MA1101R

LIVE LECTURE 3

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Topics for week 3

2.3 Inverses of Square Matrices

2.4 Elementary Matrices

2.5 Determinant

If $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{BA} = \boldsymbol{I}$

Invertible matrix

$\boldsymbol{A}$: square matrix of order n.

$\boldsymbol{A}$ is invertible

if there exists a square matrix $\boldsymbol{B}$ of order n such that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \text{ and } \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

only one

The matrix $\boldsymbol{B}$ here is called the inverse of $\boldsymbol{A}$.

We use $\boldsymbol{A}^{-1}$ to denote this unique inverse of $\boldsymbol{A}$.

A square matrix is called singular if it has no inverse.

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True or False

$A = B \Rightarrow CA = CB$ Always true

$\boxed{CA = CB \Rightarrow A = B \quad \text{True if } C \text{ is invertible}}$

Cancellation law

$C^{-1}CA = C^{-1}CB \longrightarrow IA = IB$

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Show Inverse

Given $\boldsymbol{A}$ is a square matrix and $\boldsymbol{A}^2 + \boldsymbol{A} = \boldsymbol{I}$.

Show : $\boldsymbol{A}^{-1} = \boldsymbol{A} + \boldsymbol{I}$

Proof:

Start with $\boldsymbol{A}^2 + \boldsymbol{A} = \boldsymbol{I}$

Multiply $\boldsymbol{A}^{-1}$ on both sides

$$\boldsymbol{A}^{-1}(\boldsymbol{A}^2 + \boldsymbol{A}) = \boldsymbol{A}^{-1}\boldsymbol{I}$$

$$\boldsymbol{A}^{-1}\boldsymbol{A}^2 + \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}^{-1}$$

$$(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{A} + \boldsymbol{I} = \boldsymbol{A}^{-1}$$

$$\boldsymbol{I}\boldsymbol{A} + \boldsymbol{I} = \boldsymbol{A}^{-1}$$

Conclusion : $\boldsymbol{A}^{-1} = \boldsymbol{A} + \boldsymbol{I}$

Anything wrong with the proof?

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Show Inverse

Given $\boldsymbol{A}$ is a square matrix and $\boldsymbol{A}^2 + \boldsymbol{A} = \boldsymbol{I}$.

Show : $\boldsymbol{A}^{-1} = \boldsymbol{A} + \boldsymbol{I}$ ← to show $\boldsymbol{A} + \boldsymbol{I}$ is the inverse of $\boldsymbol{A}$

To show $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$, we just need to show $\boldsymbol{BA} = \boldsymbol{I}$ or $\boldsymbol{AB} = \boldsymbol{I}$

$$\boldsymbol{A}(\boldsymbol{A}+\boldsymbol{I}) = \boldsymbol{A}^2 + \boldsymbol{A} = \boldsymbol{I}$$

algebraic manipulation use given condition

This implies
- $\boldsymbol{A}$ is invertible
- $\boldsymbol{A}^{-1} = \boldsymbol{A} + \boldsymbol{I}$

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Invertibility and matrix operations

$\boldsymbol{A}$, $\boldsymbol{B}$: two invertible matrices (same size)
a : non-zero scalar

	Matrix	Invertible?	Inverse
Scalar multiplication	$a\boldsymbol{A}$	yes	$(a\boldsymbol{A})^{-1} = (1/a)\boldsymbol{A}^{-1}$
Transpose	$\boldsymbol{A}^T$	yes	$(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$
Inverse	$\boldsymbol{A}^{-1}$	yes	$(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
Matrix multiplication	$\boldsymbol{AB}$	yes	$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

$$(\boldsymbol{AB}\ldots\boldsymbol{Z})^{-1} = \boldsymbol{Z}^{-1}\ldots\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

Note:
$\boldsymbol{A}$ and $\boldsymbol{B}$ invertible DOES NOT IMPLY $\boldsymbol{A} + \boldsymbol{B}$ is invertible
$(\boldsymbol{A}+\boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$ ← FALSE

Exercise 2 Q29

$\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices of the same size. Suppose $\boldsymbol{A} + \boldsymbol{B}$ is invertible. Show that:

(i) $\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$ is invertible and

(ii) $(\boldsymbol{A}+\boldsymbol{B})^{-1} = \boldsymbol{A}^{-1}(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})^{-1}\boldsymbol{B}^{-1}$

For (i), not easy to find a matrix $\boldsymbol{M}$ such that $(\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1})\boldsymbol{M} = \boldsymbol{I}$

For (ii), we consider the inverses of both sides:

We try to show: $(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{B}(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})\boldsymbol{A}$

