Student Number:

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*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2012-2013

MA1101R LINEAR ALGEBRA I

April/May 2013 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper contains a total of **FOUR (4)** questions and comprises **NINETEEN (19)** printed pages.
3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.
4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
Total	

Question 1 [25 marks]

Let $S = \{(1,0,1,0),\ \ (0,1,0,1),\ \ (1,1,0,0)\}$.

(i) (4 marks) Show that the set S is linearly independent.

(ii) (3 marks) What is the dimension of the vector space $V = \text{span}(S)$? Give a brief explanation.

(iii) (4 marks) Express the vector $\boldsymbol{v} = (7,-1,3,-5)$ as a linear combination of the vectors in S and write down the coordinate vector $(\boldsymbol{v})_S$.

(iv) (3 marks) Find the vector $\boldsymbol{w} \in \mathbb{R}^4$ such that the coordinate vector $(\boldsymbol{w})_S = (2,3,-6)$.

(v) (3 marks) Suppose T is another basis for V such that the transition matrix from S to T is given by $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Find the coordinate vector $(\boldsymbol{w})_T$ relative to T for the vector $\boldsymbol{w}$ in part (iv).

(vi) (4 marks) Determine the basis T in part (v), i.e. find all the vectors in T.

(vii) (4 marks) Is it possible to find a subspace U of $\mathbb{R}^4$ such that

$$V \subseteq U \subseteq \mathbb{R}^4 \text{ but } V \neq U \text{ and } U \neq \mathbb{R}^4?$$

Justify your answer.

Use the space below to write your answer and working

(i) <u>Method 1</u>. Consider the vector equation $a(1,0,1,0) + b(0,1,0,1) + c(1,1,0,0) = (0,0,0,0)$:

Equating the third component on both sides give $a = 0$.

Equating the fourth component on both sides give $b = 0$.

Equating the first component on both sides give $a + c = 0$, which implies $c = 0$.

Hence S is linearly independent.

<u>Method 2</u>. Consider the matrix form using the vectors in S as rows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{G.E.} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

Since all the rows in the row echelon form are nonzero, the set S is linearly independent.

(More working spaces for Question 1)

(ii) From part (i), S is linearly independent. Hence it is a basis for $V = \text{span}(S)$.

So $\dim V = 3$.

(iii) Consider the vector equation $a(1,0,1,0) + b(0,1,0,1) + c(1,1,0,0) = (7,-1,3,-5)$.

Equating the third component on both sides give $a = 3$.

Equating the fourth component on both sides give $b = -5$.

Equating the first component on both sides give $a + c = 7$, which implies $c = 4$.

So $(\boldsymbol{v})_S = (3,-5,4)$.

(iv) Since $(\boldsymbol{w})_S = (2,3,-6)$, we have

$\boldsymbol{w} = 2(1,0,1,0) + 3(0,1,0,1) - 6(1,1,0,0) = (-4,-3,2,3)$.

(v) We have $[\boldsymbol{w}]_T = \boldsymbol{P}[\boldsymbol{w}]_S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$.

So $(\boldsymbol{w})_T = (-4,-3,2)$.

(vi) Let $\boldsymbol{A}$ be the matrix formed by using the vectors in S as columns, and $\boldsymbol{B}$ be the matrix formed by using the vectors in T as columns.

Then $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$.

So $T = \{(1,0,0,-1), (0,1,0,1), (0,0,1,1)\}$.

(vii) Not possible.

Since $\dim V = 3$ and $\dim \mathbb{R}^4 = 4$, we have $\dim U = 3$ or 4.

If $\dim U = 3$, then $U = V$ (since $V \subseteq U$).

If $\dim U = 4$, then $U = \mathbb{R}^4$ (since $U \subseteq \mathbb{R}^4$).

(More working spaces for Question 1)

(More working spaces for Question 1)

Question 2 [25 marks]
Let $\boldsymbol{A} = (\boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \boldsymbol{a}_3 \ \ \boldsymbol{a}_4 \ \ \boldsymbol{a}_5)$ be a 4×5 matrix where $\boldsymbol{a}_i$ denotes the ith column of $\boldsymbol{A}$. Suppose the reduced row echelon form of $\boldsymbol{A}$ is given by

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) (7 marks) Write down a basis for each of the row space, column space and nullspace of the matrix $\boldsymbol{A}$.

(ii) (2 marks) Write down two vectors to extend the basis for the row space of $\boldsymbol{A}$ in part (i) to a basis for $\mathbb{R}^5$.

