MATLAB LESSON 1: MATRIX OPERATIONS AND SOLVING LINEAR SYSTEMS

MA1101R LINEAR ALGEBRA

ABSTRACT. In this laboratory session, we introduce some very basic MATLAB commands for performing matrix operations and solving linear systems.

1. INSTALLATION

In this course, we use a highly acclaimed numerical computing software MATLAB, which stands for *Matrix Laboratory*. The National University of Singapore has a Total Academic Head-count Licence for MATLAB. Students may use it for academic, research, and learning. The license allows students to install MATLAB on personally-owned computers.

1. If you are NOT using NUS network, you are required to use nVPN:

- `https://webvpn.nus.edu.sg/`

Please sign in with your NUSNET ID in the format: **nusstu\nusnetid** and password.

If you are using NUS network, please proceed to Step 2.

2. Click the link `https://sm05.stf.nus.edu.sg/studentmatlab/` to download MATLAB. It is required to sign in with your NUSNET ID in the format: **nusstu\nusnetid** and password.

3. Create a MathWorks account using your NUS email address.

(i) `https://www.mathworks.com/mwaccount/register`
- ◇ Email address: Your NUS email account (e.g., e0012345@u.nus.edu)
- ◇ Location: Singapore
- ◇ How will you use MathWorks software? Student use
- ◇ Are you at least 13 years or older? Yes

(ii) You will receive an email from `service@mathworks.com` with title "Verify Email Address". Click the link in the email to verify your account.

(iii) Finish creating your profile. Then you should be able to see the following information:
- Your account has been created and license 40707750 has been associated with your account.

4. Click the **Download** bottom to download and run the installer.

(i) When prompted, log in with your MathWorks Account (your NUS email account).

(ii) Select your licence (40707750, Student, Academic — Total Headcount).

(iii) Choose an installation folder in your PC.

(iv) Select products to install. For MA1101R, it suffices to install only MATLAB 9.6.

2. Statements

MATLAB environment behaves like a super-complex calculator. You can enter the commands at the >> command prompt. The answer appears by pressing Enter.

(i) A MATLAB statement is frequently of the form

```
>> variable = expression
```

which assigns the result of `expression` to `variable`. For example,

```
>> a = 3
a =  3
```

(ii) A MATLAB statement may have a simpler form

```
>> expression
```

in which case the result of `expression` is assigned to a special variable called `ans` (which stands for *answer*). For example,

```
>> 3 + 5
ans =  8
```

(iii) You may add a semicolon `;` at the end of the statement; then MATLAB will hide the output. For example,

```
>> b = 3;
>> b ^ 2
ans =  9
```

(iv) The command `help` will give information and usage about the specific `topic`:

```
>> help topic
```

(v) We have defined the symbol `a` as the number 3. We may remove it from the memory by using

```
>> clear a
```

or remove all variables from the memory by using

```
>> clear
```

(vi) If we want to clear the command window, use

```
>> clc
```

3. Precision

By default, MATLAB displays four decimal digits to its answers. But we can change the format for numeric display.

(i) 16 decimal digits:

```
>> format long
>> sqrt(2)
ans = 1.414213562373095
```

(ii) Rational number approximation:

```
>> format rat
>> sqrt(2)
ans = 1393/985
```

MATLAB will approximate decimals with rational numbers when you use `format rat`. Sometimes, this may cause unexpected results. Occasionally, an asterisk $*$ may appear when you expect the quantity to be 0.

(iii) 4 decimal digits (default)

```
>> format short
>> sqrt(2)
ans = 1.4142
```

4. Working with Matrices

Recall that vectors are considered as special matrices. More precisely, a column vector is an $m \times 1$ matrix, and a row vector is a $1 \times n$ matrix.

The entries of a matrix shall be entered row by row, while the entries in each row are separated by a space and the rows are separated by a semi-colon `;`. For example,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

The size of $\boldsymbol{A}$ is given by

```
>> size(A)
ans = 4 5
```

and the (i, j)-entry of $\boldsymbol{A}$ is simply given by `A(i,j)`. For example,

```
>> A(2,5)
ans = 6
```

we can generate special matrices using the following command:

(i) Zero matrix $\mathbf{0}_{m\times n}$ of size $m \times n$: $\boxed{\texttt{zeros(m,n)}}$.

```
>> zeros(2,3)
ans =  0  0  0
       0  0  0
```

(ii) Identity matrix $\boldsymbol{I}_n$ of order n: $\boxed{\texttt{eye(n)}}$.

```
>> eye(2)
ans =  1  0
       0  1
```

(iii) Diagonal matrix with diagonal entries $a_1, \ldots, a_n$: $\boxed{\texttt{diag([a1 ... an])}}$.

```
>> diag([2,3])
ans =  2  0
       0  3
```

5. Elementary Row Operations

Let $\boldsymbol{A}$ be a matrix. For example,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

```
>> A = [1 2 3 4 5; 2 3 4 5 6; 3 4 5 6 7; 4 5 6 7 8]
ans =  1  2  3  4  5
       2  3  4  5  6
       3  4  5  6  7
       4  5  6  7  8
```

The i^{th} row of $\boldsymbol{A}$ can be abstracted using $\boxed{\texttt{A(i,:)}}$. For example, the 4^{th} row is

```
>> A(4,:)
ans =  4  5  6  7  8
```

If we need more rows, indicate the indices of the rows in square brackets. For example, the following is the submatrix of $\boldsymbol{A}$ formed by the 2^{nd} and the 4^{th} rows of $\boldsymbol{A}$:

```
>> A([2,4], :)
ans =  2  3  4  5  6
       4  5  6  7  8
```

We can perform the three types of elementary row operations as follows:

(i) Multiplying the i^{th} row by a nonzero constant c: `A(i,:) = c*A(i,:)`.

```
>> A(1,:) = -2*A(1,:);
A =  -2  -4  -6  -8  -10
      2   3   4   5   6
      3   4   5   6   7
      4   5   6   7   8
```

(ii) Interchanging the ith and jth rows: `A([i,j],:) = A([j,i],:)`.

```
>> A([2,3],:) = A([3,2],:);
A =  -2  -4  -6  -8  -10
      3   4   5   6   7
      2   3   4   5   6
      4   5   6   7   8
```

(iii) Adding c times of the j^{th} row to the i^{th} row: `A(i,:) = A(i,:) + c*A(j,:)`.

```
>> A(4,:) = A(4,:) + 2*A(1,:)
A =  -2  -4  -6  -8  -10
      3   4   5   6   7
      2   3   4   5   6
      40  -3  -6  -9  -12
>> A = [1 2 3; 4 5 6]
A =   1   2   3
      4   5   6
```

6. Matrix Operations

The matrix addition, subtraction and multiplication with scalar can be evaluated using `+`, `-` and `*` respectively. For example,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}.$$

```
>> A = [1 2; 3 4];
>> B = [4 1; 2 5];
```

(i) Addition: $\boldsymbol{A}+\boldsymbol{B}$:

```
>> A + B
ans =   5   3
        5   9
```

(ii) Subtraction: $\boldsymbol{A}-\boldsymbol{B}$:

```
>> A - B
ans =  -3   1
        1  -1
```

(iii) Scalar multiplication: cA:

```
>> 3 * A
ans =   3   6
        9  12
```

We illustrate more operations using the previously defined $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}$.

(iv) Transpose $\boldsymbol{A}^{\mathrm{T}}$:

```
>> A'
ans =   1   3
        2   4
```

(v) Reduced row-echelon form of $\boldsymbol{A}$:

```
>> rref(A)
ans =   1   0
        0   1
```

(vi) Powers $\boldsymbol{A}^n$, provided that $\boldsymbol{A}$ is a square matrix and n is an integer. If $n < 0$, $\boldsymbol{A}$ needs to be invertible (that is, non-singular).

```
>> A ^ 10
ans =   4783807   6972050
        10458075   15241882
```

(vii) If $\boldsymbol{A}$ is invertible, its inverse can be evaluated using either `A^(-1)` or `inv(A)`.

```
>> A ^ (-1)
ans =   -2.0000   1.0000
        1.5000  -0.5000
>> inv(A)
ans =   -2.0000   1.0000
        1.5000  -0.5000
```

(viii) Matrix product $\boldsymbol{AB}$, provided that the sizes are matched.

```
>> A * B
ans =   8   11
        20   23
```

7. Solve Linear System

Recall that a linear system can be written in the matrix product form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A}$ is the coefficient matrix, $\boldsymbol{x}$ is the variable matrix, and $\boldsymbol{b}$ is the constant matrix. Its solution can be found from the (reduced) row-echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$.

For example, the linear system

$$\begin{cases} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 = -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 = -2 \\ 2x_1 \qquad\quad - 4x_3 + 2x_4 + x_5 = 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 = -7 \end{cases}$$

has augmented matrix

$$(\boldsymbol{A} \mid \boldsymbol{b}) = \left(\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array}\right)$$

Define the coefficient matrix

```
>> A = [2 -3 -7 5 2; 1 -2 -4 3 1; 2 0 -4 2 1; 1 -5 -7 6 2]
A =  2  -3  -7  5  2
     1  -2  -4  3  1
     2  0  -4  2  1
     1  -5  -7  6  2
```

and the constant matrix

```
>> b = [-2; -2; 3; -7]
b =  -2
     -2
     3
     -7
```

We shall find the reduced row-echelon form (RREF) of $(\boldsymbol{A} \mid \boldsymbol{b})$:

```
>> rref([A b])
ans =  1  0  -2  1  0  1
       0  1  1  -1  0  2
       0  0  0  0  1  1
       0  0  0  0  0  0
```

(Here `[A b]` is the matrix obtained by combining `A` and `b` to obtain the augmented matrix. The separator `|` should be omitted in the MATLAB command.)

