About Planar Graph

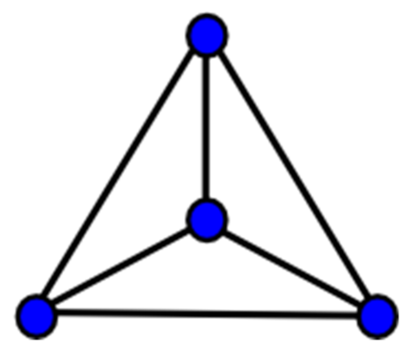
Go to https://visualgo.net/en/graphds, select Undirected/Unweighted mode, and try to (re)draw (or (re)layout) planar graph(s) discussed here/others so that the edge-crossing detector says "Cross? **No**"

Planar Graph

- Euler's Equation/Formula
  - If a graph is planar,

$$V - E + F = 1 + C$$

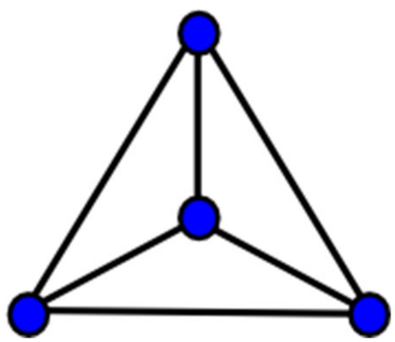
    - V = # vertices
    - E = # edges
    - F = # faces (new term: regions bounded by edges, including the outer, infinitely large region)
    - C = # components
  - V = 4, E = 6, F = 4, C = 1
    - 4 – 6 + 4 = 2 = 1 + 1

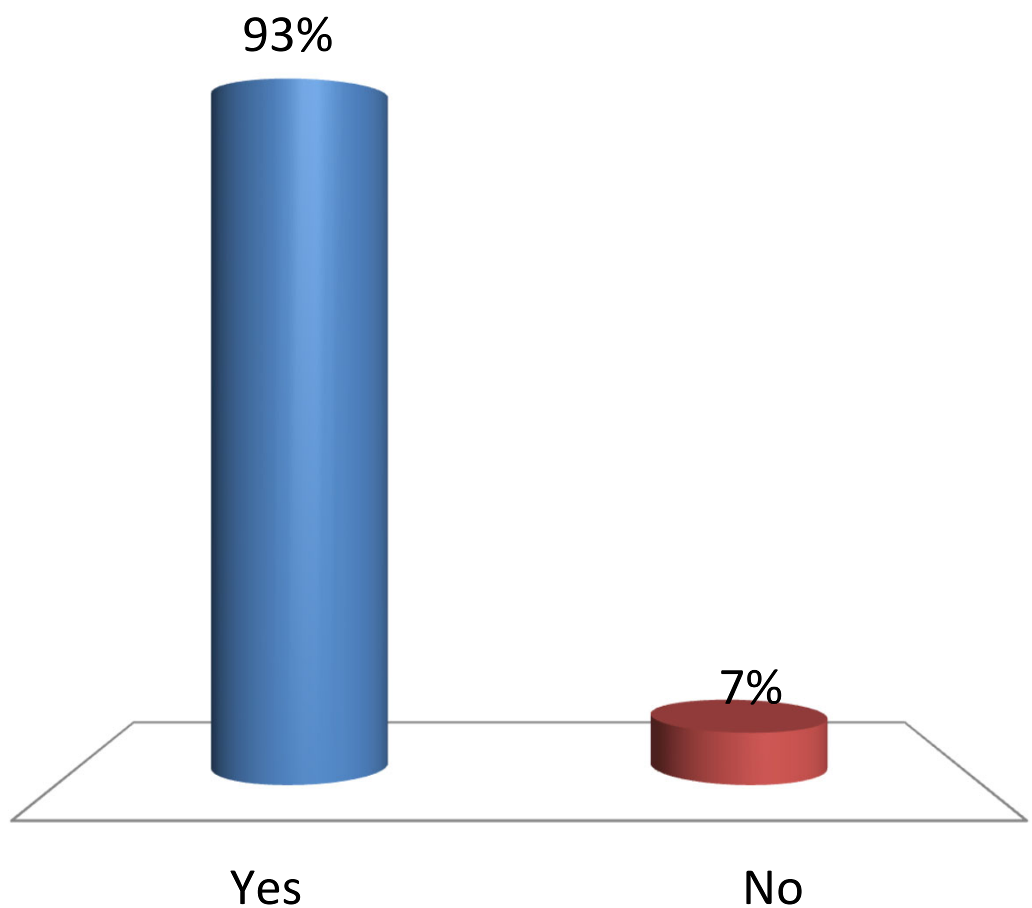


Is it a planar graph?

A. Yes

B. No

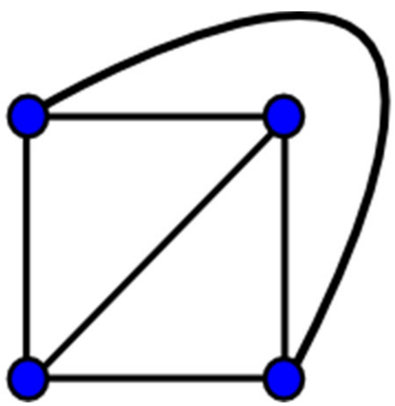


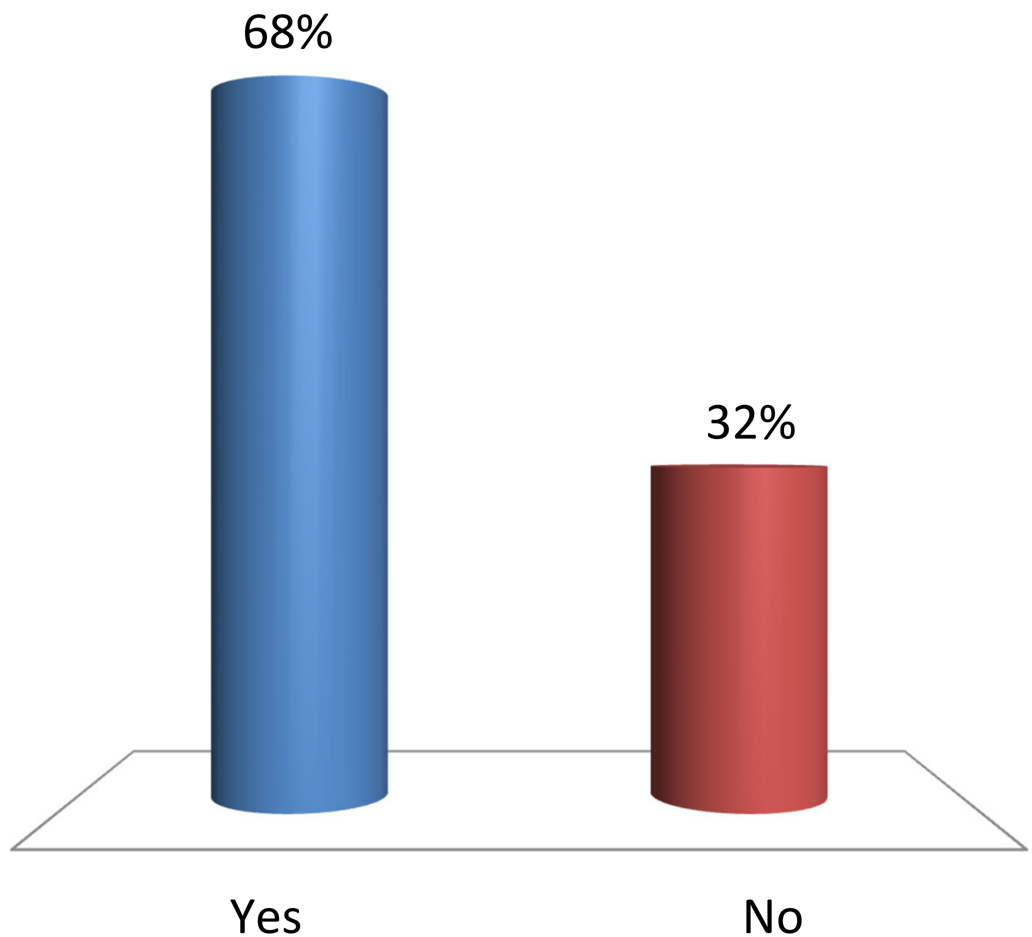


Is it a planar graph?

