Section 1.4

Gaussian Elimination

Objective

- How to use GE / GJE to solve indirect LS problems?

How to denote ERO?

Notation 1.4.9

When doing elementary row operations, we adopt the following notation:

1. cR_i

   "multiply the i^{th} row by the constant c".

2. $R_i \leftrightarrow R_j$

   "interchange the i^{th} and the j^{th} rows".

3. $R_i + cR_j$ Not the other way arnd

   "add c times of the j^{th} row to the i^{th} row".

Linear system with "unknown" constant terms

Example 1.4.10.1

What is the condition that must be satisfied by a, b, c so that the system of linear equations

$$\begin{cases} x + 2y - 3z = a \\ 2x + 6y - 11z = b \\ x - 2y + 7z = c \end{cases}$$

has at least one solution?

Turn it into REF then see if can find infinitely many solutions

Example 1.4.10.1

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 1 & -2 & 7 & c \end{array}\right) \xrightarrow{R_3 - R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{array}\right) \xrightarrow{R_3 + 2R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & 2b+c-5a \end{array}\right)$$

If $2b + c - 5a \neq 0$, system has no solution because constant column is now a pivot column

If $2b + c - 5a = 0$, system has infinitely many solns.

It has (infinitely many) solutions if and only if

$$2b + c - 5a = 0.$$

Linear system with "unknown" constant terms

Example 1.4.10.1

$$\begin{cases} x + 2y - 3z = a \\ 2x + 6y - 11z = b \\ x - 2y + 7z = c \end{cases}$$

How many solutions do these systems have?

$$\begin{cases} x + 2y - 3z = 1 \\ 2x + 6y - 11z = 1 \\ x - 2y + 7z = 1 \end{cases} \qquad \begin{cases} x + 2y - 3z = 1 \\ 2x + 6y - 11z = 2 \\ x - 2y + 7z = 1 \end{cases}$$

infinitely many solutions

sub in

$2b + c - 5a = -2$

no solution

$2b + c - 5a = 0$

It has (infinitely many) solutions if and only if

$2b + c - 5a = 0$.

Linear system with "unknown" coefficients and constant terms

Example 1.4.10.2

$$\begin{cases} x + 2y + z = 1 \\ 2x + by + 2z = 2 \\ 4x + 8y + b^2 z = 2b \end{cases}$$

Determine the values of b so that the system of linear equations has

(a) no solution

(b) a unique solution

(c) infinitely many solutions

Linear system with "unknown" coefficients and constant terms

Example 1.4.10.2

almost REF

- Depending on the value of b
- staircase might or might not appear

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & b & 2 & 2 \\ 4 & 8 & b^2 & 2b \end{array}\right) \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$$

Add -2 times of the first row to the second row.

Add -4 times of the first row to the third row.

Linear system with "unknown" coefficients and constant terms

Example 1.4.10.2

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$$

(a) The system has no solution if

the last column is a pivot column

$$\underbrace{b^2 - 4 = 0}_{b=\pm 2} \quad \text{and} \quad \underbrace{2b - 4 \neq 0}_{b \neq 2} \quad \rightarrow \therefore b = -2$$

Linear system with "unknown" coefficients and constant terms

Example 1.4.10.2

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$$

(b) The system has a unique solution if

every column is a pivot column (except the last)

$$\underbrace{b-4 \neq 0}_{b \neq 4} \text{ and } \underbrace{b^2-4 \neq 0}_{b \neq \pm 2} \Leftrightarrow b \neq 4,\ b \neq 2 \text{ and } b \neq -2$$

Linear system with "unknown" coefficients and constant terms

Example 1.4.10.2

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$$

both must be 0

- if 2b-4 = 0 then will have no solution

(c) The system has infinitely many solutions if

some columns are non-pivot columns

(i) $b - 4 = 0 \quad \rightarrow \quad b = 4$

or

(ii) $b^2 - 4 = 0$ and $2b - 4 = 0 \quad \rightarrow b = 2$

Linear system with “unknown” coefficients and constant terms

Example 1.4.10.2

$$\begin{cases} x + 2y + z = 1 \\ 2x + by + 2z = 2 \\ 4x + 8y + b^2 z = 2b \end{cases}$$

Determine the values of b so that the system of linear equations has

(a) no solution $b = -2$

(b) a unique solution $b \neq 4$, $b \neq 2$ and $b \neq -2$

(c) infinitely many solutions $b = 2$ or $b = 4$

Linear system with more than one "unknown" coefficients and constant terms

Example 1.4.10.3

Determine the values of a and b so that the system of linear equations

$$\left\{\begin{array}{rcrcrcl} ax & + & y & & & = & a \\ x & + & y & + & z & = & 1 \\ & & y & + & az & = & b \end{array}\right.$$

has

(a) no solution,

(b) a unique solution, and

(c) infinitely many solutions.

