NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING

MID-TERM TEST
AY2020/21 Semester 1

CS1231/CS1231S — DISCRETE STRUCTURES

3 October 2020 Time Allowed: 1 hour 30 minutes

INSTRUCTIONS

1. This assessment paper contains **TWENTY TWO (22)** questions (excluding question 0) in **THREE (3)** parts and comprises **EIGHT (8)** printed pages.
2. Answer **ALL** questions.
3. This is an **OPEN BOOK** assessment.
4. The maximum mark of this assessment is 50.
5. You are to submit a **single pdf file** (size $\leqslant$ 20MB) to your submission folder on LumiNUS.
6. Your submitted file should be named after your **Student Number** (eg: A1234567X.pdf) and all pages of your file should contain your Student Number as well.
7. Limit your answers to **TWO pages** if possible, or at most THREE pages.
8. Do <u>not</u> write your name in your submitted file.

——— END OF INSTRUCTIONS ———

You do not need to answer question 0. Your tutor will check its correctness.

0. (a) Does my submission folder consist of a **single pdf file**? [1 mark]
   (b) Is my file named correctly, i.e. with my **Student Number** (eg: A1234567X.pdf)? [½ mark]
   (c) Have I written my Student Number on EVERY PAGE of my submitted file? [½ mark]

Part A: Multiple Choice Questions (Total: 14 marks)

Each multiple choice question (MCQ) is worth two marks and has exactly **one** correct answer. You are advised to write your answers on a **single line** to conserve space. For example:

1 A 2 B 3 C 4 D …

Please write in **CAPITAL LETTERS**.

1. Given the following statements:

   "I" is a necessary condition for "can".
   "can" is a sufficient condition for "do".
   "can" only if "it".

   Which of the following is logically equivalent to the conjunction of the above three statements?

   A. $"I" \rightarrow "can" \wedge "do" \wedge "it"$
   B. $"can" \rightarrow "I" \wedge "do" \wedge "it"$
   C. $("can" \wedge "I") \rightarrow ("do" \vee "it")$
   D. $("I" \vee "can") \rightarrow ("do" \wedge "it")$
   E. None of (A), (B), (C), (D) is true.

2. Given the statement:

   The product of two negative real numbers is positive.

   Which of the following is the correct logical statement for the above statement?

   A. $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ (x < 0 \wedge y < 0 \rightarrow xy > 0)$
   B. $\exists x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x < 0 \wedge y < 0 \wedge xy > 0)$
   C. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x < 0 \wedge y < 0 \wedge xy > 0)$
   D. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ (x < 0 \wedge y < 0 \rightarrow xy > 0)$

3. Given the statement:

$$(\sim p \wedge \sim r) \vee \big(p \wedge \sim(\sim r \vee s)\big) \vee p \vee (\sim p \wedge q \wedge \sim r \wedge s) \vee \sim(q \vee \sim p)$$

Which of the following is logically equivalent to the above?

A. p

B. $\sim r$

C. $p \vee \sim r$

D. $\sim p \vee \sim r$

E. None of (A), (B), (C) or (D).

4. Let $A = \{1,2,3\}$ and define $f: A \to A$ by setting, for all $x \in A$,

$$f(x) = \begin{cases} 2, & \text{if } x = 1; \\ 3, & \text{if } x = 2; \\ 1, & \text{if } x = 3. \end{cases}$$

How many (distinct) elements are there in

$$\{\underbrace{f \circ f \circ \cdots \circ f}_{n-\text{many } f\text{'s}} : n \in \mathbb{Z}^+\}?$$

A. 1.

B. 2.

C. 3.

D. 4.

E. Infinitely many.

5. Let $A = \{1,2,3\}$ and define $g, h: A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

What is $(h \circ g)^{-1}$?

A. g.

B. h.

C. $g \circ h$.

D. $h \circ g$.

E. None of the above.

6. Define the function $f\colon \mathrm{Bool}^2 \to \mathrm{Bool}$ by setting, for all $p, q \in \mathrm{Bool}$,

$$f(p,q) \quad = \quad (p \vee q) \wedge {\sim}(p \wedge q).$$

(Here $\mathrm{Bool} = \{\mathbf{true}, \mathbf{false}\}$.) Which of the following statements is true?

