text{by the definition of } a_{n-1}; \\
&= a_{n-2} + (n-1) + n \\
& && \text{by the definition of } a_{n-2}; \\
&\vdots \\
&= a_1 + 2 + 3 + \cdots + (n-1) + n \\
& && \text{by the definition of } a_2; \\
&= 1 + 2 + 3 + \cdots + (n-1) + n \\
& && \text{by the definition of } a_1; \\
&= \frac{1}{2} n(n+1) && \text{by Example 7.1.3.}
\end{aligned}$$

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Recursion

What we saw

- Mathematical Induction (MI)
- Strong Mathematical Induction (Strong MI)
- Well-Ordering Principle

Now

- recursively defined sequences
- recursively defined sets
- structural induction

Why recursion

- Induction and recursion are inseparable partners.
- Recursive processes appear naturally and ubiquitously.
- Recursion is a central concept in computations.

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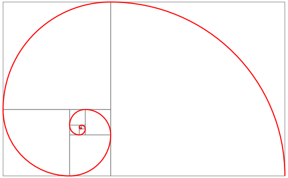

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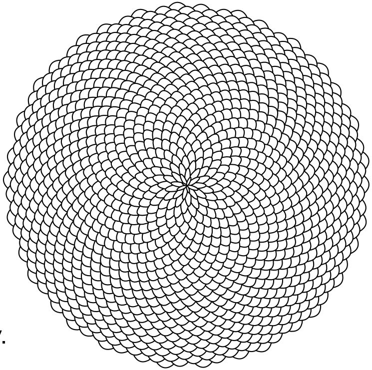


Recursively defined sequences

Terminology 7.3.1

A sequence $a_0, a_1, a_2, \ldots$ is said to be *recursively defined* if the definition of a_n involves $a_0, a_1, \ldots, a_{n-1}$ for all but finitely many $n \in \mathbb{Z}_{\geqslant 0}$.

Example 7.3.2

(1) Define $0!, 1!, 2!, \ldots$ by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$0! = 1 \quad \text{and} \quad (n+1)! = (n+1) \times n!.$$

Then $1! = 1 \times 1 = 1, \quad 2! = 2 \times 1 = 2, \quad 3! = 3 \times 2 = 6, \quad 4! = 4 \times 6 = 24, \quad \ldots.$

(2) The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n.$$

Then $F_2 = 1 + 0 = 1, \quad F_3 = 1 + 1 = 2, \quad F_4 = 2 + 1 = 3, \quad F_5 = 3 + 2 = 5, \quad \ldots.$

Recursively defined sequences

Terminology 7.3.1

A sequence $a_0, a_1, a_2, \ldots$ is said to be *recursively defined* if the definition of a_n involves $a_0, a_1, \ldots, a_{n-1}$ for all but finitely many $n \in \mathbb{Z}_{\geqslant 0}$.

Example 7.3.2

(3) Fix $r \in [0, 4]$ and $p_0 \in [0, 1]$. Define $p_1, p_2, \ldots$ by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$p_{n+1} = r(p_n - {p_n}^2).$$

Search for "logistic map".

If $r = 3$ and $p_0 = 1/2$, then

$$p_1 = 3\left(\frac{1}{2} - \left(\frac{1}{2}\right)^2\right) = \frac{3}{4}, \qquad p_2 = 3\left(\frac{3}{4} - \left(\frac{3}{4}\right)^2\right) = \frac{9}{16}, \qquad \ldots.$$

(4) Fix $a_0 \in \mathbb{Z}^+$. Define $a_1, a_2, a_3, \ldots$ by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$a_{n+1} = \begin{cases} a_n/2, & \text{if } a_n \text{ is even;} \\ 3a_n + 1, & \text{if } a_n \text{ is odd.} \end{cases}$$

Search for "Collatz Conjecture".

