GEH1027 / GEK1508 Einstein's Universe and Quantum Weirdness

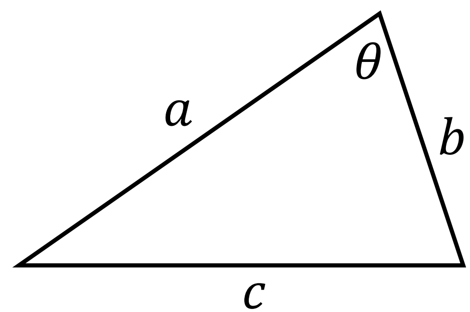
Tutorial 3

AY2020/21 Sem II

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Tutorial 3 – key ideas

- Qualitative description of Einstein's Field Equation
- Roles of the metric
- $c^2 = a^2 + b^2 - 2ab\cos\theta$



(law of cosines… generalised Pythagoras' theorem!)

1. From the discussion given in the lectures, the Einstein's famous Field Equations of General Relativity, are given as $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}$ and the coupling constant is $\kappa = \frac{8\pi G}{c^4}$.

   a) Discuss qualitatively the meaning of LHS and RHS.

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{\text{Pure geometry, or the geometry of spacetime}} = \kappa T_{\mu\nu}$$

Ricci tensor

Ricci scalar

metric

Energy-momentum tensor
= pure matter and energy

b) How many field equations are there if *time* is now treated as equal footing as *space*?

- There are 16 field equations ($\mu = 0, 1, 2, 3$ – likewise for ν)
- Recall that the Ricci tensor, metric and $T_{\mu\nu}$ are symmetric. They are symmetric in (μ, ν), i.e. $R_{\mu\nu} = R_{\nu\mu}$. Thus there are only <u>10 independent field equations</u>.

Example of symmetric matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

c) What do we mean by solving the above field equation? What are we looking for?

- We are looking for a metric that satisfies the field equation.

(i.e. search for a set of $g_{\mu\nu}$ that satisfies $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}$)

d) What is the dual role of $g_{\mu\nu}$?

- $g_{\mu\nu}$ describes a small interval in curved (or flat) spacetime.
- For gravity, it describes an interval in 4 dimensional spacetime. It is connected to the gravitational potential.

Example: for non-rotating, chargeless black hole,

$$ds^2 = -\left(1 + \frac{2V}{c^2}\right)(c\,dt)^2 + \left(1 - \frac{2V}{c^2}\right)(dx^2 + dy^2 + dz^2)$$

Note how the elements of $g_{\mu\nu}$ can be read off the metric.

The metric describes both geometry and gravity.

What happens when $m = 0$ (i.e. $V = 0$)?

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 + \frac{2V}{c^2}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2V}{c^2}\right) & 0 & 0 \\ 0 & 0 & \left(1 - \frac{2V}{c^2}\right) & 0 \\ 0 & 0 & 0 & \left(1 - \frac{2V}{c^2}\right) \end{pmatrix}$$

e) Recall the meaning of $R_{\mu\nu} = 0$?

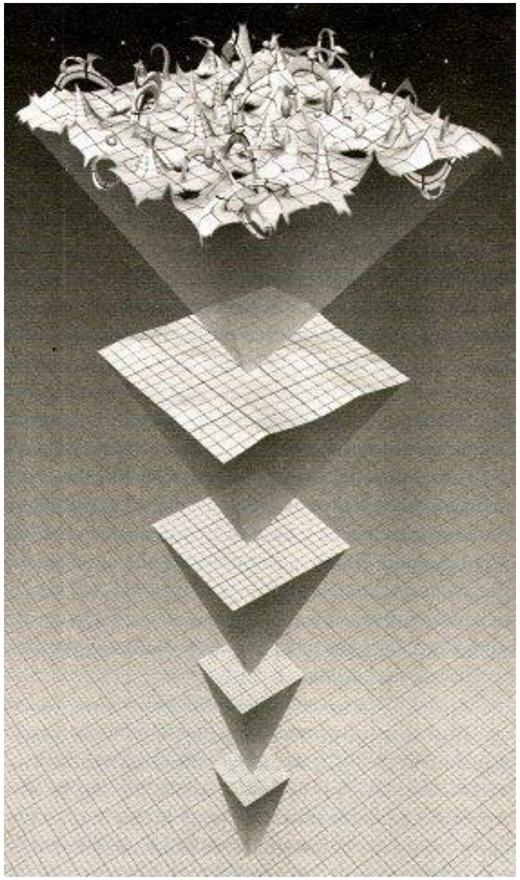
- RHS is zero, means there is no matter
- Ricci tensor equals to zero means a "flat" spacetime (can be Euclidean or Minkowski)
- Question: Is vacuum really flat? It is related to quantum theory…

