1. Let a denote a positive constant. It is known that

$$a + \frac{5}{18}a + \ldots$$

is a geometric series and it converges to the sum 2106. Find the exact value of a.

Answer 1521

$$a + \frac{5}{18}a + \cdots$$

$$= a\left(\frac{1}{1-\frac{5}{18}}\right)$$

$$= \frac{18a}{13}$$

$$\therefore \frac{18a}{13} = 2106$$

$$a = \frac{2106 \times 13}{18} = 117 \times 13$$

$$= \underline{\underline{1521}}$$

2. Use the Ratio Test to determine whether the infinite series

$$\sum_{n=1}^{\infty} \frac{8^n}{9^n - 5^n}$$

is convergent or divergent.

Answer Convergent

$$\left| \frac{\dfrac{8^{n+1}}{9^{n+1} - 5^{n+1}}}{\dfrac{8^n}{9^n - 5^n}} \right| = \frac{8(9^n - 5^n)}{9^{n+1} - 5^{n+1}}$$

$$= \frac{8}{9}\left(\frac{1 - \left(\frac{5}{9}\right)^n}{1 - 5\left(\frac{5}{9}\right)^n}\right)$$

$$\longrightarrow \frac{8}{9} \text{ when } n \to \infty \quad \left(\because \left|\frac{5}{9}\right| < 1 \quad \therefore \left(\frac{5}{9}\right)^n \to 0\right)$$

$$\because \frac{8}{9} < 1$$

$\therefore$ <u>convergent</u>

3. What is the coefficient of the term x^4 in the Taylor series of $3\cos^2 x$ at $x = 0$?
(Hint: you may want to use the formula $\cos^2 x = \frac{1+\cos 2x}{2}$ together with the standard Taylor series of $\cos t$.)

Answer: 1

$$3\cos^2 x = \frac{3}{2}(1+\cos 2x)$$

use $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$

$$= \frac{3}{2}\left(1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right)$$

$$= \frac{3}{2}\left(\cdots + \frac{16}{4!}x^4 - \cdots\right)$$

$$= \cdots + x^4 - \cdots$$

$\therefore$ Answer is $\underline{\underline{1}}$

4. It is known that one of the following four points lies on the plane $3x + 4y + 5z = 0$. Which one is it?
$(3, 4, -5)$
$(1.1.1)$
$(2, -1, 0)$
$(5, 4, -3)$

Answer: $(3, 4, -5)$

$$3 \times 3 + 4 \times 4 + 5 \times (-5) = 9 + 16 - 25 = 0$$

$\therefore$ Answer $\underline{\underline{(3, 4, -5)}}$

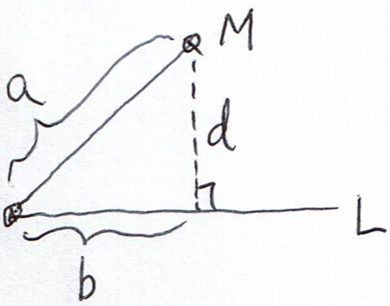
5. Let O, A, B, C denote the four points $(0,0,0)$, $(2,0,0)$, $(0,4,0)$ and $(0,0,2)$ respectively. Let M denote the mid-point of AB and let N denote the mid-point of BC. Let L denote the line that passes through O and N. Find the perpendicular distance from M to the line L. Give your answer correct to two decimal places.

Answer: 1.34

$$M = \frac{(2,0,0)+(0,4,0)}{2} = (1,2,0)$$

$$N = \frac{(0,4,0)+(0,0,2)}{2} = (0,2,1)$$

$$L : (0,0,0) + t(0,2,1)$$



dot product

$$a^2 = 1^2 + 2^2 = 5$$

$$b^2 = \left(\frac{(1,2,0)\cdot(0,2,1)}{\|(0,2,1)\|}\right)^2$$

$$= \frac{16}{5}$$

$$d = \sqrt{a^2 - b^2} = \sqrt{5 - \frac{16}{5}} = \frac{3}{\sqrt{5}}$$

$$= 1.3416\cdots$$

$$\approx \underline{\underline{1.34}}$$

6. Let L_1, L_2 denote two lines in space. It is known that L_1 passes through the point $(-1, 0, 1)$ and L_1 is perpendicular to the plane $2x - y + 7z = 1521$. It is also known that L_2 passes through the two points $(2, -4, 18)$ and $(1, -6, 21)$. If L_1 intersects L_2 at the point (a, b, c), find the exact value of $a + b + c$.

Answer: 16

$L_1 : (x, y, z) = (-1, 0, 1) + t(2, -1, 7)$

$L_2 : (x, y, z) = (2, -4, 18) + s(-1, -2, 3)$

$$L_1 \cap L_2 \Rightarrow \begin{cases} -1 + 2t = 2 - s & \text{------ } ① \\ 0 - t = -4 - 2s & \text{------ } ② \\ 1 + 7t = 18 + 3s & \text{------ } ③ \end{cases}$$

$① + ② + ③ \Rightarrow 8t = 16 \Rightarrow t = 2$

$\therefore (a, b, c) = (3, -2, 15)$

$\therefore a + b + c = \underline{\underline{16}}$

7. Let $f(x, y, z) = \sqrt{\frac{yz^2}{x}}$. Find the directional derivative of f at the point $(2, 3, 8)$ in the direction of the vector joining $(2, 3, 8)$ to $(1, 5, 21)$. Give your answer correct to two decimal places.

