

Section 2.5

Determinants

Objectives

- How do matrix operations affect determinants?
- What is the relation between invertibility and determinant?
- What is the **adjoint** of a matrix?
- What is **Cramer's rule**?

Two main results:

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

A is invertible
if and only if
 $\det(\mathbf{A}) \neq 0$

Matrices with two identical rows (columns)

Theorem 2.5.12 & Example 2.5.13

1. The determinant of a square matrix with **two identical rows** is zero.
2. The determinant of a square matrix with **two identical columns** is zero.

$$\begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}$$

$$\det = 0$$

$$\begin{pmatrix} 1 & 2 & 4 \\ -1 & 10 & 4 \\ 1 & 2 & 4 \end{pmatrix}$$

$$\det = 0$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & -3 & -3 & 9 \\ 2 & 4 & 4 & 0 \\ 0 & -2 & -2 & -1 \end{pmatrix}$$

$$\det = 0$$

Theorem 2.5.12 (Exercise 2.58)

Prove by mathematical induction.

The determinant of a square matrix with two identical rows is zero

1 x 1 matrix cannot because no identical

Base case

2x2:
$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab$$

Inductive step $k \times k \Rightarrow (k+1) \times (k+1)$

3x3:
$$\begin{vmatrix} a & b & c \\ * & * & * \\ a & b & c \end{vmatrix} = - * \begin{vmatrix} b & c \\ b & c \end{vmatrix} + * \begin{vmatrix} a & c \\ a & c \end{vmatrix} - * \begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

cofactor expansion along row 2

How does e.r.o affect determinants?

Discussion 2.5.14 & Theorem 2.5.15

$$\mathbf{A} \xrightarrow{E.R.O.} \mathbf{B}$$

What is the relation between $\det(\mathbf{A})$ and $\det(\mathbf{B})$?

E.R.O	Determinant
$\mathbf{A} \xrightarrow{kR_i} \mathbf{B}$	$\det(\mathbf{B}) = k \det(\mathbf{A})$
$\mathbf{A} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{B}$	$\det(\mathbf{B}) = -\det(\mathbf{A})$
$\mathbf{A} \xrightarrow{R_i + kR_j} \mathbf{B}$	$\det(\mathbf{B}) = \det(\mathbf{A})$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{3R_1} \mathbf{B} = \begin{pmatrix} 3 & 3 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\det(\mathbf{B}) = 3\det(\mathbf{A})$$

$$\mathbf{A} = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \mathbf{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$$

$$\det(\mathbf{B}) = -\det(\mathbf{A})$$

Using e.r.o. to find determinants

Example 2.5.17.1

working on the determinant directly

$$\begin{vmatrix} 3 & -1 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{vmatrix} \quad R_2 - R_1 = \begin{vmatrix} 3 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{vmatrix}$$

$$R_2 \leftrightarrow R_3 = \begin{vmatrix} 3 & -1 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{vmatrix} \quad R_4 - 2R_3 = \begin{vmatrix} 3 & -1 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= -3 \times 2 \times 1 \times (-1) = 6$$

Gaussian Elimination = REF = upper triangular matrix

Using e.r.o. to find determinants

Example 2.5.17.2

$$\mathbf{A} \xrightarrow{R_1 + \frac{2}{9}R_2} \mathbf{C} \xrightarrow{R_2 \leftrightarrow R_3} \mathbf{D} \xrightarrow{4R_2} \mathbf{B} = \begin{pmatrix} 5 & 0 & 8 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Find $\det(\mathbf{A})$.

$\det(\mathbf{B}) = 4 \det(\mathbf{D})$

$$\det(\mathbf{B}) = 5 \times (-2) \times 1 \times \frac{1}{3} = -\frac{10}{3}$$

outcome was $4R_2$
so to find back original need
flip: $\frac{1}{4}$

$$\det(\mathbf{A}) = \det(\mathbf{C}) = \underline{-} \det(\mathbf{D}) = -\boxed{\frac{1}{4}} \det(\mathbf{B}) = \frac{5}{6}$$

row interchange

Theorem 2.5.15

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{R_2 + kR_1} \mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + ka_{11} & a_{22} + ka_{12} & a_{23} + ka_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

the $(2, j)$ -cofactor of \mathbf{A} = the $(2, j)$ -cofactor of \mathbf{B}

$$A_{21} = B_{21} \quad A_{22} = B_{22} \quad A_{23} = B_{23}$$

Cofactor expansion along row 2 of \mathbf{B} :

$$\begin{aligned} \det(\mathbf{B}) &= (a_{21} + ka_{11})B_{21} + (a_{22} + ka_{12})B_{22} + (a_{23} + ka_{13})B_{23} \\ &= (a_{21} + ka_{11})A_{21} + (a_{22} + ka_{12})A_{22} + (a_{23} + ka_{13})A_{23} \\ &= \underbrace{(a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23})}_{\det(\mathbf{A})} + k \underbrace{(a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23})}_{\cancel{\det(\mathbf{A})}} \end{aligned}$$

