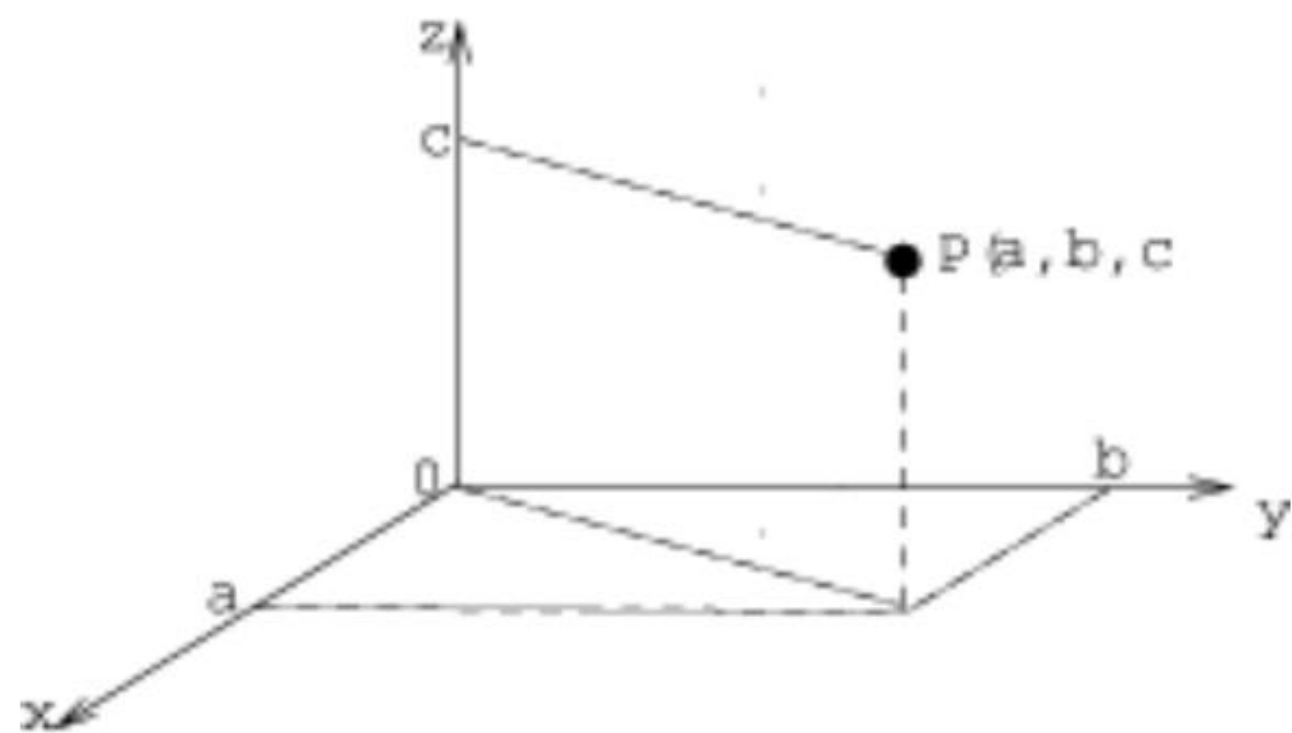


# Three Dimensional Space

# The Coordinate System of the 3D Space

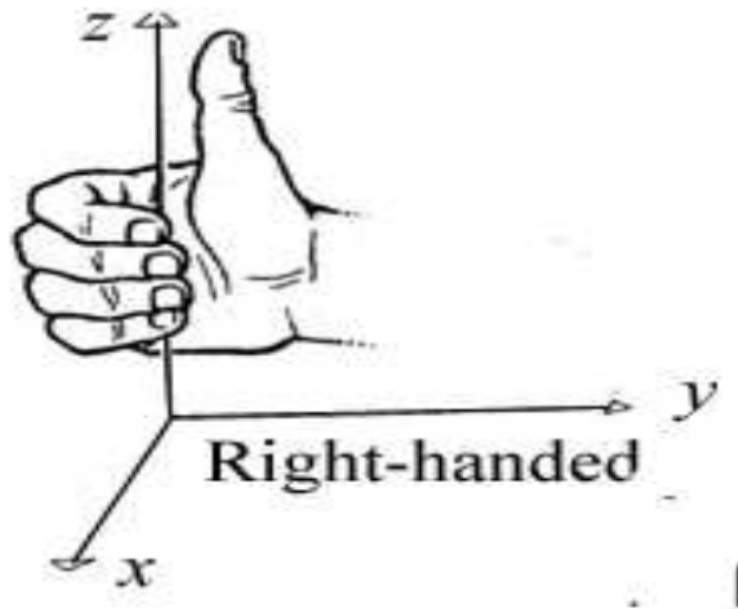
For three dimensional space, we first fix a coordinate system by choosing a point called the **origin**, and three lines, called the coordinate axes, so that each line is perpendicular to the other two. These lines are called the  $x$ -,  $y$ - and  $z$ -axes.



