

Chapter 3. Integration

3.1 Indefinite Integral

Integration can be considered as the antithesis of differentiation, and they are subtly linked by the **Fundamental Theorem of Calculus**. We first introduce indefinite integration as an “inverse” of differentiation.

3.1.1 Antiderivatives

A (differentiable) function $F(x)$ is an *antiderivative* of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f .

The set of all antiderivatives of f is

the *indefinite integral* of f with respect to x , denoted by

$$\int f(x) \, dx.$$

Terminology:

f : *integrand* of the integral x : *variable* of integration

3.1.2 Constant of Integration

Any constant function has zero derivative. Hence the antiderivatives of the zero function are all the constant functions.

If $F'(x) = f(x) = G'(x)$, then $G(x) = F(x) + C$,

where C is some constant. So

$$\int f(x)dx = F(x) + C.$$

C here is called the *constant of integration* or an *arbitrary constant*. Thus,

$$\int f(x) dx = F(x) + C$$

means the same as

$$\frac{d}{dx}F(x) = f(x).$$

In words,

indefinite integral and antiderivative (of a function) *differ by an arbitrary constant.*

3.1.3 Integral formulas

$$1. \int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1, \quad n \text{ rational}$$

$$\int 1 dx = \int dx = x + C \quad (\text{Special case, } n = 0)$$

$$2. \int \sin kx dx = -\frac{\cos kx}{k} + C$$

$$3. \int \cos kx dx = \frac{\sin kx}{k} + C$$

$$4. \int \sec^2 x \, dx = \tan x + C$$

$$5. \int \csc^2 x \, dx = -\cot x + C$$

$$6. \int \sec x \tan x \, dx = \sec x + C$$

$$7. \int \csc x \cot x \, dx = -\csc x + C$$

3.1.4 Rules for indefinite integration

$$1. \int k f(x) dx = k \int f(x) dx,$$

$k = \text{constant}$ (independent of x)

$$2. \int -f(x) dx = - \int f(x) dx$$

(Rule 1 with $k = -1$)

$$3. \int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

3.1.5 Example

Find the curve in the xy -plane which passes through the point $(9, 4)$ and whose slope at each point (x, y) is $3\sqrt{x}$.

Solution. The curve is given by $y = y(x)$, satisfying

$$(i) \quad \frac{dy}{dx} = 3\sqrt{x} \quad \text{and} \quad (ii) \quad y(9) = 4.$$

Solving (i), we get

$$y = \int 3\sqrt{x} \, dx = 3 \frac{x^{3/2}}{3/2} + C = 2x^{3/2} + C.$$

By (ii), $4 = (2)9^{3/2} + C = (2)27 + C,$

$$C = 4 - 54 = -50.$$

Hence $y = 2x^{3/2} - 50.$

3.2 Riemann Integrals

3.2.1 Area under a curve

Let $f = f(x)$ be a non-negative continuous function

$f = f(x)$ on an interval $[a, b]$.

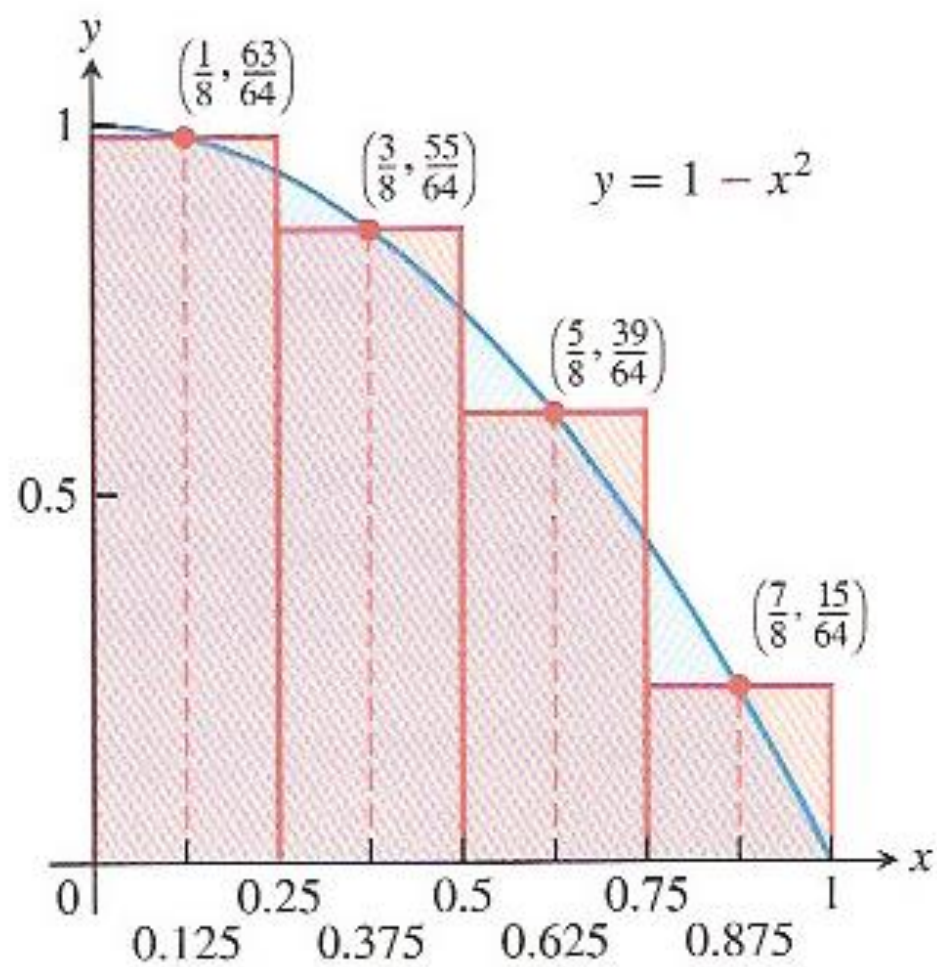
Partition $[a, b]$ into n consecutive sub-intervals $[x_{i-1}, x_i]$
 $(i = 1, 2, \dots, n)$ each of length $\Delta x = \frac{b-a}{n}$, where
we set $a = x_0$, $b = x_n$, and x_1, x_2, \dots, x_{n-1} to be
successive points between a and b with $x_k - x_{k-1} =$
 Δx .

Let c_k be any intermediate point in the sub-interval $[x_{k-1}, x_k]$.

Then the sum

$$S = \sum_{k=1}^n f(c_k) \Delta x$$

gives an approximate area under the curve of $y = f(x)$ from $x = a$ to $x = b$.



