Section 1.1

Line

Linear Systems and their solutions

Objective

- What is a linear equation and a linear system?
- What is a general solution of a LE/LS?
- What is the geometrical interpretation?
- How to find a general solution of a LE?

What is a linear equation?

Discussion 1.1.1

A line in the xy-plane

is represented algebraically by a linear equation in the variables x and/or y

e.g. $x + y = 1$

$x = 2$

$y = -3$

General form $ax + by = c$

a, b, c represent some real numbers

a and b are not both zero

If not it will be a point

What is a linear equation?

Definition 1.1.2

A linear equation in 3 variables $ax + by + cz = d$

geometrical meaning: plane

A linear equation in 4 variables $ax + by + cz + dw = e$

geometrical meaning: none

A linear equation in n variables

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Indexed Notation

variables: $x_1, x_2, \ldots, x_n$ also called the unknowns

constants: $a_1, a_2, \ldots, a_n$ and $\boxed{b}$ constant term

Coeffiencts

Example 1.1.3.1

The following are (specific) linear equations:

a) $x + 3y = 7$

b) $x_1 + 2x_2 + 2x_3 + x_4 = x_5$

c) $y = x - 0.5z + 4.5$

d) $x_1 + x_2 + \cdots + x_n = 1$

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b}$$

standard form

What is a linear equation?

Example 1.1.3.2

The following are not linear equations:

cross term (multiplied together)
hyperbola

a) $\boxed{xy = 2}$

linear in x and y if converted

b) $\underbrace{\sin(\theta)}_{x} + \underbrace{\cos(\varphi)}_{y} = 0.2$ not linear in θ and φ

c) $\boxed{x_1^{\boxed{2}}} + x_2^2 + \cdots + x_n^2 = 1$ square terms

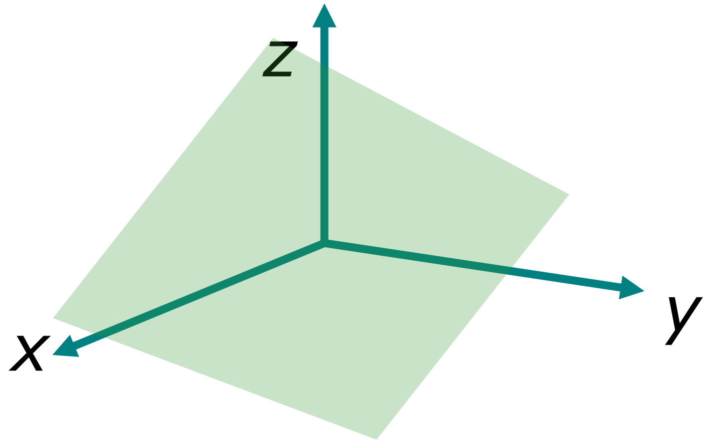
d) $x = \boxed{e^y}$ function of y

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Example 1.1.3.3

$ax + by + cz = d$ not all a, b, c are zero

represents a plane in the three dimensional space



If a, b, c all non-zero, the plane is "slanting" not parallel to any of the 3 coordinate plane

If some of a, b, c is zero, the plane is parallel to some axis

If d = 0, the plane passes through origin

What is a general solution of a LE?

Definition 1.1.4

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

variables: $x_1, x_2, \ldots, x_n$
constants: $a_1, a_2, \ldots, a_n, b$

real numbers $s_1, s_2, \ldots, s_n$

If the equation is satisfied,

$\boxed{x_1 = s_1,\ x_2 = s_2,\ \ldots,\ x_n = s_n}$ a solution of the linear equation

A linear equation has (infinitely) many solutions unless n = 1

The set of all solutions: solution set

An expression that represents all solutions: general solution

algebric expression to contain all the solutions

How to find a general solution of a LE?

Example 1.1.5.1

$4x - 2y = 1$

some solutions

$$\begin{cases} x = 1 \\ y = 1.5 \end{cases} \qquad \begin{cases} x = 1.5 \\ y = 2.5 \end{cases} \qquad \begin{cases} x = -1 \\ y = -2.5 \end{cases}$$

infinitely many solutions

- pick a random value for x
- substitute this value into the equation
- solve the value of y

general solution

- set x = t (parameter)
- substitute t for x in the equation
- express y in terms of t

represents all the different kinds of answers

$$\boxed{\begin{cases} x = t \\ y = 2t - \frac{1}{2} \end{cases}}$$

How to find a general solution of a LE?

Example 1.1.5.1

$4x - 2y = 1$

some solutions $\begin{cases} x = 1 \\ y = 1.5 \end{cases}$ $\begin{cases} x = 1.5 \\ y = 2.5 \end{cases}$ $\begin{cases} x = -1 \\ y = -2.5 \end{cases}$

general solution not a fixed answer - as long as can describe possible solutions

- set x = t (parameter)
- substitute t for x in the equation
- express y in terms of t

$$\begin{cases} x = t \\ y = 2t - \frac{1}{2} \end{cases}$$

general solution (alternative)

- set y = s (parameter)
- substitute s for y in the equation
- express x in terms of s

$$\begin{cases} x = \frac{1}{2}s + \frac{1}{4} \\ y = s \end{cases}$$

How to find a general solution of a LE?