Associated with a point  $P$  in three dimensional space is an ordered triple  $(a, b, c)$  where  $a$ ,  $b$  and  $c$  are the projections of  $P$  on the  $x$ -,  $y$ - and  $z$ -axes respectively.

This is the **Cartesian coordinate system** for three dimensional space. We also call this space the  $xyz$ -space.

By convention, we use the **right-handed coordinate system**. A right-handed coordinate system fix the orientation of the axes as follow:



# Vectors in $xyz$ -Space

A vector is measurable quantity with a *magnitude* and a *direction*. It is geometrically represented by an arrow in the  $xyz$ -space with an initial point and a terminal point. The direction of the arrow gives the direction of the vector; and the length of the arrow gives the magnitude of the vector.

# Terminologies and notations

- (1) Let  $P$  and  $Q$  be points in the  $xyz$ -space with coordinates  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  respectively.

Then the vector  $\overrightarrow{PQ}$  is algebraically given by

$$\overrightarrow{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} .$$

The vector  $\overrightarrow{OP} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$  is called the position vector of  $P$ .

(2) The zero vector in the  $xyz$ -space is  $\mathbf{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .



(3) The sum of  $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$  is

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

[Note that  $\mathbf{v}_1 + \mathbf{O} = \mathbf{O} + \mathbf{v}_1 = \mathbf{v}_1$ .]

(4) The negative of  $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$  is  $-\mathbf{v}_1 = \begin{bmatrix} -x_1 \\ -y_1 \\ -z_1 \end{bmatrix}$ .

[Note that  $\mathbf{v}_1 - \mathbf{v}_1 = -\mathbf{v}_1 + \mathbf{v}_1 = \mathbf{O}$ .]

(5) The difference  $\mathbf{v}_1 - \mathbf{v}_2$  is

$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1 + (-\mathbf{v}_2) = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ -y_2 \\ -z_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix}.$$

(6) If  $c$  is a real number, the scalar  $c\mathbf{v}_1$  of  $\mathbf{v}_1$  by  $c$  is

$$c\mathbf{v}_1 = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}.$$

If  $c > 0$ , then  $c\mathbf{v}_1$  is in the same direction as  $\mathbf{v}_1$ .

If  $d < 0$ , then  $d\mathbf{v}_1$  is in the opposite direction as

$\mathbf{v}_1$ .

(7) The magnitude of  $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$  is

$$\|\mathbf{v}_1\| = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

[Note that  $\|c\mathbf{v}_1\| = |c| \|\mathbf{v}_1\|$  for a real number  $c$ .]

# Example

Let  $P_1$ ,  $P_2$ ,  $Q_1$  and  $Q_2$  be the points  $(3, 2, -1)$ ,  $(0, 0, 0)$ ,  $(5, 5, 4)$  and  $(2, 3, 5)$  respectively.

$$\overrightarrow{P_1Q_1} = \begin{bmatrix} 5 - 3 \\ 5 - 2 \\ 4 - (-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\overrightarrow{P_2Q_2} = \begin{bmatrix} 2 - 0 \\ 3 - 0 \\ 5 - 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

Hence

$$\overrightarrow{P_1Q_1} = \overrightarrow{P_2Q_2}.$$

The magnitude of  $\overrightarrow{P_1Q_1}$  is

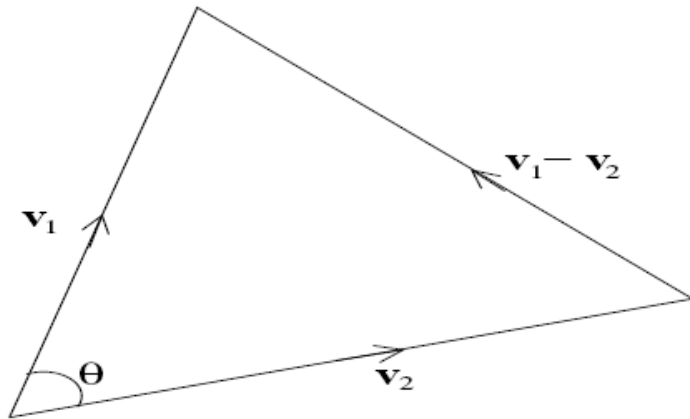
$$||\overrightarrow{P_1Q_1}|| = \sqrt{(2)^2 + (3)^2 + (5)^2} = \sqrt{38}.$$

So the magnitude of  $5\overrightarrow{P_1Q_1}$  is

$$5||\overrightarrow{P_1Q_1}|| = 5\sqrt{38}.$$

# Angle between two vectors

The angle between the nonzero vectors  $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$  is the angle  $\theta$ , ( $0 \leq \theta \leq 180^\circ$ ) as shown below.



