Section 3.4

Linear Independence

Objective

- What is a linearly independent/dependent set?
- How to show that a set is linearly (in)dependent?
- What are some conditions on linearly (in)dependent sets?

What is a redundant vector in span(S)?

Example

$S_1 = \{ (1,1,1), (1,0,-2) \}$ $S_2 = \{(1,1,1), (1,0,-2), (2,3,5)\}$

$$\mathrm{span}(S_1) \xleftrightarrow{\text{equal}} \mathrm{span}(S_2)$$

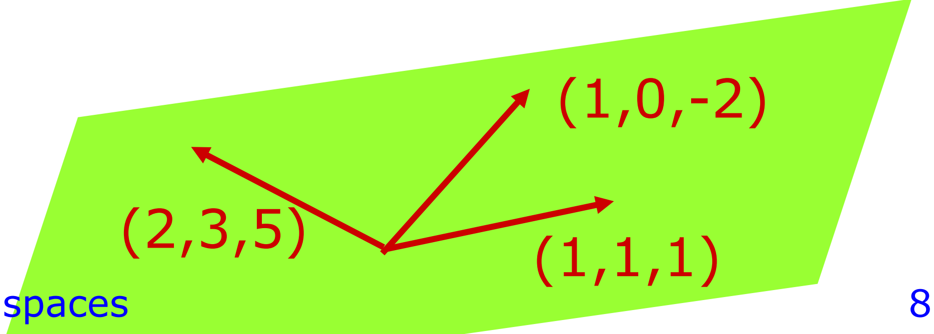
all linear combinations
a(1,1,1)+b(1,0,-2)

all linear combinations
a(1,1,1)+b(1,0,-2)+c$\boxed{(2,3,5)}$

3(1,1,1) +(-1)(1,0,-2)

Adding the vector (2, 3, 5) to S_1 does not change the linear span of S_1

There is a "redundant" vector in the span of S_2



Homogeneous vector equation

$\boldsymbol{v}_1 = (1, -2, 3),\ \boldsymbol{v}_2 = (5, 6, -1),\ \boldsymbol{v}_3 = (3, 2, 1)$

$\overset{0}{\underset{1}{c_1}}\mathbf{v}_1 + \overset{0}{\underset{1}{c_2}}\mathbf{v}_2 + \overset{0}{\underset{-2}{c_3}}\mathbf{v}_3 = \mathbf{0}$ vector equation

$c_1,\ c_2,\ c_3$ variable scalars (in **R**)

Can we find scalars c_1, c_2, c_3 that satisfies this vector equation?

Is this the only solution?

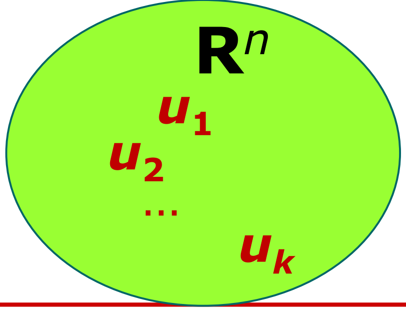
$$c_1\begin{pmatrix}1\\-2\\3\end{pmatrix} + c_2\begin{pmatrix}5\\6\\-1\end{pmatrix} + c_3\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \xrightarrow{GE} \overset{REF}{\left(\begin{array}{ccc|c}1 & 5 & 3 & 0\\0 & 16 & 8 & 0\\0 & 0 & 0 & 0\end{array}\right)}$$

homogeneous system
in variables c_1, c_2, c_3

It has infinitely many solutions

What is linearly independence?

Definition 3.4.2.1



Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbf{R}^n$.

If the vector equation

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \mathbf{0}$$

has only the trivial solution,

"Working" definition for linearly independence

set up homogeneous system, then check how many solutions

i.e. the only possible scalars are:

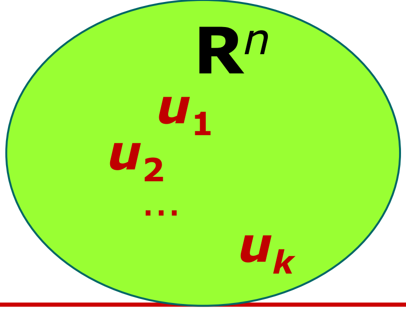
$$c_1 = 0, \; c_2 = 0, \ldots, c_k = 0$$

We say:

S is a linearly independent set and

$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent

What is linearly dependence?



Definition 3.4.2.2

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbf{R}^n$.

If the vector equation

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \mathbf{0}$$

has non-trivial solution,

"Working" definition for linearly dependence

i.e. there exists scalars $c_1, c_2, \ldots, c_n$, <u>not all</u> of them are zero

infinitely many solutions

We say:
S is a linearly dependent set and
$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly dependent

How to show that a set is linearly (in)dependent?

Example 3.4.3.1

- linear span / linear combination can equate to any vector to test
- liner independence must equate to 0 vector

Determine whether the vectors

(1, −2, 3), (5, 6, −1), (3, 2, 1)

are linearly independent.

Set up the vector equation:

$$c_1\begin{pmatrix}1\\-2\\3\end{pmatrix}+c_2\begin{pmatrix}5\\6\\-1\end{pmatrix}+c_3\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\xrightarrow[\text{matrix}]{\text{augmented}}\left(\begin{array}{ccc|c}1&5&3&0\\-2&6&2&0\\3&-1&1&0\end{array}\right)$$

$$\xrightarrow{\text{REF}}\left(\begin{array}{ccc|c}1&5&3&0\\0&16&8&0\\0&0&0&0\end{array}\right)$$

There are infinitely many solutions for c_1, c_2, c_3.

i.e. There exist non-trivial solutions.

So (1, −2, 3), (5, 6, −1), (3, 2, 1) are linearly dependent.

How to show that a set is linearly (in)dependent?

Example 3.4.3.2

Determine whether the vectors

(1, 0, 0, 1), (0, 2, 1, 0), (1, –1, 1, 1)

are linearly independent.

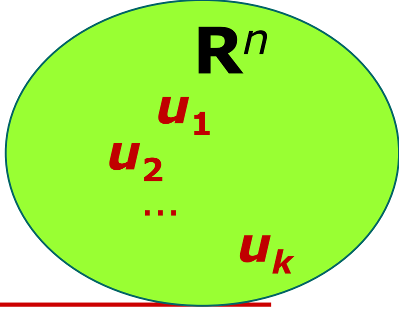
Set up the vector equation:

$$c_1\begin{pmatrix}1\\0\\0\\1\end{pmatrix}+c_2\begin{pmatrix}0\\2\\1\\0\end{pmatrix}+c_3\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c}1&0&1&0\\0&2&-1&0\\0&0&1&0\\0&0&0&0\end{array}\right)$$

Convert into augmented matrix

There is only one solution $c_1 = 0,\ c_2 = 0,\ c_3 = 0.$

So the vectors are linearly independent.



Intuitive meaning of linear dependence

Theorem 3.4.4.1

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ a set with at least 2 vectors

S is linearly dependent

if and only if

at least one vector $\boldsymbol{u}_i$ in S can be written as a linear combination of the other vectors in S

$$\underset{\substack{\uparrow \\ \text{“redundant” vector}}}{\boldsymbol{u}_i} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + \underbrace{c_{i-1}\boldsymbol{u}_{i-1} + c_{i+1}\boldsymbol{u}_{i+1}}_{\boldsymbol{u}_i \text{ is absent}} + \cdots + c_k\boldsymbol{u}_k$$

Remark 3.4.5.1

S is linearly dependent

⇔ there exists “redundant” vector in span(S)

Show a set is linearly dependent by finding a redundant vector from the set

Example 3.4.6.1

This method is not always easy

$S_1 = \{(1, 0), (0, 4), (2, 4)\} \in \mathbf{R}^2$.

