MA1521 Finals Cheatsheet v1.2 (2017-12-03)
by Julius Putra Tanu Setiaji, page 1 of 2

1 Trigo Formulae

- $\sin^2\theta+\cos^2\theta=1$, $\sin 2\theta=2\sin\theta\cos\theta$
- $\sin(A\pm B)=\sin A\cos B\pm\sin B\cos A$
- $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$
- $\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$, $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$
- $\cos 2\theta=cos^2\theta-sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$
- $\sin P+\sin Q=2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$
- $\sin P-\sin Q=2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$
- $\cos P+\cos Q=2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$
- $\cos P-\cos Q=-2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$
- $a^2=b^2+c^2-2bc\cos\theta$ and $\frac{a}{\sin a}=\frac{b}{\sin b}$

2 Functions and Limits

Existence of Limits

$\lim_{x\to a} f(x)$ only exists when:

- $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)$ (limit from left = right)
- For $a=\infty$ or $-\infty$, only if $f(x)$ **does not** oscillate

Rules of Limits

1. $\lim_{x\to a}(f\pm g)(x)=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$
2. $\lim_{x\to a} f(x)g(x)=\lim_{x\to a} f(x)\lim_{x\to a} g(x)$
3. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x)\neq 0$
4. $\lim_{x\to a} kf(x)=k\lim_{x\to a} f(x)$

Continuity

f is continuous at point $a\Leftrightarrow\lim_{x\to a} f(x)=f(a)$

L'Hôpital's Rule

Suppose:

1. f and g are differentiable
2. $f(a)=g(a)=0$
3. $g'(x)\neq 0$ for all $x\in I\setminus a$

Then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

- Use L'Hôpital's Rule for $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms.
- Common: $\lim_{x\to\frac{\pi}{2}^-}(\sin x)^{\tan x}=\lim_{x\to\frac{\pi}{2}^-} e^{\ln(\sin x)^{\tan x}}$
  $=e^{\lim_{x\to\frac{\pi}{2}^-}\tan x\ln(\sin x)}=e^{\lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\sin x)}{\cot x}}$ (now in $\frac{0}{0}$ form)
- Tips:
  1. Convert $0\cdot\infty,\infty-\infty$ by algebra manip
  2. Convert $1^\infty,\infty^0,0^0$ by first taking ln

3 Derivative

The derivative of f at point a is $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, denoted by $f'(a)$ provided the limit exists.

$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\left.\frac{dy}{dx}\right|_{x=a}$

$f'(a)=$ slope of tangent at pt a

Some properties

- $f'(a)$ exists $\Rightarrow f(x)$ is smooth ($\therefore$ continuous) at a
- $f'(a)$ does not exist at **discontinuity, corner,** and **vertical tangent.**

Since derivative is limit, if lim from left $\neq$ right, then $f'(a)$ does not exist.

Formulae

Function	Derivative
$(f(x))^n$	$nf'(x)f(x)^{n-1}$
$\sin f(x)$	$f'(x)\cos f(x)$
$\cos f(x)$	$-f'(x)\sin f(x)$
$\tan f(x)$	$f'(x)\sec^2 f(x)$
$\cot f(x)$	$-f'(x)\csc^2 f(x)$
$\sec f(x)$	$f'(x)\sec f(x)\tan f(x)$
$\csc f(x)$	$-f'(x)\csc f(x)\cot f(x)$
$a^{f}(x)$	$f'(x)a^{f(x)}\ln a$

Function	Derivative
k	0
$e^{f}(x)$	$f'(x)e^{f(x)}$
$\log_a f(x)$	$\frac{f'(x)}{f(x)\ln a}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$

Function	Derivative
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+f(x)^2}$

Rules of Differentiation

- $(kf)'(x)=kf'(x)$
- $(f\pm g)'(x)=f'(x)\pm g'(x)$
- $\frac{d}{dx}uv=u\frac{dv}{dx}+v\frac{du}{dx}$
- $\left(\frac{f}{g}\right)'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
- $\frac{d}{dx}f(g(x))=f'(g(x))g'(x)$ or $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Parametric Differentiation

Given $\begin{cases}y=u(t)\\x=v(t)\end{cases}$, we have $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

- **Second derivative**
  $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$ then do implicit differentiation w.r.t x
- **Polar equation** ($r=a\theta$): $x=r\cos\theta, y=r\sin\theta$

