GEH1027 Einstein's Universe and Quantum Weirdness

Tutorial 2

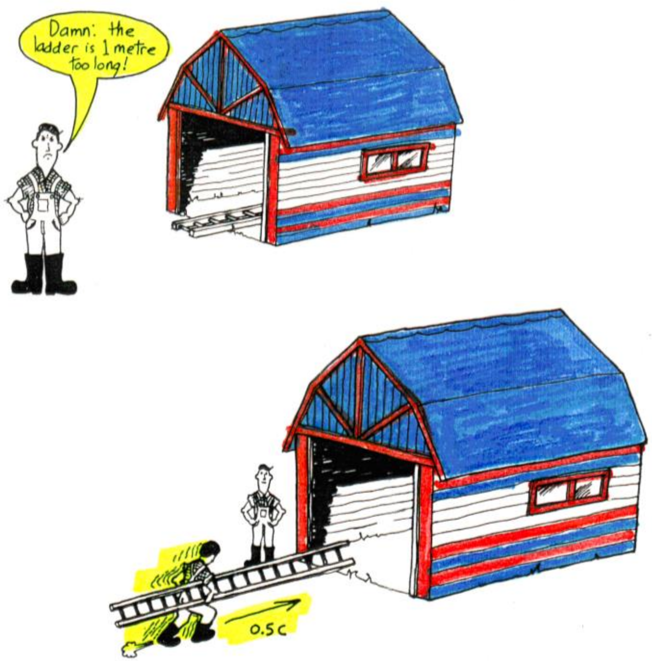
AY2020/21 Sem II

Tutorial 2 – key questions

- What is the purpose of the “irritating negative sign” in the Minkowski’s metric?
- Is spacetime interval an invariant quantity?
- Is the energy dispersion relation an invariant quantity?
- What is the main concept behind the skewing of the spacetime axis?
- Are simultaneous events relative?

Question 1

A farmer has a ladder which is 12 meters long and a barn which is only 8 meters long and so, much to the farmer's consternation, the ladder will not fit in the barn. How can he fit the ladder into the barn using the length contraction theory? See figures below.



From the farmer's frame, the ladder is moving and the barn is stationary.
Thus the ladder contracts in length.

$$L = \frac{L_0}{\gamma}$$
$$L = L_0\sqrt{1 - v^2/c^2}$$
$$8 = 12\sqrt{1 - v^2/c^2}$$
$$v = 0.75c$$

If the son is running with speed $0.75c$ or higher, the ladder can fit into the barn.
Farmer: Great! The problem is solved.

From the son’s frame, it is the barn that is moving. Thus the barn will contract in length.

Length of the barn seen by son,

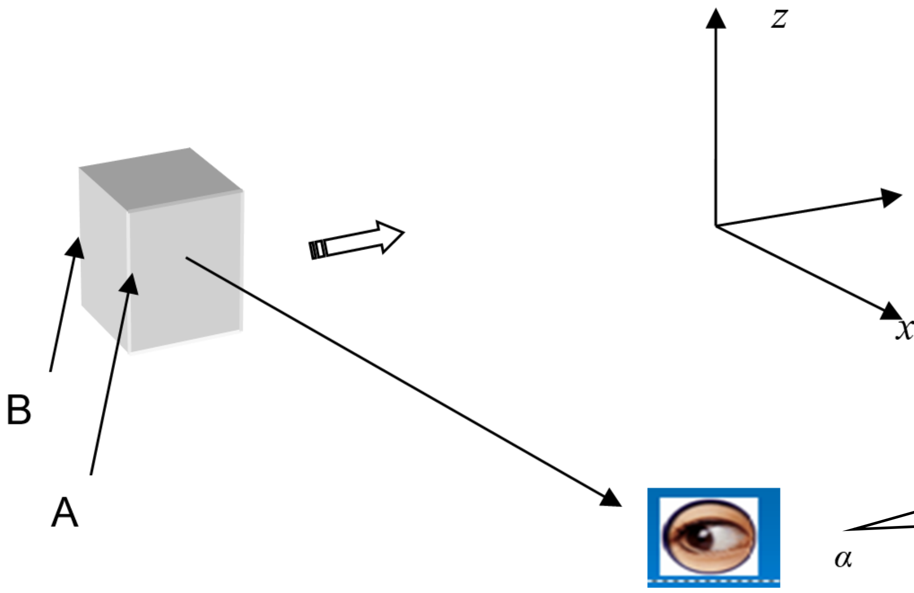
$$\begin{aligned} L_{\mathrm{barn}} &= 8\sqrt{1-0.75^2} \\ &= 5.29\ \mathrm{m} < 10\ \mathrm{m} \end{aligned}$$

Son: the ladder doesn’t fit into the barn. The problem has worsened!!

