1. Find the exact value of the radius of convergence of the power series

$$\sum_{n=0}^{\infty}\left(\frac{x}{2021}\right)^n.$$

Answer 2021

$$\left|\frac{\left(\frac{x}{2021}\right)^{n+1}}{\left(\frac{x}{2021}\right)^{n}}\right| = \left|\frac{x}{2021}\right|$$

$$\left|\frac{x}{2021}\right| < 1 \Rightarrow |x| < 2021$$

2. Let

$$\sum_{n=0}^{\infty} c_n(x-1)^n$$

denote the Taylor series of $\frac{1}{x^{20}}$ at $x = 1$. Find the exact value of c_2.

Answer 210

$$f(x) = \frac{1}{x^{20}}$$

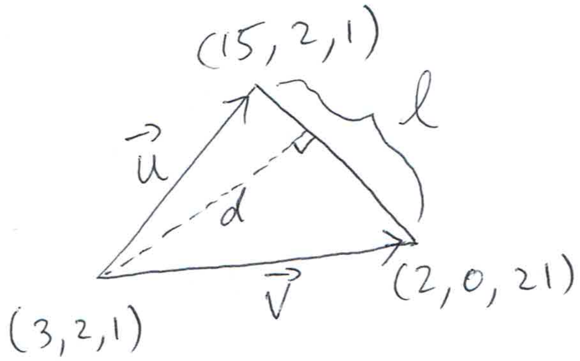
$$f'(x) = -\frac{20}{x^{21}}$$

$$f''(x) = \frac{420}{x^{22}}$$

$$c_2 = \frac{f''(1)}{2!} = \frac{420}{2!} = \underline{\underline{210}}$$

3. Let L denote the line joining the points $(15, 2, 1)$ and $(2, 0, 21)$. Find the perpendicular distance from the point $(3, 2, 1)$ to L. Give your answer correct to two decimal places.

Answer: 10.08



$$\vec{u} = (12, 0, 0)$$

$$\vec{v} = (-1, -2, 20)$$

$$l = \|(13, 2, -20)\| = \sqrt{13^2 + 2^2 + 20^2} = \sqrt{573}$$

$$\text{area of } \Delta = \frac{1}{2} \times d \times l = \frac{1}{2} \|\vec{u} \times \vec{v}\|$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 12 & 0 & 0 \\ -1 & -2 & 20 \end{vmatrix} = -240\vec{j} - 24\vec{k}$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{240^2 + 24^2} = \sqrt{58176}$$

$$\therefore \sqrt{573}\, d = \sqrt{58176}$$

$$d = \sqrt{\frac{58176}{573}} = 10.076\cdots$$

$$\approx \underline{\underline{10.08}}$$

4. Find the perpendicular distance from the point $(1, 52, 1)$ to the plane $3x - 2y + z = 2021$. Give your answer correct to two decimal places.

Answer: 566.86

$$d = \frac{|3 - 104 + 1 - 2021|}{\sqrt{3^2 + 2^2 + 1^2}}$$

$$= 566.8610\ldots$$

$$\approx \underline{\underline{566.86}}$$

5. Let $f(x,y,z) = x + y + e^z$. Find the directional derivative of f at the point $(2,3,2)$ in the direction of the vector $20\vec{i} + 2\vec{j} + \vec{k}$. Give your answer correct to two decimal places.

Answer: 1.46

$$\vec{u} = \frac{20\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{20^2+2^2+1^2}} = \frac{1}{\sqrt{405}}(20, 2, 1)$$

$$\nabla f(x,y,z) = (1, 1, e^z)$$

$$\nabla f(2,3,2) = (1,1,e^2)$$

$$D_{\vec{u}} f(2,3,2) = \nabla f(2,3,2) \cdot \vec{u}$$

$$= \frac{1}{\sqrt{405}}(20 + 2 + e^2)$$

$$= 1.4603\cdots$$

$$\approx \underline{\underline{1.46}}$$

6. Let

$$f(x) = \int_0^x te^{t^3}\,dt.$$

Using the values $\ln 2021! = 17705.86034$ and $\ln 670! = 1897.96199$,

find the value of

$$\ln f^{(2021)}(0),$$

where $f^{(2021)}(0)$ denotes the 2021st derivative of f at $x = 0$. Give your answer correct to two decimal places.

