Matriculation Number:

A								

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2019-2020

MA1521 CALCULUS FOR COMPUTING

November 2019 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** Do not write your name anywhere in this booklet. This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **FIVE (5)** questions and comprises **ELEVEN (11)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question. The marks for each question are indicated at the beginning of the question. The maximum possible total score for this examination paper is 50 marks.
4. This is a **closed book (with authorized material)** examination. Students are only allowed to bring into the examination hall **ONE** piece A4 size help-sheet which can be used on both sides.
5. Candidates may use any calculators that satisfy MOE A-Level examination guidelines. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5
Marks					

Question 1 [10 marks]

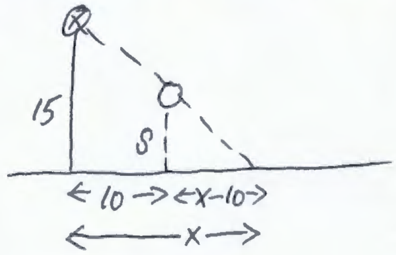
(a) A light shines from the top of a lamp post 15 metres high. At time $t = 0$ a ball is projected vertically upwards from a point on the ground 10 metres away from the foot of the lamp post. It is known that the ball moves upwards a distance of $s = 10t - 4.9t^2$ metres above the ground in t seconds for $0 \leq t \leq 1.01$. If the speed of the shadow of the ball on the ground at $t = 1$ second is equal to u metre per second, find the value of u. Give your answer correct to two decimal places.

(b) Let a denote a positive constant. It is known that the graph of $y^2 = x^2a^3 - 3x^3a^2 + 3x^4a - x^5$ has a loop and that the area bounded by this loop is equal to $\frac{2019}{1521}$. Find the value of a. Give your answer correct to two decimal places.

Answer 1(a)	0.31	**Answer 1(b)**	1.65

(Show your working below and on the next page.)

(a)



$$\frac{x}{15} = \frac{x-10}{s}$$

$$sx = 15x - 150$$

$$x = \frac{150}{15-s} = \frac{150}{15-10t+4.9t^2}$$

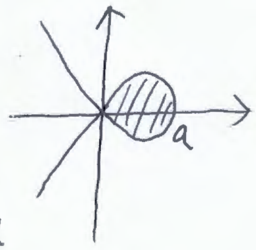
$$\frac{dx}{dt} = \frac{-150(-10+9.8t)}{(15-10t+4.9t^2)^2}$$

$$u = \left.\frac{dx}{dt}\right|_{t=1} = 0.306\cdots$$

$$\approx \underline{\underline{0.31}}$$

$$y^2 = x^2(a^3 - 3a^2x + 3ax^2 - x^3)$$

$$= x^2(a-x)^3$$



$$\frac{2019}{1521} = 2\int_0^a x(a-x)^{3/2}\,dx$$

$$= -2\int_a^0 (a-u)u^{3/2}\,du$$

(let $u = a - x$)

$$= 2\int_0^a (au^{3/2} - u^{5/2})\,du$$

$$= 2\left\{\frac{2}{5}a^{7/2} - \frac{2}{7}a^{7/2}\right\}$$

$$= \frac{8}{35}a^{7/2}$$

$$a = \left(\frac{35 \times 2019}{8 \times 1521}\right)^{2/7}$$

$$= 1.653\cdots \approx \underline{\underline{1.65}}$$

Question 2 [10 marks]

(a) Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(3^n)(n!)^2}{(2n)!}(x-5)^{2n}$.
Give your answer correct to two decimal places.

(b) Let $f(x) = \int_0^x \frac{\ln(1+t^2)}{1+t} dt$. Find the **exact value** of $f^{(7)}(0)$.

Answer 2(a)	1.15	Answer 2(b)	600

(Show your working below and on the next page.)