Question: Who is correct? Both are correct in **their respective frames**!

Question 2

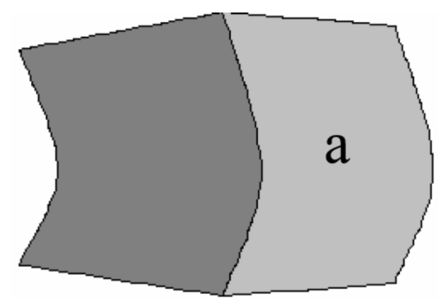
Consider a cube whose edge has a length L_0 in its own rest frame. See figure below. The cube is moving at a speed close to that of light in the y direction. How would it appear to an observer far away at α?



Aberration of light

Remember, the cube is **<u>rigid</u>**!!!

The cube will appear to be **rotated** in addition the **length contraction**.

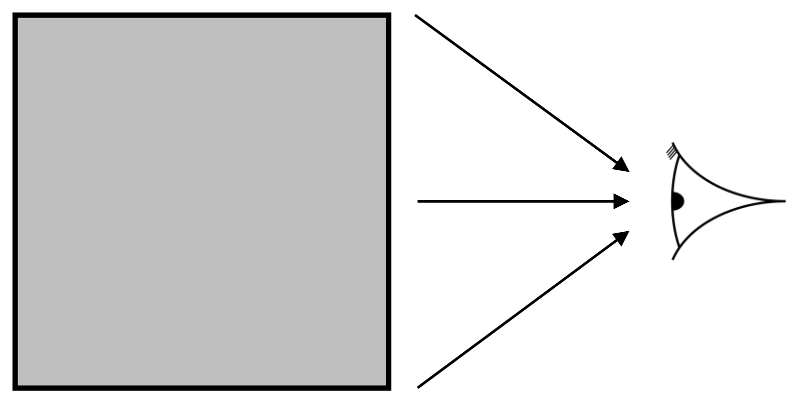


Clue : How will edges A and B appear to the observer at α ?

Question 2 (cont'd)

Hyperbolic distortion of light: Light from the top and bottom of the cube will take a longer time to travel to our eyes as compared to light from the middle, because they travel a longer distance. Thus light from the top and bottom of the cube have to be emitted some time back in the past so as to arrive at the observer with the light from the middle of the cube simultaneously. Hence the vertical line appears to be distorted.

