National University of Singapore

Semester 1, 2020/2021 MA1101R Practice Assignment 2 Solution

1. $\boldsymbol{A}$ and $\boldsymbol{B}$ are 3×3 row equivalent matrices related by the following diagram:

$$\boldsymbol{A} \xrightarrow{R_1+R_2} \quad \xrightarrow{R_1 \leftrightarrow R_3} \quad \xrightarrow{2R_3} \quad \xrightarrow{R_3-2R_2} \quad \boldsymbol{B}$$

(i) [4 marks] Write down four elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3, \boldsymbol{E}_4$ such that

$$\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{B}.$$

(ii) [2 marks] Find an invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{A} = \boldsymbol{C}\boldsymbol{B}$. (You may use MATLAB, but you need to show how $\boldsymbol{C}$ is obtained.)

(iii) [2 marks] If $\det(\boldsymbol{B}) = 12$, find $\det(\boldsymbol{A})$. You need to show how you obtain the answer.

(iv) [2 marks] If $\boldsymbol{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is the inverse of $\boldsymbol{B}$, find $\boldsymbol{A}^{-1}$. (You may use MATLAB, but you need to show how $\boldsymbol{A}^{-1}$ is obtained.)

Answer

(i) $\boldsymbol{E}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$

$\boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \boldsymbol{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$

(ii) $\boldsymbol{C} = (\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1)^{-1} = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$

(iii) $\det(\boldsymbol{A}) = \det(\boldsymbol{C}\boldsymbol{B}) = \det(\boldsymbol{C})\det(\boldsymbol{B}) = (-1/2)12 = -6.$

(iv) $\boldsymbol{A}^{-1} = (\boldsymbol{C}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{C}^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 8 & 0 & 0 \end{pmatrix}.$

2. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ a & b & c & d \end{pmatrix}$ for some real numbers a, b, c, d.

   (i) [2 marks] Find $\det(\boldsymbol{A})$. (Show your working)

   (ii) [2 marks] Write down the condition among a, b, c, d such that the homogeneous system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ does not have a non-trivial solution. (Briefly explain your answer.)

   Answer

   (i) $\det(\boldsymbol{A}) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ b & c & d \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ a & b & d \end{vmatrix}$
   $= \left(\begin{vmatrix} 1 & 1 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ b & c \end{vmatrix} \right) + a \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = d - c - b + a$

   (ii) For $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ to have no non-trivial solutions, $\boldsymbol{A}$ must be invertible and hence $\det(\boldsymbol{A}) \neq 0$. So the required condition is

   $$a - b - c + d \neq 0.$$

3. Let $U = \{(x, y, z) \mid x - y - z = 0\}$ and $V = \{(x, y, z) \mid 2x + y + z = 0\}$ be the implicit set notations representing two planes in the xyz-space.

   (i) [2 marks] Write down the explicit set notation of U.

   (ii) [2 marks] Write down the explicit set notation of $U \cap V$.

   (iii) [1 mark] Write down a vector that is parallel to the line of intersection of U and V.

   (iv) [1 mark] Is $W = \{(a, a, -a) \mid a \in \mathbb{R}\}$ a subset of V? (Briefly explain your answer.)

   Answer

   (i) Explicit set notation of U: $\{(s + t, s, t) \mid s, t \in \mathbb{R}\}$.

   (ii) Solve the system
   $$\begin{cases} x - y - z = 0 \\ 2x + y + z = 0 \end{cases}$$
   to get the general solution: $z = t, y = -t, x = 0$.
   So the explicit set notation of $U \cap V$: $\{(0, -t, t) \mid t \in \mathbb{R}\}$.

   (iii) A vector that is parallel to the line of intersection of U and V can be $(0, -1, 1)$ or any non-zero scalar multiple.

   (iv) W is not a subset of V, since $(a, a, -a)$ does not satisfy the equation $2x + y + z = 0$, which is the underlying condition of V.