CS1231/CS1231S Assignment #2

AY2020/21 Semester 1

Deadline: Wednesday, 4 November 2020, 4:00pm

IMPORTANT: Please read the instructions below

This is a graded assignment worth 10% of your final grade. Please work on it by yourself, not in a group or in collaboration with anybody. Anyone found plagiarising (submitting other's work as your own), or sending your answers to others will be penalised with a straight zero for the assignment, and if found re-committing this offence, will be referred to the disciplinary board.

You are to submit your assignment to **LumiNUS Files**. A submission folder has been created for you at Files > Assignment #2 > Your tutorial group > Your personal folder.

Your answers may be typed or handwritten. Make sure that it is legible (for example, don't use very light pencil or ink, or very small font) or marks may be deducted.

You are to submit **a SINGLE pdf file**, where each page is A4 size. Do not submit multiple files or files in other format, or we will not accept your submission.

You may test out your submission before the deadline, but make sure you remove any test files you have submitted earlier.

Late submission will NOT be accepted, as the folder will automatically close on the dot. We will set the closing time to slightly later than 4pm to provide a grace period, but in your mind, you should treat **4pm** as the deadline. If you think you might be too busy on the day of the deadline, please submit earlier. Also, avoid submitting in the last minute; if everybody does that (and we have more than 1000 students in CS1231 and CS1231S) the system may get sluggish due to the overload, or worse, it may break down, and you will miss the deadline.

Note the following as well:

- Name your pdf file with your **Student Number** (eg: A0123456M)
- To keep the submitted document short, you may submit your answers without including the questions.
- As this is an assignment given well ahead of time, we expect you to work on it early. You should submit **polished work**, not answers that are untidy or appear to have been done in a hurry, for example, with scribbling and cancellation all over the places.

To combine all pages into a single pdf document for submission, you may find the following scanning apps helpful if you intend to scan your handwritten answers:

* for Android: https://fossbytes.com/best-android-scanner-apps/

* for iphone:
https://www.switchingtomac.com/tutorials/ios-tutorials/the-best-ios-scanner-apps-to-scan-documents-images/

If you need any clarification about this assignment, please post on the **LumiNUS > Assignments** forum.

Question 0. (2 marks)

This is an administrative question (and 2 free marks). Please follow these instructions carefully.

a. Remove any unnecessary files from your submission folder, INCLUDING your midterm submission AND Assignment 1 submission, if you haven't already done so. There should be only one file in your submission folder. (1 mark)

b. Name your file with your **Student Number** (eg: A0123456M) (1 mark)

Question 1. (9 marks)

Aiken and Betty are getting married, and as their best friend you have agreed to help them run various errands large and small.

a. Your first task is to plan the seating for the "Main Table", where the happy (but severely stressed) couple and their direct family members would be sitting. As COVID-19 will finally be over by the time they get married, you are assuming a traditional table of 10 persons around a round table. This is a difficult task whose consequences can be cataclysmic, so you must proceed carefully. You begin by working out how many possible seating arrangements you can make, so that Aiken and Betty can exhaustively consider all the options. **You must show your working in order to convince Aiken and Betty, or you don't get any marks from us.**

   i. Aiken and Betty themselves, and their respective pairs of parents must sit next to each other (i.e. Aiken's dad and mum have to sit together, and likewise with Betty's). However Aiken's mum and dad do not have to sit next to Betty's mum and dad. How many ways can you arrange the couple, their parents and four of their relatives (including Uncle Pete and Aunt Jemima whom we will meet in the next part) around the table? (2 marks)

   ANSWER:
   We can treat Aiken and Betty, Aiken's parents and Betty's parents as 3 separate units, and the remainder of the relatives as 4 separate units. Hence there are 7 units. Using permutations around a circle, we get $\frac{7!}{7} = 6!$ = 720 permutations. Within each of the 3 units, there are 2 ways they can sit together, giving 2^3 = 8 ways. Therefore total # of permutations is 720 x 8 = **5,760**.

   ii. (NOTE: This part depends on you solving the earlier question about seating Aiken and Betty's relatives. Please do that part first) Aiken now tells you that his Uncle Pete and Betty's Aunt Jemima are not on the best of terms, and if they sit next to each other, World War 3 will break out. How many ways can you now arrange their relatives around this table in order to forestall this apocalypse? (3 marks)

   ANSWER:
   We find the number of ways we can put Uncle Pete and Aunt Jemima together. We thus treat them as a unit, and we get $\frac{6!}{6} = 5!$ = 120 permutations. Now there are 4 such units, with 2 ways of sitting within each unit, giving us 2^4 = 16 ways. Thus there are 16 x 120 = 1920 ways.