Side view

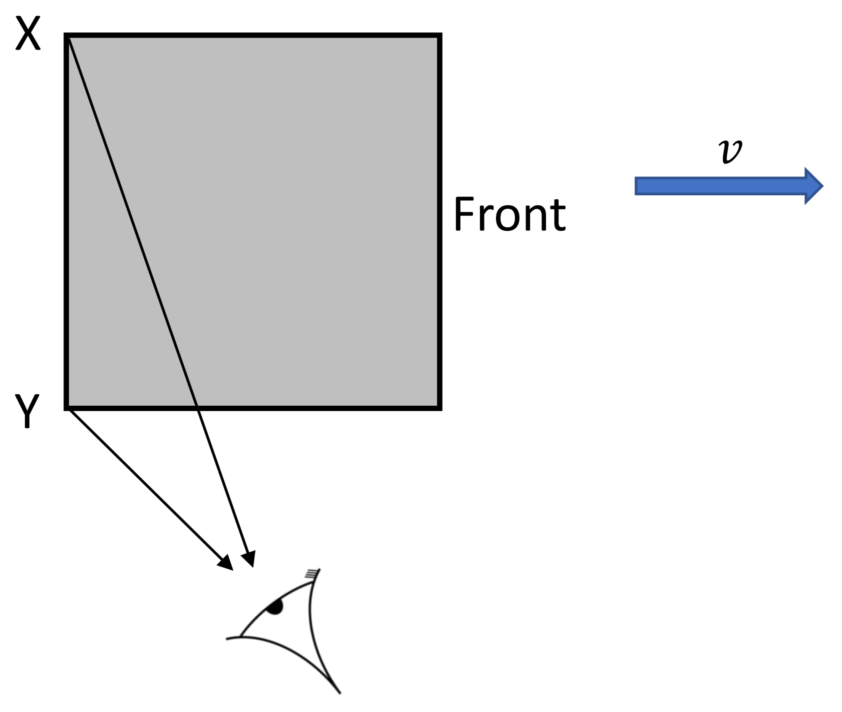


Cube is moving into/out of page

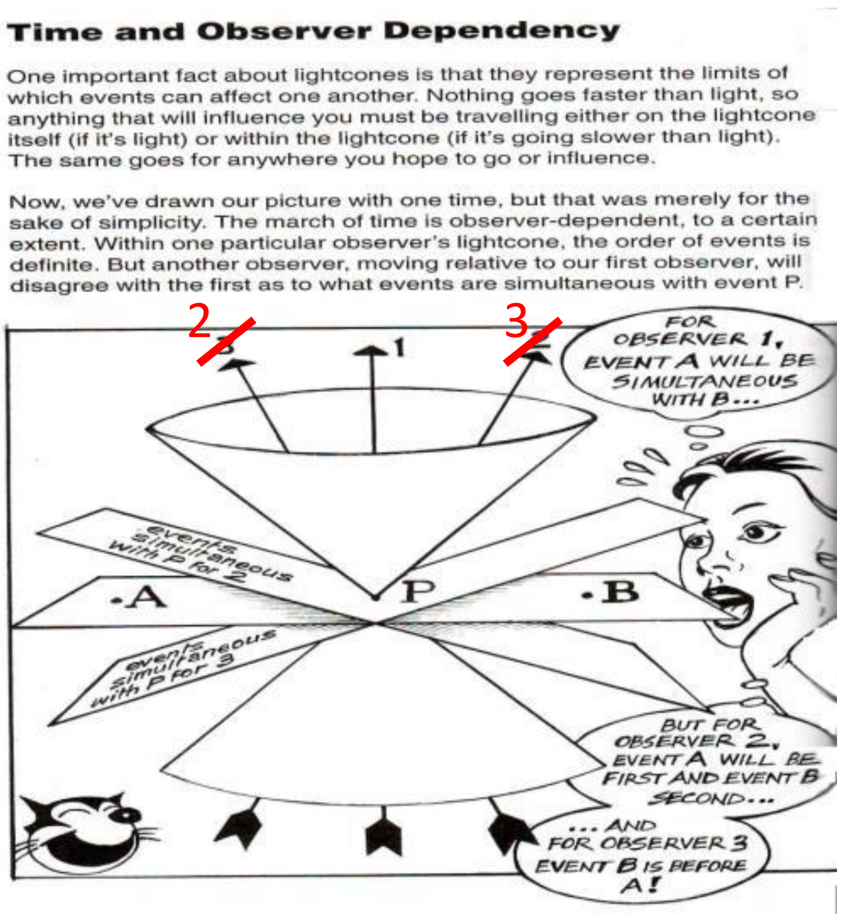
Question 2 (cont'd)

Apparent rotation: Light at X is normally blocked by the cube so you cannot see it. If the cube moves fast enough, it moves "out of the way" of this light so that you see X and all other points between X and Y. Hence, we will not only see a contracted side of the cube, but also the back of the cube. Similarly, the light emitted from the front of the cube will be blocked when the cube moves forward. Thus the front of the cube may not be visible to the observer under such circumstances.