(2, 4) is a linear combination of (1, 0) and (0, 4).

$$(2, 4) = 2(1, 0) + (0, 4)$$

(2,4) is a "redundant" vector:

$$\text{span}\{(1, 0), (0, 4), (2, 4)\} = \text{span}\{(1, 0), (0, 4)\}$$

So we can conclude that S_1 is linearly dependent.

Verification:

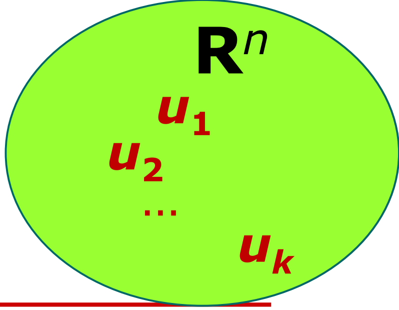
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$(0, 0) = 2(1, 0) + (0, 4) - (2, 4)$$

non-trivial scalars

connecting back to working definition
- non-zero coefficients but still back to the 0 vector

Intuitive meaning of linear independence



Theorem 3.4.4.2

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ a set with at least 2 vectors

S is linearly independent

if and only if

no vector in S can be written as a linear combination of other vectors in S

Remark 3.4.5.2

S is linearly independent

⇔ there is no "redundant" vector in span(S)

Show a set is linearly independent by showing there is no redundant vector from the set

Example 3.4.6.2

This is not an efficient method

$S_2 = \{(-1, 0, 0), (0, 3, 0), (0, 0, 7)\} \in \mathbf{R}^3$.

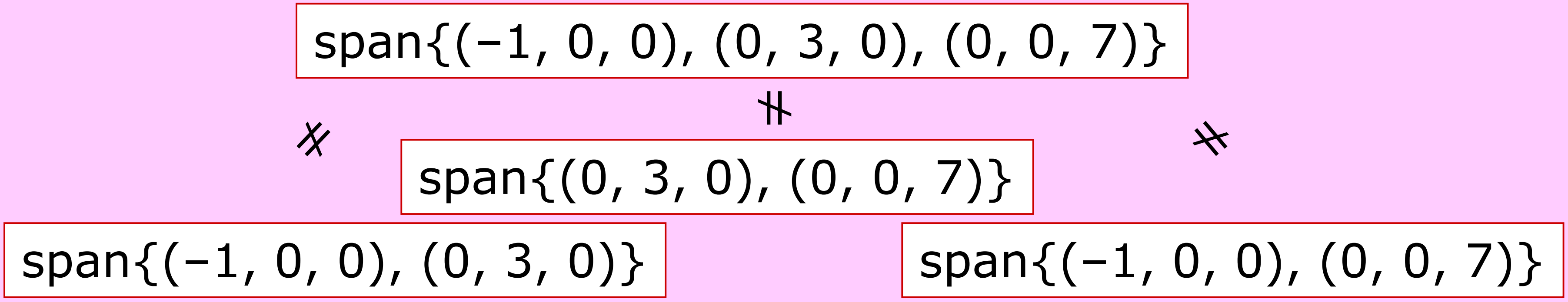
$(-1, 0, 0)$ not a lin. comb. of $(0, 3, 0)$ and $(0, 0, 7)$

$(0, 3, 0)$ not a lin. comb. of $(-1, 0, 0)$ and $(0, 0, 7)$

$(0, 0, 7)$ not a lin. comb. of $(-1, 0, 0)$ and $(0, 3, 0)$

We can conclude that S_2 is linearly independent.

There is no redundant vector in S_2:



$\text{span}\{(-1, 0, 0), (0, 3, 0), (0, 0, 7)\}$

≠ ≠ ≠

$\text{span}\{(0, 3, 0), (0, 0, 7)\}$

$\text{span}\{(-1, 0, 0), (0, 3, 0)\}$

$\text{span}\{(-1, 0, 0), (0, 0, 7)\}$

A set with one vector

properties

Example 3.4.3.3

The vector equation

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0}$$

has non-trivial solution/ only trivial solution

Let $S = \{\boldsymbol{u}\}$ be a set with one vector.

Is S linearly dependent / independent?

$$\downarrow \qquad\qquad \downarrow$$

$c\boldsymbol{u} = \mathbf{0}$ for some nonzero c / only $c = 0$

If $\boldsymbol{u} = \mathbf{0}$, then c can be non-zero.
So $S = \{\boldsymbol{u}\}$ is linearly dependent

If $\boldsymbol{u} \neq \mathbf{0}$, then c must be zero.
So $S = \{\boldsymbol{u}\}$ is linearly independent

A set with two vectors

Example 3.4.3.4

The vector equation

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \mathbf{0}$$

has non-trivial solution / only trivial solution

Let $S = \{\boldsymbol{u}, \boldsymbol{v}\}$ be a set with two vectors.

Is S linearly dependent / independent ?

$c\boldsymbol{u} + d\boldsymbol{v} = \mathbf{0}$ for c, d not both 0 / c, d both 0

$$\overbrace{\boldsymbol{u} = (-d/c)\boldsymbol{v} \quad \text{or} \quad \boldsymbol{v} = (-c/d)\boldsymbol{u}}$$

$c \neq 0$ $\qquad$ $d \neq 0$

If ***u*** and ***v*** are scalar multiples of each other, S is linearly dependent

If ***u*** and ***v*** are not scalar multiples of each other, S is linearly independent

A set with the zero-vector

Example 3.4.3.5

Let S be a finite subset of $\mathbf{R}^n$.

If $\mathbf{0} \in S$, then S is linearly dependent

Hint:

Consider the vector equation

$$c_1 \boxed{\mathbf{0}} + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \mathbf{0}$$

Show that this equation can have non-trivial solutions for $c_1, c_2, \ldots, c_k$

one of the column will not have a pivot column, then have a 0 row

A sufficient condition for linear dependence

Theorem 3.4.7 & Example 3.4.8

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbf{R}^n$.

If $k > n$, then S is linearly dependent.

If S ⊆ $\mathbf{R}^n$ and S has more than n elements, then S is linearly dependent.

1. In $\mathbf{R}^2$, a set of three or more vectors must be linearly dependent.

   more variables than eqn, so will have parameters

   {(1,2), (3,4), (5,6)} is linearly dependent
2. In $\mathbf{R}^3$, a set of four or more vectors must be linearly dependent.

   {(1,2,3), (3,4,5), (5,6,7), (7,8,9)} is linearly dependent

The proof

Theorem 3.4.7

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ in $\mathbf{R}^n$

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0}$$

vector equation

$$\begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix} \qquad \begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{pmatrix} \qquad \begin{pmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{kn} \end{pmatrix}$$

$$\begin{cases} a_{11}c_1 & + & a_{21}c_2 & + & \ldots & + & a_{k1}c_k & = & 0 \\ a_{12}c_1 & + & a_{22}c_2 & + & \ldots & + & a_{k2}c_k & = & 0 \\ \vdots & & & & & & & & \vdots \\ a_{1n}c_1 & + & a_{2n}c_2 & + & \ldots & + & a_{kn}c_k & = & 0 \end{cases}$$

Homogeneous system of n linear equations in k variables $c_1, c_2, \ldots, c_k$

The proof

$$\begin{cases} a_{11}c_1 & + & a_{21}c_2 & + & \dots & + & a_{k1}c_k & = & 0 \\ a_{12}c_1 & + & a_{22}c_2 & + & \dots & + & a_{k2}c_k & = & 0 \\ \vdots & & & & & & & & \vdots \\ a_{1n}c_1 & + & a_{2n}c_2 & + & \dots & + & a_{kn}c_k & = & 0 \end{cases}$$

Theorem 3.4.7

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}\}$

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0} \quad (*)$$

$k > n$ ⇒ more variables than equations

⇒ the system has non-trivial solutions

⇒ equation (*) has non-trivial scalars

⇒ S is linearly dependent.

