MA1101R

LIVE LECTURE 4

Q&A: log in to PollEv.com/vtpoll

Q&A: log in to PollEv.com/vtpoll

Topics for week 4

2.5 Determinant

3.1 Euclidean n-spaces

Q&A: log in to PollEv.com/vtpoll

Ways of finding determinant

- Cofactor expansion

  Express n x n determinant as a sum of (n-1)x(n-1) determinants

- Gaussian elimination

  Reduce to triangular matrix (REF)
  Effect of e.r.o. on determinants

- Special cases
  - Triangular matrices — Product of diagonal entries
  - Two identical rows/columns — det = 0
  - Zero rows/columns — det = 0

Q&A: log in to PollEv.com/vtpoll

Theorem 2.5.12 (Exercise 2 Q58)

$n \times n$

S(n) The determinant of a square matrix with two identical rows is zero.

S(2) Base case

2×2: $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab$

0

S(2) ⇒ S(3)

Inductive step $k \times k \Rightarrow (k+1) \times (k+1)$

S(k) ⇒ S(k+1)

3×3: $\begin{vmatrix} a & b & c \\ * & * & * \\ a & b & c \end{vmatrix} = -*\begin{vmatrix} b & c \\ b & c \end{vmatrix} + *\begin{vmatrix} a & c \\ a & c \end{vmatrix} - *\begin{vmatrix} a & b \\ a & b \end{vmatrix}$

0 0 0 0

cofactor expansion along row 2

$$\begin{vmatrix} a & b & c \\ * & * & * \\ a & b & c \end{vmatrix} = -*\begin{vmatrix} b & c \\ b & c \end{vmatrix} + *\begin{vmatrix} a & c \\ a & c \end{vmatrix} - *\begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

Theorem 2.5.12 (Exercise 2 Q58)

$n \times n$

S(n) The determinant of a square matrix with two identical rows is zero.

S(2)

$S(k) \Rightarrow S(k+1)$ $k = 2, 3, 4, \ldots$

i. Start with any $k+1 \times k+1$ matrix $\boldsymbol{A}$ with two identical rows: row p and row q

ii. Cofactor expansion of $\det(\boldsymbol{A})$ along row $h \neq p, q$

iii. All the $k \times k$ submatrices $\boldsymbol{M}_{hj}$ in the expansion have two identical rows

iv. By induction hypothesis S(k), $\det(\boldsymbol{M}_{hj}) = 0$ for all j.

v. This implies $\det(\boldsymbol{A}) = 0$, and hence we have S(k+1).

$$\boxed{S(2) \Rightarrow S(3) \Rightarrow S(4) \Rightarrow S(5) \ldots \Rightarrow S(n) \Rightarrow \ldots}$$

S(n) is true for all n

Q&A: log in to PollEv.com/vtpoll

Determinants and E.R.O.

E.R.O	Determinant
$\boldsymbol{A} \xrightarrow{kR_i} \boldsymbol{B}$	$\det(\boldsymbol{B}) = k\det(\boldsymbol{A})$
$\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}$	$\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$
$\boldsymbol{A} \xrightarrow{R_i + kR_j} \boldsymbol{B}$	$\det(\boldsymbol{B}) = \det(\boldsymbol{A})$

Similar for E.C.O

Q&A: log in to PollEv.com/vtpoll

What's the scalar?

$$\begin{vmatrix} 2a & 2b & 6c \\ d & e & 3f \\ g & h & 3i \end{vmatrix} = \boxed{6} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\downarrow$$

$$2\begin{vmatrix} a & b & 3c \\ d & e & 3f \\ g & h & 3i \end{vmatrix} \Rightarrow 2 \times 3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Q&A: log in to PollEv.com/vtpoll

Determinant by G.E.

$$\begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 2 & 1 & 0 & 3 \\ 0 & 4 & 1 & 2 \end{vmatrix} \xrightarrow{R_3 - R_1} \begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 4 & 1 & 2 \end{vmatrix}$$

no change

$$\xrightarrow{R_4 \leftrightarrow R_2} - \begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 1 \end{vmatrix}$$

multiply by -1

$$\xrightarrow{R_4 \leftrightarrow R_3} \begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

multiply by -1

$$= 2 \times 4 \times 3 \times 2$$

product of diagonal entries

$$\begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix} = (2\text{x}2\text{x}2) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Determinant and matrix operations

