MA1101R

LIVE LECTURE 5

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Topics for week 5

3.2 Linear Combinations and Linear Spans

3.3 Subspaces

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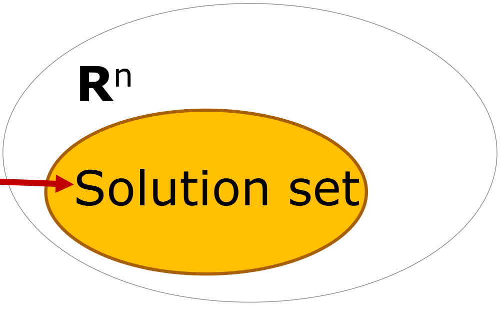
Let's revise

- 2-vectors and 3-vectors can be expressed geometrically as either points or arrows
- The set of all n-vectors is called Euclidean n-space $\mathbf{R}^n$
- Subsets of $\mathbf{R}^n$ can be expressed as set notation in implicit and explicit forms
- The solution set of a linear system with n variables is a subset of $\mathbf{R}^n$

a solution $\mathbf{x} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$



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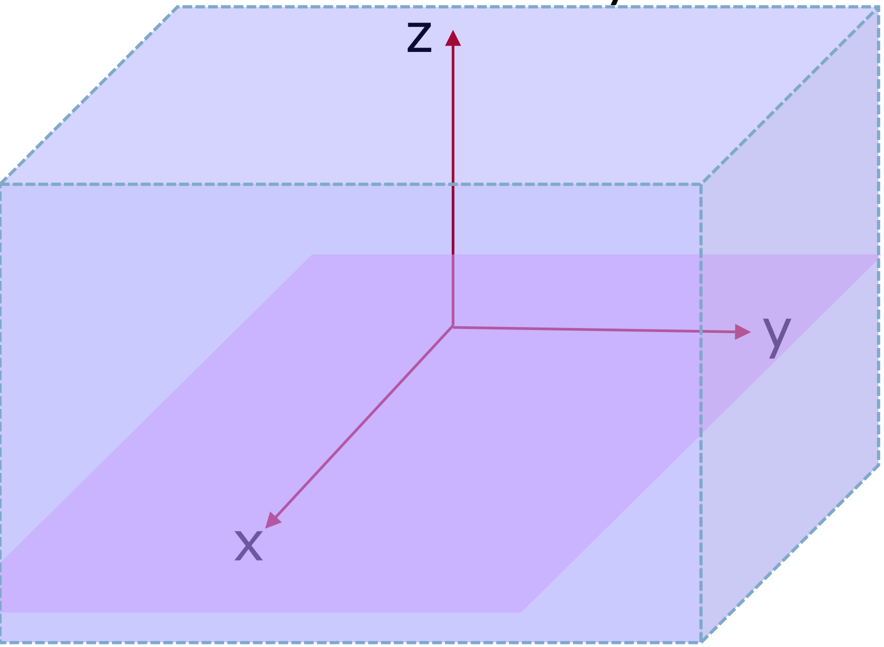
Is $\mathbf{R}^2$ a subset of $\mathbf{R}^3$?

No

Geometrically



Algebraically

$$\mathbf{R}^2 = \{ (x, y) \mid x, y \in \mathbf{R} \}$$

$$\mathbf{R}^3 = \{ (x, y, z) \mid x, y, z \in \mathbf{R} \}$$

These are called
linear combinations of (2,1,0) and (-3,0,1)

What do the following vectors have in common?