Remark 1.5.4.2:
A homogeneous system with more unknowns than equations has infinitely many solutions

Homogeneous system of n linear equations in k variables $c_1, c_2, \dots, c_k$

Geometrical meaning of linear independence

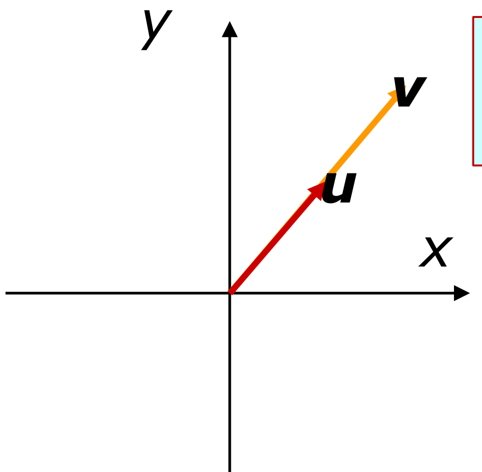
Discussion 3.4.9.1 (for two vectors)

u and ***v*** in $\mathbf{R}^2$ are linearly independent if and only if span{***u***, ***v***} = $\mathbf{R}^2$

In $\mathbf{R}^2$ (or $\mathbf{R}^3$), two vectors ***u*** and ***v*** are linearly dependent if and only if they lie on the same line.

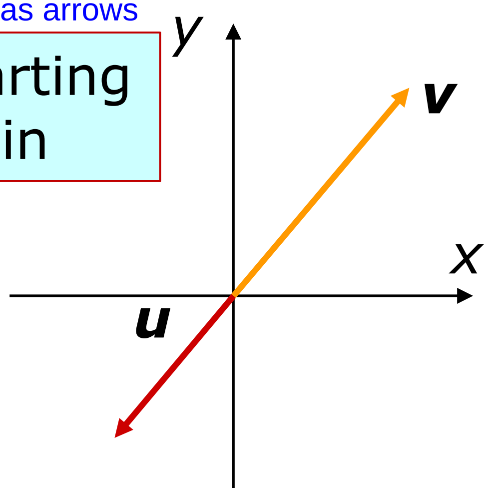
vectors as arrows

arrow starting from origin



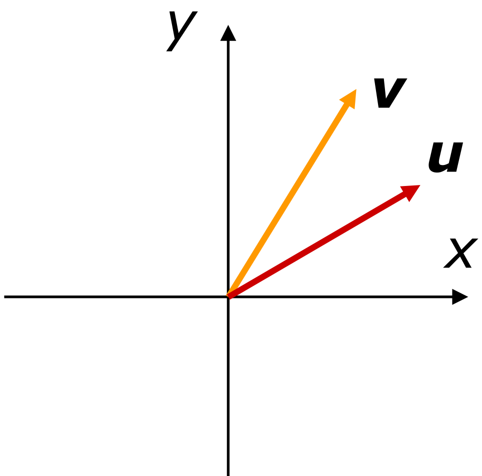
u, ***v*** are linearly dependent

span{***u***, ***v***} = line



u, ***v*** are linearly dependent

span{***u***, ***v***} = line



u, ***v*** are linearly independent

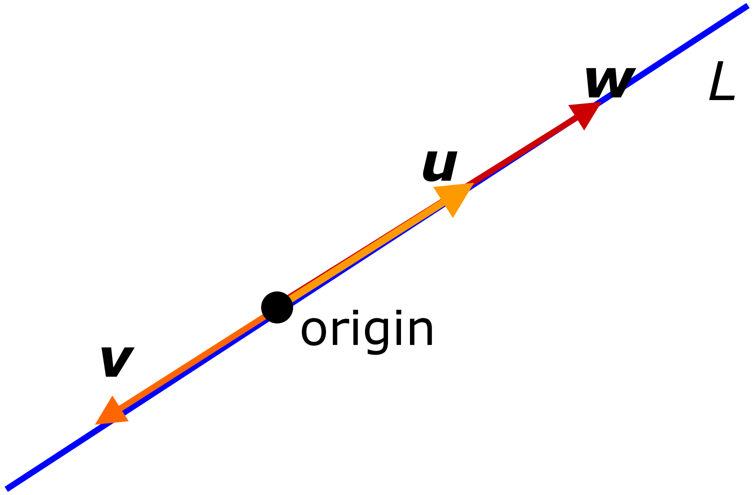
span{***u***, ***v***} = plane

Geometrical meaning of linear independence

Discussion 3.4.9.2 (for three vectors)

In $\mathbf{R}^3$,

three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly dependent if and only if they lie on the same line or same plane.



span{$\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$} = L
= span{$\boldsymbol{u}$}
= span{$\boldsymbol{v}$}
= span{$\boldsymbol{w}$}

$\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly dependent

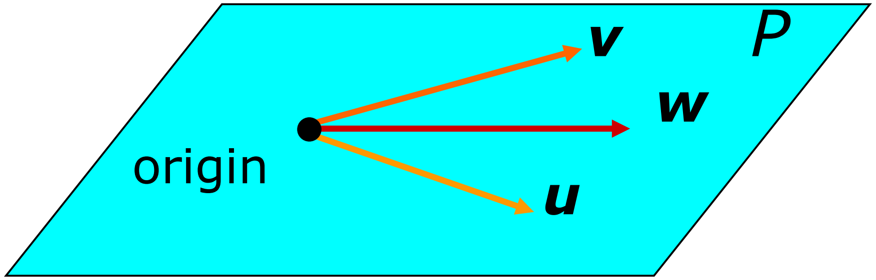
First case: same line

Geometrical meaning of linear independence

Discussion 3.4.9.2 (for three vectors)

In $\mathbf{R}^3$,

three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly dependent if and only if they lie on the same line or same plane.



$$\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = P$$
$$= \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$$
$$= \text{span}\{\boldsymbol{v}, \boldsymbol{w}\}$$
$$= \text{span}\{\boldsymbol{u}, \boldsymbol{w}\}$$

$\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly dependent

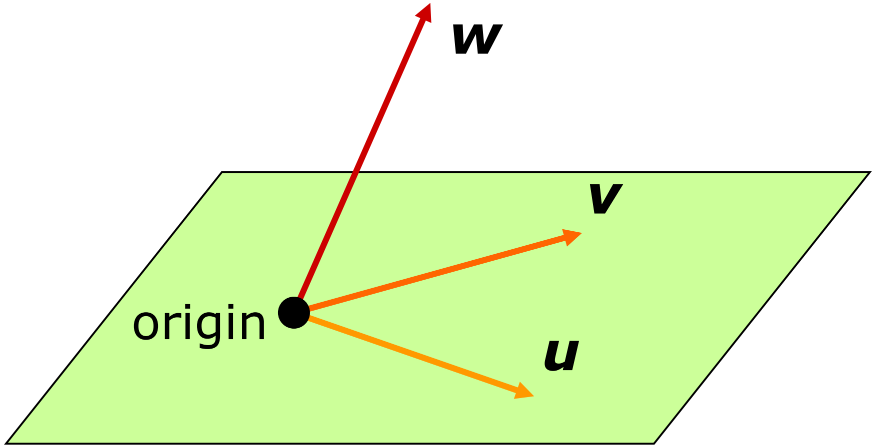
<u>Second case</u>: same plane

Geometrical meaning of linear independence

Discussion 3.4.9.2 (for three vectors)

In $\mathbf{R}^3$,

three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly dependent if and only if they lie on the same line or same plane.



$\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = \mathbf{R}^3$
$\neq \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$
$\neq \text{span}\{\boldsymbol{v}, \boldsymbol{w}\}$
$\neq \text{span}\{\boldsymbol{u}, \boldsymbol{w}\}$

$\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly independent

$\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbf{R}^3$ are linearly independent if and only if $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = \mathbf{R}^3$

How to extend a linearly independent set?

Theorem 3.4.10

$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent

$\boldsymbol{u}_{k+1}$ is not redundant

If $\boldsymbol{u}_{k+1}$ is not a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$

then $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}$ are linearly independent.