Using the difference rule, the number of seatings that do not place the two relatives together is 5,760 – 1,920 = **3,840** permutations.

ALTERNATIVE SOLUTION

This solution mirrors the approach we used in Tutorial 9 Question 7.

We ignore Uncle Pete and Aunt Jemima first. The other 8 guests are in 5 units (Aiken-Betty, Aiken's parents, Betty's parents, and the remaining two guests).

Number of ways to place these 8 guests = $\frac{5!}{5} = 4! = 24$. There are 3 two-person units, so there are $2^3 = 8$ ways of sitting within each unit. Therefore, there are 24 x 8 = 192 ways of placing these 8 guests.

These 8 guests are in 5 units so there are 5 gaps around the table for us to place Uncle Pete and Aunt Jemina. Number of ways to place Uncle Pete and Aunt Jemima = 5 x 4 = 20.

Therefore, the total number of ways = 192 x 20 = **3840**.

b. After an exhausting session of considering how to prevent an extinction-level event from occurring from poor seating choices, Aiken, Betty and their parents (6 in total) are now starving. Your next task is to buy food for them. Your options from the nearby food centre are (i) chicken rice, (ii) roast duck rice, (iii) vegetarian rice and (iv) vegetarian noodles. **Again if you do not show any working, you will not get any marks from us.**

i. How many ways can you buy food for the Starving Six? (No food for you. Sorry.) We are buying 1 dish per person. (1 mark)

ANSWER:
We are effectively binning the Starving Six into four categories: Those who want chicken rice, those who want roast duck rice, etc. We have *n*=4, *r*=6, hence

# of ways is $\binom{6+(4-1)}{6} = \binom{9}{6} = \mathbf{84}\ ways$

ii. It turns out that Aiken's parents are vegetarians. Now how many ways can you buy food for the Starving Six? Again we are buying one dish per person. (3 marks)

ANSWER:
We need to identify the illegal combinations, which are shown below, and subtract them from the answer in part (i).

Case 1: No vegetarian food, we have *n*=2, *r*=6, so $\binom{6+(2-1)}{6} = \binom{7}{6} = 7$.

Case 2: No vegetarian rice, 1 vegetarian noodles, we have *n*=2, *r*=5, so $\binom{5+(2-1)}{5} = \binom{6}{5} = 6$.

Case 3: 1 vegetarian rice, no vegetarian noodles, we have *n*=2, *r*=5, so $\binom{5+(2-1)}{5} = \binom{6}{5} = 6$.

Therefore, number of ways is 84 – 7 – 6 – 6 = **65**.

Question 2. (6 marks)

Let $a \in \mathbb{Z}_{\geqslant 2}$. Suppose that for all $m, n \in \mathbb{Z}^+$, if $a \mid mn$, then $a \mid m$ or $a \mid n$. Show that a is prime.

Answer 1:

1. Let $a \in \mathbb{Z}_{\geqslant 2}$ such that for all $m, n \in \mathbb{Z}^+$, if $a \mid mn$, then $a \mid m$ or $a \mid n$.
2. 2.1. Let m be a positive divisor of a.
   2.2. Use the definition of divisibility to find $n \in \mathbb{Z}$ such that $a = mn$.
   2.3. Then $a \mid mn$ by Example 8.1.6.
   2.4. So line 1 implies $a \mid m$ or $a \mid n$.
   2.5. Case 1: suppose $a \mid m$.
      2.5.1. Then $m = a$ or $m = -a$ by Tutorial 6 Question 1.
      2.5.2. As $a \geqslant 2$ and m is positive, this implies $m = a$.
   2.6. Case 2: suppose $a \mid n$.
      2.6.1. Line 2.2 tells us that $n \mid a$.
      2.6.2. So $n = a$ or $n = -a$ by Tutorial 6 Question 1.
      2.6.3. As $a \geqslant 2$ and m is positive, we know from line 2.2 that n is positive too.
      2.6.4. Thus $n = a$.
      2.6.5. This implies $m = \frac{a}{n} = \frac{a}{a} = 1$.
3. We know 1 and a are positive divisors of a from Example 8.1.6.
4. Lines 2 and 3 show that the only positive divisors of a are 1 and a.
5. This says a is prime as $a \geqslant 2$.

