CS1231S Chapter 9

Relations

9.1 Basics

Definition 9.1.1. Let A, B be sets.

tuple?

(1) A *relation* from A to B is a subset of $A \times B$.

(2) Let R be a relation from A to B and $(x, y) \in A \times B$. Then we may write

$$x \mathrel{R} y \text{ for } (x, y) \in R \quad \text{and} \quad x \not\mathrel{R} y \text{ for } (x, y) \notin R.$$

We read "$x \mathrel{R} y$" as "x is R-*related* to y" or simply "x is *related* to y".

Example 9.1.2. Let S be the set of all NUS students and M be the set of all modules offered by the NUS. Then "is enrolled in" is a relation from S to M. As a set, this relation is

$$\{(x, y) \in S \times M : x \text{ is enrolled in } y\}.$$

Example 9.1.3. Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3, 4\}$. Define the relation R from A to B by setting

$$x \mathrel{R} y \quad \Leftrightarrow \quad x < y.$$

Then $0 \mathrel{R} 1$ and $0 \mathrel{R} 2$, but $2 \not\mathrel{R} 1$. As a set,

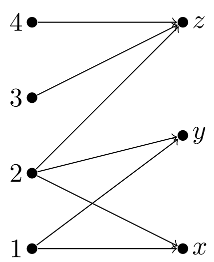
$$R = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}.$$

Arrow diagram. Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Consider the relation R from A to B defined by

$$R = \{(1, x), (1, y), (2, x), (2, y), (2, z), (3, z), (4, z)\}.$$

One can represent this relation by the following *arrow diagram*, where the existence of an arrow from a to b indicates $a \mathrel{R} b$:

arrow means there is a relation



- direction matters
- and is similar to a true/false function

discussing the overall set of relations, not just 1 arrow but all the arrows

helps with visualization

9.2 Equivalence relations

Definition 9.2.1. A *(binary) relation on a set* A is a relation from A to A.

Definition 9.2.2. Let A be a set and R be a relation on A.

(1) R is *reflexive* if $\quad \forall x \in A \ \ (x\,R\,x)$.

(2) R is *symmetric* if $\quad \forall x, y \in A \ \ (x\,R\,y \Rightarrow y\,R\,x)$. $\forall x,y \in A\ (xRy \Leftrightarrow yRx)$

(3) R is *transitive* if $\quad \forall x, y, z \in A \ \ (x\,R\,y \wedge y\,R\,z \Rightarrow x\,R\,z)$.

Example 9.2.3. Let R denote the equality relation on a set A, i.e., for all $x, y \in A$,

$$x\,R\,y \quad \Leftrightarrow \quad x = y.$$

Then R is reflexive, symmetric, and transitive.

Example 9.2.4. Let R' denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x\,R'\,y \quad \Leftrightarrow \quad x \subseteq y.$$

Then R' is reflexive, may not be symmetric (when U contains x, y such that $x \subsetneq y$), but is transitive.

Example 9.2.5. Let R denote the non-strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x\,R\,y \quad \Leftrightarrow \quad x \leqslant y.$$

Then R is reflexive, [redacted], but transitive.

Example 9.2.6. Let R' denote the strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x\,R'\,y \quad \Leftrightarrow \quad x < y.$$

Is [redacted] Is R' transitive? Is [redacted]? ✎ 9a

chapter 8

Example 9.2.7. Let R denote the divisibility relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x\,R\,y \quad \Leftrightarrow \quad x \mid y.$$

Is R reflexive? Is R transitive? Is [redacted] ✎ 9b

chapter 8

Example 9.2.8. Let $n \in \mathbb{Z}^+$ and R' denote the congruence-mod-n relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x\,R'\,y \quad \Leftrightarrow \quad x \equiv y \pmod{n}.$$

Then R' is reflexive, symmetric, and transitive by Lemma 8.6.5.

