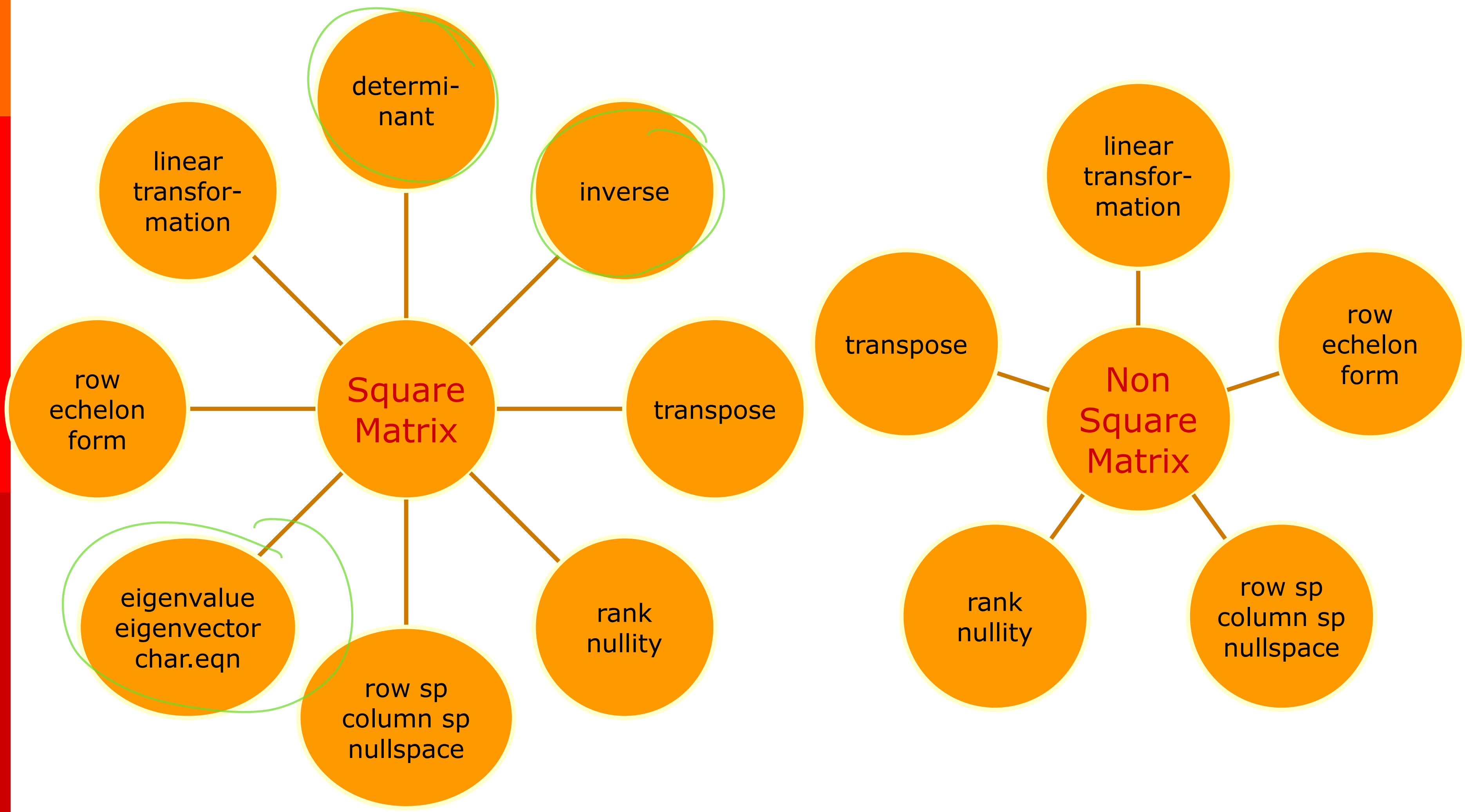
Revision Lecture

Summary

Concepts from Chapters 1 - 7

Matrix & Associated Terminologies



Many faces of linear system

Standard form

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{cases}$$

Augmented matrix form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

(A | b)

