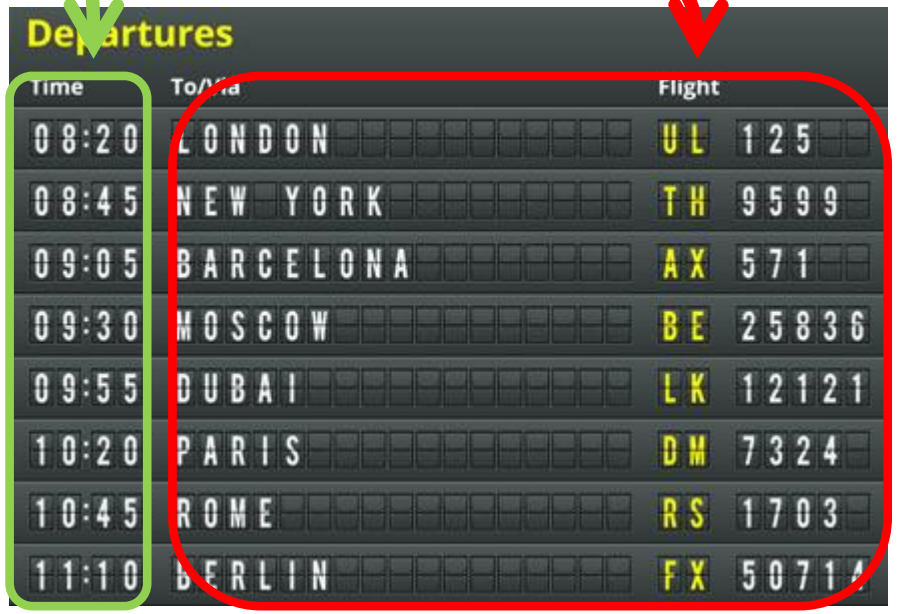
Hashing

Recap Problem: Flight Database

- One record of data:
  - Key
  - Data
- Dynamic collections
  - Can add or remove any one at anytime
- Can query the database
  - Find a particular record by the key
    - E.g. what is the flight at 09:05?
    - Or check the existence
      - Is there any flight between 09:00 to 09:20?
  - Find the one before or after
    - successor or predecessor



Dictionary ADT

void	insert(Key k, Value v)	*insert (k,v) into table*
Value	search(Key k)	*get value paired with k*
Key	**successor(Key k)**	*find next key > k*
Key	**predecessor(Key k)**	*find next key < k*
void	delete(Key k)	*remove key k (and value)*
boolean	contains(Key k)	*is there a value for k?*
int	size()	*number of (k,v) pairs*

Examples

Dictionary:	key = word value = definition
Phone Book	key = name value = phone number
Internet DNS	key = website URL value = IP address
C++ compiler	key = variable name value = type and value

Time Complexity for Each Operation

Data Structure	**Query, Modification**
Linked List	$O(n)$
Sorted Array	$O(\log n)$, $O(n)$
Unsorted Array	$O(n)$
Balanced Tree	$O(\log n)$
Can we do….?	$O(1)$

Dictionary/Symbol Tables

- Spelling correction (key=misspelled word, data=word)
- Scheme interpreter (key=variable, data=value)
- Web server
  - Lots of simultaneous network connections.
  - When a packet arrives, give it to the right process to handle the connection.
  - key=ip address, data = connection handler
- In these cases, $\mathrm{O}(\log n)$ often isn't fast enough!

Assumptions

- No duplicate keys allowed.
- No mutable keys
  - If you use an object as a key, then you can't modify that object later.

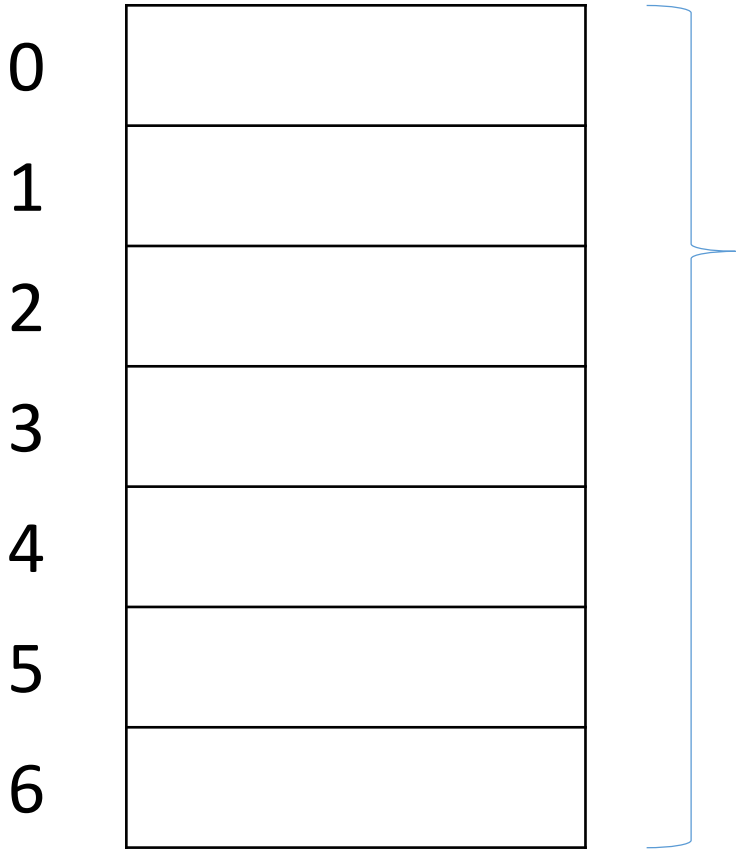
```
SymbolTable<Time, Plane> t =
                new SymbolTable<Time, Plane>();

Time  t1 = new Time(9:00);
Time  t2 = new Time(9:15);

t.insert(t1, "SQ0001");
t.insert(t2, "SQ0002");

t1.setTime(10:00);

x = new Time(9:00);
t.search(x);
```

Moral: Keys should be immutable.

Examples: Integer, String

Attempt #1: Use a table, indexed by keys

- What I do after dinner in the seven days of a week



Universe U = {0..6} of size m = 7

(key,value)

```
(0,'NetFlix')
(3,'Exercise')
(5,'Lecture')
```

Attempt #1: Use a table, indexed by keys

- What I do after dinner in the seven days of a week

0	‘NetFlix’
1	
2	
3	‘Exercise’
4	
5	‘Lecture’
6	

Example: `insert(5,'Lecture')`

Time:

insert: $O(1)$,search: $O(1)$

Direct Access Table

- Problems:
  - Too much space
  - If keys are ALL integers, then table-size > 4 billion

- What if keys are not integers?
  - Where do you put the key/value :

    "`(hippopotamus, bob)`"
  - Where do you put 3.14159?

Pythagoras said, “Everything is a number.”

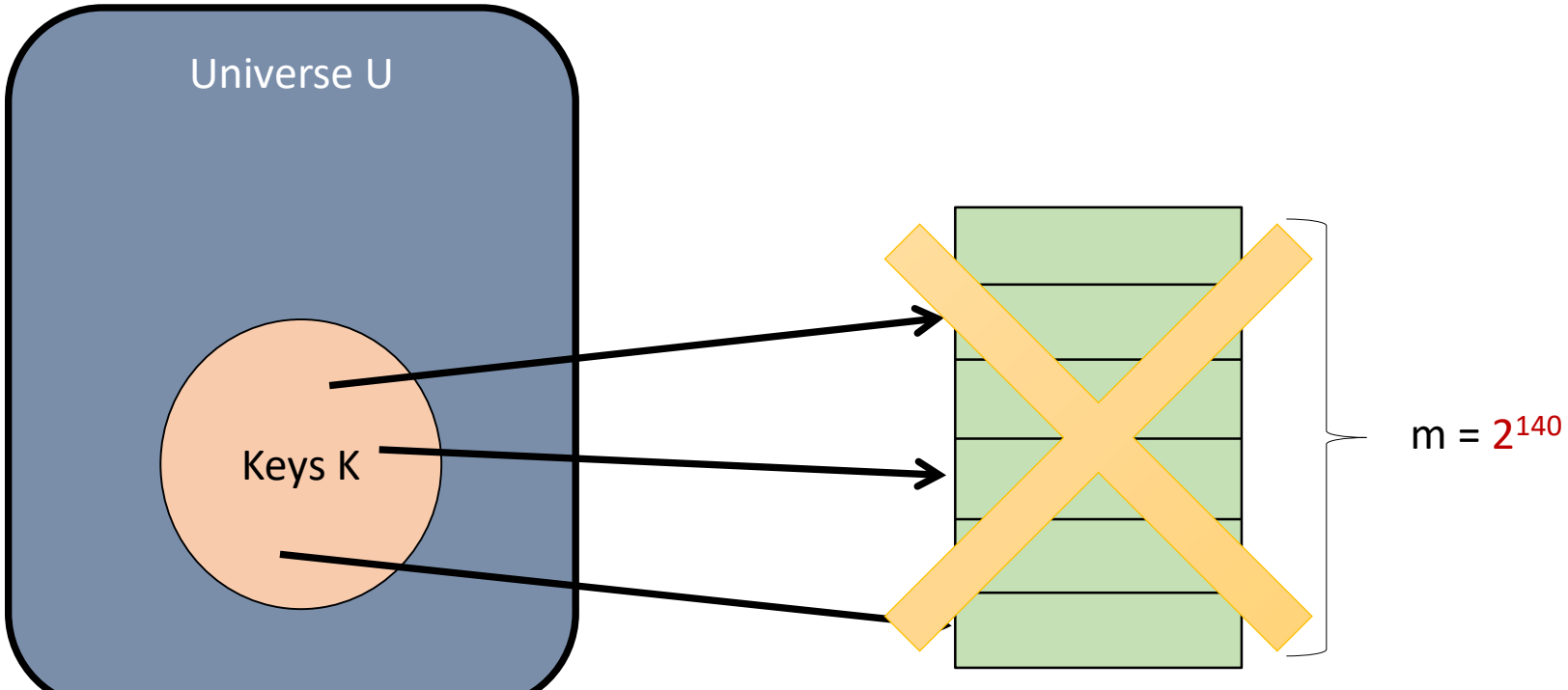
Direct Access Tables

- Pythagoras said, “Everything is a number.”
  - Everything is just a sequence of bits.
  - Treat those bits as a number.

