2. The Logic of Compound Statements (aka Propositional Logic)

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2.1 Logical Form and Logical Equivalence

- Statements; Compound Statements; Statement Form (Propositional Form)
- Logical Equivalence; Tautologies and Contradictions

2.2 Conditional Statements

- Conditional Statements; If-Then as Or
- Negation, Contrapositive, Converse and Inverse
- Only If and the Biconditional; Necessary and Sufficient Conditions

2.3 Valid and Invalid Arguments

- Argument; Valid and Invalid Arguments
- Modus Ponens and Modus Tollens
- Rules of Inference
- Fallacies

Reference: Epp's Chapter 2 The Logic of Compound Statements

At the end of this lecture, you should be able to solve this puzzle:

- You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

  a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

  b. If my glasses are on the kitchen table, then I saw them at breakfast.

  c. I did not see my glasses at breakfast.

  d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

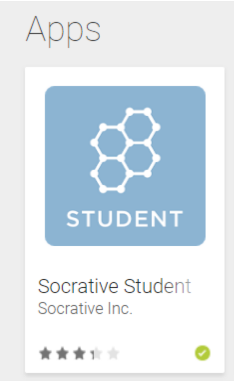
  e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

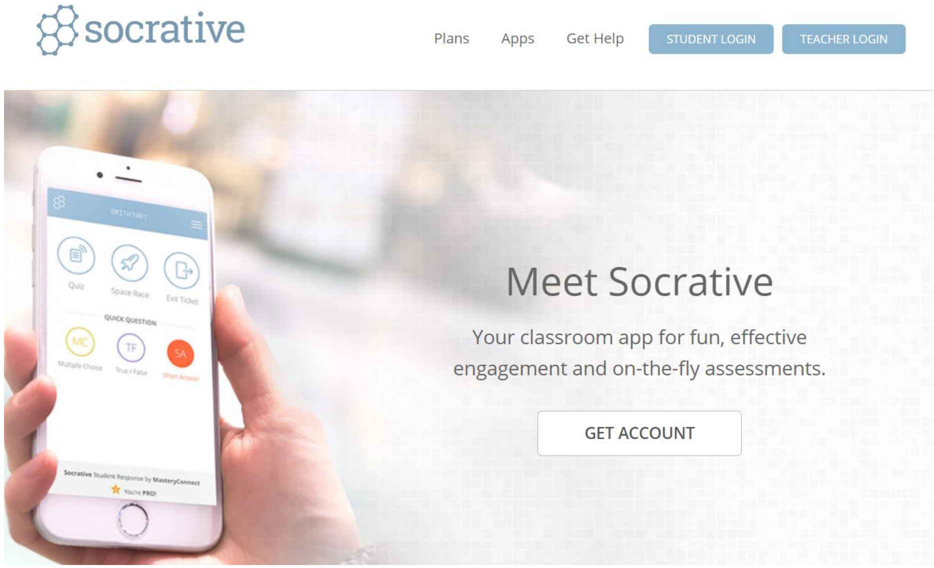
So, where are your glasses?

Another puzzle!

To be given out during lecture.

Remember to download Socrative Student Mobile App before the lecture.





Touted as the logic question that almost everyone gets wrong.

2.1 Logical Form and Logical Equivalence

Example

If Jane is a math major or Jane is a computer science major, then Jane will take MA1101R.

Jane is a computer science major.

Therefore, Jane will take MA1101R.

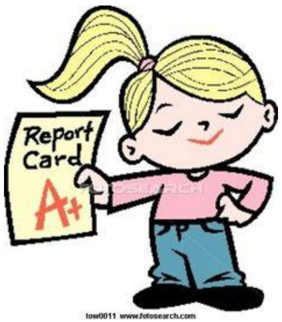
If CS1231 is easy or

____________________, then

______________________________.

I study hard.

Therefore, I will get A+ in this course.



2.1.1. Statements

If Jane is a math major or Jane is a computer science major, then Jane will take MA1101R.

Jane is a computer science major.

Therefore, Jane will take MA1101R.

Statements

Definition 2.1.1 (Statement)

A **statement** (or **proposition**) is a sentence that is true or false, but not both.