(This result gives us a way to add more vectors to a collection of linearly independent vectors.)

Outline of proof

Theorem 3.4.10

$\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent (I)

If $\boldsymbol{u_{k+1}}$ is not a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ (II)

then $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}, \boldsymbol{u_{k+1}}$ are linearly independent.

Prove by contradiction

Suppose $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}, \boldsymbol{u_{k+1}}$ are linearly dependent

Then $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_{k+1}\boldsymbol{u_{k+1}} = \boldsymbol{0}$ --(*)

for some $c_1, c_2, \cdots, c_{k+1}$ not all 0

Consider two cases: (i) $c_{k+1} = 0$ and (ii) $c_{k+1} \neq 0$

Case (i) (*) becomes $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0}$

This will contradict (I)

because u1+...+uk are linearly independent, they cannot form 0 themselves

then means 0 term comes from constant, then is contradiction since linearly dependent the constants can be anything since infinitely many solutions

Case (ii) (*) becomes $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = -c_{k+1}\boldsymbol{u_{k+1}}$

This will contradict (II)

because u(k+1) is not linear combination so cannot move to the side.

Exercise (similar to Ex 3 Q27)

Given $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent

Are $\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{w}, \boldsymbol{v} + \boldsymbol{w}$ linearly independent?

Consider $a(\boldsymbol{u} + \boldsymbol{v}) + b(\boldsymbol{u} + \boldsymbol{w}) + c(\boldsymbol{v} + \boldsymbol{w}) = \mathbf{0}$ (*)

Does (*) have non-trivial scalars for a, b, c ?

Rewrite (*): $(a+b)\boldsymbol{u} + (a+c)\boldsymbol{v} + (b+c)\boldsymbol{w} = \mathbf{0}$ (**)

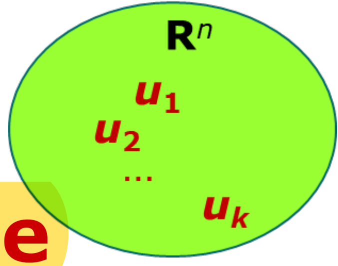
regroup

(**) has only trivial scalars for a+b, a+c, b+c

$$\left.\begin{aligned} a + b &= 0 \\ a + c &= 0 \\ b + c &= 0 \end{aligned}\right\} \text{Solve: } a = b = c = 0$$

So (*) has only trivial scalars for a, b, c

So $\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{w}, \boldsymbol{v} + \boldsymbol{w}$ are linearly independent



Linear span VS linear independence

Given that: $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ is a subset of $\mathbf{R}^n$

To Show:
$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ spans $\mathbf{R}^n$

same as: span$(S) = \mathbf{R}^n$

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \boldsymbol{v} \quad \begin{pmatrix} x \\ y \\ \vdots \\ z \end{pmatrix}$$

$\boldsymbol{v}$ is any general vector in $\mathbf{R}^n$

check whether the system is always consistent

using REF

yes → **spans**

no → does not span

To Show:
$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ is lin. indep.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \boldsymbol{0} \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\boldsymbol{0}$ is the zero vector in $\mathbf{R}^n$

because homogeneous solutions are always consistent

check whether the system has non-trivial solution

using REF

yes → lin.dep

no → **lin.indep**

Section 3.5

Bases

Objective

- What is a basis for a vector space?
- How to show that a set is a basis?
- How to find a basis for a vector space?
- What are coordinate vectors?

Discussion 3.5.1

A set V is called a vector space if:

- either V = $\mathbf{R}^n$
- or V is a subspace of $\mathbf{R}^n$.

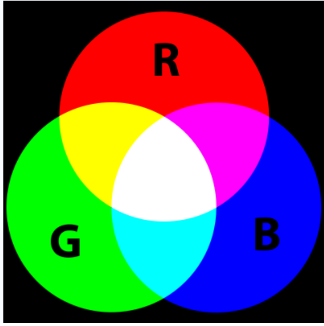
Examples

They are vector spaces

- $\mathbf{R}^3$ is a subspace of $\mathbf{R}^3$
- $\{\mathbf{0}\}$ is a subspace of $\mathbf{R}^3$
- span{ (1,2,3) } is a subspace of $\mathbf{R}^3$
- span{ (1,2,3), (2,1,4) } is a subspace of $\mathbf{R}^3$

An Analogy

Color mixing



Three primary colors: Red, Green, Blue (RGB)

$\underbrace{\text{Different color shade combination}}_{\text{e.g. 20\% Red + 45\% Green + 30\% Blue}}$ gives “all” colors

using linear algebra notation

The three primary colors span the color space:

- span{ Red, Green, Blue} = Color space

None of the three primary colors are redundant:

- {Red, Green, Blue} is linearly independent

What is a basis?

Example e.g. $(2, 3, -5) = 2\boldsymbol{e}_1 + 3\boldsymbol{e}_2 - 5\boldsymbol{e}_3$

Standard basis vectors for $\mathbf{R}^3$

$\boldsymbol{e}_1 = (1, 0, 0)$, $\boldsymbol{e}_2 = (0, 1, 0)$, $\boldsymbol{e}_3 = (0, 0, 1)$

- $\text{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} = \mathbf{R}^3$ building block
- $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is linearly independent

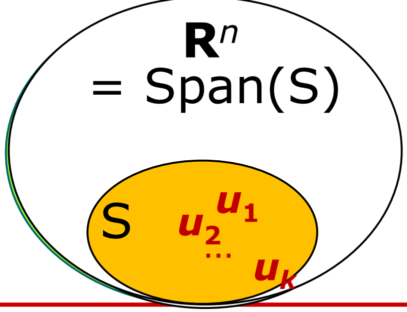
No redundant vectors

$S = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is called a basis for $\mathbf{R}^3$

S is a smallest possible subset of $\mathbf{R}^3$ so that every vector in $\mathbf{R}^3$ is a linear combination of the elements in S.

smallest possible building block of R3

What is a basis for $\mathbf{R}^n$?



Definition 3.5.4

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of $\mathbf{R}^n$.

Then S is called a basis for $\mathbf{R}^n$ if

1. S is linearly independent — no redundant vectors in S
2. S spans $\mathbf{R}^n$. — $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} = \mathbf{R}^n$

Remark 3.5.6.1

A basis for $\mathbf{R}^n$ contains the smallest possible number of vectors that can span $\mathbf{R}^n$.

Remark 3.5.6.3

$\mathbf{R}^n$ has infinitely many bases.

$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$\{(2, 0, 0), (0, 2, 0), (0, 0, 2)\}$

$\{(1, 2, 1), (2, 9, 0), (3, 3, 4)\}$

How to show that a set is a basis (for $\mathbf{R}^3$)?

Example 3.5.5.1

Show that $S = \{(1, 2, 1), (2, 9, 0), (3, 3, 4)\}$ is a basis for $\mathbf{R}^3$.

(i) S is linearly independent:

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 9 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Gaussian Elimination (details skipped) $\Rightarrow$ $c_1 = 0,\ c_2 = 0$ and $c_3 = 0$

The system only has the trivial solution.
So S is linearly independent.

How to show that a set is a basis (for $\mathbf{R}^3$)?

Example 3.5.5.1

Show $S = \{(1, 2, 1), (2, 9, 0), (3, 3, 4)\}$ is a basis for $\mathbf{R}^3$.

(ii) span(S) $= \mathbf{R}^3$:

Let (x, y, z) be any (general) vector in $\mathbf{R}^3$.

$$c_1\begin{pmatrix}1\\2\\1\end{pmatrix}+c_2\begin{pmatrix}2\\9\\0\end{pmatrix}+c_3\begin{pmatrix}3\\3\\4\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Gaussian Elimination ⇒ system is consistent for any values of x, y, z.

(details skipped)

So span (S) = $\mathbf{R}^3$.

By (i) and (ii), we conclude S is a basis for $\mathbf{R}^3$.

