NATIONAL UNIVERSITY OF SINGAPORE

CS1231 - DISCRETE STRUCTURES

(Semester 1: AY2015/16)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your Student Number (Matriculation Number) only. **Do not write your name.**
2. This assessment paper contains **TWO (2)** parts and comprises **FOURTEEN (14)** printed pages, including this page.
3. Answer **ALL** questions.
4. This is an **OPEN BOOK** assessment.
5. You are allowed to use **NUS APPROVED CALCULATORS**.
6. You may use pen or pencil, but please erase cleanly, and write legibly.
7. Please write your Student Number below.

STUDENT NUMBER: ______________________

EXAMINERS' USE ONLY			
Part	MaxScore	Mark	Remark
A	30	OCR	OCR
Q16	12		
Q17	14		
Q18	12		
Q19	12		
Subtotal Q16-Q19	50		

Part A: (30 marks) MCQ. Answer on the OCR form.

For each multiple choice question, choose the best answer and **shade** the corresponding choice on the OCR form. Each multiple choice question is worth 2 marks. No mark is deducted for wrong answers. Shade your student number (check that it is correct!) on the OCR form as well. You should use a **2B pencil**.

Q1. Find $x \in \mathbb{Z}$ such that $4x \equiv 2 \pmod 6$.

A. 1/2

B. 2

C. 3

D. 1000

E. There is no solution.

Q2. Consider the relation $\equiv_{(\text{mod } 6)}$ defined on $\mathbb{Z}$. If $x^2 \in [27]$, then x could be

A. 7

B. 8

C. 9

D. 10

E. 11

Q3. Aaron wants to set multiple choice questions for the CS1231 examination. He intends to give every student the same questions, but have each student see the questions in a different sequence. If there are 333 students in the class, what is the least number of questions Aaron must set?

A. 5

B. 6

C. 9

D. 12

E. 333

Q4. Which of the following statements is **false**?

A. $\sim(p \rightarrow \sim q) \equiv p \wedge q$

B. $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$

C. $(p \rightarrow q) \rightarrow r \equiv (p \wedge \sim q) \vee r$

D. $(p \rightarrow q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

E. None of the above.

Q5. On the island of knights (who always tell the truth) and knaves (who always lie), you meet two natives A and B. A says: "I am a knave or B is a knight". What are A and B?

A. Both A and B are knights.

B. A is a knave and B is a knight.

C. A is a knight and B is a knave.

D. Both A and B are knaves.

E. Cannot be determined.

Q6. How many spanning trees does the complete graph K_4 have?

A. 4

B. 8

C. 15

D. 16

E. 20

Q7. Five friends go to a restaurant for dinner. The restaurant offers four types of main courses, and each person will have one. What is the number of possible orders the chef could get to see? (Example of an order: "Two fish-n-chips, one portobello steak, and two carbonara pastas".)

A. 20

B. 56

C. 120

D. 126

E. None of the above.

Q8. Suppose one urn contains 5 blue balls and 5 gray balls, and a second urn contains 3 blue balls and 7 gray balls. Both urns are equally likely to be chosen. You draw a ball at random from one of the two urns. If a blue ball is drawn, what is the probability that it comes from the first urn?

A. 5/8

B. 2/5

C. 2/3

D. 3/5

E. 1/2

Q9 to Q13 refer to the following definitions. Answer each question independently.

Define functions f, g and h as follows:

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad \forall x \in \mathbb{R},\ f(x) = x^2.$$
$$g : \mathbb{N} \longrightarrow \mathbb{N}, \quad \forall x \in \mathbb{N},\ g(x) = x^2.$$
$$h : A \longrightarrow B, \quad \forall x \in A,\ h(x) = x^2.$$
$$\text{where } A = \{0, 1, 2, 3, 4\},\ \text{ and } B = \{0, 1, 4, 9, 16\}.$$

Q9. Which function is one-to-one?

A. f only.

B. g only.

C. h only.

D. g and h only.

E. All of the functions.

Q10. Which function is onto?

A. f only.

B. g only.

C. h only.

D. g and h only.

E. f and h only.

Q11. Which function has a preimage of 3?

A. f only.

B. g only.

C. f and g only.

D. g and h only.

E. All of the functions.

Q12. Let $P : C \longrightarrow D$ be a function, and let $U \subseteq C$. We denote P_U to mean the restriction of P to U. That is, $P_U : U \longrightarrow D, \quad \forall x \in U,\ P_U(x) = P(x)$.

Which of the following statements is true?

A. $\exists U \subseteq \mathbb{R}$ such that $f_U = g$

B. $\exists U \subseteq \mathbb{N}$ such that $g_U = f$

C. $\exists U \subseteq A$ such that $h_U = g$

D. $\exists U \subseteq A$ such that $h_U = f$

E. None of the above.

Q13. Which of the following statements is true?