Vector equation form

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \mathbf{b}$$

Matrix equation form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

same as linear combinations of columns of A

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \boldsymbol{Ax}$$

column space of $\boldsymbol{AB} \subseteq$ column space of $\boldsymbol{A}$

Matrix multiplication by columns

$\boldsymbol{A}$: m × n $\qquad \boldsymbol{A} = (\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \dots \;\; \boldsymbol{a}_n)$

$\boldsymbol{x}$: n × 1 $\qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$\boldsymbol{Ax} = x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n}$ $\in$ column space of $\boldsymbol{A}$

a linear combination of the columns of $\boldsymbol{A}$

$\boldsymbol{B}$: n × k $\qquad \boldsymbol{B} = (\boldsymbol{b}_1 \;\; \boldsymbol{b}_2 \;\; \dots \;\; \boldsymbol{b}_k)$

$\boldsymbol{AB} = (\boldsymbol{Ab}_1 \;\; \boldsymbol{Ab}_2 \;\; \dots \;\; \boldsymbol{Ab}_k)$

columns of $\boldsymbol{AB}$ in terms of columns of $\boldsymbol{B}$

each column of $\boldsymbol{AB}$ is a linear combination of columns of $\boldsymbol{A}$

Matrix multiplication by columns

$\boldsymbol{B} = (\boldsymbol{b}_1 \;\; \boldsymbol{b}_2 \;\; \ldots \;\; \boldsymbol{b}_k) \qquad \boldsymbol{AB} = (\boldsymbol{Ab}_1 \;\; \boldsymbol{Ab}_2 \;\; \ldots \;\; \boldsymbol{Ab}_k)$

Suppose $\boldsymbol{AB} = \mathbf{0}$.

can use when the product of 2 matrix is 0

$\Rightarrow (\boldsymbol{Ab}_1 \;\; \boldsymbol{Ab}_2 \;\; \ldots \;\; \boldsymbol{Ab}_k) = \mathbf{0}$

$\Rightarrow \boldsymbol{Ab}_1 = \mathbf{0},\;\; \boldsymbol{Ab}_2 = \mathbf{0}, \ldots, \boldsymbol{Ab}_k = \mathbf{0}$

$\Rightarrow \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k \in$ nullspace of $\boldsymbol{A}$

$\Rightarrow$ column space of $\boldsymbol{B} \subseteq$ nullspace of $\boldsymbol{A}$

There could be alternative methods to solve these problems.

What’s the use of G.E. (G.J.E.)?

1. Solve linear system
2. Find inverse of a matrix
3. Find determinant of a matrix
4. Find linear combination of a vector
5. Check linear independence of a set of vectors
6. Check whether a set of vectors spans a subspace
7. Find coordinate vectors (w.r.t. a basis)
8. Find basis for row/column space of a matrix
9. Find rank of a matrix
10. Find eigenvector of a matrix
11. Find formula for a linear transformation

What do row operations preserve?

Row equivalent matrices $\mathbf{A} \longrightarrow \longrightarrow \longrightarrow \mathbf{B}$

Preserve

- Solutions
  (of augmented matrix)
- Invertibility
- Row space
- Linear relations among columns
- Nullspace
- Rank/ Nullity

Do not preserve

- Transpose
- Column space
- Eigenvalues/ eigenvectors/ characteristic polynomials
- Determinant

but there are ways to make them related

A and **B** are row equiv.
does not mean
A and **B** are equal

Some subspaces of $\mathbf{R}^n$

- Span $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$
- Euclidean space $\mathbf{R}^n$
- Solution space of $\mathbf{Ax} = \mathbf{0}$ (homogenous system)
- Row space/ column space/ nullspace of a matrix **A**
- Eigenspace associated with eigenvalue λ of a matrix **A**
- A line in $\mathbf{R}^2$ and $\mathbf{R}^3$ that passes through the origin
- A plane in $\mathbf{R}^3$ that contains the origin
- Kernel ker(T) of a linear transformation T
- Range R(T) of a linear transformation T