- English:
  - 26 letters => 5 bits/letter
  - Longest word = 28 letters (antidisestablishmentarianism?)
  - 28 letters * 5 bits = 140 bits
  - So we can store any English text in a direct-access array of size 2^{140}.
    - ≈ number of atoms in observable universe

Hash Functions

- Problem:
  - Huge universe U of possible keys.
  - Smaller number n of actual keys.
  - We cannot have an array with size $m = 2^{140}$

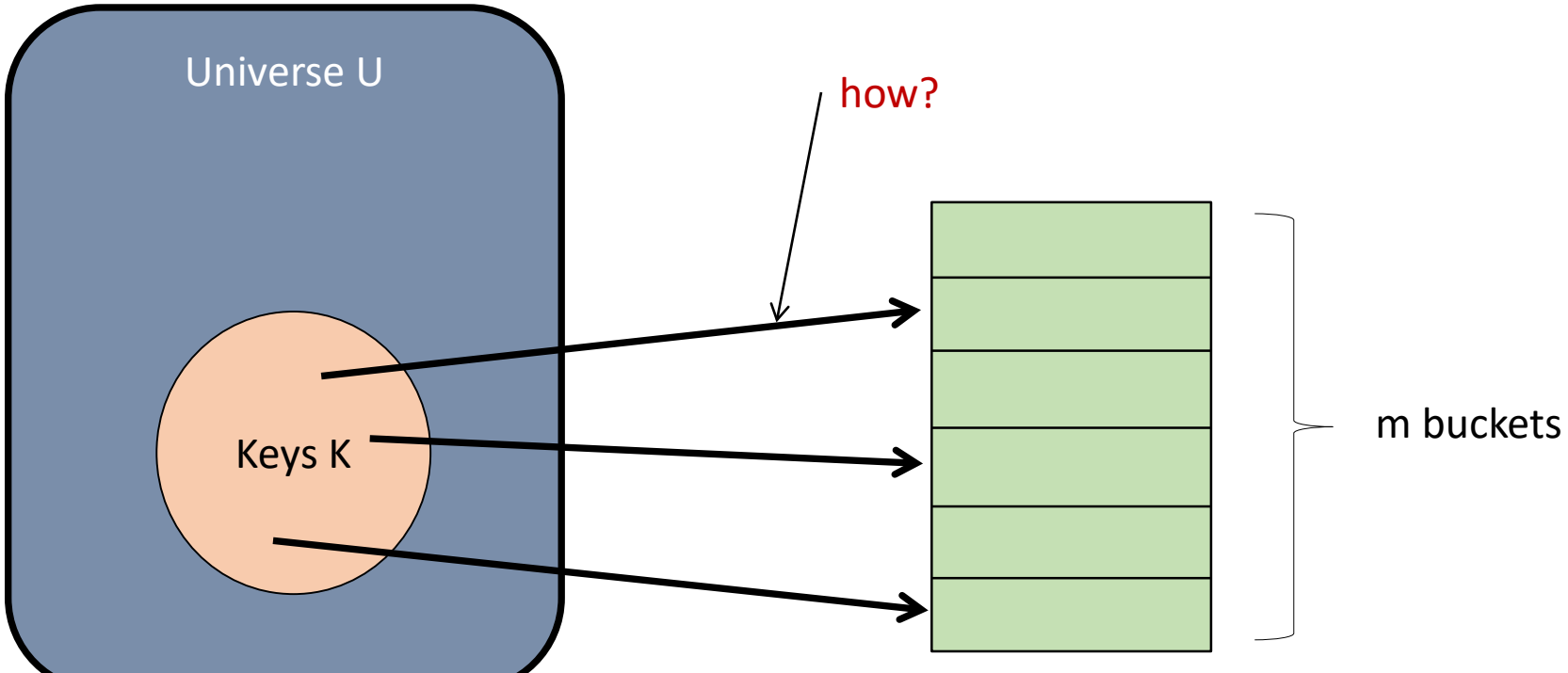


Pythagoras said, “Everything is a number.”

- English words:
  - There are possible 2^{140} English words with less than or equal to 28 letters
  - But we actually have about 1 million ($\sim 2^{20}$) of REAL English words
    - Comparing to 2^{140}

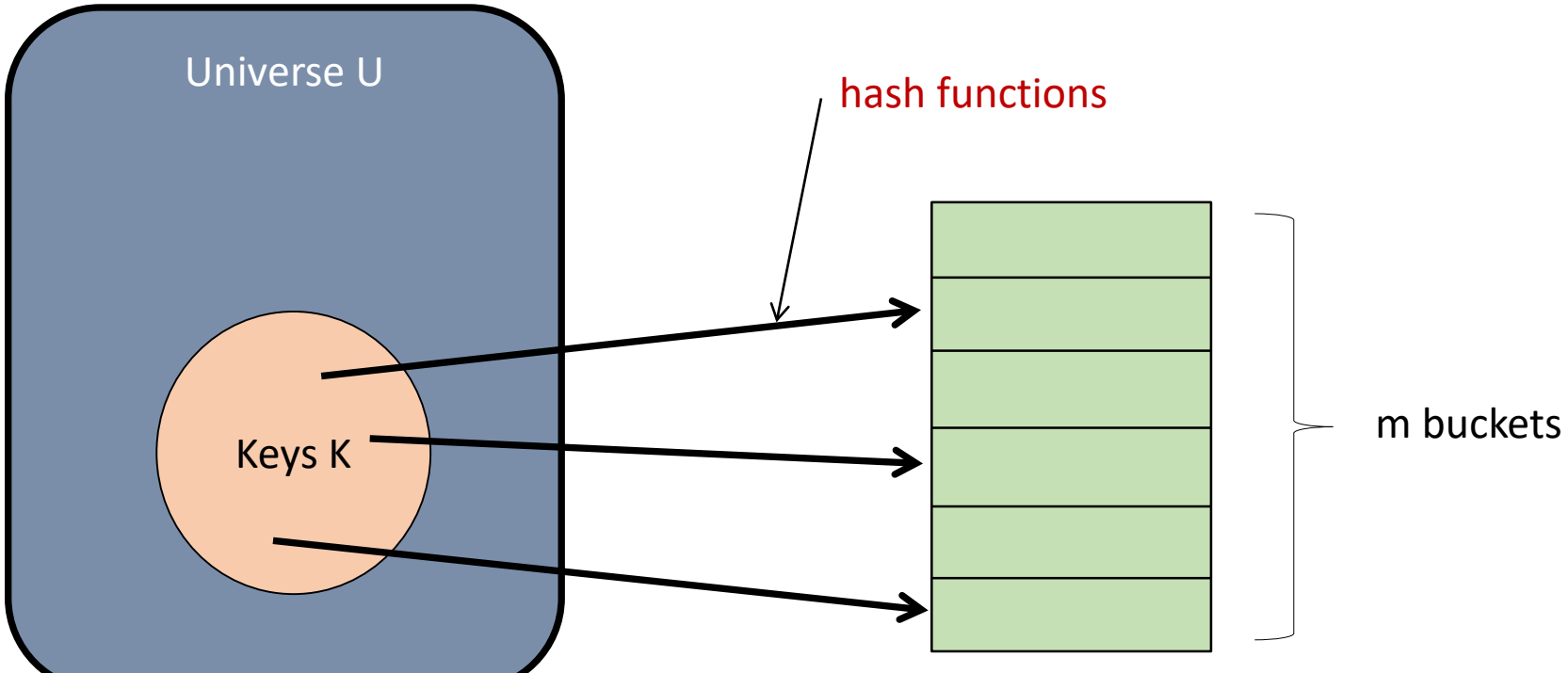
Hash Functions

- Problem:
  - Huge universe U of possible keys. e.g., $u = 2^{140}$
  - Smaller number n of actual keys. e.g., $n = 2^{20}$
  - How to put n items into, say $m \approx n$ buckets?



Hash Functions

- Define hash function $h : \mathrm{U} \rightarrow \{1..m\}$
  - Store key k in bucket $h(k)$.
  - Time complexity:
    - Time to compute h + Time to access bucket
  - For now: assume hash function cost $\mathrm{O}(1)$ to compute.



Hash Functions

- For example, the (key, value) pairs are:
  - `("pizza", "Clementi")`
  - `("coffee", "NUS")`
- For example, if the function h is the number of characters in an English word then
  - h`("pizza")` = 5
  - h`("coffee")` = 6
- What is the potential problem?