Definition 9.2.9. An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

Definition 9.2.10. Let A be a set and R be an equivalence relation on A. For each $x \in A$, the *equivalence class* of x with respect to R, denoted $[x]_R$, is defined by

$$[x]_R = \{y \in A : x\,R\,y\}.$$

[x]

When there is no risk of confusion, we may drop the subscript R and write simply $[x]$. Define $A/R = \{[x]_R : x \in A\}$.

Example 9.2.11. The equality relation R on a set A is an equivalence relation. The equivalence classes are of the form

$$[x] = \{y \in A : x = y\} = \{x\},$$

where $x \in A$. So $A/R = \{[x] : x \in A\} = \{\{x\} : x \in A\}$.

A = {1,2,3}   R = {(1,1),(2,2),(3,3),(1,2),(2,1)}

[1] = {1,2}, [2] = {1,2}, [3] = {3}

A/R = { {1,2} , {1,2} , {3} } = { {1,2} , {3} }

1 2 3

Example 9.2.12. Fix $n \in \mathbb{Z}^+$. The congruence-mod-n relation R_n on $\mathbb{Z}$ is an equivalence relation. The equivalence classes are of the form

alternative definition

$$[x] = \{y \in \mathbb{Z} : x \equiv y \pmod n\} = \{x + nk : k \in \mathbb{Z}\},$$

where $x \in \mathbb{Z}$. So $\mathbb{Z}/R_n = \{\{x + nk : k \in \mathbb{Z}\} : x \in \mathbb{Z}\} = \{[0], [1], \ldots, [n-1]\}$.

If $n = 2$, then there are two equivalence classes: $[x] = [x \bmod n]$

$$\{2k : k \in \mathbb{Z}\} \quad \text{and} \quad \{2k+1 : k \in \mathbb{Z}\}.$$

Proposition 9.2.13. Let R be an equivalence relation on a set A. The following are equivalent for all $x, y \in A$.

(i) $x \mathrel{R} y$.

(ii) $[x] = [y]$.

(iii) $[x] \cap [y] \neq \varnothing$.

alternative definitions

Proof. 1. ((i) $\Rightarrow$ (ii))

1.1. Suppose $x \mathrel{R} y$.
1.2. Then $y \mathrel{R} x$ by symmetry.
1.3. For every $z \in [x]$,

1.3.1.		$x \mathrel{R} z$	by the definition of $[x]$;
1.3.2.	$\therefore$	$y \mathrel{R} z$	by transitivity, as $y \mathrel{R} x$;
1.3.3.	$\therefore$	$z \in [y]$	by the definition of $[y]$.

1.4. This shows $[x] \subseteq [y]$.
1.5. Switching the roles of x and y, we see also that $[y] \subseteq [x]$.
1.6. So $[x] = [y]$.

2. ((ii) $\Rightarrow$ (iii)) If $[x] = [y]$, then $[x] \cap [y] = [x]$, which is nonempty because the reflexivity of R implies $x \in [x]$.
3. ((iii) $\Rightarrow$ (i))

3.1. Suppose $[x] \cap [y] \neq \varnothing$.
3.2. Take $z \in [x] \cap [y]$.
3.3. Then $x \mathrel{R} z$ and $y \mathrel{R} z$.
3.4. The latter implies $z \mathrel{R} y$ by symmetry.
3.5. So $x \mathrel{R} y$ by transitivity. □

9.3 Partitions

Definition 9.3.1. A *partition* of a set A is a set $\mathscr{C}$ of *nonempty* subsets of A such that

($\geqslant 1$) $\forall x \in A \ \exists S \in \mathscr{C} \ (x \in S)$; and every element in the original set is put into at least 1 bag

($\leqslant 1$) $\forall x \in A \ \forall S, S' \in \mathscr{C} \ \big(x \in S \wedge x \in S' \Rightarrow S = S'\big)$. every element in the original set is put into at most 1 bag

Elements of a partition are called *components* of the partition.

