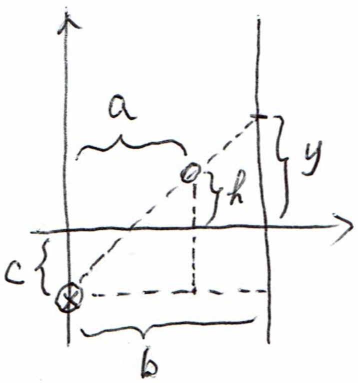
1. In this question all length measurements are in meters and time measurements are in seconds. Let a, b, c denote three positive constants with $b > a$. A light source is placed at the point $(0, -c)$. At time $t = 0$ a particle starts at the point $(a, 0)$ moving upwards along the line $x = a$ in such a way that at time t its height h from the starting point $(a, 0)$ is directly proportional to t^2. It is observed that $h = 28$ at time $t = 3$. If the line $x = b$ represents a screen and the speed of the shadow of the projection of the particle onto the screen is equal to 188 meter per second when $h = 68$, find the value of the ratio $\frac{b}{a}$. Give your answer correct to two decimal places.



$h \sim t^2 \Rightarrow h = kt^2 \Rightarrow 28 = k(3)^2 \Rightarrow k = \frac{28}{9}$

$\therefore h = \frac{28}{9}t^2 \Rightarrow dh/dt = \frac{56}{9}t$

$\therefore h = 68 \Rightarrow t = 3\sqrt{\frac{17}{7}}$

$$\frac{y+c}{b} = \frac{h+c}{a} \Rightarrow y + c = \frac{b}{a}(h+c)$$

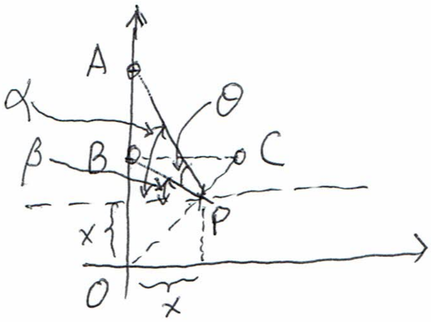
$$\therefore \frac{dy}{dt} = \frac{b}{a}\frac{dh}{dt} = \frac{56}{9}\left(\frac{b}{a}\right)t$$

$$\therefore 188 = \frac{56}{9}\left(\frac{b}{a}\right)\left(3\sqrt{\frac{17}{7}}\right)$$

$$\frac{b}{a} = \frac{3 \times 47 \times \sqrt{7 \times 17}}{14 \times 17} = 6.462\ldots$$

$$\simeq \underline{\underline{6.46}}$$

2. Let O denote the origin $(0,0)$, A denote the point $(0,2020)$, B denote the point $(0,1521)$ and C denote the point (1521,1521). Let P denote a point on the line segment OC such that P is between O and C, $P \neq O$ and $P \neq C$. Let θ denote the angle APB measured in DEGREES. Note that $0 < \theta < 90^o$. Find the maximum value of θ. Give your answer correct to two decimal places.
(Important: Leave out the degree symbol ^{o} when you enter your answer as the computer cannot recognize it. For e.g. if your answer is 1.23^o, then you just enter 1.23 for your answer.)



$$\tan\alpha = \frac{2020-x}{x}, \quad \tan\beta = \frac{1521-x}{x}$$

$$\tan\theta = \tan(\alpha-\beta)$$

$$= \frac{\frac{2020-x}{x} - \frac{1521-x}{x}}{1+\left(\frac{2020-x}{x}\right)\left(\frac{1521-x}{x}\right)}$$

$$\therefore \tan\theta = \frac{499x}{2x^2-3541x+3072420}$$

$$\sec^2\theta \frac{d\theta}{dx} = \frac{988(1536210-x^2)}{(2x^2-3541x+3072420)^2} \Rightarrow \text{critical point at}$$

$$x = \sqrt{1536210} \quad (\because 0<x<1521)$$
$$= 39\sqrt{1010}$$
$$\approx 1239.439\ldots$$

$$\frac{d\theta}{dx} \quad + \quad -$$
$39\sqrt{1010}$
$\uparrow$
max.

