CS1231S Chapter 6

Functions

6.1 Basics

Definition 6.1.1. Let A, B be sets. A *function* or a *map* from A to B is an assignment to each element of A exactly one element of B. We write $f\colon A \to B$ for "f is a function from A to B". Suppose $f\colon A \to B$.

(1) Let $x \in A$. Then $f(x)$ denotes the element of B that f assigns x to. If $y = f(x)$, then we say that f *maps* x *to* y, and we may write $f\colon x \mapsto y$.

(2) A is called the *domain* of f, and B is called the *codomain* of f.

(3) The *range* or the *image* of f is

$$\{f(x) : x \in A\} = \{y \in B : y = f(x) \text{ for some } x \in A\}.$$

Notation 6.1.2. Let A, B be sets. Often one can specify a function $f\colon A \to B$ by an expression t involving a special symbol x, sometimes called a *variable*. By this, we mean:

> f assigns to each $x_0 \in A$ the value of t when every occurrence of x in it is replaced by x_0.

In this case, we may write

$$\begin{aligned} f\colon A &\to B; \\ x &\mapsto t. \end{aligned}$$

Example 6.1.3. Consider

$$\begin{aligned} f\colon \mathbb{Z} &\to \mathbb{Z}; \\ x &\mapsto x^3 + 23x. \end{aligned}$$

Then f is the function with domain $\mathbb{Z}$ and codomain $\mathbb{Z}$ that assigns to each $x_0 \in \mathbb{Z}$ the value of ${x_0}^3 + 23x_0$. Thus $f(0) = 0^3 + 23 \times 0 = 0$ and $f(1) = 1^3 + 23 \times 1 = 24$. The range of f is

$$\{x^3 + 23x : x \in \mathbb{Z}\}.$$

Definition 6.1.4. Let A be a set. Then the *identity function* on A is the function

$$\begin{aligned} \mathrm{id}_A\colon A &\to A \\ x &\mapsto x. \end{aligned}$$

Remark 6.1.5. The domain, the codomain, and the range of id_A are all A.

Definition 6.1.6. Let absval: $\mathbb{Q} \to \mathbb{Q}$ satisfying, for every $x \in \mathbb{Q}$,

$$\operatorname{absval}(x) = \begin{cases} x, & \text{if } x \geqslant 0; \\ -x, & \text{otherwise.} \end{cases}$$

In other words, the function absval has domain $\mathbb{Q}$, codomain $\mathbb{Q}$, and

$$\forall x \in \mathbb{Q} \ \ \big((x \geqslant 0 \Rightarrow \operatorname{absval}(x) = x\big) \wedge \big({\sim}(x \geqslant 0) \Rightarrow \operatorname{absval}(x) = -x)\big).$$

Usually one writes $|x|$ for $\operatorname{absval}(x)$.

Example 6.1.7. (1) $|1| = 1$ because $1 \geqslant 0$.

(2) $|-23| = -(-23) = 23$ because ${\sim}(-23 \geqslant 0)$.

Exercise 6.1.8. Show that the range of absval is $\mathbb{Q}_{\geqslant 0}$. 6a

Definition 6.1.9. Define floor, ceil: $\mathbb{Q} \to \mathbb{Z}$ by setting, for each $x \in \mathbb{Q}$,

(1) $\operatorname{floor}(x)$ to be the largest integer y such that $y \leqslant x$; and

(2) $\operatorname{ceil}(x)$ to be the smallest integer y such that $y \geqslant x$.

Usually one writes $\lfloor x \rfloor$ and $\lceil x \rceil$ for $\operatorname{floor}(x)$ and $\operatorname{ceil}(x)$ respectively.

Example 6.1.10. (1) $\lfloor 15.11 \rfloor = 15$ and $\lceil 15.11 \rceil = 16$.

(2) $\lfloor -5.2 \rfloor = -6$ and $\lceil -5.2 \rceil = -5$.