Answer 2:

1. Let $a \in \mathbb{Z}_{\geqslant 2}$ such that a is not prime.
2. Then a is composite by Remark 8.2.2.
3. Apply Lemma 8.2.4 to find a divisor m of a such that $1 < m < a$.
4. Use the definition of divisibility to find $n \in \mathbb{Z}$ such that $a = mn$.
5. Then $a \mid mn$ by Example 8.1.6.
6. On the one hand, as $|m| = m < a = |a|$ and $a \neq 0$, we deduce from Proposition 8.1.10 that $a \nmid m$.
7. On the other hand, the fact that $m < a$ implies $n = \frac{a}{m} > \frac{a}{a} = 1$ by line 4.
8. Therefore, as $|n| = n < a = |a|$ and $a \neq 0$, we deduce from Proposition 8.1.10 that $a \nmid n$.

Question 3. (3 marks)

Use the Euclidean Algorithm to find integers x and y that satisfy

$$42x + 15y = 6.$$

Show all computational steps.

Answer 1:

Cancelling the common factor 3 on both sides of the equation gives

$$14x + 5y = 2.$$

Apply the Euclidean Algorithm to 14 and 5:

$$\begin{array}{rcll} 14 \,\underline{\mathrm{mod}}\, 5 = 4 & \leftarrow & 4 = 14 - 5 \times 2 & (1) \\ 5 \,\underline{\mathrm{mod}}\, 4 = 1 & \leftarrow & 1 = 5 - 4 \times 1 & (2) \\ 4 \,\underline{\mathrm{mod}}\, 1 = 0 & & & \end{array}$$

Hence

$$\begin{aligned} 1 &= 5 - 4 \times 1 && \text{by (2);} \\ &= 5 - (14 - 5 \times 2) && \text{by (1);} \\ &= (-1) \times 14 + 3 \times 5. \end{aligned}$$

Thus $2 = (-2) \times 14 + 6 \times 5$. So we can set $x = -2$ and $y = 6$.

Answer 2:

Apply the Euclidean Algorithm to 42 and 15:

$$\begin{array}{rcll} 42 \,\underline{\mathrm{mod}}\, 15 = 12 & \leftarrow & 12 = 42 - 15 \times 2 & (1) \\ 15 \,\underline{\mathrm{mod}}\, 12 = 3 & \leftarrow & 3 = 15 - 12 \times 1 & (2) \\ 12 \,\underline{\mathrm{mod}}\, 3 \;\; = 0 & & & \end{array}$$

Hence

$$\begin{aligned} 3 &= 15 - 12 \times 1 && \text{by (2);} \\ &= 15 - (42 - 15 \times 2) && \text{by (1);} \\ &= (-1) \times 42 + 3 \times 15. \end{aligned}$$

Thus $6 = (-2) \times 42 + 6 \times 15$. So we can set $x = -2$ and $y = 6$.

Question 4. (12 marks)

Fix $n \in \mathbb{Z}_{\geqslant 2}$. Define a relation R on $\{0,1,\ldots,n-1\}$ by setting, for all $x, y \in \{0,1,,\ldots,n-1\}$,

$$x \; R \; y \quad \Leftrightarrow \quad \exists a \in \{1,2,\ldots,n-1\} \;\; ax \equiv ay \pmod{n}.$$

(a) Show that R is reflexive and symmetric. [2 marks]

(b) Suppose $n = 4$.

