

MA 1521
Tutorial 5 Solutions

1. (a) Let $u_n = (-1)^n \frac{(x+2)^n}{n}$.

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+2)^{n+1}}{n+1} \cdot \frac{n}{(x+2)^n} \right| = |x+2|.$$

By ratio test, the power series is convergence in $|x+2| < 1$.

So the radius of convergence is 1.

- (b) Let $u_n = \frac{(3x-2)^n}{n}$.

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n} \right| = |3x-2|.$$

By ratio test, the power series is convergence in $|3x-2| < 1 \Rightarrow |x - \frac{2}{3}| < \frac{1}{3}$.

So the radius of convergence is $\frac{1}{3}$.

- (c) Let $u_n = (-1)^n (4x+1)^n$.

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(4x+1)^{n+1}}{(4x+1)^n} \right| = |4x+1|.$$

By ratio test, the power series is convergence in $|4x+1| < 1 \Rightarrow |x + \frac{1}{4}| < \frac{1}{4}$.

So the radius of convergence is $\frac{1}{4}$.

- (d) Let $u_n = \frac{(3x)^n}{n!}$.

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(3x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(3x)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3x}{n+1} \right| = 0.$$

Since the limit is less than 1 for any x , so the radius of convergence is ∞ .

- (e) Let $u_n = (nx)^n$.

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{((n+1)x)^{n+1}}{(nx)^n} \right| = \lim_{n \rightarrow \infty} \left| \left(1 + \frac{1}{n}\right)^n (n+1)x \right| = \infty$$

for all $x \neq 0$.

So the radius of convergence is 0.

- (f) Let $u_n = \frac{(4x-5)^{2n+1}}{n^{3/2}}$.

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(4x-5)^{2n+3}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{(4x-5)^{2n+1}} \right| = |4x-5|^2.$$

By ratio test, the power series is convergence in $|4x-5|^2 < 1 \Rightarrow |4x-5| < 1 \Rightarrow |x - \frac{5}{4}| < \frac{1}{4}$.

So the radius of convergence is $\frac{1}{4}$.

2. The first term of the geometric series is $a = 1$ and the common ratio is $r = -\frac{(x-3)}{2}$.

So the sum of the series is

$$\frac{a}{1-r} = \frac{1}{1+(x-3)/2} = \frac{2}{x-1}$$

provided $\left| \frac{(x-3)}{2} \right| < 1 \Rightarrow 1 < x < 5.$

3. (a)

$$\frac{x}{1-x} = x \left(\frac{1}{1-x} \right) = x \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$$

(b) Let $f(x) = \frac{1}{x^2}$.

Then $f'(x) = -\frac{2}{x^3}$, $f''(x) = \frac{3!}{x^4}$, ... and in general $f^{(n)}(x) = (-1)^n \frac{(n+1)!}{x^{n+2}}$.

So $f^{(n)}(1) = (-1)^n (n+1)!$.

The Taylor series of f at $x = 1$ is thus:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n = \sum_{n=0}^{\infty} (-1)^n (n+1) (x-1)^n.$$

- (c)

$$\begin{aligned} \frac{x}{1+x} &= \frac{(1+x)-1}{1+x} = 1 - \frac{1}{1+x} = 1 - \frac{1}{1+(x+2)-2} \\ &= 1 + \frac{1}{1-(x+2)} = 1 + \sum_{n=0}^{\infty} (x+2)^n = 2 + \sum_{n=1}^{\infty} (x+2)^n \end{aligned}$$

4. Given $f(x) = \sin x$, compute the following:

$$f(x) = \sin x \qquad f(0) = \sin 0 = 0$$

$$f^{(1)}(x) = \cos x \qquad f^{(1)}(0) = \cos 0 = 1$$

$$f^{(2)}(x) = -\sin x \qquad f^{(2)}(0) = -\sin 0 = 0$$

$$f^{(3)}(x) = -\cos x \qquad f^{(3)}(0) = -\cos 0 = -1$$

$$f^{(4)}(x) = \sin x$$

Taylor polynomial at $x = 0$ of order 3 is

$$P_3(x) = \sum_{k=0}^3 \frac{f^{(k)}(0)}{k!} x^k = 0 + \frac{1}{1!} x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 = x - \frac{1}{6} x^3.$$

$P_3(0.1)$ approximates $\sin 0.1$ as follows:

$$P_3(0.1) = 0.1 - \frac{1}{6}(0.1)^3 \approx 0.09983.$$

5. (i) We have

$$xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

Note that the radius of convergence of this power series is infinite. Therefore, we can integrate both sides from 0 to 1.

$$\int_0^1 xe^x dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^{n+1}}{n!} dx = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)}.$$

On the other hand,

$$\int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 e^x dx = e - (e - 1) = 1.$$

So we conclude $\sum_{n=0}^{\infty} \frac{1}{n!(n+2)} = 1$.

(ii) We have

$$\frac{e^x - 1}{x} = \frac{\sum_{n=1}^{\infty} \frac{x^n}{n!}}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}.$$

Note that the radius of convergence of this power series is infinite.

Differentiate both sides with respect to x , we have

$$\frac{xe^x - (e^x - 1)}{x^2} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n+1)!} = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{(n+2)!}.$$

The result now follows by setting $x = 1$.