✓ A. Yes

B. No





Is it a planar graph?

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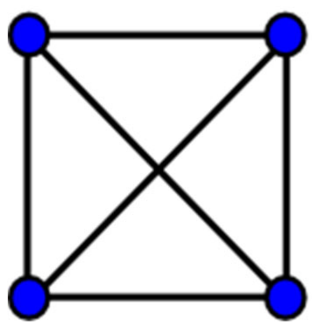
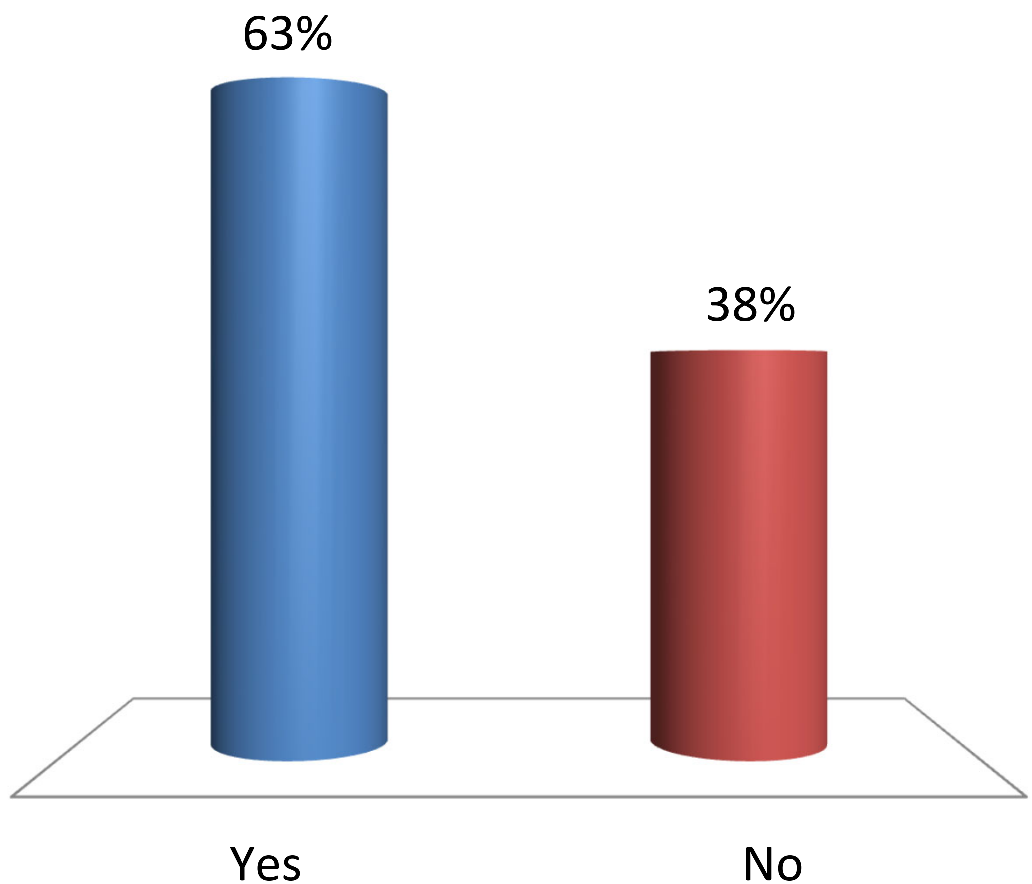
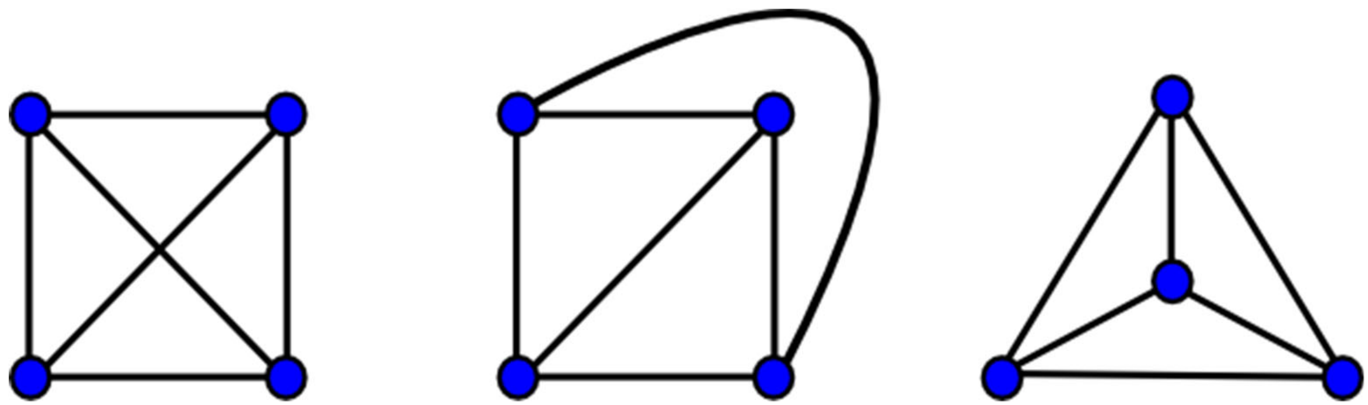
B. No





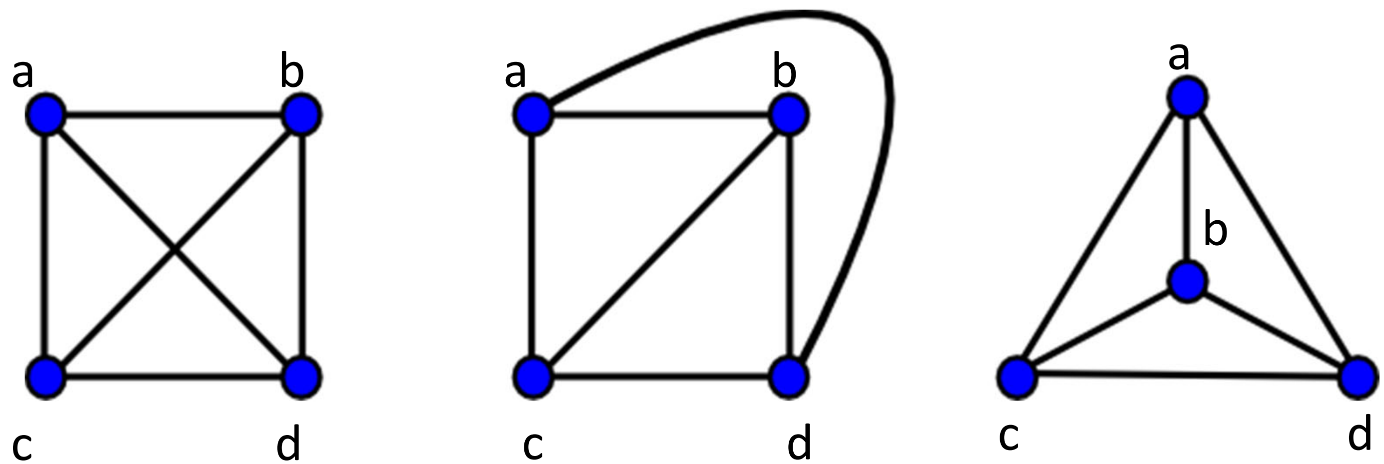
They are all the same graph

- They are all K_4 where K_4 is a complete graph with *4* vertices



In a Topologist eye

- G = {V, E}
  - V = {a, b, c, d}
  - E = { (a, b), (a, c), (a, d), (b, c), (b, d), (c, d) }



