NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 1: AY2020/21)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY THREE (23)** questions (excluding question 0) in **THREE (3)** parts and comprises **TWELVE (12)** printed pages.
2. Answer ALL questions.
3. This is an **OPEN BOOK** assessment.
4. The maximum mark of this assessment is 100.

Question	Max. mark
Q0	3
Part A: Q1 – 12	24
Part B: Q13 – 17	15
Part C: Q18	5
Part C: Q19	4
Part C: Q20	20
Part C: Q21	18
Part C: Q22	3
Part C: Q23	8
Total	**100**

5. You are to submit a single pdf file (size $\leq$ 20MB) to your submission folder on LumiNUS.
6. Your submitted file should be named after your Student Number (eg: A1234567X.pdf) and your Student Number should be written on the first page of your file.
7. <u>Do NOT write your name</u> anywhere in your submitted file.

——— END OF INSTRUCTIONS ———

You do not need to write any answer for question 0.

0. (a) Submit a **single pdf file** into the submission folder. [1 mark]

   (b) Name your pdf file with your Student Number (eg: A1234567X.pdf). [1 mark]

   (c) Write your Student Number (<u>not</u> your name!) on the first page of your file. [1 mark]

Part A: Multiple Choice Questions [Total: 12×2=24 marks]

Each multiple choice question (MCQ) is worth **TWO marks** and has exactly **one** correct answer. You are advised to write your answers on a **single line or two lines** to conserve space. For example:

1. A 2. B 3. C 4. D …

Please write in **CAPITAL LETTERS**.

1. Which of the following statements is/are true?

   (i) $0 \mid 1$.

   (ii) $1 \mid 0$.

   (iii) $-1 \underline{\text{div}}\ 12 = 0$.

   (iv) $-1 \underline{\text{div}}\ 12 = -1$.

   (v) $-1 \underline{\text{mod}}\ 12 = -1$.

   (vi) $-1 \underline{\text{mod}}\ 12 = 11$.

   A. Only (i), (iii) and (v) are true.
   B. Only (ii), (iii) and (v) are true.
   C. Only (i), (iv) and (vi) are true.
   D. Only (ii), (iv) and (vi) are true.
   E. None of A, B, C, D is correct.

2. Which of the following statements is/are true?

   (i) For every integer $n \geqslant 2$, there is a positive integer a such that a, $a+1$, $a+2$, …, $a+n$ are all composite numbers.

   (ii) For every integer $n \geqslant 2$, there is a positive integer a such that a, $a+1$, $a+2$, …, $a+n$ are all prime numbers.

   A. (i) and (ii) are both true.
   B. (i) is true but (ii) is false.
   C. (i) is false but (ii) is true.
   D. (i) and (ii) are both false.

3. Which of the following statements is/are true?

   (i) $\gcd(a, a+2) = 1$ for all integers a.

   (ii) $\gcd(a, b) = \gcd(a+b, a-b)$ for all integers a, b.

   A. (i) and (ii) are both true.
   B. (i) is true but (ii) is false.
   C. (i) is false but (ii) is true.
   D. (i) and (ii) are both false.

4. Which of the following statements is/are true?

   (i) There are $x, y, z \in \mathbb{Z}$ such that $10x + 15y - 35z = 2$.

   (ii) There are $x, y, z \in \mathbb{Z}$ such that $10x + 15y - 42z = 2$.

   A. (i) and (ii) are both true.
   B. (i) is true but (ii) is false.
   C. (i) is false but (ii) is true.
   D. (i) and (ii) are both false.

5. How many $n \in \mathbb{Z}_{\geqslant 2}$ satisfy the following sentence?

   For every $a \in \mathbb{Z}$, if $a \not\equiv 0 \pmod{n}$, then a has a multiplicative inverse modulo n.

   A. 0.
   B. 1.
   C. 2.
   D. Infinitely many.

6. Which of the following statements is/are true?

   (i) If $a, b \in \mathbb{Z}$ and $m, n \in \mathbb{Z}^+$ such that $a \equiv b \pmod{n}$, then $a \equiv b \pmod{mn}$.