Top view



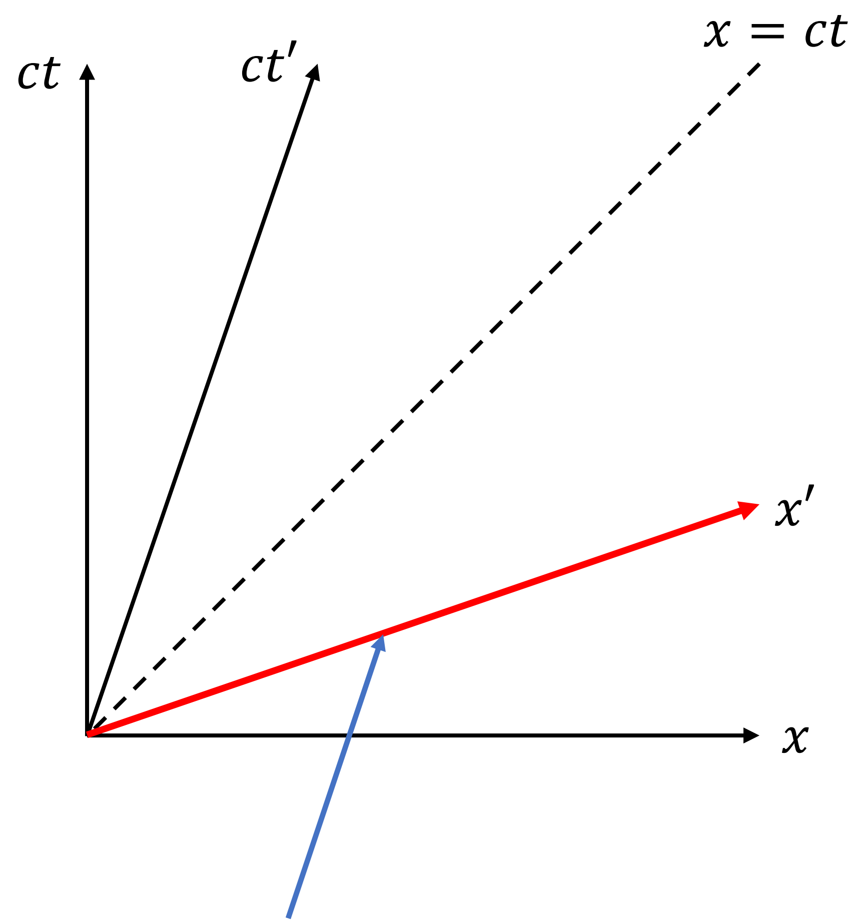
3 Use the cartoon (below) on the right to explain clearly to a friend why "simultaneous events" are observer dependent.