Answer: 15780.77

$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m = f(x) = \int_0^x t \sum_{n=0}^{\infty} \frac{t^{3n}}{n!}\,dt$$

$$= \sum_{n=0}^{\infty} \int_0^x \frac{t^{3n+1}}{n!}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{x^{3n+2}}{n!\,(3n+2)}$$

Compare coefficients for x^{2021}

$$3n+2 = 2021 \Rightarrow n = \frac{2021-2}{3} = 673$$

$$\therefore \frac{f^{(2021)}(0)}{2021!} = \frac{1}{673!\,(2021)}$$

$$\therefore \ln f^{(2021)}(0) = \ln \frac{2021!}{670! \times 673 \times 672 \times 671 \times 2021}$$

$$= \ln 2021! - \ln 670! - \ln 673 - \ln 672 - \ln 671 - \ln 2021$$

$$= 17705.86034 - 1897.96199 - 6.51174 - 6.51025$$

$$- 6.50876 - 7.61134 = 15780.767\ldots$$

$$\approx \underline{\underline{15780.77}}$$

7. Let $\vec{u}$ and $\vec{v}$ denote two vectors. If $||\vec{u}|| = 15$, $||\vec{v}|| = 21$ and $\vec{u}\cdot\vec{v} = -202.1$, where "$\cdot$" denotes the dot product, find the value of $||(\vec{u}+\vec{v}) \times (\vec{u}-\vec{v})||$ where "$\times$" denotes the cross product. Give your answer correct to two decimal places.

Answer: 483.24

Let θ = angle between $\vec{u}$ and $\vec{v}$.

$$||\vec{u}\times\vec{v}|| = ||\vec{u}||\cdot||\vec{v}|| \sin\theta = 315 \sin\theta$$

$$\vec{u}\cdot\vec{v} = ||\vec{u}||\cdot||\vec{v}|| \cos\theta = 315 \cos\theta$$

$$\therefore ||\vec{u}\times\vec{v}||^2 + (\vec{u}\cdot\vec{v})^2 = 315^2 = 99225$$

$$\therefore ||\vec{u}\times\vec{v}||^2 = 99225 - (\vec{u}\cdot\vec{v})^2$$

$$= 99225 - (-202.1)^2 = 58380.59$$

$$||(\vec{u}+\vec{v})\times(\vec{u}-\vec{v})|| = ||\vec{u}\times\vec{u} - \vec{u}\times\vec{v} + \vec{v}\times\vec{u} - \vec{v}\times\vec{v}||$$

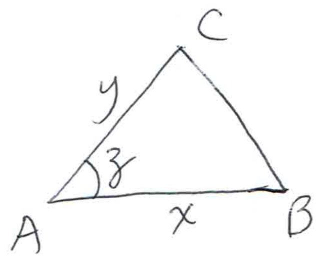
$$= ||-2\vec{u}\times\vec{v}|| = 2||\vec{u}\times\vec{v}||$$

$$= 2\times\sqrt{58380.59} = 483.2415\ldots$$

$$\approx \underline{\underline{483.24}}$$

8. In a triangle ABC we have $AB = x$ metres, $AC = y$ metres and $\angle CAB = z$ radians. If x, y, z are increasing at a rate of 0.08 metre per second, 0.06 metre per second and 0.0001 radian per second respectively, find the rate of change of the area of triangle ABC in square metres per second when $x = 1521$ metres, $y = 2021$ metres and $z = \frac{3\pi}{5}$ radians. Give your answer correct to two decimal places.