Linear system with more than one "unknown" coefficients and constant terms

Example 1.4.10.3

$$\left(\begin{array}{ccc|c} a & 1 & 0 & a \\ 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \end{array}\right)$$

$\underline{R_2 - \frac{1}{a}R_1}$

add $-1/a$ times of first row to second row

Cannot do this if $a = 0$

Need to consider two different situations:

Case 1: $a = 0$ and

Case 2: $a \neq 0$.

Case 1 $a = 0$ | Case 2 $a \neq 0$

Example 1.4.10.3

Solution **Case 1:** $a = 0$

Substitute $a = 0$ to the augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & b \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & b \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{array}\right)$$

Under the assumption $a = 0$,

- the system has no solution if $b \neq 0$;
- the system has infinitely many solutions if $b = 0$.

Case 1 $a = 0$

Case 2 $a \neq 0$

Example 1.4.10.3

Solution Case 2: $a \neq 0$

$$\left(\begin{array}{ccc|c} a & 1 & 0 & a \\ 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \end{array}\right) \xrightarrow{R_2 - \frac{1}{a}R_1} \left(\begin{array}{ccc|c} a & 1 & 0 & a \\ 0 & \frac{a-1}{a} & 1 & 0 \\ 0 & 1 & a & b \end{array}\right)$$

add $-a/(a-1)$ times of second row to third row

Cannot do this if $a = 1$

Need to consider two cases again:

Case 2a: $a = 1$ and

Case 2b: $a \neq 1$.

Case 1 $a = 0$ | Case 2 $a \neq 0$

Case 2a $a = 1$ | Case 2b $a \neq 1$

Example 1.4.10.3

Solution Case 2a: $a = 1$

Substitute $a = 1$ to the last augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & b \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Under the assumption $a = 1$,

- the system has exactly one solution.

Case 1 $a = 0$ | Case 2 $a \neq 0$

Case 2a $a = 1$ | Case 2b $a \neq 1$

Example 1.4.10.3

Solution Case 2b: $a \neq 0$ and $a \neq 1$

$$\left(\begin{array}{ccc|c} a & 1 & 0 & a \\ 0 & \frac{a-1}{a} & 1 & 0 \\ 0 & 1 & a & b \end{array}\right) \xrightarrow{R_3 - \frac{a}{a-1}R_2} \left(\begin{array}{ccc|c} a & 1 & 0 & a \\ 0 & \frac{a-1}{a} & 1 & 0 \\ 0 & 0 & \frac{a^2-2a}{a-1} & b \end{array}\right)$$

both are cfm not 0

focus on if this is 0 / non-0

- the system has no solution if $(a^2 - 2a)/(a - 1) = 0$ & $b \neq 0$ $\Leftrightarrow$ $a = 2$ & $b \neq 0$;

$\rightarrow a^2 - 2a = 0 \rightarrow (a-2)a = 0 \rightarrow a = 2$ or $a = 0$ but already considering a=0 scenario

- the system has one solution if $(a^2 - 2a)/(a - 1) \neq 0$ $\Leftrightarrow$ $a \neq 2$;
- the system has infinitely many solutions if $(a^2 - 2a)/(a - 1) = 0$ & $b = 0$ $\Leftrightarrow$ $a = 2$ & $b = 0$.

$$\begin{array}{ccc} & & \\ \text{Case 1} & & \text{Case 2} \\ a = 0 & & a \neq 0 \\ & \text{Case 2a} & & \text{Case 2b} \\ & a = 1 & & a \neq 1 \end{array}$$

Example 1.4.10.3

Answer (a)

The system has no solution:

by Case 1, $a = 0$ and $b \neq 0$

or

by Case 2b, $a \neq 0$ & $a \neq 1$ and $a = 2$ & $b \neq 0$

$$\boxed{\begin{array}{l}\text{The system has no solution if} \\ b \neq 0 \quad \text{and} \quad a = 0 \ \text{ or } \ a = 2.\end{array}}$$

Case 1 $a = 0$ | Case 2 $a \neq 0$

Case 2a $a = 1$ | Case 2b $a \neq 1$

Example 1.4.10.3

Answer (b)

The system has a unique solution:

by Case 2a, $a = 1$;

or

by Case 2b, $a \neq 0$ & $a \neq 1$ and $a \neq 2$

$$\boxed{\text{The system has a unique solution if } a \neq 0 \text{ and } a \neq 2.}$$

Case 1 $a = 0$ | Case 2 $a \neq 0$

Case 2a $a = 1$ | Case 2b $a \neq 1$

Example 1.4.10.3

Answer (c)

The system has infinitely many solutions:

by Case 1, $a = 0$ and $b = 0$

or

by Case 2b, $a \neq 0$ & $a \neq 1$ and $a = 2$ & $b = 0$

> The system has infinitely many solutions if
>
> $b = 0$ and $a = 0$ or 2.

Linear system with more than one "unknown" coefficients and constant terms

Remark on Example 1.4.10.3

$$\left(\begin{array}{ccc|c} a & 1 & 0 & a \\ 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \\ a & 1 & 0 & a \end{array}\right)$$

If we rearrange the rows of the augmented matrix in the following way:

the 2nd row at the top,

the 3rd row in the middle and

the 1st row at the bottom,

the problem will be much easier to be solved by Gaussian Elimination.