0	null
1	null
2	null
3	null
4	null
5	("pizza","Clementi")
6	("orange","NUS")
7	null
8	null
9	null

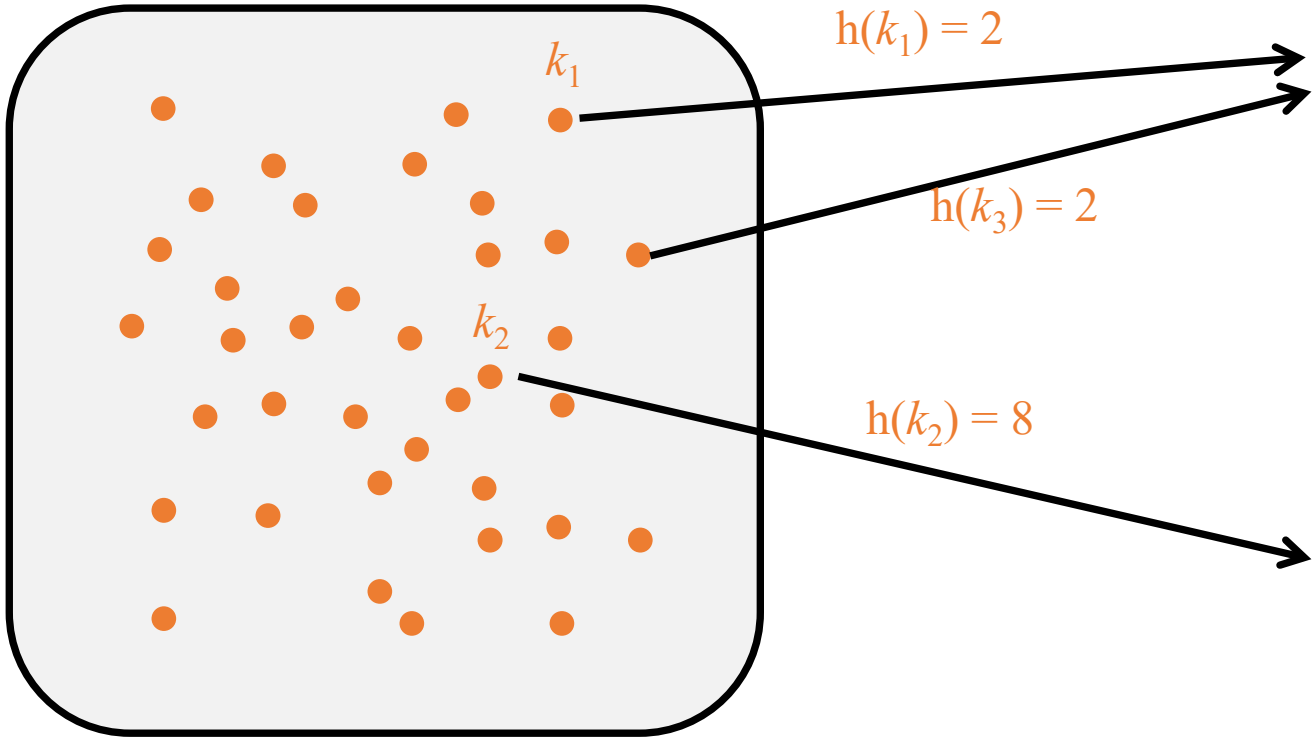
Hash Functions

insert(k_1, A)

insert(k_2, B)

insert(k_3, C)

Collision!



0	
1	null
2	**A**
3	null
4	null
5	null
6	null
7	null
8	**B**
9	null

Hash Collisions

- We say that two distinct keys k_1 and k_2 collide if:

$$h(k_1) = h(k_2)$$

- For example, if the function h is the number of characters in an English word then

$$h(\text{“}\texttt{pizza}\text{”}) = h\,(\text{“}\texttt{mango}\text{”}) = 5$$

Can we choose a hash function with no collisions?

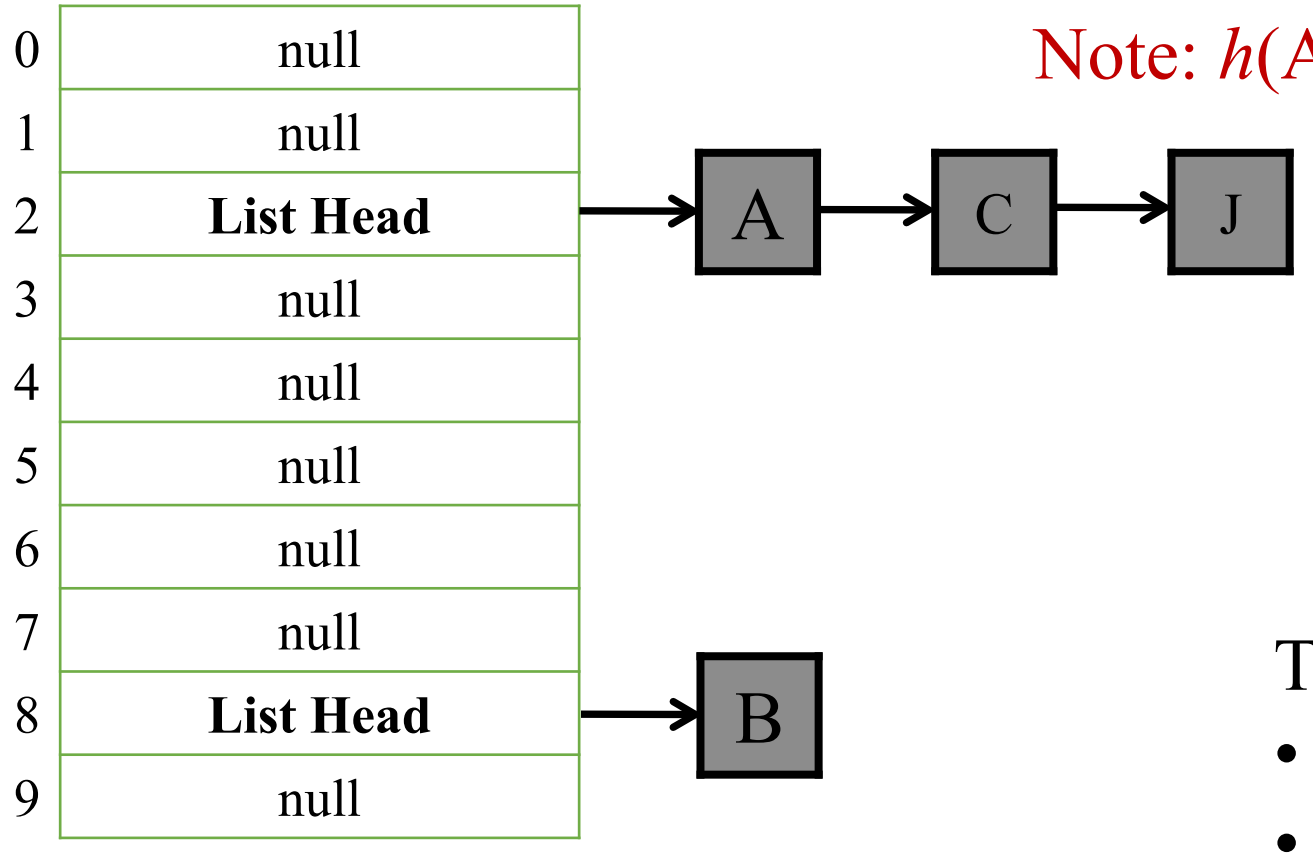
- Unavoidable!
  - The table size is smaller than the universe size.
- The pigeonhole principle says:
  - There must exist two keys that map to the same bucket.
  - Some keys must collide!

Coping with Collision

- Idea 1: choose a new, better hash functions
  - Hard to find.
  - Requires re-copying the table.
  - Eventually, there will be another collision.
- Idea 2: chaining
  - Put both items in the same bucket!
- Idea 3: open addressing
  - Find another bucket for the new item.

Chaining

- Each bucket contains a linked list of items.



Note: $h(\mathrm{A}) == h(\mathrm{C}) == h(\mathrm{J})$

Total space: $O(m + n)$

- Table size: m
- Linked list size: n

Hashing with Chaining

- Insertion
  - `insert(key, value)`
  - Calculate `h(key)`
  - Lookup `h(key)` and add `(key,value)` to the linked list.

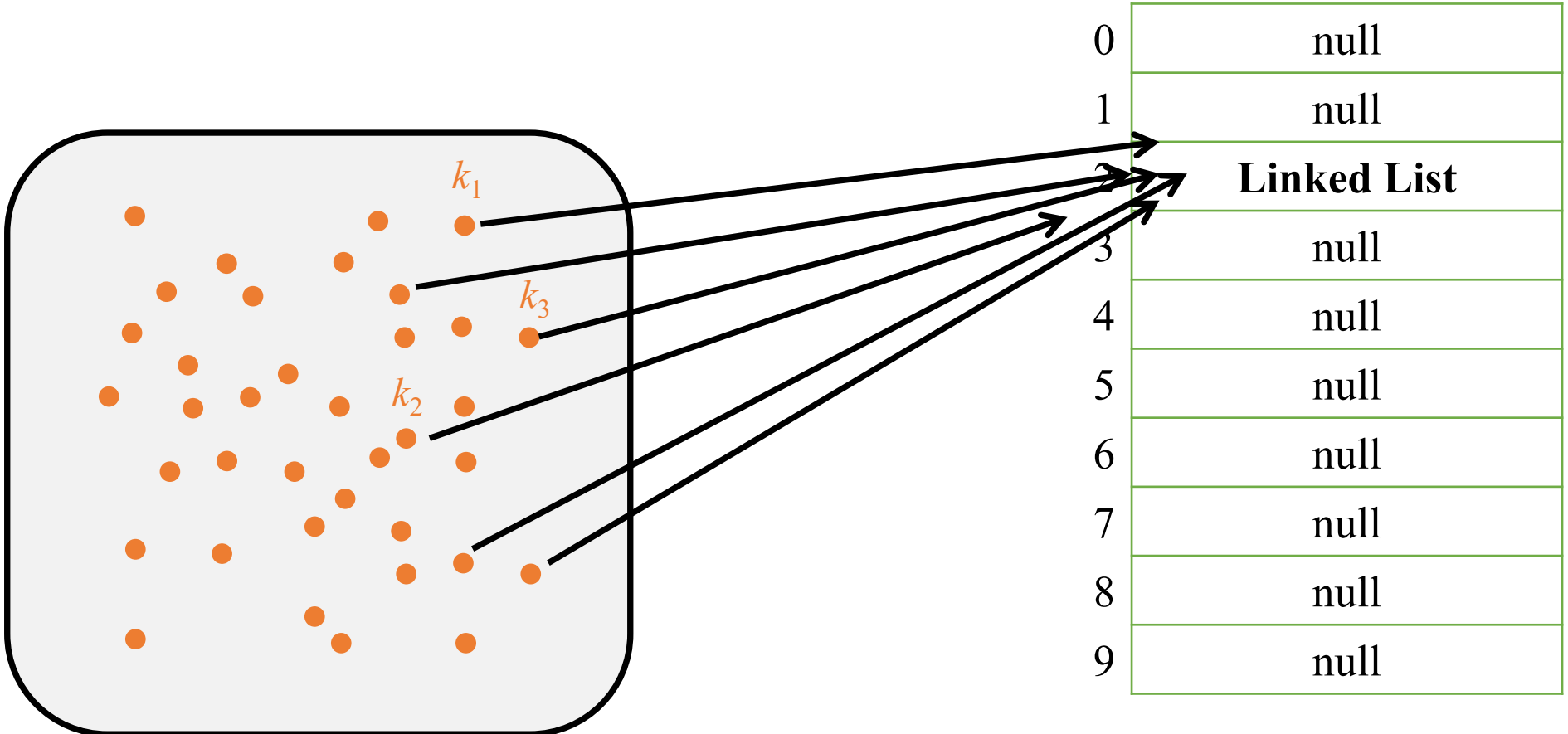
- search(key)/deletion
  - Calculate `h(key)`
  - Search for `(key,value)` in the linked list.

