Trigonometric Identities

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $\sin 0 = \cos\frac{\pi}{2} = \tan 0 = 0$

$1 + \tan^2\theta = \sec^2\theta$ $\quad$ $\sin\frac{\pi}{2} = \cos 0 = \tan\frac{\pi}{4} = 1$

$1 + \cot^2\theta = \csc^2\theta$ $\quad$ $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\sin 2\theta = 2\sin\theta\cos\theta$ $\quad$ $\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ $\quad$ $\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\sin(A \pm B) = \sin A\cos B \pm \sin B\cos A$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

$\sin P + \sin Q = 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$

$\sin P - \sin Q = 2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$

$\cos P + \cos Q = 2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$

$\cos P - \cos Q = -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$

$\frac{a}{\sin a} = \frac{b}{\sin b}$ $\quad$ $a^2 = b^2 + c^2 - 2bc\cos\theta$

Functions and Limits

Continuous: $\lim_{x\to a} f(x) = f(a)$ continuous at point a.

Continuous at every point → f is continuous.

Operations on Functions

$(f \pm g)(x) = f(x) \pm g(x)$ $\quad$ $(fg)(x) = f(x)g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $\quad$ $(f \circ g)(x) = f(g(x))$

Rules of Limits For $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = L'$:

$\lim_{x\to a}(f \pm g)(x) = L \pm L'$ $\quad$ $\lim_{x\to a}(fg)(x) = LL'$

$\lim_{x\to a}\frac{f}{g}(x) = \frac{L}{L'}, L' \neq 0$ $\quad$ $\lim_{x\to a} kf(x) = kL$

Differentiation

Derivative of f at point a (provided limit exists):

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \left.\frac{dy}{dx}\right|_{x=a}$$

Rules of Differentiation

Linearity: $(kf)'(x) = kf'(x)$

$(f \pm g)'(x) = f'(x) \pm g'(x)$

Product: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

Chain: $(f \circ g)'(x) = f'(g(x))g'(x) \equiv (f' \circ g)(x)g'(x)$

Quotient: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

Parametric Differentiation: $x = v(t),\ y = u(t)$

$$\frac{dy}{dx} = \frac{u'(t)}{v'(t)}$$

Implicit Differentiation:

Differentiate both sides w.r.t. x, solve for $\frac{dy}{dx}$.

Second Order Derivative/Higher Order Derivative:

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ $\quad$ $f^{(n)} = \frac{d^ny}{dx^n}$

Trigonometric, Exponential, Logarithmic and Inverse Trigonometric Derivatives

$\frac{d}{dx}(\sin x) = \cos x$ $\quad$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad$ $\frac{d}{dx}(\sec x) = \sec x\tan x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$ $\quad$ $\frac{d}{dx}(\csc x) = -\csc x\cot x$

$\frac{d}{dx}e^x = e^x$ $\quad$ $\frac{d}{dx}\ln x = \frac{1}{x}$

$\frac{d}{dx}a^x = a^x\ln a$ $\quad$ $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$

$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ $\quad$ $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ $\quad$ $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$ $\quad$ $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$

Maxima and Minima

$f(c)$ is **local maximum** if $f(c) \geq f(x)$ for x near c.

$f(c)$ is **local minimum** if $f(c) \leq f(x)$ for x near c.

$f(c)$ is **absolute maximum** if $f(c) \geq f(x)\ \forall x \in D$.

$f(c)$ is **absolute minimum** if $f(c) \leq f(x)\ \forall x \in D$.

Critical Point:

Interior point of domain where $f' = 0$ or does not exist.

Finding Extreme Values:

Extreme values of f at **critical points** and end-points of D.

Finding Local Extremes:

(First Derivative Test)

For **critical point** of function f, $c \in (a,b)$:

If $f'(x) > 0$ for $x \in (a,c)$ & $f'(x) < 0$ for $x \in (c,b)$, then $f(c)$ is a **local maximum**.

If $f'(x) < 0$ for $x \in (a,c)$ & $f'(x) > 0$ for $x \in (c,b)$, then $f(c)$ is a **local minimum**.

(Second Derivative Test)

If $f'(c) = 0$ & $f''(c) < 0$, then $f(c)$ is a **local maximum**.

If $f'(c) = 0$ & $f''(c) > 0$, then $f(c)$ is a **local minimum**.

Finding Absolute Extremes:

Evaluate $f(c)$ for all interior **critical points** & **end points** of domain. Largest and smallest of these values will be **absolute maximum** and **absolute minimum** respectively.

Increasing and Decreasing Functions:

For any two points x_1 and x_2 ($x_2 > x_1$) in interval I:

If $f(x_2) > f(x_1)$ or $\forall x$ on I, $f'(x) > 0$, f is **increasing** on I.

