CS2040C
Data Structures and Algorithms

All about minimum spanning trees…

Roadmap

Today: Minimum Spanning Trees

- Prim's Algorithm
- Kruskal's Algorithm
- Boruvka's Algorithm

Variations:

- Constant weight edges
- Bounded integer edge weights
- Euclidean
- Directed graphs
- Maximum Spanning Tree
- Steiner Tree

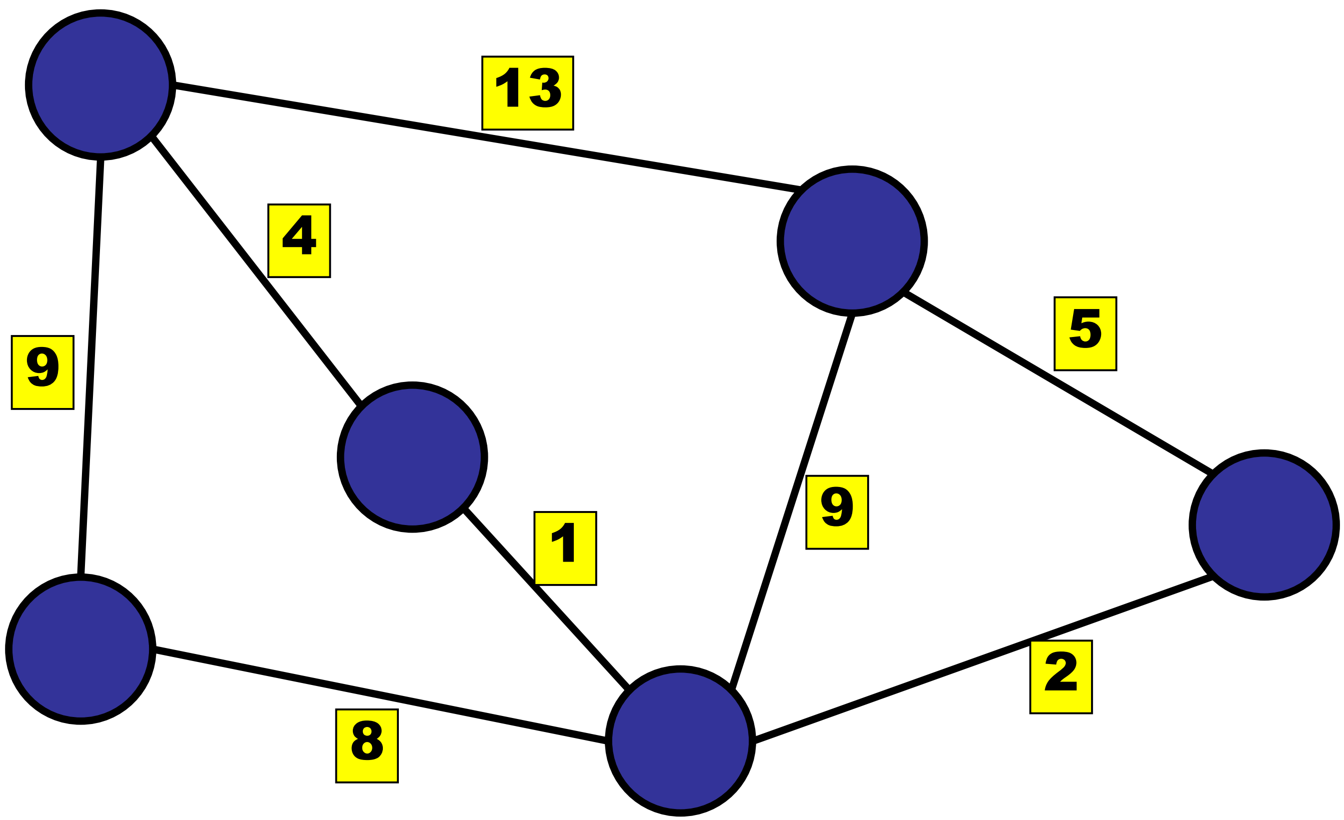
Roadmap

Minimum Spanning Trees

- **The MST Problem**
- Basic Properties of an MST
- Generic MST Algorithm
- Prim's Algorithm
- Kruskal's Algorithm
- Boruvka's Algorithm
- Variations

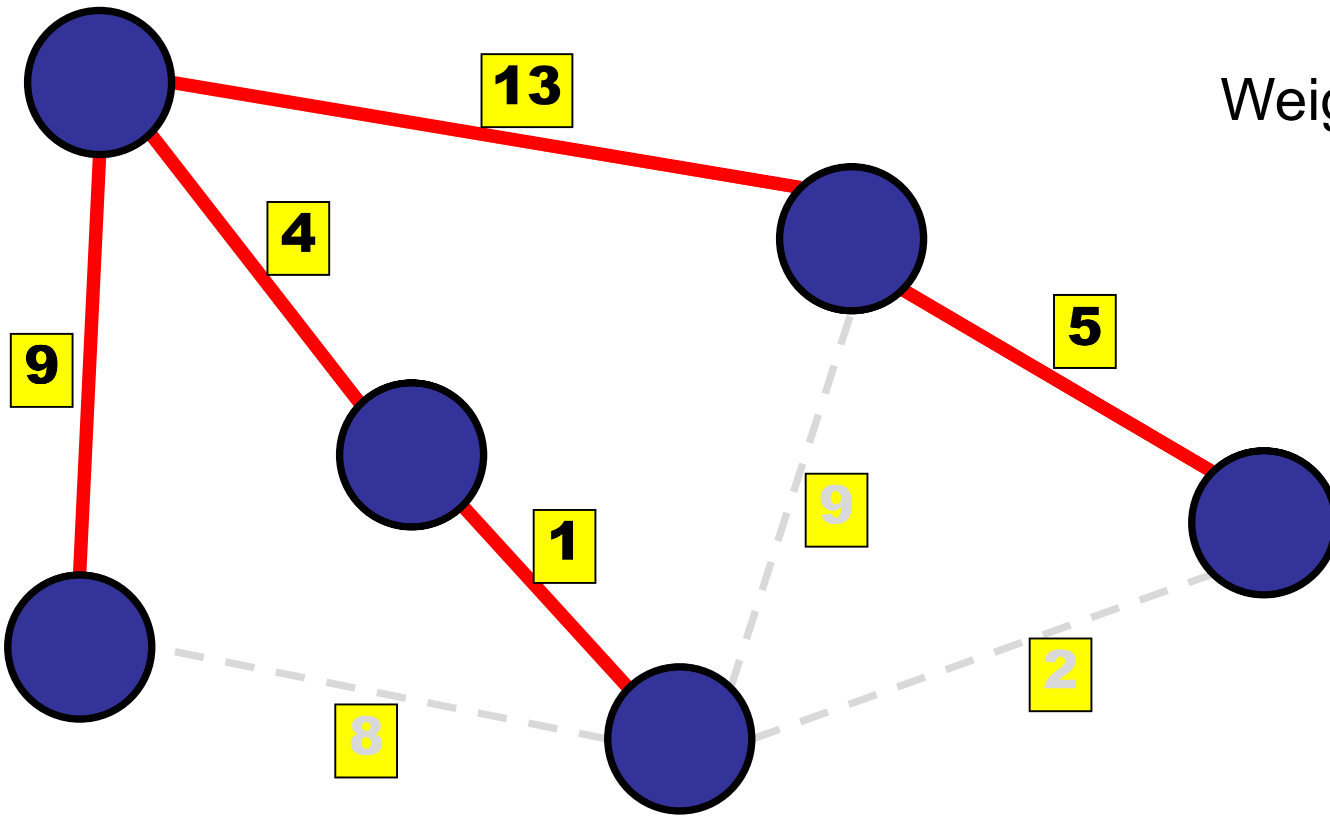
Spanning Tree

Weighted, undirected graph:



Spanning Tree

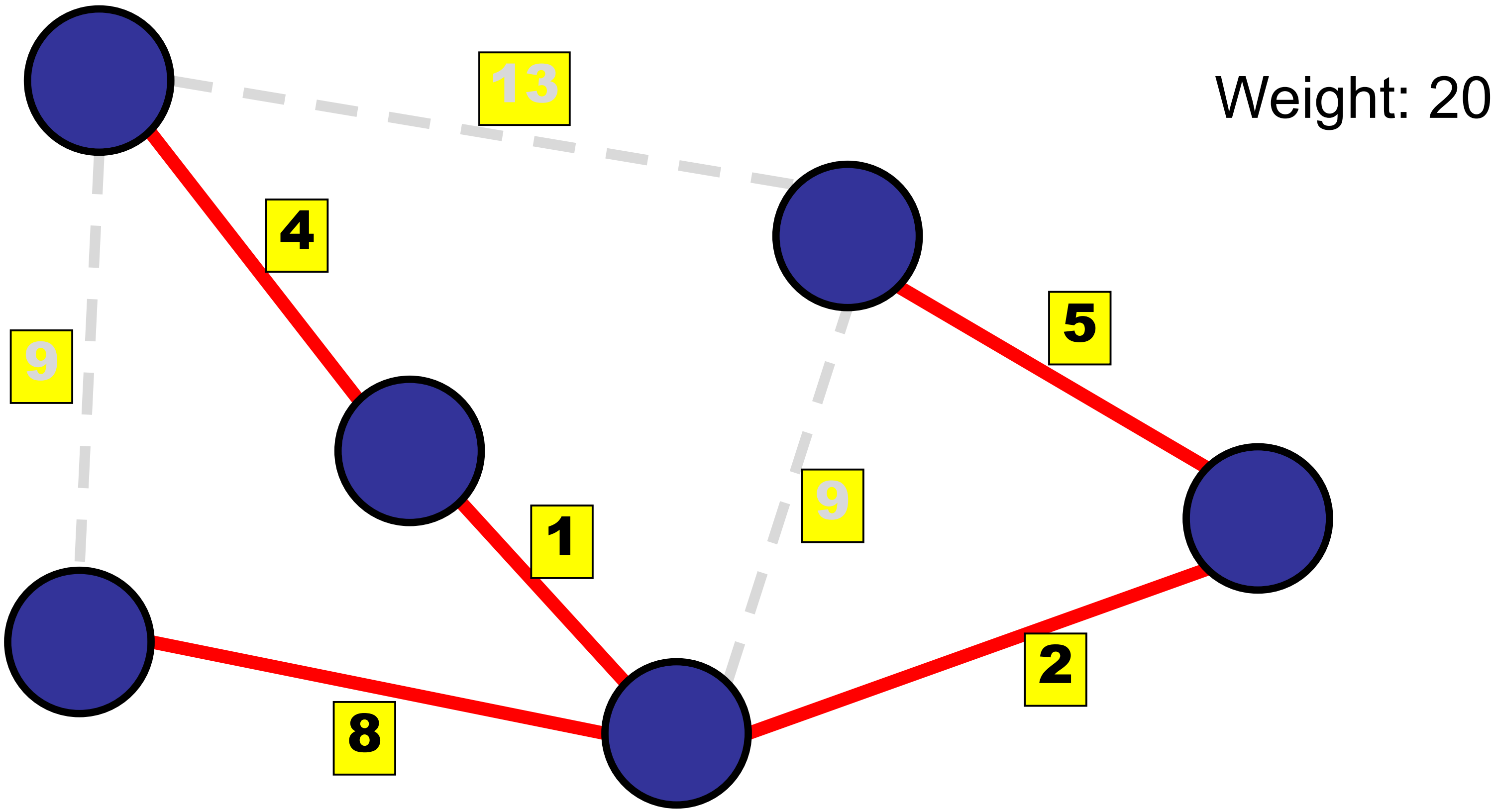
Definition: a **spanning tree** is an acyclic subset of the edges that connects all nodes



Weight: 32

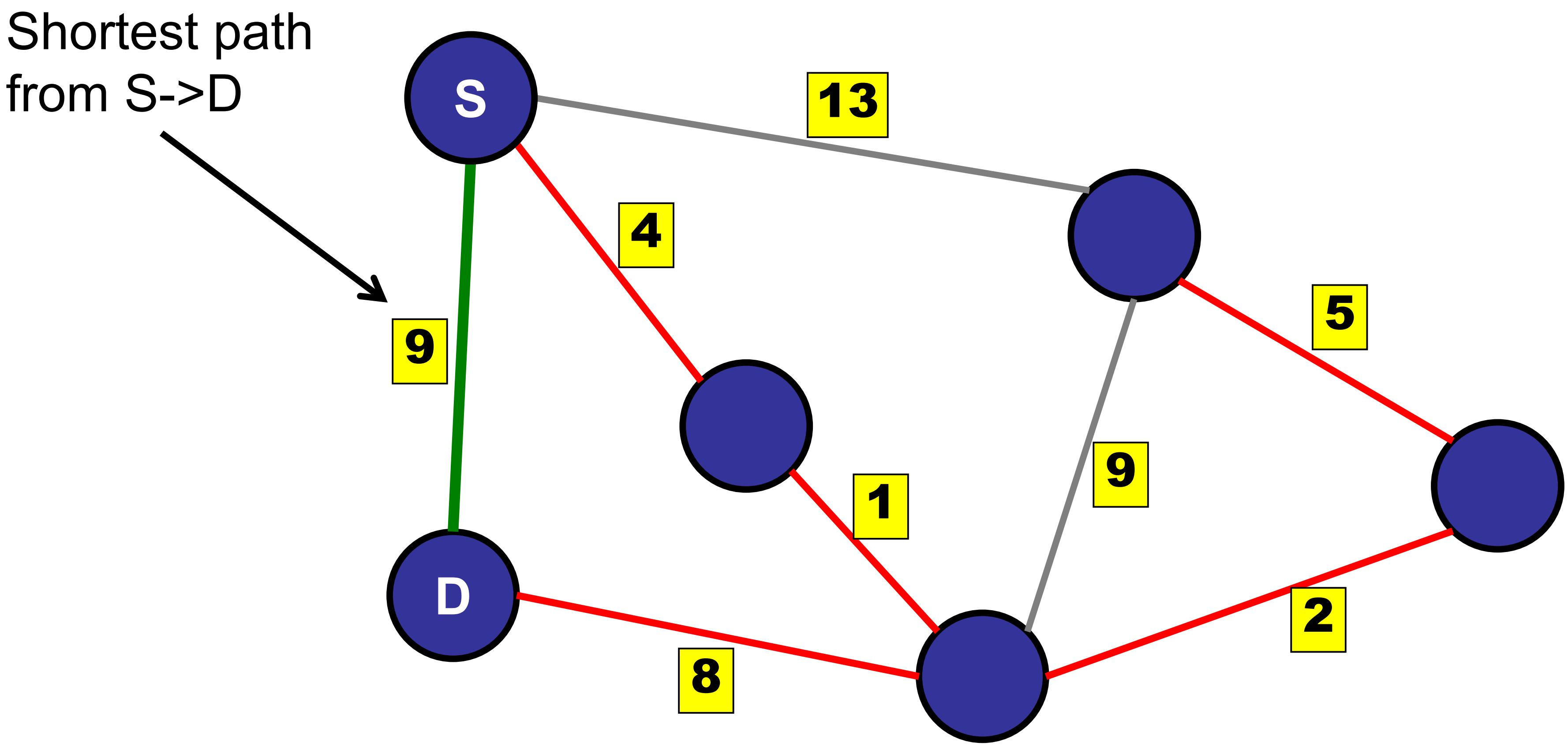
Minimum Spanning Tree

Definition: a spanning tree with minimum weight



Minimum Spanning Tree

Not the same a shortest paths:



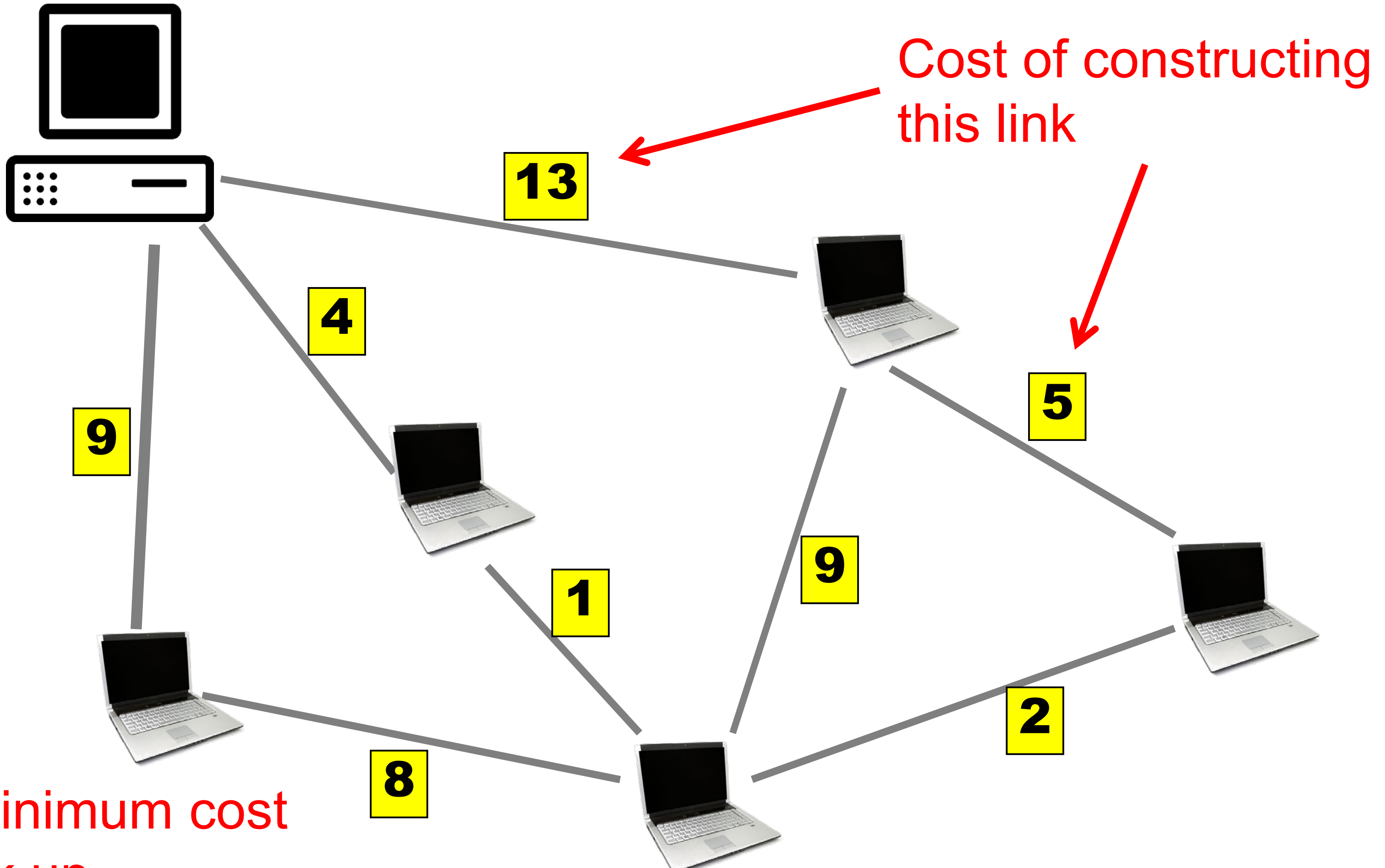
Applications of MST

Many applications:

- Network design
  - Telephone networks
  - Electrical networks
  - Computer networks
  - Ethernet autoconfig
  - Road networks
  - Bottleneck paths

Data distribution

Network:



What is the minimum cost network to link up all the users?

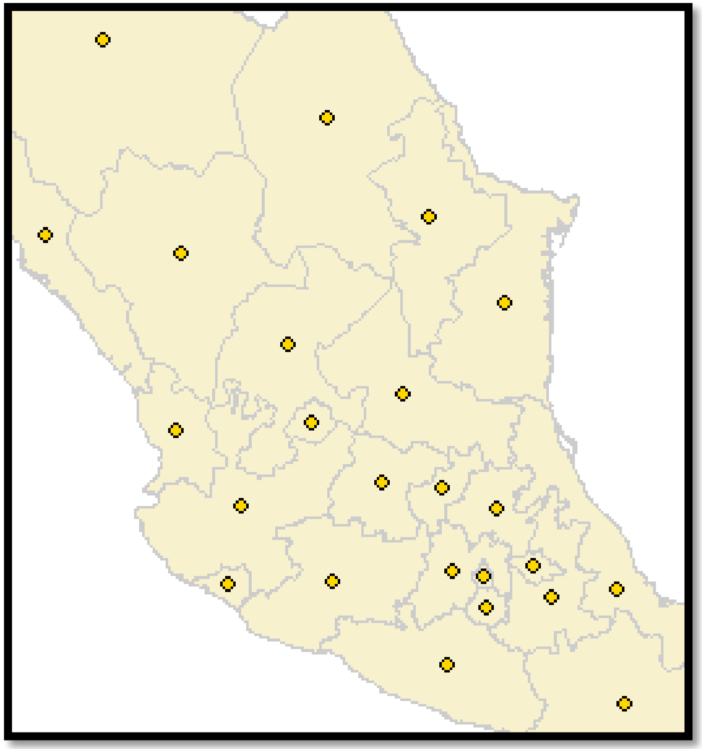
Applications of MST

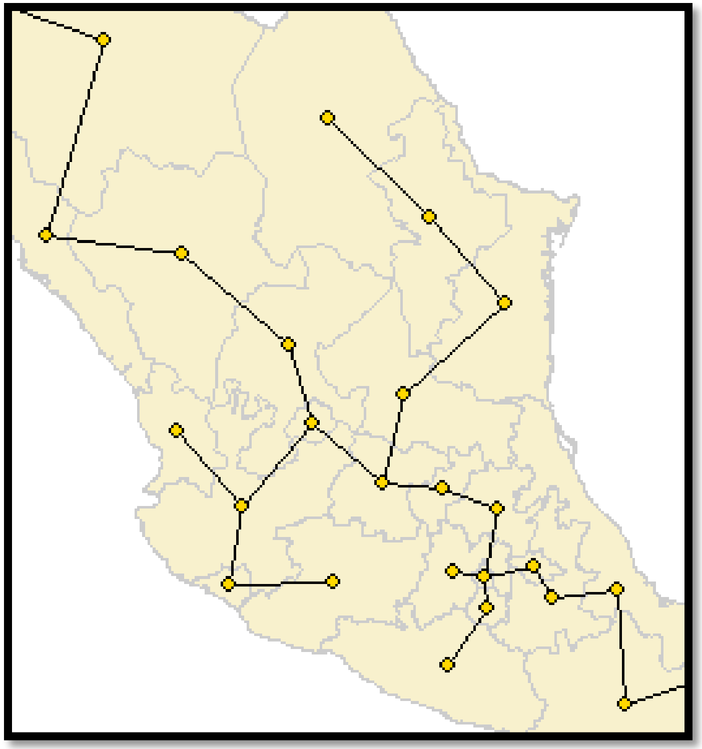
Many applications:

- Many other
  - Error correcting codes
  - Face verification
  - Cluster analysis
  - Image registration

Euclidean Minimal Spanning Tree

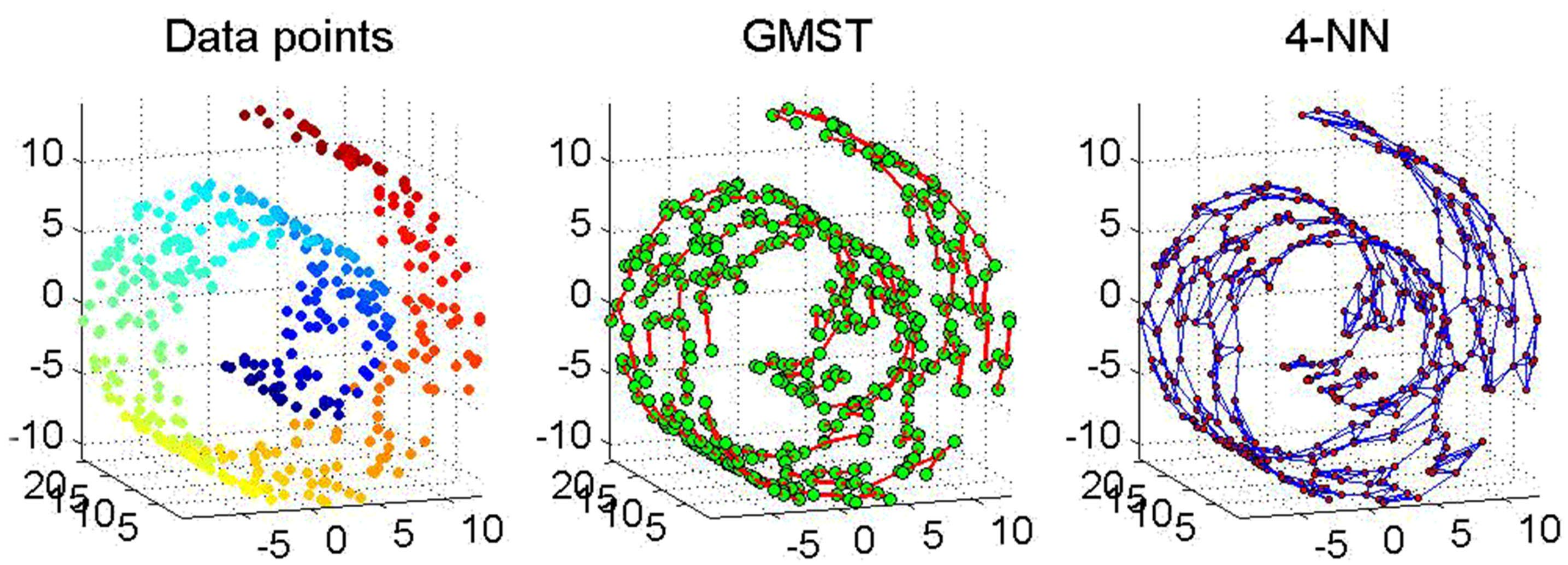
- Given point set P, $EMST(P)$ is the tree that spans P and the sum of lengths of all edges is minimal





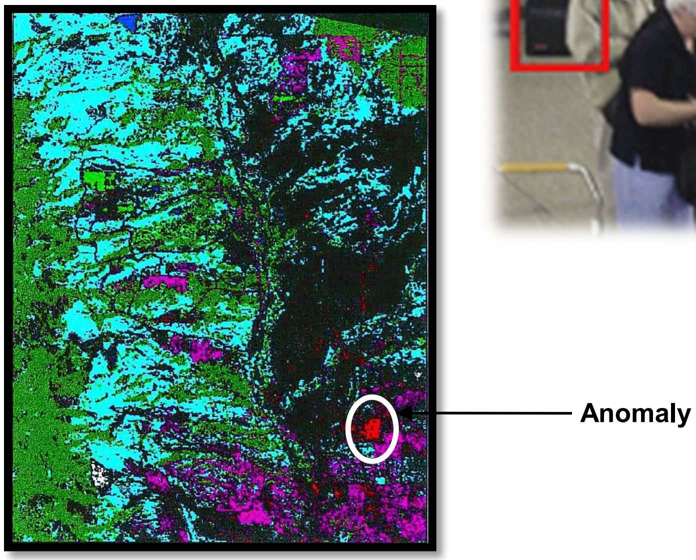
Discovering structures (mainly manifold) for high dimensional data

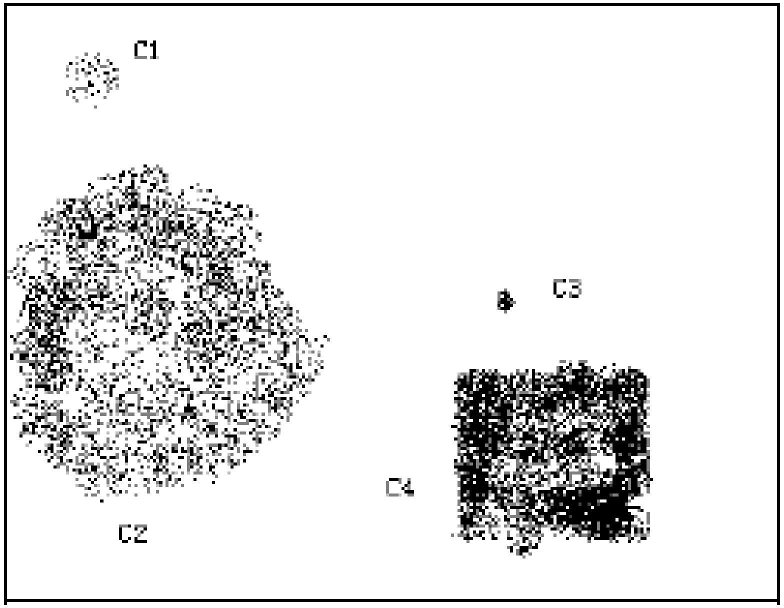
- In machine learning, pattern recognition, data mining, etc



Anomaly Detection

- Anomaly is a pattern in the data that does not conform to the expected behavior.
- E.g. Cyber intrusions, credit card fraud, air traffic safety





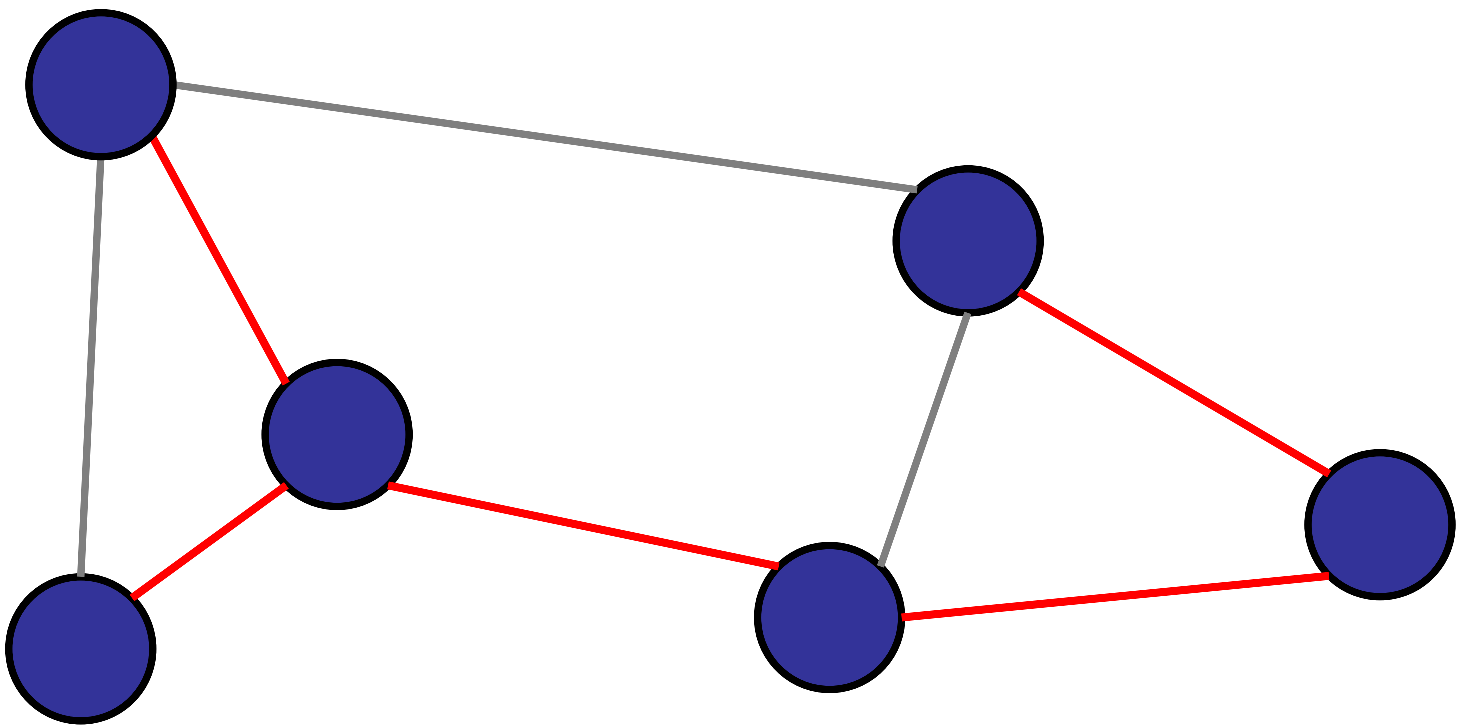
Roadmap

Minimum Spanning Trees

- The MST Problem
- **Basic Properties of an MST**
- Generic MST Algorithm
- Prim’s Algorithm
- Kruskal’s Algorithm
- Boruvka’s Algorithm
- Variations

Properties of MST

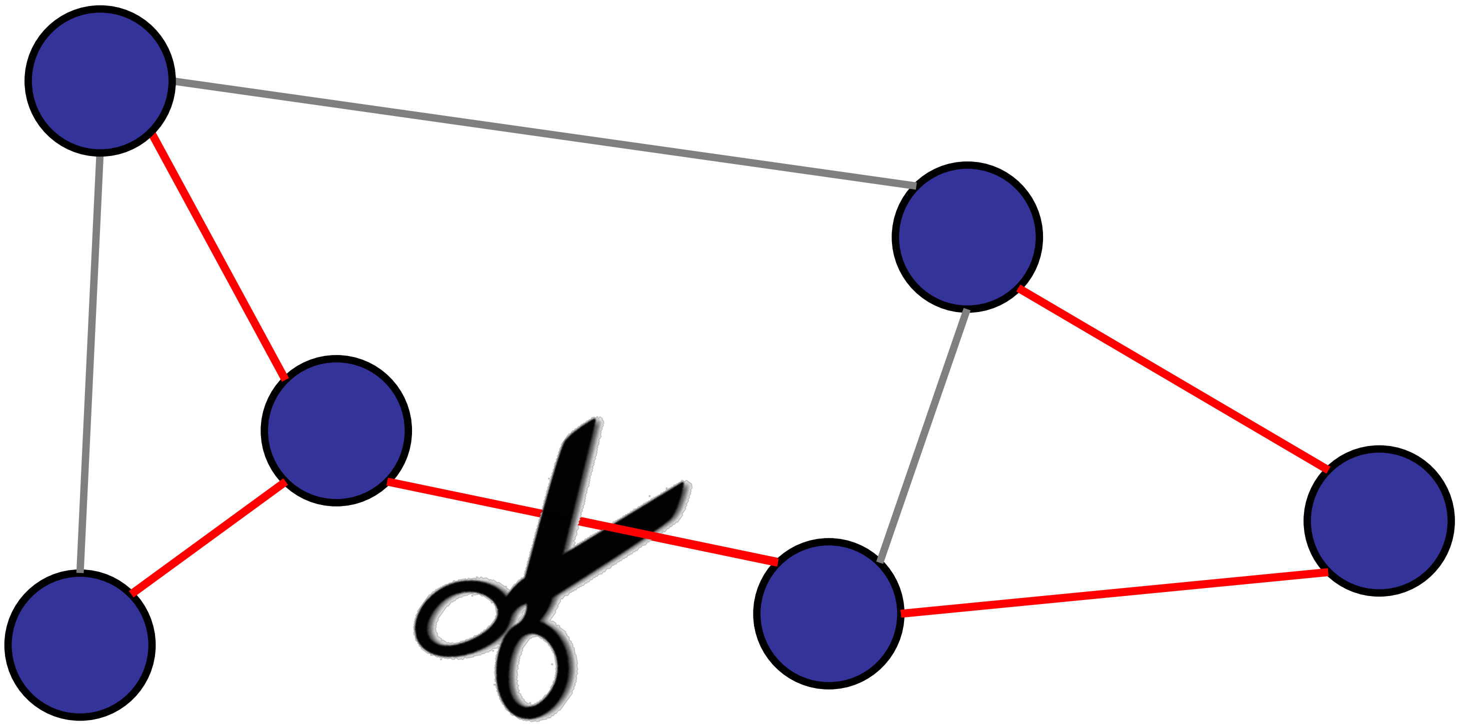
Property 1: No cycles



Why? If there were cycles, we could remove one edge and reduce the weight!

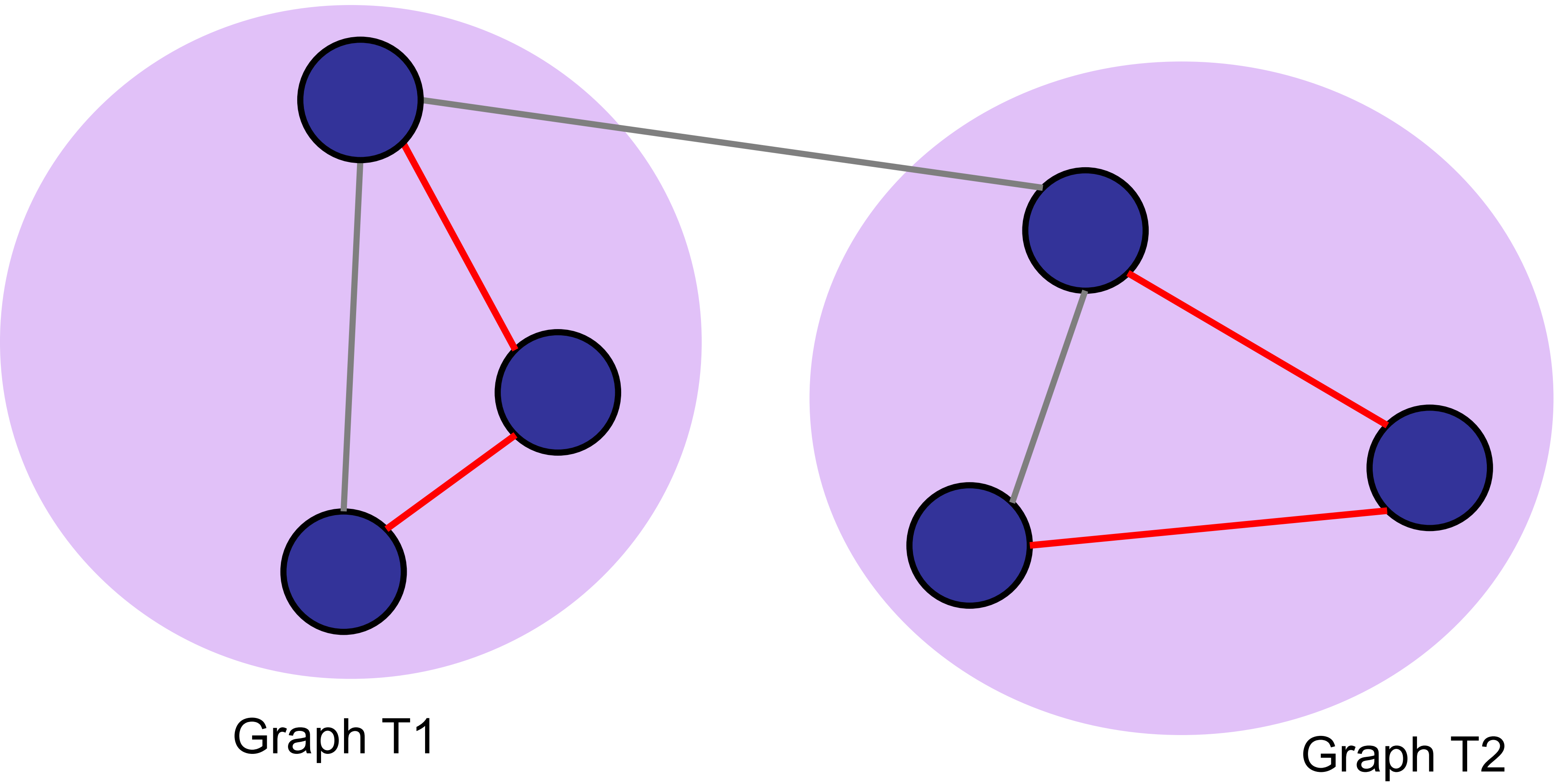
Properties of MST

What happens if you cut an MST into T1 and T2??



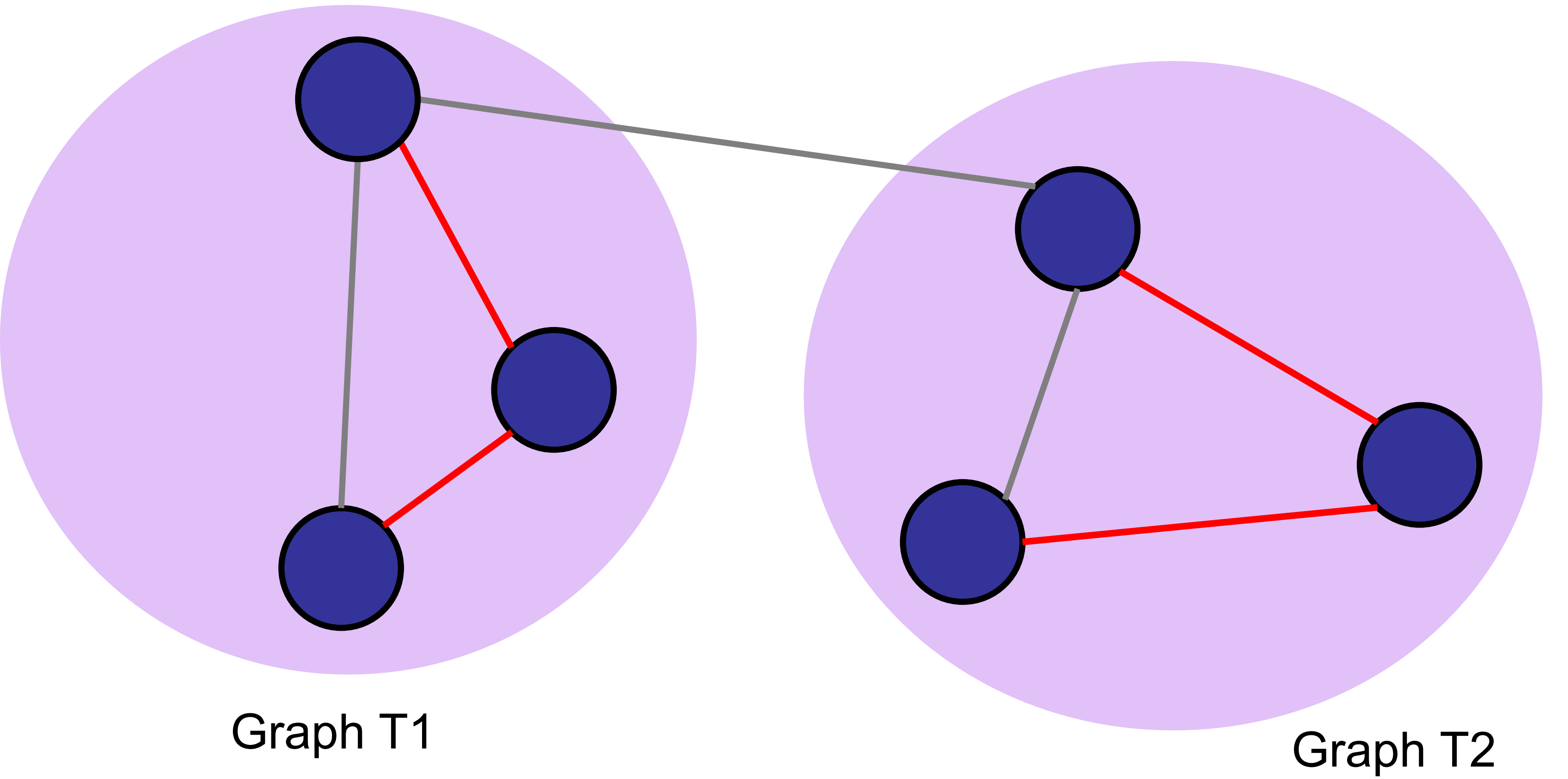
Properties of MST

What happens if you cut an MST into T1 and T2?



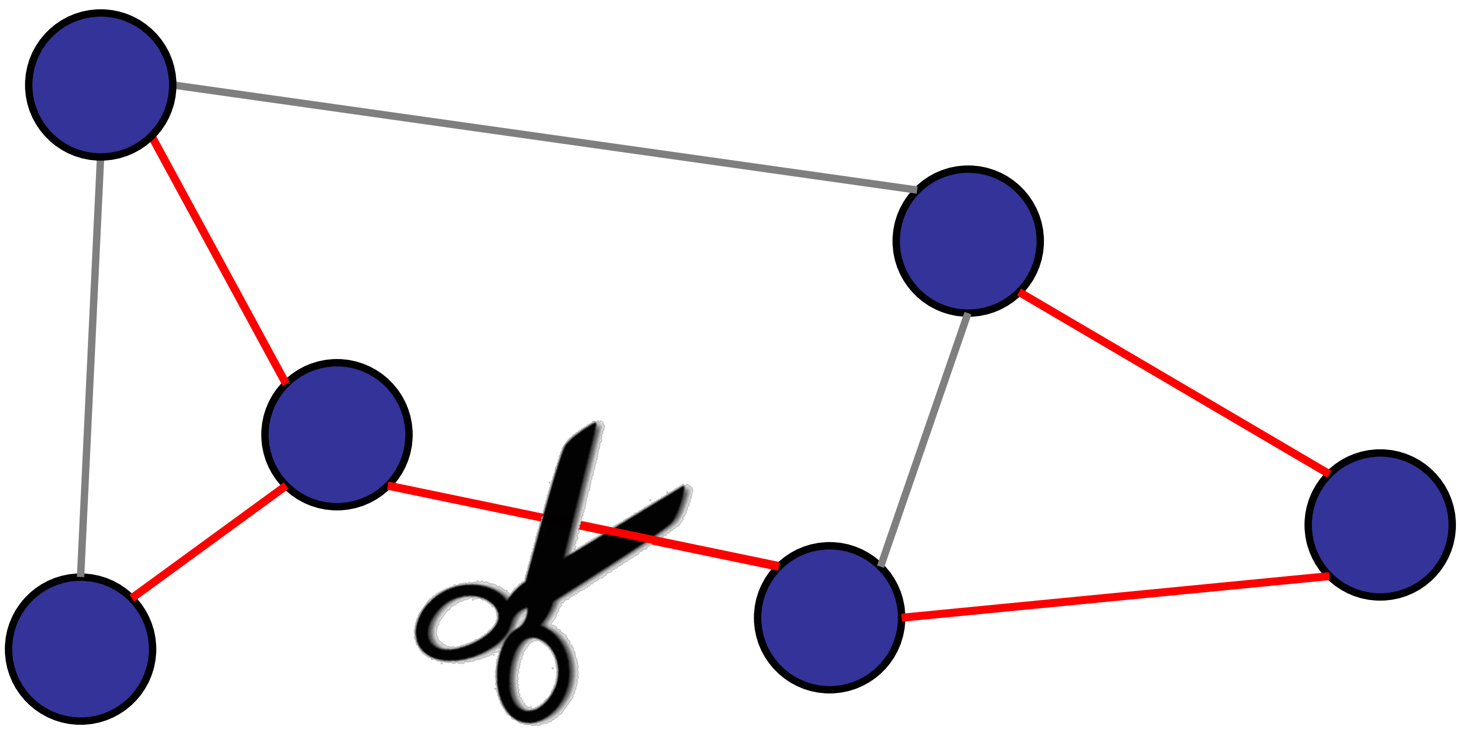
Properties of MST

Theorem: T1 is an MST and T2 is an MST.



Properties of MST

Property 2: If you cut an MST, the two pieces are both MSTs.

