Student Number:	A/U/T*								

*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2011-2012

MA1101R LINEAR ALGEBRA I

April/May 2012 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.

2. This examination paper contains a total of **FOUR (4)** questions and comprises **NINETEEN (19)** printed pages.

3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.

4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
Total	

Question 1 (a) [15 marks]

Let $\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{u_2} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, $\boldsymbol{u_3} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $\boldsymbol{u_4} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$.

(i) (3 marks) Show that $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a basis for $\mathbb{R}^3$.

(ii) (4 marks) Find the coordinate vector $[\boldsymbol{u_4}]_S$ with respect to S.

(iii) (3 marks) Prove that for all $k \in \mathbb{R}$, $[k\boldsymbol{u_4}]_S = k\,[\boldsymbol{u_4}]_S$.

(iv) (5 marks) Find a basis for $\operatorname{span}\{\boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ and determine its dimension.

Use the space below to write your answer and working

(i) Put the vectors $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ as columns in a matrix and compute the determinant.

$$\begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0 \quad \text{(since the matrix is triangular)}$$

So $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a basis for $\mathbb{R}^3$.

(ii) Solving $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} = \boldsymbol{u_4}$,

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 6 \end{array}\right) \underset{\frac{1}{3}R_3}{\overset{\frac{1}{2}R_2}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \underset{R_2 - \frac{1}{2}R_3}{\overset{R_1 - R_3}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \overset{R_1 + R_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

So $[\boldsymbol{u_4}]_S = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

(iii)

$$\boldsymbol{u_4} = 0\boldsymbol{u_1} + 1\boldsymbol{u_2} + 2\boldsymbol{u_3} \Leftrightarrow k\boldsymbol{u_4} = k \cdot 0\boldsymbol{u_1} + k \cdot 2\boldsymbol{u_2} + k \cdot \boldsymbol{u_3}$$

$$\Leftrightarrow [k\boldsymbol{u_4}]_S = \begin{pmatrix} 0 \\ k \\ 2k \end{pmatrix} = k \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = k\,[\boldsymbol{u_4}]_S$$

(More working spaces for Question 1 (a))

(iv) Putting the vectors $\boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$ as columns in a matrix.

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & 1 & 4 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow{R_2 + 2R_1} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\{\boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a basis for $\text{span}\{\boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ whose dimension is 2.

Question 1 (b) [5 marks]

The augmented matrix of a homogeneous linear system has the following reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & k_1 & 0 \\ 0 & 1 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \end{array}\right).$$

If the solution space of this system is $\text{span}\{\boldsymbol{u_3}\}$ where $\boldsymbol{u}_3$ is as in part (a), find k_1, k_2, k_3. Explain clearly how your answer is obtained.

Use the space below to write your answer and working

Since the solution space $\text{span}\{\boldsymbol{u_3}\} = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \mid t \in \mathbb{R} \right\}$ is one dimensional, the general solution can be written as

$$\begin{cases} x &= t \\ y &= t \\ z &= 3t \end{cases} \qquad t \in \mathbb{R}$$

which has only one parameter. This implies the RREF has exactly one non-pivot columns. Hence $k_3 = 0$.

From the general solution, we also have the equations $3x = z$ and $3y = z$.

In other words, we have $x - \frac{1}{3}z = 0$ which corresponds to the first row of the RREF, and $y - \frac{1}{3}z = 0$ which corresponds to the second row of the RREF.

This implies $k_1 = -\frac{1}{3}$ and $k_2 = -\frac{1}{3}$.

Question 1 (c) [5 marks]

Let S and $\boldsymbol{u}_4$ be as in part (a). Suppose $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is another basis for $\mathbb{R}^3$ such that

$$[\boldsymbol{u_4}]_T = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad [\boldsymbol{v_1}]_S = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}.$$

Find $\boldsymbol{v_3}$.

