National University of Singapore

Semester 1, 2020/2021 MA1101R Practice Assignment 4 Answer

1. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$.

   (i) [2 marks] Show that $S = \{(1,1,1,1)^T, (1,0,1,0)^T, (1,0,0,1)^T\}$ is a basis for the column space V of $\boldsymbol{A}$.

   (ii) [2 marks] Use Gram Schmidt process (Theorem 5.2.19) to convert S to an orthogonal basis T for V without normalising the resulting vectors. (Show your working. You may use MATLAB command to check your answer.)

   (iii) [2 mark] Find the coordinate vector of $\boldsymbol{w} = (8,4,12,0)^T$ with respect to the orthogonal basis T in (ii) using Theorem 5.2.8.

   (iv) [2 mark] Find the projection $\boldsymbol{p}$ of $\boldsymbol{b} = (1,2,3,4)^T$ onto V using Theorem 5.2.15.

   (v) [2 marks] Find the least squares solutions of $\boldsymbol{Ax} = \boldsymbol{b}$ using the linear system $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$, where $\boldsymbol{b}$ is the vector in (iv).

   (vi) [2 marks] Find the least squares solutions of $\boldsymbol{Ax} = \boldsymbol{b}$ by using $\boldsymbol{p}$ in part (iv) directly. (Refer to Example 5.3.9)

Answer

(i)
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{GJE} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the first three columns of $\boldsymbol{A}$ form a basis for its column space.
Hence S is a basis for V.

(ii) Let $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ where:
$\boldsymbol{v}_1 = (1,1,1,1)$
$\boldsymbol{v}_2 = (1,0,1,0) - \frac{2}{4}(1,1,1,1) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$
$\boldsymbol{v}_3 = (1,0,0,1) - \frac{2}{4}(1,1,1,1) = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

(iii) $\boldsymbol{w} = (8,4,12,0) = \frac{24}{4}(1,1,1,1) + \frac{16}{2}\left[\frac{1}{2}(1,-1,1,-1)\right] - \frac{8}{2}\left[\frac{1}{2}(1,-1,-1,1)\right]$
$(\boldsymbol{w})_T = (6, 8, -4)$

(iv) $\boldsymbol{p} = \frac{10}{4}(1,1,1,1) + \frac{-2}{2}\left[\frac{1}{2}(1,-1,1,-1)\right] + \frac{0}{2}\left[\frac{1}{2}(1,-1,-1,1)\right] = (2,3,2,3)$

(v) $\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{pmatrix}$ and $\boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 10 \\ 4 \\ 5 \\ 6 \end{pmatrix}$

Solving $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$ by Gauss Jordan elimination, we get

$$\left(\begin{array}{cccc|c} 4 & 2 & 2 & 2 & 10 \\ 2 & 2 & 1 & 0 & 4 \\ 2 & 1 & 2 & 1 & 5 \\ 2 & 0 & 1 & 2 & 6 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution is: $x_4 = t, x_3 = 0, x_2 = -1 + t, x_1 = 3 - t$.

So least squares solutions are $\begin{pmatrix} 3-t \\ -1+t \\ 0 \\ t \end{pmatrix}$ for $t \in \mathbb{R}$.

(vi) Solving the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{p}$ by Gauss Jordan elimination, we get

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 & 3 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution is: $x_4 = t, x_3 = 0, x_2 = -1 + t, x_1 = 3 - t$.

Again we get the same least squares solutions: $\begin{pmatrix} 3-t \\ -1+t \\ 0 \\ t \end{pmatrix}$ for $t \in \mathbb{R}$.

2. (Do not simply use MATLAB to obtain the answers for this question. Show your working clearly. MATLAB command can be used to check your answer.)

Let $\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$.

(i) [3 marks] Find the characteristic polynomial and all the eigenvalues of $\boldsymbol{B}$.

(ii) [4 marks] Find a basis for each eigenspace of $\boldsymbol{B}$.

(iii) [1 mark] Write down the nullspace of $\boldsymbol{B}$ without any further working. Justify your answer.

Answer

(i) The characteristic polynomial of $\boldsymbol{B}$ is given by:

$$\begin{aligned}
\det(x\boldsymbol{I}-\boldsymbol{B}) &= \begin{vmatrix} x-1 & -1 & 0 & 0 \\ -1 & x-1 & 0 & 0 \\ 0 & 0 & x-2 & -2 \\ 0 & 0 & -2 & x-2 \end{vmatrix} \\
&= (x-1)\begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-2 & -2 \\ 0 & -2 & x-2 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 \\ 0 & x-2 & -2 \\ 0 & -2 & x-2 \end{vmatrix} \\
&= (x-1)^2\begin{vmatrix} x-2 & -2 \\ -2 & x-2 \end{vmatrix} - \begin{vmatrix} x-2 & -2 \\ -2 & x-2 \end{vmatrix} \\
&= (x^2-2x)[((x-2)^2-4] \\
&= x(x-2)[x^2-4x] \\
&= x^2(x-2)(x-4)
\end{aligned}$$

So the eigenvalues of $\boldsymbol{B}$ are $\lambda = 0$ (repeated), 2 and 4.

(ii) For eigenspace associated to $\lambda = 0$, solve

$$\left(\begin{array}{cccc|c} -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution: $x_4 = t, x_3 = -t, x_2 = s, x_1 = -s$.

So a basis for the eigenspace is $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

For eigenspace associated to $\lambda = 2$, solve

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution: $x_4 = 0, x_3 = 0, x_2 = s, x_1 = s$.

So a basis for the eigenspace is $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

For eigenspace associated to $\lambda = 4$, solve

$$\left(\begin{array}{cccc|c} 3 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution: $x_4 = t, x_3 = t, x_2 = 0, x_1 = 0$.

So a basis for the eigenspace is $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

(iii) The nullspace of $\boldsymbol{B}$ is the same as its eigenspace associated to $\lambda = 0$:

$$\text{span}\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\1\end{pmatrix}\right\}.$$