(-1, 1, 1), (7, 2, -1) and (4, 2, 0)

They are all of the form:

$$s(2, 1, 0) + t(-3, 0, 1)$$

Take s = 1, t = 1

$$\boxed{1(2, 1, 0) + 1(-3, 0, 1)} = (-1, 1, 1)$$

Take s = 2, t = -1

$$\boxed{2(2, 1, 0) + (-1)(-3, 0, 1)} = (7, 2, -1)$$

Take s = 2, t = 0

$$\boxed{2(2, 1, 0) + 0(-3, 0, 1)} = (4, 2, 0)$$

$\boldsymbol{u_1} = (2, 1, 3, 1)$, $\boldsymbol{u_2} = (1, -1, 2, 2)$, $\boldsymbol{u_3} = (3, 0, 5, 1)$

Linear Combinations

$\boldsymbol{v}$ is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$

$\boldsymbol{v}$ is not a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$

can find a,b,c → $\boxed{\boldsymbol{v} = a\boldsymbol{u_1} + b\boldsymbol{u_2} + c\boldsymbol{u_3}}$ ← cannot find a,b,c

linear system has solution

$\boldsymbol{v} = (3, 3, 4, 0)$

$$(3, 3, 4, 0) = a(2, 1, 3, 1) + b(1, -1, 2, 2) + c(3, 0, 5, 1)$$

$$\begin{cases} 2a + b + 3c = 3 \\ a - b = 3 \\ 3a + 2b + 5c = 4 \\ a + 2b + c = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \\ 1 & 2 & 1 & 0 \end{array}\right) \xrightarrow{\text{G.E}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\boldsymbol{u_1} = (2, 1, 3, 1)$, $\boldsymbol{u_2} = (1, -1, 2, 2)$, $\boldsymbol{u_3} = (3, 0, 5, 1)$

Linear Combinations

v is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$

v is not a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$

can find a,b,c → $\boxed{\boldsymbol{v} = a\boldsymbol{u_1} + b\boldsymbol{u_2} + c\boldsymbol{u_3}}$ ← cannot find a,b,c

linear system has no solution

$\boldsymbol{v} = (3, 3, 4, 1)$

$$(3, 3, 4, 1) = a(2, 1, 3, 1) + b(1, -1, 2, 2) + c(3, 0, 5, 1)$$

$$\begin{cases} 2a + b + 3c = 3 \\ a - b = 3 \\ 3a + 2b + 5c = 4 \\ a + 2b + c = 1 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \\ 1 & 2 & 1 & 1 \end{array}\right) \xrightarrow{\text{G.E}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Linear System (3 forms)

$$(3, 3, 4, 0) = a(2, 1, 3, 1) + b(1, -1, 2, 2) + c(3, 0, 5, 1)$$

$$\begin{cases} 2a & + & b & + & 3c & = & 3 \\ a & - & b & & & = & 3 \\ 3a & + & 2b & + & 5c & = & 4 \\ a & + & 2b & + & c & = & 0 \end{cases}$$

standard form

$$a\begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix} + c\begin{pmatrix} 3 \\ 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

vector equation form

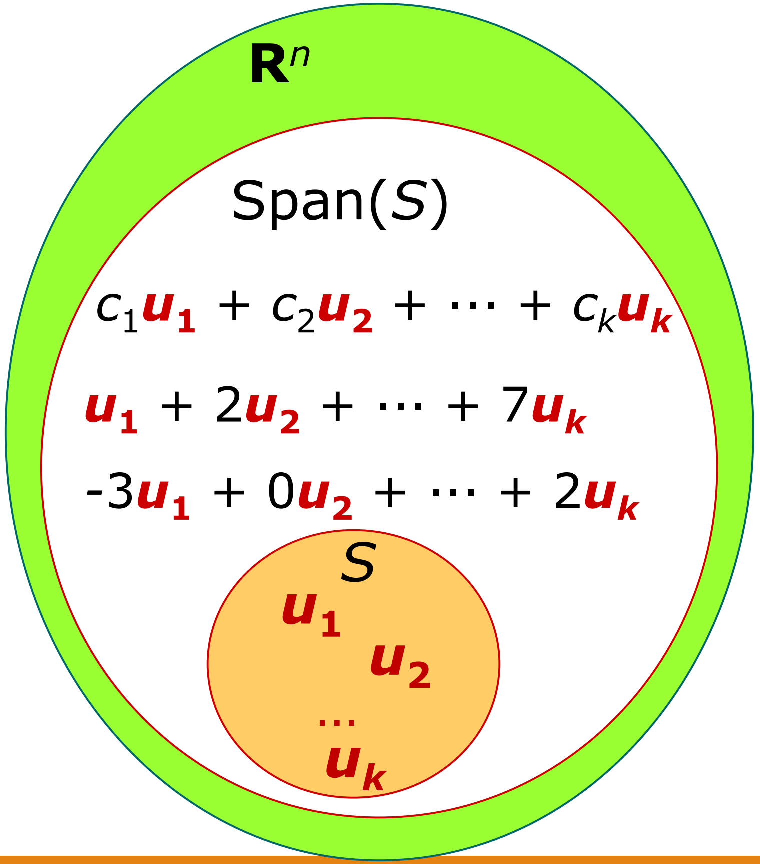
$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