Finding equation of a curve

Example 1.4.10.4

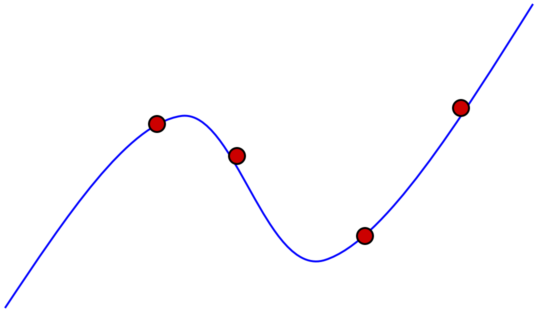
Given a cubic curve with equation

$$y = a + bx + cx^2 + dx^3,$$

where a, b, c, d are real constants, that passes through the points

$$(0, 10),\ (1, 7),\ (3, -11) \text{ and } (4, -14),$$

find the values of a, b, c, d.



4 points will determine the equation

Finding equation of a curve

Example 1.4.10.4

By substituting

$(x, y) = (0, 10), (1, 7), (3, -11)$ and $(4, -14)$

into the equation $y = a + bx + cx^2 + dx^3$,

we obtain a system of linear equations:

$$\begin{cases} a & & & & & & & = & 10 \\ a & + & b & + & c & + & d & = & 7 \\ a & + & 3b & + & 9c & + & 27d & = & -11 \\ a & + & 4b & + & 16c & + & 64d & = & -14 \end{cases}$$

where a, b, c, d are the variables

Note the role swap of notation

Finding equation of a curve

Example 1.4.10.4

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 1 & 1 & 1 & 1 & 7 \\ 1 & 3 & 9 & 27 & -11 \\ 1 & 4 & 16 & 64 & -14 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

So the solution is
$a = 10$, $b = 2$, $c = -6$ and $d = 1$.

The equation of the cubic curve is
$y = 10 + 2x - 6x^2 + x^3$.

Geometrical interpretation in 3D space

Discussion 1.4.11

LS of 3 variables
(with solutions)

REF	Solutions	**Geometrical interpretation** for 3 planes
3 non-zero rows	0 parameter	Intersect at 1 point
2 non-zero rows	1 parameter	Intersect at a line
1 non-zero row	2 parameters	Intersect at a plane
0 non-zero row zero system	3 parameters	NA

Section 1.5

Homogeneous Linear Systems

Objective

- What is a homogeneous system?
- What is a trivial / non-trivial solution of a homogeneous system?

What is a homogeneous system?

Definition 1.5.1

A system of linear equations is said to be homogeneous if it has the form

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{cases}$$

all the constant terms are zero

If a linear system has some non-zero constant terms, we say it is non-homogeneous.

What is a trivial/non-trivial solution?

Definition 1.5.1

A system of linear equations is said to be homogeneous if it has the form

$$\begin{cases} a_{11}\boxed{x_1} + a_{12}\boxed{x_2} + \dots + a_{1n}\boxed{x_n} = 0 \\ a_{21}\boxed{x_1} + a_{22}\boxed{x_2} + \dots + a_{2n}\boxed{x_n} = 0 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}\boxed{x_1} + a_{m2}\boxed{x_2} + \dots + a_{mn}\boxed{x_n} = 0 \end{cases}$$

$x_1 = 0,\ x_2 = 0,\ \dots,\ x_n = 0$ is a solution

trivial solution

Any solution other than the trivial solution is called a non-trivial solution.

Example

Consider the following homogeneous system:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$$

$x_1 = 0,\ x_2 = 0, x_3 = 0,\ x_4 = 0$ trivial solution

$x_1 = 1,\ x_2 = 0, x_3 = -1,\ x_4 = 0$ non-trivial solution

Remark: Only in a homogeneous system do we talk about trivial / non-trivial solution.

Example 1.5.2

Given a quadric surface with equation

$$ax^2 + by^2 + cz^2 = d$$

where a, b, c, d are real constants,

that passes through the points

$$(1, 1, -1), \ (1, 3, 3) \text{ and } (-2, 0, 2),$$

find a formula for the quadric surface.

$$\begin{cases} a + b + c = d \\ a + 9b + 9c = d \\ 4a + 4c = d \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & -1 & d \\ 1 & 9 & 9 & d \\ 4 & 0 & 4 & d \end{array}\right)$$

a,b,c,d are all variables, cannot just assume that d is a constant in this case

- although d was a constant previously, but after subbing in (x,y) then d is now a variable as much as a,b,c