The *exact* area A under the curve of $y = f(x)$ is achieved by letting the partition of the interval $[a, b]$ tends to infinity:

$$A = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x.$$

3.2.2 Riemann Integral

Let us continue with the notation as in the previous section and denote the limit by I .

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x = I.$$

We call I the **Riemann integral** (or **definite**

integral) of f over $[a, b]$ and we write

$$I = \int_a^b f(x) \, dx.$$

3.2.3 Terminology

$$\int_a^b f(x)dx$$

$[a, b]$: the interval of integration

a : lower limit of integration

b : upper limit of integration

x : variable of integration

$f(x)$: the integrand

x is a *dummy* variable, i.e.

$$\int_a^b f(x) \, dx = \int_a^b f(u) \, du = \int_a^b f(t) \, dt, \quad \text{etc.}$$

3.2.4 Rules of algebra for definite integrals

$$1. \int_a^a f(x) \, dx = 0$$

$$2. \int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

$$3. \int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx, \quad (\text{any constant } k)$$

$$\left(\text{In particular, } \int_a^b -f(x) \, dx = - \int_a^b f(x) \, dx \right)$$

$$4. \int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

5. If $f(x) \geq g(x)$ on $[a, b]$, then

$$\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx.$$

In particular, if $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x) \, dx \geq 0.$$

6. If f is continuous on the interval joining a , b and c , then

$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$$

3.3 The Fundamental Theorem of Calculus

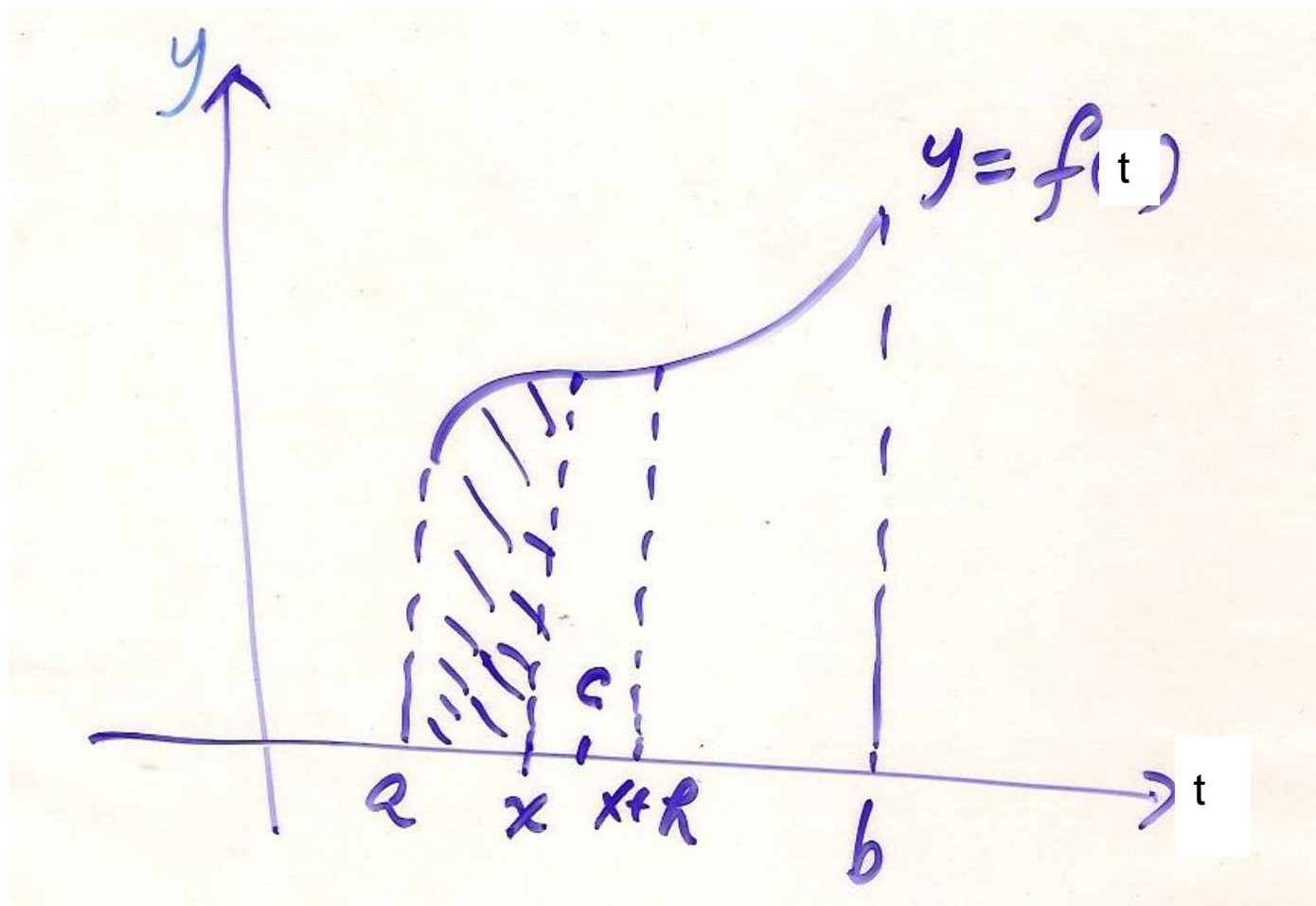
3.3.1 Part 1

If f is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t) \, dt \tag{1}$$

has a derivative at every point of $[a, b]$, and

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x). \tag{2}$$



$$F(x) = \int_a^x f(t) dt$$

$$F'(x) = \lim_{h \rightarrow 0}$$

$$\frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \rightarrow 0}$$

$$\frac{\int_a^{x+h} f(t) dt - \int_a^x f(t) dt}{h}$$

$$= \lim_{h \rightarrow 0}$$

$$\frac{\int_x^a f(t) dt + \int_a^{x+h} f(t) dt}{h}$$

$$= \lim_{h \rightarrow 0}$$

$$\frac{\int_x^{x+h} f(t) dt}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(c) h}{h}$$

c between
 x and $x+h$

$$= \lim_{h \rightarrow 0} f(c)$$

$$= f(x)$$

3.3.2 Examples

$$\frac{d}{dx} \int_{-\pi}^x \cos t \, dt = \cos x$$

$$\frac{d}{dx} \int_0^x \frac{dt}{1+t^2} = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \int_1^{x^2} \cos t \, dt = \left[\frac{d}{d(x^2)} \int_1^{x^2} \cos t \, dt \right] \frac{d(x^2)}{dx} = (\cos x^2) 2x$$

$$= \underline{\underline{2x \cos(x^2)}}$$

$$\begin{aligned} \frac{d}{dx} \int_{x^2}^{x^3} \cos t \, dt &= \frac{d}{dx} \left\{ \int_{x^2}^1 \cos t \, dt + \int_1^{x^3} \cos t \, dt \right\} \\ &= \frac{d}{dx} \left\{ - \int_1^{x^2} \cos t \, dt + \int_1^{x^3} \cos t \, dt \right\} \end{aligned}$$

$$= (-\cos x^2) 2x + (\cos x^3) (3x^2)$$

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3.3.3 Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$,

then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Set $G(x) = \int_a^x f(t) \, dt$.