- Worst case?
  - time depends on length of linked list

Reminds me of this

Hashing with Chaining

- Assume all keys hash to the same bucket!
  - Search costs $O(n)$



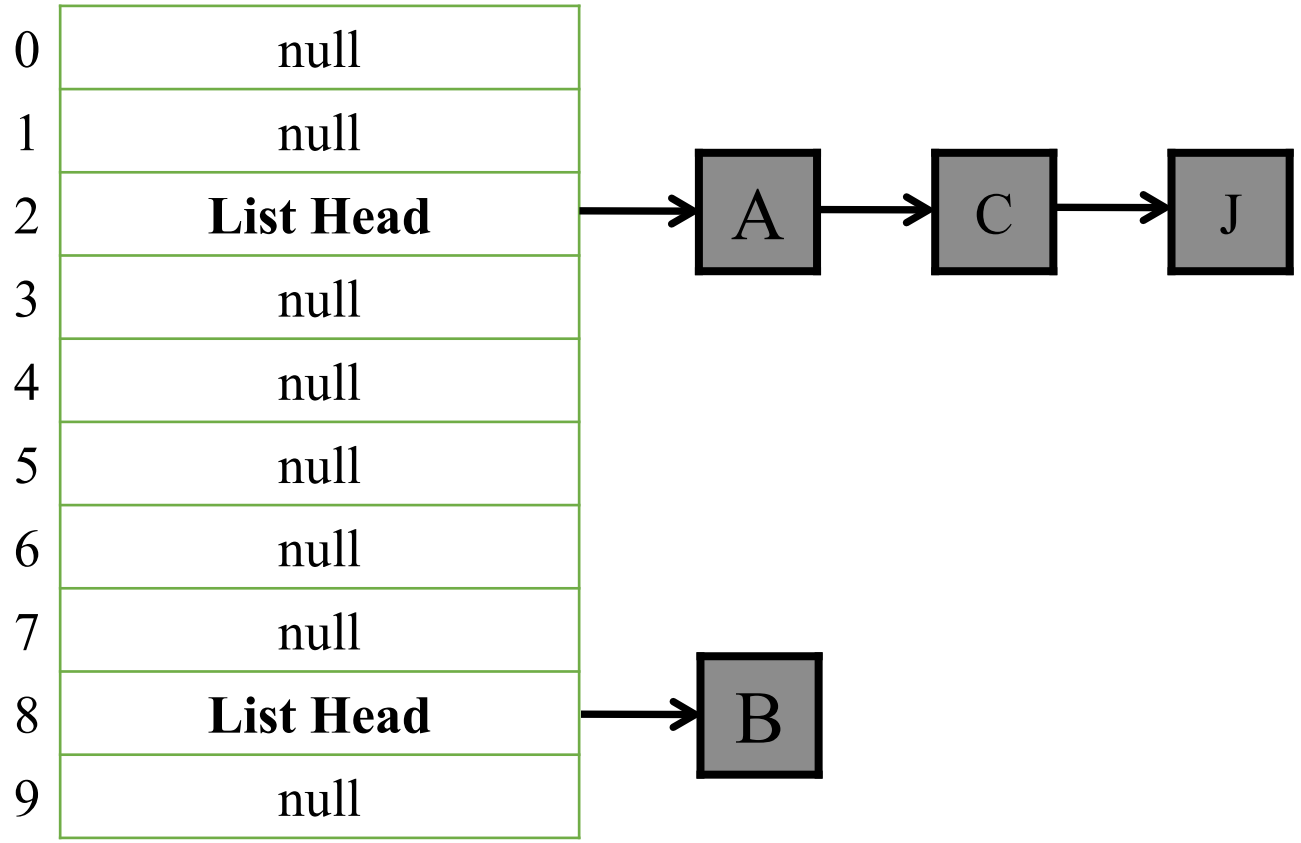
Let’s be optimistic first….

- The Simple Uniform Hashing Assumption
  - Every key is equally likely to map to every bucket.
  - Keys are mapped independently.

- Intuition:
  - Each key is put in a random bucket.
  - Then, as long as there are enough buckets, we won’t get too many keys in any one bucket.

A little probability

- What is the expected number of items in a bucket?



Let's be optimistic today.

- The Simple Uniform Hashing Assumption
  - Assume:
    - n items
    - m buckets
  - Define: load(hash table) = n/m

    = average # items / buckets.

  - Expected search time = $1 + n/m$

    hash function + array access

    linked list traversal

  - If $m > n$
    - Expected search time = $O(1)$

Hashing with Chaining

- Searching:
  - Expected search time $= 1 + n/m = \mathrm{O}(1)$
  - Worst-case search time = $\mathrm{O}(n)$

- Inserting:
  - Worst-case insertion time = $\mathrm{O}(1)$

Reality Fights Back

- Simple Uniform Hashing doesn’t exist.
- Keys are not random.
  - Lots of regularity.
  - Mysterious patterns.
- Patterns in keys can induce patterns in hash functions unless you are very careful.

Example

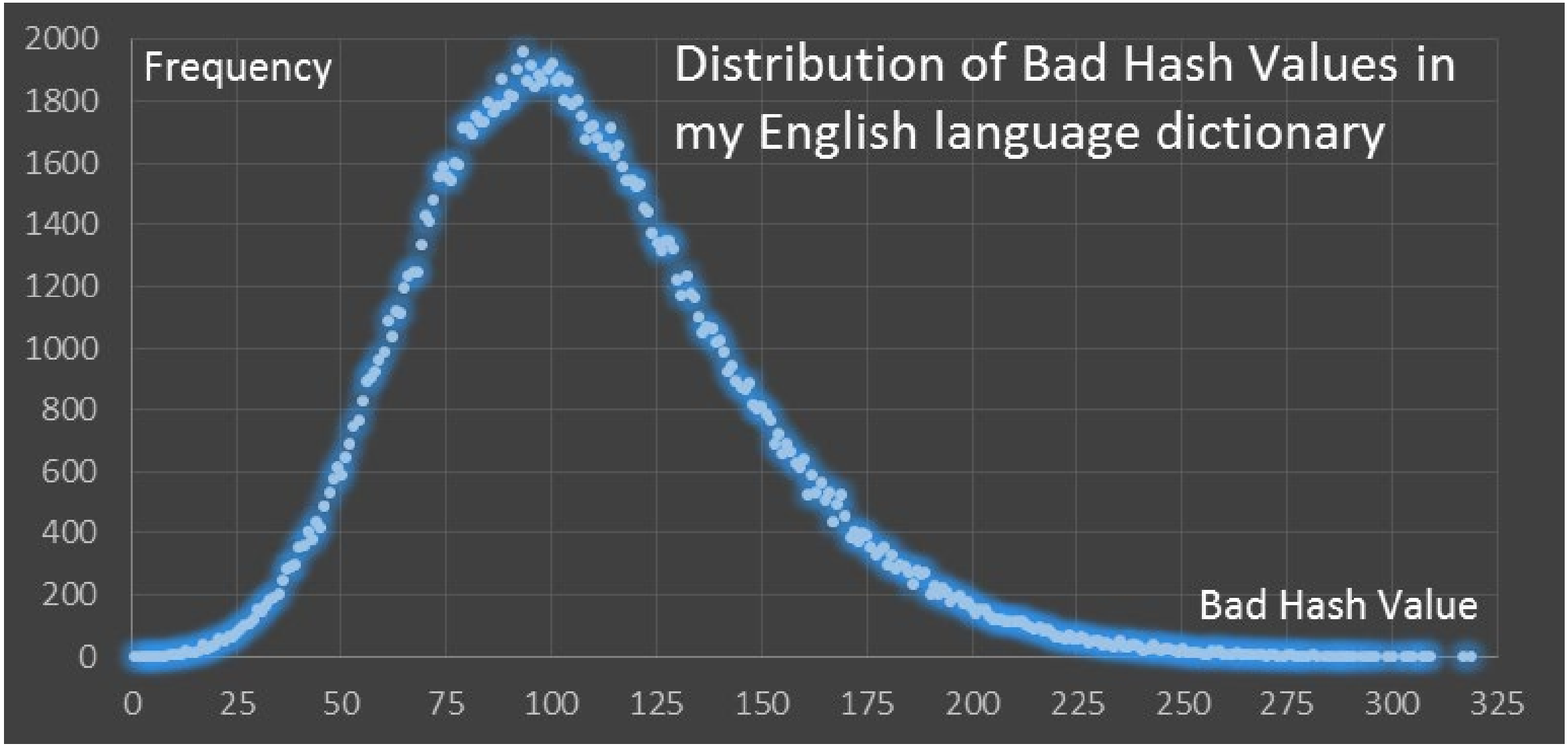
- One bucket for each letter [a..z]
- Hash function: `h(string)` = first letter.
  - E.g., `h("hippopotamus") = h`.

