1. It is known that $f(x)$ is a differentiable function and its derivative is continuous and satisfies $f'(2021) = 3042$. Find the exact value of

$$\lim_{x \to 0} \frac{f(2021 + x) - f(2021 - x)}{4x}.$$

Answer 1521

$$\lim_{x \to 0} \frac{f(2021+x) - f(2021-x)}{4x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{f'(2021+x) + f'(2021-x)}{4}$$

$$= \frac{3042 + 3042}{4}$$

$$= \underline{\underline{1521}}$$

2. Let a denote a positive constant. Let C denote the curve

$$y = ax(x-1)(x-2).$$

It is known that the normal line to C at $(1,0)$ is perpendicular to the normal line to C at $(2,0)$. Find the value of a. Give your answer correct to two decimal places.

Answer 0.71

$$\frac{dy}{dx} = a(x-1)(x-2) + ax(x-2) + ax(x-1)$$

$$x=1 \Rightarrow \frac{dy}{dx} = -a$$

$$x=2 \Rightarrow \frac{dy}{dx} = 2a$$

normal line $\perp$ $\Rightarrow$ tangent line $\perp$

$$\therefore \ -a(2a) = -1$$

$$a = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad (\because a \text{ is +ve})$$

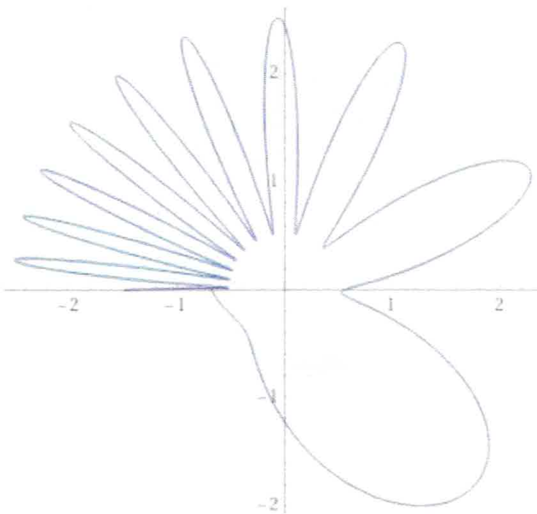
$$= 0.7071\cdots$$

$$\approx \underline{\underline{0.71}}$$

3. Let C denote the curve with equation in polar coordinates given by

$$r = \sin\left(e^{\frac{2\theta}{3}}\right) - 1.521,$$

where $0 \le \theta \le 2\pi$. Find the slope of the normal line to C at the point corresponding to $\theta = \frac{\pi}{2}$. Give your answer correct to two decimal places.
(Here is a picture of C for your reference.)



Answer 0.68

$$\frac{dr}{d\theta} = \frac{2}{3} e^{\frac{2\theta}{3}} \cos\left(e^{\frac{2\theta}{3}}\right)$$

$$y = r\sin\theta,\ x = r\cos\theta \Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

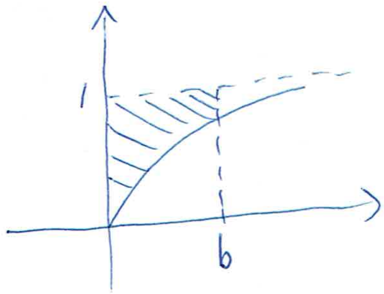
$$\theta = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{\frac{2}{3}e^{\pi/3}\cos(e^{\pi/3})}{-\{\sin(e^{\pi/3}) - 1.521\}}$$

Slope of normal at $\theta = \pi/2$ is $\dfrac{\sin(e^{\pi/3}) - 1.521}{2/3\, e^{\pi/3}\cos(e^{\pi/3})} = 0.6778\ldots$

$$\approx \underline{\underline{0.68}}$$

4. Let b denote a positive constant. Let R_b denote the region in the first quadrant bounded between the line $y = 1$ and the curve $y = \frac{e^x - 1}{e^x + 1}$ from $x = 0$ to $x = b$. Let A_b denote the area of R_b. Find the value of $\lim_{b \to \infty} A_b$. Give your answer correct to two decimal places.

Answer 1.39



$$A_b = \int_0^b \left(1 - \frac{e^x - 1}{e^x + 1}\right) dx$$

$$= 2\int_0^b \frac{1}{e^x + 1}\, dx$$

let $u = e^x + 1 \Rightarrow du = e^x dx = (u-1)\,dx$

$$\therefore A_b = 2\int_2^{e^b+1} \frac{du}{u(u-1)}$$

$$= 2\int_2^{e^b+1} \left(\frac{1}{u-1} - \frac{1}{u}\right) du$$

$$= 2\Big[\ln|u-1| - \ln|u|\Big]_2^{e^b+1}$$

$$= 2\left\{\ln \frac{e^b}{e^b+1} + \ln 2\right\}$$

$$\lim_{b \to \infty} A_b = 2\ln 2 = 1.3862\cdots$$

$$\approx \underline{\underline{1.39}}$$

5. Let a denote a positive constant. Let R denote the region in the first quadrant bounded between the curve $y = 60x^3 + 42x$ and the x-axis from $x = 0$ to $x = a$. If the area of R is equal to 1521, find the value of a. Give your answer correct to two decimal places.