Different representation of same subspace

Subspace of dimension 1

Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in \mathbf{R}^n$ $\quad \boldsymbol{v} \neq \mathbf{0}$

can add in linearly dependent vectors

- $\mathrm{Span}\{\boldsymbol{v}\} = \mathrm{Span}\{\boldsymbol{v}, 2\boldsymbol{v}\} = \mathrm{Span}\{\boldsymbol{v}, 2\boldsymbol{v}, 5\boldsymbol{v}\}$
- $\{ c\boldsymbol{v} \mid c \in \mathbf{R} \}$
- $\{(kv_1, kv_2, \ldots, kv_n) \mid k \in \mathbf{R} \}$

Explicit form

In $\mathbf{R}^2$ and $\mathbf{R}^3$

- A line that passes through the origin and parallel to $\boldsymbol{v}$

In $\mathbf{R}^2$

- $\{ (x, y) \mid ax + by = 0 \}$ a, b depend on $\boldsymbol{v}$

Implicit form

Different representation of same subspace

Subspace of dimension 2

Let $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbf{R}^n$ $\boldsymbol{u}, \boldsymbol{v} \neq \mathbf{0}$, not parallel to each other

- $\text{Span}\{\boldsymbol{u}, \boldsymbol{v}\} = \text{Span}\{3\boldsymbol{u}, 2\boldsymbol{v}\} = \text{Span}\{\boldsymbol{u}-\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}\}$

This is spanning 2 vectors

$\neq \text{Span}\{\boldsymbol{u}+\boldsymbol{v}\}$?

This is only spanning 1 vector

- $\{ c\boldsymbol{u} + d\boldsymbol{v} \mid c, d \in \mathbf{R} \}$

In $\mathbf{R}^3$

- A plane that contains the origin and the vectors $\boldsymbol{u}$, $\boldsymbol{v}$
- $\{ (x, y, z) \mid ax + by + cz = 0 \}$ a, b, c depend on $\boldsymbol{u}$, $\boldsymbol{v}$

Ways to check linear independence

$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

Standard Method:

Trying to find if only the trivial solution exists

- Form the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_n\mathbf{v}_n = 0.$

  Check whether $c_1 = 0$, $c_2 = 0$, ... $c_n = 0$ is the unique solution.

- Use the column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to form matrix $\mathbf{A}$

  *Check whether every column in the r.e.f. of **A** is a pivot column.*

- Use the row vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to form matrix $\mathbf{A}$

  *Check whether every row in the r.e.f. of **A** is non-zero.*

Ways to check linear independence

$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

Special Methods: (only work under certain circumstances)

- Use the column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to form a matrix $\mathbf{A}$.

  *If **A** is a square matrix, check whether det(**A**)* $\boxed{= 0.}$

- If there are only two vectors $\mathbf{v}_1, \mathbf{v}_2$ in the set,

  check that v_1, v_2 are $\boxed{\textit{scalar multiple of each other.}}$

- If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbf{R}^m$ and $n > m$,

  then the set is $\boxed{\textit{linearly dependent.}}$

  Converse is not true!

- If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is orthogonal and non-zero,

  then the set is $\boxed{\textit{linearly independent.}}$

Ways to check a set spans a vector space

Given S = $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$

Show

a. Span(S) = Span(T)

b. Span(S) = $\mathbf{R}^n$

c. Span(S) = W, a subspace of $\mathbf{R}^n$

Ways to check a set spans a vector space

Span(S) ⊆ W
W ⊆ Span(S)

Show that Span(S) = W, a subspace of $\mathbf{R}^n$

- Check each $\mathbf{s}_i \in W$
  - This will imply span(S) ⊆ W
- Write $W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$
- Check $(\mathbf{s}_1\ \mathbf{s}_2 \ldots \mathbf{s}_m \,|\mathbf{u}_1|\mathbf{u}_2|\ldots|\mathbf{u}_k)$ is consistent
  - This will imply W ⊆ span(S)