Common Form

If Jane is a math major or Jane is a computer science major, then Jane will take MA1101R.

Jane is a computer science major.

Therefore, Jane will take MA1101R.

If p or q, then r.

q.

Therefore, r.

Statement variables

2.1.2. Compound Statements

Logical connectives:

$\sim$ also $\neg$ *Not/negation*

$\wedge$ *and*

$\vee$ *or*

Truth tables:

p	$\sim p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 2.1.2 (Negation)

If p is a statement variable, the **negation** of p is "not p" or "it is not the case that p" and is denoted $\sim p$.

Definition 2.1.3 (Conjunction)

If p and q are statement variables, the **conjunction** of p and q is "p and q", denoted $p \wedge q$.

Definition 2.1.4 (Disjunction)

If p and q are statement variables, the **disjunction** of p and q is "p or q", denoted $p \vee q$.

- Order of operations:
  - ~ is performed first
  - $\wedge$ and $\vee$ are coequal in order of operation

$$\sim p \wedge q = (\sim p) \wedge q$$

$$p \wedge q \vee r$$

Ambiguous

- Use parentheses to override or disambiguate order of operations

$$\sim(p \wedge q)$$

Negation of $p \wedge q$

$$(p \wedge q) \vee r \qquad p \wedge (q \vee r)$$

Unambiguous

In some other modules, different symbols are used, such as $\cdot$ for conjunction and $+$ for disjunction in CS2100. Others use $\neg$ or $\overline{}$ for negation. In CS2100, conjunction is performed before disjunction. We shall follow the symbols and order of operations here.

Compound Statements: Quick Quiz

- Given:
  - h = "It is hot"
  - s = "It is sunny"
- Write logical statements for the following:

a. "It is not hot but it is sunny."

b. "It is neither hot nor sunny."

2.1.3. Statement Form (Propositional Form)

- Examples:

$\sim p \vee q$ $\quad$ $(p \vee q) \wedge \sim(p \wedge q)$ $\quad$ $(p \wedge q) \vee r$

Definition 2.1.5 (Statement Form/Propositional Form)

A **statement form** (or **propositional form**) is an expression made up of statement variables and logical connectives that becomes a statement when actual statements are substituted for the component statement variables.

- Construct the truth table for this statement form:

$$(p \lor q) \land \sim(p \land q)$$

p	q	$p \lor q$	$p \land q$	$\sim(p \land q)$	$(p \lor q) \land \sim(p \land q)$
T	T				
T	F				
F	T				
F	F				

$(p \lor q) \land \sim(p \land q)$ is also known as exclusive-or (why?)

Denoted as $p \oplus q$ or p XOR q.

2.1.4. Logical Equivalence

(1) Dogs bark and cats meow.

(2) Cats meow and dogs bark.

If (1) is true, it follows that (2) must also be true.

On the other hand, if (1) is false, it follows that (2) must also be false.

(1) and (2) are logically equivalent statements.

Definition 2.1.6 (Logical Equivalence)

Two statement forms are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables.

The logical equivalence of statement forms P and Q is denoted by $\boldsymbol{P \equiv Q}$.

- Example:

a	b	$a \wedge b$	$b \wedge a$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$a \wedge b$ and $b \wedge a$ always have the same truth values, hence they are logically equivalent.

Logical Equivalence: Double Negative Property

- Double negation:

$$\sim(\sim p) \equiv p$$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

- To show that statement forms *P* and *Q* are not logically equivalent, there are 2 ways:
  - Truth table – find at least one row where their truth values differ.
  - Find a counter example – concrete statements for each of the two forms, one of which is true and the other of which is false.

Logical Equivalence: Showing Non-equivalence

- Show that the following 2 statement forms are not logically equivalent.

$\sim(p \wedge q)$ $\sim p \wedge \sim q$

- Truth table method:

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

- Show that the following 2 statement forms are not logically equivalent.

$\sim(p \wedge q)$ $\sim p \wedge \sim q$

- Counter-example method:

Let p be the statement "$0 < 1$" and
q the statement "$1 < 0$".

$\sim(p \wedge q)$ "Not the case that both 0<1 and 1<0" which is TRUE.

