Sections 7.1 and 7.2: Induction

CS1231S Discrete Structures

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- Indicate your interest in mass consultation next Mon/Tue on "LumiNUS $>$ Poll" by tomorrow's lecture.
- I am being reviewed in this lecture.

The Sorites Paradox

- 1 grain of sand does not make a heap.
- For every $n \in \mathbb{Z}^+$, if n grains of sand do not make a heap, then $n+1$ grains of sand do not make a heap.
- Hence n grains of sand do not make a heap for any $n \in \mathbb{Z}^+$, by the Principle of Mathematical Induction.

Mathematical Induction

Why induction?

- It is a very powerful method of proof for the natural numbers $0, 1, 2, 3, \ldots$.
- In a sense, it characterizes the natural numbers (by a theorem of Dedekind and Peano).
- In the same sense, natural generalizations of induction characterize recursively defined objects.

Warning

- Induction can mean a kind of non-deductive reasoning in some areas of study.
- Our Mathematical Induction is a kind of deductive reasoning.

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The Sorites Paradox

- 1 grain of sand does not make a heap.
- For every $n \in \mathbb{Z}^+$, if n grains of sand do not make a heap, then $n + 1$ grains of sand do not make a heap.
- Hence n grains of sand do not make a heap for any $n \in \mathbb{Z}^+$, by the Principle of Mathematical Induction.

Example 7.1.3. $1+2+\cdots+n=\frac{1}{2}n(n+1)$ for all $n\in\mathbb{Z}_{\geqslant 1}$.

$$1 \qquad ========== \qquad \frac{1\times 2}{2}$$

$$\vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots$$

$$\therefore \quad \overbrace{1+2+\cdots+87}+88 = \frac{87\times 88}{2}+88=\frac{87+2}{2}\times 88 = \frac{88\times 89}{2}$$

$$\therefore \quad \overbrace{\underbrace{1+2+\cdots+87+88}}+89 = \frac{88\times 89}{2}+89=\frac{88+2}{2}\times 89 = \frac{89\times 90}{2}$$

$$\therefore \quad \overbrace{\underbrace{1+2+\cdots+87+88+89}}+90 = \frac{89\times 90}{2}+90=\frac{89+2}{2}\times 90 = \frac{90\times 91}{2}.$$

Principle 7.1.1: Mathematical Induction (MI)

can use it when have universal statement: and a particular case
relies on a previous case, which relies on a previous case

$\boxed{m \in \mathbb{Z}}$

To prove that $\forall n \in \mathbb{Z}_{\geqslant m}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(m)$ is true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant m}\ \big(P(k) \Rightarrow P(k+1)\big)$ is true.

Justification

$P(m)$	by the base step;
$P(m) \Rightarrow P(m+1)$	by the induction step with $k = m$;
$P(m+1) \Rightarrow P(m+2)$	by the induction step with $k = m+1$;
$P(m+2) \Rightarrow P(m+3)$	by the induction step with $k = m+2$;
$\vdots$	

We deduce that $P(m), P(m+1), P(m+2), \ldots$ are all true by a series of modus ponens.

Terminology 7.1.2

In the induction step, we assume we have $k \in \mathbb{Z}_{\geqslant m}$ such that $P(k)$ is true, and then show $P(k+1)$ using this assumption. In this process, the assumption that $P(k)$ is true is called the *induction hypothesis*.

proving an equation

Example 7.1.3 (again). $1+2+\cdots+n=\frac{1}{2}\,n(n+1)$ for all $n\in\mathbb{Z}_{\geqslant 1}$.

To prove that $\forall n\in\mathbb{Z}_{\geqslant 1}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(1)$ is true;

(induction step) show that $\forall k\in\mathbb{Z}_{\geqslant 1}\ \big(P(k)\Rightarrow P(k+1)\big)$ is true.

Proof

1. For each $n\in\mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition "$1+2+\cdots+n=\frac{1}{2}\,n(n+1)$".
2. (Base step) $P(1)$ is true because $1=\frac{1}{2}\times 1\times(1+1)$.
3. (Induction step)
   3.1. Let $k\in\mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., such that $1+2+\cdots+k=\frac{1}{2}\,k(k+1)$.
   3.2. Then $1+2+\cdots+k+(k+1)$
   3.3. $=\frac{1}{2}\,k(k+1)+(k+1)$ by the induction hypothesis $P(k)$;
   3.4. $=\left(\frac{k}{2}+1\right)(k+1)=\frac{k+2}{2}\,(k+1)$
   3.5. $=\frac{1}{2}\,(k+1)((k+1)+1)$.
   3.6. So $P(k+1)$ is true.
4. Hence $\forall n\in\mathbb{Z}_{\geqslant 1}\ P(n)$ is true by MI. $\square$

Terminology 7.1.4. We call this an induction *on n* as n is the active variable in it.

proving an inequality

Example 7.1.5. $n! > 2^n$ for all $n \in \mathbb{Z}_{\geqslant 4}$.

