CS1231(S) Tutorial 8: Relations
Solutions

National University of Singapore

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1. Let $A = \{1, 2, \ldots, 10\}$ and $B = \{2, 4, 6, 8, 10, 12, 14\}$. Define a relation R from A to B by setting
$$x \mathrel{R} y \quad \Leftrightarrow \quad x \text{ is prime and } x \mid y$$
for each $x \in A$ and each $y \in B$. Write down the sets R and R^{-1} in roster notation. Do not use ellipses $(\ldots)$ in your answers.

   Solution.
$$\begin{aligned} R &= \{(2,2),(2,4),(2,6),(2,8),(2,10),(2,12),(2,14),(3,6),(3,12),(5,10),(7,14)\}. \\ R^{-1} &= \{(2,2),(4,2),(6,2),(8,2),(10,2),(12,2),(14,2),(6,3),(12,3),(10,5),(14,7)\}. \end{aligned}$$

2. Let R be a relation on a set A. Show that R is symmetric if and only if $R = R^{-1}$.

   Solution.

   1. ("Only if")
      - 1.1. Suppose R is symmetric.
      - 1.2. $(\subseteq)$
        - 1.2.1. Let $x, y \in A$ such that $(x, y) \in R$.
        - 1.2.2. Then $x \mathrel{R} y$ by the definition of $x \mathrel{R} y$;
        - 1.2.3. $\therefore$ $y \mathrel{R} x$ as R is symmetric;
        - 1.2.4. $\therefore$ $x \mathrel{R^{-1}} y$ by the definition of R^{-1};
        - 1.2.5. $\therefore$ $(x, y) \in R^{-1}$ by the definition of $x \mathrel{R^{-1}} y$.
      - 1.3. $(\supseteq)$
        - 1.3.1. Let $x, y \in A$ such that $(x, y) \in R^{-1}$.
        - 1.3.2. Then $x \mathrel{R^{-1}} y$ by the definition of $x \mathrel{R^{-1}} y$;
        - 1.3.3. $\therefore$ $y \mathrel{R} x$ by the definition of R^{-1};
        - 1.3.4. $\therefore$ $x \mathrel{R} y$ as R is symmetric.
      - 1.4. So $R = R^{-1}$.
   2. ("If")
      - 2.1. Suppose $R = R^{-1}$.
        - 2.1.1. Let $x, y \in A$ such that $x \mathrel{R} y$.
        - 2.1.2. Then $(x, y) \in R$ by the definition of $x \mathrel{R} y$;
        - 2.1.3. $\therefore$ $(x, y) R^{-1}$ as $R = R^{-1}$;
        - 2.1.4. $\therefore$ $x \mathrel{R^{-1}} y$ by the definition of $x \mathrel{R^{-1}} y$;
        - 2.1.5. $\therefore$ $y \mathrel{R} x$ by the definition of R^{-1}.
      - 2.2. So R is symmetric. $\square$

3. For each of the following relations on $\mathbb{Q}$, determine if it is (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) an equivalence relation.

   (a) R is defined by setting $x \mathrel{R} y$ if and only if $xy \geqslant 0$ for all $x, y \in \mathbb{Q}$.

(b) S is defined by setting $x \; S \; y$ if and only if $xy > 0$ for all $x, y \in \mathbb{Q}$.

(c) T is defined by setting $x \; T \; y$ if and only if $|x - y| \leqslant 2$ for all $x, y \in \mathbb{Q}$.

Solution.

(a) R is reflexive and symmetric. It is not transitive because $1 \; R \; 0$ and $0 \; R \; -1$ but $1 \not R \; -1$. Since it is not transitive, it is not an equivalence relation. It is not antisymmetric because $1 \; R \; 2$ and $2 \; R \; 1$ but $1 \neq 2$.

(b) S is symmetric and transitive. It is not reflexive because $0 \not S \; 0$. Since it is not reflexive, it is not an equivalence relation. It is not antisymmetric because $1 \; S \; 2$ and $2 \; S \; 1$ but $1 \neq 2$.

(c) T is reflexive and symmetric. It is not transitive because $-2 \; T \; 0$ and $0 \; T \; 2$ but $-2 \not T \; 2$. Since it is not transitive, it is not an equivalence relation. It is not antisymmetric because $1 \; T \; 2$ and $2 \; T \; 1$ but $1 \neq 2$.

4. Define a relation R on $\mathbb{Q}$ as follows: for all $x, y \in \mathbb{Q}$,

$$x \; R \; y \quad \Leftrightarrow \quad x - y \in \mathbb{Z}.$$

(a) Show that R is an equivalence relation.

(b) Find an element a in the equivalence class $[\frac{37}{7}]$ that satisfies $0 \leqslant a < 1$.

(c) Devise a general method to find, for each given equivalence class $[x]$, where $x \in \mathbb{Q}$, an element $a \in [x]$ such that $0 \leqslant a < 1$. Justify your answer.

Solution.