$\boldsymbol{A}$ and $\boldsymbol{B}$: square matrices of order n

c a scalar

1. $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$ $\det(c\boldsymbol{A}) \neq c\det(\boldsymbol{A})$
2. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$ Multiplicative property
3. $\det(\boldsymbol{A}^{\mathrm{T}}) = \det(\boldsymbol{A})$
4. $\det(\boldsymbol{A}^{-1}) = \dfrac{1}{\det(\boldsymbol{A})}$ if $\boldsymbol{A}$ is invertible
5. $\det(\boldsymbol{A} + \boldsymbol{B}) \neq \det(\boldsymbol{A}) + \det(\boldsymbol{B})$

Q&A: log in to PollEv.com/vtpoll

Determinant and invertibility

A square matrix $\boldsymbol{A}$ is invertible

if and only if

$\det(\boldsymbol{A}) \neq 0$.

contrapositive

converse

$\boldsymbol{A}$ is invertible $\Rightarrow \det(\boldsymbol{A}) \neq 0$

$\boldsymbol{A}$ is invertible $\Leftarrow \det(\boldsymbol{A}) \neq 0$

$\boldsymbol{A}$ is not invertible $\Rightarrow \det(\boldsymbol{A}) = 0$

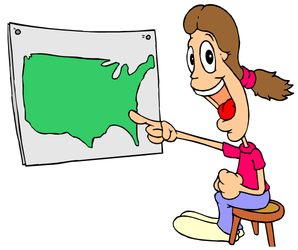
$\boldsymbol{A}$ is not invertible $\Leftarrow \det(\boldsymbol{A}) = 0$

same meaning

different meaning

different meaning

Map of LA



A **is an n×n matrix**

A **is invertible**	***A*** **is not invertible**
det $A \neq 0$	**det** $A = 0$
rref of ***A*** **is identity matrix**	**rref of** ***A*** **has a zero row**
$Ax = 0$ **has only the trivial solution**	$Ax = 0$ **has non-trivial solutions**
$Ax = b$ **has a unique solution**	$Ax = b$ **has no solution or infinitely many solutions**

to be continued

Q&A: log in to PollEv.com/vtpoll

Connecting concepts

$$\boldsymbol{A} = \begin{pmatrix} 1100 & 1101 & 1102 \\ 2020 & 2021 & 2022 \\ 9999 & 7777 & 5555 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 8 & 9 & 0 \end{pmatrix}$$

two identical rows

$\det \boldsymbol{B} = 0$

Consider system $\boldsymbol{ABx} = \boldsymbol{0}$.

How many solutions does it have?

Without finding $\boldsymbol{AB}$ and using Gaussian elimination, how can we tell this system has infinitely many solutions?

$$\det \boldsymbol{AB} = \det \boldsymbol{A} \times \det \boldsymbol{B} = 0$$

$\Rightarrow$ $\boldsymbol{AB}$ is singular

$\Rightarrow$ $\boldsymbol{ABx} = \boldsymbol{0}$ has non-trivial solutions

$\Rightarrow$ $\boldsymbol{ABx} = \boldsymbol{0}$ has infinitely many solutions

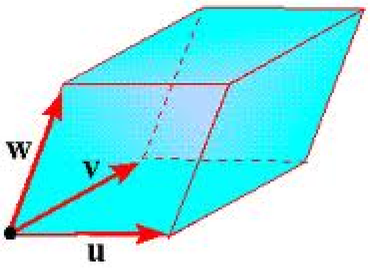
Q&A: log in to PollEv.com/vtpoll

Connection to geometry

a. Area of parallelogram (2x2)

b. Volume of parallelepiped (3x3)

c. Area of triangle (3x3)



Visualisation tools

Q&A: log in to PollEv.com/vtpoll

Adjoint

Let $\boldsymbol{A}$ be a square matrix of order n.

The adjoint of $\boldsymbol{A}$ is the $n \times n$ matrix

$$\operatorname{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^{T} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1\mathrm{n}} & A_{2\mathrm{n}} & \cdots & A_{nn} \end{pmatrix}$$

where A_{ij} is the (i, j)-cofactor of $\boldsymbol{A}$.