Example 1.1.5.2

Finding the general solution for 3 variables

$$x_1 - 4x_2 + 7x_3 = 5$$

general solution

different names

- set $x_2 = s$ and $x_3 = t$ (parameters)
- substitute s for x_2 and t for x_3 in the equation
- express x_1 in terms of s and t

$$\begin{cases} x_1 = 5 + 4s - 7t \\ x_2 = s \\ x_3 = t \end{cases}$$

n variables means n-1 parameters

Example 1.1.5.3(a)

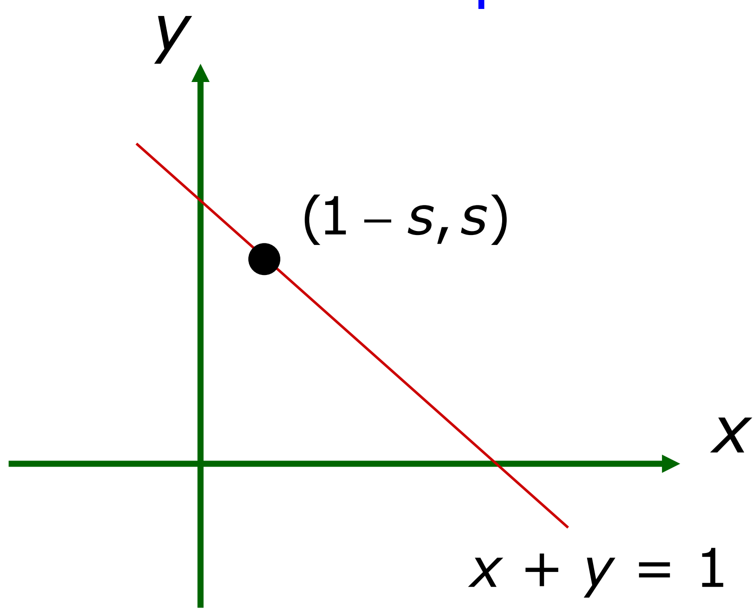
equation
$x + y = 1$

represents
a line in *xy*-plane

general solutions $\begin{cases} x = 1 - s \\ y = s \end{cases}$

Rewrite: $(x, y) = (1 - s, s)$

represents coordinates
of points on the line



Example 1.1.5.3(b)

equation

$x + y = 1$

regarded as

$x + y + 0z = 1$

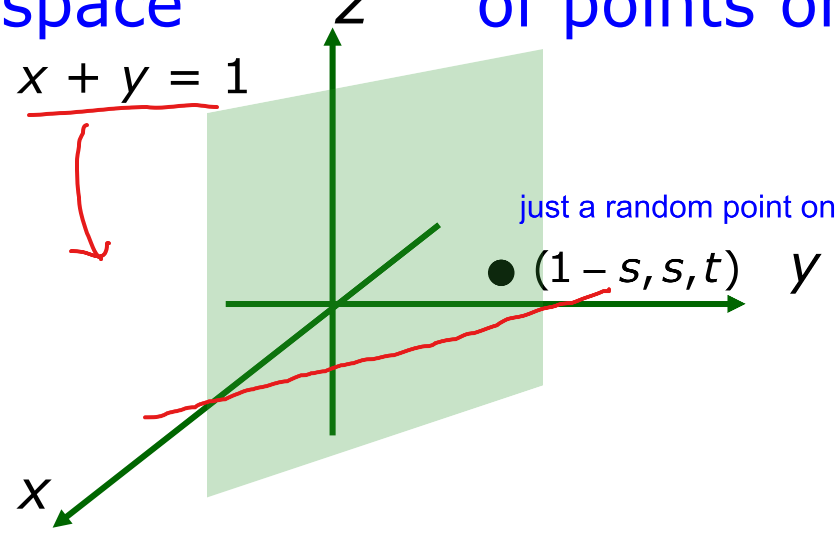
represents

a plane in 3D space

general solutions
$$\begin{cases} x &= 1 - s \\ y &= s \\ z &= t \end{cases}$$

Rewrite: $(x, y, z) = (1 - s, s, t)$

represents coordinates of points on the plane



What is a linear system?

Definition 1.1.6

Zero System
VS
Non-zero system

A system of linear equations (or a linear system)

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{cases}$$

Linear systems = Putting a few linear equations together

but must all have the same variables

m linear equations

variables

n $\boxed{\text{variables } x_1, x_2, \dots, x_n}$

$\boxed{a_{11}, a_{12}, \dots, a_{\boxed{mn}}}$ and $\boxed{b_1, b_2, \dots, b_m}$ are real constants

coefficients

double indices
m = tells us which equation
n = tells us which variable

constants

What is a general solution of a LS?

Definition 1.1.6

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

real numbers $s_1, s_2, \dots, s_n$

If all the equations are satisfied,

$$\boxed{x_1 = s_1, \ x_2 = s_2, \ \dots, \ x_n = s_n}$$

a solution of the linear system

solution set and general solution of the system are defined similarly as before.

Solution of a LS

Example 1.1.7

$$\begin{cases} 4x_1 - x_2 + 3x_3 = -1 \\ 3x_1 + x_2 + 9x_3 = -4 \end{cases}$$

$x_1 = 1,\ x_2 = 2,\ x_3 = -1$ is a solution

$x_1 = 1,\ x_2 = 8,\ x_3 = 1$ is not a solution

does not sastisfy BOTH equations

Remark 1.1.8

Not all systems of linear equations have solutions.

$$\begin{cases} x + y = 4 \\ 2x + 2y = 6 \end{cases}$$

This system has no solution both equtaions contridict with each other

every equations puts in a new contraints on the variable

more equations will reduce the number of solutions

Definition 1.1.9

A system of linear equations

no solution → **inconsistent system**

$$\begin{cases} x + y = 4 \\ 2x + 2y = 6 \end{cases}$$

at least one solution → **consistent system**

$$\begin{cases} 4x_1 - x_2 + 3x_3 = -1 \\ 3x_1 + x_2 + 9x_3 = -4 \end{cases}$$

Remark 1.1.10

IMPORTANT FEATURE OF LINEAR SYSTEM

Every system of linear equations has either

- no solution
- exactly one solution or
- infinitely many solutions

Chapter 2 will explain

Discussion 1.1.11.1

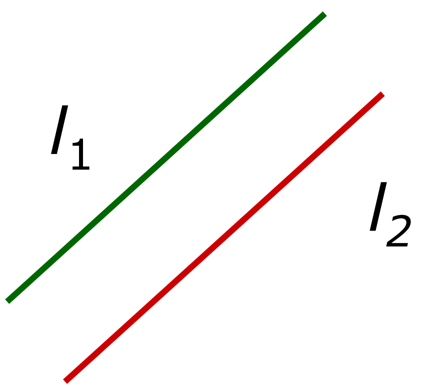
In the xy-plane, the system

$$\begin{cases} a_1x + b_1y = c_1 & (l_1) \\ a_2x + b_2y = c_2 & (l_2) \end{cases}$$

represent two straight lines.