Use the space below to write your answer and working

Let $\boldsymbol{P} = \begin{pmatrix} [\boldsymbol{v_1}]_S & [\boldsymbol{v_2}]_S & [\boldsymbol{v_3}]_S \end{pmatrix}$ be the transition matrix from T to S. So

$$\begin{aligned} \boldsymbol{P}\,[\boldsymbol{u_4}]_T = [\boldsymbol{u_4}]_S \Rightarrow \boldsymbol{P} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad &\Rightarrow \quad [\boldsymbol{v_1}]_S + 0\,[\boldsymbol{v_2}]_S + 2\,[\boldsymbol{v_3}]_S = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \\ &\Rightarrow \quad \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} + 2\,[\boldsymbol{v_3}]_S = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \\ &\Rightarrow \quad 2\,[\boldsymbol{v_3}]_S = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \\ &\Rightarrow \quad [\boldsymbol{v_3}]_S = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

Thus $\boldsymbol{v_3} = \boldsymbol{u_1} + \boldsymbol{u_2} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

Question 2 (a) [15 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$.

(i) (4 marks) Find a basis for the row space of $\boldsymbol{A}$. What is the rank of $\boldsymbol{A}$?

(ii) (3 marks) If $\boldsymbol{A}$ is the standard matrix for a linear transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^4$, determine whether $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ is in the range of T_1. Justify your answer.

(iii) (5 marks) If $\boldsymbol{A}^T$ is the standard matrix for a linear transformation $T_2 : \mathbb{R}^4 \to \mathbb{R}^3$, find a basis and determine the dimension of kernel of T_2.

(iv) (3 marks) Find two distinct vectors $\boldsymbol{v_1}, \boldsymbol{v_2}$ (that is, $\boldsymbol{v_1} \neq \boldsymbol{v_2}$) in the column space of $\boldsymbol{A}$ such that $T_2(\boldsymbol{v_1}) = T_2(\boldsymbol{v_2})$.

Use the space below to write your answer and working

(i)

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \xrightarrow[R_4 - R_1]{R_2 + R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{pmatrix} \xrightarrow[R_4 + 2R_2]{R_3 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $\{(1,0,1), (0,1,1), (0,0,2)\}$ (or any basis for $\mathbb{R}^3$) is a basis for the row space of $\boldsymbol{A}$. Rank of $\boldsymbol{A}$ is 3.

(ii) Yes. Applying the same series of elementary row operations as in (i) on

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & -2 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 4 \\ 1 & -2 & 1 & 0 \end{array}\right)$$

shows that the vector belong to the column space of $\boldsymbol{A}$ and hence the range of T_1.

(More working spaces for Question 2 (a))

(iii) Solving the linear system $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0}$.

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_2 + 2R_3]{R_1 - R_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 + R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

So a general solution is

$$\begin{cases} x_1 &= -s \\ x_2 &= -s \\ x_3 &= s, \qquad s \in \mathbb{R} \\ x_4 &= 0 \end{cases}$$

$\operatorname{Ker}(T_2) = \operatorname{span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}\right\}$ and dimension of $\operatorname{Ker}(T_2)$ is 1.

(iv) Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2$ are two distinct vectors in the column space of $\boldsymbol{A}$. Then $\boldsymbol{v}_1 - \boldsymbol{v}_2$ will also be in the column space $\boldsymbol{A}$. i.e.

$$\boldsymbol{v}_1 - \boldsymbol{v}_2 = a\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

On the other hand, suppose $T_2(\boldsymbol{v_1}) = T_2(\boldsymbol{v_2})$. Then $T_2(\boldsymbol{v_1} - \boldsymbol{v_2}) = \boldsymbol{0}$. So $\boldsymbol{v}_1 - \boldsymbol{v}_2$ will be in the kernel of T_2. By part (iii), $\boldsymbol{v}_1 - \boldsymbol{v}_2 = s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ for some non-zero s.