$\sim p \wedge \sim q$ "Not 0<1" and "not 1<0" which is FALSE.

- De Morgan's Laws:

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

Can be extended to more than two variables.

- Write negations for each of the following:

a. John is 6 feet tall and he weighs at least 200 pounds.

John is not 6 feet tall or he weighs less than 200 pounds.

b. The bus was late or Tom's watch was slow.

The bus was not late and Tom's watch was not slow.

or Neither was the bus late nor was Tom's watch slow.

2.1.5. Tautologies and Contradictions

Definition 2.1.7 (Tautology)

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement**.

Definition 2.1.8 (Contradiction)

A **contradiction** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is a **contradictory statement**.

- Logical equivalence involving tautologies and contradictions

Example: If **t** is a tautology and **c** is a contradiction, show that:

$p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$

p	$\mathbf{t}$	$\mathbf{c}$	$p \wedge \mathbf{t}$	$p \wedge \mathbf{c}$
T	T	F	T	F
F	T	F	F	F

As **t** and **c** (used in the textbook) are hard to distinguished from statement variables, we will use **true** and **false** instead.

2.1.6. Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables *p*, *q* and *r*, a tautology **true** and a contradiction **false**:

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r)$ $\equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$ $\equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge \textbf{true} \equiv p$	$p \vee \textbf{false} \equiv p$
5	Negation laws	$p \vee {\sim}p \equiv \textbf{true}$	$p \wedge {\sim}p \equiv \textbf{false}$
6	Double negative law	${\sim}({\sim}p) \equiv p$	

2.1.6. Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalences (continue)

Given any statement variables *p*, *q* and *r*, a tautology **true** and a contradiction **false**:

7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee$ **true** $\equiv$ **true**	$p \wedge$ **false** $\equiv$ **false**
9	De Morgan's laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **true** and **false**	~**true** $\equiv$ **false**	~**false** $\equiv$ **true**

Simplifying Statement Forms: Quick Quiz

- Use the laws in Theorem 2.1.1 to verify the following logical equivalence:

Remember to cite the law in every step in your workings.

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) \quad \text{(De Morgan's)}$$

2.2 Conditional Statements

2.2.1. Conditional Statements

If Jane is a math major or Jane is a computer science major, then Jane will take MA1101R.

hypothesis *conclusion*

If 4,686 is divisible by 6, then 4,686 is divisible by 3.

Conditional statement

If p, then q

$p \to q$

Logical connective:

$\rightarrow$

if-then/implies

Truth values:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Definition 2.2.1 (Conditional)

If p and q are statement variables, the **conditional** of q by p is "if p then q" or "p implies q", denoted $p \rightarrow q$.

It is false when p is true and q is false; otherwise it is true.

We called p the hypothesis (or antecedent) of the conditional and q the conclusion (or consequent).

- A conditional statement that is true by virtue of the fact that its hypothesis is false is often called **vacuously true** or **true by default**.

  p

  - “If you show up for work Monday morning, then you will get the job” q is vacuously true if you do NOT show up for work Monday morning.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- In general, when the “if” part of an if-then statement is false, the statement as a whole is said to be true, regardless of whether the conclusion is true or false.

Example #1:
A Conditional Statement with a False Hypothesis

If $0 = 1$, then $1 = 2$

Order of operations:

$\sim$	$\wedge$	$\vee$	$\rightarrow$
not	*and*	*or*	*if-then/implies*

Coequal in order

Performed first

Performed last

Conditional Statements: Example #2

Example #2: Truth Table for $p \vee \sim q \to \sim p$

$p \vee \sim q \to \sim p$ $\equiv$ $(p \vee (\sim q)) \to (\sim p)$

conclusion *hypothesis*

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$p \vee \sim q \to \sim p$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Example #3: Show that

$$p \vee q \to r \equiv (p \to r) \wedge (q \to r)$$

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$p \vee q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T	T	T	T	
T	T	F	T	F	F	F	
T	F	T	T	T	T	T	
T	F	F	T	F	T	F	
F	T	T	T	T	T	T	
F	T	F	T	T	F	F	
F	F	T	F	T	T	T	
F	F	F	F	T	T	T	

2.2.2. Representation of If-Then as Or: Implication Law

Rewrite the following statement in *if-then* form:

Either you get to work on time or you are fired.

