CS1231(S) Tutorial 5: Mathematical Induction Solutions

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1. Prove by induction that for all $n \in \mathbb{Z}_{\geqslant 1}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}\, n(n+1)(2n+1).$$

Solution.

1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition

$$\text{“ } 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}\, n(n+1)(2n+1) \text{ ”.}$$

2. (Base step) $P(1)$ is true because $1^2 = 1 = \frac{1}{6} \times 1 \times (1+1) \times (2 \times 1 + 1)$.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., that

$$\text{“ } 1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}\, k(k+1)(2k+1) \text{ ”.}$$

   - 3.2. Then $1^2 + 2^2 + \cdots + k^2 + (k+1)^2$
   - 3.3. $= \frac{1}{6}\, k(k+1)(2k+1) + (k+1)^2$ by the induction hypothesis;
   - 3.4. $= \frac{1}{6}(k+1)\big(k(2k+1) + 6(k+1)\big)$
   - 3.5. $= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$
   - 3.6. $= \frac{1}{6}(k+1)(k+2)(2k+3)$
   - 3.7. $= \frac{1}{6}(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)$.
   - 3.8. Thus $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}\ P(n)$ is true by MI. $\square$

2. Let $x \in \mathbb{R}_{\geqslant -1}$. Prove by induction that $1 + nx \leqslant (1+x)^n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

Solution.

1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition “ $1 + nx \leqslant (1+x)^n$ ”.
2. (Base step) $P(1)$ is true because $1 + 1x = 1 + x = (1+x)^1$.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., that $1 + kx \leqslant (1+x)^k$.
   - 3.2. Then $(1+x)^{k+1}$
   - 3.3. $= (1+x)^k(1+x)$
   - 3.4. $\geqslant (1+kx)(1+x)$ by the induction hypothesis, as $1 + x \geqslant 0$;
   - 3.5. $= 1 + (k+1)x + kx^2$
   - 3.6. $\geqslant 1 + (k+1)x$ as $k \geqslant 1$ and $x^2 \geqslant 0$.

3.7. Thus $P(k+1)$ is true.

4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}$ $P(n)$ is true by MI. $\square$

3. Prove by induction that 3 divides $n^3 + 11n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

Solution.

1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition " 3 divides $n^3 + 11n$ ".
2. (Base step) $P(1)$ is true because $1^3 + 11 \times 1 = 12 = 3 \times 4$.
3. (Induction step)

3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., that 3 divides $k^3 + 11k$.

3.2. Use the definition of "divides" to find $\ell \in \mathbb{Z}$ such that $k^3 + 11k = 3\ell$.

3.3. Then $(k+1)^3 + 11(k+1)$

3.4. $= (k^3 + 3k^2 + 3k + 1) + (11k + 11)$

3.5. $= (k^3 + 11k) + 3(k^2 + k + 4)$

3.6. $= 3\ell + 3(k^2 + k + 4)$ by line 3.2;

3.7. $= 3(\ell + k^2 + k + 4)$ where $\ell + k^2 + k + 4 \in \mathbb{Z}$.

3.8. Thus $P(k+1)$ is true by the definition of "divides".

4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}$ $P(n)$ is true by MI. $\square$

4. Let a be an odd integer. Prove by induction that 2^{n+2} divides $a^{2^n} - 1$ for all $n \in \mathbb{Z}_{\geqslant 1}$. (Note that $a^{b^c} = a^{(b^c)}$ by convention.)

Solution.

1. Use the definition of "odd' to find $\ell \in \mathbb{Z}$ such that $a = 2\ell + 1$.
2. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition " 2^{n+2} divides $a^{2^n} - 1$ ".
3. (Base step)

3.1. Note $a^{2^1} - 1 = a^2 - 1$

3.2. $= (a-1)(a+1)$

3.3. $= (2\ell + 1 - 1)(2\ell + 1 + 1)$ by line 1;

3.4. $= 4\ell(\ell + 1)$.

3.5. Case 1: ℓ is odd.

3.5.1. Use the definition of "odd" to find $m \in \mathbb{Z}$ such that $\ell = 2m + 1$.

3.5.2. Then $a^{2^1} - 1 = 4\ell(\ell + 1)$ by lines 3.1–3.4;

3.5.3. $= 4(2m+1)((2m+1)+1)$ by the choice of m on line 3.5.1;

3.5.4. $= 8(2m+1)(m+1)$ where $(2m+1)(m+1) \in \mathbb{Z}$.