(a)
$$\left| \frac{\frac{3^{n+1}[(n+1)!]^2}{(2n+2)!}(x-5)^{2n+2}}{\frac{3^n (n!)^2}{(2n)!}(x-5)^{2n}} \right|$$

$$= \frac{3(n+1)^2}{(2n+1)(2n+2)}|x-5|^2$$

$$\to \frac{3}{4}|x-5|^2$$

$$\frac{3}{4}|x-5|^2 < 1$$

$$\Rightarrow |x-5|^2 < \frac{4}{3}$$

$$\Rightarrow |x-5| < \frac{2}{\sqrt{3}}$$

$$\frac{2}{\sqrt{3}} = 1.154\ldots$$

$$\approx \underline{\underline{1.15}}$$

(b)
$$\int_0^x \frac{\ln(1+t^2)}{1+t} dt$$

$$= \int_0^x \left(t^2 - \frac{1}{2}t^4 + \frac{1}{3}t^6 - \ldots\right)\left(1 - t + t^2 - t^3 + t^4 - \ldots\right) dt$$

$$= \int_0^x \left\{ \ldots + \left(1 - \frac{1}{2} + \frac{1}{3}\right)t^6 + \ldots \right\} dt$$

$$= \int_0^x \left\{ \ldots + \frac{5}{6}t^6 + \ldots \right\} dt$$

$$= \ldots + \frac{5}{42}x^7 + \ldots$$

$$\therefore \frac{f^{(7)}(0)}{7!} = \frac{5}{42}$$

$$f^{(7)}(0) = \frac{5}{42} \times 7! = \underline{\underline{600}}$$

Question 3 [10 marks]

(a) Find the directional derivative of the function $f(x,y,z) = xy^2e^{\frac{2}{3}z}$ at the point $(1,2,3)$ in the direction of the vector which joins $(1,2,3)$ to $(3,2,1)$. Give your answer correct to two decimal places.

(b) Find the **exact value** of the double integral $\int\int_R (1+x)\,dA$, where R is the finite region in the third quadrant bounded above by the curve $x^2+y=0$ and bounded below by the curve $x+y^2=0$. Give your answer in the form of a fraction $\frac{m}{n}$ where m and n are positive integers without any common factors.

Answer 3(a)	6.97	Answer 3(b)	$\frac{11}{60}$

(Show your working below and on the next page.)

(a) $\nabla f = \begin{pmatrix} y^2e^{\frac{2}{3}z} \\ 2xye^{\frac{2}{3}z} \\ \frac{2}{3}xy^2e^{\frac{2}{3}z} \end{pmatrix}$

$\nabla f(1,2,3) = \begin{pmatrix} 4e^2 \\ 4e^2 \\ \frac{8}{3}e^2 \end{pmatrix}$

$\vec{u} = \dfrac{\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}}{\left\|\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}\right\|} = \frac{1}{\sqrt{8}}\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$

$D_{\vec{u}}f(1,2,3) = \nabla f(1,2,3)\cdot\vec{u}$

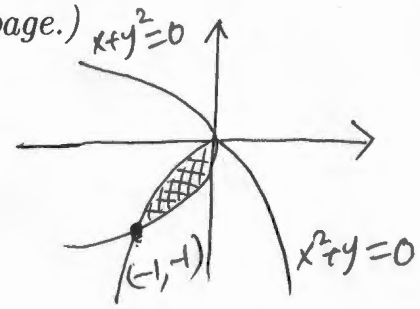
$= \sqrt{8}e^2 - \frac{2}{3}\sqrt{8}e^2$

$= \frac{1}{3}\sqrt{8}\,e^2$

$= 6.966\ldots$

$\approx \underline{6.97}$

(b)



$\iint_R (1+x)\,dA = \int_{-1}^{0}\int_{-\sqrt{-x}}^{-x^2} (1+x)\,dy\,dx$

$= \int_{-1}^{0} \left[y+xy\right]_{y=-\sqrt{-x}}^{y=-x^2} dx$

$= \int_{-1}^{0} \left[(-x^2-x^3) - (-\sqrt{-x} - x\sqrt{-x})\right] dx$

(let $x=-t$) $= \int_{1}^{0} \left[(-t^2+t^3) - (-\sqrt{t} + t\sqrt{t})\right](-dt)$