A set that is not a basis (for **R**^{4})

Example 3.5.5.3

non-example

Is $S = \{(1, 1, 1, 1), (0, 0, 1, 2), (-1, 0, 0, 1)\}$ a basis for **R**^{4} ?

A basis for **R**^{n} must have n elements

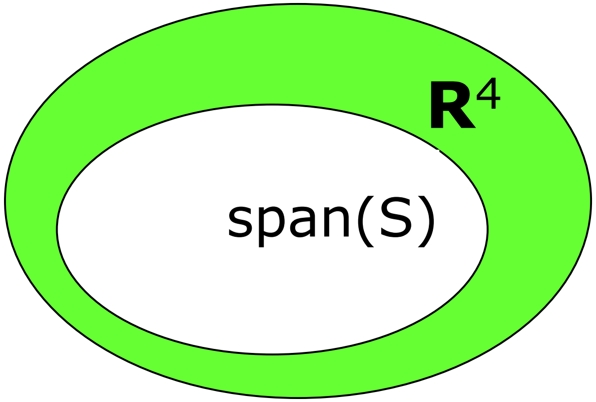
therefore might think it is a basis

S is linearly independent

number of vectors in S

span(S) $\neq$ **R**^{4} ($|S| < 4$)

So S is not a basis for **R**^{4}.



span(S) is a subspace of **R**^{4}

S is a basis for this subspace span(S)

What is a basis for a subspace of $\mathbf{R}^n$?

Definition 3.5.4

basis of a subspace

Let V be a subspace of $\mathbf{R}^n$

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ be a subset of V.

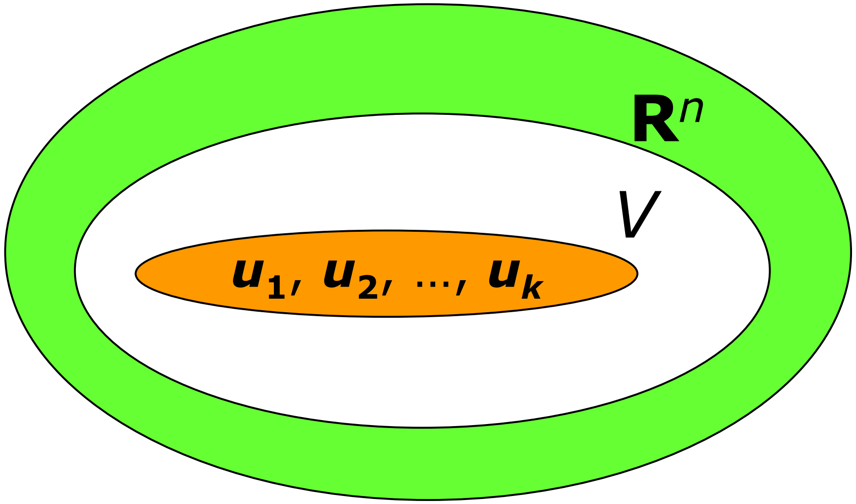
Then S is called a basis for V if

1. S is linearly independent

   no redundant vectors in S

2. S spans V.

   $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\} = V$



A set that is a basis for a subspace (of $\mathbf{R}^4$)

Example 3.5.5.2

Let V = span{(1,1,1,1), (1,-1,-1,1), (1,0,0,1)} and S = {(1, 1, 1, 1), (1, -1, -1, 1)}.

Show S a basis for V.

(i) S is linearly independent:

$$c_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Gaussian Elimination (details skipped) $\Rightarrow$ $c_1 = 0$ and $c_2 = 0$

The system only has the trivial solution.
So S is linearly independent.

Alternatively:
Just observe that (1, 1, 1, 1) and (1, -1, -1, 1) are not scalar multiple of each other, hence S is linearly indep.

A set that is a basis for a subspace (of **R**^{4})

Example 3.5.5.2

Let V = span{$\overset{u_1}{(1,1,1,1)}$, $\overset{u_2}{(1,-1,-1,1)}$, $\overset{u_3}{(1,0,0,1)}$}
and *S* = {$\underset{u_1}{(1, 1, 1, 1)}$, $\underset{u_2}{(1, -1, -1, 1)}$}.

Show S is a basis for V.

show this vector is redundant

$|S|=2 \xleftrightarrow{|\ |=2} |V|=3$

(ii) span(*S*) = V: ↔ span{$\boldsymbol{u_1}, \boldsymbol{u_2}$} $\overset{\subseteq}{\underset{\supseteq}{=}}$ span{$\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$}

Just need to show (1,0,0,1) is a linear combination of (1, 1, 1, 1) and (1, -1, -1, 1)

We can easily get

$$(1,0,0,1) = \tfrac{1}{2}\,(1,1,1,1) + \tfrac{1}{2}\,(1,-1,-1,1)$$

So (1,1,1,1), (1,-1,-1,1), (1,0,0,1) ∈ sp

By Theorem 3.2.12, showing that both ways are subsets

span{(1,1,1,1), (1,-1,-1,1), (1,0,0,1)} ⊆ span(S)

Can be omitted

A set that is not a basis for a subspace (of $\mathbf{R}^3$)

Example 3.5.5.4

Let V = span(S) where

$S = \{(1, 1, 1), (0, 0, 1), (1, 1, 0)\}$

Is S a basis for V ?

S is linearly dependent $(1, 1, 1) = (0, 0, 1) + (1, 1, 0)$

So *S* is not a basis for V   though span(*S*) = V

In general,

- if S is linearly dependent, then S is not a basis for span(S)
- if S is linearly independent, then S is a basis for span(S).

S spans span(S)

by default span(S) will be from S

How to find a basis for a subspace?

Example

$V = \{(a, a+b, b) \mid a, b \text{ in } \mathbf{R}\}$ is a subspace of $\mathbf{R}^3$

Find a basis for *V*.

Write *V* in linear span form

$$\begin{pmatrix} a \\ a+b \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Express
a general vector in *V* as
a linear combination

implicit

(1) $V = \text{span}\{(1, 1, 0), (0, 1, 1)\}$

(2) $\{(1, 1, 0), (0, 1, 1)\}$ is linearly independent

the two vectors are not scalar multiples of each other

So $\{(1, 1, 0), (0, 1, 1)\}$ is a basis for *V*

How to show that a set is a basis for a subspace?

Example

$V = \{(a, a+b, b) \mid a, b \text{ in } \mathbf{R}\}$ is a subspace of $\mathbf{R}^3$

Show that $S = \{(1, 3, 2), (1, 2, 1)\}$ is a basis for V

Check S is linearly independent

(1, 3, 2) and (1, 2, 1) are not scalar multiples of each other

Check $\boxed{\text{span}(S) = V}$ $\quad V = \text{span}\{(1, 1, 0), (0, 1, 1)\}$

no need to use x,y,z since subspace is more specific

$\text{span}\{(1,3,2), (1,2,1)\} \overset{\subseteq}{\underset{\supseteq}{=}} \text{span}\{(1,1,0), (0,1,1)\}$ (*)

To show (*), refer: Example 3.2.11

checking equality of 2 span

$$\Rightarrow \left(\begin{array}{cc|c|c} 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{array}\right) \qquad \Leftarrow \left(\begin{array}{cc|c|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

Basis for the zero space

Remark 3.5.6.2

What is a basis for the zero space $\{\mathbf{0}\}$?

$\{\mathbf{0}\} = \text{span}\{\mathbf{0}\}$

$\{\mathbf{0}\}$ is linearly dependent

So $\{\mathbf{0}\}$ is not a basis for the zero space

We regard the empty set $\emptyset$ as the basis for $\{\mathbf{0}\}$.

Uniqueness expression in terms of basis

Theorem 3.5.7

S a basis for V

S spans V

S lin. indep.

Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$

be a basis for a vector space V.

subspace of $\mathbf{R}^n$

Every vector $\boldsymbol{v}$ in V

can be expressed in the form

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

consequence of S spans V

in exactly one way.

no redundant vector

consequence of S is linearly indep.

i.e. there is a unique set of values for $c_1, c_2, \ldots, c_k$.