By the Fundamental Theorem of Calculus, Part 1,
above,

$$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

We also know that $F'(x) = f(x)$. Thus $G'(x) = F'(x)$ for $x \in [a, b]$.

Hence we have $F(x) = G(x) + c$ throughout $[a, b]$

for some constant c . Thus

$$\begin{aligned} F(b) - F(a) &= G(b) + c - (G(a) + c) \\ &= G(b) - G(a) \\ &= \int_a^b f(t) \, dt - \int_a^a f(t) \, dt \\ &= \int_a^b f(t) \, dt. \end{aligned}$$

3.3.4 Examples

$$\int_0^{\pi} \cos x \, dx = \sin x \Big|_0^{\pi} = \sin \pi - \sin 0 = 0$$

$$\int_0^2 t^2 \, dt = \frac{1}{3} t^3 \Big|_0^2 = \frac{8}{3}$$

$$\begin{aligned}\int_{-2}^2 (4 - u^2) du &= \left[4u - \frac{1}{3}u^3 \right]_{-2}^2 \\ &= \left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \\ &= \frac{32}{3}\end{aligned}$$

3.4 Integration by substitution

To evaluate $\int f(g(x))g'(x) dx$ where f and g' are continuous:

1. Set $u = g(x)$. Then $g'(x) = \frac{du}{dx}$, the given integral becomes $\int f(u) du$.
2. Integrate with respect to u .
3. Replace u by $g(x)$ in the result of step 2.

3.4.1 Examples

$$I = \int (x^2 + 2x - 3)^2 (x+1) dx$$

$$\text{Let } u = x^2 + 2x - 3$$

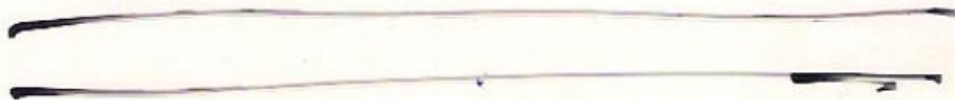
$$du = (2x + 2) dx$$

$$= 2(x+1) dx$$

$$I = \int u^2 \frac{1}{2} du = \frac{1}{2} \int u^2 du$$

$$= \frac{1}{6} u^3 + C$$

$$= \frac{1}{6} (x^2 + 2x - 3)^3 + C$$



$$I = \int \sin^4 x \cos x \, dx$$

$$\text{Let } u = \sin x$$

$$du = \cos x \, dx$$

$$I = \int u^4 du$$

$$= \frac{1}{5} u^5 + C$$

$$= \frac{1}{5} \sin^5 x + C$$

3.4.2 Substitution in definite integrals

The limits change accordingly:

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Note that in general we require $g' \geq 0$ or $g' \leq 0$ in $[a, b]$.

3.4.3 Example

$$I = \int_0^{\pi/4} \tan x \sec^2 x \, dx$$

$$\text{Let } u = \tan x$$

$$x=0 \Rightarrow u=0$$

$$x=\frac{\pi}{4} \Rightarrow u=1$$

$$du = \sec^2 x \, dx$$

$$\begin{aligned} I &= \int_0^1 u \, du \\ &= \frac{1}{2} u^2 \Big|_0^1 \\ &= \underline{\underline{\frac{1}{2}}} \end{aligned}$$

3.5 Integration by parts

Integration by parts is a technique for evaluating integrals of the form

$$\int f(x)g(x) \, dx$$

in which f can be differentiated repeatedly and g can be integrated without difficulty.

Recall the product rule

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

In differential form it becomes

$$d(uv) = u \, dv + v \, du$$

or, equivalently,

$$u \, dv = d(uv) - v \, du.$$

Thus we have the **Integration-by-parts Formula**:

$$\int u \, dv = uv - \int v \, du$$

or,

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

Example

Evaluate $I = \int x \cos x \, dx$.

Solution.

$$\begin{aligned} I &= \int x \cos x \, dx = \int x \, d(\sin x) \\ &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C \end{aligned}$$

$$\int x^2 e^x dx$$

$$\int x^2 e^x dx$$

$$= \int x^2 d(e^x)$$

$$= x^2 e^x - 2 \int x e^x dx$$

$$= x^2 e^x - 2 \int x d(e^x)$$

$$= x^2 e^x - 2x e^x + 2 \int e^x dx$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

Example

Find $\int_1^e (\ln x)^2 dx$

$$\int \underbrace{(\ln x)^2}_u \underbrace{dx}_{dv}$$

$$= x(\ln x)^2 - \int x [2 \ln x] \frac{1}{x} dx$$

$$= x(\ln x)^2 - 2 \int \underbrace{\ln x}_u \underbrace{dx}_{dv}$$

$$= x(\ln x)^2 - 2 \left\{ x \ln x - \int x \left(\frac{1}{x} \right) dx \right\}$$

$$= x(\ln x)^2 - 2x \ln x + 2x + C$$

$$\int_1^e (\ln x)^2 dx = [x(\ln x)^2 - 2x \ln x + 2x]_1^e$$

$$= e - 2e + 2e - 2$$

$$= \underline{\underline{e - 2}}$$

3.6 Area between two curves

If f_1 and f_2 are continuous functions with $f_1(x) \leq f_2(x)$ in the interval $a \leq x \leq b$, then the area of the region between the curves $y = f_1(x)$ and $y = f_2(x)$ from a to b is the integral of $f_2 - f_1$ from a to b , i.e.

$$\text{Area} = \int_a^b [f_2(x) - f_1(x)] dx. \quad (1)$$

This is the basic formula.

If the curves only cross at one or both end points of $[a, b]$, we apply (1) once to find the area. If the curves cross within the interval $[a, b]$, we need to apply (1) more than once. Thus, to find the area of the region between two curves

- (i) Sketch the curves and determine the crossing points.
- (ii) Evaluate the area(s) using (1). **Or**, integrate $|f_2 - f_1|$ over $[a, b]$.