If $f(x_2) < f(x_1)$ or $\forall x$ on I, $f'(x) < 0$, f is **decreasing** on I.

Concavity:

$y = f(x)$ concaves down on any interval where $y'' < 0$.

$y = f(x)$ concaves up on any interval where $y'' > 0$.

Points of Inflection:

c is point of inflection if f is continuous at c and **concavity** of f changes at c.

L'Hôpital's rule

f and g are continuous (differentiable) at $x = a$ (f' and g' exists), $f(a) = g(a) = 0$ and $g'(x) \neq 0$ except at $x = a$.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Use for $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms.

Convert $0 \cdot \infty, \infty - \infty$ using algebra manipulation.

Convert $1^\infty, \infty^0, 0^0$ using ln.

Integration

$$\int f(x)\,dx = F(x) + C$$

Gradient of all curves $y = F(x) + C$ at x is $f(x)$

Rules of Integration

$\int kf(x)\,dx = k\int f(x)\,dx$

$\int -f(x)\,dx = -\int f(x)\,dx$

$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Trigonometric, Exponential, Logarithmic and Inverse Trigonometric Integrals

$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$

$\int \sin kx\,dx = -\frac{\cos kx}{k} + C$

$\int \cos kx\,dx = \frac{\sin kx}{k} + C$

$\int \tan x\,dx = \ln|\sec x| + C$

$\int \sec x\,dx = \ln|\sec x + \tan x| + C$

$\int \csc x\,dx = \ln|\csc x - \cot x| + C$

$\int \cot x\,dx = \ln|\sin x| + C$

$\int \sec x\tan x\,dx = \sec x + C$

$\int \csc x\cot x\,dx = \csc x + C$

$\int \tan^2 x\,dx = \tan x - x + C$

$\int \sec^2 x\,dx = \tan x + C$

$\int \csc^2 x\,dx = -\cot x + C$

$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{a^x}{\ln a}$

$\int \ln x\,dx = x\ln x - x + C$

$\int \frac{1}{x}\,dx = \ln x + C$

Riemann (Definite) Integrals

Area under a curve f on interval $[a,b]$:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\,\triangle x$$

Rules of Definite Integrals

$\int_a^a f(x)\,dx = 0$ $\quad$ $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$f(x) \geq g(x)$ on $[a,b] \to \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

$f(x) \geq 0$ on $[a,b] \to \int_a^b f(x)\,dx \geq 0$

Continuous f on interval joining a, b and c, then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Fundamental Theorem of Calculus

If f is continuous on $[a,b]$, then

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point on $[a,b]$, and

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

If f is continuous on $[a,b]$ & F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Examples:

$$\frac{d}{dx}\int_0^{x^3}\cos t\,dt = \frac{d}{dx^3}\int_0^{x^3}\cos t\,dt \cdot \frac{dx^3}{dx} = 3x^2 \cdot \cos x^3$$

$$\frac{d}{dx}\int_{x^2}^{x^3} f(t)\,dt = \frac{d}{dx^3}\int_a^{x^3} f(t)\,dt \cdot \frac{dx^3}{dx} - \frac{d}{dx^2}\int_a^{x^2} f(t)\,dt \cdot \frac{dx^2}{dx}$$

Integration by Substitution

Let $u = g(x)$ and find $du = g'(x)\,dx$

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

For $a^2 - u^2$, sub $u = a\sin\theta$, use $1 - \sin^2\theta = \cos^2\theta$

For $a^2 + u^2$, sub $u = a\tan\theta$, use $1 + \tan^2\theta = \sec^2\theta$

For $u^2 - a^2$, sub $u = a\sec\theta$, use $\sec^2\theta - 1 = \tan^2\theta$

Integration by Parts

$$\int uv'\,dx = uv - \int u'v\,dx$$

u: Logarithmic→Inverse Trigo→Algebraic→Trigo→Exponential

Area between Curves

$A = \int_a^b (f_2(x) - f_1(x))\,dx$ $\quad$ $A = \int_c^d (g_2(y) - g_1(y))\,dy$

Volume of Solid

$V = \int_a^b \pi[f(x)]^2\,dx$ $\quad$ $V = \int_c^d \pi[g(y)]^2\,dy$

Series

Geometric Series (ar^{n-1})

$$S_n = \frac{a(1-r^n)}{1-r} \qquad S_\infty = \frac{a}{1-r}, |r| < 1$$

Rules of Series

$$\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n \qquad \sum k a_n = k \sum a_n$$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$$

Converges if $\rho < 1$, Diverges if $\rho > 1$, No Conclusion if $\rho = 1$

Convergence of Power Series

$$\sum_{n=0}^{\infty} c_n(x-a)^n \qquad \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|\left|\frac{(x-a)^{n+1}}{(x-a)^n}\right| \qquad C = \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|$$