a) l_1 and l_2 are parallel lines

The system has no solution



Discussion 1.1.11.1

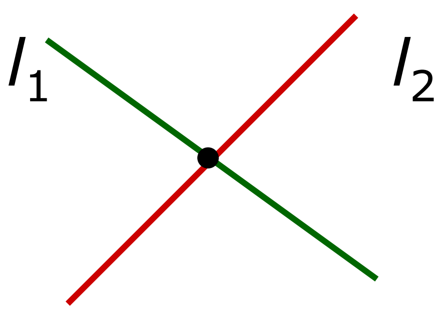
In the xy-plane, the system

$$\begin{cases} a_1 x + b_1 y = c_1 & (l_1) \\ a_2 x + b_2 y = c_2 & (l_2) \end{cases}$$

represent two straight lines.

b) l_1 and l_2 are not parallel lines.

The system has exactly one solution



Geometrical interpretation

Discussion 1.1.11.1

In the xy-plane, the system

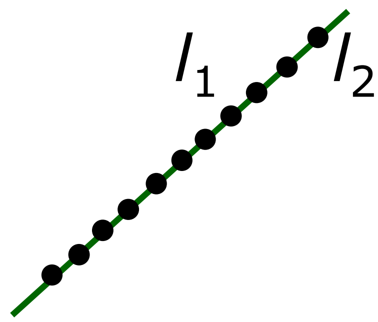
$$\begin{cases} a_1x + b_1y = c_1 & (l_1) \\ a_2x + b_2y = c_2 & (l_2) \end{cases}$$

represent two straight lines.

c) l_1 and l_2 are the same lines.

The system has infinitely many solutions

all the points are intersections



Geometrical interpretation

Discussion 1.1.11.2

In the xyz-space, the system

2 equations
3 variables

$$\begin{cases} a_1x + b_1y + c_1z = d_1 & (p_1) \\ a_2x + b_2y + c_2z = d_2 & (p_2) \end{cases}$$

represents two planes.

The system has either
no solution or infinitely many solutions.
parallel planes
the plane will intersect along a line

Section 1.2

Elementary Row Operations

Objective

- What are the three elementary row operations?
- How to perform ERO on an augmented matrix?
- What is meant by row equivalence between two augmented matrices?

What is an augmented matrix of a LS?

Definition 1.2.1

linear system

m equations
n variables

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{cases}$$

$$\Downarrow$$

augmented matrix

dont show the variables

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

rectangular array

m rows
n+1 columns

n columns of variables
+1 for the constant

What is an augmented matrix of a LS?

Example 1.2.2

Consider the system of linear equations:

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0 \end{cases}$$

The augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

take note of the sign

What are the three elementary row operations?

Definition 1.2.4

augmented matrix $\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$

Consider the following three operations on the augmented matrix:

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

These are called elementary row operations.

How to perform elementary row operations?

Definition 1.2.4

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

Multiply first row by 3

$$\left(\begin{array}{ccc|c} 3 & 3 & 6 & 27 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

Add 2 times
of first row
to second row

step 1: 1 1 2 | 9 => 2 2 4 | 18
step 2: add to the second row

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 4 & 6 & 1 & 19 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

only second row is changed

Interchange second and third rows

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 3 & 6 & -5 & 0 \\ 2 & 4 & -3 & 1 \end{array}\right)$$

Why perform ERO ?

Discussion 1.2.3

Objective is to make equations simpler to solve

Elementary row operations

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

These are the basic steps for solving linear system.

Correspond to the following action on the system

1. Multiply an equation by a nonzero constant.
2. Interchange two equations.
3. Add a multiple of one equation to another equation.

similar to normal steps in other maths

Why perform ERO ?

Example 1.2.5

example of: adding a multiple of 1 row to another row

x-2 - 2 - 2 - 6 = 0

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

Add -2 times of Equation (1) to Equation (2) to obtain Equation (4).

- 2 - 2 - 6 = 0

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

This is equivalent to adding -2 times of the first row of the matrix to the second row.

Why perform ERO ?

Example 1.2.5

x - 3   - 3   - 3   - 9   =   0

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

Add -3 times of Equation (1) to Equation (3) to obtain Equation (5).

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ \quad 6y - 9z = 3 & (5) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$$

This is equivalent to adding -3 times of the first row of the matrix to the third row.

Why perform ERO ?

Example 1.2.5

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ \quad 6y - 9z = 3 & (5) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$$

Add 6/4 times of Equation (4) to Equation (5) to obtain Equation (6).

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ \quad - 15z = 9 & (6) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

This is equivalent to adding 6/4 times of the second row of the matrix to the third row.

Why perform ERO ?

Example 1.2.5

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad\ - 4y - 4z = 4 & (4) \\ \quad\quad\quad\ - 15z = 9 & (6) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

allows for backwards substitution

row echelon form

By Equation (6), $z = -3/5.$

Substituting $z = -3/5$ into Equation (4),

$-4y - 4(-3/5) = 4 \quad \Leftrightarrow \quad y = -2/5.$

Substituting $y = -2/5$ and $z = -3/5$ into Equation (1)

$x + (-2/5) + 3(-3/5) = 0 \quad \Leftrightarrow \quad x = 11/5.$

solution for the linear system

Section 1.2

Elementary Row Operations

Objective

- What is meant by row equivalence between two augmented matrices?

