NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 1: AY2019/20)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY (20)** questions and comprises **TWELVE (12)** printed pages (including two blank pages at the end for your rough working).
2. Please write your Student Number with a pen (to prevent accidental erasure) on the first page of the **ANSWER BOOKLET**. Do not write your name.
3. This is an **OPEN BOOK** assessment.
4. You are allowed to use calculators but not programmable or graphics calculators. Other electronic devices are also not allowed.
5. Answer all questions and write your answers in the **ANSWER BOOKLET** provided.
6. You may use pencil to write your answers.
7. Submit only the **ANSWER BOOKLET** and no other document. You may keep this question paper.

——— END OF INSTRUCTIONS ———

Section A: For each multiple-choice question, write only one answer on the **Answer Booklet** provided. Two (2) marks are awarded for each correct answer and no penalty for wrong answer. [Total: 30 marks]

1. Which of the following propositional statements is a tautology?

   A. $(\sim p \vee q) \rightarrow (p \vee \sim q)$
   B. $p \rightarrow (p \wedge q)$
   C. $p \rightarrow (q \wedge \sim p)$
   D. $(p \leftrightarrow q) \vee (p \leftrightarrow \sim q)$
   E. None of the above.

2. Dirac's Theorem states that given a simple graph G with n $(n \geq 3)$ vertices:

   A sufficient condition for G to have a Hamiltonian circuit (cycle) is every vertex in G has degree at least $\lfloor n/2 \rfloor$.

   Which of the following statements is equivalent to the above statement?

   A. If G has a Hamiltonian circuit then every vertex in G has degree at least $\lfloor n/2 \rfloor$.
   B. Every vertex in G has degree at least $\lfloor n/2 \rfloor$ only if G has a Hamiltonian circuit.
   C. G has a Hamiltonian circuit only if every vertex in G has degree at least $\lfloor n/2 \rfloor$.
   D. Every vertex in G has degree at least $\lfloor n/2 \rfloor$ is a necessary condition for G not to have a Hamiltonian circuit.
   E. None of the above.

3. Consider the following statements:

   (I) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (y < x)$.
   (II) $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, (y < x)$.
   (III) $\forall x, y \in \mathbb{Q}, \big(x < y \rightarrow \exists z \in \mathbb{Q}\ (x < z < y)\big)$.

   A. Only (I) and (II) are true.
   B. Only (I) and (III) are true.
   C. Only (II) and (III) are true.
   D. All of (I), (II) and (III) are true.
   E. None of (I), (II) and (III) is true.

4. Consider the following statements:

   (I) For all sets A, B and C, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

   (II) For all sets A, B and C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

   A. Both (I) and (II) are true.
   B. (I) is true and (II) is false.
   C. (I) is false and (II) is true.
   D. Both (I) and (II) are false.
   E. Insufficient information to determine whether (I) and/or (II) are true.

5. Consider the following statements:

   (I) For all $a, b, d \in \mathbb{Z}^+$, if $d = \gcd(a, b)$ then $d = ax + by$ for some $x, y \in \mathbb{Z}$.

   (II) For all $a, b, d \in \mathbb{Z}^+$, if $d = ax + by$ for some $x, y \in \mathbb{Z}$ then $d = \gcd(a, b)$.

   A. Both (I) and (II) are true.
   B. (I) is true and (II) is false.
   C. (I) is false and (II) is true.
   D. Both (I) and (II) are false.
   E. Insufficient information to determine whether (I) and/or (II) are true.

6. Which of the following is a partition of the set P of all prime numbers?

   A. $\big\{\{p \in P : p \equiv a \pmod{4}\} : a \in \{0, 1, 2, 3\}\big\}$.
   B. $\big\{\{p \in P : p \equiv a \pmod{4}\} : a \in \{1, 2, 3\}\big\}$.
   C. $\big\{\{p \in P : p \equiv a \pmod{4}\} : a \in \{0, 1, 3\}\big\}$.
   D. $\big\{\{p \in P : p \equiv a \pmod{4}\} : a \in \{1, 3\}\big\}$.
   E. None of the above.

7. Consider the following statements:

   (I) For all $a, b, c, n \in \mathbb{Z}^+$, if $ac \equiv bc \pmod{n}$ then $a \equiv b \pmod{n}$.