matrix equation form

span(S) is the set of
all linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$

Linear Span

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ a finite collection of vectors in $\mathbf{R}^n$



$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \in \mathbf{R}^n$

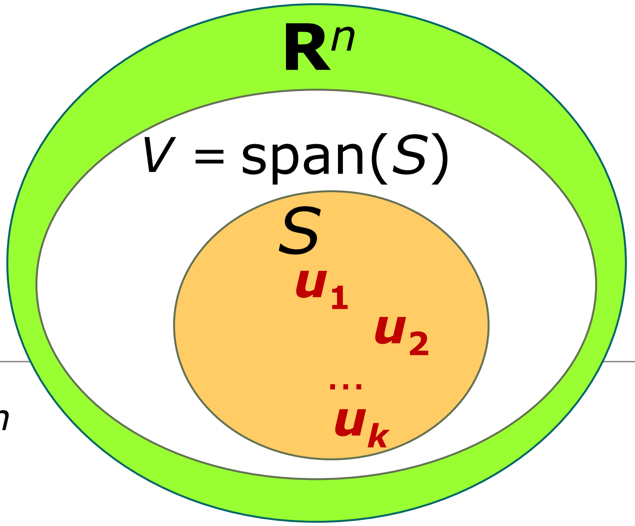
$S \subseteq \mathbf{R}^n$

$\text{span}(S) \subseteq \mathbf{R}^n$

$S \subseteq \text{span}(S)$

span(S) can be equal to $\mathbf{R}^n$ but not always.

The word "Span"



$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbf{R}^n$

Let V = span(S) = span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$

We say:

noun

- V is a linear span of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$
- V is a linear span of S

passive verb

- V is spanned by $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$
- V is spanned by S

active verb

- $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ spans V
- S spans V

Who's in the Span?

standard basis vectors for $\mathbf{R}^2$

Which of the following vectors belong to span{ (1,0), (0,1) }?

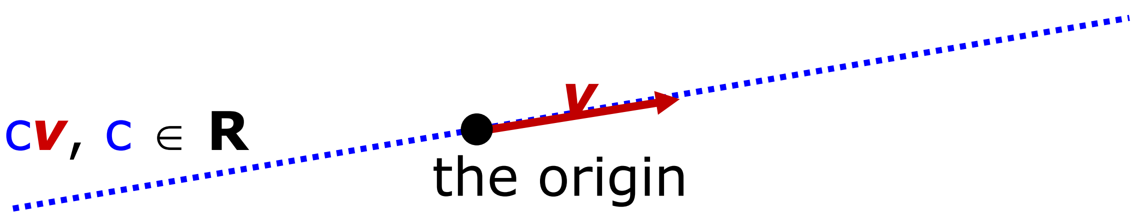
- (0,0)
- (1,0)
- (1,1)
- (1,0,0,1)
- (10.9, 2020)

Check whether each vector is a linear combination of (1,0), (0,1).

Are there vectors that do not belong to span{ (1,0), (0,1) }?

