MA1101R

LIVE LECTURE 7

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Topics for week 7

3.6 Dimensions

3.7 Transition Matrices

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Let's revise – Linear dependency

1. If the vector equation $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0}$ has only the trivial solution, then $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent
2. If the vector equation $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0}$ has a non-trivial solution, then $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly dependent
3. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are scalar multiples of each other, then $\{\boldsymbol{u}, \boldsymbol{v}\}$ is linearly dependent.
4. If S contains **0**, then S is linearly dependent.
5. If one vector in S is a linear combination of the other vectors in S, then S is linearly dependent

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Let's revise – Linear dependency & Span

6. If $\boldsymbol{u} \in \text{span}(S)$, then $S \cup \{\boldsymbol{u}\}$ is linearly $\boxed{\text{dependent}}$
7. If S is linearly independent and $\boldsymbol{u} \notin \text{span}(S)$, then $S \cup \{\boldsymbol{u}\}$ is linearly $\boxed{\text{independent.}}$
8. Let $\{\boldsymbol{u}, \boldsymbol{v}\} \in \mathbf{R}^2$.

   $\{\boldsymbol{u}, \boldsymbol{v}\}$ is linearly $\boxed{\text{independent}}$ iff $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \mathbf{R}^2$
9. Let $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \in \mathbf{R}^3$.

   $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly $\boxed{\text{independent}}$ iff $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = \mathbf{R}^3$
10. If $S \in \mathbf{R}^n$ and S has more than n elements, then S is linearly $\boxed{\text{dependent.}}$

Vector Space

$\mathbf{R}^n$ (n-space) $\supseteq$ Subspace of $\mathbf{R}^n$

Let’s revise – Bases

- A subset of a vector space V is called a basis for V if (i) span(S) = $\boxed{V}$ and (ii) S is $\boxed{\text{linearly independent}}$
- Every non-zero vector space has infinitely many different bases
- The basis for the zero space is the $\boxed{\text{empty set}}$
- All bases for the same vector space V has the same $\boxed{\text{number of vectors}}$
- Every vector in a vector space can be expressed as $\boxed{\text{linear combination}}$ of a given basis in a unique way
- S is a basis for span(S) iff S is $\boxed{\text{linearly independent}}$

Dimension

dim V

dimension of V

Let V be a vector space which has a basis
$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ with k vectors.

1. Any subset of V with more than k vectors is always linearly dependent.

   > k : too many vectors to be a basis

2. Any subset of V with less than k vectors cannot span V.

   < k : too few vectors to be a basis

All bases for a vector space have the same number of vectors

Dimension of subspaces of $\mathbf{R}^3$

- $\{\mathbf{0}\}$ basis is empty set

  dimension 0
- lines through the origin span$\{\mathbf{u}\}$

  dimension 1
- planes containing the origin span$\{\mathbf{u}, \mathbf{v}\}$

  dimension 2
- $\mathbf{R}^3$ span$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

  dimension 3

$V_1 \subseteq V_2$: we say V_1 is a subspace of V_2

Exercise 3 Q39

Give an example of a family of subspaces $V_1, V_2, \ldots, V_n$ of $\mathbf{R}^n$ such that $\dim(V_i) = i$ and $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n$.

Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \ldots, \mathbf{u}_n\}$ be a basis for $\mathbf{R}^n$

- $V_1 = \text{span}\{\mathbf{u}_1\}$ dimension 1
- $V_2 = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ dimension 2
- $V_3 = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ dimension 3
- $\vdots$
- $V_{n-1} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \ldots, \mathbf{u}_{n-1}\}$ dimension n-1
- $V_n = \mathbf{R}^n$ dimension n

Dimension of solution space

homogeneous system $\longrightarrow$ row echelon form ***R***

number of non-pivot columns in ***R***

||

number of parameters in general solution

||

number of vectors in basis for solution space

||

the dimension of the solution space

To show basis

To show S is a basis for V :

S lin. indep S spans V	or	S lin. indep $\lvert S \rvert = \dim V$	or	S spans V $\lvert S \rvert = \dim V$

If $|S| = \dim V$, then

S is linearly independent $\Leftrightarrow$ S spans V

Which ones are bases?