$n! = n \times (n-1) \times \cdots \times 1.$

To prove that $\forall n \in \mathbb{Z}_{\geqslant m}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:
(base step) show that $P(m)$ is true;
(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant m}\ \big(P(k) \Rightarrow P(k+1)\big)$ is true.

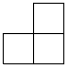
Proof

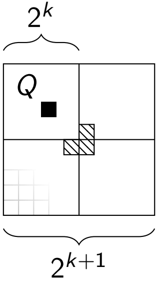
1. For each $n \in \mathbb{Z}_{\geqslant 4}$, let $P(n)$ be the proposition "$n! > 2^n$".
2. (Base step) $P(4)$ is true because $4! = 24 > 16 = 2^4$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 4}$ such that $P(k)$ is true, i.e., such that $k! > 2^k$.
   3.2. Then $(k+1)! = (k+1) \times k!$ by the definition of !;
   3.3. $> (k+1) \times 2^k$ by the induction hypothesis $P(k)$;
   3.4. $> 2 \times 2^k$ as $k + 1 \geqslant 4 + 1 > 2$;
   3.5. $= 2^{k+1}$.
   3.6. So $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 4}\ P(n)$ is true by MI. $\square$

Terminology 7.1.4. We call this an induction *on* n as n is the active variable in it.

Example 7.1.6

For all $n \in \mathbb{Z}_{\geqslant 1}$, $\overbrace{\text{if one } \square \text{ is removed from a } 2^n \times 2^n \text{ checkerboard, then the remaining } \square\text{'s can be covered by L-trominos.}}^{P(n)}$





Proof

2. (Base step) $P(1)$ is true because such a board itself is an L-tromino.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true.
   - 3.2. 3.2.1. Let B be a $2^{k+1} \times 2^{k+1}$ checkerboard with one $\square$ removed.
     - 3.2.2. Divide B into four $2^k \times 2^k$ quadrants.
     - 3.2.3. Let Q be the quadrant containing the removed $\square$.
     - 3.2.4. Remove one L-tromino from the centre of B in a way such that each quadrant other than Q has one $\square$ removed.
     - 3.2.5. We are left with four $2^k \times 2^k$ checkerboards, each with one $\square$ removed.
     - 3.2.6. By the induction hypothesis, each quadrant can be covered by L-trominos.
     - 3.2.7. Hence B can be covered by L-trominos.
   - 3.3. This shows $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}$ $P(n)$ is true by MI. $\square$

Example 7.1.7. All participants of this Zoom meeting have the same birthday.

1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition "if a Zoom meeting has exactly n participants, then all its participants have the same birthday".
2. (Base step) $P(1)$ is true because if a Zoom meeting has exactly 1 participant, then clearly all its participants have the same birthday.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true.
   - 3.2.
     - 3.2.1. Suppose a Zoom meeting has exactly $k+1$ participants.
     - 3.2.2. Pick two different participants a, b in the meeting.
     - 3.2.3. Ask a to leave the meeting.
     - 3.2.4. Since there are k people left in the meeting, by the induction hypothesis, all the remaining participants have the same birthday, including b.
     - 3.2.5. Tell a to join the meeting again, and then ask b to leave the meeting.
     - 3.2.6. Since there are k people left in the meeting, by the induction hypothesis, all the remaining participants have the same birthday, including a.
     - 3.2.7. The participants who stayed in the meeting throughout have the same birthday as both a and b.
     - 3.2.8. So a and b have the same birthday.
   - 3.3. This shows $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}$ $P(n)$ is true by MI.

No one stayed in the meeting throughout when $k = 1$.

pollev.com/wtl

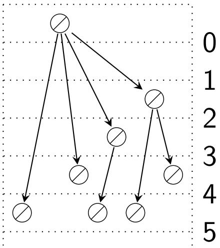
Rabbits

$F_2 = 1 + 0 = 1, \quad F_3 = 1 + 1 = 2, \quad F_4 = 2 + 1 = 3, \quad \ldots$

Definition 7.2.2

The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n.$$



Example 7.2.4

- Initially, there is one pair of newly born matched rabbits.
- Each newly born rabbit takes one month to mature.
- Each mature pair of matched rabbits produces one pair of matched rabbits per month.