A. $f \circ f$ is onto.

B. $h \circ h$ is a relation.

C. g^{-1} is a function.

D. $g \circ g$ is one-to-one.

E. None of the above.

Q14 to Q15 refer to the following definitions. Answer each question independently.

> Let $A = \{1, 2, 3, 4\}$. Since each element of $\mathcal{P}(A \times A)$ is a subset of $A \times A$, it is a binary relation on A.

Q14. Assuming each relation in $\mathcal{P}(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is *reflexive*?

A. $1/2^2$

B. $1/2^4$

C. $1/2^6$

D. $1/2^{12}$

E. $1/2^{16}$

Q15. Assuming each relation in $\mathcal{P}(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is *symmetric*?

A. $1/2^2$

B. $1/2^4$

C. $1/2^6$

D. $1/2^{12}$

E. $1/2^{16}$

Part B: (50 marks) Structured questions. Write your answer in the space provided. Marks may be deducted for unnecessary statements in proofs.

Q16. (12 marks)

(a) Define $|x|$ to be the absolute value of an integer x, that is, if x is non-negative, $|x| = x$, else $|x| = -x$.

Given the set $S = \{-9, -6, -1, 3, 5, 8\}$, for each of the following statements state whether it is true or false, with explanation.

i. (2 marks) $\exists z \in S$ such that $\forall x, y \in S,\ z > |x - y|$.

ii. (2 marks) $\exists z \in S$ such that $\forall x, y \in S,\ z < |x - y|$.

Suppose we now treat S as a general finite set with $k > 1$ elements listed from smallest to largest.

iii. (2 marks) Describe a strategy to find the desired z, if it exists, in part i.

iv. (2 marks) Describe a strategy to find the desired z, if it exists, in part ii.

Solution:

i. The maximum $|x - y| = 17$, since the largest element in S is 8, the statement is false.

ii. The minimum $|x - y| = 2$. There are 3 elements smaller than 2, hence the statement is true.

iii. The maximum $|x - y]$ is the difference between the first and last element in S. Search for an element that is larger than this difference, starting from the last element (the largest) of the set.

iv. The minimum $|x - y]$ is the smallest among all $|a_i - a_{i+1}|$ in S for $1 \leq i < k$. Search for an element that is smaller than this difference, starting from the first element (the smallest) of the set.

(b) (4 marks) Consider the equation:

$$3x^2 - 4xy + y^2 = 0.$$

For each statement below, determine whether it is true or false. If the statement is true, prove it; otherwise, disprove it.

i. For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that the equation is true.

ii. There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, the equation is true.

Solution:

i. Statement is true.
Take any real number x. Let $y = 3x$.
Then $3x^2 - 4xy + y^2 = 3x^2 - 12x^2 + 9x^2 = 0$.

ii. Statement is false.
If $x = 0$, let $y = 1$, then $3x^2 - 4xy + y^2 = 1$.
If $x \neq 0$, let $y = 0$, then $3x^2 - 4xy + y^2 = 3x^2 > 0$.

Q17. (14 marks)

(a) (4 marks) You wish to select five persons from seven men and six women to form a committee that includes at least three men.

i. In how many ways can you form the committee?

ii. If you randomly choose five persons to form the committee, what is the probability that you will get a committee with at least three men?

Solution:

i. The committee has (three men and two women), or (four men and one woman) or (five men).
$\binom{7}{3}\binom{6}{2} + \binom{7}{4}\binom{6}{1} + \binom{7}{5} = 35 \times 15 + 35 \times 6 + 21 = 756.$

ii. $\binom{13}{5} = 1287$
Hence, probability $= 756/1287 = 58.74\%$.

(b) (5 marks) In an orientation game with girls and boys, chairs are placed in a row and pupils are to sit on them, one person on each chair. The rule, however, states that no two boys are allowed to sit next to each other.

Let n, a positive integer, denote the number of chairs, and let $f(n)$ be the number of ways the chairs can be filled. For example, let G denote a girl, and B a boy. Then for $n = 3$, the chairs may be filled with GGG, GGB, GBG, BGG, or BGB, for a total of 5 ways (ie. $f(3) = 5$).

i. Write a recurrence relation for $f(n)$, including initial conditions.
ii. How many ways are there to fill ten chairs?

Solution:

i.

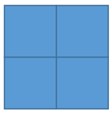
$$f(n) = \begin{cases} 2, & \text{if } n = 1. \\ 3, & \text{if } n = 2. \\ f(n-1) + f(n-2), & \text{if } n > 2. \end{cases} \tag{1}$$

ii. $f(10) = 144$

(c) (5 marks) Prove that among any five points selected inside a square with a side length of 2 units, there always exists a pair of these points that are within $\sqrt{2}$ units of each other.

(*Hint:* Apply the Pigeonhole Principle.)