Let $\sim p$ be "You get to work on time" and q be "You are fired".

$\sim p \vee q$

Also, p is "You do not get to work on time".

If you do not get to work on time, you are fired.

$p \rightarrow q$

Representation of If-Then as Or: Implication Law

$$p \to q \equiv \sim p \vee q$$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\sim p \vee q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication law

$(\sim, \wedge, \vee)$

complete set of logic

everything else is built on this

$$p \to q$$

2.2.3. Negation of a Conditional Statement

In previous slide, we have shown the Implication Law

$$p \to q \equiv {\sim}p \vee q$$

Hence, negation of a conditional statement:

$$\sim(p \to q) \equiv \sim(\sim p \vee q) \equiv \sim(\sim p) \wedge \sim q \equiv p \wedge \sim q$$

Implication law — De Morgan's law — Double negation law

$$\sim(p \to q) \equiv p \wedge \sim q$$

Negation of a Conditional Statement: Quick Quiz

- Write negation for each of the following statements:
  a. If my car is in the repair shop, then I cannot get to class.

$\sim(p \to q) \equiv p \wedge \sim q$

  b. If Sara lives in Athens, then she lives in Greece.

2.2.4. Contrapositive of a Conditional Statement

Definition 2.2.2 (Contrapositive)

The **contrapositive** of a conditional statement of the form "if p then q" is

"if $\sim q$ then $\sim p$"

Symbolically,

The contrapositive of $p \to q$ is $\sim q \to \sim p$.

$$\boxed{p \to q} \equiv \boxed{\sim q \to \sim p}$$

contrapositive

Contrapositive: Quick Quiz

- Write each of the following statements in its equivalent contrapositive form:

  a. If Howard can swim across the lake, then Howard can swim to the island.

  b. If today is Easter, then tomorrow is Monday.

2.2.5. Converse and Inverse of a Conditional Statement

Definition 2.2.3 (Converse)

The **converse** of a conditional statement "if p then q" is

"if q then p"

Symbolically,

The converse of $p \to q$ is $q \to p$.

Definition 2.2.4 (Inverse)

The **inverse** of a conditional statement "if p then q" is

"if $\sim p$ then $\sim q$"

Symbolically,

The inverse of $p \to q$ is $\sim p \to \sim q$.

Conditional statement: $p \rightarrow q$

$q \rightarrow p$ $\equiv$ $\sim p \rightarrow \sim q$

converse *inverse*

Converse and Inverse: Quick Quiz

- Write the converse and inverse of the following statements:

  a. If Howard can swim across the lake, then Howard can swim to the island.

Converse:

Inverse:

  b. If today is Easter, then tomorrow is Monday.

Converse:

Inverse:

Conditional statement and its Contrapositive, Converse and Inverse

$p \to q \equiv {\sim}q \to {\sim}p$

conditional statement — *contrapositive*

$q \to p \equiv {\sim}p \to {\sim}q$

converse — *inverse*

Note that: $p \to q \not\equiv q \to p$

2.2.6. Only If and the Biconditional

- To say "*p* only if *q*" means that *p* can take place only if *q* takes place also. That is, if *q* does not take place, then *p* cannot take place.
- Another way to say this is that if *p* occurs, then *q* must also occur (using contrapositive).

Definition 2.2.5 (Only If)

If *p* and *q* are statements,

"*p* only if *q*" means "if not *q* then not *p*"

Or, equivalently,

"if *p* then *q*"

Only If : Quick Quiz

- Rewrite the following statement in *if-then* form in two ways, one of which is the contrapositive of the other.

  John will break the world's record only if he runs the mile in under four minutes.

Version 1: If John does not run the mile in under four minutes, then John will not break the world's record.

Version 2:

Definition 2.2.6 (Biconditional)

Given statement variables p and q, the **biconditional** of p and q is "p if, and only if, q" and is denoted $p \leftrightarrow q$.

It is true if both p and q have the same truth values and is false if p and q have opposite truth values.