3.6.1 **Example**

Find area enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

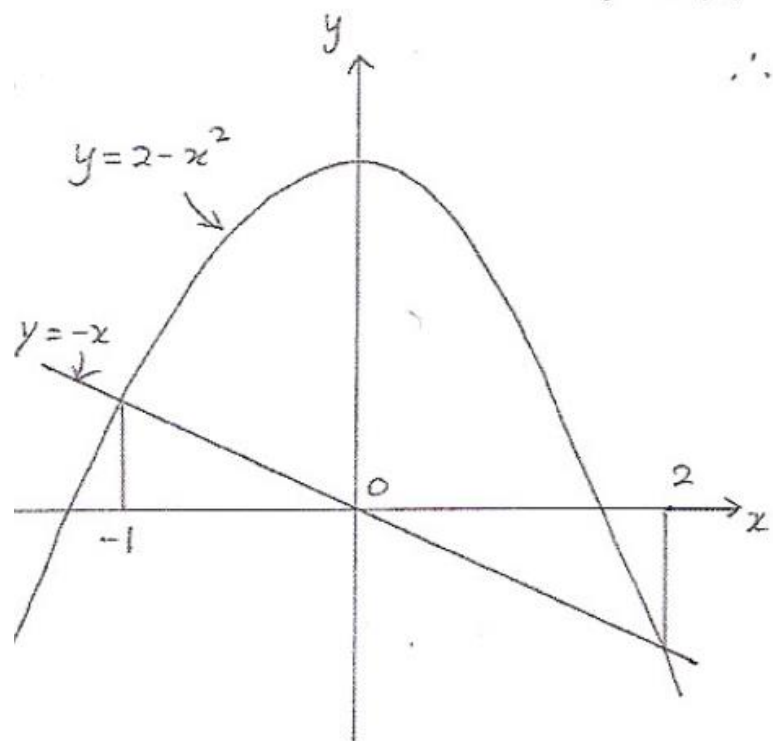
$$y = 2 - x^2, \quad y = -x$$

Points of intersection: Set $2 - x^2 = -x$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$\therefore x = -1, \quad x = 2.$$



$$\text{Area} = \int_{-1}^2 \{ (2 - x^2) - (-x) \} dx$$

$$= \int_{-1}^2 (2 - x^2 + x) dx$$

$$= \left[2x - \frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_{-1}^2$$

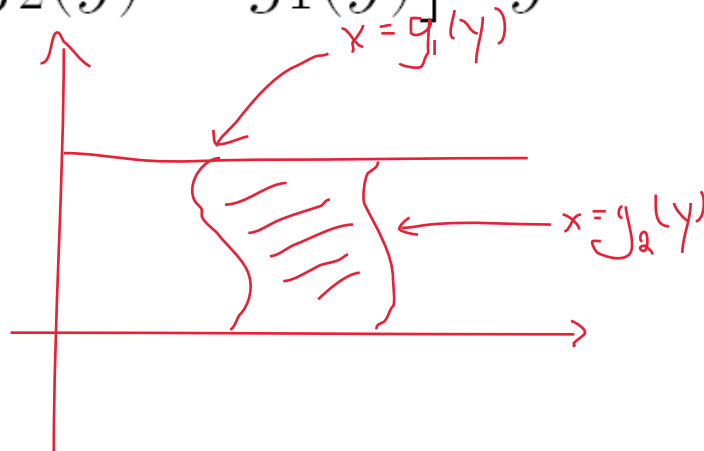
$$= \left(4 - \frac{8}{3} + 2 \right) - \left(-2 + \frac{1}{3} + \frac{1}{2} \right)$$

$$= \frac{9}{2} //$$

Remark.

Sometimes we may like to view the curve as $x = g(y)$
(instead of $y = f(x)$) when evaluating area.

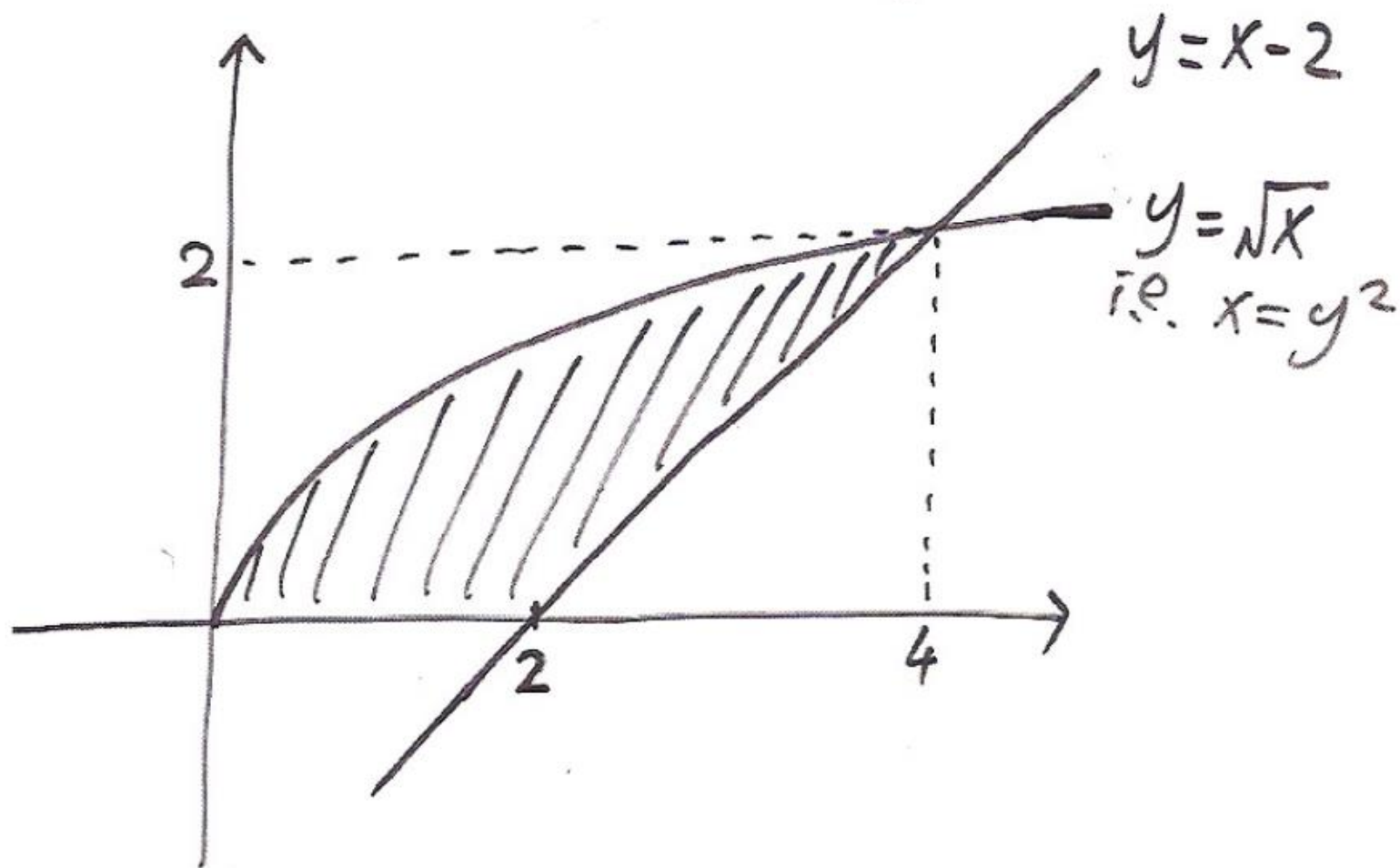
The area will be $A = \int_c^d [g_2(y) - g_1(y)] dy$.