Example 1.5.2

Given a quadric surface with equation

$$ax^2 + by^2 + cz^2 = d$$

any non-0 multiple of the equation will give the same surface

- they are all multiples of each other via t

$$\begin{cases} a + b + c - d = 0 \\ a + 9b + 9c - d = 0 \\ 4a + 4c - d = 0 \end{cases}$$

homogeneous system

General solution

$$\begin{cases} a = t \\ b = \tfrac{3}{4}t \\ c = -\tfrac{3}{4}t \\ d = t \end{cases}$$

t = 0: a = 0, b = 0, c = 0, d = 0

trivial solution

t = 4: a = 4, b = 3, c = -3, d = 4

non-trivial solution

sub t=4 because can eliminate denominator

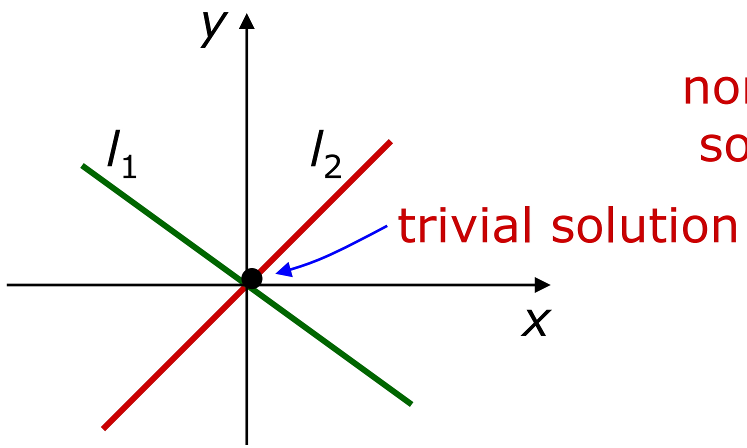
Discussion 1.5.3.1

as long as have non-trival solutions

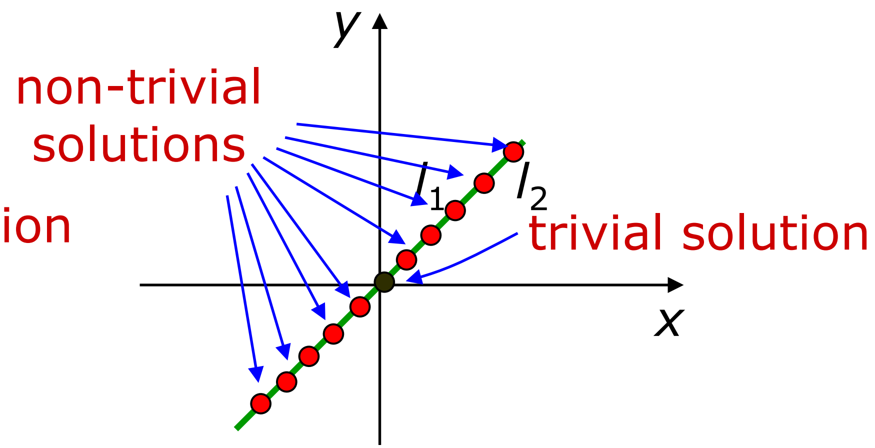
- means infinitely many solutions
- always has trival + 1 = infinitely many

$$\begin{cases} a_1 x + b_1 y = 0 & (l_1) \\ a_2 x + b_2 y = 0 & (l_2) \end{cases}$$

represent two straight lines through the origin.



exactly one solution



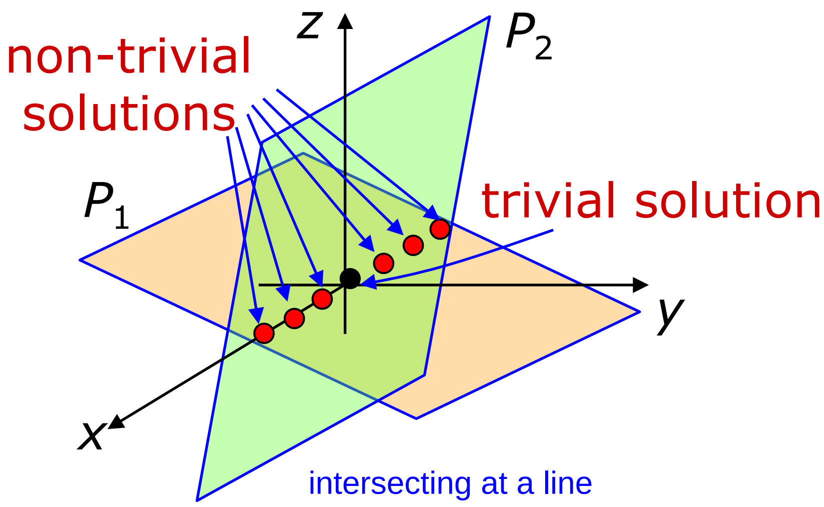
infinitely many solutions

What is a trivial/non-trivial solution?

