NATIONAL UNIVERSITY OF SINGAPORE

CS1231 - DISCRETE STRUCTURES

(Semester 1: AY2015/16)

Time Allowed: 90 minutes

INSTRUCTIONS TO CANDIDATES

1. Please write your Student Number (Matriculation Number) only. Do not write your name.
2. This assessment paper contains **TWO (2)** parts and comprises **TEN (10)** printed pages, including this page.
3. Answer **ALL** questions.
4. This is an **OPEN BOOK** assessment.
5. You are allowed to use **NUS APPROVED CALCULATORS**.
6. You may use pen or pencil, but please erase cleanly, and write legibly.
7. Please write your Student Number below.

STUDENT NUMBER: ______________________________

EXAMINER'S USE ONLY			
Part	MaxScore	Mark	Remark
A	20	OCR	OCR
Q11	4		
Q12	6		
Q13	8		
Q14	4		
Q15	8		
TOTAL	50		

Part A
(20 marks) Multiple choice questions. Answer on the OCR form.

For each multiple choice question, choose the best answer and **shade** the corresponding choice on the OCR form. Each multiple choice question is worth 2 marks. No mark is deducted for wrong answers. Shade your student number (check that it is correct!) on the OCR form as well. You should use a **2B pencil**.

Note that Appendix A on Page 10 may be useful. You may use the facts there, by citing, but without proving, them.

Q1. According to Prof. Sim, the CS1231 Message of the Day is:

A. I may do it.

B. I shall do it.

C. I can't do it.

D. I can do it.

E. I think I can do it.

Q2. Which of the following are tautologies?

(I) $\sim(p \vee q) \vee [(\sim p) \wedge q] \vee p$.

(II) $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$.

(III) $(p \to r) \wedge (q \to r) \to [(p \vee q) \to r]$.

A. None of (I), (II) or (III).

B. (I) and (II) only.

C. (I) and (III) only.

D. (II) and (III) only.

E. All of (I), (II) and (III).

Q3. John was given the following statement:

"If the product of two integers a and b is even, then, either a is even or b is even."

The following is John's proof:

1. Suppose a and b are both odd.
2. Therefore, $a = 2m + 1$ and $b = 2n + 1$ where m and n are integers.
3. Then, $ab = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$, which is odd.
4. Hence, the proof is complete.

What kind of proof did John use?

A. Proof by contradiction.

B. Proof by mathematical induction.

C. Proof by contrapositive.

D. Proof by converse.

E. Proof by universal instantiation.

Q4. Which of the following is a negation for "Acer is inside and Beau is at the pool."

A. Acer is not inside or Beau is not at the pool.

B. Acer is not inside or Beau is at the pool.

C. Acer is not inside and Beau is not at the pool.

D. Acer is inside or Beau is not at the pool.

E. Acer is inside or Beau is at the pool.

Q5. Which of the following arguments are valid?

(I) Every living thing is a plant or an animal.
David's dog is alive and it is not a plant.
All animals have hearts.
Hence, David's dog has a heart.

(II) Some scientists are not engineers.
Some astronauts are not engineers.
Hence, some scientists are not astronauts.

(III) All astronauts are scientists.
Some astronauts are engineers.
Hence, some engineers are scientists.

(IV) Some females are not mothers.
Some politicians are not females.
Hence, some politicians are not mothers.

A. (I) and (II) only.

B. (I) and (III) only.

C. (III) and (IV) only.

D. (I), (II) and (III) only.

E. (I), (III) and (IV) only.

Q6. Which of the following is the **negation** of this statement:
"There is a boy in the class such that all the girls in the class are younger than that boy."

Let $Boy(x)$ be "x is a boy", $Girl(x)$ be "x is a girl", and $Younger(x, y)$ be "x is younger than y".

A. $\exists x(Boy(x) \wedge \forall y\,(Girl(y) \rightarrow Younger(y, x)))$

B. $\forall x({\sim}Boy(x) \vee \exists y\,({\sim}Girl(y) \wedge Younger(y, x)))$

C. $\forall x({\sim}Boy(x) \vee \exists y\,(Girl(y) \wedge {\sim}Younger(y, x)))$

D. $\forall x(Boy(x) \vee \exists y\,(Girl(y) \wedge {\sim}Younger(y, x)))$

E. None of the above.

The following definitions are for the next FOUR questions (Q7 — Q10).