If dim(W) = m and S has m vectors,

- Check S is linearly independent
- Check each $\mathbf{s}_i \in W$
  - This will imply S is a basis for W
  - This will imply span(S) = W

Some facts about basis for a vector space

- Basis is not unique.
- Any non-zero vector in the subspace can be a member of a basis.

extending a basis

- Different bases for same subspace have same number of vectors.
- A basis is the largest set of linearly independent vectors in the subspace.
- A basis is the smallest set of vectors that can span the subspace.

directly related to the dimension

- Every vector can be written as a linear combination of the basis in a unique way.

Ways to find basis for a subspace V

V: Implicit form

same as solution space

derive explicit form
for general vector

usually need to
solve linear system

V: explicit form

separate the parameters

V: span {$\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ }

throw out redundant vectors

use row/column
space method

Get a basis for V

Finding bases for Well known subspaces of $\mathbf{R}^n$

- Span $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ — use row or column space method
- Euclidean space $\mathbf{R}^n$ — just use the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$
- Solution space of $\mathbf{Ax} = \mathbf{0}$ — use GE and separate parameters
- Row space/ column space of a matrix $\mathbf{A}$ — use GE to get REF
- nullspace of a matrix $\mathbf{A}$ — same as solution space of $\mathbf{Ax} = \mathbf{0}$
- Eigenspace associated with eigenvalue λ of a matrix $\mathbf{A}$ — use GE to solve $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$
- A line in $\mathbf{R}^2$ and $\mathbf{R}^3$ that passes through the origin — just find a nonzero vector on the line
- A plane in $\mathbf{R}^3$ that contains the origin — just find two non-parallel vectors on the plane
- Kernel ker(T) of a linear transformation T — same as nullspace of the standard matrix
- Range R(T) of a linear transformation T — same as column space of the standard matrix

Ways to check a set S is a basis for a vector space V

Check:
- S is linearly indep
- span(S) = V

Check:
- dim V = |S|
- S is linearly indep
- S ⊆ V

Check:
- dim V = |S|
- span(S) = V

The rest require the DIM to make life easier

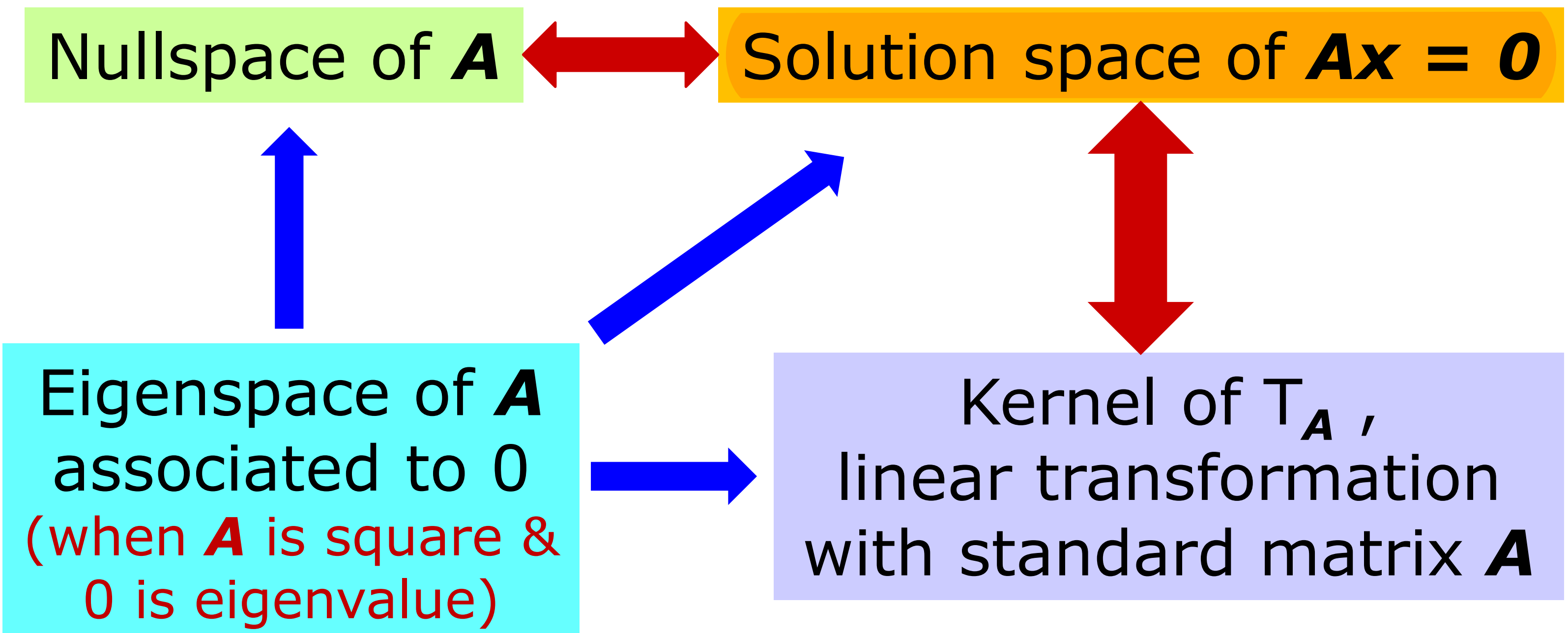
Check:
- dim V = |S|
- S is orthogonal
- S ⊆ V