Let r_n denote the number of pairs of rabbits after n months. Then for every $n \in \mathbb{Z}_{\geqslant 0}$,

$$r_0 = 1 \quad \text{and} \quad r_1 = 1 \quad \text{and} \quad r_{n+2} = r_{n+1} + r_n,$$

where the r_{n+1} comes from the rabbits already present after $(n+1)$ months, and the r_n comes from the rabbits born after $(n+1)$ months.

Observation 7.2.5

$r_n = F_{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

An upper bound for the Fibonacci sequence

To prove that $\forall n \in \mathbb{Z}_{\geqslant 1}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(1)$ is true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 1}\ \big(P(k) \Rightarrow P(k+1)\big)$ is true.

Proposition 7.2.6

$F_{n+1} \leqslant (7/4)^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

$F_0 = 0$ and $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

Proof

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition "$F_{n+1} \leqslant (7/4)^n$".
2. (Base step) $P(0)$ is true because $F_{0+1} = 1 \leqslant 1 = (7/4)^0$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(k)$ is true, i.e., such that $F_{k+1} \leqslant (7/4)^k$.
   3.2. Then $F_{(k+1)+1} = F_{k+2}$
   3.3. $= F_{k+1} + F_k$ by the definition of F_{k+2};
   3.4. $\leqslant (7/4)^k + F_k$ as $P(k)$ is true...

We have no information on how large F_k is here... unless, say, we know $P(k-1)$.

Principle 7.2.1 ($m = 1$): Strong Mathematical Induction (Strong MI)

To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(0), P(1)$ are true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(0) \wedge \cdots \wedge P(k+1) \Rightarrow P(k+2)\big)$ is true.

Justification

$P(0) \wedge P(1)$	by the base step;
$P(0) \wedge P(1) \Rightarrow P(2)$	by the induction step with $k = 0$;
$P(0) \wedge P(1) \wedge P(2) \Rightarrow P(3)$	by the induction step with $k = 1$;
$P(0) \wedge P(1) \wedge P(2) \wedge P(3) \Rightarrow P(4)$	by the induction step with $k = 2$;
$P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4) \Rightarrow P(5)$	by the induction step with $k = 3$;
$\vdots$	

We deduce that $P(0), P(1), P(2), \ldots$ are all true by a series of modus ponens. $\square$

Remark

Given the same $P(n)$, Strong MI is more likely to succeed than usual MI.

Example 7.2.6 (again). $F_{n+1} \leqslant (7/4)^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(0), P(1)$ are true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(0) \wedge \cdots \wedge P(k+1) \Rightarrow P(k+2)\big)$ is true.

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition "$F_{n+1} \leqslant (7/4)^n$".
2. (Base step) $P(0)$ and $P(1)$ are true because

$$F_{0+1} = 1 \leqslant 1 = (7/4)^0 \quad \text{and}$$
$$F_{1+1} = 1 + 0 = 1 \leqslant 7/4 = (7/4)^1.$$

3. (Induction step)

   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k+1)$ are true.

   $\vdots$

   $\vdots$

   $\vdots$

   3.8. So $P(k+2)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true by Strong MI.

$F_0 = 0$ and
$F_1 = 1$ and
$F_{n+2} = F_{n+1} + F_n$
for all $n \in \mathbb{Z}_{\geqslant 0}$.

Example 7.2.6 (again). $F_{n+1} \leqslant (7/4)^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(0), P(1)$ are true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(0) \wedge \cdots \wedge P(k+1) \Rightarrow P(k+2)\big)$ is true.

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition "$F_{n+1} \leqslant (7/4)^n$".
2. (Base step) $P(0)$ and $P(1)$ are true because ...
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k+1)$ are true.
   - 3.2. Then $F_{(k+2)+1} = F_{k+3}$
   - 3.3. $= F_{k+2} + F_{k+1}$ by the definition of F_{k+3};
   - 3.4. $\leqslant (7/4)^{k+1} + (7/4)^k$ as $P(k)$ and $P(k+1)$ are true;
   - 3.5. $= (7/4)^k(7/4+1)$
   - 3.6. $< (7/4)^k(7/4)^2$ as $7/4+1 < (7/4)^2$;
   - 3.7. $= (7/4)^{k+2}$.
   - 3.8. So $P(k+2)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true by Strong MI. $\square$

$F_0 = 0$ and
$F_1 = 1$ and
$F_{n+2} = F_{n+1} + F_n$
for all $n \in \mathbb{Z}_{\geqslant 0}$.

