CS1231(S) Tutorial 4: Functions
Solutions

National University of Singapore

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1. Which of the following formulas define a function $f\colon \mathbb{Q} \to \mathbb{Q}$?

   (a) $f(n) = \pm n$.

   (b) $f(n) = 2\sqrt{n}$.

   (c) $f(n) = \frac{1}{n^2+1}$.

   (d) $f(n) = \lfloor \sin n \rfloor$.

   Solution. Formulas (c) and (d) do, while (a) and (b) do not.

   (Here we extend the domain of floor in Definition 6.1.9(1) from $\mathbb{Q}$ to $\mathbb{R}$, because otherwise one would need to worry about whether $\sin 1$ is rational, for example, which is not the intention of this question.)

2. Let U be a set and $A \subseteq U$ such that $\varnothing \neq A \neq U$. Define the function $\chi\colon U \to \mathbb{Z}$ by setting, for all $x \in U$,

   $$\chi(x) = \begin{cases} 0, & \text{if } x \notin A; \\ 1, & \text{if } x \in A. \end{cases}$$

   Find the domain, the codomain, and the image of χ.

   Solution. The domain is U. The codomain is $\mathbb{Z}$. The image is $\{0, 1\}$.

3. Which of the functions defined in the following are injective? Which are surjective? Prove that your answers are correct. If a function defined below is both injective and surjective, then find a formula for the inverse of the function. Here denote by Bool the set $\{\textbf{true}, \textbf{false}\}$.

   $$\begin{aligned} f\colon \mathbb{Q} &\to \mathbb{Q}; \\ x &\mapsto 12x + 31, \end{aligned} \qquad \begin{aligned} g\colon \mathrm{Bool}^2 &\to \mathrm{Bool}; \\ (p, q) &\mapsto p \wedge {\sim} q, \end{aligned} \qquad \begin{aligned} h\colon \mathrm{Bool}^2 &\to \mathrm{Bool}^2; \\ (p, q) &\mapsto (p \wedge q, p \vee q), \end{aligned}$$

   $$\begin{aligned} k\colon \mathbb{Z} &\to \mathbb{Z}; \\ x &\mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases} \end{aligned}$$

   Solution.

   - 1. Note that for all $x, y \in \mathbb{Q}$,

     $$y = 12x + 31 \quad \Leftrightarrow \quad x = (y - 31)/12.$$

     2. Define $f^*\colon \mathbb{Q} \to \mathbb{Q}$ by setting, for all $y \in \mathbb{Q}$,

     $$f^*(y) = (y - 31)/12.$$

3. Then whenever $x, y \in \mathbb{Q}$,

$$y = f(x) \quad \Leftrightarrow \quad x = f^*(y).$$

4. Thus f^* is the inverse of f.
5. Hence f is both injective and surjective by Theorem 6.2.18.

- 1. $g(\mathbf{false}, \mathbf{true}) = \mathbf{false} = g(\mathbf{false}, \mathbf{false})$, where $(\mathbf{false}, \mathbf{true}) \neq (\mathbf{false}, \mathbf{false})$.
  2. So g is not injective.
  3. $g(\mathbf{true}, \mathbf{false}) = \mathbf{true}$.
  4. So every element in the codomain Bool is in the image of g by lines 1 and 3.
  5. This says g is surjective.
- 1. $h(\mathbf{true}, \mathbf{false}) = (\mathbf{false}, \mathbf{true}) = h(\mathbf{false}, \mathbf{true})$, where $(\mathbf{true}, \mathbf{false}) \neq (\mathbf{false}, \mathbf{true})$.
  2. So h is not injective.
  3. If $p, q, r \in$ Bool such that $h(p, q) = (\mathbf{true}, r)$, then

     3.1. $p \wedge q = \mathbf{true}$ by the definition of h;

     3.2. $\therefore$ $p = \mathbf{true}$

     3.3. $\therefore$ $r = p \vee q = \mathbf{true}$ by the definition of h.

