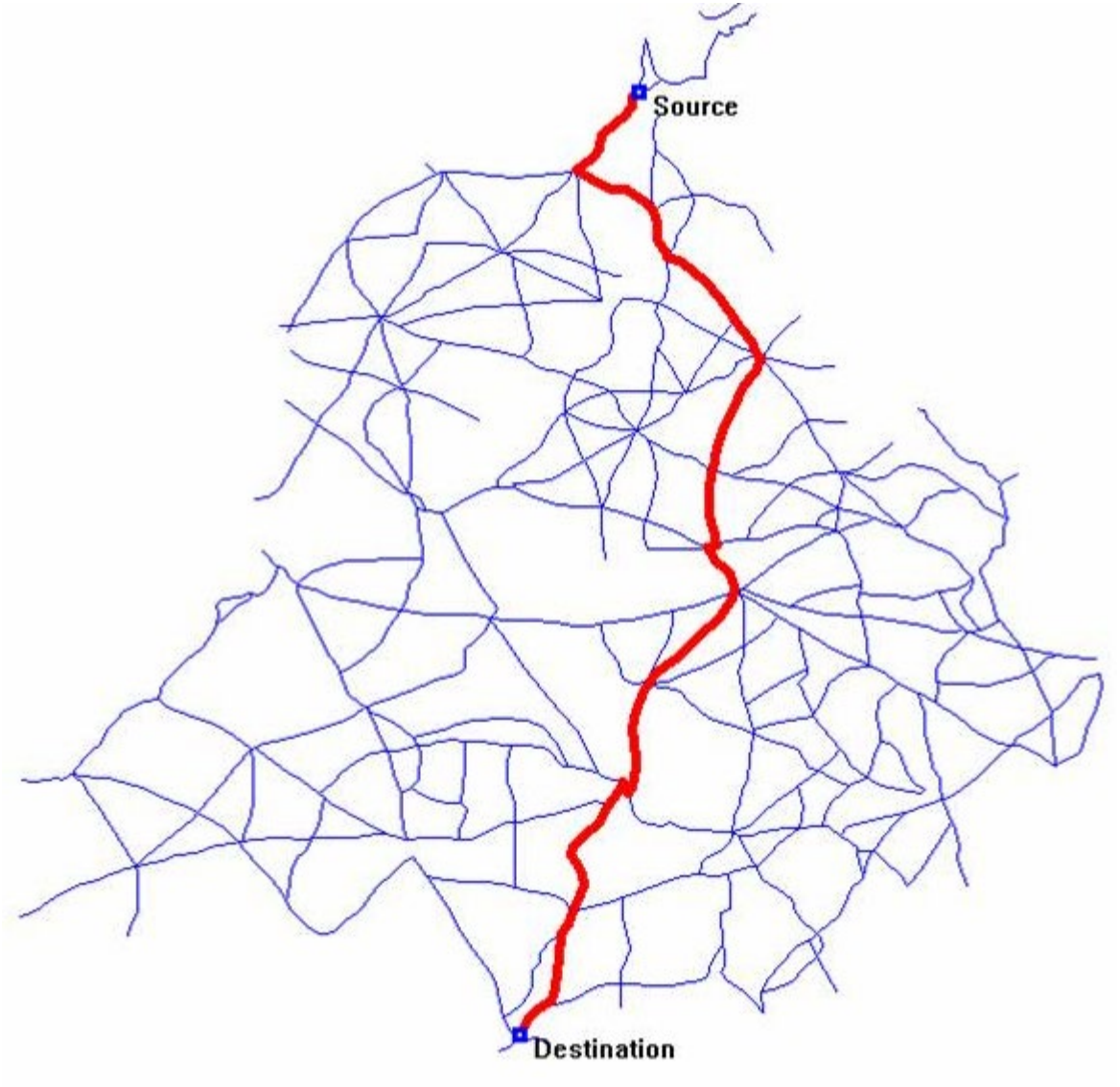
Roadmap

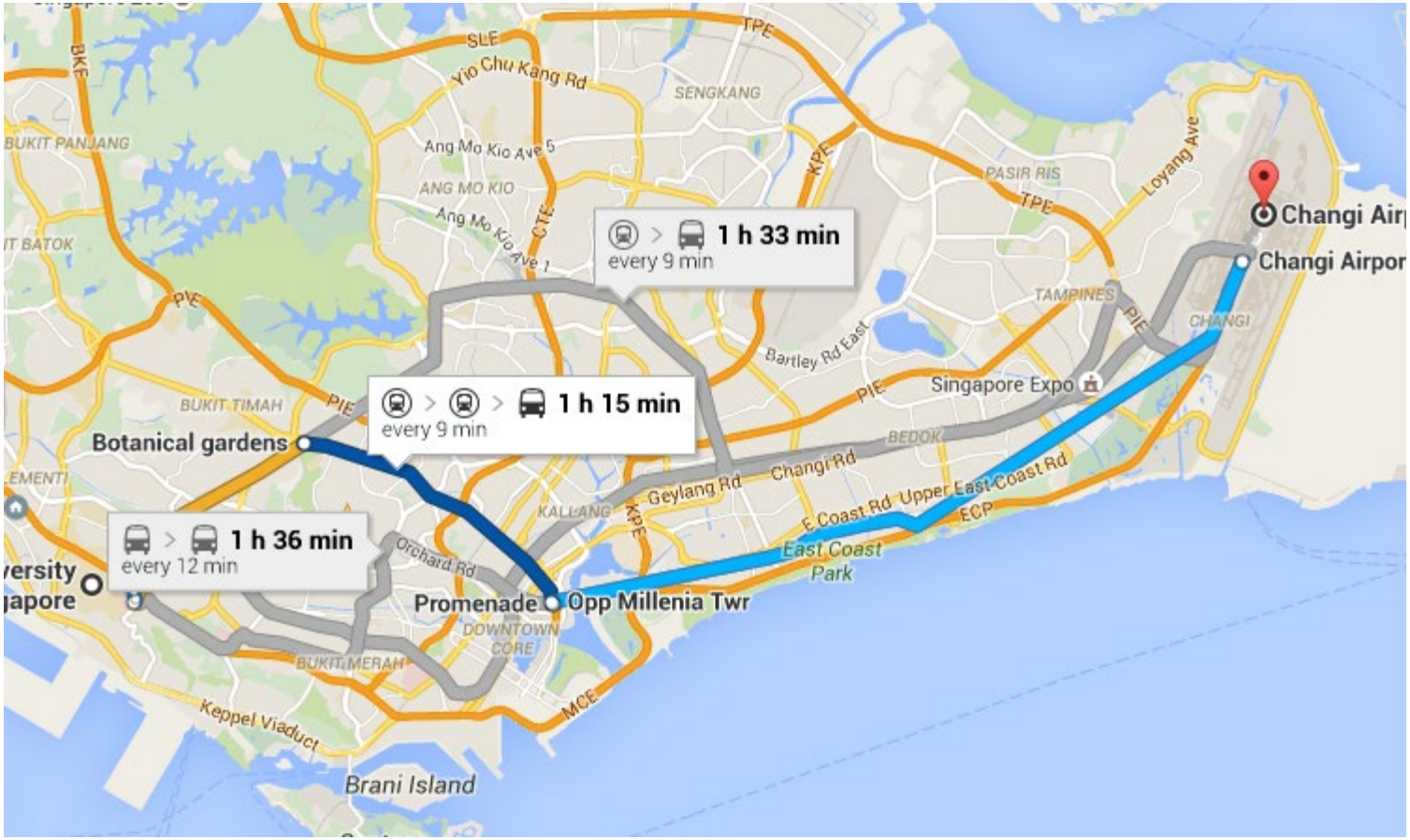
Shortest Paths

- The SSSP Problem
- Bellman-Ford

SHORTEST PATHS

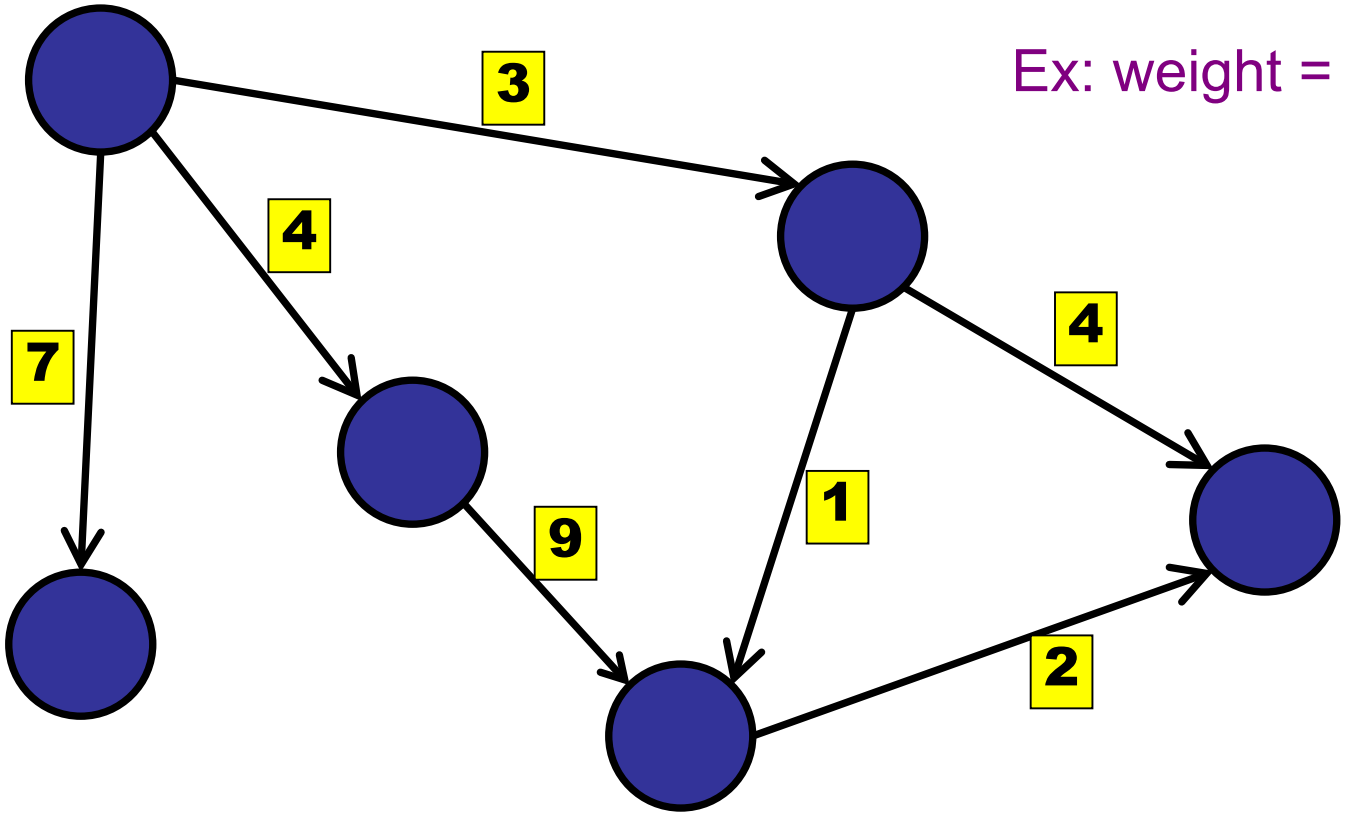


SHORTEST PATHS



Weighted Graphs

Edge weights: w(e) : E → R

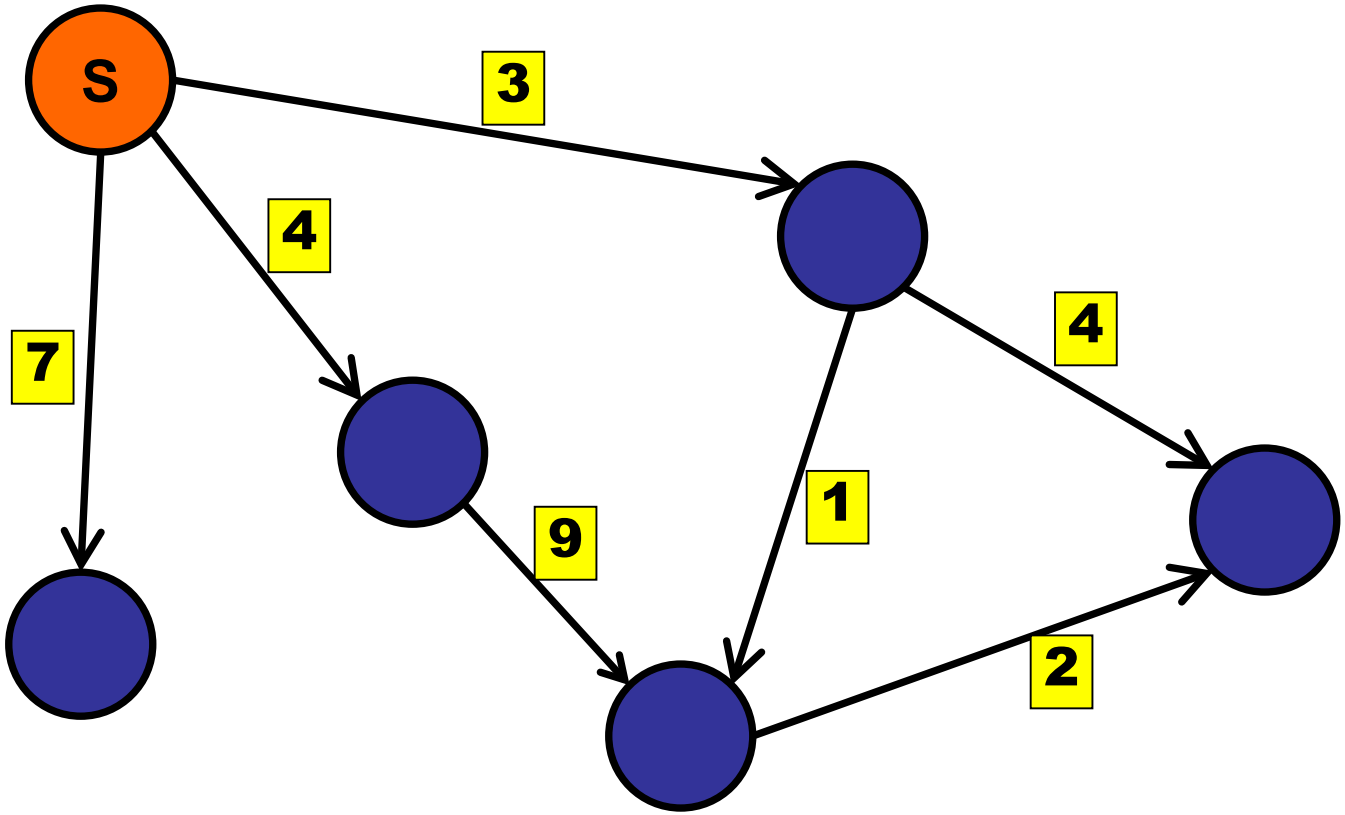


Ex: weight = distance

Adjacency list: stores weights with edge in NbrList

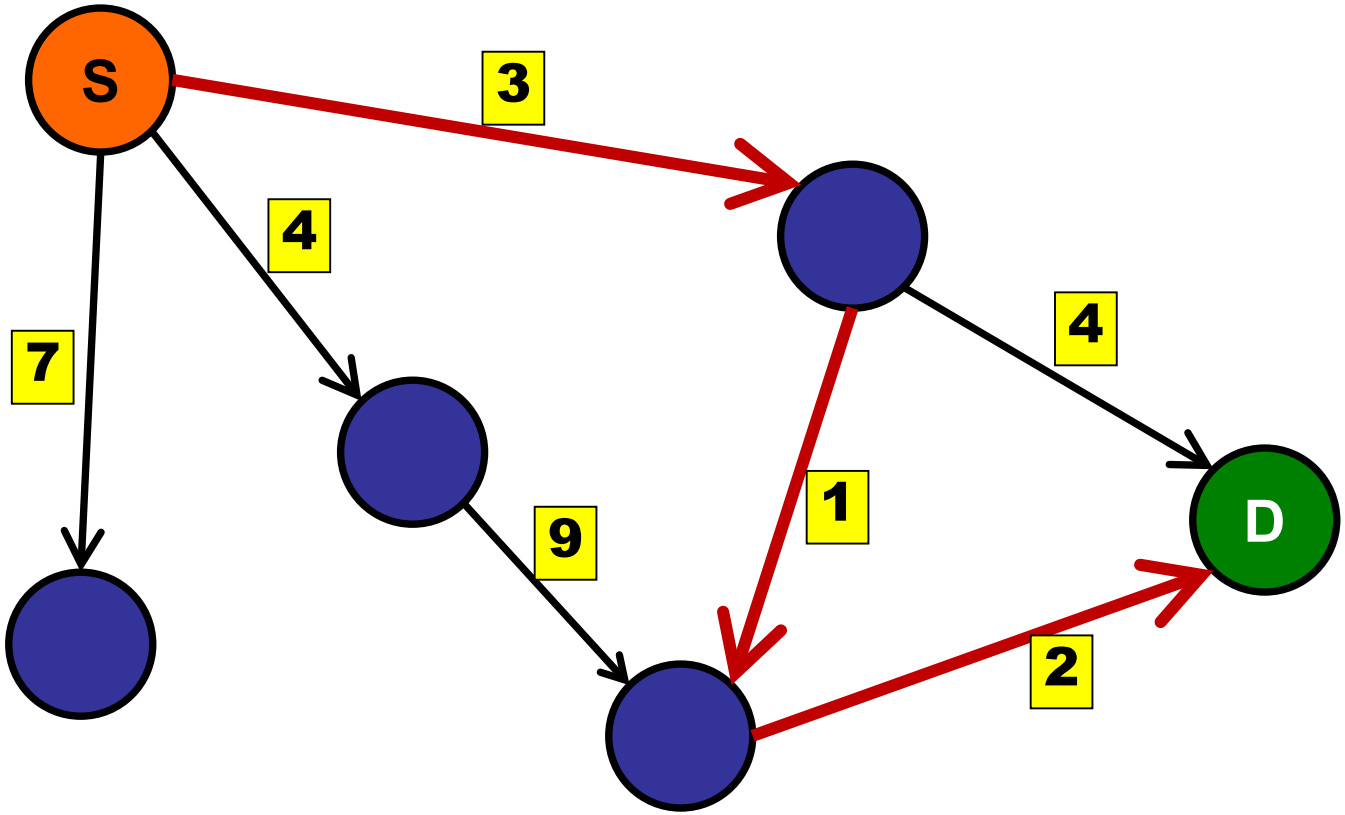
Shortest Paths

Distance from source?