(i) Draw the following diagram.

$$\begin{array}{cc} 0 & 2 \\ \\ 1 & 3 \end{array}$$

For each pair $x, y \in \{0,1,2,3\}$, draw a straight line (without arrowheads) joining x and y in your copy of the diagram if $x \neq y$ and $x \; R \; y$. Draw no other line. [2 marks]

(ii) Is R an equivalence relation in this case? Briefly justify your answer. If R is an equivalence relation, then write out all the equivalence classes in roster notation without repetition. [3 marks]

(c) Suppose $n = 6$.

(i) Draw the following diagram.

$$\begin{array}{cccc} & 0 & 3 & \\ 4 & & & 1 \\ & 2 & 5 & \end{array}$$

For each pair $x, y \in \{0,1,\ldots,5\}$, draw a straight line (without arrowheads) joining x and y in your copy of the diagram if $x \neq y$ and $x\ R\ y$. Draw no other line. [2 marks]

(ii) Is R an equivalence relation in this case? Briefly justify your answer. If R is an equivalence relation, then write out all the equivalence classes in roster notation without repetition. [3 marks]

Explicit references to Lemma 8.6.5 and Proposition 8.6.13 may be omitted in your proofs and arguments.

Answer:

(a) 1. (Reflexivity)

1.1. Let $x \in \{0,1,\ldots,n-1\}$.

1.2. The reflexivity of congruence-mod-n tells us $x \equiv x \pmod{n}$.

1.3. Thus $1x \equiv 1x \pmod{n}$.

1.4. This shows $x\ R\ x$.

2. (Symmetry)

2.1. Let $x, y \in \{0,1,\ldots,n-1\}$ such that $x\ R\ y$.

2.2. The definition of R gives $a \in \{1,2,\ldots,n-1\}$ such that $ax \equiv ay \pmod{n}$.

2.3. The symmetry of congruence-mod-n then tells us $ay \equiv ax \pmod{n}$.

2.4. So $y\ R\ x$.

(b) (i)

$0 — 2$

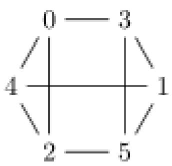
$1 — 3$

(ii) Yes.

The relation R is transitive (and is hence an equivalence relation) in this case because if $x, y \in \{0,1,2,3\}$ such that $x\ R\ y$ and $y\ R\ z$, then we can see from the diagram for (i) that either $x, y, z \in \{0,2\}$ or $x, y, z \in \{1,3\}$, and in either case we have $x\ R\ z$.

The equivalence classes are $\{0,2\}$ and $\{1,3\}$.

(c) (i)



(ii) No.

The relation R is not transitive in this case. For instance, we see from the diagram in (i) that $0\ R\ 3$ and $3\ R\ 1$ but $\sim(0\ R\ 1)$.

Explanations for the diagrams:

The following are the multiplication tables for $n = 4$ and $n = 6$.

$xy \pmod 4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$xy \pmod 6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

There is a line between x and y in the diagram if and only if $x \neq y$ and there is an entry in a non-0 column in the row for x that is the same as the corresponding entry in the row for y.

Question 5. (5 marks)

Let $\mathcal{C}$ be a partition of a set A. Show that there exist a set B and a surjection $f: A \to B$ such that

$$\mathcal{C} = \big\{\{x \in A : f(x) = y\} : y \in B\big\}.$$

Answer:

1. Let A be a set and $\mathcal{C}$ be a partition of A.
2. Define the function $f: A \to \mathcal{C}$ by setting, for each $x \in A$,

   $f(x)$ to be the unique $S \in \mathcal{C}$ such that $x \in S$.

   This function is well-defined as $\mathcal{C}$ is a partition of A.
3. We claim that f is surjective.

   3.1. Let $S \in \mathcal{C}$.

   3.2. Then S is a nonempty subset of A as $\mathcal{C}$ is a partition of A.

   3.3. Pick an element $x \in S$.

   3.4. Then $f(x) = S$ by the definition of f.
4. For all $S \in \mathcal{C}$ and all $x_0 \in A$,

$$x_0 \in S \quad \Leftrightarrow \quad f(x_0) = S \quad \text{by the definition of } f;$$

$$\Leftrightarrow \quad x_0 \in \{x \in A : f(x) = S\}.$$

5. So $S = \{x \in A : f(x) = S\}$ for all $S \in \mathcal{C}$.
6. This implies $\mathcal{C} = \{\{x \in A : f(x) = S\} : S \in \mathcal{C}\}$.