Applying the law of cosines to this triangle, we obtain

$$||\mathbf{v}_1 - \mathbf{v}_2||^2 = ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2||\mathbf{v}_1|| ||\mathbf{v}_2|| \cos \theta. \quad (1)$$

Now LHS of (1)  $||\mathbf{v}_1 - \mathbf{v}_2||^2$  is given by

$$\begin{aligned} & (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\ &= x_1^2 + x_2^2 + y_1^2 + y_2^2 + z_1^2 + z_2^2 - 2(x_1x_2 + y_1y_2 + z_1z_2) \\ &= ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2(x_1x_2 + y_1y_2 + z_1z_2). \end{aligned}$$



If we substitute this expression in (1) and solve for  $\cos \theta$ , we obtain

$$\cos \theta = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{||\mathbf{v}_1|| ||\mathbf{v}_2||} \quad (2)$$

# Scalar or dot product

The **scalar product** or **dot product** of the vec-

tors  $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

is defined by

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2 + z_1z_2.$$

Thus we can rewrite (2), where  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are nonzero vectors, as

$$\cos \theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{||\mathbf{v}_1|| \ ||\mathbf{v}_2||}, \quad (0 \leq \theta \leq 180^0)$$

and notice that

$$\mathbf{v}_1 \text{ and } \mathbf{v}_2 \text{ are perpendicular} \iff \mathbf{v}_1 \cdot \mathbf{v}_2 = 0.$$

## Example

If  $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ , then

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(-1) + (4)(2) + (5)(3) = 21.$$

Also

$$\|\mathbf{v}_1\| = \sqrt{2^2 + 4^2 + 5^2} = \sqrt{45},$$

$$\|\mathbf{v}_2\| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}.$$

Hence

$$\cos \theta = \frac{21}{\sqrt{45}\sqrt{14}} = \frac{\sqrt{7}}{\sqrt{10}}.$$

Thus  $\theta$  is approximately  $33^{\circ}13'$ .

The vectors  $\mathbf{w}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$  and  $\mathbf{w}_2 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$  are perpendicular since their dot product

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(4) + (-5)(2) + (1)(2) = 0.$$

# Properties of scalar product

If  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_3$  are vectors in  $xyz$ -space and  $c$  is a real number, then

(a)  $\mathbf{v}_1 \cdot \mathbf{v}_1 = ||\mathbf{v}_1||^2 \geq 0.$

$$(b) \quad \mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1.$$

$$(c) \quad (\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_3.$$

$$(d) \quad (c\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot (c\mathbf{v}_2) = c(\mathbf{v}_1 \cdot \mathbf{v}_2).$$



# Unit vector

A **unit vector** in  $xyz$ -space is a vector of magnitude or length 1. The vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are unit vectors along the positive  $x$ -,  $y$ - and  $z$ -axes respectively.

Notice that every vector  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

For example,

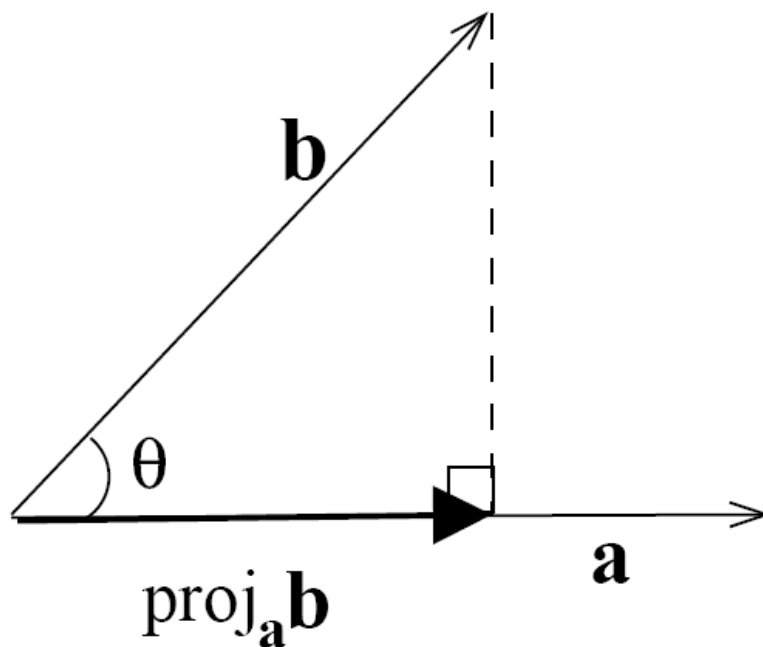
$$\mathbf{w} = \begin{bmatrix} 4 \\ -5 \\ 22 \end{bmatrix} = 4\mathbf{i} - 5\mathbf{j} + 22\mathbf{k}.$$