The words *if and only if* are sometimes abbreviated *iff*.

$p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Only If and the Biconditional

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Order of operations:

$\sim$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
not	*and*	*or*	*if-then/implies*	*if and only if*

Performed first: *not*

Coequal in order: *and*, *or*

Coequal in order: *if-then/implies*, *if and only if* — Performed last

Biconditional : Quick Quiz

- Rewrite the following statement as a conjunction of two if-then statements.

  This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

2.2.7. Necessary and Sufficient Conditions

Definition 2.2.7 (Necessary and Sufficient Conditions)

If *r* and *s* are statements,

"*r* is a sufficient condition for *s*"	means	"if *r* then *s*"
"*r* is a necessary condition for *s*"	means	"if not *r* then not *s*" or "if *s* then *r*"

- In other words, to say "*r* is a sufficient condition for *s*" means that the occurrence of *r* is *sufficient* to guarantee the occurrence of *s*.

- On the other hand, to say "r is a necessary condition for s" means that if r does not occur, then s cannot occur either: The occurrence of r is necessary to obtain the occurrence of s.
- Note that due to the equivalence between a statement and its contrapositive:

r is a necessary condition for s also means "if s then r".

- Consequently,

r is a necessary and sufficient condition for s means "r, if and only if, s".

2.3 Valid and Invalid Arguments

2.3.1. Valid and Invalid Arguments

Argument: a sequence of statements ending in a conclusion.

If Socrates is a man, then Socrates is mortal.	If p, then q
Socrates is a man.	p
• Socrates is mortal.	• q

Abstract form

An argument form is called valid if, and only if, whenever statements are substituted that make all the premises true, the conclusion is also true.

Definition 2.3.1 (Argument)

An **argument** (**argument form**) is a sequence of statements (statement forms). All statements in an argument (argument form), except for the final one, are called **premises** (or **assumptions** or **hypothesis**). The final statement (statement form) is called the **conclusion**. The symbol •, which is read "therefore", is normally placed just before the conclusion.

To say that an argument form is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true.

Example:

If p, then q	*premises*
p	
• q	*conclusion*

When an argument is valid and its premises are true, the truth of the conclusion is said to be inferred or deduced from the truth of the premises.

If a conclusion "ain't necessarily so", then it isn't a valid deduction.

2.3.2. Determining Validity or Invalidity

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a critical row.
   - If there is a critical row in which the conclusion is false $\Rightarrow$ the argument form is invalid.
   - If the conclusion in every critical row is true $\Rightarrow$ the argument form is valid.

Determining Validity or Invalidity: Example #1

$p \to q \lor {\sim}r$

$q \to p \land r$

$\bullet\ p \to r$

Invalid argument

p	q	r	${\sim}r$	$q \lor {\sim}r$	$p \land r$	$p \to q \lor {\sim}r$	$q \to p \land r$	$p \to r$
						premises		*conclusion*
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Critical rows

2.3.3. Modus Ponens and Modus Tollens

- Syllogism: An argument form consisting of two premises and a conclusion.
- A famous form of syllogism is called modus ponens:

If p then q
p
• q

Modus Ponens and Modus Tollens

- Modus ponens is a valid form of argument.

p	q	$p \to q$	p	q
T	T	T	T	T
T	F	F	T	
F	T	T	F	
F	F	T	F	

$p \to q$

p

$\bullet\ q$

- Modus tollens is another valid form of argument.

If p then q
$\sim q$
$\bullet \sim p$

Modus Ponens and Modus Tollens: Quick Quiz

- Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

a. If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes.

- ____

b. If 870,232 is divisible by 6, then it is divisible by 3.
870,232 is not divisible by 3.

- ____

2.3.4. Additional Valid Argument Forms: Rules of Inference

- A rule of inference is a form of argument that is valid.
  - Thus modus ponens and modus tollens are both rules of inference.
- Other rules of inference:
  1. Generalization
  2. Specialization
  3. Elimination
  4. Transitivity
  5. Proof by Division into Cases

2.3.4.1. Rules of Inference: Generalization

- The following argument forms are valid.

p	q
$\bullet\ p \vee q$	$\bullet\ p \vee q$

- Example:

Anton is a junior.
• (More generally) Anton is a junior or Anton is a senior.