Example 9.3.2. The set $A = \{1, 2, 3\}$ has the following partitions: essentially a smaller set of a set

$$\{\{1\},\{2\},\{3\}\}, \quad \{\{1\},\{2,3\}\}, \quad \{\{2\},\{1,3\}\}, \quad \{\{3\},\{1,2\}\}, \quad \{\{1,2,3\}\}.$$

Example 9.3.3. The congruence-mod-2 relation gives rise to the following partition of $\mathbb{Z}$:

$$\{\{2k : k \in \mathbb{Z}\}, \{2k+1 : k \in \mathbb{Z}\}\}.$$

Theorem 9.3.4. Let R be an equivalence relation on a set A. Then A/R is a partition of A.

iff

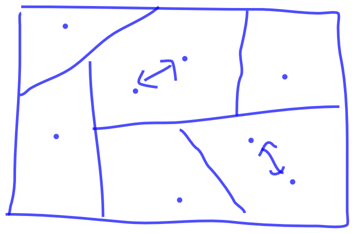
Proof. 1. ($\geqslant 1$)

Proving that all will be inside

1.1. Let $x \in A$.
1.2. Then $x\ R\ x$ by reflexivity.
1.3. So $x \in [x] \in A/R$.

2. ($\leqslant 1$) By Proposition 9.2.13, for all $[x], [y] \in A/R$, if $[x] \cap [y] \neq \varnothing$, then $[x] = [y]$. ☐

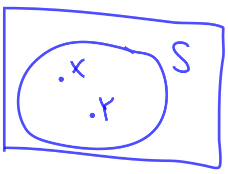
Theorem 9.3.5. Let $\mathscr{C}$ be a partition of a set A. Then there is an equivalence relation R on A such that $A/R = \mathscr{C}$.



Proof. 1. Define a relation R on A, by setting, for all $x, y \in A$,

$$x\ R\ y \quad \Leftrightarrow \quad x, y \in S \text{ for some } S \in \mathscr{C}.$$

2. (Reflexivity)
   2.1. Let $x \in A$.
   2.2. Axiom ($\geqslant 1$) for partitions gives $S \in \mathscr{C}$ such that $x \in S$.
   2.3. So $x\ R\ x$.
3. (Symmetry)
   3.1. Let $x, y \in A$ such that $x\ R\ y$.
   3.2. Find $S \in \mathscr{C}$ such that $x, y \in S$.
   3.3. Then $y, x \in S \in \mathscr{C}$ and thus $y\ R\ x$.
4. (Transitivity)
   4.1. Let $x, y, z \in A$ such that $x\ R\ y$ and $y\ R\ z$.
   4.2. Use the definition of R to find $S, S' \in \mathscr{C}$ such that $x, y \in S$ and $y, z \in S'$.
   4.3. Then $y \in S \cap S'$.
   4.4. So $S = S'$ by axiom ($\leqslant 1$) for partitions.
   4.5. Thus $x, z \in S$, making $x\ R\ z$.
5. So R is an equivalence relation.
6. 6.1. Let $x \in S \in \mathscr{C}$.



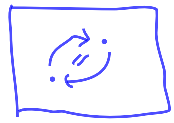
   6.2. $S \subseteq [x]$ because x is related to all the elements of S by the definition of R.
   6.3. Let $y \in [x]$. Then $x\ R\ y$ by the definition of $[x]$.
   6.4. So the definition of R gives some $S' \in \mathscr{C}$ such that $x, y \in S'$.
   6.5. Since $x \in S \cap S'$, we deduce that $S = S'$ by axiom ($\leqslant 1$) for partitions.
   6.6. Hence $y \in S' = S$.
   6.7. Since the choice of $y \in [x]$ was arbitrary, we infer that $[x] \subseteq S$.
   6.8. Thus $[x] = S$.