One sees that the 1st, the 2nd and the 5th columns are pivot columns.

Set $x_3 = s$ and $x_4 = t$ to be arbitrary parameters, and solve other variables:

$$x_1 = 2s - t + 1, \quad x_2 = -s + t + 2, \quad x_5 = 1.$$

Indeed, we can verify that $\boldsymbol{x} = \begin{pmatrix} 2s - t + 1 \\ -s + t + 2 \\ s \\ t \\ 1 \end{pmatrix}$ is a solution.

We first declare that s and t are parameters.

```
>>  syms s t
```

Then define

```
>>  x = [2*s-t+1; -s+t+2; s; t; 1]
x =  2*s - t + 1
     t - s + 2
     s
     t
```

Note that $\boldsymbol{x}$ is a solution if and only if $\boldsymbol{Ax} = \boldsymbol{b}$. So we evaluate $\boldsymbol{Ax}$ and compare it with $\boldsymbol{b}$:

```
>>  A * x
ans =  -2
       -2
       3
       -7
```

8. Activities

8.1. **Activity 1.** Enter the following commands in MATLAB window and observe the outputs. Describe what MATLAB has done.

```
>>  x = [1 2 3]
>>  b = [1; 2; 3]
>>  A = [1 2 pi; 0.1 5 6; 7 8 1/2]
>>  format rat
```

```
>> A
>> format short
>> A
>> format long
>> A
>> format
>> A
>> A1 = [1 0; 2 1; 3 2; 4 3]
>> 2 * 3
>> ans
>> 0.3 * A
>> ans
>> B = ans * b
>> A * x
>> A * x'
```

8.2. **Activity 2.** Input the following three matrices.

$$\boldsymbol{A} = \begin{pmatrix} 7 & -2 & 0 & -2 \\ -3 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 16 & -3 & 0 & -2 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} -2 & 5 & 17 & -2 \\ 2 & 6 & -15 & -1 \\ -2 & 6 & 19 & -2 \\ 1 & 2 & -6 & 0 \end{pmatrix}, \qquad \boldsymbol{C} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Using MATLAB, compute the products $(\boldsymbol{AB})^{-1}$, $(\boldsymbol{BA})^{-1}$, $\boldsymbol{A}^{-1}\boldsymbol{B}^{-1}$ and $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. What are the relations between these matrices?

(ii) Compute $\boldsymbol{C}^{-1}$. What result do you get? Why?

(iii) Compute $(\boldsymbol{A}+\boldsymbol{B})^{-1}$ and $\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1}$. Are these matrices equal?

(iv) Compute the products $(\boldsymbol{AB})^{\mathrm{T}}$, $(\boldsymbol{BA})^{\mathrm{T}}$, $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}$ and $\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$. What are the relations between these matrices?

(v) Compute $(\boldsymbol{A}^{\mathrm{T}})^{-1}$ and $(\boldsymbol{A}^{-1})^{\mathrm{T}}$. Are these matrices equal? Is this relation true for any invertible matrix $\boldsymbol{A}$?

(vi) Compute $\boldsymbol{C}^2, \boldsymbol{C}^3$ and $\boldsymbol{C}^4$. What do you observe? Can you generalize this observation to upper triangular matrices of order n with all the diagonal entries 0?

8.3. **Activity 3.**

1. Consider the following linear system:

$$\begin{cases} x + y + 2z = 1 \\ 3x + 6y - 5z = -1 \\ 2x + 4y + 3z = 0 \end{cases}$$

   (i) Enter the coefficient matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 6 & -5 \\ 2 & 4 & 3 \end{pmatrix}$ and constant matrix $\boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

   (ii) Type `inv(A) * b` to get a solution for this system.

   (iii) Type `rref([A b])` to get the reduced row-echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$, and then solve the system.

2. Try to use the two different methods to solve the following linear system:

$$\begin{cases} x + 2y + z = 1 \\ x + 2y + 2z = 1 \\ 2x + 4y + z = 2 \end{cases}$$

   (*Hint*: Only one method works.)

3. For each of the linear systems in Question 1.16 in the textbook,

   (i) Use `inv(A) * b` (if applicable) or `rref([A b])` to find its general solution (if any).

   (ii) Verify that the solution (if any) that you found in (i) is a solution to the linear system by checking if $\boldsymbol{Ax} = \boldsymbol{b}$.

MATLAB LESSON 2: VECTOR SPACES AND REDUCED ROW-ECHELON FORM

ABSTRACT. In this laboratory session, we will learn how to use the `rref` command to better understand and solve problems related to concepts in vector space, including linear combinations, linear spans, linear independence, bases and dimensions. This worksheet covers content in Chapter 3 from Section 3.1 to 3.7.

Type `format rat`. Throughout the entire worksheet, we will use the rational format to read the entries of matrices.

1. LINEAR COMBINATIONS

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. A vector $\boldsymbol{u} \in \mathbb{R}^n$ is a **linear combination** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ if there exist numbers $c_1, c_2, \ldots, c_k$ such that

$$\boldsymbol{u} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k.$$

View each $\boldsymbol{u}_i$ and $\boldsymbol{v}$ as column vectors, and write $\boldsymbol{A} = \begin{pmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_k\end{pmatrix}$. Then $\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ if and only if the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{v}$ is consistent.

For example, let $\boldsymbol{u}_1 = (1,0,1,2,3)$, $\boldsymbol{u}_2 = (2,1,-1,1,0)$, $\boldsymbol{u}_3 = (1,1,-2,-1,-3)$ and $\boldsymbol{u}_4 = (1,2,3,1,1)$. To see whether $\boldsymbol{u} = (2,0,0,1,0)$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$:

(i) Input $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ and $\boldsymbol{u}$ as column vectors in MATLAB. For example,

```
>>  u1 = [1; 0; 1; 2; 3]
u1 =  1
      0
      1
      2
      3
```

(ii) Define the 5×4 matrix $\boldsymbol{A}$ whose columns are $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ and $\boldsymbol{u}_4$:

```
>>  A = [u1 u2 u3 u4]
A =  1    2     1    1
     0    1     1    2
     1   -1    -2    3
     2    1    -1    1
     3    0    -3    1
```

(iii) Find the reduced row-echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{u})$ to check the consistency of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{u}$:

```
>> rref([A u])
ans =   1     0    -1     0     0
        0     1     1     0     0
        0     0     0     1     0
        0     0     0     0     1
        0     0     0     0     0
```

Since the last column of the reduced row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{u})$ is pivot, the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{u}$ is inconsistent. Therefore, $\boldsymbol{u}$ is not a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$.

Repeat the same argument for $\boldsymbol{v} = (-2, -1, 1, -1, 0)$.

```
>> v = [-2; -1; 1; -1; 0]
v =  -2
     -1
     1
     -1
     0
>> rref([A v])
ans =   1     0    -1     0     0
        0     1     1     0    -1
        0     0     0     1     0
        0     0     0     0     0
        0     0     0     0     0
```

Since the last column of the reduced row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{v})$ is non-pivot, the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{v}$ is consistent. Therefore, $\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$.

2. Linear Independence

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of vectors in $\mathbb{R}^n$. Then S is said to be **linearly independent** if the linear system $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$ has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$.

View each $\boldsymbol{v}_i$ and $\boldsymbol{0}$ as column vectors, and write $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix}$. Then S is linearly independent if and only if the homogeneous linear system $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution.

For example, let $\boldsymbol{v}_1 = (1, 0, 2, 0, 3)$, $\boldsymbol{v}_2 = (1, 1, 0, 2, 2)$, $\boldsymbol{v}_3 = (1, -3, 8, -6, 6)$, $\boldsymbol{v}_4 = (1, 2, 3, 4, 1)$, $\boldsymbol{v}_5 = (0, -1, 1, -2, 1)$, $\boldsymbol{v}_6 = (1, 1, 1, 1, 1)$.

(i) Input $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_6$ and $\boldsymbol{w} = \mathbf{0}$ as column vectors in MATLAB. For example,

```
>> v1 = [1; 0; 2; 0; 3]
v1 =  1
      0
      2
      0
      3
```

(ii) Define the 5×6 matrix $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 & \boldsymbol{v}_5 & \boldsymbol{v}_6 \end{pmatrix}$.

```
>> B = [v1 v2 v3 v4 v5 v6]
B =  1    1    1    1     0    1
     0    1   -3    2    -1    1
     2    0    8    3     1    1
     0    2   -6    4    -2    1
     3    2    6    1     1    1
```

(iii) Find the reduced row-echelon form of the augmented matrix $(\boldsymbol{B} \mid \mathbf{0})$ of the homogeneous linear system $\boldsymbol{Bx} = \mathbf{0}$. (Recall that $\boldsymbol{w} = \mathbf{0}$ is defined in Step (i).)

```
>> rref([B w])
ans =  1    0    4     0    4/5    0    0
       0    1   -3     0   -3/5    0    0
       0    0    0     1   -1/5    0    0
       0    0    0     0     0     1    0
       0    0    0     0     0     0    0
```

Since the last column is non-pivot and the 3rd and 5th columns are non-pivot, the homogeneous linear system $\boldsymbol{Ax} = \mathbf{0}$ has infinitely many non-trivial solutions (with 2 arbitrary parameters). As a conclusion, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$ is a linearly dependent set.

Note that if $\boldsymbol{R}$ is the reduced row-echelon form of $\boldsymbol{B}$, then $(\boldsymbol{R} \mid \mathbf{0})$ is the reduced row-echelon form of $(\boldsymbol{B} \mid \mathbf{0})$, and vice versa. Therefore, we can drop the last zero column and simply check whether the reduced row echelon form $\boldsymbol{R}$ of $\boldsymbol{B}$ has non-pivot columns:

```
>> R = rref(B)
R =  1    0    4     0    4/5    0
     0    1   -3     0   -3/5    0
     0    0    0     1   -1/5    0
     0    0    0     0     0     1
     0    0    0     0     0     0
```

Since the 3rd and 5th columns of the reduced row-echelon form of $\boldsymbol{B}$ are non-pivot, the homogeneous linear system $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}$ has infinitely many non-trivial solutions. We also conclude that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$ is a linearly dependent set.