  4. So $(\mathbf{true}, \mathbf{false})$ in the codomain is not in the image of h.
  5. Thus h is not surjective.
- (For this question, we implicitly assume that every integer is either odd or even, but not both. This will be proved in Corollary 8.1.22.)
  1. We first show that if x is an even integer, then $k(x)$ is even.
     1.1. Let x be an even integer.
     1.2. Then $k(x) = x$ by the definition of k.
     1.3. So $k(x)$ is even.
  2. Next we show that if x is an odd integer, then $k(x)$ is odd.
     2.1. Let x be an odd integer.
     2.2. Then $k(x) = 2x - 1 = 2(x - 1) + 1$, where $x - 1$ is an integer.
     2.3. So $k(x)$ is odd.
  3. Since every integer is either even or odd but not both, lines 1 and 2 tell us that, for every $x \in \mathbb{Z}$,
     3.1. x is even if and only if $k(x)$ is even; and
     3.2. x is odd if and only if $k(x)$ is odd.
  4. Now we show that k is injective.
     4.1. Let $x, x' \in \mathbb{Z}$ such that $k(x) = k(x')$.
     4.2. Case 1: $k(x)$ is even.
          4.2.1. Then both x and x' are even by line 3.1.
          4.2.2. So $x = k(x) = k(x') = x'$ by the definition of k.
     4.3. Case 2: $k(x)$ is odd.
          4.3.1. Then both x and x' are odd by line 3.2.
          4.3.2. So $2x - 1 = k(x) = k(x') = 2x' - 1$ by the definition of k.
          4.3.3. Thus $x = x'$.
     4.4. Since $k(x)$ is either even or odd, we conclude that $x = x'$ in any case.
  5. Finally, we show that k is not surjective.
     5.1. We prove this by contradiction.
          5.1.1. Suppose k is surjective.
          5.1.2. Note 3 is in the codomain $\mathbb{Z}$.
          5.1.3. Use the surjectivity of k to find $x \in \mathbb{Z}$ such that $k(x) = 3$.
          5.1.4. Note $k(x) = 3 = 2 \times 1 + 1$ is odd.

5.1.5. So x is odd by line 3.2.
5.1.6. Thus $3 = k(x) = 2x - 1$ by the choice of x and the definition of k.
5.1.7. Solving gives $x = (3 + 1)/2 = 2 = 2 \times 1$, which is even.
5.1.8. This contradicts line 5.1.5 as no integer is both even and odd.

5.2. Hence k is not surjective.

4. Let $f\colon B \to C$.

   (a) Suppose f is injective. Show that $g \circ f$ is injective whenever g is an injective function with domain C.

   (b) Suppose we have a function g with domain C such that $g \circ f$ is injective. Show that f is injective.

   Solution.

   (a)
   1. Suppose f is injective.
   2. Let g be an injective function with domain C.
   3. Take $x, x' \in B$ such that $(g \circ f)(x) = (g \circ f)(x')$.
   4. Then $\quad g(f(x)) = g(f(x'))$ by the definition of $g \circ f$;
   5. $\quad \therefore \quad f(x) = f(x')$ as g is injective;
   6. $\quad \therefore \quad x = x'$ as f is injective.

   (b)
   1. Suppose g is a function with domain C such that $g \circ f$ is injective.
   2. Let $x, x' \in B$ such that $f(x) = f(x')$.
   3. Then $(g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x')$ by the definition of $g \circ f$.
   4. So $x = x'$ as $g \circ f$ is injective by the choice of g.

5. Let $f\colon B \to C$.

   (a) Suppose f is surjective. Show that $f \circ h$ is surjective whenever h is a surjective function with codomain B.

   (b) Suppose we have a function h with codomain B such that $f \circ h$ is surjective. Show that f is surjective.

   Solution.

   (a)
   1. Suppose f is surjective.
   2. Let h be a surjective function with codomain B.
   3. Take any $y \in C$.
   4. Apply the surjectivity of f to find $x \in B$ such that $y = f(x)$.
   5. Apply the surjectivity of h to find w in the domain of h such that $x = h(w)$.
   6. Then $y = f(x) = f(h(w)) = (f \circ h)(w)$ by the definition of $f \circ h$.

   (b)
   1. Suppose h is a function with codomain B such that $f \circ h$ is surjective.
   2. Take any $y \in C$.
   3. Apply the surjectivity of $f \circ h$ to find w in the domain of h such that $y = (f \circ h)(w)$.
   4. Let $x = h(w)$.
   5. Then $x \in B$ and $y = (f \circ h)(w) = f(h(w)) = f(x)$ by the definition of $f \circ h$.

6. Let $A = \{1, 2, 3\}$. The *order* of a bijection $f\colon A \to A$ is defined to be the least $n \in \mathbb{Z}^+$ such that
$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$

Define functions $g, h\colon A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

Find the orders of g, h, $g \circ h$, and $h \circ g$.

Solution. The orders are respectively 2, 2, 3 and 3.

7. Let A, B, C be sets. Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for all bijections $f\colon A \to B$ and all bijections $g\colon B \to C$.

   Solution.