(iii) (3 marks) Find a 5×5 matrix <u>without zero rows</u> or <u>repeating rows</u> that has the same row space as $\boldsymbol{A}$.

(iv) (3 marks) Is $\{\boldsymbol{a}_3, \ \boldsymbol{a}_4, \ \boldsymbol{a}_5\}$ a basis for the column space of $\boldsymbol{A}$? Justify your answer.

(v) (4 marks) By pre-multiplying $\boldsymbol{A}$ with an invertible 4×4 matrix $\boldsymbol{B}$, is it necessary that $\boldsymbol{BA}$ has the same row space as $\boldsymbol{A}$? Justify your answer.

(vi) (3 marks) Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be a linear transformation with $\boldsymbol{A}$ above as the standard matrix. Suppose we are given

$$T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}, \quad T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 5 \\ 1 \\ 0 \end{pmatrix}, \quad T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find $T\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{pmatrix}\right)$.

(vii) (3 marks) For the linear transformation T in part (vi), is there enough information to determine its formula? Justify your answer.

Use the next three pages to write your answer and working

(Working spaces for Question 2)

(i) row space basis: $\{(1,0,0,1,0),(0,1,0,0,1),(0,0,1,1,1)\}$.
column space basis: $\{\boldsymbol{a}_1,\ \ \boldsymbol{a}_2,\ \ \boldsymbol{a}_3\}$.

Nullspace. From $\boldsymbol{R}$, we get the general solution of $\boldsymbol{Ax}=\boldsymbol{0}$:

$$\begin{pmatrix} -s \\ -t \\ -s-t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

So a basis is given by $\left\{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

(ii) From $\boldsymbol{R}$, we see that "pivots" are missing in the 4th and 5th columns. So we extend the basis of the row space to $\mathbb{R}^5$ by adding $(0,0,0,1,0)$ and $(0,0,0,0,1)$.

(iii) We can find such a matrix by using the nonzero rows $(1,0,0,1,0),(0,1,0,0,1),(0,0,1,1,1)$ of $\boldsymbol{R}$, together with two more rows, which can be any linear combinations of these three rows.

Examples:

$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 & 3 \end{pmatrix}$ (use scalar multiples of rows 1 and 2),

$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \end{pmatrix}$ (use additions of rows 1 and 2, rows 1 and 3 resp.) etc.

(More working spaces for Question 2)

(iv) Yes. Since the 3rd, 4th, 5th columns of $\boldsymbol{R}$ are linearly independent, we have $\boldsymbol{a}_3, \ \boldsymbol{a}_4, \ \boldsymbol{a}_5$ are also linearly independent.

As $\text{rank}(\boldsymbol{A}) = 3$, we conclude that $\{\boldsymbol{a}_3, \ \boldsymbol{a}_4, \ \boldsymbol{a}_5\}$ is a basis for the column space of $\boldsymbol{A}$.

(v) Yes. $\boldsymbol{B}$ is invertible, implies $\boldsymbol{B}$ is a product of elementary matrices $\boldsymbol{E}_n \cdots \boldsymbol{E}_1$.

So $\boldsymbol{A}$ is row equivalent to $\boldsymbol{E}_n \cdots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{B}\boldsymbol{A}$.

Hence row space of $\boldsymbol{A}$ is equal to row space of $\boldsymbol{B}\boldsymbol{A}$.

(vi) $$T\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{pmatrix}\right) = T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) + 2T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) + 3T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$$
$$= \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 5 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \\ 8 \\ 5 \end{pmatrix}.$$

(vii) Yes.

The three given images of T are the first three columns $\boldsymbol{a}_1, \ \boldsymbol{a}_2, \ \boldsymbol{a}_3$ of the standard matrix $\boldsymbol{A}$.

From the reduced row echelon form $\boldsymbol{R}$, we can determine $\boldsymbol{a}_4, \ \boldsymbol{a}_5$, columns 4 and 5 of $\boldsymbol{A}$. (More explicitly, $\boldsymbol{a}_4 = \boldsymbol{a}_1 + \boldsymbol{a}_3$, and $\boldsymbol{a}_5 = \boldsymbol{a}_2 + \boldsymbol{a}_3$.)

Hence we can find $\boldsymbol{A}$ completely and hence the formula of T.

(More working spaces for Question 2)

Question 3 (a) [13 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$.

(i) (4 marks) Find the characteristic polynomial of $\boldsymbol{A}$. Hence, or otherwise, show that the eigenvalues of $\boldsymbol{A}$ are -2 and 4.