This is the same as showing: $\boldsymbol{B}^{-1}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{A}^{-1} = (\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})$

A and **B** are invertible matrices of the same size. Suppose **A** + **B** is invertible. Show that:

(i) $\mathbf{A}^{-1} + \mathbf{B}^{-1}$ is invertible and

(ii) $(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1}(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}\mathbf{B}^{-1}$

Exercise 2 Q29

(ii) Is the same as showing: $\boldsymbol{B}^{-1}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{A}^{-1} = (\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})$

$$\boldsymbol{B}^{-1}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}$$

$$= \boldsymbol{B}^{-1}\boldsymbol{I} + \boldsymbol{I}\boldsymbol{A}^{-1}$$

$$= \boldsymbol{B}^{-1} + \boldsymbol{A}^{-1}$$

$$= \boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$$

This shows part (ii)

To show (i), note that $\boldsymbol{B}^{-1}$, $\boldsymbol{A}+\boldsymbol{B}$ and $\boldsymbol{A}^{-1}$ are all invertible

so the product $\boldsymbol{B}^{-1}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{A}^{-1}$ is invertible

This means $\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$ is invertible

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Elementary matrices

A square matrix is called an elementary matrix
if it can be obtained from an identity matrix
by performing a single elementary row operation.

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2R_2} \boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{2R_2} \boldsymbol{B} = \boldsymbol{E}_1\boldsymbol{A}$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{R_2 \leftrightarrow R_3} \boldsymbol{C} = \boldsymbol{E}_2\boldsymbol{A}$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 + 2R_1} \boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{R_3 + 2R_1} \boldsymbol{D} = \boldsymbol{E}_3\boldsymbol{A}$$

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Which are elementary?

Which of the following are elementary matrices ?

$0R_2$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$R_1 + 2R_3$

$R_3 + R_1$

$R_1 \leftrightarrow R_3$

$R_2 \leftrightarrow R_1$

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\boxed{\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$-R_1$

All elementary matrices are invertible

The inverse of an elementary matrix is also an elementary matrix

Inverse of elementary matrix

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2R_2} \boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{E}_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{E}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 + 2R_1} \boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \longrightarrow \boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{E}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

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Gaussian elimination

$$\boldsymbol{A} \longrightarrow \quad \longrightarrow \quad \longrightarrow \quad \cdots \longrightarrow \boldsymbol{R}$$ REF

$\boldsymbol{R}$ is obtained from $\boldsymbol{A}$ by pre-multiplying $\boldsymbol{A}$ with

a series of elementary matrices

$$\boldsymbol{E}_n \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{R}$$

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True or false

Given $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent

I. There is an invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{CA} = \boldsymbol{B}$ True

II. There is an invertible matrix $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{DB}$ True

$$\boldsymbol{A} \longrightarrow \quad \longrightarrow \quad \longrightarrow \quad \cdots \quad \longrightarrow \boldsymbol{B}$$

$$\boldsymbol{E}_n \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{B} \quad \Rightarrow \boldsymbol{CA} = \boldsymbol{B} \quad \Rightarrow \boldsymbol{A} = \boldsymbol{C}^{-1}\boldsymbol{B}$$

Let $\boldsymbol{C} = \boldsymbol{E}_n \ldots \boldsymbol{E}_2 \boldsymbol{E}_1$

Let $\boldsymbol{D} = \boldsymbol{C}^{-1}$

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Finding inverse matrix

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{G.J.E.}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\boxed{(\,\boldsymbol{A} \mid \boldsymbol{I}\,) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} (\,\boldsymbol{I} \mid \boldsymbol{A}^{-1}\,)}$$

$$\boldsymbol{A} \xrightarrow{\text{GJE}} \boldsymbol{I} \qquad\qquad \boldsymbol{I} \xrightarrow{\text{GJE}} \boldsymbol{A}^{-1}$$

$$\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I} \Longrightarrow \boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{I} = \boldsymbol{A}^{-1}$$

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Let’s revise

- An elementary matrix can be obtained by performing exactly one e.r.o. on the identity matrix.
- The action of an e.r.o. on ***A*** is the same as pre-multiplying an elementary matrix on ***A***
- There are three types of elementary matrices
- All elementary matrices are invertible
- The inverse of an elementary matrix is an elementary matrix.

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A very important theorem

Let $\boldsymbol{A}$ be a square matrix.