Time and Observer Dependency

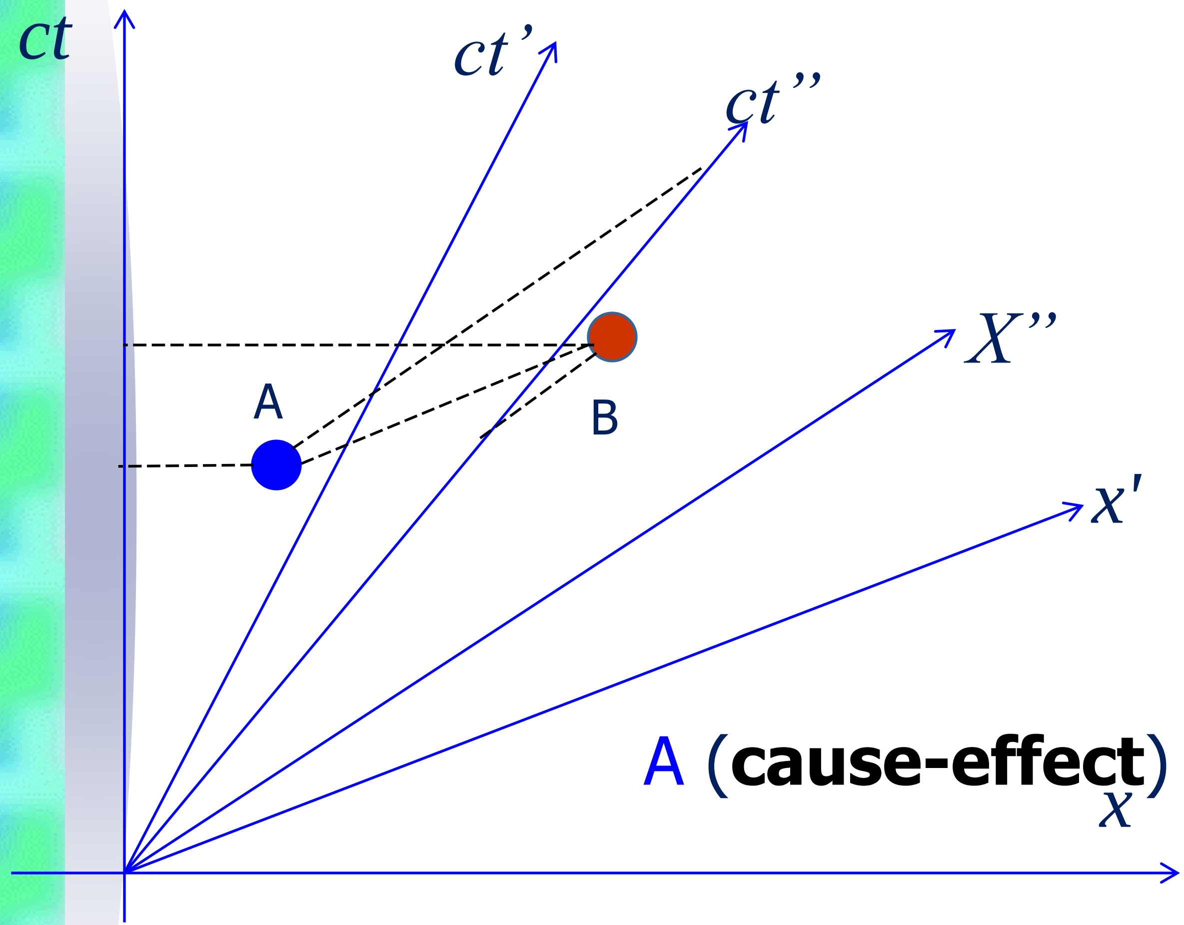
One important fact about lightcones is that they represent the limits of which events can affect one another. Nothing goes faster than light, so anything that will influence you must be travelling either on the lightcone itself (if it's light) or within the lightcone (if it's going slower than light). The same goes for anywhere you hope to go or influence.

Now, we've drawn our picture with one time, but that was merely for the sake of simplicity. The march of time is observer-dependent, to a certain extent. Within one particular observer's lightcone, the order of events is definite. But another observer, moving relative to our first observer, will disagree with the first as to what events are simultaneous with event P.



Events that take place on a line that is parallel to this x' axis are simultaneous events in the primed frame.

A Space-time diagram Illustration ?



A (**cause-effect**) Question ?

In reference frame ct' vs x' events A and B happened at the same time.

In reference frame ct vs x events A happened before event B.

What about in reference frame ct'' vs x''? **Anything peculiar ?**

Question 4

If modified relativistic quantities are given as $m = \gamma m_0$, $p = \gamma m_0 v$ and $E = \gamma m_0 c^2$. Show that $E^2 = (m_0 c^2)^2 + (pc)^2$, where p is the momentum, v is the speed and m_0 is the rest mass of moving object under consideration. **Note that this formula may be used for all particles whether they are massive or massless.**

Can you comment on this equation and what can you learn?

Starting with $E = \gamma m_0 c^2$, squaring both sides we get $E^2 = \gamma^2 m_0^2 c^4$.

Taylor expand to infinite number of terms:

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} = 1 + (-1)\left(-\frac{v^2}{c^2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{v^2}{c^2}\right)^2 + \cdots$$

$$= 1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6} + \cdots$$

Question 4 (cont'd)

$$\boxed{\gamma^2 = 1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6} + \cdots}$$

Thus,

$$E^2 = m_0^2 c^4 \left(1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6} + \cdots \right)$$

$$= m_0^2 c^4 + m_0^2 c^4 \left(\frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6} + \cdots \right)$$

$$= m_0^2 c^4 + m_0^2 c^4 \left(\frac{v^2}{c^2} \right) \underbrace{\left(1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \cdots \right)}_{\gamma^2}$$

$$= m_0^2 c^4 + m_0^2 v^2 \gamma^2 c^2 = m_0^2 c^4 + p^2 c^2$$

Alternate method:

$$p^2 = \gamma^2 m_0^2 v^2$$

$$p^2 = \left(\frac{1}{1-\frac{v^2}{c^2}}\right) m_0^2 v^2$$

$$p^2\left(1-\frac{v^2}{c^2}\right) = m_0^2 v^2$$

$$p^2 = (m_0^2 c^2 + p^2)\frac{v^2}{c^2}$$

$$\Rightarrow \frac{v^2}{c^2} = \frac{p^2}{m_0^2 c^2 + p^2}$$

$$1-\frac{v^2}{c^2} = 1-\frac{p^2}{m_0^2 c^2 + p^2} = \frac{m_0^2 c^2}{m_0^2 c^2 + p^2}$$

$$\frac{1}{1-\frac{v^2}{c^2}} = \gamma^2 = \frac{m_0^2 c^2 + p^2}{m_0^2 c^2}$$

Finally, using $E = \gamma m_0 c^2$,

$$E^2 = \gamma^2 m_0^2 c^4$$

$$= \left(\frac{m_0^2 c^2 + p^2}{m_0^2 c^2}\right) m_0^2 c^4$$

$$E^2 = m_0^2 c^4 + p^2 c^2$$

4(b) From $E^2 = (m_0c^2)^2 + (pc)^2$ above show that if $v \ll c$, we will obtain $E \approx m_0c^2 + \frac{p^2}{2m_0}$.

Can you comment on what is new here as compared to Newtonian energy?

$$E = \sqrt{(m_0c^2)^2 + (pc)^2}$$

$$= m_0c^2\left(1 + \frac{p^2c^2}{m_0^2c^4}\right)^{\frac{1}{2}}$$

$$\approx m_0c^2\left(1 + \frac{1}{2} \cdot \frac{p^2c^2}{m_0^2c^4}\right)$$

Binomial expansion:
$(1 + x)^n = 1 + nx + \cdots$

$$= m_0c^2 + \frac{p^2}{2m_0}$$

There is a rest mass term as compared to the Newtonian energy!

Note:

$$\frac{p^2}{2m} = \frac{m^2v^2}{2m} = \frac{1}{2}mv^2$$

This is only true for slow moving systems.

Question 5

Recall Lorentz Transformations: $t' = \left(t - \frac{vx}{c^2}\right)\gamma$, $x' = (x - vt)\gamma$, $y' = y$, $z' = z$. The 'prime' coordinates represent the moving frame.

a) Show that $t' = \gamma(t - \beta x)$, $x' = \gamma(x - \beta t)$, $y' = y$, $z' = z$, if c is set to unity (for convenience).

$$t' = \left(t - \frac{vx}{c^2}\right)\gamma$$

$$= \left(t - \frac{1}{c}\frac{v}{c}x\right)\gamma$$

$$= (t - \beta x)\gamma$$

$$x' = \gamma(x - vt)$$

$$= \gamma\left(x - v\left(\frac{c}{c}\right)t\right)$$

$$= \gamma(x - \beta ct)$$

$$= \gamma(x - \beta t)$$

Take note:

- $\beta \equiv \frac{v}{c}$
- "c is set to unity" means $c = 1$.

5(b) Check this invariant quantity $-t^2 + x^2 + y^2 + z^2 = -t'^2 + x'^2 + y'^2 + z'^2$.

Can you think of another invariant quantity where 2 different observers may agree on?

Recall:

$\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}}$ and $(1-\beta^2) = \left(1 - \frac{v^2}{c^2}\right)$

Thus, $\gamma^2(1-\beta^2) = 1$

$$
\begin{aligned}
s'^2 &= -t'^2 + x'^2 + y'^2 + z'^2 \\
&= -[\gamma(t-\beta x)]^2 + [\gamma(x-\beta t)]^2 + y^2 + z^2 \\
&= -\gamma^2[t^2 - 2\beta xt + \beta^2 x^2] + \gamma^2[x^2 - 2\beta xt + \beta^2 t^2] + y^2 + z^2 \\
&= -t^2\gamma^2[1-\beta^2] + x^2\gamma^2[1-\beta^2] + y^2 + z^2 \\
&= -t^2 + x^2 + y^2 + z^2 = s^2
\end{aligned}
$$

We have shown that the prime and unprimed frames share the same spacetime interval, i.e. $s'^2 = s^2$. This means that the spacetime interval s^2 is an invariant quantity.