3. Redundant Vectors

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a set of vectors in $\mathbb{R}^n$, and let $V = \mathrm{span}(S)$. If S is linearly independent, then S is a **basis** for V. If S is linearly dependent, then some of the vectors in S are redundant to generate the vector space V.

For example, let $\boldsymbol{v}_1 = (1,0,2,0,3)$, $\boldsymbol{v}_2 = (1,1,0,2,2)$, $\boldsymbol{v}_3 = (1,-3,8,-6,6)$, $\boldsymbol{v}_4 = (1,2,3,4,1)$, $\boldsymbol{v}_5 = (0,-1,1,-2,1)$, $\boldsymbol{v}_6 = (1,1,1,1,1)$ as in Section 2.

View each $\boldsymbol{v}_i$ as column vectors, and write $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 & \boldsymbol{v}_5 & \boldsymbol{v}_6 \end{pmatrix}$. We have found the reduced row-echelon form $\boldsymbol{R}$ of $\boldsymbol{B}$:

```
>>  R

R =  1    0    4     0    4/5    0
     0    1    -3    0    -3/5   0
     0    0    0     1    -1/5   0
     0    0    0     0      0    1
     0    0    0     0      0    0
```

Since the 3rd and 5th columns of $\boldsymbol{R}$ are non-pivot; $\boldsymbol{v}_3$ and $\boldsymbol{v}_5$ are redundant vectors to span V, i.e., $V = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4, \boldsymbol{v}_6\}$.

Moreover, by observing the entries in the 3rd column $\begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and 5th column $\begin{pmatrix} 4/5 \\ -3/5 \\ -1/5 \\ 0 \\ 0 \end{pmatrix}$ of $\boldsymbol{R}$,

$$\boldsymbol{v}_3 = 4\boldsymbol{v}_1 - 3\boldsymbol{v}_2 \quad \text{and} \quad \boldsymbol{v}_5 = \tfrac{4}{5}\boldsymbol{v}_1 - \tfrac{3}{5}\boldsymbol{v}_2 - \tfrac{1}{5}\boldsymbol{v}_4.$$

Verify the above relations:

```
>>  v3 - (4*v1 - 3*v2)

ans =  0
       0
       0
       0
```

```
        0
>>  v5 - (4/5*v1 - 3/5*v2 - 1/5*v4)
ans =   -1/18014398509481984
        0
        0
        0
        -1/2251799813685248
```

The 1^{st} and 5^{th} entries are supposed to be 0. The nonzero values displayed are due to rounding errors.

4. Linear Spans

Let $S = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ and $T = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l\}$ be subsets of vectors in $\mathbb{R}^n$.

Let $U = \text{span}(S)$ and $V = \text{span}(V)$. Then

(i) $U \subseteq V$ if and only if every vector in S is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l$.
(ii) $V \subseteq U$ if and only if every vector in T is a linear combination of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$.

For example, let $U = \text{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ and $V = \text{span}\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4\}$, where

$$\boldsymbol{c}_1 = (1,1,2,2,3), \quad \boldsymbol{c}_2 = (1,0,2,0,3), \quad \boldsymbol{c}_3 = (1,1,1,1,1),$$

and

$$\boldsymbol{d}_1 = (3,2,5,3,7), \quad \boldsymbol{d}_2 = (0,0,1,1,2), \quad \boldsymbol{d}_3 = (2,2,1,1,0), \quad \boldsymbol{d}_4 = (1,-1,3,-1,5).$$

(i) Input $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ and $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ as column vectors in MATLAB. For example,

```
>>  c1 = [1; 1; 2; 2; 3]
c1 =  1
      1
      2
      2
      3
```

(ii) Form the matrices $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 \end{pmatrix}$. For example,

```
>>  C = [c1 c2 c3]
C =  1    1    1
     1    0    1
     2    2    1
     2    0    1
     3    3    1
```

(iii) In order to check whether $V \subseteq U$, we shall check if each $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ is a linear combination of $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$; i.e., if the linear systems $\boldsymbol{Cx} = \boldsymbol{d}_i$, $i = 1,2,3,4$, are consistent.

One may check one by one. Alternatively, consider $(\boldsymbol{C} \mid \boldsymbol{D}) = (\boldsymbol{C} \mid \boldsymbol{d}_1 \mid \boldsymbol{d}_2 \mid \boldsymbol{d}_3 \mid \boldsymbol{d}_4)$:

```
>>  rref([C D])
ans =   1     0     0     1     1     -1     0
        0     1     0     1     0      0     2
        0     0     1     1     -1     3     -1
        0     0     0     0     0      0     0
        0     0     0     0     0      0     0
```

The columns corresponding to $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ are all non-pivot. So $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ are linear combinations of $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$. In fact, observing the entries of the column vectors,

$$\boldsymbol{d}_1 = \boldsymbol{c}_1 + \boldsymbol{c}_2 + \boldsymbol{c}_3, \quad \boldsymbol{d}_2 = \boldsymbol{c}_1 - \boldsymbol{c}_3, \quad \boldsymbol{d}_3 = -\boldsymbol{c}_1 + 3\boldsymbol{c}_3, \quad \boldsymbol{d}_4 = 2\boldsymbol{c}_2 - \boldsymbol{c}_3.$$

Hence, $V \subseteq U$.

(iv) In order to check whether $U \subseteq V$, we shall check if each $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ is a linear combination of $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$. Similarly, consider $(\boldsymbol{D} \mid \boldsymbol{C}) = (\boldsymbol{D} \mid \boldsymbol{c}_1 \mid \boldsymbol{c}_2 \mid \boldsymbol{c}_3)$:

```
>>  rref([D C])
ans =   1     0     0     2        0      1      0
        0     1     0     -9/2     3/2    -2     1/2
        0     0     1     -5/2     1/2    -1     1/2
        0     0     0     0        0      0      0
        0     0     0     0        0      0      0
```

The columns corresponding to $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are all non-pivot. So $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are linear combinations of $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$. In fact,

$$\boldsymbol{c}_1 = \tfrac{3}{2}\boldsymbol{d}_2 + \tfrac{1}{2}\boldsymbol{d}_3, \quad \boldsymbol{c}_2 = \boldsymbol{d}_1 - 2\boldsymbol{d}_2 - \boldsymbol{d}_3, \quad \boldsymbol{c}_3 = \tfrac{1}{2}\boldsymbol{d}_2 + \tfrac{1}{2}\boldsymbol{d}_3.$$

Hence, $U \subseteq V$. We conclude that $U = V$.

Suppose we use the same U and V except $\boldsymbol{d}_4$ is replaced by $\boldsymbol{e}_4 = (1,-1,3,-1,0)$.

(i) Input $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ and $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{e}_4$ as column vectors. In fact, we just need to define $\boldsymbol{e}_4$:

```
>>  e4 = [1; -1; 3; -1; 0]
e4 =   1
       -1
       3
       -1
       0
```

(ii) Form the matrices $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 \end{pmatrix}$ and $\boldsymbol{E} = \begin{pmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{e}_4 \end{pmatrix}$. Indeed, we just need to define $\boldsymbol{E}$:

```
>> E = [d1 d2 d3 e4]
E =  3    0    2    1
     2    0    2   -1
     5    1    1    3
     3    1    1   -1
     7    2    0    0
```

(iii) Check the consistency of $\boldsymbol{Cx} = \boldsymbol{d}_i$, $i = 1,2,3$, and $\boldsymbol{Cx} = \boldsymbol{e}_4$:

```
>> rref([C E])
ans =  1    0    0    1    1    -1    0
       0    1    0    1    0     0    0
       0    0    1    1   -1    3    0
       0    0    0    0    0     0    1
       0    0    0    0    0     0    0
```

Since the column corresponding to $\boldsymbol{e}_4$ is pivot, the system $\boldsymbol{Cx} = \boldsymbol{e}_4$ is inconsistent; so $\boldsymbol{e}_4 \notin \operatorname{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\} = U$. Consequently, $V \not\subseteq U$.

(iv) Check the consistency of $\boldsymbol{Ex} = \boldsymbol{c}_i$, $i = 1,2,3$.

```
>> rref([E C])
ans =  1    0    0    0    0      1      0
       0    1    0    0    3/2   -2     1/2
       0    0    1    0    1/2   -1     1/2
       0    0    0    1    0      0      0
       0    0    0    0    0      0      0
```

Since the columns corresponding to $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are all non-pivot, the systems $\boldsymbol{Ex} = \boldsymbol{c}_i$, $i = 1,2,3$, are all consistent; so $\boldsymbol{c}_i \in \operatorname{span}\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{e}_4\} = V$, $i = 1,2,3$. Consequently, $U \subseteq V$.

5. Base and Dimensions

Let S be a subset of vectors in $\mathbb{R}^n$. Then S is a **basis** for a vector space V if (i) $V = \operatorname{span}(S)$ and (ii) S is linearly independent. For this case, the number of vectors in S, $|S|$, is called the **dimension** of V, denoted by $\dim(V)$.

5.1. Find Basis from Generating Set.

Let $S = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4\}$, where

$$\boldsymbol{g}_1 = (1,1,1,1,1), \quad \boldsymbol{g}_2 = (1,-1,2,3,0), \quad \boldsymbol{g}_3 = (-1,-3,0,1,-2), \quad \boldsymbol{g}_4 = (0,1,1,-1,-1).$$

Let $V = \text{span}(S)$. Then S is a basis for V if and only if S is linearly independent.

