NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2018-2019

MA1521 CALCULUS FOR COMPUTING

May 2019 Time allowed: 2 hours

Question 1 [10 marks]

(a) (Multiple Choice Question)

Let $f(x) = 1384\,(1 - 2x)\,e^{3x}$, $-\infty < x < \infty$. Find the absolute maximum value of f. Give your answer correct to the nearest integer.

(A) 1521 (B) 1001 (C) 2001

(b) Let m and n denote two positive even integers with $m < n$. It is known that the area of the region between the graphs of $y = 2\cos x$ and $y = \sin 2x$ from $x = m\pi$ to $x = (n+1)\pi$ is equal to 8554. Find the **exact value** of $n - m$.

(2) $f'(x) = 1384\{-2e^{3x} + 3(1-2x)e^{3x}\}$

$= 1384e^{3x}(1-6x)$

$f'(x) = 0 \Rightarrow x = \frac{1}{6}$

$f'(x) > 0$ when $x < \frac{1}{6}$

$f'(x) < 0$ when $x > \frac{1}{6}$

$\therefore$ absolute maximum

$f(x) = f(\frac{1}{6})$

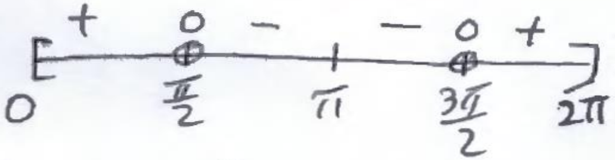
$= 1384(1-\frac{1}{3})e^{\frac{1}{2}}$

$= 1521.2\cdots$

$\approx \underline{\underline{1521}}$

(b) Consider first $0 \le x \le 2\pi$.

$2\cos x - \sin 2x = 2\cos x(1-\sin x)$

$=$ 

$\left(\because (1-\sin x) \geq 0\right)$

let $A_1 =$ area from 0 to π

$A_2 =$ area from π to 2π

$A =$ area from 0 to 2π

Then $A_1 = \int_0^{\pi/2}(2\cos x - \sin 2x)\,dx - \int_{\pi/2}^{\pi}(2\cos x - \sin 2x)\,dx = 2$

$A_2 = -\int_{\pi}^{3\pi/2}(2\cos x - \sin 2x)\,dx + \int_{3\pi/2}^{2\pi}(2\cos x - \sin 2x)\,dx = 6$

$A = A_1 + A_2 = 8$

$\because$ m and n are even

$\therefore$ there are $\frac{n-m}{2}$ interval of length 2π

from $m\pi$ to $n\pi$

$\therefore$ area from $m\pi$ to $n\pi = 8 \times \frac{n-m}{2}$

area from $n\pi$ to $(n+1)\pi = A_1 = 2$

$\therefore 8 \times \frac{n-m}{2} + 2 = 8554 \Rightarrow n-m = \underline{\underline{2138}}$

Question 2 [10 marks]

(a) Let $P(x)$ denote the degree two Taylor polynomial of the function $\ln(2+\tan x)$ at $x=0$. Find the value of $P\left(\frac{9}{10}\right)$. Give your answer correct to two decimal places.

(b) Find the directional derivative of the function $f(x,y,z)=4xyz-2x^2+y^2+z^2+321$ at the point $(1,1,2)$ in the direction of the vector which joins $(2,3,1)$ to $(1,2,3)$. Give your answer correct to two decimal places.