$$\max\theta = \tan^{-1}\frac{499 \times 39\sqrt{1010}}{2(39\sqrt{1010})^2 - 3541\times 39\sqrt{1010} + 3072420}$$

$$\approx \tan^{-1} 0.352212 \simeq 19.4028\ldots \text{ degree}$$

$$\simeq \underline{19.40}$$

3. Let $f(x) = 2\left(\sin\left(\frac{\pi}{4} - 864x\right)\right)\left(\sin\left(\frac{\pi}{4} + 288x\right)\right)$. Find the value of the definite integral

$$\int_0^{10\pi} |f(x)|dx,$$

where $|z|$ denotes the absolute value of z. Give your answer correct to two decimal places.

$$I = \int_0^{10\pi} |f(x)|\,dx = \int_0^{10\pi} \left| \cos(-1152x) - \cos\left(\frac{\pi}{2} - 576x\right) \right| dx$$

$$= \int_0^{10\pi} |\cos 1152x - \sin 576x|\, dx$$

let $\theta = 576x \Rightarrow I = \frac{1}{576}\int_0^{5760\pi} |\cos 2\theta - \sin\theta|\, d\theta$

$\because \cos 2\theta - \sin\theta$ is 2π-periodic

$$\therefore I = \left(\frac{5760}{2}\right)\frac{1}{576}\int_0^{2\pi} |\cos 2\theta - \sin\theta|\, d\theta = 5\int_0^{2\pi} |\cos 2\theta - \sin\theta|\, d\theta$$

$\cos 2\theta - \sin\theta = 0 \Rightarrow \cos 2\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow 2\theta = 2n\pi \pm \left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ for $0 \leq \theta \leq 2\pi$

$\cos 2\theta - \sin\theta$: $+$ on $\left(0, \frac{\pi}{6}\right)$, $-$ on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$, $+$ on $\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, $+$ on $\left(\frac{3\pi}{2}, 2\pi\right)$

$$\therefore I = 5\left\{\int_0^{\frac{\pi}{6}} - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} + \int_{\frac{5\pi}{6}}^{2\pi} (\cos 2\theta - \sin\theta)\, d\theta\right\}$$

$$= 5\left\{\frac{3\sqrt{3}}{4} - 1 - \left(\frac{-3\sqrt{3}}{2}\right) + \left(\frac{1+3\sqrt{3}}{4}\right)\right\}$$

$$= 15\sqrt{3} = 25.980\ldots \approx \underline{\underline{25.98}}$$

4. Let

$$f(x) = \frac{x+1}{x^2+2x+7}.$$

Find the value of $f^{(9)}(-1)$ (i.e. the 9-th derivative of f at $x=-1$). Give your answer correct to two decimal places.

$$f(x) = \frac{(x+1)}{(x+1)^2+6}$$

$$= \frac{1}{6}(x+1)\frac{1}{1+\frac{1}{6}(x+1)^2}$$

$$= \frac{1}{6}(x+1)\sum_{n=0}^{\infty}(-1)^n\frac{1}{6^n}(x+1)^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{6^{n+1}}(x+1)^{2n+1}$$

$$= \cdots + \frac{(-1)^4}{6^{4+1}}(x+1)^{8+1} + \cdots$$

$$\therefore \frac{f^{(9)}(-1)}{9!} = \frac{1}{6^5}$$

$$f^{(9)}(-1) = \frac{9!}{6^5} = \frac{362880}{7776}$$

$$= 46.666\cdots$$

$$\approx \underline{\underline{46.67}}$$

5. It is known that the power series

$$\sum_{n=0}^{\infty} c_n x^n$$

has a positive radius of convergence larger than $\frac{1}{20}$. It is also known that $c_0 = 1, c_1 = 1$ and the equation

$$c_{n+2} = 18c_{n+1} - 28c_n$$

holds for all non-negative integers $n = 0, 1, 2, 3, \ldots$. If

$$\sum_{n=1}^{\infty} \frac{c_n}{(68)^n} = \frac{a}{b}$$

where a and b are two positive integers with no common factors, find the exact value of $a + b$.