(i) Input $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ as column vectors in MATLAB.

(ii) Define the matrix $\boldsymbol{G} = \begin{pmatrix} \boldsymbol{g}_1 & \boldsymbol{g}_2 & \boldsymbol{g}_3 & \boldsymbol{g}_4 \end{pmatrix}$:

```
>>  G = [g1 g2 g3 g4]
G =  1     1    -1     0
     1    -1    -3     1
     1     2     0     1
     1     3     1    -1
     1     0    -2    -1
```

(iii) Find the reduced row-echelon form of $\boldsymbol{G}$:

```
>>  rref(G)
ans =  1     0    -2     0
       0     1     1     0
       0     0     0     1
       0     0     0     0
       0     0     0     0
```

The 1st, 2nd and 4th columns are pivot, while the 3rd column is non-pivot. We conclude that

(i) $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_4\}$ is a basis for V.

(ii) $\dim(V) = 3$.

Moreover, by observing the entries of the 3rd column, $\boldsymbol{g}_3 = -2\boldsymbol{g}_1 + \boldsymbol{g}_2$.

5.2. **Check Basis.**

If $V = \text{span}(S)$, in order to check whether a set of vectors T is a basis for V, we shall verify:

(i) T is linearly independent;

(ii) $V \subseteq \text{span}(T)$, i.e., every vector in S is a linear combination of vectors in T; and

(iii) $\text{span}(T) \subseteq V$, i.e., every vector in T is a linear combination of vectors in S.

We use the set S and vector space V as in Section 5.1:

Let $V = \text{span}(S)$, where $S = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4\}$,

$$\boldsymbol{g}_1 = (1,1,1,1,1), \quad \boldsymbol{g}_2 = (1,-1,2,3,0), \quad \boldsymbol{g}_3 = (-1,-3,0,1,-2), \quad \boldsymbol{g}_4 = (0,1,1,-1,-1).$$

Set $T = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3\}$, where

$$\boldsymbol{h}_1 = (2,0,3,4,1), \quad \boldsymbol{h}_2 = (1,0,3,2,-1), \quad \boldsymbol{h}_3 = (1,2,2,0,0).$$

In the following, we check whether T is a basis for V:

(i) Input $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ as column vectors, and define $\boldsymbol{H} = \begin{pmatrix} \boldsymbol{h}_1 & \boldsymbol{h}_2 & \boldsymbol{h}_3 \end{pmatrix}$. Find the reduced row-echelon form of $\boldsymbol{H}$:

```
>> rref(H)
ans =  1     0     0
       0     1     0
       0     0     1
       0     0     0
       0     0     0
```

Since the columns are all pivot, $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ are linearly independent.

(ii) In order to check the consistency of $\boldsymbol{H}\boldsymbol{x} = \boldsymbol{g}_i$, $i = 1, \ldots, 4$, we find the reduced row-echelon form of $(\boldsymbol{H} \mid \boldsymbol{g}_1 \mid \boldsymbol{g}_2 \mid \boldsymbol{g}_3 \mid \boldsymbol{g}_4) = (\boldsymbol{H} \mid \boldsymbol{G})$:

```
>> rref([H G])
ans =  1     0     0     1/2     1/2    -1/2    -1/2
       0     1     0    -1/2     1/2     3/2     1/2
       0     0     1     1/2    -1/2    -3/2     1/2
       0     0     0       0       0       0       0
       0     0     0       0       0       0       0
```

Since the columns corresponding to $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4$ are all non-pivot, each $\boldsymbol{g}_i$ is a linear combination of $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$. Hence, $V = \operatorname{span}(S) \subseteq \operatorname{span}(T)$.

(iii) In order to check the consistency of $\boldsymbol{G}\boldsymbol{x} = \boldsymbol{h}_i$, $i = 1, 2, 3$, we find the reduced row-echelon form of $(\boldsymbol{G} \mid \boldsymbol{h}_1 \mid \boldsymbol{h}_2 \mid \boldsymbol{h}_3) = (\boldsymbol{G} \mid \boldsymbol{H})$:

```
>> rref([G H])
ans =  1     0    -2     0     1     0     1
       0     1     1     0     1     1     0
       0     0     0     1     0     1     1
       0     0     0     0     0     0     0
       0     0     0     0     0     0     0
```

Since the columns corresponding to $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ are all non-pivot, each $\boldsymbol{h}_i$ is a linear combination of $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4$. Hence, $\operatorname{span}(T) \subseteq \operatorname{span}(S)$.

Therefore, we conclude that T is a basis for V.

5.3. **Find Coordinate Vector.**

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a basis for a vector space V. Then every vector in V can be uniquely represented as a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. Precisely, for any $\boldsymbol{v} \in V$, there exist unique numbers $c_1, c_2, \dots, c_k \in \mathbb{R}$ such that

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k.$$

Then the column vector $(c_1, c_2, \dots, c_k)$ is called the **coordinate vector** of $\boldsymbol{v}$ relative to S, denoted by $(\boldsymbol{v})_S$.

Using the same definition as in Sections 5.1 and 5.2, $T = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3\}$ is a basis for V. In the following, we find the coordinate vector of $\boldsymbol{h} = (-1, -3, 0, 1, -2)$ relative to S.

(i) Input $\boldsymbol{h}$ as a column vector in MATLAB.

```
>>  h = [-1; -3; 0; 1; -2]
h =   -1
      -3
      0
      1
      -2
```

(ii) Solve the linear system $\boldsymbol{Hx} = \boldsymbol{h}$ (recall that $\boldsymbol{H} = \begin{pmatrix}\boldsymbol{h}_1 & \boldsymbol{h}_2 & \boldsymbol{h}_3\end{pmatrix}$).

```
>>  rref([H h])
ans =   1     0     0     -1/2
        0     1     0     3/2
        0     0     1     -3/2
        0     0     0     0
        0     0     0     0
```

Observing the entries in the column corresponding to $\boldsymbol{h}$, we obtain $\boldsymbol{h} = -\frac{1}{2}\boldsymbol{h}_1 + \frac{3}{2}\boldsymbol{h}_2 - \frac{3}{2}\boldsymbol{h}_3$. Hence, $(\boldsymbol{h})_T = \left(-\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}\right)$.

6. Practices

1. Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5, \boldsymbol{u}_6\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$, where

$$\begin{aligned} \boldsymbol{u}_1 &= (1,1,1,1,1,1), & \boldsymbol{u}_2 &= (1,0,1,0,1,0), & \boldsymbol{u}_3 &= (1,1,1,0,0,0), \\ \boldsymbol{u}_4 &= (1,1,0,0,1,1), & \boldsymbol{u}_5 &= (1,1,0,1,1,0), & \boldsymbol{u}_6 &= (1,0,0,1,0,0), \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{v}_1 &= (1,-1,2,0,1,2), & \boldsymbol{v}_2 &= (-1,2,0,1,2,1), & \boldsymbol{v}_3 &= (2,-3,2,-1,-1,1), \\ \boldsymbol{v}_4 &= (0,1,2,1,-1,2), & \boldsymbol{v}_5 &= (1,2,1,-1,2,0), & \boldsymbol{v}_6 &= (1,3,3,0,1,2). \end{aligned}$$

(i) Determine the relation between $\operatorname{span}(S)$ and $\operatorname{span}(T)$, i.e., whether (a) $\operatorname{span}(S) \subseteq \operatorname{span}(T)$ and (b) $\operatorname{span}(T) \subseteq \operatorname{span}(S)$.

(ii) Let $V = \text{span}(T)$. Find a basis T' for V consisting of vectors in T. Express the redundant vectors in T as linear combinations of vectors in T'. Moreover, write down their coordinate vectors with respect to T'.

(iii) Determine whether S is a basis for $\mathbb{R}^6$.

2. Use MATLAB to solve Questions 8, 9, 11, 12, 26(a), 32, 33, 34, 40(a)(b)(c), 46, 47 in Chapter 3.

MATLAB LESSON 3: ROW SPACE, COLUMN SPACE, NULLSPACE, DOT PRODUCT AND ORTHOGONAL SETS

ABSTRACT. In this laboratory session, we will learn how to use MATLAB commands to solve problems related to concepts on row space, column space, nullspace, as well as dot product and orthogonal sets. Some new commands that you will be introduced here include `rank`, `null`, `norm` and `orth`.

1. ROW SPACE AND COLUMN SPACE

1.1. Row Space. Let $\boldsymbol{A} = (a_{ij})$ be an $m \times n$ matrix. Let $\boldsymbol{r}_i = \begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix}$ be the i^{th} row of $\boldsymbol{A}$. Then $\boldsymbol{r}_i \in \mathbb{R}^n$ and

$$\operatorname{span}\{\boldsymbol{r}_1, \dots, \boldsymbol{r}_m\}$$

is a subspace of $\mathbb{R}^n$, called the *row space* of $\boldsymbol{A}$. It is proved that

(i) Elementary row operation preserves the row space. Precisely, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent, then they have the same row space.
(ii) For a matrix in row echelon form, its nonzero rows form a basis for its row space.

Therefore, if $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$, then the nonzero rows of $\boldsymbol{R}$ form a basis for the row space of $\boldsymbol{A}$.

For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 1 & 2 & 8 \\ 2 & 8 & 2 & 3 & 12 \\ 3 & 12 & 3 & -1 & -4 \\ 4 & 16 & -1 & -4 & -16 \end{pmatrix}$.

(i) Input $\boldsymbol{A}$ in MATLAB.

```
>> A = [1 4 1 2 8; 2 8 2 3 12; 3 12 3 -1 -4; 4 16 -1 -4 -16];
```

(ii) Find the reduced row-echelon form of $\boldsymbol{A}$.

```
>> rref(A)
ans =  1    4    0    0    0
       0    0    1    0    0
       0    0    0    1    4
       0    0    0    0    0
```

We conclude that the row space of $\boldsymbol{A}$ has a basis $\{(1,4,0,0,0),(0,0,1,0,0),(0,0,0,1,4)\}$. In particular, the dimension of the row space of $\boldsymbol{A}$ is 3.