(ii) (4 marks) Find a basis for each of the eigenspaces of $\boldsymbol{A}$.

(iii) (2 marks) Is $\boldsymbol{A}$ diagonalizable? Justify your answer.

(iv) (3 marks) Find a square matrix $\boldsymbol{B}$ such that $\boldsymbol{B}^3 = \boldsymbol{A}$. (You may leave your answer as a product of matrices.)

Use the space below to write your answer and working

(i) Characteristic polynomial of $\boldsymbol{A}$ is given by

$$\begin{vmatrix} \lambda - 1 & +3 & -3 \\ -3 & \lambda + 5 & -3 \\ -6 & 6 & \lambda - 4 \end{vmatrix} = \cdots = \lambda^3 - 12\lambda - 16 = (\lambda + 2)^2(\lambda - 4)$$

So the eigenvalues of $\boldsymbol{A}$ are -2 and 4.

(ii) For $\lambda = -2$, we solve

$$\left(\begin{array}{ccc|c} -3 & +3 & -3 & 0 \\ -3 & 3 & -3 & 0 \\ -6 & 6 & -6 & 0 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s - t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

So a basis to the eigenspace is $\{(1, 1, 0), (-1, 0, 1)\}$.

(More working spaces for Question 3a)

(ii) (cont) For $\lambda = 4$, we solve

$$\left(\begin{array}{ccc|c} 3 & +3 & -3 & 0 \\ -3 & 9 & -3 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{pmatrix} = t \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$.

A basis for the eigenspace is $\{(1, 1, 2)\}$.

(iii) Yes. $\boldsymbol{A}$ is 3×3 matrix. From part (ii), we see that there are three linearly independent eigenvectors (two associated to eigenvalue -2, and one associated to eigenvalue 4).

(iv) From above, we can write $\boldsymbol{A}$ as

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1}$$

So for matrix $\boldsymbol{B}$ such that $\boldsymbol{B}^3 = \boldsymbol{A}$, we have

$$\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \sqrt[3]{-2} & 0 & 0 \\ 0 & \sqrt[3]{-2} & 0 \\ 0 & 0 & \sqrt[3]{4} \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1}$$

Question 3 (b) [6 marks]

Find the least squares solution of the linear system

$$\begin{cases} x \phantom{{}+y} + 2z = 1 \\ \phantom{x+{}} y + 3z = 0 \\ -x + y + z = 0 \\ - y - 3z = 1 \,. \end{cases}$$

Use the space below to write your answer and working

Write the system as $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & -1 & -3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 3 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 3 & 7 \\ 1 & 7 & 23 \end{pmatrix}$$

$$\boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 3 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

By solving $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}^T\boldsymbol{b}$:

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ -1 & 3 & 7 & -1 \\ 1 & 7 & 23 & -1 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{ccc|c} 1 & 0 & 2 & \frac{2}{5} \\ 0 & 1 & 3 & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So the least squares solution of the system is

$$x = \frac{2}{5} - 2t, \quad y = -\frac{1}{5} - 3t, \quad z = t.$$

Question 3 (c) [6 marks]

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same size. Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{AB}$ associated with eigenvalue λ.

(i) If $\lambda \neq 0$, show that $\boldsymbol{Bx}$ is an eigenvector of $\boldsymbol{BA}$ with eigenvalue λ.

(ii) If $\lambda = 0$, is $\boldsymbol{Bx}$ an eigenvector of $\boldsymbol{BA}$ with eigenvalue λ? Justify your answer.

Use the space below to write your answer and working

(i) First of all, since $\lambda \neq 0$, $\boldsymbol{ABx} \neq \boldsymbol{0}$, so $\boldsymbol{Bx} \neq \boldsymbol{0}$.

Given $\boldsymbol{ABx} = \lambda \boldsymbol{x}$. Pre-multiplying by $\boldsymbol{B}$, we have

$$\boldsymbol{BABx} = \boldsymbol{B}\lambda\boldsymbol{x} \Rightarrow \boldsymbol{BA}(\boldsymbol{Bx}) = \lambda\boldsymbol{Bx}$$

Hence $\boldsymbol{Bx}$ is an eigenvector of $\boldsymbol{BA}$.

(ii) Not necessary.

For example, take $\boldsymbol{B} = \boldsymbol{0}$ and $\boldsymbol{x}$ any nonzero vector. Then $\boldsymbol{ABx} = \boldsymbol{0}$.

So $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{AB}$ associated with eigenvalue 0.

But $\boldsymbol{Bx} = \boldsymbol{0}$, which is not an eigenvector.

