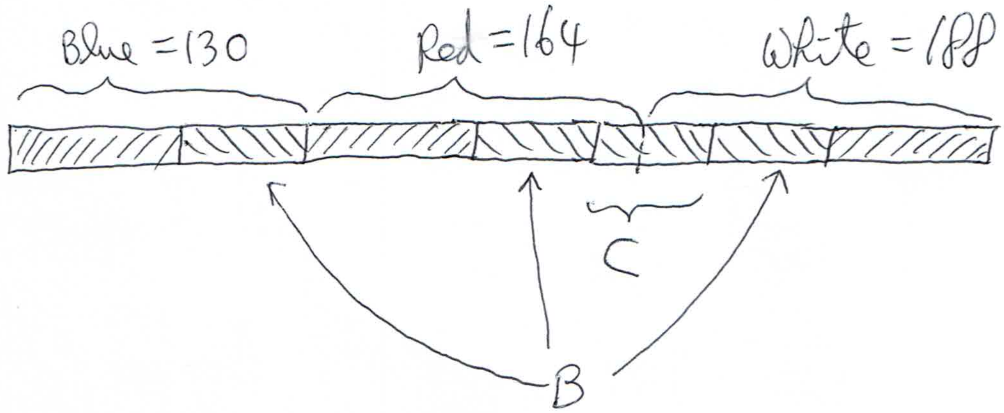
1. A has 130 blue marbles, 164 red marbles and 188 white marbles. A first gives B n blue marbles, n red marbles and n white marbles. A then gives C some red and some white marbles (not necessary the same number) so that the total number of marbles that A gives C is equal to n. After this A has the same number of blue, red and white marbles left. Find the total number of marbles that A has left.

Answer 114

This is a primary school problem, we use primary school method to solve it (i.e. use a model).



C has $188 + 164 - 130 - 130 = 92$

A is left with $(130 - 92) \times 3 = \underline{\underline{114}}$

2. Let a denote a positive constant. A bathtub has a hot water tap and a cold water tap on top and a drain at the bottom. When both the hot water tap and the drain are closed the cold water tap can fill an empty tub in a minutes. When both the cold water tap and the drain are closed the hot water tap can fill an empty tub in 15 minutes. When both hot and cold water taps are closed, the drain can drain a full tup of water empty in 21 minutes. If when both hot and cold water taps are turned on and the drain is open the empty tub gets fill up with water in 20.21 minutes, find the value of a. Give your answer correct to two decimal places.

Answer 32.86

Let V = volume of bathtub.

$\therefore$ input rate of cold water tap $= V/a$

input rate of hot water tap $= V/15$

output rate of drain $= V/21$

$\therefore$ net input rate $= \frac{V}{a} + \frac{V}{15} - \frac{V}{21}$

$$\therefore \frac{V}{\frac{V}{a} + \frac{V}{15} - \frac{V}{21}} = 20.21$$

$$\frac{1}{20.21} = \frac{1}{a} + \frac{1}{15} - \frac{1}{21}$$

$$\frac{1}{a} = \frac{1}{20.21} + \frac{1}{21} - \frac{1}{15} = \frac{315 + 303.15 - 424.41}{20.21 \times 21 \times 15}$$

$$\therefore a = \frac{20.21 \times 21 \times 15}{193.74} = 32.8592\cdots \simeq \underline{\underline{32.86}}$$

3. Let S denote the set of all postive integers x which satisfy all the following three conditions:
(i) $1 \leq x \leq 1521$,
(ii) x leaves a remainder equal to 2 when x is divided by 7,
(iii) x leaves a remainder equal to 3 when x is divided by 8.
Find the sum of all the elements in S.

Answer 21033

Let m = smallest element in S.

(iii) $\Rightarrow$ the possible values of m are:

$$3,\ 11,\ 19,\ 27,\ 35,\ 43,\ 51,$$

remainder when divided by 7 $\quad 3,\ 4,\ 5,\ 6,\ 0,\ 1,\ 2,$

$\therefore$ (ii) $\Rightarrow m = 51$.

Let n = any element in S.