3.6.2 Example

Find area of the region in the first quadrant bounded by $y = \sqrt{x}$ and $y = x - 2$.

View the curve as $x = f(y)$



$$\text{Area} = \int_0^2 \{ (y+2) - (y^2) \} dy$$

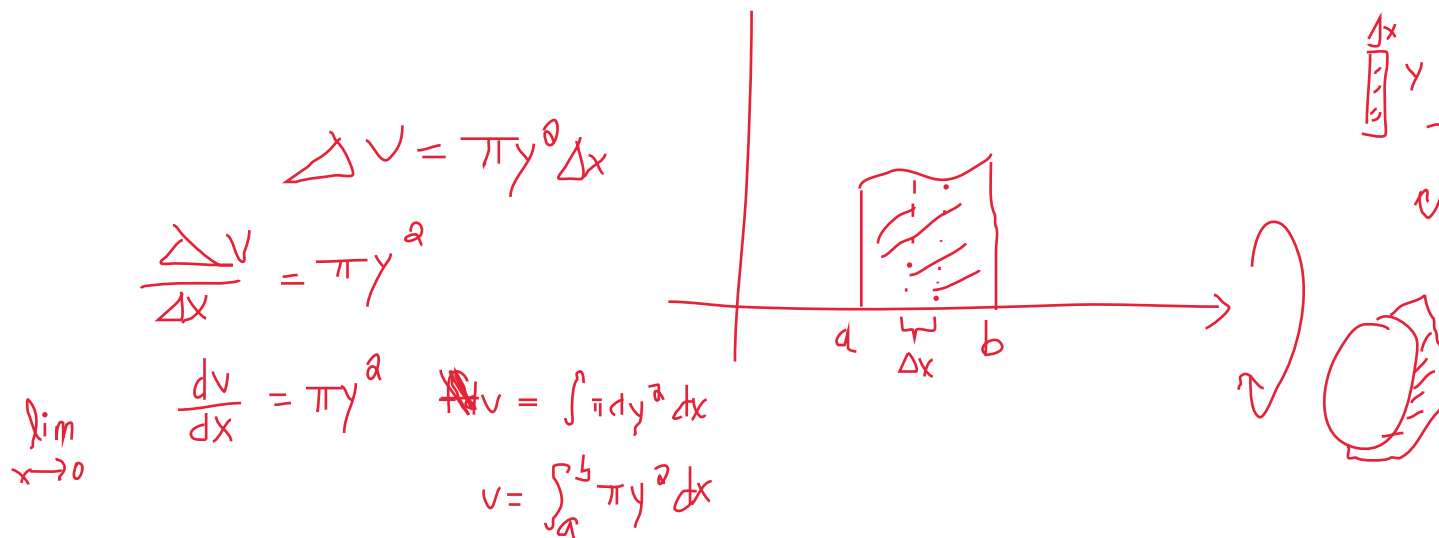
$$= \left[\frac{1}{2} y^2 + 2y - \frac{1}{3} y^3 \right]_0^2$$

$$= 2 + 4 - \frac{8}{3}$$

$$= \frac{10}{3}$$

3.7 Volume of solids of revolution

In general, solids of revolutions are solids which are generated by revolving plane regions about x - or y -axis.