(3) $\lfloor 23 \rfloor = 23$ and $\lceil 23 \rceil = 23$.

(4) As $\lfloor x \rfloor = x = \lceil x \rceil$ for all $x \in \mathbb{Z}$,

$$\{\lfloor x \rfloor : x \in \mathbb{Q}\} = \mathbb{Z} = \{\lceil x \rceil : x \in \mathbb{Q}\}.$$

Exercise 6.1.11. Let $x \in \mathbb{Q}$. 6b

(1) Show that $\lfloor x \rfloor$ is the unique $y \in \mathbb{Z}$ such that $y \leqslant x < y + 1$.

(2) Show that $\lceil x \rceil$ is the unique $y \in \mathbb{Z}$ such that $y - 1 < x \leqslant y$.

Definition 6.1.12. A *sequence* is a function a whose domain is $\mathbb{Z}$, $\mathbb{Z}^+$ or $\{x \in \mathbb{Z} : k \leqslant x \leqslant \ell\}$ for some $k, \ell \in \mathbb{Z}$. If a is a sequence, then we may write a_n for $a(n)$.

Example 6.1.13 (harmonic sequence)**.** Let a be the sequence $\mathbb{Z}^+ \to \mathbb{Q}$ defined by setting $a_n = 1/n$ for every $n \in \mathbb{Z}^+$. Then $a_1, a_2, a_3, \ldots$ are respectively

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Example 6.1.14. Let b be the sequence $\{x \in \mathbb{Z} : 2 \leqslant x \leqslant 5\} \to \mathbb{Z}^+$ defined by setting $b_n = 3n$ for every $n \in \{x \in \mathbb{Z} : 2 \leqslant x \leqslant 5\}$. Then b_2, b_3, b_4, b_5 are respectively

$$6, 9, 12, 15.$$

Question 6.1.15. Why does the following sentence *not* define a function? 6c

$$\text{Define } f \colon \mathbb{Q} \to \mathbb{Q} \text{ by setting } f(x) = 2^x \text{ for all } x \in \mathbb{Q}.$$

Question 6.1.16. Why does the following sentence *not* define a function? 6d

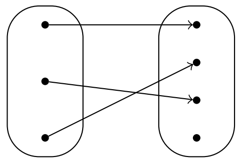
$$\text{Define } g \colon \mathbb{Q} \to \mathbb{Q} \text{ by setting } g(x) = \frac{x^2 + 1}{x^2 + 2x + 1} \text{ for all } x \in \mathbb{Q}.$$

Question 6.1.17. Why does the following sentence *not* define a function? 6e

$$\text{Define } h\colon \mathbb{Q} \to \mathbb{Z} \text{ by setting } h(m/n) = m \text{ for all } m, n \in \mathbb{Z} \text{ where } n \neq 0.$$

Terminology 6.1.18. A function is *well-defined* if its definition ensures that every element of the domain is assigned exactly one element of the codomain.

Arrow diagrams.



The figure above represents a function in the following sense.

- The dots on the left denote the elements of the domain.
- The dots on the right denote the elements of the codomain.
- An arrow from a left dot to a right dot indicates that the left dot is assigned the right dot.

Since every dot on the left is joined to exactly one dot on the right in the figure above, this function is well-defined.

Definition 6.1.19. Two functions $f\colon A \to B$ and $g\colon C \to D$ are *equal* if

(1) $A = C$ and $B = D$; and

(2) $f(x) = g(x)$ for all $x \in A$.

In this case, we write $f = g$.

Example 6.1.20. Let $f\colon \{0,2\} \to \mathbb{Z}$ and $g\colon \{0,2\} \to \mathbb{Z}$ defined by setting, for all $x \in \{0,2\}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = x^2.$$

Then $f = g$ because their domains are the same, their codomains are the same, and $f(x) = g(x)$ for every $x \in \{0,2\}$.

Example 6.1.21. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Q}$ defined by setting, for all $x \in \mathbb{Z}$,

$$f(x) = x^3 = g(x).$$

Then $f \neq g$ because they have different codomains.