Combining the above, we have

$$a\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & -s \\ -1 & 1 & 0 & -s \\ 0 & 1 & 1 & s \\ 1 & -2 & 1 & 0 \end{array}\right) \xrightarrow{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -s \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & s \\ 1 & -2 & 1 & 0 \end{array}\right)$$

which gives an inconsistent system in view of row 2 and 3, since $s \neq 0$.

So we conclude that such a pair of $\boldsymbol{v}_1, \boldsymbol{v}_2$ does not exist.

Question 2 (b) [5 marks]

Let $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & x \\ 0 & 1 & 1 & 0 \\ 1 & 0 & x+1 & -1 \end{pmatrix}$ and $\boldsymbol{C}$ be a 5×4 matrix of full rank.

Find all values of x such that $\boldsymbol{B}$ and $\boldsymbol{C}$ have the same row space.

Justify your answer.

Use the space below to write your answer and working

Since $\boldsymbol{C}$ is of full rank, it has rank 4 and hence its row space is $\mathbb{R}^4$.

So we just need to find all x such that $\text{rank}(\boldsymbol{B}) = 4$.

$$\boldsymbol{B} \xrightarrow{R_4 - R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & x \\ 0 & 1 & 1 & 0 \\ 0 & 0 & x & -1 \end{pmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x \end{pmatrix}.$$

So the values of x for $\text{rank}(\boldsymbol{B}) = 4$ are all $x \in \mathbb{R} - \{0\}$.

Question 2 (c) [5 marks]

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be the matrices in part (a) and (b) respectively.
Show that for all values of x, the column space of $\boldsymbol{A}$ is a subset of the row space of $\boldsymbol{B}$.

Use the space below to write your answer and working

We investigate the column space of $\boldsymbol{A}$ by looking at the row space of $\boldsymbol{A}^T$.
From part (a(iii)), we observe that

$$\text{column space of } \boldsymbol{A} \ = \text{row space of } \boldsymbol{A}^T \ = \text{span}\{(1,0,1,0),(0,1,1,0),(0,0,0,1)\}.$$

Denote the vectors $\boldsymbol{u_1} = (1,0,1,0), \boldsymbol{u_2} = (0,1,1,0), \boldsymbol{u_3} = (0,0,0,1)$ so the column space of $\boldsymbol{A}$ is $\text{span}\{\boldsymbol{u_1},\boldsymbol{u_2},\boldsymbol{u_3}\}$. From part (b), we observe that $\boldsymbol{B}$ is row equivalent to

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x \end{pmatrix}.$$

So the row space of $\boldsymbol{B}$ is the same as the row space of $\boldsymbol{R}$. Note that $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are the first two rows of $\boldsymbol{R}$ and thus belong to the row space of $\boldsymbol{B}$.

- If $x \neq 0$, the row space of $\boldsymbol{B}$ is $\mathbb{R}^4$ and clearly the column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^4$.

- If $x = 0$, then $\boldsymbol{u_3}$ is the negative of the third row of $\boldsymbol{R}$ and thus belong to the row space of $\boldsymbol{B}$.

Thus in either case, $\text{span}\{\boldsymbol{u_1},\boldsymbol{u_2},\boldsymbol{u_3}\}$ is always a subspace of the row space of $\boldsymbol{B}$ for all values of x.

Question 3 (a) [15 marks]

(i) (4 marks) Find the characteristic polynomial of the symmetric matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

and show that the eigenvalues of $\boldsymbol{A}$ are 2 and 4.

(ii) (5 marks) Find a basis for each of the eigenspaces of $\boldsymbol{A}$.

(iii) (3 marks) Find an <u>orthogonal</u> matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P}$ is a <u>diagonal</u> matrix.

(iv) (3 marks) Find a symmetric matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^2 = \boldsymbol{A}$. (You may leave your answer as a product of matrices.)

Use the space below to write your answer and working

(i) The characteristic polynomial of $\boldsymbol{A}$ is given by

$$\det(\lambda\boldsymbol{I}-\boldsymbol{A}) = \begin{vmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-3 & 1 \\ 0 & 1 & \lambda-3 \end{vmatrix} = (\lambda-2)[(\lambda-3)^2-1] = (\lambda-2)(\lambda^2-6\lambda+8) = (\lambda-2)^2(\lambda-4).$$

Hence the eigenvalues of $\boldsymbol{A}$ are 2 and 4.