Predicates $P(x), Q(x, y)$ and $R(x, y, z)$ are defined as follows:

$$P(x) = (\ x > 1 \ \wedge \ \forall y \in \mathbb{N}\,(y \mid x \ \rightarrow \ (y = 1 \vee y = x))\ ),\ \forall x \in \mathbb{N}.$$
$$Q(x, y) = (\ x \mid y\ ),\ \forall x, y \in \mathbb{N}.$$
$$R(x, y, z) = (\ z \mid x \wedge z \mid y \wedge \forall u \in \mathbb{N}\,((u \mid x \wedge u \mid y) \rightarrow u \leq z)\ ),\ \forall x, y, z \in \mathbb{N}.$$

Solution: Another way of stating predicates $P(x), R(x, y, z)$ is:
$P(x) = (\ x$ is a prime $)$, and $R(x, y, z) = (\ z = \gcd(x, y)\ )$.

Q7. Which of the following statements is true?

(I) $Q(0, 5)$ (II) $Q(3, 0)$ (III) $Q(21, 7)$ (IV) $Q(7, 9)$

A. (II) only.
B. (I) and (II) only.
C. (II) and (III) only.
D. (I) and (IV) only.
E. None of (I), (II), (III) or (IV).

Q8. Which integer x makes $P(x)$ true?

(I) $x = 1$ (II) $x = 2$ (III) $x = 9$ (IV) $x = 97$

A. (I) only.
B. (II) only.
C. (II) and (IV) only.
D. (III) and (IV) only.
E. All of (I), (II), (III) and (IV).

Q9. Which pair of x, y makes true the statement: $\sim Q(x, y) \wedge R(x, y, y)$?

(I) $x = 2, y = 2$ (II) $x = 2, y = 6$ (III) $x = 0, y = 4$ (IV) $x = 4, y = 0$

A. (I) only.
B. (III) only.
C. (II) and (III) only.
D. (I) and (IV) only.
E. None of (I), (II), (III) or (IV).

Q10. Which pair of x, y makes **false** the statement: $R(x, y, 1) \rightarrow (P(x) \wedge P(y) \wedge x \neq y)$?

(I) $x = 3, y = 5$ (II) $x = 2, y = 6$ (III) $x = 7, y = 7$ (IV) $x = 5, y = 6$

A. (I) only.
B. (IV) only.
C. (II) and (III) only.
D. (II) and (III) and (IV) only.
E. None of (I), (II), (III), or (IV).

Part B
(30 marks) Structured questions. Write your answer in the space provided.

Q11. (4 marks) For each of the following statements, indicate whether the statement is true or false and justify your answer.

(a) (2 marks) $\forall$ integers a, $\exists$ an integer b such that $a + b = 0$.

Solution: It is true.

Proof. (Direct proof)

1. Let $b = -1 \times a = -a$
2. $b \in \mathbb{Z}$ by closure property.
3. Then $a + b = a + (-a) = 0$.

■

(b) (2 marks) $\exists$ an integer a such that $\forall$ integers b, $a + b = 0$.

Solution: It is false. One possible proof:

Proof. (Counterexample)

1. Let $b = -a + 1$
2. $b \in \mathbb{Z}$ by closure property.
3. Then $a + b = a + (-a + 1) = 1 \neq 0$.

■

Q12. (6 marks) You are given the following English statements:

1. All swimmers are able to swim across the river.
2. No archers are short-sighted.
3. Patrick wears glasses.
4. Everybody is either an archer or a swimmer.

(a) (2 marks) Rewrite each of the above sentences into formal statements, using quantifiers wherever appropriate, and well-named predicates. You may assume that the domain is the set of people, which may be omitted in your statements. You may use the logically equivalent form in your statements.

Solution:

1. $\forall x, Swimmer(x) \rightarrow CrossRiver(x)$. [rule 1]
2. $\forall x, ShortSighted(x) \rightarrow {\sim}Archer(x)$. (or its contrapositive statement) [rule 2]
3. $WearGlasses(Patrick)$. [fact 3]
4. $\forall x, Archer(x) \vee Swimmer(x)$. [rule 4]

(b) (2 marks) There is a missing statement above. Adding that missing statement would allow you to answer this question "Is Patrick able to swim across the river?" Write down the missing statement (as a formal quantified statement) and the conclusion (in English) about Patrick.

Solution:

Missing statement: $\forall x, WearGlasses(x) \rightarrow ShortSighted(x)$. [rule 5]

Conclusion: Patrick is able to swim across the river.

(c) (2 marks) Show your proof to derive the conclusion about Patrick in part (b) above.

Solution:

1. $ShortSighted(Patrick)$. [fact 6: from fact 3 and rule 5]
2. $\sim Archer(Patrick)$. [fact 7: from fact 6 and rule 2]
3. $Swimmer(Patrick)$. [fact 8: from fact 7 and rule 4]
4. $CrossRiver(Patrick)$. [from fact 8 and rule 1]

Q13. (8 marks) On the island of knights (who always tell the truth) and knaves (who always lie), you meet three natives A, B, and C, who address you as follows:

A: At least one of us is a knave.
B: At most two of us are knaves.

What are A, B and C? In your derivation, assume that A is a knave first. Later in your derivation, again assume that B is a knave first.