Solution: Suppose we divide the square into four smaller square regions each of side length 1 unit:

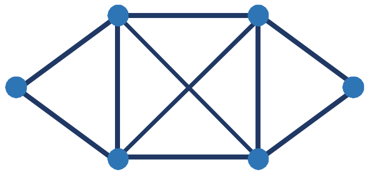


These smaller square regions are our "pigeonholes". Now, if one selects 5 points (our "pigeons") inside the larger square, the pigeonhole principle requires that 2 of these points be in at least one of the smaller squares. Since the diagonal of the smaller square is $\sqrt{2}$ units, we know there exists a pair of points out of the 5 points that will be within $\sqrt{2}$ units of each other.

Q18. (12 marks)

(a) (4 marks) Draw a graph with six vertices and ten edges, and where each vertex has a degree of two or four.

Solution:



(b) (4 marks) Let $E(K_n)$ denote the number of edges in a complete graph K_n of n vertices. Write an explicit formula for $E(K_n)$ and prove it.

Solution: $E(K_n) = n(n-1)/2$.
Prove by mathematical induction.

1. K_1 has no edges. Also, $E(K_1) = 1 \times 0/2 = 0$. This proves the base case.
2. Assume $E(K_i) = i(i-1)/2$, for $i \leq n$.
3. For K_{i+1}, there are additional i edges from the new vertex to each of the i vertices.
   $E(K_{i+1}) = E(K_i) + i = i(i-1)/2 + i = (i(i-1) + 2i)/2 = i^2 + i/2 =$
   $(i+1)i/2$.

(c) (4 marks) Prove the following:
A connected graph G has an Euler path if, and only if, exactly two vertices have odd degree.

Solution:

1. Forward direction:
   Suppose first that G has an Euler path starting at vertex v and ending at vertex w. Add a new edge to the graph with endpoints v and w, forming G'. Then G' has an Euler circuit, and by Theorem 10.2.2 (If a graph has an Euler circuit, then every vertex of the graph has positive even degree).

   The degrees of v and w in G are therefore odd, while all others are even.
2. Backward direction:
   Suppose the degrees of v and w in G are odd, while all other vertices have even degree. Add a new edge e to the graph with endpoints v and w, forming G'. Every vertex in G' has even degree now, so by Theorem 10.2.2 there is an Euler circuit which we can write as "$v, e_1, e_2, ..., w, e, v$" so that "$v, e_1, e_2, ..., w$" is an Euler path.

Q19. (12 marks)

(a) An integer sequence u_n is defined recursively as:

$$u_{n+1} = \begin{cases} 1, & \text{if } n = 0, \\ (n+1)^2\, u_n, & \text{if } n > 0. \end{cases}$$

i. (3 marks) Determine u_2, u_3 and u_4.

Solution:

$$u_2 = 2^2\, u_1 = 1^2 \cdot 2^2.$$
$$u_3 = 3^2\, u_2 = 1^2 \cdot 2^2 \cdot 3^2.$$
$$u_4 = 4^2\, u_3 = 1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2.$$

ii. (3 marks) Prove, by Mathematical Induction, that $u_n = (n!)^2$, for $n \in \mathbb{Z}^+$.

Solution:

Proof. (By Mathematical Induction)

1. Let $P(n) = (\ u_n = (n!)^2\ )$, for all $n \in \mathbb{Z}^+$.
2. By definition $u_1 = 1$.
3. Also, $(1!)^2 = 1$, so $P(1)$ is true.
4. For any $k \in \mathbb{Z}^+$:
   a. Assume $P(k)$ is true; that is, $u_k = (k!)^2$.
      i. Now, $u_{k+1} = (k+1)^2\, u_k$, by the recurrence relation.
      ii. Thus $u_{k+1} = (k+1)^2 \times (k!)^2$, by the Inductive hypothesis.
      iii. Thus $u_{k+1} = ((k+1)!)^2$, by basic algebra; so $P(k+1)$ is true.
5. Hence, by Mathematical Induction, the statement is true. ■

(b) Another integer sequence, s_n, is defined recursively as:

$$s_n = \begin{cases} 0, & \text{if } n = 0, \\ s_{n-1} + (-1)^n \, n^2, & \text{if } n > 0. \end{cases}$$

i. (3 marks) Determine s_1, s_2, s_3 and s_4.

Solution: $s_1 = -1, s_2 = 3, s_3 = -6, s_4 = 10$

ii. (3 marks) Find an explicit formula for s_n. (You do not need to prove the formula using Mathematical Induction; instead, just explain how you obtain it.)

Solution: Continue to determine a few more terms using the recurrence relation:

$$s_5 = -15, s_6 = 21, s_7 = -28$$

This pattern is reminiscent of the triangle numbers $\triangle_n$:

n	0	1	2	3	4	5	6	7
$\triangle_n$	0	1	3	6	10	15	21	28

From Section 5.3.4 of the lecture note on sequences, the formula for triangle numbers is: $\triangle_n = n(n+1)/2$ for all $n \in \mathbb{Z}^+$.

Hence $s_n = (-1)^n \, n(n+1)/2$.

END OF PAPER