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$

$c_{n+2} = 18c_{n+1} - 28c_n \Rightarrow \sum_{n=0}^{\infty} c_{n+2} x^{n+2} = \sum_{n=0}^{\infty} 18c_{n+1} x^{n+2} - \sum_{n=0}^{\infty} 28c_n x^{n+2}$

$\Rightarrow f(x) - x - 1 = 18x(f(x) - 1) - 28x^2 f(x)$

$\Rightarrow f(x) = \dfrac{1-17x}{1-18x+28x^2}$

$\dfrac{a}{b} = \left(\sum_{n=0}^{\infty} \dfrac{c_n}{68^n}\right) - 1 = f\left(\frac{1}{68}\right) - 1 = \dfrac{1-\frac{17}{68}}{1-\frac{18}{68}+\frac{28}{68^2}} - 1$

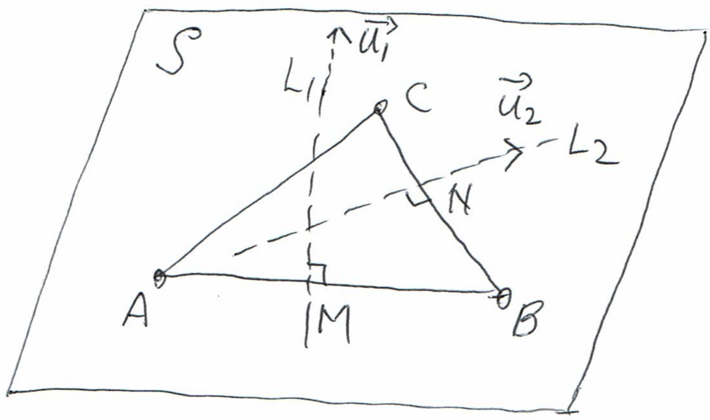
$= \dfrac{867}{857} - 1 = \dfrac{10}{857}$

$\therefore a+b = 10 + 857 = \underline{\underline{867}}$

6. Let A, B and C denote the three points $(101, 0, 0)$, $(0, 202, 0)$ and $(0, 0, 303)$ respectively. Let S denote the plane that passes through the three points A, B and C. Let M denote the mid-point of the line segment AB and let N denote the mid-point of the line segment BC. A line L_1 is drawn on the plane S such that L_1 passes through M and L_1 is perpendicular to AB. Another line L_2 is drawn on the plane S such that L_2 passes through N and L_2 is perpendicular to BC. If (a, b, c) denote the point of intersection of L_1 and L_2, find the value of $a + b + c$. Give your answer correct to two decimal places.

$S: \frac{x}{101} + \frac{y}{202} + \frac{z}{303} = 1$

$\Rightarrow 6x + 3y + 2z = 606$



$M = \left(\frac{101}{2}, 101, 0\right)$

$N = \left(0, 101, \frac{303}{2}\right)$

$\overrightarrow{AB} = (-101, 202, 0)$

$\overrightarrow{BC} = (0, -202, 303)$

$\vec{u_1} = (6, 3, 2) \times (-101, 202, 0) = (-404, -202, 1515)$

$\vec{u_2} = (6, 3, 2) \times (0, -202, 303) = (1313, -1818, -1212)$

$L_1: \left(\frac{101}{2}, 101, 0\right) + t(-404, -202, 1515)$

$L_2: \left(0, 101, \frac{303}{2}\right) + s(1313, -1818, -1212)$

$L_1 \cap L_2 \Rightarrow \frac{101}{2} - 404t = 1313s,\ 101 - 202t = 101 - 1818s,\ 1515t = \frac{303}{2} - 1212s$

$\Rightarrow s = \frac{1}{98},\ t = \frac{9}{98}$

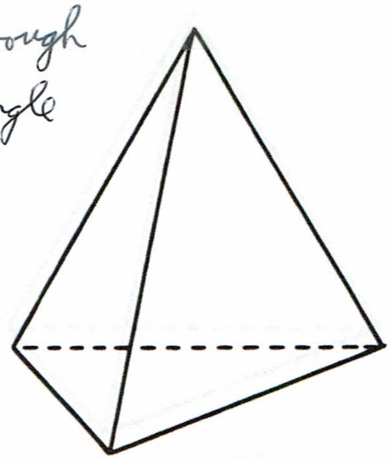
$\Rightarrow (a, b, c) = \left(\frac{1313}{98}, \frac{4040}{49}, \frac{13635}{98}\right)$