Definition 6.1.22. Let $f\colon A \to B$ and $g\colon B \to C$. Then $g \circ f\colon A \to C$ such that for every $x \in A$,

$$(g \circ f)(x) = g(f(x)).$$

We read $g \circ f$ as "g composed with f", or "g circle f".

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"'] & & C \end{tikzcd}$$

Figure 6.1: Function composition

Note 6.1.23. For $g \circ f$ to be well-defined, the codomain of f must equal the domain of g.

Example 6.1.24. Let $f\colon A \to B$.

(1) $(f \circ \mathrm{id}_A)(x) = f(\mathrm{id}_A(x)) = f(x)$ for all $x \in A$. So $f \circ \mathrm{id}_A = f$.

(2) $(\mathrm{id}_B \circ f)(x) = \mathrm{id}_B(f(x)) = f(x)$ for all $x \in A$. So $\mathrm{id}_B \circ f = f$.

Example 6.1.25. Let $f, g\colon \mathbb{Z} \to \mathbb{Z}$ such that for every $x \in \mathbb{Z}$,

$$f(x) = 3x \quad \text{and} \quad g(x) = x + 1.$$

Then for every $x \in \mathbb{Z}$,

$$(g \circ f)(x) = g(f(x)) = g(3x) = 3x + 1 \quad \text{and} \quad (f \circ g)(x) = f(g(x)) = f(x + 1) = 3(x + 1).$$

Note $(g \circ f)(0) = 1 \neq 3 = (f \circ g)(0)$.

$$\begin{tikzcd}
& x \arrow[rr, mapsto, "f"] & & 3x \arrow[dd, mapsto, "g"] \\
x \arrow[dd, mapsto, "g"'] & \mathbb{Z} \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{Z} \arrow[d, "g"] & \\
& \mathbb{Z} \arrow[r, "f"'] & \mathbb{Z} & 3x+1 \\
x+1 \arrow[rr, mapsto, "f"'] & & 3(x+1) &
\end{tikzcd}$$

Figure 6.2: A non-commuting square

Theorem 6.1.26 (associativity of function composition)**.** Let $f\colon A \to B$ and $g\colon B \to C$ and $h\colon C \to D$. Then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, dashed, "g \circ f"'] & B \arrow[d, "g"] \arrow[dr, dashed, "h \circ g"] & \\
& C \arrow[r, "h"'] & D
\end{tikzcd}$$

Figure 6.3: All paths from A to D are the same

Proof.
1. The domains of $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are both A.
2. The codomains of $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are both D.
3. For every $x \in A$,

$$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x))) = h((g \circ f)(x)) = (h \circ (g \circ f))(x). \quad \square$$

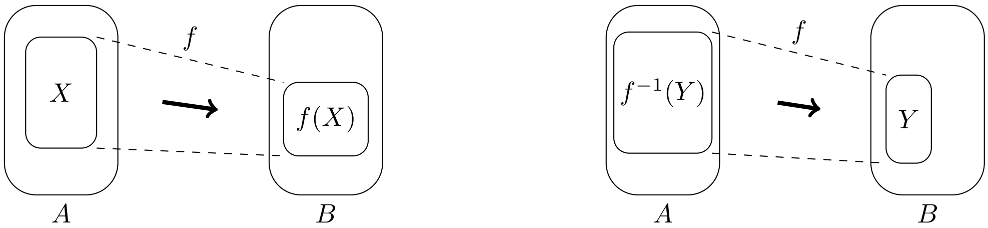


Figure 6.4: Setwise image and setwise preimage

6.2 Injectivity and surjectivity

Definition 6.2.1. Let $f\colon A \to B$.

(1) If $X \subseteq A$, then let $f(X) = \{y \in B : y = f(x) \text{ for some } x \in X\} = \{f(x) : x \in X\}$.