If $a_0 = 1$, then $a_1 = 3 \times 1 + 1 = 4, \quad a_2 = 4/2 = 2, \quad a_3 = 2/2 = 1, \quad \ldots.$

Well definition of the Fibonacci sequence

Definition 7.2.2

The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n.$$

Proposition 7.3.4

There is a unique sequence $a_0, a_1, a_2, \ldots$ satisfying, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$a_0 = 0 \quad \text{and} \quad a_1 = 1 \quad \text{and} \quad a_{n+2} = a_{n+1} + a_n.$$

Proof sketch

- (Existence) Show by induction on n that a_n exists for every $n \in \mathbb{Z}_{\geqslant 0}$.
- (Uniqueness) Let $b_0, b_1, b_2, \ldots$ and $c_0, c_1, c_2, \ldots$ be sequences such that

$$\begin{aligned} b_0 &= 0 \quad \text{and} \quad b_1 = 1 \quad \text{and} \quad b_{n+2} = b_{n+1} + b_n; \quad \text{and} \\ c_0 &= 0 \quad \text{and} \quad c_1 = 1 \quad \text{and} \quad c_{n+2} = c_{n+1} + c_n. \end{aligned}$$

  Show by induction on n that $b_n = c_n$ for every $n \in \mathbb{Z}_{\geqslant 0}$. □

Principle 7.1.1: Mathematical Induction (MI)

To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(0)$ is true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(k) \Rightarrow P(k+1)\big)$ is true.

Justification

$P(0)$	by the base step;
$P(0) \Rightarrow P(1)$	by the induction step with $k = 0$;
$P(1) \Rightarrow P(2)$	by the induction step with $k = 1$;
$\vdots$	

We deduce that $P(0), P(1), P(2), \ldots$ are all true by a series of modus ponens.

Ultimate reason

(1) $0 \in \mathbb{Z}_{\geqslant 0}$.

(2) If $x \in \mathbb{Z}_{\geqslant 0}$, then $x + 1 \in \mathbb{Z}_{\geqslant 0}$.

(3) Membership for $\mathbb{Z}_{\geqslant 0}$ can always be demonstrated by (finitely many) successive applications of the clauses above.

Recursive definition of $\mathbb{Z}_{\geqslant 0}$

Example 7.3.6

$$
\begin{aligned}
& & 0 &\in \mathbb{Z}_{\geqslant 0} & &\text{by (1).} \\
&\therefore & 1 &\in \mathbb{Z}_{\geqslant 0} & &\text{by (2) and the previous line.} \\
&\therefore & 2 &\in \mathbb{Z}_{\geqslant 0} & &\text{by (2) and the previous line.}
\end{aligned}
$$

Theorem 7.3.5 and Terminology 7.3.8

It can be shown that $\mathbb{Z}_{\geqslant 0}$ is the unique set satisfying (1)–(3). So we can view (1)–(3) as a definition of $\mathbb{Z}_{\geqslant 0}$. This is called a *recursive definition* of $\mathbb{Z}_{\geqslant 0}$.

Remark 7.3.7

(1) and (2) are true when $\mathbb{Z}_{\geqslant 0}$ is changed to $\mathbb{Q}$, but (3) is not.

Ultimate reason

(1) $0 \in \mathbb{Z}_{\geqslant 0}$. (*base clause*)

(2) If $x \in \mathbb{Z}_{\geqslant 0}$, then $x + 1 \in \mathbb{Z}_{\geqslant 0}$. (*recursion clause*)

(3) Membership for $\mathbb{Z}_{\geqslant 0}$ can always be demonstrated by (finitely many) successive applications of the clauses above. (*minimality clause*)

Recursive definition of $2\mathbb{Z}_{\geqslant 1}$

Example 7.3.9

The set $2\mathbb{Z}_{\geqslant 1}$ of all positive even integers can be defined recursively as follows.

(1) $2 \in 2\mathbb{Z}_{\geqslant 1}$. (base clause)

(2) If $x \in 2\mathbb{Z}_{\geqslant 1}$, then $x + 2 \in 2\mathbb{Z}_{\geqslant 1}$. (recursion clause)

(3) Membership for $2\mathbb{Z}_{\geqslant 1}$ can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Theorem 7.3.10 (Structural induction over $2\mathbb{Z}_{\geqslant 1}$)

To prove that $\forall n \in 2\mathbb{Z}_{\geqslant 1}\ \ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(2)$ is true; and

(induction step) show that $\forall x \in 2\mathbb{Z}_{\geqslant 1}\ \ \big(P(x) \Rightarrow P(x+2)\big)$ is true.

Question 7.3.11

Define a set S recursively as follows.