Shortest Paths

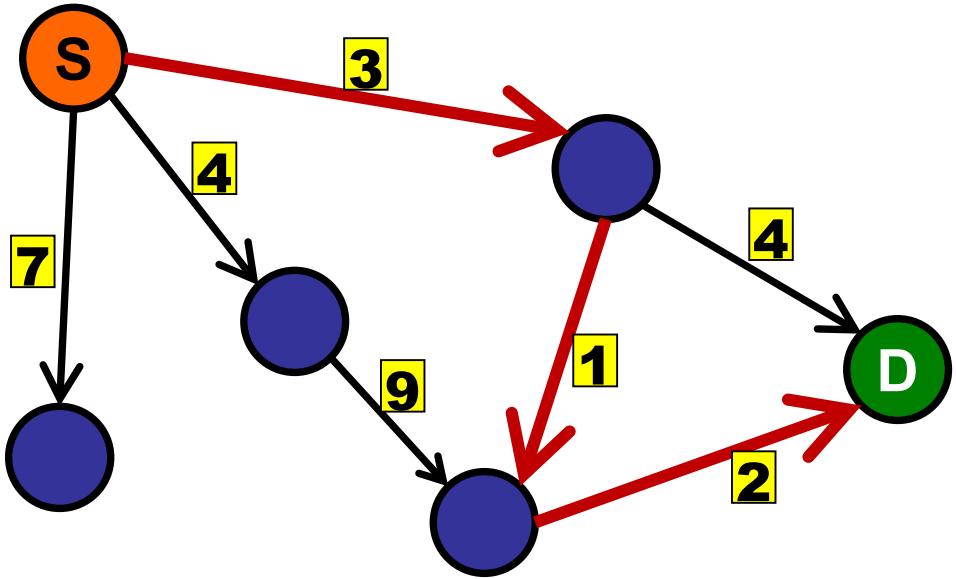
Distance from source?



Shortest Paths

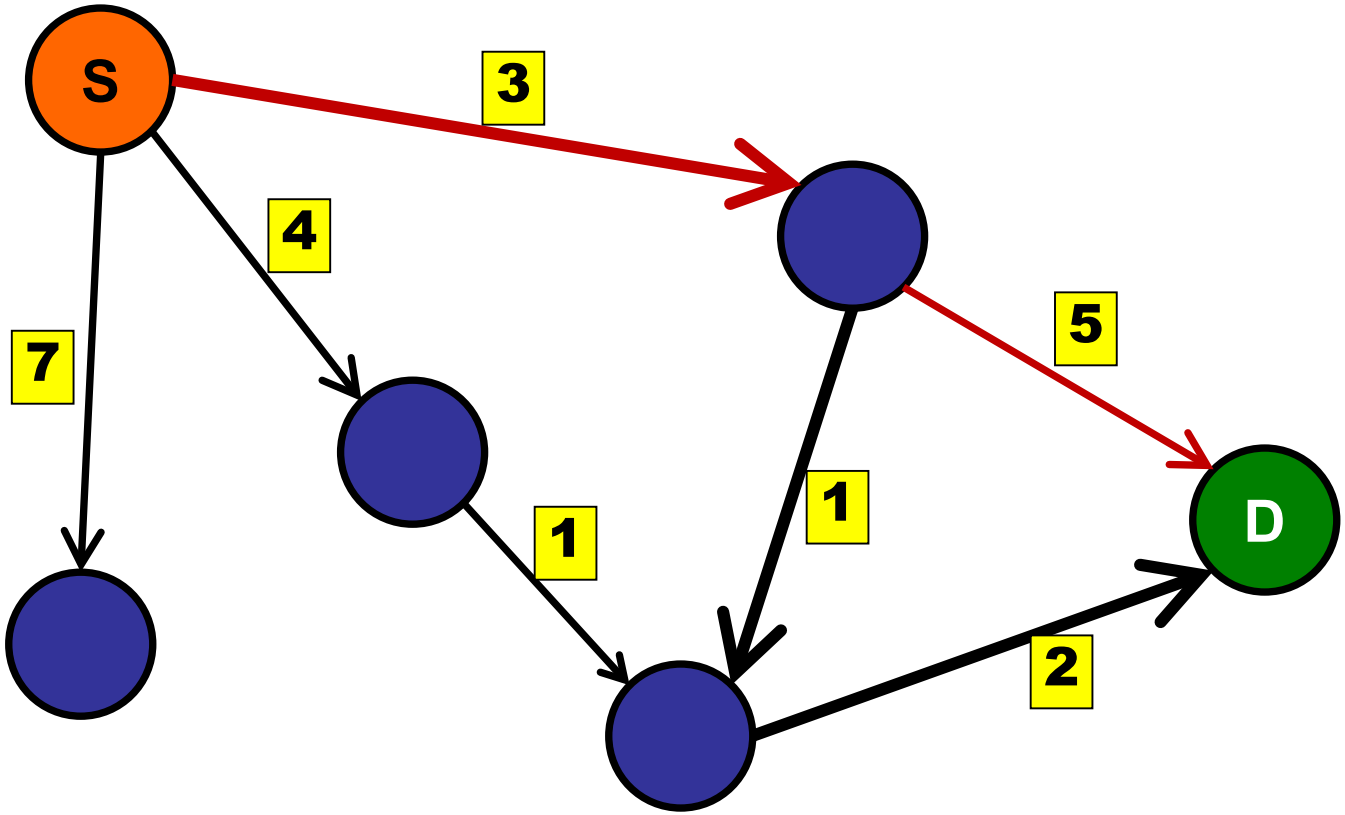
Questions:

- How far is it from S to D?
- What is the shortest path from S to D?
- Find the shortest path from S to every node.
- Find the shortest path between every pair of nodes.



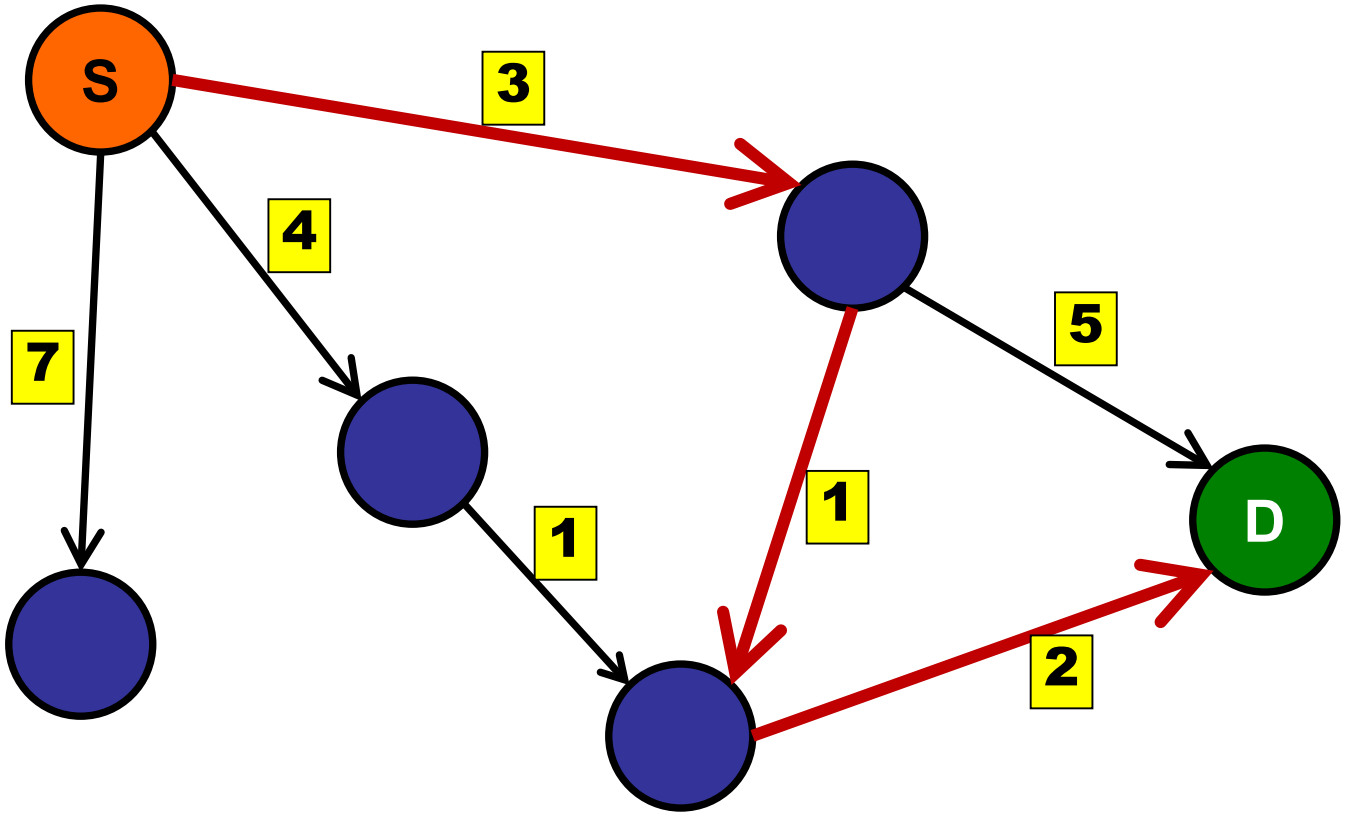
Shortest Paths

Common mistake: “Why can’t I use BFS?”



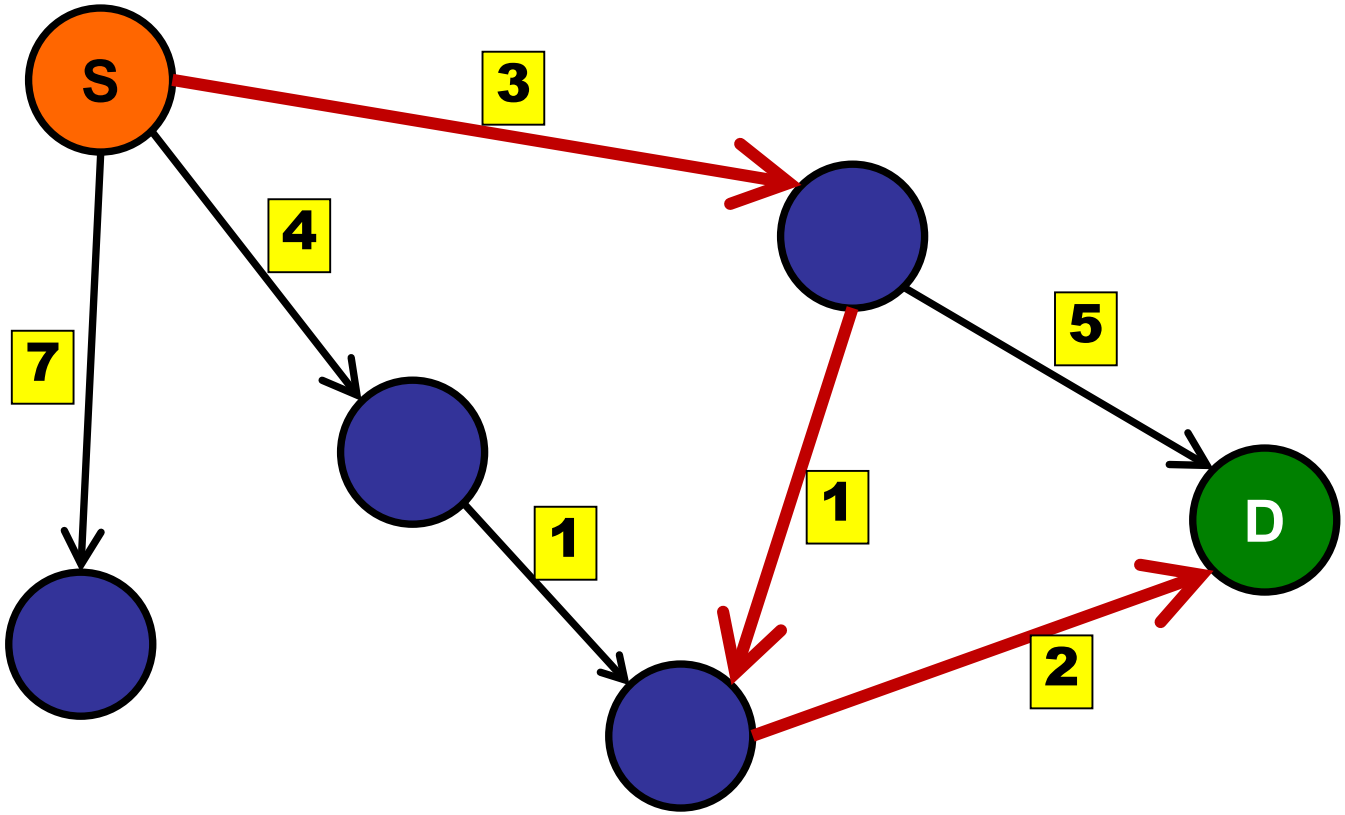
Shortest Paths

Common mistake: “Why can’t I use BFS?”