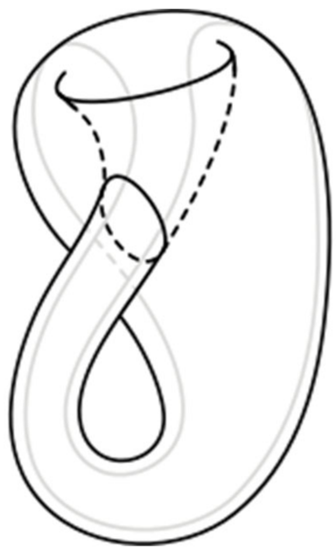
- Topology cares about connectivity

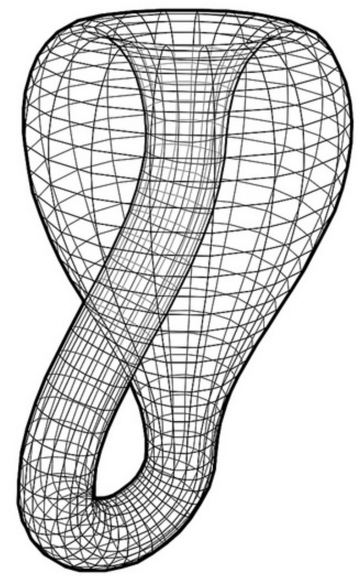
Geometry vs Topology

- Geometry
  - deals with shapes and relative **positions** and **sizes** of figures, and properties of space such as **curvature**.
- Topology
  - studies the properties of space that are preserved under continuous deformations, this means stretching and bending but not cutting or gluing.

(Topology) Definition

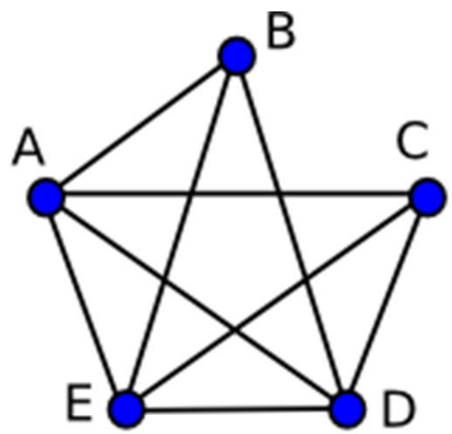
- An **embedding** of an *abstract simplicial complex* is how you draw it out in a d-dimension space



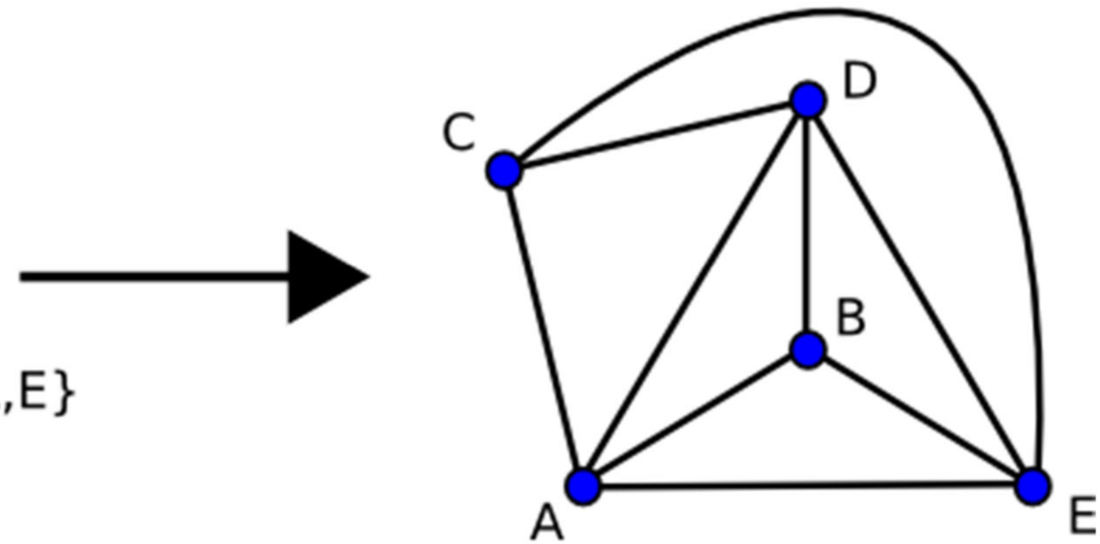


- A graph is **planar** if there **exists** an embedding in a plane

Embedding



A: {A,C}, {A,D}, {A,B}, {A,E}
B: {B,A}, {B,D}, {B,E}
C: {C,A}, {C,E}, {C,D}
D: {D,C}, {D,E}, {D,B}, {D,A}
E: {E,A}, {E,B}, {E,D}, {E,C}



- A straight line embedding is an embedding with all straight lines as edges
  - Exercise: Re-draw the graph above so that we have a true straight line embedding especially with edge {C, E}, illustration on https://visualgo.net/en/graphds

About “Exist”

- Three good friends, an engineer, a mathematician and a computer scientist, are driving on a highway that is in the middle of no where. Suddenly one of the tires went flat and they have no spare tire.

Maths vs CS vs Engineering

- Engineer

  “Let’s use bubble gum to patch the tire and use the strew to inflate it again”

- Computer Scientist

  “Let’s remove the tire, put it back, and see if it can fix itself again”

- Mathematician

  “I can prove that there is a good tire **<u>exists</u>** in somewhere this continent”