- Bad hash function: many fewer words start with the letter x than start with the letter s.

Example

- One bucket for each number from [1..26*28]
- Hash function: `h(string)` = sum of the letters.
  - E.g., `h("hat")` = 8 + 1 + 20 = 29.

- Bad hash function: lots of words collide, and you don't get a uniform distribution (since most words are short).


Distribution of Bad Hash Values in my English language dictionary
Frequency
Bad Hash Value
2000
1800
1600
1400
1200
1000
800
600
400
200
0
0
25
50
75
100
125
150
175
200
225
250
275
300
325

Moral of the Story

- Don’t design your own hash functions.
  - Ever.
- Unless you really need to.

- But pretty good hash functions do exist…
  - Optimism pays off!

Designing Hash Functions (If you really have to)

- Two common hashing techniques…
  - Division Method
  - Multiplication Method

Division Method

- $h(k) = k \bmod m$
  - For example: if $m = 7$, then $h\ (17) = 3$
  - For example: if $m = 20$, then $h\ (100) = 0$
  - For example: if $m = 20$, then $h\ (97) = 17$

- Two keys k_1 and k_2 collide when:

$$k_1 \bmod m = k_2 \bmod m$$

- Collision unlikely if keys are random.

Division Method Problem: Regularity

- What if k and m has a common divisor d?

$$k = k \bmod m + i * m$$

divisible by d divisible by d

- Implies that $h(k) = k \bmod m$ is divisible by d.
- For all those key values that are divisible by d, by what fraction of the hash table will they utilize?

$$1/d$$

0	**A**
1	null
2	null
$d = 3$	**B**
4	null
5	null
$2d = 6$	**C**
7	null
8	null
$3d = 9$	**D**

Division Method

- $h(k) = k \bmod m$
- Choose m = prime number
- Division method is popular (and easy), but not always the most effective.
  - Division is slow.

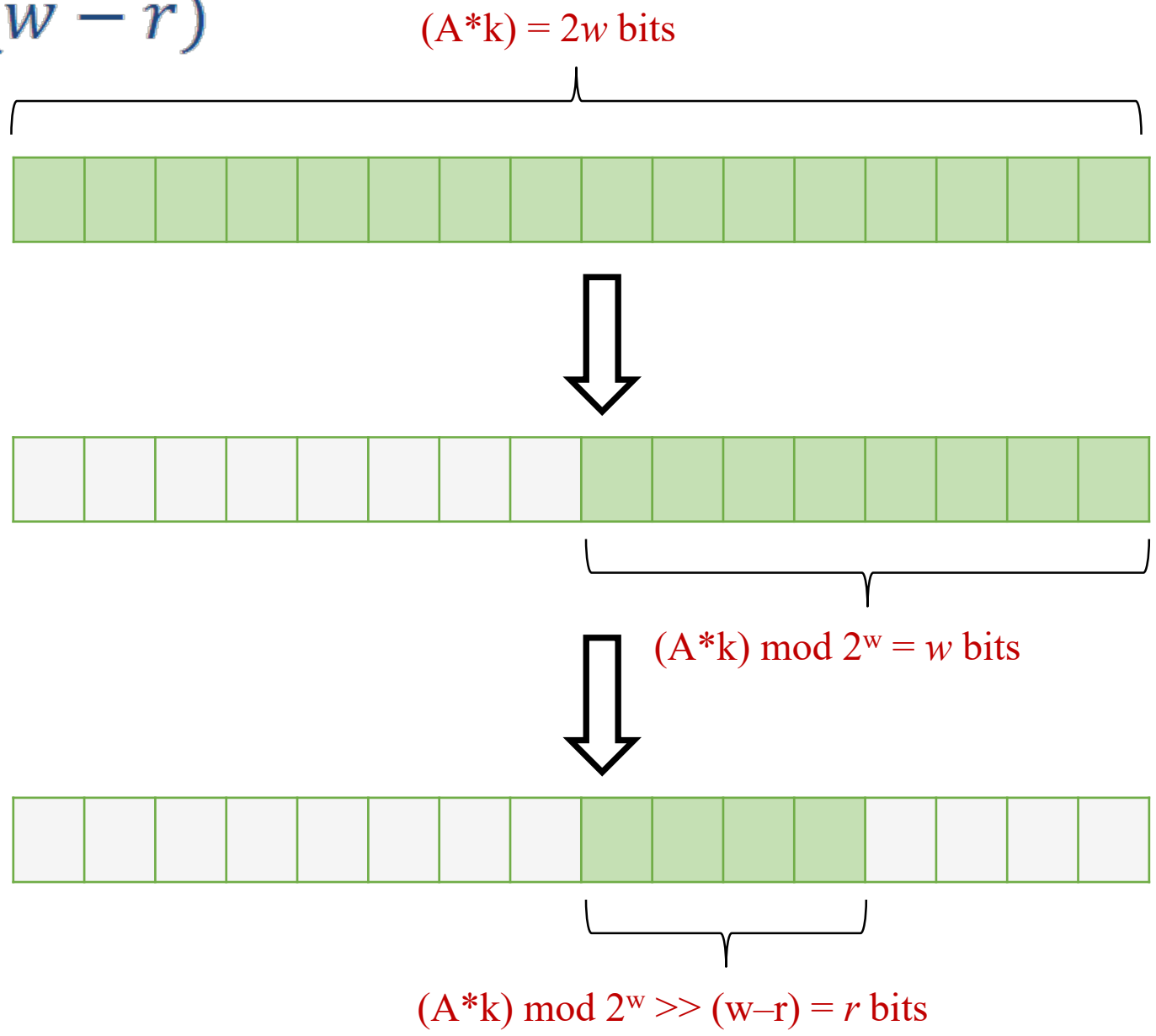
Multiplication Method

- Fix table size: $m = 2^r$, for some constant r.
- Fix word size: w, size of a key in bits.
- Fix (odd) constant A.

$$h(k) = (Ak) \bmod 2^w \gg (w - r)$$

$$h(k) = (Ak) \bmod 2^w \gg (w - r)$$

- A and k are w bits integers



Multiplication Method

- Faster than Division Method
  - Multiplication, shifting faster than division

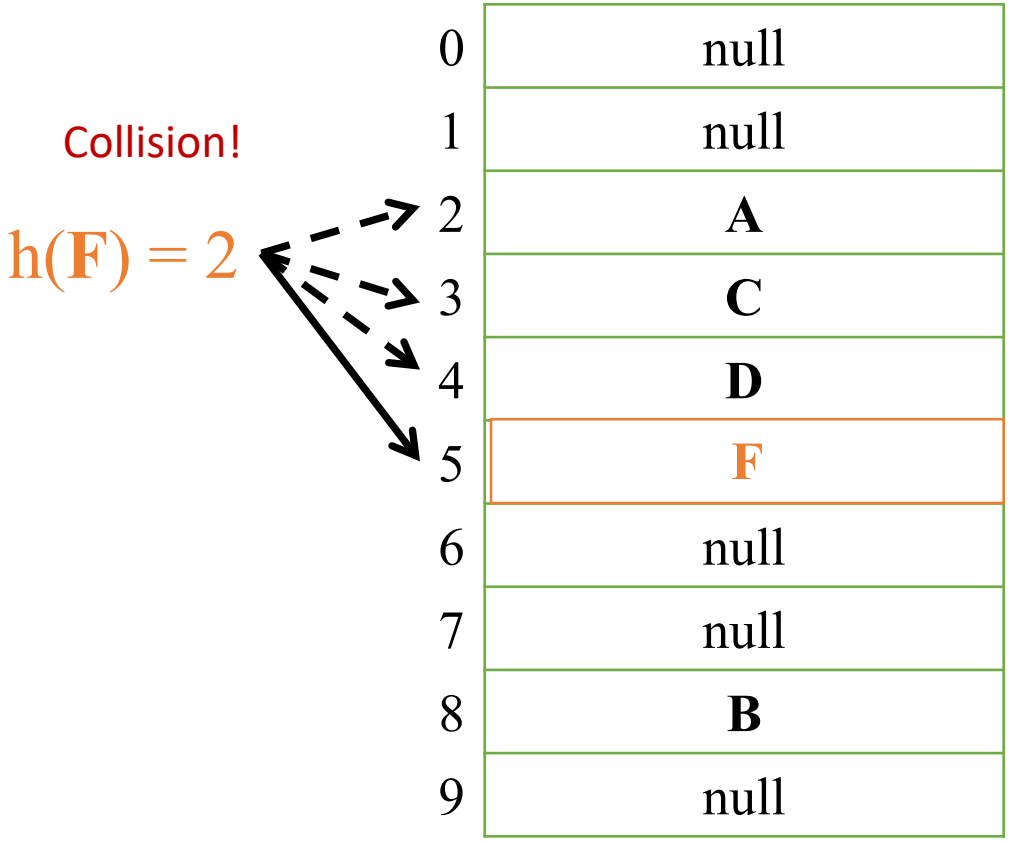
- Works reasonably well when *A* is an odd integer > 2^{w-1}
  - Odd: if it is even, then lose at least one bit's worth
  - Big enough: use all the bits in *A*.

Coping with Collision

- Idea 1: choose a new, better hash functions
  - Hard to find.
  - Requires re-copying the table.
  - Eventually, there will be another collision.
- Idea 2: chaining
  - Put both items in the same bucket!
- Idea 3: open addressing
  - Find another bucket for the new item.

Open Addressing

- Advantages:
  - No linked lists!
  - All data directly stored in the table.
  - One item per slot.
- On collision
  - Probe a sequence of buckets until you find an empty one.