Geometrical meaning of Span

- span{$\boldsymbol{v}$} $\boldsymbol{v} \neq \mathbf{0}$
- all scalar multiples $c\boldsymbol{v}$, $c \in \mathbf{R}$



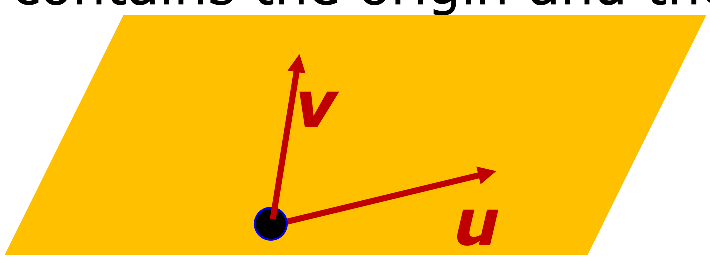
In $\mathbf{R}^2$ and $\mathbf{R}^3$

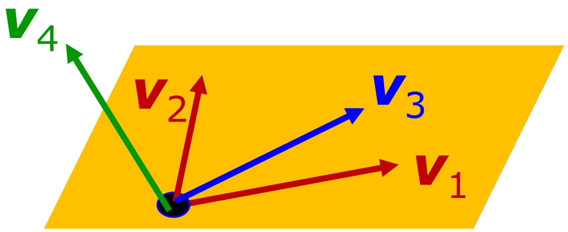
- A line that passes through the origin and parallel to $\boldsymbol{v}$

- span{$\boldsymbol{u}$, $\boldsymbol{v}$} $\boldsymbol{u}, \boldsymbol{v} \neq \mathbf{0}$, not parallel to each other
- all linear combinations $c\boldsymbol{u} + d\boldsymbol{v}$, $c, d \in \mathbf{R}$

In $\mathbf{R}^2$ and $\mathbf{R}^3$

- A plane that contains the origin and the vectors $\boldsymbol{u}$, $\boldsymbol{v}$





Geometrical meaning of Span

$\boldsymbol{v_1} = (2, 1, 3)$, $\boldsymbol{v_2} = (1, -1, 2)$, $\boldsymbol{v_3} = (3, 0, 5)$, $\boldsymbol{v_4} = (1, 2, 4)$

span$\{\boldsymbol{v_1}\}$ represents a line

span$\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ represents a plane

span$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ represents the same plane

span$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ represents the entire $\mathbf{R}^3$

Set relations between spans

- Show span(S) ⊆ span(T)
- Show span(S) = span(T)
- Show span(S) ≠ span(T)

Show span(S) ⊆ span(T)

$S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$ $T = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_m\}$ column form

Every vector of S is a linear combination of vectors in T

$(\ \mathbf{t}_1\ \mathbf{t}_2\ \ldots\ \mathbf{t}_m\ |\mathbf{s}_1|\mathbf{s}_2|\ldots|\mathbf{s}_n\ )$

Check that this multiple-augmented matrix is consistent

REF has no pivot columns among the augmented columns

Show span(S) = span(T)

$S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$ $T = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_m\}$ column form

Check $\text{span}(S) \subseteq \text{span}(T)$ AND $\text{span}(T) \subseteq \text{span}(S)$

Every vector of S is a linear combination of vectors in T

$$(\ \mathbf{t}_1\ \mathbf{t}_2\ \ldots\ \mathbf{t}_m\ |\mathbf{s}_1|\mathbf{s}_2|\ldots|\mathbf{s}_n\ )$$

And every vector of T is a linear combination of vectors in S

$$(\ \mathbf{s}_1\ \mathbf{s}_2\ \ldots\ \mathbf{s}_n\ |\mathbf{t}_1|\mathbf{t}_2|\ldots|\mathbf{t}_m\ )$$

Check that both multiple-augmented matrices are consistent

Show span(S) $\neq$ span(T)

$S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$ $T = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_m\}$ column form

Check span(S) $\not\subseteq$ span(T) OR span(T) $\not\subseteq$ span(S)

Some vector of S is not a linear combination of vectors in T

$$(\mathbf{t}_1\ \mathbf{t}_2\ \ldots\ \mathbf{t}_m \mid \mathbf{s}_1 \mid \mathbf{s}_2 \mid \ldots \mid \mathbf{s}_n)$$