Implicit Differentiation

Differentiate w.r.t. to var, then multiply by $\frac{d\langle var\rangle}{dx}$ Common: $y=x^x\iff\ln y=x\ln x$

Higher Order Derivatives

The n-th derivative is denoted by $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$

Maxima and Minima

- $f(c)$ **is Local Maximum** if $f(c)\geq f(x)$ for x near c
- $f(c)$ **is Local Minimum** if $f(c)\leq f(x)$ for x near c
- $f(c)$ **is abs maximum** if $f(c)\geq f(x)\forall x\in$ domain
- $f(c)$ **is abs minimum** if $f(c)\leq f(x)\forall x\in$ domain
- **Critical Point** :
  Let f be a function with domain D. An interior point (not end-point) c in D is called a **Critical Point** of f if $f'(c)=0$ of $f'(c)$ does not exist.
- **Method to find extreme values of** f :
  Check critical points of f, end-points of domain D

Method to Find Local Extreme values

A function may not have a local extreme at a critical pt. Check using 1st/2nd derivative tests.

- **1st Derivative Test** :
  Assume $c\in(a,b)$ is a critical point of f
  1. $f'(x)>0$ for $x\in(a,c)$ and $f'(x)<0$ for $x\in(c,b)$, then f is a **local maximum**
  2. $f'(x)<0$ for $x\in(a,c)$ and $f'(x)>0$ for $x\in(c,b)$, then f is a **local minimum**
- **2nd Derivative Test** :
  $f'(c)=0\begin{cases}f''(c)<0\iff f\text{ has local max at }c\\f''(c)>0\iff f\text{ has local min at }c\end{cases}$
  Note: if $f'(c)=0$ and $f''(c)=0$ then 2nd derivative test fails. Use 1st derivative test.

Method to Find Absolute Extreme Values

1. Find all critical points c in the interior
2. Evaluate $f(c)$, where c is a critical or end point
3. The largest and smallest of these values will be abs max & min respectively

Increasing and Decreasing Functions

Test for Monotonic Functions ($f:I$ (interval) $\to\mathbb{R}$):

- $f'(x)>0$ for any x in $I\Rightarrow f$ is **increasing** on I
- $f'(x)<0$ for any x in $I\Rightarrow f$ is **decreasing** on I

Concavity

$\begin{cases}f''(x)<0\Leftrightarrow f'(x)\text{ is decreasing}\Leftrightarrow\text{Concave Down}\\f''(x)>0\Leftrightarrow f'(x)\text{ is increasing}\Leftrightarrow\text{Concave Up}\end{cases}$

Points of Inflection

Let $f:I\to\mathbb{Z}$ and $c\in I$.
c is a pt of inflection of f if f is continuous at c and the concavity of f changes at c.
In another word: c is pt of inflection $\to f''(c)=0$ (but not the reverse – c is a pt of inflection only if $f''(c)$ crosses from (+) to (-) and vice versa.)

4 Integration

Indefinite Integral

Denoted by $\int f(x)dx=F(x)+C$

Geometrical Interpretation

All curves $y=F(x)+C$ s.t. their slopes at x are $f(x)$

Rules of Indefinite Integration

1. $\int kf(x)dx=k\int f(x)dx$
2. $\int -f(x)dx=-\int f(x)dx$
3. $\int[f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx$

Integral Formulae

Function	Integral
$\int\cot x dx$	$\ln(\sin x)+C$
$\int\sec x\tan x dx$	$\sec x+C$
$\int\csc x\cot x dx$	$\csc x+C$
$\int\sec^2 x dx$	$\tan x+C$
$\int\csc^2 x dx$	$-\cot x+C$
$\int x^n dx$	$\frac{x^{n+1}}{n+1}+C, n\neq -1, n$ rational
$\int\frac{1}{\sqrt{a^2-x^2}}dx$	$\sin^{-1}(\frac{x}{a})+C$
$\int\frac{1}{a^2+x^2}dx$	$\frac{1}{a}\tan^{-1}(\frac{x}{a})+C$
$\int 1dx=\int dx$	$x+C$
$\int e^x dx$	e^x+C
$\int a^x dx$	$\frac{a^x}{\ln a}$
$\int\ln x dx$	$x\ln x-x+C$
$\int\frac{1}{x}dx$	$\ln x+C$
$\int\sin kx dx$	$-\frac{\cos kx}{k}+C$
$\int\cos kx dx$	$\frac{\sin kx}{k}+C$
$\int\tan x dx$	$\ln(\sec x)+C$ **or** $-\ln(\sec x)+C$
$\int\tan^2 x dx$	$\tan x-x+C$
$\int\sec x dx$	$\ln(\sec x+\tan x)+C$
$\int\csc x dx$	$\ln(\csc x-\cot x)+C$