Probing

- Find the next position that's empty to insert a key
- If the current slot with index i in the table is occupied, try slot i + 1
- It is the same way to say
  - originally let's try $h(F)$
  - if it collides, try $(h(F) + 1) \bmod m$
  - if it collides again, try $(h(F) + 2) \bmod m$
  - if it collides again, try $(h(F) + 3) \bmod m$
  - …

After 1 collision

After 2 collision

After 3 collision

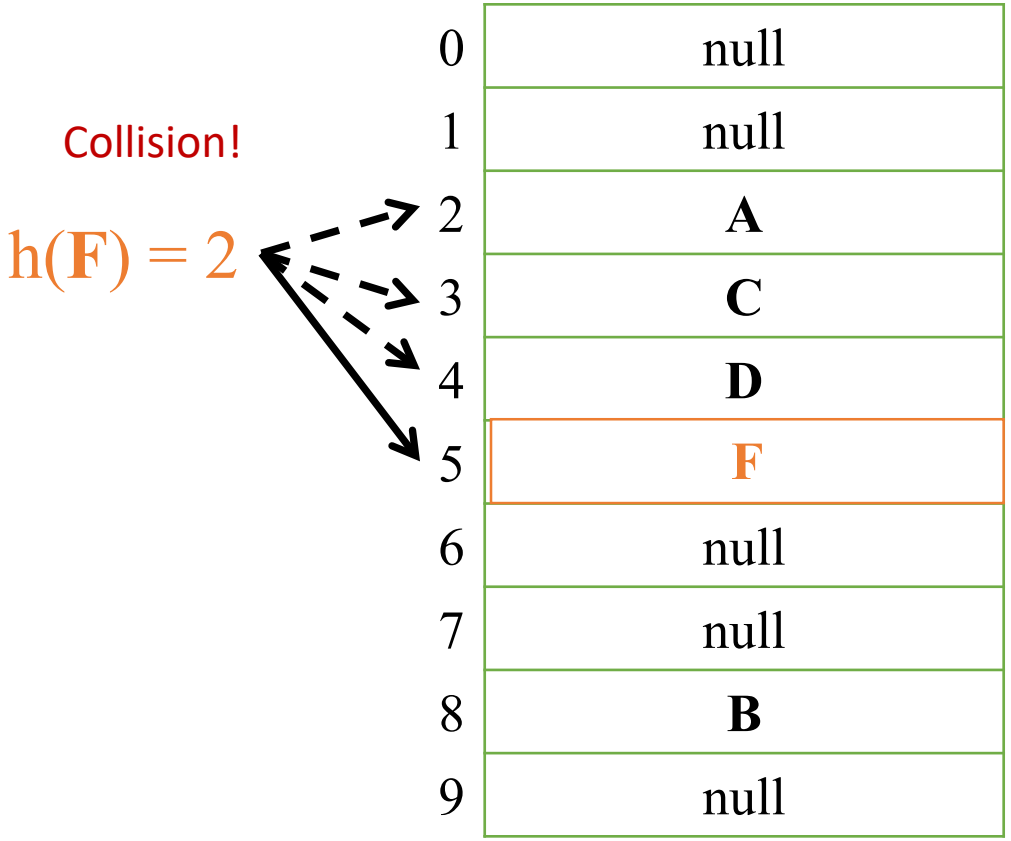
Collision!

$h(\mathbf{F}) = 2$

0	null
1	null
2	**A**
3	**C**
4	**D**
5	**F**
6	null
7	null
8	**B**
9	null

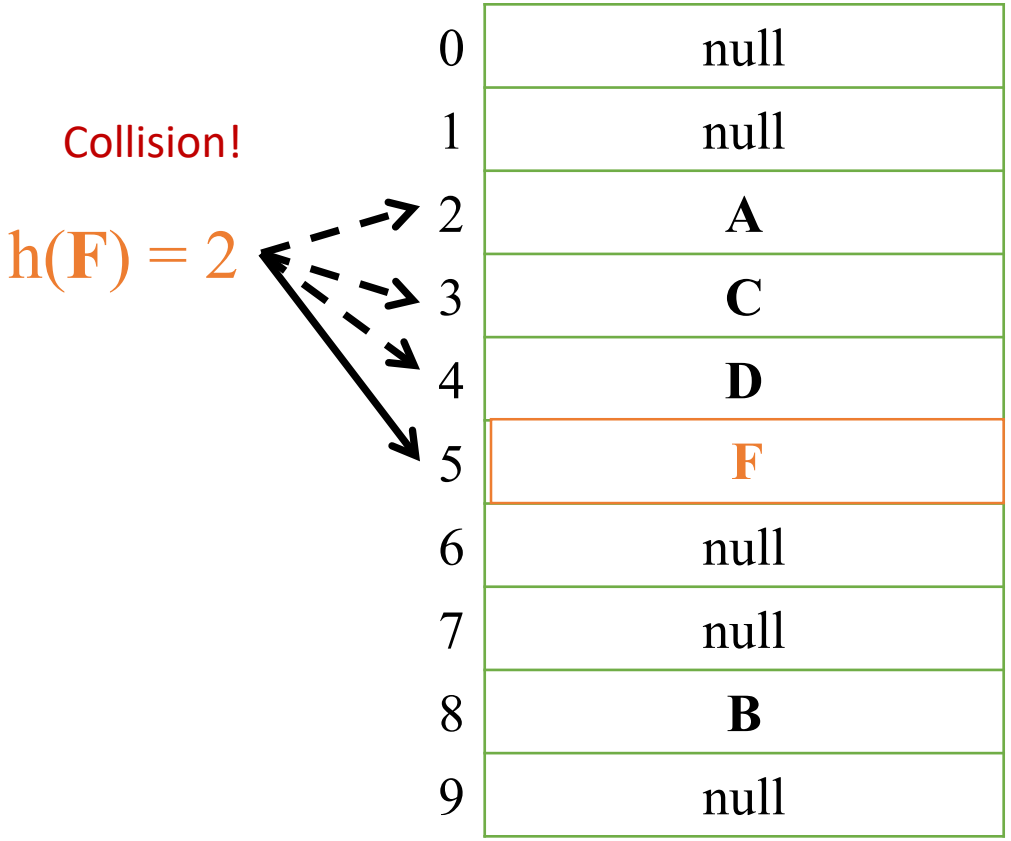
Probing

- Find the next position that's empty to insert a key
- If the current slot with index i in the table is occupied, try slot i + 1
- It is the same way to say
  - originally let's try $h(F)$
  - if it collides, try $(h(F) + i) \bmod m$
    - After i collisions
- Or if it collides, try $(h(F) + f(i)) \bmod m$
  - After i collisions
  - for $f(i) = i$



Probing

- Find the next position that's empty to insert a key
- originally let's try $h(F)$
- if it collides, try $(h(F) + f(i)) \bmod \mathrm{m}$
  - After *i* collisions
  - for $f(i) = i$
- We have freedom to change $f(i)$ into other functions
- For $f(i) = i$, it is called ***Linear Probing***



Probing

- For $f(i) = i^2$, it is called ***<u>Quadratic Probing</u>***
- Originally let's try h(F)
- if it collides, try $(h(F) + 1^2) \bmod m$
- if it collides again, try $(h(F) + 2^2) \bmod m$
- if it collides again, try $(h(F) + 3^2) \bmod m$
- …
- After i collisions, try $(h(F) + i^2) \bmod m$

Collision!

h(**F**) = 2

0	null
1	null
2	**A**
3	**C**
4	**D**
5	null
6	**F**
7	null
8	**B**
9	null

Some Probing Functions

- Let's redefine the hashing function to be $h(\text{key}, i)$ for i is the number of collisions
- Linear probing:

$$h(\text{key},i) = h(\text{key}) + i$$

- Quadratic probing

$$h(\text{key},i) = h(\text{key}) + i^2$$

- Double hashing (for another hashing function g)

$$h(\text{key},i) = h(\text{key}) + i \times g(\text{key})$$

How do we search now?

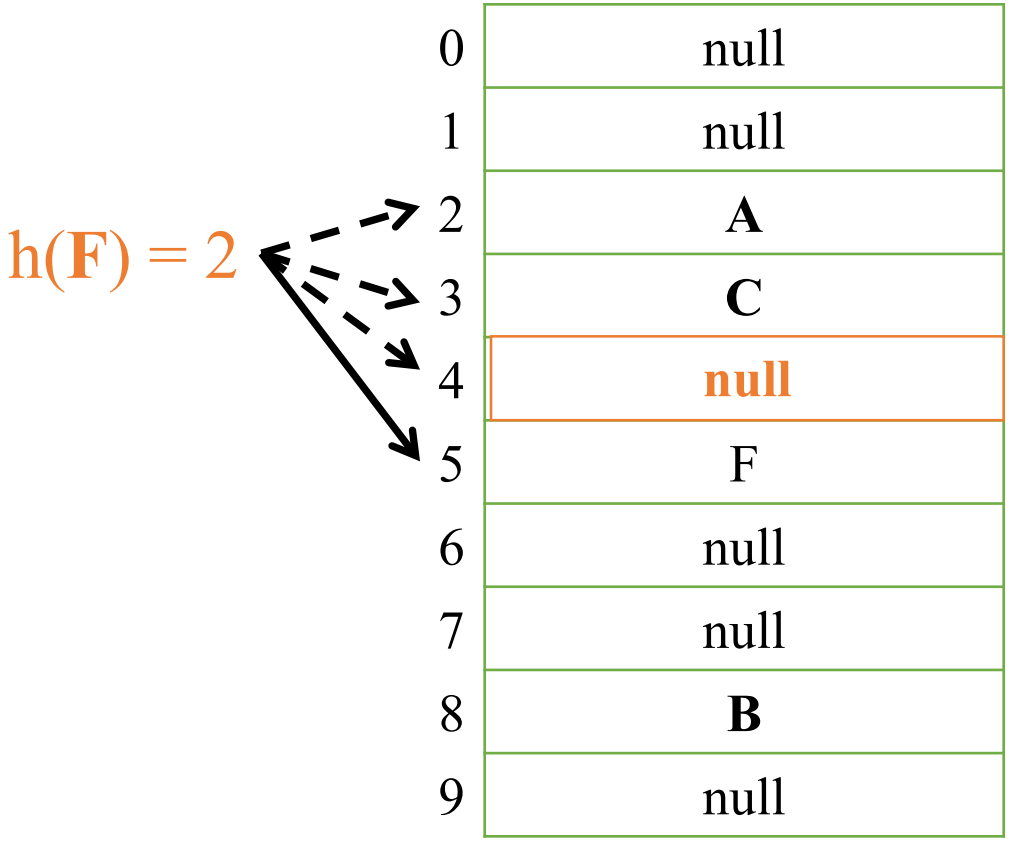
```
int i = 0;
while (i <= m) {
      int bucket = h(key, i);
      if (T[bucket] == null) // Empty bucket!
            return key-not-found;
      if (T[bucket].key == key) // Full bucket.
                  return T[bucket].data;
      i++;
}
return key-not-found; // Exhausted entire table.
```