3.5.5. So 2^{1+2} divides $a^{2^1} - 1$ as $8 = 2^{1+2}$.

3.6. Case 2: ℓ is even.

3.6.1. Use the definition of "even" to find $m \in \mathbb{Z}$ such that $\ell = 2m$.

3.6.2. Then $a^{2^1} - 1 = 4\ell(\ell + 1)$ by lines 3.1–3.4;

3.6.3. $= 4(2m)(2m+1)$ by the choice of m on line 3.6.1;

3.6.4. $= 8m(2m+1)$ where $m(2m+1) \in \mathbb{Z}$.

3.6.5. So 2^{1+2} divides $a^{2^1} - 1$ as $8 = 2^{1+2}$.

3.7. Since ℓ is either odd or even, we conclude that 2^{1+2} divides $a^{2^1} - 1$ in all cases.

3.8. So $P(1)$ is true.

4. (Induction step)

4.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., that 2^{k+2} divides $a^{2^k} - 1$.

4.2. Use the definition of "divides" to find $m \in \mathbb{Z}$ such that $a^{2^k} - 1 = 2^{k+2}m$.

4.3. Then $a^{2^{k+1}} - 1 = a^{2^k \times 2} - 1$

4.4. $= (a^{2^k})^2 - 1$

4.5. $= (a^{2^k} - 1)(a^{2^k} + 1)$

4.6. $= (2^{k+2}m)\big((2^{k+2}m + 1) + 1\big)$ by the choice of m;

4.7. $= 2^{k+3}m(2^{k+1}m + 1)$ where $m(2^{k+1}m + 1) \in \mathbb{Z}$.

4.8. Thus $P(k+1)$ is true by the definition of "divides".

5. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}\ P(n)$ is true by MI. $\square$

5. Prove by induction that

$$\forall n \in \mathbb{Z}_{\geqslant 8}\ \exists x, y \in \mathbb{Z}_{\geqslant 0}\ (n = 3x + 5y).$$

(As a consequence, any integer-valued transaction over 8 dollars can be carried out using only 3-dollar and 5-dollar coins.)

Solution.

1. For each $n \in \mathbb{Z}_{\geqslant 8}$, let $P(n)$ be the proposition " $\exists x, y \in \mathbb{Z}_{\geqslant 0}\ (n = 3x + 5y)$ ".
2. (Base step) $P(8)$ is true because $8 = 3 \times 1 + 5 \times 1$.
3. (Induction step)

   3.1. Let $k \in \mathbb{Z}_{\geqslant 8}$ such that $P(k)$ is true.

   3.2. Find $x, y \in \mathbb{Z}_{\geqslant 0}$ such that $k = 3x + 5y$.

   3.3. Case 1: $y > 0$.

   3.3.1. Then $k + 1 = (3x + 5y) + 1$ by the choice of x, y;

   3.3.2. $= 3(x + 2) + 5(y - 1)$ where $x + 2 \in \mathbb{Z}_{\geqslant 0}$.

   3.3.3. As $y > 0$, we know $y - 1 \in \mathbb{Z}_{\geqslant 0}$.

   3.3.4. So $P(k+1)$ is true.

   3.4. Case 2: $y = 0$.

   3.4.1. Then $k = 3x + 3 \times 0 = 3x$

   3.4.2. $\therefore\ x = k/3 \geqslant 8/3$ as $k \geqslant 8$;

   3.4.3. $\therefore\ x \geqslant \lceil 8/3 \rceil = 3$ as $x \in \mathbb{Z}$.

   3.4.4. Thus $k + 1 = 3x + 1 = 3(x - 3) + 5 \times 2$, where $x - 3 \in \mathbb{Z}_{\geqslant 0}$.

   3.4.5. So $P(k+1)$ is true.

   3.5. Thus $P(k+1)$ is true in all cases.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}\ P(n)$ is true by MI. $\square$

Alternative solution.

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition " $\exists x, y \in \mathbb{Z}_{\geqslant 0}\ (n + 8 = 3x + 5y)$ ".
2. (Base step)

   2.1. $P(0)$ is true because $0 + 8 = 8 = 3 \times 1 + 5 \times 1$.

   2.2. $P(1)$ is true because $1 + 8 = 9 = 3 \times 3 + 5 \times 0$.

   2.3. $P(2)$ is true because $2 + 8 = 10 = 3 \times 0 + 5 \times 2$.
3. (Induction step)

   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k+2)$ is true.

   3.2. Apply $P(k)$ to find $x, y \in \mathbb{Z}_{\geqslant 0}$ such that $k + 8 = 3x + 5y$.

   3.3. Then $(k + 3) + 8 = (k + 8) + 3$

   3.4. $= (3x + 5y) + 3$ by the choice of x, y;

   3.5. $= 3(x + 1) + 5y$ where $x + 1, y \in \mathbb{Z}_{\geqslant 0}$.

