Matriculation Number: MA1521

A								

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2017-2018

MA1521 CALCULUS FOR COMPUTING

May 2018 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** Do not write your name anywhere in this booklet. This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **FIVE (5)** questions and comprises **ELEVEN (11)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question. The marks for each question are indicated at the beginning of the question. The maximum possible total score for this examination paper is 50 marks.
4. This is a **closed book (with authorized material)** examination. Students are only allowed to bring into the examination hall **ONE** piece A4 size help-sheet which can be used on both sides.
5. Candidates may use any calculators that satisfy MOE A-Level examination guidelines. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5
Marks					

Question 1 [10 marks]

(a) (Multiple Choice Question)

The line $y = mx + c$ is the normal line to the curve $y = x^3$ at the point $(1, 1)$. Find the value of c.

(A) $\frac{4}{3}$ (B) $\frac{5}{3}$ (C) $\frac{1}{3}$

(b) Let a denote a positive constant. Let R denote the finite domain in the first quadrant bounded between the curve $y^2 = x^2\sqrt{a^2 - x^2}$, the part of the x-axis with $0 \leq x \leq \frac{1}{2}a$, and the line $x = \frac{1}{2}a$. If the volume of the solid generated by rotating R one complete round about the x-axis is equal to 567, find the value of a. Give your answer correct to two decimal places.

Answer 1(a)	A	Answer 1(b)	8.28

(Show your working below and on the next page.)

(a) $\left.\frac{dy}{dx}\right|_{x=1} = 3$

$m = -\frac{1}{3}$

$1 = -\frac{1}{3} + c$

$\Rightarrow c = \frac{4}{3}$

(b) $567 = \pi \int_0^{\frac{1}{2}a} x^2 \sqrt{a^2 - x^2}\, dx$

$= \pi \int_0^{\frac{\pi}{6}} a^4 \sin^2\theta \cos^2\theta \, d\theta$ (let $x = a\sin\theta$)

$= \frac{\pi}{4} \int_0^{\frac{\pi}{6}} a^4 \sin^2 2\theta \, d\theta$

$= \frac{\pi a^4}{4} \int_0^{\frac{\pi}{6}} \frac{1 - \cos 4\theta}{2} \, d\theta$

$= \frac{\pi a^4}{8} \left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{6}}$

$= \frac{\pi a^4}{8}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$

$\therefore a = \sqrt[4]{\frac{567 \times 8}{\pi\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)}} = 8.280\ldots$

≈ 8.28

Question 2 [10 marks]

(a) Find the **exact value** of the radius of convergence of the power series $\sum_{n=0}^{\infty}\left(\frac{n+1521}{5^n}\right)(x-3)^n$.

(b) Let $f(x)=\frac{4x-7}{x^2-5x+4}$. Find the value of $f^{(6)}(2)$, where $f^{(6)}(2)$ denotes the sixth derivative of f evaluated at the point $x=2$. Give your answer correct to the nearest integer.

Answer 2(a)	5	Answer 2(b)	703

(Show your working below and on the next page.)

(a) $\left|\dfrac{\frac{n+1+1521}{5^{n+1}}(x-3)^{n+1}}{\frac{n+1521}{5^n}(x-3)^n}\right|$

$=\frac{1}{5}\left(\frac{n+1522}{n+1521}\right)|x-3|$

$\to \frac{1}{5}|x-3|$

$\frac{1}{5}|x-3|<1$

$\Rightarrow |x-3|<\underline{\underline{5}}$

(b) $f(x)=\dfrac{4x-7}{x^2-5x+4}=\dfrac{1}{x-1}+\dfrac{3}{x-4}$

$=\dfrac{1}{x-2+1}+\dfrac{3}{x-2-2}$

$=\dfrac{1}{1-\{-(x-2)\}}-\dfrac{3}{2}\left\{\dfrac{1}{1-\left(\frac{x-2}{2}\right)}\right\}$

$=\sum_{n=0}^{\infty}(-1)^n(x-2)^n-\frac{3}{2}\sum_{n=0}^{\infty}\frac{(x-2)^n}{2^n}$

$=\sum_{n=0}^{\infty}\left\{(-1)^n-\frac{3}{2^{n+1}}\right\}(x-2)^n$

$\therefore \dfrac{f^{(6)}(2)}{6!}=(-1)^6-\dfrac{3}{2^7}$

$f^{(6)}(2)=6!\left\{1-\frac{3}{2^7}\right\}$

$=703.1\ldots$

$\approx \underline{\underline{703}}$

Question 3 [10 marks]

(a) It is known that the equation $z^3y + zx^2 = 42$ defines z as a function of the two variables x and y. As a consequence the equation $w = z^2y$ defines w as a function of x and y. Find the value of $\frac{\partial w}{\partial x}$ when $x = 5$, $y = -1$, $z = 2$. Give your answer correct to two decimal places.