Example Suppose $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a basis for $\mathbf{R}^3$.

Then $3\boldsymbol{u_1} + 5\boldsymbol{u_2} + 2\boldsymbol{u_3} \neq 2\boldsymbol{u_1} + 4\boldsymbol{u_2} + 6\boldsymbol{u_3}$

Proof of uniqueness

Theorem 3.5.7

Every vector $\boldsymbol{v}$ in V can be expressed in the form

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

in exactly one way.

Express $\boldsymbol{v}$ as two linear combinations:

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} \quad (1)$$

$$\boldsymbol{v} = d_1\boldsymbol{u_1} + d_2\boldsymbol{u_2} + \cdots + d_k\boldsymbol{u_k} \quad (2)$$

(1) – (2):

$$\Rightarrow (c_1 - d_1)\boldsymbol{u_1} + (c_2 - d_2)\boldsymbol{u_2} + \cdots + (c_k - d_k)\boldsymbol{u_k} = \boldsymbol{0}$$

Given S is linearly independent

$$\Rightarrow c_1 - d_1 = 0,\ c_2 - d_2 = 0,\ \ldots,\ c_k - d_k = 0$$

everything must be trival

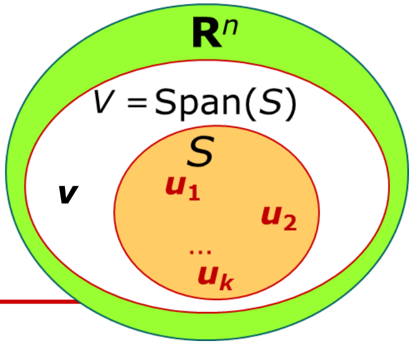
$$\Rightarrow c_1 = d_1,\ c_2 = d_2,\ \ldots,\ c_k = d_k.$$

So the expression is unique.

What are coordinate vectors?

Definition 3.5.8

fixed order



$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$: a basis for a vector space V

$\boldsymbol{v}$: a vector in V $\in \mathbf{R}^n$

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k$$ (unique expression)

$(\boldsymbol{v})_S = (c_1, c_2, \ldots, c_k)$ $\in \mathbf{R}^k$

$c_1, c_2, \ldots, c_k$ are called the coordinates of $\boldsymbol{v}$ relative to the basis S

Form the vector $(c_1, c_2, \ldots, c_k)$ in $\mathbf{R}^k$

This is called the coordinate vector of $\boldsymbol{v}$ relative to S

Denote this vector by $(\boldsymbol{v})_S$

depends on basis S

How to find coordinate vectors?

Example 3.5.9.1

Let $S = \{(1, 2, 1), (2, 9, 0), (3, 3, 4)\}$.

S is a basis for $\mathbf{R}^3$.

(a) Find the coordinate vector of $\boldsymbol{v} = (5, -1, 9)$ relative to S.

$$\boldsymbol{v} \longrightarrow (\boldsymbol{v})_S\,?$$

(b) Find a vector $\boldsymbol{w}$ in $\mathbf{R}^3$ such that $(\boldsymbol{w})_S = (-1, 3, 2)$.

$$\boldsymbol{w}\,? \longleftarrow (\boldsymbol{w})_S$$

How to find coordinate vectors?

$$\begin{tikzcd} \boldsymbol{v} \arrow[r] & (\boldsymbol{v})_S \\ \boldsymbol{w} & (\boldsymbol{w})_S \arrow[l] \end{tikzcd}$$

Example 3.5.9.1

(a) Solving the equation

$$a(1, 2, 1) + b(2, 9, 0) + c(3, 3, 4) = (5, -1, 9)$$

set up LS and use GE etc

we obtain $a = 1$, $b = -1$, $c = 2$.

i.e. $\boldsymbol{v} = 1(1, 2, 1) - (2, 9, 0) + 2(3, 3, 4)$.

So $(\boldsymbol{v})_S = (1, -1, 2)$.

(b) $(\boldsymbol{w})_S = (-1, 3, 2)$

substitution

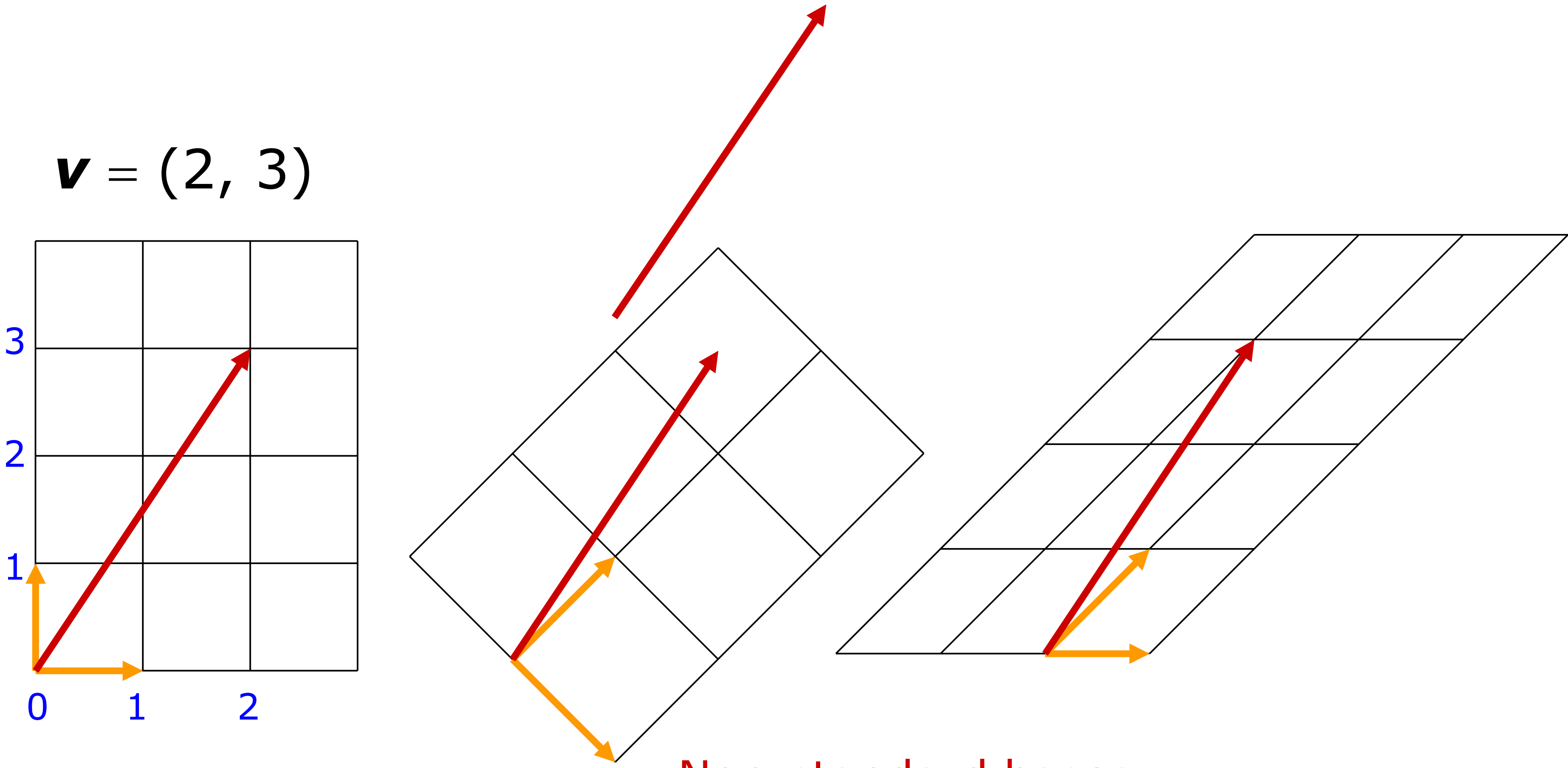
$$\boldsymbol{w} = a(1, 2, 1) + b(2, 9, 0) + c(3, 3, 4)$$