Alternative answer:

1. Let A be a set and $\mathcal{C}$ be a partition of A.
2. For each $S \in \mathcal{C}$, pick an element $a_S \in S$. This is possible because partitions by definition consist of nonempty sets.
3. Define $B = \{a_S : S \in \mathcal{C}\}$ and $f: A \to B$ by setting, for each $x \in A$,

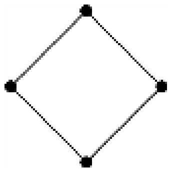
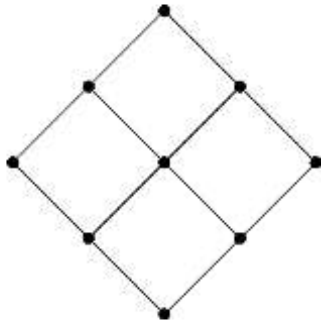
   $f(x)$ to be the unique a_S such that $x \in S$.

   This function is well-defined as $\mathcal{C}$ is a partition of A.
4. We claim that f is surjective.
   - 4.1. Let $y \in B$.
   - 4.2. Use the definition of B to find $S \in \mathcal{C}$ such that $y = a_S$.
   - 4.3. Note that $a_S \in S$ by line 2.
   - 4.4. So $f(a_S) = a_S = y$ by the definition of f and line 4.2.
5. For all $S \in \mathcal{C}$ and all $x_0 \in A$,

   $$x_0 \in S \quad \Leftrightarrow \quad f(x_0) = a_S \quad \text{by the definition of } f;$$
   $$\Leftrightarrow \quad x_0 \in \{x \in A : f(x) = a_S\}.$$
6. So $S = \{x \in A : f(x) = a_S\}$ for all $S \in \mathcal{C}$.
7. This implies $\mathcal{C} = \{\{x \in A : f(x) = y\} : y \in B\}$ by the definition of B.

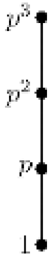
Question 6. (3 marks)

For each $n \in \mathbb{Z}^+$, let $D_n = \{d \in \mathbb{Z}^+ : d \mid n\}$. Write down $a, b, c \in \mathbb{Z}^+$ such that the following are respectively Hasse diagrams for D_a, D_b, and D_c with respect to the partial order "divides":

(a)  (b)  (c) 

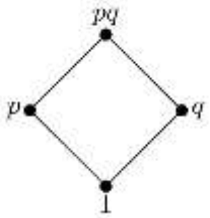
Answer:

(a) The prime factorization of a must be of the form p^3, where p is a prime, so that the Hasse diagram is labelled as follows:



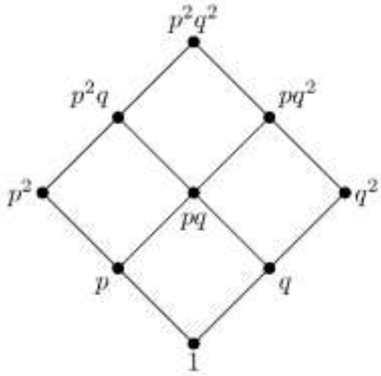
For instance, one can set a to be $2^3 = 8$, or $3^3 = 27$, or $5^3 = 125$, ….

(b) The prime factorization of b must be of the form pq, where p, q are distinct primes, so that the Hasse diagram is labelled as follows:



For instance, one can set b to be $2 \times 3 = 6$, or $2 \times 5 = 10$, or $3 \times 5 = 15$, ….

(c) The prime factorization of c must be of the form p^2q^2, where p, q are distinct primes, so that the Hasse diagram is labelled as follows:



For instance, one can set c to be $2^2 3^2 = 36$, or $2^2 5^2 = 100$, or $3^2 5^2 = 225$, ….