The unit vector with the same direction as  $\mathbf{w}$  is

$$\begin{aligned}\frac{1}{\|\mathbf{w}\|}\mathbf{w} &= \frac{1}{\sqrt{4^2 + 5^2 + 22^2}}(4\mathbf{i} - 5\mathbf{j} + 22\mathbf{k}) \\ &= \frac{4}{\sqrt{525}}\mathbf{i} - \frac{5}{\sqrt{525}}\mathbf{j} + \frac{22}{\sqrt{525}}\mathbf{k}.\end{aligned}$$

# Projection

The **projection** of a vector **b** onto a vector **a**, denoted by  $\text{proj}_{\mathbf{a}}\mathbf{b}$  is illustrated below.



From the definition of the scalar product, we have

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos \theta.$$

Therefore the length of the projection of  $\mathbf{b}$  onto  $\mathbf{a}$  is

$$||\text{proj}_{\mathbf{a}} \mathbf{b}|| = ||\mathbf{b}|| \cos \theta = \frac{(\mathbf{a} \cdot \mathbf{b})}{||\mathbf{a}||}.$$

So

$$\begin{aligned}\text{proj}_{\mathbf{a}}\mathbf{b} &= (||\text{proj}_{\mathbf{a}}\mathbf{b}||) \cdot (\text{unit vector along } \mathbf{a}) \\ &= \frac{(\mathbf{a} \cdot \mathbf{b})}{||\mathbf{a}||} \left( \frac{\mathbf{a}}{||\mathbf{a}||} \right) = \frac{(\mathbf{a} \cdot \mathbf{b})}{||\mathbf{a}||^2} \mathbf{a}.\end{aligned}$$

## Example

Find the projection of  $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$  onto the vector  $\mathbf{b} = \mathbf{i} + \mathbf{j}$ .

**Solution:** The length of the projection of  $\mathbf{a}$  onto  $\mathbf{b}$  is

$$\frac{(\mathbf{a} \cdot \mathbf{b})}{\|\mathbf{b}\|} = \frac{(2\mathbf{i} + 5\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})}{\sqrt{1^2 + 1^2}} = \frac{7}{\sqrt{2}}.$$



A unit vector along **b** is

$$\frac{\mathbf{i} + \mathbf{j}}{||\mathbf{i} + \mathbf{j}||} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}.$$

Hence the projection of **a** onto **b** is

$$\frac{7}{\sqrt{2}} \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \frac{7}{2}\mathbf{i} + \frac{7}{2}\mathbf{j}.$$

# Vector Product

If  $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ ,

then their **vector product** or **cross product** is

the vector

$$\begin{aligned}
 \mathbf{v}_1 \times \mathbf{v}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \\
 &= (y_1 z_2 - y_2 z_1) \mathbf{i} - (x_1 z_2 - x_2 z_1) \mathbf{j} + (x_1 y_2 - x_2 y_1) \mathbf{k}.
 \end{aligned}$$

For example, if  $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$ ,  
then their vector product is the vector

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix} = -\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}.$$

# Properties of vector product

Let  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_3$  be vectors in  $xyz$ -space, and let  $c$  be a real number. Then

(a)  $\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1.$

$$(b) \quad \mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_3.$$

$$(c) \quad (\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_3 + \mathbf{v}_2 \times \mathbf{v}_3.$$

$$(d) \quad c(\mathbf{v}_1 \times \mathbf{v}_2) = (c\mathbf{v}_1) \times \mathbf{v}_2 = \mathbf{v}_1 \times (c\mathbf{v}_2).$$

$$(e) \quad \mathbf{v}_1 \times \mathbf{v}_1 = \mathbf{0}.$$

$$(f) \quad \mathbf{0} \times \mathbf{v}_1 = \mathbf{v}_1 \times \mathbf{0} = \mathbf{0}.$$