2.3.4.2. Rules of Inference: Specialization

- The following argument forms are valid.

$p \wedge q$	$p \wedge q$
$\bullet\ p$	$\bullet\ q$

Allows you to discard extraneous information to concentrate on the particular property of interest.

- Example:

Ana knows numerical analysis and Ana knows graph algorithms.
• (In particular) Ana knows graph algorithms.

So if you are looking for someone who knows graph algorithms to work with you on a project, and you discover that Ana knows both numerical analysis and graph algorithms, would you invite her to work with you on your project?

2.3.4.3. Rules of Inference: Elimination

- The following argument forms are valid.

$p \vee q$
$\sim q$
$\bullet\ p$

$p \vee q$
$\sim p$
$\bullet\ q$

When you have two possibilities and you can rule one out, the other must be the case.

- Example:

Suppose you know that for a particular number x,

$$x - 3 = 0 \text{ or } x + 2 = 0$$

If you also know that x is not negative, then $x \neq -2$, so by elimination you can conclude that $x = 3$.

2.3.4.4. Rules of Inference: Transitivity

- The following argument form is valid.

$$p \rightarrow q$$
$$q \rightarrow r$$
$$\bullet\ p \rightarrow r$$

Many arguments in mathematics contain chains of if-then statements.

From the fact that one statement implies a second and the second implies the third, you can conclude that the first statement implies the third.

- Example:

If 18,486 is divisible by 18, then 18,486 is divisible by 9.

If 18,486 is divisible by 9, then the sum of the digits of 18,486 is divisible by 9.

• If 18,486 is divisible by 18, then the sum of the digits of 18,486 is divisible by 9.

2.3.4.5. Rules of Inference: Proof by Division into Cases

- The following argument form is valid.

$p \vee q$
$p \to r$
$q \to r$
$\bullet\ r$

It often happens that you know one thing or another is true. If you can show that in either case a certain conclusion follows, then this conclusion must also be true.

- Example:

Suppose you know that *x* is a nonzero real number.
The trichotomy property of the real numbers says that any number is positive, negative, or zero. Thus (by elimination) you know that *x* is positive or negative.
You can deduce that $x^2 > 0$ by arguing as follows:

x is positive or *x* is negative.
If *x* is positive, then $x^2 > 0$.
If *x* is negative, then $x^2 > 0$.
$\bullet\ x^2 > 0$.

2.3.4.6. Rules of Inference: Example

- You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:
    a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.
    b. If my glasses are on the kitchen table, then I saw them at breakfast.
    c. I did not see my glasses at breakfast.
    d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
    e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

So, where are your glasses?

Rules of Inference: Example

Let

- *RK* = I was reading the newspaper in the kitchen.
- *GK* = My glasses are on the kitchen table.
- *SB* = I saw my glasses at breakfast.
- *RL* = I was reading the newspaper in the living room.
- *GC* = My glasses are on the coffee table.

Here is a sequence of steps you might use to reach the answer, together with the rules of inference that allow you to draw the conclusion of each step:

1. $RK \to GK$ by (a)
   $GK \to SB$ by (b)
   - $RK \to SB$ by transitivity

2.3.5. Fallacies

- A fallacy is an error in reasoning that results in an invalid argument.
- Three common fallacies:
  1. Using **ambiguous premises**, and treating them as if they were unambiguous.
  2. **Circular reasoning** (assuming what is to be proved without having derived it from the premises)
  3. **Jumping to a conclusion** (without adequate grounds)

2.3.5.1. Fallacies: Converse Error

- Example:

If Zeke is a cheater, then Zeke sits in the back row.
Zeke sits in the back row.
• Zeke is a cheater.

$p \rightarrow q$	$q \rightarrow p$
q	q
$\bullet\ p$	$\bullet\ p$

Converse error is also known as the fallacy of affirming the consequence.

2.3.5.2. Fallacies: Inverse Error

- Example:

If interest rates are going up, stock market prices will go down.
Interest rates are not going up.
• Stock market prices will not go down.

