NATIONAL UNIVERSITY OF SINGAPORE

CS1231S DISCRETE STRUCTURES

(Semester 2: AY2019/2020)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. This assessment paper contains **FOUR** questions and comprises **FOUR** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated in brackets.
3. Write your answers on your own paper.

EXAMINER'S USE ONLY		
Question	Marks	Score
Q1	7	
Q2	8	
Q3	21	
Q4	14	
Total	50	

1. For a set X, the identity function is $i_X : X \to X$ such that $i_X(x) = x$ for all $x \in X$.

   (i) Give an example of a function $f : \{a, b\} \to \{a, b\}$ such that $f \neq i_{\{a,b\}}$ and f is bijective. [1 mark]

   (ii) Give an example of a function $g : \{a, b, c\} \to \{a, b, c\}$ such that $g \neq i_{\{a,b,c\}}$ and $g \circ g$ is bijective. [2 marks]

   (iii) Suppose $h : X \to X$ is a function such that $h \circ h$ is 1-1 (injective). Prove that h is 1-1. [2 marks]

   (iv) Suppose $h : X \to X$ is a function such that $h \circ h$ is onto (surjective). Prove that h is onto. [2 marks]

2. Recall from the Assignment and Quiz2 the equivalence relation $\approx$ on $\mathbb{R}$ defined by

$$\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ x \approx y \leftrightarrow \lfloor x \rfloor = \lfloor y \rfloor.$$

   The equivalence classes are $I_n = \{x \in \mathbb{R} \mid n \leq x < n + 1\}$, where $n \in \mathbb{Z}$.

   (i) Let $k \in \mathbb{Z}$. Explain why, if $k \in I_n$, then $k = n$. [2 marks]

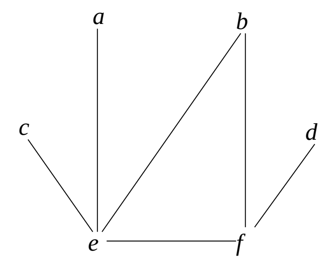
   (ii) Let $\mathcal{I} = \{I_n \mid n \in \mathbb{Z}\}$. Prove that $\mathcal{I}$ is countable. [3 marks]

   (iii) It is known that $\mathbb{R}$ is uncountable. Prove that I_n is uncountable for every $n \in \mathbb{Z}$. [3 marks]

3. Let G be an undirected graph, and $G = (V, E)$. Recall that a subgraph of the form $(\{x_1, \ldots, x_p\}, \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{p-1}, x_p\}\})$ is called a **path** between x_1 and x_p in G, and this path has **length** $p - 1$.

   Now, for integer $k \geq 1$, define
   $E_k = \{\{x, y\} \mid x \neq y$ and there is a path of length k in G between x and $y\}$,
   and let $G_k = (V, E_k)$. Note that $E_1 = E$ and $G_1 = G$.

   Thus, for the undirected graph in Figure 1,
   we have $\{e, b\} \in E_1$, $\{e, d\} \in E_2$, $\{e, d\} \in E_3$, etc.



(V, E)

Figure 1

   (i) List the elements of V and E for Figure 1. [2 marks]

   (ii) What is the length of the longest path in Figure 1? [1 mark]

   (iii) Draw all spanning trees for the graph in Figure 1. [3 marks]

   (iv) Draw G_2, G_3, G_4 and G_5 for the graph in Figure 1. [4 marks]

   (v) Identify all (if any) cyclic graphs in (iv). [1 mark]

   (vi) Identify all (if any) connected graphs in (iv). [1 mark]

   (vii) Among G_2, G_3, G_4 and G_5, which (if any) are trees? [1 mark]

   (viii) In (iv), how many connected components does G_4 have? [1 mark]

   (ix) For this part, consider any G (not just the one in Figure 1). Prove that G is connected if and only if $\{x, y\} \in \bigcup_{k=1}^{\infty} E_k$ for every $x, y \in V$ such that $x \neq y$. [2 marks]

   (x) Determine the number of graphs (with the same V) that are isomorphic to the graph in Figure 1. [5 marks]

4. Let T be a rooted binary tree of height h. For $h \geq 1$, we call T a **strand** if and only if the following holds:

   (I) there is exactly one leaf and one parent at every level ℓ, for $1 \leq \ell \leq h-1$ and

   (II) there are exactly two leaves at level h.

   Figure 2 below illustrates three strands T_1, T_2 and T_3.

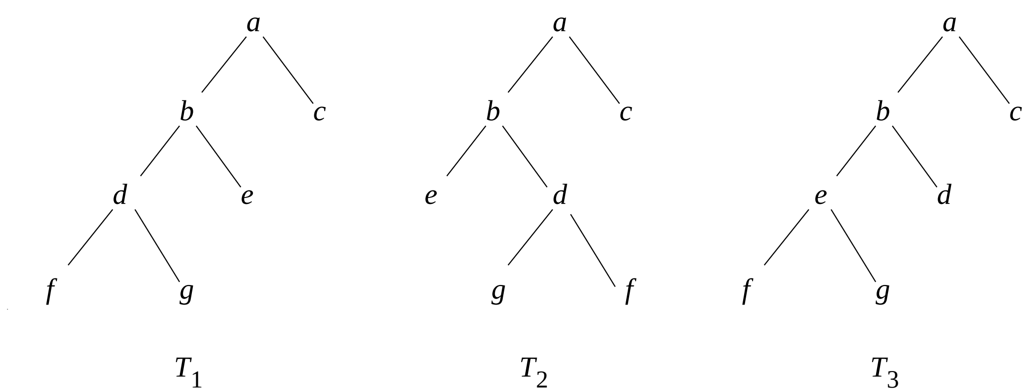


Figure 2

   (i) Is $T_1 = T_2$? Is $T_2 = T_3$? Justify your answers. [2 marks]

   (ii) Prove that, for any $h \geq 1$, a strand of height h has $2h+1$ nodes. [2 marks]

   Let $N(1) = 3$ and, for $h > 1$, let $N(h)$ be the number of different strands of height h, whose nodes are $\{v_1, v_2, \ldots, v_{2h+1}\}$.

   (iii) Prove that $N(h) = 2(2h+1)hN(h-1)$ for integer $h > 1$. [5 marks]

   (iv) Use induction to prove that $N(h) = \frac{(2h+1)!}{2}$ for every positive integer h. [5 marks]

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