$$= (11, 31, 7).$$

Geometrical meaning of coordinate vectors

Example 3.5.9.2

$\boldsymbol{v} = (2, 3)$



Standard basis

$S_1 = \{(1, 0), (0, 1)\}$

Non-standard bases

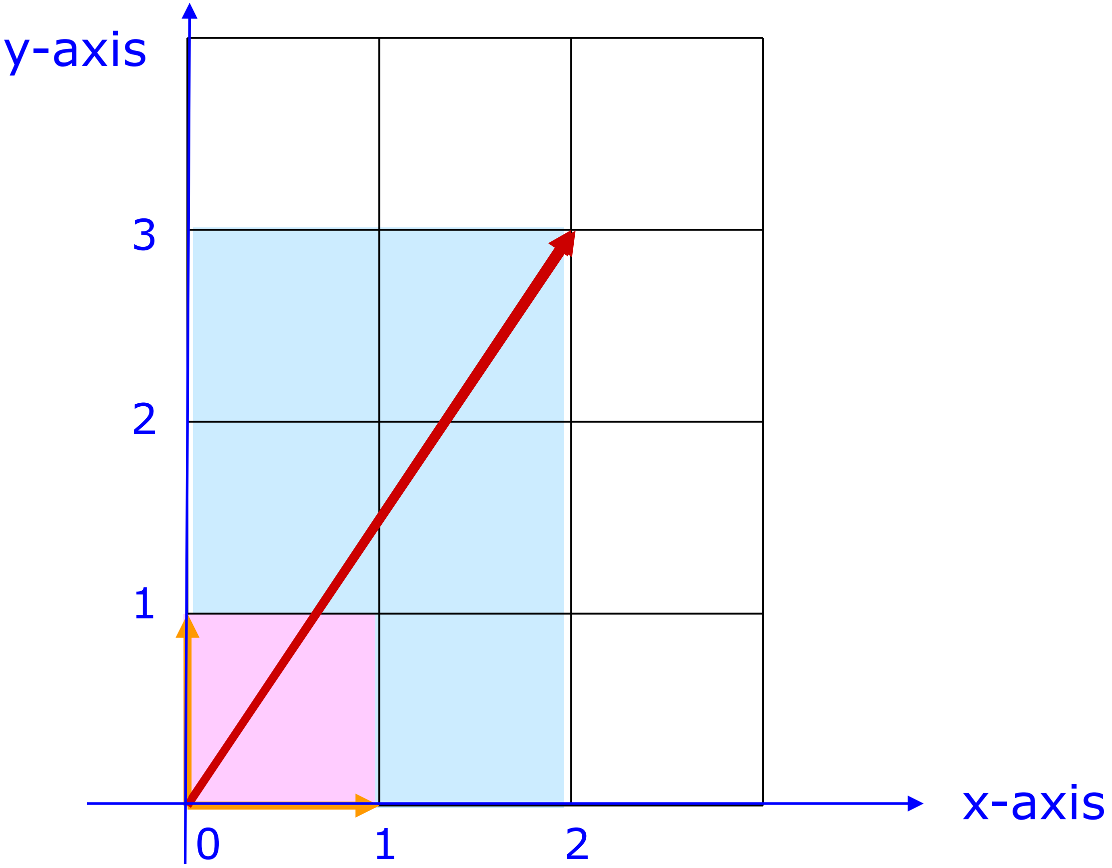
$S_2 = \{(1, -1), (1, 1)\}$

$S_3 = \{(1, 0), (1, 1)\}$

$S_1 = \{(1, 0), (0, 1)\}$

Example 3.5.9.2(a)

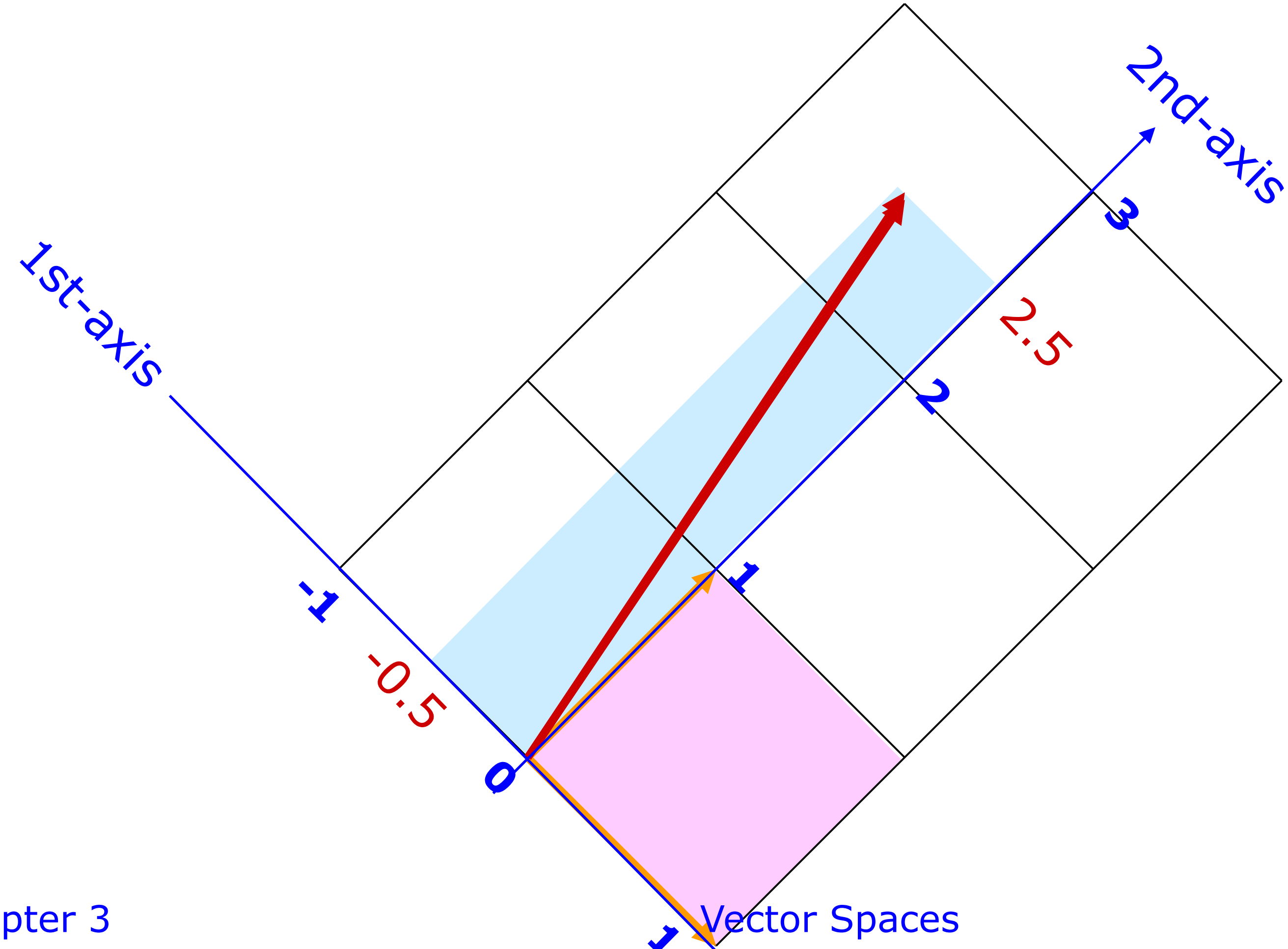
$\boldsymbol{v} = (2, 3) = 2(1, 0) + 3(0, 1) \Rightarrow \quad (\boldsymbol{v})_{S_1} = \underline{(2, 3)}$



$S_2 = \{(1, -1), (1, 1)\}$

Example 3.5.9.2(b)

