CS1231S Chapter 7

Induction and recursion

7.1 Mathematical Induction

Principle 7.1.1 (Mathematical Induction (MI))**.** Let $m \in \mathbb{Z}$. To prove that $\forall n \in \mathbb{Z}_{\geqslant m}\ \ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(m)$ is true; and

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant m}\ \ \big(P(k) \Rightarrow P(k+1)\big)$ is true.

Justification. The two steps ensure the following are true:

$$\begin{array}{ll} P(m) & \text{by the base step;} \\ P(m) \Rightarrow P(m+1) & \text{by the induction step with } k = m; \\ P(m+1) \Rightarrow P(m+2) & \text{by the induction step with } k = m+1; \\ P(m+2) \Rightarrow P(m+3) & \text{by the induction step with } k = m+2; \\ \vdots & \end{array}$$

We deduce that $P(m), P(m+1), P(m+2), \ldots$ are all true by a series of modus ponens. □

Terminology 7.1.2. In the induction step, we assume we have $k \in \mathbb{Z}_{\geqslant m}$ such that $P(k)$ is true, and then show $P(k+1)$ using this assumption. In this process, the assumption that $P(k)$ is true is called the *induction hypothesis*.

Example 7.1.3. $1 + 2 + \cdots + n = \frac{1}{2}\,n(n+1)$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

Proof.
1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition "$1 + 2 + \cdots + n = \frac{1}{2}\,n(n+1)$".
2. (Base step) $P(1)$ is true because $1 = \frac{1}{2} \times 1 \times (1+1)$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., such that
   $$1 + 2 + \cdots + k = \frac{1}{2}\,k(k+1).$$
   3.2. Then $1 + 2 + \cdots + k + (k+1)$
   3.3. $= \dfrac{1}{2}\,k(k+1) + (k+1)$ by the induction hypothesis $P(k)$;
   3.4. $= \Big(\dfrac{k}{2} + 1\Big)(k+1) = \dfrac{k+2}{2}\,(k+1)$
   3.5. $= \dfrac{1}{2}\,(k+1)((k+1)+1)$.
   3.6. So $P(k+1)$ is true.

4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1} \;\; P(n)$ is true by MI. □

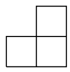
Terminology 7.1.4. We call the proof above an induction *on* n because n is the active variable in it.

Example 7.1.5. $n! > 2^n$ for all $n \in \mathbb{Z}_{\geqslant 4}$, where $n! = n \times (n-1) \times \cdots \times 1$.

Proof. 1. For each $n \in \mathbb{Z}_{\geqslant 4}$, let $P(n)$ be the proposition "$n! > 2^n$".

2. (Base step) $P(4)$ is true because $4! = 24 > 16 = 2^4$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 4}$ such that $P(k)$ is true, i.e., such that
   $$k! > 2^k.$$
   3.2. Then $(k+1)! = (k+1) \times k!$ by the definition of !;
   3.3. $> (k+1) \times 2^k$ by the induction hypothesis $P(k)$;
   3.4. $> 2 \times 2^k$ as $k + 1 \geqslant 4 + 1 > 2$;
   3.5. $= 2^{k+1}$.
   3.6. So $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 4} \;\; P(n)$ is true by MI. □

Example 7.1.6. An *L-tromino* is the following L-shape formed by three squares of the checkerboard:



For all $n \in \mathbb{Z}_{\geqslant 1}$, if one square is removed from a $2^n \times 2^n$ checkerboard, then the remaining squares can be covered by L-trominos.

Proof. 1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition
   if one square is removed from a $2^n \times 2^n$ checkerboard, then the remaining squares can be covered by L-trominos.

2. (Base step) $P(1)$ is true because such a board itself is an L-tromino.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true.
   3.2. 3.2.1. Let B be a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed.
   3.2.2. Divide B into four $2^k \times 2^k$ quadrants.
   3.2.3. Let Q be the quadrant containing the removed square.
   3.2.4. Remove one L-tromino from the centre of B in a way such that each quadrant other than Q has one square removed.
   3.2.5. We are left with four $2^k \times 2^k$ checkerboards, each with one square removed.
   3.2.6. By the induction hypothesis, each quadrant can be covered by L-trominos.
   3.2.7. Hence B can be covered by L-trominos.
   3.3. This shows $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1} \;\; P(n)$ is true by MI. □

Example 7.1.7. All participants in this Zoom meeting have the same birthday. ✎ 7a

Proof. 1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition
   if a Zoom meeting has exactly n participants, then all its participants have the same birthday.