Shortest Paths

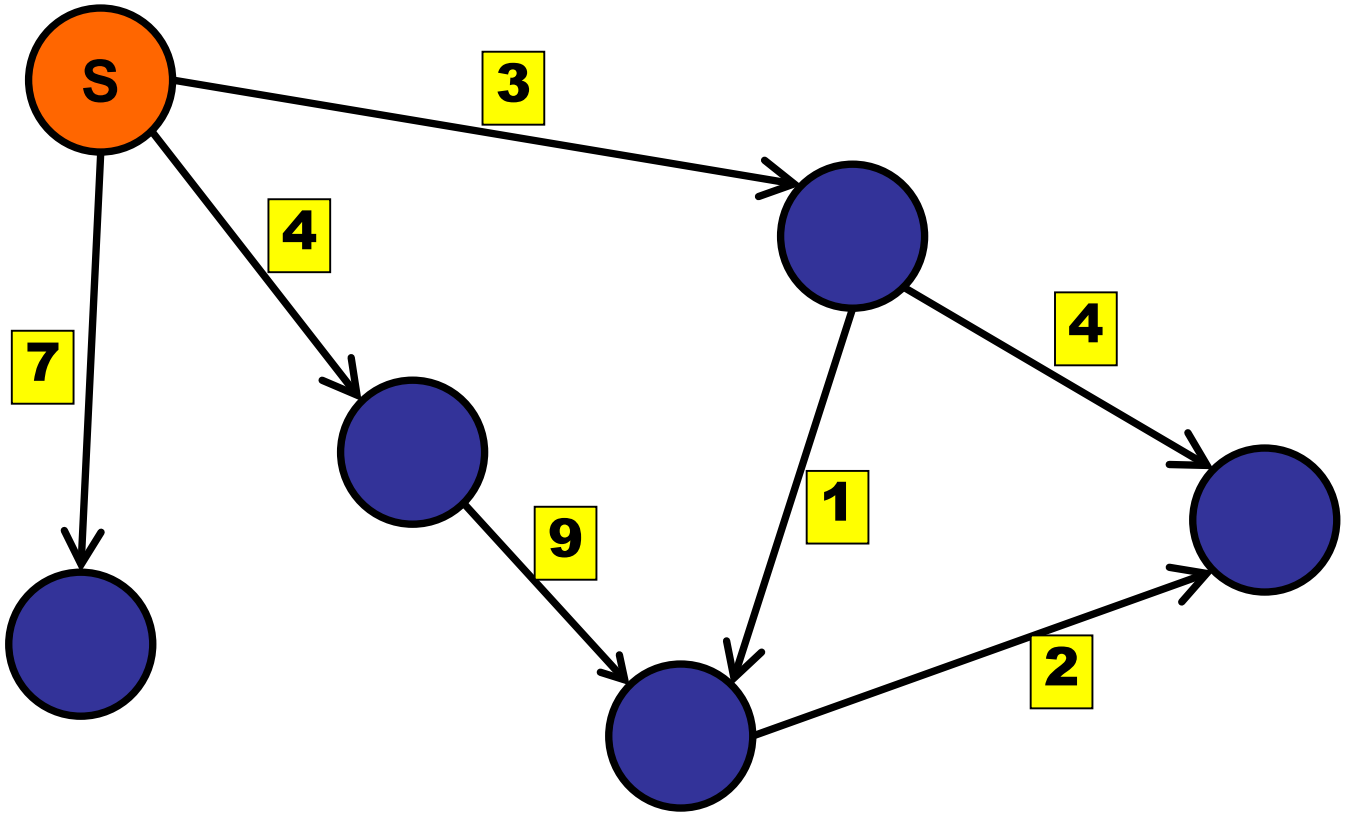
Common mistake: “Why can’t I use BFS?”



BFS finds minimum number of HOPS not minimum DISTANCE.

Shortest Paths

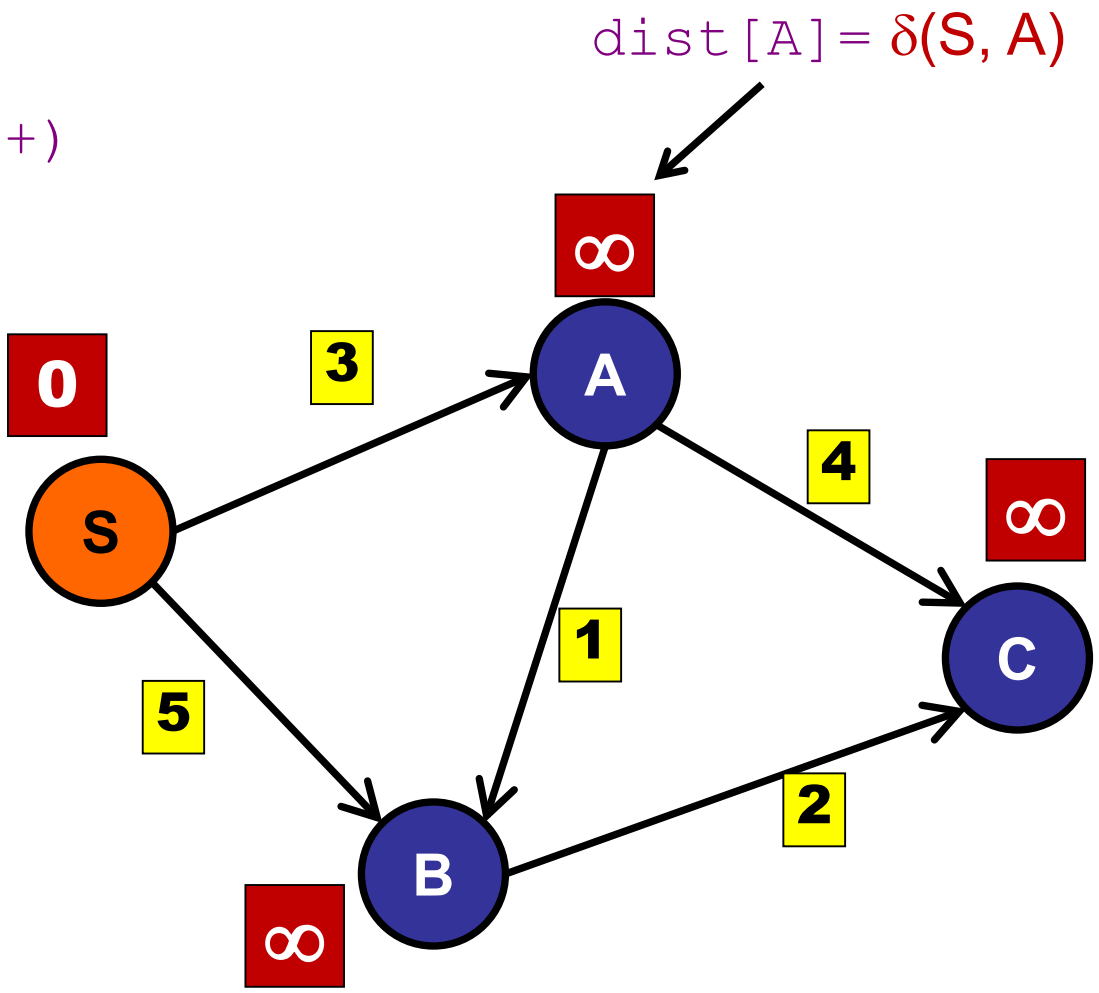
Notation: $\delta(u,v)$ = distance from u to v



Shortest Paths

Maintain estimate for each distance:

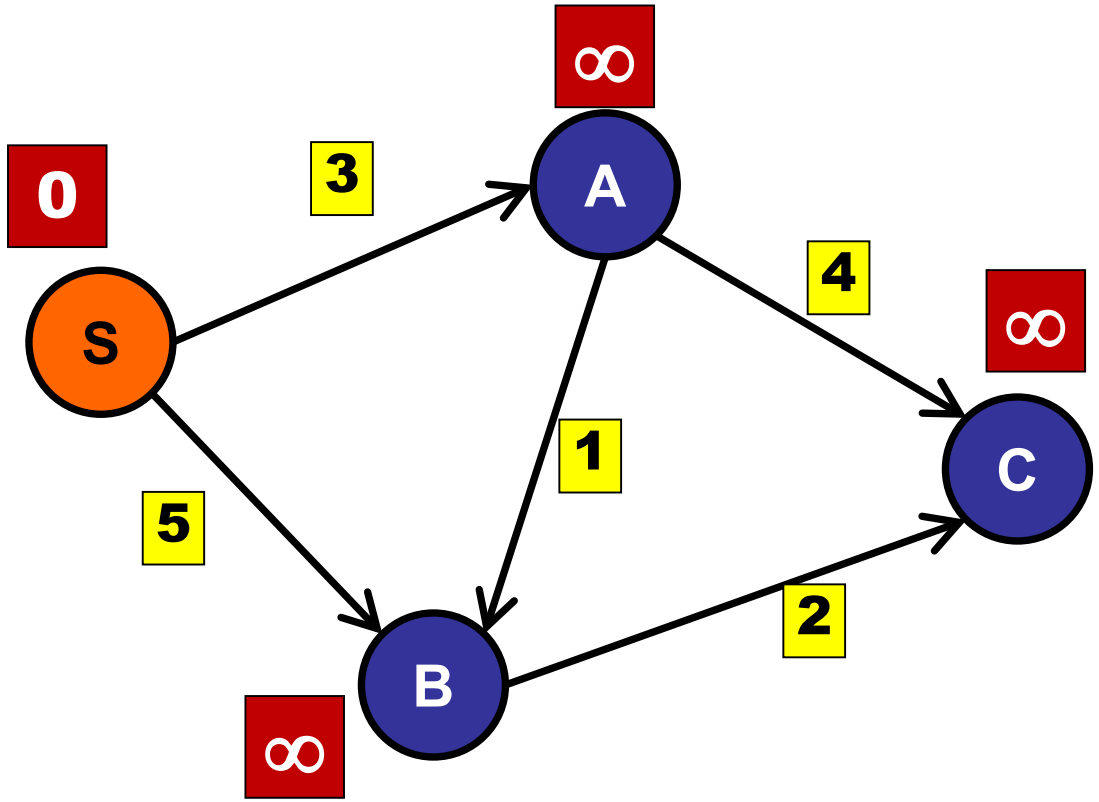
```
int dist[numNode];

for(i=0;i<numNode;i++)

  dist = infinity;

dist[start] = 0;
```



Shortest Paths

Maintain estimate for each distance:

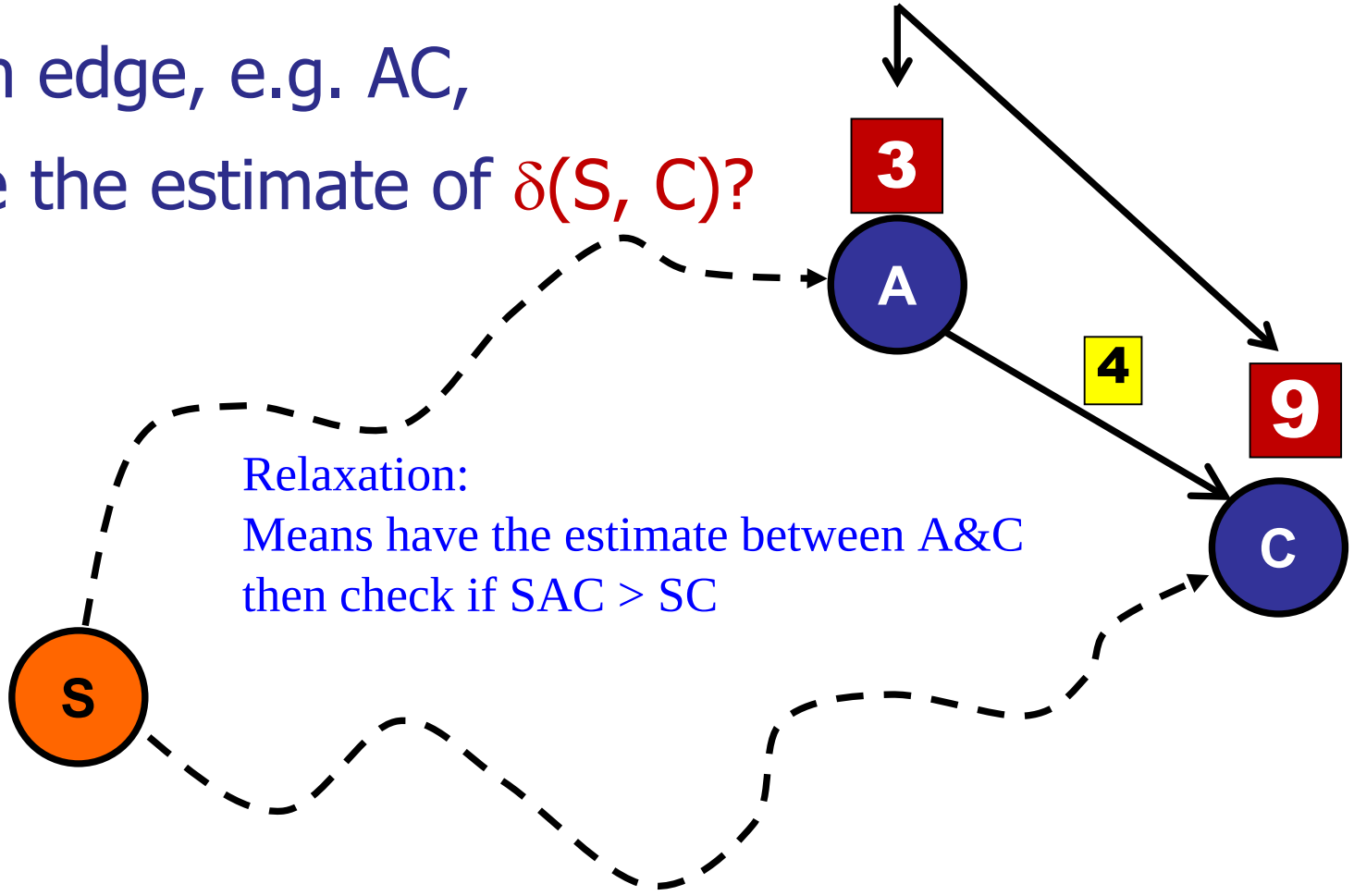
- Reduce estimate
- Invariant: estimate ≥ **actual shortest distance**



Shortest Paths

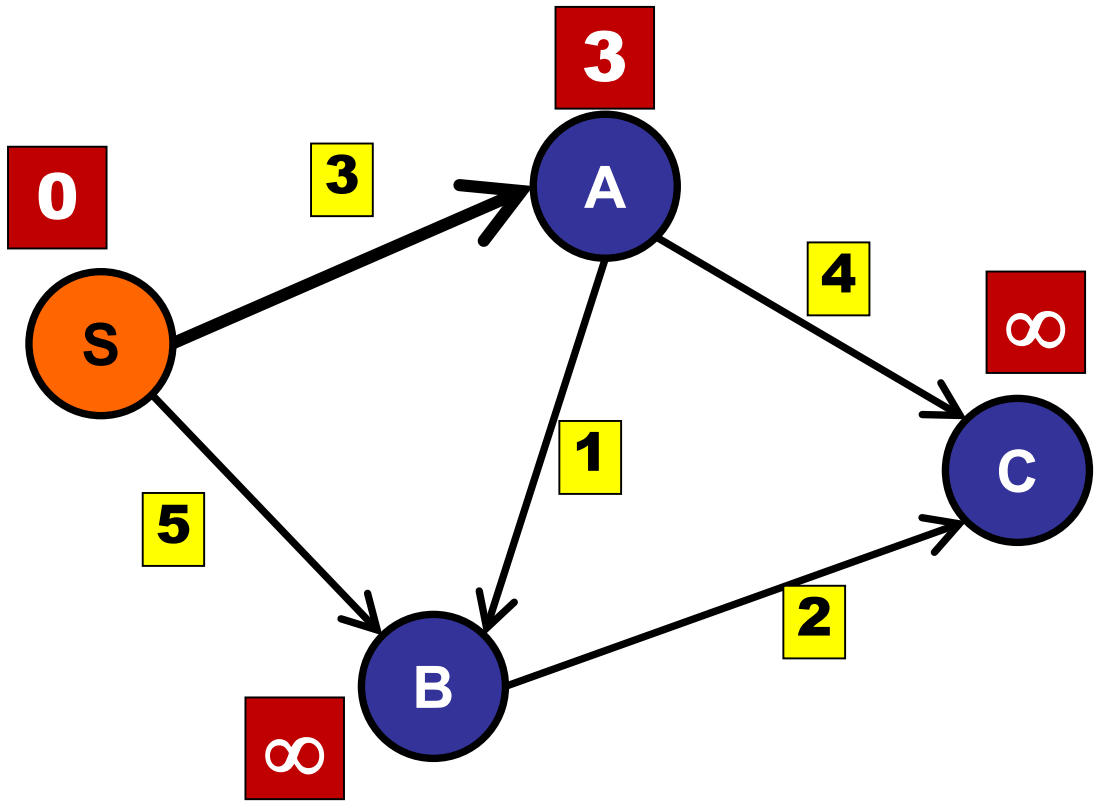
Maintain estimate for each distance:

- Lets say we have some estimate already
- For each edge, e.g. AC,
- Improve the estimate of $\delta(S, C)$?