Question 4 (a) [13 marks]

Let $V = \{(w, x, y, z) \mid w - x + y - z = 0\}$.

(i) (3 marks) Write down the vector space V explicitly. Hence, find a basis for V.

(ii) (6 marks) Use the Gram-Schmidt process to find an orthogonal basis for V.

(iii) (2 marks) Extend the set obtained in (ii) to an orthogonal basis for $\mathbb{R}^4$.

(iv) (2 marks) Find the projection of $(2, -2, 2, -2)$ onto V.

Use the space below to write your answer and working

(i) $V = \{(s - t + r, s, t, r) \mid s, t, r \in \mathbb{R}\}$. To find a basis, we separate the parameters from the explicit form:

$$(s - t + r, s, t, r) = s(1, 1, 0, 0) + t(-1, 0, 1, 0) + r(1, 0, 0, 1)$$

and get a basis $\{(1, 1, 0, 0), (-1, 0, 1, 0), (1, 0, 0, 1)\}$.

(ii) For short, let us denote the three vectors in part (i) by $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

Applying Gram-Schmidt process to this basis, we get

$$\begin{aligned}
\boldsymbol{v}_1' &= \boldsymbol{v}_1 = (1, 1, 0, 0) \\
\boldsymbol{v}_2' &= \boldsymbol{v}_2 - \frac{\boldsymbol{v}_1' \cdot \boldsymbol{v}_2}{\boldsymbol{v}_1' \cdot \boldsymbol{v}_1'}\boldsymbol{v}_1' = (-1, 0, 1, 0) + \frac{1}{2}(1, 1, 0, 0) = \frac{1}{2}(-1, 1, 2, 0) \\
\boldsymbol{v}_3' &= \boldsymbol{v}_3 - \frac{\boldsymbol{v}_1' \cdot \boldsymbol{v}_3}{\boldsymbol{v}_1' \cdot \boldsymbol{v}_1'}\boldsymbol{v}_1' - \frac{\boldsymbol{v}_2' \cdot \boldsymbol{v}_3}{\boldsymbol{v}_2' \cdot \boldsymbol{v}_2'}\boldsymbol{v}_2' \\
&= (1, 0, 0, 1) - \frac{1}{2}(1, 1, 0, 0) + \frac{1}{3}\left(\frac{1}{2}(-1, 1, 2, 0)\right) = \frac{1}{6}(2, -2, 2, 6)
\end{aligned}$$

So an orthogonal basis can be given by $\{(1, 1, 0, 0), \frac{1}{2}(-1, 1, 2, 0), \frac{1}{6}(2, -2, 2, 6)\}$.

(More working spaces for Question 4a)

(iii) To extend any basis for V to a basis for $\mathbb{R}^4$, we just need one more vector. We can get this vector from the coefficients of the equation $w - x + y - z = 0$, namely $(1, -1, 1, -1)$. In fact this vector is orthogonal to every vector in V. Hence we can use it to extend the orthogonal basis for V in part (ii) to an orthogonal basis for $\mathbb{R}^4$.

(iv) Since $(2, -2, 2, -2) = 2(1, -1, 1, -1)$ is orthogonal to V, the projection of $(2, -2, 2, -2)$ onto V is the zero vector.

Question 4 (b) [6 marks]

Let $\boldsymbol{A}$ be a square matrix of order n such that for any $\boldsymbol{u} \in \mathbb{R}^n$,

$$\|\boldsymbol{A}\boldsymbol{u}\| = \|\boldsymbol{u}\|.$$

(i) Prove that $\boldsymbol{A}\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}$ for any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.

(ii) Using (i) or otherwise, prove that $\boldsymbol{A}$ is an orthogonal matrix.

Use the space below to write your answer and working

(i) Applying the condition $\|\boldsymbol{A}\boldsymbol{u}\| = \|\boldsymbol{u}\|$ to the vector $\boldsymbol{u}+\boldsymbol{v}$ and squaring both sides, we have