(1) $1 \in S$. (base clause)

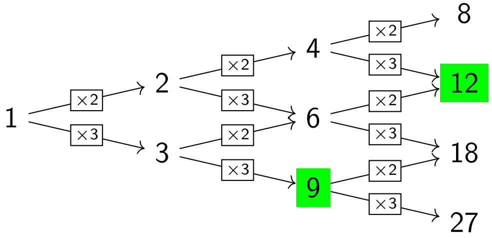
(2) If $x \in S$, then $2x \in S$ and $3x \in S$. (recursion clause)

(3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the numbers $9, 10, 11, 12, 13$ are in S? Which are not?

Answer

$9, 12 \in S \quad \text{and} \quad 10, 11, 13 \notin S.$



Observation

$S = \{2^a 3^b : a, b \in \mathbb{Z}_{\geqslant 0}\}$.

Structural induction over S

To prove that $\forall n \in S\ \ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(1)$ is true; and

(induction step) show that $\forall x \in S\ \ \big(P(x) \Rightarrow P(2x) \wedge P(3x)\big)$ is true.

Well-formed formulas in propositional logic

Let Σ be a nonempty set.

Definition 7.3.12

Define the set $\mathrm{WFF}(\Sigma)$ recursively as follows.

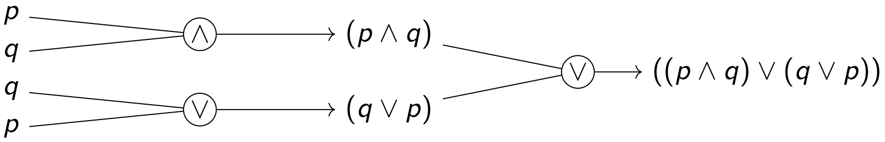
(1) Every element p of Σ is in $\mathrm{WFF}(\Sigma)$. (base clause)

(2) If x, y are in $\mathrm{WFF}(\Sigma)$, then $\sim x$ and $(x \wedge y)$ and $(x \vee y)$ are in $\mathrm{WFF}(\Sigma)$. (recursion clause)

(3) Membership for $\mathrm{WFF}(\Sigma)$ can always be demonstrated by (finitely many) successive applications of the clauses above. (minimality clause)

Example 7.3.13

Let $\Sigma = \{p, q\}$. Then



Well-formed formulas in propositional logic

Let Σ be a nonempty set.

Definition 7.3.12

Define the set WFF(Σ) recursively as follows.

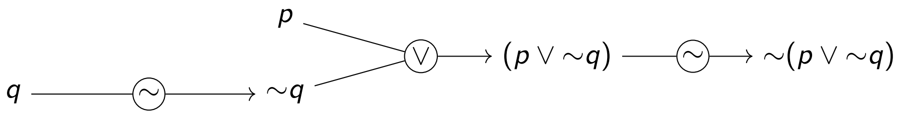
(1) Every element p of Σ is in WFF(Σ). (base clause)

(2) If x, y are in WFF(Σ), then $\sim x$ and $(x \wedge y)$ and $(x \vee y)$ are in WFF(Σ). (recursion clause)

(3) Membership for WFF(Σ) can always be demonstrated by (finitely many) successive applications of the clauses above. (minimality clause)

Example 7.3.14

Let $\Sigma = \{p, q\}$. Then



Positive well-formed formulas

Let Σ be a nonempty set.

Definition 7.3.16

Define the set $\mathrm{WFF}^+(\Sigma)$ recursively as follows.

(1) Every element p of Σ is in $\mathrm{WFF}^+(\Sigma)$. (base clause)

(2) If x, y are in $\mathrm{WFF}^+(\Sigma)$, then $\cancel{\sim x}$ and $(x \wedge y)$ and $(x \vee y)$ are in $\mathrm{WFF}^+(\Sigma)$. (recursion clause)

(3) Membership for $\mathrm{WFF}^+(\Sigma)$ can always be demonstrated by (finitely many) successive applications of the clauses above. (minimality clause)

Question (informal)

Is every element of $\mathrm{WFF}(\Sigma)$ equivalent to one that does not involve $\sim$?

Question (formal)

Is it true that $\forall x \in \mathrm{WFF}(\Sigma)\ \exists y \in \mathrm{WFF}^+(\Sigma)\ \ y \equiv x$?

Positive well-formed formulas

Let Σ be a nonempty set.

Definition 7.3.16

Define the set $\mathrm{WFF}^+(\Sigma)$ recursively as follows.