   (ii) If $a, b \in \mathbb{Z}$ and $m, n \in \mathbb{Z}^+$ such that $a \equiv b \pmod{mn}$, then $a \equiv b \pmod{n}$.

   A. (i) and (ii) are both true.
   B. (i) is true but (ii) is false.
   C. (i) is false but (ii) is true.
   D. (i) and (ii) are both false.

7. Which of the following statements is/are true whenever $n \in \mathbb{Z}_{\geqslant 2020}$ and $R_1, R_2, \ldots, R_n$ are relations on $\mathbb{Z}$?

(i) If $\bigcap_{i=1}^{n} R_i$ is reflexive, then each R_i is reflexive.

(ii) If $\bigcap_{i=1}^{n} R_i$ is symmetric, then each R_i is symmetric.

(iii) If $\bigcap_{i=1}^{n} R_i$ is transitive, then each R_i is transitive.

A. (i) but not (ii) and not (iii).
B. (i) and (ii) but not (iii).
C. (i), (ii) and (iii).
D. None of (i), (ii), (iii).
E. None of A, B, C, D is correct.

8. Define a relation R on $\mathbb{Z}^+ \times \mathbb{Z}^+$ by setting, for all $a, b, c, d \in \mathbb{Z}^+$,

$$(a, b)\ R\ (c, d) \quad \Leftrightarrow \quad ab \leqslant cd.$$

Which of the following statements is true about R?

A. R is antisymmetric and transitive.
B. R is antisymmetric but not transitive.
C. R is transitive but not antisymmetric.
D. R is neither antisymmetric nor transitive.

9. Which of the following statements are true with respect to any partial order?

(i) If x is a minimal element, then x is a smallest element.

(ii) If x is a smallest element, then x is a minimal element.

(iii) If there is a minimal element, then there is a unique minimal element.

(iv) If there is a smallest element, then there is a unique smallest element.

A. (i) and (iii) but not (ii) and not (iv).
B. (ii) and (iv) but not (i) and not (iii).
C. (i) and (iv) but not (ii) and not (iii).
D. (ii) and (iii) but not (i) and not (iv).
E. None of A, B, C, D is correct.

10. A standard die has six sides showing the numbers 1 through 6. What is the expected sum for rolling three fair standard dice once?

A. 6
B. 7
C. 9
D. 10.5
E. 12

11. Let a, b, c, d, e, f be statement variables and the following statement is given:

$$(\sim b \wedge c \wedge d) \vee \big((\sim a \vee b \vee \sim e \vee f) \wedge c\big) \vee \sim(d \wedge e) \vee c$$

If each statement variable is randomly assigned a value of true or false, what is the probability that the above statement evaluates to **false**?

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. $\frac{1}{8}$

D. $\frac{1}{16}$

E. $\frac{1}{32}$

12. The adjacency matrix A for a directed graph $G = (V, E)$ where $V = \{1,2,3,4\}$ is given below. Rows are numbered 1 through 4 from top to bottom, columns numbered 1 through 4 from left to right. An element A_{ij} is 1 when there is an edge from vertex i to vertex j, and zero when there is no edge from vertex i to vertex j.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

How many **walks of length 3** are there from vertex 1 to vertex 4?

A. 0

B. 4

C. 6

D. 7

E. 8

Part B: Multiple Response Questions [Total: 5×3=15 marks]

Each multiple response question (MRQ) is worth **THREE marks** and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, B, C are the correct answers, write A, B, C.

Only if you get all the answers correct will you be awarded three marks. **No partial credit will be given for partially correct answers.**

You are advised to write your answers on **a single line** to conserve space. For example:

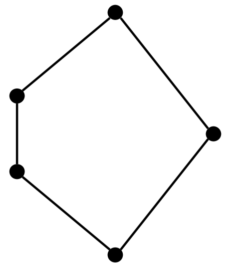
13. A,B 14. B,D 15. C 16. A,B,C,D 17. B,D,C

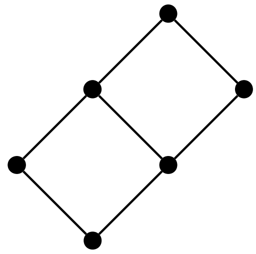
Please write in **CAPITAL LETTERS**.