2. (Base step) $P(1)$ is true because if a Zoom meeting has exactly 1 participant, then clearly all its participants have the same birthday.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true.

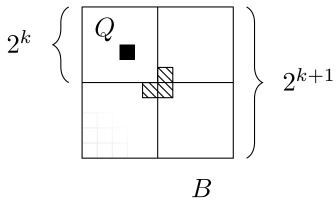


Figure 7.1: Covering a checkerboard with L-trominos

3.2. 3.2.1. Suppose a Zoom meeting has exactly $k+1$ participants.
   3.2.2. Pick two different participants a, b in the meeting.
   3.2.3. Ask a to leave the meeting.
   3.2.4. Since there are k people left in the meeting, by the induction hypothesis, all the remaining participants have the same birthday, including b.
   3.2.5. Tell a to join the meeting again, and then ask b to leave the meeting.
   3.2.6. Since there are k people left in the meeting, by the induction hypothesis, all the remaining participants have the same birthday, including a.
   3.2.7. The participants who stayed in the meeting throughout have the same birthday as both a and b.
   3.2.8. So a and b have the same birthday.

3.3. This shows $P(k+1)$ is true.

4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1} \;\; P(n)$ is true by MI.

7.2 Strong Mathematical Induction

Principle 7.2.1 (Strong Mathematical Induction (Strong MI))**.** To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(0), P(1), \ldots, P(m)$ are true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 0} \;\; \big(P(0) \wedge P(1) \wedge \cdots \wedge P(k+m) \Rightarrow P(k+m+1)\big)$ is true

for some $m \in \mathbb{Z}_{\geqslant 0}$.

Justification. The two steps ensure the following are true:

$P(0) \wedge P(1) \wedge \cdots \wedge P(m)$	by the base step;
$P(0) \wedge P(1) \wedge \cdots \wedge P(m) \Rightarrow P(m+1)$	by the induction step with $k=0$;
$P(0) \wedge P(1) \wedge \cdots \wedge P(m) \wedge P(m+1) \Rightarrow P(m+2)$	by the induction step with $k=1$;
$P(0) \wedge P(1) \wedge \cdots \wedge P(m) \wedge P(m+1) \wedge P(m+2) \Rightarrow P(m+3)$	by the induction step with $k=2$;
$\vdots$	

We deduce that $P(0), P(1), P(2), P(3), \ldots$ are all true by a series of modus ponens. □

Definition 7.2.2. The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by setting

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n$$

for each $n \in \mathbb{Z}_{\geqslant 0}$.

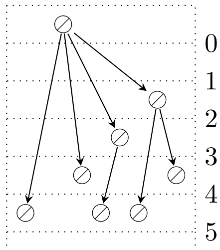


Figure 7.2: Rabbits

Example 7.2.3. $F_2 = 1+0 = 1, \quad F_3 = 1+1 = 2, \quad F_4 = 2+1 = 3, \quad F_5 = 3+2 = 5, \quad \ldots.$

Example 7.2.4.
- Initially, there is one pair of newly born matched rabbits.
- Each newly born rabbit takes one month to mature.
- Each mature pair of matched rabbits produces one pair of matched rabbits per month.

Let r_n denote the number of pairs of rabbits after n months. Then for every $n \in \mathbb{Z}_{\geqslant 0}$,

$$r_0 = 1 \quad \text{and} \quad r_1 = 1 \quad \text{and} \quad r_{n+2} = r_{n+1} + r_n,$$

where the r_{n+1} comes from the rabbits already present after $(n+1)$ months, and the r_n comes from the rabbits born after $(n+1)$ months.