$(-1)^{i+j} \det(\boldsymbol{M}_{\mathrm{ij}})$

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$(\,\boldsymbol{A} \mid \boldsymbol{I}\,) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} (\,\boldsymbol{I} \mid \boldsymbol{A}^{-1}\,)$$

Formula for inverse matrix

$$\boxed{\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\operatorname{adj}(\boldsymbol{A})}$$

For smaller matrices
Use for proof

$$\det(\boldsymbol{A})\boldsymbol{I} = \boldsymbol{A}\operatorname{adj}(\boldsymbol{A})$$

$$\boldsymbol{A}\operatorname{adj}(\boldsymbol{A}) = \begin{pmatrix} \det(\mathbf{A}) & 0 & \dots & 0 \\ 0 & \det(\mathbf{A}) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \det(\mathbf{A}) \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\,\mathrm{adj}(\boldsymbol{A})$$

Finding inverse using adjoint

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = \begin{pmatrix} 1/a & -1/a & (be-cd)/adf \\ 0 & 1/d & -e/df \\ 0 & 0 & 1/f \end{pmatrix}$$

upper triangular

upper triangular

$$\frac{1}{adf}\begin{pmatrix} \begin{vmatrix} d & e \\ 0 & f \end{vmatrix} & -\begin{vmatrix} 0 & e \\ 0 & f \end{vmatrix} & \begin{vmatrix} 0 & d \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} d & e \\ 0 & f \end{vmatrix} & \begin{vmatrix} a & c \\ 0 & f \end{vmatrix} & -\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} b & c \\ d & e \end{vmatrix} & -\begin{vmatrix} a & c \\ 0 & e \end{vmatrix} & \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} \end{pmatrix}^T = \frac{1}{adf}\begin{pmatrix} df & -df & be-cd \\ 0 & af & -ae \\ 0 & 0 & ad \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\,\mathrm{adj}(\boldsymbol{A})$$

$$\mathrm{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T$$

Exercise 2 Q60

A n x n invertible matrix

a) Show that adj(**A**) is invertible

$\mathrm{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{A}^{-1}$

Since $\mathbf{A}^{-1}$ is invertible and $\det(\mathbf{A}) \neq 0$, so adj(**A**) is invertible.

b) Find det(adj(**A**)) and $\mathrm{adj}(\mathbf{A})^{-1}$

$\det(\mathrm{adj}(\mathbf{A})) = \det(\det(\mathbf{A})\mathbf{A}^{-1}) = \det(\mathbf{A})^n \det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{n-1}$

$\mathrm{adj}(\mathbf{A})^{-1} = (\det(\mathbf{A})\mathbf{A}^{-1})^{-1} = \det(\mathbf{A})^{-1}(\mathbf{A}^{-1})^{-1} = \det(\mathbf{A})^{-1}\mathbf{A}$

c) What is adj(adj(**A**))?

$\mathrm{adj}(\mathrm{adj}(\mathbf{A})) = \det(\mathrm{adj}(\mathbf{A}))\ \mathrm{adj}(\mathbf{A})^{-1}$

$= \det(\mathbf{A})^{n-1}\det(\mathbf{A})^{-1}\mathbf{A} = \det(\mathbf{A})^{n-2}\mathbf{A}$

Q&A: log in to PollEv.com/vtpoll

Cramer's Rule

Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is a linear system where $\boldsymbol{A}$ is an n x n invertible matrix.

Let $\boldsymbol{A_i}$ be the matrix obtained from $\boldsymbol{A}$ by replacing the i^{th} column of $\boldsymbol{A}$ by $\boldsymbol{b}$.

Then the system has a unique solution

$$\boldsymbol{x} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} \det(\boldsymbol{A}_1) \\ \det(\boldsymbol{A}_2) \\ \vdots \\ \det(\boldsymbol{A}_n) \end{pmatrix}$$

$$x_1 = \frac{\det(\boldsymbol{A_1})}{\det(\boldsymbol{A})}$$

$$x_2 = \frac{\det(\boldsymbol{A_2})}{\det(\boldsymbol{A})}$$

$$x_n = \frac{\det(\boldsymbol{A}_n)}{\det(\boldsymbol{A})}$$

Which is correct?

$$\boldsymbol{x} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} \det(\boldsymbol{A}_1) \\ \det(\boldsymbol{A}_2) \\ \vdots \\ \det(\boldsymbol{A}_n) \end{pmatrix}$$

Suppose $\boldsymbol{Ax} = \boldsymbol{0}$ is a linear system with infinitely many solutions, where $\boldsymbol{A}$ is an n x n matrix.