Answer: 72.79



Let W = Area of ΔABC

Then $W = (xy \sin z)/2$

Using Chain Rule

$$\frac{dW}{dt} = \left(\frac{\partial W}{\partial x}\frac{dx}{dt} + \frac{\partial W}{\partial y}\frac{dy}{dt} + \frac{\partial W}{\partial z}\frac{dz}{dt}\right)/2$$

$$= \left\{(y \sin z)\frac{dx}{dt} + (x \sin z)\frac{dy}{dt} + (xy \cos z)\frac{dz}{dt}\right\}/2$$

When $x = 1521$, $y = 2021$, $z = \frac{3\pi}{5}$

$$\frac{dW}{dt} = \left\{\left(2021 \sin \frac{3\pi}{5}\right)(0.08) + \left(1521 \sin \frac{3\pi}{5}\right)(0.06) + \left(1521 \times 2021 \times \cos \frac{3\pi}{5}\right)(0.0001)\right\}/2$$

$$= (153.766817 + 86.793417 - 94.990000)/2$$

$$= 72.785\cdots \approx \underline{\underline{72.79}}$$

9. Let $f(x, y, z)$ denote a differentiable function of three variables. It is known that $\frac{\partial f}{\partial y} = 9$ at the point $(1, 5, 21)$ and that the maximum value of the directional derivative of f at the point $(1, 5, 21)$ occurs in the direction of the vector $2\vec{\imath} + 3\vec{\jmath} + 8\vec{k}$. Find the maximum value of the directional derivative of f at the point $(1, 5, 21)$. Give your answer correct to two decimal places.

Answer: 26.32

at $(1,5,21)$ let $\frac{\partial f}{\partial x} = a$, $\frac{\partial f}{\partial z} = b$

$\therefore \nabla f(1,5,21) = (a, 9, b)$

$\because \nabla f(1,5,21) \parallel (2,3,8)$

$\therefore \frac{a}{2} = \frac{9}{3} = \frac{b}{8} \Rightarrow a = 6,\ b = 24$

$\therefore \nabla f(1,5,21) = (6, 9, 24)$

$$\|\nabla f(1,5,21)\| = \sqrt{6^2+9^2+24^2}$$
$$= \sqrt{36+81+576}$$
$$= \sqrt{693}$$
$$= 26.3248\cdots$$
$$\approx \underline{\underline{26.32}}$$

10. Let

$$\sum_{n=0}^{\infty} a_n$$

denote a convergent geometric series with $a_0 = 1$ and sum to infinity equals to 1521. Find the value of

$$\ln\left(\sum_{n=1}^{\infty} n a_n\right).$$

Give your answer correct to two decimal places.

Answer: 14.65

$\because \sum a_n$ is geometric and $a_0 = 1$

$\therefore a_n = r^n$ where r = common ratio

$\sum a_n = 1521 \Rightarrow \frac{1}{1-r} = 1521 \Rightarrow r = 1 - \frac{1}{1521} = \frac{1520}{1521}$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} r^n x^n = \sum_{n=0}^{\infty} (rx)^n$

$|rx| < 1 \Leftrightarrow |x| < \frac{1}{r} = \frac{1521}{1520}$

$\therefore$ When $|x| < \frac{1521}{1520}$

we have $f(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-rx}$

$\therefore f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \frac{r}{(1-rx)^2}$ holds for $|x| < \frac{1521}{1520}$

$\because 1 < \frac{1521}{1520}$, $\therefore x = 1 \Rightarrow \sum_{n=1}^{\infty} n a_n = \frac{r}{(1-r)^2}$

$\therefore \sum_{n=1}^{\infty} n a_n = 1520 \times 1521 = 2311920$

$\therefore \ln\left(\sum_{n=1}^{\infty} n a_n\right) = \ln 2311920 = 14.6535\ldots$

$\approx \underline{\underline{14.65}}$