(a) $\ln(2+\tan 0) = \ln 2$

$$\frac{d}{dx}\ln(2+\tan x)\Big|_0 = \frac{\sec^2 x}{2+\tan x}\Big|_0 = \frac{1}{2}$$

$$\frac{d}{dx}\left(\frac{\sec^2 x}{2+\tan x}\right)\Big|_0$$

$$= \frac{2\sec^2 x\tan x(2+\tan x) - \sec^4 x}{(2+\tan x)^2}\Big|_0$$

$$= -\frac{1}{4}$$

$$\therefore P(x) = \ln 2 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$P(0.9) = \ln 2 + \frac{0.9}{2} - \frac{0.9^2}{8}$$

$$= 1.041\cdots$$

$$\approx \underline{\underline{1.04}}$$

(b)

$$\nabla f = (4yz - 4x,\ 4xz + 2y,\ 4xy + 2z)$$

$$\nabla f(1,1,2) = (4, 10, 8)$$

$$\vec{u} = \frac{(1,2,3) - (2,3,1)}{\|(1,2,3) - (2,3,1)\|}$$

$$= \frac{(-1, -1, 2)}{\sqrt{6}}$$

$$D_{\vec{u}} f(1,1,2) = \frac{-4 - 10 + 16}{\sqrt{6}}$$

$$= \frac{2}{\sqrt{6}}$$

$$= 0.816\cdots$$

$$\approx \underline{\underline{0.82}}$$

Question 3 [10 marks]

(a) It is known that the function $f(x, y) = 3xy - x^2 - y^3 - 5$ has exactly one local maximum point at (a, b). If $a+b = \frac{m}{n}$ where m and n are two positive integers without any common factors, find the **exact value** of $m+n$.

(b) The region R lies above the paraboloid $z = 4 - x^2 - y^2$ and below the paraboloid $z = 8 - 3x^2 - 3y^2$. Find the volume of R. Give your answer correct to two decimal places.

(a) $f_x = 3y - 2x = 0 \Rightarrow x = \frac{3}{2}y$

$f_y = 3x - 3y^2 = 0 \Rightarrow \frac{9}{2}y - 3y^2 = 0$

$\Rightarrow y = 0$ or $\frac{3}{2}$

$\therefore (0,0), \left(\frac{9}{4}, \frac{3}{2}\right)$

$f_{xx} = -2$, $f_{xy} = 3$

$f_{yy} = -6y$

$f_{xx}f_{yy} - f_{xy}^2 = 12y - 9$

$= \begin{cases} -\text{ve at } (0,0) \\ +\text{ve at } \left(\frac{9}{4}, \frac{3}{2}\right) \end{cases}$

$\therefore \left(\frac{9}{4}, \frac{3}{2}\right)$ is the loc. max.

$a + b = \frac{9}{4} + \frac{3}{2} = \frac{15}{4}$

$m + n = 15 + 4 = \underline{\underline{19}}$

(b) $4-x^2-y^2=8-3x^2-3y^2$

$\Rightarrow 2x^2+2y^2=4$

$\Rightarrow x^2+y^2=2$

$$Vol=\iint_{x^2+y^2\leq 2}\{(8-3x^2-3y^2)-(4-x^2-y^2)\}\,dA$$

$$=\int_0^{2\pi}\int_0^{\sqrt{2}}(4-2r^2)\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[2r^2-\frac{1}{2}r^4\right]_0^{\sqrt{2}}d\theta$$

$$=4\pi$$

$$=12.566\cdots$$

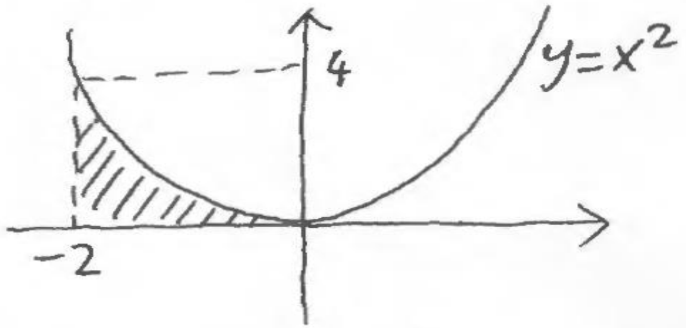
$$\approx \underline{\underline{12.57}}$$

Question 4 [10 marks]

(a) Evaluate $\int_{-2}^{0}\left(\int_{0}^{x^2} e^{\left(y-\frac{1}{3}y^{\frac{3}{2}}\right)}dy\right)dx$. Give your answer correct to two decimal places.