   A. f is not well defined, i.e., f does not satisfy the definition of a function.

   B. f is a function that is neither injective nor surjective.

   C. f is an injection but not a surjection.

   D. f is a surjection but not an injection.

   E. f is a bijection.

7. Let A, B, C be sets. Which of the following may not be a subset of $(A \setminus C) \cup (B \setminus C)$?

   A. $(A \cap B) \setminus C$.

   B. $(A \cup B) \setminus C$.

   C. $(A \setminus B) \setminus C$.

   D. $(A \setminus C) \setminus B$.

   E. $A \cap B \cap C$.

Part B: Multiple Response Questions [Total: 24 marks]

Each multiple response question (MRQ) is worth two marks and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, B, C are the correct answers, write A, B, C. Only if you get all the answers correct will you be awarded two marks. **No partial credit will be given for partially correct answers.**

You are advised to write your answers on **at most two lines** to conserve space. For example:

8 A,B   9 B,D   10 C   11 A,B,C,D   …

Please write in **CAPITAL LETTERS**.

8. Let the domain of discourse be the set of real numbers and define $P(x,y)$ and $Q(x,y)$ as follows:

$$P(x,y)\colon xy = 0$$

$$Q(x,y)\colon x/y = 1$$

Which of the following is/are FALSE?

   A. $\forall x\, \forall y\, P(x,y) \equiv \forall x\, \forall y\, Q(x,y)$

   B. $\forall x\, \exists y\, P(x,y) \equiv \forall x\, \exists y\, Q(x,y)$

   C. $\exists x\, \forall y\, P(x,y) \equiv \exists x\, \forall y\, Q(x,y)$

   D. $\exists x\, \exists y\, P(x,y) \equiv \exists x\, \exists y\, Q(x,y)$

9. Which of the following is/are FALSE for some predicates $P(x)$ and $Q(x)$?
   A. $\forall x\ (P(x) \vee Q(x)) \Leftrightarrow \forall x\ P(x) \vee \forall x\ Q(x)$
   B. $\forall x\ (P(x) \wedge Q(x)) \Leftrightarrow \forall x\ P(x) \wedge \forall x\ Q(x)$
   C. $\exists x\ (P(x) \vee Q(x)) \Leftrightarrow \exists x\ P(x) \vee \exists x\ Q(x)$
   D. $\exists x\ (P(x) \wedge Q(x)) \Leftrightarrow \exists x\ P(x) \wedge \exists x\ Q(x)$

10. Let the domain of discourse be the set of real numbers. Which of the following is/are FALSE?
    A. $\forall x\ (x^2 > 0)$
    B. $\exists!\, x\ (x^2 + 2x - 3 = 0)$
    C. $\exists x\ (x^2 + x + 1 = 0)$
    D. $\forall x\ (x \neq 0 \rightarrow x^2 \geq 1)$

11. Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{-1, 0, 1\}$. Which of the following is/are FALSE?
    A. $\forall x, y \in A\ (x + y \in A)$
    B. $\forall x, y \in B\ (xy \in B)$
    C. $\exists x \in A\ \forall y \in A\ \exists z \in \mathbb{Z}\ (y = xz)$
    D. $\forall x \in A\ \forall y \in B\ \exists z \in \mathbb{Z}\ (x + y < z < xy)$

12. Which of the following set(s) is/are equal to

$$\{3n + 1 : n \in \mathbb{Z}\}?$$

    A. $\{3n : n \in \mathbb{Z}\} \cup \{1\}$.
    B. $\{3n - 1 : n \in \mathbb{Z}\}$.
    C. $\{3n - 2 : n \in \mathbb{Z}\}$.
    D. The domain of the function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying $f(n) = 3n + 1$ for all $n \in \mathbb{Z}$.
    E. The codomain of the function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying $f(n) = 3n + 1$ for all $n \in \mathbb{Z}$.

13. Let $S = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$ and $R = \{\mathrm{A}, 2,3,4,5,6,7,8,9,10, \mathrm{J}, \mathrm{Q}, \mathrm{K}\}$. Which of the following **set(s)** contain

$$\{(\spadesuit, \mathrm{A}), (\spadesuit, \mathrm{K}), (\spadesuit, \mathrm{Q}), (\spadesuit, \mathrm{J}), (\spadesuit, 10)\}$$

as an **element**?