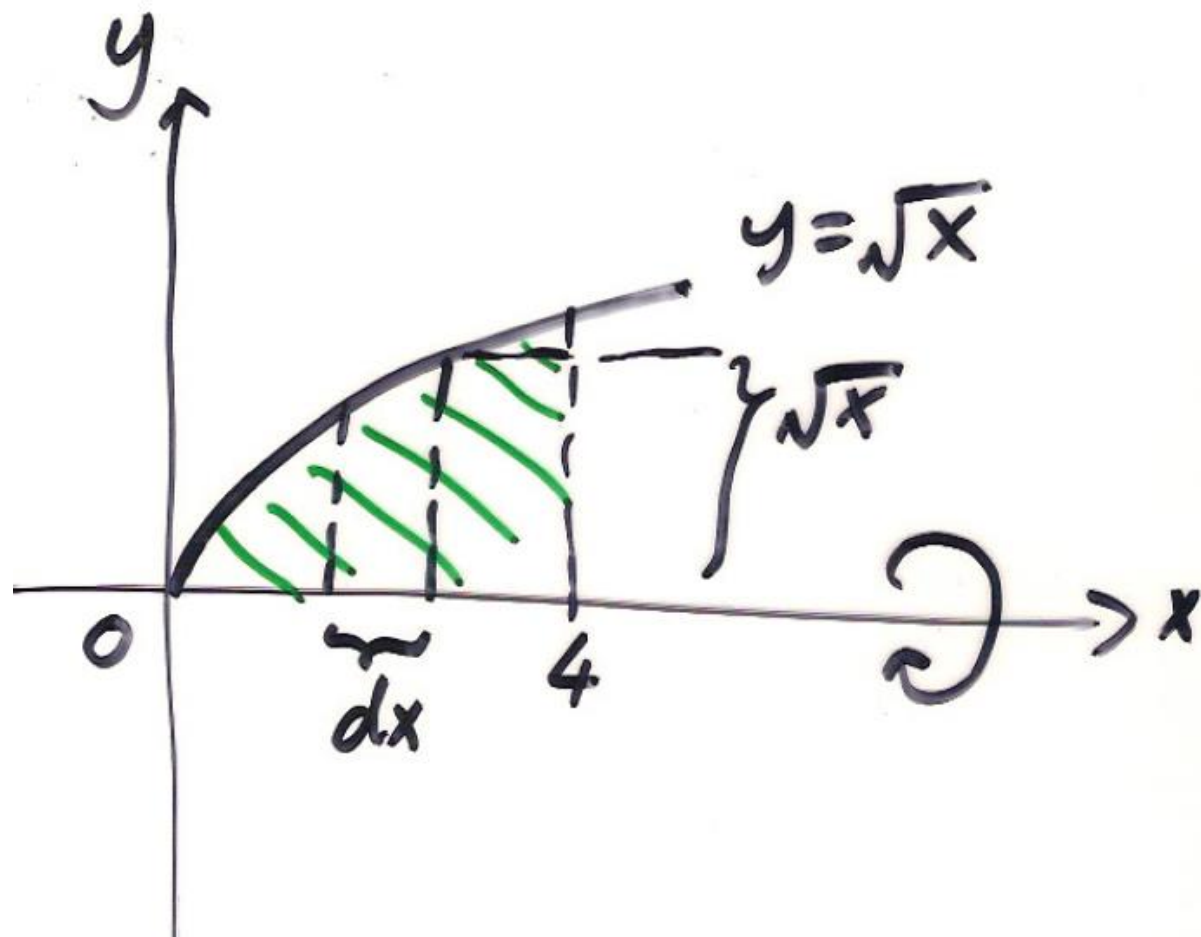
3.7.1 Revolution about x -axis

The volume of a solid generated by revolving *about the* x -axis the region between the graph of a continuous function $y = f(x)$ and the x -axis from $x = a$ to $x = b$ is

$$\text{Volume} = \int_a^b \pi [f(x)]^2 dx.$$

3.7.2 Example

The region between $y = \sqrt{x}$, $0 \leq x \leq 4$, and the x -axis is revolved about the x -axis. Find the volume of the solid generated.