(b)At time $t = 0$ a tank contains 20 pounds of salt dissolved in 120 gallons of water. Assume that water containing 0.5 pound of salt per gallon is entering the tank at a rate of 4 gallons per minute and the well stirred solution is leaving the tank at the same rate. Find the amount of salt in the tank at time $t = 16$ minutes. Give your answer in pounds correct to two decimal places.

(a)



$$\int_{-2}^{0}\int_{0}^{x^2} e^{(y-\frac{1}{3}y^{3/2})}\,dy\,dx$$

$$=\int_{0}^{4}\int_{-2}^{-\sqrt{y}} e^{(y-\frac{1}{3}y^{3/2})}\,dx\,dy$$

$$=\int_{0}^{4}(2-\sqrt{y})\,e^{(y-\frac{1}{3}y^{3/2})}\,dy$$

$$=\left[2e^{(y-\frac{1}{3}y^{3/2})}\right]_{0}^{4}$$

$$=2(e^{\frac{4}{3}}-1)$$

$$=5.587\cdots$$

$$\approx \underline{\underline{5.59}}$$

(b) $\frac{dQ}{dt} = 2 - \frac{4Q}{120} = 2 - \frac{1}{30}Q$

$$\frac{dQ}{dt} + \frac{1}{30}Q = 2$$

$$R = e^{\int \frac{1}{30} dt} = e^{\frac{1}{30}t}$$

$$Q = e^{-\frac{1}{30}t} \int 2e^{\frac{1}{30}t} dt$$

$$= e^{-\frac{1}{30}t} \{ 60 e^{\frac{1}{30}t} + C \}$$

$$= 60 + Ce^{-\frac{1}{30}t}$$

$Q(0) = 20 \Rightarrow C = -40$

$$Q(16) = 60 - 40e^{-\frac{16}{30}}$$

$$= 36.534\cdots$$

$$\approx \underline{\underline{36.53}}$$

Question 5 [10 marks]

(a) Let $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{y^3}{x^2}, \quad \text{with} \quad x > 0, y > 0 \quad \text{and } y(1) = \sqrt{\frac{5}{7}}.$$

Find the value of $y(\frac{3}{2})$. Give your answer correct to two decimal places.

(b) The growth of the sandhill crane population follows a logistic model with a birth rate per capita of 10% per year. Initially at time $t = 0$ there were 1521 sandhill cranes. It is known that at time $t = 10$ years there were 2019 sandhill cranes. How many sandhill cranes will there be after a very long time? Give your answer correct to the nearest integer.

(a) Let $z = y^{1-3} = y^{-2}$

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$-\frac{y^3}{2}\frac{dz}{dx} + \frac{2}{x}y = \frac{y^3}{x^2}$$

$$\frac{dz}{dx} - \frac{4}{x}z = -\frac{2}{x^2}$$

$$R = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = \frac{1}{x^4}$$

$$z = x^4\int -\frac{2}{x^6}dx = x^4\left\{\frac{2}{5x^5} + C\right\}$$

$$= \frac{2}{5x} + Cx^4$$

$$y = \frac{1}{\sqrt{\frac{2}{5x} + Cx^4}}$$

$$y(1) = \sqrt{\frac{5}{7}} \Rightarrow C = 1$$

$$y\left(\frac{3}{2}\right) = \frac{1}{\sqrt{\frac{4}{15} + \left(\frac{3}{2}\right)^4}} = 0.433\cdots$$

$$\approx \underline{\underline{0.43}}$$

(b)

$$N = \frac{N_\infty}{1+\left(\frac{N_\infty}{\hat{N}}-1\right)e^{-\beta t}}$$

$$2019 = \frac{N_\infty}{1+\left(\frac{N_\infty}{1521}-1\right)e^{-1}}$$

$$N_\infty = \frac{2019 - \frac{2019}{e}}{1 - \frac{2019}{1521e}}$$

$$= 2494.2\ldots$$

$$\approx \underline{\underline{2494}}$$