1. Let a and b denote two non-zero constants. Let

$$f(x) = ax^3 - bx^2.$$

It is known that $f(x)$ has a point of inflection at $(1, 8)$. Find the exact value of the product ab.

Answer

2. Let L denote the line which passes through the two points $(1, 1, 2)$ and $(9, 7, 4)$. Let S denote a plane which is perpendicular to L. If the point $(2, 6, 9)$ lies on S, find the perpendicular distance from the point $(15, 2, 1)$ to S. Give your answer correct to two decimal places.

Answer

3. Let a denote a positive constant. For each $t > 0$, let R_t denote the bounded region in the first quadrant bounded by the curve $y = \frac{\sqrt{x}}{1+x^2}$, the x-axis, and the line $x = t$. Let $V(t)$ denote the volume of the solid of revolution generated by rotating R_t one complete round about the x-axis. If

$$\frac{V(a)}{\lim\limits_{t\to\infty} V(t)} = \frac{1521}{2020},$$

Find the value of a. Give your answer correct to two decimal places.

Answer

4. In this question, time t is measured in days. Let a denote a positive constant. At time $t = 0$ you started an experiment with a certain amount of a radio active substance X. At time $t = 10$ you had 2020 mg of X left. At time $t = 20$ you had 1521 mg of X left. If the half-life of X is a days, find the value of a. Give your answer correct to two decimal places.

Answer

5. Let $f(x)$ denote a differentiable function which is defined for all $x > 0$. It is known that the equation

$$\frac{d}{dx}\int_{\sqrt{x}}^{1521} f(t)dt = -(\sqrt{x})(e^{x})$$

holds for all $x > 0$. Find the value of $f'(2)$ (i.e. the derivative of $f(x)$ at $x = 2$). Give your answer correct to two decimal places.

Answer

6. Let a denote a positive constant. In this question the scale for the coordinate axes is in metres (so for example the point $(5, 0)$ is located on the positive x-axis at a distance 5 metres from the origin). Two points A on the positive y-axis and B on the positive x-axis are chosen so that all the three points A, B and $(8, 9)$ lie on the same straight line. A particle moves from A to B in the following way: first it moves from A along the y-axis to the origin at a speed of 2 metre per minute and then it moves from the origin to B along the x-axis at a speed of 3 metre per minute. The two points A and B are chosen so that the particle can finish the trip in the least amount of time. If this least amount of time is equal to a minutes, find the value of a. Give your answer correct to two decimal places.

Answer

7. Let a denote a positive constant. Let L denote the tangent line to the curve $y^2 = 2ax$ at the point $(2a, 2a)$. Let R denote the finite region in the first quadrant bounded by the line L, the curve $x^2 + y^2 = 2ax$, the y-axis and the line $x = 2a$. It is known that $\iint_R y dA = 898$. Find the value of a. Give your answer correct to two decimal places.

Answer

8. Let a denote a positive constant. A tank contains 100 litres of salt solution with a salt concentration of 10 gram per litre. Starting at $t = 0$ minute, a solution with salt concentration of a gram per litre is added to the tank at a rate of 5 litre per minute, and the resulting well-stirred mixture is drained out at the same rate. If the salt concentration in the tank is 15.21 gram per litre when $t = 32$ minutes, find the value of a. Give your answer correct to two decimal places.

Answer

9. Let $f(t)$ denote a differentiable function of one variable. The function F is a differentiable function of three variables defined by the equation

$$F(x, y, z) = f(x^2 + y^2 + z^2).$$

Let $\vec{u}$ denote the vector joining the point $(20, 2, 0)$ to the point $(1, 5, 21)$. If the directional derivative of F at the point $(6, 5, 6)$ in the direction of the vector $\vec{u}$ is equal to 88, find the value of the derivative of $f(t)$ at $t = 97$. Give your answer correct to two decimal places.

Answer

10. Let R denote the rectangle $OABC$ on the xy-plane with $O = (0,0,0)$, $A = (20,0,0)$, $B = (20,21,0)$ and $C = (0,21,0)$. The rectangle R is then folded along the diagonal AC so that the three verices O, A, C remain at their original position but the face ABC is now perpendicular to the xy-plane and the new position of B which is now denoted by B' is above the xy-plane. Find the perpendicular distance from B' to the plane $6x + 17y + 6z = 0$. Give your answer correct to two decimal places.

Answer:

11. Let a, b, c, u, v, w denote six constants with $a \neq 0$ and $a \neq u$. It is known that when these six numbers are written in the order a, u, v, b, w, c they form six consecutive terms of an arithmetic progression. It is also known that the infinite geometric series

$$S = a + b + c + ...$$

with the first three terms given by a, b, c respectively is a convergent geometric series with sum to infinity given by $S = 639$. Find the exact value of

$$a + b + c + u + v + w.$$

Answer

12. Let a denote a positive constant. Let R^* denote the three dimensional space minus the origin $O = (0,0,0)$. (This means that R^* contains every point of the three dimensional space except the origin.) Let $f(x,y,z)$ denote a differentiable function of three variables defined on R^* in the following way: at a point $P = (x,y,z)$ in R^* we draw a line L passing through O and P, then we construct the point Q on L so that the direction from O to Q is the same as the direction from O to P and $||\vec{OQ}||$ times $||\vec{OP}||$ is equal to a^2; if (u,v,w) denotes the coordinates of the point Q, then we define $f(x,y,z) = u + v + w$. If the maximum rate of increasing of f at $(1,2,3)$ is equal to 689, find the value of a. Give your answer correct to two decimal places.

Answer

13. Let $w = w(x,y)$ and $z = z(x,y)$ denote two differentiable functions of two variables. (Note: here the notation $w = w(x,y)$ means that w is a function of the variables x and y and similar meaning for $z = z(x,y)$.) It is known that w and z satisfy the two equations $z = f(x,y,w)$ and $w = g(x,y,z)$ where f and g are two given differentiable functions of three variables. It is also known that when $x = 1, y = 2$, we have $z = 1521, w = 2020, \frac{\partial f}{\partial x} = 5, \frac{\partial f}{\partial y} = 6, \frac{\partial f}{\partial w} = \frac{1}{2}, \frac{\partial g}{\partial x} = 2, \frac{\partial g}{\partial y} = 9, \frac{\partial g}{\partial z} = \frac{5}{3}$. Find the exact value of $\frac{\partial z}{\partial x} + \frac{\partial w}{\partial x}$ when $x = 1, y = 2$.

Answer:

14. Let a, b and c denote three constants. It is known that the power series $f(x) = 8 + ax + bx^2 + cx^3 + 129x^4 + \cdots$ has a positive radius of convergence. If the second derivative of f satisfies $f''(x) = (f(x))^2$, find the value of

$$\lim_{x\to\infty} \left(\frac{f(\frac{1}{x})}{f(0)}\right)^x .$$

Give your answer correct to two decimal places.

Answer

15. Let a denote a positive constant with $a > 1$. Let S denote the part of the surface of $x^2 + y^2 - 2z = 0$ which lies between the two planes $z = 1$ and $z = a$. If the area of S is equal to 828, find the value of a. Give your answer correct to two decimal places.

Answer