$$\text{Vol} = \int_0^4 \pi (\sqrt{x})^2 dx$$

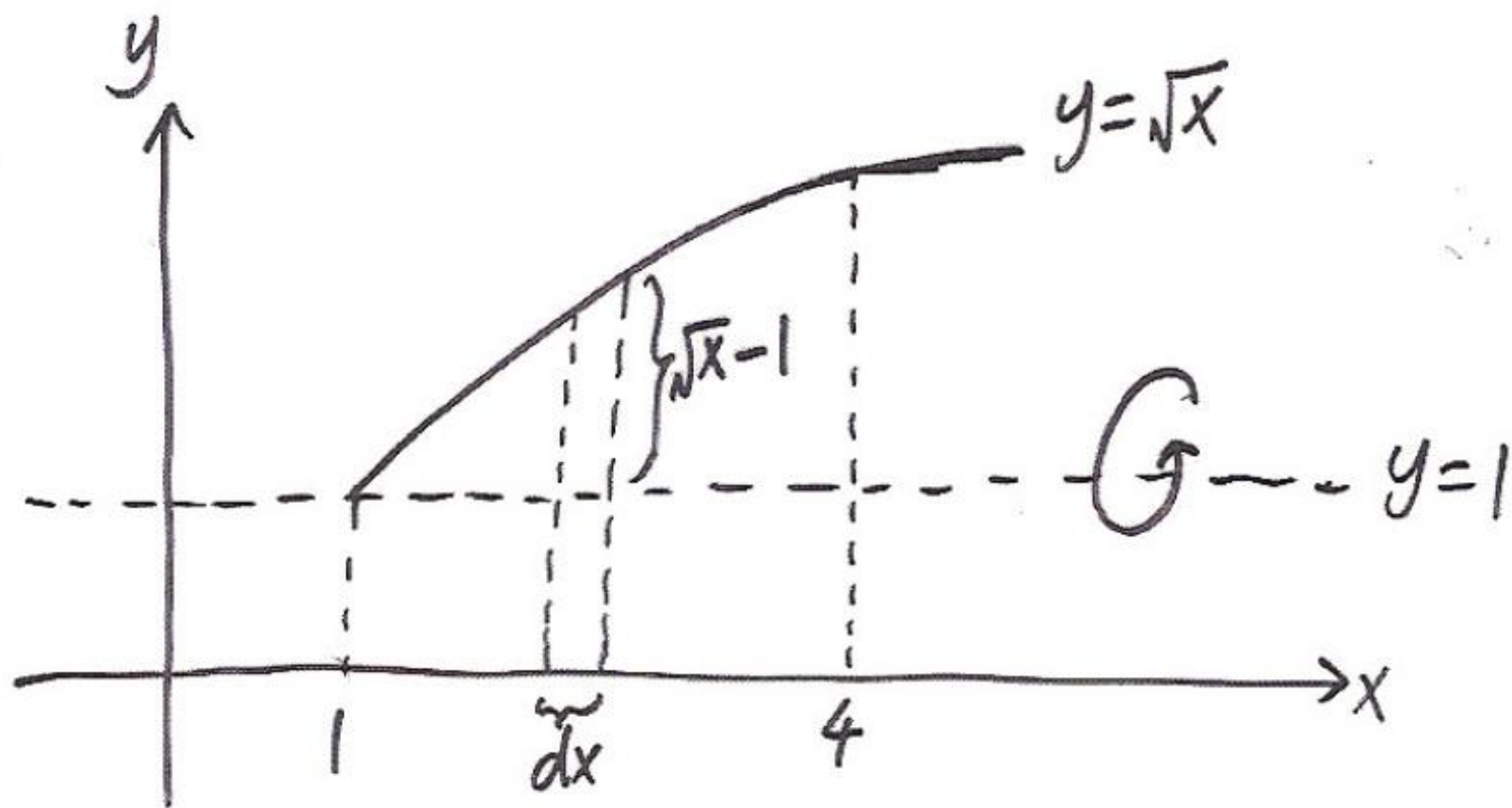
$$= \pi \int_0^4 x dx$$

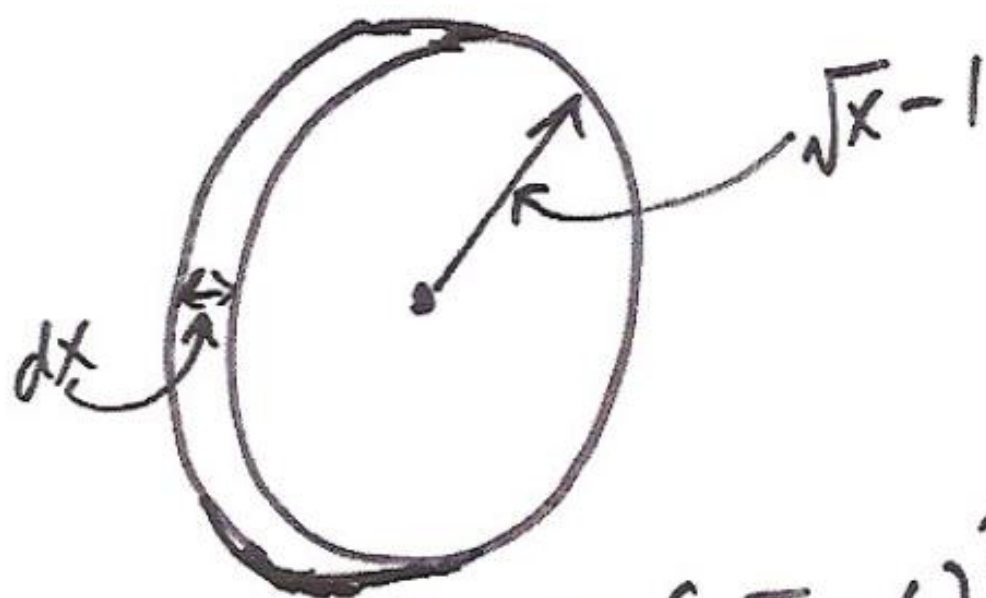
$$= \pi \frac{1}{2} x^2 \Big|_0^4$$

$$= \underline{\underline{8\pi}}$$

3.7.3 Example

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$ and $x = 4$ about the line $y = 1$.





$$dV = \pi (\sqrt{x}-1)^2 dx$$

$$\text{Vol.} = \int_1^4 \pi (\sqrt{x} - 1)^2 dx$$

$$= \int_1^4 \pi (x - 2\sqrt{x} + 1) dx$$

$$= \pi \left[\frac{1}{2}x^2 - \frac{4}{3}x^{\frac{3}{2}} + x \right]_1^4$$

$$= \frac{7}{6} \pi$$

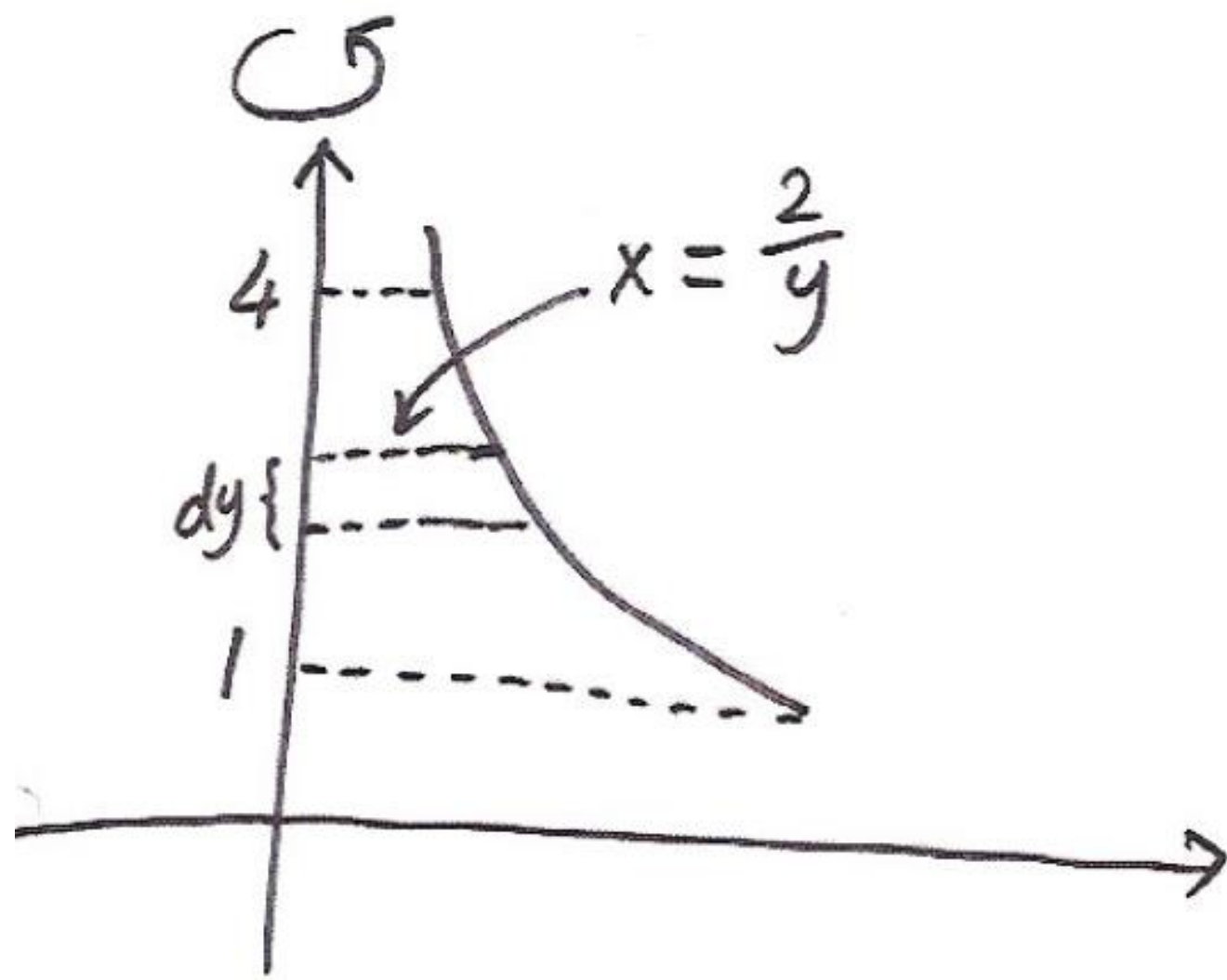
3.7.4 Revolution about y -axis

The volume of a solid generated by revolving about the y -axis the region between the graph of $x = g(y)$ and the y -axis from $y = c$ to $y = d$ is

$$\text{Volume} = \int_c^d \pi [g(y)]^2 dy.$$

3.7.5 Example

The region between the curve $x = \frac{2}{y}$, $1 \leq y \leq 4$ and the y -axis is revolved about the y -axis to generate a solid. Find its volume.