Repeat the same procedure for $\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 3 & 4 & 8 & 16 & 32 \\ 4 & 1 & 7 & 7 & 19 & 31 \end{pmatrix}$.

(i) Input $\boldsymbol{B}$ in MATLAB.

```
>> B = [1 1 1 1 1 1; 1 -1 1 -1 1 -1; 1 3 4 8 16 32; 4 1 7 7 19 31];
```

(ii) Find the reduced row-echelon form of $\boldsymbol{B}$.

```
>> rref(B)
ans =  1    0    0    0    -4    -8
       0    1    0    0    0      0
       0    0    1    0    5      8
       0    0    0    1    0      1
```

So the row space of $\boldsymbol{B}$ has a basis $\{(1,0,0,0,-4,-8),(0,1,0,0,0,0),(0,0,1,0,5,8),(0,0,0,1,0,1)\}$. In particular, the dimension of the row space of $\boldsymbol{B}$ is 4. On the other hand, the row space of $\boldsymbol{B}$ is spanned by the four rows of $\boldsymbol{B}$; so the rows of $\boldsymbol{B}$ are linearly independent, and they form a basis for the row space of $\boldsymbol{B}$.

1.2. **Column Space.** Let $\boldsymbol{A} = (a_{ij})$ be an $m \times n$ matrix. Let $\boldsymbol{c}_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ be the j^{th} column of $\boldsymbol{A}$. Then $\boldsymbol{c}_j \in \mathbb{R}^m$, and

$$\operatorname{span}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$$

is a subspace of $\mathbb{R}^m$, called the *column space* of $\boldsymbol{A}$. It is proved that

(i) Elementary row operation preserves the linear relations of the columns of a matrix. Precisely, suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent.
   - If there is a linear relation among the columns of $\boldsymbol{A}$, then the same relation holds among the columns of $\boldsymbol{B}$.
   - If a set of columns of $\boldsymbol{A}$ are linearly independent (resp. span the column space of $\boldsymbol{A}$, form a basis for the column space of $\boldsymbol{A}$), then the corresponding set of columns of $\boldsymbol{B}$ are linearly independent (resp. span the column space of $\boldsymbol{A}$, form a basis for the column space of $\boldsymbol{A}$).

(ii) For a matrix in row-echelon form, its pivot columns form a basis for its column space.

Therefore, if $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$, then the columns of $\boldsymbol{A}$ which correspond to the pivot columns of $\boldsymbol{R}$ form a basis for the column space of $\boldsymbol{A}$.

We use the same matrices as in Section 1.1.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 1 & 2 & 8 \\ 2 & 8 & 2 & 3 & 12 \\ 3 & 12 & 3 & -1 & -4 \\ 4 & 16 & -1 & -4 & -16 \end{pmatrix} \text{ has the reduced row-echelon form } \begin{pmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the reduced row-echelon form, the 1^{st}, 3^{rd} and 4^{th} columns are pivot. Then the column space of $\boldsymbol{A}$ has a basis formed by the 1^{st}, 3^{rd} and 4^{th} columns of $\boldsymbol{A}$:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \\ 4 \end{pmatrix} \right\}.$$

In particular, the dimension of the column space of $\boldsymbol{A}$ is 3.

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 3 & 4 & 8 & 16 & 32 \\ 4 & 1 & 7 & 7 & 19 & 31 \end{pmatrix} \text{ has the reduced row-echelon form } \begin{pmatrix} 1 & 0 & 0 & 0 & -4 & -8 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 & 8 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

In the reduced row-echelon form, the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} columns are pivot. Then the column space of $\boldsymbol{B}$ has a basis formed by the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} columns of $\boldsymbol{B}$:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 8 \\ 7 \end{pmatrix} \right\}.$$

In particular, the dimension of the column space of $\boldsymbol{B}$ is 4.

1.3. **Rank.** In the previous examples, the row space and the column space of $\boldsymbol{A}$ have the same dimension; and the same conclusion holds for $\boldsymbol{B}$.

In general, suppose $\boldsymbol{A}$ is a matrix with row-echelon form $\boldsymbol{R}$. Then

$$\begin{aligned} \text{the dimension of the row space of } \boldsymbol{A} &= \text{the number of nonzero rows of } \boldsymbol{R} \\ &= \text{the number of pivot points of } \boldsymbol{R} \\ &= \text{the number of pivot columns of } \boldsymbol{R} \\ &= \text{the dimension of the column space of } \boldsymbol{A}. \end{aligned}$$

The common number is called the *rank* of $\boldsymbol{A}$, denoted by $\text{rank}(\boldsymbol{A})$.

In MATLAB, we can use a simple command `rank` to find the rank of a matrix:

```
>> rank(A)
```

```
ans =   3
>>  rank(B)
ans =   4
```

2. Bases for Vector Spaces

2.1. **Basis for Linear Span.** (This is a revision from Lesson 2.)

Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$. There are two methods to find a basis for $V = \text{span}(S)$.

2.1.1. *Use Row Vectors.*

View each $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ as a row vector. Then the nonzero rows of any row-echelon form of the matrix $\begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_k \end{pmatrix}$ form a basis for V.

For example, let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$, where

$$\boldsymbol{v}_1 = (1,1,1,1,1),\ \boldsymbol{v}_2 = (1,-1,1,-1,1),\ \boldsymbol{v}_3 = (2,0,2,0,2),\ \boldsymbol{v}_4 = (1,-2,4,-8,16),\ \boldsymbol{v}_5 = (0,4,-6,16,30).$$

(i) Input $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5$ into MATLAB as row vectors:

```
>>  v1 = [1 1 1 1 1];   v2 = [1 -1 1 -1 1];   v3 = [2 0 2 0 2];
>>  v4 = [1 -2 4 -8 16];   v5 = [0 4 -6 16 -30];
```

(ii) Find the reduced row-echelon form of the matrix $\begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_5 \end{pmatrix}$.

```
>>  rref([v1; v2; v3; v4; v5])
ans =   1     0     0     2     -4
        0     1     0     1     0
        0     0     1     -2     5
        0     0     0     0     0
        0     0     0     0     0
```

Its nonzero rows $\{(1,0,0,2,-4), (0,1,0,1,0), (0,0,1,-2,5)\}$ form a basis for $V = \text{span}(S)$. Note that the vectors in the basis are not necessarily in S.

2.1.2. *Use Column Vectors.*

View each $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ as column vectors. Find the pivot columns of any row-echelon form of the matrix $\begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k \end{pmatrix}$. Then the corresponding vectors in S form a basis S' for V. Note that $S' \subseteq S$.

(i) Input $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5$ into MATLAB as column vectors. In Section 2.1.1, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5$ are defined as vectors. Their transposes $\boldsymbol{v}_1^{\mathrm{T}}, \ldots, \boldsymbol{v}_5^{\mathrm{T}}$ (`v1'`, ..., `v5'`) are the required column vectors.

(ii) Find the reduced row-echelon form of the matrix $\begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_5 \end{pmatrix}$.

```
>> rref([v1' v2' v3' v4' v5'])
ans =  1    0    1    0    1
       0    1    1    0    1
       0    0    0    1   -2
       0    0    0    0    0
       0    0    0    0    0
```

Its 1st, 2nd and 4th columns are pivot. Then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\} = \{(1,1,1,1,1), (1,-1,1,-1,1), (1,-2,4,-8,16)\}$ form a basis for V. Note that every vector in this basis is taken from S.

2.2. **Extension of Linearly Independent Set.**

Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a linearly independent subset of $\mathbb{R}^n$. Then S may be extended to a basis for $\mathbb{R}^n$ by adding $n-k$ vectors.

(i) View each $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ as a row vector.

(ii) Find a row-echelon form for the matrix $\begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_k \end{pmatrix}$.

(iii) Note that exactly k columns of the row-echelon form are pivot. Insert $n-k$ rows properly (e.g., rows of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$) such that all columns are pivot.

For example, in Section 2.1.2 we have found a linearly independent subset of $\mathbb{R}^5$:

$$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\} = \{(1,1,1,1,1), (1,-1,1,-1,1), (1,-2,4,-8,16)\}.$$

Recall that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5$ as defined as row vectors in Section 2.1.1. Then the reduced row-echelon form of the matrix $\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_4 \end{pmatrix}$ by using

```
>> rref([v1; v2; v4])
ans =  1    0    0    2   -4
       0    1    0    1    0
       0    0    1   -2    5
```

The 1st, 2nd and the 3rd columns are pivot. Then we can add rows $(0,0,0,1,0)$ and $(0,0,0,0,1)$ to make the 4th and 5th columns pivot. Therefore,

$$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\} \cup \{(0,0,0,1,0), (0,0,0,0,1)\}$$

is a basis for $\mathbb{R}^5$.

3. Nullspaces

Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then the solution set of the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ is always a subspace of $\mathbb{R}^n$, called the *nullspace* of $\boldsymbol{A}$.

We use the same matrices as in Section 1.1.

$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 1 & 2 & 8 \\ 2 & 8 & 2 & 3 & 12 \\ 3 & 12 & 3 & -1 & -4 \\ 4 & 16 & -1 & -4 & -16 \end{pmatrix}$ has the reduced row-echelon form $\begin{pmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Assume that the variables are x_1, x_2, x_3, x_4, x_5. Since the 2^{nd} and the 5^{th} columns of the reduced row-echelon form are non-pivot, set $x_2 = s$ and $x_5 = t$ as arbitrary parameters. We get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -4s \\ s \\ 0 \\ -4t \\ t \end{pmatrix} = s \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 1 \end{pmatrix}.$$

Then the nullspace of $\boldsymbol{A}$ has a basis

$$\left\{ \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

MATLAB can provide a basis for the nullspace of $\boldsymbol{A}$ using the same way by `null(A,'r')`.*

```
>>  null(A, 'r')
ans =  -4     0
        1     0
        0     0
        0    -4
        0     1
```

The columns of the answer form a basis for the nullspace of $\boldsymbol{A}$.