What is row equivalence ?

Definition 1.2.6

Two augmented matrices are row equivalent (to each other)

something is equal to something

if one can be obtained from the other by a series of elementary row operations.

doesnt matter how many operations

In example 1.2.5,

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

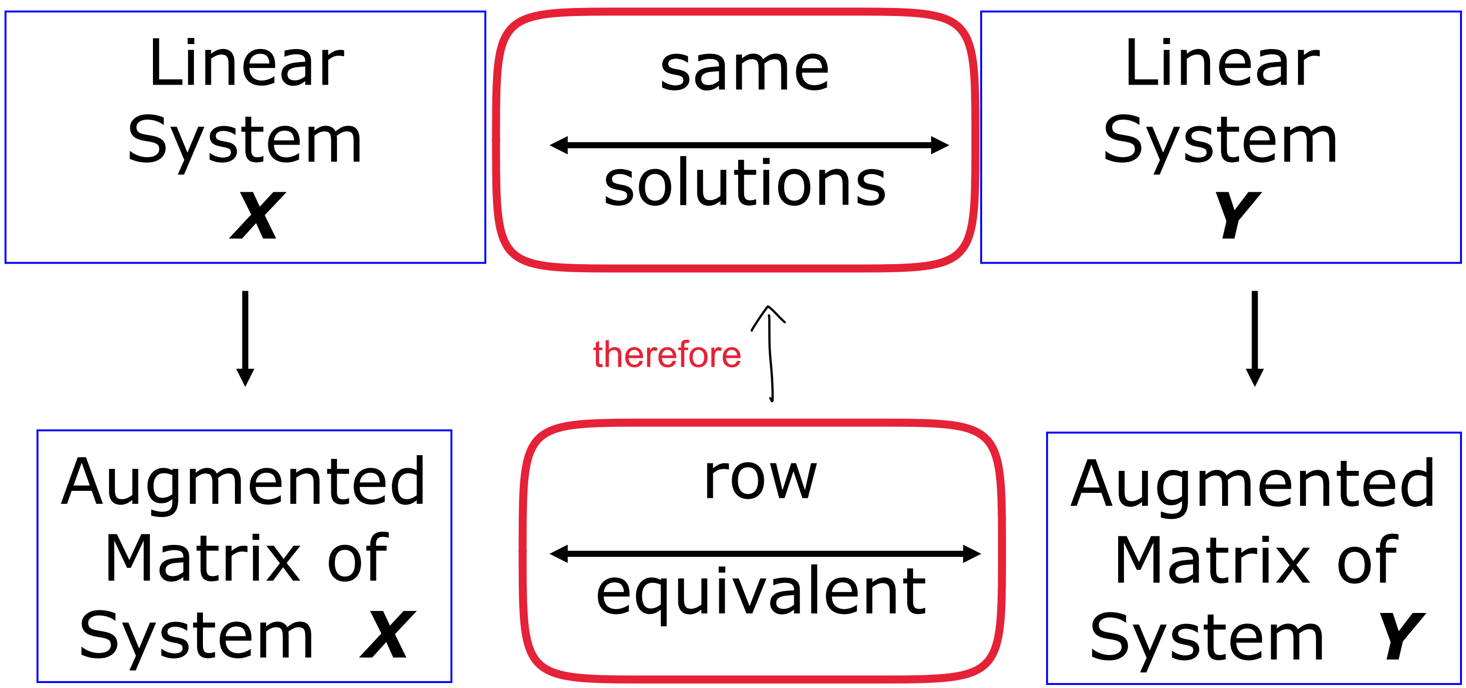
Any 2 of the augmented matrices are row equivalent

1st and 4th are row equivalent

What can we say about 2 row equivalent LS ?

Theorem 1.2.7

If augmented matrices of two linear systems are row equivalent, then the two systems have the same set of solutions.



What can we say about 2 row equivalent LS ?

Example 1.2.8

All augmented matrices in Example 1.2.5 are row equivalent.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

So all systems of linear equations in Example 1.2.5 have the same solution.

1st
$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases}$$

2nd
$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases}$$

3rd
$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ 6y - 9z = 3 & (5) \end{cases}$$

4th
$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ -15z = 9 & (6) \end{cases}$$

Remark 1.2.9

means there is a proof

To see why Theorem 1.2.7 is true,
we only need to check that
every elementary row operation applied to an augmented matrix
will not change the solution set of the corresponding linear system.

x+y=1 > 2x+2y=2
relationship of x&y are still similar

1. Multiply a row by a nonzero constant
2. Interchange two rows

interchanging the position of 2 equation
will not change solution

3. Add a multiple of one row to another

1: multiplying equation by constant = no change
2: adding together = no change

Section 1.3

Row-Echelon Forms

Objective

- How to identify a row-echelon form (REF) and a reduced row-echelon form (RREF)?
- How to use REF / RREF to get solutions of linear system?
- How to tell the number of solutions from REF?

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times \qquad \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \checkmark$$

How to identify REF?