Minkowski spacetime:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

Euclidean space:

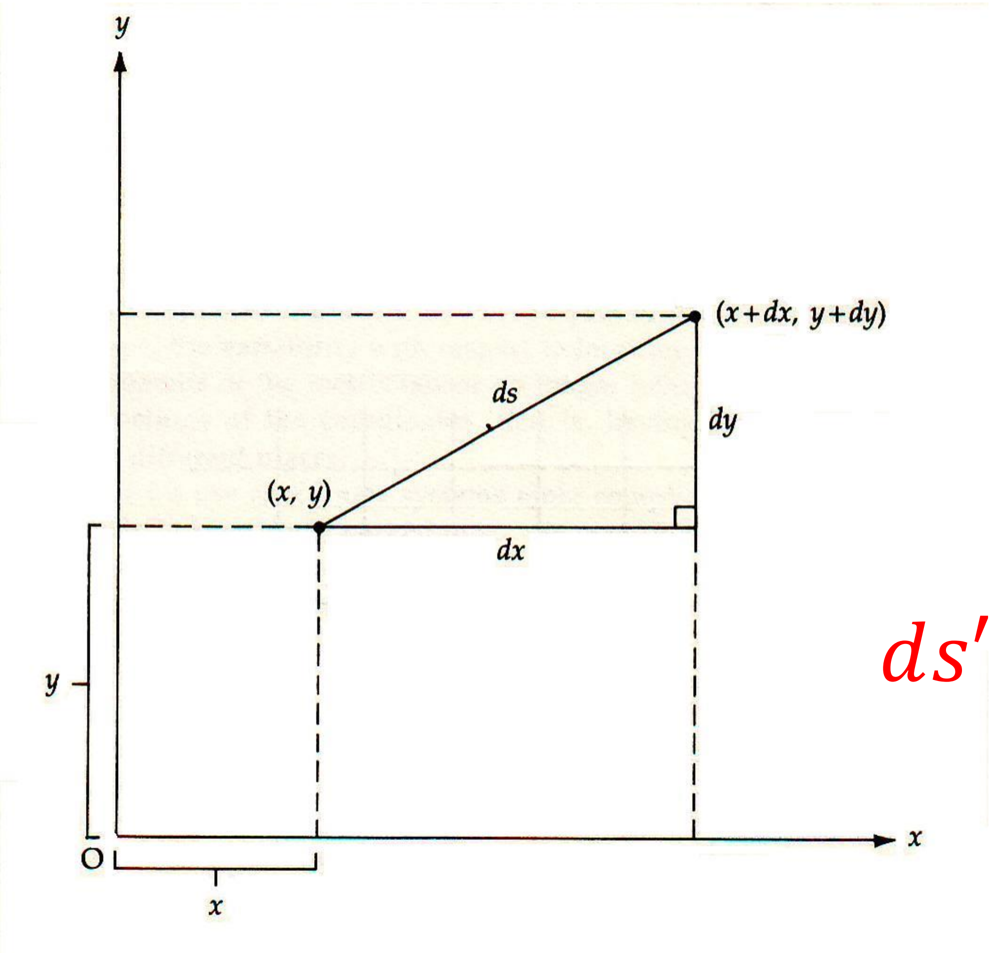
$$dr^2 = dx^2 + dy^2 + dz^2$$



Quantum fluctuation:

- Temporary change in amount of energy ΔE in a point in space
- Uncertainty principle, $\Delta E \Delta t \geq \frac{\hbar}{2}$
- Can "borrow" energy ΔE as long as it is "returned" in Δt… Really strange!