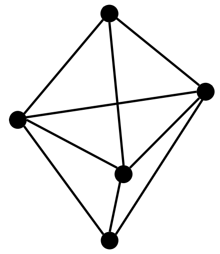
13. Which of the following is/are true for all predicates $P(x)$ on a nonempty set D?
    - A. $\forall x \in D\ (P(x)) \rightarrow \exists x \in D\ (P(x))$.
    - B. $\left(\exists x \in D\ (P(x)) \rightarrow \forall x \in D\ (P(x))\right)$ if $|D| = 1$.
    - C. $\left(\forall x \in D\ (P(x)) \rightarrow \forall x \in E\ (P(x))\right)$ if $E \subseteq D$.
    - D. $\left(\exists x \in D\ (P(x)) \rightarrow \exists x \in E\ (P(x))\right)$ if $E \subseteq D$.

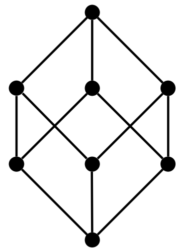
14. Which of the following is/are true?
    - A. For all $a, b, c, d, n \in \mathbb{Z}_{\geqslant 2}$, if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$.
    - B. For all $a, b, c, d, n \in \mathbb{Z}_{\geqslant 2}$, if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.
    - C. For all $a, b, c, d, n \in \mathbb{Z}_{\geqslant 2}$, if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a^c \equiv b^d \pmod{n}$.
    - D. For all $a, b, c, d, n \in \mathbb{Z}_{\geqslant 2}$, if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $\gcd(a, c) \equiv \gcd(b, d) \pmod{n}$.

15. For each $n \in \mathbb{Z}^+$, let $D_n = \{d \in \mathbb{Z}^+ : d \mid n\}$. Which of the following can be a Hasse diagram for D_n with respect to the partial order “divides” for some $n \in \mathbb{Z}^+$?

A. 

B. 

C. 

D. 

16. With respect to the partial order “divides” on the set $A = \{2,4,6,18,21,36,42,180\}$, which of the following is/are true?

A. 180 is a largest element.

B. 180 is a maximal element.

C. 42 is a largest element.

D. 42 is a maximal element.

17. Which of the following is/are true of the partial order “divides” on the set $A = \{2,4,6,18,21,36,42,180\}$ in Question 16?

A. It has a linearization $\preccurlyeq^*$ such that $4 \preccurlyeq^* 18$.

B. It has a linearization $\preccurlyeq^*$ such that $18 \preccurlyeq^* 4$.

C. It has a linearization $\preccurlyeq^*$ such that $6 \preccurlyeq^* 36$.

D. It has a linearization $\preccurlyeq^*$ such that $36 \preccurlyeq^* 6$.

Part C: There are 6 questions in this part [Total: 58 marks]

18. [Total: 5 marks]

Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$. Define a relation R on A by setting, for all $(a, b), (c, d) \in A$,

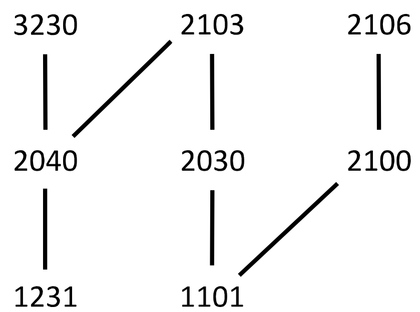
$$(a, b)\ R\ (c, d) \quad \Leftrightarrow \quad ab = cd.$$

(a) Prove that R is an equivalence relation on A. [3 marks]

(b) Write down the equivalence classes $[(1,1)]$ and $[(4,3)]$ in roster notation. [2 marks]

19. [Total: 4 marks]

Consider the partial order R on $M = \{1101, 1231, 2030, 2040, 2100, 2103, 2106, 3230\}$ with the following Hasse diagram.