Shortest Paths

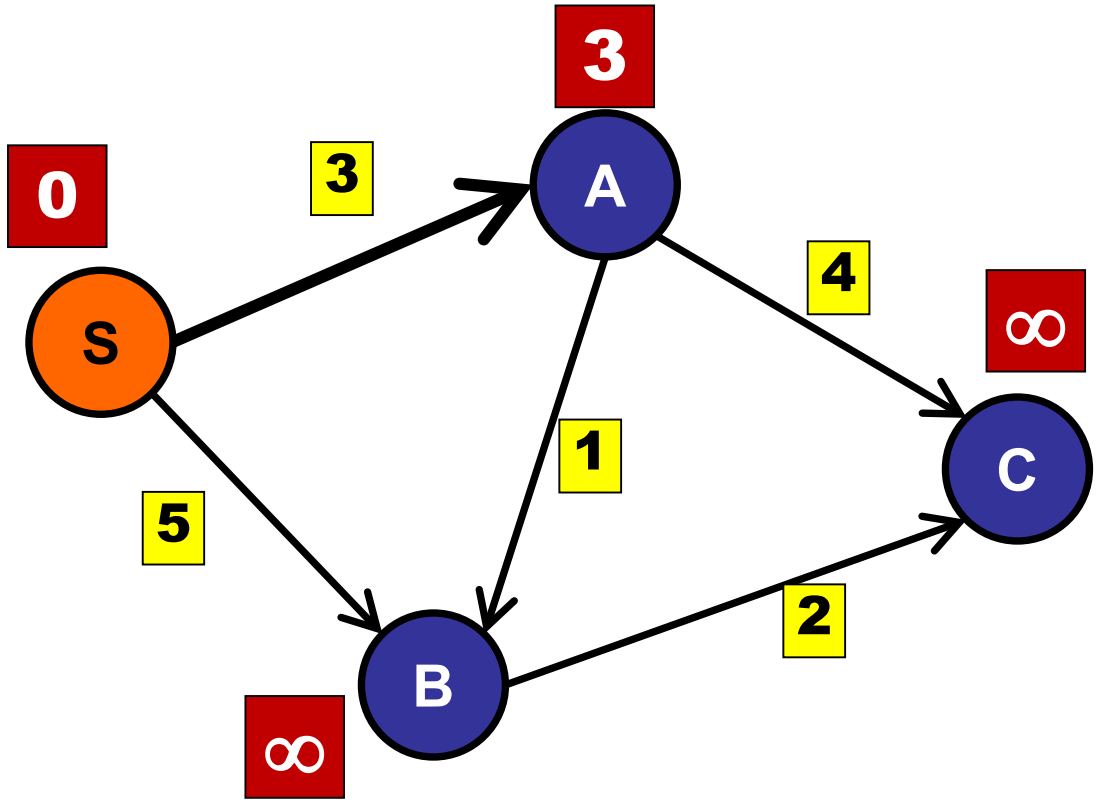
```
relax(int u, int v){
    if (dist[v] > dist[u] + weight(u,v))
        dist[v] = dist[u] + weight(u,v);
}
```



Shortest Paths

Maintain estimate for each distance:

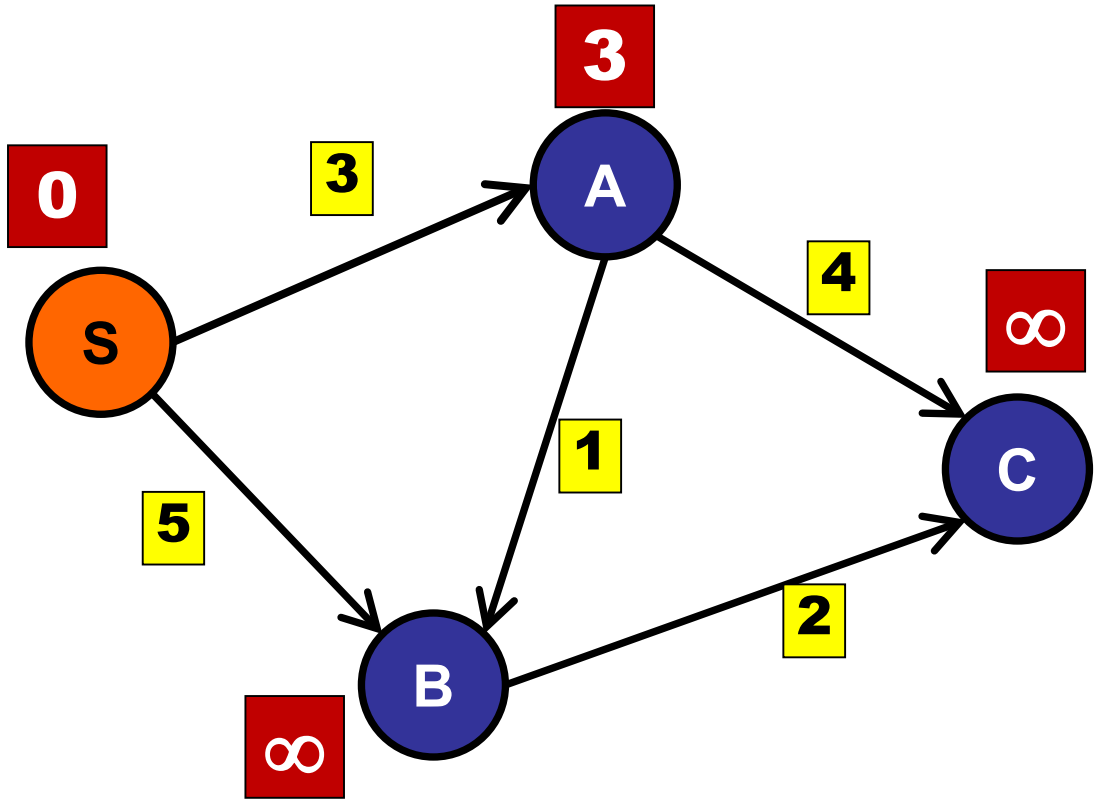
relax(S, A)



Shortest Paths

Maintain estimate for each distance:

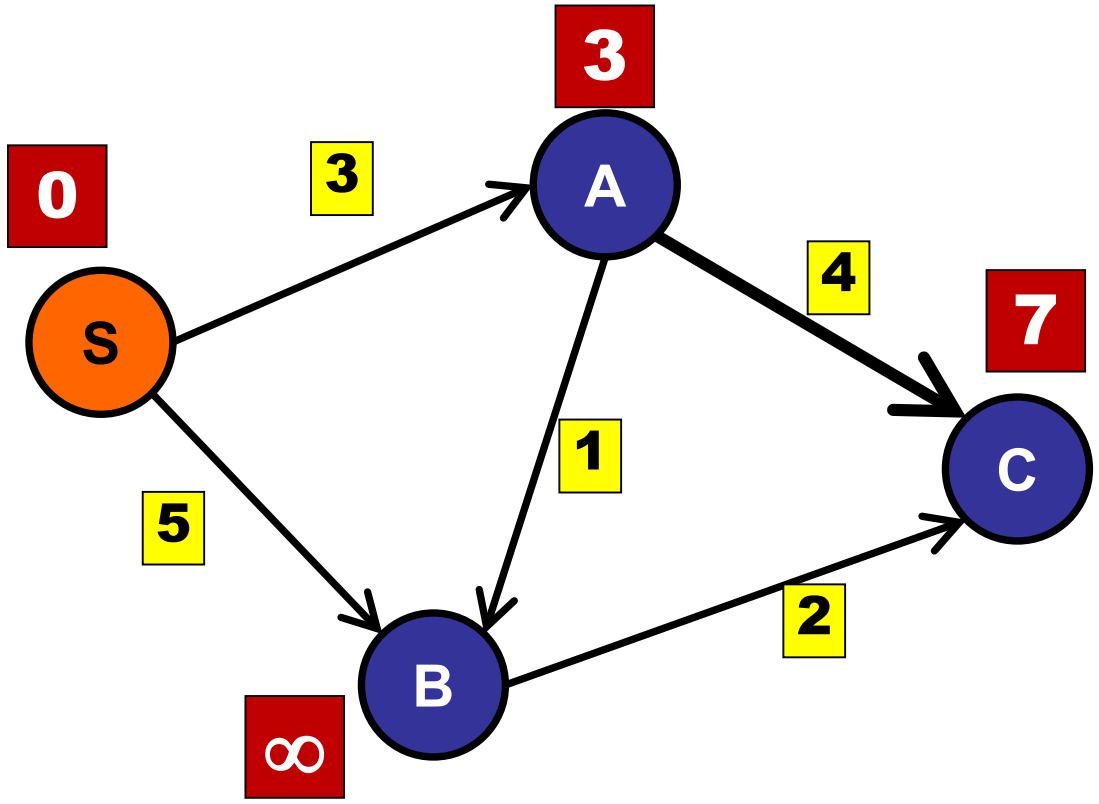
relax(A, C)



Shortest Paths

Maintain estimate for each distance:

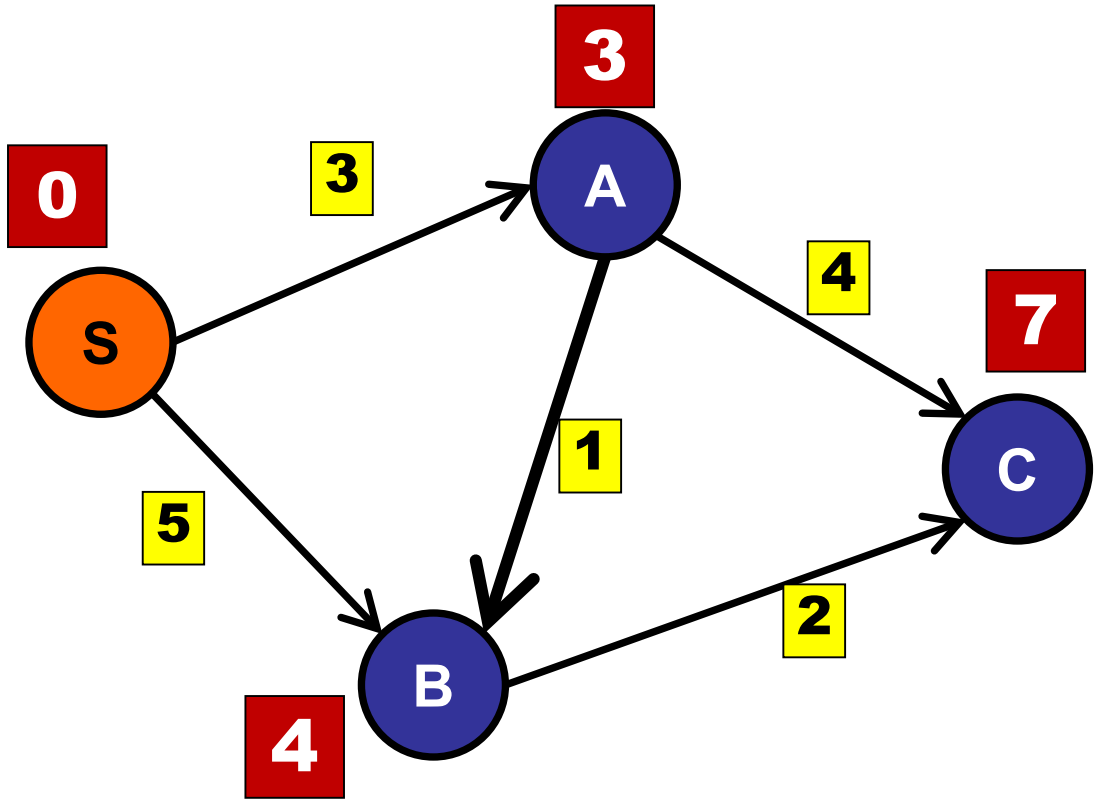
relax(A, C)



Shortest Paths

Maintain estimate for each distance:

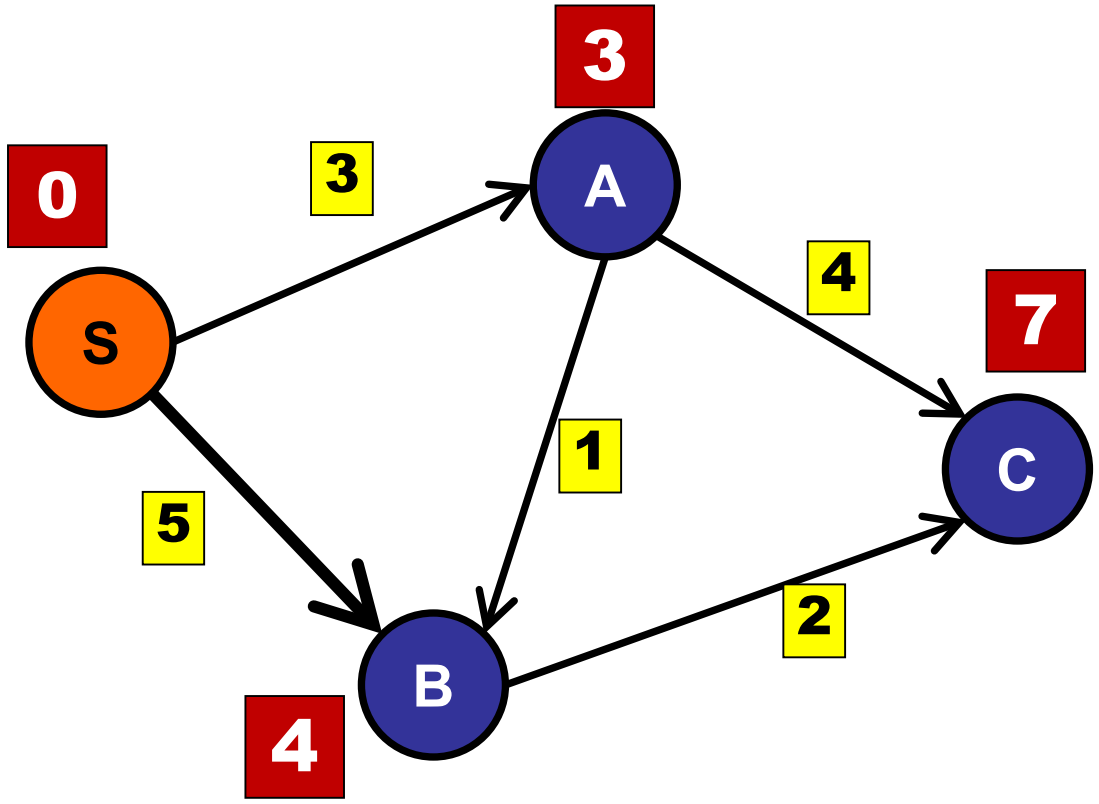
relax(A, B)



Shortest Paths

Maintain estimate for each distance:

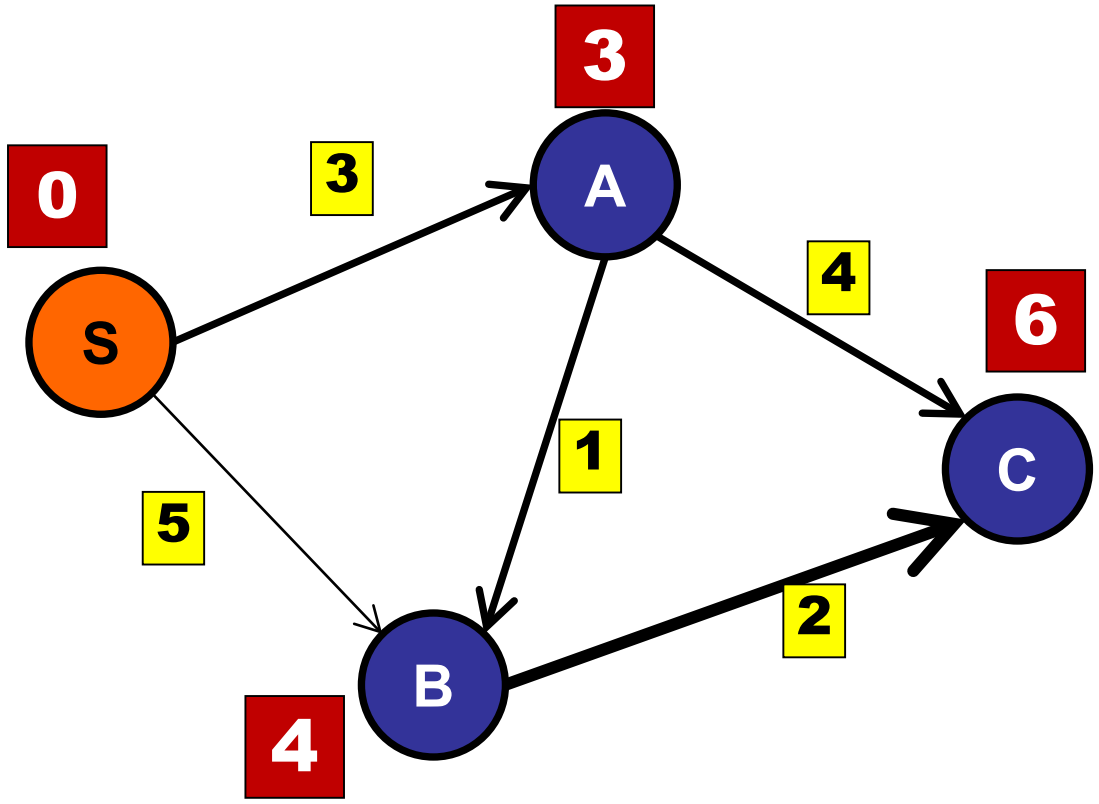
relax(S, B) will not update B



Shortest Paths

Maintain estimate for each distance:

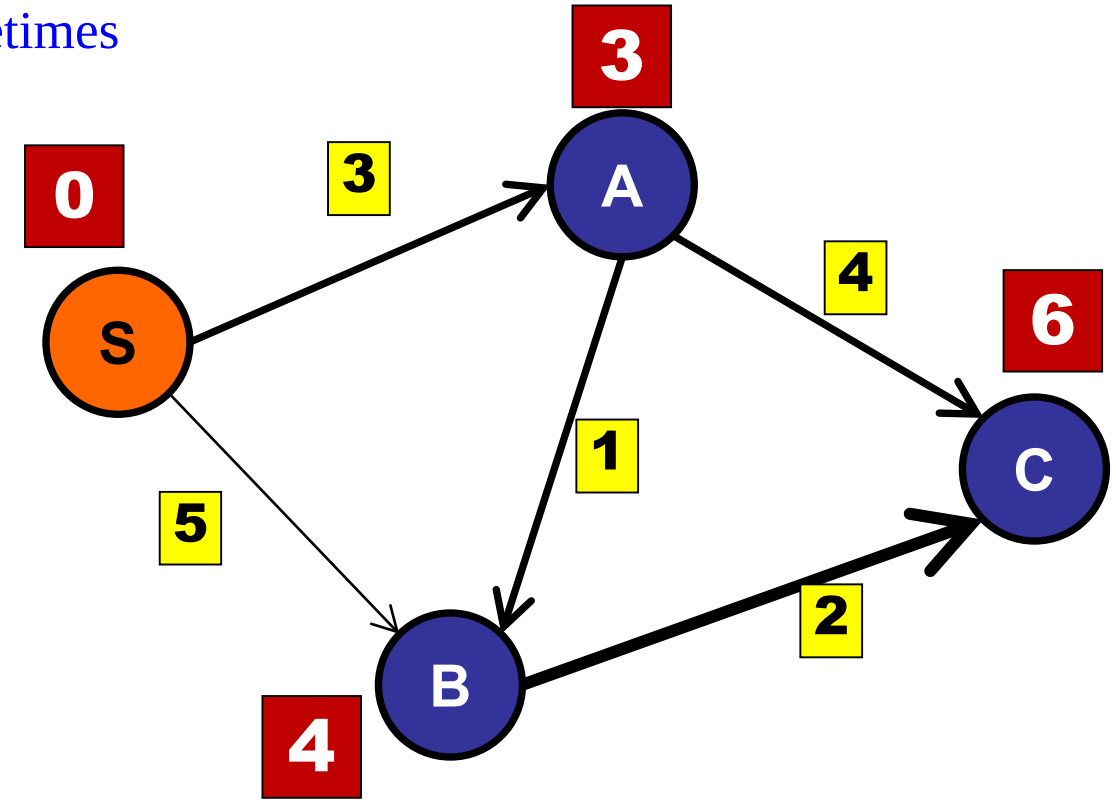
relax(B, C)



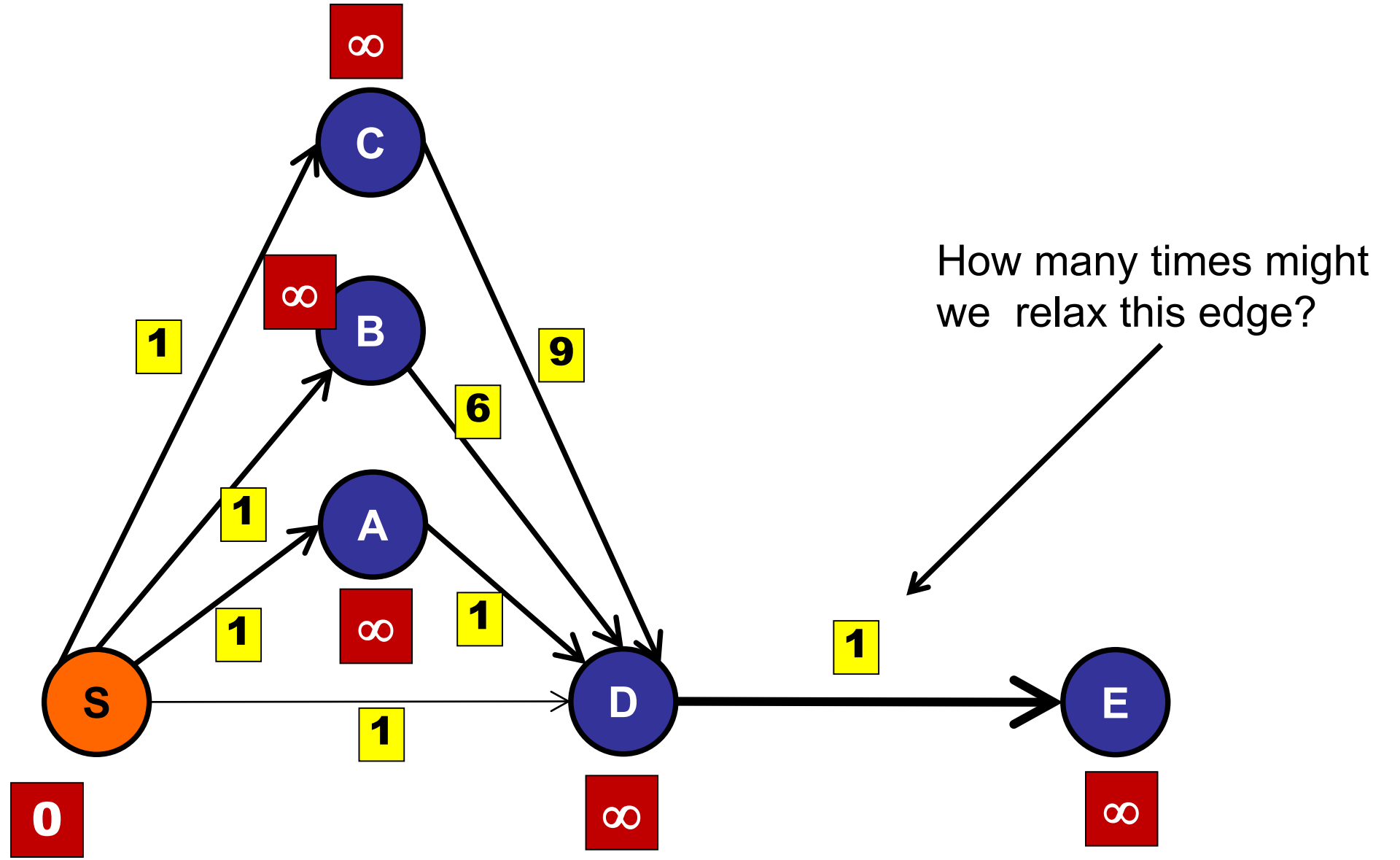
Shortest Paths

```
for (every edge e in a graph)
    relax(e)
```