The following statements are equivalent

1. $\boldsymbol{A}$ is invertible.
2. The linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
3. The reduced row-echelon form of $\boldsymbol{A}$ is an identity matrix.
4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

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REF and invertible matrix

To check whether a square matrix is invertible:

$n \times n$

- Look at the RREF
  - RREF = $\boldsymbol{I}$ implies invertible — n non-zero rows
  - RREF ≠ $\boldsymbol{I}$ implies not invertible — < n non-zero rows
- Look at REF
  - REF has no zero row implies invertible — n non-zero rows
  - REF has zero rows implies not invertible — < n non-zero rows

1. $\boldsymbol{A}$ is invertible.
2. The linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
3. The reduced row-echelon form of $\boldsymbol{A}$ is an identity matrix.
4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

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Exercise 2 Q44(b)

$\boldsymbol{A}$ is m x n and $\boldsymbol{B}$ is n x m.
If m > n, can $\boldsymbol{AB}$ be invertible?

So $\boldsymbol{AB}$ is NOT invertible

$$\boldsymbol{Bx} = \boldsymbol{0}$$

This homogeneous system has m variables and n equations

The system has infinitely many solutions

So it has a non-trivial solution, say $\boldsymbol{x} = \boldsymbol{u}$

$$\boldsymbol{Bu} = \boldsymbol{0}$$

$$\boldsymbol{ABu} = \boldsymbol{A0} = \boldsymbol{0}$$

So $\boldsymbol{ABx} = \boldsymbol{0}$ has a non-trivial solution $\boldsymbol{u}$

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Product of elementary matrices

Express $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$ as a product of elementary matrices

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{-R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 - R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

$$\boldsymbol{E}_5\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{I}$$

$$\boldsymbol{A} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\boldsymbol{E}_3^{-1}\boldsymbol{E}_4^{-1}\boldsymbol{E}_5^{-1}\boldsymbol{I}$$

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Elementary column operations

Perform e.c.o. C to a matrix $\boldsymbol{A}$ is the same as post-multiply a certain elementary matrix $\boldsymbol{E}$ to $\boldsymbol{A}$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2C_2} \boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{2C_2} \boldsymbol{B} = \boldsymbol{AE}_1$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_3} \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{C_2 \leftrightarrow C_3} \boldsymbol{C} = \boldsymbol{AE}_2$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_3 + 2C_1} \boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{C_3 + 2C_1} \boldsymbol{D} = \boldsymbol{AE}_3$$

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True or false

We can reduce a matrix $\boldsymbol{A}$ to REF by performing a series of elementary column operations (e.c.o).

False

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

If $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{BA} = \boldsymbol{I}$

Exercise 2 Q45

$R_1, R_2, \ldots, R_n$ are e.r.o. corresponding to some elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_n$.
$C_1, C_2, \ldots, C_n$ are e.c.o. corresponding to the same elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_n$.

If

$$\boldsymbol{A} \xrightarrow{R_1} \xrightarrow{R_2} \cdots \xrightarrow{R_n} \boldsymbol{I}$$

then

$$\boldsymbol{A} \xrightarrow{C_n} \xrightarrow{C_{n-1}} \cdots \xrightarrow{C_1} \boldsymbol{I}$$

$\boxed{\text{True}}$ or false?

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Determinant

For a 2 x 2 matrix, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

For a 3 x 3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\boldsymbol{M}_{11}} - a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\boldsymbol{M}_{12}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\boldsymbol{M}_{13}}$$

cofactor expansion along row 1

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Cofactor expansion

$$\det(\mathbf{A}) = a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}$$

cofactor expansion along row 1

$$A_{ij} = (-1)^{i+j}\det(\boldsymbol{M_{ij}})$$ (i, j)-cofactor of $\boldsymbol{A}$

$$\det(\mathbf{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

cofactor expansion along row i

for any $j = 1, 2, \dots, n$

$$\det(\mathbf{A}) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

cofactor expansion along column j

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Finding determinant

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

cofactor expansion along column 3

$$\det(\boldsymbol{A}) = 2 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 2\times 1 \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2\times 1(2\times 2 - 2\times 1) = 2$$

cofactor expansion along column 1

If ***A*** is a triangular matrix,
then the determinant of ***A*** is equal to
the product of the diagonal entries of ***A***.

Same determinant?

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

(A) All 3 have same determinant
(B) Only ***A*** and ***B***
(C) Only ***B*** and ***C***
(D) Only ***A*** and ***C***
(E) All 3 have different determinant

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Announcement

- Practice Session
  - Practice 1 this week
  - No meeting in week 4
  - Practice 2 week 5
- MATLAB
  - Worksheet 1 this week (own pace)
- Textbook exercise
  - Exercise 1 solution in LumiNUS > Files
- Online quiz 3
  - Due this Sunday