h(**F**) = 2

0	null
1	null
2	**A**
3	**C**
4	**D**
5	**F**
6	null
7	null
8	**B**
9	null

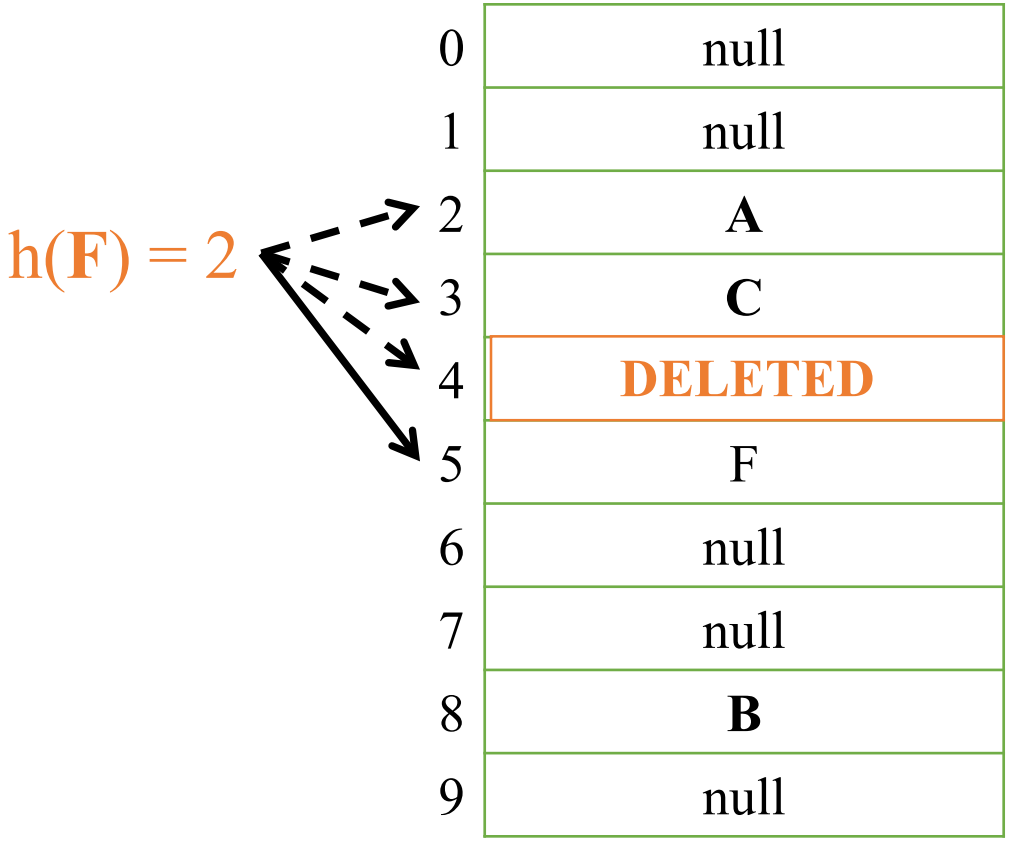
Open Addressing with Probing

- Now we can
  - Insert
  - Search
- How about deleting an item?
  - Just set the slot to “null”?
  - E.g. `delete(D)`?
  - Problem?



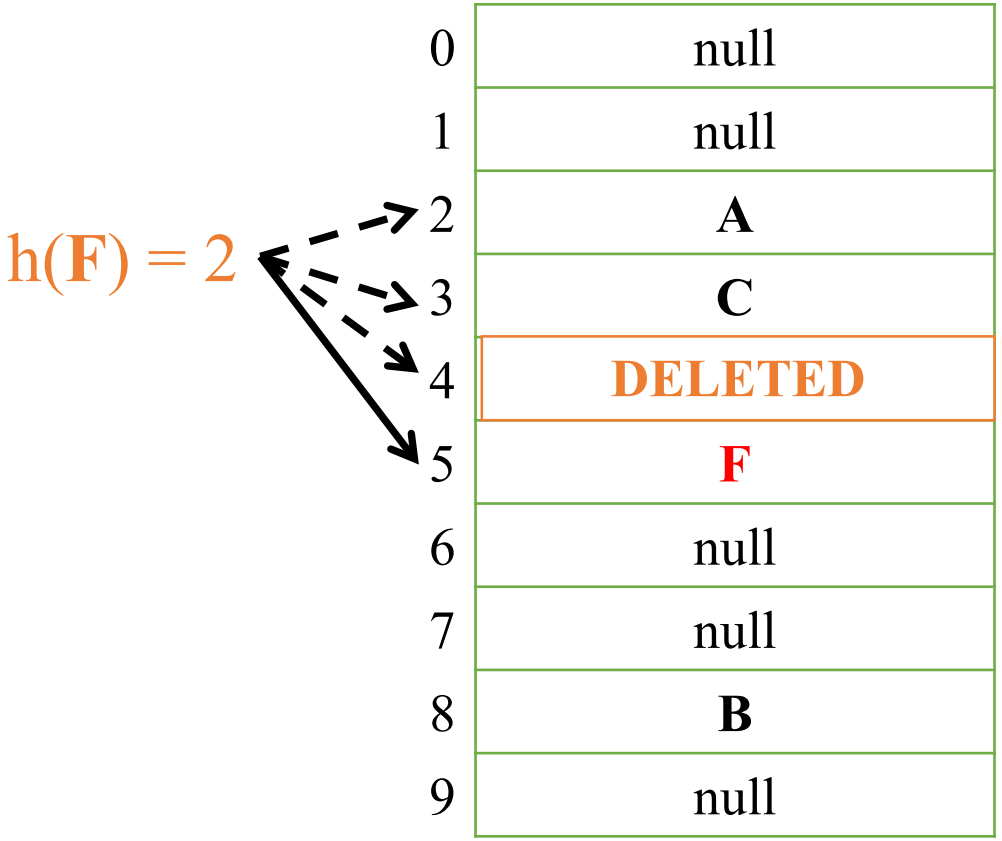
Open Addressing with Probing

- Deletion:
  - Set the slot to be “DELETED”
- For Searching, we can use the previous algorithm
  - Note that we will stop searching when we met a “null” or we found the key ONLY
- How about insertion now?
  - We can replace an item on the slot of “DELETED”



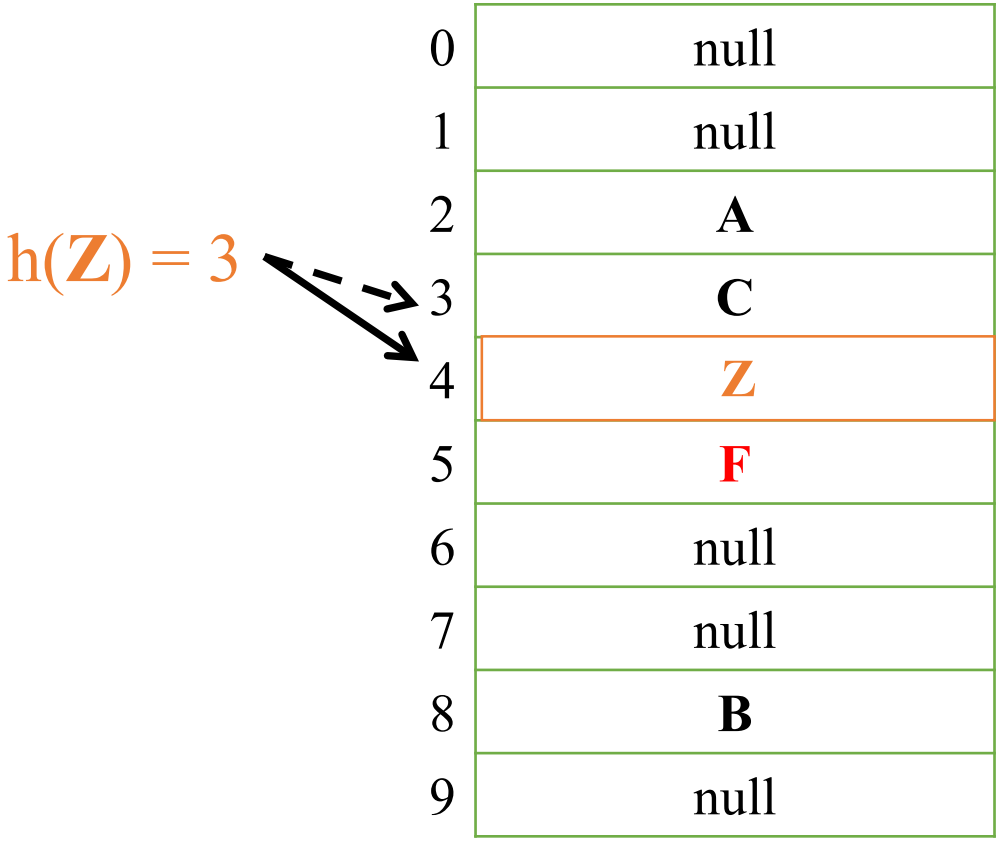
Deletion Example

- Assuming $h(F,0) = 2$
- Delete(D)
- Search for F
  - Same



Deletion Example

- Assuming $h(F,0) = 2$
- Delete(D)
- Search for F
  - Same
- Now insert Z for $h(Z) = 3$
  - Collided with C
  - Look for $h(Z,1) = 4$
    - Insert Z into slot 4



Two Properties of Good Hashing Functions

1. $h(\text{key}, i)$ must be able to reach all slots
   - For every bucket j, there is some i such that:

$$h(\text{key}, i) = j$$

   - For linear probing: true!
2. Simple Uniform Hashing Assumption
   - Every key is equally likely to be mapped to every bucket, independently of every other key.
   - An “Art”
   - Could be either with good math or empirical proof

Performance

Performance

- Under uniform hashing assumption, what is the expected time for insertion after we inserted m items into a table of size m?
  a) O(1)
  b) O(log n)
  c) O(n)
  d) O(n^2)
  e) None of the above.