Observation 7.2.5. $r_n = F_{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

Example 7.2.6. $F_{n+1} \leqslant (7/4)^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

Proof.
1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition "$F_{n+1} \leqslant (7/4)^n$".
2. (Base step) $P(0)$ and $P(1)$ are true because
$$F_{0+1} = 1 \leqslant 1 = (7/4)^0 \quad \text{and} \quad F_{1+1} = 1+0 = 1 \leqslant 7/4 = (7/4)^1.$$
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k+1)$ are true.
   3.2. Then $F_{(k+2)+1} = F_{k+3}$
   3.3. $= F_{k+2} + F_{k+1}$ by the definition of F_{k+3};
   3.4. $\leqslant (7/4)^{k+1} + (7/4)^k$ as $P(k)$ and $P(k+1)$ are true;
   3.5. $= (7/4)^k(7/4+1)$
   3.6. $< (7/4)^k(7/4)^2$ as $7/4+1 = 11/4 < 49/16 = (7/4)^2$;
   3.7. $= (7/4)^{k+2}$.
   3.8. So $P(k+2)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0} \;\; P(n)$ is true by Strong MI. □

Theorem 7.2.7 (Strong MI, alternative formulation)**.** To prove that $\forall n \in \mathbb{Z}_{\geqslant 0} \; P(n)$ is true, where each $P(n)$ is a proposition, it suffices to show that

$$\forall \ell \in \mathbb{Z}_{\geqslant 0} \;\; \big(\forall i \in \mathbb{Z}_{\geqslant 0} \;\; \big(i < \ell \Rightarrow P(i)\big) \Rightarrow P(\ell)\big) \tag{$*$}$$

is true.

Proof.
1. Suppose $(*)$ is true.
2. (Base step)
   2.1. Applying $(*)$ to $\ell = 0$ tells us $\forall i \in \mathbb{Z}_{\geqslant 0} \;\; \big(i < 0 \Rightarrow P(i)\big) \Rightarrow P(0)$ is true.

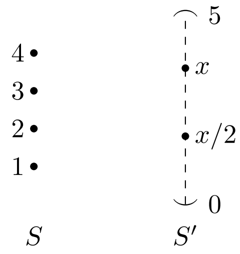


Figure 7.3: A difference between $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Q}_{\geqslant 0}$

2.2. $\forall i \in \mathbb{Z}_{\geqslant 0} \quad \big(i < 0 \Rightarrow P(i)\big)$ is true trivially.
2.3. So $P(0)$ is true.
3. (Induction step)
3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k)$ is true.
3.2. Then $\forall i \in \mathbb{Z}_{\geqslant 0} \quad \big(i < k+1 \Rightarrow P(i)\big)$ is true.
3.3. So $(*)$ applied to $\ell = k+1$ tells us $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0} \quad P(n)$ is true by Strong MI. □

Example 7.2.8. (1) $S = \{x \in \mathbb{Z}_{\geqslant 0} : 0 < x < 5\}$ has smallest element 1.

(2) $S' = \{x \in \mathbb{Q}_{\geqslant 0} : 0 < x < 5\}$ has no smallest element because if $x \in S'$, then $x/2 \in S'$ and $x/2 < x$.

Theorem 7.2.9 (Well-Ordering Principle)**.** Every nonempty subset of $\mathbb{Z}_{\geqslant 0}$ has a smallest element.

Proof. 1. Let $S \subseteq \mathbb{Z}_{\geqslant 0}$ with no smallest element.
2. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition "$n \notin S$".
3. (Base step)
3.1. 3.1.1. Suppose $0 \in S$.
3.1.2. Then 0 is the smallest element of S as $S \subseteq \mathbb{Z}_{\geqslant 0}$.
3.1.3. This contradicts our assumption that S has no smallest element on line 1.
3.2. So $0 \notin S$.
3.3. Thus $P(0)$ is true.
4. (Induction step)
4.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k)$ are true, i.e., that $0, 1, \ldots, k \notin S$.
4.2. 4.2.1. Suppose $k+1 \in S$.
4.2.2. Then $k+1$ is the smallest element of S by the induction hypothesis as $S \subseteq \mathbb{Z}_{\geqslant 0}$.
4.2.3. This contradicts our assumption that S has no smallest element on line 1.
4.3. So $k+1 \notin S$.
4.4. Thus $P(k+1)$ is true.
5. Hence $\forall n \in \mathbb{Z}_{\geqslant 0} \quad P(n)$ is true by Strong MI.
6. This implies $S = \varnothing$ as $S \subseteq \mathbb{Z}_{\geqslant 0}$. □