$$\text{Vol.} = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy$$

$$= 4\pi \int_1^4 \frac{1}{y^2} dy$$

$$= 4\pi \left(\frac{3}{4}\right)$$

$$= \underline{\underline{3\pi}}$$

More Examples

Example

$$\int_0^1 \frac{1}{1+x^{\frac{1}{3}}} dx$$

Solution: Using direct substitution with $u = 1 + x^{\frac{1}{3}}$, and $du = \frac{1}{3}x^{\frac{-2}{3}} dx$, so $dx = 3x^{\frac{2}{3}} du = 3(u - 1)^2 du$. When $x = 0$, $u = 1$ and when $x = 1$, $u = 2$. We have that:

$$\begin{aligned}\int_0^1 \frac{1}{1 + x^{\frac{1}{3}}} dx &= \int_1^2 \frac{3(u - 1)^2}{u} du = \int_1^2 \left(3u - 6 + \frac{3}{u}\right) du \\&= \left(\frac{3}{2}u^2 - 6u + 3 \ln |u|\right) \Big|_1^2 \\&= (6 - 12 + 3 \ln 2) - \left(\frac{3}{2} - 6 + 3 \ln 1\right) = -\frac{3}{2} + 3 \ln 2 \\ \Rightarrow \int_0^1 \frac{1}{1 + x^{\frac{1}{3}}} dx &= -\frac{3}{2} + 3 \ln 2.\end{aligned}$$

Example

Evaluate

$$\int_{\frac{1}{e}}^e |\ln x| \, dx$$

$$\int_{1/e}^e |\ln x| dx = \int_{1/e}^1 -\ln x dx + \int_1^e \ln x dx$$

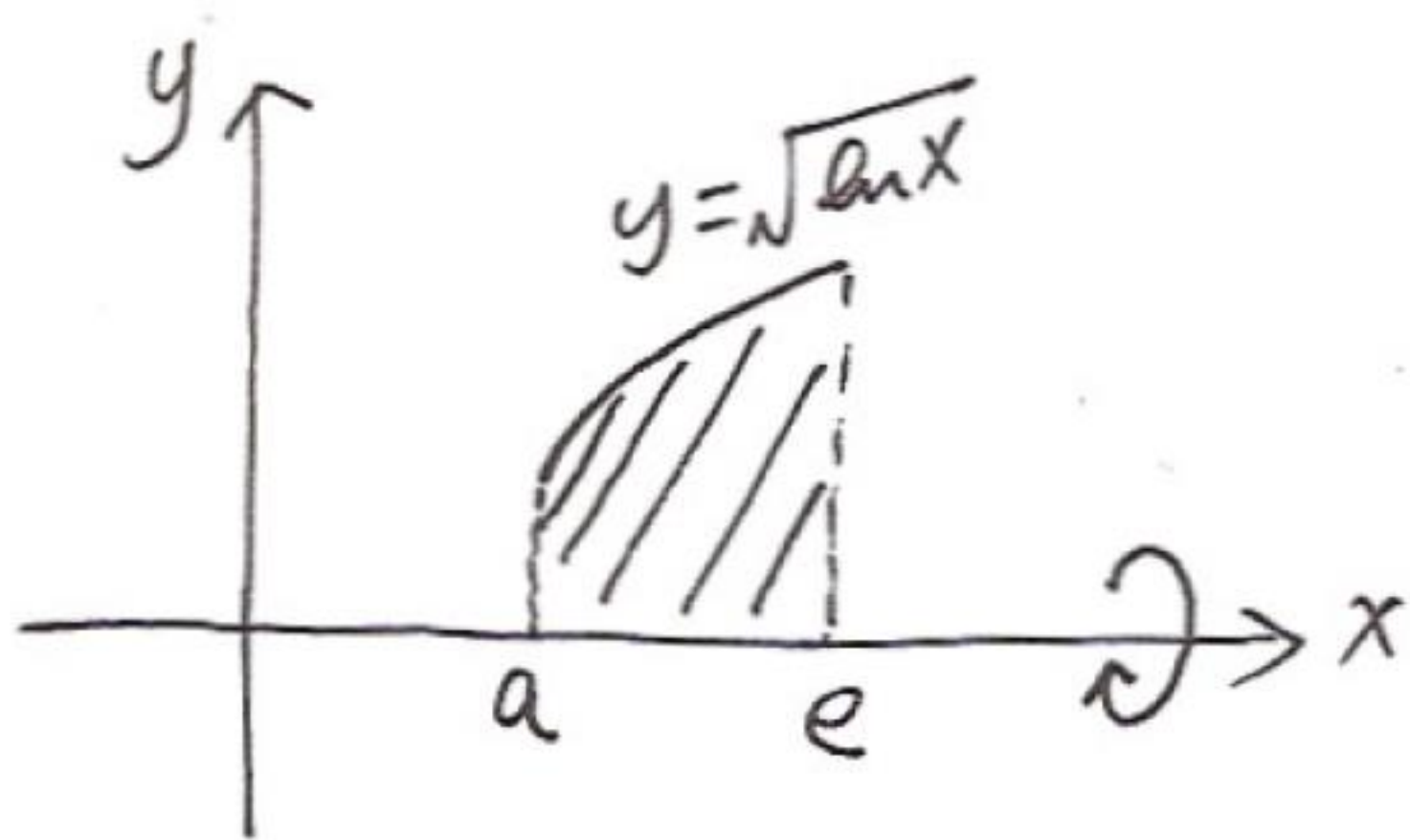
$$= -x \ln x \Big|_{1/e}^1 + \int_{1/e}^1 dx + x \ln x \Big|_1^e - \int_1^e dx$$

$$= -\frac{1}{e} + 1 - \frac{1}{e} + e - e + 1$$

$$= 2 - \frac{2}{e} = \underline{\underline{2(1 - e^{-1})}}$$

Example

Let a be a positive constant and $1 < a < e$. Let R denote the finite region in the first quadrant bounded by the curve $y = \sqrt{\ln x}$, the x -axis, the line $x = a$ and the line $x = e$. Find the **exact value** of the volume of the solid formed by revolving R one complete round about the x -axis. Leave your answer in terms of a .



$$\text{Vol} = \int_a^e \pi y^2 dx$$

$$= \pi \int_a^e \ln x dx$$

$$= \pi \left[x \ln x - x \right]_a^e$$

$$= \underline{\underline{\pi (a - a \ln a)}}$$