(a) Write down all partitions $\mathcal{C}_0$ of M with at most 3 components of which all the components are *chains* with respect to R, i.e.,

$$\forall S \in \mathcal{C}_0\ \ \forall x, y \in S\ \ (x\ R\ y\ \text{ or }\ y\ R\ x).$$

(b) Write down all partitions $\mathcal{C}_1$ of M with at most 3 components of which all the components are *antichains* with respect to R, i.e.,

$$\forall S \in \mathcal{C}_1\ \ \forall x, y \in S\ \sim(x\ R\ y\ \text{ or }\ y\ R\ x).$$

No justification is needed for this question.

20. Counting and Probability [Total: 20 marks]

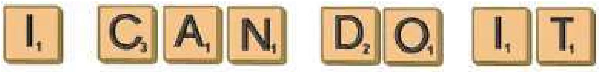
(a) How many solutions for x, y and z does the following equation have, given that $x, y, z \in \mathbb{Z}_{\geq 3}$?

$$x + y + z = 88.$$

Write your answer as a single number. Working is not required for this question.

[3 marks]

(b) Out of 8 Scrabble® tiles 'I', 'C', 'A', 'N', 'D', 'O', 'I', and 'T',



(i) how many ways can you select 4 tiles so that there are duplicate letters?

(ii) how many ways can you select 4 tiles so that there are no duplicate letters?

For example, the selections ('I', 'C', 'A', 'N') and ('C', 'N', 'I', 'A') are considered the same.

Write your answers as single numbers. Working is not required for this question.

[3 marks]

(c) The binomial expansion of $(a+b)^n$ for any real numbers a and b and non-negative integer n is shown below:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{r} a^{n-r} b^r + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n.$$

The first term of the expansion is a^n, the second term is $\binom{n}{1} a^{n-1} b^1$, etc.

For what value of x does the fifth term in the binomial expansion of the following formula equal $\mathbf{105}$?

$$\left(\frac{1}{2\sqrt{x}} - \frac{1}{2}\right)^{10}$$

Write your answer as a fraction. Working is not required for this question.

[3 marks]

(d) In this question, we assume that there are no parallel edges in a graph.

(i) How many directed graphs on three vertices a, b, c are there?

(ii) A loop is an edge that connects a vertex to itself. How many directed graphs on three vertices a, b, c with at least a loop are there?

Write your answers as single numbers. Working is not required. [3 marks]

20. (continue…)

(e) In 1995, Mars© introduced the blue M&M's to replace the tan M&M's. Before that, the colour mix in a bag of plain M&M's was 30% brown, 20% yellow, 20% red, 10% green, 10% orange and 10% tan. After the replacement, the colour mix was 24% blue, 16% green, 20% orange, 14% yellow, 13% red and 13% brown.

Your friend Aiken, being a big fan of M&M's, has two bags of plain M&M's, one from 1994 and the other from 1996. He won't tell you which is which, but he gives you one M&M from each bag. One is yellow and one is green. What is the probability that the **yellow M&M came from the 1994 bag,** assuming the M&M's are picked randomly?

You may express your answer as a fraction or a decimal number correct to 3 significant digits. [4 marks]

(f) 21 students took an exam and their scores sum to 200. If the scores are non-negative integers, prove that there are two students with the same score. [4 marks]

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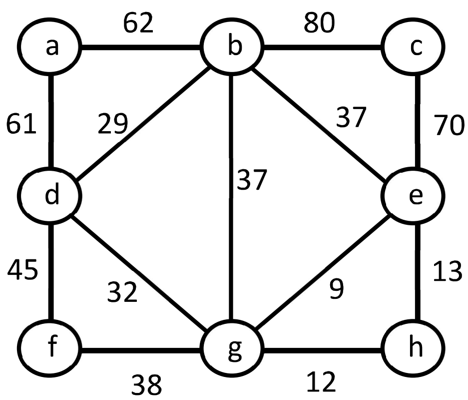
21. Graphs and Trees [Total: 18 marks]

(a) An undirected graph is given below, where the value on each edge indicates the weight of that edge:



Write out the weight of a **minimum spanning tree (MST)** of the graph and list out all the edges in the MST, in non-decreasing order of the weights of the edges. An undirected edge connecting vertices u and v is denoted by $\{u, v\}$. Do not draw any diagram in your answer. [3 marks]

21. (continue…)

(b) A simple graph is a graph with no loops and parallel edges. Given a simple undirected graph $G = (V, E)$, the **complement graph** of G is a simple undirected graph $\bar{G} = (V, \bar{E})$

where $\bar{E} = \{\{u, v\}: u, v \in V \wedge u \neq v \wedge \{u, v\} \notin E\}$.