$A/R = \{[x] : x \in A\}$ where $[x] = \{y \in A : xRy\}$

7. Block 6 shows that if $x \in S \in \mathscr{C}$, then $[x] = S$.
8. 8.1. Let $[x] \in A/R$.
   8.2. Use axiom ($\geqslant 1$) for partitions to find $S \in \mathscr{C}$ such that $x \in S$.
   8.3. Then line 7 implies $[x] = S \in \mathscr{C}$.
9. Since the choice of $[x] \in A/R$ was arbitrary, we infer that $A/R \subseteq \mathscr{C}$.
10. 10.1. Let $S \in \mathscr{C}$.
    10.2. Then $S \neq \varnothing$ as $\mathscr{C}$ is a partition.
    10.3. Take $x \in S$. (by line 6.1)
    10.4. Then line 7 implies $S = [x] \in A/R$.
11. Since the choice of $S \in \mathscr{C}$ was arbitrary, we infer that $\mathscr{C} \subseteq A/R$.
12. Hence $A/R = \mathscr{C}$. ☐

order is for comparing 2 objects
partial because not all objects are comparable

9.4 Partial orders



Definition 9.4.1. Let A be a set and R be a relation on A.

(1) R is *antisymmetric* if $\forall x, y \in A \ \ (x \mathrel{R} y \wedge y \mathrel{R} x \Rightarrow x = y)$.

(2) R is a *(non-strict) partial order* if R is reflexive, antisymmetric, and transitive.

(3) Suppose R is a partial order. Let $x, y \in A$. Then x, y are *comparable (under R)* if

$$x \mathrel{R} y \quad \text{or} \quad y \mathrel{R} x.$$

(4) R is a *(non-strict) total order* if R is a partial order and $\forall x, y \in A \ \ (x \mathrel{R} y \vee y \mathrel{R} x)$.

comparable

Note 9.4.2. A total order is always a partial order.

Example 9.4.3. Let R denote the non-strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x \leqslant y.$$

Then R is antisymmetric. In fact, it is a total order.

Example 9.4.4. Let R' denote the strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

xRy ^ yRx is false, therefore implication is vacuously true

$$x \mathrel{R'} y \quad \Leftrightarrow \quad x < y.$$

Is R' antisymmetric? [redacted] ✎ 9c

Example 9.4.5. Let R denote the equality relation on a set A, i.e., for all $x, y \in A$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x = y.$$

Then R is antisymmetric. It is a partial order, but [redacted]

Example 9.4.6. Fix $n \in \mathbb{Z}^+$. Let R' denote the congruence-mod-n relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x \mathrel{R'} y \quad \Leftrightarrow \quad x \equiv y \pmod{n}.$$

Then [redacted] because $0 \mathrel{R'} n$ and $n \mathrel{R'} 0$ but $0 \neq n$.

Example 9.4.7. Let R denote the divisibility relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x \mid y.$$

[redacted] Is R a partial order? Is R a total order? ✎ 9d

Example 9.4.8. Let R' denote the divisibility relation on $\mathbb{Z}^+$, i.e., for all $x, y \in \mathbb{Z}^+$,

$$x \mathrel{R'} y \quad \Leftrightarrow \quad x \mid y.$$

will become antisymmetric if only positive numbers

Is R antisymmetric? Is R a partial order? [redacted] $\rightarrow 2 \nmid 3 \wedge 3 \nmid 2$ ✎ 9e

not divisible

Example 9.4.9. Let R denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x \subseteq y.$$

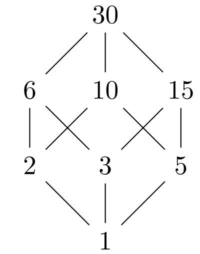
Then R is antisymmetric. It is always a partial order, but it may not be a total order.

Notation 9.4.10. We often use $\preccurlyeq$ to denote a partial order. In this case, we write $x \prec y$ for $x \preccurlyeq y \wedge x \neq y$.

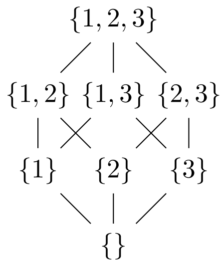
Definition 9.4.11. Let $\preccurlyeq$ be a partial order on a set A. A *Hasse diagram* of $\preccurlyeq$ satisfies the following condition for all $x, y \in A$:

If $x \prec y$ and no $z \in A$ is such that $x \prec z \prec y$, then x is placed below y and there is a line joining x to y.