$\therefore a + b + c = \frac{11514}{49} = 234.979\cdots \approx \underline{\underline{234.98}}$

7. Let n denote a positive constant. Let S denote a tetrahedron (i.e. a triangular pyramid; for your reference a picture of an example of such a solid is shown below) with its four vertices at the points $(-6, -6, -(\frac{2020}{1521})^n)$, $(9, 6, 3)$, $(-6, 0, -9)$, and $(6, 0, -6)$. If the volume of S is equal to 297, find the value of n. Give your answer correct to two decimal places.

Let P = plane through base triangle

Let d = distance from apex to base triangle



$\vec{u} = (9, 6, 3) - (-6, 0, -9)$
$= (15, 6, 12)$

$\vec{v} = (9, 6, 3) - (6, 0, -6)$
$= (3, 6, 9)$

$\vec{u} \times \vec{v} = (-18, -99, 72)$

$P: -18x - 99y + 72z = (6, 0, -6) \cdot (-18, -99, 72) = -540$

Area of base triangle $= \frac{1}{2} \| \vec{u} \times \vec{v} \| = \frac{1}{2} (18^2 + 99^2 + 72^2)^{1/2}$

$$d = \frac{\left| -18(-6) - 99(-6) + 72\left(-\left(\frac{2020}{1521}\right)^n\right) + 540 \right|}{(18^2 + 99^2 + 72^2)^{1/2}}$$

$$297 = \frac{1}{3} \left\{ \frac{1}{2} (18^2 + 99^2 + 72^2)^{1/2} \right\} \left\{ \frac{\left| 108 + 594 - 72\left(\frac{2020}{1521}\right)^n + 540 \right|}{(18^2 + 99^2 + 72^2)^{1/2}} \right\}$$

$$\therefore \pm 297 = 207 - 12 \left(\frac{2020}{1521}\right)^n$$

$$\therefore n = \frac{\ln 504 - \ln 12}{\ln 2020 - \ln 1521} = 13.173\ldots$$

$$\approx \underline{\underline{13.17}}$$

8. Let S denote a plane. It is known that S passes through the point $(10, 15, 5)$ and that the vector joining $(10.15, 5)$ to $(18, 8, 11)$ is perpendicular to S. Let $f(x, y, z)$ denote a differentiable function of three variables defined in the following way: at the point (x, y, z) we draw a line L passing through this point and perpendicular to the plane S; if (a, b, c) denotes the point of intersection of L and S, then we define $f(x, y, z) = a + b + c$. Find the directional derivative of f at the point $(1001, 1521, 2020)$ in the direction of the vector joining $(1001, 1521, 2020)$ to $(1000, 1522, 2021)$. Give your answer correct to two decimal places.

$$(18, 8, 11) - (10, 15, 5) = (8, -7, 6)$$

$$\therefore S : 8x - 7y + 6z = (8, -7, 6) \cdot (10, 15, 5) = 5$$

$$L : (x, y, z) + t(8, -7, 6) = (x + 8t, y - 7t, z + 6t)$$

$$L \cap S \Rightarrow 8(x + 8t) - 7(y - 7t) + 6(z + 6t) = 5$$

$$8x - 7y + 6z + t(64 + 49 + 36) = 5$$

$$t = \frac{5 - 8x + 7y - 6z}{149}$$

$$\therefore f(x, y, z) = x + 8t + y - 7t + z + 6t$$

$$= x + y + z + \frac{7(5 - 8x + 7y - 6z)}{149}$$

$$= \frac{1}{149}(93x + 198y + 107z + 35)$$

$$\nabla f = \frac{1}{149}(93, 198, 107)$$

$$\vec{u} = \{(1000, 1522, 2021) - (1001, 1521, 2020)\} / \sqrt{3}$$

$$= \frac{1}{\sqrt{3}}(-1, 1, 1)$$

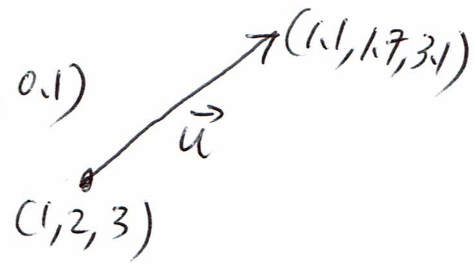
$$\nabla f(1001, 1521, 2020) \cdot \vec{u} = \frac{1}{149\sqrt{3}}(-93 + 198 + 107) = 0.8214\ldots$$