   (II) For all $a, b, c, n \in \mathbb{Z}^+$, if $a \equiv b \pmod{n}$ then $ac \equiv bc \pmod{n}$.

   A. Both (I) and (II) are true.
   B. (I) is true and (II) is false.
   C. (I) is false and (II) is true.
   D. Both (I) and (II) are false.
   E. Insufficient information to determine whether (I) and/or (II) are true.

8. Let R be a relation on a non-empty set A. Consider the following statements:

   (I) If R is reflexive, then R^{-1} is reflexive.

   (II) If R is symmetric, then R^{-1} is symmetric.

   (III) If R is transitive, then R^{-1} is transitive.

   A. Only (II) is true.
   B. Only (I) and (II) are true.
   C. Only (II) and (III) are true.
   D. (I), (II) and (III) are all true.
   E. The truth of (I) and/or (II) and/or (III) depends on A and/or R.

9. Consider the following statements:

   (I) For all relations R and S on $\mathbb{Z}$, if $R \subseteq S$ and S is symmetric, then R is symmetric.

   (II) For all relations R and S on $\mathbb{Z}$, if $R \subseteq S$ and S is anti-symmetric, then R is anti-symmetric.

   A. Both (I) and (II) are true.
   B. (I) is true and (II) is false.
   C. (I) is false and (II) is true.
   D. Both (I) and (II) are false.
   E. Insufficient information to determine whether (I) and/or (II) are true.

10. Let $\preccurlyeq$ be a partial order on a non-empty set A, and let $a \in A$. Consider the following statements:

   (I) If a is the smallest element, then whenever $x \in A$ is a minimal element, we have $x = a$.

   (II) If whenever $x \in A$ is a minimal element we have $x = a$, then a is the smallest element.

   A. Both (I) and (II) are necessarily true.
   B. (I) is necessarily true but the truth of (II) depends on A and $\preccurlyeq$.
   C. (II) is necessarily true but the truth of (I) depends on A and $\preccurlyeq$.
   D. The truth of both (I) and (II) depends on A and $\preccurlyeq$.
   E. None of the above.

11. In how many ways can the letters of the word "CREATE" be arranged such that the vowels always stay together? Vowels are A, E, I, O and U.

    A. 72
    B. 120
    C. 144
    D. 240
    E. 360

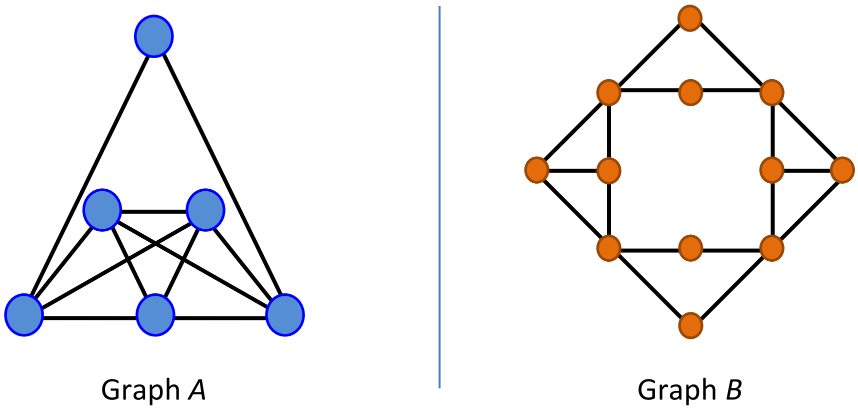
12. Suppose a random sample of 2 lightbulbs is selected from a group of 6 bulbs in which 2 are defective, what is the expected value of the number of defective bulbs in the sample?

    A. 1/18
    B. 1/3
    C. 8/9
    D. 2/3
    E. 14/15

13. What is the coefficient of x^7 in the expansion of $(x + 3)^9$?

    A. 7
    B. 36
    C. 108
    D. 320
    E. None of the above.

14. Given these two graphs *A* and *B* shown below:



Graph *A* Graph *B*

An Eulerian graph is one that contains an Eulerian circuit (cycle). A Hamiltonian graph is one that contains a Hamiltonian circuit (cycle).

Which of the following statements is true?