Riemann (Definite) Integrals

Riemann sum on f on $[a,b]\approx\sum_{k=1}^n f(c_k)\Delta x$

Exact area $=\lim_{n\to\infty}\sum_{k=1}^n f(c_k)\Delta x$

Riemann Integral of f **over** $[a,b]$:

$\int_a^b f(x)dx=\lim_{n\to\infty}\sum_{k=1}^n f(c_k)\Delta x$

Rules of Definite Integrals

1. $\int_a^a f(x)dx=0$, $\int_a^b kf(x)dx=k\int_a^b f(x)dx$
2. $\int_a^b f(x)dx=-\int_b^a f(x)dx$
3. $\int_a^b[f(x)\pm g(x)]=\int_a^b f(x)\pm\int_a^b g(x)$
4. If $f(x)\geq g(x)$ on $[a,b]$, then $\int_a^b f(x)dc\geq\int_a^b g(x)dx$
   If $f(x)\geq 0$ on $[a,b]$, then $\int_a^b f(x)dx\geq 0$
5. If f is continuous on the interval joining a,b and c, then $\int_a^b f(x)dx+\int_b^c f(x)dx=\int_a^c f(x)dx$

Fundamental Thm of Calculus

$F'(x)=f(x)$ If F is an antiderivative of f on $[a,b]$, then $\int_a^b F'(x)dx=\int_a^b f(x)dx=F(b)-F(a)$

x' Let f be continuous on $[a,b]$. Then $\frac{d}{dx}\int_a^x f(t)dt=f(x)$

Note the 2 x's: on $\frac{d}{dx}$ and $\int_a^x$ and $f(t)$ is indep of x

1. $\frac{d}{dx}\int_0^2 t^2dt=0$, $\frac{d}{dx}\int_0^x\sin\sqrt{t}dt=\sin\sqrt{x}$
2. $\frac{d}{dx}\left(\int_1^{x^4}\frac{t}{\sqrt{t^3+2}}dt\right)=\frac{d}{dx^4}\left(\int_1^{x^4}\frac{t}{\sqrt{t^3+2}}dt\right)\frac{dx^4}{dx}$
   $=\frac{x^4}{\sqrt{(x^4)^3+2}}(4x^3)=\frac{4x^7}{\sqrt{x^12+2}}$
3. $\frac{d}{dx}\int_x^a f(t)dt=-\frac{d}{dx}\int_a^x f(t)dt$
4. $\frac{d}{dx}\int_{x^2}^{x^4} f(t)dt=\frac{d}{dx}\int_a^{x^4} f(t)dt-\frac{d}{dx}\int_a^{x^2} f(t)dt$

Integration Methods

- **Integration by Substitution** :
  Use the form $\int f(g(x))dg(x)$ OR use a dummy variable to get to a form in the Integral Formulae (taking into account chain rule)

Integral	Sub	Use identity
a^2-u^2	$u=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$
a^2+u^2	$u=a\tan\theta$	$1+\tan^2\theta=\sec^2\theta$
u^2-a^2	$u=a\sec\theta$	$sec^2\theta-1=\tan^2\theta$

- **Integration by Part** :
  $\int uv'dx=uv-\int u'vdx$
  Choose u by LIATE (Logarithmic, Inverse trigo, Algebraic, Trigo, Exponential)

Area between 2 curves

$A=\int_a^b(g(x)-f(x))dx$ provided $g(x)$ is above $f(x)$

Volume of a solid

Volume (around x-axis) $=\int_a^b\pi y^2dx$

5 Series

Geometric Series

$\sum_{r=1}^n ar^{n-1}=a\frac{1-r^n}{1-r}$

$\sum_{r=1}^\infty ar^{n-1}=\frac{a}{1-r}$ if $|r|<1$, diverges otherwise

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Rules on Series
$\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$, $\sum(ka_n) = k\sum a_n$