Example 9.4.12. Consider $\{d \in \mathbb{Z}^+ : d \mid 30\}$ partially ordered by the divisibility relation $|$. A Hasse diagram is as follows:

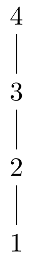


Example 9.4.13. Consider $\mathcal{P}(\{1,2,3\})$ partially ordered by the inclusion relation $\subseteq$. A Hasse diagram is as follows:



Hasse diagram will reveal the internal structure

Example 9.4.14. Consider $\{1,2,3,4\}$ partially ordered by the non-strict less-than relation $\leqslant$. A Hasse diagram is as follows:



9.5 Linearization

Definition 9.5.1. Let $\preccurlyeq$ be a partial order on a set A, and $c \in A$.

(1) c is a *minimal element* if

$$\forall x \in A \quad (x \preccurlyeq c \Rightarrow c = x).$$

finding the element which is at the top of the diagram

(2) c is a *maximal element* if

nothing that is strictly above it

$$\forall x \in A \quad (c \preccurlyeq x \Rightarrow c = x).$$

(3) c is the *smallest element* (or the *minimum element*) if

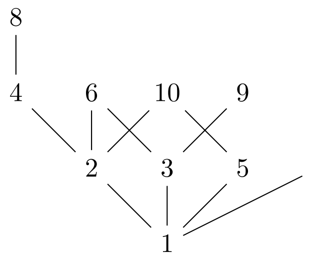
$$\forall x \in A \quad (c \preccurlyeq x).$$

(4) c is the *largest element* (or the *maximum element*) if

must be at the top of everything else

$$\forall x \in A \quad (x \preccurlyeq c).$$

Example 9.5.2. The divisibility relation $|$ on $\{1, 2, \ldots, 10\}$ is represented by the Hasse diagram



- The only minimal element is 1.
- The maximal elements are $6, 7, 8, 9, 10$.
- The smallest element is 1.
- There is no largest element.

Example 9.5.3. (1) $\mathbb{Q}^+$ under the non-strict less-than relation $\leqslant$ has neither a minimal element nor a maximal element.

(2) $\mathbb{Z}^+$ under the non-strict less-than relation $\leqslant$ has a smallest element but no maximal element.

Definition 9.5.4. A *well-order* on a set A is a total order on A with respect to which every nonempty subset of A has a smallest element.

Lemma 9.5.5. Consider a partial order $\preccurlyeq$ on a set A.

(1) A smallest element is minimal.

(2) There is at most one smallest element.

Proof. (1)
1. Let c be a smallest element.
2. Take any $x \in A$ such that $x \preccurlyeq c$.
3. By smallestness, we know $c \preccurlyeq x$ too.
4. So $c = x$ by antisymmetry.

(2)
1. Let c, c' be smallest elements.
2. Then $c \preccurlyeq c'$ and $c' \preccurlyeq c$ by the smallestness of c and c' respectively.
3. So $c = c'$ by antisymmetry. □

Exercise 9.5.6. Show the statements analogous to Lemma 9.5.5 for largest and maximal elements. ✎ 9f

Proposition 9.5.7. With respect to any partial order $\preccurlyeq$ on a nonempty finite set A, one can find a minimal element.

Proof.
1. Take any $c_0 \in A$. This is possible since $A \neq \varnothing$.
2. If c_0 is not minimal, then find $c_1 \in A$ such that $c_1 \prec c_0$.
3. Continue this process: if c_n is not minimal, then find $c_{n+1} \in A$ such that $c_{n+1} \prec c_n$.
4. Note that $c_{n+1} \neq c_i$ for any $i \in \{0, 1, \ldots, n\}$ because if $i \in \{0, 1, \ldots, n\}$ such that $c_{n+1} = c_i$, then
   4.1. $c_n \prec c_{n-1} \prec \cdots \prec c_i = c_{n+1}$;
   4.2. so $c_n \preccurlyeq c_{n+1}$ by transitivity;
   4.3. so $c_n = c_{n+1}$ by antisymmetry as $c_{n+1} \prec c_n$;
   4.4. so we have a contradiction with $c_{n+1} \prec c_n$.
5. Since A is finite, this process must end, say with c_n.