(2) If $Y \subseteq B$, then let $f^{-1}(Y) = \{x \in A : y = f(x) \text{ for some } y \in Y\}$.

We call $f(X)$ the *(setwise) image* of X, and $f^{-1}(Y)$ the *(setwise) preimage* of Y under f.

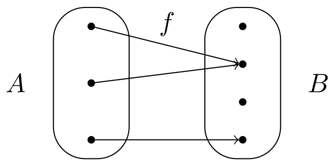
Remark 6.2.2. If $f\colon A \to B$, then $f(A)$ is the range/image of f.

Example 6.2.3. Define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$.

(1) If $X = \{-1, 0, 1\}$, then $g(X) = \{g(-1), g(0), g(1)\} = \{1, 0, 1\} = \{0, 1\}$.

(2) If $Y = \{0, 1, 2\}$, then $g^{-1}(Y) = \{0, -1, 1\}$.

Note 6.2.4. In general, we cannot make f^{-1} operate on elements instead of subsets.



Definition 6.2.5. Let $f\colon A \to B$.

(1) f is *surjective* or *onto* if
$$\forall y \in B \;\; \exists x \in A \;\; (y = f(x)).$$
A *surjection* is a surjective function.

(2) f is *injective* or *one-to-one* if
$$\forall x, x' \in A \;\; (f(x) = f(x') \Rightarrow x = x').$$
An *injection* is an injective function.

(3) f is *bijective* if it is both surjective and injective, i.e.,
$$\forall y \in B \;\; \exists! x \in A \;\; (y = f(x)).$$
A *bijection* is a bijective function.

Example 6.2.6. The function $f\colon \mathbb{Q} \to \mathbb{Q}$, defined by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$, is surjective.

Proof. 1. Take any $y \in \mathbb{Q}$.
2. Let $x = (y - 1)/3$.

3. Then $x \in \mathbb{Q}$ and $f(x) = 3x + 1 = y$. ☐

Remark 6.2.7. (1) A function is surjective if and only if its codomain is equal to its range.

(2) A function $f \colon A \to B$ is *not* surjective if and only if

$$\exists y \in B \ \ \forall x \in A \ \ (y \neq f(x)).$$

Example 6.2.8. As in Example 6.2.3, define $g \colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not surjective.

Proof. 1. Note $g(x) = x^2 \geqslant 0 > -1$ for all $x \in \mathbb{Z}$.
2. So $g(x) \neq -1$ for all $x \in \mathbb{Z}$, although $-1 \in \mathbb{Z}$. ☐

Example 6.2.9. As in Example 6.2.6, define $f \colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$. Then f is injective.

Proof. 1. Let $x, x' \in \mathbb{Q}$ such that $f(x) = f(x')$.
2. Then $3x + 1 = 3x' + 1$.
3. So $x = x'$. ☐

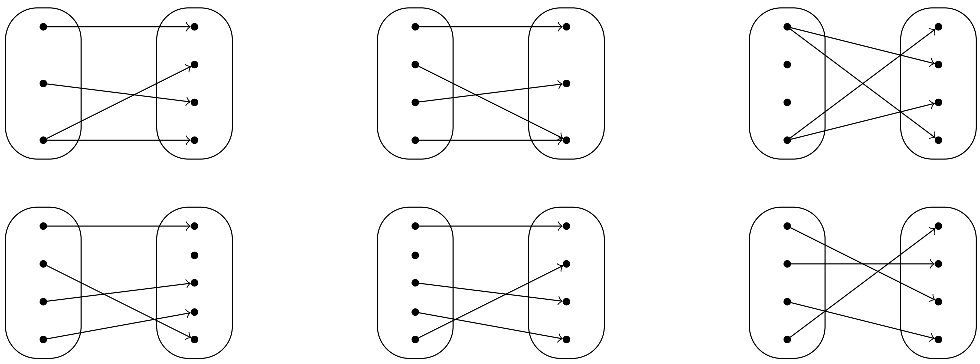
Remark 6.2.10. A function $f \colon A \to B$ is *not* injective if and only if

$$\exists x, x' \in A \ \ (f(x) = f(x') \wedge x \neq x').$$

Example 6.2.11. As in Example 6.2.3, define $g \colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not injective.