Ratio Test
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$, the series $\begin{cases}\text{converges if} & \rho < 1\\ \text{diverges if} & \rho > 1\\ \text{no conclusion if} & \rho = 1\end{cases}$

p-series
$\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}\text{diverges} & 0 \le p \le 1\\ \text{converges} & p > 1\end{cases}$

Radius of convergence (R)
Use the **Ratio Test** to find **range** of convergence of **Power Series** about $x = a$, $\sum_{n=0}^{\infty} c_n(x-a)^n$

1. $R = 0$, converges only at a
2. $R = h$, converges in $(a-h, a+h)$ but diverges outside
3. $R = \infty$, converges at every x

Differentiation and Integration of Power Series
Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, a-h < x < a+h$ where h is Radius of Convergence, then for $a-h < x < a+h$,
$f'(x) = \sum_{n=0}^{\infty}\frac{d}{dx}(c_n(x-a)^n) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
$f''(x) = \sum_{n=1}^{\infty} nc_n\frac{d}{dx}(x-a)^{n-1} = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}$
Note lower bound of sum increases by 1
$\int_0^x f(x)dx = \int_0^x \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$
The radius of convergence is h after diff and integ

Taylor Series of f at a
$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$

MacLaurin Series
Taylor series of f at 0, i.e. $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

List of common MacLaurin Series

1. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, -1 < x < 1, R = 1$
2. $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n, -1 < x < 1, R = 1$
3. $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} x^2 n, -1 < x < 1, R = 1$
4. $ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}, -1 < x < 1, R = 1$
5. $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}, -\infty < x < \infty, R = \infty$
6. $\cos x = \sum_{n=1}^{\infty}\frac{(-1)^n x^2 n}{(2n)!}, -\infty < x < \infty, R = \infty$
7. $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, -\infty < x < \infty, R = \infty$
8. $\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}, -1 \le x \le 1, R = 1$
9. $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, -1 < x < 1, R = 1$
0. $\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2}, -1 < x < 1, R = 1$
1. $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n, -1 < x < 1, R = 1$
2. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots, -1 < x < 1, R = 1$

Taylor Polynomials
The n-th order Taylor Polynomial of f at a
$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$
It gives a good polynomial approxn of order n

Taylor's Theorem
$f(x) = P_n(x) + R_n(x)$ where
$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for $a < c < x$.
$R_n(x)$ is **remainder of order** n or **error term**

6 Vector

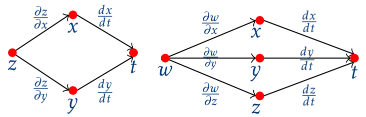
Dot Product
$v_1 = \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}, v_2 = \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$, $v_1 \cdot v_2 = x_1x_2 + y_1y_2 + z_1z_2$
$\cos\theta = \frac{v_1\cdot v_2}{\|v_1\|\,\|v_2\|}$, Projection of b onto $a = \frac{b\cdot a}{\|a\|^2}a$
Commut, assoc, distr, and $v_1 \cdot v_1 = \|v_1\|^2$

Cross Product
$v_1 \times v_2 = (y_1z_2 - y_2z_1)\mathbf{i} - (x_1z_2 - x_2z_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}$ Area of parallellogram $= \|v_1 \times v_2\| = \|v_1\|\,\|v_2\|\,\sin\theta$
Distr, assoc, but $v_1 \times v_2 = -v_2 \times v_1$ and $v_1 \times v_1 = O$

7 Functions of Several Variables

Partial Derivatives
of $z = f(x,y)$ w.r.t. x is denoted by $\left.\frac{\partial z}{\partial x}\right|_{(a,b)}$ or $f_x(a,b)$
Method: Fix the other variable (Note: $f_{xy} = f_{yx}$)

Chain Rule
$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$ AND $\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$



$z = f(x,y)$,
$x = x(t), y = y(t)$
$w = f(x,y,z)$,
$x = x(t)$,
$y = y(t), z = z(t)$

Directional Derivative
$f_x(a,b)$ is rate of change of f along direction of x-axis
Directional derivative of f at (a,b) in direction of unit vector $u = u_1\mathbf{i} + u_2\mathbf{j}$ is $D_uf(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$
or $D_uf(a,b,c) = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3$
$df = D_uf(a,b)\cdot dt$ (normal $\cdot$ multiplication) measures change in f (df) when we move a small distance dt, and u is the unit directional vector of the change and (a,b) is the original pt