Answer: 1.64

$$\nabla f(x,y,z) = \frac{1}{2}\left(\frac{yz^2}{x}\right)^{-1/2}\left(-\frac{yz^2}{x^2}, \frac{z^2}{x}, \frac{2yz}{x}\right)$$

$$\nabla f(2,3,8) = \frac{1}{2\sqrt{96}}(-48, 32, 24)$$

$$\vec{u} = \frac{(1,5,21)-(2,3,8)}{\|(1,5,21)-(2,3,8)\|} = \frac{1}{\sqrt{174}}(-1, 2, 13)$$

dot product

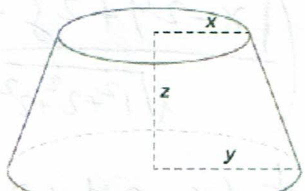
$$D_{\vec{u}} f(2,3,8) = \nabla f(2,3,8) \cdot \vec{u}$$

$$= \frac{1}{2\sqrt{96}\sqrt{174}}(48 + 64 + 312)$$

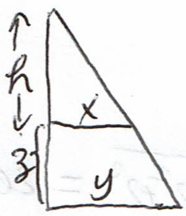
$$= \frac{212}{\sqrt{96 \cdot 174}} = 1.64031\ldots$$

$$\simeq \underline{\underline{1.64}}$$

8. The picture below shows the frustrum of a right circular cone where x is the top radius, y is the base radius and z is the height of the frustrum. If x is increasing at a rate of 1.521 m/s (i.e. metre per second), y is increasing at a rate of 2.02 m/s and z is decreasing at a rate of 5.99 m/s, estimate the rate of change of its volume in cubic metre per second when $x = 15$ metres, $y = 20$ metres and $z = 30$ metres. Give your answer correct to two decimal places.



Answer: 77.23



$$\frac{h}{x} = \frac{h+z}{y} \Rightarrow yh = xh + xz$$

$$\Rightarrow h = \frac{xz}{y-x}$$

$$V = \frac{1}{3}\pi y^2(z+h) - \frac{1}{3}\pi x^2 h$$

$$= \frac{1}{3}\pi\{y^2 z + (y^2 - x^2)h\} = \frac{1}{3}\pi\{y^2 z + (y+x)xz\}$$

$$= \frac{1}{3}\pi(x^2 + xy + y^2)z$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left\{(2x+y)z\frac{dx}{dt} + (x+2y)z\frac{dy}{dt} + (x^2+xy+y^2)\frac{dz}{dt}\right\}$$

$\therefore$ at the given instant

$$\frac{dV}{dt} = \frac{1}{3}\pi\{1500 \times 1.521 + 1650 \times 2.02 + 925 \times (-5.99)\}$$

$$= \frac{1}{3}\pi\{2281.5 + 3333 - 5540.75\}$$

$$= 77.2308\ldots \approx \underline{\underline{77.23}}$$

9. Let a denote a positive constant. Let S denote the plane $x - 2y + 2z = a$. Let $f(x, y, z)$ denote a function of three variables defined in the following way: for each point (x, y, z), let d denote the perpendicular distance from this point to S; then we define $f(x, y, z) = d^2$. If the maximum rate of increasing of f at the point $(1, 2, 3)$ is equal to 2020, find the exact value of a.

Answer: 3033

$$f(x,y,z) = \left(\frac{|x-2y+2z-a|}{\sqrt{1^2+2^2+2^2}}\right)^2$$

$$= \frac{1}{9}(x-2y+2z-a)^2$$

$$\nabla f(x,y,z) = \frac{2}{9}(x-2y+2z-a)(1,-2,2)$$

$$\nabla f(1,2,3) = \frac{2}{9}(3-a)(1,-2,2)$$

$$\|\nabla f(1,2,3)\| = 2020 \Rightarrow \frac{2}{9}|3-a|\sqrt{1^2+2^2+2^2} = 2020$$

$$|3-a| = 3030$$

$$3-a = \pm 3030$$

$$a = 3033 \text{ or } -3027$$

$$\because a > 0 \quad \therefore \underline{a = 3033}$$

10. Let (r, θ) denote the polar coordinate system and (x, y) denote the Cartesian coordinate system of a two dimensional plane. Recall that they are related by the following two equations:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Find the value of

$$\frac{\partial^2\theta}{\partial y \partial x}$$

when $r = 0.7$ and $\theta = \frac{2\pi}{5}$. Give your answer correct to two decimal places.

Answer: 1.65

$$\tan\theta = \frac{y}{x}$$

Differentiate with respect to y on both sides:

$$\sec^2\theta \frac{\partial\theta}{\partial y} = \frac{1}{x} \Rightarrow \frac{\partial\theta}{\partial y} = \frac{1}{x\sec^2\theta} = \frac{1}{x\left(1+\frac{y^2}{x^2}\right)}$$

$$\therefore \frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2}$$

$$\therefore \frac{\partial^2\theta}{\partial x\partial y} = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(y^2+x^2)^2}$$

$$\therefore \frac{\partial^2\theta}{\partial y\partial x} = \frac{\partial^2\theta}{\partial x\partial y} = \frac{y^2-x^2}{(y^2+x^2)^2} = \frac{r^2\sin^2\theta - r^2\cos^2\theta}{r^4}$$

$$= \frac{-\cos 2\theta}{r^2} = \frac{-\cos\frac{4\pi}{5}}{(0.7)^2} = 1.6510\ldots$$

$$\approx \underline{\underline{1.65}}$$