   3.6. Thus $P(k+3)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true by Strong MI. $\square$

6. Prove by induction that every positive integer can be written as a sum of *distinct* non-negative integer powers of 2, i.e.,

$$\forall n \in \mathbb{Z}_{\geqslant 1}\ \exists \ell \in \mathbb{Z}_{\geqslant 1}\ \exists i_1, i_2, \ldots, i_\ell \in \mathbb{Z}_{\geqslant 0}\ (i_1 < i_2 < \cdots < i_\ell \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}).$$

(Hint: think in terms of binary representations.)

Solution. Since the question asks for the proof that a property holds for all positive integers, the following form of Strong Mathematical Induction is more convenient than Principle 7.2.1 in the notes. One readily sees that a similar variant works over $\mathbb{Z}_{\geqslant m_0}$ for any $m_0 \in \mathbb{Z}$. (All such variants can be used in your work in this module.)

Strong Mathematical Induction (Strong MI) over $\mathbb{Z}_{\geqslant 1}$. To prove that $\forall n \in \mathbb{Z}_{\geqslant 1}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(1), P(2), \ldots, P(m+1)$ are true;

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant 1}\ \big(P(1) \wedge P(2) \wedge \cdots \wedge P(k+m) \Rightarrow P(k+m+1)\big)$ is true

for some $m \in \mathbb{Z}_{\geqslant 0}$.

1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition

   $$\text{“ } \exists \ell \in \mathbb{Z}_{\geqslant 1}\ \exists i_1, i_2, \ldots, i_\ell \in \mathbb{Z}_{\geqslant 0}\ (i_1 < i_2 < \cdots < i_\ell \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}) \text{ ”}.$$

2. (Base step) $P(1)$ is true because $1 = 2^0$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(1), P(2), \ldots, P(k)$ is true.
   3.2. Find $m \in \mathbb{Z}$ such that $k+1 = 2m$ or $k+1 = 2m+1$. This is possible because $k+1$ is either odd or even.
   3.3. Note $\quad 2m \leqslant k+1 \quad$ as $k+1 = 2m$ or $k+1 = 2m+1$;
   3.4. $\qquad\qquad \leqslant k+k \quad$ as $k \geqslant 1$;
   3.5. $\qquad\qquad = 2k$.
   3.6. So $m \leqslant k$.
   3.7. Also $\quad 2m+1 \geqslant k+1 \quad$ as $k+1 = 2m$ or $k+1 = 2m+1$;
   3.8. $\quad \therefore \quad 2m \geqslant k \geqslant 1$
   3.9. $\quad \therefore \quad m \geqslant \dfrac{1}{2}$
   3.10. $\quad \therefore \quad m \geqslant \left\lceil \dfrac{1}{2} \right\rceil = 1 \quad$ as $m \in \mathbb{Z}$.
   3.11. By lines 3.6 and 3.10, we know that $P(m)$ is true by the induction hypothesis.
   3.12. Apply $P(m)$ to find $\ell \in \mathbb{Z}_{\geqslant 1}$ and $i_1, i_2, \ldots, i_\ell \in \mathbb{Z}_{\geqslant 0}$ such that

   $$i_1 < i_2 < \cdots < i_\ell \quad \text{and} \quad m = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}.$$

   3.13. Case 1: $k+1 = 2m$.
      3.13.1. Then $\quad k+1 = 2m$
      3.13.2. $\qquad = 2(2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}) \quad$ by the choice of $i_1, i_2, \ldots, i_\ell$;
      3.13.3. $\qquad = 2^{i_1+1} + 2^{i_2+1} + \cdots + 2^{i_\ell+1}$.
      3.13.4. Also $i_1 + 1 < i_2 + 1 < \cdots < i_\ell + 1$ as $i_1 < i_2 < \cdots < i_\ell$.
      3.13.5. So $P(k+1)$ is true.
   3.14. Case 2: $k+1 = 2m+1$.
      3.14.1. Then $\quad k+1 = 2m+1$
      3.14.2. $\qquad = 2(2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}) + 1 \quad$ by the choice of $i_1, i_2, \ldots, i_\ell$;
      3.14.3. $\qquad = 2^0 + 2^{i_1+1} + 2^{i_2+1} + \cdots + 2^{i_\ell+1}$.
      3.14.4. Also $0 < i_1 + 1 < i_2 + 1 < \cdots < i_\ell + 1$ as $0 \leqslant i_1 < i_2 < \cdots < i_\ell$.
      3.14.5. So $P(k+1)$ is true.
   3.15. Thus $P(k+1)$ is true in any case.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}\ P(n)$ is true by Strong MI. $\square$

7. Show that $F_{n+4} = 3F_{n+2} - F_n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

   Solution.