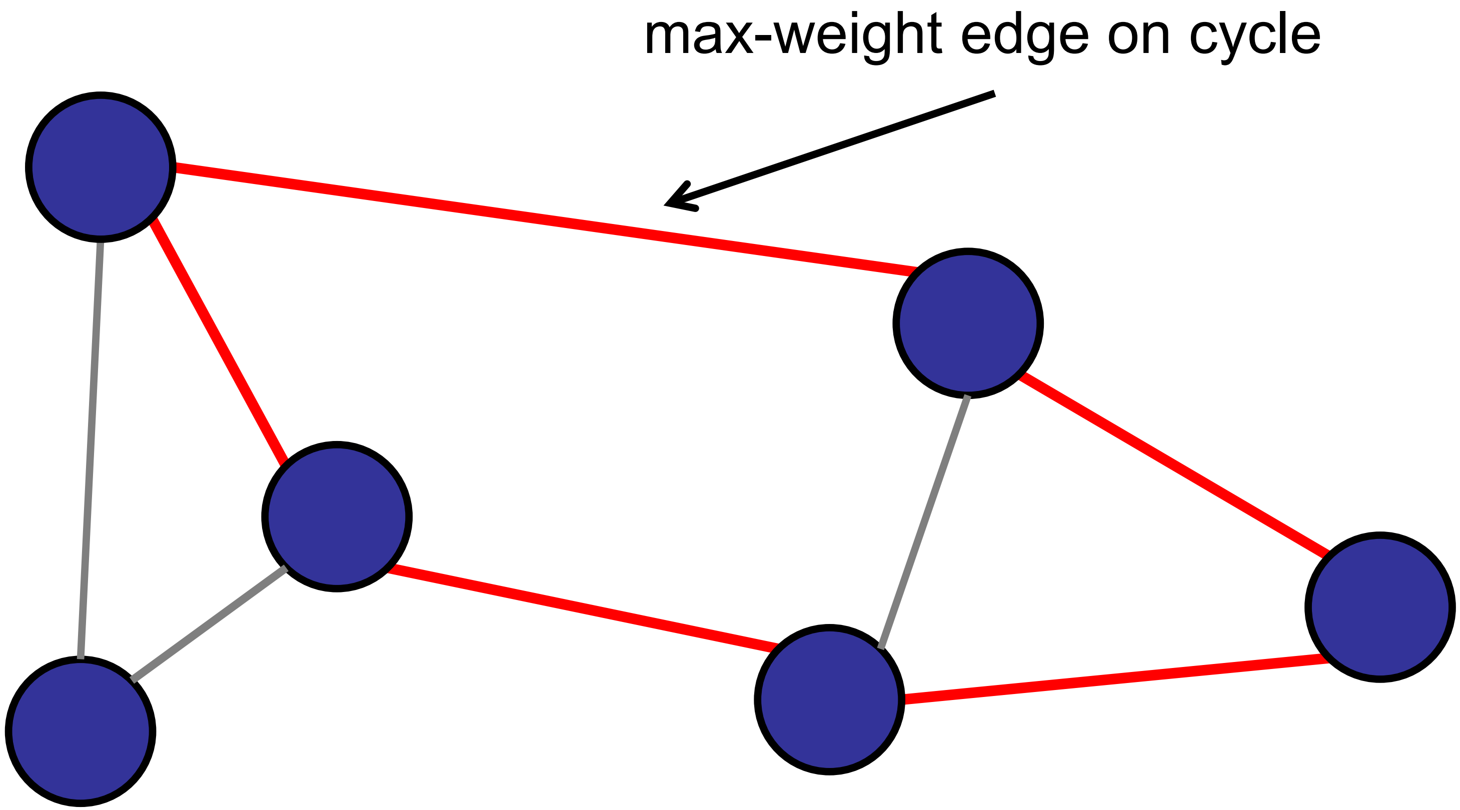


Overlapping sub-problems! Dynamic programming? Yes, but better…

Properties of MST

Property 3: Cycle property

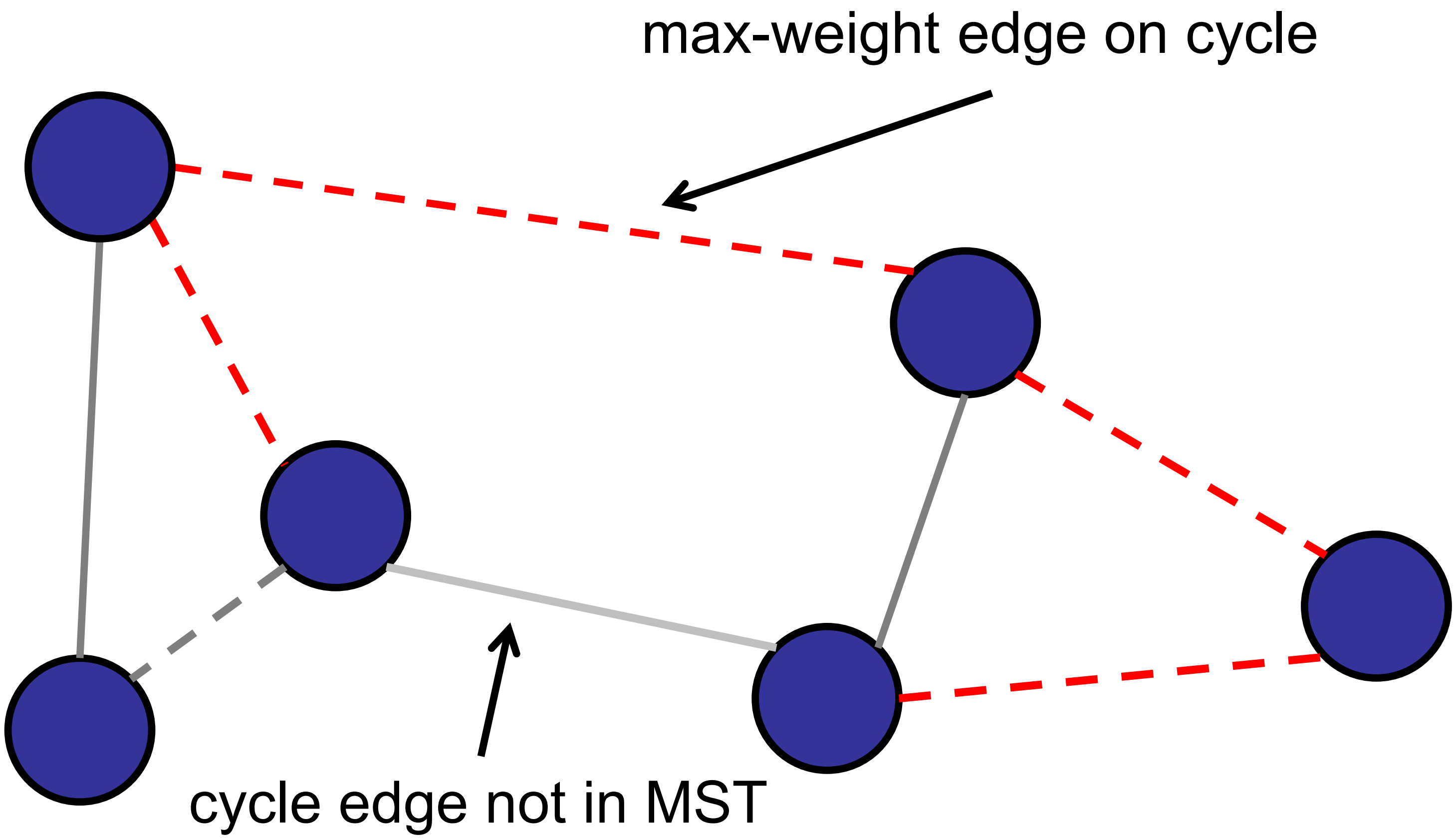
For every cycle, the maximum weight edge is ***not*** in the MST.



Properties of MST

Proof: Cut-and-paste

Assume heavy edge is in the MST.

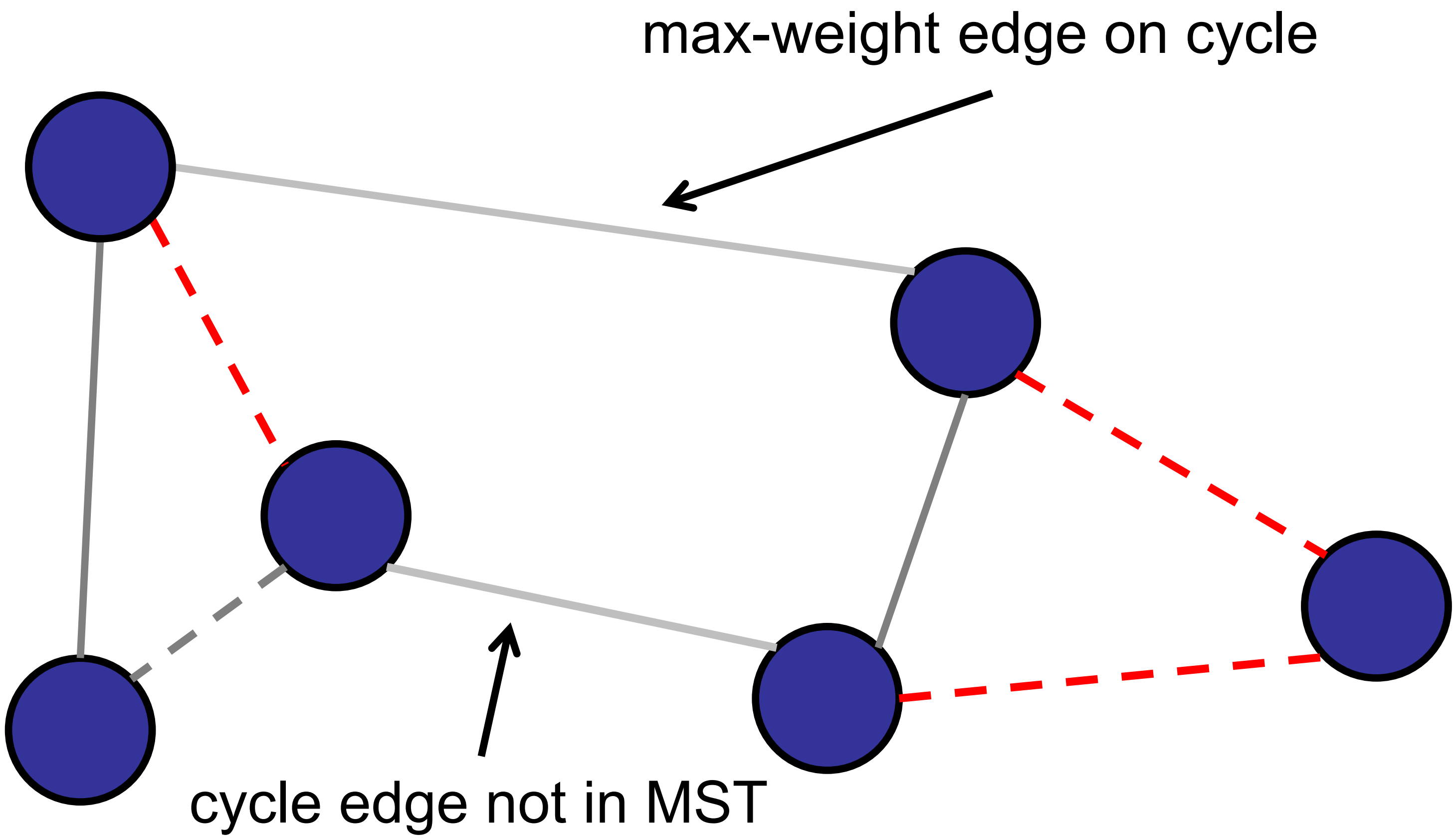


Properties of MST

Proof: Cut-and-paste

Assume heavy edge is in the MST.

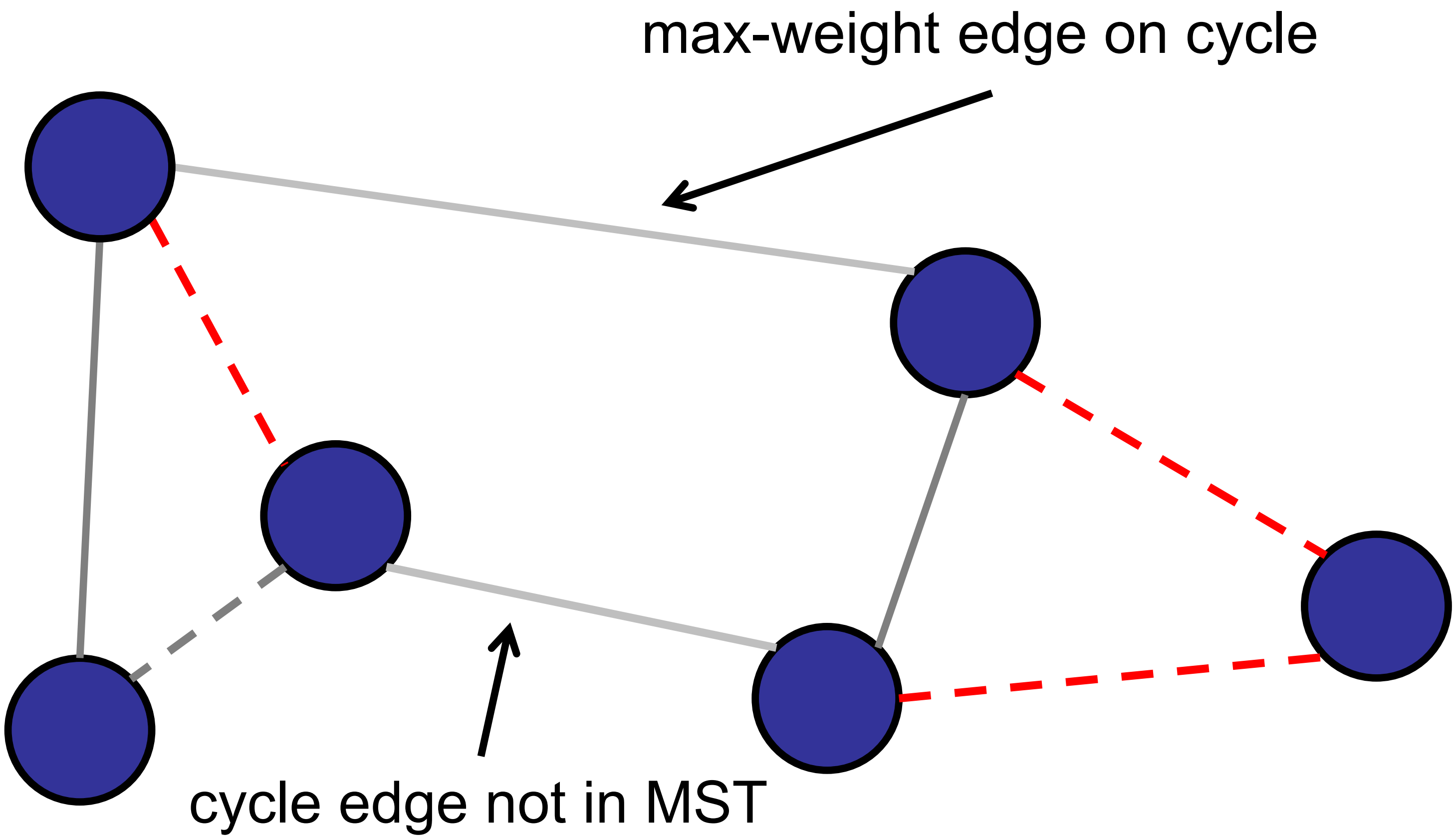
Remove max-weight edge; cuts graph.



Properties of MST

Proof: Cut-and-paste

There exists another cycle edge that crosses the cut. (Even # of cycle edges across cut.)

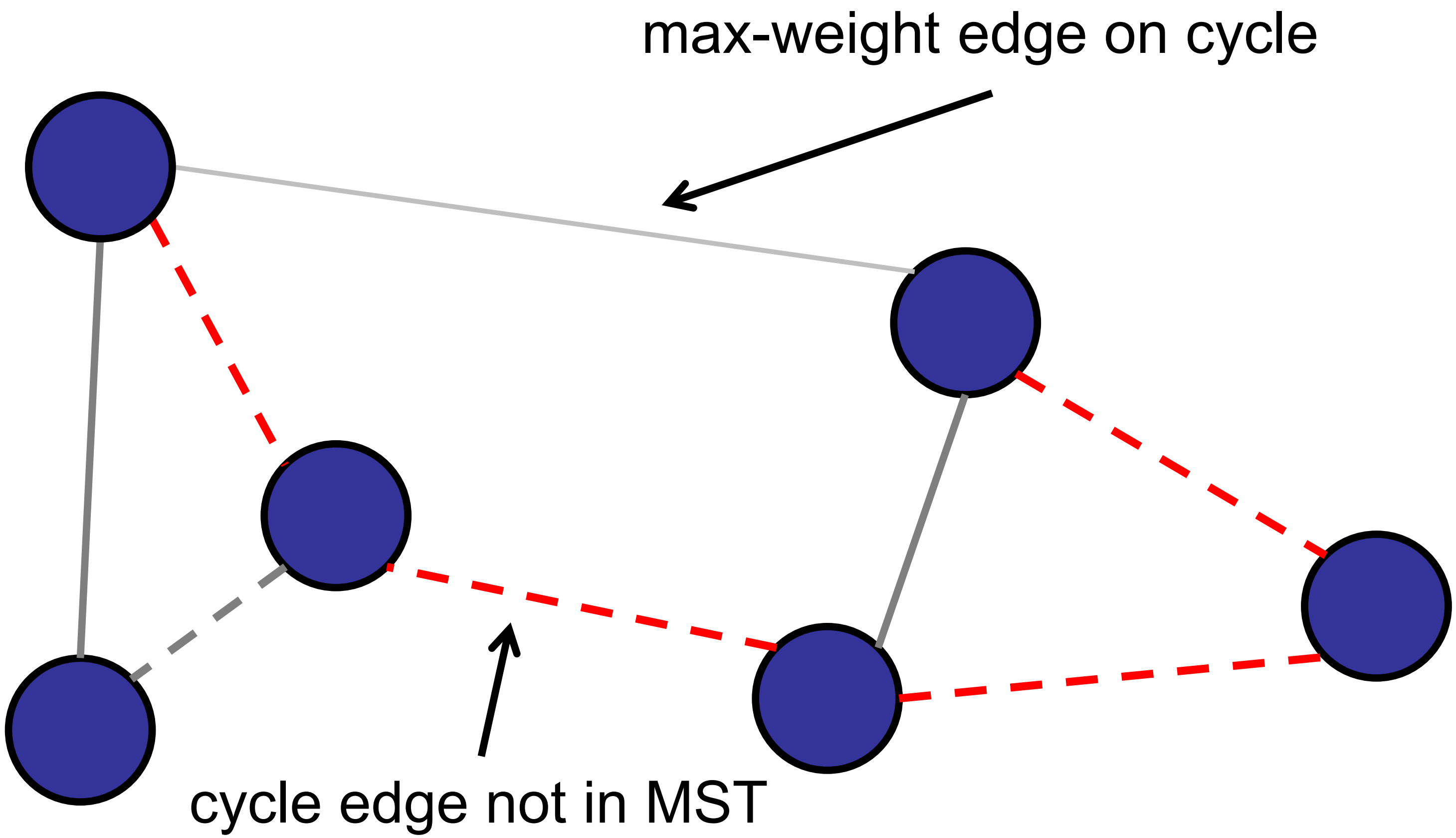


Properties of MST

Proof: Cut-and-paste

Replace heavy edge with lighter edge.

Still a spanning tree: Property 2.

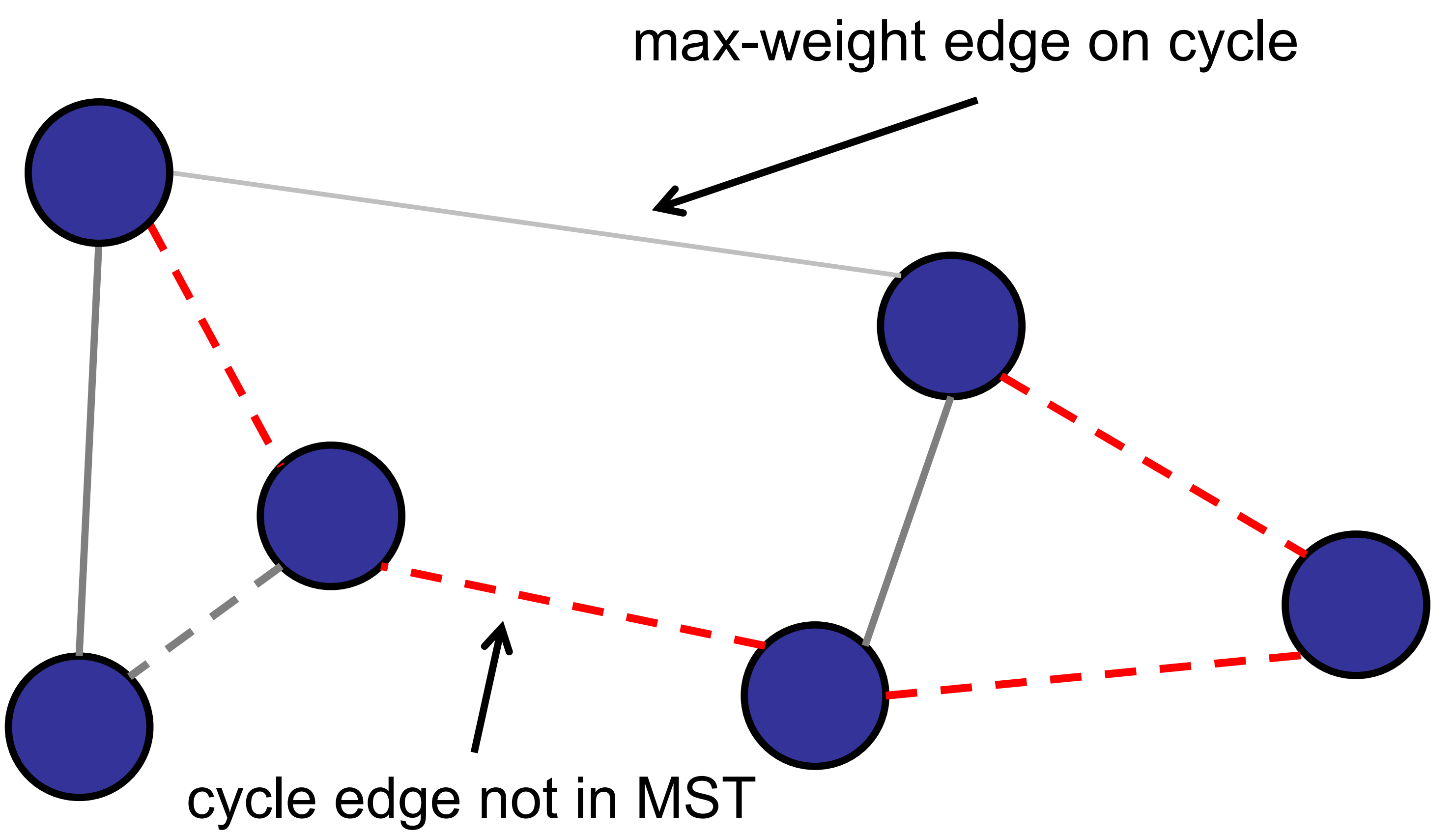


Properties of MST

Proof: Cut-and-paste

Replace heavy edge with lighter edge.

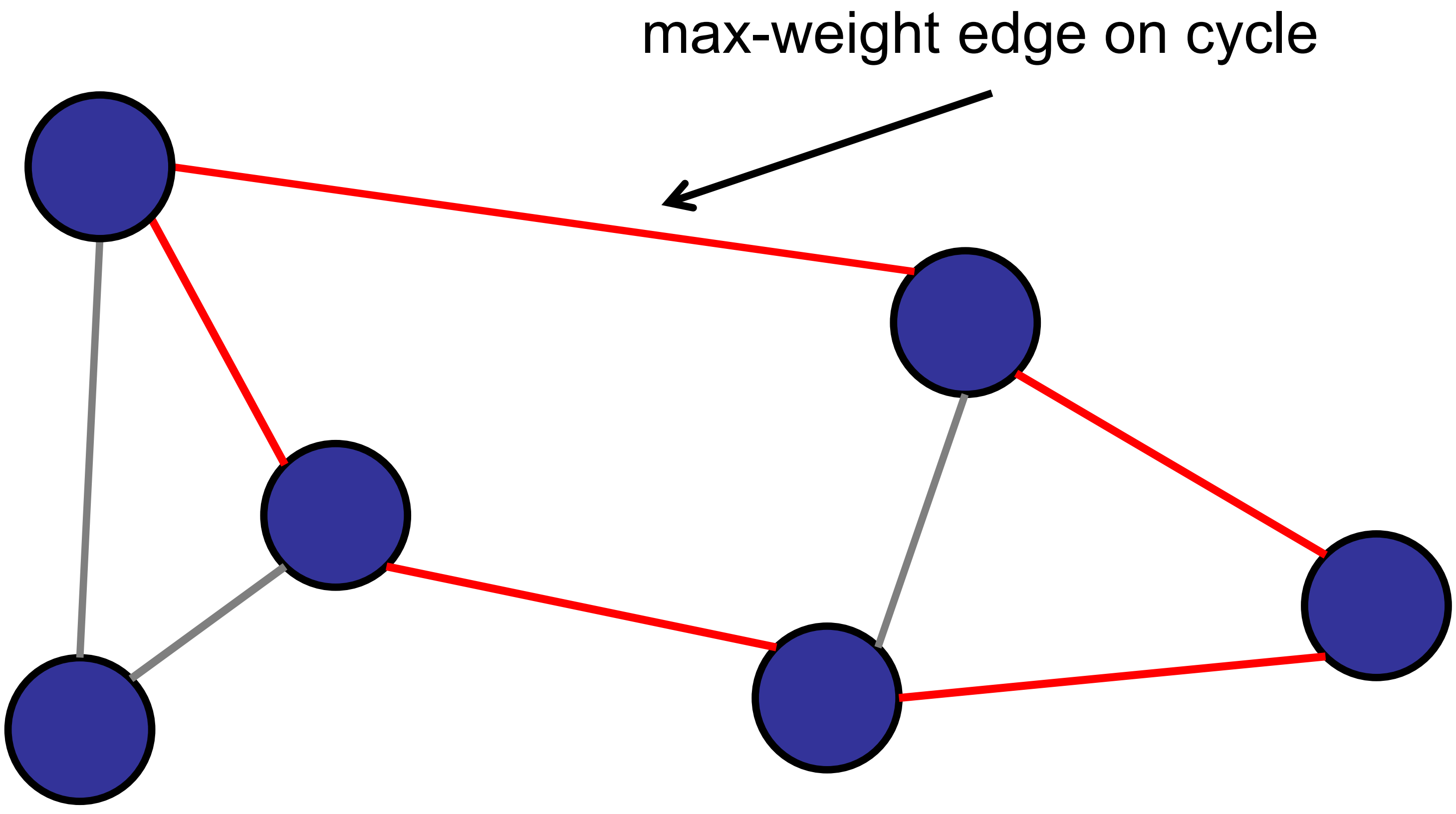
Less weight! Contradiction...



Properties of MST

Property 3: Cycle property

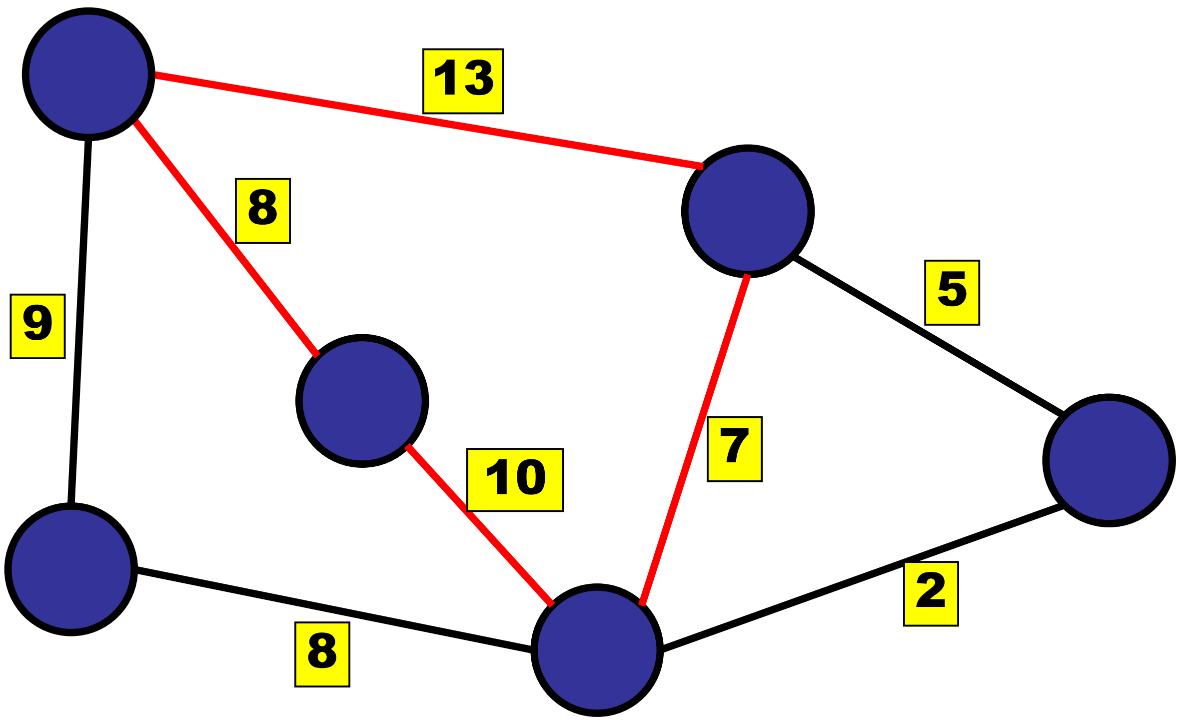
For every cycle, the maximum weight edge is ***not*** in the MST.



Properties of MST

Property 3: False Cycle property

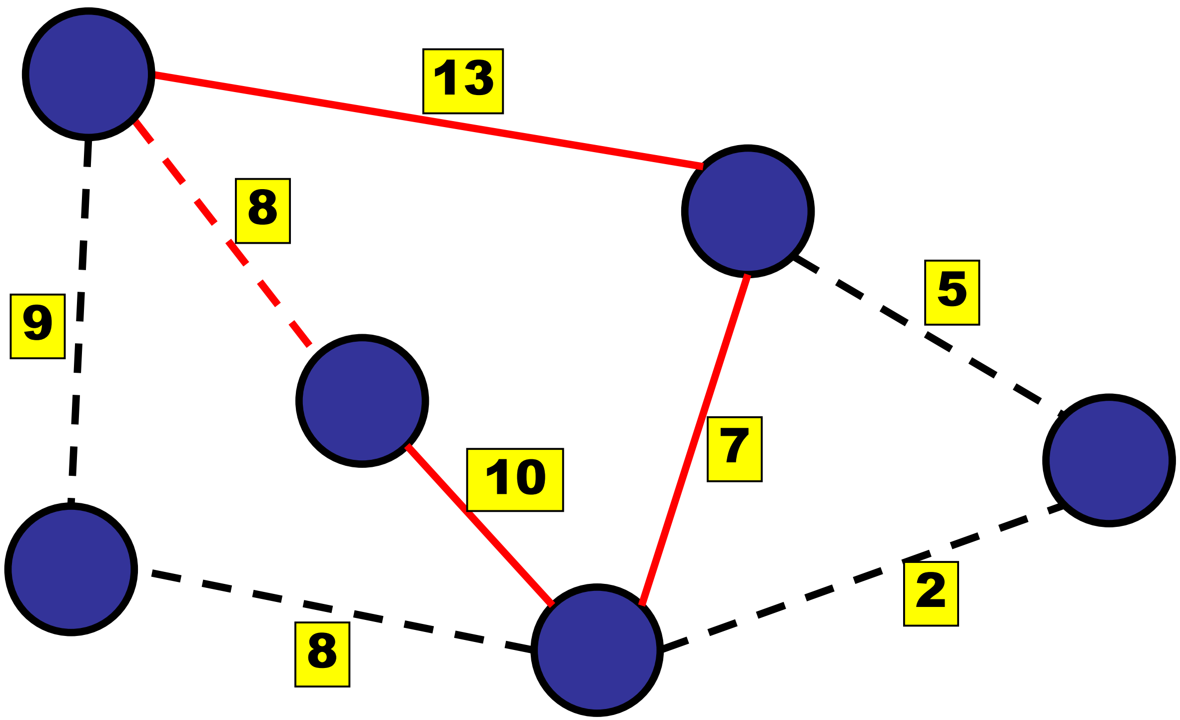
For every cycle, the minimum weight edge ***may or may not*** be in the MST.



Properties of MST

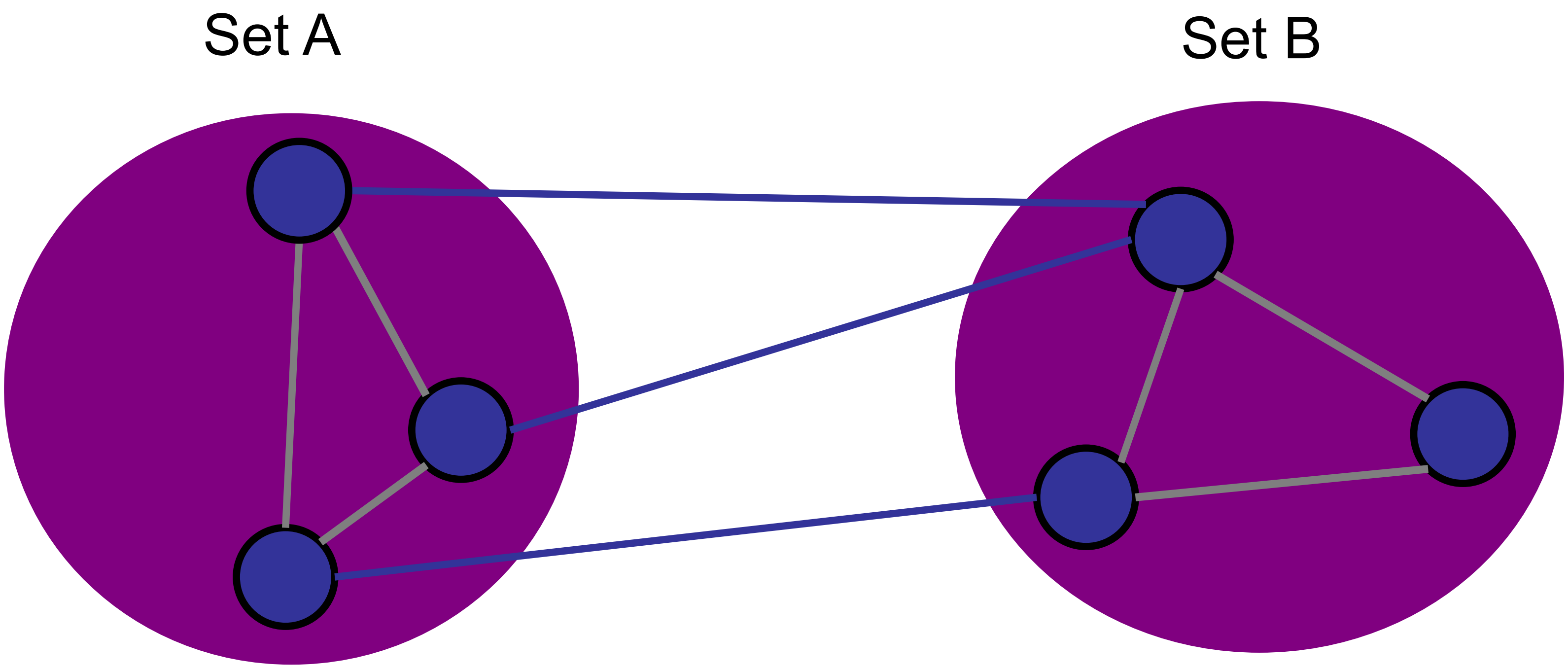
Property 3: False Cycle property

For every cycle, the minimum weight edge ***may or may not*** be in the MST



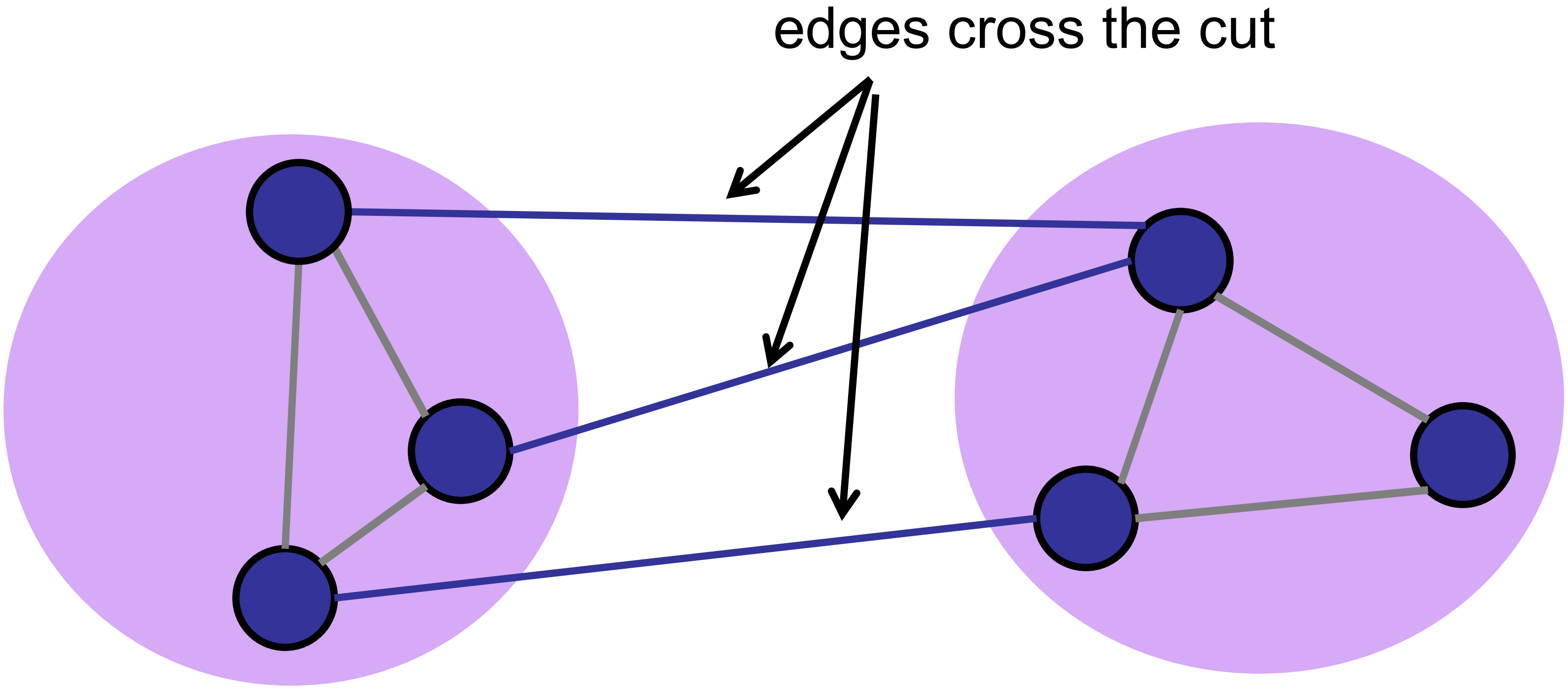
Properties of MST

Definition: A *cut* of a graph G=(V,E) is a partition of the vertices V into two disjoint subsets.



Properties of MST

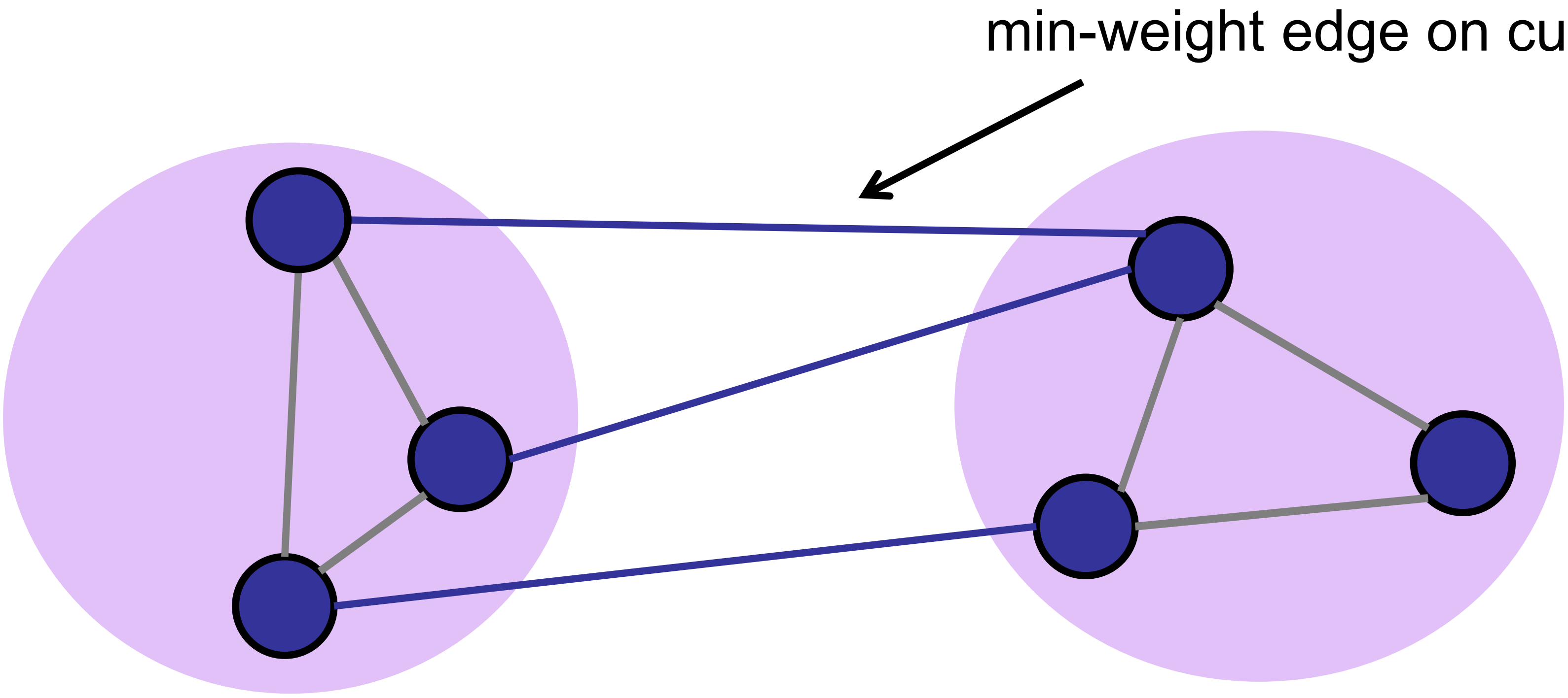
Definition: An edge *crosses a cut* if it has one vertex in each of the two sets.



Properties of MST

Property 4: Cut property

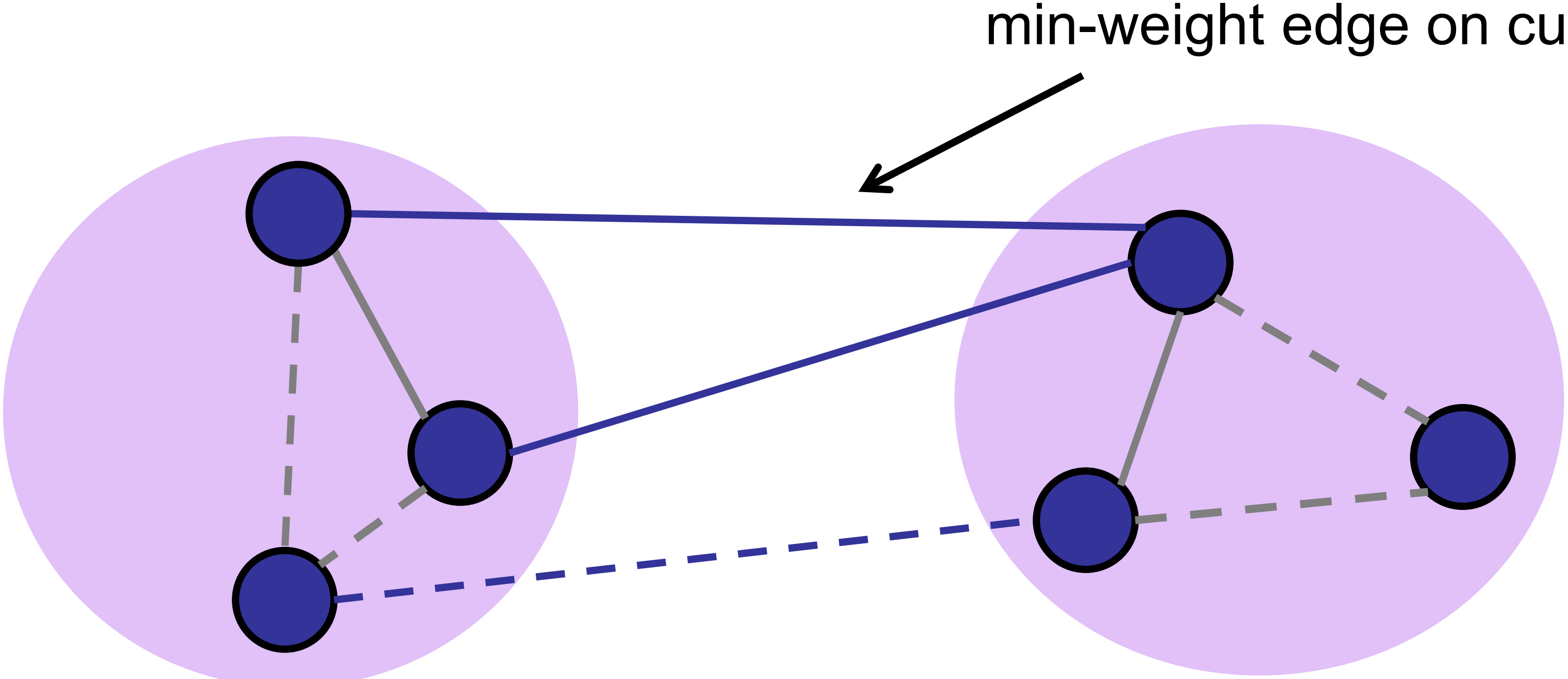
For every partition of the nodes, the minimum weight edge across the cut ***is*** in the MST.



Properties of MST

Proof: Cut-and-paste

Assume not.

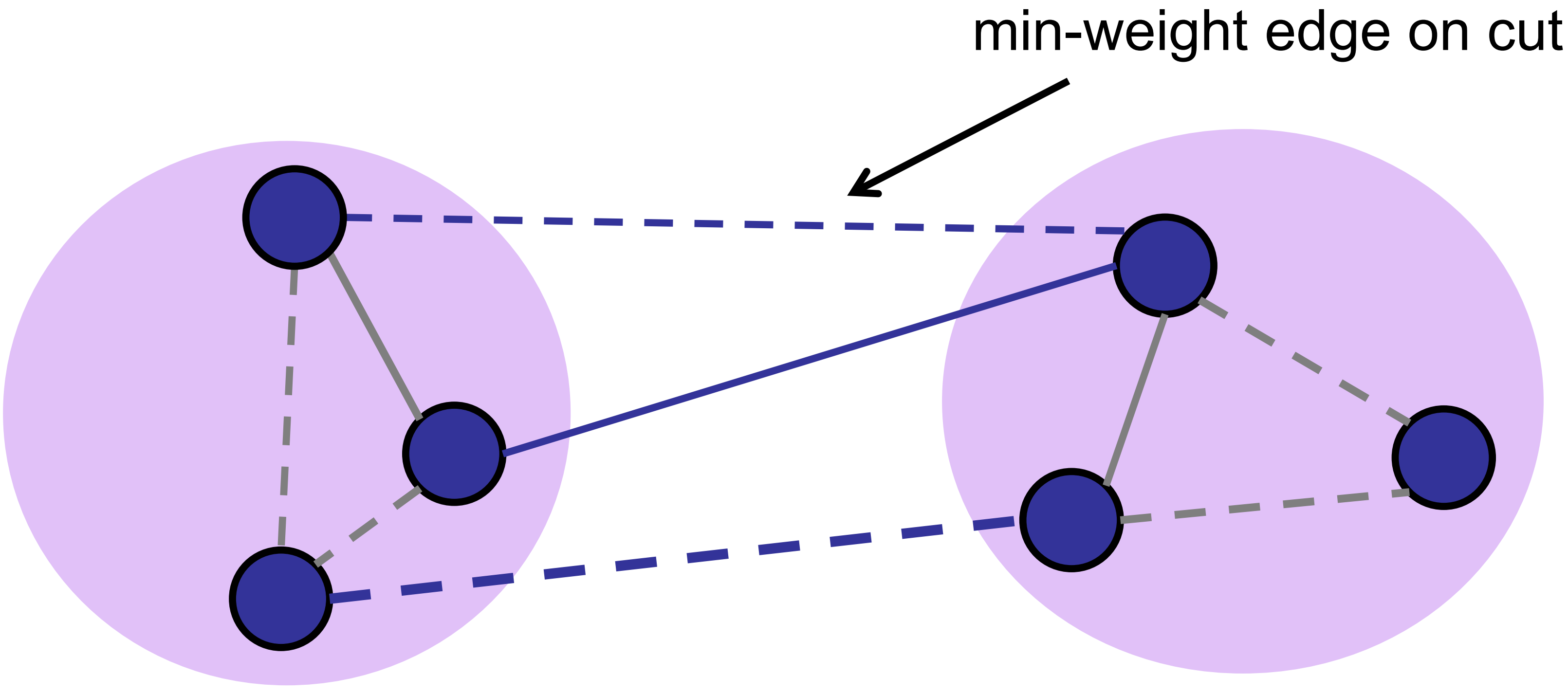


Properties of MST

Proof: Cut-and-paste

Assume not.

Add min-weight edge on cut.



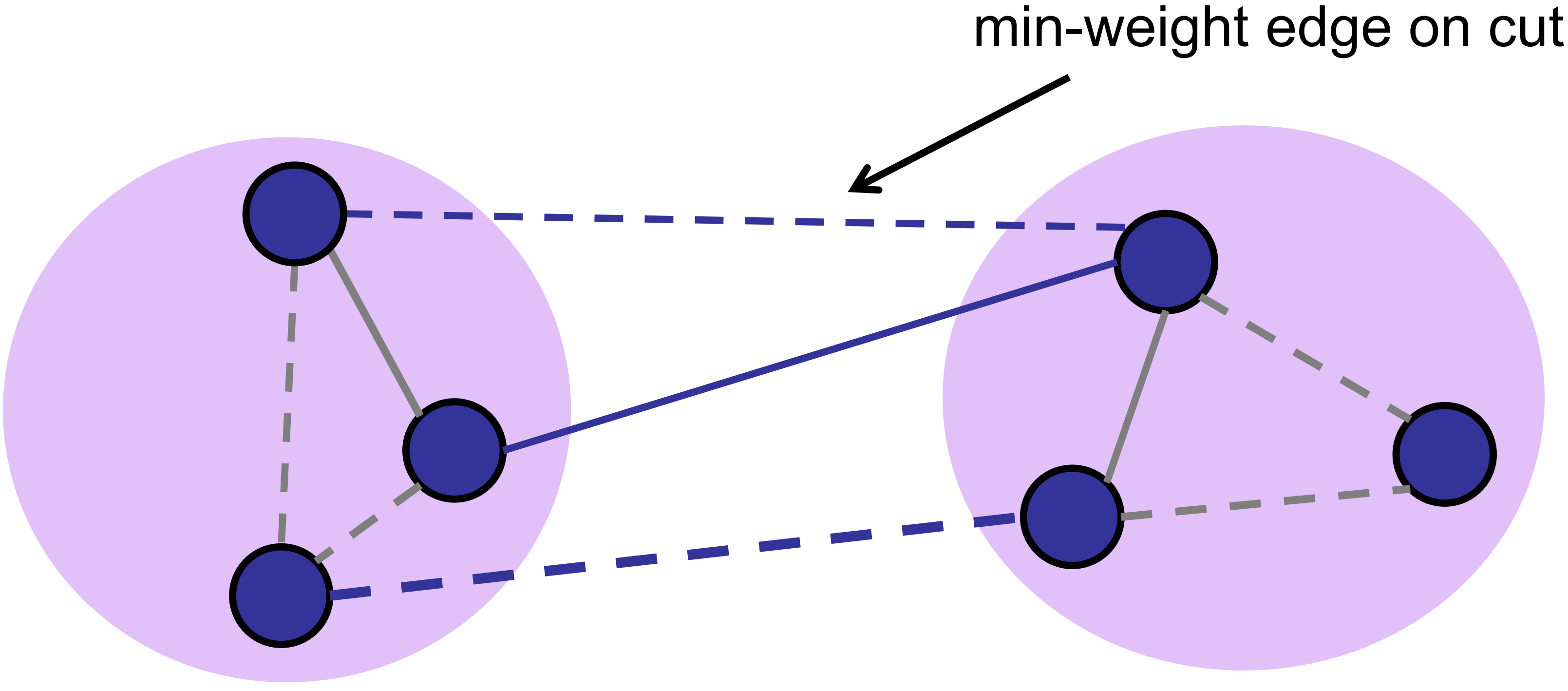
Properties of MST

Proof: Cut-and-paste

Assume not.

Add min-weight edge on cut.

Oops, creates a cycle!