(ii) To find a basis for the eigenspaces E_2:

$$(2\boldsymbol{I}-\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ t \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

So a basis for E_2 is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$.

To find a basis for the eigenspaces E_4:

$$(4\boldsymbol{I}-\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ -t \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

So a basis for E_4 is $\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$.

(More working spaces for Question 3 (a))

(iii) We note that the union of bases for E_2 and E_4 in part (ii) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is an orthogonal set.

So we can get the required orthogonal matrix $\boldsymbol{P}$ by normalizing the set above:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

i.e. $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$

(iv) $\boldsymbol{A} = \boldsymbol{PDP}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$

By letting $\boldsymbol{M} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$, we have $\boldsymbol{M}^2 = \boldsymbol{D}$ and hence

$\boldsymbol{A} = \boldsymbol{PDP}^T = \boldsymbol{PM}^2\boldsymbol{P}^T = (\boldsymbol{PMP}^T)(\boldsymbol{PMP}^T) = \boldsymbol{C}^2.$

We check that $\boldsymbol{C}^T = (\boldsymbol{PMP}^T)^T = ((\boldsymbol{P}^T)^T\boldsymbol{M}^T\boldsymbol{P}^T) = (\boldsymbol{PMP}^T) = \boldsymbol{C}.$

So $\boldsymbol{C}$ is a symmetric matrix.

Question 3 (b) [**5 marks**]

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ be a basis for $\mathbb{R}^4$ and $\boldsymbol{B}$ a 4×4 matrix such that:

$$\boldsymbol{B}\boldsymbol{u}_1 = \boldsymbol{u}_2, \quad \boldsymbol{B}\boldsymbol{u}_2 = \boldsymbol{u}_1, \quad \boldsymbol{B}\boldsymbol{u}_3 = \boldsymbol{u}_4, \quad \boldsymbol{B}\boldsymbol{u}_4 = \boldsymbol{u}_3.$$

Find <u>all</u> eigenvalues of $\boldsymbol{B}$ and determine whether $\boldsymbol{B}$ is diagonalizable.
Justify your answers.

Use the space below to write your answer and working

Observe that:
$\boldsymbol{B}(\boldsymbol{u_1} + \boldsymbol{u_2}) = \boldsymbol{B}\boldsymbol{u}_1 + \boldsymbol{B}\boldsymbol{u}_2 = \boldsymbol{u}_2 + \boldsymbol{u}_1$.
So $\boldsymbol{u_1} + \boldsymbol{u_2}$ is an eigenvector with eigenvalue 1.
$\boldsymbol{B}(\boldsymbol{u_1} - \boldsymbol{u_2}) = \boldsymbol{B}\boldsymbol{u}_1 - \boldsymbol{B}\boldsymbol{u}_2 = \boldsymbol{u}_2 - \boldsymbol{u}_1 = -(\boldsymbol{u_1} - \boldsymbol{u_2})$.
So $\boldsymbol{u_1} - \boldsymbol{u_2}$ is an eigenvector with eigenvalue -1.
$\boldsymbol{B}(\boldsymbol{u_3} + \boldsymbol{u_4}) = \boldsymbol{B}\boldsymbol{u}_3 + \boldsymbol{B}\boldsymbol{u}_4 = \boldsymbol{u}_4 + \boldsymbol{u}_3$.
So $\boldsymbol{u_3} + \boldsymbol{u_4}$ is an eigenvector with eigenvalue 1.
$\boldsymbol{B}(\boldsymbol{u_3} - \boldsymbol{u_4}) = \boldsymbol{B}\boldsymbol{u}_3 - \boldsymbol{B}\boldsymbol{u}_4 = \boldsymbol{u}_4 - \boldsymbol{u}_3 = -(\boldsymbol{u_3} - \boldsymbol{u_4})$.
So $\boldsymbol{u_3} - \boldsymbol{u_4}$ is an eigenvector with eigenvalue -1.
We claim that $\{\boldsymbol{u_1} + \boldsymbol{u_2}, \boldsymbol{u_1} - \boldsymbol{u_2}, \boldsymbol{u_3} + \boldsymbol{u_4}, \boldsymbol{u_3} - \boldsymbol{u_4}\}$ is linearly independent, and hence $\boldsymbol{B}$ is diagonalizable with eigenvalues 1 and -1.
To prove our claim, let
$a(\boldsymbol{u_1} + \boldsymbol{u_2}) + b(\boldsymbol{u_1} - \boldsymbol{u_2}) + c(\boldsymbol{u_3} + \boldsymbol{u_4}) + d(\boldsymbol{u_3} - \boldsymbol{u_4}) = \boldsymbol{0}$.
which gives $(a+b)\boldsymbol{u_1} + (a-b)\boldsymbol{u_2} + (c+d)\boldsymbol{u_3} + (c-d)\boldsymbol{u_4} = \boldsymbol{0}$.
Since $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ is linearly independent, we have