   1. For all $x \in A$ and all $z \in C$,

1.1.		$z = (g \circ f)(x)$	
1.2.	$\Leftrightarrow$	$z = g(f(x))$	by the definition of $g \circ f$;
1.3.	$\Leftrightarrow$	$g^{-1}(z) = f(x)$	by the definition of g^{-1};
1.4.	$\Leftrightarrow$	$f^{-1}(g^{-1}(z)) = x$	by the definition of f^{-1};
1.5.	$\Leftrightarrow$	$(f^{-1} \circ g^{-1})(z) = x$	by the definition of $f^{-1} \circ g^{-1}$.

   2. So $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ by the definition of $(g \circ f)^{-1}$.

8. Fix sets A, B. Define the *graph* of a function $f\colon A \to B$ to be

   $$\{(x, y) \in A \times B : y = f(x)\}.$$

   (a) Assuming $A \neq \varnothing$, find a subset $S \subseteq A \times B$ that cannot be the graph of any function $A \to B$.

   (b) Show that a subset $S \subseteq A \times B$ is the graph of a function $A \to B$ if and only if

   $$\forall x \in A \quad \exists! y \in B \quad (x, y) \in S.$$

   Solution.

   (a) We claim that $S = \varnothing$ works.
   1. We prove this by contradiction.
      1.1. Suppose $f\colon A \to B$ whose graph is S.
      1.2. Since $A \neq \varnothing$, it has an element, say x.
      1.3. Then $(x, f(x)) \in S$ by the definition of graphs.
      1.4. This contradicts the fact that $S = \varnothing$.
   2. So S cannot be the graph of any function $A \to B$.

   (b)
   1. ("Only if")
      1.1. Suppose S is the graph of a function $f\colon A \to B$.
      1.2. Pick any $x \in A$.
      1.3. ("Existence part")
         1.3.1. $f(x) \in B$ as B is the codomain of f.
         1.3.2. As S is the graph of f, we know $(x, f(x)) \in S$.
         1.3.3. So $(x, y) \in S$ for some $y \in B$.
      1.4. ("Uniqueness part")
         1.4.1. Let $y \in B$ such that $(x, y) \in S$.
         1.4.2. As S is the graph of f, we know $y = f(x)$.
      1.5. So there is a unique $y \in B$ such that $(x, y) \in S$.
   2. ("If")

2.1. Suppose $\forall x \in A\ \exists! y \in B\ (x, y) \in S$.
2.2. Define $f\colon A \to B$ by setting $f(x)$ to be the unique $y \in B$ such that $(x, y) \in S$, for every $x \in A$.
2.3. This function is well-defined by line 2.1.
2.4. By the definition of f, for all $(x, y) \in A \times B$,

$$(x, y) \in S \quad \Leftrightarrow \quad y = f(x).$$

2.5. So S is indeed the graph of f.

9. Let $f\colon A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$.

(a) Compare the sets X and $f^{-1}(f(X))$. Is one always a subset of the other? Justify your answer.

(b) Compare the sets Y and $f(f^{-1}(Y))$. Is one always a subset of the other? Justify your answer.

Solution.

(a) First we show it is always the case that $X \subseteq f^{-1}(f(X))$.
1. Let $x \in X$.
2. Then $f(x) \in f(X)$ by the definition of $f(X)$.
3. So $x \in f^{-1}(f(X))$ by the definition of $f^{-1}(f(X))$.

Next we show it is possible that $f^{-1}(f(X)) \not\subseteq X$.
1. Consider $f\colon \{-1, 1\} \to \{0\}$ where $f(-1) = 0 = f(1)$, and $X = \{1\}$.
2. Note $f(X) = \{f(1)\} = \{0\}$.
3. Since $f(-1) = 0$, we know $-1 \in f^{-1}(\{0\}) = f^{-1}(f(X))$.
4. As $-1 \notin \{1\} = X$, we deduce that $f^{-1}(f(X)) \not\subseteq X$.

(b) First we show it is always the case that $f(f^{-1}(Y)) \subseteq Y$.
1. Take any $y \in f(f^{-1}(Y))$.
2. Then the definition of $f(f^{-1}(Y))$ gives some $x \in f^{-1}(Y)$ such that $y = f(x)$.
3. Now as $x \in f^{-1}(Y)$, we get $y' \in Y$ which makes $y' = f(x)$.
4. Since f is a function, this implies $y = f(x) = y' \in Y$, as required.

Next we show it is possible that $Y \not\subseteq f(f^{-1}(Y))$.
1. Consider $f\colon \{0\} \to \{-1, 1\}$ where $f(0) = 1$, and $Y = \{-1\}$.
2. Note that no $x \in \{0\}$ makes $f(x) = -1$.
3. So $f^{-1}(Y) = \varnothing$ by the definition of $f^{-1}(Y)$.
4. This entails $f(f^{-1}(Y)) = \varnothing \not\supseteq \{-1\} = Y$.