6. c_n must be minimal for this process to end. $\square$

Exercise 9.5.8. Convince yourself that the statement analogous to Proposition 9.5.7 is true for maximal elements. 9g

Theorem 9.5.9. Let A be a set and $\preccurlyeq$ be a partial order on A. Then there exists a total order $\preccurlyeq^*$ on A such that for all $x, y \in A$,

$$x \preccurlyeq y \quad \Rightarrow \quad x \preccurlyeq^* y.$$

Proof for finite A (due to Kahn 1962). 1. Consider the following process.

(1) Set $A_0 := A$ and $i := 0$.

(2) If $A_i = \varnothing$, then stop, else

  (2.1) use Proposition 9.5.7 to find a minimal element c_i of A_i with respect to $\preccurlyeq \cap (A_i \times A_i)$;

  (2.2) set $A_{i+1} := A_i \setminus \{c_i\}$.

(3) Set $i := i + 1$, and go back to step (2).

2. Since A is finite, this process stops.
3. Let $c_0, c_1, \dots, c_{n-1}$ be the sequence of elements of A produced.
4. We know $A = \{c_0, c_1, \dots, c_{n-1}\}$ because the process stopped after picking out these elements.
5. Define $\preccurlyeq^*$ on A by setting, for all $i, j \in \{0, 1, \dots, n-1\}$,

$$c_i \preccurlyeq^* c_j \quad \Leftrightarrow \quad i \leqslant j.$$

6. Let $i, j \in \{0, 1, \dots, n-1\}$ such that $c_i \prec c_j$.
7. 7.1. Suppose $i > j$.
   7.2. The minimality of c_j in A_j means $\forall x \in A_j \ (x \preccurlyeq c_j \Rightarrow c_j = x)$.
   7.3. Since $c_i \preccurlyeq c_j$ and $c_j \neq c_i$, we deduce that $c_i \notin A_j$.
   7.4. As $A_j \supseteq A_{j+1} \supseteq \cdots \supseteq A_i$, this implies $c_i \notin A_i$, which is a contradiction.
8. So $i \leqslant j$.
9. Thus $c_i \preccurlyeq^* c_j$ by the definition of $\preccurlyeq^*$. $\square$

Example 9.5.10. Consider $\{d \in \mathbb{Z}^+ : d \mid 30\}$ partially ordered by the divisibility relation $\mid$ as in Example 9.4.12.

- Set $A_0 := \{d \in \mathbb{Z}^+ : d \mid 30\}$.
- 1 is the only minimal element of A_0. Set $c_0 := 1$ and $A_1 := A_0 \setminus \{1\}$.
- $2, 3, 5$ are the minimal elements of A_1. Set $c_1 := 3$ and $A_2 := A_1 \setminus \{3\}$.
- $2, 5$ are the minimal elements of A_2. Set $c_2 := 2$ and $A_3 := A_2 \setminus \{2\}$.
- $5, 6$ is the only minimal element of A_3. Set $c_3 := 5$ and $A_4 := A_3 \setminus \{5\}$.
- $6, 10, 15$ are the minimal elements of A_4. Set $c_4 := 6$ and $A_5 := A_4 \setminus \{6\}$.
- $10, 15$ are the minimal elements of A_5. Set $c_5 := 15$ and $A_6 := A_5 \setminus \{15\}$.
- 10 is the only minimal element of A_6. Set $c_6 := 10$ and $A_7 := A_6 \setminus \{10\}$.
- 30 is the only (minimal) element of A_7. Set $c_7 := 30$ and $A_8 := A_7 \setminus \{30\}$.
- $A_8 = \varnothing$ and so we stop.

A linearization is $1 \preccurlyeq^* 3 \preccurlyeq^* 2 \preccurlyeq^* 5 \preccurlyeq^* 6 \preccurlyeq^* 15 \preccurlyeq^* 10 \preccurlyeq^* 30$.

extending from partial order to total order