Performance of Open Addressing

- Chaining:
  - When $m == n$, we can still add new items to the hash table.
  - We can still search efficiently.

- Open addressing:
  - When $m == n$, the table is full.
  - We cannot insert any more items.
  - We cannot search efficiently.

Performance of Open Addressing

- Define:
  - Load: $\alpha = n / m$
  - Assume $\alpha < 1$.
- Claim:
- For n items, in a table of size m, assuming uniform hashing, the expected cost of an operation is:

$$\leq \frac{1}{1-\alpha}$$

- Example: if ($\alpha = 90\%$), then E[# probes] = 10

Performance of Open Addressing

- First probe: probability that first bucket is full is:

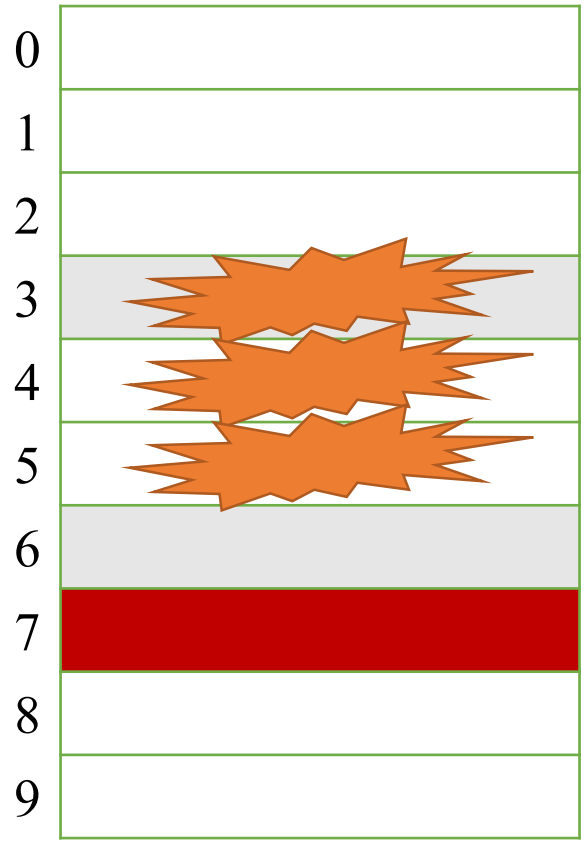
$$n / m$$

- Second probe: if first bucket is full, then the probability that the second bucket is also full:

$$(n - 1) / (m - 1)$$

- Third probe: probability is full:

$$(n - 2) / (m - 2)$$



Performance of Open Addressing

- Expected #probes

$$1 + \frac{n}{m}\left(\text{Expected cost of remaining probes}\right)$$

First probe

Probability of collision on first probe

Performance of Open Addressing

- Expected #probes

$$1 + \frac{n}{m}\left(\text{Expected cost of remaining probes}\right)$$

$$1 + \frac{n}{m}\left(1 + \frac{n-1}{m-1}\left(\text{Expected cost of remaining probes}\right)\right)$$

First probe

Probability of collision on first probe

Probability of collision on second probe

Performance of Open Addressing

- Expected #probes

$$1 + \frac{n}{m}\left(1 + \frac{n-1}{m-1}\left(1 + \frac{n-2}{m-2}\left(1 + \frac{n-3}{m-3}\right)(\ldots)\right)\right)$$

- Note that

$$\frac{n-i}{m-i} \leq \frac{n}{m} \leq \alpha$$

Performance of Open Addressing

- Expected #probes

$$1 + \frac{n}{m}\left(1 + \frac{n-1}{m-1}\left(1 + \frac{n-2}{m-2}\left(1 + \frac{n-3}{m-3}\right)(\dots)\right)\right)$$

$$\leq 1 + \alpha\left(1 + \alpha\left(1 + \alpha(1+\alpha)(\dots)\right)\right)$$

$$\leq 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \cdots$$

$$\leq \frac{1}{1-\alpha}$$

Performance of Open Addressing

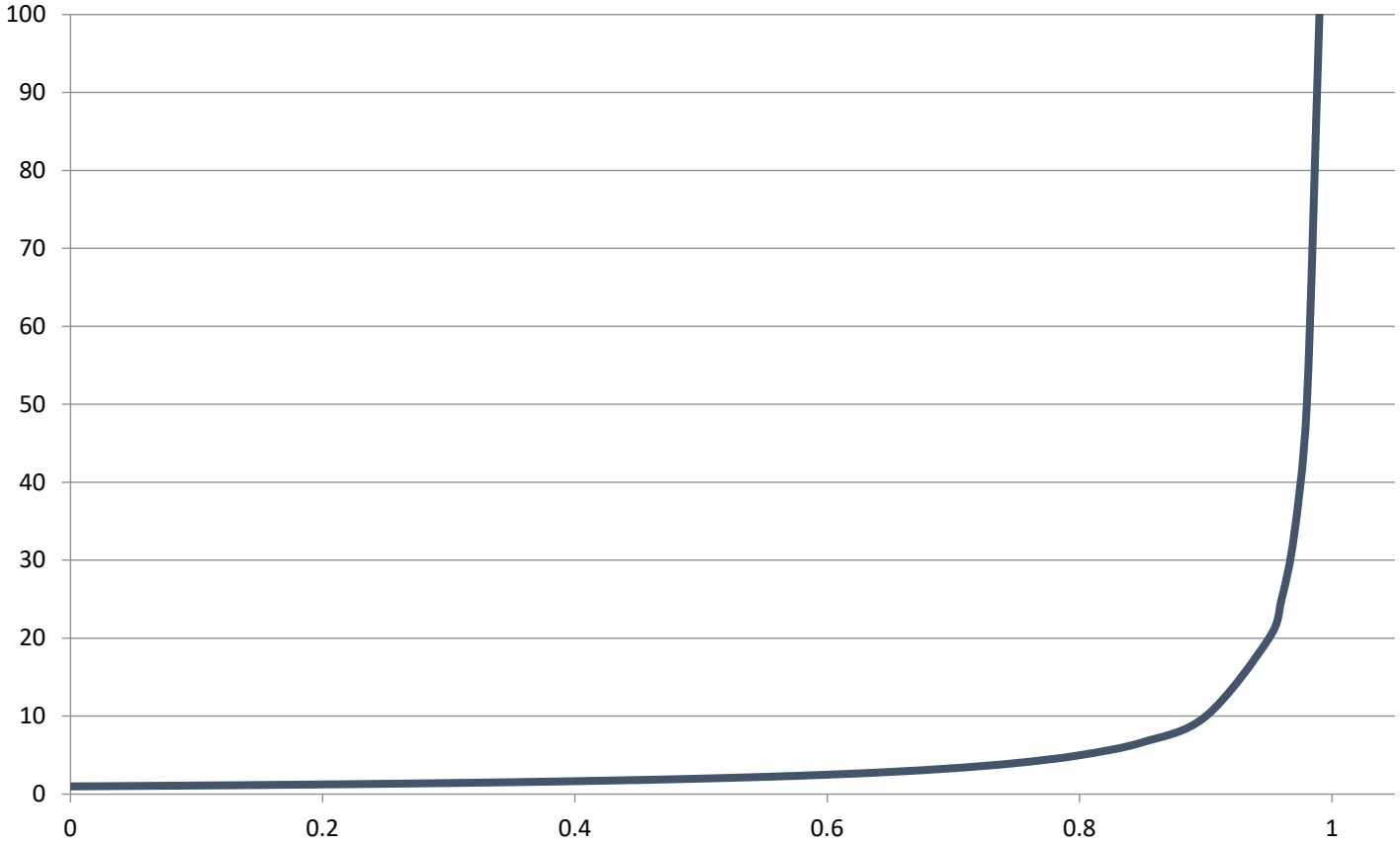
- Define:
  - Load: $\alpha = n / m$
  - Assume $\alpha < 1$.
- Claim:
- For n items, in a table of size m, assuming uniform hashing, the expected cost of an operation is:

$$\leq \frac{1}{1-\alpha}$$

- Example: if ($\alpha = 90\%$), then E[# probes] = 10

Open Addressing Advantages

- Saves space
  - Empty slots vs. linked lists.
- Rarely allocate memory
  - No new list-node allocations.
- Better cache performance
  - Table all in one place in memory
  - Fewer accesses to bring table into cache.
  - Linked lists can wander all over the memory.

Open Addressing Disadvantages

- More sensitive to choice of hash functions.
  - Clustering is a common problem.
  - See issues with linear probing.
- More sensitive to load.
  - Performance degrades badly as $\alpha \rightarrow 1$.