(b) Let $\mathbf{u}$ denote a unit vector in space which starts at the origin and ends at a point in the first octant (i.e. the part of space where all three coordinates are positive). It is known that $\mathbf{u}$ makes an angle of $\frac{\pi}{4}$ with the (x, y) plane and that the vector projection of $\mathbf{u}$ onto the (x, y) plane makes an angle $\frac{\pi}{3}$ with the positive x-axis. Let $f(x, y, z) = xyz$. Find the directional derivative of f at the point $(1, 2, 3)$ in the direction $\mathbf{u}$. Give your answer correct to two decimal places.

Answer 3(a)	6.15	**Answer 3(b)**	5.37

(Show your working below and on the next page.)

(a) $3z^2 z_x y + z_x x^2 + 2zx = 0$

$\Rightarrow z_x = \dfrac{-2zx}{3z^2y + x^2}$

$w_x = 2z z_x y$

$= \dfrac{-4xyz^2}{3z^2y + x^2}$

$x=5,\ y=-1,\ z=2$

$\Rightarrow w_x = \dfrac{80}{-12+25}$

$= \dfrac{80}{13}$

$= 6.153\ldots$

$\approx \underline{\underline{6.15}}$

(b) $\vec{u} = \cos\frac{\pi}{4}\cos\frac{\pi}{3}\vec{i} + \cos\frac{\pi}{4}\sin\frac{\pi}{3}\vec{j} + \sin\frac{\pi}{4}\vec{k}$

$= \dfrac{1}{2\sqrt{2}}\vec{i} + \dfrac{\sqrt{3}}{2\sqrt{2}}\vec{j} + \dfrac{1}{\sqrt{2}}\vec{k}$

$\nabla f = yz\vec{i} + xz\vec{j} + xy\vec{k}$

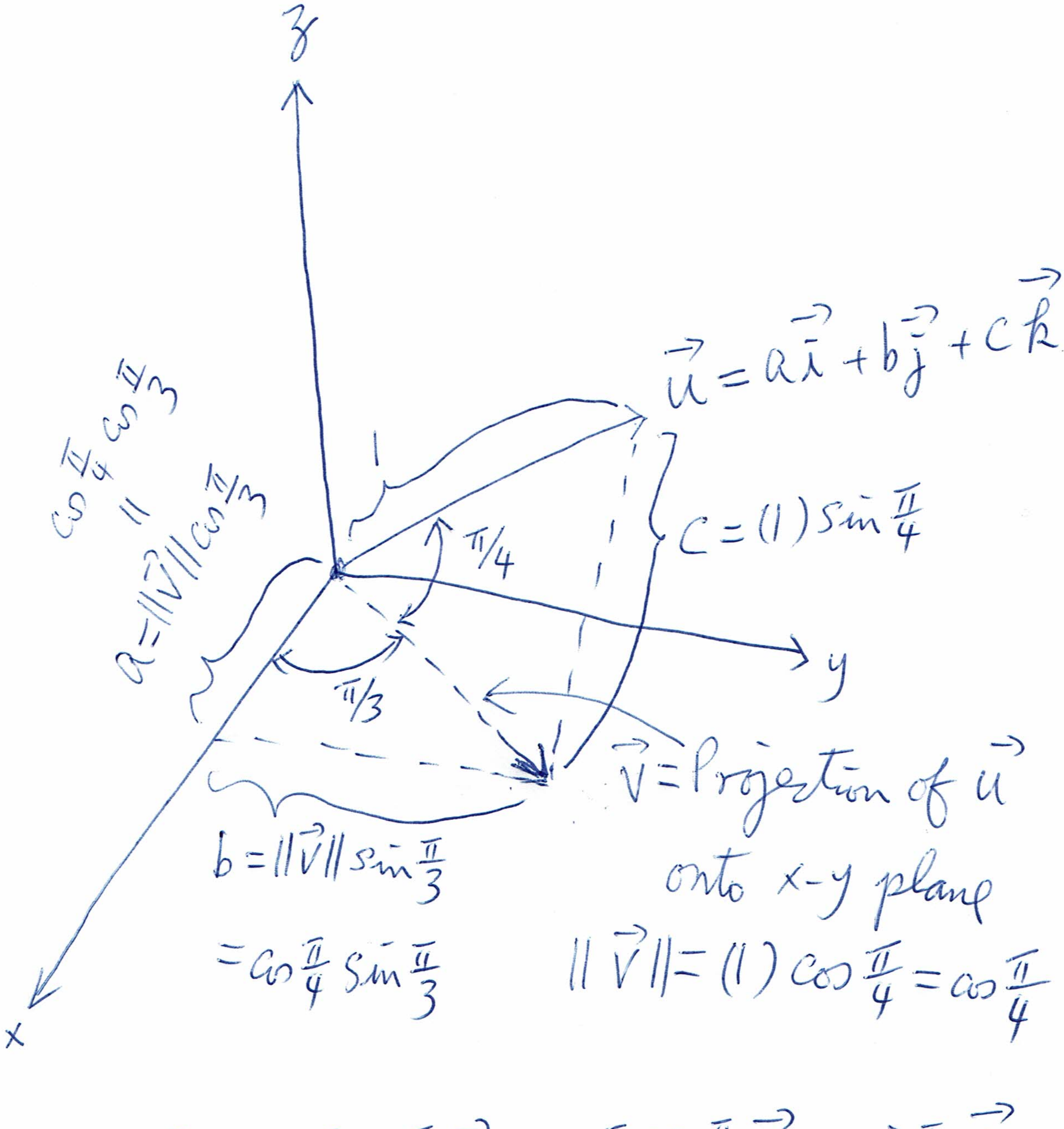
$\nabla f(1,2,3) = 6\vec{i} + 3\vec{j} + 2\vec{k}$