→ orthogonal basis for V

Rank of a matrix A

- Dimension of row/ column space of **A**.
- No. of non-zero rows in r.e.f. of **A**
- No. of pivot columns in r.e.f. of **A**
- No. of columns – nullity of **A**
- Max. number of linearly independent columns in **A**.
- Max. number of linearly independent rows in **A**.

Nullspace of a matrix



Orthogonal basis

Basis — Linear independent + span

Gram-Schmidt

Orthogonal Basis — basis + orthogonal

Normalizing

Orthonormal Basis — orthogonal basis + norm 1

Applications

- Find coordinate vectors
- Find projection onto a subspace
- Find transition matrix
- Give orthogonal matrix

Ways to find coordinate vectors

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ basis

$\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_n$ $\qquad (\boldsymbol{v})_S = (c_1\ c_2\ \ldots\ c_n)$

- Use Gaussian elimination
  - Convert vector equation into linear system
- Use orthogonal basis
  - If S is orthogonal, c_i can be found using dot product
- Use transition matrix
  - Transform from one coordinate to another

Finding Least Squares solutions

If a linear system $\mathbf{Ax} = \mathbf{b}$ is consistent

- A least squares solution $\mathbf{x}_0$ of the system is an actual solution of $\mathbf{Ax} = \mathbf{b}$ itself

If a linear system $\mathbf{Ax} = \mathbf{b}$ is inconsistent

- A least squares solution $\mathbf{x}_0$ of the system is given by the actual solution of $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$

A least squares solution $\mathbf{x}_0$ of $\mathbf{Ax} = \mathbf{b}$:

- is the best approximation to a solution of the system $\quad \mathbf{Ax}_0 \underset{\text{closest}}{\approx} \mathbf{b} \quad \underset{\text{smallest}}{||\mathbf{Ax}_0 - \mathbf{b}||}$
- $\mathbf{Ax}_0$ = projection of $\mathbf{b}$ onto column space of $\mathbf{A}$

$\mathbf{x}_0 \neq$ projection of $\mathbf{b}$

Ways to find projection onto subspaces

Project $\mathbf{v}$ onto subspace $\mathbf{W}$

- If you have an orthonormal basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r\}$ for W
  - Use the formula

$$\boxed{\text{Projection} = (\boldsymbol{v} \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 + (\boldsymbol{v} \cdot \boldsymbol{u}_2)\boldsymbol{u}_2 + \ldots + (\boldsymbol{v} \cdot \boldsymbol{u}_r)\boldsymbol{u}_r}$$

