CS1231/CS1231S: Discrete Structures

Tutorial #10: Counting and Probability II

(Week 12: 2 – 6 November 2020)

I. Discussion Questions

You are strongly encouraged to discuss D1 – D3 on LumiNUS forum. No answers will be provided.

D1. Suppose a random sample of 2 lightbulbs is selected from a group of 8 bulbs in which 3 are defective, what is the expected value of the number of defective bulbs in the sample? Let X represent of the number of defective bulbs that occur on a given trial, where $X = 0,1,2$. Find $E[X]$.

D2. How many **injective functions** are there from a set A with m elements to a set B with n elements, where $m \leq n$?

D3. How many **surjective functions** are there from a 5-element set A to a 3-element set B?

II. Tutorial Questions

1. Your organization has 6 designers, 12 business consultants, and 20 programmers. How many possible teams of 5 members can you have if:
   a. The team is made up completely of programmers.
   b. The team must have at least 1 programmer.
   c. The team must have at least 2 programmers, at least 1 designer and at least 1 business consultant.

2. You are the Director of Research at your company and you have \$25m to spend. There are 15 projects that require funding. Funding amounts are in units of \$1m, though projects may not necessarily receive funding (i.e. they get \$0).
   a. How many ways can you fund the 15 projects?
   b. The Chief Executive Officer insists that you must provide exactly \$3m for one particular project, and at least \$2m for each of five other particular projects. How many ways can you fund the 15 projects?

3. Think of a set with $m+n$ elements as composed of two parts, one with m elements and the other with n elements. Give a **combinatorial argument** to show that

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} \qquad \text{... (A)}$$

where $m, n \in \mathbb{Z}^+$, $r \le m$ and $r \le n$.

Call the above equation (A). Using equation (A), prove that for all integers $n \ge 0$,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

4. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

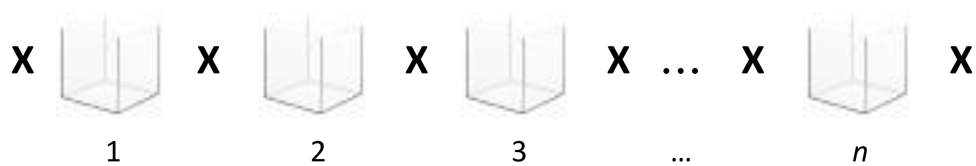
5. Let's revisit Question 5 of Tutorial #8:

   Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   Last week, the answer given was

   For k $(1 \le k \le n)$ consecutively numbered boxes that contain white balls, there are $n-k+1$ ways. Therefore, total number of ways is $\sum_{k=1}^{n}(n-k+1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?



6. You meet a hustler on the street who lets you toss 3 separate coins once each for \$2. If you get 3 heads, you win \$10. If you get 2 heads (not in a row), you win \$5, if you get 2 heads (in a row), you win \$1. Otherwise you win nothing. If you play this game many, many times, how much would you win overall per game?

7. The hustler now has two loaded coins with probability of 0.7 of getting tails, and one fair coin with probability of 0.5 of getting heads or tails. Is there a particular arrangement of coins (e.g. FLL, where F=fair and L=loaded) that he should use to maximize his profits? Explain why your choice works.

8. One urn contains 10 red balls and 25 green balls, and a second urn contains 22 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a loaded coin with probability 0.4 of landing heads up and probability of 0.6 of landing tails up. If the coin lands heads up, the first urn is chosen; otherwise, the second urn is chosen. Then a ball is picked at random from the chosen urn.

   Write your answers correct to three significant figures.

   (a) What is the probability that the chosen ball is green?

   (b) If the chosen ball is green, what is the probability that it was picked from the first urn?

9. (AY2015/16 Semester 1 exam question)

   Let $A = \{1, 2, 3, 4\}$. Since each element of $P(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. ($P(S)$ denotes the powerset of S.)

   Assuming each relation in $P(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

   Can you generalize your answer to any set A with n elements?

10. Let's revisit Question 1 of Tutorial #8:

    > In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?

    The solution given last week uses a possibility tree to depict the 15 ways. Now, let's approach this problem using combination.

    Let us define a function $W(a, b)$ to be the number of ways the tournament can be completed if team A has to win a more games to win, while team B has to win b more games to win. Hence,

$$W(a, b) = \begin{cases} 1, & \text{if } a = 0 \text{ or } b = 0; \\ W(a, b-1) + W(a-1, b), & \text{if a} > 0 \text{ and } b > 0. \end{cases}$$

    We may express $W(a, b)$ as a simple combination formula as follows:

$$W(a, b) = \binom{a+b}{a}.$$

    Verify the above.

    Now, let's denote the function $T(n, k)$ to be the number of ways the tournament can be completed, given that the first team to win n games wins the tournament, and team A wins the first k ($k \leq n$) games.

    Derive a simple combination formula for $T(n, k)$ (hint: relate function T to function W), and hence solve $T(4, 2)$ which is the problem in question 1 of tutorial #8.