$= \int_0^1 -t^2+t^3+t^{1/2}-t^{3/2}\,)\,dt$

$= \left[-\frac{1}{3}t^3+\frac{1}{4}t^4+\frac{2}{3}t^{3/2}-\frac{2}{5}t^{5/2}\right]_0^1$

$= -\frac{1}{3}+\frac{1}{4}+\frac{2}{3}-\frac{2}{5} = \underline{\frac{11}{60}}$

Question 4 [10 marks]

(a) Let k denote a positive constant. Let R denote the plane circular disc region centred at the origin with radius k on the xy-plane. Let D denote the solid region under the surface of the function $z = e^{-\left(\frac{x^2+y^2}{k^2}\right)}$ and over the region R. If the volume of D equals 101, find the value of k. Give your answer correct to two decimal places.

(b) You started an experiment with 100 mg of a radioactive substance X which has a half life of 30 minutes. After 0.82 hour, you had m mg of X left. Find the value of m. Give your answer correct to the nearest integer.

Answer 4(a)	7.13	Answer 4(b)	32

(Show your working below and on the next page.)

(a)

$$101 = \iint_R e^{-\left(\frac{x^2+y^2}{k^2}\right)} dA$$

$$= \int_0^k \int_0^{2\pi} e^{-\frac{r^2}{k^2}} r\,dr\,d\theta$$

$$= 2\pi \left[-\frac{k^2}{2} e^{-\frac{r^2}{k^2}} \right]_0^k$$

$$= 2\pi \left(\frac{k^2}{2} - \frac{k^2}{2} e^{-1} \right)$$

$$= \pi k^2 (1 - e^{-1})$$

$$k = \sqrt{\frac{101}{\pi(1-e^{-1})}}$$

$$= 7.131\ldots$$

$$\approx \underline{\underline{7.13}}$$

(b)

$$m = 100 e^{-\frac{\ln 2}{0.5} \times 0.82}$$

$$= 32.08\ldots$$

$$\approx \underline{\underline{32}}$$

Question 5 [10 marks]

(a) Let r denote a positive constant. At time $t = 0$ a tank contains 100 grams of salt dissolved in 100 litres of water. Assume that water containing 3 grams of salt per litre is entering the tank at a rate of r litre per minute and that the well stirred solution is draining from the tank at the same rate. It is known that at time $t = 45$ minutes, there are 200 grams of salt in the tank. Find the value of r. Give your answer correct to two decimal places.

(b) Let $y(x)$ be the solution of the differential equation

$$x\frac{dy}{dx} + 2y = \frac{\cos x}{x}, \quad \text{with } x > 0 \text{ and } y(\pi) = 1.$$

Find the value of $y(\frac{\pi}{6})$. Give your answer correct to two decimal places.

Answer 5(a)	1.54	Answer 5(b))	37.82

(Show your working below and on the next page.)

(a) $\frac{dQ}{dt} = 3r - \frac{Q}{100}r$

$= -\frac{r}{100}(Q-300)$

$\frac{dQ}{Q-300} = -\frac{r}{100}dt$

$\ln|Q-300| = -\frac{r}{100}t + C$

$Q - 300 = Ae^{-\frac{rt}{100}}$

$Q(0)=100 \Rightarrow A = -200$

$Q = 300 - 200e^{-\frac{rt}{100}}$

$200 = 300 - 200e^{-\frac{45r}{100}}$

$e^{-\frac{45r}{100}} = \frac{1}{2}$

$r = \frac{100\ln 2}{45} = 1.540\ldots$

$\approx \underline{\underline{1.54}}$

(b) $\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}$

$R = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$y = \frac{1}{x^2}\int x^2 \frac{\cos x}{x^2}dx$

$= \frac{1}{x^2}(\sin x + C)$

$1 = \frac{1}{\pi^2}(\sin\pi + C)$

$C = \pi^2$

$y = \frac{1}{x^2}(\sin x + \pi^2)$

$y(\frac{\pi}{6}) = \frac{36}{\pi^2}(\frac{1}{2} + \pi^2)$

$= 37.823\ldots$

$\approx \underline{\underline{37.82}}$