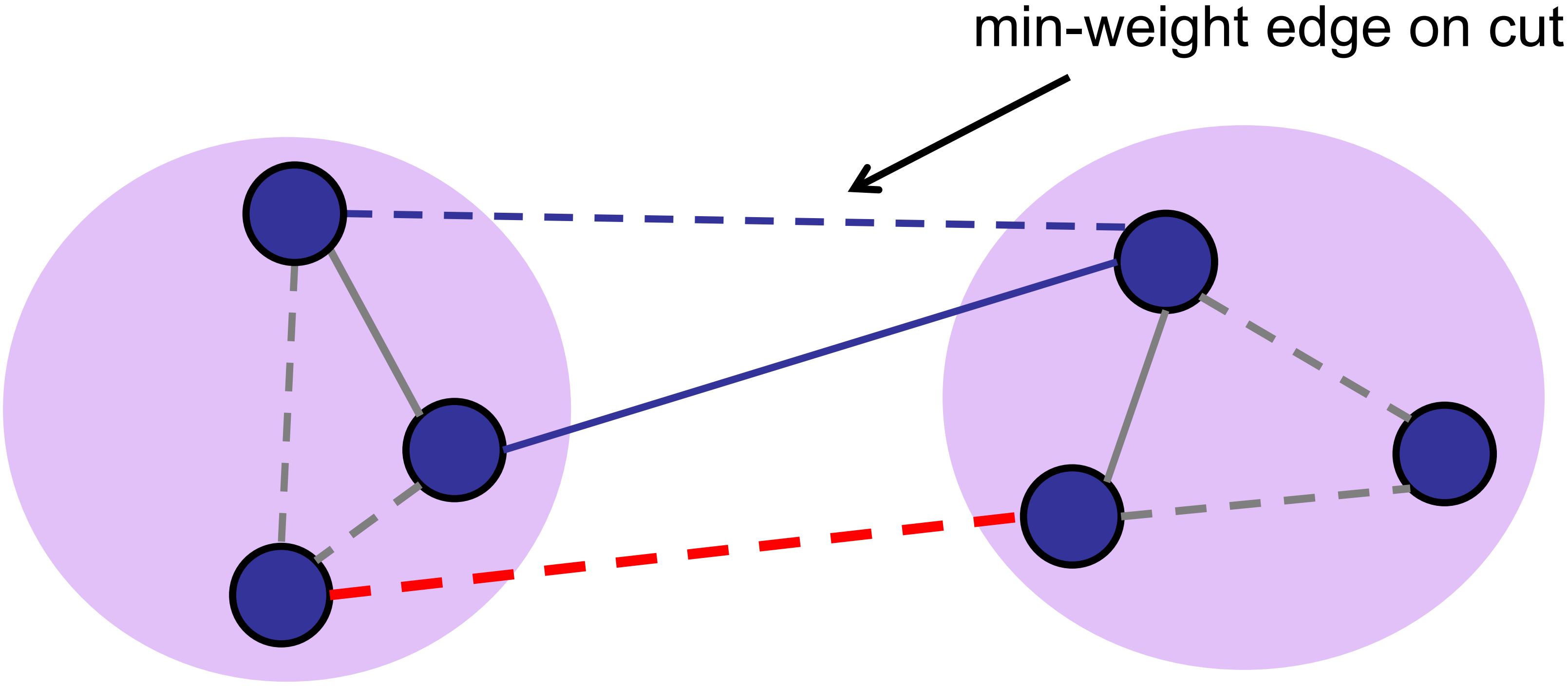
Properties of MST

Proof: Cut-and-paste

Assume not.

Add min-weight edge on cut.

Remove heaviest edge on cycle.



Properties of MST

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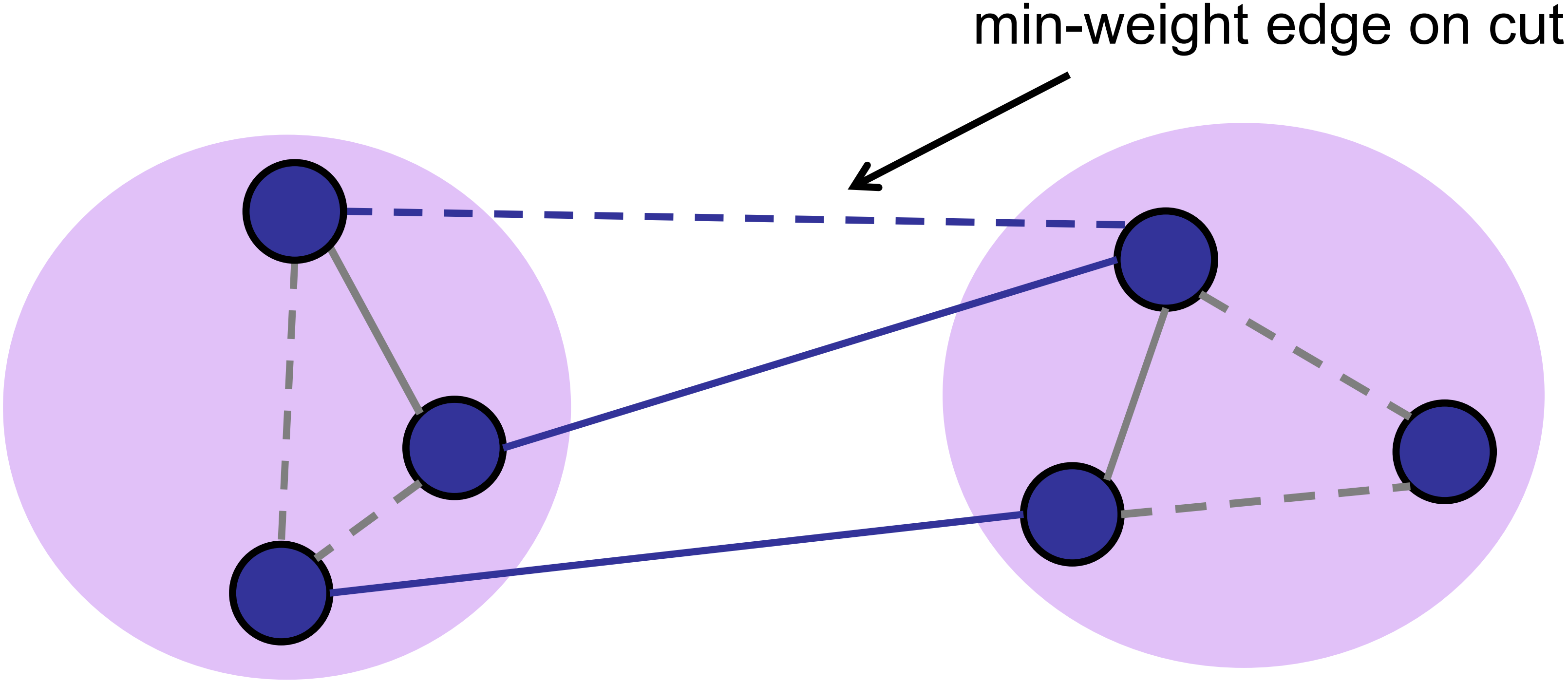
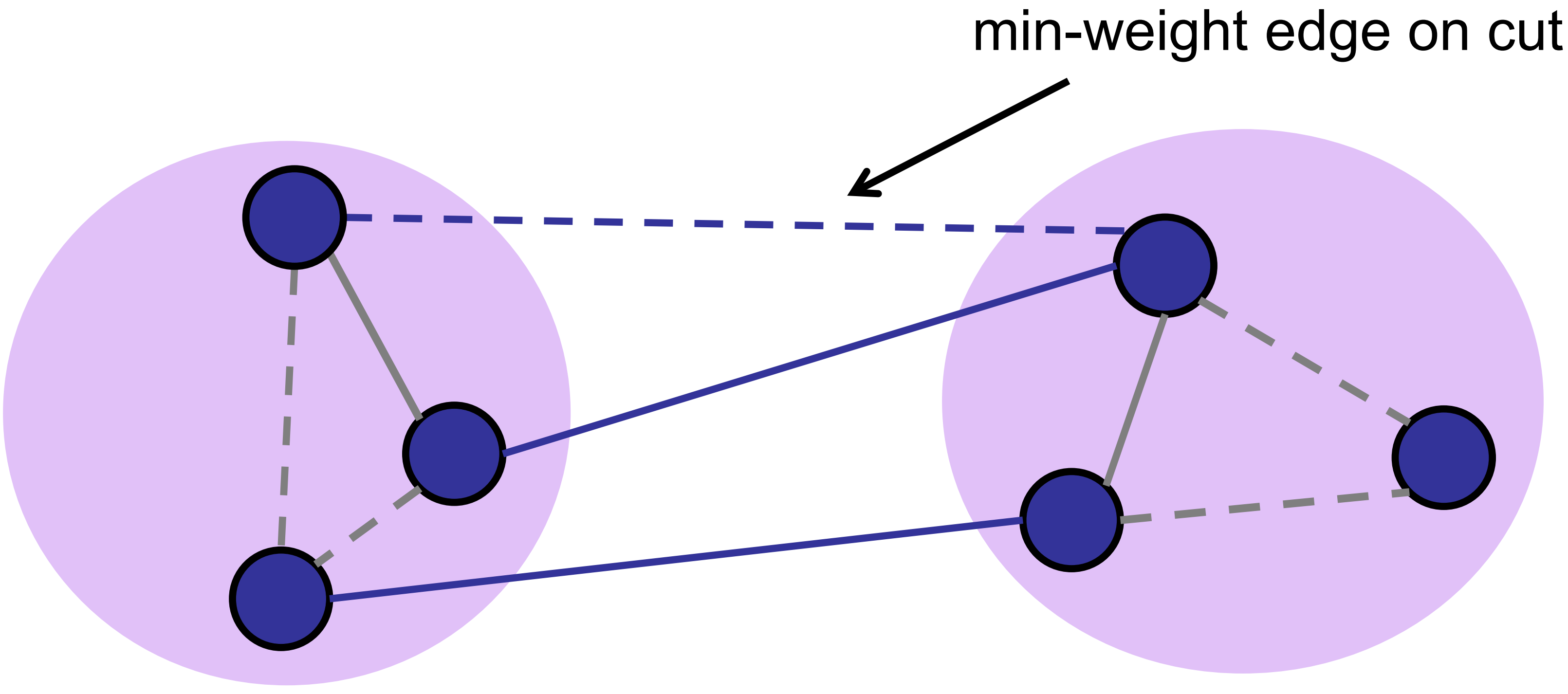


Properties of MST

Proof: Cut-and-paste

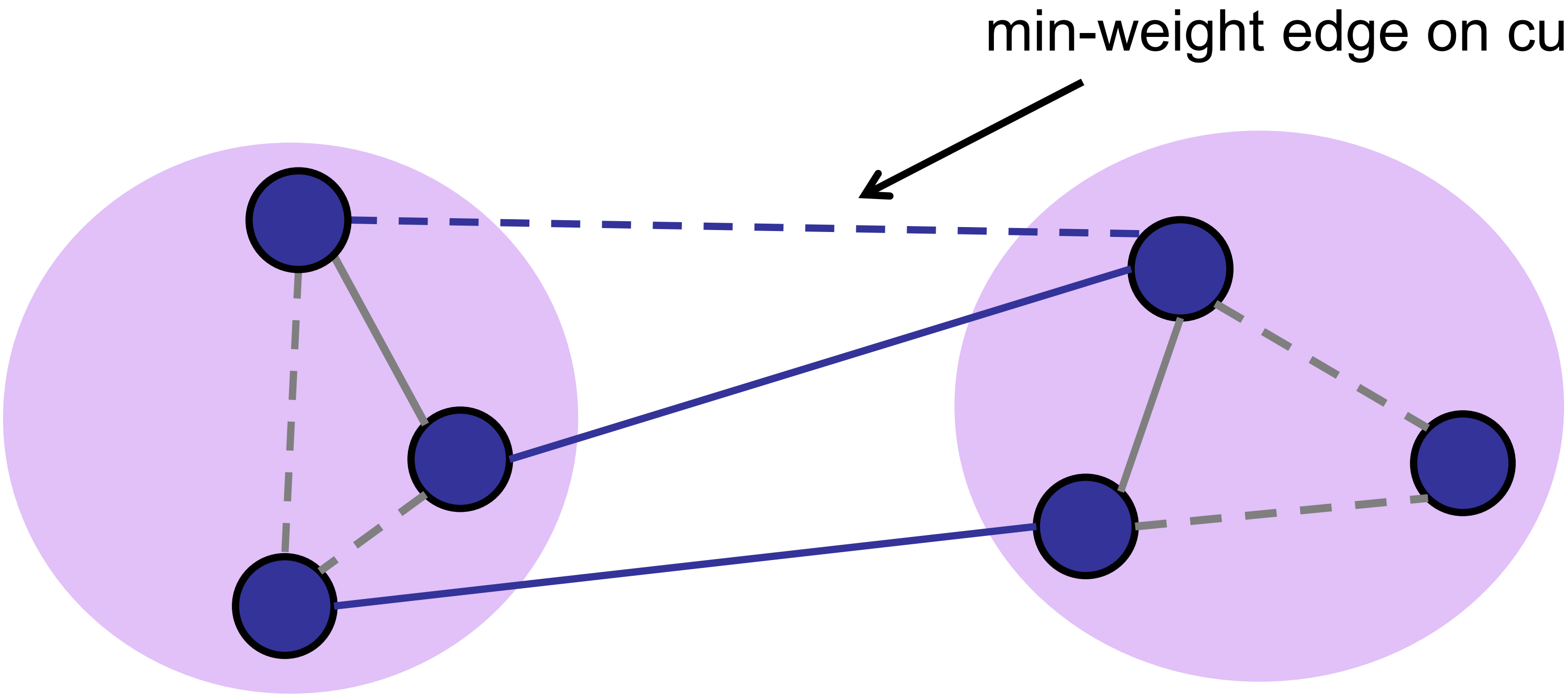
Result: a new spanning tree.



Properties of MST

Proof: Cut-and-paste

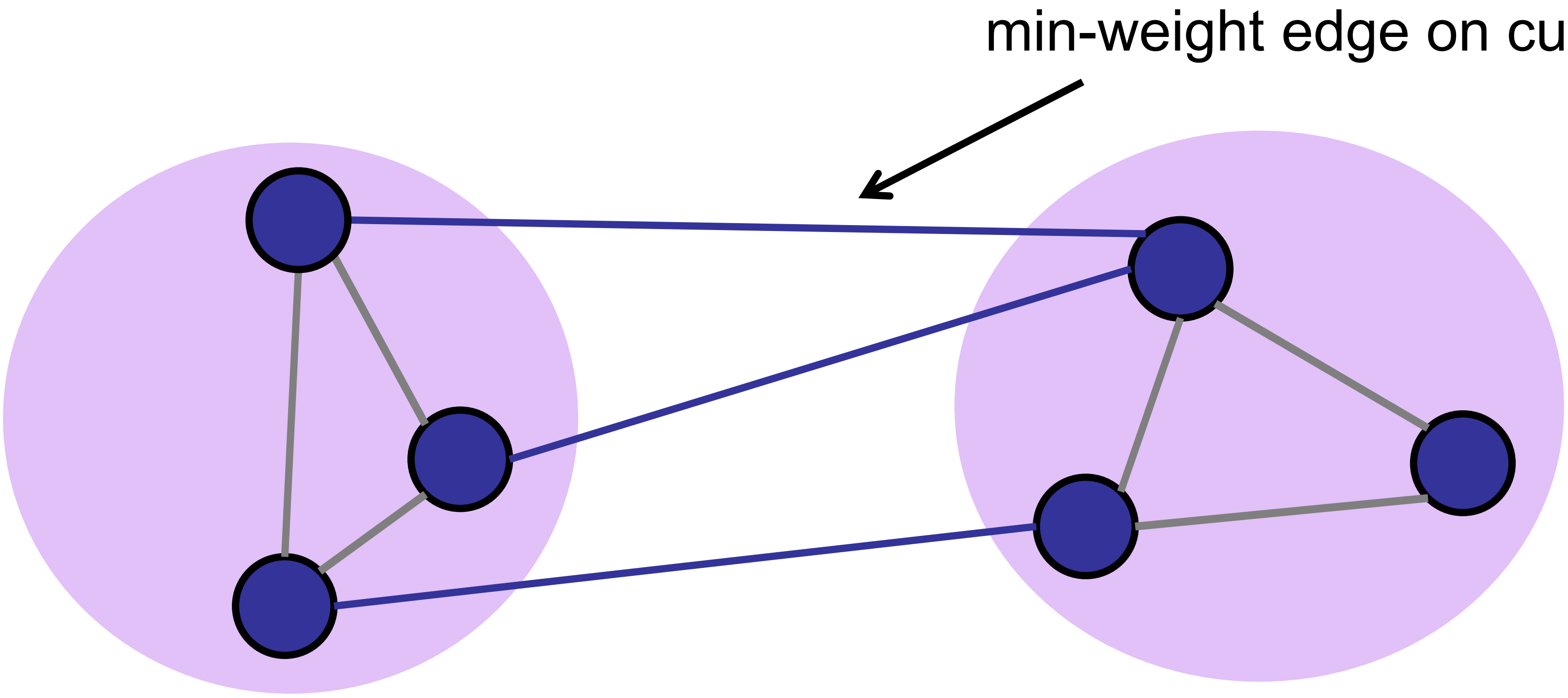
Less weight: replaced heavier edge with lighter edge.



Properties of MST

Property 4: Cut property

For every partition of the nodes, the minimum weight edge across the cut *is* in the MST.



Properties of MST

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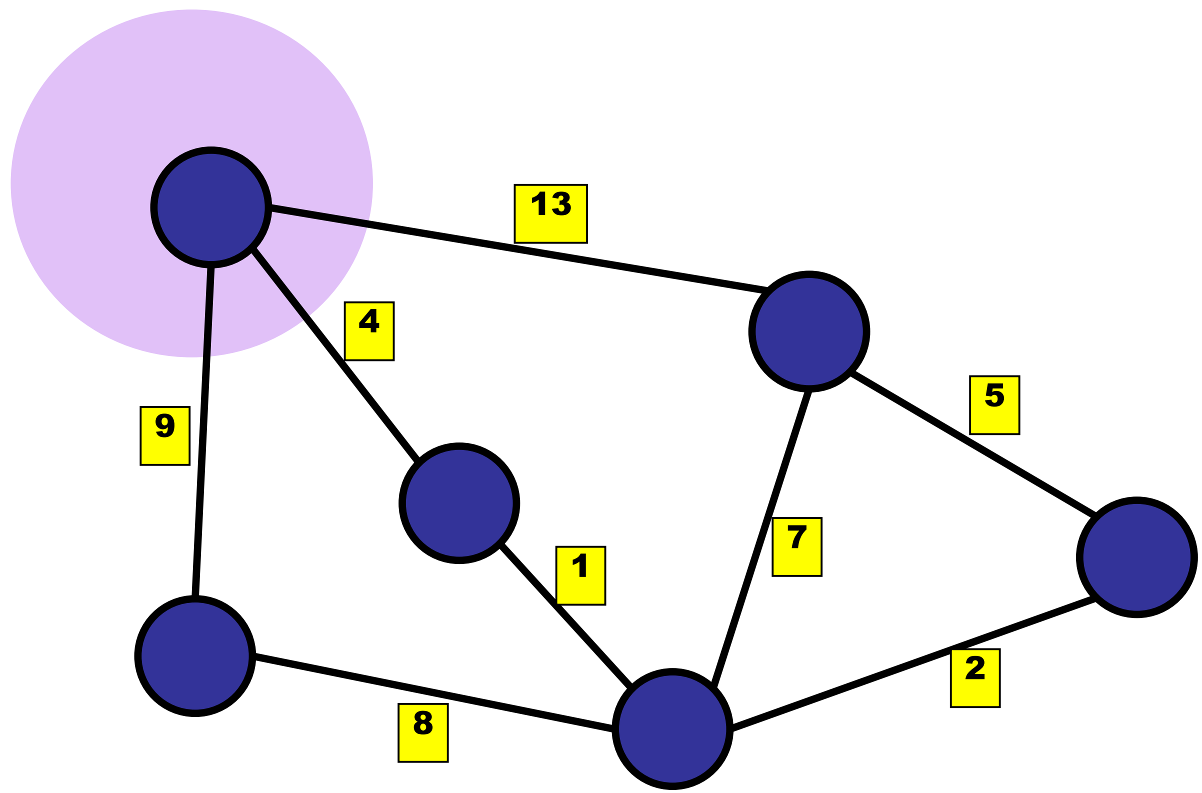
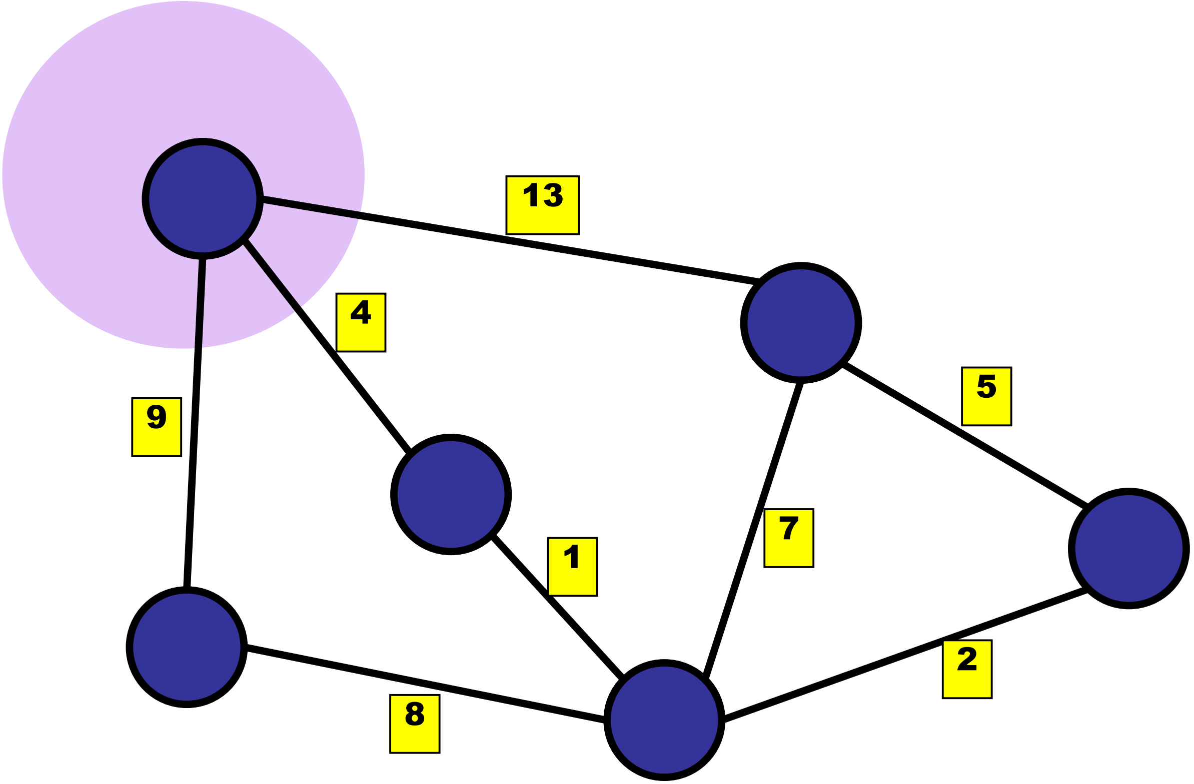
For every partition of the nodes, the minimum weight edge across the cut *is* in the MST.



Properties of MST

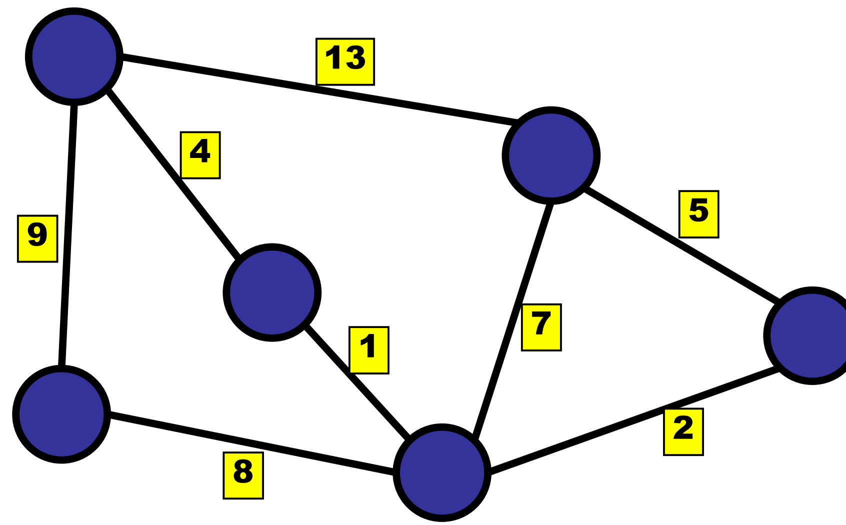
Property 4**b**: Cut property

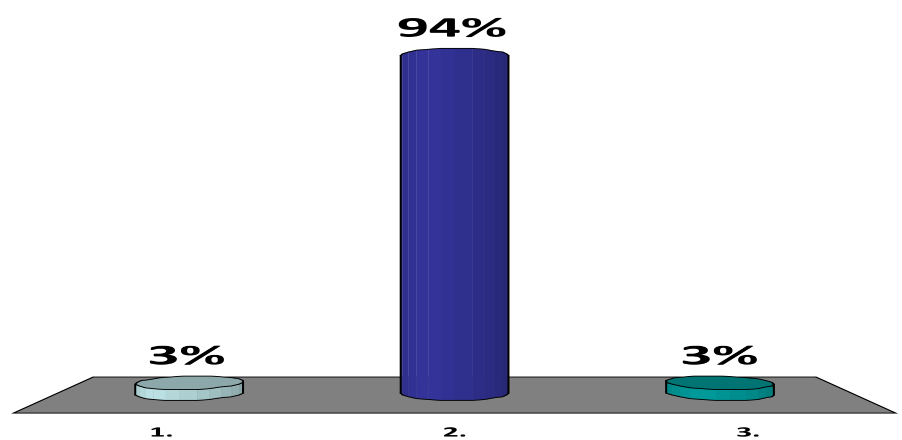
For every vertex, the minimum outgoing edge is always part of the MST



For every vertex, the maximum outgoing edge is never part of the MST.

1. True
2. False
3. I don't know.

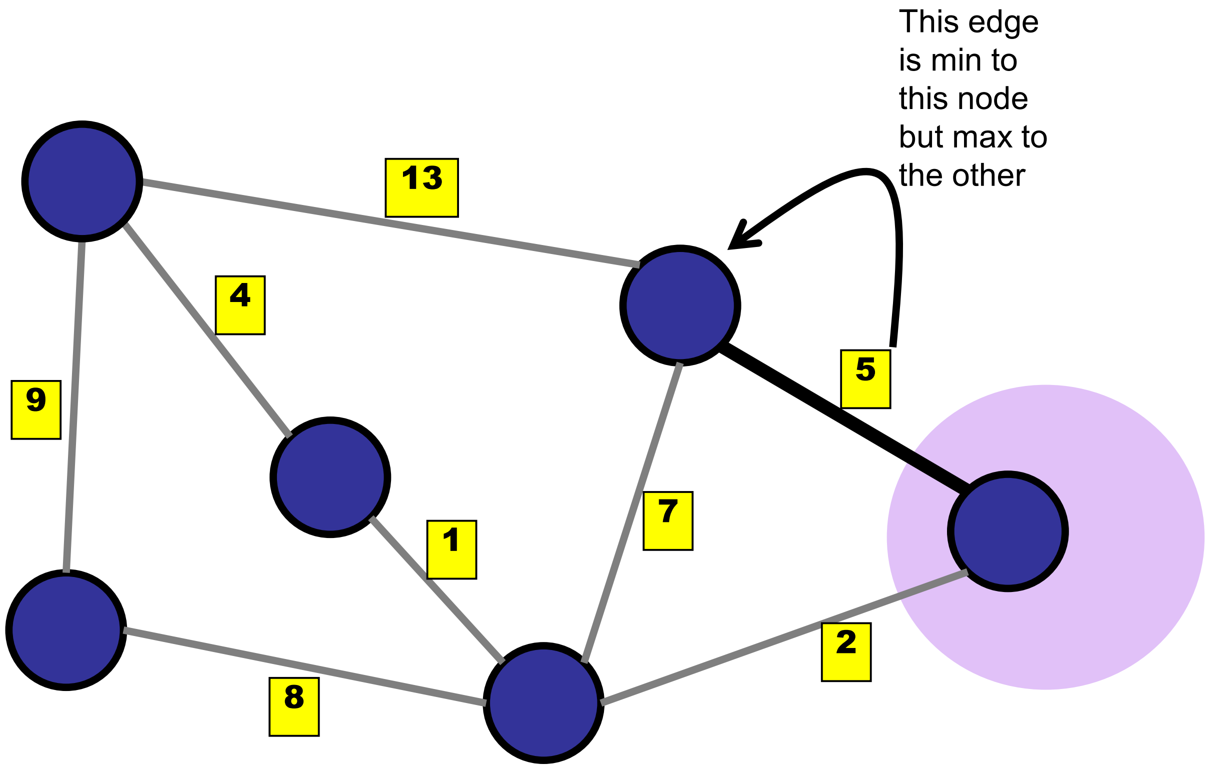




Properties of MST

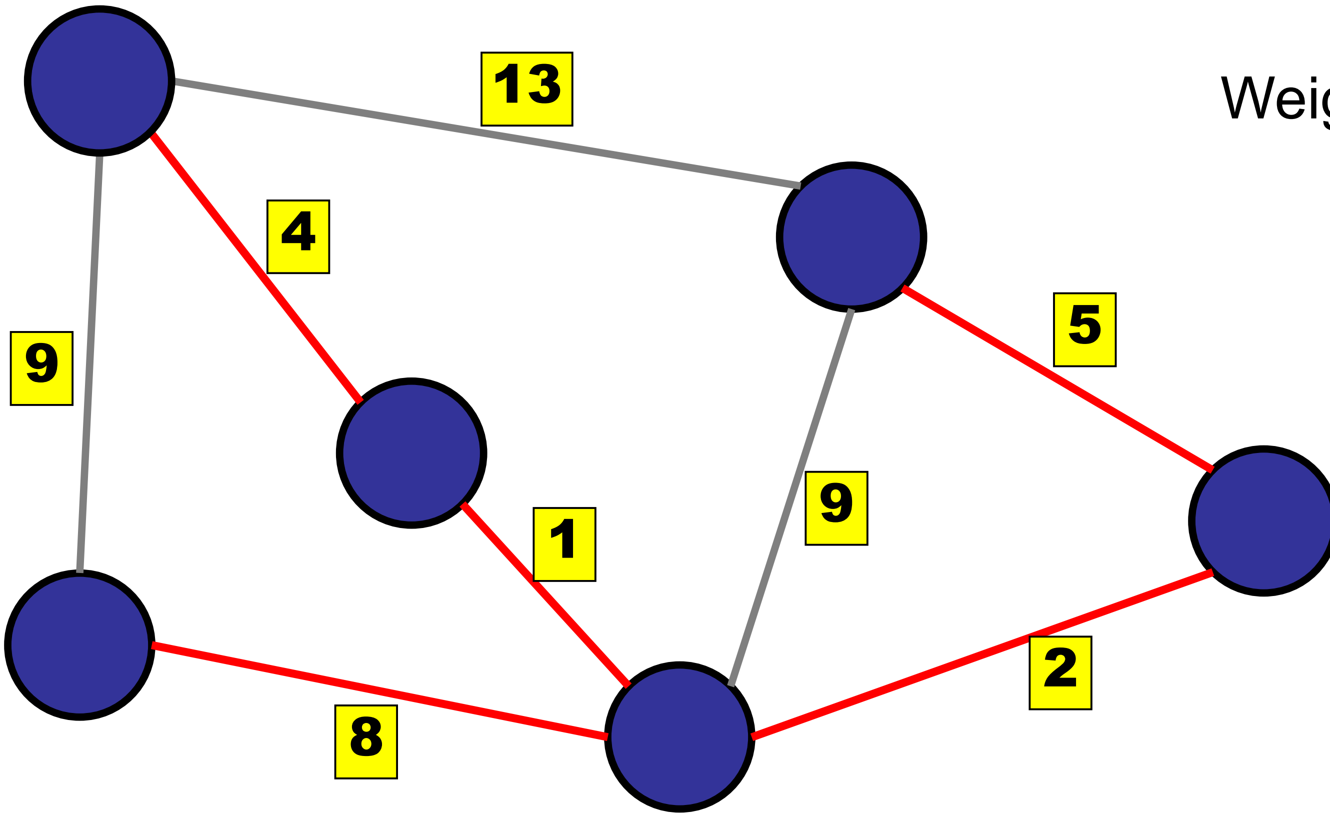
Property 4: Cut property

For every partition of the nodes, the minimum weight edge across the cut *is* in the MST.



Minimum Spanning Tree

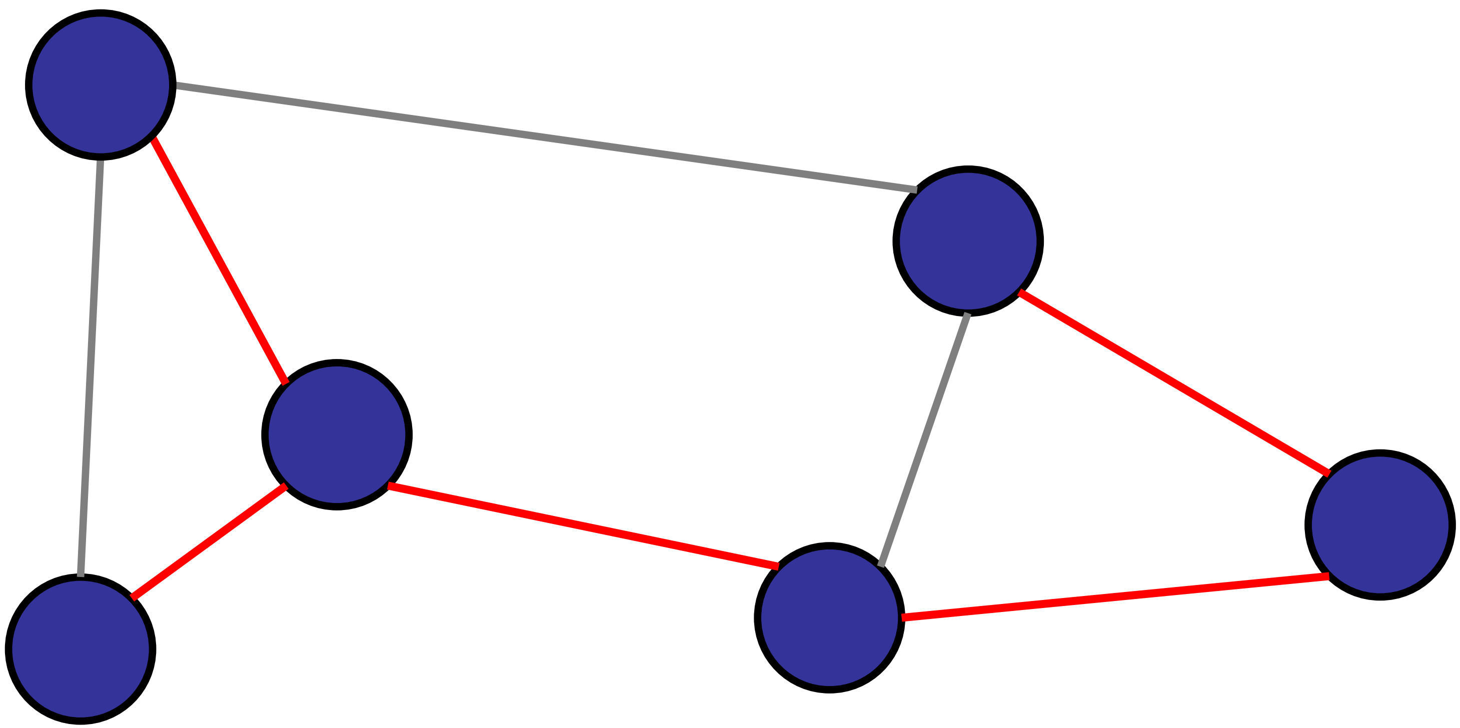
Definition: a spanning tree with minimum weight



Weight: 20

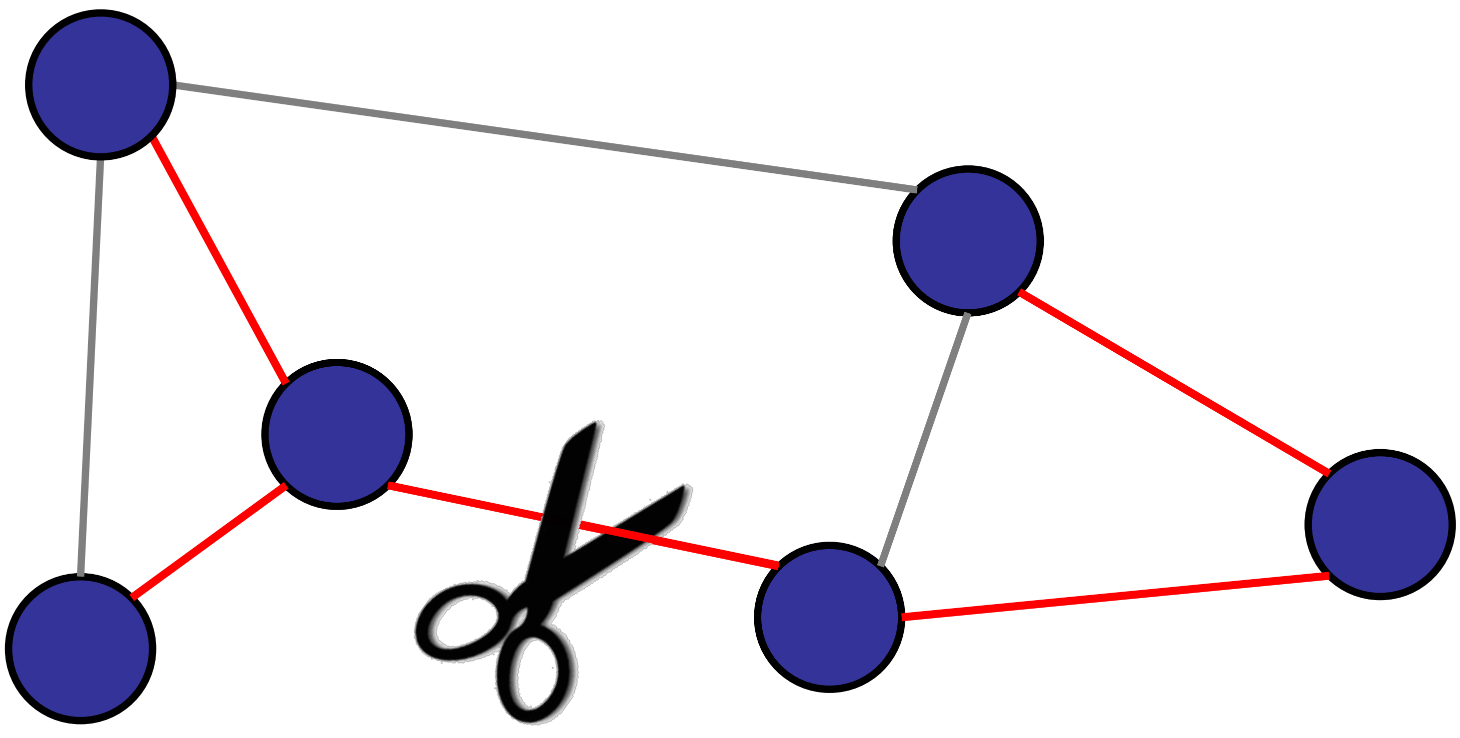
Properties of MST

Property 1: No cycles



Properties of MST

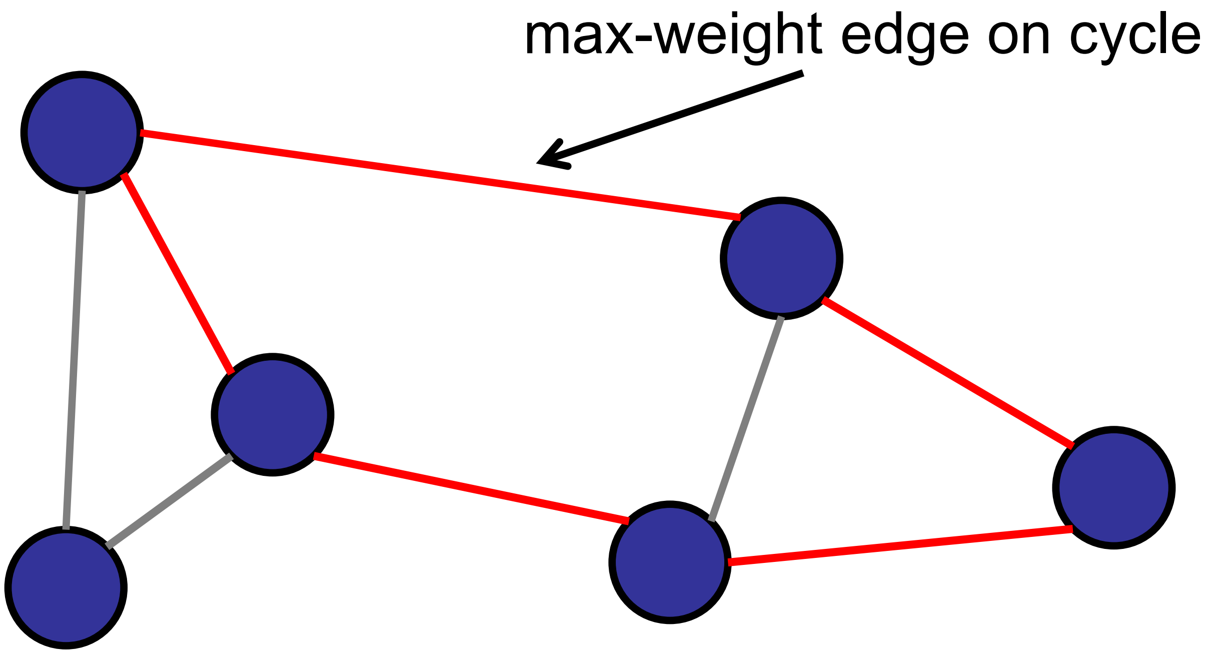
Property 2: If you cut an MST, the two pieces are both MSTs.



Properties of MST

Property 3: Cycle property

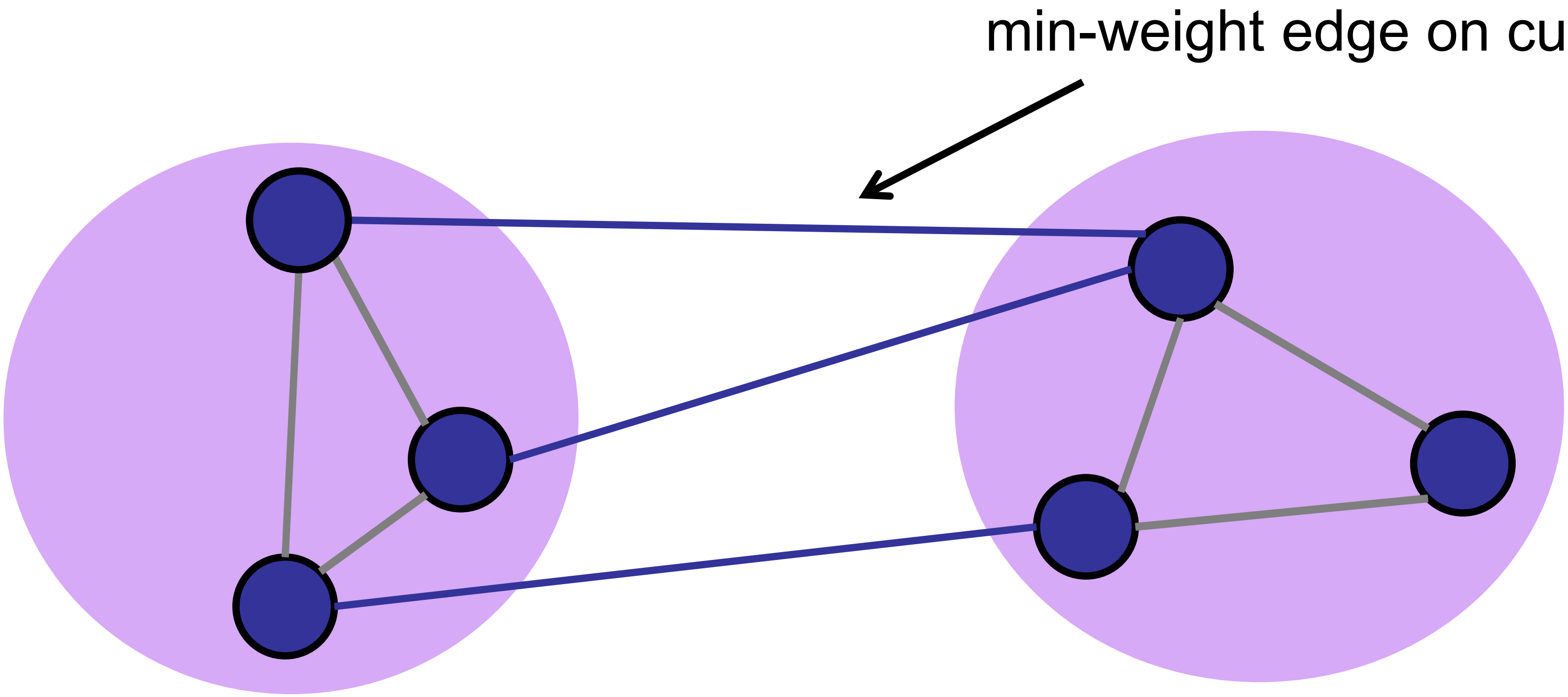
For every cycle, the maximum weight edge is *not* in the MST.



Properties of MST

Property 4: Cut property

For every cut D, the minimum weight edge that crosses the cut *is* in the MST.



Property of MST

- No cycles
- If you cut an MST, the two pieces are both MSTs.
- Cycle property
  - For every cycle, the maximum weight edge is not in the MST.
- Cut property
  - For every cut D, the minimum weight edge that crosses the cut is in the MST.

Roadmap

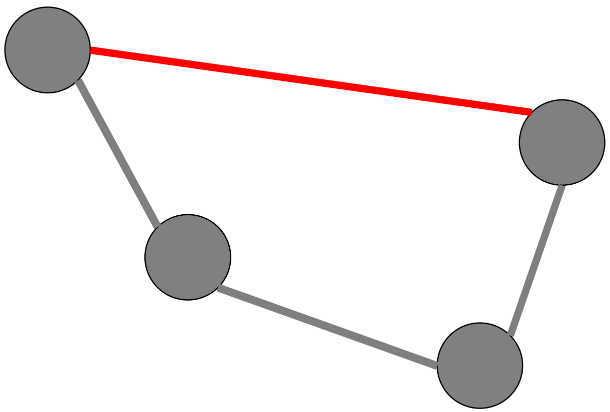
Minimum Spanning Trees

- The MST Problem
- Basic Properties of an MST
- **Generic MST Algorithm**
- Prim's Algorithm
- Kruskal's Algorithm
- Boruvka's Algorithm
- Variations

Generic MST Algorithm

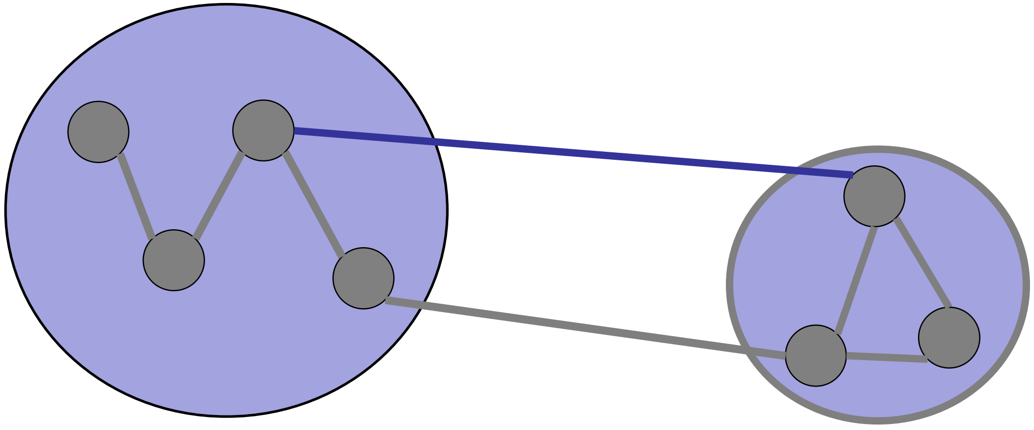
Red rule: (Property 3)

If C is a cycle with no red arcs, then color the max-weight edge in C red.



Blue rule: (Property 4)

If D is a cut with no blue arcs, then color the min-weight edge in D blue.



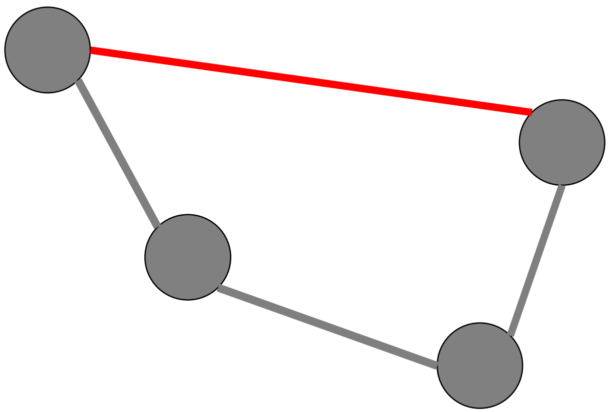
Generic MST Algorithm

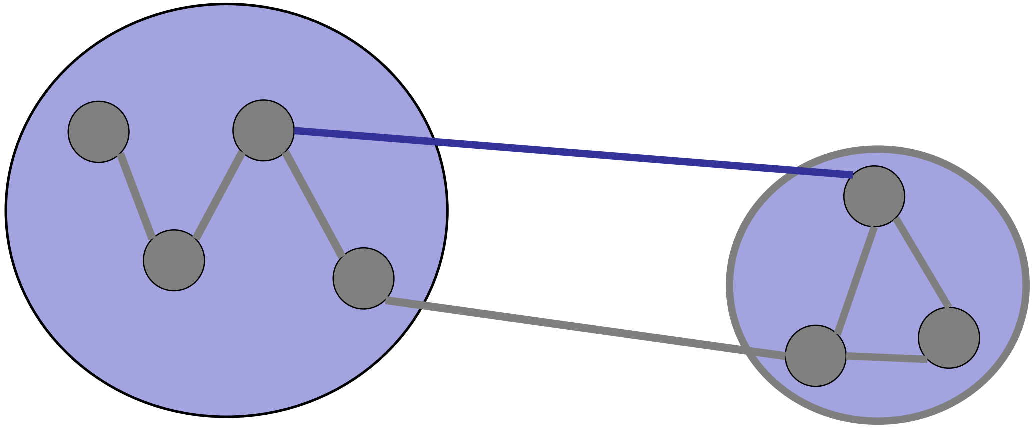
Greedy Algorithm:

Repeat:

Apply red rule or blue rule to an arbitrary edge.