(1) Every element p of Σ is in $\mathrm{WFF}^+(\Sigma)$. (base clause)

(2) If x, y are in $\mathrm{WFF}^+(\Sigma)$, then $\cancel{\sim x}$ and $(x \wedge y)$ and $(x \vee y)$ are in $\mathrm{WFF}^+(\Sigma)$. (recursion clause)

(3) Membership for $\mathrm{WFF}^+(\Sigma)$ can always be demonstrated by (finitely many) successive applications of the clauses above. (minimality clause)

Theorem 7.3.18 (Structural induction over $\mathrm{WFF}^+(\Sigma)$)

To prove that $\forall x \in \mathrm{WFF}^+(\Sigma)\ \ P(x)$ is true, where each $P(x)$ is a proposition, it suffices to:

(base step) show that $P(p)$ is true for every $p \in \Sigma$;

(induction step) show that $\forall x, y \in \mathrm{WFF}^+(\Sigma)\ \ \big(P(x) \wedge P(y) \Rightarrow P((x \wedge y)) \wedge P((x \vee y))\big)$.

All-false assignments

Let Σ be a nonempty set.

Lemma 7.3.19

If $x \in \mathrm{WFF}^+(\Sigma)$, then assigning **false** to all the elements of Σ makes x evaluate to **false**.

Theorem 7.3.20

$\exists x \in \mathrm{WFF}(\Sigma)\ \forall y \in \mathrm{WFF}^+(\Sigma)\ \ y \not\equiv x$.

Proof

1. Take $p \in \Sigma$. This is possible since $\Sigma \neq \varnothing$.
2. Pick any $y \in \mathrm{WFF}^+(\Sigma)$.
3. Assigning **false** to all the elements of Σ makes y evaluate to **false** by Lemma 7.3.19, but it makes $\sim p$ evaluate to $\sim$**false** $\equiv$ **true**.
4. So $y \not\equiv \sim p$. $\square$

All-false assignments

Let Σ be a nonempty set.

Lemma 7.3.19

If $x \in \mathrm{WFF}^+(\Sigma)$, then $\overbrace{\text{assigning **false** to all the elements of } \Sigma \text{ makes } x \text{ evaluate to **false**}}^{P(x)}$.

If $x \in \mathrm{WFF}^+(\Sigma)$, then assigning **false** to all the elements of Σ makes x evaluate to **false**.

Proof

2. (Base step) $P(p)$ is true for every $p \in \Sigma$ because assigning **false** to all the elements of Σ in particular assigns **false** to p.
3. (Induction step)
   3.1. Let $x, y \in \mathrm{WFF}^+(\Sigma)$ such that $P(x)$ and $P(y)$ are true, i.e.,

   assigning **false** to all the elements of Σ makes both x and y evaluate to **false**.

   3.2. Then assigning **false** to all the elements of Σ must make $(x \wedge y)$ and $(x \vee y)$ evaluate to **false** by the induction hypothesis because

   $$\textbf{false} \wedge \textbf{false} \equiv \textbf{false} \equiv \textbf{false} \vee \textbf{false}.$$

   3.3. So $P((x \wedge y))$ and $P((x \vee y))$ are true.
4. Hence $\forall x \in \mathrm{WFF}^+(\Sigma)\ \ P(x)$ is true by structural induction over $\mathrm{WFF}^+(\Sigma)$. $\square$

Summary

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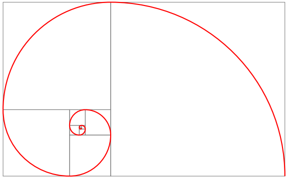
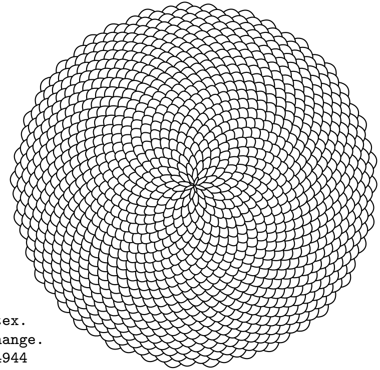


What we saw

- how to recursively define a set using base clauses, recursion clauses, and the minimality clause
- how to formulate and use structural induction over recursively defined sets
- how to recursively define functions on a recursively defined set

Next

- integers



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