Or some vector of T is not a linear combination of vectors in S

$$(\mathbf{s}_1\ \mathbf{s}_2\ \ldots\ \mathbf{s}_n \mid \mathbf{t}_1 \mid \mathbf{t}_2 \mid \ldots \mid \mathbf{t}_m)$$

Check that at least one of the multiple-augmented matrices is inconsistent

Show span(S) = $\mathbf{R}^n$

$S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ column form

Check span(S) $\subseteq \mathbf{R}^n$ AND $\mathbf{R}^n \subseteq$ span(S)

- span(S) $\subseteq \mathbf{R}^n$ is automatic (nothing to check)
- To check $\mathbf{R}^n \subseteq$ span(S)
  - Take a general vector $\mathbf{x} \in \mathbf{R}^n$
  - Check $(\ \mathbf{s}_1\ \mathbf{s}_2\ \ldots\ \mathbf{s}_k\ |\mathbf{x})$ is consistent

It is enough to check:
REF of $(\ \mathbf{s}_1\ \mathbf{s}_2\ \ldots\ \mathbf{s}_k)$ has no zero row

$S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ $\quad \mathbf{s}_1 = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix}, \mathbf{s}_2 = \begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{pmatrix}, \ldots, \mathbf{s}_k = \begin{pmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{kn} \end{pmatrix}.$

Show span(S) = $\mathbf{R}^n$

Consider the linear system

general vector $\mathbf{x} \in \mathbf{R}^n$

$$\underbrace{\left(\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{k1} & x \\ a_{12} & a_{22} & \cdots & a_{k2} & y \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{kn} & z \end{array}\right)}_{\mathbf{A}} \xrightarrow{\text{G.E.}} \underbrace{\left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \ddots & * & \vdots & \vdots \\ 0 & \cdots & 0 & * & * \end{array}\right)}_{\mathbf{R}} \text{ REF}$$

R has no zero row
$\Rightarrow$ system is always consistent
$\Rightarrow$ span$\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\} = \mathbf{R}^n$

R has a zero row
$\Rightarrow$ system may be inconsistent
$\Rightarrow$ span$\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\} \neq \mathbf{R}^n$

If $k < n$, then span$\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\} \neq \mathbf{R}^n$

If $k \geq n$, span$\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ may or may not be equal to $\mathbf{R}^n$

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Exercise 3 Q13

Suppose span$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = \mathbf{R}^3$.

Determine which sets below span $\mathbf{R}^3$ as well.

$S_1 = \{\boldsymbol{u}, \boldsymbol{v}\}$

$S_2 = \{\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{u}\}$

$S_4 = \{\boldsymbol{u}, \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}\}$

$S_1 = \{\boldsymbol{u}, \boldsymbol{v}\}$ has only two vectors, and cannot span $\mathbf{R}^3$.

In fact span$\{\boldsymbol{u}, \boldsymbol{v}\}$ represents a plane in the 3D space.

$S_2 = \{\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{u}\}$ cannot span $\mathbf{R}^3$.

$\boldsymbol{w} - \boldsymbol{u} = -(\boldsymbol{u} - \boldsymbol{v}) - (\boldsymbol{v} - \boldsymbol{w}) \in$ span$\{\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{w}\}$

The 3 vectors are on the same plane, so span(S_2) represents a plane.

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Exercise 3 Q13

Suppose $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = \mathbf{R}^3$.

Determine which sets below span $\mathbf{R}^3$ as well.

$S_1 = \{\boldsymbol{u}, \boldsymbol{v}\}$

$S_2 = \{\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{u}\}$

$S_4 = \{\boldsymbol{u}, \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}\}$

$S_4 = \{\boldsymbol{u}, \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}\}$

$$\boldsymbol{v} = (\boldsymbol{u} + \boldsymbol{v}) - \boldsymbol{u}$$

$$\boldsymbol{w} = (\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}) - (\boldsymbol{u} + \boldsymbol{v})$$

$\Rightarrow \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \text{span}(S_4)$

$\Rightarrow \text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \subseteq \text{span}(S_4)$ So $\text{span}(S_4) = \mathbf{R}^3$.