Proof of part 3

To prove: $\det(\mathbf{B}) = \det(\mathbf{A})$

Theorem 2.5.15

shown = 0

$$\det(\mathbf{B}) = \det(\mathbf{A}) + k(a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23})$$

entries on row 1

cofactors for row 2

replace row 2 of \mathbf{A} by row 1

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{aligned} A_{21} &= A'_{21} \\ A_{22} &= A'_{22} \\ A_{23} &= A'_{23} \end{aligned}$$

$$\mathbf{A}' = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

cofactor expansion (\mathbf{A}') along row 2

$$\begin{aligned} \det(\mathbf{A}') &= a_{11}A'_{21} + a_{12}A'_{22} + a_{13}A'_{23} \\ &= a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} \\ &= 0 \end{aligned}$$

two identical rows

In terms of elementary matrices

Theorem 2.5.15 (part 4)

\mathbf{A} : nxn square matrix

	$\det(\mathbf{E})$	e.r.o.	Determinant
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}$	k	$\mathbf{A} \xrightarrow{kR_i} \mathbf{B}$	$\det(\mathbf{B}) = k \det(\mathbf{A})$ $= \det(\mathbf{E})\det(\mathbf{A})$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	-1	$\mathbf{A} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{B}$	$\det(\mathbf{B}) = -\det(\mathbf{A})$ $= \det(\mathbf{E})\det(\mathbf{A})$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}$	1	$\mathbf{A} \xrightarrow{R_i + kR_j} \mathbf{B}$	$\det(\mathbf{B}) = \det(\mathbf{A})$ $= \det(\mathbf{E})\det(\mathbf{A})$

\mathbf{E} : nxn elementary matrix

$$\mathbf{EA} = \mathbf{B} \Rightarrow \det(\mathbf{EA}) = \det(\mathbf{B}) = \det(\mathbf{E})\det(\mathbf{A})$$

Pre-multiplying a matrix with elementary matrices

Remark

For any square matrix \mathbf{A} and elementary matrix \mathbf{E} :

$$\det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A})$$

$$\begin{aligned}\det(\mathbf{E}_2\mathbf{E}_1\mathbf{A}) &= \det(\mathbf{E}_2)\det(\mathbf{E}_1\mathbf{A}) \\ &= \det(\mathbf{E}_2)\det(\mathbf{E}_1)\det(\mathbf{A})\end{aligned}$$

$$\det(\mathbf{E}_k\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \det(\mathbf{E}_k)\cdots\det(\mathbf{E}_2)\det(\mathbf{E}_1)\det(\mathbf{A})$$

In particular $\downarrow \mathbf{I} \neq \mathbf{A} = \mathbf{I}, \det(\mathbf{I}) = 1$

$$\det(\mathbf{E}_k\cdots\mathbf{E}_2\mathbf{E}_1) = \det(\mathbf{E}_k)\cdots\det(\mathbf{E}_2)\det(\mathbf{E}_1)$$

Note: We have **not yet** proved that

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

Remark 2.5.18

Since $\det(\mathbf{A}) = \det(\mathbf{A}^T)$ for any square matrix \mathbf{A} , Theorem 2.5.15 is still true if we change “rows” to “columns”.

- We have 3 corresponding elementary column operations.
- Column operations have same effect as post-multiplying an elementary matrix.

$$\mathbf{A} \xrightarrow{\text{C: column operation}} \mathbf{B} \qquad \mathbf{B} = \mathbf{A}\mathbf{E}$$

Remark 2.5.18 = (Theorem 2.5.15)^T

A : nxn square matrix

E : nxn elementary matrix

Elementary column operation	Determinant
$\mathbf{A} \xrightarrow{kC_i} \mathbf{B}$	$\det(\mathbf{B}) = k \det(\mathbf{A})$
$\mathbf{A} \xrightarrow{C_i \leftrightarrow C_j} \mathbf{B}$	$\det(\mathbf{B}) = -\det(\mathbf{A})$
$\mathbf{A} \xrightarrow{C_i + kC_j} \mathbf{B}$	$\det(\mathbf{B}) = \det(\mathbf{A})$

$$\det(\mathbf{AE}) = \det(\mathbf{A})\det(\mathbf{E})$$

Determinant and invertibility

Theorem 2.5.19

A square matrix \mathbf{A} is invertible
if and only if
 $\det(\mathbf{A}) \neq 0$.

contrapositive

converse

\mathbf{A} is invertible $\Rightarrow \det(\mathbf{A}) \neq 0$

\mathbf{A} is invertible $\Leftarrow \det(\mathbf{A}) \neq 0$

\mathbf{A} is not invertible $\Rightarrow \det(\mathbf{A}) = 0$

\mathbf{A} is not invertible $\Leftarrow \det(\mathbf{A}) = 0$

different
meaning

different
meaning

same
meaning

The proof

A square matrix **A** is invertible
if and only if
 $\det(\mathbf{A}) \neq 0$.