(ii) $\Rightarrow n = 7k+2 \Rightarrow n-m = 7k-49 = 7(k-7)$
$\Rightarrow n-m$ divisible by 7.

Similarly $n-m$ divisible by 8

$\because$ 7 and 8 have no common factors

$\therefore$ $n-m$ divisible by $7 \times 8 = 56$

$\therefore$ S is an arithmetic progression: $51+56(k-1)$
$k = 1, 2, 3, \dots$

$51+56(k-1) < 1521 \Rightarrow k < 27.25$
$\Rightarrow k = 27$

$\therefore$ Sum $= \dfrac{51+\{51+56\times(27-1)\}}{2} \times 27 = \underline{\underline{21033}}$

4. You have some green marbles and some yellow marbles in a box. It is known that the total number of marbles in the box is less than 1600. If the probability that two marbles are taken out from the box at random are of the same colour is equal to $\frac{1}{2}$, find the largest possible number of green marbles that you can have in the box.

Answer 780

Let ${}^nC_r$ denote the binomial coefficient $\binom{n}{r}$.

i.e. $${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

Let g = number of green marbles

y = number of yellow marbles

$$\therefore \frac{{}^gC_2 + {}^yC_2}{{}^{(g+y)}C_2} = \frac{1}{2}$$

$$\therefore \frac{g(g-1) + y(y-1)}{(g+y)(g+y-1)} = \frac{1}{2}$$

$$2g^2 - 2g + 2y^2 - 2y = g^2 + 2gy + y^2 - g - y$$

$$g^2 - 2gy + y^2 = g + y < 1600$$

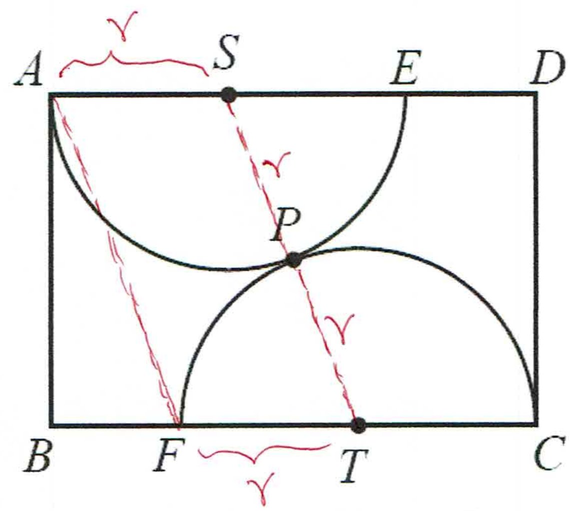
$$\therefore (g-y)^2 = g + y < 40^2$$

g largest possible $\Rightarrow g - y = 39$

$$\therefore g + y = 39^2 = 1521$$

$$g = \frac{1521 + 39}{2} = \underline{\underline{780}}$$

5. In the diagram below $ABCD$ is a rectangle. The semi-circles with diameters AE and FC each have radius r, have centres S and T, and touch at a single point P, as shown. If $AB = 1521$ and $BC = 2021$, find the value of r. Give your answer correct to two decimal places.



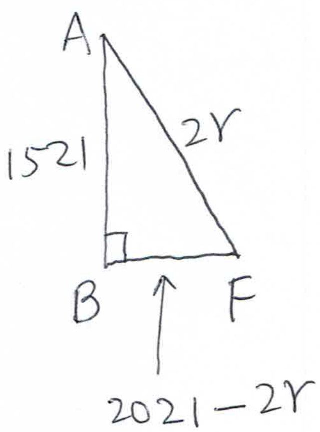
Answer 791.43

Observe that SPT is a straight line.

$\because AS // FT$ and $AS = FT$

$\therefore AFTS$ is a parallelogram

$\therefore AF = ST = 2r$



$$\therefore 1521^2 + (2021-2r)^2 = (2r)^2$$

$$1521^2 + 2021^2 - 8084r + 4r^2 = 4r^2$$

$$\therefore r = \frac{1521^2 + 2021^2}{8084}$$

$$= \frac{2313441 + 4084441}{8084} = 791.4252\ldots$$

$$\approx \underline{\underline{791.43}}$$