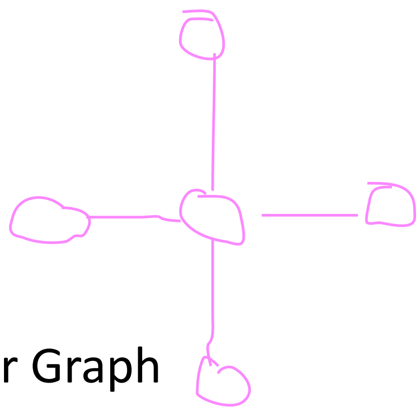
Planar Graph

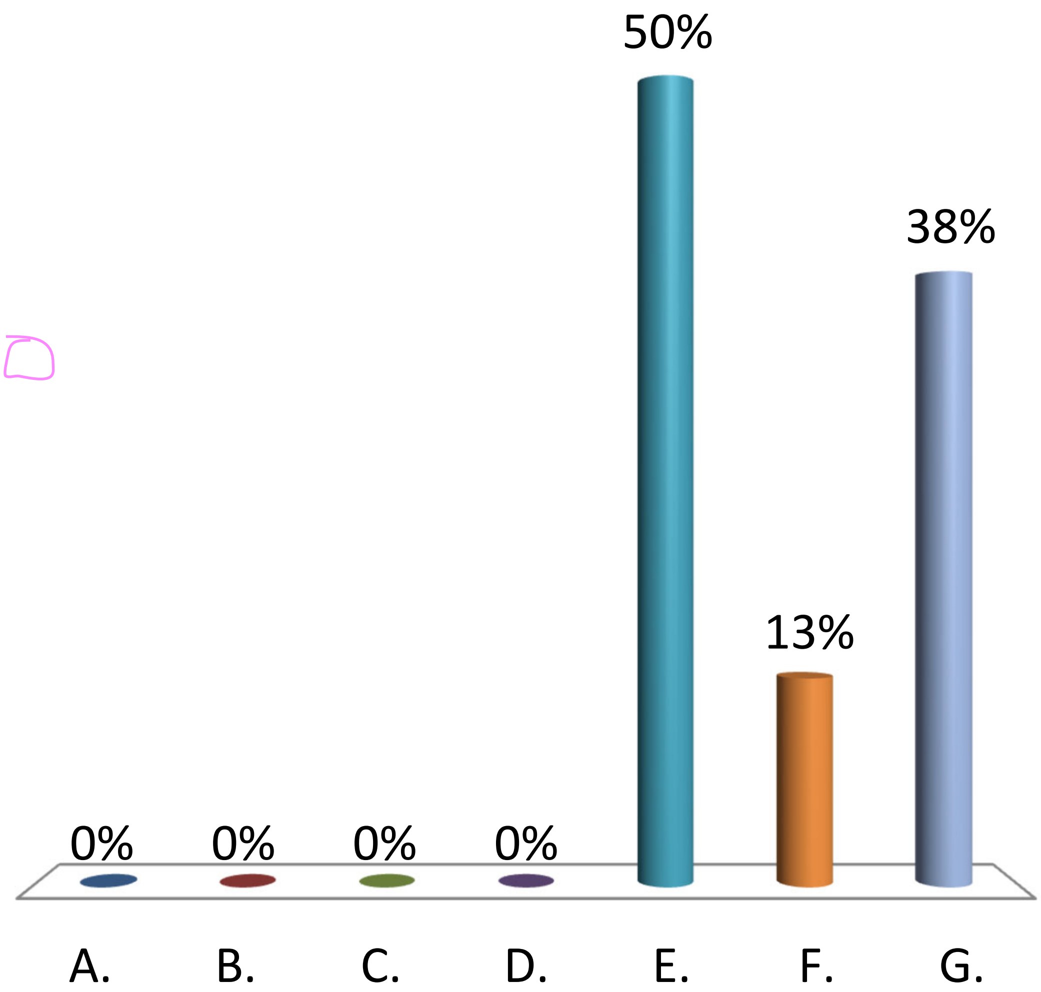
- Topologyists say,
  - If there exists an embedding for a graph, it's planar
  - If a graph is planar, $V - E + F = 1 + C$

- Geometrists say
  - If a graph is planar, how do I draw it on a plane?

What is the max degree of a planar graph?

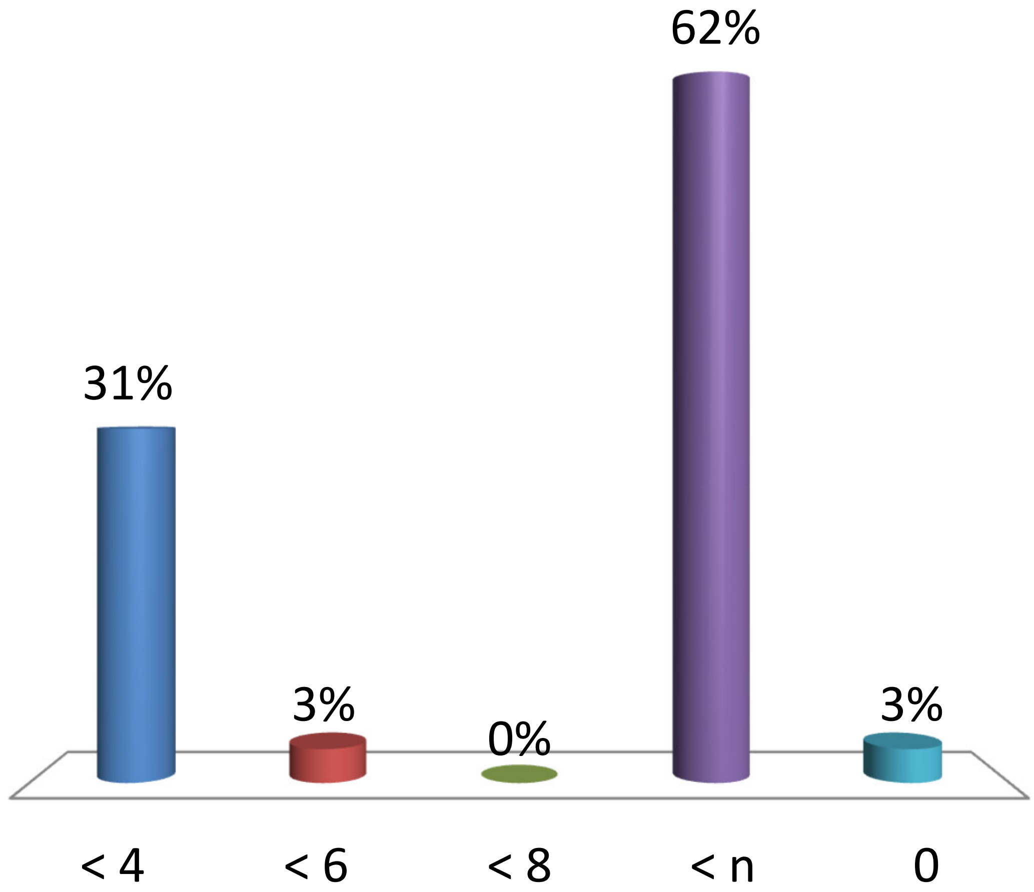
A. 2
B. 4
C. 6
D. 8
E. V -1 ✓ e.g., a Star Graph
F. O(V^2)
G. To infinity and beyond!





What is the average degree of a planar graph?

A. < 4
B. < 6
C. < 8
D. < n
E. 0



Before drawing, let’s prove this

- The average degree of a node in a planar graph is less than 6

- Assuming maximally connected
  - Namely, a planar graph with the max. no. of edges
  - This implies that every face is a triangle

Average degree of a node in a planar graph is less than 6

- $V - E + F = 1 + C$
- $C = 1$, assuming a connected planar graph, thus, $V - E + F = 2$
- Every face has at least 3 edges and every edge touches exactly 2 faces, i.e., $3F \leq 2E$ or $F \leq \frac{2E}{3}$, thus, $V - E + \frac{2E}{3} \geq 2$
- Thus, $E \leq 3V - 6$, so planar graph is a sparse graph
- Average degree = $\frac{2E}{V} \leq \frac{2(3V-6)}{V} \leq \frac{6V-12}{V} \leq 6 - \frac{12}{V} < 6$

Algorithm to draw a planar graph

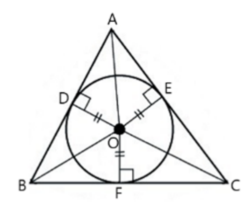
- Assuming G is maximally connected
- Repeat while G has more than 3 vertices
  - There is a vertex u in G with degree k_u < 6
    - i.e., 3, 4, or 5
  - G := G – $\{u\}$
  - If the degree of u is more than 3, add an artificial edge in G such that G is maximally connected
  - Push u onto a stack S, together with
    - the k_u neighbors of u
    - the artificial edges added

Algorithm to draw a planar graph

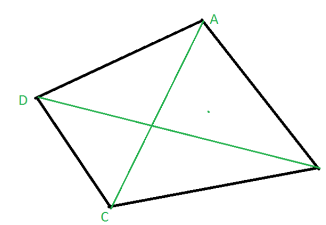
- G has three vertices as a triangle, just draw it
- Repeat while S is not empty
  - Pop u from S
  - Remove the artificial edges added
  - Draw u into the
    - Triangle, or
    - Quadrilateral, or
    - Pentagon

Can we always insert u without creating an intersection?