Answer 3.06

$$1521 = \int_0^a (60x^3 + 42x)\,dx$$

$$= 15a^4 + 21a^2$$

$$5a^4 + 7a^2 - 507 = 0$$

$$a^2 = \frac{-7 + \sqrt{49 + 10140}}{10} = \frac{-7 + \sqrt{10189}}{10}$$

$$a = \sqrt{\frac{-7 + \sqrt{10189}}{10}} = 3.0649\ldots$$

$$\approx \underline{\underline{3.06}}$$

6. Let C denote the curve

$$(x^2 + y^2)^2 - 8(x^2 - y^2) - 1 = 0.$$

Let L denote the tangent line to C at the point $(2, 1)$. Find the distance from the origin to L. Give your answer correct to two decimal places.

Answer 1.41

$$2(x^2+y^2)(2x+2yy') - 8(2x-2yy') = 0$$

$$x=2,\ y=1 \Rightarrow 10(4+2y') - 8(4-2y') = 0$$

$$8 + 36y' = 0$$

$$y' = -\frac{2}{9}$$

$$L:\ \frac{y-1}{x-2} = -\frac{2}{9} \Rightarrow 9y - 9 = -2x + 4$$

$$\Rightarrow 2x + 9y - 13 = 0$$

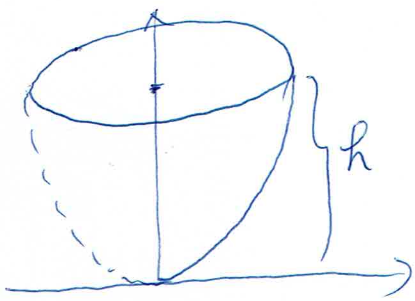
distance from $(0,0)$ to L is

$$\frac{13}{\sqrt{2^2+9^2}} = \frac{13}{\sqrt{85}} = 1.410\ldots$$

$$\approx \underline{\underline{1.41}}$$

7. Let a and k denote two positive constants. In this question the scale for the coordinates on the xy-plane are measured in metres. (So for example, the point (0,2) represents the point on the positive part of the y-axis at a distance 2 metres from the origin.) An empty water tank in the shape of a parabolic bowl is constructed by rotating the part of the curve $y = x^2$ from the origin to $(\sqrt{a}, a)$ one complete round about the y-axis. At time $t = 0$ minutes, a tap is turned on and the tank is being filled up with water at a constant rate of k cubic metres per minute. It is observed that at time $t = 15.21$ minutes the water level in the tank has just risen to the rim of the tank and at that moment the water level is rising at a rate of 20.21 metres per minute. Find the value of a. Give your answer correct to two decimal places.

Answer 614.79



$$V = \int_0^h \pi x^2\,dy = \int_0^h \pi y\,dy = \frac{1}{2}\pi h^2$$

$$k = \frac{dV}{dt} = \pi h \frac{dh}{dt}$$

$$h = a \Rightarrow V = \frac{1}{2}\pi a^2, \quad k = \pi a (20.21)$$

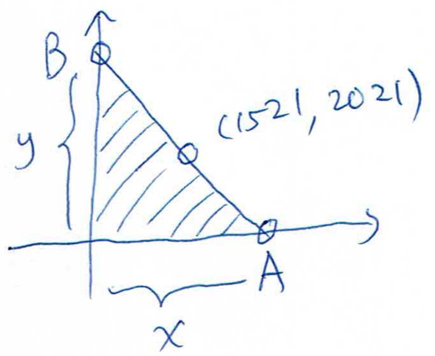
$$\therefore \quad 15.21 = \frac{\frac{1}{2}\pi a^2}{k} = \frac{\frac{1}{2}\pi a^2}{\pi a (20.21)} = \frac{a}{2 \times 20.21}$$

$$\therefore \quad a = 2 \times 15.21 \times 20.21 = 614.7882$$

$$\approx \underline{\underline{614.79}}$$

8. Let a denote a positive constant. Let O denote the origin of the xy-plane. Let P denote the point $(1521, 2021)$. A point A on the positive x-axis and a point B on the positive y-axis are chosen so that the three points A, P, B all lie on the same straight line. If the smallest possible area of the triangle OAB is equal to a, find the value of $\ln a$. Give your answer correct to two decimal places.