## Direction of $\mathbf{v}_1 \times \mathbf{v}_2$

Let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be two (non-parallel) vectors which determine a plane  $\Pi$ . i.e.  $\Pi$  is the plane that contains both  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

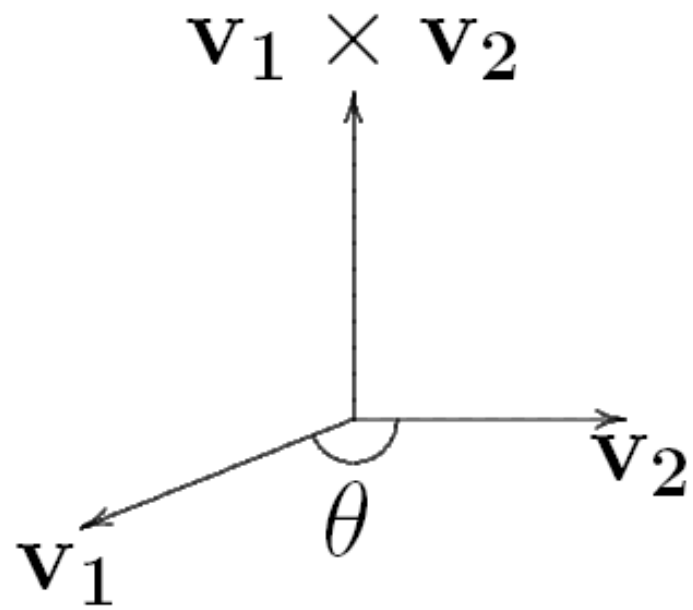


Using the definition of vector product, we can check that

$$(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_1 = 0 \quad \text{and} \quad (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_2 = 0$$

i.e.  $\mathbf{v}_1 \times \mathbf{v}_2$  is perpendicular to  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

Hence  $\mathbf{v}_1 \times \mathbf{v}_2$  is perpendicular to the plane  $\Pi$ .



## Magnitude of $\mathbf{v}_1 \times \mathbf{v}_2$

Let  $\theta$  be the angle between  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

We have

$$||\mathbf{v}_1 \times \mathbf{v}_2|| = ||\mathbf{v}_1|| ||\mathbf{v}_2|| \sin \theta.$$

# Lines in 3D Space

## Vector equation of a line

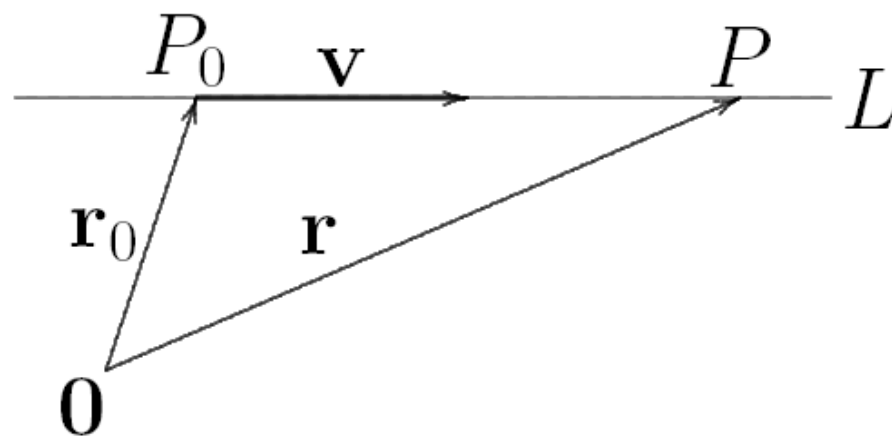
Let  $L$  be a line passing through a point  $P_0$  with position vector  $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$  and parallel to a vector  $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ . Then any point  $P$  on  $L$  has

position vector

$$\begin{aligned}\overrightarrow{OP} = \mathbf{r} &= \mathbf{r}_0 + t\mathbf{v} \\ &= (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \quad (3)\end{aligned}$$

for some  $t \in \mathbf{R}$ .

(3) is called a **vector equation** of the line  $L$ .



# Parametric equation of a line

Writing

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

the vector equation (3) becomes

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).$$

Equating the three components, we get

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

These are called the **parametric equations** of the line  $L$  due to the parameter  $t$  in the equations.



## Example

The points  $A$  and  $B$  have position vectors

$$-3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \quad \text{and} \quad \mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

respectively. Write down the parametric equations of the line passing through  $A$  and  $B$ .