$$\approx \underline{\underline{0.82}}$$

9. Let $f(x,y,z)$ denote a differentiable function of three variables. It is known that $f(1,2,3) = 11$, $f(1.1,1.7,3.1) = 16$, $f(1.2,2.2,3.3) = 18$ and $f(0.9,2.1,2.9) = 20$. Using directional derivatives, estimate the maximum rate of change of f at the point $(1,2,3)$. Give your answer correct to two decimal places.

Let $\nabla f(1,2,3) = (a,b,c)$

$\vec{u} = (1.1, 1.7, 3.1) - (1,2,3) = (0.1, -0.3, 0.1)$



$16-11 = \left\{(a,b,c)\cdot\frac{\vec{u}}{\|\vec{u}\|}\right\}\|\vec{u}\|$

$5 = 0.1a - 0.3b + 0.1c$

$a - 3b + c = 50$ ----- ①

Similarly $2a + 2b + 3c = 70$ ----- ②

$-a + b - c = 90$ ----- ③

① + ③ $\Rightarrow -2b = 140 \Rightarrow b = -70$

① + ② + ③ $\Rightarrow 2a + 3c = 210$

③ $\Rightarrow a + c = -160$

$\therefore a = -690$

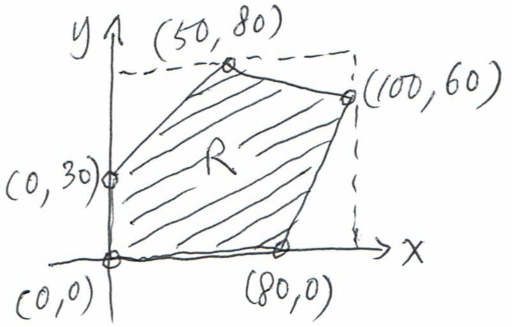
$c = 530$

$\therefore \nabla f(1,2,3) = (-690, -70, 530)$

$\|\nabla f(1,2,3)\| = \sqrt{690^2 + 70^2 + 530^2} = 872.868\ldots$

$\approx \underline{\underline{872.87}}$

10. Let a denote a positive constant. Let S denote a pentagon on the plane $18x + 38y - az + 1521 = 0$. It is known that the projection of S on the xy-plane is another pentagon with vertices at $(0,0,0)$, $(80,0,0)$, $(100,60,0)$, $(50,80,0)$, and $(0,30,0)$. If the area of S is equal to 8888, find the value of a. Give your answer correct to two decimal places.



Area of R

$$= 100 \times 80 - \frac{20 \times 60}{2} - \frac{50 \times 20}{2} - \frac{50 \times 50}{2}$$

$$= 5650$$

$$z = \frac{18}{a}x + \frac{38}{a}y + \frac{1521}{a}$$

$$\Rightarrow \sqrt{1 + z_x^2 + z_y^2} = \left(1 + \frac{18^2}{a^2} + \frac{38^2}{a^2}\right)^{1/2} = \frac{\sqrt{a^2 + 18^2 + 38^2}}{a}$$

$$8888 = \iint_R \frac{\sqrt{a^2+18^2+38^2}}{a}\, dA = \frac{\sqrt{a^2+18^2+38^2}}{a} \times 5650$$

$$a = \left(\frac{18^2 + 38^2}{\left(\frac{8888}{5650}\right)^2 - 1}\right)^{1/2} = \frac{1768 \times 5650^2}{8888^2 - 5650^2}$$

$$= \left(\frac{56438980000}{47074044}\right)^{1/2} = 34.625\ldots$$

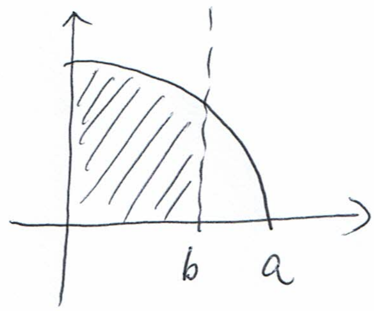
$$\approx \underline{\underline{34.63}}$$