Gradient Vector
Denoted by $\nabla f = f_x\mathbf{i} + f_y\mathbf{j}$ where
$\nabla f(a,b)\cdot u = D_uf(a,b) = \|\nabla f(a,b)\|\cos\theta$
$D_uf(a,b) > 0$ and max when $\cos\theta = 1 \iff \theta = 0°$
$D_uf(a,b) < 0$ and min when $\cos\theta = -1 \iff \theta = 180°$

Max and Min Values
Critical Points - First Derivative Test
f has a local max or min at $(a,b) \wedge f_x$ exists $\wedge f_y$ exists $\to f_x = 0 \wedge f_y = 0$ (But not the converse)
Second Derivative Test
Disriminant $= f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$

1. $D > 0 \wedge f_{xx}(a,b) > 0 \to f$ has a local min at (a,b)
2. $D > 0 \wedge f_{xx}(a,b) < 0 \to f$ has a local max at (a,b)
3. $D < 0 \to f$ has a saddle-point at (a,b)
4. $D = 0 \to$ no conclusion

8 Ordinary Differential Equation (ODE)

No Crossing Principle: solution curves do not cross each other

First order ODE
There is only 1 soln for initial value problem with 1st order ODE. Intersection point of 2 curves is the initial pt.

1. $\frac{dy}{dx} = \frac{M(x)}{N(y)} \iff \int M(x)dx = \int N(y)dy$
2. $y' = f(\frac{y}{x}) \Leftrightarrow$ Let $v = \frac{y}{x}, f(v) = y' \Leftrightarrow \frac{dv}{f(v)-v} = \frac{dx}{x}$
3. $y' = \frac{ax+by+c}{a_1x+b_1y+c_1} \Leftrightarrow$ Let $u = ax + by$
4. $\frac{dy}{dx} + p(x)y = Q(x) \Leftrightarrow ye^{\int p(x)dx} = \int Q(x)e^{\int p(x)dx}dx$
5. **Bernoulli eqn** $y' + p(x)y = q(x)y^n \Leftrightarrow$
   Let $z = y^{1-n} \Leftrightarrow z' + (1-n)p(x)z = (1-n)q(x)$

Scenarios
Radioactive decay: $\frac{dx}{dt} = kx$, $x(t) = x(0)e^{-\frac{\ln 2}{\tau}t}$
Uranium-Thorium: $\frac{T}{U} = \frac{k_U}{k_T - k_U}(1 - e^{-(k_T - k_U)t})k_N = \frac{\ln 2}{\tau_N}$
Cooling/Heating: $\int\frac{dT}{T-T_0} = \int kdt$, $T(t) - T_0 = (T(0) - T_0)e^{kt}$
Retarded fall: $m\frac{dv}{dt} = mg - bv^2$,
$v = k\frac{1+ce^{-pt}}{1-ce^{-pt}}, k^2 = \frac{mg}{b}, c = \frac{v(0)-k}{v(0)+k}, p = \frac{2kb}{m}$

Hyperbolic Functions
$\cosh x = \frac{e^x+e^{-x}}{2}, \sinh x = \frac{e^x+e^{-x}}{2}, \tanh x = \frac{\sinh x}{\cosh x} = \frac{1-e^{-2x}}{1+e^{-2x}}$

Second order linear ODE
Form: $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = F(x)$
homogeneous $\Leftrightarrow F(x) = 0$, else non-homogeneous

Homogeneous 2nd order linear ODE
Linearly dependent $\Leftrightarrow \forall x\exists c$ s.t. $u(x) = cv(x)$
y_1, y_2 are lin. indep. solns $\Rightarrow$ a general soln is $y = c_1y_1 + c_2y_2$
y_1, y_2 are NOT lin. indep. solns $\Rightarrow y = c_1y_1 + c_2y_2$ is a soln but not a general soln
$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$ has the trivial soln $y = 0$ and non-trivial soln: Let $y = e^{\lambda x}$, solve $\lambda^2 + A\lambda + B = 0$, general soln is: (PS: Reverse is $A = -(\lambda_1 + \lambda_2), B = \lambda_1\lambda_2$)