**Solution:** The position vectors of  $A$  and  $B$  are

$$\overrightarrow{OA} = -3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \quad \overrightarrow{OB} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

respectively. So the line is parallel to the vector

$$\overrightarrow{AB} = (\mathbf{i} - \mathbf{j} + 4\mathbf{k}) - (-3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}.$$

If we use the position vector of  $A$  as  $\mathbf{r}_0$ , the vector equation is given by

$$\mathbf{r} = (-3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + t(4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}). \quad (4)$$

Alternatively, if we use the position vector of  $B$  as  $\mathbf{r}_0$ , the vector equation is given by

$$\mathbf{r} = (\mathbf{i} - \mathbf{j} + 4\mathbf{k}) + s(4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}). \quad (5)$$

To get the parametric equations of the line, let

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

and substitute in the LHS of equation (4) or (5).

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (-3 + 4t)\mathbf{i} + (2 - 3t)\mathbf{j} + (-3 + 7t)\mathbf{k}.$$

Hence

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

## Example.

Given the following lines whose vector equations are

$$L_1 : \mathbf{r} = \mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$

$$L_2 : \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_2 \left( 3\mathbf{i} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k} \right) \text{ and}$$

$$L_3 : \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_3(3\mathbf{i} + \mathbf{j}).$$

(a) Find the position vector of the point of intersection of  $L_1$  and  $L_2$ .

(b) Show that  $L_1$  and  $L_3$  are skew, i.e. do not intersect each other.

(a) Eliminating  $\mathbf{r}$  from the vector equations of  $L_1$  and  $L_2$ , we get

$$\mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + \mathbf{j}) + t_2 \left( 3\mathbf{i} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k} \right).$$

Hence it follows that

$$t_1 = 1 + 3t_2, \quad 2t_1 = 1 + \frac{9}{2}t_2, \quad 3t_1 = \frac{9}{2}t_2$$

from which we obtain

$$t_1 = -1, \quad t_2 = -2/3.$$

Putting  $t_1 = -1$  into the vector equation of  $L_1$ , we obtain

$$\mathbf{r} = \mathbf{i} + (-1)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = -2\mathbf{j} - 3\mathbf{k}.$$



So the position vector of the point of intersection  $P$   
of the two lines:

$$\overrightarrow{OP} = -2\mathbf{j} - 3\mathbf{k}.$$

(b) Eliminating  $\mathbf{r}$  from the vector equations of  $L_1$  and  $L_3$ , we get

$$\mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + \mathbf{j}) + t_3(3\mathbf{i} + \mathbf{j}).$$

Hence it follows that

$$t_1 = 1 + 3t_3, \quad 2t_1 = 1 + t_3, \quad 3t_1 = 0$$

Solving the first two equations above gives  $t_1 = 2/5$  but the last equation says  $t_1 = 0$ , thus there is a contradiction. So there is no solution to the equations and we conclude that  $L_1$  and  $L_3$  do not intersect.

