Tutorial 9

Exercise 5

32. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be an orthogonal matrix of order n and let $\boldsymbol{u}$, $\boldsymbol{v}$ be any two vectors in $\mathbb{R}^n$. Show that

(a) $||\boldsymbol{u}|| = ||\boldsymbol{A}\boldsymbol{u}||$;

(b) $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{A}\boldsymbol{v})$; and

(c) the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to the angle between $\boldsymbol{A}\boldsymbol{u}$ and $\boldsymbol{A}\boldsymbol{v}$.

33. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be an orthogonal matrix of order n and let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ be a basis for $\mathbb{R}^n$.

(a) Show that $T = \{\boldsymbol{A}\boldsymbol{u_1}, \boldsymbol{A}\boldsymbol{u_2}, \ldots, \boldsymbol{A}\boldsymbol{u_n}\}$ is a basis for $\mathbb{R}^n$.

(b) If S is orthogonal, show that T is orthogonal.

(c) If S is orthonormal, is T orthonormal?

Exercise 6

1. For each of the following, (i) find the characteristic equation of $\boldsymbol{A}$; (ii) find all the eigenvalues of $\boldsymbol{A}$; and (iii) find a basis for the eigenspace associated with each eigenvalue of $\boldsymbol{A}$.

$$\text{(h)} \quad \boldsymbol{A} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 2 & -1 & 0 \end{pmatrix},$$

$$\text{(j)} \quad \boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. Let $\boldsymbol{A}$ be a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{A}$.

(a) Show that if λ is an eigenvalue of $\boldsymbol{A}$, then $\lambda = 0$ or 1.

(b) Find all 2×2 matrices $\boldsymbol{A}$ such that $\boldsymbol{A}^2 = \boldsymbol{A}$ and $\boldsymbol{A}$ has eigenvalues 0 and 1.

8. Let $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ be a basis for $\mathbb{R}^n$ and let $\boldsymbol{A}$ be an $n \times n$ matrix such that $\boldsymbol{A}\boldsymbol{u_i} = \boldsymbol{u_{i+1}}$ for $i = 1, 2, \ldots, n-1$ and $\boldsymbol{A}\boldsymbol{u_n} = \boldsymbol{0}$. Show that the only eigenvalue of $\boldsymbol{A}$ is 0 and find all the eigenvectors of $\boldsymbol{A}$.