Theorem 2.5.19

$$\mathbf{A} \xrightarrow{R_1} \xrightarrow{R_2} \dots \xrightarrow{R_k} \mathbf{B} \text{ (RREF)}$$

$$\mathbf{E}_k \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{B}$$

$$\det(\mathbf{E}_k \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A}) = \det(\mathbf{B})$$

$$\underbrace{\det(\mathbf{E}_k) \cdots \det(\mathbf{E}_2) \det(\mathbf{E}_1)}_{\text{non-zero}} \det(\mathbf{A}) = \det(\mathbf{B})$$

A invertible \Rightarrow RREF **B** = **I** $\Rightarrow \det(\mathbf{B}) = 1 \Rightarrow \det(\mathbf{A}) \neq 0$

A not invertible \Rightarrow RREF **B** has zero row
 $\Rightarrow \det(\mathbf{B}) = 0 \Rightarrow \det(\mathbf{A}) = 0$

Using determinant to check invertibility

Remark 2.5.21

check invertibility

Theorem 2.5.19

Remark 2.4.10

determinant

row echelon form

- When the determinant is easy to get
- Connecting concepts

In practice

What is the **actual value** of determinant for?

Determinant and matrix operations

Theorem 2.5.22

A and **B** : square matrices of order n

c a scalar

$$\det(c\mathbf{A}) \neq c \det(\mathbf{A})$$

1. $\det(c\mathbf{A}) = c^n \det(\mathbf{A})$ because multiplying c into a matrix
- every row will multiply by c

2. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ Multiplicative property

3. if **A** is invertible, then

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} \quad \text{inverse the determinant}$$

4. $\det(\mathbf{A} + \mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})$

WARNING

Proof of part 2

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

$$\det(\mathbf{EB}) = \det(\mathbf{E})\det(\mathbf{B})$$

multiply singular

matrix to another will
create singular matrix

Theorem 2.5.22

Case 1: \mathbf{A} is singular

By Theorem 2.4.14

\mathbf{AB} is singular

$$\Rightarrow \det(\mathbf{A}) = 0$$

$$\Rightarrow \det(\mathbf{A}) \det(\mathbf{B}) = 0$$

$$\Rightarrow \det(\mathbf{AB}) = 0$$

Case 2: \mathbf{A} is invertible

Theorem 2.4.7: (1) implies (4)

$$\Rightarrow \mathbf{A} = \mathbf{E}_1 \mathbf{E}_2 \cdots \mathbf{E}_k \text{ (product of elementary matrices)}$$

$$\det(\mathbf{AB}) = \det(\mathbf{E}_1 \mathbf{E}_2 \cdots \mathbf{E}_k \mathbf{B})$$

$$= \det(\mathbf{E}_1) \det(\mathbf{E}_2) \cdots \det(\mathbf{E}_k) \det(\mathbf{B})$$

$$= \det(\mathbf{E}_1 \mathbf{E}_2 \cdots \mathbf{E}_k) \det(\mathbf{B})$$

$$= \det(\mathbf{A}) \det(\mathbf{B})$$

What is adjoint?

Definition 2.5.24

Let \mathbf{A} be a square matrix of order n .

The **adjoint** of \mathbf{A} is the $n \times n$ matrix

$$\text{adj}(\mathbf{A}) = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

where A_{ij} is the (i, j) -cofactor of \mathbf{A} .

$$(-1)^{i+j} \det(\mathbf{M}_{ij})$$

What is adjoint?

Example 2.5.26.2

$$\mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\text{adj}(\mathbf{B}) = \begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} -3 & 0 & 1 \\ 3 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}^T = \begin{pmatrix} -3 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

What is adjoint for?

Theorem 2.5.25

Let \mathbf{A} be a square matrix.

If \mathbf{A} is invertible, then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A})$$

Example 2.5.26.2

$$\mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix} \quad \det(\mathbf{B}) = -2 \quad \text{adj}(\mathbf{B}) = \begin{pmatrix} -3 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$
$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \text{adj}(\mathbf{B}) = -\frac{1}{2} \begin{pmatrix} -3 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

The proof

Theorem 2.5.25

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad \text{adj}(\mathbf{A}) = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

the (i, i) -entry of $\mathbf{A}[\text{adj}(\mathbf{A})]$

diagonal entries

$$= a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} = \det(\mathbf{A})$$

cofactor expansion along row i

the (i, j) -entry of $\mathbf{A}[\text{adj}(\mathbf{A})]$

with $i \neq j$,

$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \dots + a_{in}A_{jn}$$

non-diagonal entries

$$= 0$$

see the proof of Theorem 2.5.15.3.