- Triangle
  - Trivial, put u anywhere inside the triangle
    - For a better aesthetic, maybe put u in the center of incircle of the triangle



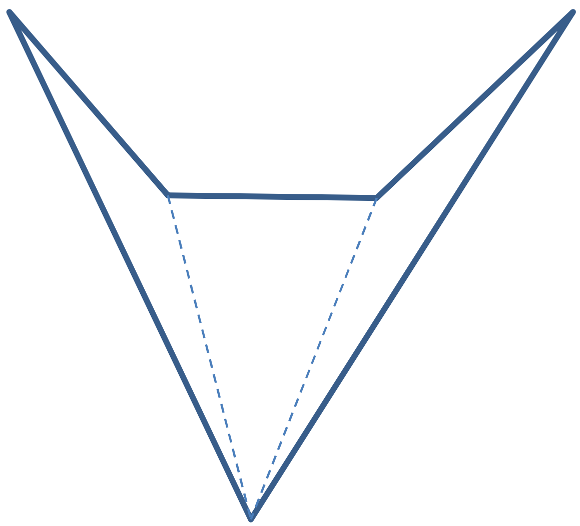
- Quadrilateral
  - Draw u on the intersection of the two diagonals



- Pentagon
  - Proof?

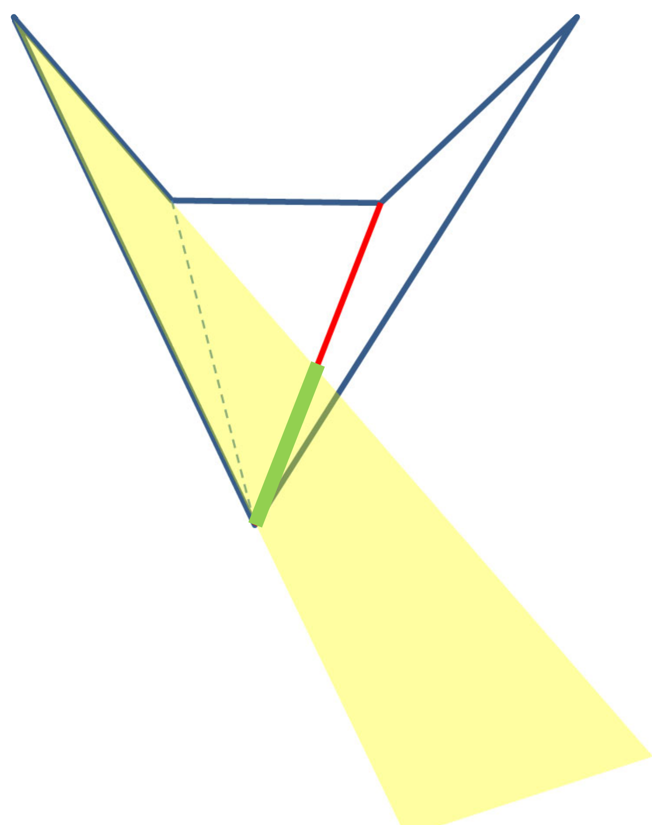
Constructive Proof (1)

- We can always divide a pentagon into three triangles



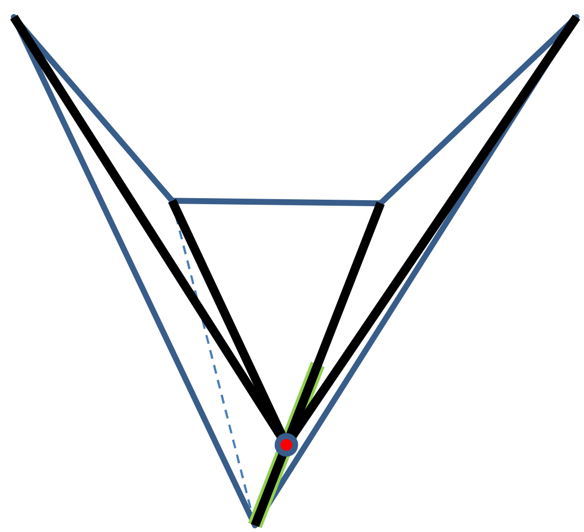
Constructive Proof (2)

- On one of the division line, you can project the other triangle onto it



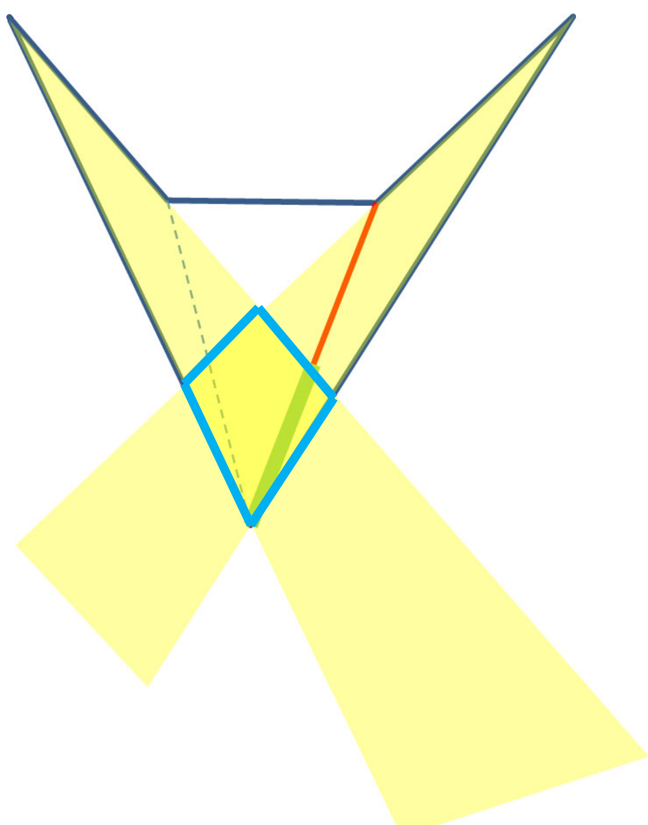
Constructive Proof (3)