   1. $F_{n+4} = F_{n+3} + F_{n+2} \quad$ by the definition of F_{n+4};
   2. $\qquad = (F_{n+2} + F_{n+1}) + F_{n+2} \quad$ by the definition of F_{n+3};
   3. $\qquad = 2F_{n+2} + F_{n+1}$
   4. $\qquad = 3F_{n+2} - F_{n+2} + F_{n+1}$
   5. $\qquad = 3F_{n+2} - (F_{n+1} + F_n) + F_{n+1} \quad$ by the definition of F_{n+2};
   6. $\qquad = 3F_{n+2} - F_n$.

8. Show by induction that $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

Solution.

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition

$$\text{“ } F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n \text{ ”.}$$

2. (Base step)

2.1. Since $F_0 = 0$ and $F_1 = 1$,

$$F_{0+1}^2 - F_{0+1}F_0 - F_0^2 = 1^2 - 1 \times 0 - 0^2 = 1 = (-1)^0.$$

2.2. So $P(0)$ is true.

3. (Induction step)

3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(k)$ is true, i.e., that

$$F_{k+1}^2 - F_{k+1}F_k - F_k^2 = (-1)^k.$$

3.2. Then $F_{(k+1)+1}^2 - F_{(k+1)+1}F_{k+1} - F_{k+1}^2$

3.3. $= F_{k+2}^2 - F_{k+2}F_{k+1} - F_{k+1}^2$

3.4. $= (F_{k+1} + F_k)^2 - (F_{k+1} + F_k)F_{k+1} - F_{k+1}^2$ by the definition of F_{k+2};

3.5. $= F_{k+1}^2 + 2F_{k+1}F_k + F_k^2 - F_{k+1}^2 - F_kF_{k+1} - F_{k+1}^2$

3.6. $= -(F_{k+1}^2 - F_{k+1}F_k - F_k)$

3.7. $= -(-1)^k$ by the induction hypothesis;

3.8. $= (-1)^{k+1}$.

3.9. Thus $P(k+1)$ is true.

4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}$ $P(n)$ is true by MI. □

9. Let $a_0, a_1, a_2, \ldots$ be the sequence satisfying

$$a_0 = 0, \quad a_1 = 2, \quad a_2 = 7, \quad \text{and} \quad a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. Prove by induction that $a_n < 3^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

Solution.

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition “ $a_n < 3^n$ ”.

2. (Base step)

2.1. $P(0)$ is true because $a_0 = 0 < 1 = 3^0$.

2.2. $P(1)$ is true because $a_1 = 2 < 3 = 3^1$.

2.3. $P(2)$ is true because $a_2 = 7 < 9 = 3^2$.

3. (Induction step)

3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k+2)$ are true.

3.2. $P(k), P(k+1), P(k+2)$ are true means

$$a_k < 3^k \quad \text{and} \quad a_{k+1} < 3^{k+1} \quad \text{and} \quad a_{k+2} < 3^{k+2}.$$

3.3. Then $a_{k+3} = a_{k+2} + a_{k+1} + a_k$ by the definition of a_{k+3};

3.4. $< 3^{k+2} + 3^{k+1} + 3^k$ by the induction hypothesis;

3.5. $< 3^{k+2} + 3^{k+2} + 3^{k+2}$

3.6. $= 3 \times 3^{k+2} = 3^{k+3}$.

3.7. Thus $P(k+3)$ is true.

4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}$ $P(n)$ is true by Strong MI. □

10. Define a set S recursively as follows.

(a) $2 \in S$. (base clause)

(b) If $x \in S$, then $3x \in S$ and $x^2 \in S$. (recursion clause)

(c) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the numbers $0, 6, 15, 16, 36$ are in S? Which are not?

Solution.

Structural induction over S. To prove that $\forall n \in S \; P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(2)$ is true; and

(induction step) show that $\forall x \in S \; \big(P(x) \Rightarrow P(3x) \wedge P(x^2)\big)$ is true.

- We know $0 \notin S$ because all $x \in S$ satisfy $x \geqslant 2$, as one can show by structural induction over S.
- $$\begin{array}{rrl} & 2 \in S & \text{by the base clause.} \\ \therefore & 6 \in S & \text{by the recursion clause with } x = 2 \text{ and the previous line.} \\ \therefore & 36 \in S & \text{by the recursion clause with } x = 6 \text{ and the previous line.} \end{array}$$
- $$\begin{array}{rrl} & 2 \in S & \text{by the base clause.} \\ \therefore & 4 \in S & \text{by the recursion clause with } x = 2 \text{ and the previous line.} \\ \therefore & 16 \in S & \text{by the recursion clause with } x = 4 \text{ and the previous line.} \end{array}$$
- We know $15 \notin S$ because no $x \in S$ is odd, as one can show by structural induction over S.