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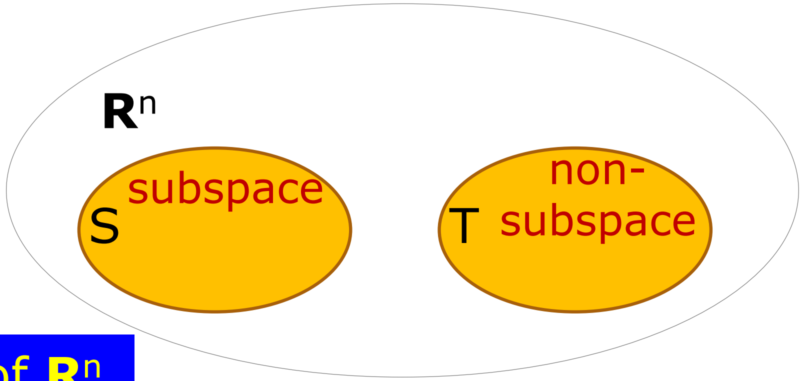
Subspaces

S is a subset of $\mathbf{R}^n$

T is a subset of $\mathbf{R}^n$



Two types of subsets of $\mathbf{R}^n$

Subspaces	Non-subspaces
• Can be written as linear span $S = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_k\}$ • Satisfy closure properties	• Cannot be written as linear span $T \neq \text{span}\{\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_k\}$ • Violate closure properties

Subspaces of $\mathbf{R}^3$

$\mathbf{R}^3$

Subspaces

- span$\{\mathbf{0}\}$

just one point (origin)

- span$\{\mathbf{v}_1\}$

a line that passes through origin

- span$\{\mathbf{v}_1, \mathbf{v}_2\}$

a plane that contains the origin

- span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

the entire 3D space

Non-subspaces

- A point that is not the origin
- Line or plane that does not contain the origin
- A (space) curve or a bended surface (e.g. paraboloid)
- A rectangular block
- etc

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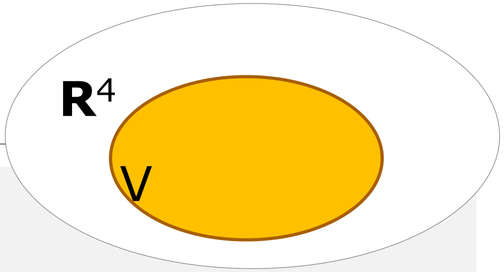
Subspaces (Example 1)



$V = \{ (a, b, a+b, 0) \mid a, b \in \mathbf{R} \}$

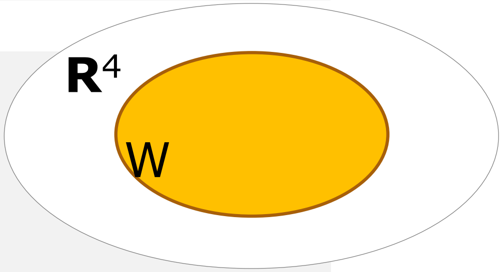
Is V a subspace of $\mathbf{R}^4$?

Can we write $V = \text{span}\{ \mathbf{u}_1, \mathbf{u}_2, \ldots\}$?

general vector: $(a, b, a+b, 0) = a(1, 0, 1, 0) + b(0, 1, 1, 0)$

So $V = \text{span}\{ (1, 0, 1, 0), (0, 1, 1, 0) \}$

$\Rightarrow$ V is a subspace



$W = \{ (a, b, a+b, 1) \mid a, b \in \mathbf{R} \}$

Is W a subspace of $\mathbf{R}^4$?