Proof. Note $g(1) = 1^2 = 1 = (-1)^2 = g(-1)$, although $1 \neq -1$. ☐

Question 6.2.12. Which of the arrow diagrams below represent a function from the LHS set to the RHS set? Amongst those that represent a function, which ones represent injections, which ones represent surjections, and which ones represent bijections? 6f



Definition 6.2.13. Let $f \colon A \to B$. Then $g \colon B \to A$ is an *inverse* of f if

$$\forall x \in A \ \ \forall y \in B \ \ \big(y = f(x) \quad \Leftrightarrow \quad x = g(y)\big).$$

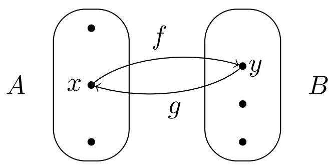


Figure 6.5: An inverse of a function

Example 6.2.14. As in Example 6.2.9, define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$. Note that for all $x, y \in \mathbb{Q}$,

$$y = 3x + 1 \quad \Leftrightarrow \quad x = (y - 1)/3.$$

Let $g\colon \mathbb{Q} \to \mathbb{Q}$ such that $g(y) = (y - 1)/3$ for all $y \in \mathbb{Q}$. Then the equivalence above tells us

$$\forall x, y \in \mathbb{Q} \ \ (y = f(x) \Leftrightarrow x = g(y)).$$

So g is an inverse of f.

Note 6.2.15. We have no guarantee of a description of an inverse of a general function that is much different from what is given by the definitions.

Proposition 6.2.16 (uniqueness of inverses)**.** If g, g' are inverses of $f\colon A \to B$, then $g = g'$.

Proof.
1. Note $g, g'\colon B \to A$.
2. Since g, g' are inverses of f, for all $x \in A$ and all $y \in B$,

$$x = g(y) \quad \Leftrightarrow \quad y = f(x) \quad \Leftrightarrow \quad x = g'(y).$$

3. So $g = g'$. $\square$

Definition 6.2.17. The inverse of a function f is denoted f^{-1}.

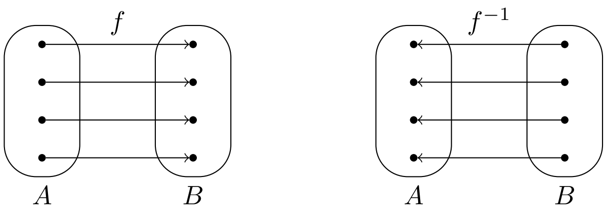


Figure 6.6: A bijective function and its inverse

Theorem 6.2.18. A function $f\colon A \to B$ is bijective if and only if it has an inverse.

Proof.
1. ("If")
   1.1. Suppose f has an inverse, say $g\colon B \to A$.
   1.2. We first show injectivity.
      1.2.1. Let $x, x' \in A$ such that $f(x) = f(x')$.
      1.2.2. Define $y = f(x) = f(x')$.
      1.2.3. Then $x = g(y)$ and $x' = g(y)$ as g is an inverse of f.
      1.2.4. Thus $x = x'$.
   1.3. Next we show surjectivity.
      1.3.1. Let $y \in B$.
      1.3.2. Define $x = g(y)$.
      1.3.3. Then $y = f(x)$ as g is an inverse of f.
2. ("Only if")
   2.1. Suppose f is bijective.
   2.2. Then $\forall y \in B \ \ \exists! x \in A \ \ (y = f(x))$.
   2.3. Define the function $g\colon B \to A$ by setting $g(y)$ to be the unique $x \in A$ such that $y = f(x)$ for all $y \in B$.
   2.4. This g is well-defined and is an inverse of f by the definition of inverse functions.