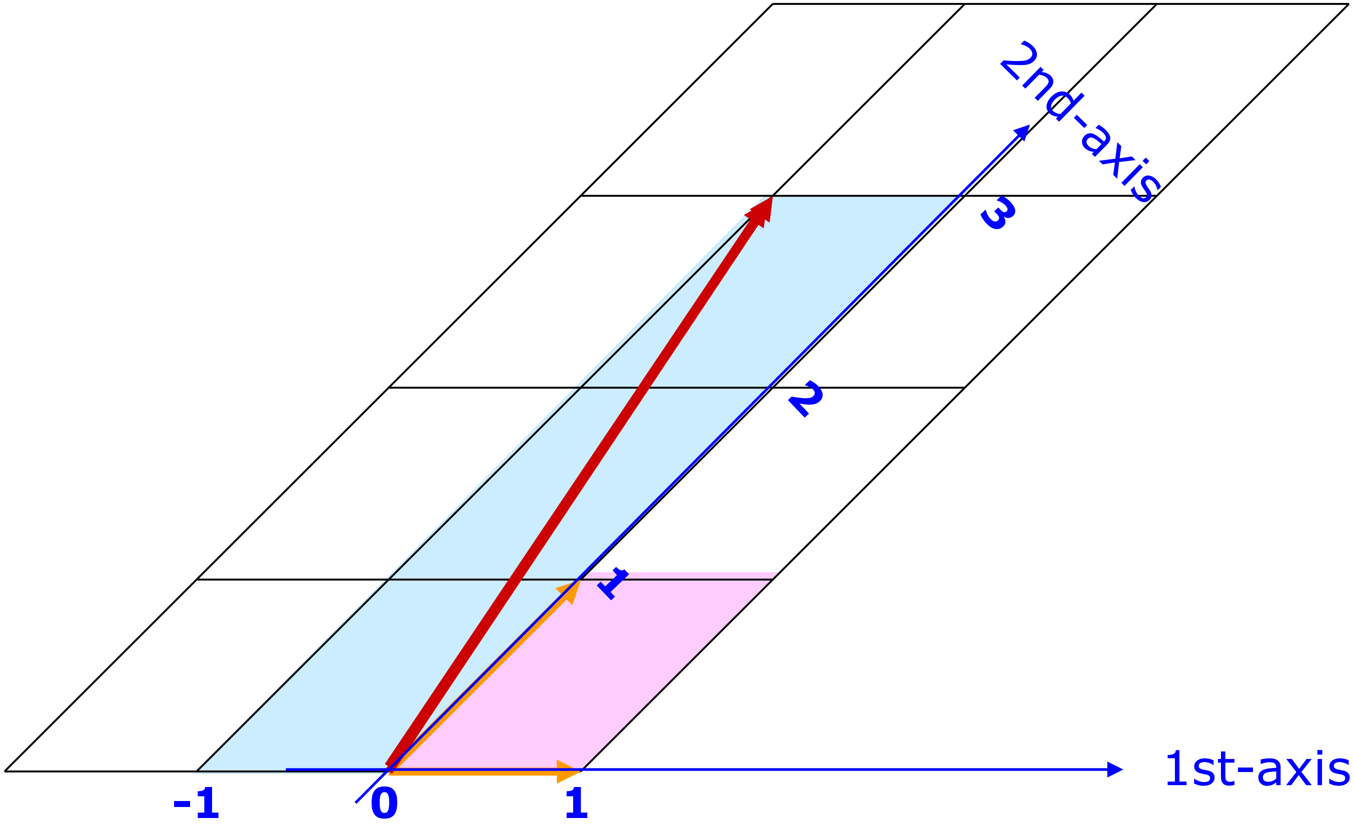
$$\boldsymbol{v} = (2, 3) = -\tfrac{1}{2}(1, -1) + \tfrac{5}{2}(1, 1) \Rightarrow (\boldsymbol{v})_{S_2} = \underline{\left(-\tfrac{1}{2}, \tfrac{5}{2}\right)}$$



$S_3 = \{(1, 0), (1, 1)\}$

Example 3.5.9.2(c)

$\boldsymbol{v} = (2, 3) = -(1, 0) + 3(1, 1) \Rightarrow \quad (\boldsymbol{v})_{S_3} = \underline{(-1, 3)}$



Coordinate vectors with respect to standard basis

Example 3.5.9.3 (Standard Basis for $\boldsymbol{R}^n$)

If S is the standard basis for $\boldsymbol{R}^n$, then $(\boldsymbol{u})_S = \boldsymbol{u}$

$$S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$$

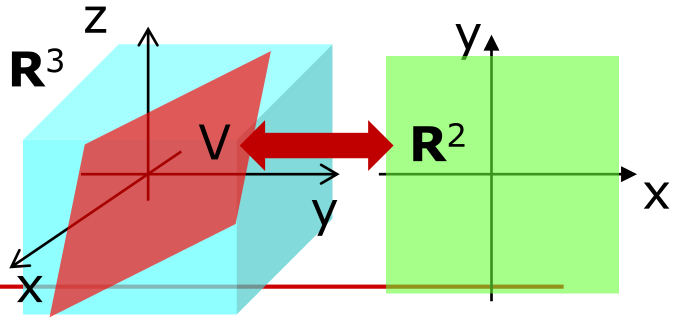
$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0, 0)$$
$$\mathbf{e}_2 = (0, 1, 0, \ldots, 0, 0)$$
$$\ldots$$
$$\mathbf{e}_n = (0, 0, 0, \ldots, 0, 1)$$

For a vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ in $\mathbf{R}^n$

$$\boldsymbol{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \cdots + u_n\mathbf{e}_n$$

$$(\boldsymbol{u})_S = (u_1, u_2, \ldots, u_n)$$

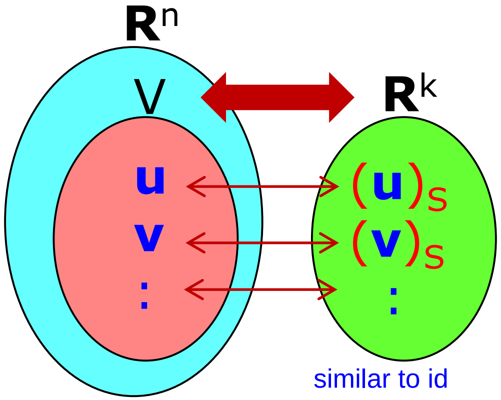
Properties of coordinate vectors



Remark 3.5.10

Some useful rules about coordinate vectors:

Let S be a basis for a vector space V.



similar to id

1. For any $\mathbf{u}$, $\mathbf{v} \in V$,
   $\mathbf{u} = \mathbf{v}$ if and only if $(\mathbf{u})_S = (\mathbf{v})_S$.

2. For any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \in V$ and $c_1, c_2, \ldots, c_r \in \mathbf{R}$,
   $(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r)_S$
   $= c_1(\mathbf{v}_1)_S + c_2(\mathbf{v}_2)_S + \ldots \quad + c_r(\mathbf{v}_r)_S$.

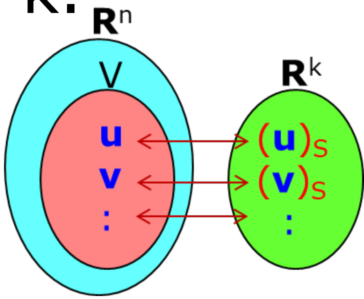
corresponds to a certain number if viewed in R2, even though it is not in the same R3 space

coordinate vector of linear combination
= linear combination of coordinate vectors

help to break up the brackets

Some preserving properties of coordinate vectors

Theorem 3.5.11



S be a basis for a vector space V with |S| = k.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \in V$. Then

1. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are linearly dependent (resp. independent) in V if and only if $(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S$ are linearly dependent (resp. independent) in $\mathbf{R}^k$;
2. $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = V$ if and only if $\text{span}\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S\} = \mathbf{R}^k$.

$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a basis for V
if and only if
$\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S\}$ is a basis for $\mathbf{R}^k$