Other invariant quantities:

1. Speed of light in vacuum
2. Mass, m
3. $E^2 - p^2c^2 = m^2c^4$

5(c) Discuss (with the Tutor) the significance of 5(b) above.

Can you comment in this equation and what can you learn?

- The "irritating minus sign" is to ensure the invariance of the space-time interval s^2.

- The Pythagoras' theorem in 3+1 dimensions is an invariant quantity. (This is related to the LHS of Einstein's field equation, that is geometry)

For x' axis, we set $ct' = 0$.

$$0 = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$t = \frac{vx}{c^2}$$

$$\boxed{ct = \frac{v}{c}x}$$

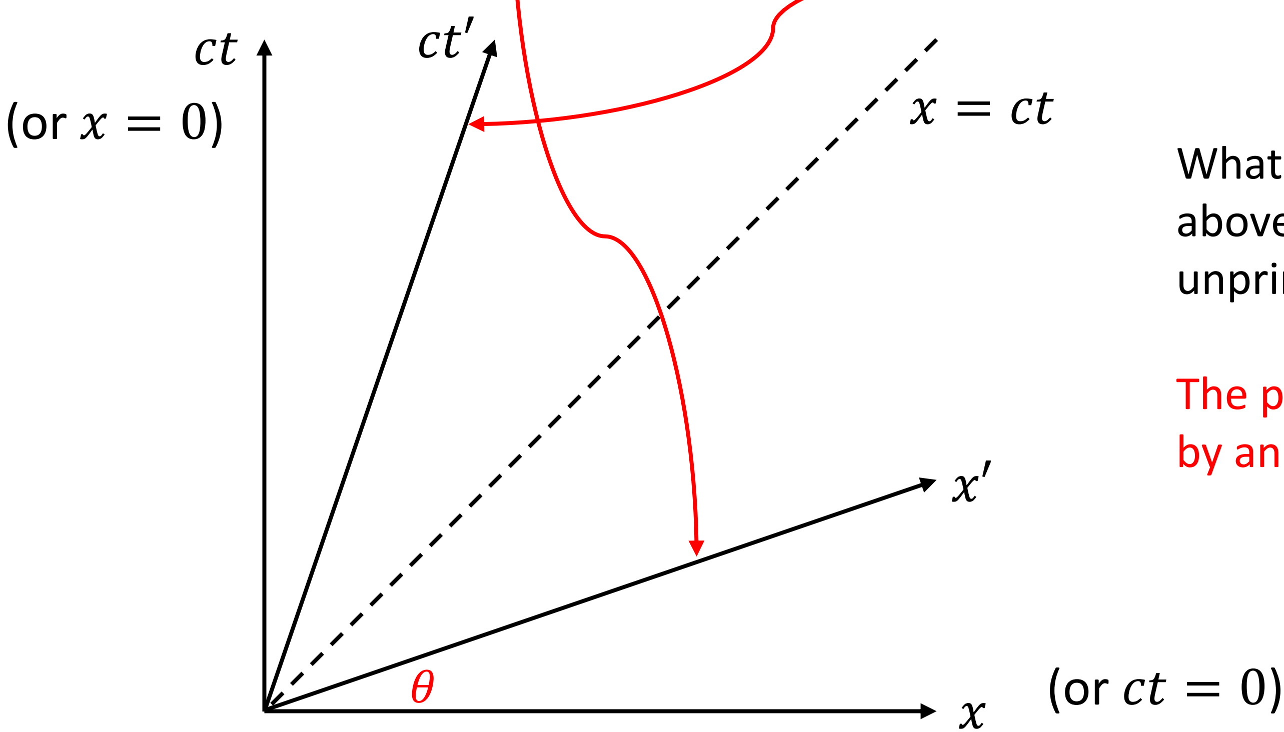
For ct' axis, we set $x' = 0$.

$$0 = \gamma(x - vt)$$

$$x = vt$$

$$\frac{1}{v}x = t$$

$$\boxed{ct = \frac{c}{v}x}$$



What can you learn from the above exercise regarding primed / unprimed axes?

The primed axes "rotate inward" by an angle θ such that

$$\tan\theta = \frac{v}{c}$$