Definition 1.3.1

An augmented matrix is said to be in **row-echelon form** if it has the following 2 properties:

always be in the augmented matrix

1. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

just move all the 0 rows below

$$\left(\begin{array}{cccc|c} * & * & \cdots & \cdots & * \\ \vdots & \vdots & & & \vdots \\ * & * & \cdots & \cdots & * \\ \hline 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{nonzero rows} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \text{zero rows (if any)} \end{array}$$

How to identify REF?

$$\begin{pmatrix} 0 & 0 & 0 & \underline{1} & 3 \\ 0 & \underline{1} & \underline{2} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times \qquad \begin{pmatrix} 0 & \underline{1} & \underline{2} & 0 & 1 \\ 0 & 0 & 0 & \underline{1} & \underline{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \checkmark$$

Definition 1.3.1

2. In any two successive $\boxed{\text{non-zero}}$ rows, the first nonzero number in the lower row occurs farther to the right than the first nonzero number in the higher row.

essentially is to form the staircase

$$\left(\begin{array}{cccccc|c} \underline{0 \quad 0} & \otimes & * & \cdots & \cdots & & * \\ \underline{0 \quad \cdots \quad \cdots} & & \otimes & * & \cdots & & \cdots \end{array}\right) \begin{matrix} \\ \end{matrix}\Big\}\ \text{two successive rows}$$

$\boxed{\text{leading entries}}$ leading entries no need be same

numbers after leading entry are not essential

How to identify REF?

Definition 1.3.1

Combining properties 1 and 2:

width of stairs does not matter

BUT
height of stairs matter

$$\left(\begin{array}{ccccccc|c} 0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & \otimes & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \text{nonzero rows} \\ \left.\begin{array}{c} \\ \end{array}\right\} \text{zero rows (if any)} \end{array}$$

everything below the staircase is 0 row

leading entries

This is a row-echelon form (REF)

How to identify REF?

Definition 1.3.1

columns that contain pivot points called pivot columns

$$\left(\begin{array}{cccccc|c|c} 0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & \otimes & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right)$$

nonzero rows

zero rows (if any)

leading entries also called pivot point

How to identify RREF?

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \checkmark$$

Definition 1.3.1

An augmented matrix is said to be in reduced row-echelon form (RREF)

special type of REF

so must already have REF properties

if it is in row-echelon form and has the following properties:

3. The leading entry of every nonzero row is 1.

$$\left(\begin{array}{ccccccc|c} 0 & 1 & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & 1 & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right)$$

How to identify RREF?

$$\begin{pmatrix} 0 & 1 & \boxed{2} & 0 & 0 \\ 0 & 0 & 1 & \boxed{3} & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times \qquad \begin{pmatrix} 0 & 1 & \boxed{0} & \boxed{0} & 0 \\ 0 & 0 & 1 & \boxed{0} & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \checkmark$$

Definition 1.3.1

4. In each pivot column, except the pivot point, all other entries are zeros.

main focus is above the pivot point/leading entry must be 0

non pivot columns no restrictions

$$\left(\begin{array}{cccccccc|c} 0 & 1 & \cdots & 0 & * & 0 & \cdot & * & * \\ 0 & 0 & 0 & 1 & * & \vdots & \cdot & * & * \\ 0 & \vdots & \cdots & 0 & \ddots & 0 & \cdot & * & * \\ 0 & \vdots & \cdots & \vdots & 0 & 1 & & * & * \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdot & 0 & 0 \end{array}\right)$$

pivotal columns

How to identify (R)REF?

Remark 1.3.2

In this module

Properties 1 + 2: REF

Properties 1 + 2 + 3 + 4: RREF

In some textbooks

Properties 1 + 2 + 3: REF

Properties 1 + 2 + 3 + 4: RREF

How to use REF / RREF to get solutions?

Discussion 1.3.4

If the augmented matrix of a linear system is in REF or RREF,

we can get the solutions of the system easily.

$$\left(\begin{array}{ccccccc|c} 0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & \otimes & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccccccc|c} 0 & 1 & \cdots & 0 & * & 0 & * & * \\ 0 & \cdots & 0 & 1 & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & 0 & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right)$$

REF RREF

How to use REF / RREF to get solutions?

Example 1.3.5.1

$$\begin{array}{cccc} x1 & x2 & x3 & \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \end{array}$$

= only 1 solution

RREF

$$\begin{cases} x_1 & & & = 1 \\ & x_2 & & = 2 \\ & & x_3 & = 3 \end{cases}$$

index notation

The system has only one solution:

$$x_1 = 1,\ x_2 = 2,\ x_3 = 3.$$

standard form

$\therefore$ RREF almost will providce immediate answers

How to use REF / RREF to get solutions?

Example 1.3.5.2

more variables than equations

means need to intro parameters

$$\left(\begin{array}{ccccc|c} 0 & 2 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right)$$

REF

$$\begin{cases} 2x_2 + 2x_3 + x_4 - 2x_5 = 2 \\ x_3 + x_4 + x_5 = 3 \\ 2x_5 = 4 \end{cases}$$

introduce parameters
no. of variables - no. of equations

non-pivot columns

use variables that correspond to non-pivot column as parameters

x_1 = free parameter s

x_4 = free parameter t

$x_5 = 2$

$x_3 = 1 - t$

$x_2 = 2 + (1/2)t$

The general solution is

$$\begin{cases} x_1 = s \\ x_2 = 2 + \frac{1}{2}t \\ x_3 = 1 - t \\ x_4 = t \\ x_5 = 2 \end{cases}$$

back substitution

The system has infinitely many solutions.

How to use REF / RREF to get solutions?

Example 1.3.5.3

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & -2 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} x_1 - x_2 + 3x_4 = -2 \\ x_3 + 2x_4 = 5 \\ \cancel{0 = 0} \end{cases}$$

RREF

non-pivot columns

need to intro 2 parameters

The general solution is

$$\begin{cases} x_1 = -2 + s - 3t \\ x_2 = s \\ x_3 = 5 - 2t \\ x_4 = t \end{cases}$$

The system has infinitely many solutions

Example 1.3.5.4

$$\boxed{\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}$$

zero system

no leading entry so everything is a non-pivot column

The general solution is

$$\begin{cases} x_1 = r \\ x_2 = s \\ x_3 = t \end{cases}$$

The system has infinitely many solutions

How to use REF / RREF to get solutions?