1. $C = 0 < 1$, Converges $\forall x$, Radius of Convergence, $r = \infty$.
2. $C = \infty > 1$, Diverges $\forall x$ except at $x = a, r = 0$.
3. $C \neq 0$ or $C \neq \infty$, Converges for $C|x-a| < 1, r = \frac{1}{C}$

Differentiation and Integration of Power Series

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad a-h < x < a+h \qquad r = h$$

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \qquad r = h$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2} \qquad r = h$$

$$\int \sum_{n=0}^{\infty} c_n(x-a)^n dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \qquad r = h$$

Taylor Series

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n +$$

Maclaurin Series (Taylor Series at $x = 0$)

Range: $-1 < x < 1$, Radius $r = 1$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n \qquad \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \qquad \frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} \qquad (1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots$$

Range: $-\infty < x < \infty$, Radius $r = \infty$:

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \qquad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Range: $-1 \le x < 1$, Radius $r = 1$:

$$\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

Taylor Polynomials

n^{th} order Taylor Polynomial of f at $x = a$:

$$\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Best polynomial approximation of degree n.

Vectors

Dot Product

$$\mathbf{v_1} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad \begin{array}{l} \mathbf{v_1}\cdot\mathbf{v_2} = x_1x_2 + y_1y_2 + z_1z_2 \\ \mathbf{v_1}\cdot\mathbf{v_2} = \|\mathbf{v_1}\|\|\mathbf{v_2}\|\cos\theta,\ 0 \le \theta \le \pi \end{array}$$

Properties of Dot Product

$$\mathbf{v_1}\cdot\mathbf{v_1} = \|\mathbf{v_1}\|^2 \ge 0 \qquad \mathbf{v_1}\cdot\mathbf{v_2} = \mathbf{v_2}\cdot\mathbf{v_1}$$

$$(\mathbf{v_1}+\mathbf{v_2})\cdot\mathbf{v_3} = \mathbf{v_1}\cdot\mathbf{v_3} + \mathbf{v_2}\cdot\mathbf{v_3}$$

$$(c\mathbf{v_1})\cdot\mathbf{v_2} = \mathbf{v_1}\cdot(c\mathbf{v_2}) = c(\mathbf{v_1}\cdot\mathbf{v_2})$$

Unit Vector

$$\frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

Partial Differentiation

First Order Partial Derivatives

Differentiating $f(x,y)$ w.r.t x at (a,b) (fix y as constant):

$$f_x(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}$$

Higher Order Partial Derivatives

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} \qquad f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

Chain Rule

For $w = f(x(s,t), y(s,t), z(s,t))$:

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Directional Derivative

Directional derivative of f at (x,y,z) in direction of unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$:

$$D_u f(x,y,z) = f_x(x,y,z)\cdot u_1 + f_y(x,y,z)\cdot u_2 + f_z(x,y,z)\cdot u_3$$

Measures change in value of f, df, when we move a small distance, dt, from point (x,y,z) in the direction of vector $\mathbf{u}$.

$$df = D_u f(x,y,z)\cdot dt$$

Gradient Vector

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$$

$$\nabla f(x,y,z)\cdot\mathbf{u} = D_u f(x,y,z) = \|\nabla f(x,y,z)\|\cos\theta$$

f **increases** most rapidly in direction $\nabla f(x,y,z)$.

f **decreases** most rapidly in direction $-\nabla f(x,y,z)$.

$D_u f(x,y,z) > 0$ and **largest** when $\theta = 0, \cos\theta = 1$.

$D_u f(x,y,z) < 0$ and **smallest** when $\theta = \pi, \cos\theta = -1$.

Maximum and Minimum Values

$f(x,y)$ has **local maximum** at (a,b) if $f(x,y) \le f(a,b)$ for all points (x,y) near (a,b). $f(a,b)$ is the **local maximum value**.

$f(x,y)$ has **local minimum** at (a,b) if $f(x,y) \ge f(a,b)$ for all points (x,y) near (a,b). $f(a,b)$ is the **local minimum value**.

f may have **local maximum** or **minimum** at (a,b) if $f_x(a,b)$ or $f_y(a,b)$ does not exist or if $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Second Derivative Test (Type of Critical Point)

Let $f_x(x,y) = 0, f_y(a,b) = 0$ to find $x = a$ and $y = b$.

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

$D > 0$ and $f_{xx}(a,b) > 0$, f has **local minimum** at (a,b).

$D > 0$ and $f_{xx}(a,b) < 0$, f has **local maximum** at (a,b).

$D < 0$, f has **saddle point** at (a,b).

$D = 0$, no conclusion.