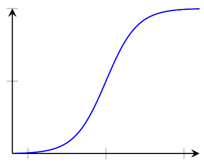
1. 2 real roots: $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
2. 1 real root: $y = c_1e^{\lambda_1 x} + c_2xe^{\lambda_1 x}$
3. 2 complex roots $(a + ib)$: $y = e^{ax}(c_1\cos bx + c_2\sin bx)$

9 Mathematical Modelling (B = birth rate, D = death rate)

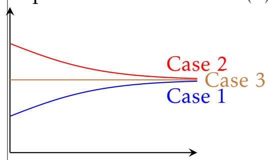
Malthusian Population Growth
$N(t) = N(0)e^{kt}$ where $k = B - D$. Conditions:

1. $k > 0$ $(B > D)$: popn explosion ($e^{kt} \to \infty, N(t) \to \infty$ as $t \to \infty$)
2. $k = 0$ $(B = D)$: stable ($N(t) = N(0)$ for all t)
3. $k < 0$ $(B < D)$: extinction ($e^{kt} \to 0, N(t) \to 0$ as $t \to \infty$)

Logistic Growth Model

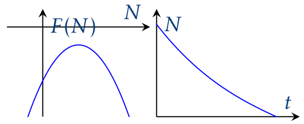


Eqn: $\frac{dN}{dt} = (B - D)N, N(0) = \hat{N}, N_\infty = \frac{B}{s}$
$\frac{dN}{dt} = (B - D)N = (B - sN)N = BN - sN^2$ where s is a small number compared to B.
$\frac{dN}{dt} = 0$ when $N \approx \frac{B}{s}$ (population stops growing)
This constant $\frac{B}{s}$ is called carrying capacity, sustainable population, or logistic equilibrium population. Or that the population stabilises at $\frac{B}{s}$
$N(t) = \frac{B}{s + \left(\frac{B}{N_0} - s\right)e^{-Bt}} = \frac{N_\infty}{1 + (\frac{N_\infty}{N_0} - 1)e^{-Bt}}$
$\lim_{t\to\infty} N(t) = \frac{B}{s}$
Case 1: $B - sN(t) > 0\ \forall t$ **(Popn < sustainable popn)**
Logistic curve increasing
Case 2: $B - sN(t) < 0$ **at all** t **(Popn > sustainable popn)**
Logistic curve decreasing
Case 3: $B - sN(t) = 0$ **at all** t **(At sustainable popn)**
Population constant at $N(0)$

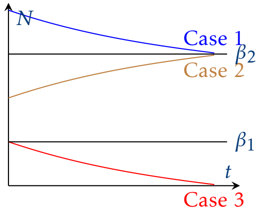


Harvesting
$\frac{dN}{dt} = BN - sN^2 - E$ where E is fish caught per year.
DO NOT ATTEMPT TO SOLVE THE ODE. They will just ask to draw graph.
Method:

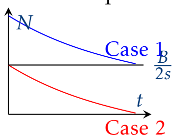
1. Let $F(N) = \frac{dN}{dt} = -sN^2 + BN - E$
2. Discriminant $= B^2 - 4(-s)(-E) = B^2 - 4sE$
3. Cases:
   (a) $D < 0$: No equiblirium soln (Popn is decreasing to extinction)
   Note: $-s < 0$, shape is $\cap$, $F(N) \neq 0$



   (b) $D > 0$: 2 equilibrium solns
   Solve $F(N)$ for β_1, β_2 where $\beta_1 < \beta_2 < \frac{B}{s}$
   There are 3 possible cases:



   $\frac{B}{s} = \beta_1 + \beta_2$, $\frac{E}{s} = \beta_1\beta_2$
   β_2 is stable ($N(0)$ slightly diff from β_2, popn will still tend to β_2). β_1 is not stable ($N(0)$ slightly diff from β_1 will not tend to β_1)
   (c) $D = 0$: 1 equilibrium solns



   Suppose $N(0) > \frac{B}{2s}$ then max. harvesting w/o extinction $E = \frac{B^2}{4s}$

PS: more precise curves, follow the original logistic growth model graph (S-shaped) increasing: gentle-steep-gentle, decreasing: steep-gentle-steep

Additional Notes

- If the question is in powers above 2, e.g. $\frac{dN}{dt} = aN^4 + bN^3 + cN^2 + dN + e$, the same rule about the graph still applies: the stable populations are the solutions to $aN^4 + bN^3 + cN^2 + dN + e = 0$
- If there is no harvesting, then $N = 0$ is also a solution.