Discussion 1.5.3.2

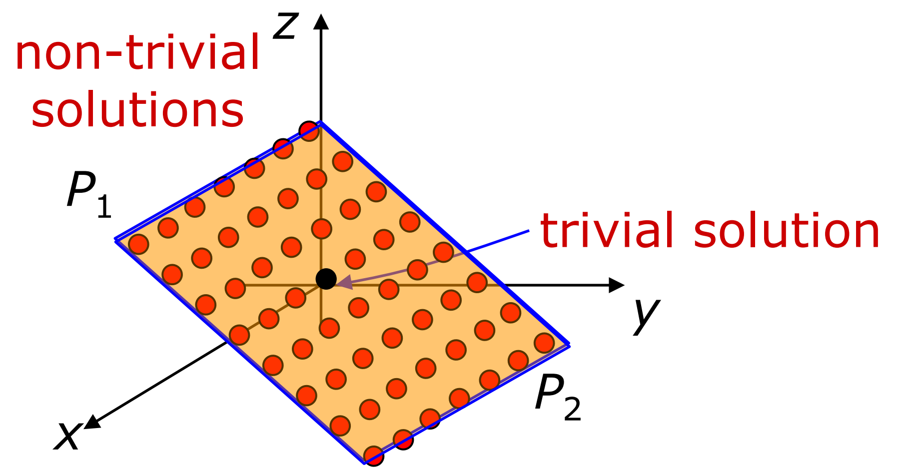
$$\begin{cases} a_1x + b_1y + c_1z = 0 & (P_1) \\ a_2x + b_2y + c_2z = 0 & (P_2) \end{cases}$$

represent two planes through the origin.



intersecting at a line
and passes origin

infinitely many solutions



intersecting at a plane

infinitely many solutions

How many solutions does a homogeneous solution have?

homogeneous solution only has 2 possibilities

- unlike general linear system
- always have trivial solution

Remark 1.5.4

1. A homogeneous system of linear equations has either only the trivial solution or infinitely many solutions in addition to the trivial solution.
2. A homogeneous system of linear equations with more variables than equations has infinitely many solutions.

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$

2 equation

3 variables

infinitely many solutions

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = 0$$

3 equations

4 variables

infinitely many solutions

Section 2.1

Introduction to Matrices

Objective

- What are the size, entries, order of a matrix?
- What are diagonal, identity, symmetric, triangular matrices?
- How to express matrices using (i, j)-entries?

What are the size and entries of a matrix?

Summary 2.1.1-2.1.5

Matrix

$$A = \begin{array}{cl} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \text{row} \\ \text{column} & \end{array}$$

can be simplified as $\boldsymbol{A} = (a_{ij})_{m \times n}$ or (a_{ij})

number of rows is *m*   number of columns is *n*

<u>We say</u>: The size of the matrix ***A*** is *m* x *n*

A is an *m* x *n* matrix

a_{ij} denotes the number in the i^{th} row and j^{th} column.

<u>We say</u>: a_{ij} is the (*i*, *j*)-entry of the matrix ***A***

What are the size and entries of a matrix?

Example 2.1.6

1. $\boldsymbol{A} = (a_{ij})_{2\times 3}$ where $a_{ij} = i + j$

interpreting the algebra and recovering the matrix

$$\boldsymbol{A} = \begin{array}{c} \begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array} \\ \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \\ \begin{array}{ccc} a_{21} & a_{22} & a_{23} \end{array} \end{array}$$

2. $\boldsymbol{B} = (b_{ij})_{3\times 2}$ where $b_{ij} = \begin{cases} 1 & \text{if } i+j \text{ is even} \\ -1 & \text{if } i+j \text{ is odd} \end{cases}$

$$\boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Learn how to describe various types of matrices in terms of (i, j)-entries

What are the order and diagonal of a square matrix?

Summary 2.1.7-2.1.8

Square matrices

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

diagonal of A

same number of rows and columns

$\boldsymbol{A}$ is an n x n matrix

$\boldsymbol{A} = (a_{ij})$ is a square matrix of order n

a_{11}, a_{22}, …, a_{nn} are called the diagonal entries

only from left to right

a_{ij}, $i \neq j$, are called the non-diagonal entries

What are diagonal, scalar, identity matrices?

How to express them using (i, j)-entries?

Summary 2.1.7-2.1.8

Types of square matrices

Diagonal matrix	all non-diagonal entries are zero	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$a_{ij} = 0$ whenever $i \neq j$
Scalar matrix	diagonal matrix with all diagonal entries the same	$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	$a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ c & \text{if } i = j \end{cases}$
Identity matrix $\boldsymbol{I}_n$	diagonal matrix with all diagonal entries equal 1 scalar matrix but simplified	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ I_3	$a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

What are symmetric and triangular matrices?

How to express them using (i, j)-entries?

Summary 2.1.7-2.1.8

Types of square matrices

Zero matrix $\mathbf{0}_{m \times n}$	all entries equal to zero can be non-square	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$a_{ij} = 0$ for all i, j
Symmetric matrix	k^{th} row "equal" k^{th} column for all k	$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix}$	$a_{ij} = a_{ji}$ for all i, j
Upper triangular matrix	all entries below diagonals are zero	$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix}$	$a_{ij} = 0$ for all $i > j$
Lower triangular matrix	all entries above diagonals are zero	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 2 \end{pmatrix}$	$a_{ij} = 0$ for all $i < j$

Section 2.2

Matrix Operations

Objective

- How to perform matrix addition & multiplication, scalar multiplication and transpose?
- How to express these operations using (i, j)-entries?
- What are some properties of these operations?
- What are some different ways to express matrix multiplication?
- How to express LS in matrix equation form?