Then Cramer's Rule will

1. gives the trivial solution
2. gives a non-trivial solution
3. gives the general solution
4. not give any solution

Q&A: log in to PollEv.com/vtpoll

Chapter 3 (n-vector)

- An n-vector has the form $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$
- Do not write $\{u_1, u_2, \ldots, u_i, \ldots, u_n\}$
- We can identify it as a 1 x n matrix or n x 1 matrix:

$$\boldsymbol{u} = (u_1\ u_2\ \ldots\ u_n) \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

- The set of all n-vectors of real numbers is called the Euclidean n-space and is denoted by $\mathbf{R}^n$.
- We can perform addition on two n-vectors $\boldsymbol{u} + \boldsymbol{v}$
- We can perform scalar multiplication on a vector $c\boldsymbol{v}$

Q&A: log in to PollEv.com/vtpoll

Dimension 2 and 3

A 2-vector $\boldsymbol{u} = (u_1, u_2)$ in $\mathbf{R}^2$ can be represented as

a point or an arrow in the xy-plane

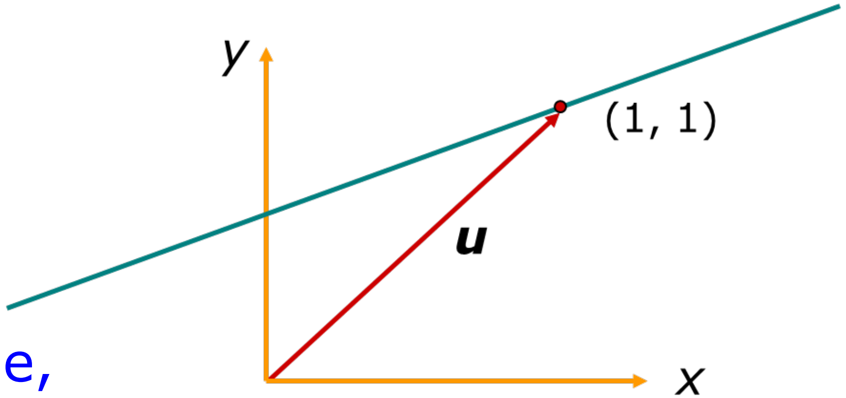
A 3-vector $\boldsymbol{u} = (u_1, u_2, u_3)$ in $\mathbf{R}^3$ can be represented as

a point or an arrow in the xyz-space

Line with equation: $2y - x = 1$

A solution: $x = 1, y = 1$

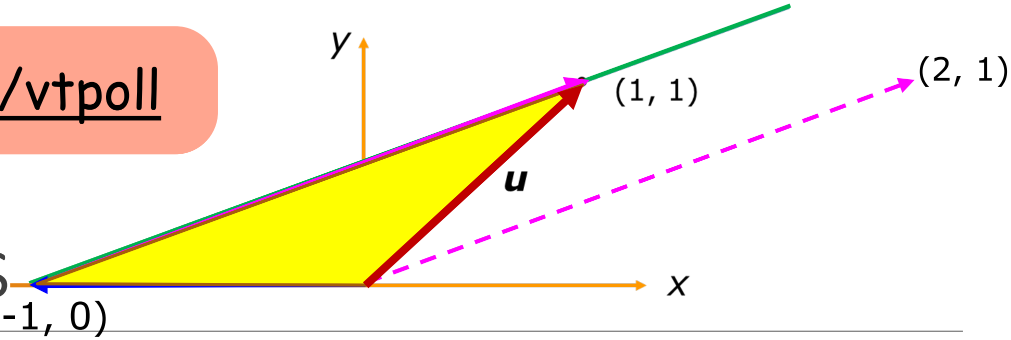
Does (1, 1) lie on the line?



The point (1, 1) lies on the line,
but the arrow (1, 1) does not lie on the line

Q&A: log in to PollEv.com/vtpoll

Points vs Arrows



The point (1, 1) lies on the line: $2y - x = 1$

but the arrow (1, 1) does not lie on the line

We treat (1, 1) as a point when
the line is regarded as a subset of points in the XY-plane

We treat (1, 1) as an arrow when
- we perform vector addition and scalar multiplication
- we use it to indicate direction

General solution: $(x, y) = (2t - 1, t)$

$$= t\underbrace{\boxed{(2, 1)}}_{\text{an arrow parallel to the line}} + \underbrace{\boxed{(-1, 0)}}_{\text{a point on the line}}$$