The proof

Theorem 2.5.25

$$\mathbf{A} \operatorname{adj}(\mathbf{A}) = \begin{pmatrix} \det(\mathbf{A}) & 0 & \dots & 0 \\ 0 & \det(\mathbf{A}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \det(\mathbf{A}) \end{pmatrix} = \det(\mathbf{A}) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \det(\mathbf{A}) \mathbf{I}$$

because $\det(\mathbf{A})$ is a scalar

$$\Rightarrow \mathbf{A} \left[\frac{1}{\det(\mathbf{A})} \operatorname{adj}(\mathbf{A}) \right] = \mathbf{I}$$

$$\Rightarrow \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \operatorname{adj}(\mathbf{A})$$

Using adjoint to find inverse

Remark

find inverse

Theorem 2.5.25

Discussion 2.4.8

adjoint

for smaller size

reduced row echelon form

for larger size

give explicit formula
for inverse

What is Cramer's rule?

Example 2.5.28 (Cramer's rule)

Use **Cramer's rule** to solve the system of linear equations

$$\begin{cases} x + y + 3z = 0 \\ 2x - 2y + 2z = 4 \\ 3x + 9y = 3 \end{cases}$$

Rewrite the linear system as

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 1 & 3 \\ 4 & -2 & 2 \\ 3 & 9 & 0 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 0 \end{pmatrix} \quad \mathbf{A}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \\ 3 & 9 & 3 \end{pmatrix}$$

What is Cramer's rule?

$$\begin{cases} x + y + 3z = 0 \\ 2x - 2y + 2z = 4 \\ 3x + 9y = 3 \end{cases}$$

Example 2.5.28

Cramer's rule says:

$$X = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 0 & 1 & 3 \\ 4 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \frac{132}{60} = 2.2$$

$$Y = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \frac{-24}{60} = -0.4$$

$$Z = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \\ 3 & 9 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \frac{-36}{60} = -0.6$$

this gives
the unique
solution of
the system

What is Cramer's rule?



Theorem 2.5.27 (Cramer's Rule)

Suppose $\mathbf{Ax} = \mathbf{b}$ is a linear system where \mathbf{A} is an $n \times n$ invertible matrix.

terms and conditions

Let \mathbf{A}_i be the matrix obtained from \mathbf{A} by replacing the i th column of \mathbf{A} by \mathbf{b} .

Then the system has a unique solution

$$\mathbf{x} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} \det(\mathbf{A}_1) \\ \det(\mathbf{A}_2) \\ \vdots \\ \det(\mathbf{A}_n) \end{pmatrix}$$

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})}$$

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})}$$

$$x_n = \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A})}$$

cannot be 0

The proof

Proof of Theorem 2.5.27

To prove:

$$X_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \Leftrightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A})\mathbf{b}$$

because A is invertible

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$x_1 = \frac{1}{\det(\mathbf{A})} (b_1 A_{11} + b_2 A_{21} + \dots + b_n A_{n1})$$

$$x_i = \frac{1}{\det(\mathbf{A})} (b_1 A_{1i} + b_2 A_{2i} + \dots + b_n A_{ni})$$

for $i = 1, 2, \dots, n$

To show

The proof

To prove:

$$X_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$$

Proof of Theorem 2.5.27

entries on column i

$$X_i = \frac{1}{\det(\mathbf{A})} (b_1 A_{1i} + b_2 A_{2i} + \dots + b_n A_{ni}) = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$$

cofactors for column i

cofactor expansion of \mathbf{A}_i along column i

$$\det(\mathbf{A}_i) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$$

SLIDE 27

Section 3.1

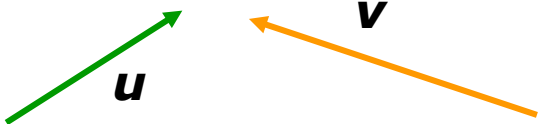
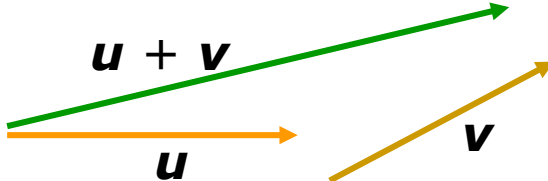

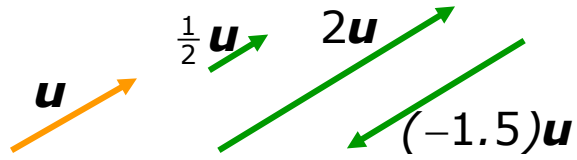
Euclidean n-Spaces

Objectives

- What is an n -vector?
- What are some operations on n -vectors?
- What is a Euclidean n -space \mathbf{R}^n ?
- How to express subsets of \mathbf{R}^n ?