Example 1.3.5.5

$$\left(\begin{array}{cc|c} 3 & 1 & 4 \\ 0 & 2 & 1 \\ \boxed{0 \quad 0} & & \boxed{1} \end{array}\right)$$

$$\begin{cases} 3x_1 + x_2 = 4 \\ 2x_2 = 1 \\ \boxed{0 = 1} \end{cases}$$

usually is after performing ERO on the original equations

This system is $\boxed{\text{inconsistent}}$, i.e. no solution.

Recall:
Any linear system has
- no solution
- exactly one solution
- infinitely many solutions

Section 1.4

Gaussian Elimination

Objective

- What are Gaussian elimination and Gauss-Jordan elimination?
- How to use GE / GJE to reduce an augmented matrix to a REF / RREF ?

Row echelon form of augmented matrix

Definition 1.4.1

Gaussian Elimination is an algorithm
to reduce an augmented matrix
to a row-echelon form
by using elementary row operations.

Carl Friedrich Gauss
(1777-1855)

$$\begin{pmatrix} \text{Augmented} \\ \text{matrix} \end{pmatrix} \xrightarrow{\text{e.r.o.}} \begin{pmatrix} \text{row - echelon} \\ \text{form} \end{pmatrix}$$

Systematic process

information for the computer to do

How to use GE to reduce a matrix to REF?

Algorithm 1.4.2

Step 1: Locate the leftmost column that does not consist entirely of zero.

Example A

$$\begin{pmatrix} 0 & 3 & \ldots & \ldots \\ 1 & -2 & \ldots & \ldots \\ 4 & 0 & \ldots & \ldots \end{pmatrix}$$

the first nonzero column

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \ldots \\ 0 & 2 & -3 & \ldots \\ 0 & -1 & 6 & \ldots \end{pmatrix}$$

the first nonzero column

Algorithm 1.4.2

Step 2: Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

Example A

$$\begin{pmatrix} 0 & 3 & \ldots & \ldots \\ 1 & -2 & \ldots & \ldots \\ 4 & 0 & \ldots & \ldots \end{pmatrix}$$

Interchange the 1st row with the 2nd row.

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \ldots \\ 0 & 2 & -3 & \ldots \\ 0 & -1 & 6 & \ldots \end{pmatrix}$$

No action is needed

How to use GE to reduce a matrix to REF?

Algorithm 1.4.2

Step 3: For each row below the top row, add a suitable multiple of the top row to it so that the entry below the leading entry of the top row becomes zero.

Example A

$$\begin{pmatrix} 1 & -2 & \cdots & \cdots \\ 0 & 3 & \cdots & \cdots \\ 4 & 0 & \cdots & \cdots \end{pmatrix}$$

Add – 4 times of the 1st row to the 3rd row so that the entry marked by becomes 0.

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \cdots \\ 0 & 2 & -3 & \cdots \\ 0 & -1 & 6 & \cdots \end{pmatrix}$$

Add – 2/3 times of the 1st row to the 2nd row so that the entry marked by becomes 0.

Add 1/3 times of the 1st row to the 3rd row so that the entry marked by becomes 0.

Algorithm 1.4.2

Step 4: Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains.

Example A

$$\begin{pmatrix} \boxed{1 & -2 & \ldots & \ldots} \\ 0 & 3 & \ldots & \ldots \\ 0 & 8 & \ldots & \ldots \end{pmatrix}$$

Cover the 1st row and work on the remaining rows.

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \ldots \\ 0 & 0 & -11/3 & \ldots \\ 0 & 0 & 19/3 & \ldots \end{pmatrix}$$

Cover the 1st row and work on the remaining rows.

Continue in this way until the entire matrix is in row-echelon form.

What is Gauss-Jordan elimination?

Algorithm 1.4.3

zam ERO until become RREF

Gauss-Jordan Elimination is an algorithm to reduce an augmented matrix to the reduce row-echelon form by using elementary operations.

$$\overbrace{\underbrace{\begin{pmatrix}\text{Augmented}\\ \text{matrix}\end{pmatrix} \xrightarrow{\text{e.r.o.}} \begin{pmatrix}\text{row - echelon}\\ \text{form}\end{pmatrix}}_{\text{Gaussian Elimination}} \xrightarrow{\text{e.r.o.}} \begin{pmatrix}\text{reduce row}\\ \text{echelon form}\end{pmatrix}}^{\text{Gauss-Jordan Elimination}}$$

Algorithm 1.4.3

Given an augmented matrix, use Algorithm 1.4.1 to reduce it to a row-echelon form.

Example A

$$\begin{pmatrix} 0 & 3 & \ldots & \ldots \\ 1 & -2 & \ldots & \ldots \\ 4 & 0 & \ldots & \ldots \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & -2 & \ldots & \ldots \\ 0 & 3 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix}$$

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \ldots \\ 0 & 2 & -3 & \ldots \\ 0 & -1 & 6 & \ldots \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 3 & 1 & \ldots \\ 0 & 0 & -11/3 & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix}$$

How to use GJE to reduce a matrix to RREF?

Algorithm 1.4.3

Step 5: Multiply a suitable constant to each row so that all the leading entries becomes 1.

Example A

$$\begin{pmatrix} 1 & -2 & \dots & \dots \\ 0 & 3 & \dots & \dots \\ 0 & 0 & \dots & \dots \end{pmatrix}$$

No action is needed.

Multiply the 2nd row by 1/3 so that the entry marked by ● becomes 1.

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \dots \\ 0 & 0 & -11/3 & \dots \\ 0 & 0 & 0 & \dots \end{pmatrix}$$

Multiply the 1st row by 1/3 so that the entry marked by ● becomes 1.