Solution:

Proof. (Direct proof)

1. Suppose A is a knave.
   1. Therefore what A says is false. *(by definition of knave)*
   2. Therefore none of them is a knave. *(negation of what A says)*
   3. Contradiction. *(contradict with the supposition that A is a knave)*
2. Therefore A is a knight. *(negation of supposition)*
3. Suppose B is a knave.
   1. Therefore what B says is false. *(by definition of knave)*
   2. Therefore all of them are knaves. *(negation of what B says)*
   3. Contradiction. *(contradict with the supposition that A is a knight)*
4. Therefore B is a knight. *(negation of supposition)*
5. Since both A and B are knights, C must be a knave. *(according to what A says)*

■

Q14. (4 marks) In Q9 of Tutorial 1, you proved a shortcut to test for divisibility by 9. Here, you will prove a shortcut to test for divisibility by 5. That is, prove that any non-negative integer n is divisible by 5 if, and only if, its rightmost decimal digit is 0 or 5.

Solution:

Proof. (Direct proof)

1. As in Q9 of Tutorial 1, let the decimal representation of $n \in \mathbb{Z}^+$ be $d_k d_{k-1} \ldots d_1 d_0$.
2. Then, by definition, $n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \ldots + d_1 \cdot 10 + d_0$.
3. $= 10 \cdot \underbrace{\left(d_k \cdot 10^{k-1} + d_{k-1} \cdot 10^{k-2} + \ldots + d_1\right)}_{M} + d_0$.
4. So $n = 5 \cdot 2M + d_0$, where M is an integer by the closure property.
5. Clearly, $5 \mid 5 \cdot 2M$.
6. If $d_0 = 0$ or $d_0 = 5$: (forward direction)
   1. Then clearly $5 \mid d_0$.
   2. And thus $5 \mid n$ by Theorem 4.1.1.
7. Re-writing: $d_0 = n - 5 \cdot 2M$.
8. Conversely, if $5 \mid n$: (backward direction)
   1. Then $5 \mid d_0$ by Theorem 4.1.1.
   2. Thus $d_0 = 0$ or $d_0 = 5$, because $0 \leq d_0 \leq 9$ and 5 does not divide any other digits in this range.
9. Hence, $5 \mid n$ iff $d_0 = 0$ or $d_0 = 5$. ■

Q15. (8 marks) Find a positive integer n such that: (i) its prime factorization contains no repeated prime factors; and (ii) for any prime p, $p \mid n \longleftrightarrow (p-1) \mid n$.

Be sure to clearly explain and justify how you obtain n.
(Note: the list of primes in Appendix A may be useful.)

Solution:

1. Let S be the multiset (ie. duplicates allowed) of prime factors of n. (We seek to determine all the members of S.)
2. By the unique prime factorization theorem, we may list the primes in S from smallest to largest: $p_1, p_2, \ldots, p_k$, for some $k \in \mathbb{Z}^+$.
3. By Property (i), this list is distinct, ie. $p_i \neq p_j$ for all $i, j \in 1, 2, \ldots k$.
4. Property (ii) implies that a prime p is in S if the prime factors of $(p-1)$ are also in S.
5. This means that every prime p (except 2) in S may be written as 1 + the product of some of the distinct primes in S which are smaller than p.
6. This gives us a way to determine the primes in S:
7. Clearly, $2 \in S$ because $1 \mid n$ and $1+1=2$ is prime.
8. Using Line 5., this means $3 \in S$, because $3 = 1 + 2$, and $2 \in S$ and 3 is prime.
9. Using Line 5. again, this means $7 \in S$, because $7 = 1 + 2 \cdot 3$ and both $2, 3 \in S$ and 7 is prime.
10. Using Line 5. yet again, this means $43 \in S$, because $43 = 1 + 2 \cdot 3 \cdot 7$, and $2, 3, 7 \in S$ and 43 is prime.
11. No other primes are in S, because this process of writing a prime as 1 + product of some smaller primes stops after 43. For example, $2 \cdot 3 \cdot 7 \cdot 43 + 1 = 1807 = 13 \cdot 139$ is not a prime; $2 \cdot 7 + 1 = 15$ is not a prime; $3 \cdot 7 + 1 = 22$ is not a prime. By exhaustive checking, no other primes can be generated in this manner from $2, 3, 7, 43$.
12. No other primes are in S because they cannot satisfy both Properties (i) and (ii).
13. Hence $n = 2 \cdot 3 \cdot 7 \cdot 43 = 1806$.

Appendix A

List of primes less than 100

2,	3,	5,	7,	11,	13,	17,	19,	23,	29,
31,	37,	41,	43,	47,	53,	59,	61,	67,	71,
73,	79,	83,	89,	97.					

END OF PAPER