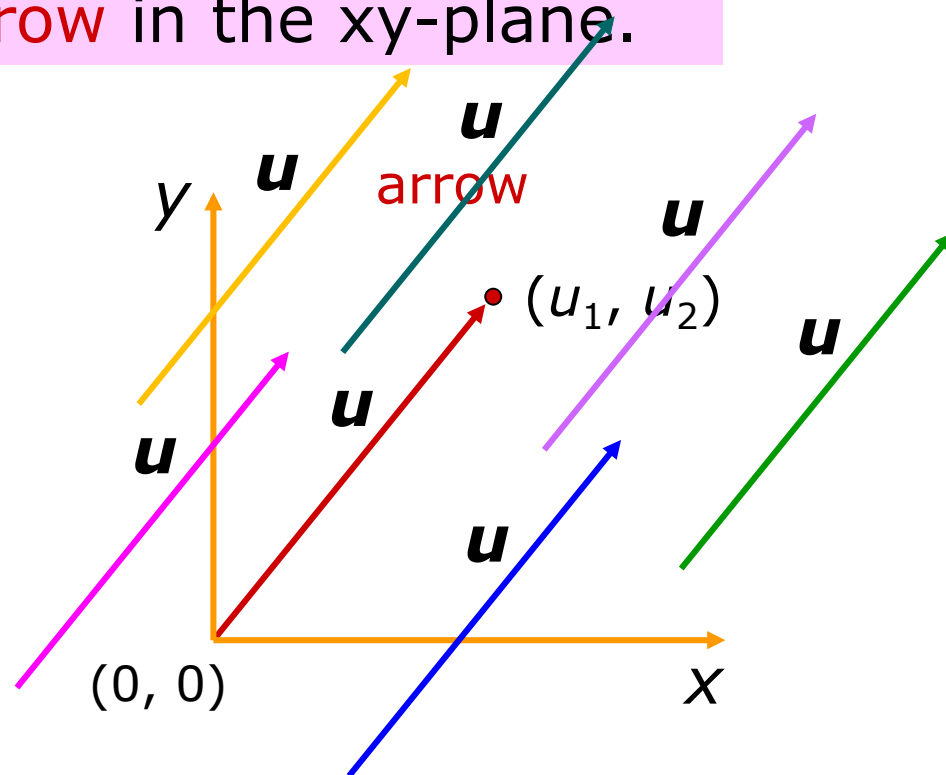
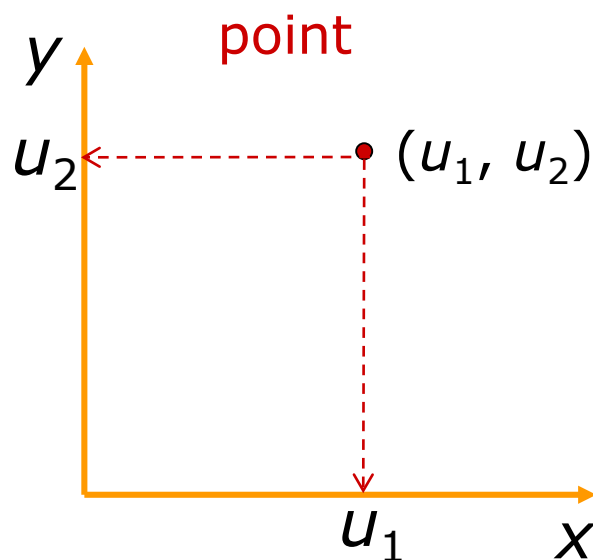
What is a vector?

Discussion 3.1.1 - 3.1.2 (Vectors)

Notation	geometric	algebraic (2-dimension) (3-dimension)
Vector \mathbf{u} Vector \mathbf{v}		(u_1, u_2) (v_1, v_2) (u_1, u_2, u_3) (v_1, v_2, v_3)
Addition $\mathbf{u} + \mathbf{v}$		$(u_1 + v_1, u_2 + v_2)$ $(u_1 + v_1, u_2 + v_2, u_3 + v_3)$
Negative $-\mathbf{u}$		$(-u_1, -u_2)$ $(-u_1, -u_2, -u_3)$
Scalar multiple $a\mathbf{u}$		(au_1, au_2) (au_1, au_2, au_3)

Discussion 3.1.2.1

Geometrically, (u_1, u_2) can represent either a **point** and an **arrow** in the xy -plane.



Similarly for (u_1, u_2, u_3) in the xyz -space.