$V = \text{span}\{(1,0,0), (0,1,0), (1,1,0)\}$

I. $\{(1,0,0), (0,1,0)\}$ Yes
II. $\{(1,0,0), (1,-1,0)\}$ Yes
III. $\{(1,0,0), (0,0,1)\}$ No

Observe:

- $V = \text{span}\{(1,0,0), (0,1,0)\}$
- $\{(1,0,0), (0,1,0)\}$ is a basis for V
- ➢ So dim V = 2
- (1,0,0), (1,-1,0) are linearly independent vectors in V
- ➢ So $\{(1,0,0), (1,-1,0)\}$ is a basis for V

Deriving bases

V a vector space, and S, T are finite subsets of V.

- Suppose span(S) = V.

  We can find S′ ⊆ S such that S′ is a basis for V.

- Suppose T is a linearly independent subset of V.

  We can find T ⊆ T′ such that T′ is a basis for V.

Techniques in chapter 4

Dimensions give the "size" of subspaces

Let U and V be subspaces of $\mathbf{R}^n$

(i) If $U \subseteq V$, then $\dim(U) \leq \dim(V)$

(ii) If $U \subseteq V$ and $U \neq V$, then $\dim(U) < \dim(V)$

True or false

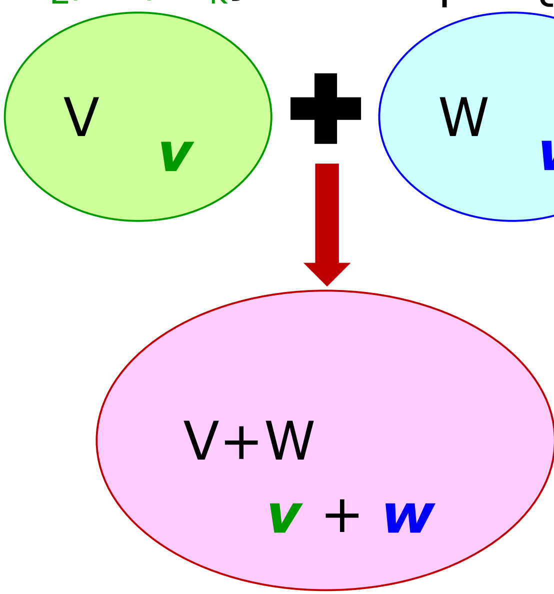
- If $\dim(U) = \dim(V)$, then $U = V$ False
- If $\dim(U) \leq \dim(V)$, then $U \subseteq V$ False
- If $U \subseteq V$ and $\dim(U) = \dim(V)$, then $U = V$ True

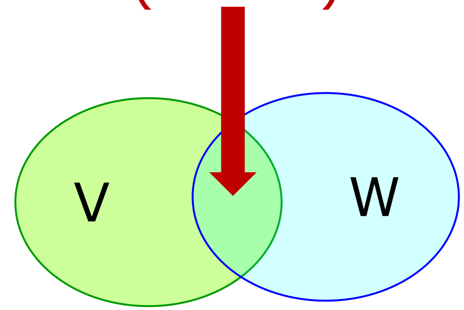
Exercise 3 Q43

V, W subspaces of $\mathbf{R}^n$. Show that:

$$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$$

span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ span$\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_h\}$





span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k , \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_h\}$

Exercise 3 Q43

V, W subspaces of $\mathbf{R}^n$. Show that:

$$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$$

Simple example in $\mathbf{R}^3$:

V, W: two lines through origin

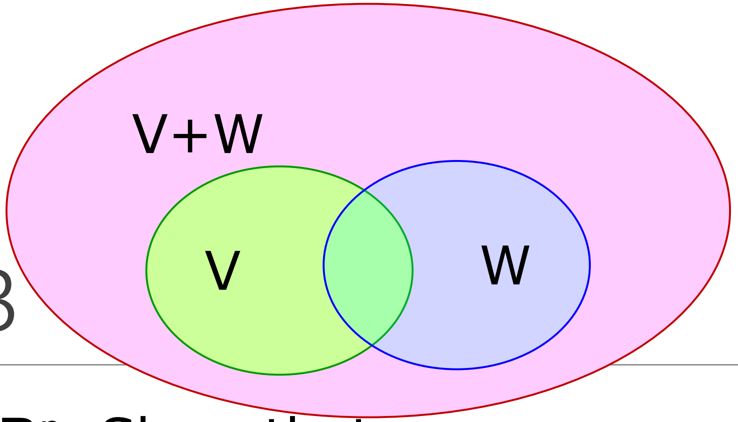
(i) If V, W represent the same line ℓ,

then $V \cap W = \ell$ and $V + W = \ell$

(ii) If V, W represent two different lines ℓ_1 and ℓ_2,

then $V \cap W = \{\mathbf{0}\}$ and $V + W =$ plane containing ℓ_1 and ℓ_2

Exercise 3 Q43



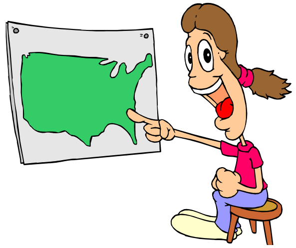
V, W subspaces of $\mathbf{R}^n$. Show that:

$$\dim(V + W) = \underbrace{\dim(V)}_{k} + \underbrace{\dim(W)}_{h} - \underbrace{\dim(V \cap W)}_{m}$$

Idea of proof:

- Start with a basis $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ for $V \cap W$
- Extend S to a basis for V: $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}_{m+1}, \ldots, \mathbf{v}_k\}$
- Extend S to a basis for W: $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{w}_{m+1}, \ldots, \mathbf{w}_h\}$
- $\mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}_{m+1}, \ldots, \mathbf{v}_k, \mathbf{w}_{m+1}, \ldots, \mathbf{w}_h\} = V + W$
- Show $T = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}_{m+1}, \ldots, \mathbf{v}_k, \mathbf{w}_{m+1}, \ldots, \mathbf{w}_h\}$ is linearly independent (exercise)
- Then T is a basis for $V + W$, and $\dim(V+W) = k + h - m$

Map of LA



A is an n×n matrix

A is invertible	chapter 2	***A*** is not invertible
det ***A*** $\neq$ 0	chapter 2	det ***A*** = 0
rref of ***A*** is identity matrix	chapter 1	rref of ***A*** has a zero row
Ax = ***0*** has only the trivial solution	chapter 1	***Ax*** = ***0*** has non-trivial solutions
Ax = ***b*** has a unique solution	chapter 1	***Ax*** = ***b*** has no solution or infinitely many solutions
Columns (rows) of ***A*** are linearly independent	chapter 3	Columns (rows) of ***A*** are linearly dependent
a basis for $\mathbf{R}^n$		not a basis for $\mathbf{R}^n$

to be continued

Transition matrix

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$
two bases for a vector space V.

Given $\boldsymbol{w} \in V$

$$\begin{tikzcd}
(\boldsymbol{w})_S \arrow[r] \arrow[rr, bend right] & \boldsymbol{w} \arrow[r] & (\boldsymbol{w})_T
\end{tikzcd}$$

Is there a direct method?

does not depend on $\boldsymbol{w}$

$[\boldsymbol{w}]_T = \boldsymbol{P}\,[\boldsymbol{w}]_S$ for some fixed k x k matrix $\boldsymbol{P}$

transition matrix

Finding transition matrix

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$

two bases for a vector space V.

1. Express each $\boldsymbol{u}_\mathbf{i}$ as linear combination of $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$
2. Form the (column) coordinate vectors w.r.t. T

$$[\boldsymbol{u}_1]_T = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{pmatrix} \quad [\boldsymbol{u}_2]_T = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{pmatrix} \quad \cdots \quad [\boldsymbol{u}_k]_T = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{kk} \end{pmatrix}$$

3. Form the matrix $\boldsymbol{P} = ([\boldsymbol{u_1}]_T \ [\boldsymbol{u_2}]_T \ \ldots \ [\boldsymbol{u_k}]_T)$

$$\boldsymbol{P} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

transition matrix from S to T

4. $\boldsymbol{P}[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$ for any vector $\boldsymbol{w}$ in V.

Finding transition matrix

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$
two bases for a vector space V.

$$\begin{array}{cccccc} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{array}$$

$$\left(\begin{array}{ccc|c|c|c} 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right)$$

$$\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} [\boldsymbol{u}_1]_T & [\boldsymbol{u}_2]_T & [\boldsymbol{u}_3]_T \end{pmatrix}$$

$\boldsymbol{P}$: the transition matrix from S to T

The transition matrix from T to S
is given by $\boldsymbol{P}^{-1}$

Exercise 3 Q48

$V = \{(x, y, z) \mid 2x - y + z = 0\}$

Similar argument for T

$S = \{(0,1,1), (1,2,0)\}$ $T = \{(1,1,-1), (1,0,-2)\}$

a) Show that both S and T are bases for V.

b) Find the transition matrix from T to S and the transition matrix from S to T.