7.3 Recursion

7.3.1 Recursively defined sequences

Terminology 7.3.1. A sequence $a_0, a_1, a_2, \ldots$ is said to be *recursively defined* if the definition of a_n involves $a_0, a_1, \ldots, a_{n-1}$ for all but finitely many $n \in \mathbb{Z}_{\geqslant 0}$.

Example 7.3.2. (1) Define $0!, 1!, 2!, \ldots$ by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$0! = 1 \quad \text{and} \quad (n+1)! = (n+1) \times n!.$$

Then $1! = 1 \times 1 = 1, \quad 2! = 2 \times 1 = 2, \quad 3! = 3 \times 2 = 6, \quad 4! = 4 \times 6 = 24, \quad \ldots.$

(2) The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ was defined in Definition 7.2.2 by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n.$$

Then $F_2 = 1 + 0 = 1, \quad F_3 = 1 + 1 = 2, \quad F_4 = 2 + 1 = 3, \quad F_5 = 3 + 2 = 5, \quad \ldots.$

(3) Fix $r \in [0, 4]$ and $p_0 \in [0, 1]$. Define $p_1, p_2, \ldots$ by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$p_{n+1} = r(p_n - {p_n}^2).$$

If $r = 3$ and $p_0 = 1/2$, then

$$p_1 = 3\Big(\frac{1}{2} - \Big(\frac{1}{2}\Big)^2\Big) = \frac{3}{4}, \quad p_2 = 3\Big(\frac{3}{4} - \Big(\frac{3}{4}\Big)^2\Big) = \frac{9}{16}, \quad \ldots.$$

(4) Fix $a_0 \in \mathbb{Z}^+$. Define $a_1, a_2, a_3, \ldots$ by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$a_{n+1} = \begin{cases} a_n/2, & \text{if } a_n \text{ is even;} \\ 3a_n + 1, & \text{if } a_n \text{ is odd.} \end{cases}$$

If $a_0 = 1$, then $a_1 = 3 \times 1 + 1 = 4, \quad a_2 = 4/2 = 2, \quad a_3 = 2/2 = 1, \quad \ldots.$

Exercise 7.3.3. Let $a_1 = 1$ and $a_{n+1} = a_n + (n+1)$ for all $n \in \mathbb{Z}_{\geqslant 1}$. Find a general formula for a_n in terms of n that does not involve $a_0, a_1, \ldots, a_{n-1}$. ✎ 7b

Proposition 7.3.4. There is a unique sequence $a_0, a_1, a_2, \ldots$ satisfying, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$a_0 = 0 \quad \text{and} \quad a_1 = 1 \quad \text{and} \quad a_{n+2} = a_{n+1} + a_n.$$

Proof. For the purpose of this proof, let us call a sequence $b_0, b_1, \ldots, b_{n-1}$ a *partial sequence* if for all $i \in \mathbb{Z}_{\geqslant 0}$ with $i < n$,

$$b_i = \begin{cases} 0, & \text{if } i = 0; \\ 1, & \text{if } i = 1; \\ b_{i-1} + b_{i-2}, & \text{if } i \geqslant 2. \end{cases}$$

1. First, we claim that there is a partial sequence of length n for every $n \in \mathbb{Z}_{\geqslant 0}$.
   1.1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition

   "there is a partial sequence of length n".

