Tutorial 10

Exercise 6

9. For each matrix $\boldsymbol{A}$ in Question 6.1, (i) determine whether $\boldsymbol{A}$ is diagonalizable; and (ii) if $\boldsymbol{A}$ is diagonalizable, find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

(a) $\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$, (b) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$,

(c) $\boldsymbol{A} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$, (d) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

(e) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, (f) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 9 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

(g) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, (h) $\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 2 & -1 & 0 \end{pmatrix}$,

(i) $\boldsymbol{A} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, (j) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

16. A square matrix $(a_{ij})_{n\times n}$ is called a *stochastic matrix* if all the entries are non-negative and the sum of entries of each column is 1, i.e. $a_{1i}+a_{2i}+\cdots+a_{ni} = 1$ for $i = 1, 2, \ldots, n$.

   (a) Let $\boldsymbol{A}$ be a stochastic matrix.

      (i) Show that 1 is an eigenvalue of $\boldsymbol{A}$.

      (ii) If λ is an eigenvalue of $\boldsymbol{A}$, then $|\lambda| \leq 1$.

   (b) Let $\boldsymbol{B} = \begin{pmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.95 & 0.05 \\ 0 & 0.05 & 0.95 \end{pmatrix}$.

      (i) Is $\boldsymbol{B}$ a stochastic matrix?

      (ii) Find a 3×3 invertible matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{B}$.

20. Following the procedure discussed in Example 6.2.11.2, solve the following recurrence relations.

   (a) $a_n = 3a_{n-1} - 2a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$.

   (b) $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 1$ and $a_1 = 0$.

24. For each of the following, find a matrix $\boldsymbol{P}$ that orthogonally diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P}$.

(e) $\boldsymbol{A} = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 0 \end{pmatrix}$,

(g) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

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29. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Show that $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$.

(b) Let $\boldsymbol{v} = (a, b, c, d)^{\mathrm{T}}$ be a nonzero vector. Show that if $\boldsymbol{v} \cdot \boldsymbol{u} = 0$, then $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$.

(c) Suppose $\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & a_1 & a_2 & a_3 \\ \frac{1}{2} & b_1 & b_2 & b_3 \\ \frac{1}{2} & c_1 & c_2 & c_3 \\ \frac{1}{2} & d_1 & d_2 & d_3 \end{pmatrix}$ is an orthogonal matrix. Find $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P}$.