How to perform matrix addition, scalar multiplication?

Summary 2.2.1 - 2.2.5

Let $\boldsymbol{A} = (a_{ij})_{m \times n}$ $\boldsymbol{B} = (b_{ij})_{m \times n}$ and c a real constant. scalar

Matrix Equality entirely the same	$\boldsymbol{A} = \boldsymbol{B}$	$\boldsymbol{A}$ and $\boldsymbol{B}$ have same size and same corresponding entries	$a_{ij} = b_{ij}$ for all i, j
Matrix Addition	$\boldsymbol{A} + \boldsymbol{B}$	addition of corresponding entries of $\boldsymbol{A}$ and $\boldsymbol{B}$	$(a_{ij} + b_{ij})_{m \times n}$
Matrix subtraction	$\boldsymbol{A} - \boldsymbol{B}$	subtraction of corresponding entries of $\boldsymbol{A}$ and $\boldsymbol{B}$	$(a_{ij} - b_{ij})_{m \times n}$ add to the same entry
Scalar multiplication	$c\boldsymbol{A}$	multiply every entry of $\boldsymbol{A}$ by scalar c	$(ca_{ij})_{m \times n}$
Negative of matrix	$-\boldsymbol{A}$	attach negative sign to every entry of $\boldsymbol{A}$	$(-a_{ij})_{m \times n}$

What are some properties of these operations?

Summary 2.2.6 - 2.2.7

Properties

- On matrix addition and scalar multiplication
- Theorem 2.2.6
- Similar to ordinary numbers operations
- Commutative Law: $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
- Associative Law: $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$
- Zero matrix behaves like number "0" in matrix addition

Definition 2.2.8 & Example 2.2.9.1

Matrix Multiplication

size must be compatible

$$\underset{2 \times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \underset{3 \times 2}{\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}}$$

1-1 entry; 1-2 entry; 2-1 entry; 2-2 entry

$$= \underset{2 \times 2}{\begin{pmatrix} 1\times1+2\times2+3\times-1=2 & 1\times1+2\times3+3\times-2=1 \\ 4\times1+5\times2+6\times-1=8 & 4\times1+5\times3+6\times-2=7 \end{pmatrix}}$$

$$= \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}$$

How to perform matrix multiplication?

Definition 2.2.8 (Matrix Multiplication)

Let $\boldsymbol{A} = (a_{ij})_{m \times p}$ and $\boldsymbol{B} = (b_{ij})_{p \times n}$ be two matrices.

follows the row of A and column of B

The product $\boldsymbol{AB}$ is an $m \times n$ matrix

its (*i*, *j*)-entry is

1 entry

$$a_{i1}b_{1j} + \boxed{a_{i2}b_{2j}} + \dots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

summation notation

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix}$$

How to perform matrix multiplication?

Remark 2.2.10.1

We can only multiply two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ (in the manner $\boldsymbol{AB}$) when the number of columns of $\boldsymbol{A}$ is equal to the number of rows of $\boldsymbol{B}$.

$$\boldsymbol{A} = (a_{ij})_{m\times p} \quad \text{and} \quad \boldsymbol{B} = (b_{ij})_{p\times n}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \qquad \cancel{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}}$$

Remark 2.2.10.2-4

Different from ordinary numbers multiplication

The matrix multiplication is not commutative.

i.e. $\boldsymbol{AB} \neq \boldsymbol{BA}$ in general, even if the product exist.

$\boldsymbol{AB}$: pre-multiplication of $\boldsymbol{A}$ to $\boldsymbol{B}$ AB

$\boldsymbol{BA}$: post-multiplication of $\boldsymbol{A}$ to $\boldsymbol{B}$ BA

$\boldsymbol{AB} = \boldsymbol{0}$ does not imply $\boldsymbol{A} = \boldsymbol{0}$ or $\boldsymbol{B} = \boldsymbol{0}$.

What are some properties of matrix multiplication?

Theorem 2.2.11.1-3

Similar to ordinary numbers multiplication

cannot swap their order, must be A then B then C

1. $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$ Associative Law
2. $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{AB}_1 + \boldsymbol{AB}_2$
   $(\boldsymbol{C}_1 + \boldsymbol{C}_2)\,\boldsymbol{A} = \boldsymbol{C}_1\boldsymbol{A} + \boldsymbol{C}_2\boldsymbol{A}$ Distributive Law
3. $c(\boldsymbol{AB}) = (c\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(c\boldsymbol{B})$ c is a scalar

To prove these properties,
check LHS and RHS have same size
and same corresponding entries

What are some properties of matrix multiplication?

Theorem 2.2.11.4

Similar to ordinary numbers multiplication

Let $\boldsymbol{A}$ be a $m \times n$ matrix.