   1.2. (Base step) $P(0)$ is true because the empty sequence is trivially a partial sequence of length 0.
   1.3. (Induction step)
      1.3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(k)$ is true.
      1.3.2. This gives a partial sequence $b_0, b_1, \ldots, b_{k-1}$ of length k.
      1.3.3. Define

$$b_k = \begin{cases} 0, & \text{if } k = 0; \\ 1, & \text{if } k = 1; \\ b_{k-1} + b_{k-2}, & \text{if } k \geqslant 2. \end{cases}$$

1.3.4. Then $b_0, b_1, \ldots, b_k$ is a partial sequence of length $k+1$ by the choice of b_k and because $b_0, b_1, \ldots, b_{k-1}$ is a partial sequence.
1.3.5. So $P(k+1)$ is true.
1.4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0} \;\; P(n)$ is true by MI.
2. If $b_0, b_1, \ldots, b_{m-1}$ and $c_0, c_1, \ldots, c_{n-1}$ are partial sequences with $m \leqslant n$, then

$$\begin{aligned} b_0 &= 0 = c_0, \\ b_1 &= 1 = c_1, \\ b_2 &= b_1 + b_0 = c_1 + c_0 = c_2, \\ b_3 &= b_2 + b_1 = c_2 + c_1 = c_3, \\ &\vdots \\ b_{m-1} &= b_{m-2} + b_{m-3} = c_{m-2} + c_{m-3} = c_{m-1}. \end{aligned}$$

3. For each $n \in \mathbb{Z}_{\geqslant 0}$, define a_n to be the nth element of any partial sequence of length at least n.
4. Then the sequence $a_0, a_1, a_2, \ldots$ is well defined by lines 1 and 2.
5. This sequence $a_0, a_1, a_2, \ldots$ is what we want because it agrees with all the partial sequences, and the conditions in the definition of partial sequences match with the required conditions.
6. Let $b_0, b_1, b_2, \ldots$ be a sequence satisfying, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$b_0 = 0 \quad \text{and} \quad b_1 = 1 \quad \text{and} \quad b_{n+2} = b_{n+1} + b_n.$$

7. We show that $a_n = b_n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.
7.1. Let $n \in \mathbb{Z}_{\geqslant 0}$.
7.2. Note that $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ are partial sequences.
7.3. So $a_n = b_n$ by line 2. □

7.3.2 Recursively defined sets

Theorem 7.3.5. $\mathbb{Z}_{\geqslant 0}$ is the unique set with the following properties.

(1) $0 \in \mathbb{Z}_{\geqslant 0}$. (base clause)

(2) If $x \in \mathbb{Z}_{\geqslant 0}$, then $x + 1 \in \mathbb{Z}_{\geqslant 0}$. (recursion clause)

(3) Membership for $\mathbb{Z}_{\geqslant 0}$ can always be demonstrated by (finitely many) successive applications of the clauses above. (minimality clause)

Example 7.3.6.

$$\begin{array}{rll} & 0 \in \mathbb{Z}_{\geqslant 0} & \text{by (1).} \\ \therefore & 1 \in \mathbb{Z}_{\geqslant 0} & \text{by (2) and the previous line.} \\ \therefore & 2 \in \mathbb{Z}_{\geqslant 0} & \text{by (2) and the previous line.} \end{array}$$

Remark 7.3.7. (1) and (2) are true when $\mathbb{Z}_{\geqslant 0}$ is changed to $\mathbb{Q}$, but (3) is not.

Terminology 7.3.8. Theorem 7.3.5 gives a *recursive definition* of $\mathbb{Z}_{\geqslant 0}$.

Example 7.3.9. The set $2\mathbb{Z}_{\geqslant 1}$ of all positive even integers can be defined recursively as follows.

(1) $2 \in 2\mathbb{Z}_{\geqslant 1}$. (base clause)

(2) If $x \in 2\mathbb{Z}_{\geqslant 1}$, then $x + 2 \in 2\mathbb{Z}_{\geqslant 1}$. (recursion clause)

(3) Membership for $2\mathbb{Z}_{\geqslant 1}$ can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