Q&A: log in to PollEv.com/vtpoll

Set notation for subsets of $\mathbf{R}^n$

Implicit form

$$\left\{ \begin{array}{c} \text{general n-tuple} \\ (u_1, u_2, \dots, u_n) \end{array} \;\middle|\; \begin{array}{c} \text{conditions satisfied by} \\ u_1, u_2, \dots, u_n \end{array} \right\}$$

$$S = \{ (u_1, u_2, u_3, u_4) \mid u_1 = 0 \text{ and } u_2 = u_4 \}$$

Explicit form

Not always possible to express in explicit form

$$\left\{ \begin{array}{c} \text{n-tuples with} \\ \text{explicit form} \end{array} \;\middle|\; \begin{array}{c} \text{range of parameters} \\ \text{appearing on the left} \end{array} \right\}$$

$$S = \{ (0, a, b, a) \mid a, b \in \mathbf{R} \}$$

Don’t write $\{a, b \in \mathbf{R} \mid (0, a, b, a) \}$

defined by
(i) a point (a, b, c) on the line, and
(ii) an arrow (u, v, w) parallel to the line

Explicit form:
(a, b, c) + t(u, v, w)

Set notations for lines and planes

Lines in xy-plane

Implicit form: $\{ (x, y) \mid ax + by = c \}$

Explicit form: $\left\{ \left(\dfrac{c - bt}{a}, \ t \right) \,\middle|\, t \in \mathbf{R} \right\}$

Planes in xyz-space

Implicit form: $\{ (x, y, z) \mid ax + by + cz = d \}$

Explicit form: $\left\{ \left(\dfrac{d - bs - ct}{a}, \ s, \ t \right) \,\middle|\, s, t \in \mathbf{R} \right\}$

Lines in xyz-space

Implicit form: { (*x*, *y*, *z*) | eqn of two planes } Does not exist

Explicit form: { (general solution) | 1 parameter }

Q&A: log in to PollEv.com/vtpoll

Exercise 3 Q3

Which of these subsets of $\mathbf{R}^3$ are the same?

- A = a line passes through the origin and (9,9,9) geometrical
- $B = \{(k, k, k) \mid k \in \mathbf{R}\}$ Explicit form
- $C = \{(a, b, c) \mid a = b = c\}$
- $D = \{(x, y, z) \mid 2x - y - z = 0\}$
- $E = \{(a, b, c) \mid 2a - b - c = 0 \text{ and } a + b + c = 0\}$
- $F = \{(u, v, w) \mid 2u - v - w = 0 \text{ and } 3u - 2v - w = 0\}$

Implicit form

Exercise 3 Q3

A = a line passes through the origin and (9,9,9)

B = $\{(k, k, k) \mid k \in \mathbf{R}\}$

C = $\{(a, b, c) \mid a = b = c\}$

D = $\{(x, y, z) \mid 2x - y - z = 0\}$

E = $\{(a, b, c) \mid 2a - b - c = 0 \text{ and } a + b + c = 0\}$

F = $\{(u, v, w) \mid 2u - v - w = 0 \text{ and } 3u - 2v - w = 0\}$

A: contains the point (0,0,0) and parallel to the arrow (9,9,9)

Explicit form of A: $(0,0,0) + t(9,9,9) = (9t,9t,9t) = (s, s, s)$

Explicit form of C: (a, a, a)

So A, B, C are the same subset of $\mathbf{R}^3$.

D represents a plane while A represents a line.

So D is not the same subset as A, B, C.

Vectors of the form (k, k, k) satisfies the equation $2x - y - z = 0$.

This means the line (represented by A, B, C) lies on the plane D.

So A, B, C $\subseteq$ D

Exercise 3 Q3

A = a line passes through the origin and (9,9,9)

B = $\{(k, k, k) \mid k \in \mathbf{R}\}$

C = $\{(a, b, c) \mid a = b = c\}$

D = $\{(x, y, z) \mid 2x - y - z = 0\}$

E = $\{(a, b, c) \mid 2a - b - c = 0 \text{ and } a + b + c = 0\}$

F = $\{(u, v, w) \mid 2u - v - w = 0 \text{ and } 3u - 2v - w = 0\}$

E represents a line (intersection of 2 planes).

(k, k, k) satisfies the equation $2a - b - c = 0$ but not $a + b + c = 0$

This means this line of intersection of the 2 planes is not the same line as A, B, C

So E is not the same subset as A, B, C, D.

F represents a line (intersection of 2 planes).

(k, k, k) satisfies both equations $2u - v - w = 0$ and $3u - 2v - w = 0$

This means this line of intersection of the 2 planes is the same line as A, B, C

So F is the same subset as A, B, C, but not D.

Q&A: log in to PollEv.com/vtpoll

Announcement

- Practice Session
  - Practice 2 next week
  - Practice 1 scores in submission folder
- MATLAB
  - Intro video in LumiNUS > Conferencing > Expired
- Textbook exercise
  - Exercise 2 (part 1) solution in LumiNUS > Files
- Online quiz 4
  - Due this Sunday