$\square$

Note 6.2.19. Let $f\colon A \to B$.

(1) If $X \subseteq A$, then $f(X) = \{f(x) : x \in X\}$, which is a set. If $x \in A$, then $f(x) \in B$.

(2) If $Y \subseteq B$, then $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$, which exists even when the inverse function f^{-1} does not. If $y \in B$ and f^{-1} exists, then $f^{-1}(y) \in A$.

Question 6.2.20. As in Example 6.2.3, define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Which of the following are true statements? 6g

- $g(0) = 0$.
- $g(0) = \{0\}$.
- $g(\{0\}) = 0$.
- $g(\{0\}) = \{0\}$.
- $g^{-1}(0) = 0$.
- $g^{-1}(0) = \{0\}$.
- $g^{-1}(\{0\}) = 0$.
- $g^{-1}(\{0\}) = \{0\}$.

6.3 Cardinality

Definition 6.3.1 (Cantor). (1) Two set A, B are said to have the *same cardinality* if there is a bijection $A \to B$.

(2) A set is *countable* if it is finite or it has the same cardinality as $\mathbb{Z}_{\geqslant 0}$.

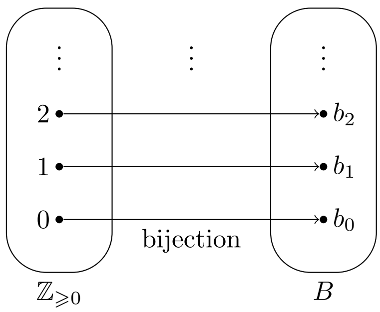


Figure 6.7: A countable set B

Note 6.3.2. An infinite set B is countable if and only if

there is a sequence $b_0, b_1, b_2, \ldots \in B$ in which every element of B appears exactly once.

Example 6.3.3. (1) $\mathbb{Z}_{\geqslant 0}$ is countable because each element of $\mathbb{Z}_{\geqslant 0}$ is listed exactly once in the sequence

$$0, 1, 2, 3, 4, \ldots.$$

(2) The set $E = \{2x : x \in \mathbb{Z}_{\geqslant 0}\}$ is countable because each element of E is listed exactly once in the sequence

$$0, 2, 4, 6, 8, 10, 12, \ldots.$$

Note that $E \subsetneq \mathbb{Z}_{\geqslant 0}$, but E and $\mathbb{Z}_{\geqslant 0}$ have the same cardinality.

(3) $\mathbb{Z}$ is countable because each element of $\mathbb{Z}$ is listed exactly once in the sequence

$$0, 1, -1, 2, -2, 3, -3, \ldots .$$

Note that $\mathbb{Z}$ is the union of two disjoint infinite sets $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Z}^-$, but $\mathbb{Z}$ and $\mathbb{Z}_{\geqslant 0}$ have the same cardinality.

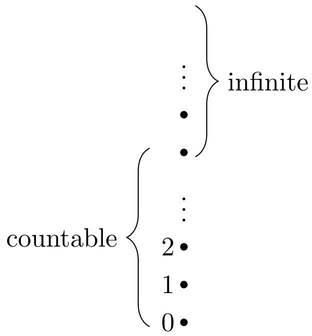


Figure 6.8: The smallest cardinalities

Proposition 6.3.4. Any subset A of a countable set B is countable.

Proof. 1. If A is finite, then A is countable by definition.

2. So suppose A is infinite.
   2.1. Then B is infinite too as $A \subseteq B$.
   2.2. Use the countability of B to find a sequence $b_0, b_1, b_2, \ldots$ in which every element of B appears exactly once.
   2.3. Taking away those terms in the sequence that are not in A, we are left with a subsequence in which every element of A appears exactly once.
   2.4. So A is countable. □

Proposition 6.3.5. Every infinite set B has a countable infinite subset.