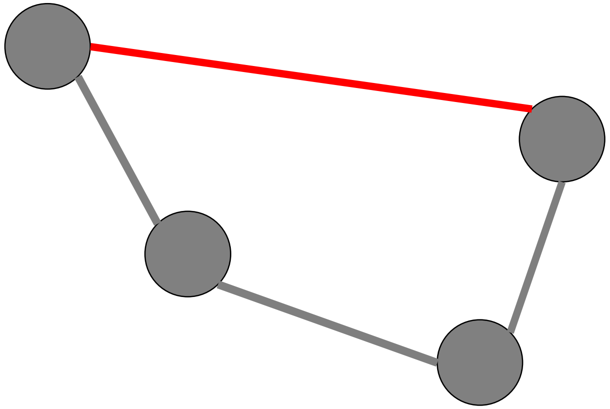
until no more edges can be colored.

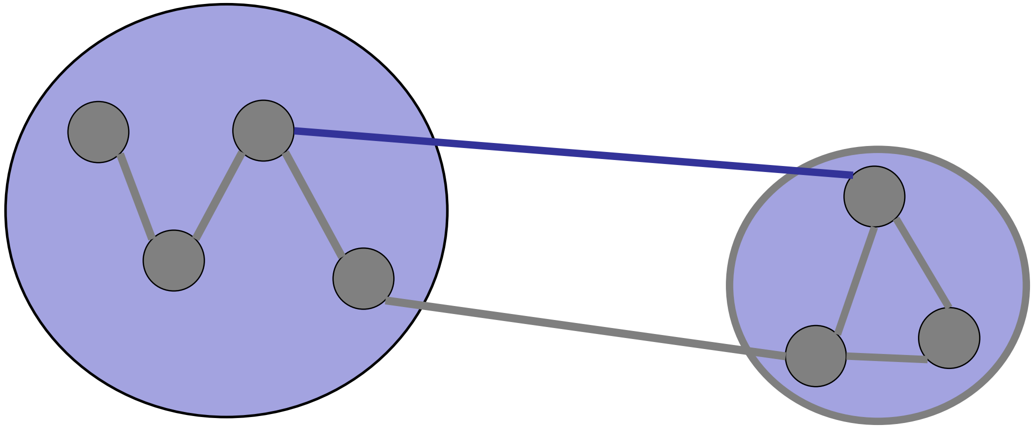




Generic MST Algorithm

Claim: On termination, the blue edges are an MST.



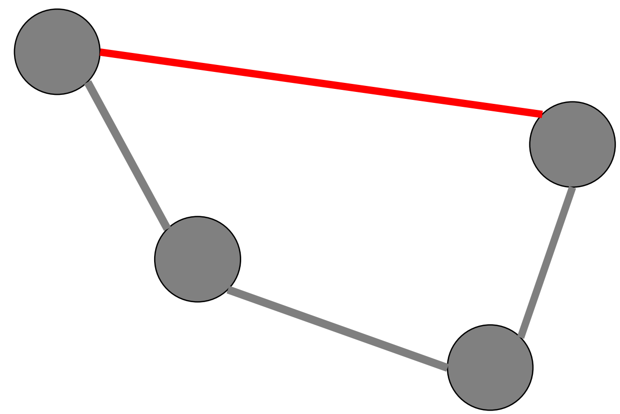


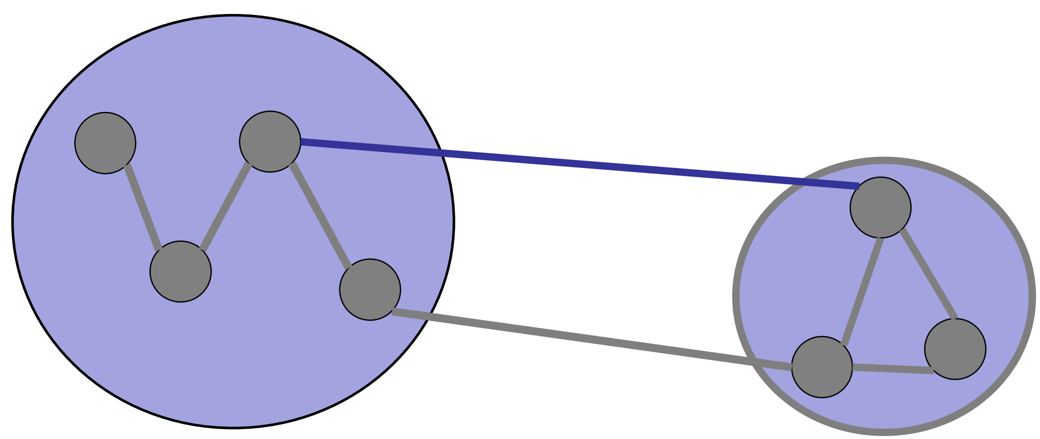
Generic MST Algorithm

Claim: On termination, the blue edges are an MST.

On termination:

1. Every cycle has a red edge.
   No blue cycles.



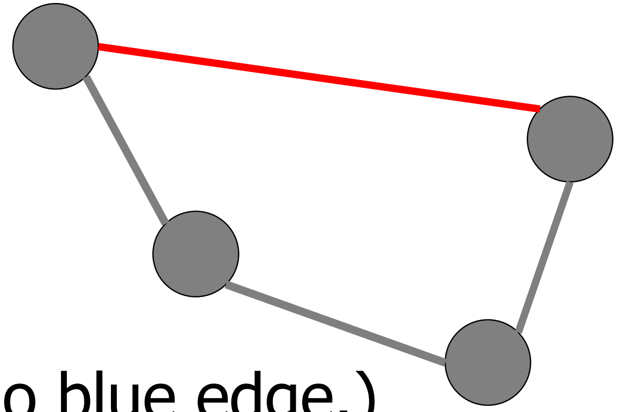


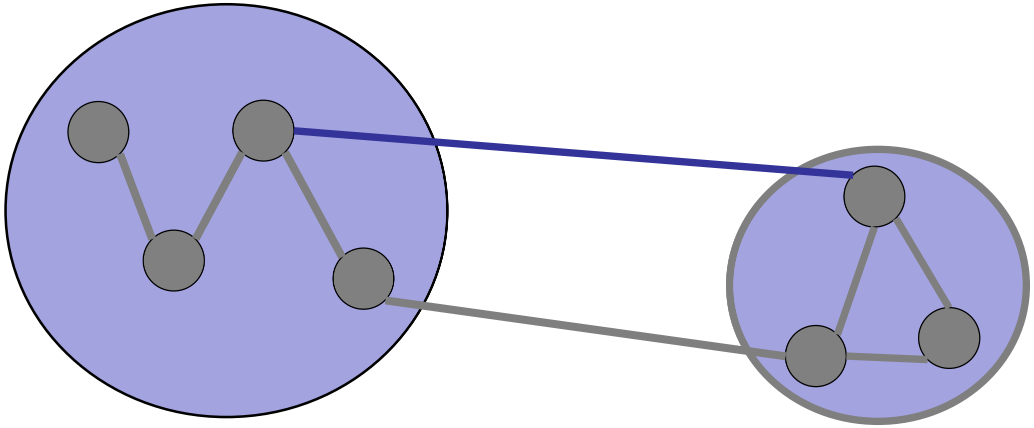
Generic MST Algorithm

Claim: On termination, the blue edges are an MST.

On termination:

1. Every cycle has a red edge.
   No blue cycles.
2. Blue edges form a tree.
   (Otherwise, there is a cut with no blue edge.)



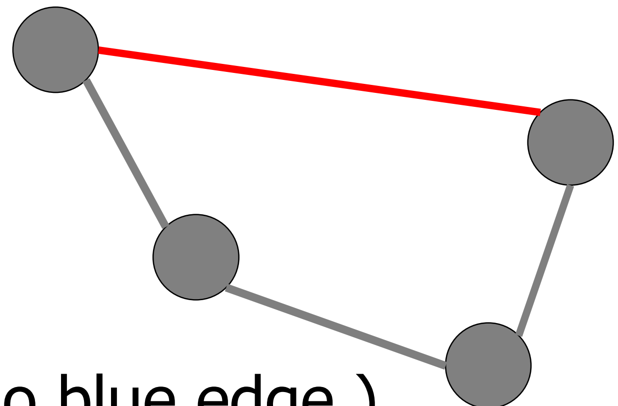


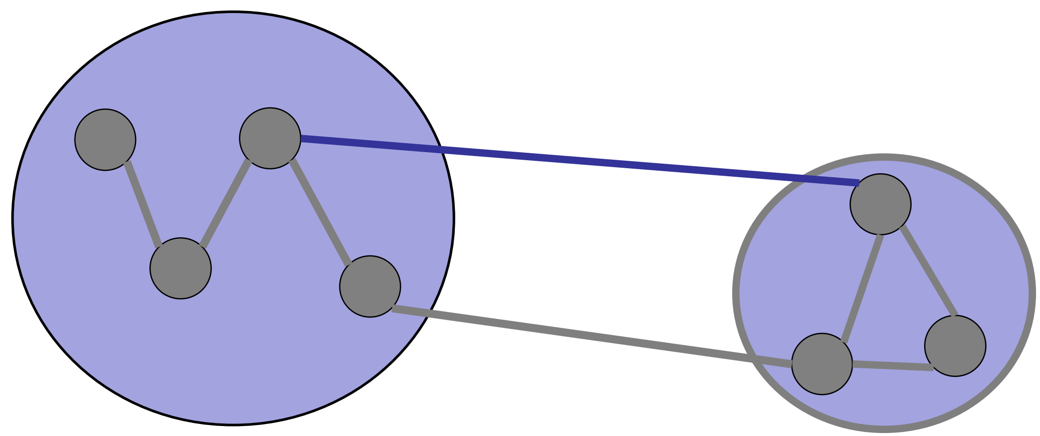
Generic MST Algorithm

Claim: On termination, the blue edges are an MST.

On termination:

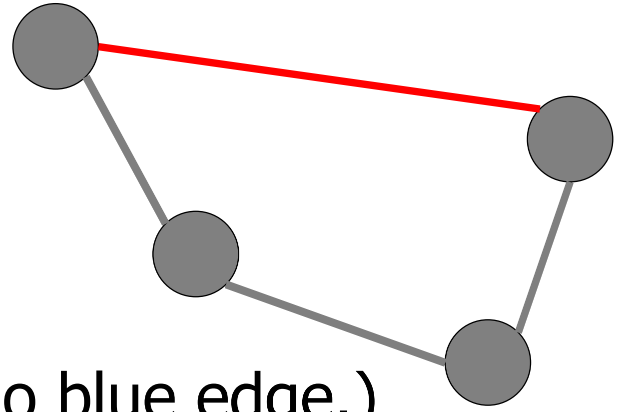
1. Every cycle has a red edge.
   No blue cycles.
2. Blue edges form a tree.
   (Otherwise, there is a cut with no blue edge.)
3. Every edge is colored.





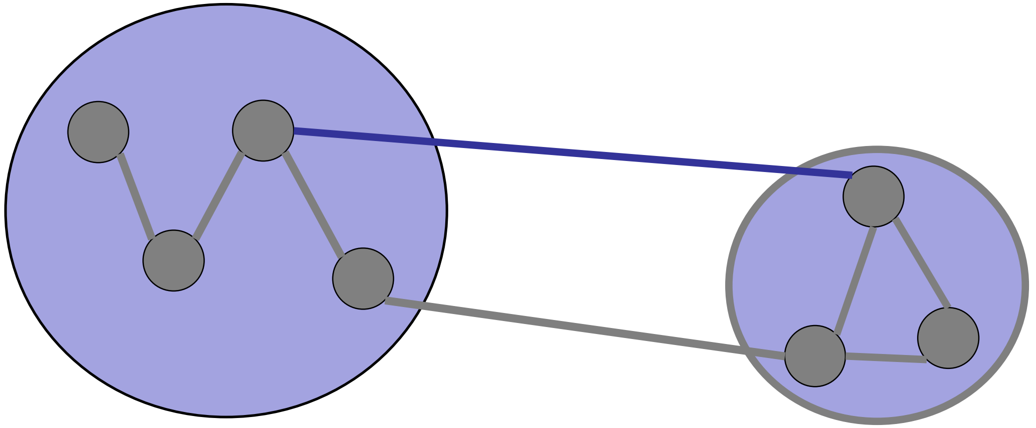
Generic MST Algorithm

Claim: On termination, the blue edges are an MST.



On termination:

1. Every cycle has a red edge.
   No blue cycles.
2. Blue edges form a tree.
   (Otherwise, there is a cut with no blue edge.)
3. Every edge is colored.
4. Every blue edge is in
   the MST (Property 4).



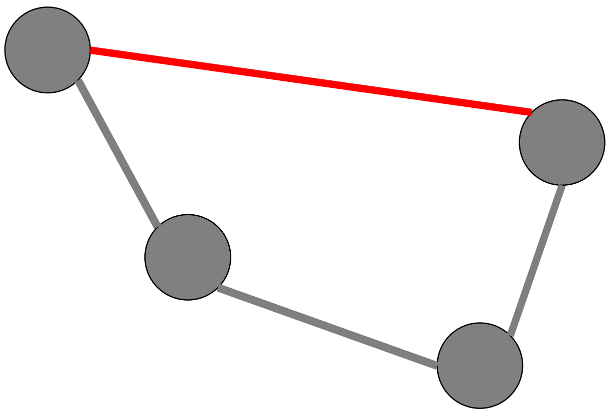
Generic MST Algorithm

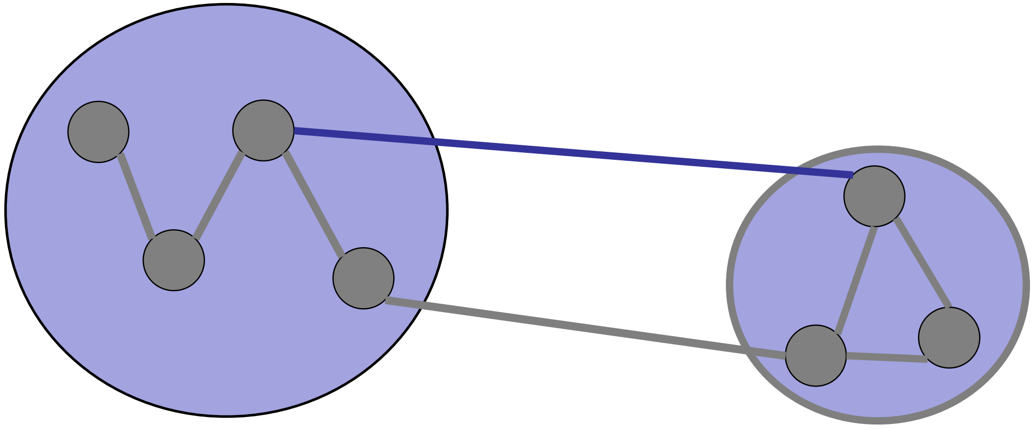
Greedy Algorithm:

Repeat:

Apply red rule or blue rule to an arbitrary edge.

until no more edges can be colored.





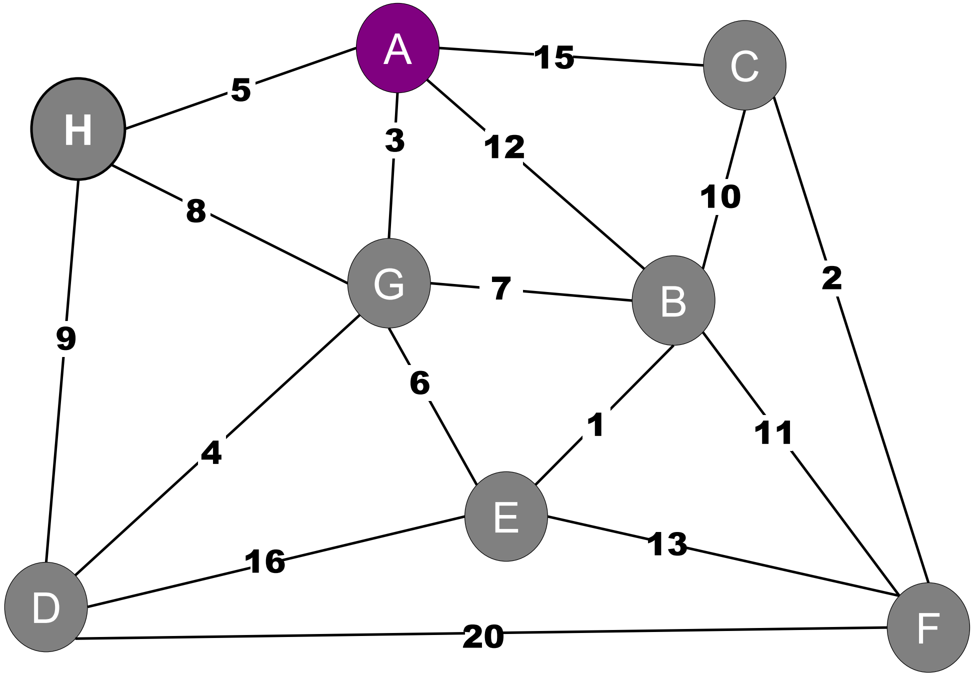
Roadmap

Minimum Spanning Trees

- The MST Problem
- Basic Properties of an MST
- Generic MST Algorithm
- **Prim’s Algorithm**
- Kruskal’s Algorithm
- Boruvka’s Algorithm
- Variations

Prim’s Algorithm

Prim’s Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

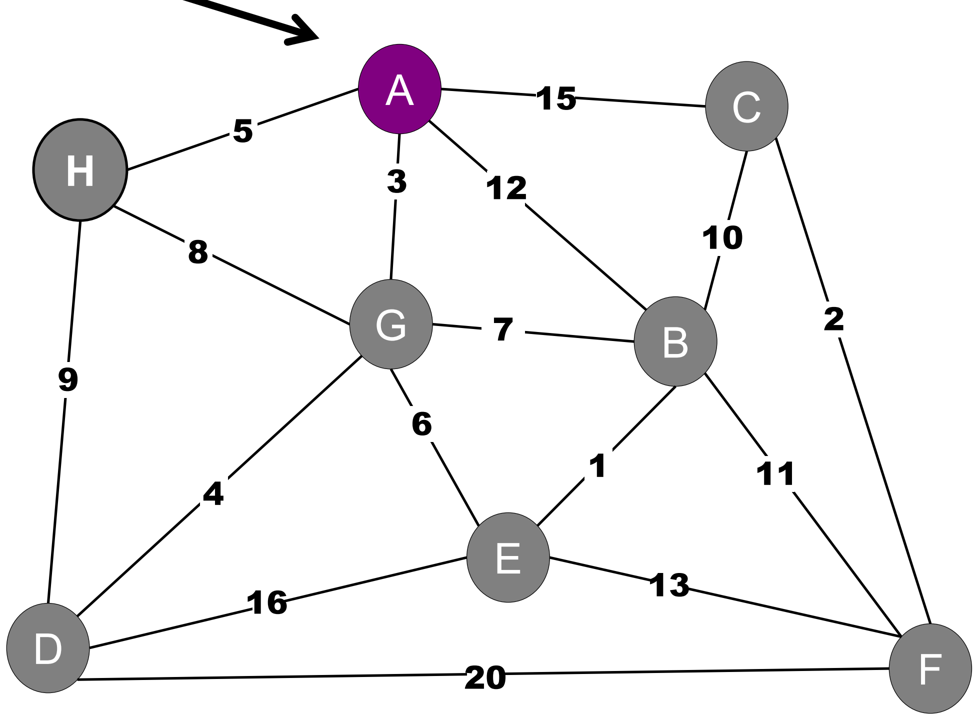


Prim's Algorithm

Prim's Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

Basic idea:

- S : set of nodes connected by blue edges. (An MST of a subgraph S)
- Initially: S = {A}

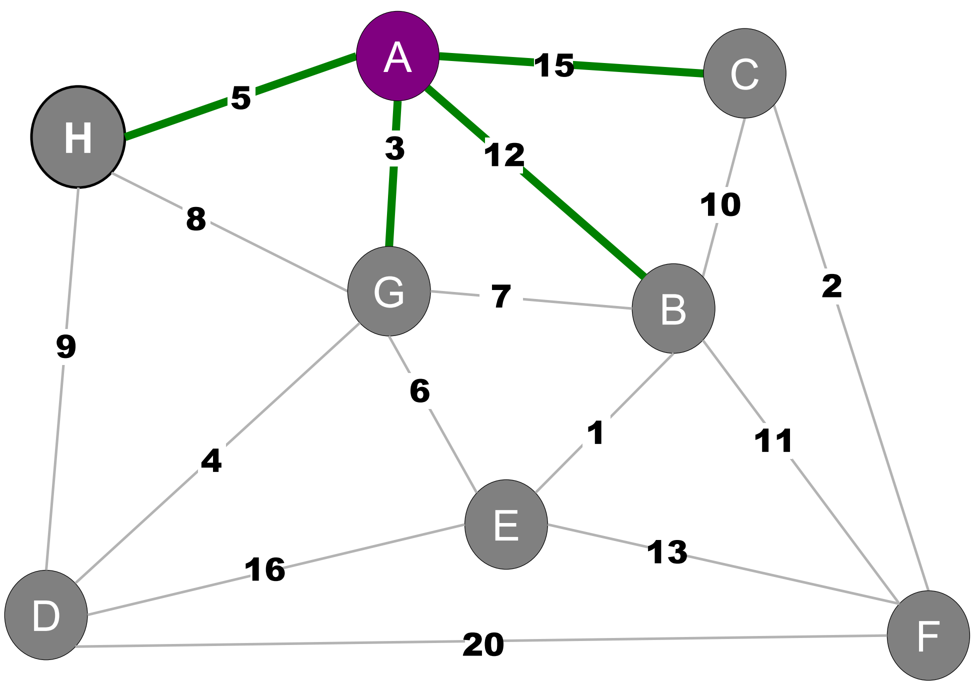


Prim's Algorithm

Prim's Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

Basic idea:

- S : set of nodes connected by blue edges.
- Initially: S = {A}
- Identify cut: {S, V–S}



Prim's Algorithm

Prim's Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

Basic idea:

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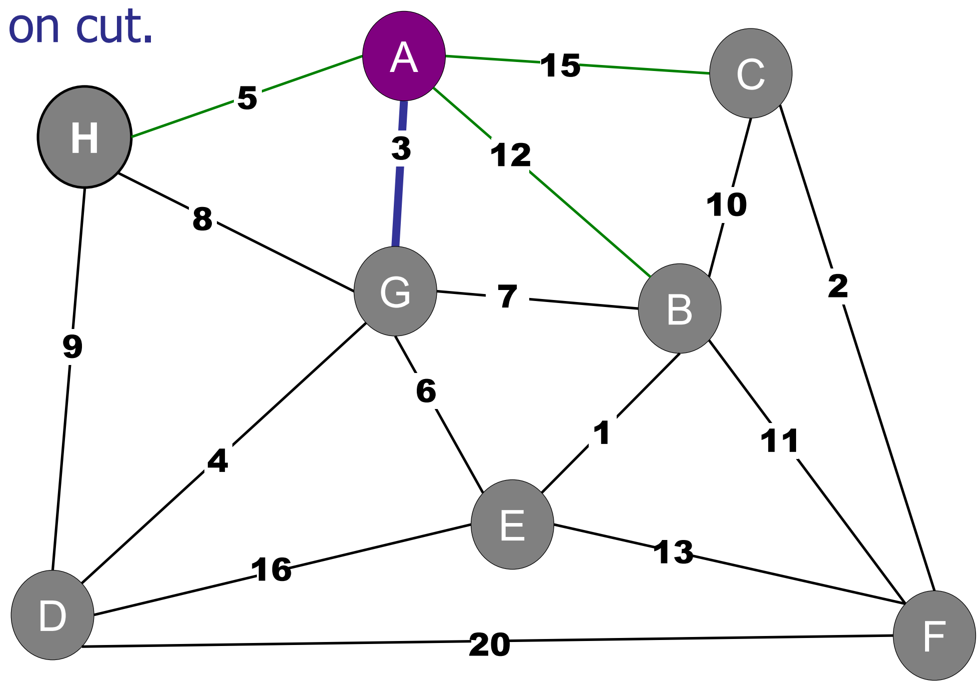
- Find minimum weight edge on cut.



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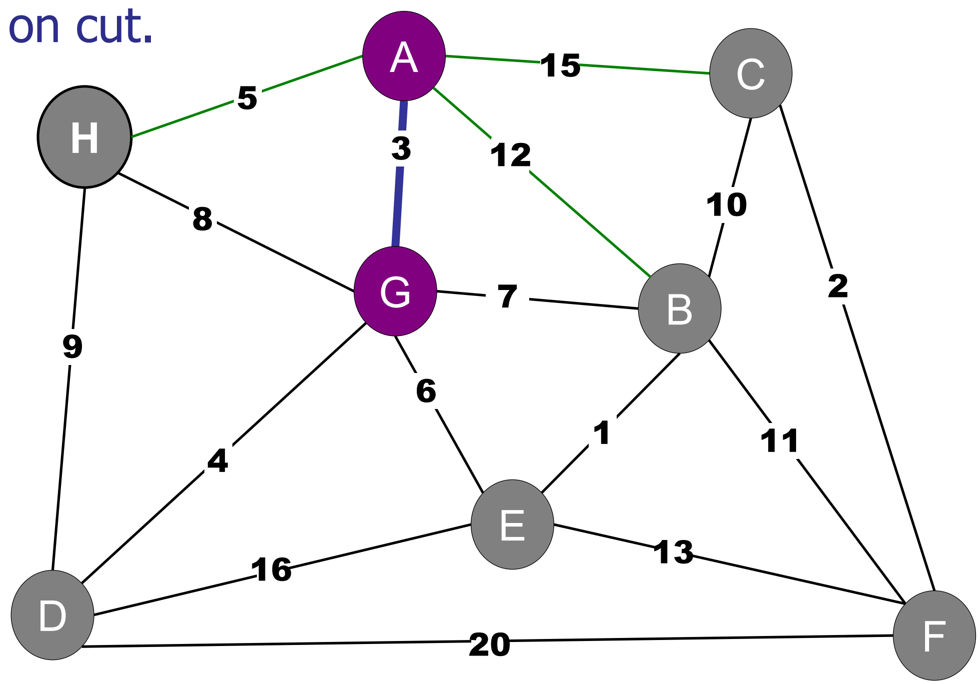
- Find minimum weight edge on cut.
- Add new node to S.



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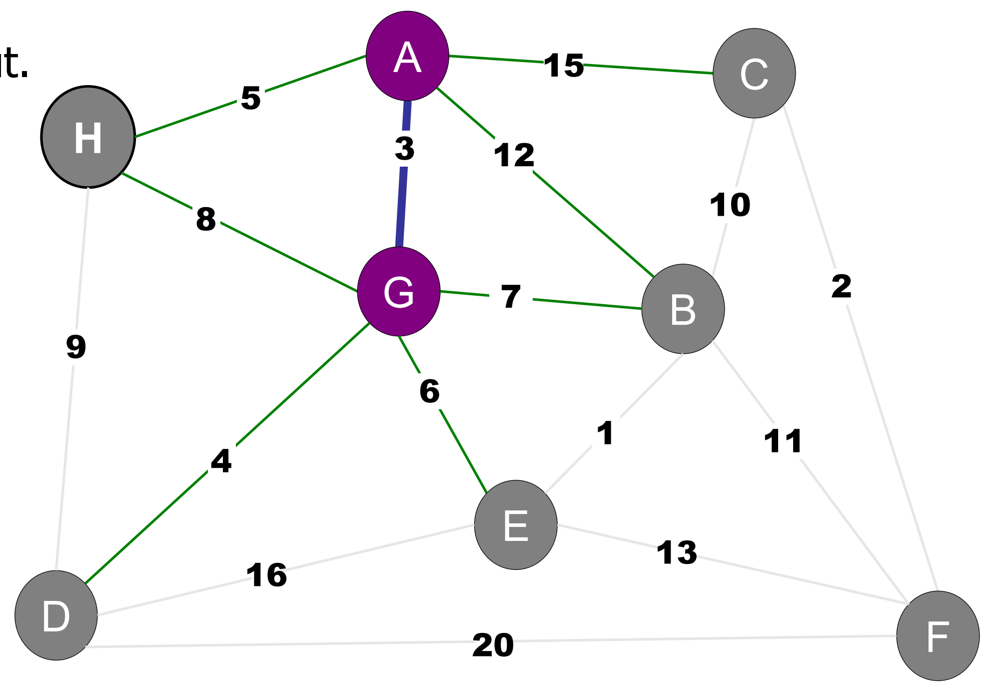

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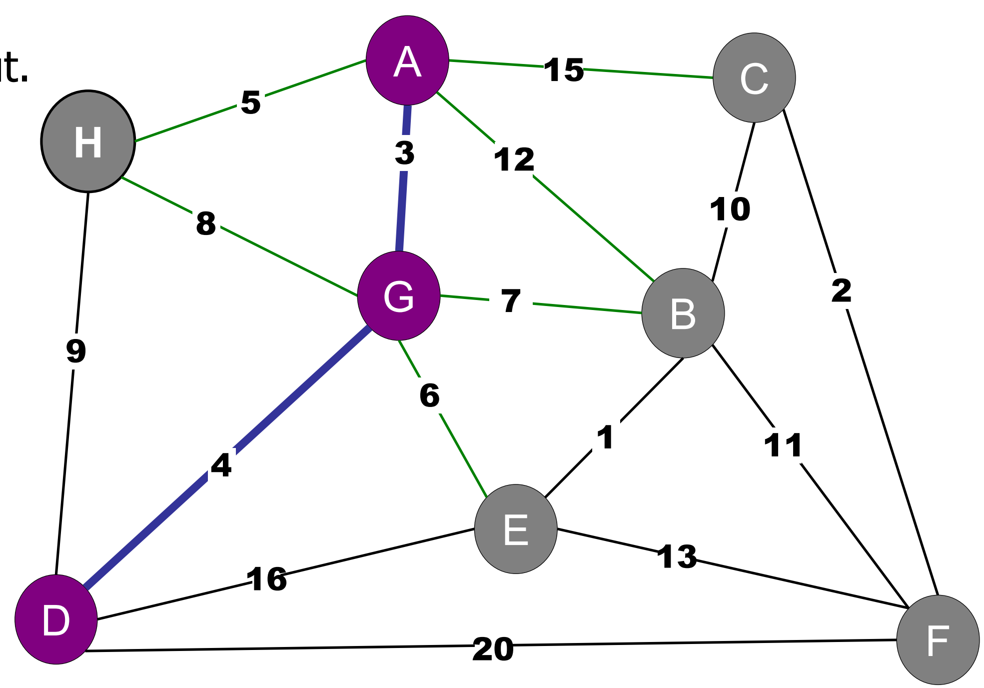

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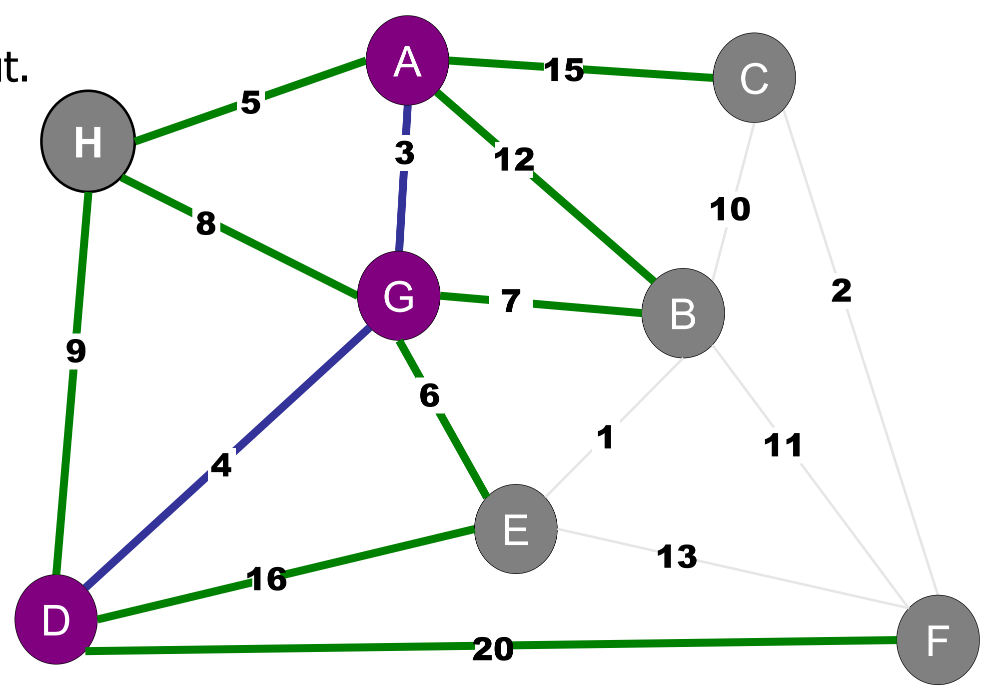
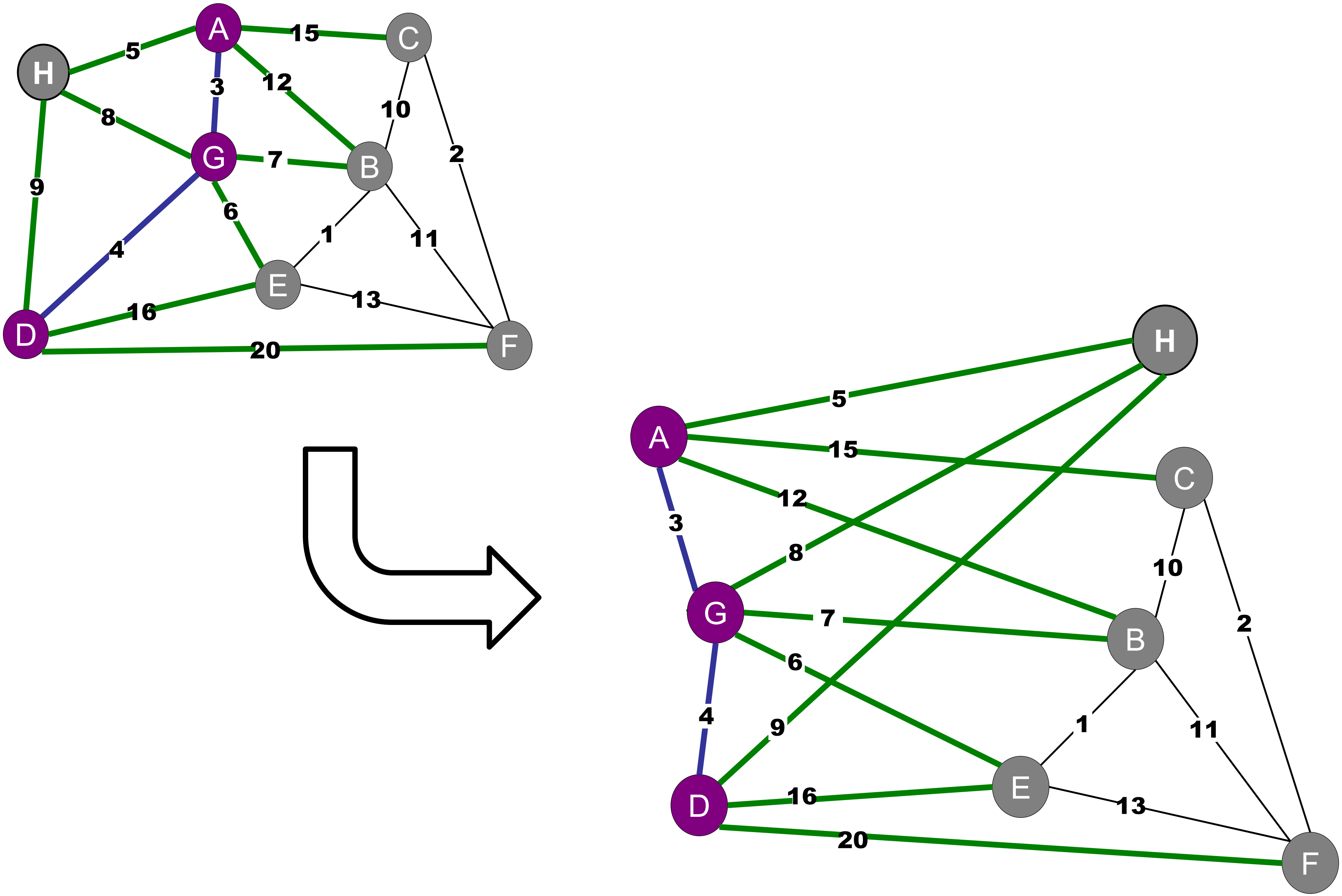


Prim’s Algorithm



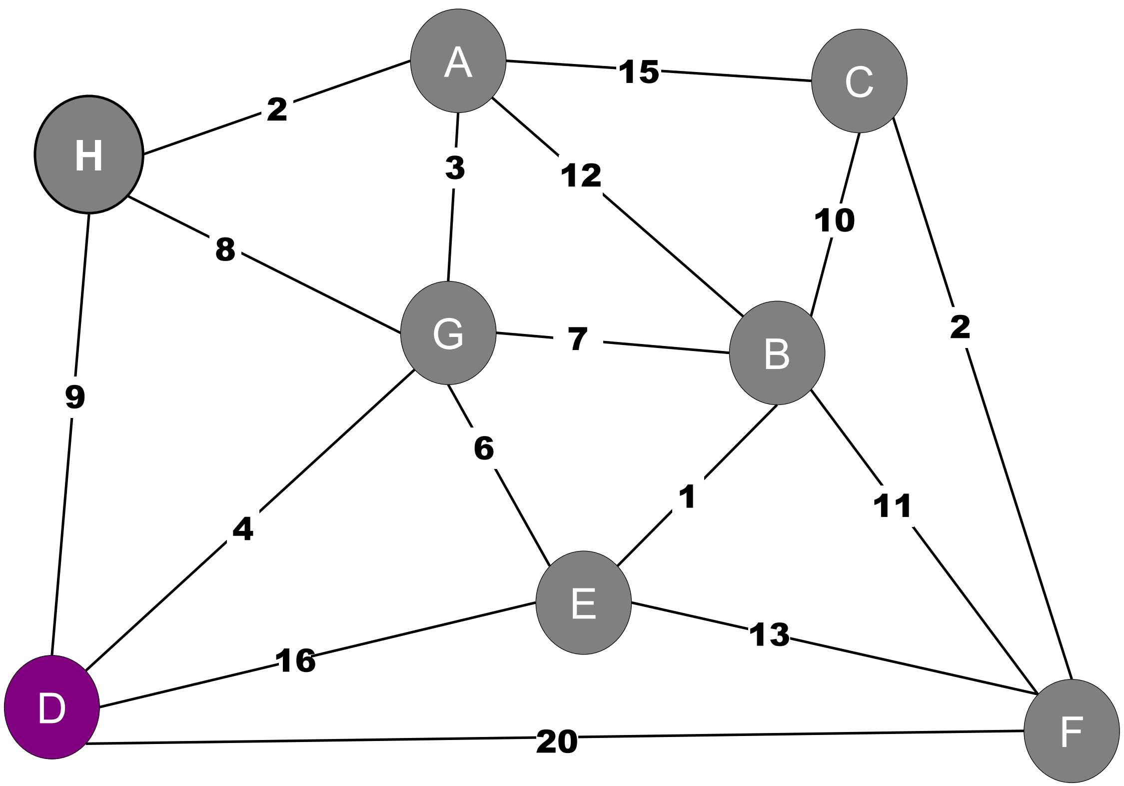
Prim's Algorithm: Initialization

```
// Initialize priority queue
PriorityQueue pq = new PriorityQueue();
for (Node v : G.V()) {
        pq.insert(v, INFTY);
}

pq.decreaseKey(start, 0);
// Initialize set S
HashSet<Node> S = new HashSet<Node>();
S.put(start);

// Initialize parent hash table
HashMap<Node,Node> parent = new HashMap<Node,Node>();
parent.put(start, null);
```

Prim's Algorithm

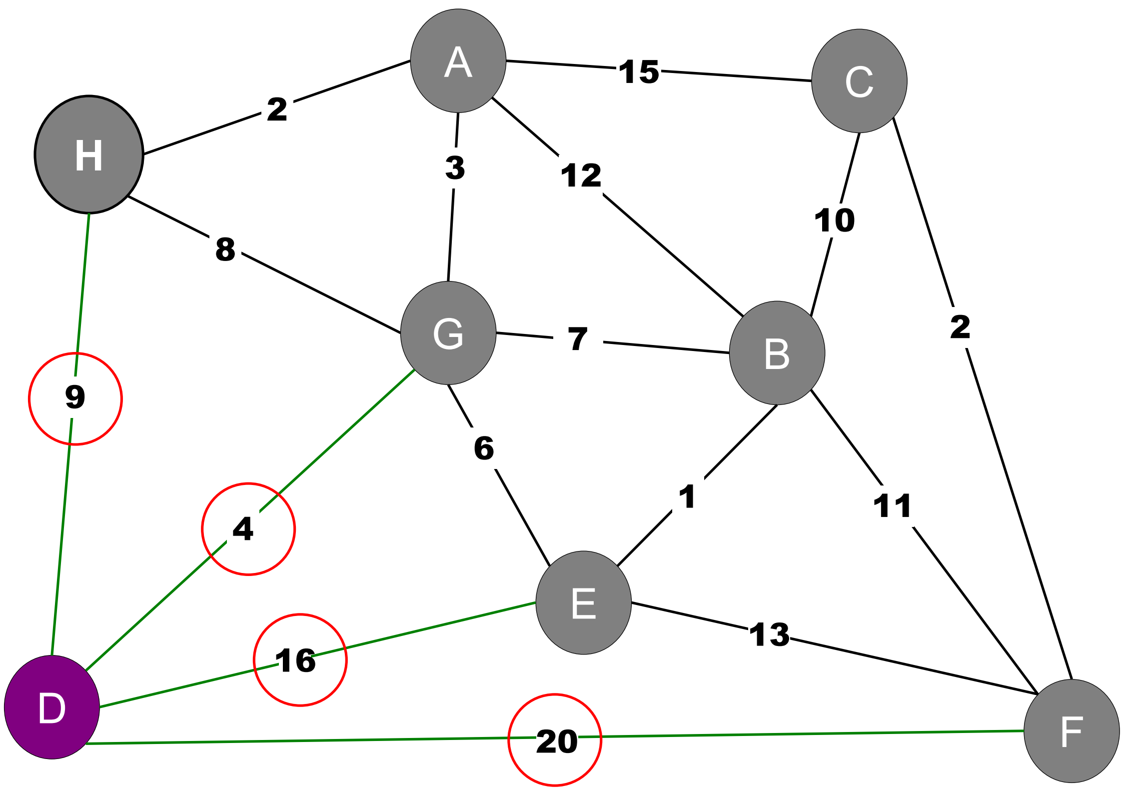
```
while (!pq.isEmpty()){
    Node v = pq.deleteMin();
    S.put(v);
    for each (Edge e : v.edgeList()){
        Node w = e.otherNode(v);
        if (!S.get(w)) {
            pq.decreaseKey(w, e.getWeight());
            parent.put(w, v);
        }
    }
}
```

Prim's Example



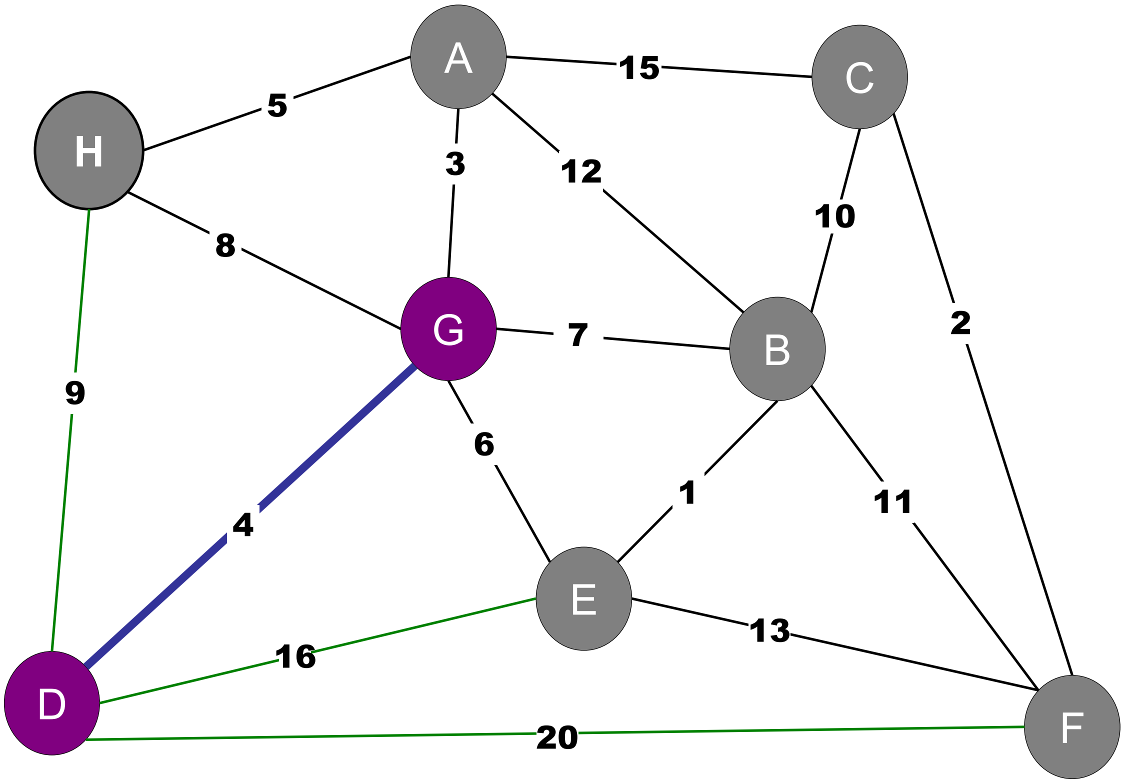
Vertex	Weight
D	0

Prim's Example



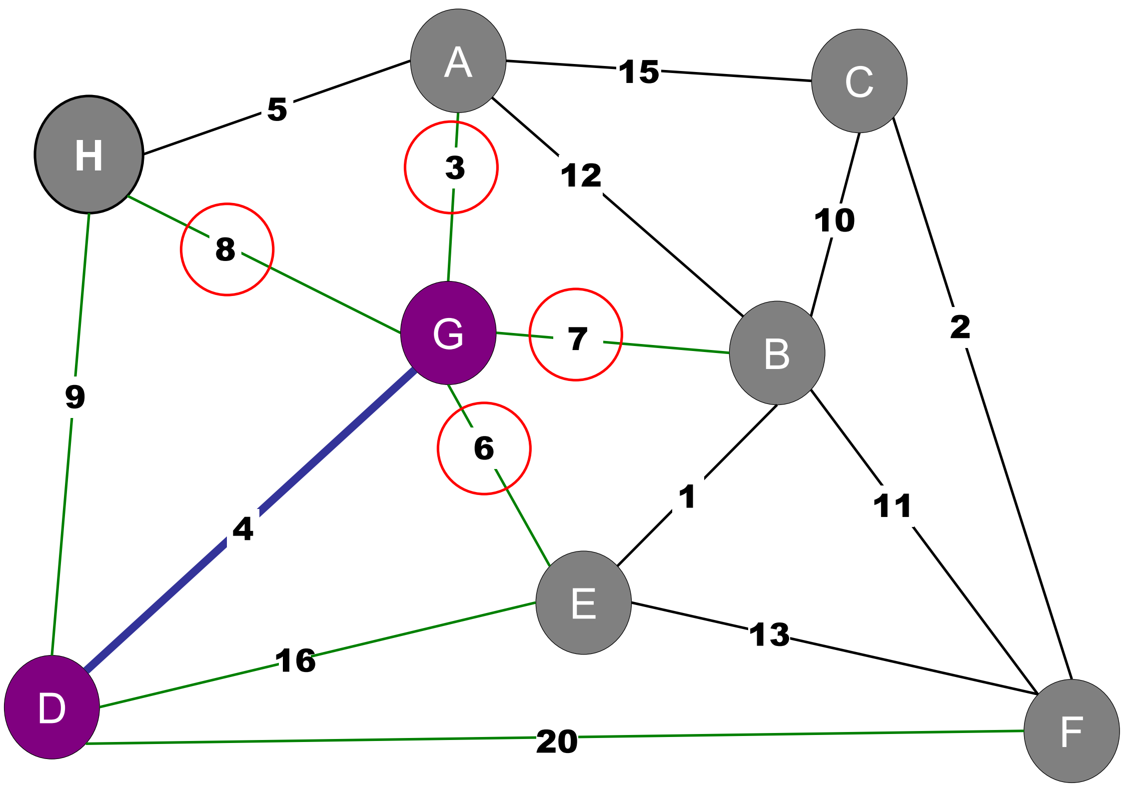
Vertex	Weight
G	4
H	9
E	16
F	20

Prim's Example



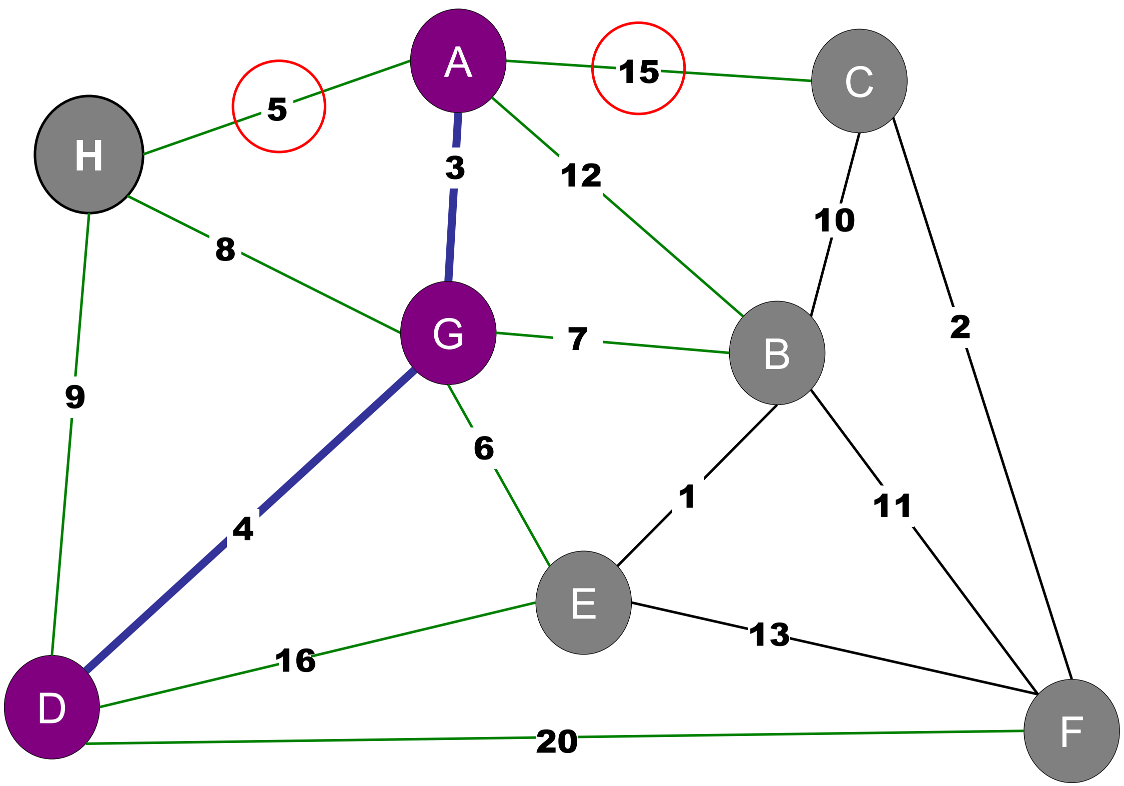
Vertex	Weight
H	9
E	16
F	20

Prim's Example



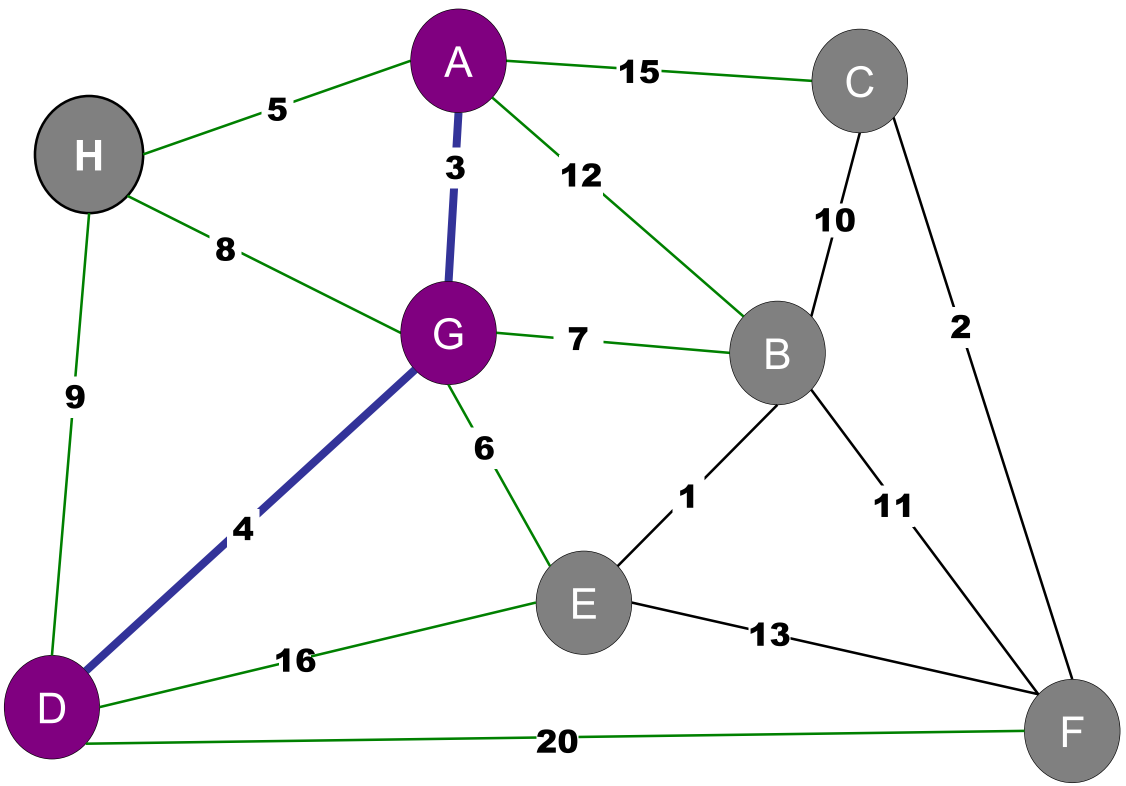
Vertex	Weight
A	**3**
E	**16->6**
B	**7**
H	**9->8**
F	20