$\nabla f(1,2,3)\cdot\vec{u}$

$= \dfrac{6}{2\sqrt{2}} + \dfrac{3\sqrt{3}}{2\sqrt{2}} + \dfrac{2}{\sqrt{2}}$

$= 5.372\ldots$

$\approx \underline{\underline{5.37}}$



$$\therefore \vec{u} = \cos\frac{\pi}{4}\cos\frac{\pi}{3}\,\vec{i} + \cos\frac{\pi}{4}\sin\frac{\pi}{3}\,\vec{j} + \sin\frac{\pi}{4}\,\vec{k}$$

Question 4 [10 marks]

(a) A fossilized bone is found to contain 85% of the original amount of Carbon-14. We know that the half-life of Carbon-14 is 5600 years. Find the age of this fossilized bone. Give your answer in years correct to the nearest integer.

(b)Let y be a positive function which satisfies the differential equation

$$\frac{1}{2}\frac{dy}{dx} + y\tan x = (\cos x)(\sqrt{y}),$$

with $y(0) = 4$. Find the value of $y(\frac{\pi}{4})$. Give your answer correct to two decimal places.

Answer 4(a)	1313	Answer 4(b)	3.88

(Show your working below and on the next page.)

(a) $\frac{dQ}{dt} = -kQ$

$k = \frac{\ln 2}{5600}$

$Q = Q_0 e^{-kt}$

$0.85 = e^{-kt}$

$\Rightarrow \ln 0.85 = -kt$

$\Rightarrow t = \frac{-\ln 0.85}{k}$

$= \frac{-5600 \ln 0.85}{\ln 2}$

$= 1313.0\ldots$

$\approx \underline{\underline{1313}}$

(b) Let $z = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$

$\frac{dz}{dx} = \frac{1}{2}\frac{1}{\sqrt{y}}\frac{dy}{dx}$

$\therefore \sqrt{y}\frac{dz}{dx} + y\tan x = (\cos x)\sqrt{y}$

$\frac{dz}{dx} + (\tan x) z = \cos x$

$R = e^{\int \tan x\, dx} = e^{-\ln \cos x} = \frac{1}{\cos x}$

$z = \cos x \int \frac{1}{\cos x}\cos x\, dx = (\cos x)(x + c)$

$\sqrt{y} = (\cos x)(x + c)$

$y(0) = 4 \Rightarrow c = \sqrt{4} = 2$

$y = (\cos^2 x)(x+2)^2$

$y(\frac{\pi}{4}) = \frac{1}{2}(\frac{\pi}{4}+2)^2$

$= 3.879\ldots \approx \underline{\underline{3.88}}$

Question 5 [10 marks]

(a) The lion population at the Ngorongoro Conservation Area in northern Tanzania follows a logistic growth model with a birth rate per capita of 10%. Initially at time $t = 0$, there were L lions. At time $t = 10$ years, there were $\frac{3}{8}aL$ lions, where a denotes a positive constant. After a long time, the lion population settled down to its carrying capacity of aL lions. Find the value of a. Give your answer correct to two decimal places.

~~(b) Let $w = w(x, y)$ denote a function of two variables x and y. If $w(x, y)$ is the answer that you get by applying the method of separation of variables to solve the partial differential equation $x^2(\frac{\partial w}{\partial x}) = y\frac{\partial w}{\partial y}$, with $x > 0$, $y > 0$ and $w(1,1) = \frac{3}{e^2}$, find the value of $w(3,3)$. Give your answer correct to two decimal places.~~

Answer 5(a)	5.53	Answer 5(b))	13.86

(Show your working below and on the next page.)

(a) We have

$$\frac{3}{8}aL = \frac{aL}{1+\left(\frac{aL}{L}-1\right)e^{-1}}$$

$$\therefore a = \frac{5e+3}{3}$$

$$= 5.530\ldots$$

$$\approx \underline{\underline{5.53}}$$

(b) Let $w(x,y) = X(x)Y(y)$

$$x^2X'Y = yXY'$$

$$\frac{x^2X'}{X} = \frac{yY'}{Y} = k$$

$$\frac{X'}{X} = \frac{k}{x^2} \Rightarrow X = C_1e^{-\frac{k}{x}}$$

$$\frac{Y'}{Y} = \frac{k}{y} \Rightarrow Y = C_2y^k$$

$$\therefore w = ce^{-\frac{k}{x}}y^k$$

$$\frac{3}{e^2} = w(1,1) = ce^{-k}$$

$$\Rightarrow c = 3,\ k = 2$$

$$\therefore w(x,y) = 3e^{-\frac{2}{x}}y^2$$

$$w(3,3) = 3e^{-\frac{2}{3}} \cdot 3^2$$

$$= 13.862\ldots \approx \underline{\underline{13.86}}$$