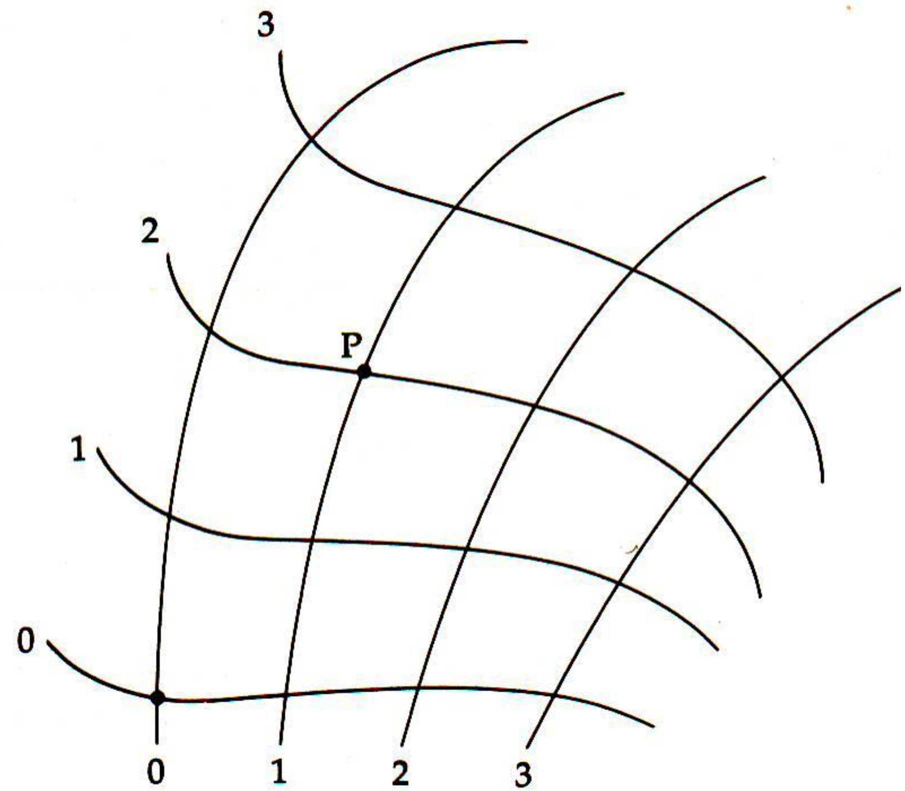
2. Consider the 2 dimensional Euclidean figures below.
   a) Write an expression for ds^2 in terms of dx and dy for extreme left figure.
   b) Write an expression for ds^2 if x and y axes both have twice the number of lines as before



$$ds^2 = dx^2 + dy^2$$

$$ds'^2 = \left(\frac{1}{2}dx\right)^2 + \left(\frac{1}{2}dy\right)^2$$

c) Write the most general expression for ds^2 in terms of dx and dy for the extreme right figure.



$$ds^2 = g_{xx}dx^2 + g_{yy}dy^2 + g_{xy}dxdy + g_{yx}dydx$$

Note that the g's are some functions!

3(a) Given a metric tensor where components: $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$, also that $g_{\mu\nu} = 0$ when $\mu \neq \nu$, show that $g_{\mu\nu}g^{\mu\nu} = 4$.

$$
\begin{aligned}
g_{\mu\nu}g^{\mu\nu} &= g_{0\nu}g^{0\nu} + g_{1\nu}g^{1\nu} + g_{2\nu}g^{2\nu} + g_{3\nu}g^{3\nu} \\
&= g_{00}g^{00} + g_{10}g^{10} + g_{20}g^{20} + g_{30}g^{30} \\
&+ g_{01}g^{01} + g_{11}g^{11} + g_{21}g^{21} + g_{31}g^{31} \\
&+ g_{02}g^{02} + g_{12}g^{12} + g_{22}g^{22} + g_{32}g^{32} \\
&+ g_{03}g^{03} + g_{13}g^{13} + g_{23}g^{23} + g_{33}g^{33} \\
&= (1)(1) + (-1)(-1) + (-1)(-1) + (-1)(-1) \\
&= 4
\end{aligned}
$$

Physicists not only make discoveries, they also invent the language (mathematics) and symbols to describe their theories.
Another example: Newton invented Calculus when trying to understand gravity.

3(b) Recall the Standard Einstein's Field Equation, $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$.

As mentioned in the lecture, note that $R = g^{\alpha\beta} R_{\alpha\beta}$.

Show that if $T_{\alpha\beta} = 0$ then $R_{\alpha\beta} = 0$.

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0$$

Multiply both sides by the metric

$$\Rightarrow g^{\alpha\beta} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) = 0$$

$$\Rightarrow g^{\alpha\beta} R_{\alpha\beta} - \left[\frac{1}{2} g^{\alpha\beta} g_{\alpha\beta} \right] R = 0$$

$g^{\alpha\beta} R_{\alpha\beta} = R$

$g^{\mu\nu} g_{\mu\nu} = 4$

$$\Rightarrow R - \frac{1}{2}(4) R = 0$$

$$\Rightarrow R - 2R = 0$$

$$\Rightarrow -R = 0$$

$$\Rightarrow R_{\alpha\beta} = 0$$

Recall from lecture, Einstein questioned $R_{\mu\nu} = 0$
(or possibility of pure geometry analysis, with no matter on RHS)