11. Let a and b denote two positive constants such that $a > b$ and $a + b = 88$. Let R denote the finite region in the first quadrant of the xy-plane bounded by the x-axis, the y-axis, the circle $x^2 + y^2 = a^2$, and the line $x = b$. It is known that the surface area of the portion of the cylinder $x^2 + z^2 = a^2$ above R is equal to 1521. Find the exact value of $a^2 + b^2$.

$$x^2 + z^2 = a^2$$

$$\Rightarrow z = \sqrt{a^2 - x^2}$$

$$\Rightarrow z_x = \frac{-x}{\sqrt{a^2 - x^2}}, \quad z_y = 0$$



$$\sqrt{1 + z_x^2 + z_y^2} = \frac{a}{\sqrt{a^2 - x^2}}$$

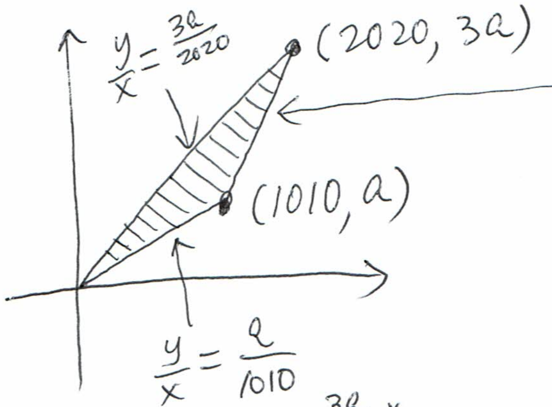
$$1521 = \int_0^b \int_0^{\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2}} \, dy \, dx = ab$$

$$\therefore a^2 + b^2 = (a+b)^2 - 2ab$$

$$= 88^2 - 3042$$

$$= \underline{\underline{4702}}$$

12. Let a denote a positive constant. Let R denote the finite triangular plane region on the xy-plane with verices at $(0,0)$, $(1010, a)$ and $(2020, 3a)$. Let D denote the solid region under the hyperbolic paraboloid $z = xy$ and over the plane region R. If the volume of D is equal to 8^8, find the value of a. Give your answer correct to two decimal places.



$$\frac{y-a}{x-1010} = \frac{3a-a}{2020-1010} = \frac{2a}{1010}$$

$$\Rightarrow y = \frac{2a}{1010}x - a$$

$$8^8 = \int_0^{1010}\int_{\frac{a}{1010}x}^{\frac{3a}{2020}x} xy\,dy\,dx + \int_{1010}^{2020}\int_{\frac{2a}{1010}x-a}^{\frac{3a}{2020}x} xy\,dy\,dx$$

$$= \int_0^{1010} \frac{5a^2x^3}{8160800}\,dx + \int_{1010}^{2020}\left\{\frac{9a^2x^3}{8160800} - \frac{1}{2}x\left(\frac{2ax}{1010} - a\right)^2\right\}dx$$

$$= \frac{255025}{8}(5a^2) + \frac{255025}{24}(61a^2)$$

$$= \frac{4845475}{6}a^2$$

$$a = \sqrt{\frac{8^8 \times 6}{4845475}} = 4.5579\ldots$$

$$\approx \underline{\underline{4.56}}$$

13. Let a and n denote two positive constants with $n < \frac{3}{2}$. A perfectly spherical rain drop with volume a at time $t = 0$ second falls through very dry air and it evaporates in such a way that it always keeps its perfectly spherical shape and that the rate of reduction of its volume is directly proportional to the n-th power of its surface area. It is observed that the volume of the rain drop is equal to $\frac{1}{25}a$ at time $t = 30$ seconds and that the raindrop completely disappears at time $t = 80$ seconds. Find the value of n. Give your answer correct to two decimal places.