A. Graphs *A* and *B* are both Eulerian and Hamiltonian.
B. Graph *A* is both Eulerian and Hamiltonian; graph *B* is neither Eulerian nor Hamiltonian.
C. Graph *A* is Eulerian but not Hamiltonian; graph *B* is neither Eulerian nor Hamiltonian.
D. Graph *A* is Eulerian but not Hamiltonian; graph *B* is Hamiltonian but not Eulerian.
E. Graphs *A* and *B* are Hamiltonian but not Eulerian.

15. What is the maximum number of edges that a bipartite graph with n vertices can have?

A. $\lfloor n^2 \rfloor$
B. $\left\lfloor \frac{n^2}{2} \right\rfloor$
C. $\left\lfloor \frac{n^2}{4} \right\rfloor$
D. $\left\lfloor \frac{n^2-1}{4} \right\rfloor$
E. $\left\lfloor \frac{n(n-1)}{4} \right\rfloor$

Section B: Write your answers on the **Answer Booklet** provided.

16. **[Total: 10 marks]**

    (a) Find gcd(98069, 30629). [2 marks]

    (b) Find the base 9 representation of $(53292)_{10}$. [2 marks]

    (c) Find a positive integer that has exactly 5 positive divisors. [2 marks]

    (d) Let $A = \{0,1,2,\cdots,11\}$. For each $a \in A$, define $m_a\colon A \to A$ by $m_a(x) = ax \ \mathbf{mod}\ 12$. Find an $a \in A - \{1\}$ such that m_a is bijective. [2 marks]

    (e) Find all integers x such that $34x \equiv 89 \pmod{113}$. Express your answer in the form $x \equiv a \pmod{n}$, where $0 \le a < n$. [2 marks]

17. **[Total: 12 marks]**

    Let P be a partial order on a non-empty set A. Let R be another relation on A, and suppose that $R \subseteq P$. Let $\tilde{R}$ be the reflexive closure of R and let T be the transitive closure of $\tilde{R}$. Prove that:

    (a) T is a partial order on A. [6 marks]

    (b) If T' is another partial order on A such that $R \subseteq T'$, then $T \subseteq T'$. [6 marks]

    Recall from Tutorial 8 that the reflexive closure of a relation is the smallest reflexive relation on the same set that contains this relation as a subset. Similarly, the transitive closure of a relation is the smallest transitive relation on the same set that contains this relation as a subset.

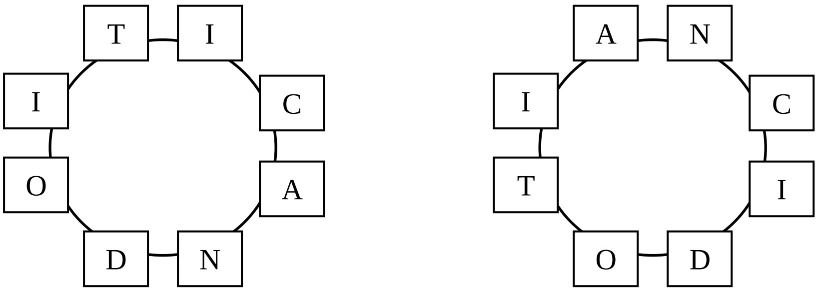
18. **Counting and Probability [Total: 20 marks]**

(a) How many non-negative integer solutions for a, b and c does the following equation have?

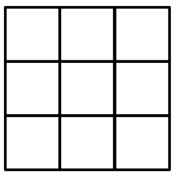
$$a + b + c = 100.$$ [2 marks]

(b) Four children are to be selected from 6 boys and 4 girls to participate in a training camp. There must be at least one boy among the selected children. In how many ways can the children be selected? [3 marks]

(c) You want to lay the letter tiles of these four words "I", "CAN", "DO", "IT" in a circular arrangement. The letters in the groups "CAN", "DO" and "IT" must be kept together in each group, but the letters within each group may be arranged in any order within that group. Also, no two similar letters should be placed next to each other. In how many ways can this be done? The diagram below shows two possible arrangements. [4 marks]



(d) Each of the 9 cells in the 3×3 grid below is to be filled with the number -1, 0 or 1. Prove that among the 3 row-sums, 3 column-sums and 2 diagonal-sums, there are two sums that are equal in value. [3 marks]



(e) It is known that 0.3% of the population are sufferers of a certain disease. The probability of a sufferer tested positive by a diagnostic test is 98%, while the probability of a non-sufferer tested negative by the test is 95%.