$p \to q$
$\sim p$
$\bullet \sim q$

$\sim p \to \sim q$
$\sim p$
$\bullet \sim q$

2.3.5.3. Fallacies: A Valid Argument with a False Premise and a False Conclusion

- The argument below is valid by modus ponens. But its major premise is false, and so is its conclusion.

If Joseph Schooling is a Singaporean, then Joseph Schooling is a badminton player.

Joseph Schooling is a Singaporean.

• Joseph Schooling is a badminton player.

This premise is false!

2.3.5.4. Fallacies: An Invalid Argument with True Premises and a True Conclusion

- The argument below is invalid by the converse error, but it has a true conclusion.

If Singapore is a garden city, then Singapore has lots of trees.

Singapore has lots of trees.

• Singapore is a garden city.

2.3.5.5. Fallacies: Sound and Unsound Arguments

Definition 2.3.2 (Sound and Unsound Arguments)

An argument is called **sound** if, and only if, it is valid and all its premises are true.

An argument that is not sound is called **unsound**.

2.3.6. Contradictions and Valid Arguments

The concept of logical contradiction can be used to make inferences through a technique of reasoning called the contradiction rule. Suppose p is some statement whose truth you wish to deduce.

Contradiction Rule

If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true.

Contradictions and Valid Arguments: Example – Contradiction Rule

Show that the following argument form is valid:

$\sim p \to \textbf{false}$

$\bullet\ p$

			premise	*conclusion*
p	$\sim p$	**false**	$\sim p \to \textbf{false}$	p
T	F	F	T	T
F	T	F	F	

Only one critical row, and in this row the conclusion is true. Hence this form of argument is valid.

Contradictions and Valid Arguments: Example – Contradiction Rule

- The contradiction rule is the logical heart of the method of proof by contradiction.
- A slight variation also provides the basis for solving many logical puzzles by eliminating contradictory answers:

> If an assumption leads to a contradiction, then that assumption must be false.

2.3.7. Summary of Rules of Inference

Table 2.3.1

Rule of inference		
Modus Ponens	$p \to q$ p $\bullet\ q$	
Modus Tollens	$p \to q$ $\sim q$ $\bullet\ \sim p$	
Generalization	p $\bullet\ p \vee q$	q $\bullet\ p \vee q$
Specialization	$p \wedge q$ $\bullet\ p$	$p \wedge q$ $\bullet\ q$
Conjunction	p q $\bullet\ p \wedge q$	

2.3.7. Summary of Rules of Inference

Table 2.3.1 (cont'd)

Rule of inference		
Elimination	$p \vee q$ $\sim q$ • p	$p \vee q$ $\sim p$ • q
Transitivity	$p \to q$ $q \to r$ • $p \to r$	
Proof by Division Into Cases	$p \vee q$ $p \to r$ $q \to r$ • r	
Contradiction Rule	$\sim p \to$ **false** • p	

Next week’s lectures

3. The Logic of Quantified Statements

$\forall \qquad \exists$

Past year's midterm questions.

Q1. John was given the following statement:

"If the product of two integers a and b is even, then either a is even or b is even."

The following is John's proof:

1. Suppose a and b are both odd.
2. Therefore, $a = 2m + 1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$ (by definition of odd numbers).
3. Then, $ab = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$, which is odd.
4. Hence, the proof is complete.

What kind of proof did John use?

Proof by …

Past year's midterm questions.

Q2. Which of the following statements is/are logically equivalent to $p \leftrightarrow q$?

(I) $(\sim p \vee q) \wedge (p \vee \sim q)$

(II) $(\sim p \wedge \sim q) \vee (p \wedge q)$

(III) $(\sim p \vee \sim q) \wedge (p \vee q)$

(IV) $(\sim p \wedge q) \vee (p \wedge \sim q)$

Past year's midterm questions.

Q3. What is/are the missing premise(s) to make the following argument valid?

$p \to r$

$q \to s$

(Some missing premise(s))

$\bullet\ (p \vee q) \to (r \wedge s)$

(I) $p \vee q$

(II) $(p \wedge q) \to (r \wedge s)$

(III) $p \to s$

(IV) $q \to r$

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