Tutorial 5

Exercise 3

17. Give an example of a 2×3 matrix $\boldsymbol{A}$, if possible, such that the solution space of the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ is

    (a) $\mathbb{R}^3$.

    (b) the plane $\{ (x, y, z) \mid 2x + 3y - z = 0 \}$.

    (c) the line $\{ (t, 2t, 3t) \mid t \in \mathbb{R} \}$.

    (d) the zero subspace.

19. Let U, V and W be the three planes defined in Question 3.4. Is $U \cap V$ a subspace of $\mathbb{R}^3$? Is $V \cap W$ a subspace of $\mathbb{R}^3$? Justify your answers.

$$U = \{ (x, y, z) \mid 2x - y + 3z = 0 \}, \quad V = \{ (x, y, z) \mid 3x + 2y - z = 0 \},$$
$$W = \{ (x, y, z) \mid x - 3y - 2z = 1 \}.$$

25. For each of the sets S_1 to S_6 in Question 3.9, determine whether the set is linearly independent

    (a) $S_1 = \{(1, 1, -1),\ (-2, 2, 1)\}$.

    (b) $S_2 = \{(1, 1, -1),\ (-2, -2, 2)\}$.

    (c) $S_3 = \{(1, 1, -1),\ (-2, 2, 1),\ (1, 5, -2)\}$.

    (d) $S_4 = \{(1, 1, -1),\ (-2, 2, 1),\ (4, 0, 3)\}$.

    (e) $S_5 = \{(1, 1, -1),\ (-2, 2, 1),\ (1, 5, -2)\ (0, 8, -2)\}$.

    (f) $S_6 = \{(1, 1, -1),\ (-2, 2, 1),\ (4, 0, 3),\ (2, 6, -3)\}$.

27. In Question 3.13, suppose $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine which of the sets S_1 to S_5 are linearly independent.

$$S_1 = \{\boldsymbol{u},\ \boldsymbol{v}\}, \quad S_2 = \{\boldsymbol{u} - \boldsymbol{v},\ \boldsymbol{v} - \boldsymbol{w},\ \boldsymbol{w} - \boldsymbol{u}\}, \quad S_3 = \{\boldsymbol{u} - \boldsymbol{v},\ \boldsymbol{v} - \boldsymbol{w},\ \boldsymbol{u} + \boldsymbol{w}\},$$
$$S_4 = \{\boldsymbol{u},\ \boldsymbol{u} + \boldsymbol{v},\ \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}\}, \quad S_5 = \{\boldsymbol{u} + \boldsymbol{v},\ \boldsymbol{v} + \boldsymbol{w},\ \boldsymbol{u} + \boldsymbol{w},\ \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}\}.$$

29. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^3$ such that $V = \operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ and $W = \operatorname{span}\{\boldsymbol{u}, \boldsymbol{w}\}$ are planes in $\mathbb{R}^3$. Find $V \cap W$ if

    (a) $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent.

    (b) $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are not linearly independent.

30. (All vectors in this question are written as column vectors.) Let $\boldsymbol{u_1},\ \boldsymbol{u_2},\ \ldots,\ \boldsymbol{u_k}$ be vectors in $\mathbb{R}^n$ and $\boldsymbol{P}$ a square matrix or order n.

    (a) Show that if $\boldsymbol{P}\boldsymbol{u_1},\ \boldsymbol{P}\boldsymbol{u_2},\ \ldots,\ \boldsymbol{P}\boldsymbol{u_k}$ are linearly independent, then $\boldsymbol{u_1},\ \boldsymbol{u_2},\ \ldots,\ \boldsymbol{u_k}$ are linearly independent.

    (b) Suppose $\boldsymbol{u_1},\ \boldsymbol{u_2},\ \ldots,\ \boldsymbol{u_k}$ are linearly independent.

        (i) Show that if $\boldsymbol{P}$ is invertible, then $\boldsymbol{P}\boldsymbol{u_1},\ \boldsymbol{P}\boldsymbol{u_2},\ \ldots,\ \boldsymbol{P}\boldsymbol{u_k}$ are linearly independent.

        (ii) If $\boldsymbol{P}$ is not invertible, are $\boldsymbol{P}\boldsymbol{u_1},\ \boldsymbol{P}\boldsymbol{u_2},\ \ldots,\ \boldsymbol{P}\boldsymbol{u_k}$ linearly independent?