Multiply the 2nd row by – 3/11 so that the entry marked by ● becomes 1.

How to use GJE to reduce a matrix to RREF?

Algorithm 1.4.3

Step 6: Beginning with the last nonzero row and working upward, add a suitable multiples of each row to the rows above to introduce zeros above the leading entries.

Example C

$$\begin{pmatrix} 1 & -2 & \cdots & \cdots & -4 & \cdots \\ & 1 & \cdots & \cdots & 3 & \cdots \\ & & \ddots & \cdots & \cdots & \cdots \\ & & & 1 & 2 & \cdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots \end{pmatrix}$$

do from bottom to top

Add 4 times of the last row to the 1st row so the entry marked by becomes 0.

Add – 3 times of the last row to the 2nd row so the entry marked by becomes 0.

Add – 2 times of the last row to the next row so the entry marked by becomes 0.

-2R4 + R3 > R3
since everything before the leading entry = 0 > only affects the value above

How to use GJE to reduce a matrix to RREF?

Algorithm 1.4.3

Step 6: Beginning with the last nonzero row and working upward, add a suitable multiples of each row to the rows above to introduce zeros above the leading entries.

Example C

$$\begin{pmatrix} 1 & -2 & \cdots & \cdots & 0 & \cdots \\ & 1 & \cdots & \cdots & 0 & \cdots \\ & & \ddots & \cdots & \cdots & \cdots \\ & & & 1 & 0 & \cdots \\ & & & & 1 & \cdots \end{pmatrix}$$

Apply the same process to the next pivot column on the left

How to use GJE to reduce a matrix to RREF?

Example 1.4.4

$$\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 2 & 8 \\ 1 & 2 & 4 & 5 & 3 & -9 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$$

Gaussian Elimination

Gauss-Jordan Elimination

$$\left(\begin{array}{ccccc|c} \boxed{1} & 2 & \boxed{4} & 5 & \boxed{3} & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 0 & -29 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$$

Do we have to strictly follow the steps ?

Remark 1.4.5.2

In the actual implementation of the algorithms, the steps mentioned in Algorithm 1.4.2 and Algorithm 1.4.3 are usually modified to avoid the round-off errors during the computation

many entries in a matrix
small change in entry can impact solution very big
from solution to no solution

Ill-conditioned matrix

very sensitive matrix

See Exercise 1 Q21

Additional remarks

Modification in GE

Example

Standard

$$\begin{pmatrix} 4 & 3 & \ldots & \ldots \\ 1 & -2 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 3 & \ldots & \ldots \\ 0 & -11/4 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix}$$

Variation

$$\begin{pmatrix} 1 & -2 & \ldots & \ldots \\ 4 & 3 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & \ldots & \ldots \\ 0 & 11 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix}$$

swap first even though gaussian no need to follow strictly

make life easy

Is the REF/RREF of a matrix unique ?