---

* `null(A,'r')` is a "rational" basis for the nullspace obtained from the reduced row echelon form.

$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 3 & 4 & 8 & 16 & 32 \\ 4 & 1 & 7 & 7 & 19 & 31 \end{pmatrix}$ has the reduced row-echelon form $\begin{pmatrix} 1 & 0 & 0 & 0 & -4 & -8 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 & 8 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$.

Assume that the variables are $x_1, x_2, x_3, x_4, x_5, x_6$. Since the 5$^{\text{th}}$ and 6$^{\text{th}}$ columns of the reduced row-echelon form are non-pivot. Set $x_5 = s$ and $x_6 = t$ as arbitrary parameters. We get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 4s+8t \\ 0 \\ -5s-8t \\ -t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 8 \\ 0 \\ -8 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Then the nullspace of $\boldsymbol{B}$ has a basis

$$\left\{ \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \\ -8 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Alternatively, use `null(B,'r')`:

```
>> null(B, 'r')
ans =   4       8
        0       0
        -5     -8
        0     -1
        1       0
        0       1
```

4. Dot Product and Norm

4.1. **Dot Product.** Let $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ be vectors in $\mathbb{R}^n$. Their *dot product* is defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + \cdots + u_n v_n = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

If both $\boldsymbol{u}$ and $\boldsymbol{v}$ are defined as row vectors, then $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$.

If both $\boldsymbol{u}$ and $\boldsymbol{v}$ are defined as column vectors, then $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}$.

For example, let $\boldsymbol{u} = (1, 2, 3, 4, 5)$ and $\boldsymbol{v} = (1, 0, 1, -1, 2)$.

```
>> u = [1 2 3 4 5];  v = [1 0 1 -1 2];
>> u * v'
ans =  10
```

Alternatively, MATLAB provides a command `dot` for dot product, regardless whether the vectors are defined as row or column vectors.

```
>> dot(u, v)
ans =  10
>> dot(u', v)
ans =  10
>> dot(u, v')
ans =  10
>> dot(u', v')
ans =  10
```

4.2. **Norm.** The norm of a vector $\boldsymbol{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ is defined by

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + \cdots + v_n^2} = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}.$$

Using $\boldsymbol{u}$ and $\boldsymbol{v}$ as defined in Section 4.1, their norms can be evaluated by

```
>> sqrt(dot(u, u))
ans =  7.4162
>> sqrt(dot(v, v))
ans =  2.6458
```

Alternatively, in MATLAB `norm` can be used to find the norm of a vector.

```
>> norm(u)
ans =  7.4162
```

Note that the norm of a vector is usually irrational, and the output is in floating-point. We can use `sym` to define a vector as *symbolic* object, and use `norm` to get the exact value of the norm. For example,

```
>> u = sym([1 2 3 4 5])
u =  [1, 2, 3, 4, 5]
>> norm(u)
ans =  55^(1/2)
```

5. Orthonormal Basis

5.1. **Orthogonality.**

5.1.1. *Orthogonal Set.* A set of vectors $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ in $\mathbb{R}^n$ is said to be an *orthogonal* set if

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0 \quad \text{for all } i \neq j.$$

View each vector as a row vector and consider $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_k \end{pmatrix}$. Then $\boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} \boldsymbol{v}_1^{\mathrm{T}} & \cdots & \boldsymbol{v}_k^{\mathrm{T}} \end{pmatrix}$ and

$$\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} \boldsymbol{v}_1\boldsymbol{v}_1^{\mathrm{T}} & \cdots & \boldsymbol{v}_1\boldsymbol{v}_k^{\mathrm{T}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{v}_k\boldsymbol{v}_1^{\mathrm{T}} & \cdots & \boldsymbol{v}_k\boldsymbol{v}_k^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_1 \cdot \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_1 \cdot \boldsymbol{v}_k \\ \vdots & \ddots & \vdots \\ \boldsymbol{v}_k \cdot \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k \cdot \boldsymbol{v}_k \end{pmatrix}.$$

Hence, $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is orthogonal if and only if $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$ is a diagonal matrix.

For example, consider a set of vectors $\{(1,1,1,1), (1,0,-1,0), (1,-1,1,-1)\}$ in $\mathbb{R}^4$.

(i) Define the matrix $\boldsymbol{C}$ whose rows are the given vectors.

```
>> C = [1 1 1 1; 1 0 -1 0; 1 -1 1 -1];
```

(ii) Evaluate $\boldsymbol{C}\boldsymbol{C}^{\mathrm{T}}$.

```
>> C * C'
ans =  4    0    0
       0    2    0
       0    0    4
```

Since $\boldsymbol{C}\boldsymbol{C}^{\mathrm{T}}$ is a diagonal matrix, the given set of vectors is orthogonal.

5.1.2. *Orthonormal Set.* A set of vectors $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ in $\mathbb{R}^n$ is said to be an *orthonormal* set if

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

View each vector as a row vector and consider $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_k \end{pmatrix}$. Then $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is orthonormal if and only if $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}_k$, the identity matrix of order k.

For example, consider a set of vectors $\left\{(\cos\frac{\pi}{3}, 0, \sin\frac{\pi}{3}), (0,1,0), (\sin\frac{\pi}{3}, 0, -\cos\frac{\pi}{3})\right\}$ in $\mathbb{R}^3$.

(i) Define the matrix $\boldsymbol{D}$ whose rows are the given vectors.

```
>> D = [cos(pi/3) 0 sin(pi/3); 0 1 0; sin(pi/3) 0 -cos(pi/3)];
```

(ii) Evaluate $\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}$.

```
>> D * D'
ans =  1.0000         0    0.0000
            0    1.0000         0
       0.0000         0    1.0000
```

Since $\boldsymbol{C}\boldsymbol{C}^{\mathrm{T}} = \boldsymbol{I}_3$, the given set of vectors is orthonormal.

5.2. **Orthogonal and Orthonormal Basis.** If $V = \text{span}(S)$, and S is an orthonormal set, then S is linearly independent, and S is called an *orthonormal basis* for V.

In MATLAB, `orth` can be used to get an *orthonormal basis* for the column space of a matrix.

Suppose $V = \text{span}(S)$, where $S = \{(1,1,1,1), (1,1,0,0), (0,1,1,0)\}$.

(i) Define matrix $\boldsymbol{E} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ whose columns are the vectors in the spanning set.

```
>> E = [1 1 0; 1 1 1; 1 0 1; 1 0 0];
```

or

```
>> E = [1 1 1 1; 1 1 0 0; 0 1 1 0]';
```

(ii) Use `orth` to get an orthonormal basis for the column space of $\boldsymbol{E}$, i.e., for V.

```
>> orth(E)
ans =  -0.4835    0.7071     -0.1273
       -0.6635    0.0000      0.5565
       -0.4835   -0.7071     -0.1273
       -0.3035   -0.0000     -0.8111
```

The columns of the resulting matrix form an orthonormal basis

$$\{(-0.4835, -0.6635, -0.4835, -0.3035), (0.7071, 0, -0.7071, 0), (-0.1273, 0.5565, -0.1273, -0.8111)\}$$

for the vector space V.

We shall note the followings:

(i) The spanning set S is not necessarily linearly independent. The number of columns of the resulting matrix shall equal to $\dim(V)$.
(ii) The classical Gram-Schmidt process is *numerically unstable*; so the function `orth` in MATLAB uses a *modified Gram-Schmidt process*[†] to generate orthonormal basis.
(iii) The result is in floating-point (or rational expression if we set `format rat`).

In order to get the exact form of the orthonormal basis, we can use `sym` to define the matrix as *symbolic* object (Ref: Section 4.2). By dealing with symbolic objects, there is no longer numerical error; then `orth` uses the classical Gram-Schmidt process to generate orthonormal basis.

Again, suppose $V = \text{span}(S)$, where $S = \{(1,1,1,1), (1,1,0,0), (0,1,1,0)\}$.

(i) Define matrix $\boldsymbol{E}$ whose columns are the vectors in S as an symbolic object.

```
>> E = sym([1 1 0; 1 1 1; 1 0 1; 1 0 0]);
E =  [1, 1, 0]
     [1, 1, 1]
     [1, 0, 1]
     [1, 0, 0]
```

(ii) Use `orth` to get an orthonormal basis for the column space of $\boldsymbol{E}$, i.e., for V.

```
>> orth(E)
ans =  [1/2, 1/2, -1/2]
       [1/2, 1/2, 1/2]
       [1/2, -1/2, 1/2]
       [1/2, -1/2, -1/2]
```

The columns of the resulting matrix form an orthonormal basis

$$\{(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}), (-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2})\}$$

for the vector space V.

The command `orth` also allows us to generate an orthogonal basis in symbolic form by "skipping the normalization" in Gram-Schmidt process:

[†]Your may refer to the session *Numerical Stability* on Gram-Schmidt process in WIKIPEDIA: https://en.wikipedia.org/wiki/Gram-Schmidt_process.

```
>> orth(E, 'skipnormalization')
ans =  [1, 1/2, -1/2]
       [1, 1/2, 1/2]
       [1, -1/2, 1/2]
       [1, -1/2, -1/2]
```

6. Practices

1. Let $M = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(i) Find an orthonormal basis for the column space of M.

(ii) Find an orthonormal basis for the row space of M.