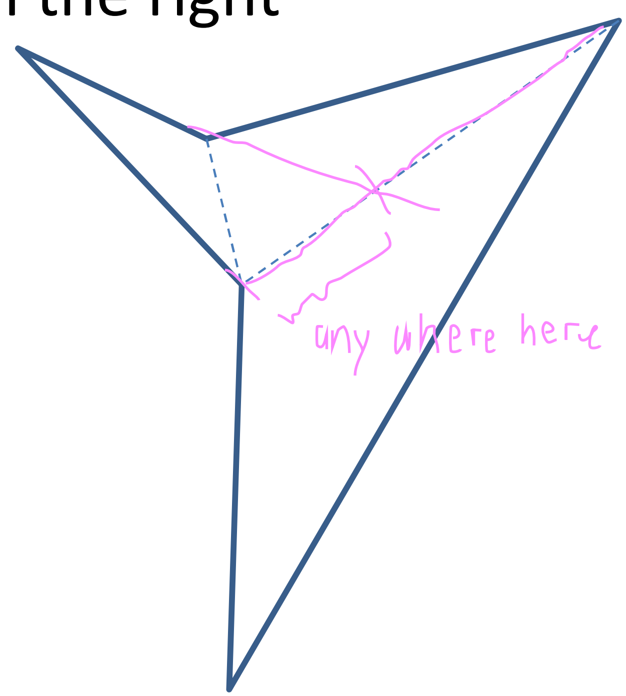
- And you can add a new point on this area and connect all the vertices of the pentagon without crossing



Constructive Proof (4)

- In fact, any point inside this blue region can see all 5 vertices via a straight line (another big Computational Geometry: the Art Gallery Problem)
- Exercise: Do the same on the figure on the right





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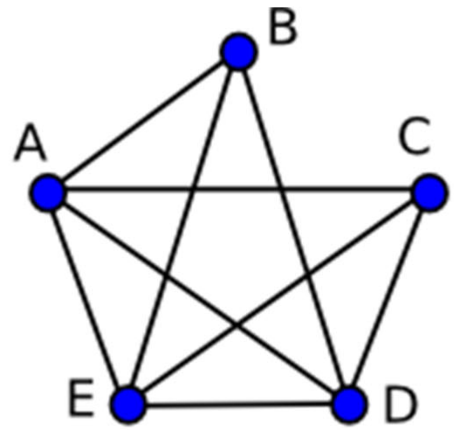
Example



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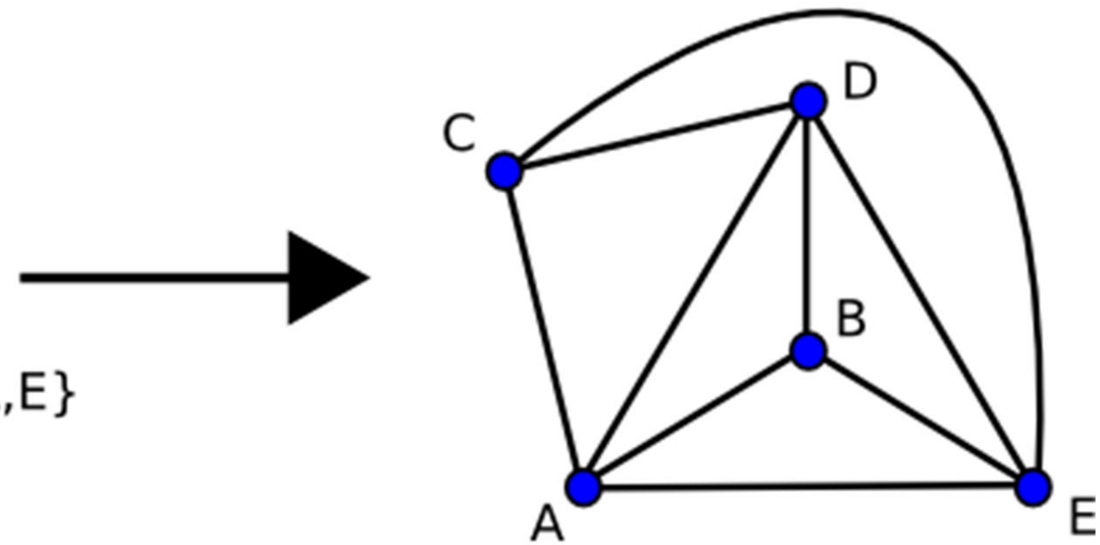


Example

- Remove A
  - Degree of A = 4
  - A's neighbor: B C D E
  - Add an artificial edge
    - E.g. C B

A → C D B E
B → A D E
C → A D E
D → C E B A
E → A B D C

- Stack:

Node	Neighbors	Artificial edge
A	B C D E	C B

Example

- Remove B
  - Degree of B = 3
  - B's neighbor: C D E

B → C D E
C → B D E
D → C E B
E → B D C

- Stack:

Node	Neighbors	Artificial edge
B	C D E	
A	B C D E	C B

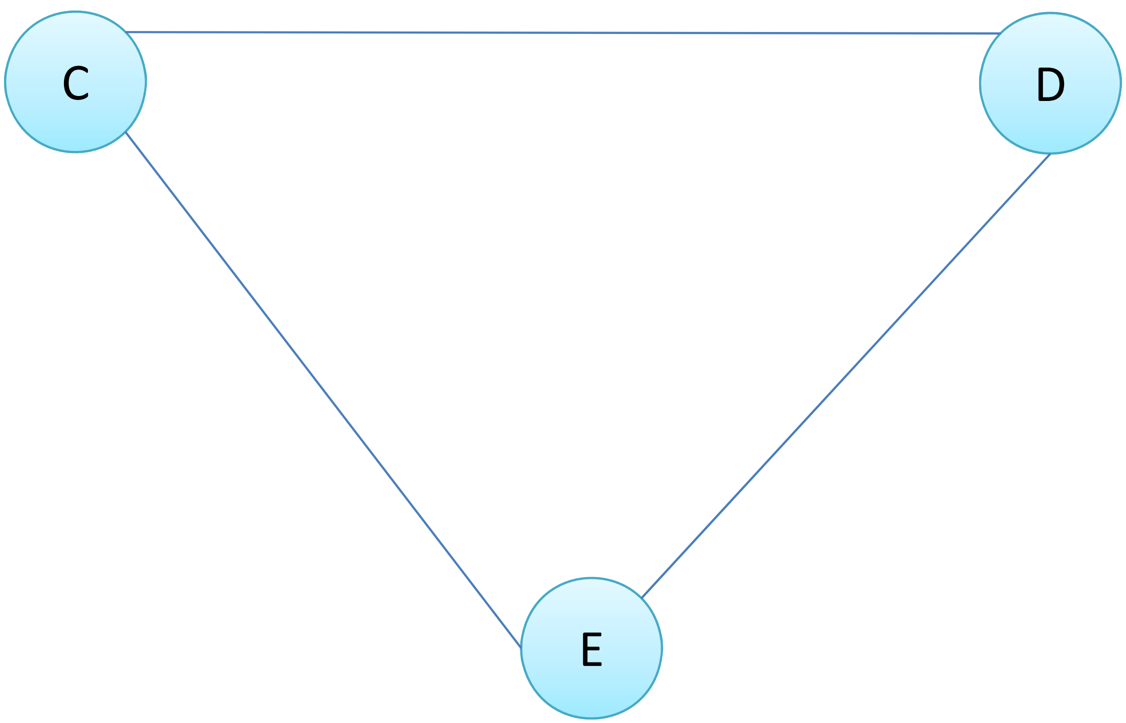
Example

- Three vertices left
- Just draw it

C → D E

D → C E

E → D C



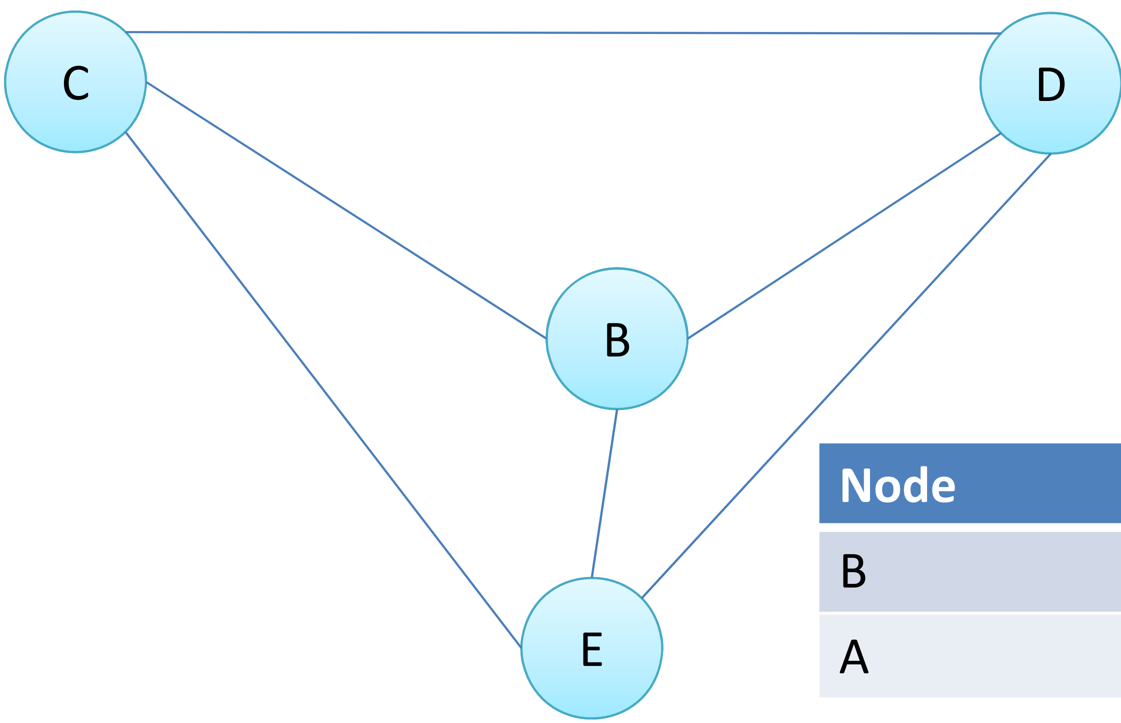
Example

- Pop B
  - B's neighbor: C D E

C → D E

D → C E

E → D C



Node	Neighbors	Artificial edge
B	C D E	
A	B C D E	C B

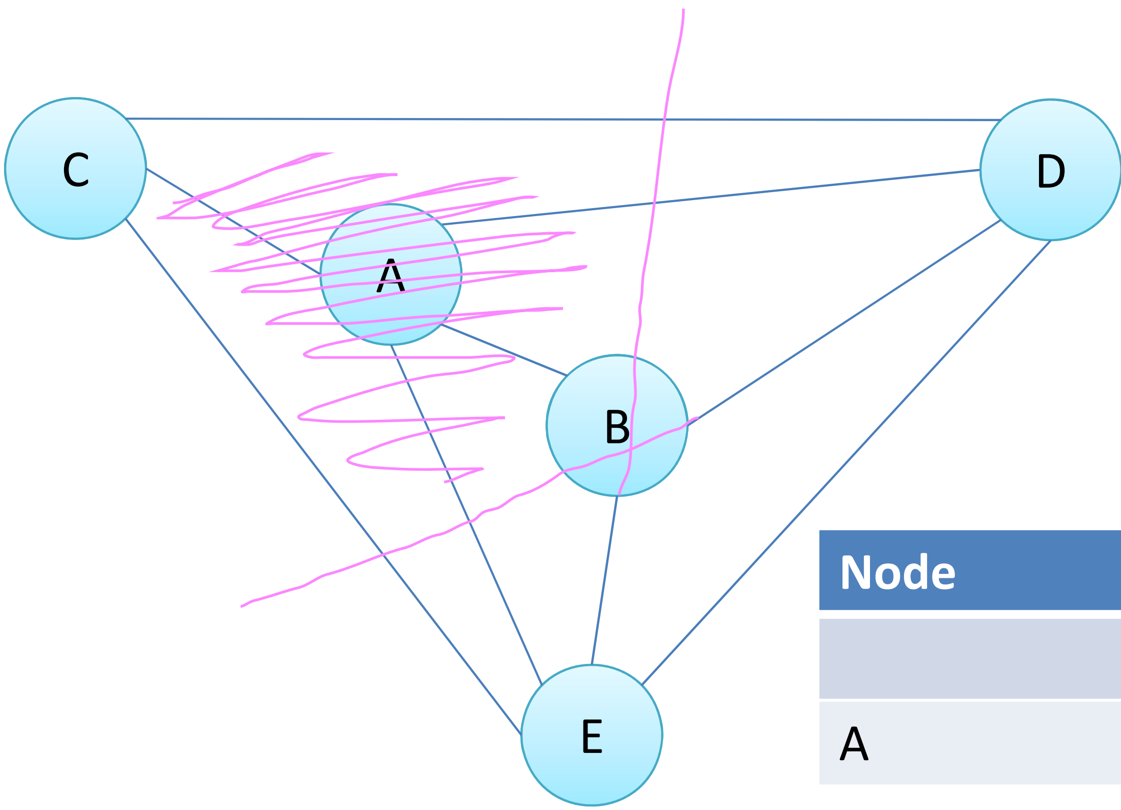
Example

- Pop A
  - Remove artificial edge CB
  - A's neighbor: B C D E

C → D E

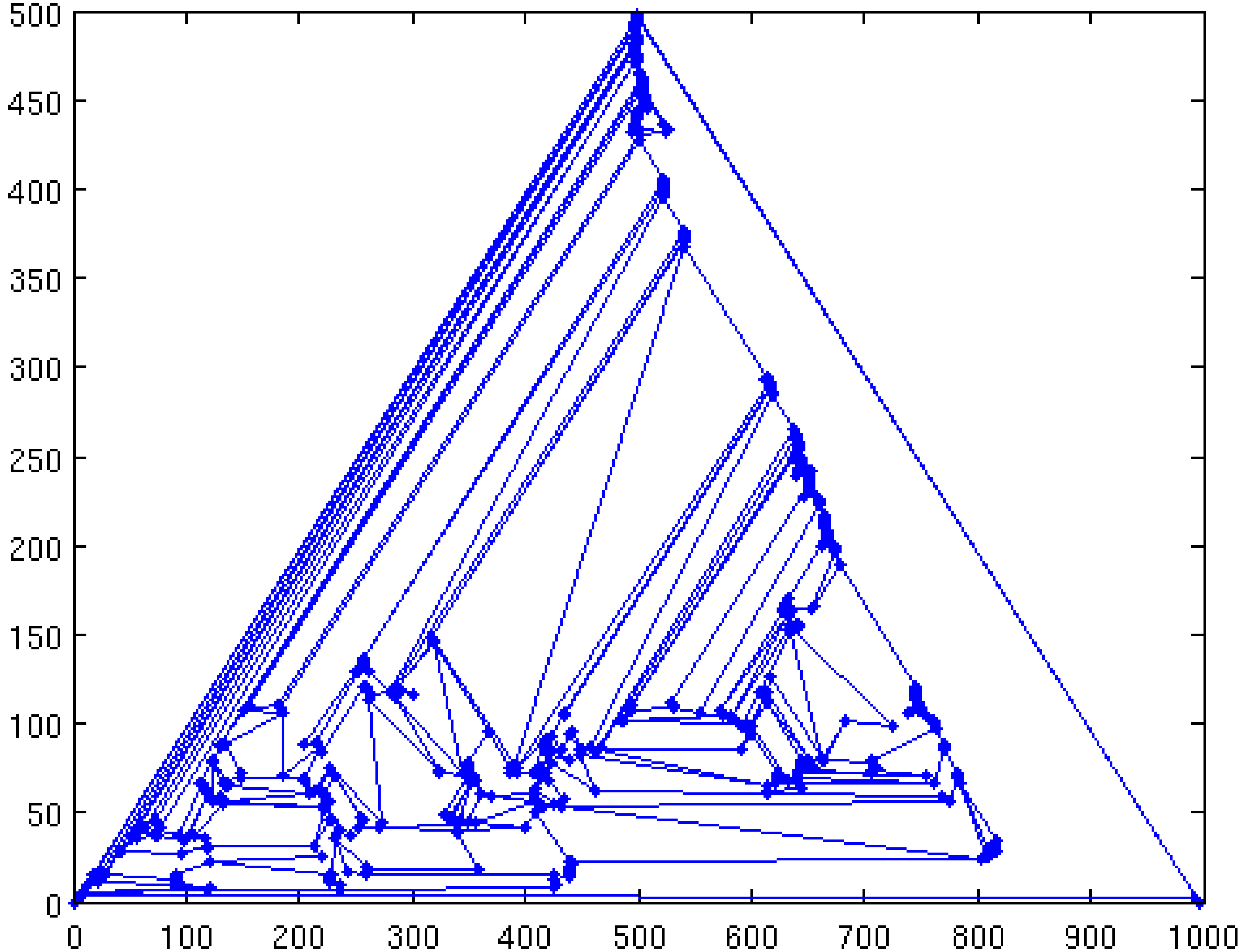
D → C E

E → D C



Node	Neighbors	Artificial edge
A	B C D E	C B

But in general, not so good looking

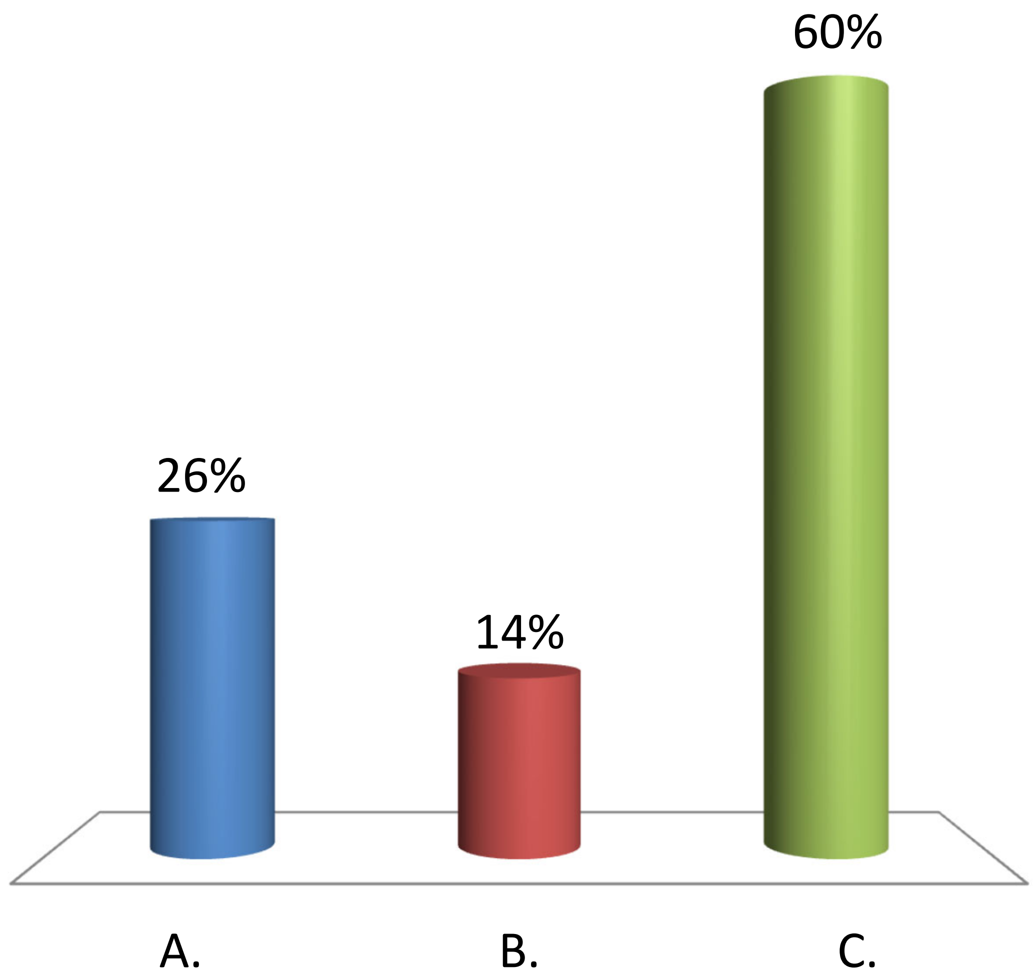


Can I perform vertex coloring as the same way?

- A vertex k-coloring on a graph colors the vertices with k different colors
  - And if two nodes share the same edge, the two nodes must have different colors

How can we do a 6-coloring on a planar graph?

A. Impossible
B. Too difficult, let me go home and think
C. Too easy, a piece of cake



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    - Pentagon
  - Just color u with a different color!

About 4-coloring for Planar Graph

- Given a planar Graph
- Can you color each vertex with four colors only provided that each neighbor has a different color?

History

- 1852 when Francis Guthrie, while trying to color the map of counties of England . Conjecture appeared in a letter from Augustus De Morgan
- 'Proof' by Kempe in 1879, Tait in 1880
  - Incorrectness was pointed out by Heawood in 1890
  - Petersen in 1891
- Confirmed by Appel and Haken in 1976
