Date No.

Classical Kinetic Energy

$$K = \int F\,dx$$
$$= \int_0^v m_0 \frac{dv}{dt}\,dx$$
$$= \int_0^v m_0\,dv\,\frac{dx}{dt}$$
$$= \int_0^v m_0\,v\,dv$$
$$= m_0 \int_0^v v\,dv$$
$$= m_0\left(\tfrac{1}{2}v^2\right)$$
$$K = \tfrac{1}{2}m_0 v^2 \quad \checkmark$$

Relativistic

$$K = \int F\,dx$$
$$= \int_0^v \frac{d}{dt}(mv)\,dx$$
$$= \int_0^v d(mv)\,\frac{dx}{dt}$$
$$= \int_0^v (m\,dv + v\,dm)\,v$$
$$= \int_0^v (mv\,dv + v^2\,dm)$$

but what is $(mv\,dv + v^2 dm)$?

$$\therefore \quad K = \int_{M_0}^{M} c^2\,dm$$
$$= c^2 \int_{M_0}^{M} dm$$
$$K = mc^2 - m_0c^2$$
$$\boxed{K = Mc^2 - M_0c^2}$$

Consider $m = \gamma m_0$

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$m^2\left(1-\frac{v^2}{c^2}\right) = m_0^2$$
$$m^2(c^2-v^2) = m_0^2c^2$$
$$m^2c^2 - m^2v^2 = m_0^2c^2$$
$$2mc^2\,dm - [2mv^2\,dm + 2m^2v\,dv] = 0$$
$$c^2\,dm - [v^2\,dm + mv\,dv] = 0$$
$$\therefore \quad c^2\,dm = v^2\,dm + mv\,dv$$

Rearrange :- $Mc^2 = M_0c^2 + K$

$$\boxed{\underbrace{E}_{\text{total energy}} = \underbrace{M_0c^2}_{\text{Rest energy}} + \underbrace{K}_{\text{Kinetic Energy}}}$$

So $\boxed{E = mc^2}$