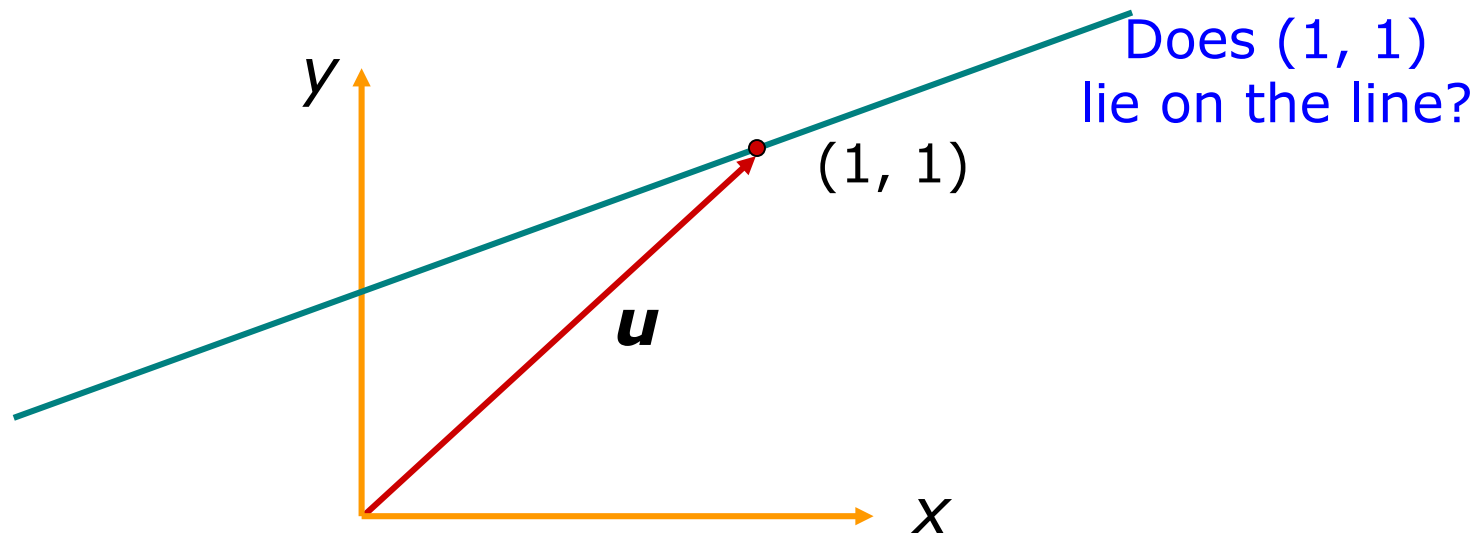
thinking about a set of points

- can lie on a line to describe the points

Point or Arrow?

Geometrically, (u_1, u_2) can represent either a **point** and an **arrow** in the xy -plane.

Linear equation: $2y - x = 1$ A solution: $x = 1, y = 1$



The **point** $(1, 1)$ lies on the line,
but the **arrow** $(1, 1)$ does not lie on the line

What is an n-vector?

(u_1, u_2) 2-vector
 (u_1, u_2, u_3) 3-vector

Definition 3.1.3

***n*-vector** $(u_1, u_2, \dots, u_{j'}, \dots, u_n) \neq \{u_1, u_2, \dots, u_{j'}, \dots, u_n\}$

where u_1, u_2, \dots, u_n are real numbers

j^{th} **component** (or j^{th} **coordinate**) of the n-vector

Always think/view an n-vector
as a **SINGLE object**
and not **n numbers**

n -vectors as matrices

Notation 3.1.5

vector need commas

We can identify an n -vector (u_1, u_2, \dots, u_n) with a $1 \times n$ matrix $(u_1 \ u_2 \ \dots \ u_n)$ (row vector)

or an $n \times 1$ matrix $\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ (column vector).

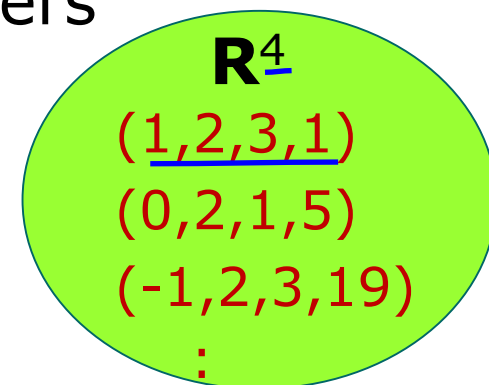
Which form to use depends on the context.

Definitions and properties of vector operations are similar to matrix operations (see 3.1.3 to 3.1.6)

What is a Euclidean n -space ?

Definition 3.1.7

The set of all n -vectors of real numbers is called the **Euclidean n -space** and is denoted by \mathbf{R}^n .



u is an element in set \mathbf{R}^n

$$\mathbf{u} \in \mathbf{R}^n \longleftrightarrow \mathbf{u} \text{ is an } n\text{-vector} \longleftrightarrow \mathbf{u} = (u_1, u_2, \dots, u_n)$$

Euclidean 2-space \mathbf{R}^2

all the 2-vectors (as points) in **xy-plane**

Euclidean 3-space \mathbf{R}^3

all the 3-vectors (as points) in **xyz-space**

How to express subsets of \mathbf{R}^n ?