Check both $(0,1,1)$, $(1,2,0)$ satisfy the equation $2x - y + z = 0$

Also $\{(0,1,1), (1,2,0)\}$ is linearly independent

So $\text{span}\{(0,1,1), (1,2,0)\} = V$

So S is a basis for V.

a) Show that both S and T are bases for V.

b) Find the transition matrix from T to S and the transition matrix from S to T.

Exercise 3 Q48

V = {(x, y, z) | 2x – y + z = 0}

S = {(0,1,1), (1,2,0)} T = {(1,1,-1), (1,0,-2)}

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & -2 \end{array}\right) \xrightarrow{\text{G.J.E.}} \left(\begin{array}{cc|cc} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The transition matrix from *T* to *S* is ***P*** = $\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$

The transition matrix from *S* to *T* is ***P***$^{-1}$ = $\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$

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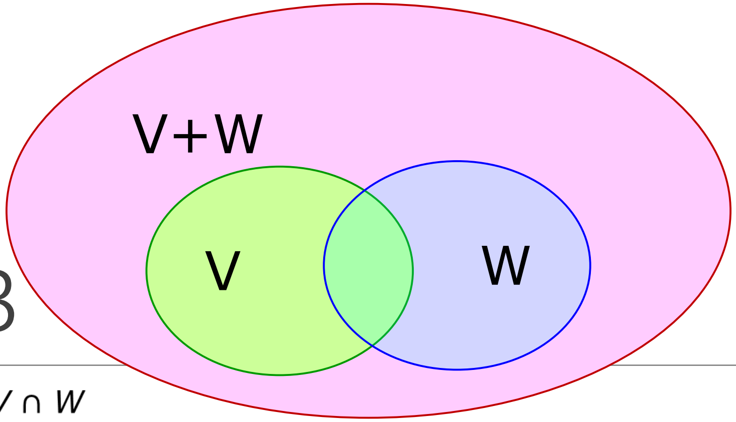
Announcement

- **Homework 1**
  - Deadline: 2 October (this Friday)
  - Submission folder will close at 11.59pm
  - Declaration form
- **Online quiz 7**
  - Due this Sunday

Exercise 3 Q43



- basis $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ for $V \cap W$
- basis for V: $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_k$
- basis for W: $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m, \mathbf{w}_{m+1}, \dots, \mathbf{w}_h$
- Show $T = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_k, \mathbf{w}_{m+1}, \dots, \mathbf{w}_h\}$ is linearly independent

$$c_1\mathbf{u}_1 + c_1\mathbf{u}_2 + \dots + c_m\mathbf{u}_m + d_{m+1}\mathbf{v}_{m+1} + \dots + d_k\mathbf{v}_k + e_{m+1}\mathbf{w}_{m+1} + \dots + e_h\mathbf{w}_h = \mathbf{0}$$

$$\underbrace{c_1\mathbf{u}_1 + c_1\mathbf{u}_2 + \dots + c_m\mathbf{u}_m + d_{m+1}\mathbf{v}_{m+1} + \dots + d_k\mathbf{v}_k}_{\text{in } V} = \underbrace{-e_{m+1}\mathbf{w}_{m+1} - \dots - e_h\mathbf{w}_h}_{\text{in } W} \quad (*)$$

in $V \cap W$

$$-e_{m+1}\mathbf{w}_{m+1} - \dots - e_h\mathbf{w}_h = f_1\mathbf{u}_1 + f_2\mathbf{u}_2 + \dots + f_m\mathbf{u}_m$$

$$f_1\mathbf{u}_1 + f_2\mathbf{u}_2 + \dots + f_m\mathbf{u}_m + e_{m+1}\mathbf{w}_{m+1} + \dots + e_h\mathbf{w}_h = \mathbf{0} \quad (**)$$

$$(**) \Rightarrow f_1 = f_2 = \dots = f_m = e_{m+1} = \dots = e_h = 0$$

$$(*) \Rightarrow c_1 = c_2 = \dots = c_m = d_{m+1} = \dots = d_k = 0$$