**MA1521 Tutorial 11 Solutions**

Question 1.

(1a)

$$y' + (1+\frac{1}{x})y = \frac{1}{x}e^{-x}$$

Integrating factor is $\exp\int(1+\frac{1}{x}) = \exp(x+\ln x) = xe^{x}$ (in general, $y'+P(x)y \Rightarrow$ multiply by $\exp\int P \Rightarrow y'e^{\int P} + Pye^{\int P} = \frac{d}{dx}(ye^{\int P}))$

So $\frac{d}{dx}(yxe^{x}) = 1(= xe^{x} \times \frac{1}{x}e^{-x})$

$\Rightarrow yxe^{x} = x + c \Rightarrow y = e^{-x} + cx^{-1}e^{-x}$

(1b)

$$\exp\int -(1+\frac{3}{x}) = \exp(-x - 3\ln|x|) = \frac{1}{x^3}e^{-x}$$

$$\Rightarrow \frac{d}{dx}(y\frac{1}{x^3}e^{-x}) = (x+2)\frac{1}{x^3}e^{-x}$$

$$\frac{y}{x^3}e^{-x} = \int\frac{e^{-x}}{x^2} + 2\int\frac{e^{-x}}{x^3} + c$$

$$= \int\frac{e^{-x}}{x^2} + \frac{-e^{-x}}{x^2} - \int\frac{e^{-x}}{x^2} + c$$

$$= \frac{-e^{-x}}{x^2} + c$$

$y = -x + cx^3e^{x}$

since $y(1) = e - 1 = -1 + ce \Rightarrow c = 1$

$$y = -x + x^3e^{x}$$

(1c)

This kind is called a Bernoulli equation -- set

$z = y^2 \qquad z' = 2yy' \qquad y' = z'/2y$

$$\frac{z'}{2y} + y + \frac{x}{y} = 0 \Rightarrow \frac{1}{2}z' + z + x = 0 \Rightarrow z' + 2z = -2x \Rightarrow \frac{d}{dx}(e^{2x}z) = -2xe^{+2x}$$

$$\Rightarrow ze^{2x} = (-x + \frac{1}{2})e^{2x} + c \Rightarrow y^2 = \frac{1}{2} - x + ce^{-2x}$$

(1d)

Since $2yy' = (y^2)'$ we define $Y = y^2$, $Y' + (1 - \frac{1}{x})Y = xe^x$, $\exp\int(1 - \frac{1}{x}) = \frac{1}{x}e^x$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}e^x Y\right) = e^{2x} \Rightarrow \frac{1}{x}e^x Y = \frac{1}{2}e^{2x} + c$$

$$\Rightarrow y^2 = \frac{1}{2}xe^x + cxe^{-x}$$

Question 2.

The constant C has units of 1/time. Solving the equation [either as linear first order or as a separable equation] we get

$$P = M - Me^{-Ct}$$

Clearly P will approach M more rapidly if C is large; that is, C measures how rapidly the student is able to learn. Thus the equation indeed expresses the idea that the student's performance improves more slowly as she approaches **her** maximum possible performance.

Question 3.

The constant K measures the rapidity with which the rumour will spread. It depends on how interesting the rumour is, how much the people like to gossip, etc. The right hand side of the equation is designed to be small both near R = 1 and near R = 1300, when indeed the rumour can be expected to spread slowly either because not enough or too many people have heard it.

We have

$$\frac{dR}{dt} - 1300KR = -KR^2.$$

This is a Bernoulli equation as discussed in the notes. We solve it by defining Z = 1/R, which transforms the equation into a linear one:

$$\frac{dZ}{dt} + 1300KZ = K,$$

with solution

$$\frac{1}{R} = \frac{1}{1300} + C\exp(-1300Kt).$$

The problem says that this highly interesting rumour was started by one person, so R(0) = 1. Thus C = 1299/1300. Hence

$$\frac{1}{R} = \frac{1}{1300} + \frac{1299}{1300}\exp(-1300Kt).$$

Of course as t tends to infinity, R tends to 1300.

Question 4.

Suppose we write the equation governing the Uranium as

$$\frac{dU}{dt} = -k_U U$$

where U represents the number of Uranium atoms, etc, as usual. The half-life of Uranium can be used to compute the decay rate constant, as in the lecture notes, and similarly for Thorium. From the lecture notes we have

$$\frac{T}{U} = \frac{k_U}{k_T - k_U}[1 - \exp((k_U - k_T)t)]$$

and so if T/U is 0.1 we know everything in this equation except t. Solving for t, you should find that the answer is approximately 40 thousand years.

Question 5

Following the standard equations for the Malthus Model

$$
\begin{aligned}
&N = \hat{N}e^{kt}; N(0) = 10000 = \hat{N} \\
&N(2.5) = 10000e^{2.5k} = 11000 \\
&\Rightarrow e^{2.5k} = 1.1 \Rightarrow k = \frac{1}{2.5}\ell n(1.1) \\
&\qquad\qquad\qquad\quad = 0.0381 \\
&N(10) = 10000e^{10k} = 10000e^{10(0.0381)} \approx 14600 \\
&20000 = 10000e^{kt} \to t = \frac{1}{k}\ell n(2) \\
&\qquad\qquad\qquad\qquad\quad = 18.18 \text{ hours}
\end{aligned}
$$

Question 6.

Recall that the solution of the logistic equation is

$$
N = \frac{B}{s + \left(\frac{B}{\hat{N}} - s\right)e^{-Bt}} = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right)e^{-Bt}}
$$

Here $\hat{N} = 200$, $B = 1.5$, so at $t = 2$ we have

$$
\begin{aligned}
&360 = \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right)e^{-1.5\times 2}} \\
&\Rightarrow 360 + \frac{360}{200}e^{-3}N_\infty - 360e^{-3} = N_\infty \\
&N_\infty = \frac{360(1 - e^{-3})}{1 - \frac{360}{200}e^{-3}} \approx 376 \\
&N(3) = \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right)e^{-4.5}} \approx 372
\end{aligned}
$$