(iii) Find an orthonormal basis for the nullspace of M.

2. Use MATLAB to solve Exercises 4.1, 4.2, 4.3, 4.5, 4.7, 4.11, 4.16, 5.1, 5.2, 5.6, 5.10, 5.11, 5.12, 5.13, 5.14, 5.15, 5.17.

MATLAB LESSON 4: EIGENVALUES, EIGENVECTORS AND DIAGONALIZATION

ABSTRACT. In this laboratory session, we will learn how to use MATLAB commands to solve problems related to concepts on eigenvalues, eigenvectors, diagonalization, as well as large power of matrices. The new commands that you will be introduced here include `charpoly`, `roots` and `eig`.

Throughout the lesson, we illustrate using the matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ -5 & 0 & -5 & -5 & 0 & -3 \end{pmatrix}.$$

```
>> A = [2 0 0 0 0 0; 0 -3 0 0 0 0; 0 0 1 0 0 0; 0 0 1 2 0 0; 0 0 0 0 -3 0;
   -5 0 -5 -5 0 -3];
```

1. EIGENVALUE AND EIGENVECTOR

Let $\boldsymbol{A}$ be a square matrix of order n. If there exists a constant λ and a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$, then λ is called an *eigenvalue* of $\boldsymbol{A}$, and $\boldsymbol{v}$ is an *eigenvector* of $\boldsymbol{A}$ associated to λ.

1.1. Characteristic Polynomial.

The *characteristic polynomial* of $\boldsymbol{A}$ is the polynomial given by

$$p_{\boldsymbol{A}}(\lambda) = \det(\lambda \boldsymbol{I}_n - \boldsymbol{A}). \tag{1}$$

Note that the degree of the characteristic polynomial is n and its leading coefficient is 1.

MATLAB provides several ways to find the characteristic polynomial of $\boldsymbol{A}$.

(a) `charpoly(A)` gives a *vector* with $n+1$ components which are the coefficients of the characteristic polynomial in *descending* order. For example,

```
>> charpoly(A)
ans =  1     4     -10     -40     45     108     -108
```

The output means that the characteristic polynomial of $\boldsymbol{A}$ is (in variable λ)

$$p_{\boldsymbol{A}}(\lambda) = \lambda^6 + 4\lambda^5 - 10\lambda^4 - 40\lambda^3 + 45\lambda^2 + 108\lambda - 108.$$

(b) `charpoly(A, lambda)` gives the characteristic polynomial in variable `lambda`. In order to use λ as the variable for the characteristic polynomial of $\boldsymbol{A}$, we shall use `syms` to declare it as a symbolic object:

```
>> syms lambda;
```

Then type

```
>> charpoly(A, lambda)
ans =  lambda^6 + 4*lambda^5 - 10*lambda^4 - 40 * lambda^3 + 45*lambda^2
+ 108*lambda - 108
```

(c) We can also use the definition (1) to find the characteristic polynomial. Recall that `eye(n)` generates the identity matrix of order n. In this example, $\boldsymbol{A}$ has order 6. So we use

```
>> syms lambda;
>> det(lambda*eye(6) - A)
ans =  (lambda - 1) * (lambda - 2)^2 * (lambda + 3)^3
```

Hence, the characteristic polynomial of $\boldsymbol{A}$ is

$$p_{\boldsymbol{A}}(\lambda) = (\lambda - 1)(\lambda - 2)^2(\lambda + 3)^3.$$

1.2. **Eigenvalues.** The eigenvalues of a square matrix $\boldsymbol{A}$ are precisely all the roots to the characteristic polynomial of $\boldsymbol{A}$.

(a) `solve` can be used to find the roots of an equation or a function.*

In particular, if the characteristic equation is found in polynomial forms using methods (b) or (a) in Section 1.1, then `solve` can be used to generate the eigenvalues:

```
>> det(lambda*eye(6) - A);
>> solve(ans)
ans =  -3
       -3
       -3
       1
       2
       2
```

or

```
>> charpoly(A, lambda);
>> solve(ans)
ans =  -3
       -3
```

*The roots of a function $f(x)$ are the roots of the equation $f(x) = 0$.

```
-3
1
2
2
```

(b) `roots` computes numerically the roots of the polynomial whose coefficients are the components of the input vector in ascending order.

In particular, if the characteristic polynomial is generated as a vector using the method (c) in Section 1.1, then `roots` can be used to compute the eigenvalues:

```
>>  roots(charpoly(A))
ans =  -3.0000 + 0.0000i†
       -3.0000 + 0.0000i
       -3.0000 + 0.0000i
       2.0000 + 0.0000i
       2.0000 - 0.0000i
       1.0000 + 0.0000i
```

(c) MATLAB provides a simple commands `eig` to produce the eigenvalue of $\boldsymbol{A}$ as a column vector:

```
>>  eig(A)
ans =  -3
       2
       -3
       2
       1
       -3
```

Using any of these three methods, we see that the eigenvalues of $\boldsymbol{A}$ are -3, 2 and 1, with -3 and 2 being repeated eigenvalues.

1.3. **Eigenvectors.** Let λ be an eigenvalue of a matrix $\boldsymbol{A}$. Then the eigenvectors of $\boldsymbol{A}$ associated to λ are precisely all nonzero vectors in the nullspace of $\lambda\boldsymbol{I} - \boldsymbol{A}$. For this reason, the nullspace of $\lambda\boldsymbol{I} - \boldsymbol{A}$ is also called the eigenspace of $\boldsymbol{A}$ associated to λ, denoted by $E_{\boldsymbol{A},\lambda}$.

In our example,

(i) $\lambda = -3$:

```
>>  -3*eye(6) - A;
```

† `roots` searches the roots of a polynomial in complex numbers and the answers are given in floating points. Here i is the *imaginary unit* such that $i^2 = -1$. If the coefficient for i is 0 or very small, we may assume that the answer is a real number.

```
>> null(ans, 'r')
ans =   0     0     0
        1     0     0
        0     0     0
        0     0     0
        0     1     0
        0     0     1
```

Then the eigenspace E_{-3} of $\boldsymbol{A}$ associated to eigenvalue -3 has a basis

$$\{(0,1,0,0,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\},$$

which are linearly independent eigenvectors of $\boldsymbol{A}$ associated to the eigenvalue -3.

(ii) $\lambda = 2$:

```
>> 2*eye(6) - A;
>> null(ans, 'r')
ans =   -1     -1
        0     0
        0     0
        1     0
        0     0
        0     1
```

Then the eigenspace E_2 of $\boldsymbol{A}$ associated to eigenvalue 2 has a basis

$$\{(-1,0,0,1,0,0),(-1,0,0,0,0,1)\},$$

which are linearly independent eigenvectors of $\boldsymbol{A}$ associated to the eigenvalue 2.

(iii) $\lambda = 1$:

```
>> 1*eye(6) - A;
>> null(ans, 'r')
ans =   0
        0
        -1
        1
        0
        0
```

Then the eigenspace E_1 of $\boldsymbol{A}$ associated to eigenvalue 1 has a basis

$$\{(0,0,-1,1,0,0)\}.$$

So there is only one linearly independent eigenvector of $\boldsymbol{A}$ associated to the eigenvalue 1.

2. Diagonalization and Its Applications

2.1. Diagonalization.

A square matrix $\boldsymbol{A}$ is said to be *diagonalizable* if there exists an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$.

Let $\boldsymbol{v}_i$ be the i^{th} column of $\boldsymbol{P}$ and λ_i the i^{th} diagonal entry of $\boldsymbol{D}$. Then $\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$, i.e., $\boldsymbol{v}_i$ is an eigenvector of $\boldsymbol{A}$ associated to the eigenvalue λ_i. It follows that

(i) The diagonal entries of $\boldsymbol{D}$ are the eigenvalues of $\boldsymbol{A}$.
(ii) The columns of $\boldsymbol{P}$ are the corresponding eigenvectors of $\boldsymbol{A}$.

This is the criterion for a square matrix to be diagonalizable:

> *A square matrix of order n is diagonalizable if and only if it has n linearly independent eigenvectors.*

It is known that the eigenvectors of a square matrix associated to distinct eigenvalues are linearly independent. Therefore,

> *A square matrix of order n is diagonalizable if and only if the sum of the dimensions of its eigenspaces equals n.*

In Section 1.3, we have found that the sum of the dimensions of eigenspaces of $\boldsymbol{A}$ is

$$\dim(E_{-3}) + \dim(E_2) + \dim(E_1) = 3 + 2 + 1 = 6,$$

which equals the order of $\boldsymbol{A}$. Then $\boldsymbol{A}$ can be diagonalized by the matrix $\boldsymbol{P}$ whose columns are the vectors in the bases for the eigenspaces. The diagonal entries of the diagonal matrix $\boldsymbol{D}$ are the corresponding eigenvalues.

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{D} = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can verify that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$:

```
>> P = [0 0 0 -1 -1 0; 1 0 0 0 0 0; 0 0 0 0 0 -1; 0 0 0 1 0 1; 0 1 0 0 0 0;
   0 0 1 0 1 0];
>> inv(P) * A * P
ans =  -3     0     0     0     0     0
        0    -3     0     0     0     0
```

```
0   0   -3   0   0   0
0   0   0    2   0   0
0   0   0    0   2   0
0   0   0    0   0   1
```

We may use either of the following ways to determine whether a square matrix is diagonalizable and diagonalize it if the matrix is diagonalizable.

