National University of Singapore

Semester 1, 2020/2021 MA1101R Practice Assignment 1 Solution

1. A certain linear system has the augmented matrix

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 0 & 2 & 2 & c \end{array}\right)$$

for some real numbers a, b, c.

(i) [3 marks] Reduce the augmented matrix to a row echelon form using <u>two</u> elementary row operations (show the two e.r.o. in your working.)

(ii) [3 marks] Write down the condition in terms of a, b, c (if possible) for the system to have (a) no solution; (b) only one solution; (c) infinitely many solutions.

(iii) [2 marks] If the above linear system is a homogeneous system in variables x, y, z (in that order), write down a general solution of this system.

Answer

(i)

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 0 & 2 & 2 & c \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & b \\ 0 & 1 & 1 & a \\ 0 & 2 & 2 & c \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & b \\ 0 & 1 & 1 & a \\ 0 & 0 & 0 & c-2a \end{array}\right)$$

(ii) (a) no solution: $c - 2a \neq 0$;
(b) only one solution: not possible;
(c) infinitely many solutions: $c - 2a = 0$.

(iii) For homogeneous system, $a = b = c = 0$.
Set $z = t$. Then $y = -t$ and $x = -t$.
So the general solution is given by:

$$\begin{cases} x & = & -t \\ y & = & -t \\ z & = & t \end{cases}$$

2. [4 marks] Let

$\boldsymbol{A} = (a_{ij})_{2\times 3}$ with $a_{ij} = 2i - j$ and $\boldsymbol{B} = (b_{ij})_{3\times 2}$ with $b_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ 2 & \text{if } j = 2 \end{cases}$.

Write down $\boldsymbol{A}$ and $\boldsymbol{B}$ explicitly.

Answer

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix}.$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

3. Given that the following linear system is consistent:

$$\begin{cases} x & + & y & = & 1 \\ x & - & y & = & 1 \\ x & - & 3y & = & 1 \\ 3x & + & y & = & 3 \end{cases}$$

(i) [1 mark] Write the linear system in matrix equation form $\boldsymbol{Ax} = \boldsymbol{b}$.

(ii) [2 marks] Compute $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}^T\boldsymbol{b}$ for $\boldsymbol{A}$ and $\boldsymbol{b}$ in part (i).

(iii) [2 marks] Pre-multiply $\boldsymbol{A}^T$ on both sides of the matrix equation in (i), derive the solution of the linear system <u>without using Gaussian Elimination</u>. Show your working.

Answer

(i)

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

(ii) $\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$ and

$\boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$

(iii) Pre-multiplying $\boldsymbol{Ax} = \boldsymbol{b}$ by $\boldsymbol{A}^T$ on both sides gives: $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.
By (ii), this gives

$$\begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

Multiply out the matrices on the left hand side, we get

$$12x = 12 \text{ and } 12y = 0.$$

This gives the solution $x = 1$ and $y = 0$.

4. [3 marks] Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & a & b & 1 \\ 0 & a & b & 1 \\ 0 & 0 & b & 1 \end{array}\right)$$

for some real numbers a and b.
Suppose the linear system is inconsistent and $a \neq 0$. Find the reduced row echelon form of the above augmented matrix. Show how you derive your answer.

Answer

Since $a \neq 0$, for the system to be inconsistent, row 3 of the augmented matrix must be $(0 \ \ 0 \ \ 0 \ \ 1)$. This means b must be 0.

For $b = 0$, the augmented matrix reduces to

$$\left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & a & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{a}R_2} \left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - \frac{1}{a}R_3} \left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - R_3} \left(\begin{array}{ccc|c} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - aR_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

where the last augmented matrix is the reduced row echelon form.