## Example

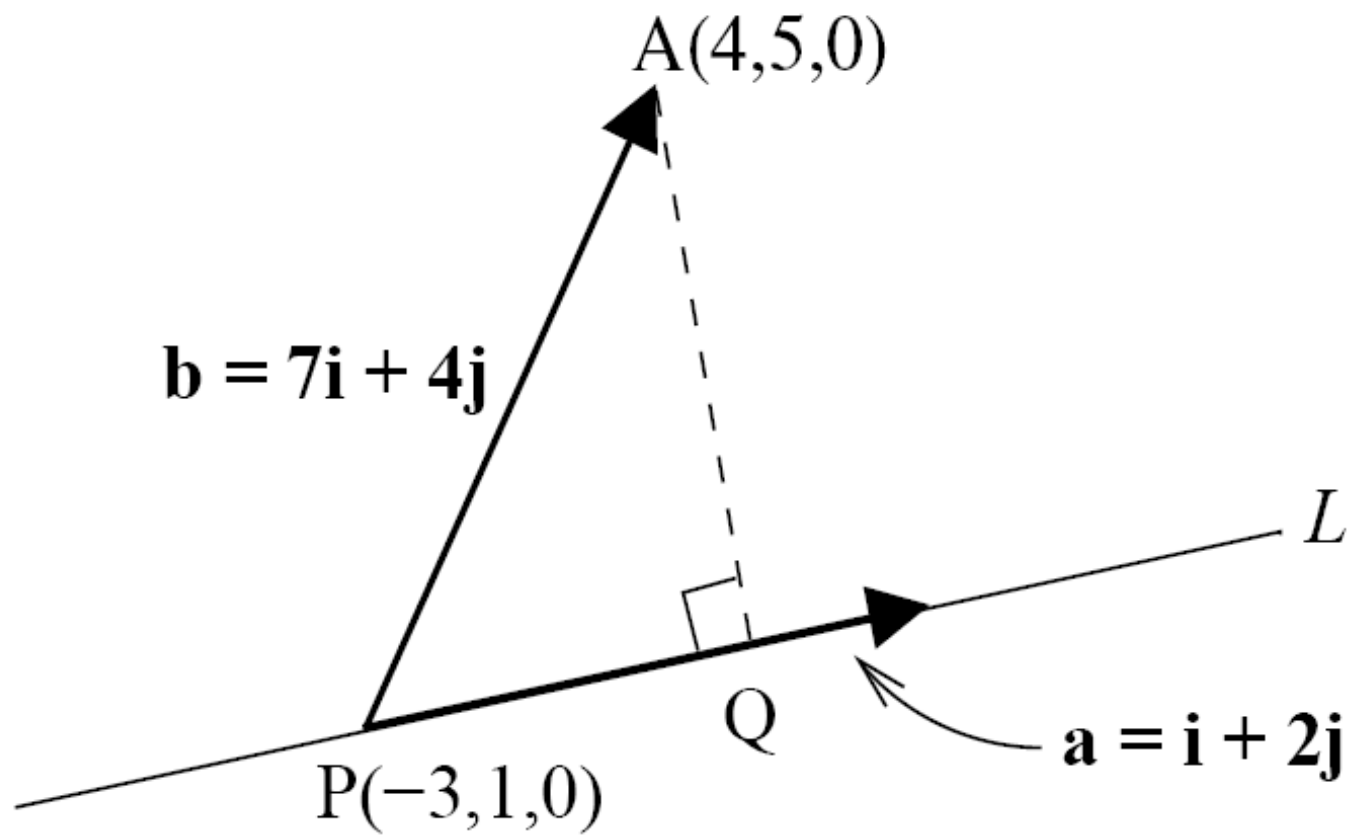
Find the shortest distance from the point  $A$  with position vector  $4\mathbf{i} + 5\mathbf{j}$  to the line  $L$  whose vector equation is

$$\mathbf{r} = (-3\mathbf{i} + \mathbf{j}) + t(\mathbf{i} + 2\mathbf{j}).$$

**Solution:**  $L$  passes through the point  $P(-3, 1, 0)$

and is parallel to the vector  $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ . Let  $\mathbf{b}$  be the vector

$$\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = (4\mathbf{i} + 5\mathbf{j}) - (-3\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 4\mathbf{j}.$$



From Section 6.2.8, the length of the projection of  $\mathbf{b}$  onto  $\mathbf{a}$  is

$$|PQ| = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|} = \frac{(\mathbf{i} + 2\mathbf{j}) \cdot (7\mathbf{i} + 4\mathbf{j})}{\sqrt{1^2 + 2^2}} = \frac{15}{\sqrt{5}}.$$

Now the shortest distance from  $A$  to  $L$  is given by

$$|AQ|.$$

Applying Pythagoras theorem on the right triangle

$APQ$ ,

$$|AP|^2 = |PQ|^2 + |AQ|^2,$$

we get

$$\begin{aligned} |AQ| &= \sqrt{||\mathbf{b}||^2 - \left(\frac{15}{\sqrt{5}}\right)^2} \\ &= \sqrt{7^2 + 4^2 - \frac{15^2}{5}} \\ &= 2\sqrt{5}. \end{aligned}$$