Combining the base step and the induction step

To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(0)$ are true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(0) \wedge \cdots \wedge P(k) \Rightarrow P(k+1)\big)$ is true.

Theorem 7.2.7 (Strong MI, alternative formulation)

To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to show that

$$\forall \ell \in \mathbb{Z}_{\geqslant 0}\ \big(\forall i \in \mathbb{Z}_{\geqslant 0}\ \big(i < \ell \Rightarrow P(i)\big) \Rightarrow P(\ell)\big). \tag{$*$}$$

Idea of proof

Applying $(*)$ to $\ell = 0$ gives

$$\underbrace{\forall i \in \mathbb{Z}_{\geqslant 0}\ \underbrace{\big(\overbrace{i < 0}^{\text{false}} \Rightarrow P(i)\big)}_{\text{true}}}_{\text{true}} \Rightarrow P(0).$$

Thus $P(0)$ is true by modus ponens. □

Well-Ordering Principle

To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(0)$ are true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(0) \wedge \cdots \wedge P(k) \Rightarrow P(k+1)\big)$ is true.

Theorem 7.2.9 (Well-Ordering Principle)

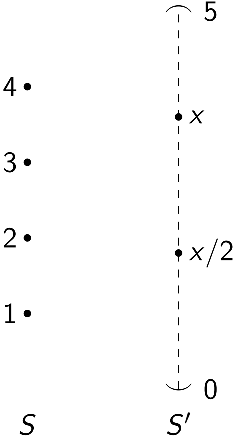
characteristic of natural numbers

Every nonempty subset of $\mathbb{Z}_{\geqslant 0}$ has a smallest element.

Example 7.2.8

(1) $S = \{x \in \mathbb{Z}_{\geqslant 0} : 0 < x < 5\}$ has smallest element 1.

(2) $S' = \{x \in \mathbb{Q}_{\geqslant 0} : 0 < x < 5\}$ has no smallest element because if $x \in S'$, then $x/2 \in S'$ and $x/2 < x$.



prove by contrapositive

Proof of "Every nonempty subset of $\mathbb{Z}_{\geqslant 0}$ has a smallest element."

To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(0)$ are true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(0) \wedge \cdots \wedge P(k) \Rightarrow P(k+1)\big)$ is true.

1. Let $S \subseteq \mathbb{Z}_{\geqslant 0}$ with no smallest element.
2. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition " $n \notin S$ ".
3. (Base step)
   - 3.1. If $0 \in S$, then 0 is the smallest element of S as $S \subseteq \mathbb{Z}_{\geqslant 0}$, which contradicts our assumption that S has no smallest element.
   - 3.2. So $0 \notin S$ and thus $P(0)$ is true.
4. (Induction step)
   - 4.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k)$ are true, i.e., that $0, 1, \ldots, k \notin S$.
   - 4.2. If $k+1 \in S$, then $k+1$ is the smallest element of S by the induction hypothesis as $S \subseteq \mathbb{Z}_{\geqslant 0}$, which contradicts our assumption that S has no smallest element.
   - 4.3. So $k+1 \notin S$ and thus $P(k+1)$ is true.
5. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true by Strong MI.
6. This implies $S = \varnothing$ as $S \subseteq \mathbb{Z}_{\geqslant 0}$. $\square$

$$\begin{array}{r}
\vdots \\
k+1 \bullet \notin S \\
k \bullet \notin S \\
\vdots \\
1 \bullet \notin S \\
0 \bullet \notin S
\end{array}$$

Summary

$m \in \mathbb{Z}$

Principle 7.1.1 (Mathematical Induction (MI))

To prove that $\forall n \in \mathbb{Z}_{\geqslant m}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(m)$ is true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant m}\ \big(P(k) \Rightarrow P(k+1)\big)$ is true.

Principle 7.2.1 (Strong Mathematical Induction (Strong MI), where $m = 1$)

To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(0), P(1)$ are true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(0) \wedge \cdots \wedge P(k+1) \Rightarrow P(k+2)\big)$ is true.

Theorem 7.2.9 (Well-Ordering Principle)

Every nonempty subset of $\mathbb{Z}_{\geqslant 0}$ has a smallest element.

Next: recursion

Every element of $\mathbb{Z}_{\geqslant 0}$ is interesting.

Suppose not. By the Well-Ordering Principle, there is a smallest uninteresting non-negative integer. The smallest uninteresting integer is *highly* interesting. This is the required contradiction. $\square$