$$\begin{aligned}
& \|\boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v})\|^2 = \|\boldsymbol{u}+\boldsymbol{v}\|^2 \\
\Rightarrow\ & \boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v}) \cdot \boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v}) = (\boldsymbol{u}+\boldsymbol{v}) \cdot (\boldsymbol{u}+\boldsymbol{v}) \\
\Rightarrow\ & (\boldsymbol{A}\boldsymbol{u}+\boldsymbol{A}\boldsymbol{v}) \cdot (\boldsymbol{A}\boldsymbol{u}+\boldsymbol{A}\boldsymbol{v}) = (\boldsymbol{u}+\boldsymbol{v}) \cdot (\boldsymbol{u}+\boldsymbol{v}) \\
\Rightarrow\ & \boldsymbol{A}\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{u} + 2\boldsymbol{A}\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{v} + \boldsymbol{A}\boldsymbol{v} \cdot \boldsymbol{A}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{u} + 2\boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{v} \\
\Rightarrow\ & 2\boldsymbol{A}\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{v} = 2\boldsymbol{u} \cdot \boldsymbol{v} \quad (\text{since } \boldsymbol{A}\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{u} \ \text{ and } \boldsymbol{A}\boldsymbol{v} \cdot \boldsymbol{A}\boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{v}) \\
\Rightarrow\ & \boldsymbol{A}\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}
\end{aligned}$$

(ii) Consider the standard basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ of $\mathbb{R}^n$.

From part (i), $\boldsymbol{A}\boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$.

Rewrite the dot product as matrix multiplication, we get $\boldsymbol{e}_i^T \boldsymbol{A}^T \boldsymbol{A}\boldsymbol{e}_j = \boldsymbol{e}_i^T \boldsymbol{e}_j$.

The left hand side is the (i,j)-entry of $\boldsymbol{A}^T\boldsymbol{A}$, while the right hand side is either 1 or 0 depending on whether $i = j$ or $i \neq j$.

In other word, we have $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$, which means $\boldsymbol{A}$ is an orthogonal matrix.

Question 4 (c) [6 marks]

Let $\boldsymbol{A}$ be a square matrix of order n such that $\boldsymbol{A}^2 = \boldsymbol{A}$.

(i) Prove that $\boldsymbol{A}$ is diagonalizable.

(Hint: First show that the only possible eigenvalues of $\boldsymbol{A}$ are 0 and 1.)

(ii) Prove that $\text{rank}(\boldsymbol{A}) = \text{tr}(\boldsymbol{A})$.
(Here $\text{tr}(\boldsymbol{A})$ denotes the trace of $\boldsymbol{A}$, which is given by the sum of the diagonal entries of $\boldsymbol{A}$.)

Use the space below to write your answer and working

(i) Let λ be an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{u}$ an associated eigenvector. Then

$$\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u} = \boldsymbol{A}^2\boldsymbol{u} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{u}) = \boldsymbol{A}(\lambda\boldsymbol{u}) = \lambda(\boldsymbol{A}\boldsymbol{u}) = \lambda^2\boldsymbol{u}.$$

Since $\boldsymbol{u} \neq \boldsymbol{0}$, we have $\lambda = \lambda^2$, i.e., $\lambda = 0$ or $\lambda = 1$.

Let $r = \texttt{rank}(\boldsymbol{A})$. Let $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r\}$ be a basis for the column space of $\boldsymbol{A}$. So for each i, there exists $\boldsymbol{x}_i$ such that $\boldsymbol{u}_i = \boldsymbol{A}\boldsymbol{x}_i$. Then

$$\boldsymbol{u}_i = \boldsymbol{A}\boldsymbol{x}_i = \boldsymbol{A}^2\boldsymbol{x}_i = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x}_i) = \boldsymbol{A}\boldsymbol{u}_i.$$

Since $\boldsymbol{u}_i \neq \boldsymbol{0}$, $\boldsymbol{u}_i$ is an eigenvector of $\boldsymbol{A}$ associated to $\lambda = 1$.

Note that $\text{nullity}(\boldsymbol{A}) = n - r$. Let $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{n-r}\}$ be a basis for the nullspace of $\boldsymbol{A}$. Each $\boldsymbol{v}_j$ is an eigenvector of $\boldsymbol{A}$ associated to $\lambda = 0$.

Therefore $\boldsymbol{A}$ is diagonalizable by the matrix $\boldsymbol{P} = (\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_r \ \boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_{n-r})$.

Furthermore, $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} \boldsymbol{I}_r & 0 \\ 0 & 0 \end{pmatrix}$.

(ii)

$$\text{tr}(\boldsymbol{A}) = \text{tr}(\boldsymbol{A}\boldsymbol{P}\boldsymbol{P}^{-1}) = \text{tr}(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}) = \text{tr}\begin{pmatrix} \boldsymbol{I}_r & 0 \\ 0 & 0 \end{pmatrix} = r = \text{rank}(\boldsymbol{A}).$$

(More working spaces. Please indicate the question numbers clearly.)

(More working spaces. Please indicate the question numbers clearly.)

[END OF PAPER]