Prim's Example



Vertex	Weight
H	**8->5**
E	6
B	7
C	**15**
F	20

Prim's Example



Vertex	Weight
H	5
E	6
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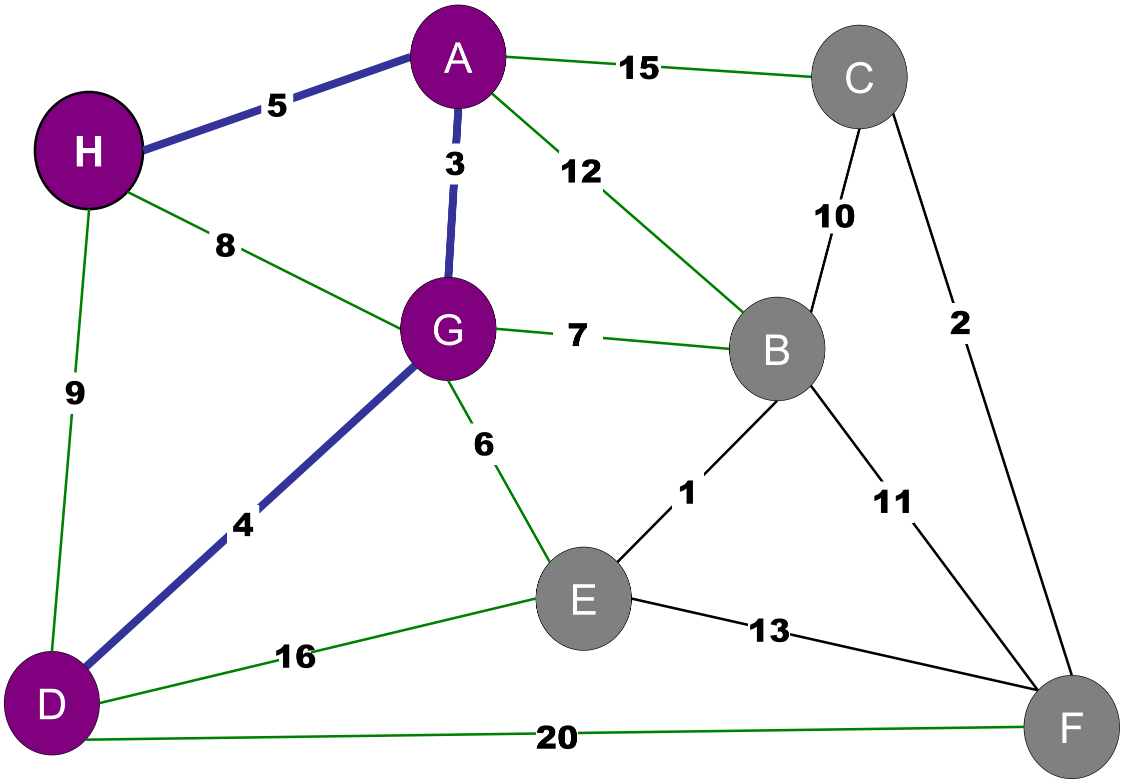
Prim's Example



Vertex	Weight
E	6
B	7
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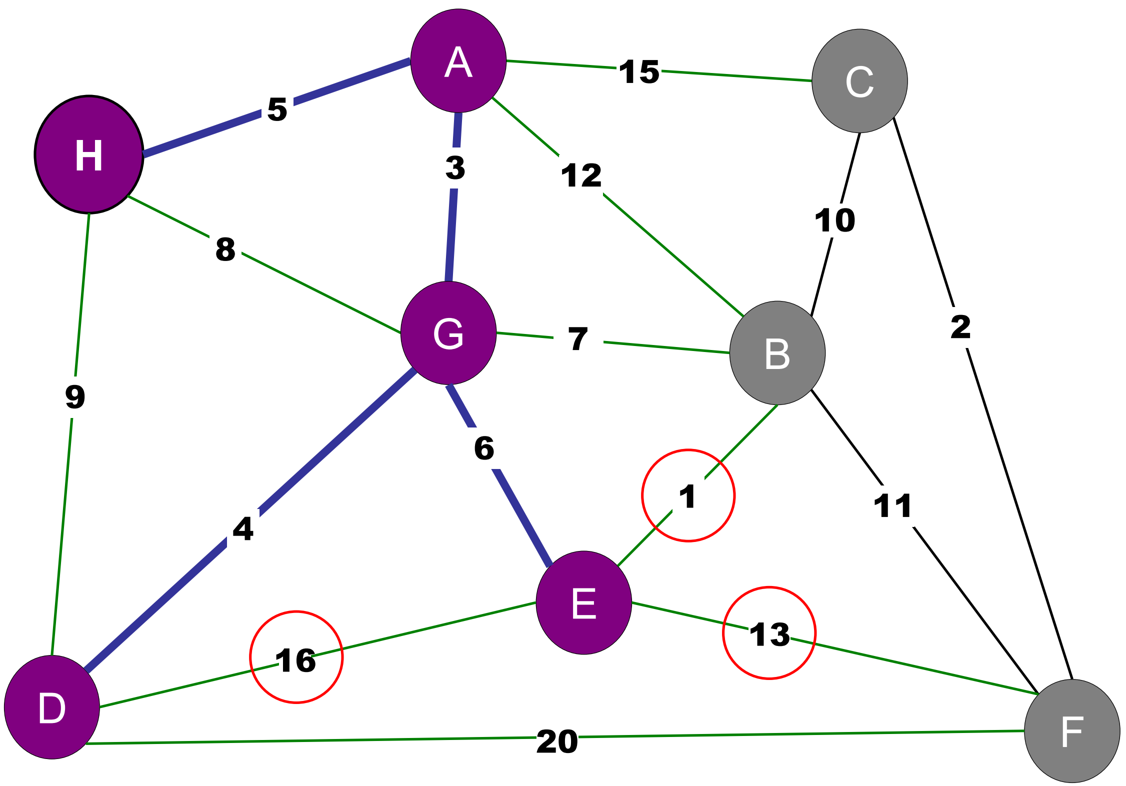
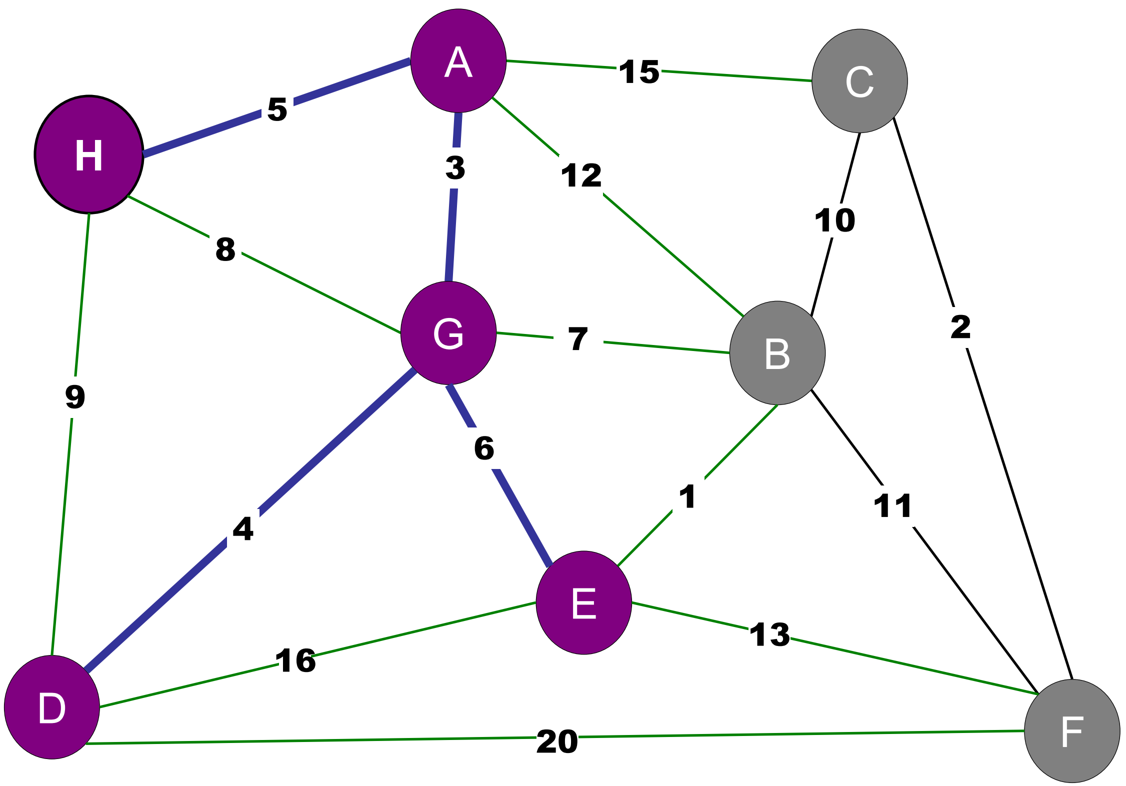
F	20

Prim's Example



Vertex	Weight
B	**7->1**
C	15
F	**20->13**

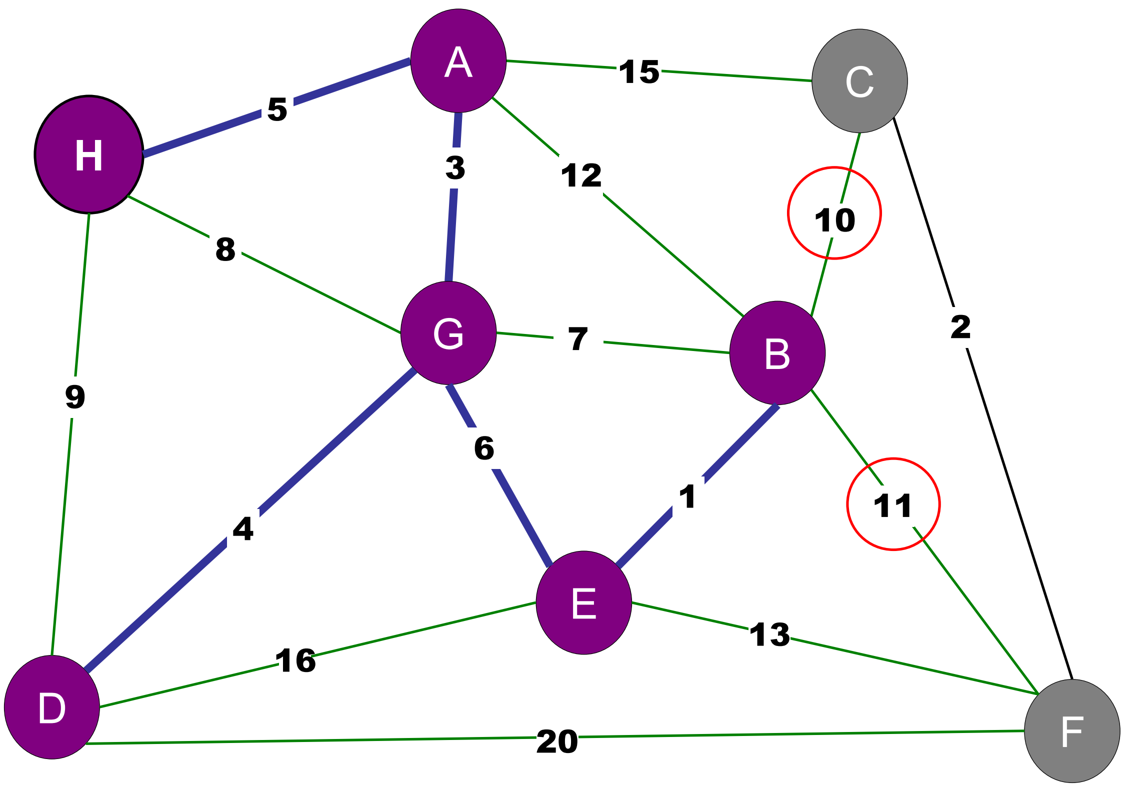
Prim's Example



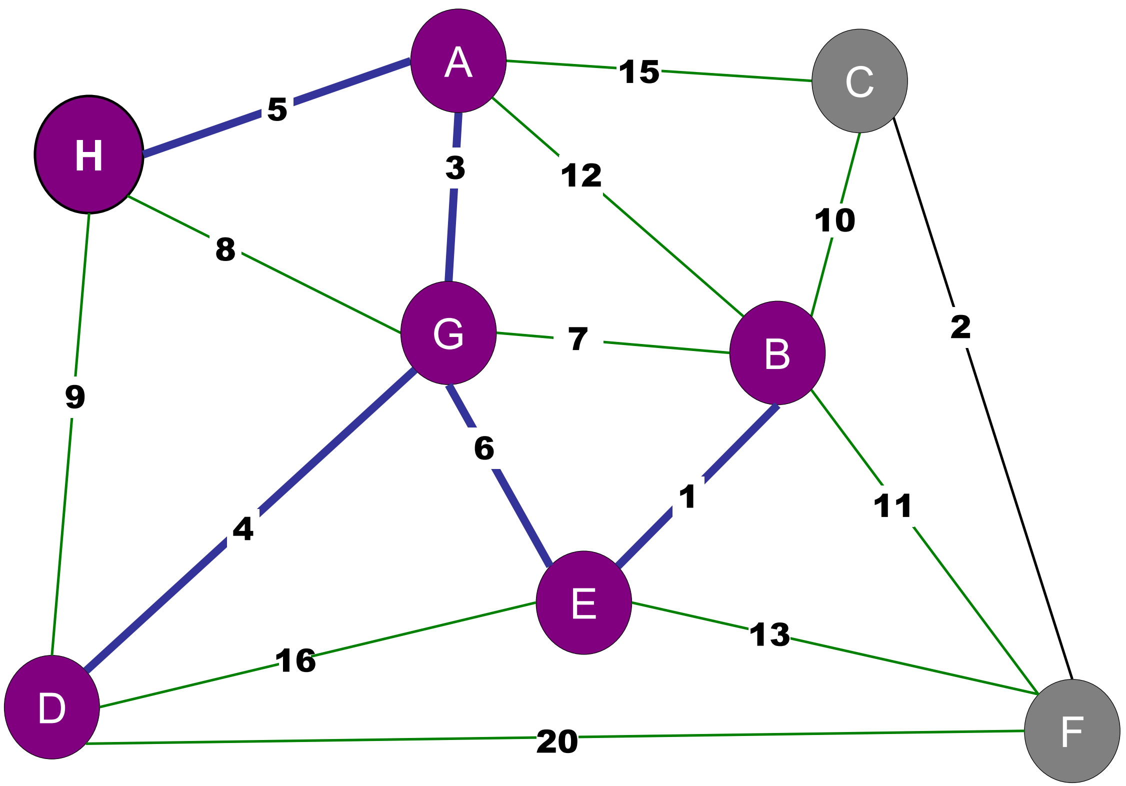
Vertex	Weight
B	1
C	15
F	13

Prim's Example

Vertex	Weight
C	15->10
F	13->11



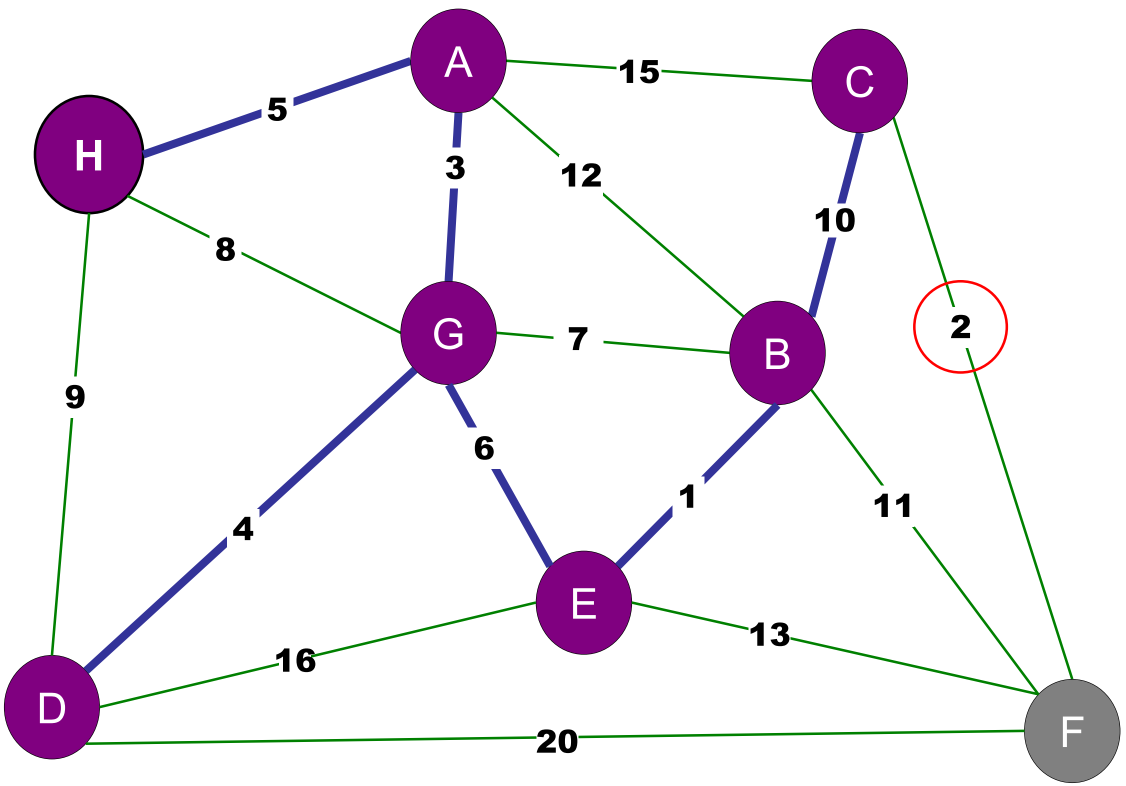
Prim's Example



Vertex	Weight
C	10
F	11

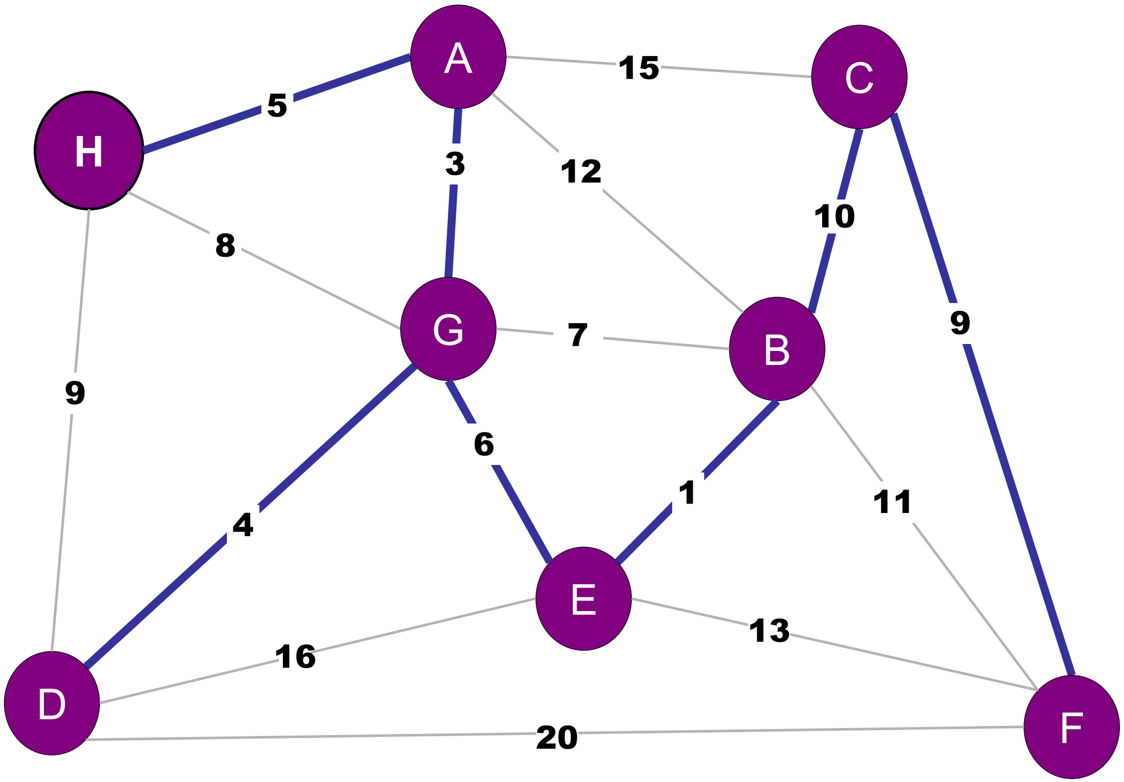
Prim's Example

Vertex	Weight
F	11->2



Prim's Example

Vertex	Weight



Prim's Algorithm

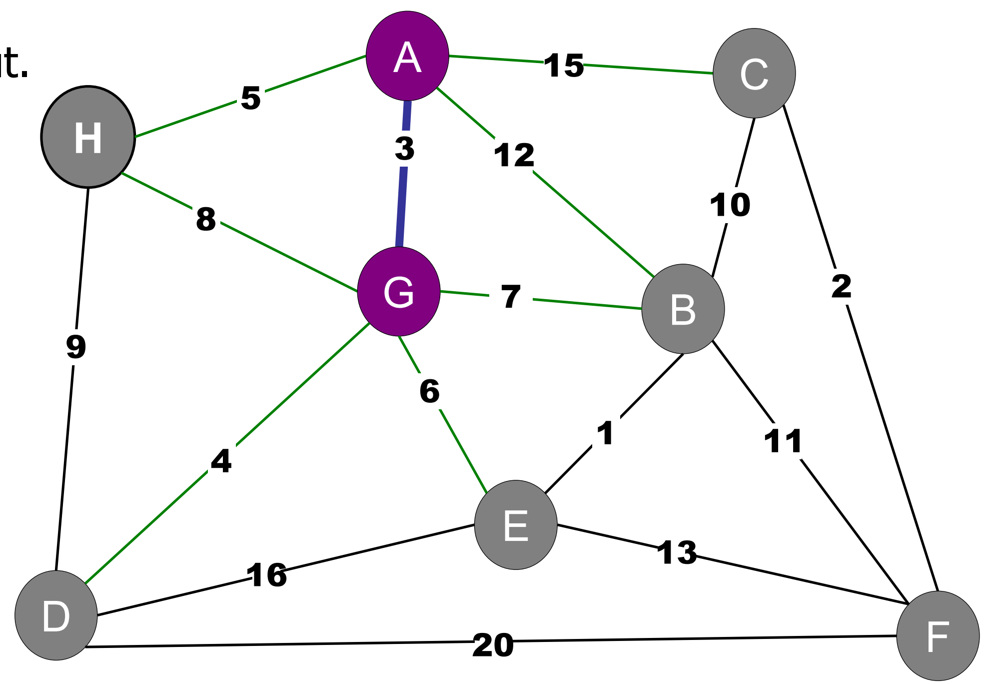
Prim's Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

Basic idea:

- S : set of nodes connected by blue edges.
- Initially: S = {A}
- Repeat:
  - Identify cut: {S, V–S}
  - Find minimum weight edge on cut.
  - Add new node to S.

Proof:

- Each added edge is the lightest on some cut.
- Hence each edge is in the MST.



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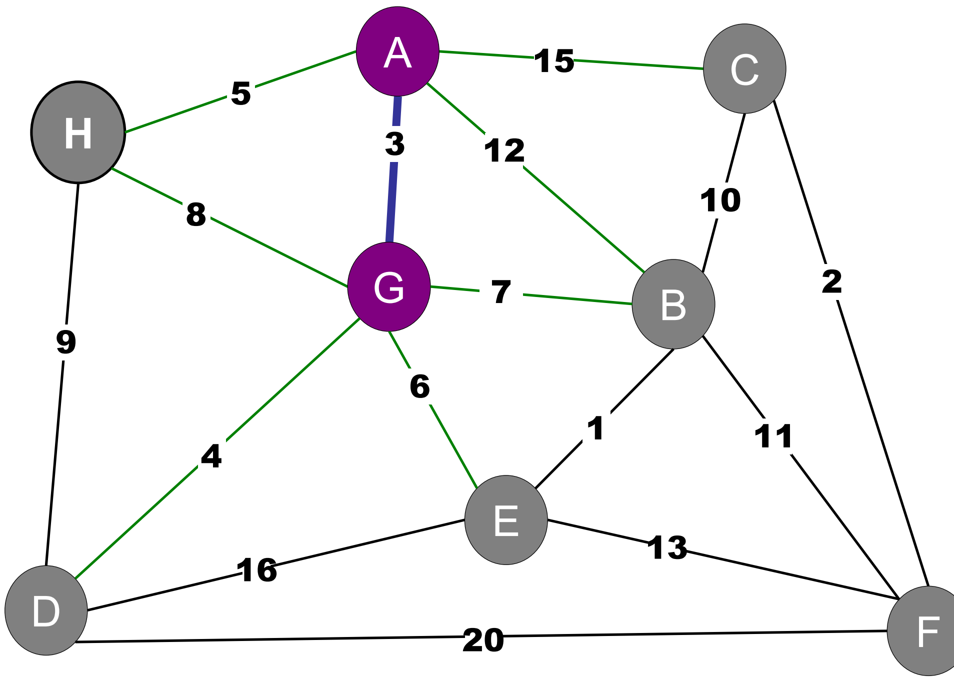

Analysis:

- Each vertex added/removed once from the priority queue: O(V log V)
- Each edge => one decreaseKey: O(E log V).

Two Algorithms

Prim's Algorithm.

Basic idea:

- Maintain a set of visited nodes.
- Greedily grow the set by adding node connected via the **lightest** edge.
  - Use Priority Queue to order nodes by **edge weight**.

Dijkstra's Algorithm.

Basic idea:

- Maintain a set of visited nodes.
- Greedily grow the set by adding neighboring node that is **closest to the source**.
  - Use Priority Queue to order nodes by **distance**.

Roadmap

Minimum Spanning Trees

- The MST Problem
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- **Kruskal’s Algorithm**
- Boruvka’s Algorithm
- Variations

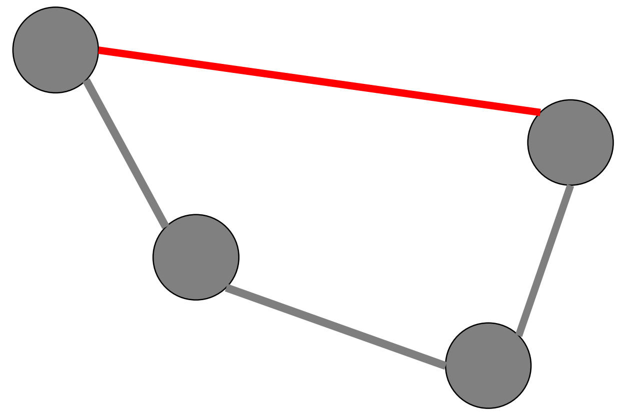
Generic MST Algorithm

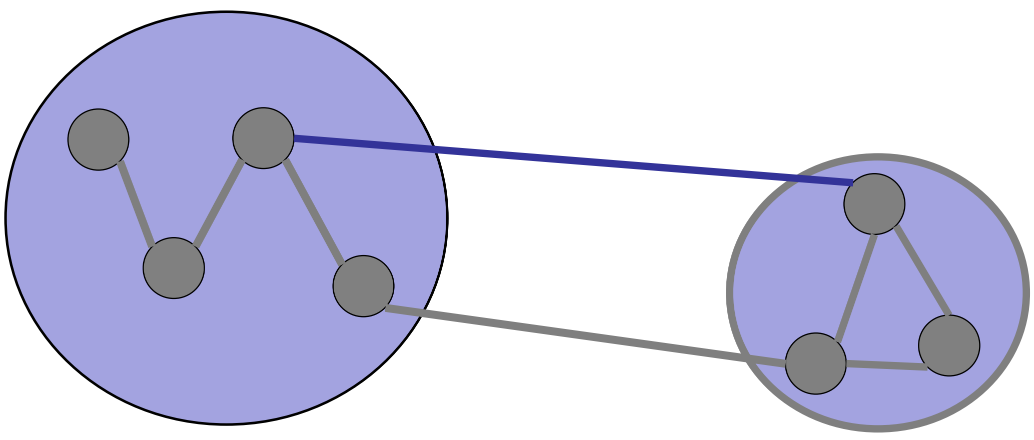
Greedy Algorithm:

Repeat:

Apply red rule or blue rule to an arbitrary edge.

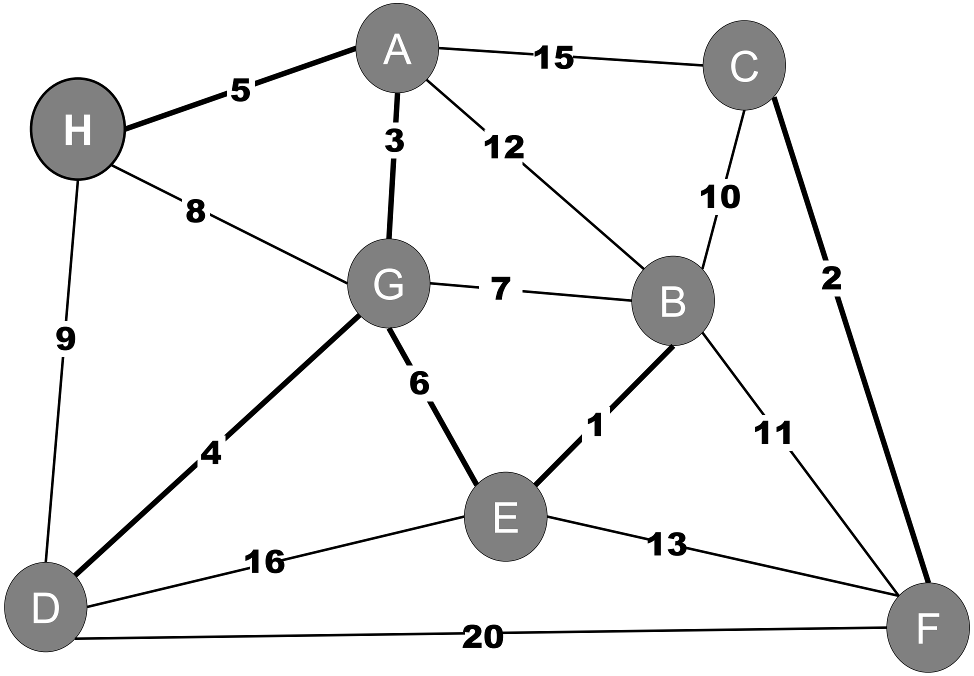
until no more edges can be colored.





Kruskal's Algorithm

Kruskal's Algorithm. (Kruskal 1956)

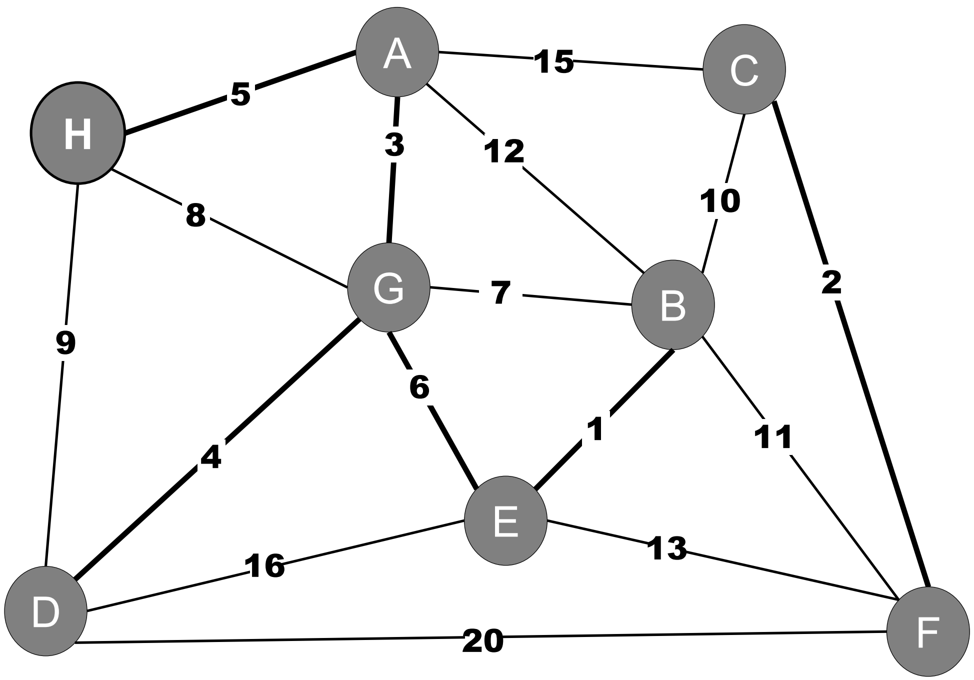


Kruskal's Algorithm

Kruskal's Algorithm. (Kruskal 1956)

Basic idea:

- Sort edges by weight.
- Consider edges in ascending order:
  - If both endpoints are in the **same** blue tree, then color the edge red.
  - Otherwise, color the edge blue.



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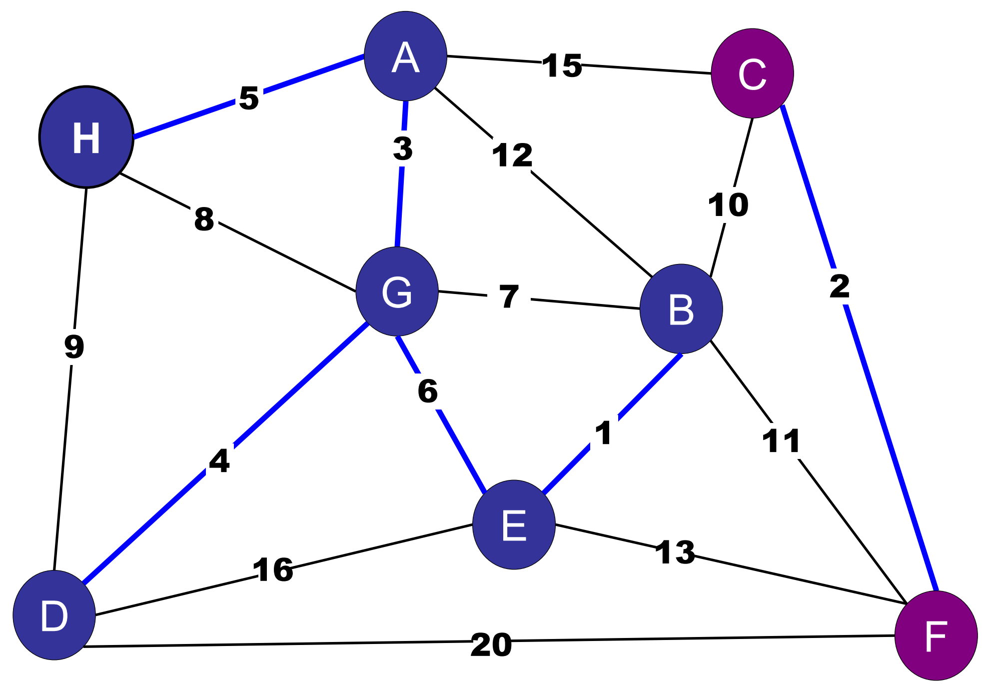

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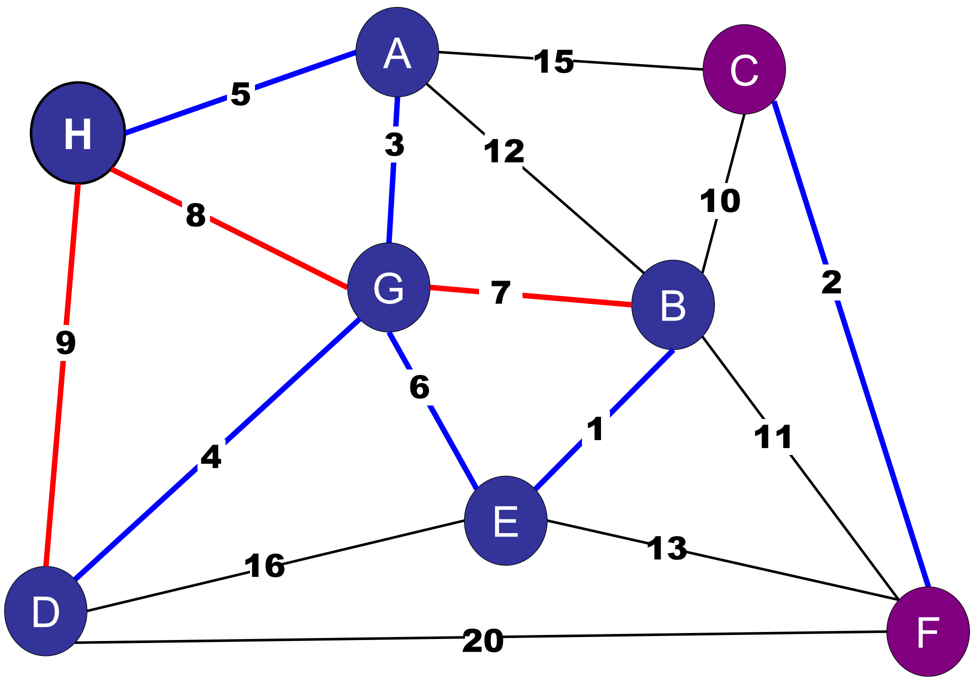

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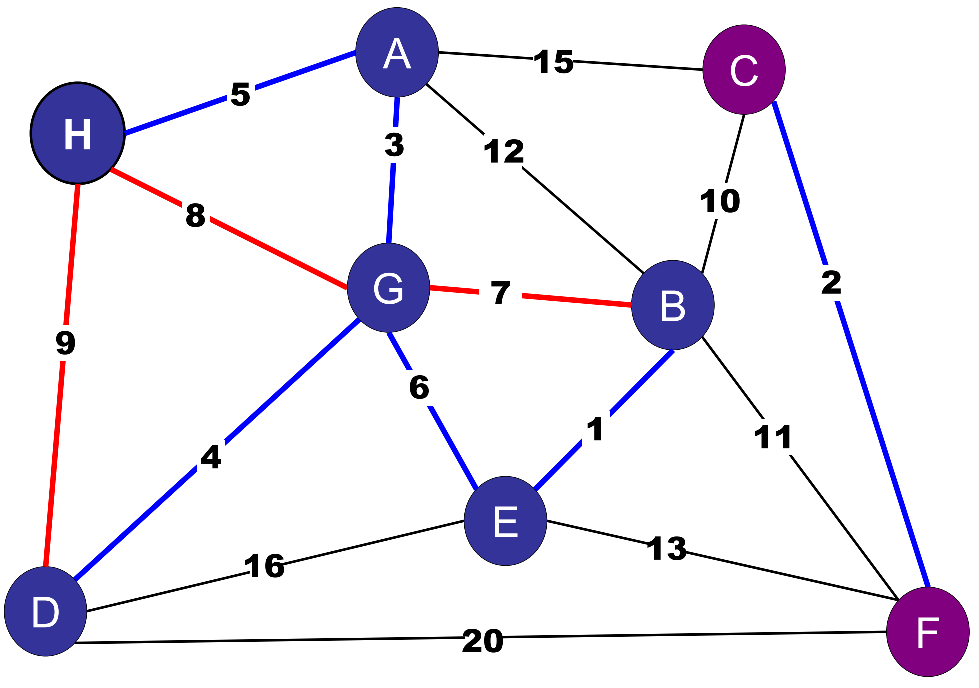


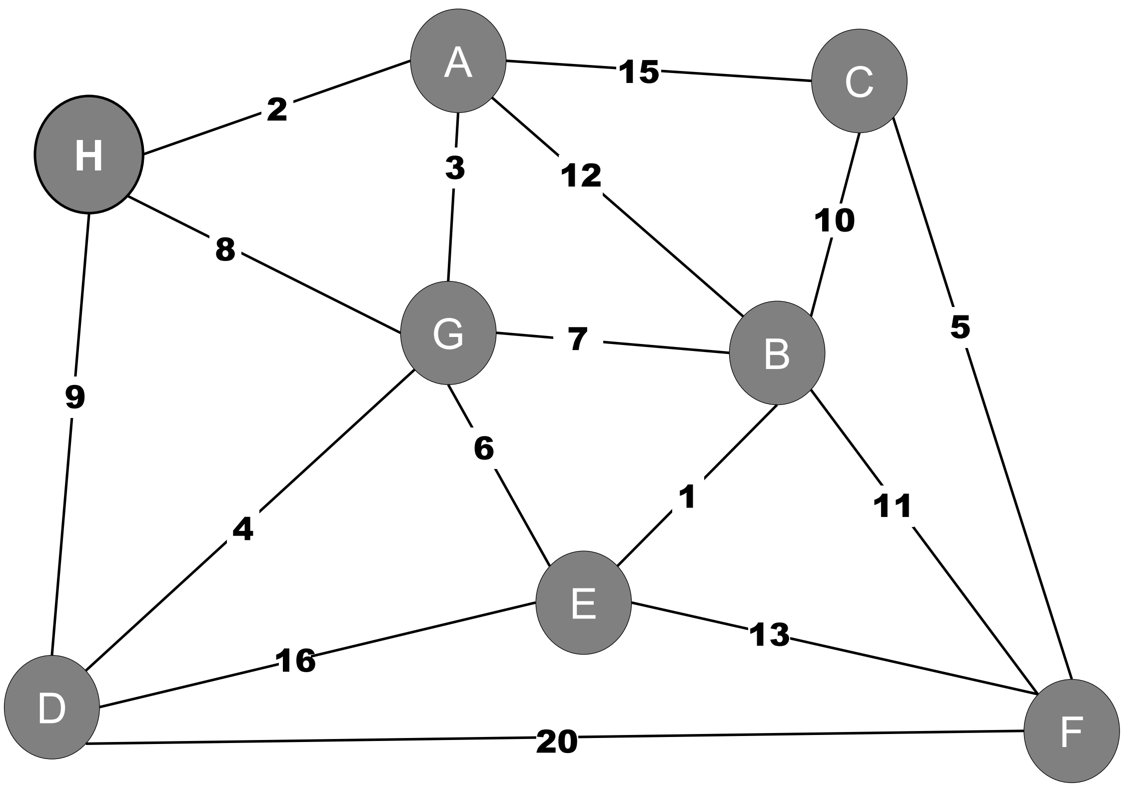
Data structure:

- Union-Find
- Connect two nodes if they are in the same blue tree.