(a) Note that in Section 1.3, the eigenvectors are the columns of the output matrices. We can simply put them together to get $\boldsymbol{P}$:

```
>> V1 = null(-3*eye(6) - A, 'r');
>> V2 = null(2*eye(6) - A, 'r');
>> V3 = null(1*eye(6) - A, 'r');
```

Then

```
>> P = [V1 V2 V3]
P =  0   0   0   -1   -1   0
     1   0   0    0    0   0
     0   0   0    0    0  -1
     0   0   0    1    0   1
     0   1   0    0    0   0
     0   0   1    0    1   0
```

In general, we can always generate a matrix $\boldsymbol{P}$ whose columns are linearly independent eigenvectors of $\boldsymbol{A}$ regardless whether $\boldsymbol{A}$ is diagonalizable. But $\boldsymbol{A}$ is diagonalizable if and only if there are enough linearly independent eigenvectors, i.e., $\boldsymbol{P}$ is a square matrix.

(b) There is a direct way to obtain $\boldsymbol{P}$ using `eig`. We shall

(i) Declare the matrix $\boldsymbol{A}$ as a symbolic object, say $\boldsymbol{A}_1$.

```
>> A1 = sym(A);
```

(ii) Declare the output matrices $\boldsymbol{P}_1$ and $\boldsymbol{D}_1$:

```
>> [P1,D1] = eig(A1)
P1 =  [0,  -1,  -1,  0,  0,  0]
      [0,   0,   0,  1,  0,  0]
      [-1,  0,   0,  0,  0,  0]
      [1,   1,   0,  0,  0,  0]
      [0,   0,   0,  0,  1,  0]
      [0,   0,   1,  0,  0,  1]

D1 =  [1,   0,   0,  0,  0,  0]
```

```
[0,  2,  0,  0,  0,  0]
[0,  0,  2,  0,  0,  0]
[0,  0,  0,  -3,  0,  0]
[0,  0,  0,  0,  -3,  0]
[0,  0,  0,  0,  0,  -3]
```

Note that the linearly independent eigenvectors of $\boldsymbol{A}_1$ (= $\boldsymbol{A}$) are still represented by the columns of $\boldsymbol{P}_1$ and the diagonal entries of the diagonal matrix $\boldsymbol{D}_1$ are the corresponding eigenvalues. Note that $\boldsymbol{P}_1$ (respectively $\boldsymbol{D}_1$) can be obtained from $\boldsymbol{P}$ (respectively $\boldsymbol{D}$) by permuting the columns. Again, $\boldsymbol{A}$ is diagonalizable only if $\boldsymbol{P}_1$ is a square matrix.

Remark. Note that the matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ are not unique:

(i) Eigenvectors may be replaced by nonzero constant multiples.
(ii) If an eigenspace has dimension ≥ 2, the eigenvectors can be recombined through linear combinations.
(iii) The columns of $\boldsymbol{P}$ may be permutated; then the diagonal entries of $\boldsymbol{D}$ shall be permutated accordingly.

2.2. **Powers of Diagonalizable Matrices.** Let $\boldsymbol{A}$ be a diagonalizable matrix. Let $\boldsymbol{P}$ be an invertible matrix such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$ is a diagonal matrix. For any positive integer m,

$$\boldsymbol{A}^m = \boldsymbol{P}\boldsymbol{D}^m\boldsymbol{P}^{-1}.$$

2.2.1. *Markov Chain.* A **Markov process** is a system which has a finite set of states $1, \ldots, n$. At any instant the system is in a definite state and over a fixed period of time it changes to another state.

Example. In a large city, the soft-drink market was 100% dominated by brand A. Four months ago, two new brands B and C were introduced to the market. According to the market research, for each month, about 1% and 2% of the customers of brand A switch to brands B and C respectively; about 1% and 2% of the customers of brand B switch to brands A and C respectively; and about 2% and 2% of the customers of brand C switch to brands A and B respectively.

Let a_n, b_n, c_n be the market share of brand A, B, C, respectively in the n^{th} month. Then

$$\begin{cases} a_{n+1} = 0.97a_n + 0.01b_n + 0.02c_n, \\ b_{n+1} = 0.01a_n + 0.97b_n + 0.02c_n, \\ c_{n+1} = 0.02c_n + 0.02b_n + 0.96c_n. \end{cases}$$

Let $\boldsymbol{M} = \begin{pmatrix} 0.97 & 0.01 & 0.02 \\ 0.01 & 0.97 & 0.02 \\ 0.02 & 0.02 & 0.96 \end{pmatrix}$ and $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$. Then $\boldsymbol{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{M}\boldsymbol{x}_n = \boldsymbol{x}_{n+1}$. Using the same argument as in Example 6.1.1 of the textbook,

$$\boldsymbol{x}_n = \boldsymbol{M}^n \boldsymbol{x}_0.$$

Input $\boldsymbol{M}$ and $\boldsymbol{x}_0$ in MATLAB:

```
>> M = [0.97 0.01 0.02; 0.01 0.97 0.02; 0.02 0.02 0.96];
>> x0 = [1; 0; 0];
```

We can answer the following questions:

(i) Compute the present market share of the three brands of soft drink.

This is four months after brands B and C are introduced. So we are asked for $\boldsymbol{x}_4 = \boldsymbol{M}^4 \boldsymbol{x}_0$:

```
>> x4 = M^4 * x0
x4 =  0.8881
      0.0388
      0.0731
```

(ii) Compute the market shares of the three brands of soft drink one year after brand B and C are introduced.

We are asked for $\boldsymbol{x}_{12} = \boldsymbol{M}^{12} \boldsymbol{x}_0$:

```
>> x12 = M^12 * x0
x12 =  0.7190
       0.1063
       0.1747
```

(iii) Diagonalize $\boldsymbol{M}$ and get a formula for $\boldsymbol{x}_n$:

```
>> M1 = sym(M);
>> [P D] = eig(M1)
P =  [1,  -1/2,  -1]
     [1,  -1/2,   1]
     [1,     1,   0]

D =  [1,     0,       0]
     [0,  47/50,      0]
     [0,     0,   24/25]
```

Then $\boldsymbol{x}_n = \boldsymbol{M}^n \boldsymbol{x}_0 = \boldsymbol{P}\boldsymbol{D}^n \boldsymbol{P}^{-1} \boldsymbol{x}_0$:

```
>> syms n
>> x(n) = P * D^n * inv(P) * x0
```

```
x(n) =  (24/25)^n/2 + (47/50)^n/6 + 1/3
        (47/50)^n/6 - (24/25)^n/2 + 1/3
        1/3 - (47/50)^n/3
```

We conclude that

$$\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(0.96)^n + \frac{1}{6}(0.94)^n + \frac{1}{3} \\ \frac{1}{6}(0.94)^n - \frac{1}{2}(0.96)^n + \frac{1}{3} \\ -\frac{1}{3}(0.94)^n + \frac{1}{3} \end{pmatrix}.$$

(iv) Use part (iii) to estimate the market shares in the long run if the trend continues. Will the market shares stabilize in the long run?

Using basic concepts in limits, when n is large ($n \to \infty$), 0.96^n and 0.94^n both tend to 0 ($0.96^n \to 0$ and $0.94^n \to 0$); and we have

$$\boldsymbol{x}_{\text{long run}} \approx \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Formally,

$$\lim_{n\to\infty} \boldsymbol{x}_n = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \qquad \text{or} \qquad n \to \infty \Rightarrow \boldsymbol{x}_n \to \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

In MATLAB, we may use

```
>>  limit(x(n), n, inf)
ans =  1/3
       1/3
       1/3
```

2.2.2. *Recursive Sequences.* A sequence (a_n) may be defined recursively by

$$a_0 = a, \quad a_1 = b, \quad \text{and} \quad a_{n+1} = \alpha a_{n-1} + \beta a_n, \quad n \geq 1.$$

One may use matrix to find a general formula for a_n.

Note that

$$\begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ \alpha a_n + \beta a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$$

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$. Then $\boldsymbol{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\boldsymbol{x}_{n+1} = \boldsymbol{A}\boldsymbol{x}_n$. Consequently,

$$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0.$$

It reduces to the problem of finding powers of $\boldsymbol{A}$.

Example. A sequence (b_n) is defined by

$$b_0 = 0, \quad b_1 = 1, \quad \text{and} \quad b_{n+1} = 3b_{n-1} + 2b_n, \quad n \geq 1.$$

Let $\boldsymbol{y}_n = \begin{pmatrix} b_n \\ b_{n+1} \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$. Then $\boldsymbol{y}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{y}_n = \boldsymbol{B}^n \boldsymbol{y}_0$.

(i) Define $\boldsymbol{y}_0$ and input $\boldsymbol{B}$ in MATLAB as symbolic object:

```
>> y0 = [0; 1];
>> B = sym([0 1; 3 2]);
```

(ii) Find $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{D}$. In this example, $\boldsymbol{B}$ is diagonalizable.

```
>> [P D] = eig(B)
P =  [-1,  1/3]
     [1,     1]
D =  [-1,    0]
     [0,     3]
```

(iii) $\boldsymbol{y}_n = \boldsymbol{B}^n \boldsymbol{y}_0 = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}\boldsymbol{y}_0$:

```
>> y(n) = P * D^n * inv(P) * y0
y(n) =  3^n/4 - (-1)^n/4
        (-1)^n/4 + (3*3^n)/4
```

(iv) By definition, b_n is the 1st component of $\boldsymbol{y}_n$: $b_n = \frac{1}{4}[3^n - (-1)^n]$.

3. Exercises

1. For each of the following matrices,

$$\boldsymbol{B} = \begin{pmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ -5 & 0 & -5 & -5 & 0 & -3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{C} = \begin{pmatrix} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ -5 & 0 & -6 & -5 & 0 & -3 \end{pmatrix},$$

(i) find the characteristic polynomial,

(ii) find the eigenvalues,

(iii) find the bases for eigenspaces,

(iv) determine if the matrix is diagonalizable.

2. Use MATLAB to solve Exercises 1, 2, 6, 7(a)(b), 10, 11, 16(b), 17, 18, 20, 21 in Chapter 6 of the textbook.