Example 3.1.8.1

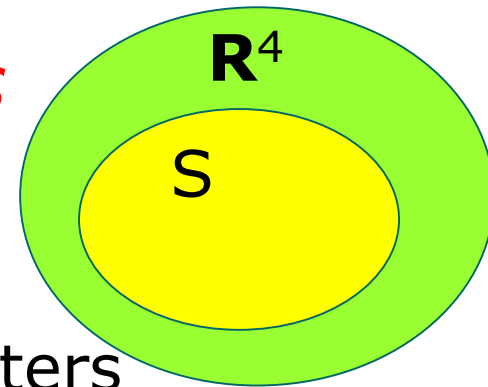
Set notation

→ $S = \{ \underbrace{(u_1, u_2, u_3, u_4)}_{\substack{\text{type of elements} \\ \text{is 4-vector}}} \mid \underbrace{u_1 = 0 \text{ and } u_2 = u_4}_{\substack{\text{conditions on the components} \\ \text{determine whether they are in green or yellow zone}}} \}$ **implicit form**

$(0, 0, 0, 0), (0, 1, 5, 1), (0, \pi, -3, \pi) \in S$

$(0, 2, 2, 3), (1, 1, 1, 1) \notin S$

general form $(0, a, b, a)$ a, b : parameters



→ $S = \{ (0, a, b, a) \mid a, b \in \mathbf{R} \}$ **explicit form**

take the conditions and put directly

How to express subsets of \mathbf{R}^n ?

Set notation for subsets of \mathbf{R}^n

Implicit form

$$\left\{ \begin{array}{l} \text{general n-tuple} \\ (u_1, u_2, \dots, u_n) \end{array} \middle| \begin{array}{l} \text{conditions satisfied} \\ \text{by } u_1, u_2, \dots, u_n \end{array} \right\}$$

$$S = \{ (u_1, u_2, u_3, u_4) \mid u_1 = 0 \text{ and } u_2 = u_4 \}$$

Explicit form Not always possible to express in explicit form
- condition too complicated etc.

$$\left\{ \begin{array}{l} \text{n-tuples with} \\ \text{explicit form} \end{array} \middle| \begin{array}{l} \text{range of parameters} \\ \text{appearing on the left} \end{array} \right\}$$

$$S = \{ (0, a, b, a) \mid a, b \in \mathbf{R} \}$$

Don't write $\{a, b \in \mathbf{R} \mid (0, a, b, a)\}$

Solution set as a subset of \mathbf{R}^n

Example 3.1.8.2

$$\begin{cases} x + y + z = 0 \\ x - y + 2z = 1 \end{cases}$$

$$x = 2, y = -1, z = -1$$

infinitely many solutions
because of parameters

- solution SET

general solution:

3 variables

solution set

subset of \mathbf{R}^3

$(2, -1, -1)$

a 3-vector

$$Ax = b \quad x = \begin{pmatrix} a \\ -1 \\ -1 \end{pmatrix}$$

Explicit form

$$\{ (0.5 - 1.5t, -0.5 + 0.5t, t) \mid t \in \mathbf{R} \}$$

Implicit form

$$\{ (x, y, z) \mid x + y + z = 0 \text{ and } x - y + 2z = 1 \}$$

can find linear system from condition

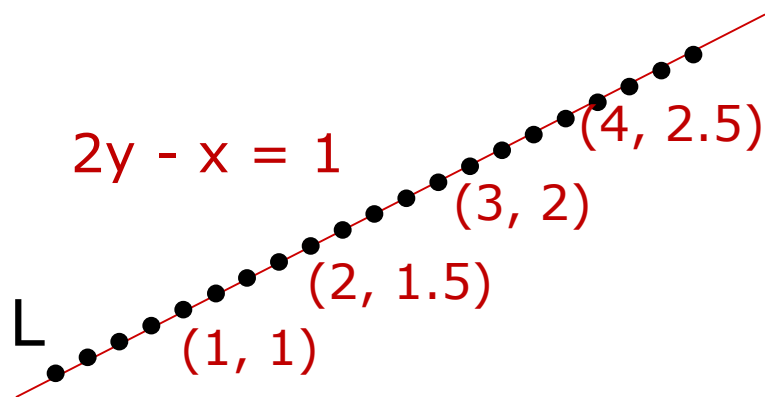
Vector spaces

Line and plane as subsets of \mathbf{R}^2 and \mathbf{R}^3

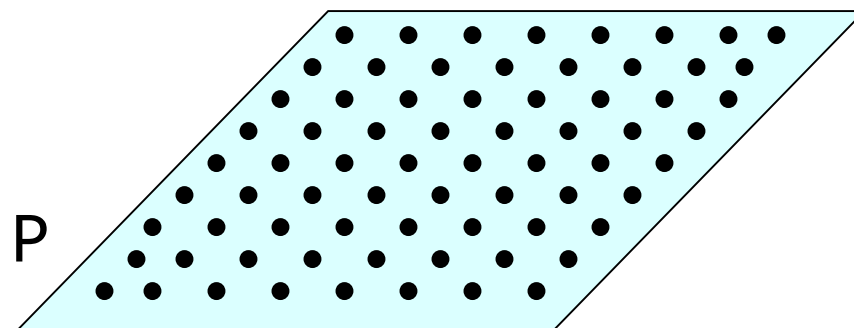
Example 3.1.8.3

A line/plane in the xy-plane/ xyz-space can be regarded as a collection of points.