$\Rightarrow$ W is not a subspace

$(1,1,2,1) \in W$, $(1,0,1,1) \in W$

but $(1,1,2,1) + (1,0,1,1) = (2,1,3,2) \notin W$

closure property under addition not satisfied

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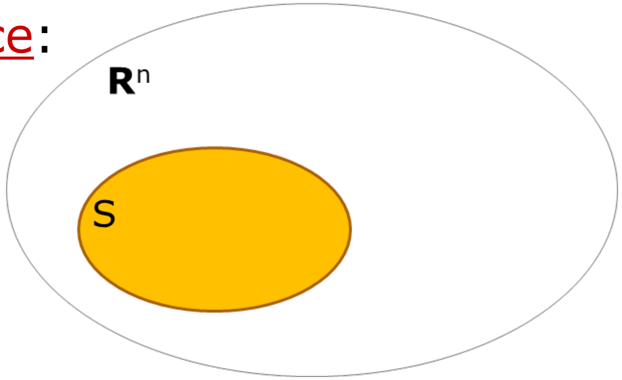
Testing Subspace

To test whether a subset S of $\mathbf{R}^n$ is a subspace:



If S can be expressed as a linear span

- ➢ then S is a subspace

If one of the conditions below occurs:

- the zero vector is not in S
- you can find $\mathbf{u}, \mathbf{v} \in S$ but $\mathbf{u} + \mathbf{v} \notin S$ — give examples of $\mathbf{u}$ and $\mathbf{v}$
- you can find $\mathbf{v} \in S$ and a scalar c such that $c\mathbf{v} \notin S$

➢ then S is not a subspace — give examples of $\mathbf{v}$ and c

S is not a subspace:

- if the zero vector is not in S
- if there are **u**, **v** $\in$ S but **u** + **v** $\notin$ S
- if there is a **v** $\in$ S and a scalar c such that c**v** $\notin$ S

Subspaces (Example 2)

Which of the following subsets of $\mathbf{R}^2$ are subspaces of $\mathbf{R}^2$?

The set of all vectors that are scalar multiples of (2,3)

c(2,3)

span{(2,3)} → it is a subspace

The set of all vectors of the form (a, a^2 + 1) for any a.

does not contain zero vector (0,0) → it is not a subspace

The set of all vectors that has either 0 in the first or second component

(0,1) and (1,0) both belong to the set,
but (0,1)+(1,0) = (1,1) does not.

→ it is not a subspace

The set of all vectors where both components are non-negative

(1,1) belongs to the set,
but (-1)(1,1) = (-1,-1) does not.

→ it is not a subspace

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Solution set of linear system

Ax = b linear system with n variables

system is homogeneous
Ax = 0

Solution space

solution set is a subspace of $\mathbf{R}^n$

system is non-homogeneous
Ax = b

solution set is not a subspace of $\mathbf{R}^n$

If **u**, **v** are solutions of **Ax = 0**,

- **u** + **v** also a solution
- c**u** also a solution

If **u**, **v** are solutions of **Ax = b**,

- **u** + **v** not a solution
- c**u** not a solution

Exercise 3 Q22

Let $\boldsymbol{A}$ be a fixed n×n matrix.

Show that $\{\, \boldsymbol{u} \in \mathbf{R}^n \mid \boldsymbol{Au} = \boldsymbol{u} \}$ is a subspace of $\mathbf{R}^n$.

$$\{\, \boldsymbol{u} \in \mathbf{R}^n \mid \boldsymbol{Au} = \boldsymbol{u} \} \;=\; \{\, \boldsymbol{u} \in \mathbf{R}^n \mid (\boldsymbol{A} - \boldsymbol{I})\boldsymbol{u} = \boldsymbol{0} \}$$

Implicit set notation with underlying condition: $\boldsymbol{Au} = \boldsymbol{u}$

$$\boldsymbol{Au} = \boldsymbol{u} \Leftrightarrow \boldsymbol{Au} - \boldsymbol{u} = \boldsymbol{0} \quad \Leftrightarrow (\boldsymbol{A} - \boldsymbol{I})\boldsymbol{u} = \boldsymbol{0}$$

subspace of $\mathbf{R}^n$

Solution set of the homogeneous system $(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$

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Announcement

- Practice Session
  - Practice 2 this week
- Online quiz 5
  - Due this Sunday
- Homework 1
  - will be published this Friday
  - Deadline: 2 October (week 7)
- MATLAB
  - Worksheet 2 next week