Suppose the test is administered to a person randomly chosen from the population, answer the following parts, writing your answer correct to <u>3 significant figures</u>.

(i) What is the probability that the test result of the person will be positive?

(ii) If the test result is positive, what is the probability that the person is a sufferer?

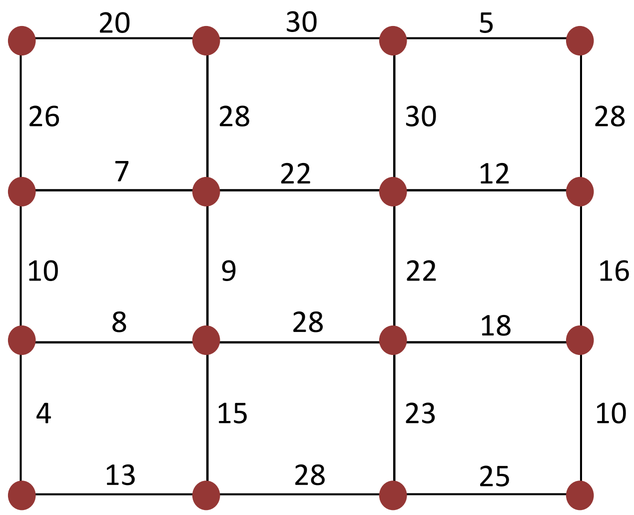
(iii) What is the probability that the person will be misclassified? A person is misclassified if he has the disease but is tested negative, or he does not have the disease but is tested positive.

[8 marks]

19. Graphs and Trees [Total: 20 marks]

(a) How many faces (regions) does a connected planar graph with 100 vertices and 123 edges contain? [2 marks]

(b) Given the following graph, mark out clearly on the **Answer Booklet** its minimum spanning tree. [4 marks]



(c) The post-order traversal and in-order traversal of a binary tree with 12 vertices are given below:

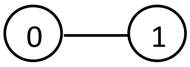
Post-order: S W V P M H E R K C X A

In-order: P W S V A E H M X R C K

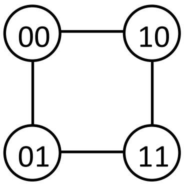
Draw the binary tree. The root has been drawn for you. [4 marks]

19. (continue…)

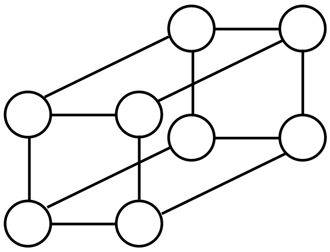
(d) A hypercube graph Q_n may be constructed by creating a vertex for each n-digit binary number (leading zeroes are added if necessary to form n digits), with two vertices adjacent when their binary representations <u>differ in a single digit</u>. The following figures show Q_1, Q_2, Q_3 and Q_4. $N_v(G)$ and $N_e(G)$ denote the number of vertices and number of edges, respectively, in graph G. The vertices in Q_1 and Q_2 are also labelled as shown in the figures.



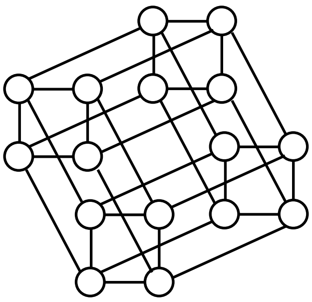
Graph Q_1
$N_v(Q_1) = 2$
$N_e(Q_1) = 1$



Graph Q_2
$N_v(Q_2) = 4$
$N_e(Q_2) = 4$



Graph Q_3
$N_v(Q_3) = 8$
$N_e(Q_3) = 12$



Graph Q_4
$N_v(Q_4) = 16$
$N_e(Q_4) = 32$

(i) Fill in the vertex labels for Q_3 on the **Answer Booklet**. Note that three of the vertices have been labelled 000, 001 and 101 for you. [2 marks]

(ii) What is $N_v(Q_n)$? [3 marks]

(iii) What is $N_e(Q_n)$? [5 marks]

20. Let $n \in \mathbb{Z}^+$ with $n \geq 3$, and let $A = \{0, 1, \cdots, n-1\}$.

Prove that there exists an $m \in A$ such that $m \not\equiv a^2 \pmod{n}$ for any $a \in \mathbb{Z}$. [8 marks]

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