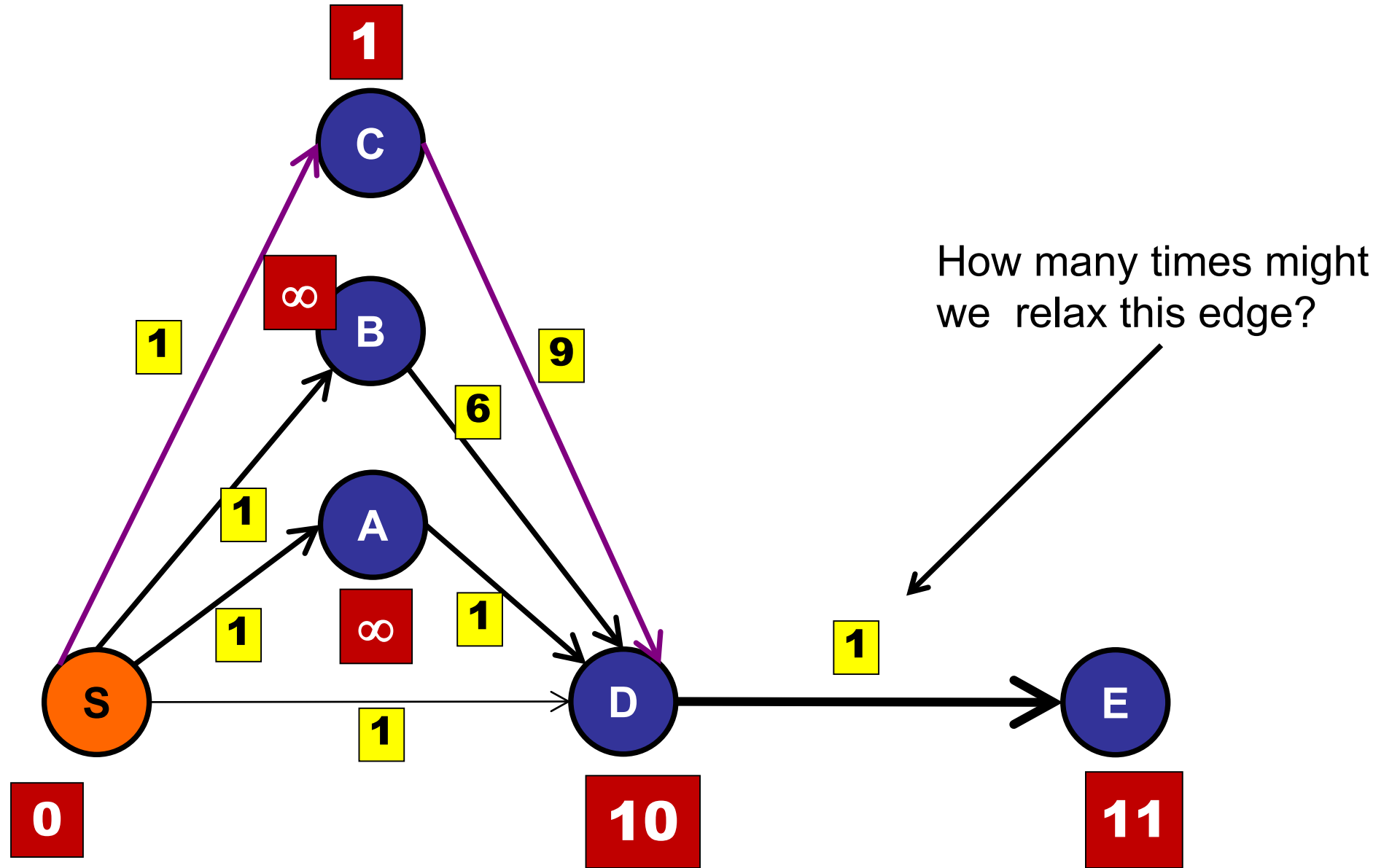
But this algo only works sometimes



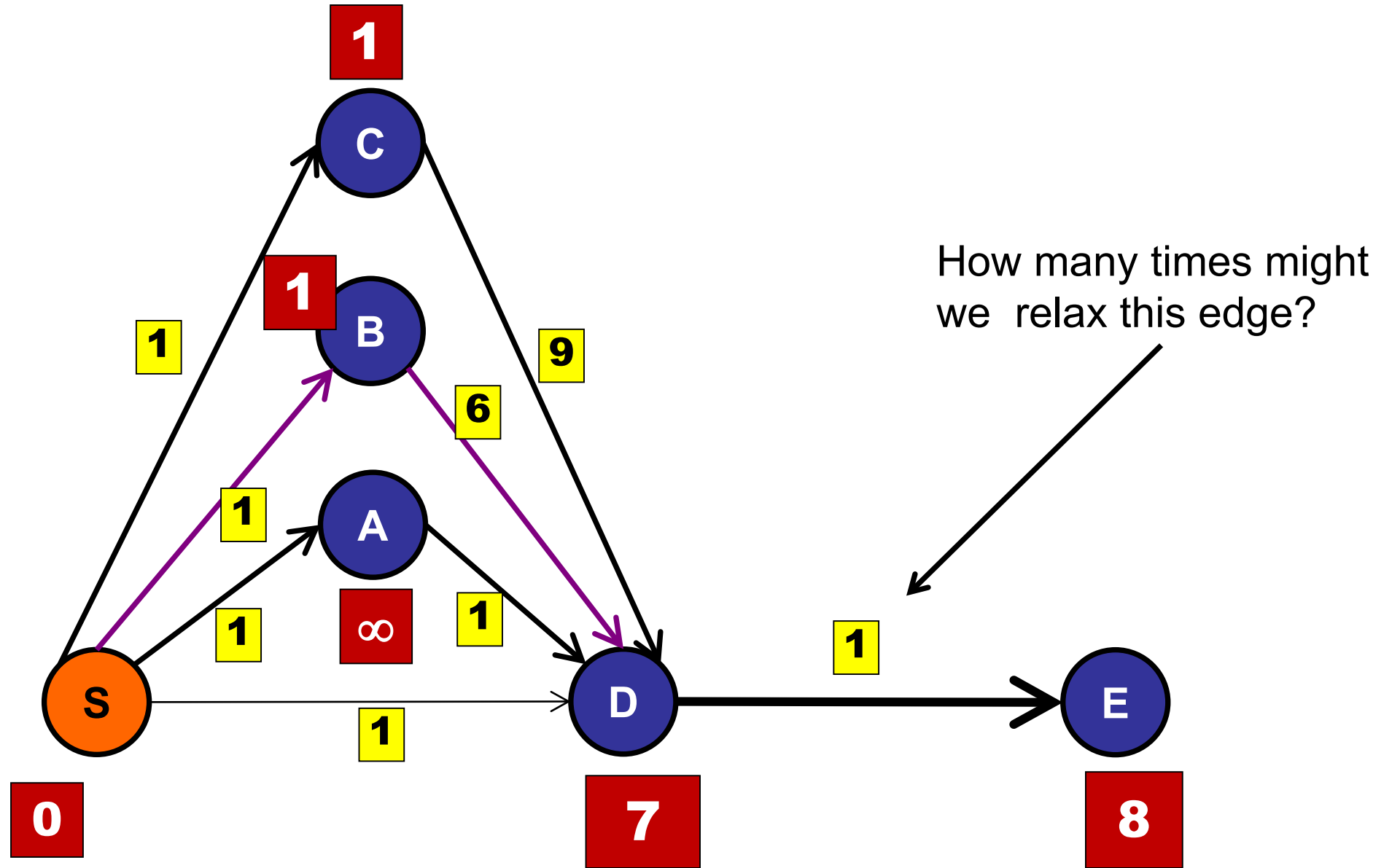
Shortest Paths



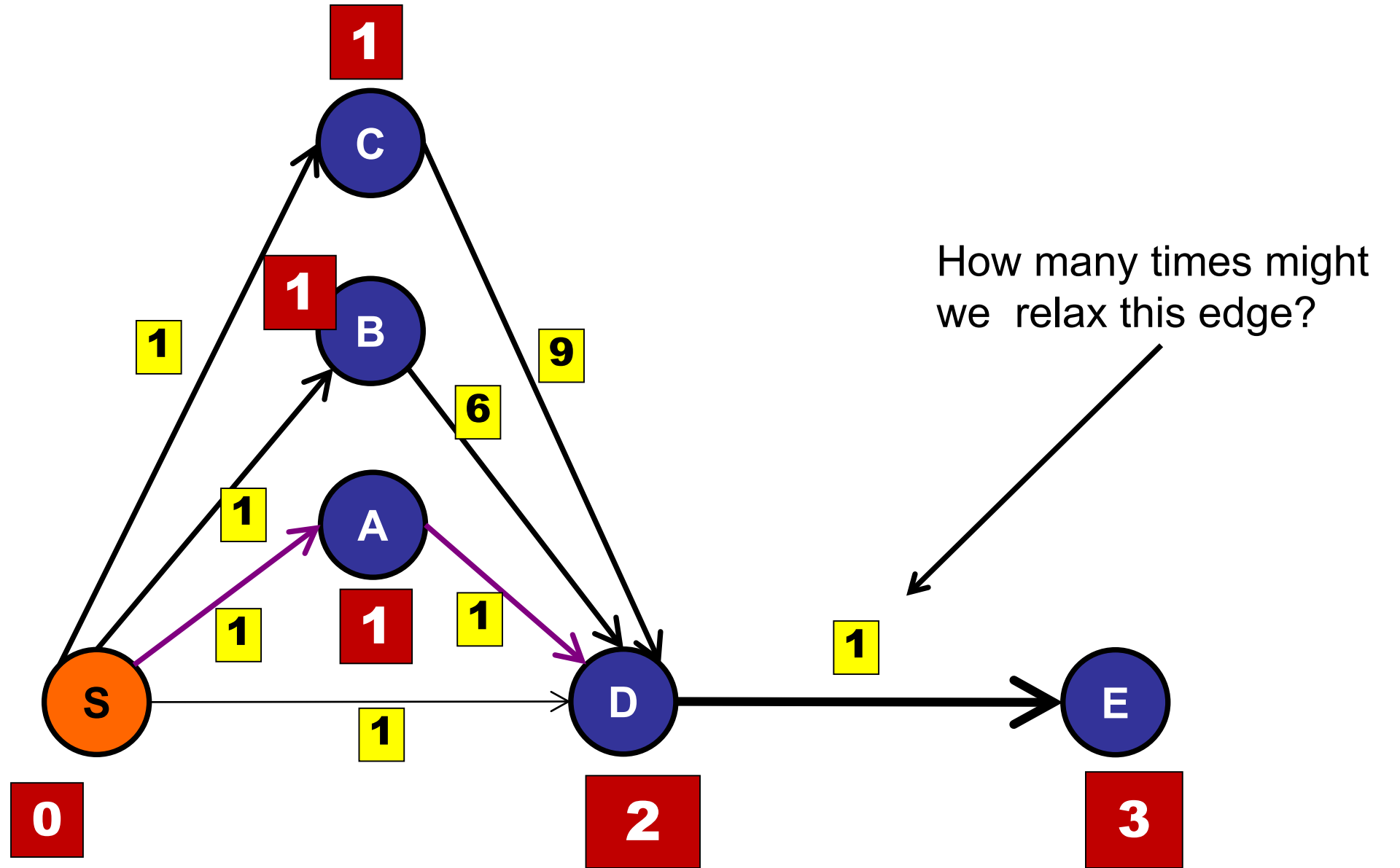
Shortest Paths



Shortest Paths

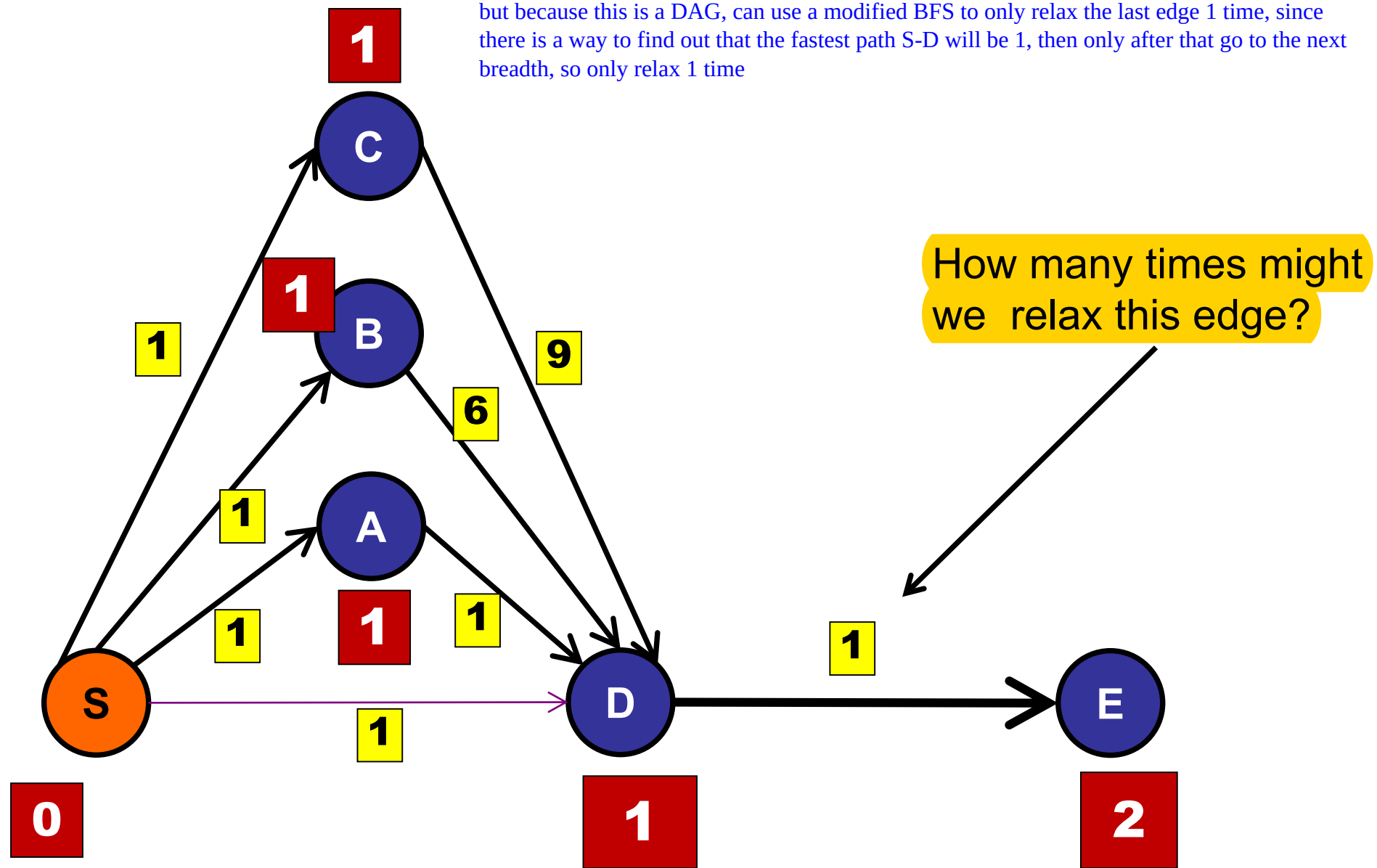


Shortest Paths



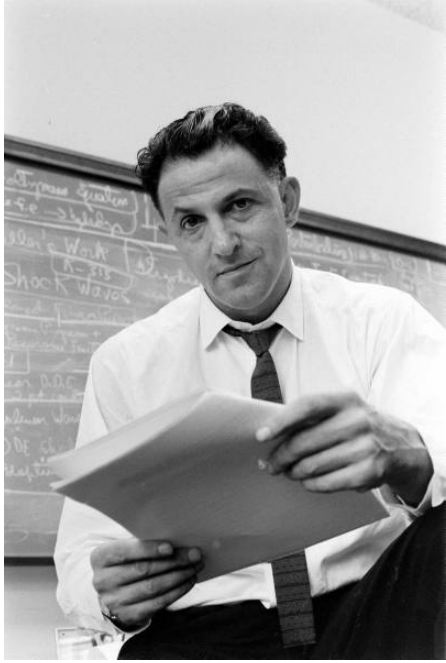
Shortest Paths

but because this is a DAG, can use a modified BFS to only relax the last edge 1 time, since there is a way to find out that the fastest path S-D will be 1, then only after that go to the next breadth, so only relax 1 time



Bellman-Ford

```
for (i=0; i<numNode; i++) V
    for (every edge e in graph) E
        relax(e)
```



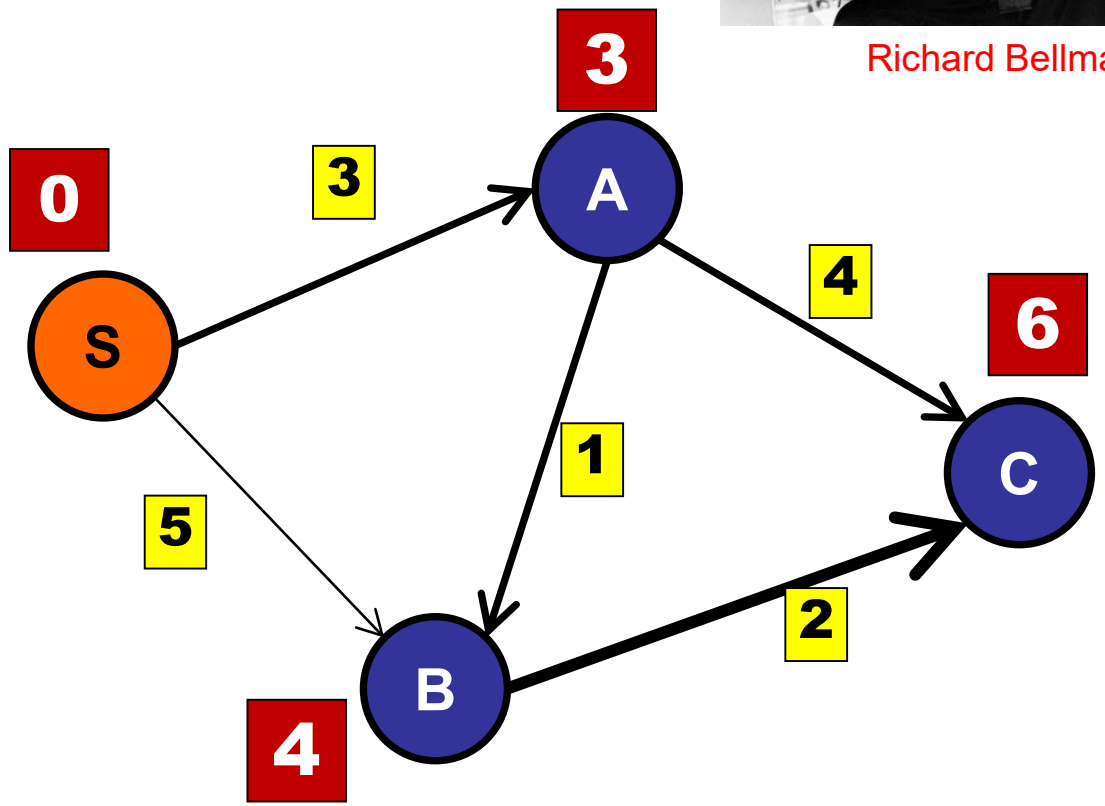
Richard Bellman

slow but very useful and simple

$O(VE)$

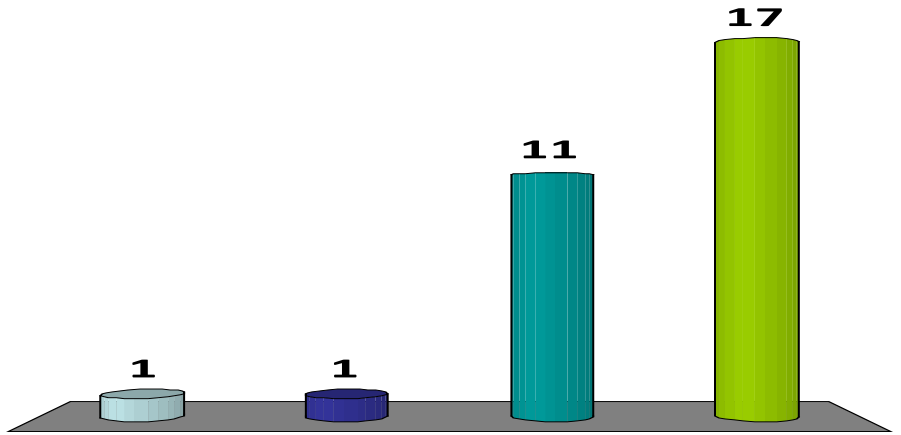
If $E = V^2$ (worst)

$O(V^3)$



When can you terminate early?

1. When a relax operation has no effect.
2. When two consecutive relax operations have no effect.
3. When an entire sequence of |E| relax operations have no effect.
4. Never. Only after |V| complete iterations.



Bellman-Ford

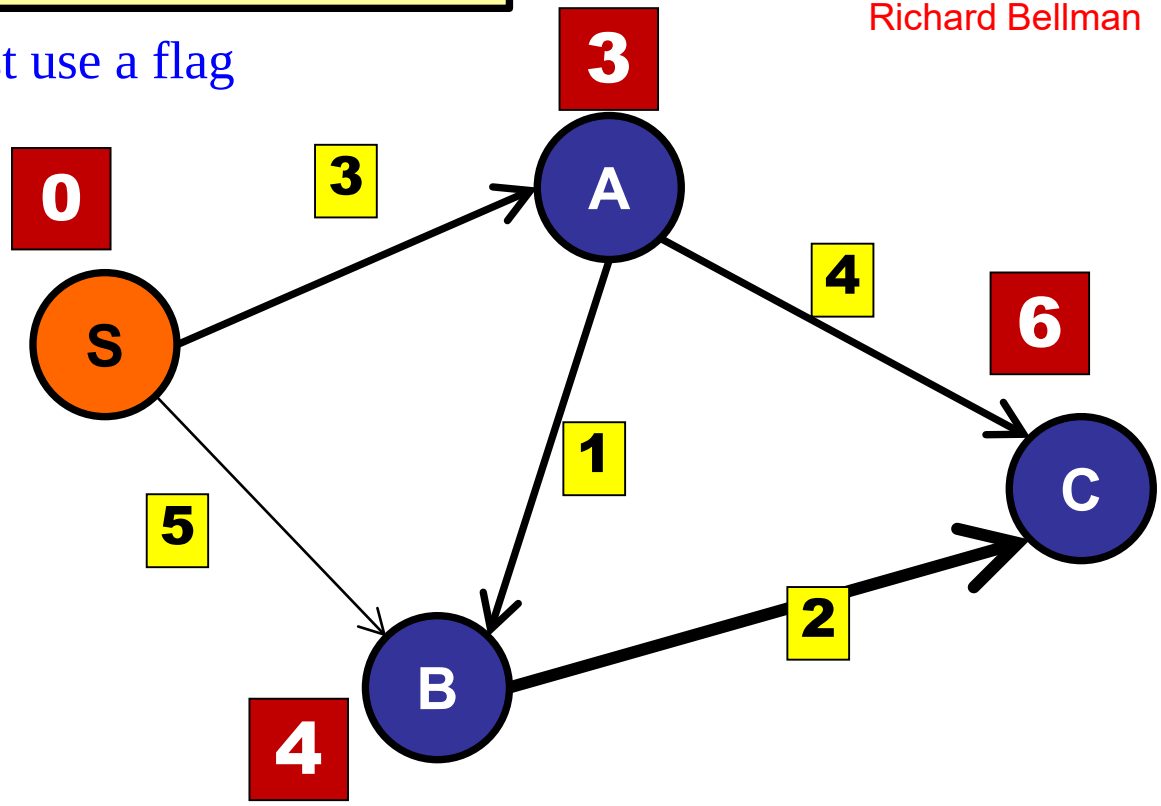
```
for (i=0; i<numNode; i++)
```

```
for (every edge e in graph)

        relax(e)
```

can just use a flag

Terminate early if no more improvement

Richard Bellman



Bellman-Ford

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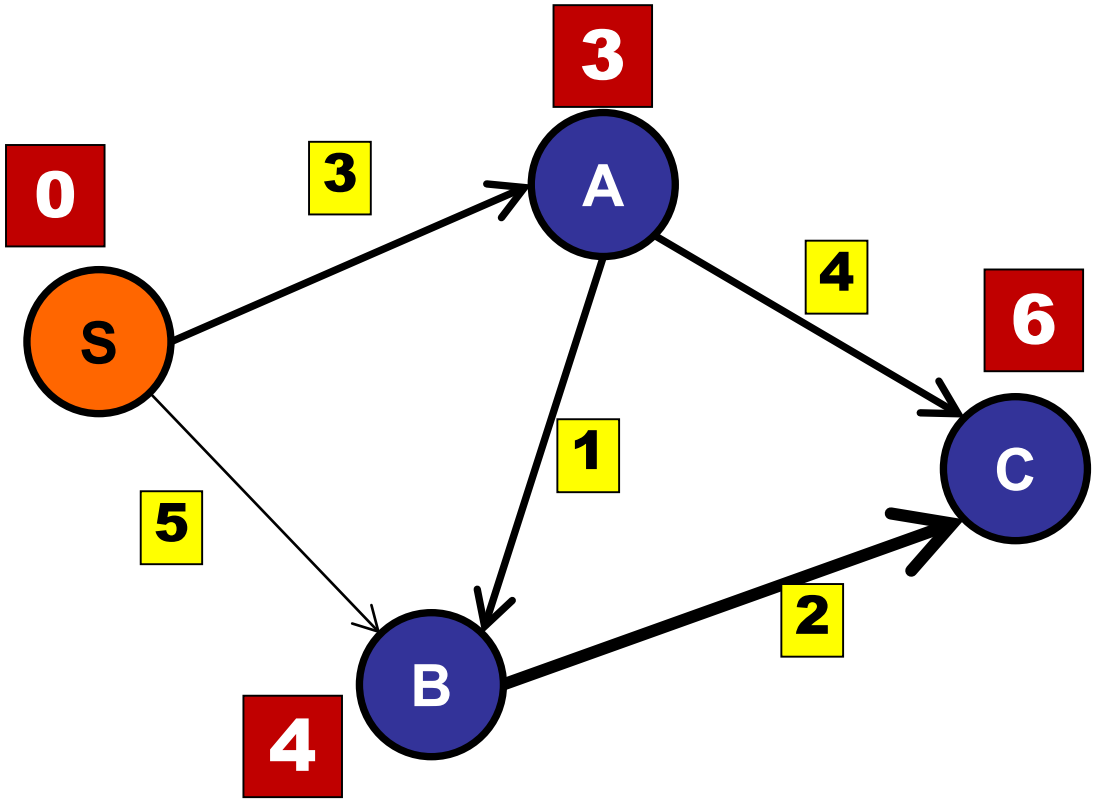

Bellman-Ford

Why does this work?