- $\boldsymbol{A}\mathbf{0}_{n\times q} = \mathbf{0}_{m\times q}$ and $\mathbf{0}_{p\times m}\boldsymbol{A} = \mathbf{0}_{p\times n}$
- $\boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A}$

Zero matrix behaves like number "0" in matrix multiplication

Identity matrix behaves like number "1" in matrix multiplication

What are the powers of a matrix?

Definition 2.2.12

Similar to ordinary numbers multiplication

A : square matrix
n : nonnegative integer

We define $\boldsymbol{A}^n$ as follows:

$$\boldsymbol{A}^n = \underbrace{\boldsymbol{AA} \ldots \boldsymbol{A}}_{\text{n times}} \qquad n \geq 1$$

$$\mathbf{A}^0 = \mathbf{I}$$

Remark 2.2.14

1. $\boldsymbol{A}^r \boldsymbol{A}^s = \boldsymbol{A}^{r+s}$ Similar to ordinary number

already a new matrix product

2. $(\boldsymbol{AB})^n \neq \boldsymbol{A}^n \boldsymbol{B}^n$ Different from ordinary number

$$\underbrace{(\boldsymbol{AB})\,(\boldsymbol{AB})\,\ldots\,(\boldsymbol{AB})}_{\text{n times}}$$

$$\underbrace{\boldsymbol{AA}\,\ldots\,\boldsymbol{A}}_{\text{n times}}\;\underbrace{\boldsymbol{BB}\,\ldots\,\boldsymbol{B}}_{\text{n times}}$$

Other ways to "zip" a matrix

Notation 2.2.15

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix}$$

"unzipped"

"partially zipped"

- This does not represent a number

"zipped"

$$\boldsymbol{A} = (a_{ij})_{m\times p} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{pmatrix} \begin{array}{l} 1^{st} \text{ row of } \boldsymbol{A} \\ 2^{nd} \text{ row of } \boldsymbol{A} \\ \\ m^{th} \text{ row of } \boldsymbol{A} \end{array}$$

a1 represents the whole row

zipped along the rows

$\boldsymbol{a}_i$ is a 1 x p row matrix

$$\boldsymbol{a}_1 = (a_{11} \quad a_{12} \quad \ldots \quad a_{1p})$$

$$\boldsymbol{a}_2 = (a_{21} \quad a_{22} \quad \ldots \quad a_{2p})$$

$$\vdots$$

$$\boldsymbol{a}_m = (a_{m1} \quad a_{m2} \quad \ldots \quad a_{mp})$$

Other ways to "zip" a matrix

Notation 2.2.15

$$\boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix}$$

representing column matrix

$$\boldsymbol{B} = (b_{ij})_{p\times n} = (\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \cdots \quad \boldsymbol{b}_n)$$

1st column of B

2nd column of B

nth column of B

zipped along the columns

$\boldsymbol{b}_i$ is a p x 1 column matrix

$$\boldsymbol{b}_1 = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{pmatrix} \qquad \boldsymbol{b}_2 = \begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{p2} \end{pmatrix} \qquad \cdots \qquad \boldsymbol{b}_n = \begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{pn} \end{pmatrix}$$

What are some different ways to express matrix multiplication?

Notation 2.2.15

Example 2.2.16

$$\begin{array}{cc} & A \\ \begin{array}{c} a_1 \\[6pt] a_2 \\[6pt] a_3 \end{array} & \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \\ & 3\times 2 \end{array} \begin{array}{c} B \quad 2\times 3 \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\ b_1 \quad b_2 \quad b_3 \end{array} = \overset{AB}{\begin{pmatrix} \underset{a_1b_1}{5} & \underset{a_1b_2}{7} & \underset{a_1b_3}{9} \\ \underset{a_2b_1}{14} & \underset{a_2b_2}{19} & \underset{a_2b_3}{24} \\ \underset{a_3b_1}{-9} & \underset{a_3b_2}{-12} & \underset{a_3b_3}{-15} \end{pmatrix}}$$

General Form

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} \quad \mathbf{B} = (\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n) \Rightarrow \mathbf{AB} = \begin{pmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \cdots & \mathbf{a}_1\mathbf{b}_n \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \cdots & \mathbf{a}_2\mathbf{b}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_m\mathbf{b}_1 & \mathbf{a}_m\mathbf{b}_2 & \cdots & \mathbf{a}_m\mathbf{b}_n \end{pmatrix}$$

What are some different ways to express matrix multiplication?

Notation 2.2.15

Example 2.2.16

3×2 & 2×1

$\downarrow$

3×1

$= Ab_1$

$$\underset{\boldsymbol{A}}{\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}} \begin{array}{c} \boldsymbol{B} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\ \boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \boldsymbol{b}_3 \end{array} = \begin{array}{c} \boldsymbol{AB} \\ \begin{pmatrix} 5 & 7 & 9 \\ 14 & 19 & 24 \\ -9 & -12 & -15 \end{pmatrix} \\ \boldsymbol{Ab}_1 \quad \boldsymbol{Ab}_2 \quad \boldsymbol{Ab}_3 \end{array}$$

$\boldsymbol{A}(\,j$ th column of $\boldsymbol{B}) = j$ th column of $\boldsymbol{AB}$

$$\boxed{\mathbf{AB} = (\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \cdots \quad \mathbf{Ab}_n)}$$

What are some different ways to express matrix multiplication?