Proof. 1. Keep choosing elements $b_0, b_1, b_2, \ldots$ from B. When we choose b_n, where $n \in \mathbb{Z}_{\geqslant 0}$, we can always make sure $b_n \neq b_i$ for any $i < n$, because otherwise B is equal to the finite set $\{b_0, b_1, \ldots, b_{n-1}\}$, which is a contradiction.

2. The result is a countable infinite set $\{b_0, b_1, b_2, \ldots\} \subseteq B$. □

Theorem 6.3.6 (Cantor 1877)**.** $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is countable.

Proof sketch.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
(0,4) \arrow[u] & (1,4) \arrow[l] & (2,4) \arrow[l] & (3,4) \arrow[l] & (4,4) \arrow[l] & \cdots \\
(0,3) \arrow[r] & (1,3) \arrow[r] & (2,3) \arrow[r] & (3,3) \arrow[d] & (4,3) \arrow[u] & \cdots \\
(0,2) \arrow[u] & (1,2) \arrow[l] & (2,2) \arrow[l] & (3,2) \arrow[d] & (4,2) \arrow[u] & \cdots \\
(0,1) \arrow[r] & (1,1) \arrow[d] & (2,1) \arrow[u] & (3,1) \arrow[d] & (4,1) \arrow[u] & \cdots \\
(0,0) \arrow[u] & (1,0) \arrow[r] & (2,0) \arrow[u] & (3,0) \arrow[r] & (4,0) \arrow[u] & \cdots
\end{tikzcd}$$

The figure above describes a sequence in which every element of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ appears exactly once. $\square$

Corollary 6.3.7. $\mathbb{Q}$ is countable.

Proof sketch. We can view $\frac{m}{n} \in \mathbb{Q}$ as the pair (m, n), where $n \neq 0$ and $\gcd(m, n) = 1$. $\square$

Theorem 6.3.8 (Cantor 1891)**.** Let A be a countable infinite set. Then $\mathcal{P}(A)$ is not countable.

Proof. We prove this by contradiction.

1. Suppose $\mathcal{P}(A)$ is countable.
2. We know $\mathcal{P}(A)$ is infinite because A is infinite and $\{a\} \in \mathcal{P}(A)$ for every $a \in A$.
3. Use the countability of $\mathcal{P}(A)$ to find a sequence $B_0, B_1, B_2, \ldots \in \mathcal{P}(A)$ in which every element of $\mathcal{P}(A)$ appears exactly once.
4. Use the countability of A to find a sequence $a_0, a_1, a_2, \ldots \in A$ in which every element of A appears exactly once.
5. Define $B = \{a_i : a_i \notin B_i\}$.
6. Note that $B \subseteq A$ since $a_0, a_1, a_2, \ldots \in A$.
7. 7.1. Let $i \in \mathbb{Z}_{\geqslant 0}$.
   7.2. If $a_i \notin B_i$, then $a_i \in B$ by the definition of B.
   7.3. if $a_i \in B_i$, then $a_i \notin B$ by the definition of B because no $j \neq i$ makes $a_j = a_i$ by the choice of $a_0, a_1, a_2, \ldots$.
   7.4. In either case, we know $B \neq B_i$.
8. This contradicts line 3 that every element of $\mathcal{P}(A)$ appears in $B_0, B_1, B_2, \ldots$. $\square$

$$
\begin{array}{ccccccc}
 & a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\
B_0 & \boxed{\notin} & \in & \notin & \notin & \notin & \cdots \\
B_1 & \in & \boxed{\notin} & \in & \notin & \in & \cdots \\
B_2 & \notin & \in & \boxed{\in} & \notin & \in & \cdots \\
B_3 & \notin & \notin & \in & \boxed{\notin} & \notin & \cdots \\
B_4 & \in & \notin & \in & \in & \boxed{\notin} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
B & \in & \in & \notin & \in & \in & \cdots
\end{array}
$$

Figure 6.9: Illustration of Cantor's diagonal argument

Corollary 6.3.9. $\mathbb{R}$ is not countable.