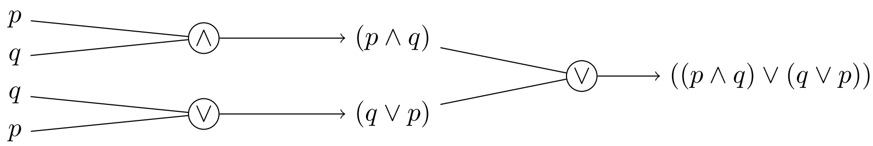


Figure 7.4: The construction of $((p \wedge q) \vee (q \vee p))$

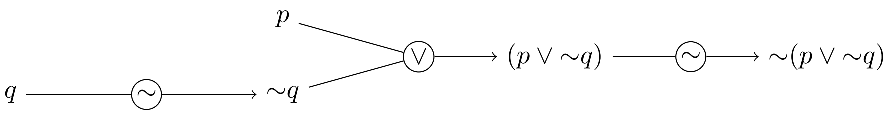


Figure 7.5: The construction of $\sim(p \vee \sim q)$

Theorem 7.3.10 (Structural induction over $2\mathbb{Z}_{\geqslant 1}$)**.** To prove that $\forall n \in 2\mathbb{Z}_{\geqslant 1}\ \ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(2)$ is true; and

(induction step) show that $\forall x \in 2\mathbb{Z}_{\geqslant 1}\ \ \big(P(x) \Rightarrow P(x+2)\big)$ is true.

Question 7.3.11. Define a set S recursively as follows. ✐ 7c

(1) $1 \in S$. (base clause)

(2) If $x \in S$, then $2x \in S$ and $3x \in S$. (recursion clause)

(3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the numbers $9, 10, 11, 12, 13$ are in S? Which are not?

Definition 7.3.12. Designate a nonempty set Σ whose elements will be used as propositional variables. Define the set $\mathrm{WFF}(\Sigma)$ recursively as follows.

(1) Every element p of Σ is in $\mathrm{WFF}(\Sigma)$. (base clause)

(2) If x, y are in $\mathrm{WFF}(\Sigma)$, then $\sim x$ and $(x \wedge y)$ and $(x \vee y)$ are in $\mathrm{WFF}(\Sigma)$. (recursion clause)

(3) Membership for $\mathrm{WFF}(\Sigma)$ can always be demonstrated by (finitely many) successive applications of the clauses above. (minimality clause)

Example 7.3.13. Let $\Sigma = \{p, q\}$. Then

$$\begin{array}{rrl} & p, q \in \mathrm{WFF}(\Sigma) & \text{by (1);} \\ \therefore & (p \wedge q), (q \vee p) \in \mathrm{WFF}(\Sigma) & \text{by (2) and the previous line;} \\ \therefore & ((p \wedge q) \vee (q \vee p)) \in \mathrm{WFF}(\Sigma) & \text{by (2) and the previous line.} \end{array}$$

Example 7.3.14. Let $\Sigma = \{p, q\}$. Then

$$\begin{array}{rrl} & p, q \in \mathrm{WFF}(\Sigma) & \text{by (1);} \\ \therefore & \sim q \in \mathrm{WFF}(\Sigma) & \text{by (2) and the previous line;} \\ \therefore & (p \vee \sim q) \in \mathrm{WFF}(\Sigma) & \text{by (2) and the two previous lines;} \\ \therefore & \sim(p \vee \sim q) \in \mathrm{WFF}(\Sigma) & \text{by (2) and the previous line.} \end{array}$$

Theorem 7.3.15 (Structural induction over $\mathrm{WFF}(\Sigma)$)**.** To prove that $\forall x \in \mathrm{WFF}(\Sigma)\ \ P(x)$ is true, where each $P(x)$ is a proposition, it suffices to:

(base step) show that $P(p)$ is true for every $p \in \Sigma$;

(induction step) show that

$$\forall x, y \in \mathrm{WFF}(\Sigma) \quad \big(P(x) \wedge P(y) \Rightarrow P({\sim}p) \wedge P((x \wedge y)) \wedge P((x \vee y))\big).$$

Definition 7.3.16. Designate a nonempty set Σ whose elements will be used as propositional variables. Define the set $\mathrm{WFF}^+(\Sigma)$ recursively as follows.