Notation 2.2.15

Example 2.2.16

A $1\times 2 \; X \; 2\times 3$ $\quad B \quad\quad AB$

$$\begin{array}{c} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{array} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 7 & 9 \\ 14 & 19 & 24 \\ -9 & -12 & -15 \end{pmatrix} \begin{array}{c} \boldsymbol{a}_1\boldsymbol{B} \\ \boldsymbol{a}_2\boldsymbol{B} \\ \boldsymbol{a}_3\boldsymbol{B} \end{array}$$

(i th row of $\boldsymbol{A}$) $\boldsymbol{B}$ = i th row of $\boldsymbol{AB}$

$$\boxed{\mathbf{AB} = \begin{pmatrix} \mathbf{a}_1\mathbf{B} \\ \mathbf{a}_2\mathbf{B} \\ \vdots \\ \mathbf{a}_m\mathbf{B} \end{pmatrix}}$$

How to express LS in matrix equation form?

Example 2.2.18

$$\begin{cases} 4x + 5y + 6z = 1 \\ x - y = 2 \\ y - z = 3 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} 4 & 5 & 6 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

matrix equation form

vs augmented matrix

$$\Leftrightarrow \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} y + \begin{pmatrix} 6 \\ 0 \\ -1 \end{pmatrix} z = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

vector equation form

How to express LS in matrix equation form?

Remark 2.2.17

Consider the system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

rewrite the system using the matrix multiplication:

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\text{variable matrix}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\text{constant matrix}}$$

How to express LS in matrix equation form?

Example 2.2.18

$$\begin{cases} 4x + 5y + 6z = 1 \\ x - y = 2 \\ y - z = 3 \end{cases}$$

don't confuse matrix equation form
with augmented matrix

$$\begin{pmatrix} 4 & 5 & 6 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \left(\begin{array}{ccc|c} 4 & 5 & 6 & 1 \\ 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right)$$

matrix equation form   augmented matrix

A concise notation for linear system

Remark 2.2.17

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\boldsymbol{b}}$$

$\boxed{x_1 = u_1 \quad x_2 = u_2 \quad \cdots \quad x_n = u_n}$

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

We can represent the linear system as $\boldsymbol{Ax} = \boldsymbol{b}$

representing a linear system

A solution of the linear system

is represented by an n x 1 column matrix.

$\boldsymbol{u}$ is a solution of $\boldsymbol{Ax} = \boldsymbol{b}$

if and only if $\boldsymbol{Au} = \boldsymbol{b}$

Remark 2.2.17

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{cases}$$

Linear system can also be written in vector equation form:

$$\underbrace{\underbrace{\begin{pmatrix} a_{11} \\ a_{21} \\ \\ a_{m1} \end{pmatrix} x_1}_{\text{scalar multiplication}} + \underbrace{\begin{pmatrix} a_{12} \\ a_{22} \\ \\ a_{m2} \end{pmatrix}}\, x_2 + \dots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \\ a_{mn} \end{pmatrix} x_n}_{\text{matrix addition}} = \begin{pmatrix} b_1 \\ b_2 \\ \\ b_m \end{pmatrix}$$

use of this form in chapter 3

How to perform matrix transpose?

Summary 2.2.19 - 2.2.20

Let $\boldsymbol{A} = (a_{ij})_{m \times n}$

Matrix Transpose	$\boldsymbol{A}^T$ (or $\boldsymbol{A}^t$)	interchanging the rows and columns of $\boldsymbol{A}$	$\boldsymbol{A}^T = (a_{ji})_{n \times m}$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \qquad \boldsymbol{A}^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$$

The transpose operator interchanges *i* and *j* of the entries. i & j represent row & column, so swap to ji

Relation between transpose and symmetric matrix

Remark 2.2.21

2. A square matrix is symmetric if and only if

$$\boldsymbol{A} = \boldsymbol{A}^T.$$

proof that a matrix is symmetric

The transpose operator does not change a symmetric matrix. another definition of symmetric matrix

We can determine whether an (implicit) matrix $\boldsymbol{A}$ is symmetric by checking whether $\boldsymbol{A} = \boldsymbol{A}^T$.

What are some properties of transpose?

Theorem 2.2.22

Let $\boldsymbol{A}$ be an $m \times n$ matrix.

1. $(\boldsymbol{A}^T)^T = \boldsymbol{A}$
2. If $\boldsymbol{B}$ is an $m \times n$ matrix, then $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$.
3. If a is a scalar, then $(a\boldsymbol{A})^T = a\boldsymbol{A}^T$.
4. If $\boldsymbol{B}$ is an $n \times p$ matrix, then $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$.

AB: row i of A x column j of B

$(AB)^T$: row j of A x column i of B

$B^T A^T$: row i of B^T x column j of A^T

"column" i of B x "row" j of A