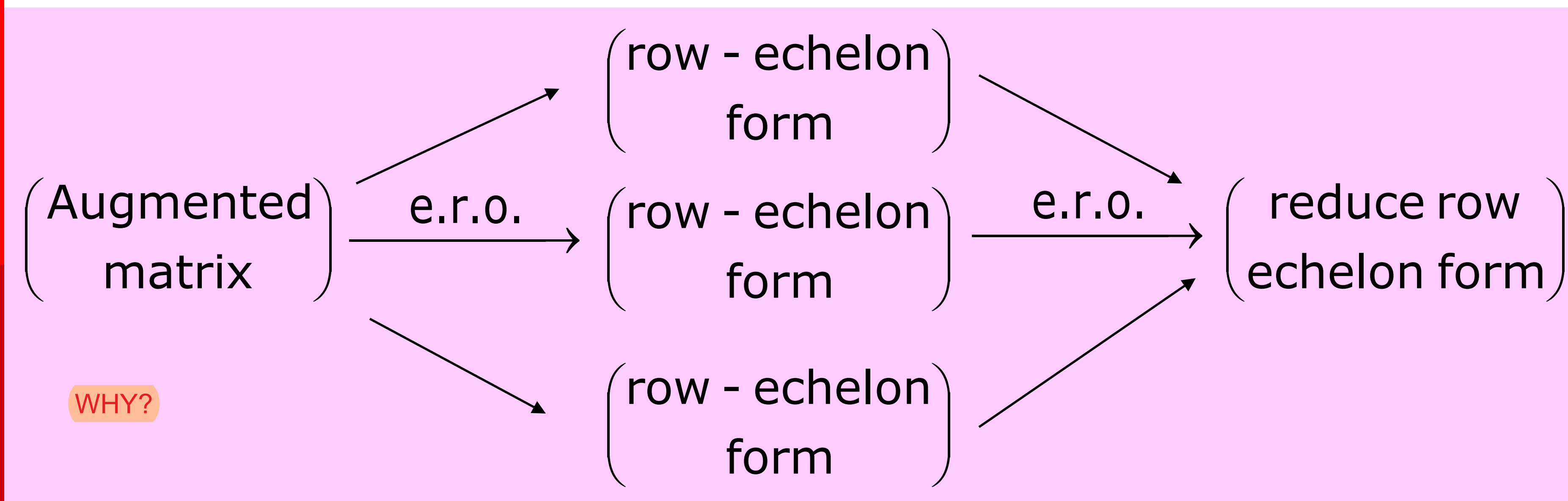
Remark 1.4.5.1

why RREF over REF

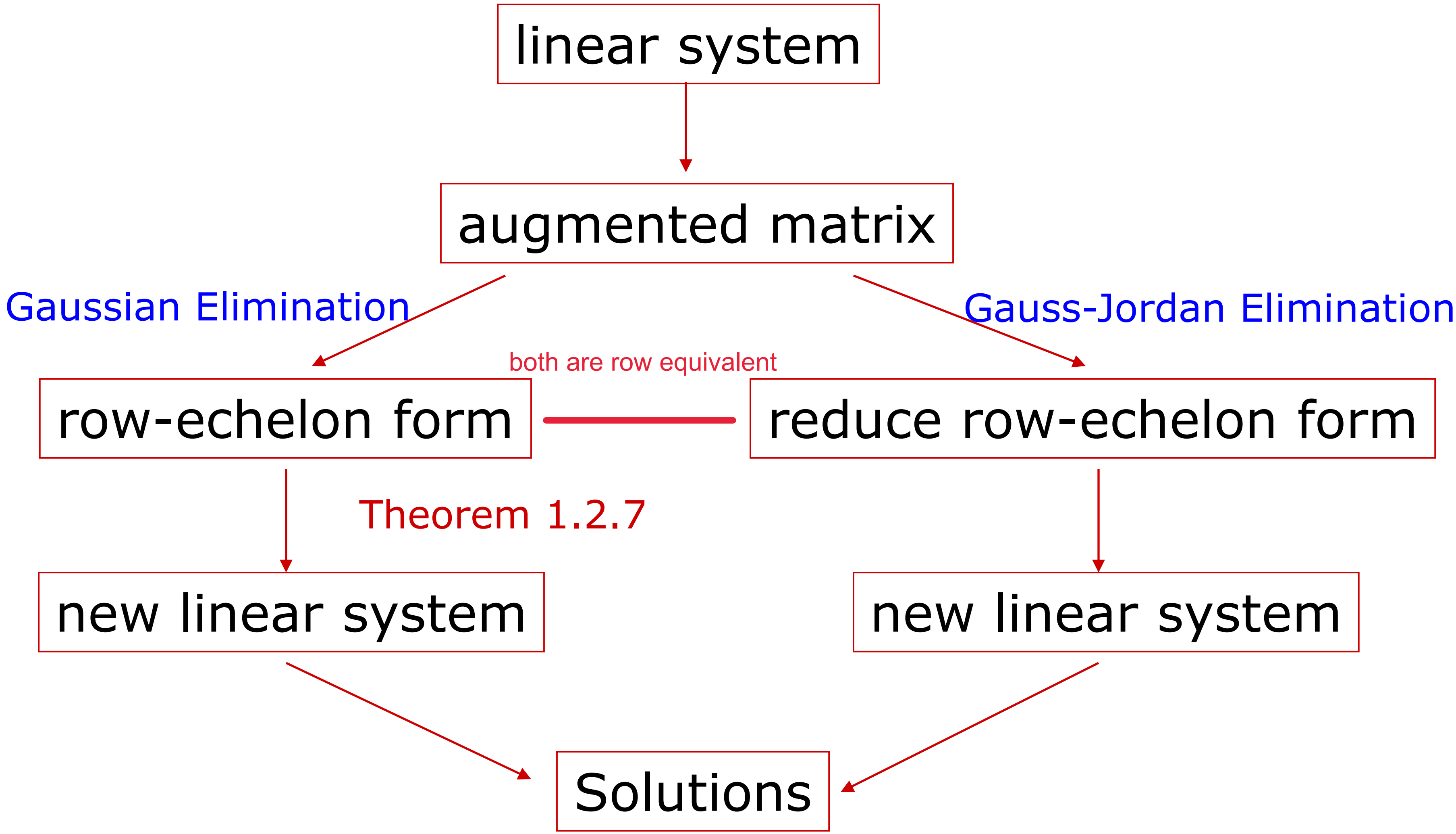
Every matrix can have **many different** row-echelon forms.

to check whether 2 augmented matrices are row equivalent, check RREF can help to check answers with other people

> Every matrix has a **unique** reduced row-echelon form



Discussion 1.4.6



How to tell the number of solutions from REF?

Remark 1.4.8.1

inconsistent system

A linear system has no solution if:

REF has a row with nonzero last entry but zero elsewhere.

The last column of REF is a pivot column.

If constant column is a pivot column,
then system will have no solution

$$\left(\begin{array}{ccccccc|c} 0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & \otimes & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes \end{array}\right) \text{ e.g. } \left(\begin{array}{ccc|c} 3 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

How to tell the number of solutions from REF?

Remark 1.4.8.2

A linear system has exactly one solution if:

every column of REF is a pivot column, except the last column.

because no need to introduce parameter

no. of variables = no. of equations

$$\left(\begin{array}{cccc|c} \otimes & \cdots & * & \cdots & * \\ \cdots & \otimes & * & \cdots & * \\ \cdots & \cdots & \ddots & \cdots & * \\ \cdots & \cdots & 0 & \otimes & * \\ \cdots & \cdots & \cdots & \cdots & 0 \end{array}\right)$$

Remark 1.4.8.2

In other words, a consistent linear system has exactly one solution if:

# of variables in LS = # of nonzero rows in REF

e.g. $$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 4 & -1 & 2 \\ 0 & 0 & 0 & -1 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right)$$

4 variables = 4 equations

How to tell the number of solutions from REF?

Remark 1.4.8.3

A linear system has infinitely many solutions if:

there is a non-pivot column in the REF, other than the last column.

at least 1 non-pivot column = intro parameters

$$\left(\begin{array}{ccccccc|c} 0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & \otimes & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right)$$

How to tell the number of solutions from REF?

Remark 1.4.8.3

In other words, a consistent linear system has infinitely many solutions if:

# of variables in LS > # of nonzero rows in REF

e.g. $$\left(\begin{array}{cccc|c} 5 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

4 variables > 3 equations

4 variables > 3 equations
with 1 hidden variable x1

Finding a REF/RREF first then find if inconsistent/consistent