MA 1521

Tutorial 1 Solutions

1. Note that $(g \circ f)(x) = \sqrt{|3 - \frac{6}{x}|}$ and $(f \circ g)(x) = \frac{6}{\sqrt{|3-x|}}$.

2. (a) $y = \dfrac{ax+b}{cx+d}$, $\quad y' = \dfrac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \dfrac{ad - bc}{(cx+d)^2}$ (use quotient rule)

   (b) $y = \sin^n x \cos mx$, $\quad y' = n \sin^{n-1} x \cos x \cos mx - m \sin^n x \sin mx$
   (use product rule and chain rule)

   (c) $y = e^{x^2+x^3}$ $\quad y' = e^{x^2+x^3}(2x + 3x^2)$ (use chain rule)

   (d) $y = x^3 - 4(x^2 + e^2 + \ln 2)$, $\quad y' = 3x^2 - 8x$ (note that e^2 and $\ln 2$ are constants)

   Similarly, we find the derivatives in (e) - (h).

   (e) $-2 \sin \theta (\cos \theta - 1)^{-2}$ (use quotient and chain rule)

   (f) $\sqrt{t} \sec^2(2\sqrt{t}) + \tan(2\sqrt{t})$ (use product and chain rule)

   (g) $\frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}} \cos(\theta + \sqrt{\theta+1})$ (use chain rule)

   (h) $4 \tan x \sec x - \csc^2 x$ (use quotient rule)

3. Let $V_c(t)$ be the volume of coffee in the cone at time t and $V_p(t)$ be the volume of coffee in the pot at time t.

   Note that the rate of volume change in the cone $\dfrac{dV_c}{dt}$ is equal the rate of volume change in the pot $\dfrac{dV_p}{dt}$.
   Let $h_c(t)$ be the level of coffee in the cone at time t and $h_p(t)$ be the level of coffee in the pot at time t.

   (a) $V_p =$ base area $\times\, h_p = 9\pi h_p$.

$$\begin{aligned} \frac{dV_p}{dt} &= 9\pi \frac{dh_p}{dt} \\ \Rightarrow \quad 10 &= 9\pi \frac{dh_p}{dt} \\ \Rightarrow \quad \frac{dh_p}{dt} &= \frac{10}{9\pi} \end{aligned}$$

(b) $V_c = \dfrac{1}{3}\text{base area} \times h_c = \dfrac{1}{3}\pi r^2 h_c = \dfrac{1}{3}\pi(\dfrac{h_c}{2})^2 h_c = \dfrac{\pi h_c^3}{12}$.

Note that the base radius r of the cone is half that of the height h_c.

$$\begin{aligned}
\frac{dV_c}{dt} &= \frac{\pi h_c^2}{4}\frac{dh_c}{dt} \\
\Rightarrow \quad 10 &= \frac{\pi 5^2}{4}\frac{dh_c}{dt} \\
\Rightarrow \quad \frac{dh_c}{dt} &= \frac{8}{5\pi}
\end{aligned}$$

4. (a) $x^{2/3} + y^{2/3} = a^{2/3}$. Differentiating the equality we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0.$$

Since $0 < x < a$ and $0 < y$, we have

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{y^{1/3}}{x^{1/3}} = -\frac{\sqrt{a^{2/3} - x^{2/3}}}{x^{1/3}} = -\sqrt{(\frac{a}{x})^{2/3} - 1}; \\
\frac{d^2y}{dx^2} &= -\frac{1}{2}\frac{1}{\sqrt{(\frac{a}{x})^{2/3} - 1}}(-\frac{2}{3})a^{2/3}x^{-5/3} = \frac{a^{2/3}}{3x^{5/3}\sqrt{(\frac{a}{x})^{2/3} - 1}} = \frac{a^{2/3}}{3x^{4/3}\sqrt{a^{2/3} - x^{2/3}}}.
\end{aligned}$$

(b) $y = (\sin x)^{\sin x}$, $0 < x < \frac{\pi}{2}$, so $\sin x > 0$.

$$\begin{aligned}
\ln y &= \sin x \ln\sin x, \quad \frac{y'}{y} = \cos x \ln\sin x + \cos x, \quad y' = y(1 + \ln\sin x)\cos x, \\
y'' &= y'(1 + \ln\sin x)\cos x + y\left[(1 + \ln\sin x)(-\sin x) + \frac{\cos^2 x}{\sin x}\right] \\
&= y(1 + \ln\sin x)^2\cos^2 x + y\left[\frac{\cos^2 x}{\sin x} - (1 + \ln\sin x)\sin x\right].
\end{aligned}$$

Hence

$$\begin{aligned}
y' &= (\sin x)^{\sin x}(1 + \ln\sin x)\cos x, \\
y'' &= (\sin x)^{\sin x}\left[(1 + \ln\sin x)^2\cos^2 x + \frac{\cos^2 x}{\sin x} - (1 + \ln\sin x)\sin x\right].
\end{aligned}$$

(c) $x = a\cos t$, $y = a\sin t$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{-a\sin t} = -\cot t, \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(-\cot t)}{\frac{dx}{dt}} = \frac{\frac{1}{\sin^2 t}}{-a\sin t} = -\frac{1}{a\sin^3 t}.
\end{aligned}$$