- Again by Robertson, Sanders, Seymour and Thomas (1996)

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A NEW PROOF OF THE FOUR-COLOUR THEOREM

NEIL ROBERTSON, DANIEL P. SANDERS, PAUL SEYMOUR, AND ROBIN THOMAS

(Communicated by Ronald Graham)

Abstract. The four-colour theorem, that every loopless planar graph admits a vertex-colouring with at most four different colours, was proved in 1976 by Appel and Haken, using a computer. Here we announce another proof, still using a computer, but simpler than Appel and Haken's in several respects.

for two reasons:

(i) part of the A&H proof uses a computer, and cannot be verified by hand, and
(ii) even the part of the proof that is supposed to be checked by hand is extraordinarily complicated and tedious, and as far as we know, no one has made a complete independent check of it.

graph admits a 4-colouring, that is, a mapping $c : V(G) \to \{0, 1, 2, 3\}$ such that $c(u) \neq c(v)$ for every edge of G with ends u and v. This was conjectured by F. Guthrie in 1852, and remained open until a proof was found by Appel and Haken [3], [4], [5] in 1976.

Unfortunately, the proof by Appel and Haken (briefly, A&H) has not been fully accepted. There has remained a certain amount of doubt about its validity, basically for two reasons:

(i) part of the A&H proof uses a computer, and cannot be verified by hand, and
(ii) even the part of the proof that is supposed to be checked by hand is extraordinarily complicated and tedious, and as far as we know, no one has made a complete independent check of it.