Kruskal's Algorithm

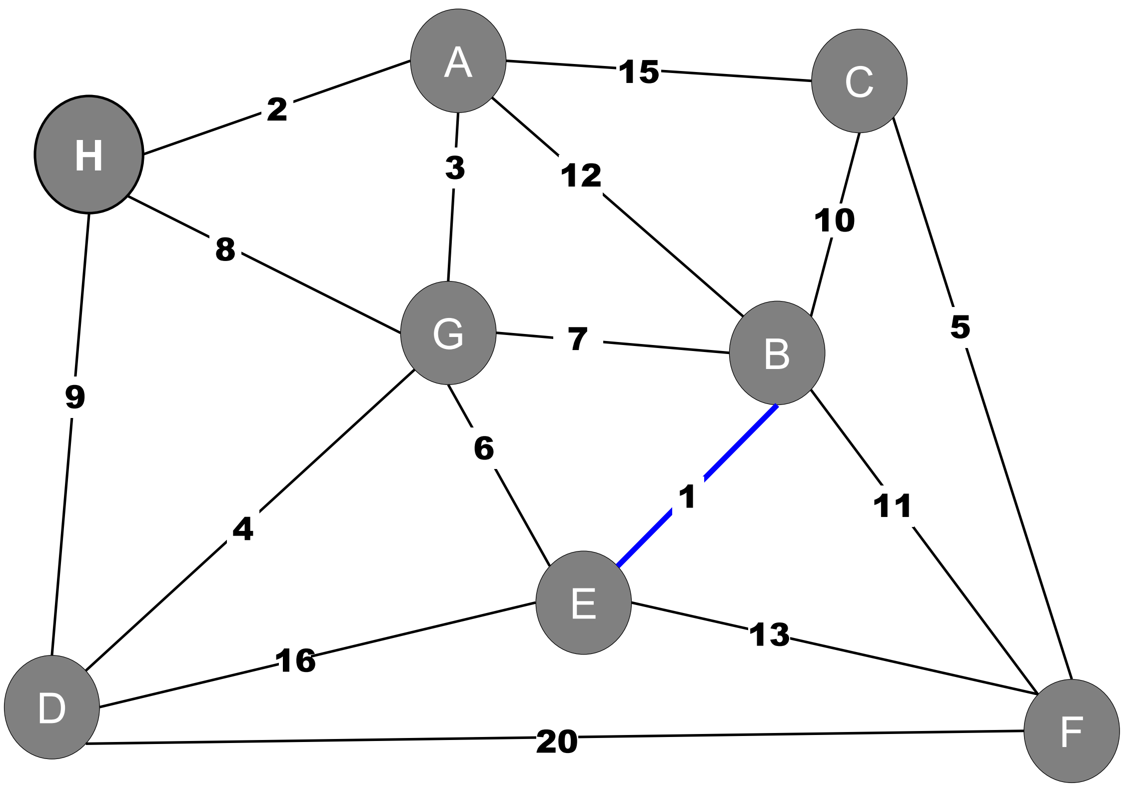
```
// Sort edges and initialize
Edge[] sortedEdges = sort(G.E());
ArrayList<Edge> mstEdges = new ArrayList<Edge>();
UnionFind uf = new UnionFind(G.V());


// Iterate through all the edges, in order
for (int i=0; i<sortedEdges.length; i++) {
         Edge e = sortedEdges[i]; // get edge
         Node v = e.one(); // get node endpoints
         Node w = e.two();

         if (!uf.find(v,w)) { // in the same tree?
               mstEdges.add(e); // save edge
               uf.union(v,w); // combine trees
         }
}
```

Kruskal's Example



Weight	Edge
1	(E,B)
2	(A,H)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,H)
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Kruskal's Example



Weight	Edge
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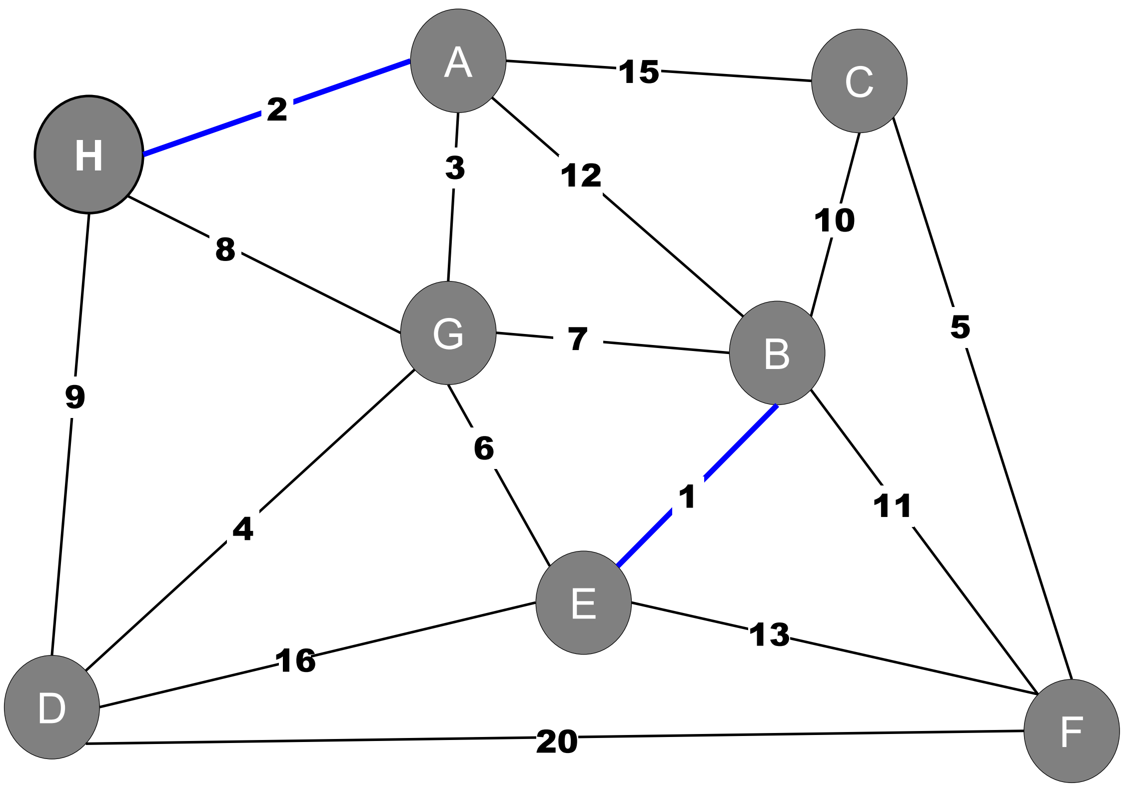
Kruskal's Example



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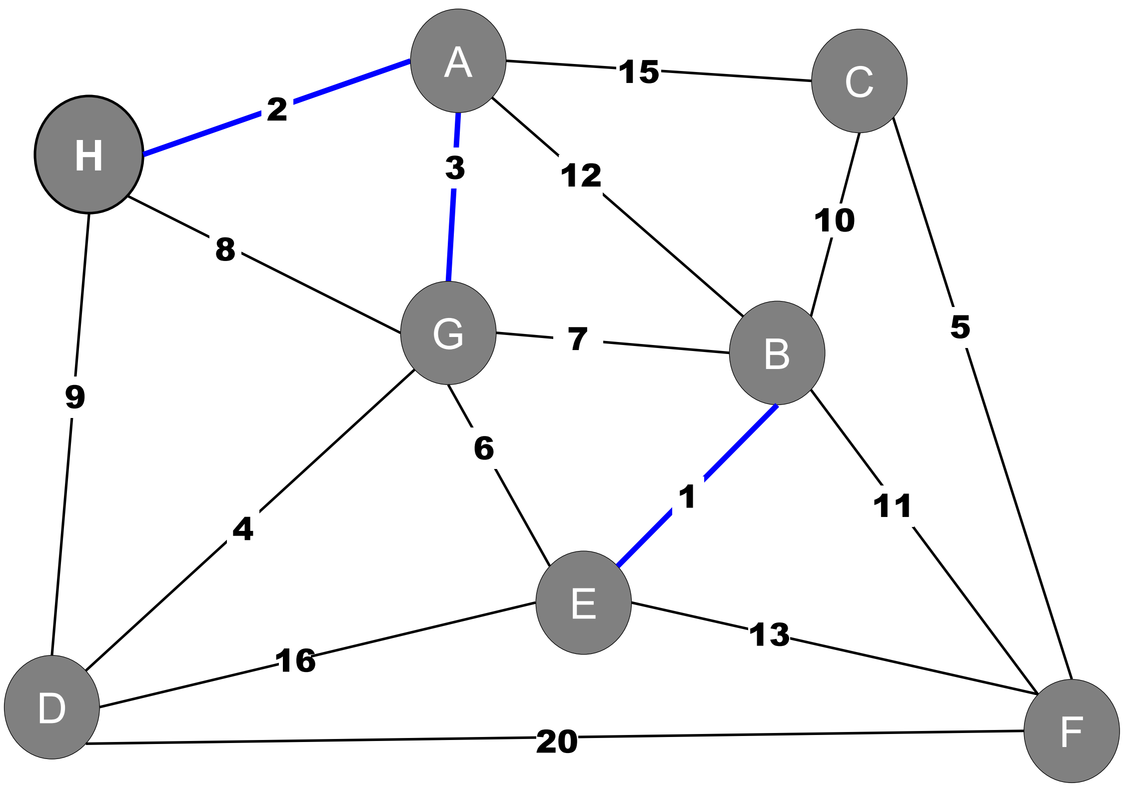
Kruskal's Example



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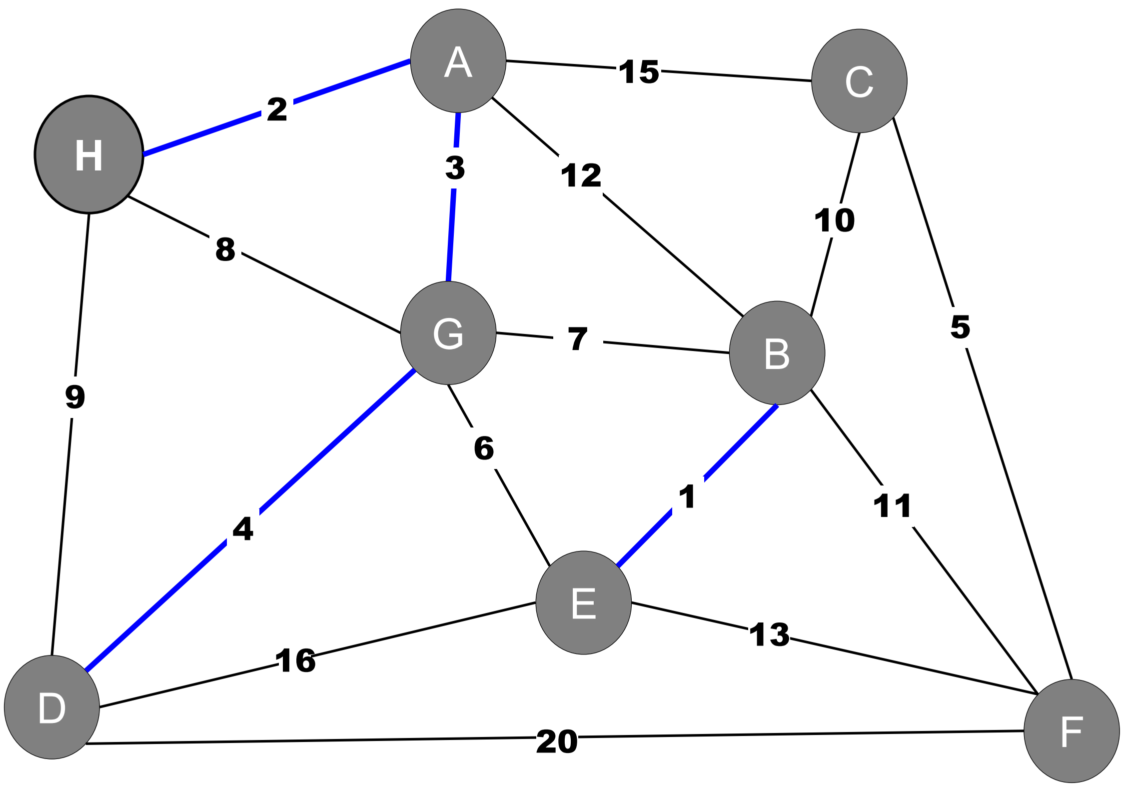
Kruskal's Example



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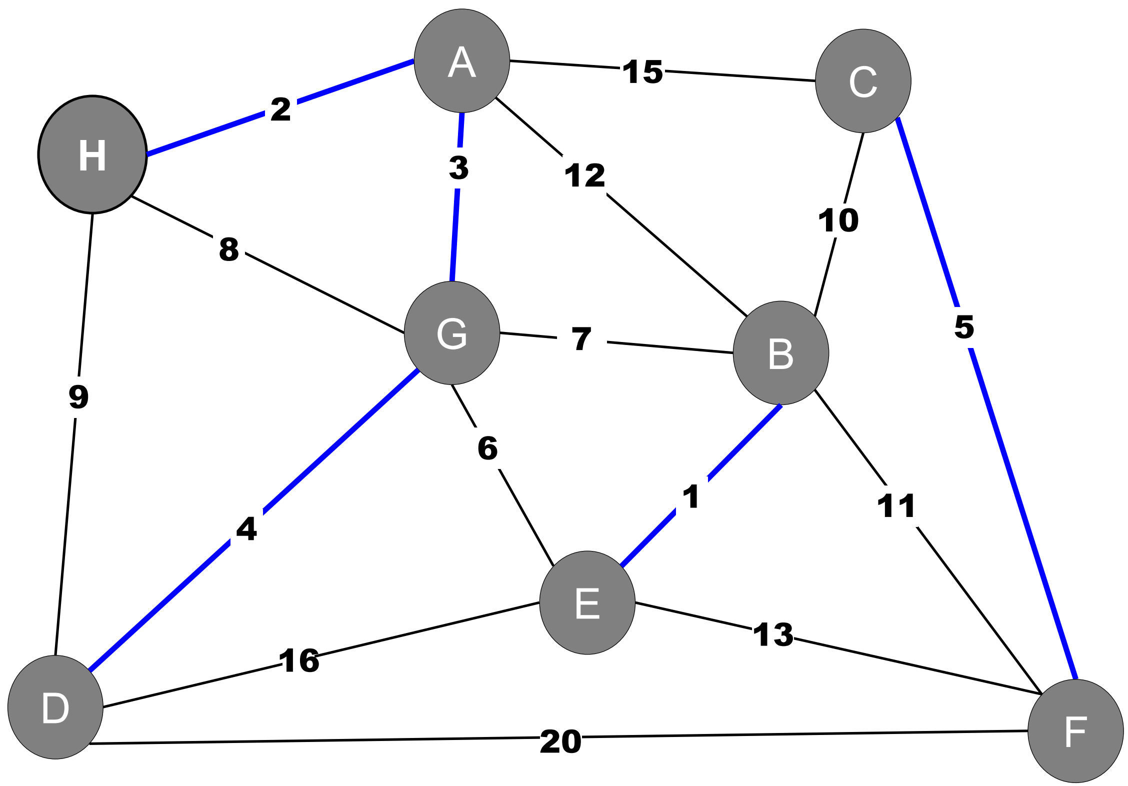
Kruskal's Example



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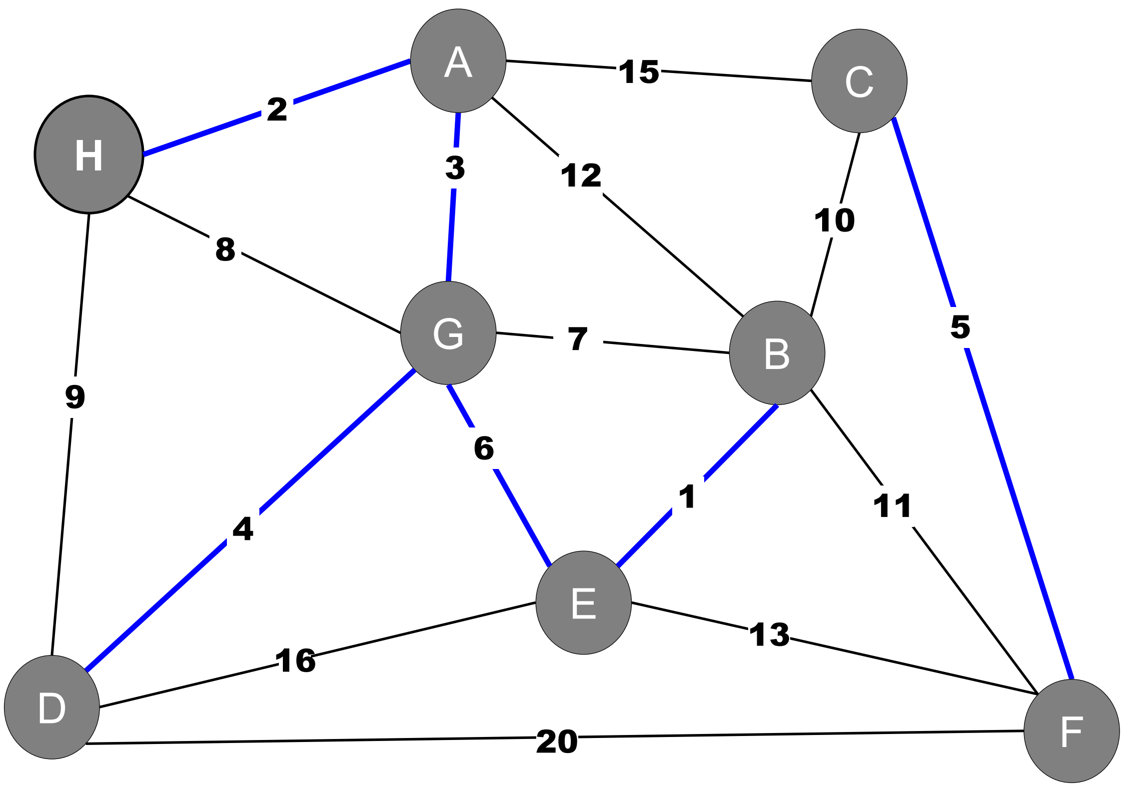
Kruskal's Example



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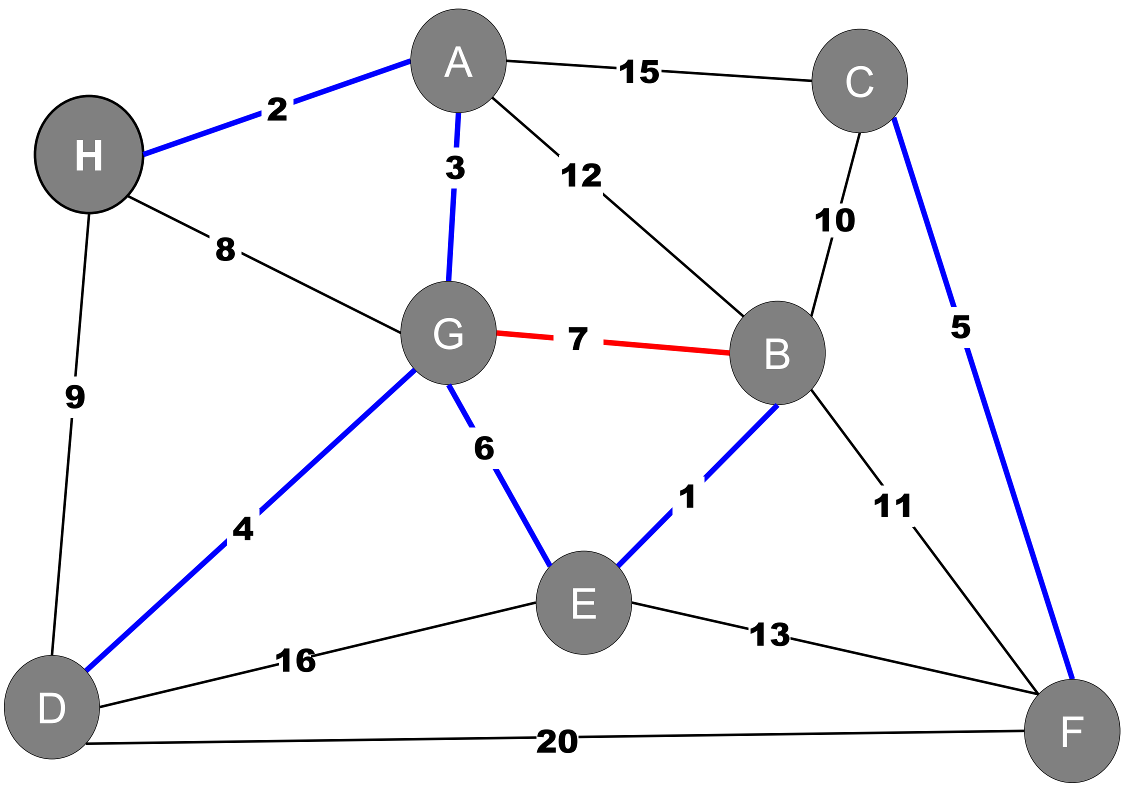
Kruskal's Example



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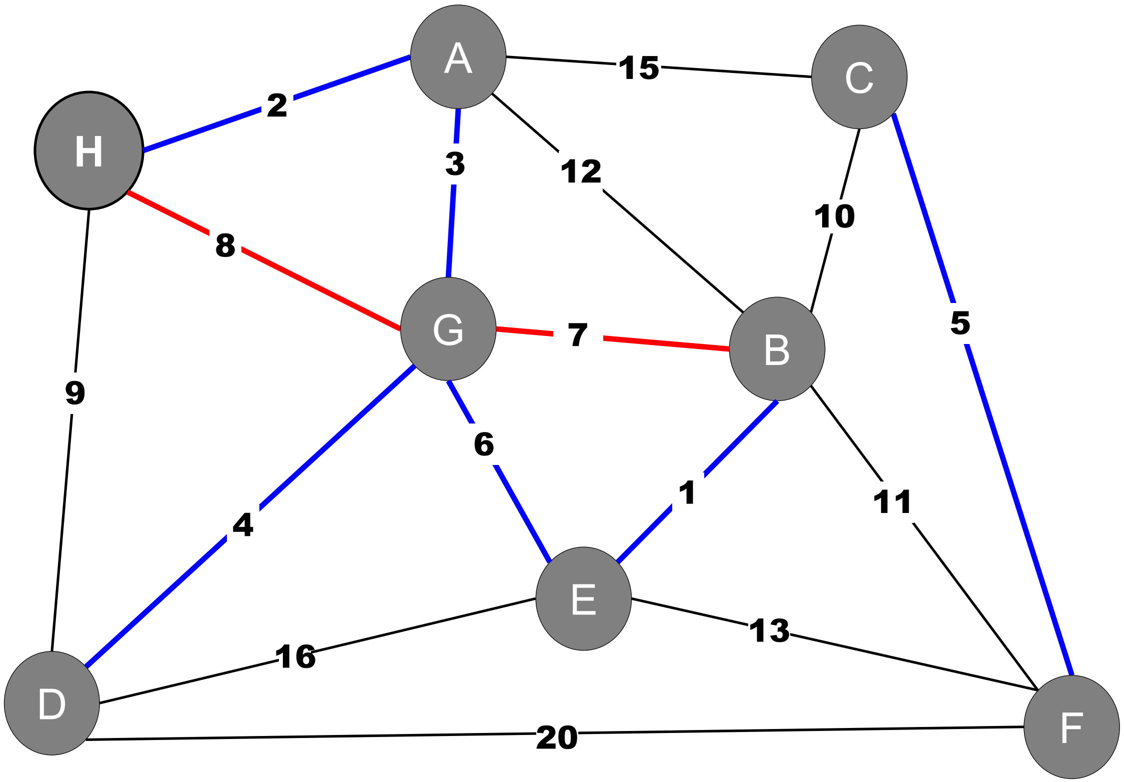
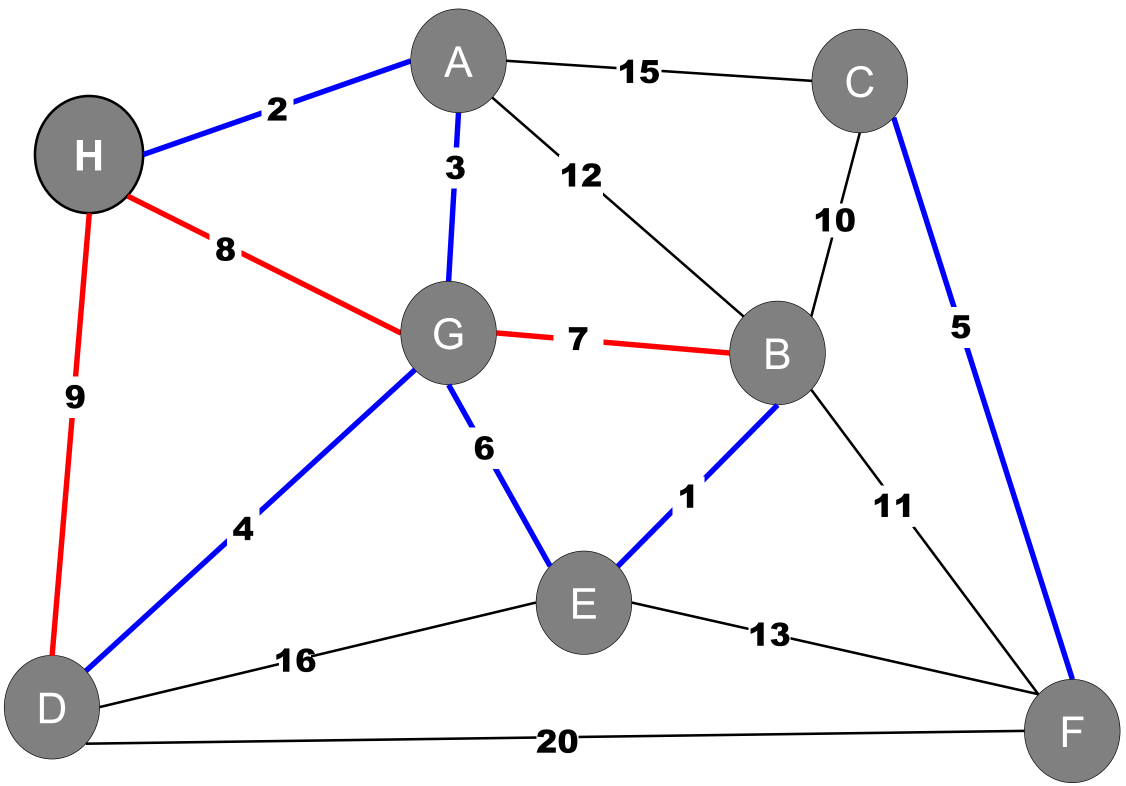
20	(D,F)

Kruskal's Example



Weight	Edge
1	(E,B)
2	(A,H)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	**(G,H)**
9	(D,H)
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Kruskal's Example



Weight	Edge
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3	(A,G)
4	(D,G)
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6	(E,G)
7	(B,G)
8	(G,H)
9	**(D,H)**
10	(B,C)
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15	(A,C)
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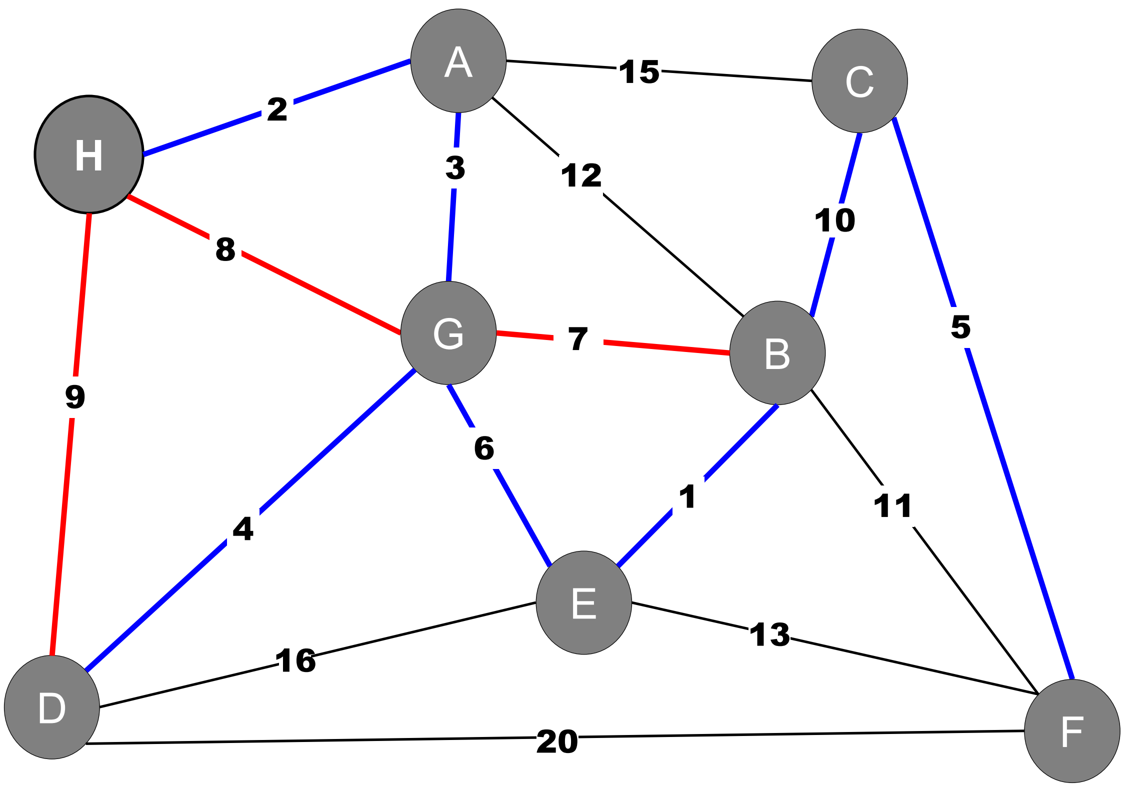
20	(D,F)

Kruskal’s Example



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9	(D,H)
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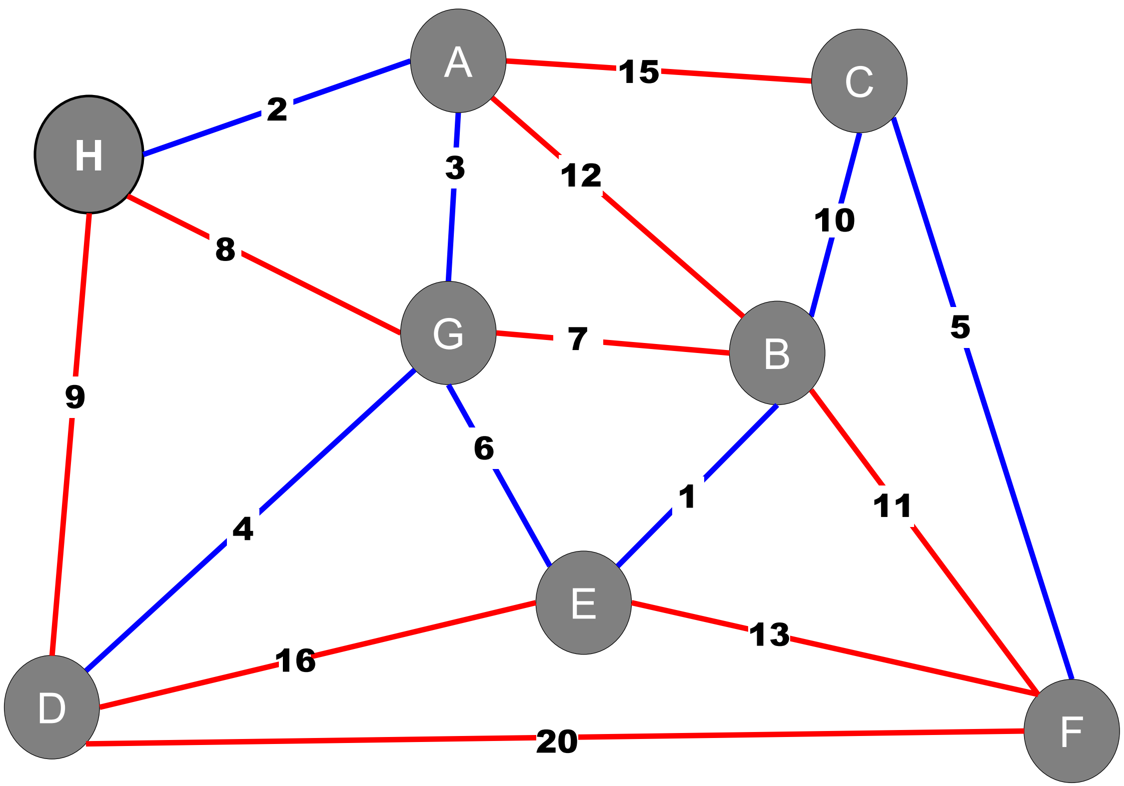
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Kruskal's Example



Weight	Edge
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3	(A,G)
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7	(B,G)
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11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Kruskal's Algorithm

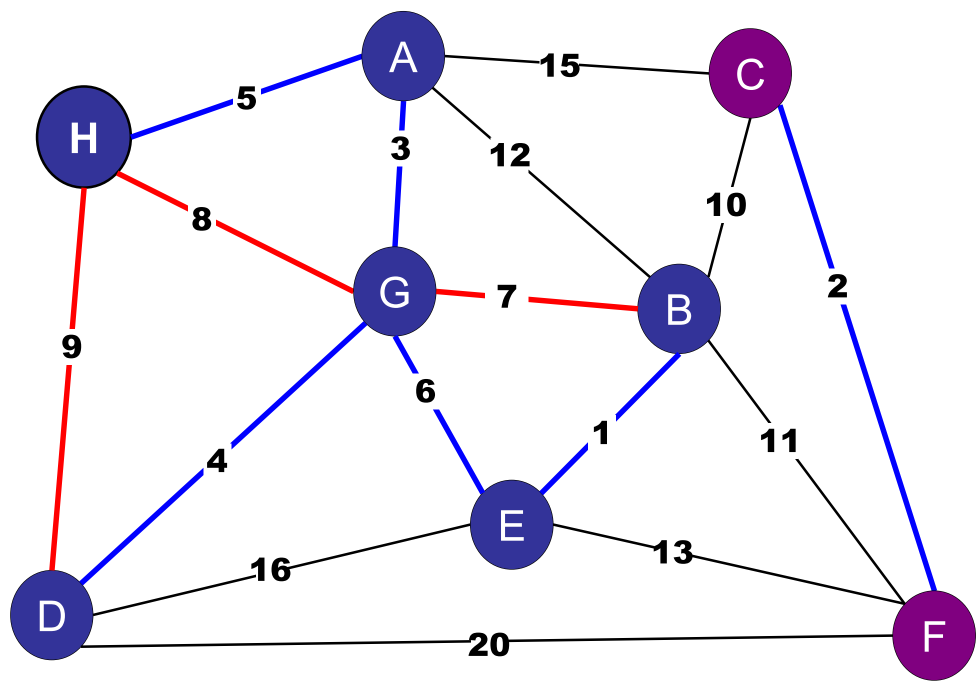
Kruskal's Algorithm. (Kruskal 1956)

Basic idea:

- Sort edges by weight.
- Consider edges in ascending order:
  - If both endpoints are in the **same** blue tree, then color the edge red.
  - Otherwise, color the edge blue.

Proof:

- Each added edge crosses a cut.
- Each edge is the lightest edge across the cut: all other lighter edges across the cut have already been considered.



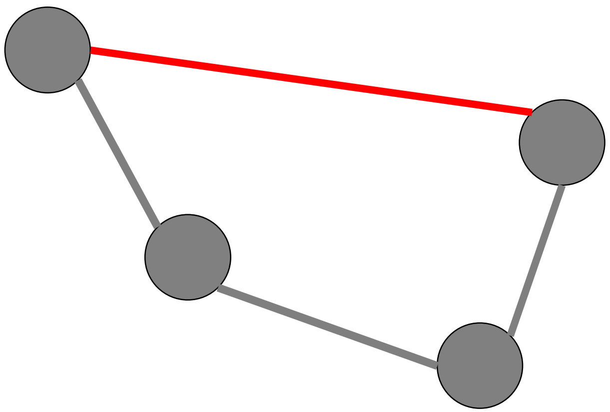
Generic MST Algorithm

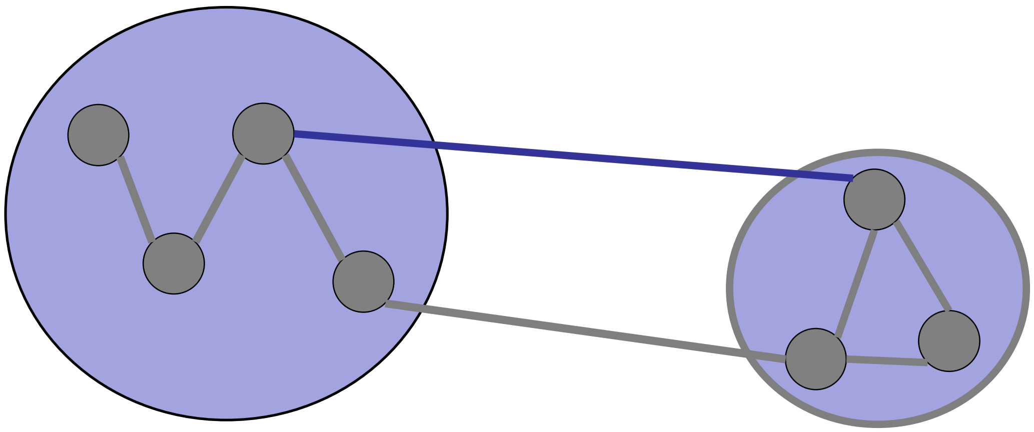
Greedy Algorithm:

Repeat:

Apply red rule or blue rule to an arbitrary edge.

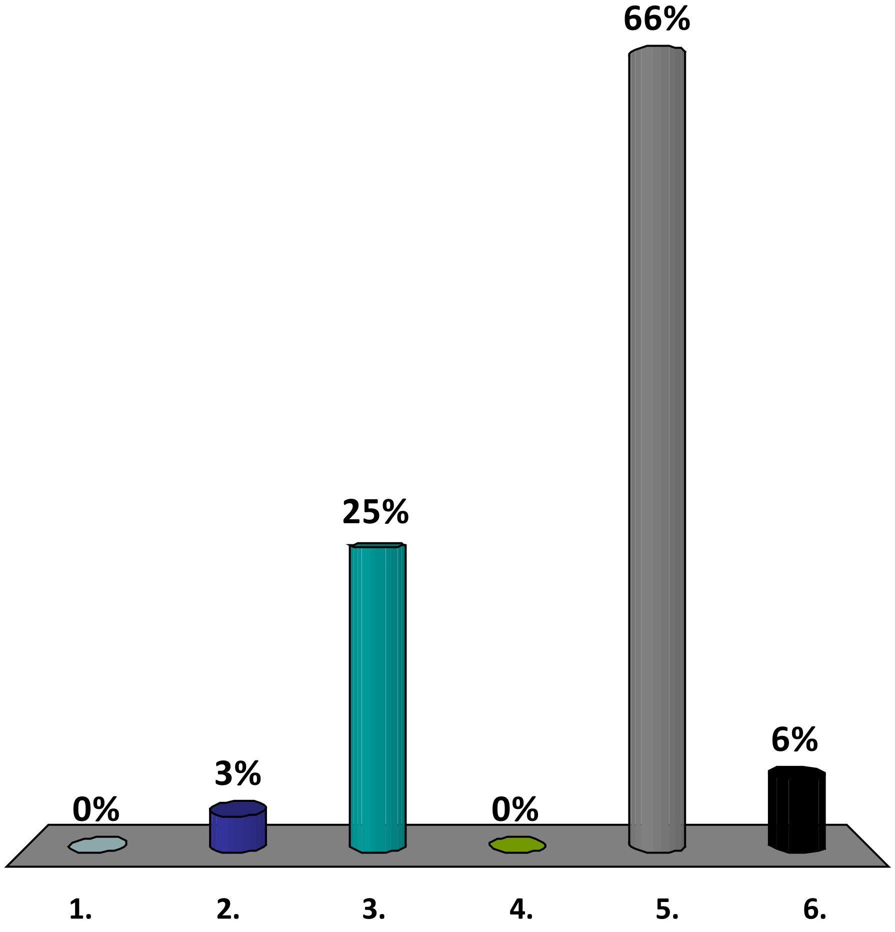
until no more edges can be colored.





What is the overall running time of Kruskal's Algorithm on a connected graph?

1. O(V)
2. O(E)
3. O(E α)
4. O(V α)
5. O(E log V)
6. O(V log E)



Kruskal's Algorithm

Kruskal's Algorithm. (Kruskal 1956)

Basic idea:

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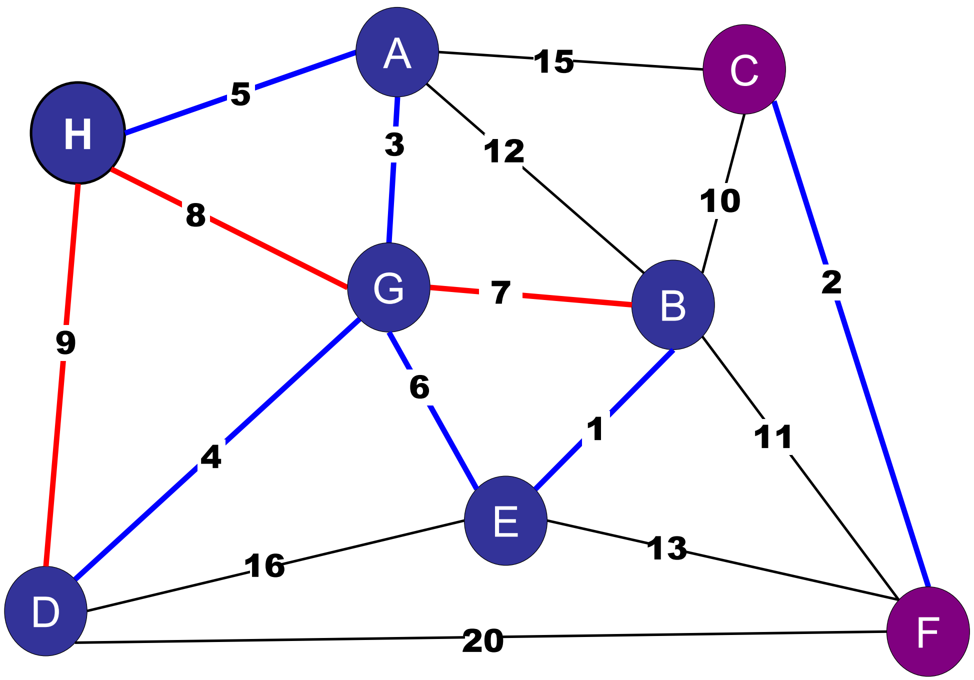


Performance:

- Sorting: $O(E \log E) = O(E \log V)$
- For E edges:
  - Find: $O(\alpha)$ or $O(\log V)$
  - Union: $O(\alpha)$ or $O(\log V)$

Roadmap

Minimum Spanning Trees

- The MST Problem
- Basic Properties of an MST
- Generic MST Algorithm
- Prim’s Algorithm
- Kruskal’s Algorithm
- **Boruvka’s Algorithm**
- Variations

MST Algorithms

Classic:

- Prim’s Algorithm
- Kruskal’s Algorithm

Modern requirements:

- Parallelizable
- Faster in “good” graphs (e.g., planar graphs)
- Flexible

Boruvka's Algorithm

Origin: 1926

- Otakar Boruvka
- Improve the electrical network of Moravia

Based on generic algorithm:

- Repeat: add all "obvious" blue edges.
- Very simple, very flexible.

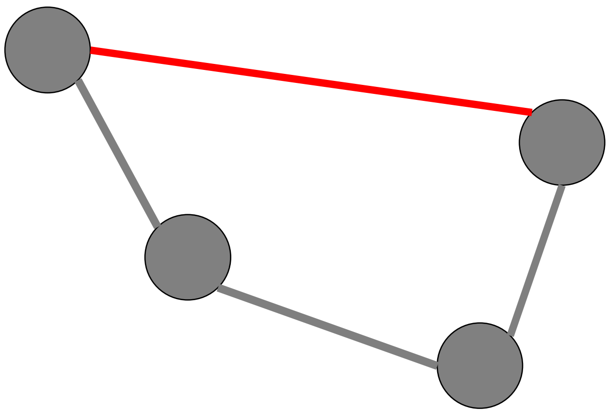
Generic MST Algorithm

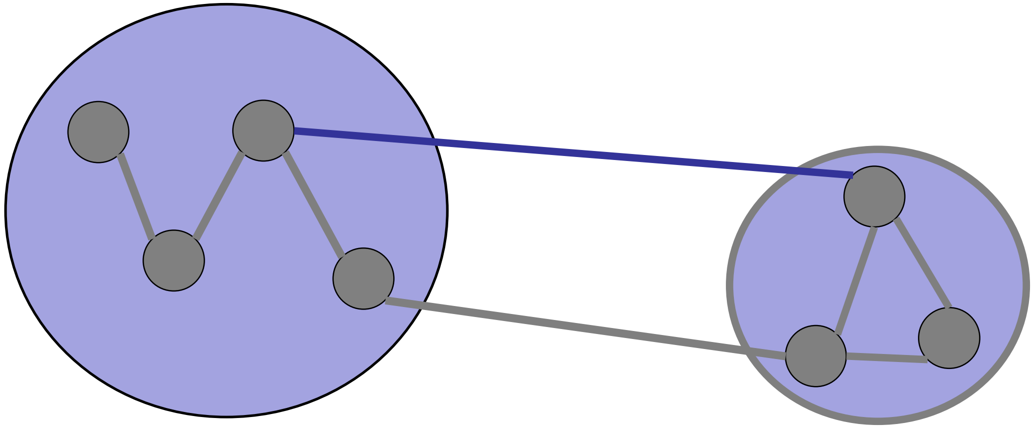
Greedy Algorithm:

Repeat:

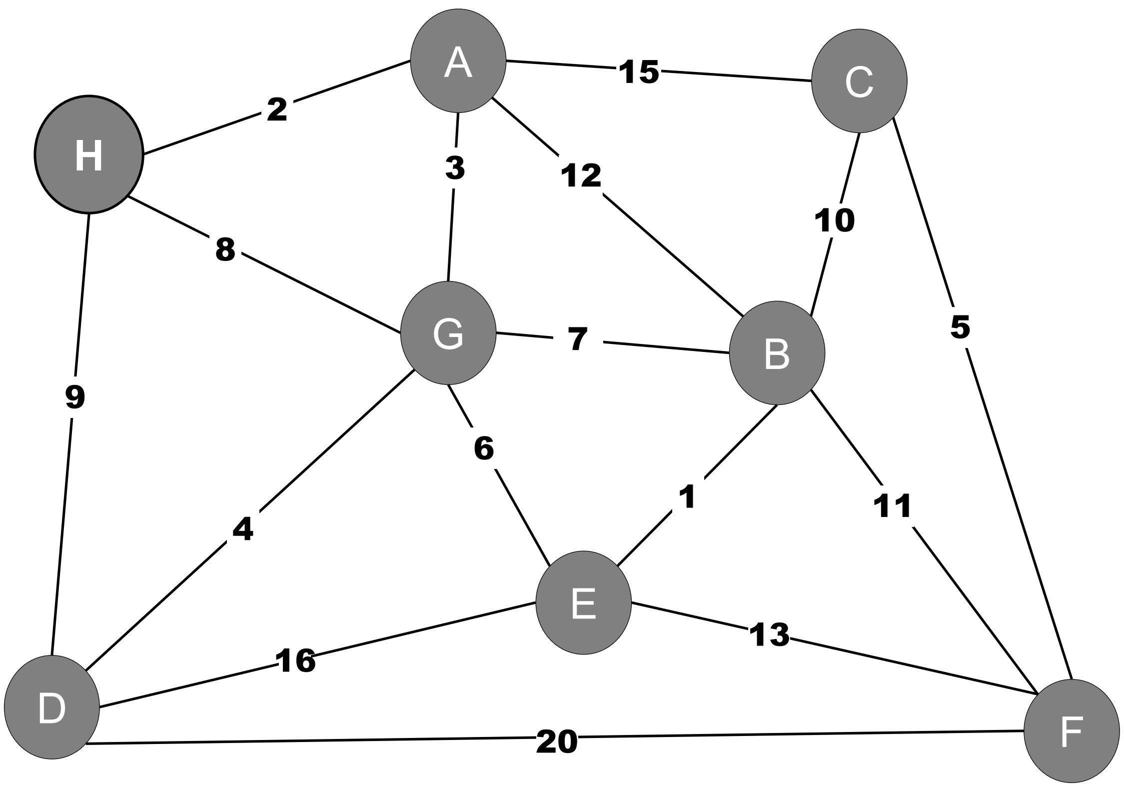
Apply red rule or blue rule to an arbitrary edge.

until no more edges can be colored.





Boruvka's Example

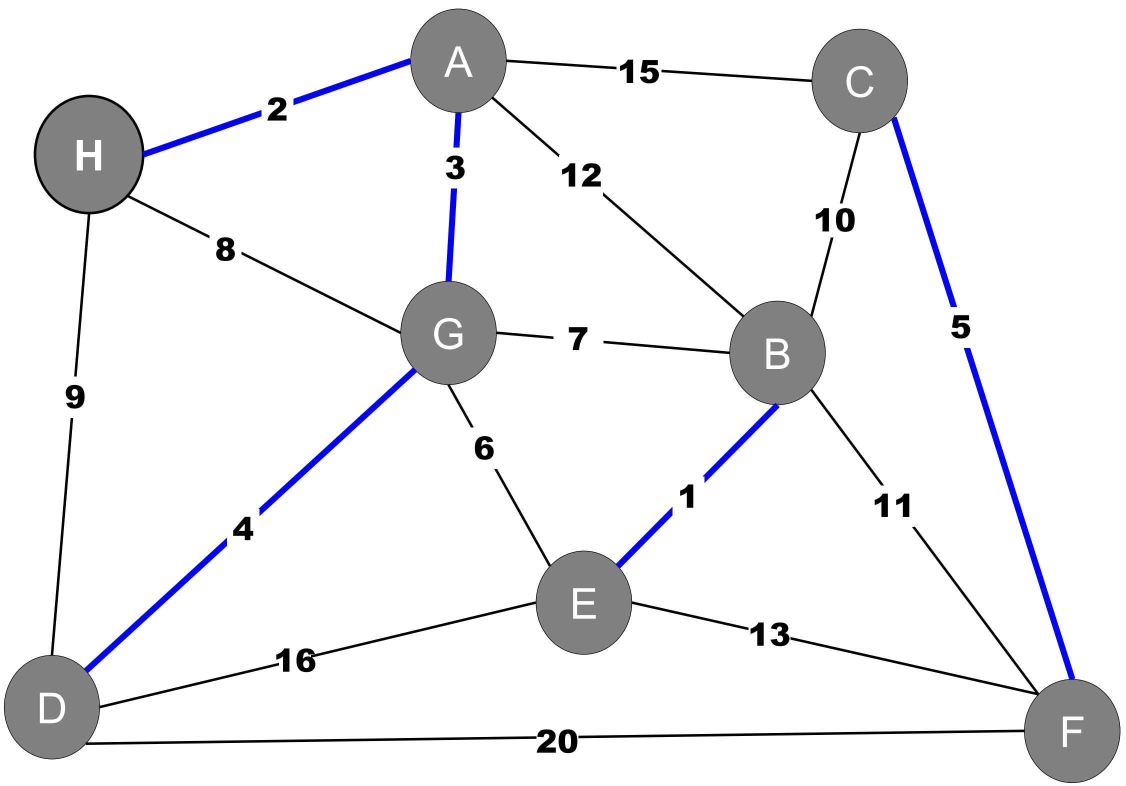


Which edges are "obviously" in the MST?