Bellman-Ford

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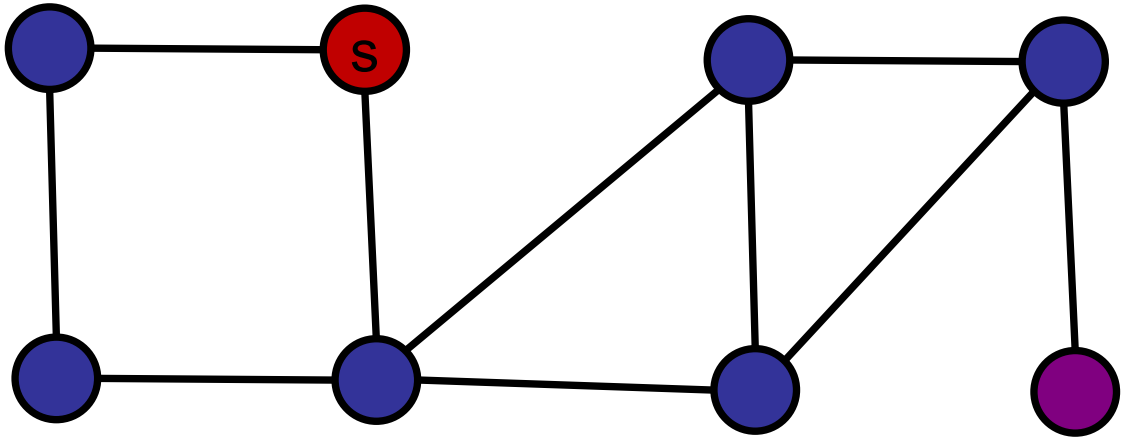

Bellman-Ford

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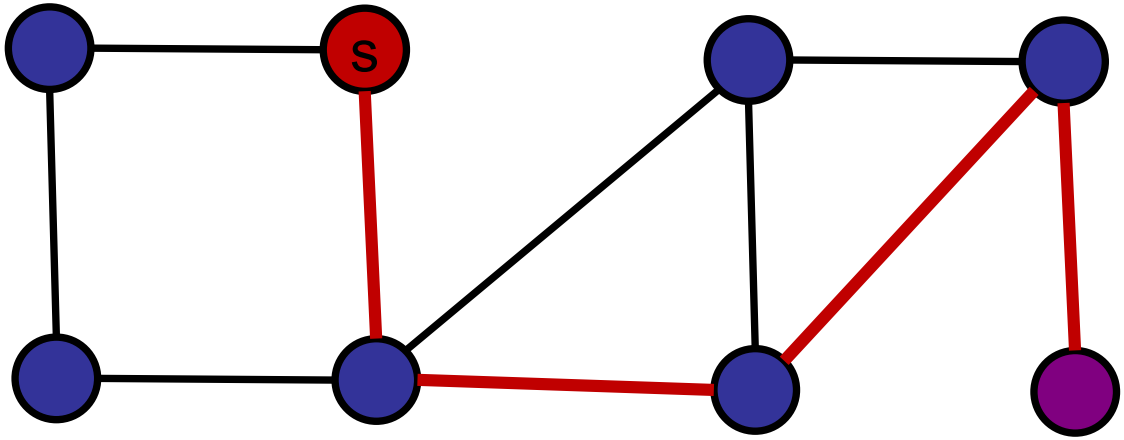
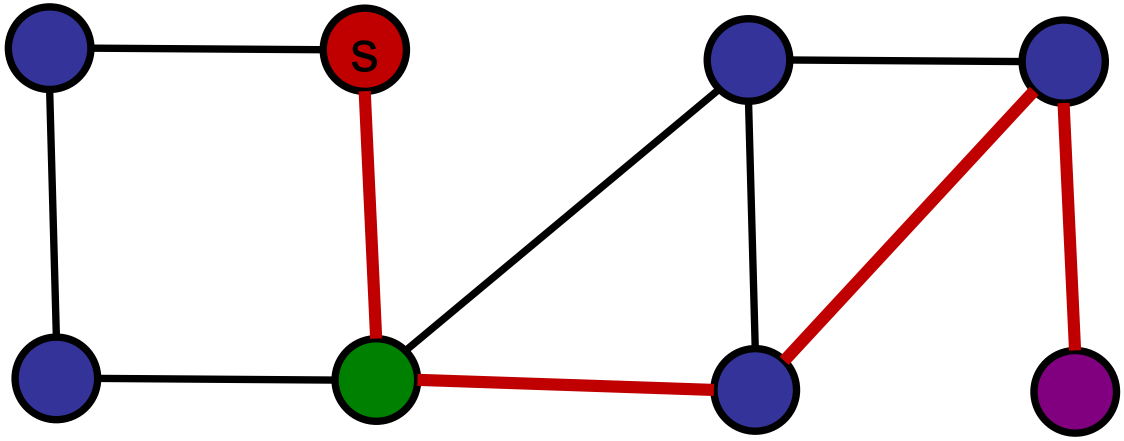


Look at minimum weight path from S to D.

(Path is simple: no loops.)

Bellman-Ford

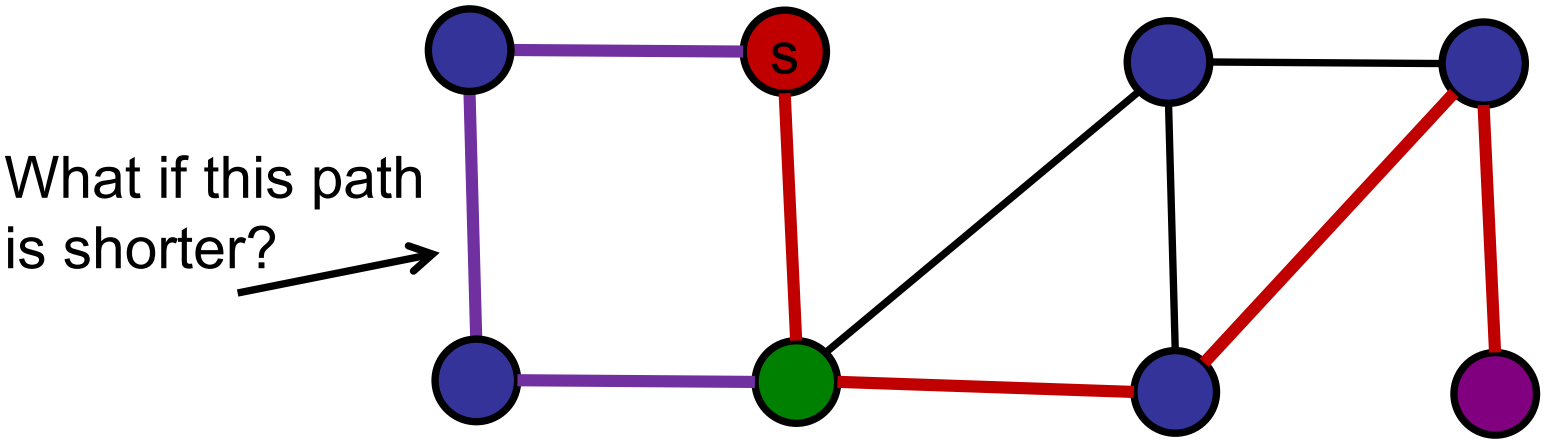
Why does this work?



After 1 iteration, 1 hop estimate is correct.

Bellman-Ford

Why does this work?



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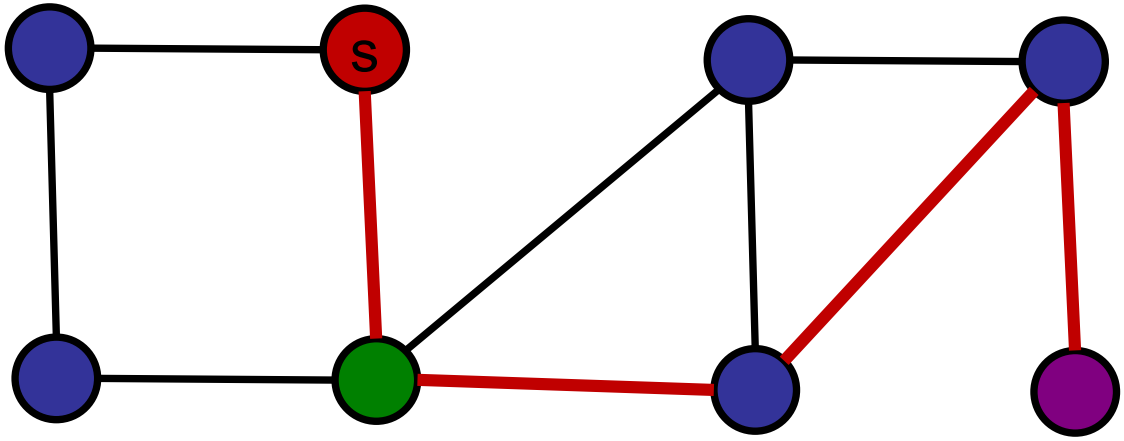

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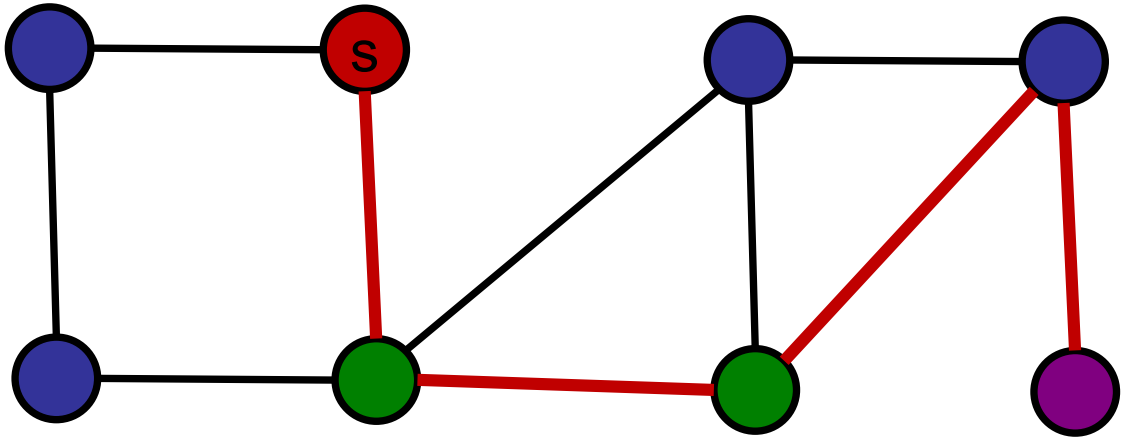
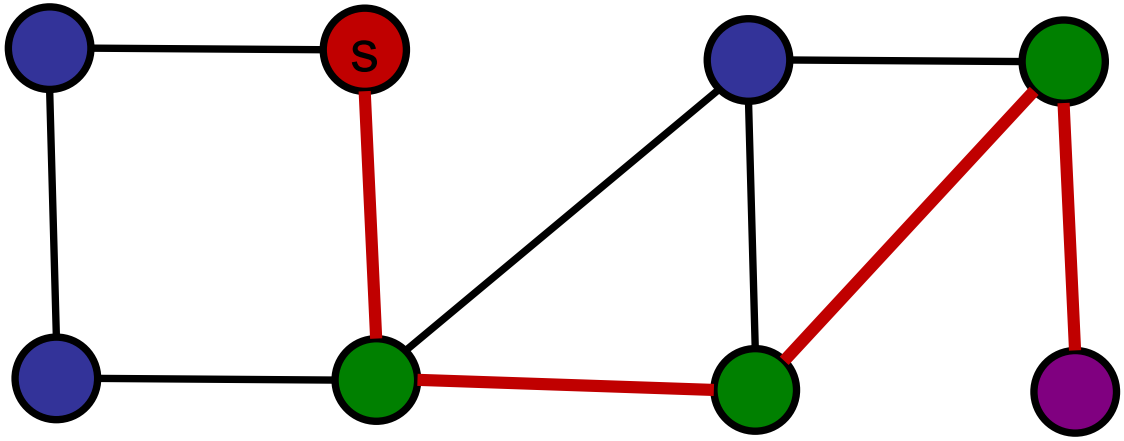


After 2 iterations, 2 hop estimate is correct.

Bellman-Ford

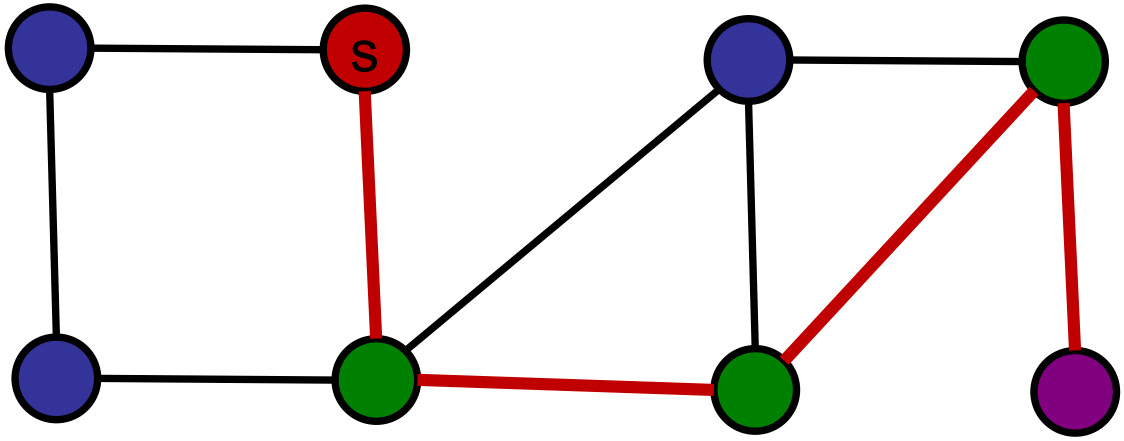
Why does this work?



After 3 iterations, 3 hop estimate is correct.

Bellman-Ford

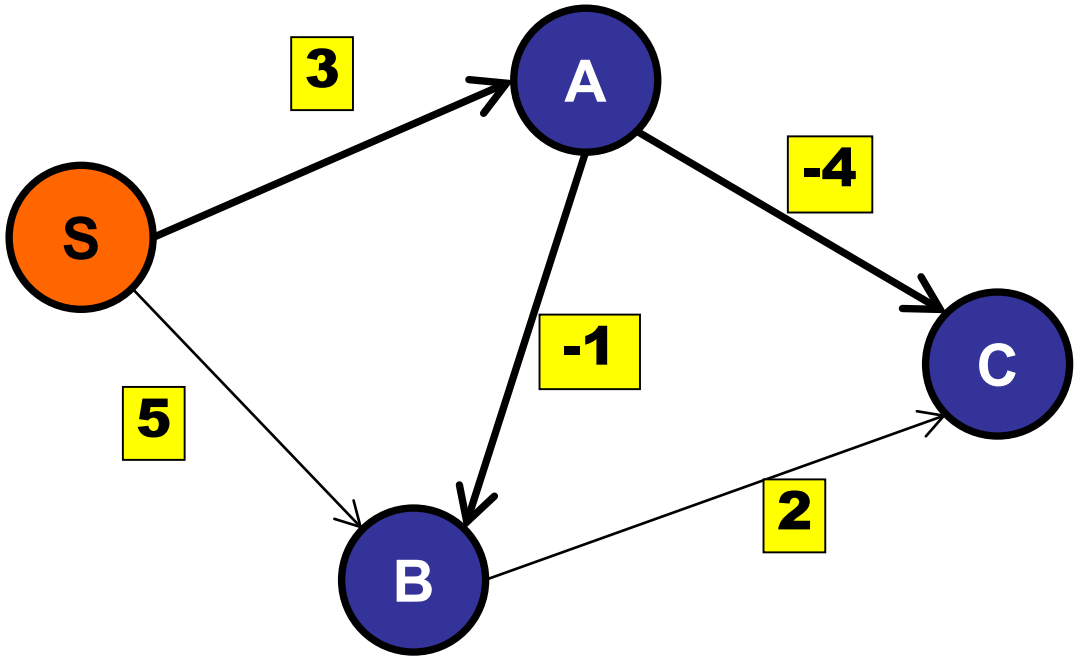
Why does this work?