$$a+b=0, \quad a-b=0, \quad c+d=0, \quad c-d=0$$

which implies $a=b=c=d=0$.
Hence $\{\boldsymbol{u_1} + \boldsymbol{u_2}, \boldsymbol{u_1} - \boldsymbol{u_2}, \boldsymbol{u_3} + \boldsymbol{u_4}, \boldsymbol{u_3} - \boldsymbol{u_4}\}$ is linearly independent.

Question 3 (c) [5 marks]

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two diagonalizable 3×3 matrices, both having exactly two eigenvalues 1 and -1.
Suppose 2 and -2 are not eigenvalues of $\boldsymbol{A}+\boldsymbol{B}$. Show that $\boldsymbol{A}+\boldsymbol{B}$ is singular.

Use the space below to write your answer and working

Let $E_{A,1}$ and $E_{A,-1}$ be the two eigenspaces of $\boldsymbol{A}$ associated to eigenvalues 1 and -1 respectively.
Since $\boldsymbol{A}$ is diagonalizable, the dimensions of the two eigenspaces are 1 and 2.
Let $E_{B,1}$ and $E_{B,-1}$ be the two eigenspaces of $\boldsymbol{B}$ associated to eigenvalues 1 and -1 respectively.
Similarly, since $\boldsymbol{B}$ is diagonalizable, the dimensions of the two eigenspaces are 1 and 2.
If $\dim E_{A,1} = 2 = \dim E_{B,1}$, then these two spaces have non-trivial intersection. i.e. there is some non-zero vector $\boldsymbol{u}$ which is in both $E_{A,1}$ and $E_{B,1}$.
Then $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u} = 2\boldsymbol{u}$. This implies 2 is an eigenvalue of $\boldsymbol{A}+\boldsymbol{B}$, which is not the case.
Likewise, if $\dim E_{A,-1} = 2 = \dim E_{B,-1}$, then -2 is an eigenvalue of $\boldsymbol{A}+\boldsymbol{B}$, which is again not the case.
Hence, we must have either $\dim E_{A,1} = 2 = \dim E_{B,-1}$, or $\dim E_{A,-1} = 2 = \dim E_{B,1}$.
In the first case, let $\boldsymbol{u}$ be the common non-zero vector in $E_{A,1}$ and $E_{B,-1}$.
Then $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u} = 0\boldsymbol{u}$. This implies 0 is an eigenvalue of $\boldsymbol{A}+\boldsymbol{B}$, which implies $\boldsymbol{A}+\boldsymbol{B}$ is singular.
We get the same result for the second case.

Question 4 (a) [15 marks]

Let $\boldsymbol{u}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\boldsymbol{u}_3 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$.

(i) (3 marks) Show that the subspace $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ of $\mathbb{R}^3$ is orthogonal to $\boldsymbol{u}_3$.

