MA 1521

Tutorial 4 Solutions

1. (a) Observe that $\sec^2 x > 0$ and $-4\sin^2 x \le 0$ on $[-\pi/3,\ \pi/3]$.

$$\begin{aligned}
\text{Area } &= \int_{-\pi/3}^{\pi/3} \left[\frac{1}{2}\sec^2 x - (-4\sin^2 x)\right] dx \\
&= \left[\frac{1}{2}\tan x + \int (2 - 2\cos 2x)\, dx\right]_{-\pi/3}^{\pi/3} \\
&= \tan\frac{\pi}{3} + (2x - \sin 2x)\Big|_{-\pi/3}^{\pi/3} \\
&= \sqrt{3} + \frac{4}{3}\pi - 2\sin\frac{\pi}{3} = \frac{4}{3}\pi.
\end{aligned}$$

(b) The points of intersection: $x = x^2/4$ implies $x = 0$ or $x = 4$. Hence the points of intersection are $(0,0)$ and $(4,4)$.

Note that $y = x^2/4 \Leftrightarrow x = 2\sqrt{y}$.

$$\text{The required area } = \int_0^1 \left[2\sqrt{y} - (y)\right] dy = \left[\frac{4}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^1 = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}.$$

(c) We have that $(2-x)-(4-x^2) = x^2-x-2 = (x+1)(x-2)$

is negative if and only if $x \in (-1,2)$.

Hence

$$\begin{aligned}
\text{Area } &= \int_{-2}^{3} \big|(2-x)-(4-x^2)\big|\, dx \\
&= \Big[\int_{-2}^{-1} + \int_{2}^{3}\Big](x^2-x-2)\, dx + \int_{-1}^{2} -(x^2-x-2)\, dx \\
&= \Big[\int_{-2}^{3} - 2\int_{-1}^{2}\Big](x^2-x-2)\, dx \\
&= \Big[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\Big]_{-2}^{3} - 2\Big[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\Big]_{-1}^{2} \\
&= \frac{1}{3}\big[(27+8)-2(8+1)\big] - \frac{1}{2}\big[(9-4)-2(4-1)\big] - 2\big[5-2(3)\big] \\
&= \frac{1}{3}17 + \frac{1}{2} + 2 = \frac{49}{6}.
\end{aligned}$$

2. (a) The parabola and the line meet at (x,y) with $3 = y^2+1$, i.e. at $(3, \pm\sqrt{2})$.

By formula,

$$\begin{aligned}
\text{Volume } &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi\big[(y^2+1)-3\big]^2\, dy = \pi\int_{-\sqrt{2}}^{\sqrt{2}} \big[y^4-4y^2+4\big]\, dy \\
&= \pi\Big[\frac{1}{5}y^5 - \frac{4}{3}y^3 + 4y\Big]_{-\sqrt{2}}^{\sqrt{2}} \\
&= \pi\, 2\Big[\frac{1}{5}4\sqrt{2} - \frac{4}{3}2\sqrt{2} + 4\sqrt{2}\Big] = \frac{64}{15}\sqrt{2}\pi.
\end{aligned}$$

(b) The parabola and the line meet at (x,y) with $x^2 = 2x$, i.e. at $(0,0)$ and $(2,4)$.

Now $y = 2x \Leftrightarrow x = y/2$ and $y = x^2 \Leftrightarrow x = \sqrt{y}$, while $\sqrt{y} - (y/2) = \sqrt{y}\,(1-\sqrt{y}/2)$ is positive for $y \in (0,4)$.

So $x = \sqrt{y}$ is the outer curve and $x = y/2$ is the inner curve. Hence,

$$\begin{aligned}
\text{volume } &= \text{volume of space enclosed by outer shell} - \text{volume of hole enclosed by inner shell} \\
&= \int_0^4 \pi\sqrt{y}^2\, dy - \int_0^4 \pi\Big(\frac{y}{2}\Big)^2 dy = \pi\frac{1}{2}\big[4^2-0^2\big] - \pi\frac{1}{4}\frac{1}{3}\big[4^3-0^3\big] = \frac{8}{3}\pi.
\end{aligned}$$

3. Let $I = \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$.

Apply the substitution $x = a\sin\theta$, we have

$I = \int_0^{\frac{\pi}{2}} \frac{a\cos\theta}{a\sin\theta + a\cos\theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta} d\theta$.

Now observe that $\cos\theta = \frac{1}{2}(\cos\theta+\sin\theta) + \frac{1}{2}(\cos\theta - \sin\theta)$. Therefore

$I = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}(\cos\theta+\sin\theta)+\frac{1}{2}(\cos\theta-\sin\theta)}{\sin\theta+\cos\theta} d\theta = \frac{\pi}{4} + \big[\frac{1}{2}\ln|\sin\theta+\cos\theta|\big]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$.

4. Solving the simultaneous equations $y^2 = x + 4a^2$ and $x - ay + 2a^2 = 0$ by eliminating x we have $y^2 - ay - 2a^2 = 0$ and so $y = -a$ or $y = 2a$.

   Area $= \int_{-a}^{2a} \left[\left(ay - 2a^2\right) - \left(y^2 - 4a^2\right)\right] dy = \left[\frac{1}{2}ay^2 + 2a^2y - \frac{1}{3}y^3\right]_{-a}^{2a} = \frac{9}{2}a^3.$

5. Volume $= \int_0^{\frac{\pi}{4}} \pi \left(\sqrt{\tan x}\right)^2 dx = \left[-\pi \ln \cos x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{2} \ln 2.$