$$V = \frac{4}{3}\pi r^3, \quad A = 4\pi r^2, \quad \frac{dV}{dt} = -bA^n$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 4\pi r^2 \frac{dr}{dt} = -b(4\pi r^2)^n$$

$$\Rightarrow \frac{dr}{dt} = -k r^{2n-2}$$

$$\therefore r^{2-2n} dr = -k\,dt$$

$$\therefore \frac{1}{3-2n} r^{3-2n} = -kt + C \quad (\because n < \frac{3}{2}, \therefore 3-2n \neq 0)$$

Let $r(0) = r_0 \Rightarrow C = \frac{1}{3-2n} r_0^{3-2n}$

$$\therefore kt = \frac{1}{3-2n}\left(r_0^{3-2n} - r^{3-2n}\right)$$

$$V(30) = \frac{1}{25}a = \frac{1}{25}\left(\frac{4}{3}\pi r_0^3\right) \Rightarrow \frac{4}{3}\pi r_{30}^3 = \frac{1}{25}\left(\frac{4}{3}\pi r_0^3\right)$$

$$\Rightarrow r_{30} = \frac{1}{25^{1/3}} r_0$$

$$\therefore 30k = \frac{1}{3-2n} r_0^{3-2n}\left(1 - \frac{1}{25^{(3-2n)/3}}\right) \text{ ---- ①}$$

$$V(80) = 0 \Rightarrow r_{80} = 0$$

$$\Rightarrow 80k = \frac{1}{3-2n} r_0^{3-2n} \text{ ------ ②}$$

$$\frac{①}{②} \Rightarrow \frac{3}{8} = 1 - 25^{(2n-3)/3} \Rightarrow n = \frac{1}{2}\left\{\frac{3(\ln 5 - \ln 8)}{\ln 25} + 3\right\}$$

$$= 1.2809\cdots$$

$$\approx \underline{\underline{1.28}}$$

14. Let a denote a positive constant. Let $y(x)$ denote the solution to the differential equation

$$\frac{dy}{dx} = \frac{x^2 + axy + y^2}{xy}$$

with $x > 0$, $y > 0$, $y(1) = \frac{1}{a}$ and $y(2) = \frac{2020}{a}$. Find the value of a. Give your answer correct to two decimal places.

Let $y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}x + u \Rightarrow x\frac{du}{dx} + u = \frac{1 + au + u^2}{u}$

$$\Rightarrow x\frac{du}{dx} = \frac{1+au}{u} \Rightarrow \frac{u}{1+au}du = \frac{1}{x}dx$$

$$\Rightarrow \frac{1+au-1}{1+au}du = \frac{a}{x}dx$$

$$\Rightarrow \left(1 - \frac{1}{1+au}\right)du = \frac{a}{x}dx$$

$\therefore u - \frac{1}{a}\ln(1+au) = a\ln x + C \quad (\because x, y, a > 0)$

$\therefore \frac{y}{x} - \frac{1}{a}\ln\left(1 + \frac{ay}{x}\right) = a\ln x + C$

$y(1) = \frac{1}{a} \Rightarrow \frac{1}{a} - \frac{1}{a}\ln 2 = C$

$\therefore \frac{y}{x} - \frac{1}{a}\ln\left(1 + \frac{ay}{x}\right) = a\ln x + \frac{1}{a} - \frac{1}{a}\ln 2$

$y(2) = \frac{2020}{a} \Rightarrow \frac{1010}{a} - \frac{1}{a}\ln 1011 = a\ln 2 + \frac{1}{a} - \frac{1}{a}\ln 2$

$\therefore 1010 - \ln 1011 = a^2\ln 2 + 1 - \ln 2$

$\therefore a = \sqrt{\frac{1009 + \ln 2 - \ln 1011}{\ln 2}} = 38.0354\ldots$

$\approx \underline{\underline{38.04}}$

15. (Optional tie breaker. Will be used in the event that the number of "A-" grades or above based on the scores of the previous 14 questions together with the Online Test and the Online Assignment exceeds the quota)

Let a, b and c denote three positive constants with $b > c$. Let $y(x)$ denote the solution to the differential equation

$$\frac{dy}{dx} = \cos(a + bx + cy)$$

with $y(0) = -\frac{a}{c}$ and $y(1) = -\frac{b}{c}$.