- If you do not have an orthonormal basis for W
  - Find any basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r\}$ for W
  - Form matrix $\mathbf{A}$ using $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ as column vectors
  - Solve the system $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{v}$, say $\mathbf{x}_0$

$$\boxed{\text{projection} = \mathbf{A}\mathbf{x}_0}$$

least squares solution of $\mathbf{Ax} = \mathbf{v}$

Properties of an orthogonal matrix A

no such thing as orthonormal matrix

- $\mathbf{A}\mathbf{A}^T = \mathbf{I}$
- $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \boxed{\mathbf{A}^T}$
- The rows of **A** form an orthonormal basis
- The columns of **A** form an orthonormal basis
- The transition matrix between two orthonormal bases is an orthogonal matrix

D = PAP^T

Ways to find eigenvalues

- Solve $\boxed{\text{characteristic}}$ equation $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$
- If an eigenvector **u** is given, multiply it by the matrix: $\mathbf{A}\mathbf{u} = \lambda \mathbf{u}$
- If the matrix is triangular, take the $\boxed{\text{diagonal}}$ entries
- If you are given the diagonalization of $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, then eigenvalues are given by the $\boxed{\text{diagonals}}$ of **D**.
- If λ is an eigenvalue of **A**, then
  - $\boxed{\lambda}$ is an eigenvalue of $\mathbf{A}^T$
  - $\boxed{\lambda^n}$ is an eigenvalue of $\mathbf{A}^n$
  - $\boxed{\lambda^{-1}}$ is an eigenvalue of $\mathbf{A}^{-1}$ (when **A** is invertible)

Ways to find eigenvectors

- Solve the homogeneous system $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$
- Look for vectors $\mathbf{u}$ such that $\mathbf{Au} = k\mathbf{u}$
- If you are given the diagonalization of $\mathbf{A} = \mathbf{PDP}^{-1}$, then eigenvectors are given by the columns of $\mathbf{P}$.
- If you are given the eigenspace E_λ, any nonzero vector in it is an eigenvector.
- If $\mathbf{u}$ is an eigenvector w.r.t. λ, then $\boxed{k\mathbf{u}}$ is also an eigenvector w.r.t. λ, for any $k \neq 0$.
- If $\mathbf{u}, \mathbf{v}$ are eigenvectors w.r.t. λ, then $\boxed{s\mathbf{u} + t\mathbf{v}}$ is also an eigenvector w.r.t. λ, for any s, t not both 0.

Diagonalization of a matrix

Diagonalizable Matrix **A**

$\mathbf{P^{-1}AP = D}$ (diagonal)

n linear independence eigenvectors

$\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$

$$\boldsymbol{P} = (\boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \;\; \dots \;\; \boldsymbol{v}_n)$$

Invertible matrix

multiplicity of λ_i

Characteristic polynomial

$$(x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2}\cdots(x-\lambda_k)^{r_k}$$

$$\mathbf{D} = \begin{pmatrix} \overbrace{\lambda_1 \;\ddots\; \lambda_1}^{r_1} & & & \\ & \overbrace{\lambda_2 \;\ddots\; \lambda_2}^{r_2} & & \\ & & \ddots & \\ & & & \overbrace{\lambda_k \;\ddots\; \lambda_k}^{r_k} \end{pmatrix}$$

Orthogonal symmetric

Diagonalization of a matrix

Symmetric Matrix **A**

$$P^{T}AP = D \text{ (diagonal)}$$

n orthonormal eigenvectors

$\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$

$$\boldsymbol{P} = (\boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \dots \ \boldsymbol{v}_n)$$

orthogonal matrix

multiplicity of λ_i

Characteristic polynomial

$$(x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2}\cdots(x-\lambda_k)^{r_k}$$

$$\mathbf{D} = \begin{pmatrix} \overbrace{\lambda_1 \ \ddots \ \lambda_1}^{r_1} & & & \\ & \overbrace{\lambda_2 \ \ddots \ \lambda_2}^{r_2} & & \\ & & \ddots & \\ & & & \overbrace{\lambda_k \ \ddots \ \lambda_k}^{r_k} \end{pmatrix}$$

When is an nxn matrix A diagonalizable?