A. $S \times R$.

B. $\mathcal{P}(S \times R)$.

C. $S \cup R$.

D. $\mathcal{P}(S \cup R)$.

14. Let $M_k = \{n \in \mathbb{Z} : n = km \text{ for some } m \in \mathbb{Z}\}$ for each $k \in \mathbb{Z}$. Which of the following contain(s) the number 100 as an element?

A. $M_2 \times M_3 \times M_5$.

B. $(M_2 \setminus M_3) \setminus M_5$.

C. $M_2 \cup M_3 \cup M_5$.

D. $M_2 \cap M_3 \cap M_5$.

15. Which of the following **cannot** be $|A \times (B \cup \mathcal{P}(C))|$ for any choice of mutually disjoint nonempty finite sets A, B, C?

A. 1.

B. 2.

C. 3.

D. 4.

E. 5.

16. Which of the following function(s) $f: \mathbb{Z} \to \mathbb{Z}$ defined below satisfies/satisfy

$$\forall X \subseteq \mathbb{Z} \quad f^{-1}\big(f(X)\big) \subseteq X?$$

A. $f: x \mapsto x^2$.

B. $f: \mapsto 3x + 1$.

C. $f: x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$

D. $f: x \mapsto \left\lfloor \frac{x}{2} \right\rfloor$.

17. Which of the following function(s) $f: \mathbb{Z} \to \mathbb{Z}$ defined below satisfies/satisfy

$$\forall Y \subseteq \mathbb{Z} \quad Y \subseteq f(f^{-1}(Y))?$$

   A. $f: x \mapsto x^2$.

   B. $f: \mapsto 3x + 1$.

   C. $f: x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$

   D. $f: x \mapsto \left\lfloor \frac{x}{2} \right\rfloor$.

18. Define a set S recursively as follows.

   - $1 \in S$. (base clause)
   - If $x \in S$, then $2x \in S$ and $x - 3 \in S$. (recursion clause)
   - Membership for S can always be demonstrated by (finitely many) successive applications of the clauses above. (minimality clause)

   Which of the following is/are in S?

   A. 3

   B. 5

   C. 7

   D. 9

19. Let $A_0, A_1, A_2, \ldots$ be the sequence of sets satisfying

$$A_0 = \emptyset \quad \text{and} \quad A_{n+1} = A_n \cup \{A_n\}.$$

   Which of the following is/are true?

   A. $\big\{\{\emptyset\}\big\} \in A_{100}$.

   B. $\big\{\{\emptyset\}\big\} \subseteq A_{100}$.

   C. $\big\{\emptyset, \{\emptyset\}\big\} \in A_{100}$.

   D. $\big\{\emptyset, \{\emptyset\}\big\} \subseteq A_{100}$.

Part C: There are 3 questions in this part [Total: 10 marks]

20. Given statement variables p, q and r, is the following statement a tautology?

$$\big((p \to q) \wedge (q \to r)\big) \vee (p \to r) \to (r \to p)$$

You need to show your working. Merely stating that it is a tautology or it is not, or giving a counterexample (in the case it is not) without showing that it leads to false is not sufficient and no mark will be awarded. No justification in your steps are needed if you use the definitions of conjunction, disjunction and conditional statement. Do not use truth table. [3 marks]

21. In Appendix A (Properties of the Real Numbers) of Epp's book, the following theorem is given and you may quote it without proof:

T25. Suppose a and b are real numbers, if $ab > 0$, then both a and b are positive or both are negative.

Prove: $\forall x \in \mathbb{R}\,\big((x^2 > x) \to (x < 0) \vee (x > 1)\big)$

You are to give justification wherever appropriate. [3 marks]

22. Let $a_0, a_1, a_2, \ldots$ be the sequence satisfying

$$a_0 = 1 \quad \text{and} \quad a_{n+1} = 2a_n + 2$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. Show by induction that

$$a_n = 2^{n+1} + 2^n - 2$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. [4 marks]

=== END OF PAPER ===