(i) Find a formula to calculate a in terms of b and c. Give your formula in terms of funcitons that we have learned in this semester and give your formula in its simplest form.

(ii) For the particular case when $b = 2$ and $c = 1$, what is the largest possible value that a can take if it is known that $a < 100$. Give your answer correct to two decimal places.

For this question you must write up your complete solution clearly on not more than two pages (more credit will be given if you can present your complete solution clearly on one page) and then scan or photograph it and convert it into ONE PDF file (if you use two pages you must combine them into one file), name your file with your matriculation number (for e.g. A0112345B) and then submit your file into the "Online Exam Submission" folder in Files at the MA1521 module website on LumiNUS before 11:15am on 5 May 2020. Make sure you name your file correctly with your matriculation number before you submit so that you will get the proper credit (if any).

15). (i) $\frac{dy}{dx} = \cos(a+bx+cy)$, $y(0) = -\frac{a}{c}$, $y(1) = -\frac{b}{c}$, $b > c > 0$, $a > 0$

Let $u = a+bx+cy \Rightarrow \frac{du}{dx} = b + c\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{c}\left(\frac{du}{dx} - b\right)$

$\therefore \frac{du}{dx} - b = c\cos u \Rightarrow \frac{du}{b+c\cos u} = dx$

$$\therefore \frac{du}{b\left(\cos^2\frac{u}{2} + \sin^2\frac{u}{2}\right) + c\left(\cos^2\frac{u}{2} - \sin^2\frac{u}{2}\right)} = dx$$

$$\Rightarrow \frac{2\sec^2\frac{u}{2}\, d\left(\frac{u}{2}\right)}{(b+c) + (b-c)\tan^2\frac{u}{2}} = dx \Rightarrow \frac{2d\left(\tan\frac{u}{2}\right)}{\left(\sqrt{\frac{b+c}{b-c}}\right)^2 + \tan^2\frac{u}{2}} = (b-c)dx$$

$$\Rightarrow 2\sqrt{\frac{b-c}{b+c}}\tan^{-1}\left(\sqrt{\frac{b-c}{b+c}}\tan\frac{a+bx+cy}{2}\right) = (b-c)x + k$$

$$y(0) = -\frac{a}{c} \Rightarrow k = 0 \Rightarrow \frac{2}{\sqrt{b^2-c^2}}\tan^{-1}\left(\sqrt{\frac{b-c}{b+c}}\tan\frac{a+bx+cy}{2}\right) = x$$

$$y(1) = -\frac{b}{c} \Rightarrow \frac{2}{\sqrt{b^2-c^2}}\tan^{-1}\left(\sqrt{\frac{b-c}{b+c}}\tan\frac{a}{2}\right) = 1$$

$$\therefore \tan\frac{a}{2} = \sqrt{\frac{b+c}{b-c}}\tan\frac{\sqrt{b^2-c^2}}{2} \Rightarrow \frac{a}{2} = n\pi + \tan^{-1}\left(\sqrt{\frac{b+c}{b-c}}\tan\frac{\sqrt{b^2-c^2}}{2}\right)$$

$n = 0, 1, 2, 3, \ldots$

$$\therefore \underline{\underline{a = 2n\pi + 2\tan^{-1}\left(\sqrt{\frac{b+c}{b-c}}\tan\frac{\sqrt{b^2-c^2}}{2}\right), \quad n = 0, 1, 2, 3, \ldots}}$$

(ii) $b = 2, c = 1 \Rightarrow a = 2n\pi + 2\tan^{-1}\left(\sqrt{3}\tan\frac{\sqrt{3}}{2}\right)$

$= 2n\pi + 2.2287119\ldots$

$\therefore a < 100 \Rightarrow 2n\pi + 2.2287119\ldots < 100$

$\Rightarrow n < 15.560\ldots \Rightarrow n = 15$

$\therefore a = 30\pi + 2.2287119\ldots = 96.4764\ldots$

$\approx \underline{\underline{96.48}}$