Answer 15.63

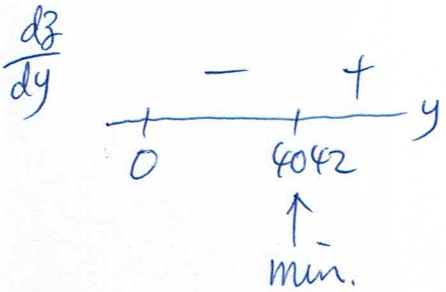


$$\frac{1521}{x} = \frac{y-2021}{y}$$

$$x = \frac{1521y}{y-2021}$$

$$\text{area} = z = \frac{1}{2}xy = \frac{1521y^2}{2(y-2021)} = \frac{1521}{2}\left(y + 2021 + \frac{2021^2}{y-2021}\right)$$

$$\frac{dz}{dy} = \frac{1521}{2}\left(1 - \frac{2021^2}{(y-2021)^2}\right) = \frac{1521}{2}\left\{\frac{y(y-4042)}{(y-2021)^2}\right\}$$



$\Rightarrow z = \min$ when $y = 4042$

$$\Rightarrow a = \frac{1521 \times 4042^2}{2 \times (4042 - 2021)} = 1521 \times 4042$$

$$\therefore \ln a = 15.6316\ldots$$

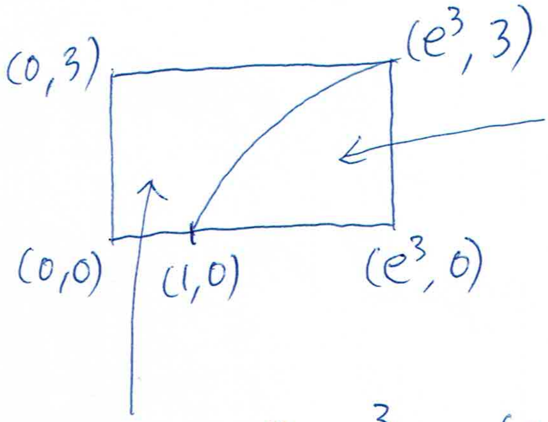
$$\simeq \underline{\underline{15.63}}$$

9. Let r, A, B, C denote four positive constants. Let R denote the rectangle in the first quadrant with vertices at

$$(0,0), \quad (e^3,0), \quad (e^3,3), \quad (0,3).$$

The curve $y = \ln x$ divides R into two parts, a bigger part which has an area equal to A and a smaller part which has an area equal to B. Let C denote the area of a circle with radius r. If $A - B = C$, find the value of r. Give your answer correct to two decimal places.

Answer 2.65



$$\text{area} = \int_1^{e^3} \ln x \, dx$$

$$= x\ln x \Big|_1^{e^3} - \int_1^{e^3} x\left(\tfrac{1}{x}\right) dx$$

$$= 3e^3 - e^3 + 1 = 2e^3 + 1$$

$$\text{area} = 3 \cdot e^3 - (2e^3+1)$$

$$= e^3 - 1$$

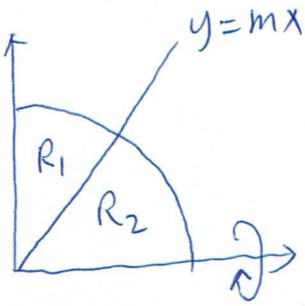
$$\therefore A = 2e^3+1, \quad B = e^3 - 1$$

$$\therefore \pi r^2 = (2e^3+1) - (e^3-1) = e^3 + 2$$

$$r = \sqrt{\frac{e^3+2}{\pi}} = 2.6514\ldots \approx \underline{\underline{2.65}}$$

10. Let a and m denote two positive constants. Let R denote the finite region in the first quadrant bounded by the circle $x^2 + y^2 = a^2$ and the two coordinates axes. The line $y = mx$ divides R into two parts: R_1 and R_2. If the volume of the solid of revolution generated by rotating R_1 one complete round about the x-axis equals the volume of the solid of revolution generated by rotating R_2 one complete round about the x-axis, find the value of m. Give your answer correct to two decimal places.

Answer 1.73



$$\begin{cases} x^2+y^2=a^2 \\ y=mx \end{cases} \Rightarrow (1+m^2)x^2=a^2$$
$$\Rightarrow x = \frac{a}{\sqrt{1+m^2}}$$

$$V_2 = V_1 = \int_0^{a/\sqrt{1+m^2}} \pi\{(a^2-x^2) - m^2x^2\}\,dx$$
$$= \pi\left[a^2x - \frac{1}{3}x^3 - \frac{1}{3}m^2x^3\right]_0^{a/\sqrt{1+m^2}}$$
$$= \frac{\pi a^3}{\sqrt{1+m^2}}\left(1 - \frac{1}{3}\left(\frac{1}{1+m^2}\right) - \frac{1}{3}\left(\frac{m^2}{1+m^2}\right)\right)$$
$$= \frac{2\pi a^3}{3\sqrt{1+m^2}}$$

$\because V_1 + V_2 = \frac{2}{3}\pi a^3$ ($\because$ $R_1 + R_2$ rotate about x-axis $= \frac{1}{2}$ of a ball of radius a)

$\therefore V_1 = \frac{1}{3}\pi a^3 \Rightarrow \frac{2}{\sqrt{1+m^2}} = 1 \Rightarrow 1+m^2 = 4$
$\Rightarrow m = \sqrt{3} = 1.7320\ldots$
≈ 1.73