All the min outgoing edges! (Property 4b)

Weight	Edge
1	(E,B)
2	(C,F)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,G)
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Boruvka's Example

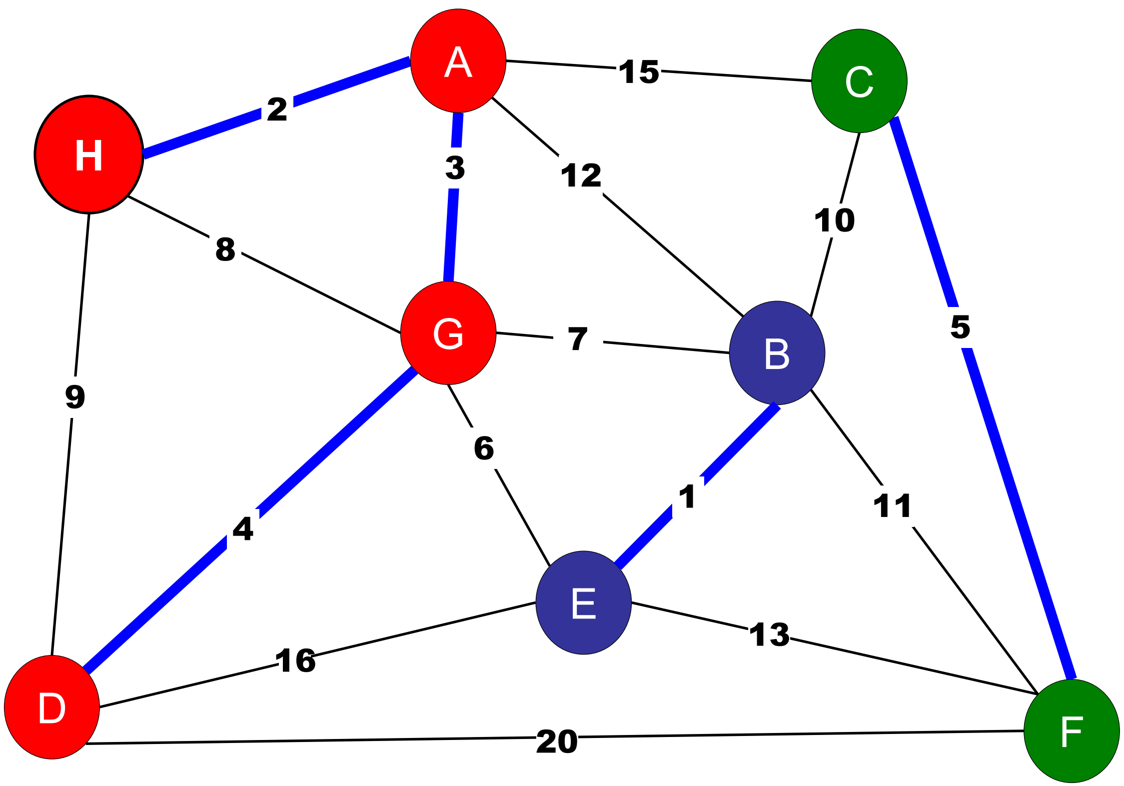


Weight	Edge
1	(E,B)
2	(C,F)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,G)
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

For every node: add minimum adjacent edge.

Add at least n/2 edges.

Boruvka's Example



Weight	Edge
1	(E,B)
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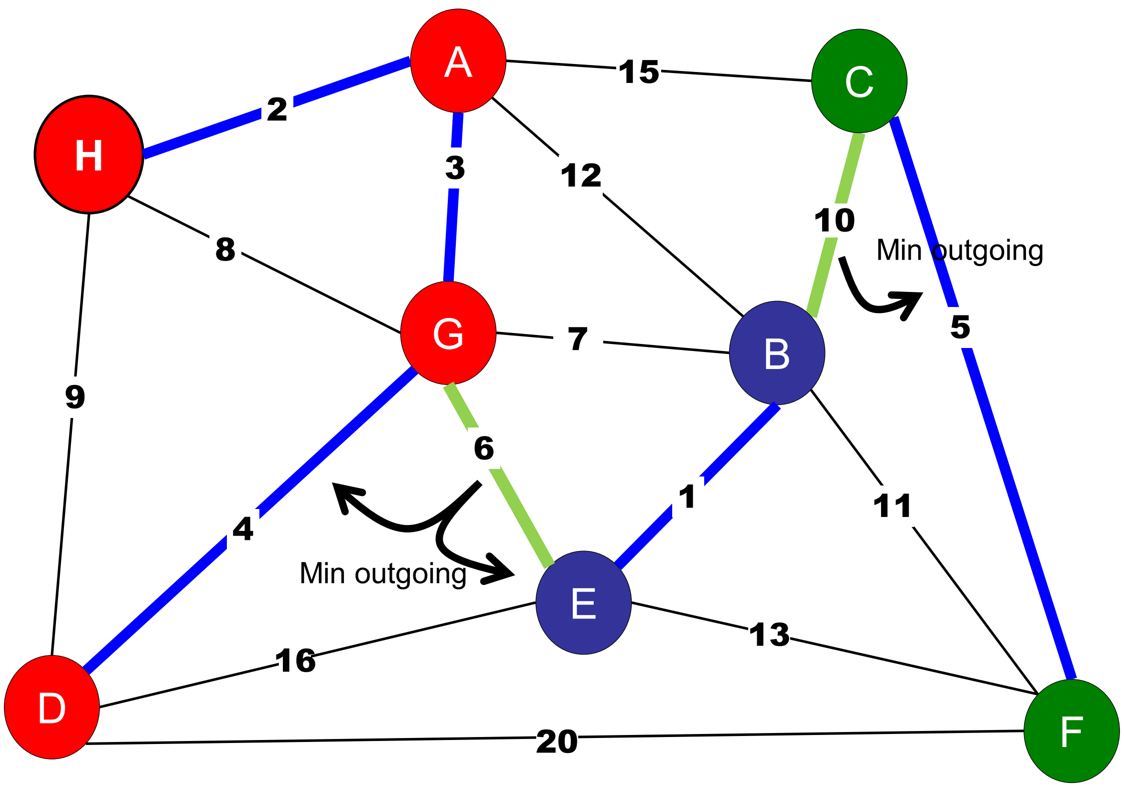
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,G)
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Look at connected components…

At most n/2 connected components.

Boruvka's Example



Repeat: for every connected components, add minimum outgoing edge.

Weight	Edge
1	(E,B)
2	(C,F)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,G)
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15	(A,C)
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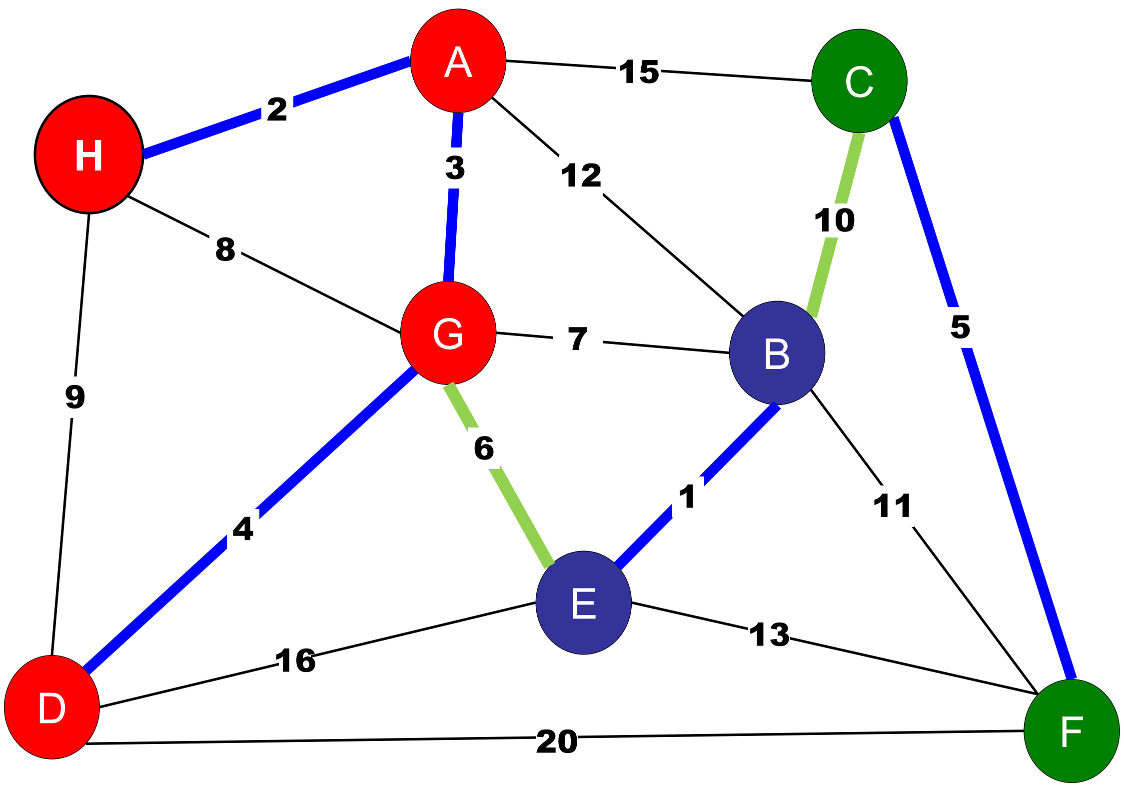
Boruvka's Example



Repeat: for every connected components, add minimum outgoing edge.

Weight	Edge
1	(E,B)
2	(C,F)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
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16	(D,E)
20	(D,F)

Boruvka's Algorithm

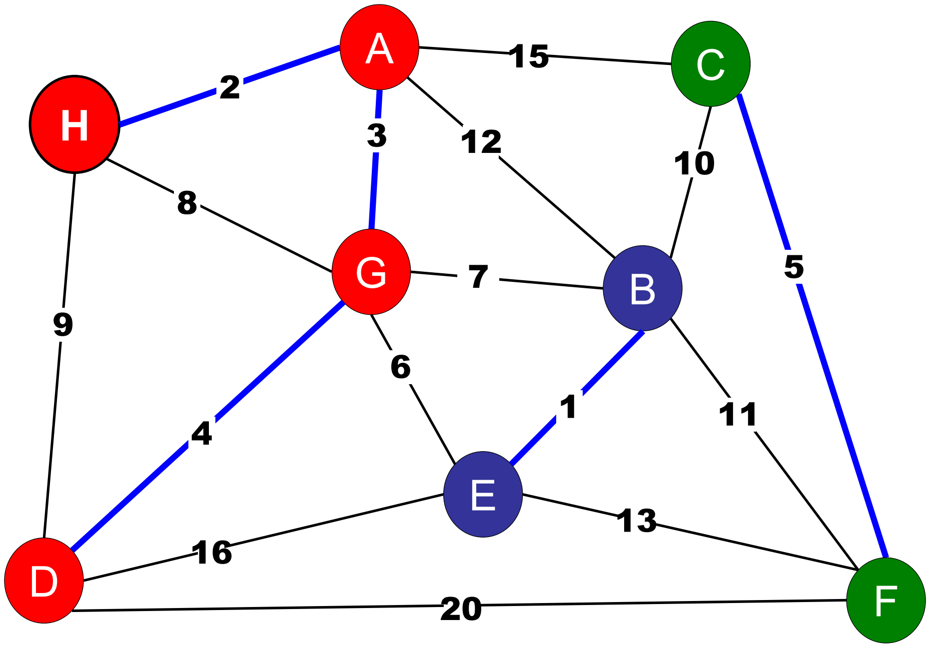
Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

One "Boruvka" Step:

- For each connected component, search for the minimum weight outgoing edge.
- Add selected edges.
- Merge connected components.



Boruvka’s Algorithm

Boruvka’s Algorithm

Initially:

– Create n connected components, one for each node in the graph.

For each node: store a component identifier.

One “Boruvka” Step:

– For each connected component, search for the minimum weight outgoing edge.

– Add selected edges.

– Merge connected components.

Boruvka's Algorithm

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Initially:

- Create n connected components, one for each node in the graph.

One "Boruvka" Step:

- For each connected component, search for the minimum weight outgoing edge.
- Add selected edges.
- Merge connected components.

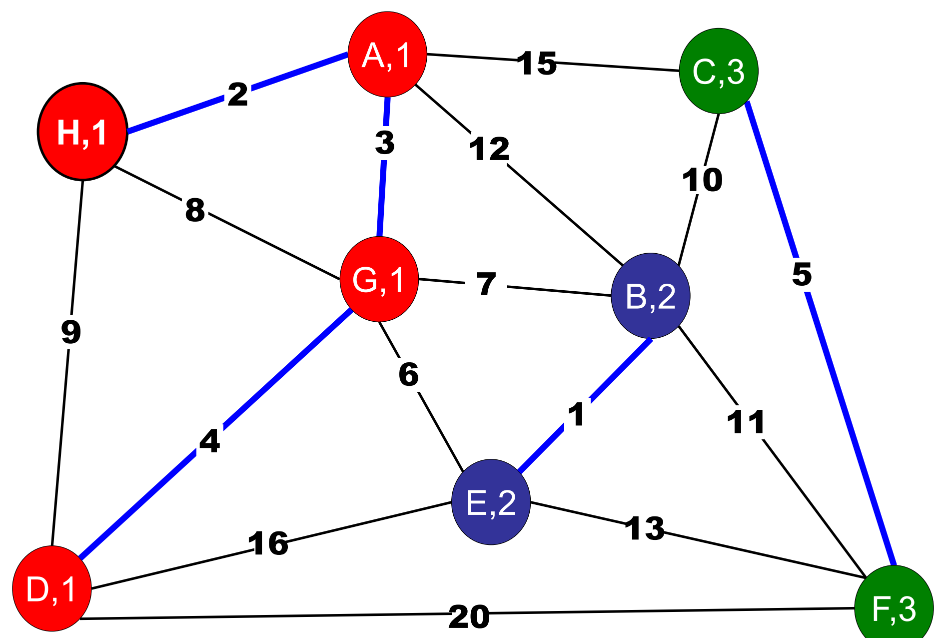
For each node: store a component identifier.

DFS or BFS:

Check if edge connects two components.

Remember minimum cost edge connected to each component.

Component	1	2	3
Min cost edge	(G,E), 6	(G,E), 6	(B,C), 10
To be merged	1 and 2	1 and 2	2 and 3



Boruvka's Algorithm

For each node: store a component identifier.

Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

DFS or BFS:

Check if edge connects two components.

Remember minimum cost edge connected to each component.

One "Boruvka" Step:

- For each connected component, search for the minimum weight outgoing edge.
- Add selected edges.
- Merge connected components.

Scan every node:

Compute new component ids.

Update component ids.

Mark added edges.

Component	**1**	**2**	**3**
Min cost edge	(G,E), 6	(G,E), 6	(B,C), 10
To be merged	1 and 2	1 and 2	2 and 3
New ID:	1	1	1

Boruvka's Algorithm

Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

One "Boruvka" Step: O(V+E)

- For each connected component, search for the minimum weight outgoing edge.
- Add selected edges.
- Merge connected components.

For each node: O(V)
store a component identifier.

DFS or BFS: O(V + E)
Check if edge connects two components.

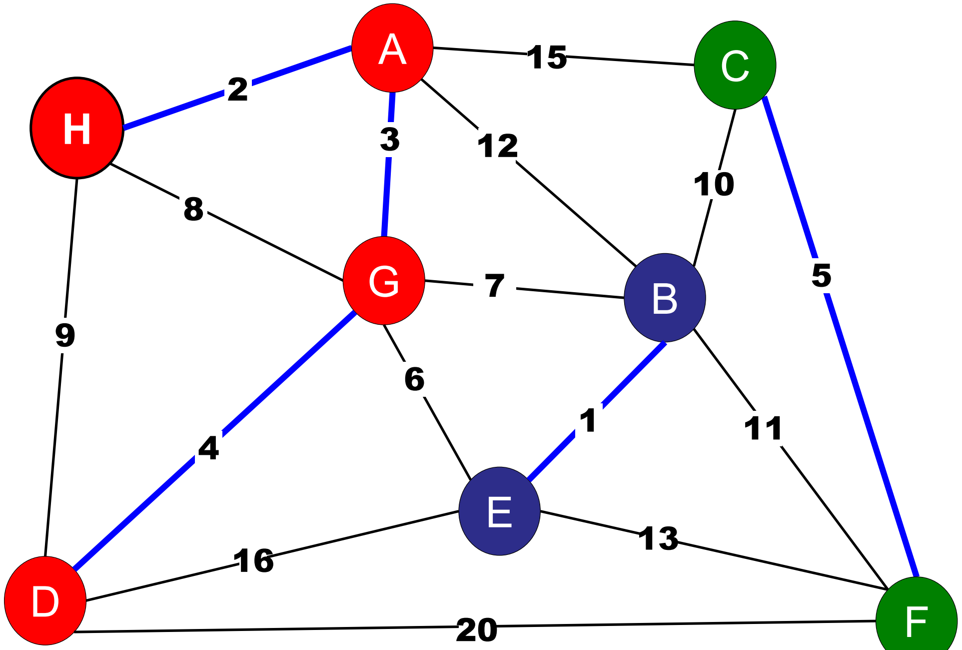
Remember minimum cost edge connected to each component.

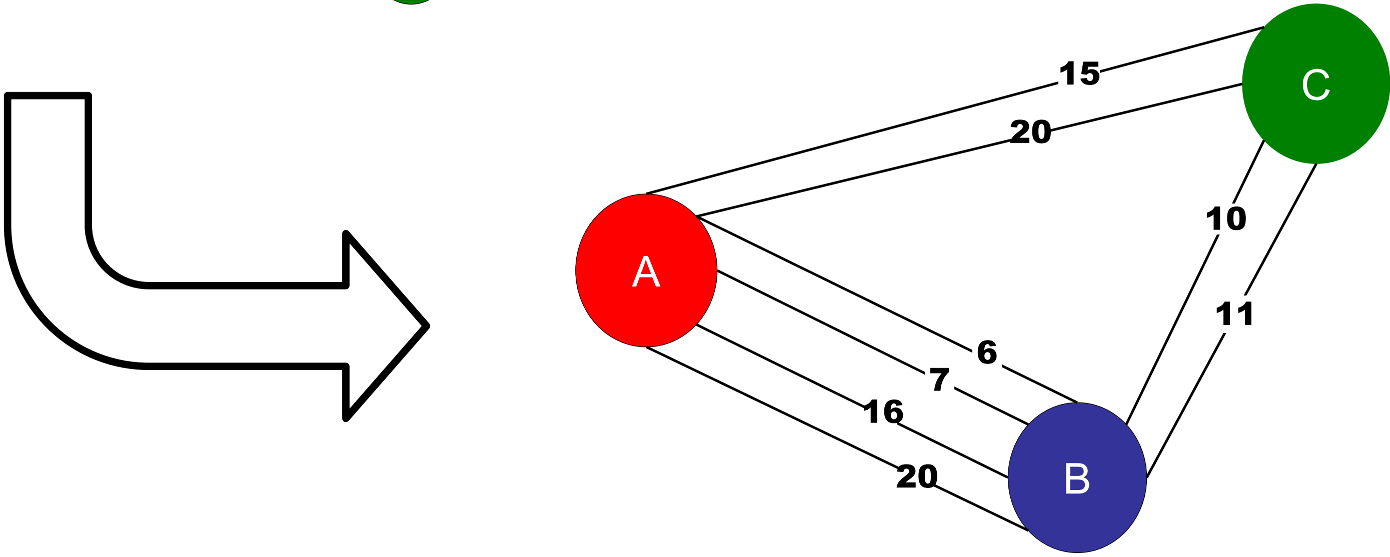
Scan every node: O(V)
Computer new component ids.

Update component ids.

Mark added edges.

Boruvka's Example: Contraction





Boruvka's Algorithm

Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

In each "Boruvka" Step: O(V+E)

- Assume k components, initially.
- At least k/2 edges added.

Count edges:

Each component adds one edge.

Some choose same edge.

Each edge is chosen by at most two different components.

Boruvka's Algorithm

Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

In each "Boruvka" Step: O(V+E)

- Assume k components, initially.
- At least k/2 edges added.
- At least k/2 components merge.

Merging:
Each edge merges two components

Boruvka's Algorithm

Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

In each "Boruvka" Step: O(V+E)

- Assume k components, initially.
- At least k/2 edges added.
- At least k/2 components merge.
- At end, at most k/2 components remain.

Boruvka's Algorithm

Boruvka's Algorithm

Initially:

n components

At each step:

k components ➔ k/2 components.

Termination:

1 component

Conclusion:

At most O(log V) Boruvka steps.

Total time:

O((E+V)log V) = O(E log V)

Boruvka's Algorithm

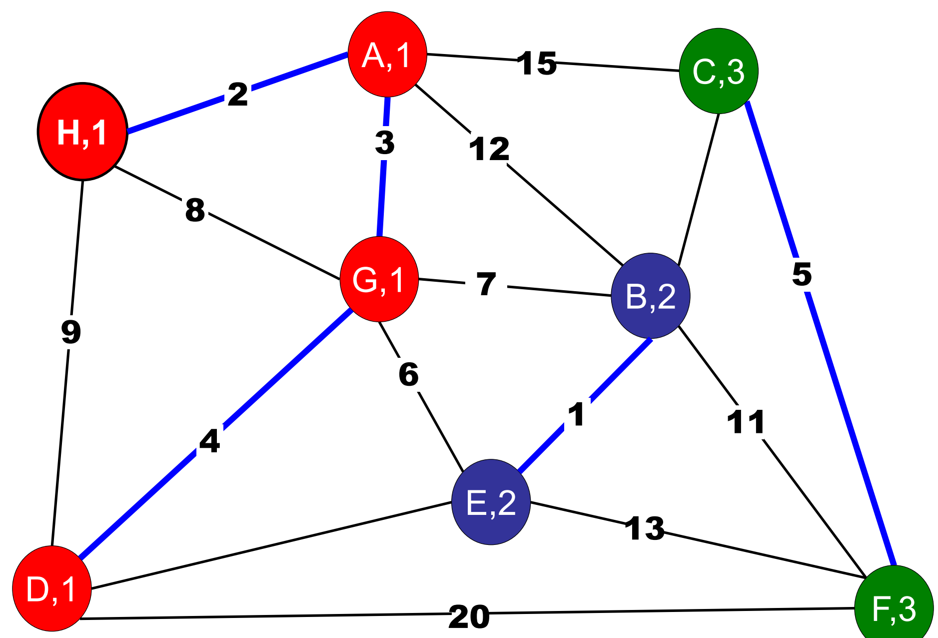
Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

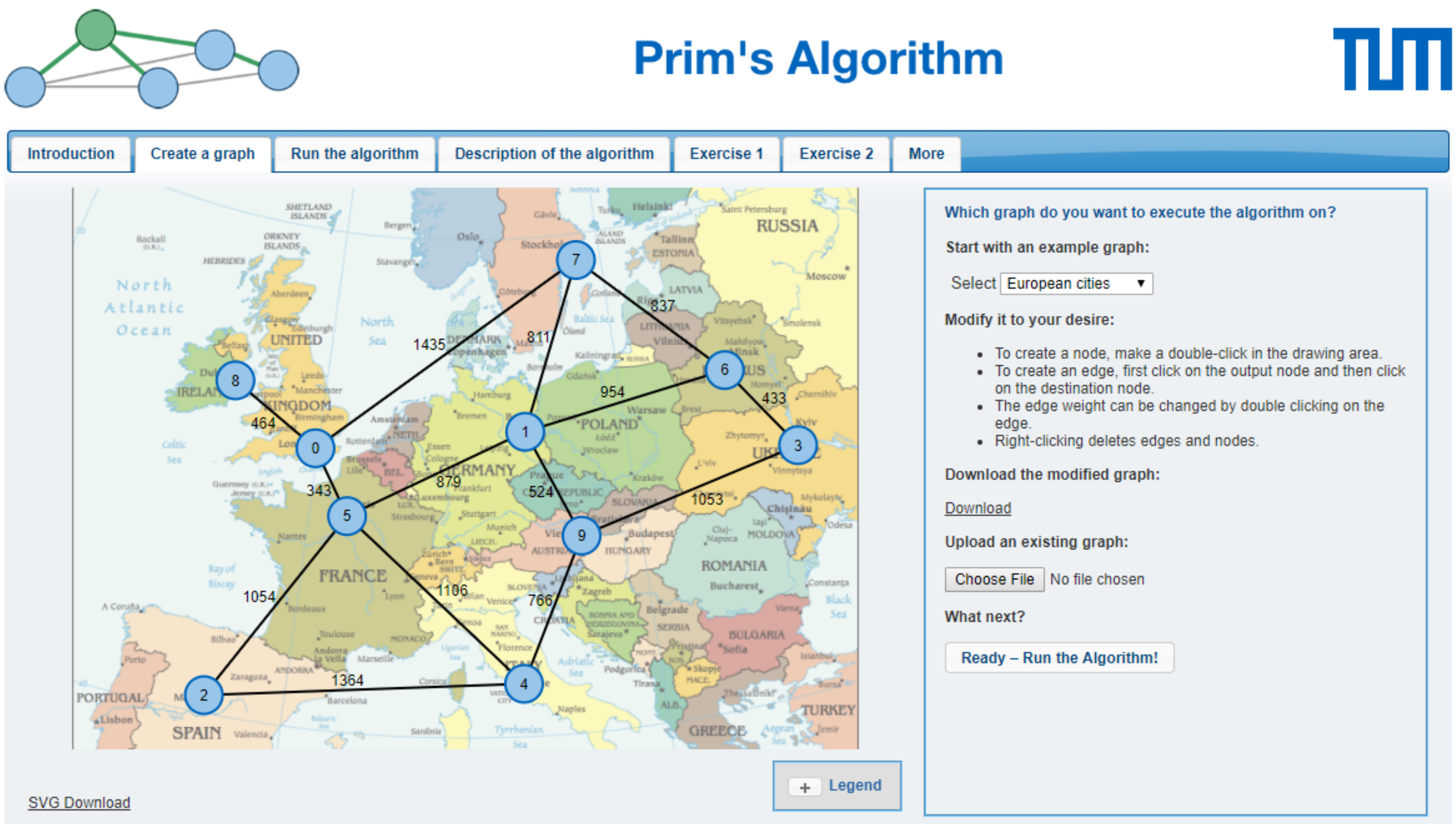
One "Boruvka" Step: O(V+E)

- For each connected component, search for the minimum weight outgoing edge.
- Add selected edges.
- Merge connected components.



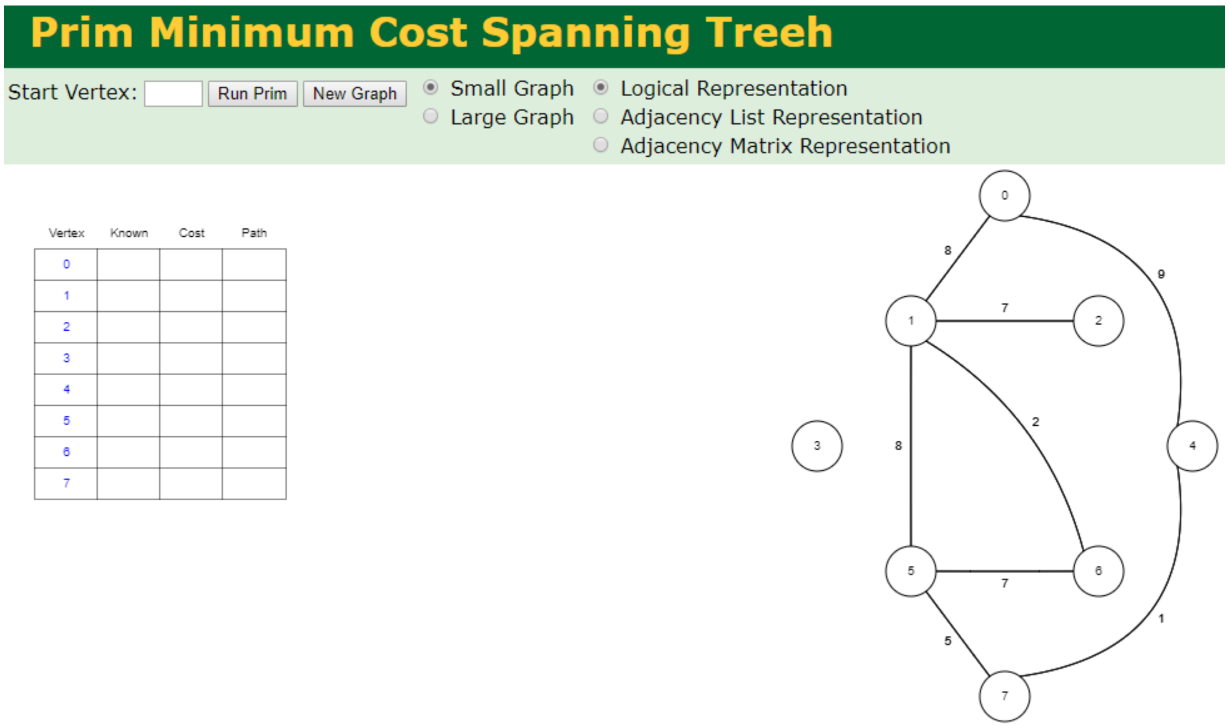
Websites to have fun with MST

- https://www-m9.ma.tum.de/graph-algorithms/mst-prim/index_en.html



Websites to have fun with MST

- https://www.cs.usfca.edu/~galles/visualization/Prim.html



- Kruskal's
  - https://www.cs.usfca.edu/~galles/visualization/Kruskal.html

Roadmap

So far:

Minimum Spanning Trees

- Prim’s Algorith
- Kruskal’s Algorithm
- Boruvka’s Algorithm

Minimum Spanning Tree Summary

Classic greedy algorithms: O(E log V)

- Prim's (Priority Queue)
- Kruskal's (Union-Find)
- Boruvka's

Best known: $O(m\,\alpha(m, n))$

- Chazelle (2000)

Holy grail and major open problem: $O(m)$

Minimum Spanning Tree Summary

Classic greedy algorithms: $O(E \log V)$

- Prim's (Priority Queue)
- Kruskal's (Union-Find)
- Boruvka's

Best known: $O(m\ \alpha(m, n))$

- Chazelle (2000)

Holy grail and major open problem: $O(m)$

- Randomized: Karger-Klein-Tarjan (1995)
- Verification: Dixon-Rauch-Tarjan (1992)

Roadmap

Today: Minimum Spanning Trees

- Prim’s Algorithm
- Kruskal’s Algorithm
- Boruvka’s Algorithm

Variations:

- Constant weight edges
- Bounded integer edge weights
- Euclidean
- Directed graphs
- Maximum Spanning Tree
- Steiner Tree

MST Variants

What if all the edges have the same weight?

- Depth-First-Search or Breadth-First-Search

MST Variants

What if all the edges have the same weight?

- Depth-First-Search or Breadth-First-Search
- An MST contains exactly (V-1) edges.
- Every spanning tree contains (V-1) edges!
- Thus, any spanning tree you find with DFS/BFS is a minimum spanning tree.

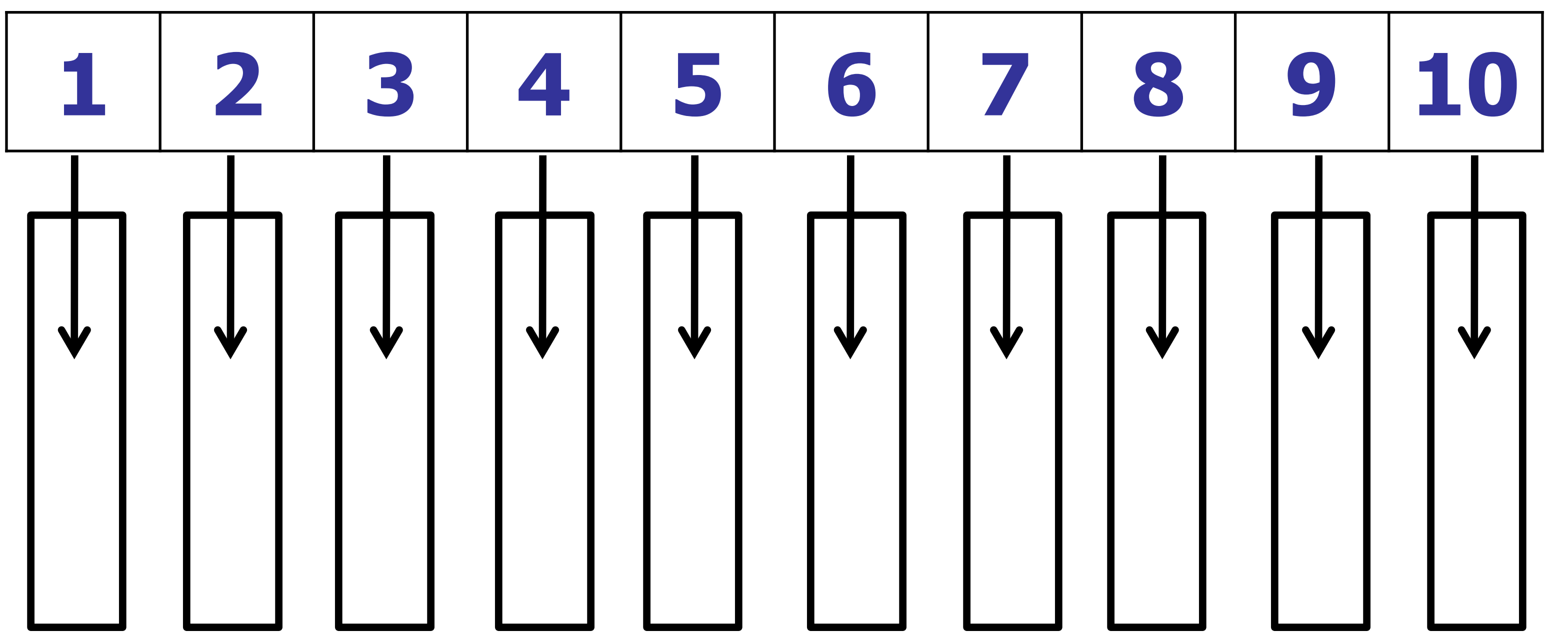
Kruskal's Variants

What if all the edges have weights from {1..10}?

Kruskal's Variants

What if all the edges have weights from {1..10}?

Idea: Use an array of size 10



slot A[j] holds a linked list of edges of weight j

Kruskal's Variants

What if all the edges have weights from {1..10}?

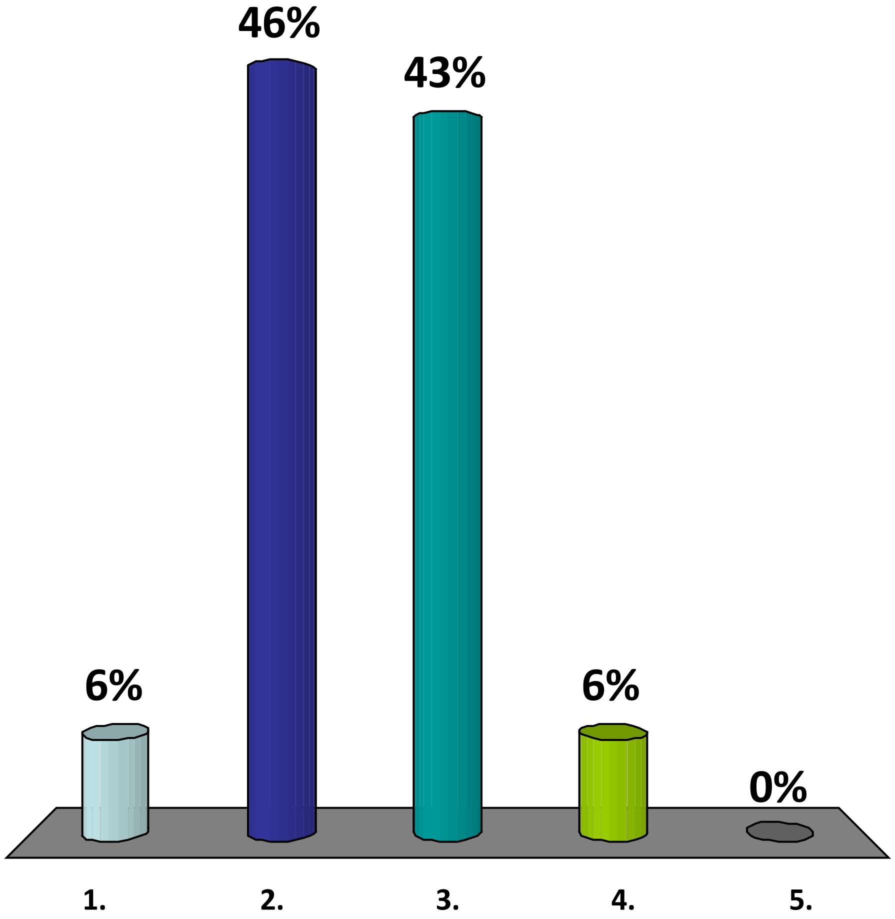
Idea: Use an array of size 10

- Putting edges in array of linked lists: $O(E)$
- Iterating over all edges in ascending order: $O(E)$
- Checking whether to add an edge: $O(\alpha(V))$
- Union two components: $O(\alpha(V))$

Total: $O(\alpha(V)E)$

What is the running time of (modified) Prim's if all the edge weights are in {1..10}?

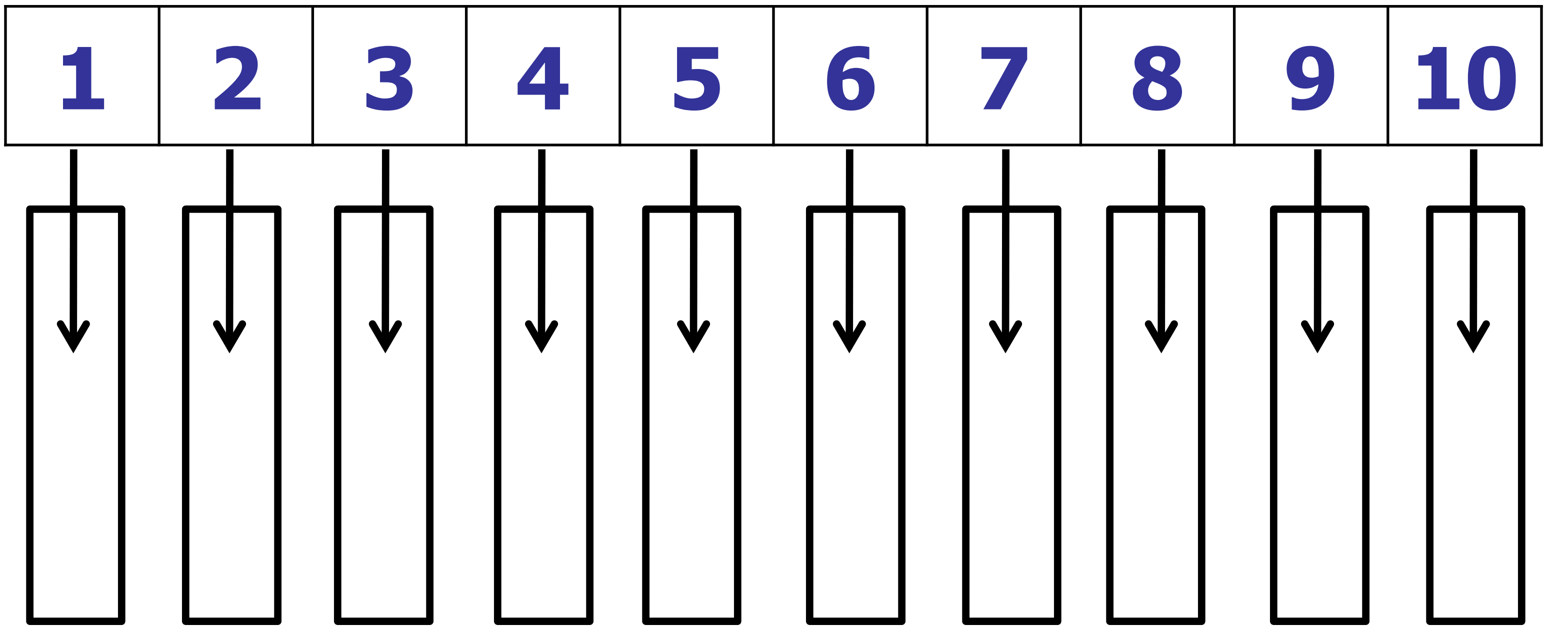
1. O(V)
2. O(E)
3. O(E log V)
4. O(V log E)
5. O(EV)



Prim’s Variants

What if all the edges have weights from {1..10}?

Idea: Use an array of size 10 as a Priority Queue

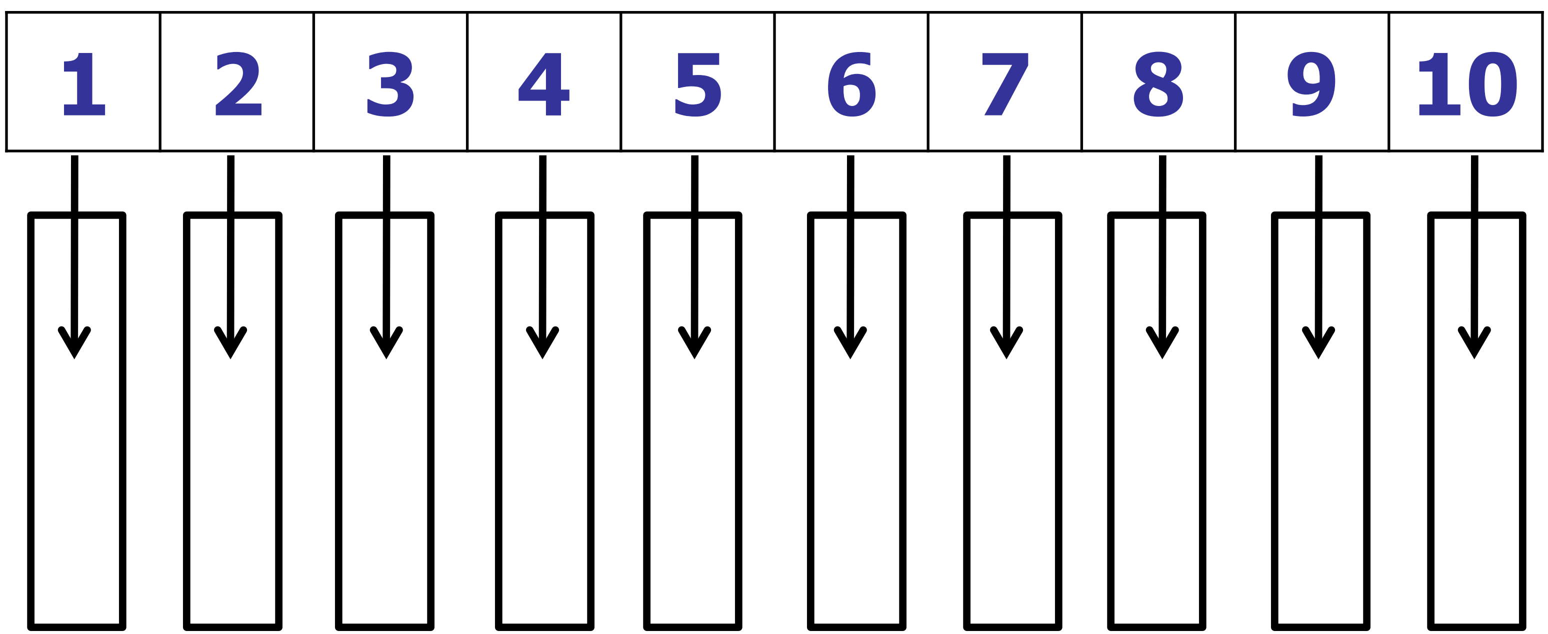


slot A[j] holds a linked list of **nodes** of weight j

Prim's Variants

What if all the edges have weights from {1..10}?

Idea: Use an array of size 10 as a Priority Queue



decreaseKey: move node to new linked list

Prim's Variants

What if all the edges have weights from {1..10}?

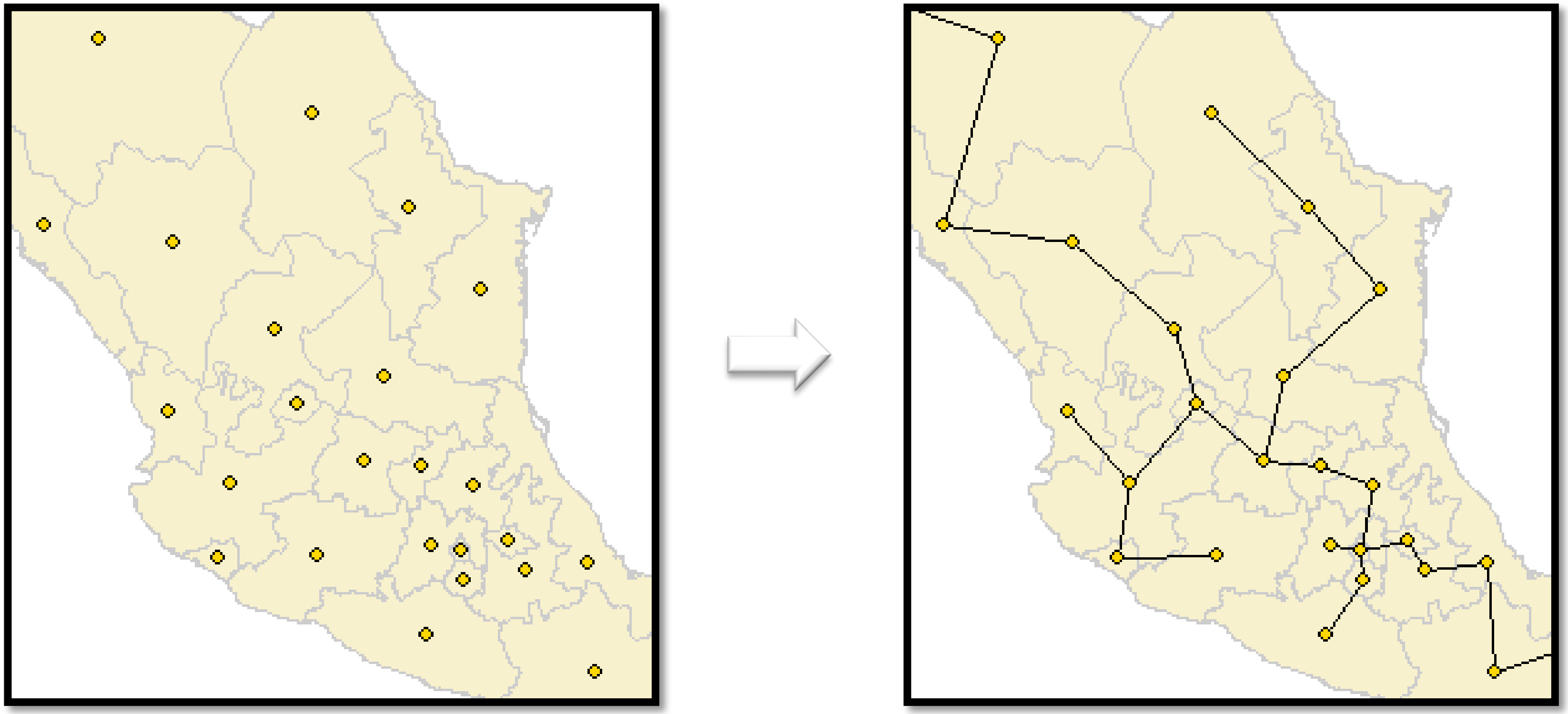
Idea: Use an array of size 10

- Inserting/Removing nodes from PQ: O(V)
- decreaseKey: O(E)

Total: O(V + E) = O(E)

Euclidean Minimal Spanning Tree

- Given point set P, $EMST(P)$ is the tree that spans P and the sum of lengths of all edges is minimal



EMST: Naïve solution

- Compute a complete graph of P with each edge equal to the Euclidean distance
  - $O(n^2)$
- Then run MST
  - $O(n^2 \log n)$

- Any better solution?
  - $O(n \log n)$ by Delaunay Triangulation
  - Come to my computational Geometry Class

Roadmap

Today: Minimum Spanning Trees

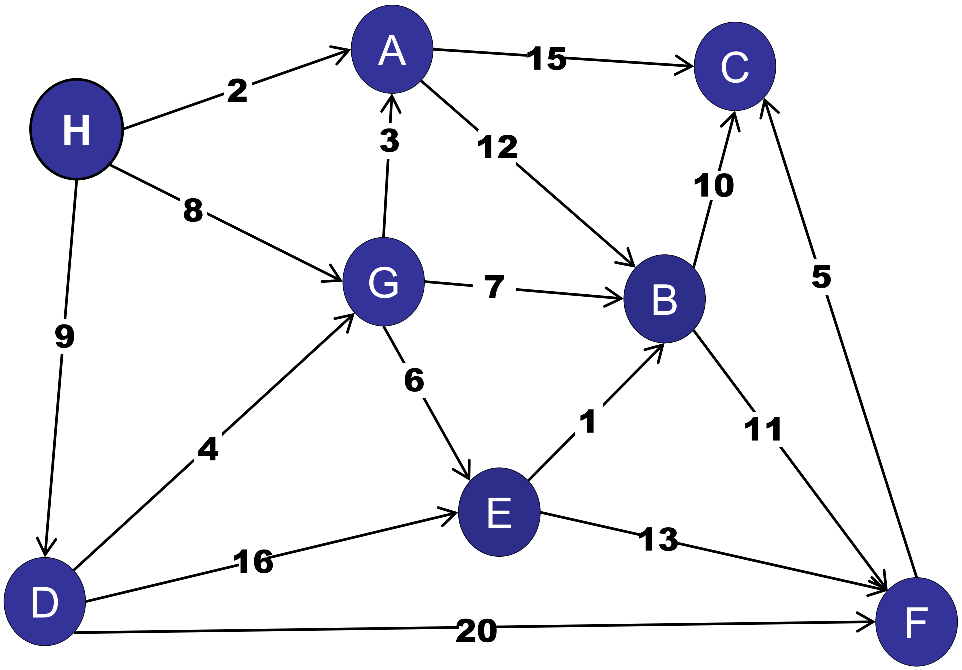
- Prim's Algorithm
- Kruskal's Algorithm
- Boruvka's Algorithm

Variations:

- Constant weight edges
- Bounded integer edge weights
- Euclidean
- Directed graphs
- Maximum Spanning Tree
- Steiner Tree

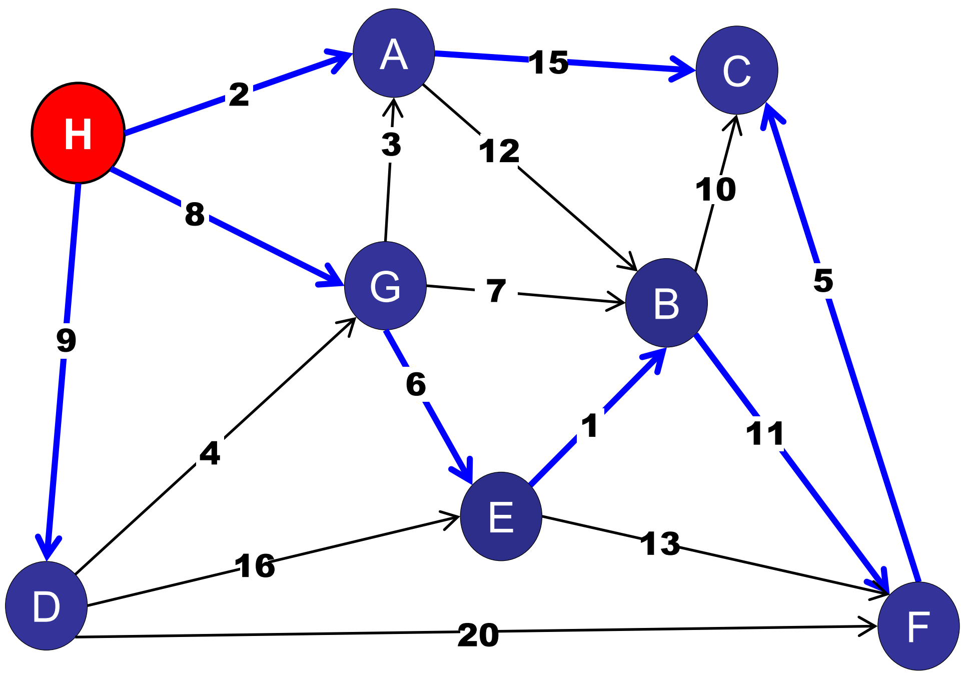
Directed Minimum Spanning Tree

What if the edges are directed?