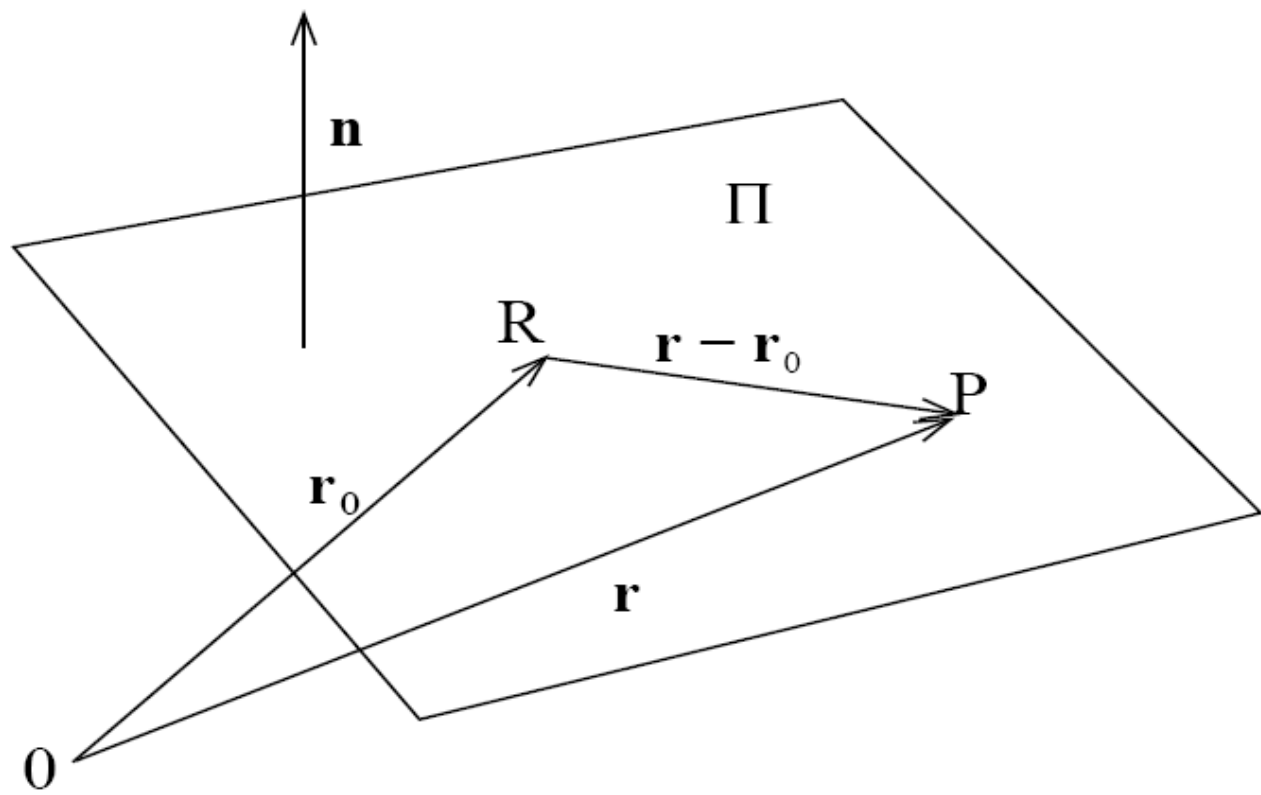
# Planes in 3D Space

Suppose we wish to find the vector equation of a plane  $\Pi$  passing through a given point  $R$  with position vector  $\mathbf{r}_0$  relative to the origin  $O$  and such that  $\Pi$  has  $\mathbf{n}$  as a normal vector to it. Let  $P$  be a general point in the plane with position vector  $\mathbf{r}$ . Then

$\overrightarrow{RP} = \mathbf{r} - \mathbf{r}_0$  is a vector lying in the plane, and perpendicular to the normal vector  $\mathbf{n}$ .

Hence

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0.$$



# Cartesian Equation of a plane

Let us write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k},$$

and

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k},$$

so that

$$\mathbf{r} - \mathbf{r}_0 = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

and

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = a(x - x_0) + b(y - y_0) + c(z - z_0).$$

Therefore, the vector equation of the plane can be written in the form

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0.$$

The Cartesian equation of a plane passing through a point  $(x_0, y_0, z_0)$  and with normal vector  $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  is

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

## Example

Find the equation of the plane passing through the point  $(0, 2, -1)$  normal to the vector  $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ .



**Solution:** The required equation is

$$3x + 2y - z = 3(0) + 2(2) - (-1), \quad \text{or}$$

$$3x + 2y - z = 5.$$

## Example

Find the vector equation of the plane passing through the points  $A(0, 0, 1)$ ,  $B(2, 0, 0)$  and  $C(0, 3, 0)$ .

**Solution:** The following vector is perpendicular to the plane:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

The plane passes through  $(0, 0, 1)$ . So an equation of the plane is

$$3x + 2y + 6z = 3(0) + 2(0) + 6(1),$$

or

$$3x + 2y + 6z = 6.$$

# Distance from a point to a plane

The shortest distance from a point  $S (x_0, y_0, z_0)$  to a plane  $\Pi : ax + by + cz = d$ , is given by

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} \quad (6)$$

## Example

Find the distance of the point  $(2, -3, 4)$  to the plane

$$x + 2y + 3z = 13.$$

**Solution:** Using (6), we have  $(x_0, y_0, z_0) = (2, -3, 4)$  and  $a = 1, b = 2, c = 3$ .

So the distance is

$$\frac{|1(2) + 2(-3) + 3(4) - 13|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{5}{\sqrt{14}}$$