A graph G is **self-complementary** if and only if G and $\bar{G}$ are isomorphic.

Suppose a self-complementary simple undirected graph G has n vertices. How many edges does G have? Working is not required for this question. [2 marks]

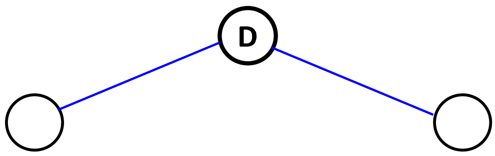
(c) Given the definition of a self-complementary graph in part (b) above, is there a self-complementary simple undirected graph with $4k + 2$ vertices, where $k \in \mathbb{Z}^+$? Explain your answer. [4 marks]

(d) The **in-order traversal** and **post-order traversal** of a binary tree with 11 vertices are given below:

In-order: **U C A N D O I T Y E H**

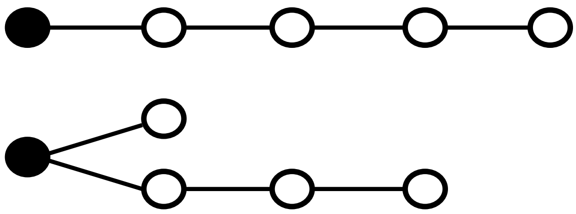
Post-order: **C U N A T Y I H E O D**

Draw the binary tree (clearly!). The root of the tree is **D** as shown below. [4 marks]



(e) For two rooted trees to be isomorphic under a permutation π, the image $\pi(r)$ of a root r must also be a root.

Draw (clearly!) a list of rooted trees on 5 vertices such that every rooted tree on 5 vertices is isomorphic to exactly one in your list. Draw the roots as solid black circles and the rest of the vertices as hollow circles. Two such trees are already shown below and you do not need to re-draw them in your answers. [5 marks]



22. [Total: 3 marks]

Write down a positive integer n (in the usual base-10 representation) and digits x, y, z such that the base-9 representation of n is $(xyz)_9$ and the base-6 representation of n is $(zyx)_6$. (Note that x and z must both be nonzero here.) No working is required for this question.

23. [Total: 8 marks]

For each $n \in \mathbb{Z}_{\geqslant 2}$, define an equivalence relation R_n on the set $A_n = \{1, 2, \ldots, n-1\}$ by setting, for all $a, b \in A_n$,

$$a \; R_n \; b \qquad \Leftrightarrow \qquad \forall k \in \mathbb{Z}^+ \left(a^k \equiv 0 \pmod{n} \;\Leftrightarrow\; b^k \equiv 0 \pmod{n} \right).$$

(a) Is it true that for all $n \in \mathbb{Z}_{\geqslant 2}$, if there is exactly one equivalence class with respect to R_n, then n is prime? If your answer is "yes", then give a proof. If your answer is "no", then give a counterexample.

(b) Is it true that for all $n \in \mathbb{Z}_{\geqslant 2}$, if n is prime, then there is exactly one equivalence class with respect to R_n? If your answer is "yes", then give a proof. If your answer is "no", then give a counterexample.

Explicit references to Lemma 8.6.5 and Proposition 8.6.13 may be omitted in your proof(s). You may use the following basic fact without proof.

Proposition P. Let p be a prime number and $a \in \{0, 1, 2, \ldots, p-1\}$. Then the following are equivalent.

(i) $\gcd(a, p) = 1$.
(ii) $a \neq 0$.
(iii) $a \not\equiv 0 \pmod{n}$.

=== END OF PAPER ===