Directed Minimum Spanning Tree

A rooted spanning tree:



Every node is reachable on a path from the root.

No cycles.

Directed Minimum Spanning Tree

Harder problem:

- Cut property does not hold.
- Cycle property does not hold.
- Generic MST algorithm does not work.

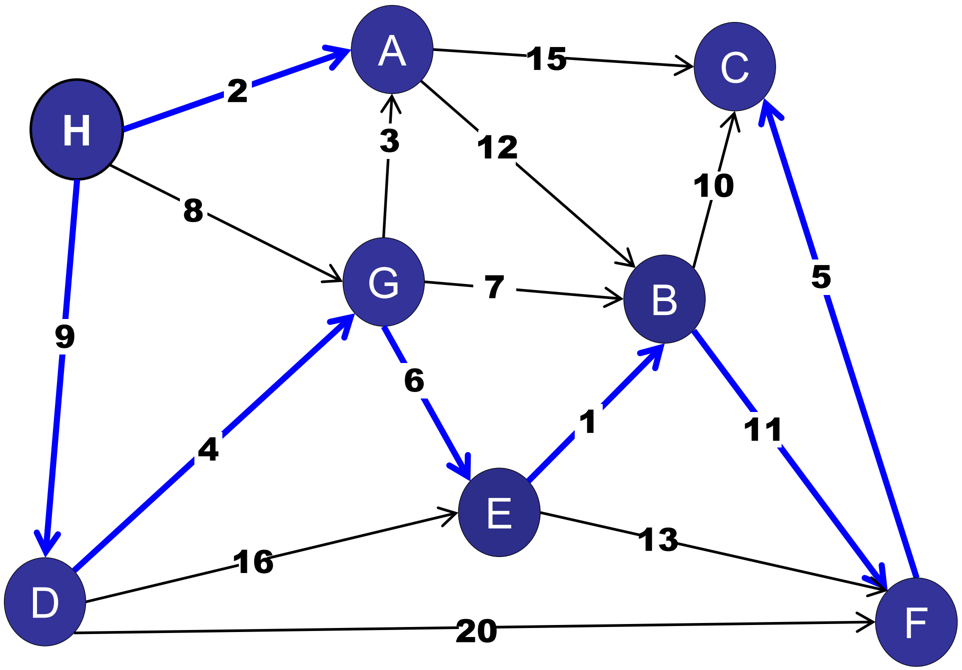
Prim's, Kruskal's, Boruvka's do not work.

See CS3230 / CS5234 for more details…

Directed Minimum Spanning Tree

For a directed acyclic graph with one "root":

For every node except the root: add minimum weight incoming edge.



Directed Minimum Spanning Tree

For a directed acyclic graph with one "root":

For every node except the root: add minimum weight incoming edge.

Observations:

- No cycles (since acyclic graph).
- Each edge is chosen only once.

Tree:
V nodes
V – 1 edges
No cycles

Directed Minimum Spanning Tree

For a directed acyclic graph with one "root":

For every node except the root: add minimum weight incoming edge.

Observations:

- No cycles (since acyclic graph).
- Each edge is chosen only once.

Tree:
V nodes
V – 1 edges
No cycles

- Every node has to have at least one incoming edge in the MST, so this is the minimum spanning tree.

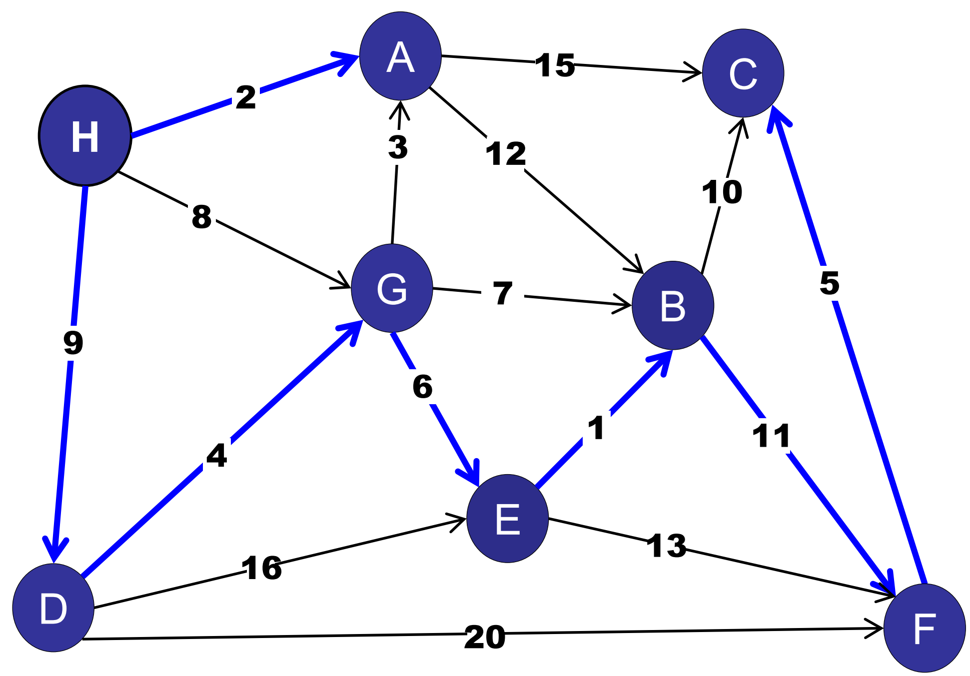
Directed Minimum Spanning Tree

For a directed acyclic graph with one "root":

For every node except the root: add minimum weight incoming edge.

Conclusion: Minimum Spanning Tree

O(E) time



Roadmap

Today: Minimum Spanning Trees

- Prim’s Algorithm
- Kruskal’s Algorithm
- Boruvka’s Algorithm

Variations:

- Constant weight edges
- Bounded integer edge weights
- Euclidean
- Directed graphs
- Maximum Spanning Tree
- Steiner Tree

Maximum Spanning Tree

A MaxST is a spanning tree of maximum weight.

How do you find a MaxST?

Maximum Spanning Tree

Reweighting a spanning tree:

- What happens if you add a constant k to the weight of every edge?

Kruskal's Algorithm

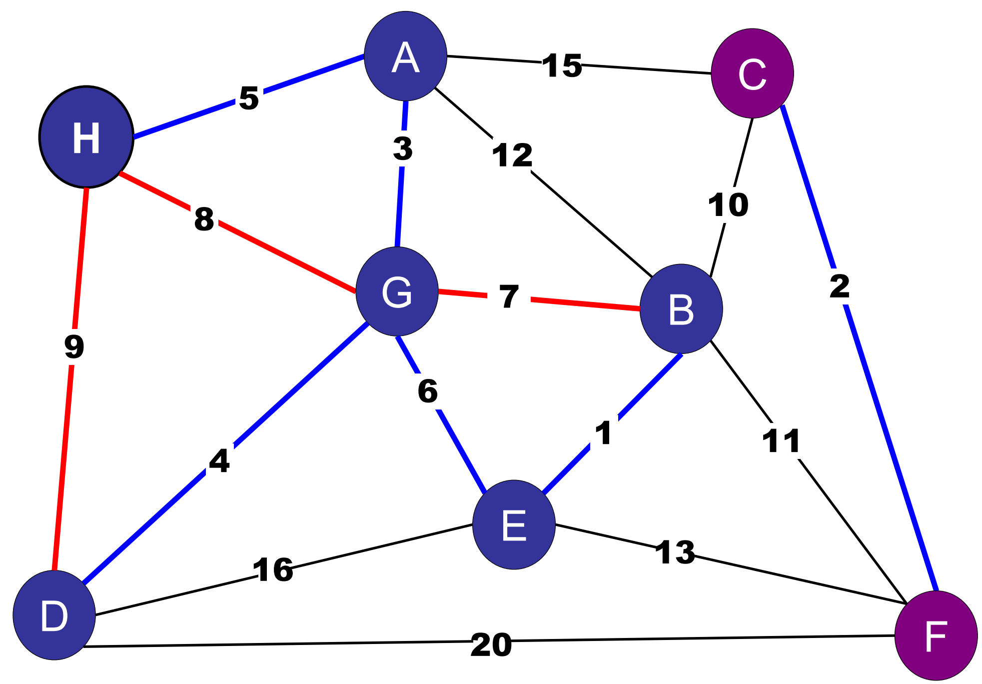
Kruskal's Algorithm. (Kruskal 1956)

Basic idea:

- Sort edges by weight.
- Consider edges in ascending order:
  - If both endpoints are in the **same** blue tree, then color the edge red.
  - Otherwise, color the edge blue.

What matters?

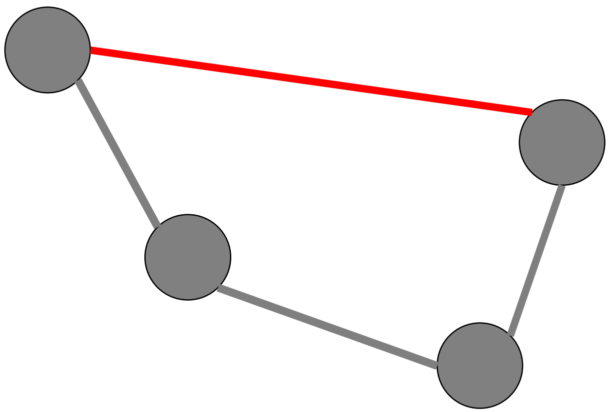
- Relative edge weights.
- Absolute edge weights have no impact.



Generic MST Algorithm

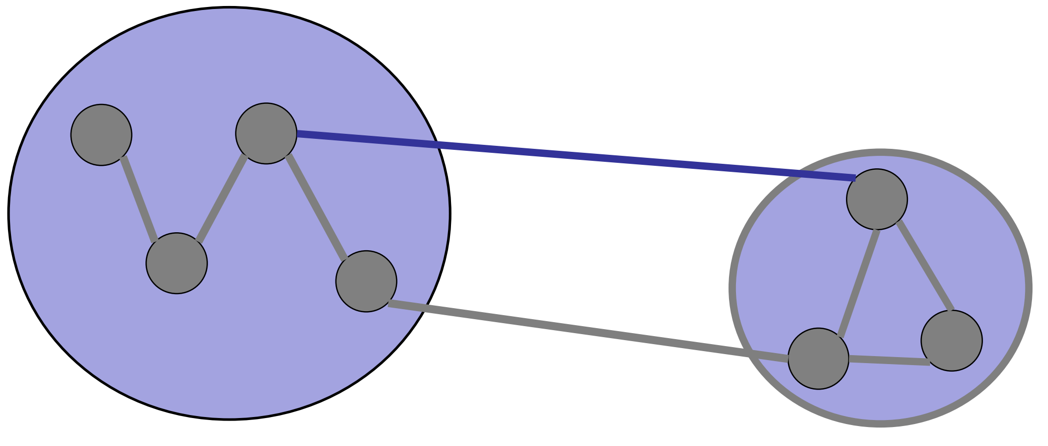
Red rule:

If C is a cycle with no red arcs, then color the max-weight edge in C red.



Blue rule:

If D is a cut with no blue arcs, then color the min-weight edge in D blue.



Maximum Spanning Tree

Reweighting a spanning tree:

- What happens if you add a constant k to the weight of every edge?

No change!

We can add or subtract weights without effecting the MST.

Maximum Spanning Tree

MST with negative weights?

Maximum Spanning Tree

MST with negative weights?

No problem!

1. Reweight MST by adding a big enough value to each edge so that it is positive.

2. Actually, no need to reweight. Only relative edge weights matter, so negative weights have no bad impact.

Maximum Spanning Tree

A MaxST is a spanning tree of maximum weight.

How do you find a MaxST?

Easy!

1. Multiply each edge weight by -1.
2. Run MST algorithm.
3. MST that is “most negative” is the maximum.

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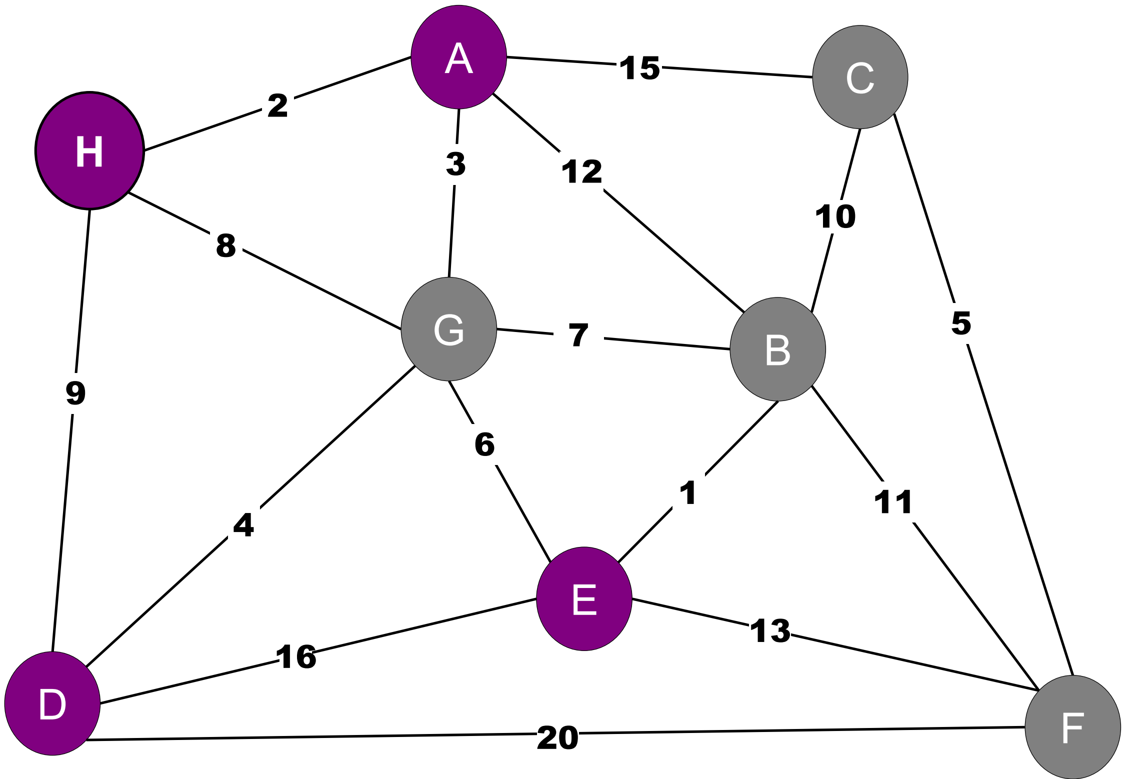
Steiner Tree Problem

What if I want a minimum spanning tree of a subset of the vertices?

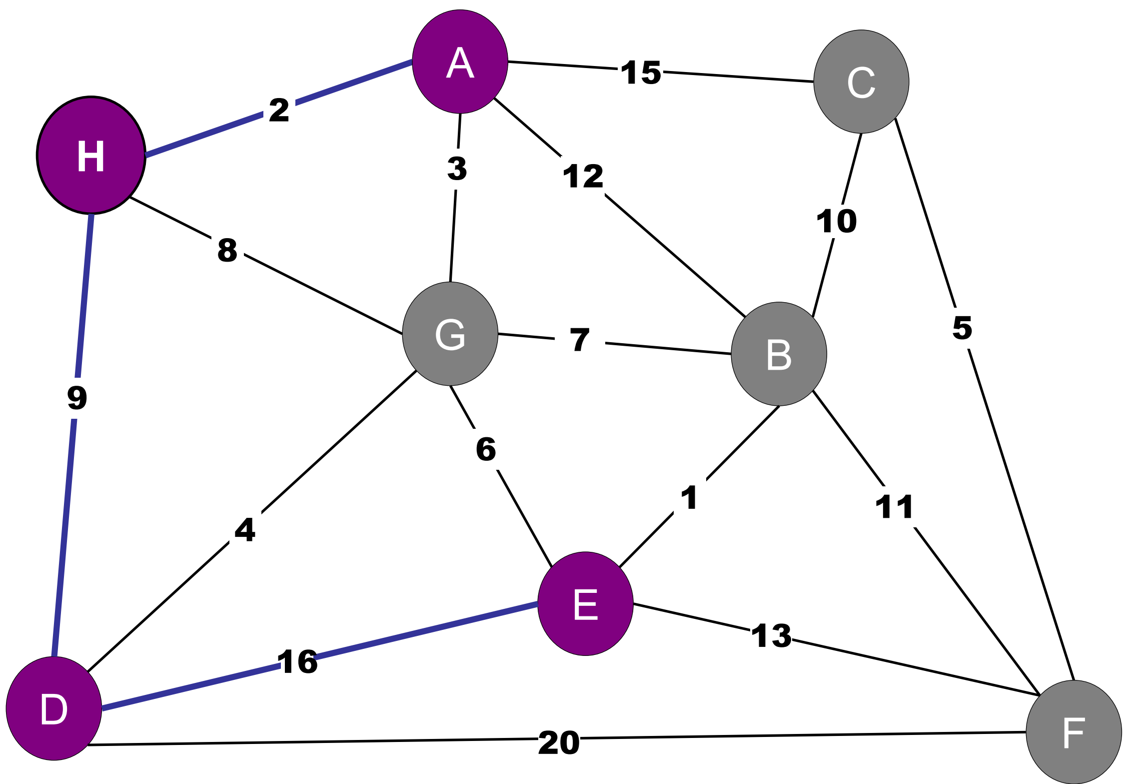


Steiner Tree Problem

What if I want a minimum spanning tree of a subset of the vertices?

1. Just use the sub-graph.

weight = 27

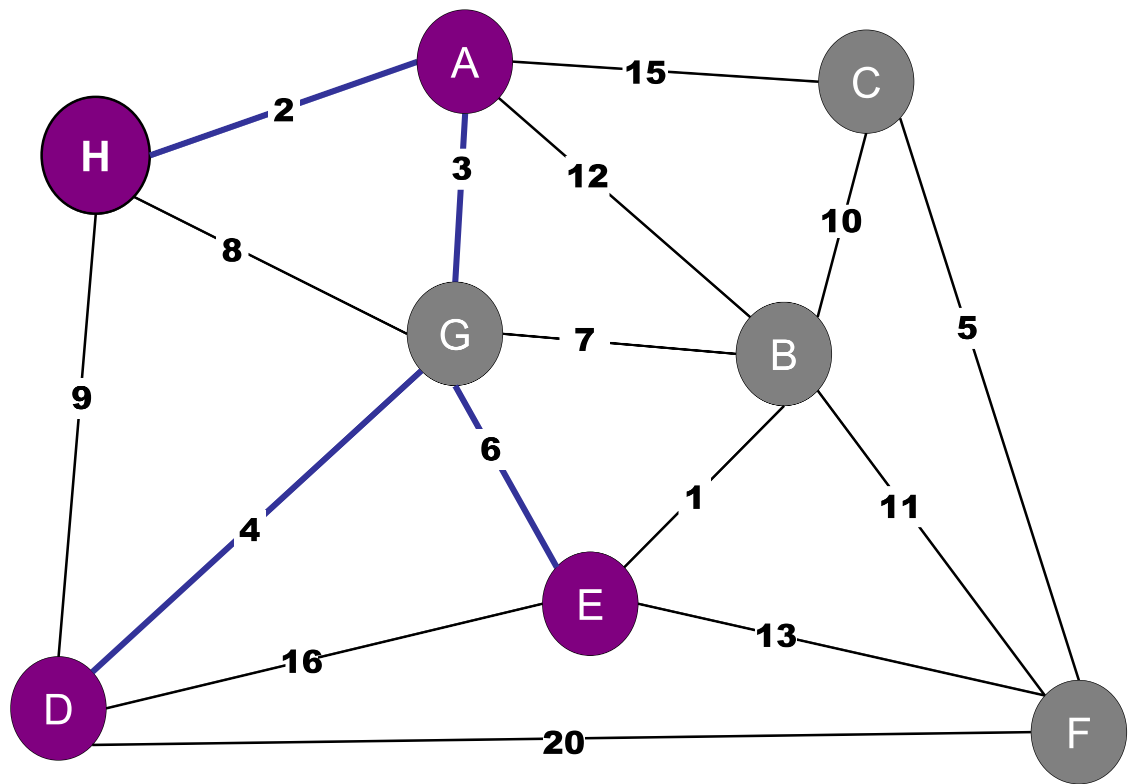


Steiner Tree Problem

What if I want a minimum spanning tree of a subset of the vertices?

1. Just use the sub-graph.
2. Use other nodes.

weight = 15

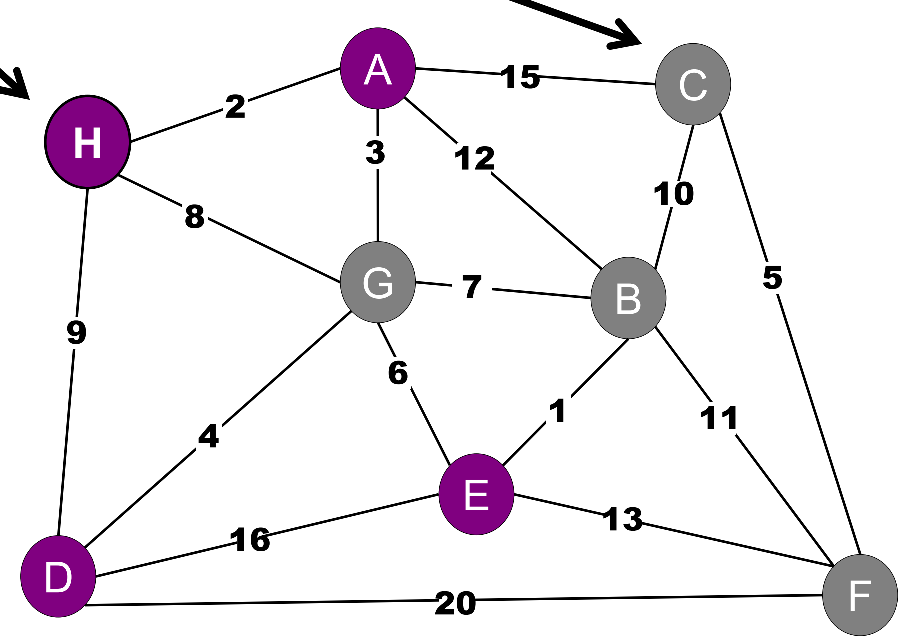


Steiner Tree Problem

What is the minimum spanning tree of a subset of the vertices?

- Steiner nodes
- Required nodes

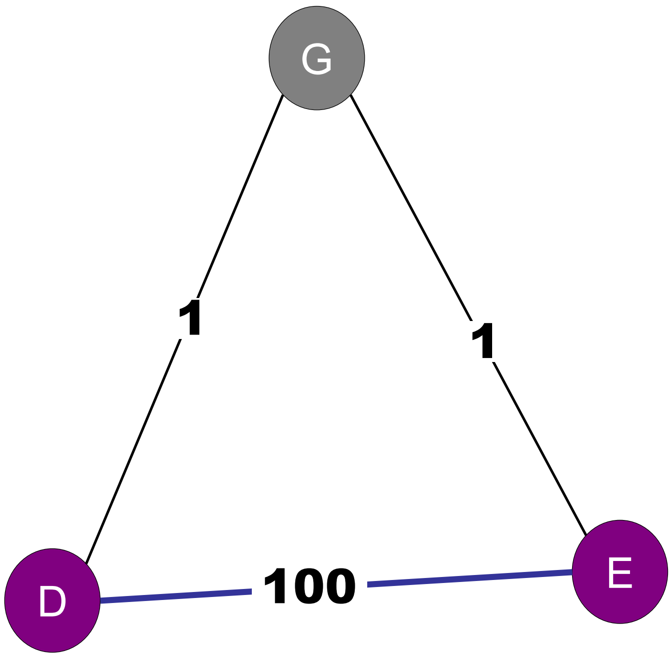
Find spanning tree of required vertices. You may include Steiner vertices, optionally.



Steiner Tree Problem

Just computing MST doesn't work:

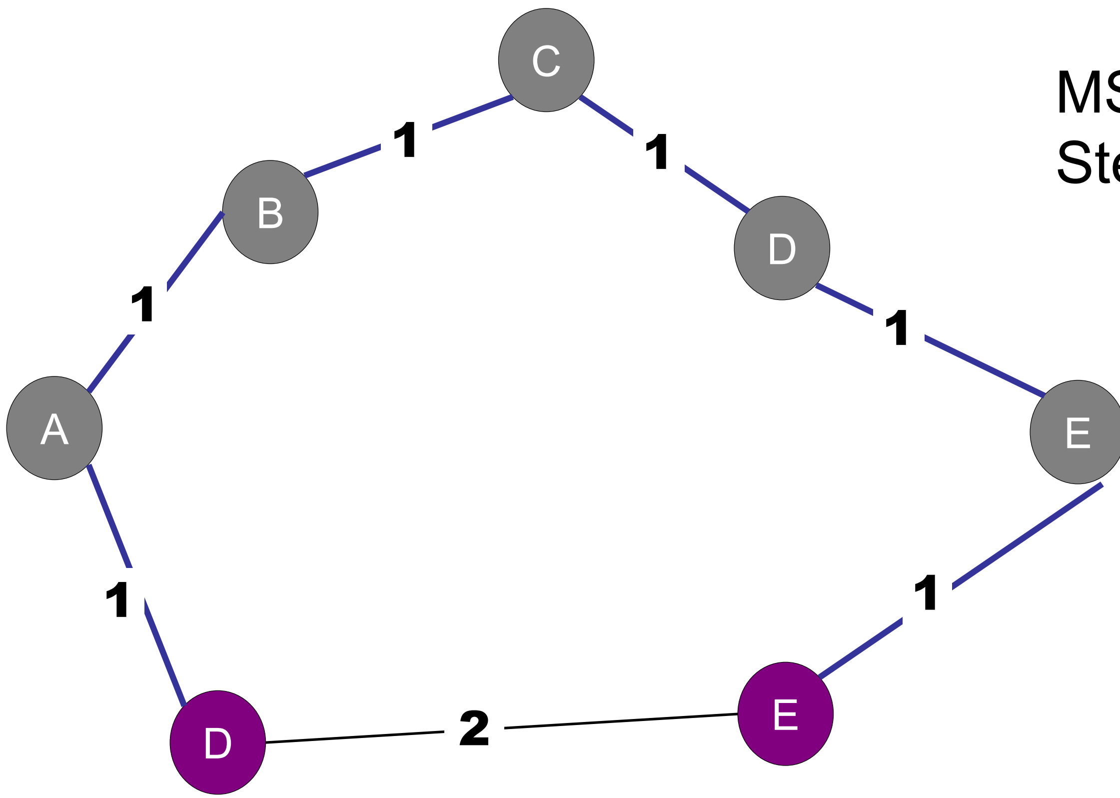
1. Computing MST with no Steiner nodes.



Steiner Tree Problem

Just computing MST doesn't work:

2. Computing MST with all Steiner nodes.



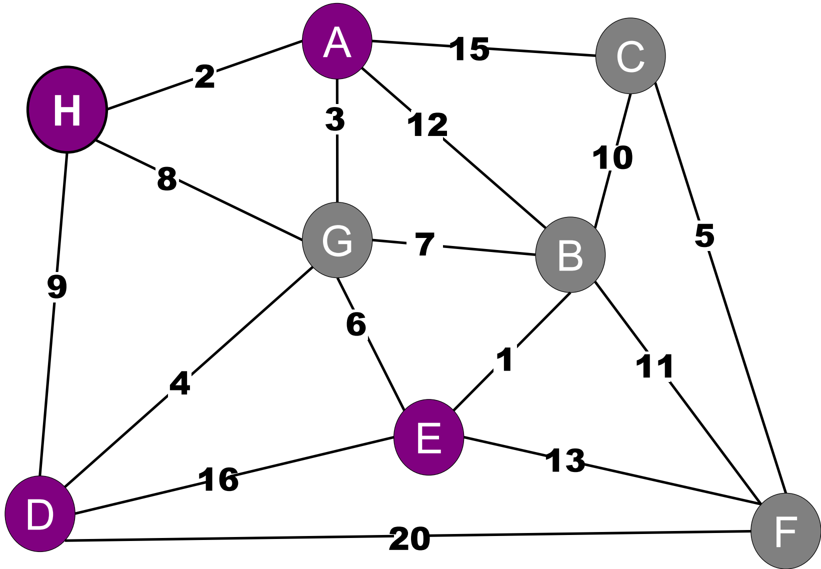
MST = 6
Steiner Tree = 2

Steiner Tree Problem

What is the minimum spanning tree of a subset of the vertices?

Bad News: NP-Hard

No efficient (polynomial) time algorithm (unless P = NP).



Steiner Tree Problem

What is the minimum spanning tree of a subset of the vertices?

Good News: Efficient approximation algorithms

Algorithm SteinerMST guarantees:

- OPT(G) = minimum cost Steiner Tree
- T = output of SteinerMST
- T < 2*OPT(G)

Steiner Tree Problem

Algorithm SteinerMST guarantees:

- OPT(G) = minimum cost Steiner Tree
- T = output of SteinerMST
- T < 2*OPT(G)

Example:

- Optimal Steiner Tree has cost 50.
- Our algorithm always outputs a solution with cost < 100.

Steiner Tree Problem

Algorithm SteinerMST:

1. For every pair of required vertices (v,w), calculate the shortest path from (v to w).
   - Use Dijkstra V times.
   - Or wait until we cover All-Pairs-Shortest-Paths next time.

Steiner Tree Problem

Example: Step 1

Shortest Paths:

(A,H) = 2

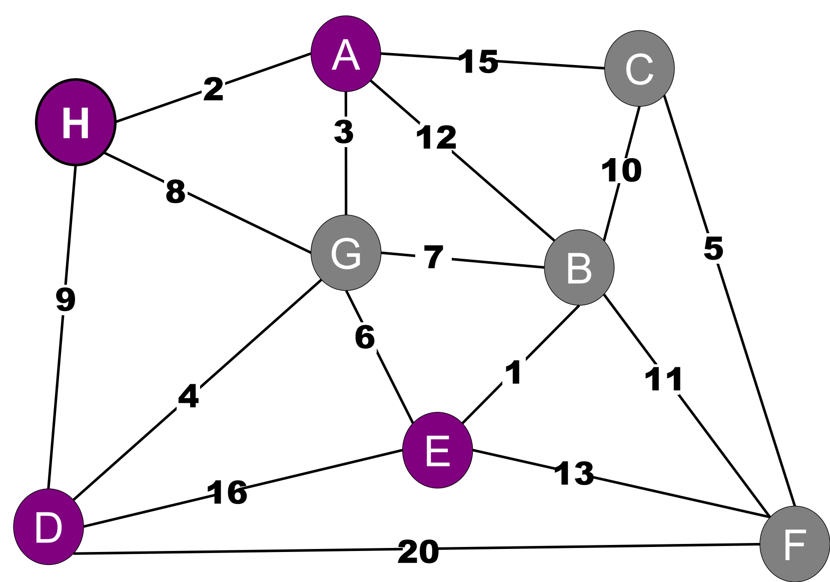
(A,D) = 7

(A,E) = 9

(H,D) = 9

(H,E) = 11

(D,E) = 10



Steiner Tree Problem

Algorithm SteinerMST:

1. For every required vertex (v,w), calculate the shortest path from (v to w).
2. Construct new graph on required nodes.
    - V = required nodes
    - E = shortest path distances

Steiner Tree Problem

Example: Step 2

Shortest Paths:

(A,H) = 2

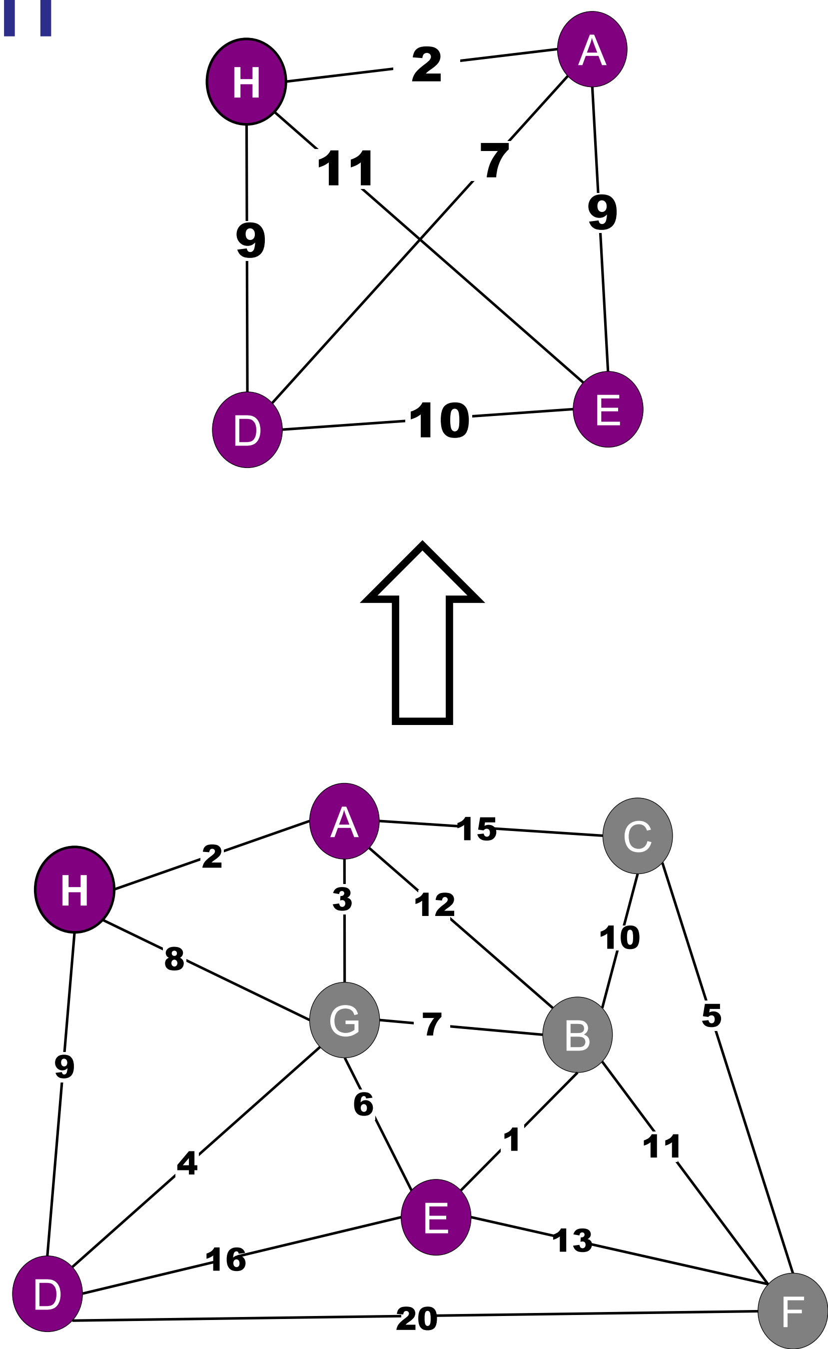
(A,D) = 7

(A,E) = 9

(H,D) = 9

(H,E) = 11

(D,E) = 10



Steiner Tree Problem

Algorithm SteinerMST:

1. For every required vertex (v,w), calculate the shortest path from (v to w).
2. Construct new graph on required nodes.
3. Run MST on new graph.
   - Use Prim's or Kruskal's
   - MST gives edges on new graph

Steiner Tree Problem

Example: Step 3

Shortest Paths:

(A,H) = 2

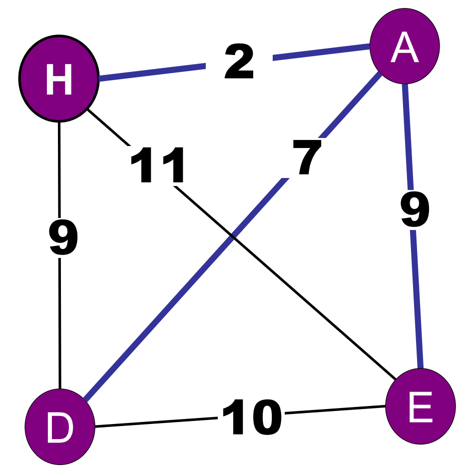
(A,D) = 7

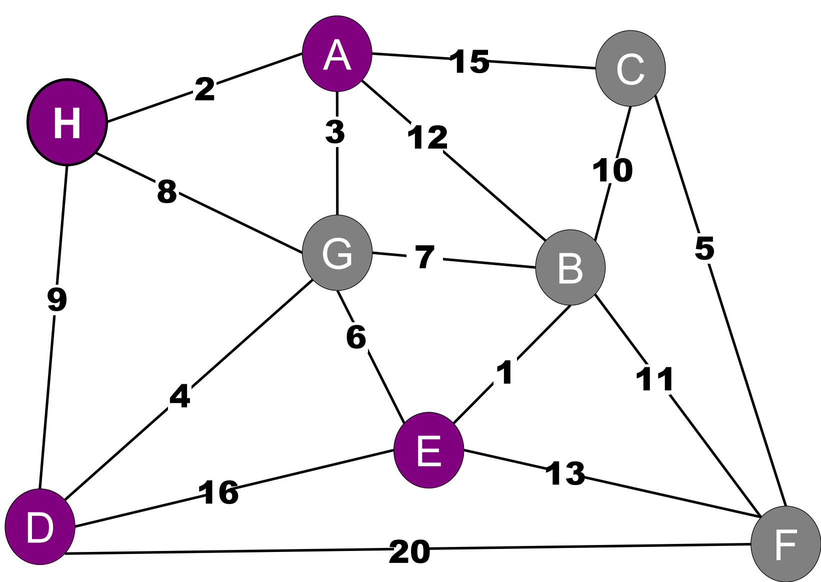
(A,E) = 9

(H,D) = 9

(H,E) = 11

(D,E) = 10





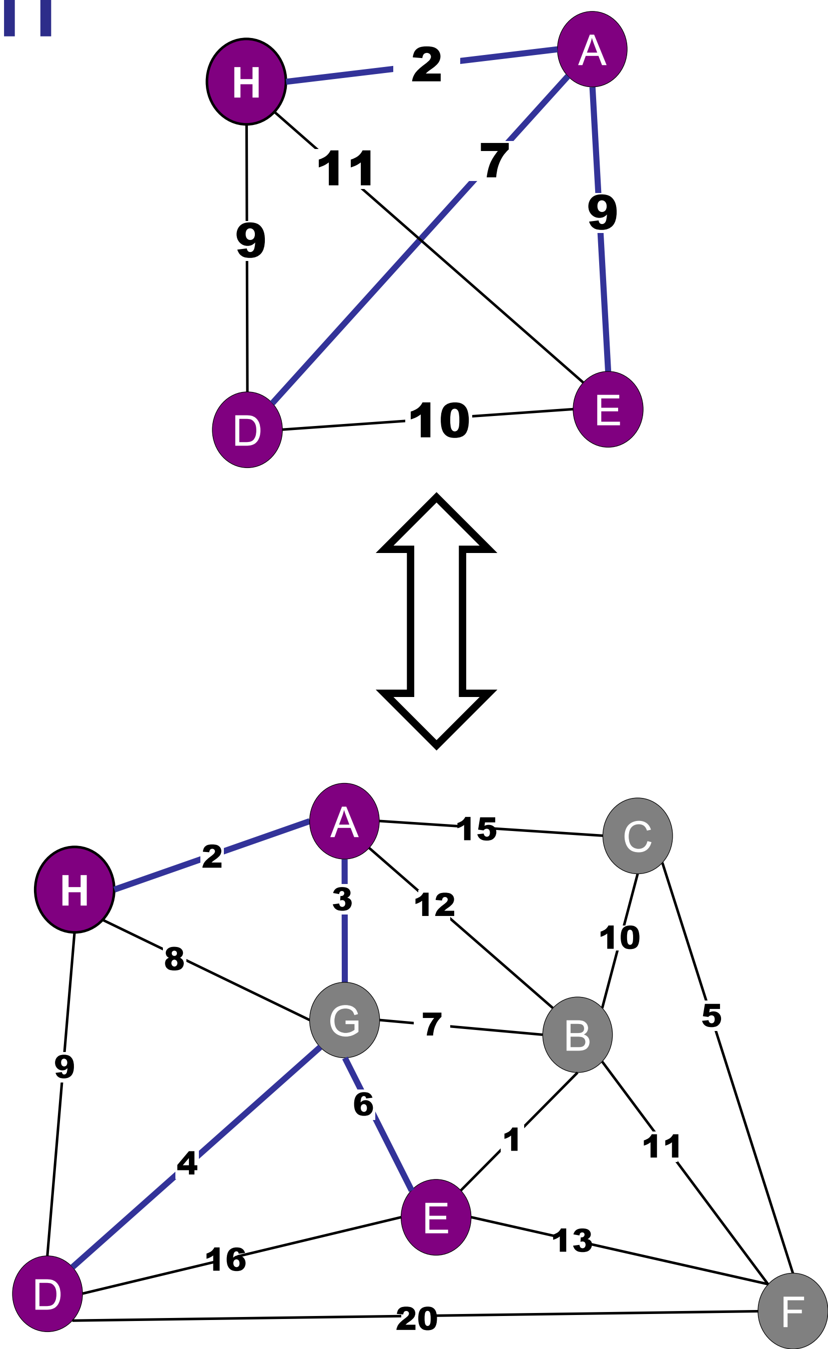
Steiner Tree Problem

Algorithm SteinerMST:

1. For every required vertex (v,w), calculate the shortest path from (v to w).
2. Construct new graph on required nodes.
3. Run MST on new graph.
4. Map new edges back to original graph.
   - Use shortest path discovered in Step 1.
   - Add these edges to Steiner MST.
   - Remove duplicates.

Steiner Tree Problem

Example: Step 4



Shortest Paths:

(A,H) = 2

(A,D) = 7

(A,E) = 9

(H,D) = 9

(H,E) = 11

(D,E) = 10

Steiner Tree Problem

Algorithm SteinerMST:

1. For every required vertex (v,w), calculate the shortest path from (v to w).
2. Construct new graph on required nodes.
3. Run MST on new graph.
4. Map new edges back to original graph.

Note: Does NOT guarantee optimal Steiner tree.

Steiner Tree Problem

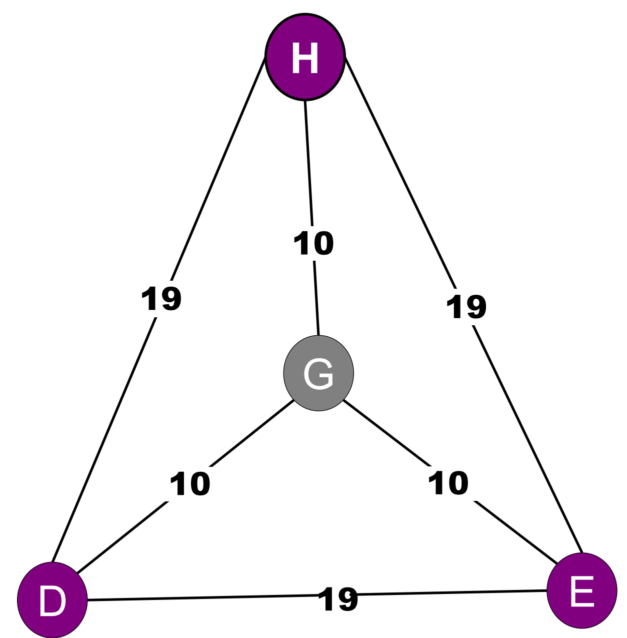
Example:

Shortest Paths:

(D,H) = 19

(D,E) = 19

(E,H) = 19



Steiner Tree Problem

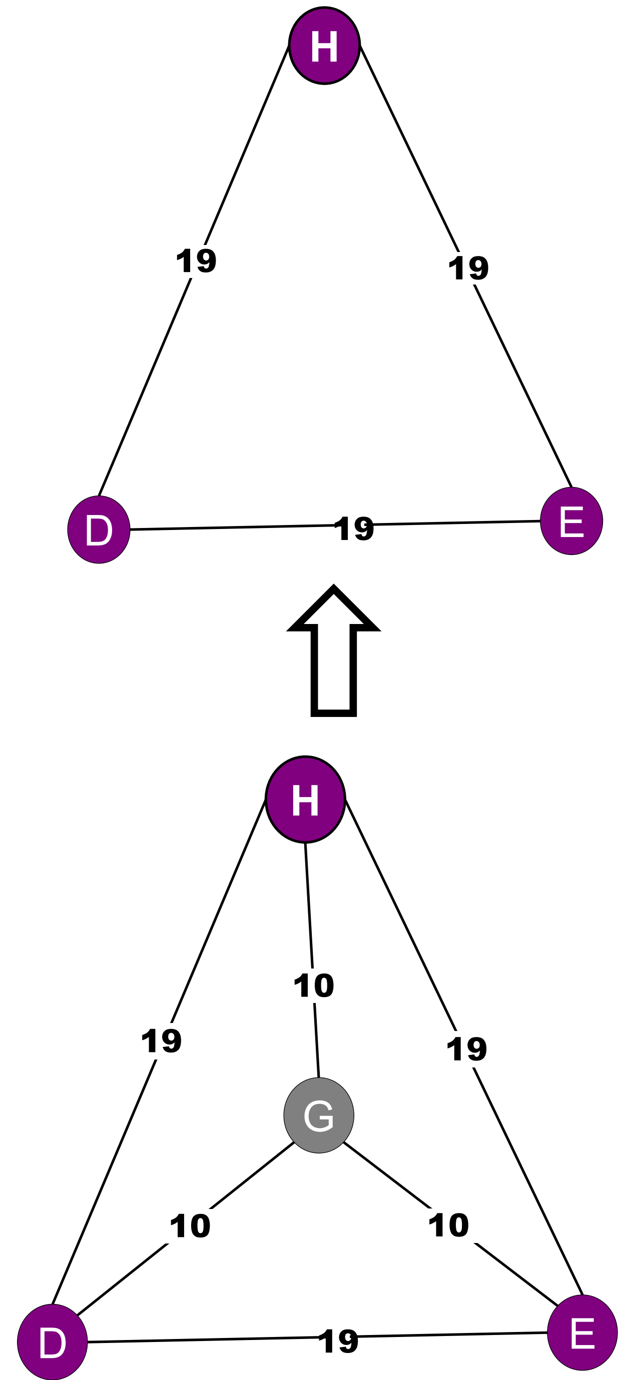
Example:

Shortest Paths:

(D,H) = 19

(D,E) = 19

(E,H) = 19



Steiner Tree Problem

Example:

Shortest Paths:

(D,H) = 19

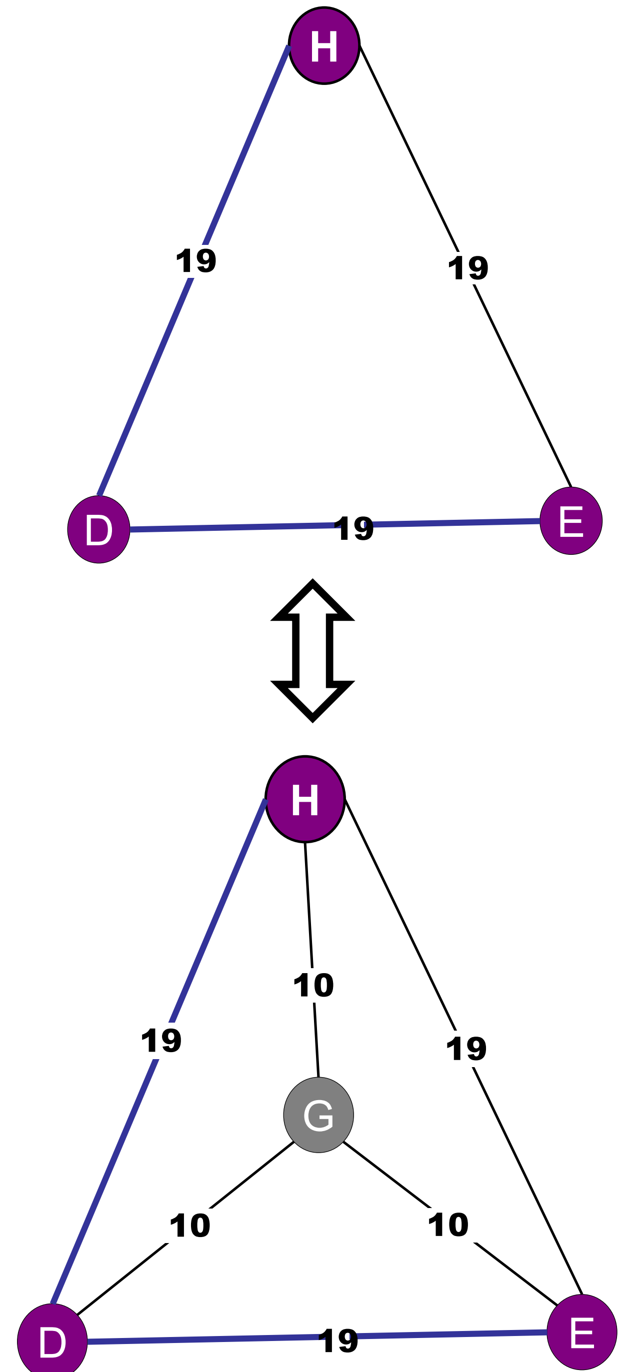
(D,E) = 19

(E,H) = 19

Cost = 38:

OPT Steiner = 30

Challenge: bigger gap!



Steiner Tree Problem

Algorithm SteinerMST:

1. For every required vertex (v,w), calculate the shortest path from (v to w).
2. Construct new graph on required nodes.
3. Run MST on new graph.
4. Map new edges back to original graph.

Note: Does NOT guarantee optimal Steiner tree.

Best known approximation: 1.55