- When **A** is a diagonal matrix
- When **A** is a symmetric matrix
- When **A** has n distinct eigenvalues

Sufficient conditions

- When **A** has n linearly independent eigenvectors
- When dim E_λ = multiplicity of λ for every eigenvalue λ of **A**

Equivalent conditions

To show that a matrix is not diagonalizable:

Find one eigenvalue such that

dim $E_{\lambda i}$ < multiplicity of λ_i

Char. Poly = $(x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2}\cdots(x-\lambda_k)^{r_k}$

Powers of square matrices

Diagonal matrix: $\mathbf{D}^k$ is easy to compute

Diagonalisable matrix: $\mathbf{A}^k$

- Find **P** such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ is diagonal.
- Compute $\mathbf{D}^k$.
- $\mathbf{D}^k = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^k = \mathbf{P}^{-1}\mathbf{A}^k\mathbf{P}$.
- $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\,\mathbf{P}^{-1}$.

tends to infinity
(in the long run)

Find eigenvalues: **D**
Find eigenspace: **P**
Find $\mathbf{P}^{-1}$
Multiply $\mathbf{P}\mathbf{D}^k\,\mathbf{P}^{-1}$

Completely determine a Linear transformation

$T: \mathbf{R}^n \rightarrow \mathbf{R}^m$ linear transformation

There are three ways to completely determine T

Formula of T is given

Images of a basis $T(u_1), T(u_2), \ldots T(u_n)$ is given

Standard matrix of T is given

Linear transformation vs Subspaces

$T: \mathbf{R}^n \to \mathbf{R}^m$
linear transformation

U is a subspace of $\mathbf{R}^n$

Linearity conditions

(i) T $\boxed{\text{preserves}}$ addition
Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$.
Then $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v})$.

(ii) T $\boxed{\text{preserves}}$ scalar mult
Let $\mathbf{u} \in \mathbf{R}^n$, $c \in \mathbf{R}$.
Then $T(c\mathbf{u}) = cT(\mathbf{u})$.

(iii) T $\boxed{\text{preserves}}$ zero vector
$T(\mathbf{0}) = \mathbf{0}$

If one of (i), (ii), (iii) is violated,
T is not a linear transformation

Closure Properties

(a) U is $\boxed{\text{closed}}$ under addition
Let $\mathbf{u}, \mathbf{v} \in U$.
Then $\mathbf{u}+\mathbf{v} \in U$.

(b) U is $\boxed{\text{closed}}$ under scalar mult
Let $\mathbf{u} \in U$, $c \in \mathbf{R}$.
Then $c\mathbf{u} \in U$.

(c) U $\boxed{\text{contains}}$ the zero vector
$\mathbf{0} \in U$

If one of (a), (b), (c) is violated,
U is not a subspace of $\mathbf{R}^n$

Linear transformation vs standard matrix

Linear Transformation	Standard matrix
$T: \mathbf{R}^n \rightarrow \mathbf{R}^m$ $S: \mathbf{R}^m \rightarrow \mathbf{R}^k$	$\mathbf{A}$ is an $m \times n$ matrix $\mathbf{B}$ is an $k \times m$ matrix
$T(\mathbf{u})$	$\mathbf{Au}$
$T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$	columns of $\mathbf{A}$
$R(T)$	column space of $\mathbf{A}$
$Ker(T)$	nullspace of $\mathbf{A}$
$S \circ T$	$\mathbf{BA}$

Finding range and kernel

To find R(T):

- $R(T) = \{T(\mathbf{u})$ formula $\mid \mathbf{u} \in V\ \}$

OR

- $\text{span}\{T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)\}$
  where $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for V

OR

- use column space of **A**
  where **A** is the standard matrix of T

To find ker(T):

- set formula = **0** and solve this homogeneous system
  the general solution gives ker(T)

OR

- use nullspace of **A**