(ii) (3 marks) Find an orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ for V such that $\boldsymbol{v}_1$ is parallel to $\boldsymbol{u}_1$.

(iii) (3 marks) Find the projection of $\boldsymbol{w}$ onto V.

(iv) (3 marks) Find the equation of a plane that is perpendicular to V and contains $\boldsymbol{w}$.

(v) (3 marks) Write down two orthogonal matrices both having $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as its first two columns respectively.

Use the space below to write your answer and working

(i) We just need to check $\boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = 0$ and $\boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = 0$:

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} = 2 \cdot 2 + 0 \cdot (-1) + 1 \cdot (-4) = 0$$

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} = 1 \cdot 2 + 2 \cdot (-1) + 0 \cdot (-4) = 0.$$

(ii) Apply Gram Schmidt to $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$:

$\boldsymbol{u}_1' = \boldsymbol{u}_1$;

$$\boldsymbol{u}_2' = \boldsymbol{u}_2 - \frac{\boldsymbol{u}_1 \cdot \boldsymbol{u}_2}{\|\boldsymbol{u}_1\|^2}\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ 10 \\ -2 \end{pmatrix}$$

So $\boldsymbol{v}_1 = \dfrac{1}{\|\boldsymbol{u}_1'\|}\boldsymbol{u}_1' = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_2 = \dfrac{1}{\|\boldsymbol{u}_2'\|}\boldsymbol{u}_2' = \dfrac{1}{\sqrt{105}}\begin{pmatrix} 1 \\ 10 \\ -2 \end{pmatrix}$

(More working spaces for Question 4 (a))

(iii) The projection $\boldsymbol{p}$ is given by the formula:

$$[\boldsymbol{v}_1 \cdot \boldsymbol{w}]\boldsymbol{v}_1 + [\boldsymbol{v}_2 \cdot \boldsymbol{w}]\boldsymbol{v}_2$$

$$= \left[\frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \begin{pmatrix}0\\0\\2\end{pmatrix}\right]\frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\1\end{pmatrix} + \left[\frac{1}{\sqrt{105}}\begin{pmatrix}1\\10\\-2\end{pmatrix} \cdot \begin{pmatrix}0\\0\\2\end{pmatrix}\right]\frac{1}{\sqrt{105}}\begin{pmatrix}1\\10\\-2\end{pmatrix}$$

$$= \frac{2}{5}\begin{pmatrix}2\\0\\1\end{pmatrix} + \frac{-4}{105}\begin{pmatrix}1\\10\\-2\end{pmatrix} = \frac{1}{105}\begin{pmatrix}80\\-40\\50\end{pmatrix}$$

(iv) We can take this plane to be the subspace

$$\text{span}\{\boldsymbol{w}, \boldsymbol{p}\} \quad \text{or simply} \quad \text{span}\{(0,0,2)^T, (8,-4,5)^T\}.$$

The equation of this plane has the form: $ax + by + cz = 0$.

On substituting $(0, 0, 2)$, we get $c = 0$.

On substituting $(8, -4, 5)$, we get $8a - 4b = 0 \Rightarrow a = t, b = 2t$ for a parameter t.

Hence the equation can be given by $x + 2y = 0$.

(v) From (i), we know that $\boldsymbol{u}_3$ is orthogonal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, so we can take the third column of the matrix to be

$$\frac{1}{\|\boldsymbol{u}_3\|}\boldsymbol{u}_3 = \pm\frac{1}{\sqrt{21}}\begin{pmatrix}2\\-1\\-4\end{pmatrix}$$

Question 4 (b) [5 marks]

Find the least squares solutions of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Use the space below to write your answer and working

Compute $\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 3 & 2 \\ 2 & 2 & 4 \end{pmatrix}$

and $\boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Solving $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$ by Gaussian elimination:

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ -1 & 3 & 2 & -1 \\ 2 & 2 & 4 & 0 \end{array}\right) \xrightarrow{G.E.} \left(\begin{array}{ccc|c} 1 & -3 & -2 & 1 \\ 0 & 4 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

we have $x_3 = t, \quad x_2 = \frac{1}{4}(-1-4t) = -\frac{1}{4} - t, \quad x_1 = 1 + 3x_2 + 2x_3 = \frac{1}{4} - t$

or $\boldsymbol{x} = \begin{pmatrix} \frac{1}{4} - t \\ -\frac{1}{4} - t \\ t \end{pmatrix}$ with parameter $t \in \mathbb{R}$.

