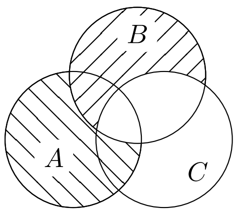
Answers to selected exercises

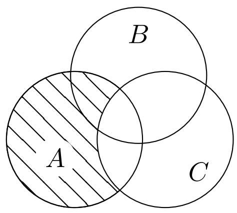
5a, page 3

- $\{1\} \in C$ but $\{1\} \not\subseteq C$;
- $\{2\} \notin C$ but $\{2\} \subseteq C$;
- $\{3\} \in C$ and $\{3\} \subseteq C$; and
- $\{4\} \notin C$ and $\{4\} \not\subseteq C$.

5b, page 7



$(A \setminus B) \cup (B \setminus C)$



$A \setminus C$

No. For a counterexample, let $A = C = \varnothing$ and $B = \{1\}$. Then

$$(A \setminus B) \cup (B \setminus C) = \varnothing \cup \{1\} = \{1\} \neq \varnothing = A \setminus C.$$

6a, page 9

We show $\{|x| : x \in \mathbb{Q}\} = \mathbb{Q}_{\geqslant 0}$.

Proof. 1. $(\subseteq)$

1.1. Let $z \in \{|x| : x \in \mathbb{Q}\}$.
1.2. Find $x \in \mathbb{Q}$ such that $z = |x|$.
1.3. 1.3.1. Case 1: Suppose $x \geqslant 0$.
  1.3.2. Then $z = |x| = x \in \mathbb{Q}_{\geqslant 0}$ by the definition of $|\cdot|$.
1.4. 1.4.1. Case 2: Suppose $x < 0$.
  1.4.2. Then $z = |x| = -x$ by the definition of $|\cdot|$.
  1.4.3. As $x < 0$, we know $-x > 0$.
  1.4.4. As $x \in \mathbb{Q}$, we know $-x \in \mathbb{Q}$.
  1.4.5. Thus $z = -x \in \mathbb{Q}_{\geqslant 0}$.
1.5. In either case, we have $z \in \mathbb{Q}_{\geqslant 0}$.

2. $(\supseteq)$

2.1. Let $z \in \mathbb{Q}_{\geqslant 0}$.
2.2. Then the definition of $|\cdot|$ implies $z = |z| \in \{|x| : x \in \mathbb{Q}\}$. $\square$

✐ 6b, page 9

We only show (1) here. The proof of (2) is similar.

Proof. 1. We know $\lfloor x \rfloor \in \mathbb{Z}$ by the definition of $\lfloor x \rfloor$.
2. As $\lfloor x \rfloor + 1 \in \mathbb{Z}$ and $\lfloor x \rfloor + 1 > \lfloor x \rfloor$, it follows from the maximality condition in the definition of $\lfloor \cdot \rfloor$ that $\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1$.
3. Now let us show uniqueness: let $y \in \mathbb{Z}$ such that $y \leqslant x < y + 1$.
4. 4.1. Suppose $\lfloor x \rfloor < y$.
   4.2. Then $\lfloor x \rfloor + 1 \leqslant y$ as $\lfloor x \rfloor, y \in \mathbb{Z}$.
   4.3. This implies $x < \lfloor x \rfloor + 1 \leqslant y \leqslant x$ by line 2 and line 3, which is a contradiction.
5. So $\lfloor x \rfloor \geqslant y$.
6. Similarly, one shows $\lfloor x \rfloor \leqslant y$.
7. Thus $\lfloor x \rfloor = y$. □

✐ 6c, page 9

It does not make f assign any value in the codomain $\mathbb{Q}$ to the element $1/2$ of the domain $\mathbb{Q}$. (Recall $2^{1/2} = \sqrt{2} \notin \mathbb{Q}$.)

✐ 6d, page 9

It does not make g assign any value in the codomain $\mathbb{Q}$ to the element -1 of the domain $\mathbb{Q}$.

✐ 6e, page 9

It makes $h(1/2) = 1 \neq 2 = h(2/4)$, although $1/2 = 2/4$.

✐ 6f, page 13

Only the top-middle, the bottom-left and the bottom-right diagrams represent functions. Of these, the first is a surjection but not an injection; the second is an injection but not a surjection; and the third is both an injection and a surjection. Thus only the last one is a bijection.

✐ 6g, page 14

The following are true statements:

$$g(0) = 0, \quad g(\{0\}) = \{0\}, \quad g^{-1}(\{0\}) = \{0\}.$$

Each of the following has a type mismatch error because there is a number on one side of the equation, and a set on the other side:

$$g(0) = \{0\}, \quad g(\{0\}) = 0, \quad g^{-1}(\{0\}) = 0.$$

The following refer to the inverse of g, which does not exist in view of Example 6.2.8 or Example 6.2.11, and Theorem 6.2.18:

$$g^{-1}(0) = 0, \quad g^{-1}(0) = \{0\}.$$