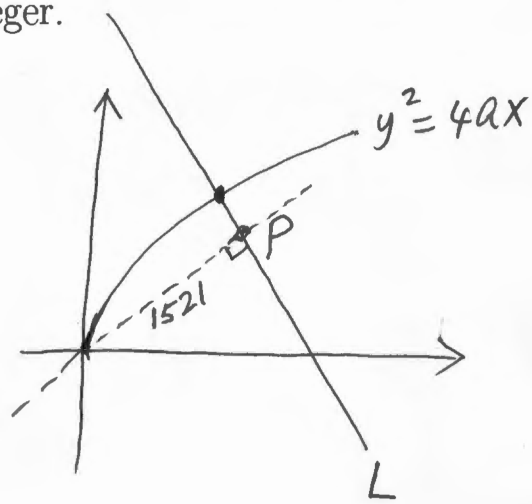
1. Let a denote a positive constant. Let L denote the normal line to the parabola $y^2 = 4ax$ at the point $(a, 2a)$. If the distance from the origin to L is equal to 1521. Find the value of a. Give your answer correct to the nearest integer.



$$2y\frac{dy}{dx} = 4a$$

$$x=a,\ y=2a \Rightarrow 4a\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = 1$$

$$l:\ y = x \text{ ---- } ①$$

$$L:\ \frac{y-2a}{x-a} = -1 \text{ ---- } ②$$

$$① \,\&\, ② \Rightarrow \frac{x-2a}{x-a} = -1$$

$$\Rightarrow x-2a = -x+a \Rightarrow x = \frac{3}{2}a$$

$$\therefore P = \left(\frac{3}{2}a, \frac{3}{2}a\right)$$

$$\left(\frac{3}{2}a\right)^2 + \left(\frac{3}{2}a\right)^2 = 1521^2$$

$$\frac{9}{2}a^2 = 1521^2$$

$$a = \sqrt{\frac{2\times 1521^2}{9}} = 717.006\cdots$$

$$\approx \underline{\underline{717}}$$

2. Let a denote a positive constant. If $x = \sin t$ and $y = \sin 5t$ and the equation $\left(1 - x^2\right) \frac{d^2y}{dx^2} - x\frac{dy}{dx} + ay = 0$ holds for all values of t, find the **exact value** of a.

$$\frac{dy}{dt} = 5\cos 5t$$

$$\frac{dx}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{5\cos 5t}{\cos t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-25\sin 5t\cos t + 5\cos 5t\sin t}{\cos^2 t\ (\cos t)}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + ay = 0$$

$$\Rightarrow \cos^2 t\left(\frac{-25\sin 5t\cos t + 5\cos 5t\sin t}{\cos^2 t\cos t}\right)$$

$$- \sin t\,\frac{5\cos 5t}{\cos t} + a\sin 5t = 0$$

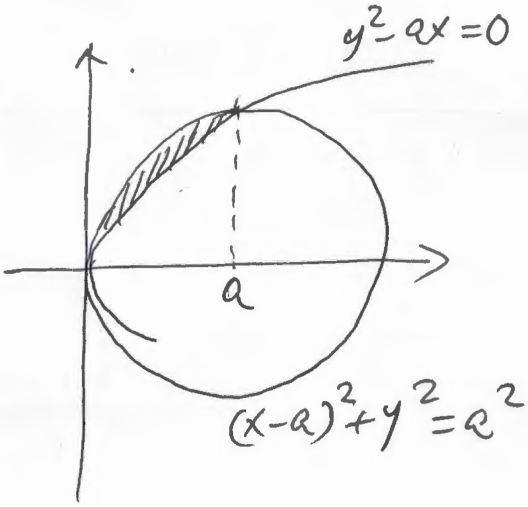
$$\Rightarrow (-25 + a)\sin 5t = 0$$

$$\Rightarrow \underline{\underline{a = 25}}$$

3. Let a denote a positive constant. Let R denote the finite region in the first quadrant bounded between the two curves $x^2-2ax+y^2=0$ and $y^2-ax=0$. If the volume of revolution of R one round about the x-axis is equal to 1521, find the value of a. Give your answer correct to two decimal places.

$$x^2-2ax+y^2=0 \Leftrightarrow x^2-2ax+a^2+y^2=a^2$$
$$\Leftrightarrow (x-a)^2+y^2=a^2$$

$$\begin{cases} x^2-2ax+y^2=0 \\ y^2-ax=0 \end{cases} \Rightarrow x^2-ax=0 \Rightarrow x=0 \text{ or } x=a$$



$$\pi\int_0^a \{(2ax-x^2)-ax\}dx=1521$$

$$\left[\frac{1}{2}ax^2-\frac{1}{3}x^3\right]_0^a=\frac{1521}{\pi}$$

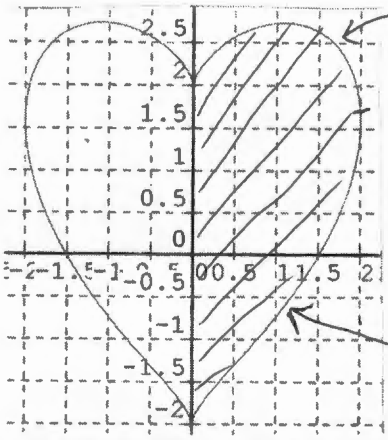
$$\frac{1}{6}a^3=\frac{1521}{\pi}$$

$$a=\left(\frac{6\times 1521}{\pi}\right)^{1/3}$$

$$=14.268\ldots$$

$$\approx \underline{\underline{14.27}}$$

4. Let a denote a positive constant. On Valentine's Day this year my wife sent me this gift: $x^2+(y-x^{2/3})^2=a^2$. When I plotted its graph, I got a curve that looks like the one in the picture below.



$y = x^{2/3} + \sqrt{a^2-x^2}$
$0 \le x \le a$

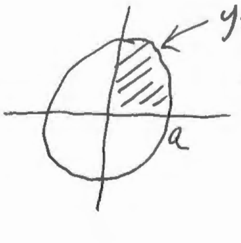
$y = x^{2/3} - \sqrt{a^2-x^2}$
$0 \le x \le a$

Being a maths guy I calculated the area bounded by this curve and found that it equals 2020. What is the value of a? Give your answer correct to two decimal places.

$$x^2+(y-x^{2/3})^2=a^2 \Rightarrow y = x^{2/3} \pm \sqrt{a^2-x^2}$$

$$\text{Area} = 2\int_0^a \left\{ (x^{2/3}+\sqrt{a^2-x^2}) - (x^{2/3}-\sqrt{a^2-x^2}) \right\} dx$$ (refer to graph above)

$$\therefore 2020 = 4\int_0^a \sqrt{a^2-x^2}\, dx$$



$y = \sqrt{a^2-x^2}$
$0 \le x \le a$
a

Note that for the circle $x^2+y^2=a^2$

$$\text{area} = 4\int_0^a \sqrt{a^2-x^2}\, dx = \pi a^2.$$

$$\therefore 2020 = \pi a^2$$

$$\therefore a = \sqrt{\frac{2020}{\pi}} = 25.357\ldots$$

$$\approx \underline{\underline{25.36}}$$