a collection of vectors



L is a subset of \mathbf{R}^2



P is a subset of \mathbf{R}^3

Set notations of lines and planes

Example 3.1.8.3

Lines in xy-plane

Implicit form: $\{ (x, y) \mid \text{equation of line } ax + by = c \}$

Explicit form: $\{ \left(\frac{c - bt}{a}, t \right) \mid \text{general solution } t \in \mathbf{R} \}$
1 parameter

Planes in xyz-space

Implicit form: $\{ (x, y, z) \mid \text{equation of plane } ax + by + cz = d \}$

Explicit form: $\{ \left(\frac{d - bs - ct}{a}, s, t \right) \mid s, t \in \mathbf{R} \}$
2 parameters

Lines in xyz-space

Implicit form: $\{ (x, y, z) \mid \text{eqn of the line} \} ?$

Explicit form: $\{ (\text{general solution}) \mid 1 \text{ parameter} \} ?$

Line as a subset of \mathbf{R}^3

Example 3.1.8.2 (revisited)

$$\begin{cases} x + y + z = 0 \\ x - y + 2z = 1 \end{cases} \quad \begin{array}{l} \text{two planes} \\ \text{that intersect at a line} \end{array}$$

solution set (explicit form)

$$\{ (0.5 - 1.5t, -0.5 + 0.5t, t) \mid t \in \mathbf{R} \}$$

This represents a line in the xyz-space

$$(0.5, -0.5, 0) + (-1.5t, 0.5t, t)$$

$$(0.5, -0.5, 0) + t(-1.5, 0.5, 1)$$

a point on the line

think as point

the direction of the line

think as arrow

Line as a subset of \mathbf{R}^3

Example 3.1.8.3(c)

Set notation (explicit)

$$\{ (\underline{a_0 + at}, \underline{b_0 + bt}, \underline{c_0 + ct}) \mid t \text{ in } \mathbf{R} \} \quad t: \text{parameter}$$

constant term and parameter term

a_0, b_0, c_0, a, b, c are fixed real numbers

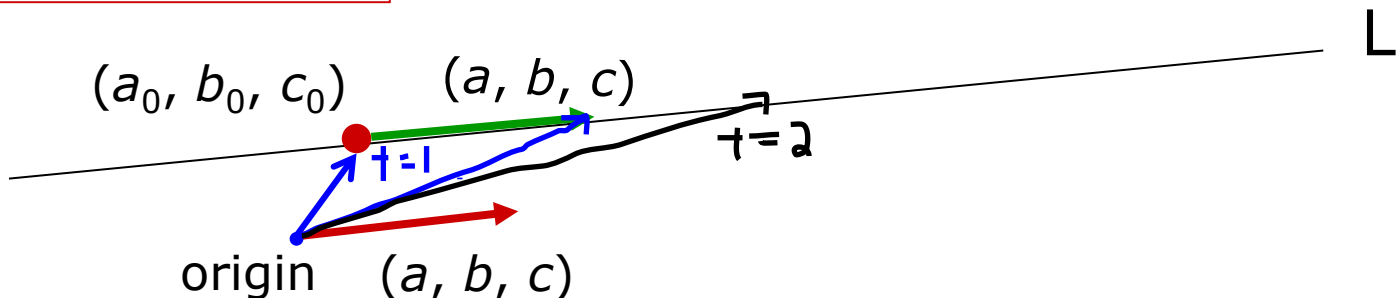
a, b, c not all zero

$$\{ (a_0, b_0, c_0) + t(a, b, c) \mid t \text{ in } \mathbf{R} \}$$

Think of arrow

a point on the line

the direction of the line



Line as a subset of \mathbf{R}^3

Example 3.1.8.3(c)

Set notation (Implicit) $\{ (x, y, z) \mid \text{eqn of the line} \} ?$

A line in \mathbf{R}^3 cannot be represented by a single linear equation.

in \mathbf{R}^3 , a single equation is a plane

But it can be regarded as

the intersection of two planes P_1 and P_2 .

form line

Suppose the equations of the two planes are given by

$$P_1: a_1x + b_1y + c_1z = d_1 \quad P_2: a_2x + b_2y + c_2z = d_2$$

Set notation (implicit)

$$\{ (x, y, z) \mid a_1x + b_1y + c_1z = d_1 \text{ and } a_2x + b_2y + c_2z = d_2 \}$$

refer to 3.1.8.2

Number of elements in a set

Notation 3.1.9 & Example 3.1.10

For a finite set S , we denote the number of elements of S by $|S|$



$$S_1 = \{ 1, 2, 3, 4 \} \quad |S_1| = 4$$

$$S_2 = \{ \underbrace{(1, 2, 3, 4)}_{\text{grouped together}} \} \quad |S_2| = 1$$

$$S_3 = \{ (1, 2, 3), (2, 3, 4) \} \quad |S_3| = 2$$