After 4 iterations, D estimate is correct.

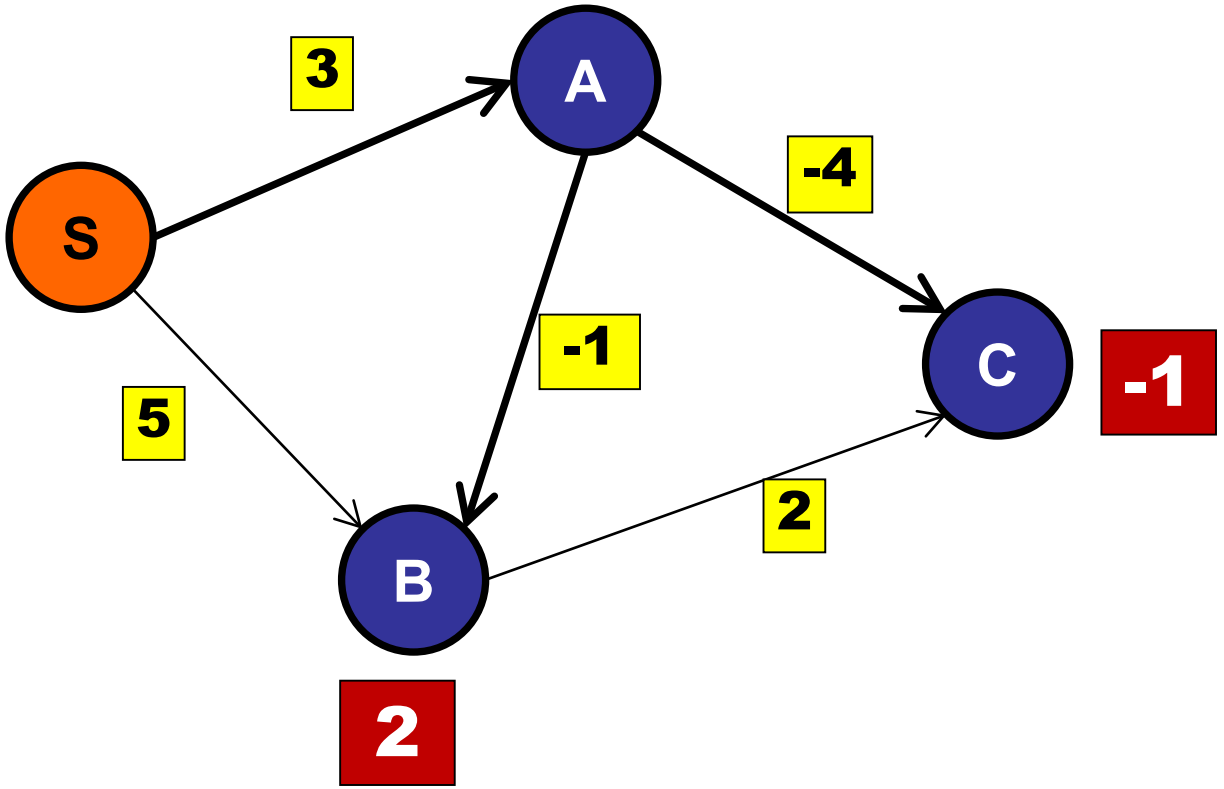
Bellman-Ford

What if edges have negative weight?



Bellman-Ford

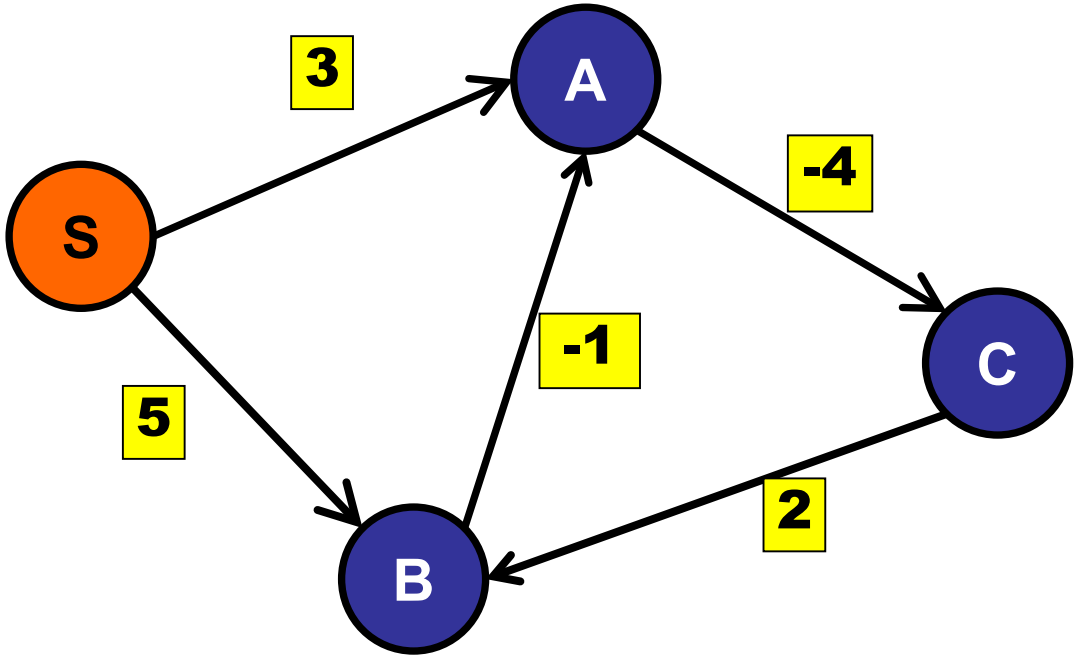
What if edges have negative weight?



No problem!

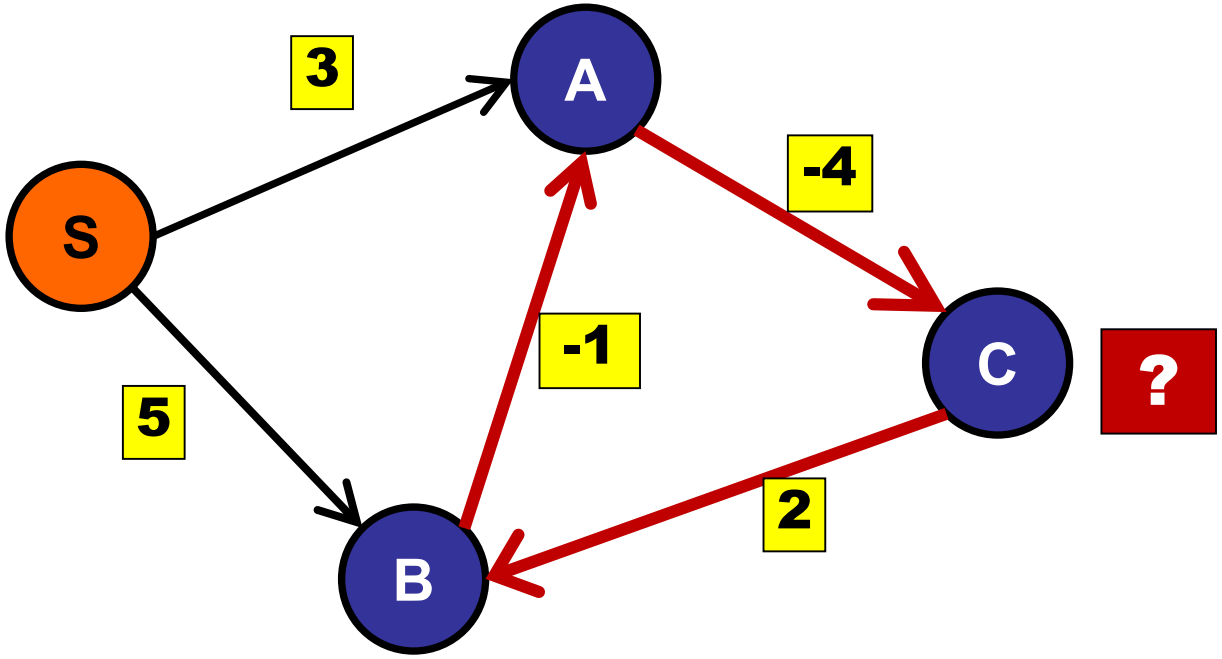
Bellman-Ford

What if edges have negative weight?



Bellman-Ford

What if edges have negative weight?

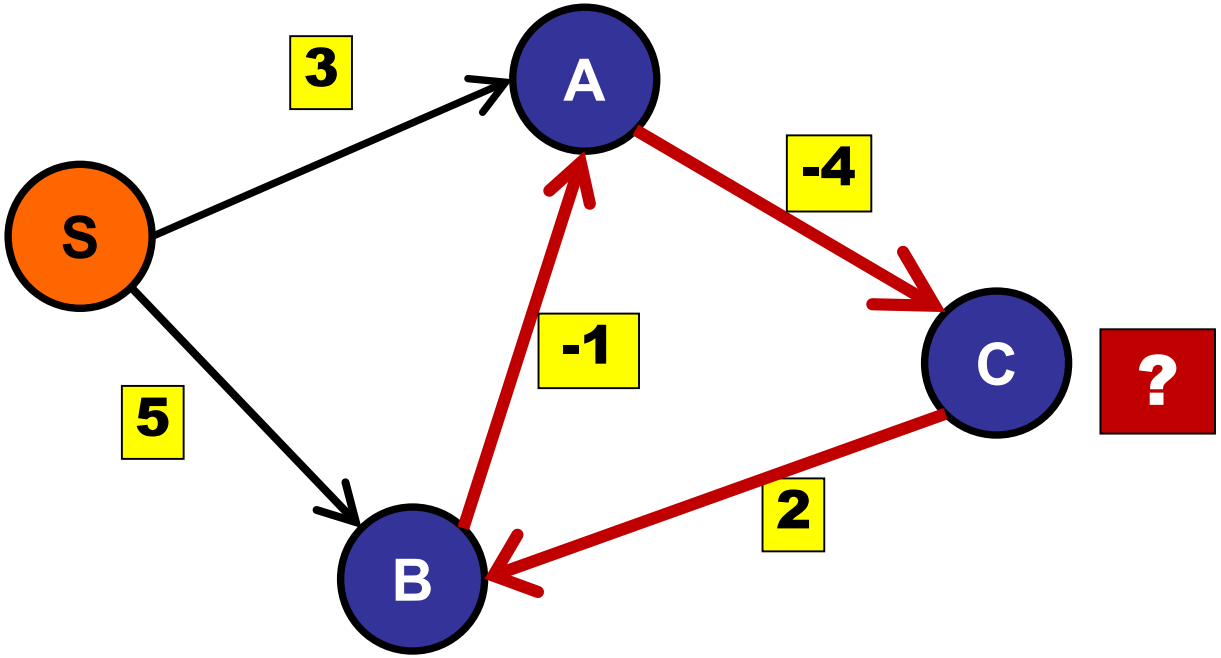


d(S,C) is infinitely negative!

Negative weight cycles

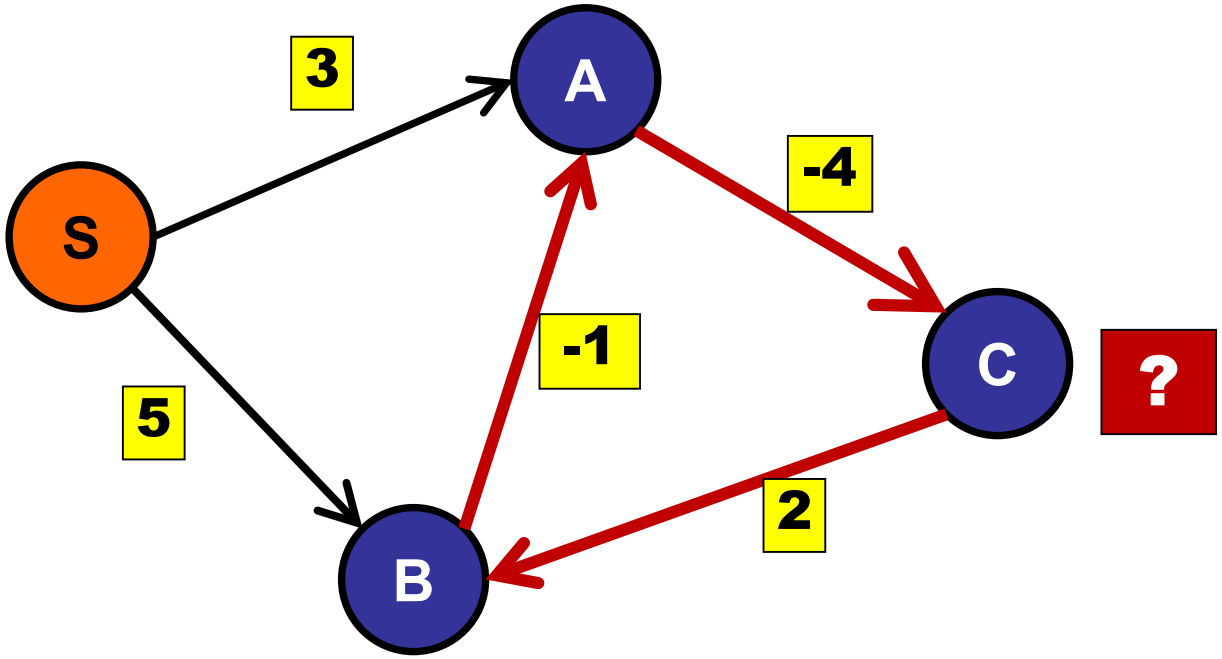
How to detect negative weight cycles?

This will create an infinite negative cycle



Negative weight cycles

How to detect negative weight cycles?

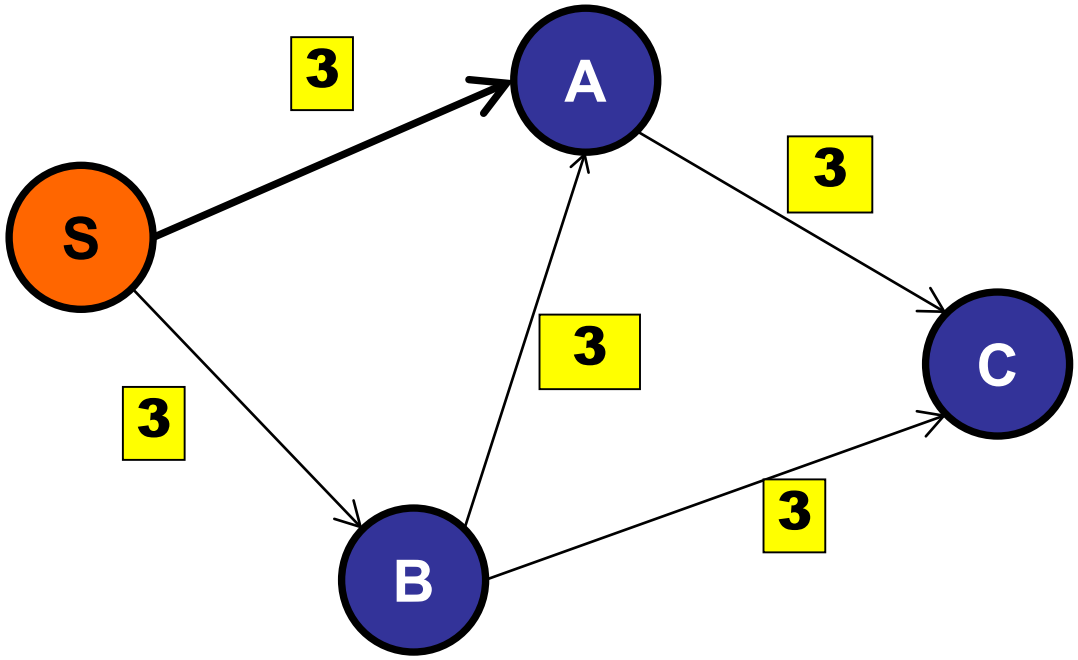


Run Bellman-Ford for |V|+1 iterations.

If an estimate changes in the last iteration…
then negative weight cycle.

Bellman-Ford

Special case: all edges have the same weight.



Use regular Breadth-First Search.

Bellman-Ford Summary

Basic idea:

- Repeat |V| times: relax every edge
- Stop when "converges".
- O(VE) time.

Special issues:

- If negative weight-cycle: impossible.
- Use Bellman-Ford to detect negative weight cycle.
- If all weights are the same, use BFS.