Question 4 (c) [5 marks]

Show that every invertible matrix $\boldsymbol{A}$ can be written as $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}$ where $\boldsymbol{B}$ is an orthogonal matrix and $\boldsymbol{C}$ is an upper triangular matrix.

Use the space below to write your answer and working

Let $\boldsymbol{A} = (\boldsymbol{a}_1\ \boldsymbol{a}_2\ \cdots \boldsymbol{a}_n)$ where $\boldsymbol{a}_i$'s are the columns of $\boldsymbol{A}$.

Then $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ form a basis for $\mathbb{R}^n$.

By applying Gram Schmidt process on this basis, we can get an orthonormal basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ as follow:

$$\begin{cases} \boldsymbol{b}_1 = c_{01}\boldsymbol{a}_1 \\ \boldsymbol{b}_2 = c_{02}\boldsymbol{a}_2 + c_{12}\boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_n = c_{0n}\boldsymbol{a}_n + c_{1n}\boldsymbol{b}_1 + c_{2n}\boldsymbol{b}_2 + \cdots + c_{n-1,n}\boldsymbol{b}_{n-1} \end{cases} \qquad (*)$$

where c_{ij} are some scalars.

We can rewrite $(*)$ as

$$\begin{cases} \boldsymbol{a}_1 = c'_{11}\boldsymbol{b}_1 \\ \boldsymbol{a}_2 = c'_{12}\boldsymbol{b}_1 + c'_{22}\boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{a}_n = c'_{1n}\boldsymbol{b}_1 + c'_{2n}\boldsymbol{b}_2 + c'_{3n}\boldsymbol{b}_3 + \cdots + c'_{nn}\boldsymbol{b}_n \end{cases} \qquad (**)$$

We can rewrite $(**)$ in matrix form:

$$\boldsymbol{a}_1 = (\boldsymbol{b}_1\ \boldsymbol{b}_2\ \cdots \boldsymbol{b}_n)\begin{pmatrix} c'_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \boldsymbol{a}_2 = (\boldsymbol{b}_1\ \boldsymbol{b}_2\ \cdots \boldsymbol{b}_n)\begin{pmatrix} c'_{12} \\ c'_{22} \\ \vdots \\ 0 \end{pmatrix}, \cdots \boldsymbol{a}_n = (\boldsymbol{b}_1\ \boldsymbol{b}_2\ \cdots \boldsymbol{b}_n)\begin{pmatrix} c'_{1n} \\ c'_{2n} \\ \vdots \\ c'_{nn} \end{pmatrix}$$

or simply $(\boldsymbol{a}_1\ \boldsymbol{a}_2\ \cdots \boldsymbol{a}_n) = (\boldsymbol{b}_1\ \boldsymbol{b}_2\ \cdots \boldsymbol{b}_n)\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1n} \\ 0 & c'_{22} & \cdots & c'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c'_{nn} \end{pmatrix}$.

The matrix $\boldsymbol{B} = (\boldsymbol{b}_1\ \boldsymbol{b}_2\ \cdots \boldsymbol{b}_n)$ is orthogonal and the matrix $\boldsymbol{C} = \begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1n} \\ 0 & c'_{22} & \cdots & c'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c'_{nn} \end{pmatrix}$

is upper triangular and we have $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}$.

(More working spaces. Please indicate the question numbers clearly.)

(More working spaces. Please indicate the question numbers clearly.)

[END OF PAPER]