(1) Every element p of Σ is in $\mathrm{WFF}^+(\Sigma)$. (base clause)

(2) If x, y are in $\mathrm{WFF}^+(\Sigma)$, then $(x \wedge y)$ and $(x \vee y)$ are in $\mathrm{WFF}^+(\Sigma)$. (recursion clause)

(3) Membership for $\mathrm{WFF}^+(\Sigma)$ can always be demonstrated by (finitely many) successive applications of the clauses above. (minimality clause)

Example 7.3.17. Let $\Sigma = \{p, q\}$. The argument in Example 7.3.13 shows that $((p \wedge q) \vee (q \vee p)) \in \mathrm{WFF}^+(\Sigma)$.

Theorem 7.3.18 (Structural induction over $\mathrm{WFF}^+(\Sigma)$)**.** To prove that $\forall x \in \mathrm{WFF}^+(\Sigma)\ \ P(x)$ is true, where each $P(x)$ is a proposition, it suffices to:

(base step) show that $P(p)$ is true for every $p \in \Sigma$;

(induction step) show that

$$\forall x, y \in \mathrm{WFF}^+(\Sigma) \quad \big(P(x) \wedge P(y) \Rightarrow P((x \wedge y)) \wedge P((x \vee y))\big).$$

Lemma 7.3.19. Let Σ be a nonempty set. If $x \in \mathrm{WFF}^+(\Sigma)$, then assigning **false** to all the elements of Σ makes x evaluate to **false**.

Proof. 1. For each $x \in \mathrm{WFF}^+(\Sigma)$, let $P(x)$ be the proposition
assigning **false** to all the elements of Σ makes x evaluate to **false**.
2. (Base step) $P(p)$ is true for every $p \in \Sigma$ because assigning **false** to all the elements of Σ in particular assigns **false** to p.
3. (Induction step)
   3.1. Let $x, y \in \mathrm{WFF}^+(\Sigma)$ such that $P(x)$ and $P(y)$ are true, i.e.,
   assigning **false** to all the elements of Σ makes both x and y evaluate to **false**.
   3.2. Then assigning **false** to all the elements of Σ must make $(x \wedge y)$ and $(x \vee y)$ evaluate to **false** by the induction hypothesis because **false** $\wedge$ **false** $\equiv$ **false** $\equiv$ **false** $\vee$ **false**.
   3.3. So $P((x \wedge y))$ and $P((x \vee y))$ are true.
4. Hence $\forall x \in \mathrm{WFF}^+(\Sigma)\ \ P(x)$ is true by structural induction over $\mathrm{WFF}^+(\Sigma)$. □

Theorem 7.3.20. The set $\{\wedge, \vee\}$ is not a complete set of propositional connectives. In other words, for every nonempty set Σ,

$$\exists x \in \mathrm{WFF}(\Sigma) \quad \forall y \in \mathrm{WFF}^+(\Sigma) \quad y \not\equiv x.$$

Proof. 1. Take $p \in \Sigma$. This is possible since $\Sigma \neq \varnothing$.
2. Pick any $y \in \mathrm{WFF}^+(\Sigma)$.
3. Assigning **false** to all the elements of Σ makes y evaluate to **false** by Lemma 7.3.19, but it makes ${\sim}p$ evaluate to ${\sim}$**false** $\equiv$ **true**.
4. So $y \not\equiv {\sim}p$. □

Remark 7.3.21. Recall that $\mathbb{Z}_{\geqslant 0}$ can be recursively defined using the clauses in Theorem 7.3.5. The version of structural induction over $\mathbb{Z}_{\geqslant 0}$ corresponding to this recursive definition is precisely Mathematical Induction (with $m = 0$). Thus Mathematical Induction is actually an instance of structural induction.

Remark 7.3.22. Recursive definition of sets described in this subsection and recursive definitions of sequences described in Subsection 7.3.1 are different but related kinds of recursion. The former defines a *set*, while the latter defines an infinite *sequence*. Recall from Definition 6.1.12 that an infinite sequence is essentially a function whose domain is $\mathbb{Z}_{\geqslant 0}$. Recursive definitions of sequences exploit recursive definitions of $\mathbb{Z}_{\geqslant 0}$, e.g., that given by Theorem 7.3.5, to define functions with domain $\mathbb{Z}_{\geqslant 0}$. Similarly, recursive definitions can exploit the recursive definition of another set S to define functions with domain S.