(a)
1. ("Reflexivity")
   1.1. Let $x \in \mathbb{Q}$.
   1.2. Then $x - x = 0 \in \mathbb{Z}$.
   1.3. So $x \; R \; x$.
2. ("Symmetry")
   2.1. Let $x, y \in \mathbb{Q}$ such that $x \; R \; y$.
   2.2. Then $x - y \in \mathbb{Z}$ by the definition of R.
   2.3. So $y - x = -(x - y) \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under taking negatives.
   2.4. This implies $y \; R \; x$ by the definition of R.
3. ("Transitivity")
   3.1. Let $x, y, z \in \mathbb{Q}$ such that $x \; R \; y$ and $y \; R \; z$.
   3.2. Then $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$ by the definition of R.
   3.3. So $x - z = (x - y) + (y - z) \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under addition.
   3.4. This implies $x \; R \; z$ by the definition of R.
4. Since R is reflexive, symmetric and transitive, it is an equivalence relation. $\square$

(b) Note that $\frac{37}{7} = 5\frac{2}{7}$. Thus $\frac{37}{7} - \frac{2}{7} = 5 \in \mathbb{Z}$. This implies $\frac{37}{7} \; R \; \frac{2}{7}$ and hence $\frac{2}{7} \in [\frac{37}{7}]$.

(c) Let $x \in \mathbb{Q}$. Take $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$ such that $x = m/n$. Without loss of generality, we may assume $n > 0$. Define $a = (m \mathbin{\underline{\mathrm{mod}}} n)/n$. Then we know $0 \leqslant a < 1$ because $0 \leqslant m \mathbin{\underline{\mathrm{mod}}} n < n$ by the definition of $m \mathbin{\underline{\mathrm{mod}}} n$. In addition,

$$x - a = \frac{m}{n} - \frac{m \mathbin{\underline{\mathrm{mod}}} n}{n} = \frac{m - (m \mathbin{\underline{\mathrm{mod}}} n)}{n} = m \mathbin{\underline{\mathrm{div}}} n \in \mathbb{Z}$$

as $m = n(m \mathbin{\underline{\mathrm{div}}} n) + (m \mathbin{\underline{\mathrm{mod}}} n)$. Thus $x \; R \; a$ and so $a \in [x]$.

5. Let A, B be nonempty sets and f be a surjection $A \to B$. Show that $\mathscr{C}$ is a partition on A, where

$$\mathscr{C} = \big\{ \{x \in A : f(x) = y\} : y \in B \big\}.$$

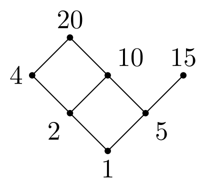
Solution.

1. We claim that each element of $\mathscr{C}$ is nonempty.
   1.1. Let $S \in \mathscr{C}$.
   1.2. Use the definition of $\mathscr{C}$ to find $y_0 \in B$ such that $S = \{x \in A : f(x) = y_0\}$.
   1.3. Use the surjectivity of f to find $x_0 \in A$ such that $f(x_0) = y_0$.
   1.4. Then $x_0 \in S$ by the choice of y_0.
   1.5. In particular, the set S is nonempty.
2. ($\geqslant 1$)
   2.1. Let $x_0 \in A$.
   2.2. Define $y_0 = f(x_0)$ and $S = \{x \in A : f(x) = y_0\} \in \mathscr{C}$.
   2.3. Then $x_0 \in S$ as $f(x_0) = y_0$.
3. ($\leqslant 1$)
   3.1. Let $x_0 \in A$ and $S, S' \in \mathscr{C}$ such that $x_0 \in S$ and $x_0 \in S'$.
   3.2. Use the definition of $\mathscr{C}$ to find $y, y' \in B$ such that $S = \{x \in A : f(x) = y\}$ and $S' = \{x \in A : f(x) = y'\}$.
   3.3. Then $f(x_0) = y$ and $f(x_0) = y'$ as $x_0 \in S$ and $x_0 \in S'$.
   3.4. This implies $y = y'$ by the functionality of f.
4. So $\mathscr{C}$ is a partition of A. $\square$

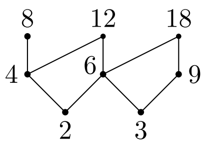
6. Consider the "divides" relation on each of the following sets of integers. For each of these, draw a Hasse diagram, find all largest, smallest, maximal and minimal elements, and a linearization.

   (a) $A = \{1, 2, 4, 5, 10, 15, 20\}$.
   (b) $B = \{2, 3, 4, 6, 8, 9, 12, 18\}$.

   Solution.

   (a)
   

   1 is the only minimal element and is the smallest element. 15 and 20 are maximal elements. There is no largest element. There are many linearizations; the easiest one is probably $\leqslant$ on A.

   (b)
   

   2 and 3 are minimal elements. 8, 12 and 18 are maximal elements. There is no largest element. there is no smallest element. There are many linearizations; the easiest one is probably $\leqslant$ on B.

7. **Definition.** Let $\preccurlyeq$ be a partial order on a set P, and $a, b \in P$.

   - We say a, b are *comparable* if $a \preccurlyeq b$ or $b \preccurlyeq a$.
   - We say a, b are *compatible* if there exists $c \in P$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

   (a) Is it true that, in all partially ordered sets, any two comparable elements are compatible? Justify your answer.
   (b) Is it true that, in all partially ordered sets, any two compatible elements are comparable? Justify your answer.

   Solution.

(a) Yes. If a and b are comparable, then either $a \preccurlyeq b$ or $b \preccurlyeq a$. In the former case, we have $a \preccurlyeq b$ and $b \preccurlyeq b$ by the symmetry of $\preccurlyeq$, and so a and b are compatible. In the latter case, we have $a \preccurlyeq a$ and $b \preccurlyeq a$ by the symmetry of $\preccurlyeq$, and so a and b are compatible.

(b) No. Consider the "divides" relation $|$ on $\mathbb{Z}^+$. This is a partial order on $\mathbb{Z}^+$. We know $2 \mid 6$ and $3 \mid 6$. So 2 and 3 are compatible. However, we also know that $2 \nmid 3$ and $3 \nmid 2$. So 2 and 3 are not comparable.