Ordinary Differential Equation

First Order Linear ODE

1. $\dfrac{dy}{dx} = \dfrac{M(x)}{N(y)} \to \int M(x)\,dx = \int N(y)\,dy$
2. $y' = f\left(\dfrac{y}{x}\right) \to v = \dfrac{y}{x},\ y' = v + xv' \to v + xv' = f(v)$
   $\to \frac{1}{f(v)-v}dv = \frac{1}{x}dx \to$ Solve for $v \to$ Solve for y
3. $y' = f(ax+by+c)$, f is continuous and $b \neq 0$
   $\to$ Sub $u = ax + by + c$
4. $y' + P(x)y = Q(x) \to R = e^{\int P\,dx} \to y = \dfrac{1}{R}\int RQ\,dx$
5. $y' + Py = Qy^n \to z = y^{1-n} \to z' + (1-n)Pz = Q(1-n)$

Radioactive decay: $\frac{dx}{dt} = kx, x = x_0e^{kt}, k = -\frac{\ln 2}{t_{\frac{1}{2}}}$

Uranium-Thorium: $\frac{T}{U} = \frac{k_U}{k_T - k_U}\left(1 - e^{-(k_T-k_U)t}\right), k_N = \frac{\ln 2}{t_{\frac{1}{2}}}$

Temperature: $\int\frac{dT}{T-T_E} = \int k\,dt, T(t) - T_E = (T_0 - T_E)e^{kt}$

Descend (Air Resistance): $m\frac{dv}{dt} = mg - bv^2$

Terminal Velocity $k = \sqrt{\frac{mg}{b}}, A = \frac{v_0-k}{v_0+k}, v = k\frac{1+Ae^{-Bt}}{1-Ae^{-Bt}}, B = \frac{2kb}{m}$

Second Order Linear ODE

$y'' + p(x)y' + q(x)y = F(x)$, homogeneous if $F(x) = 0$

General Solution of Homogeneous 2nd Order ODE

$y'' + Ay' + By = 0$, A, B are constants

Linearly independent if $y_1 \neq ky_2$ for all constants k. General Solution: $y = c_1y_1 + c_2y_2$, y_1, y_2 are **linearly independent**.

$y = e^{\lambda x}$ is a solution if λ is a solution of $\lambda^2 + A\lambda + B = 0$:

1. 2 Distinct Real Roots ($e^{\lambda_1 x}, e^{\lambda_2 x}$): $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
2. 1 Real Root (Double Root λ):
   $y = c_1e^{\lambda x} + c_2xe^{\lambda x}$ ($e^{\lambda x}, xe^{\lambda x}$ are linearly independent)
3. 2 Complex Roots ($\alpha \pm \beta i$): $y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$

Mathematical Modelling

Malthus Model (B = Birth Rate, D = Death Rate)

$$N(t) = N_0e^{kt},\ k = B - D \text{ (Constant)}$$

1. $k > 0$ ($B > D$): Explosion ($t \to \infty, e^{kt} \to \infty, N(t) \to \infty$)
2. $k = 0$ ($B = D$): Stable ($\forall t,\ N(t) = N_0$)
3. $k < 0$ ($B < D$): Extinction ($t \to \infty, e^{kt} \to 0, N(t) \to 0$)

Logistic Model (B constant, $D = SN$, S constant)

$$\frac{dN}{dt} = BN - DN = BN - SN^2$$

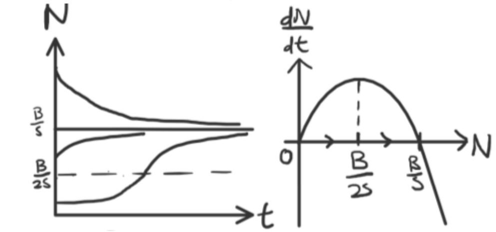
Equilibrium Solutions: $\frac{dN}{dt} = 0,\ N = 0,\ N = \frac{B}{S}$ (Carrying Cap.)

$$N(t) = \frac{N_\infty}{1 + \left(\frac{N_\infty}{N_0} - 1\right)e^{-Bt}} = \frac{B}{S + \left(\frac{B}{N_0} - S\right)e^{-Bt}}$$

Case 1: $B - SN(t) > 0$ (Population < Stable): Increases

Case 2: $B - SN(t) < 0$ (Population > Stable): Decreases

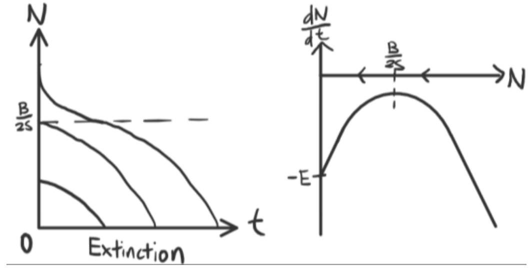
Case 3: $B - SN(t) = 0$ (Population = Stable): Constant



Harvesting Model

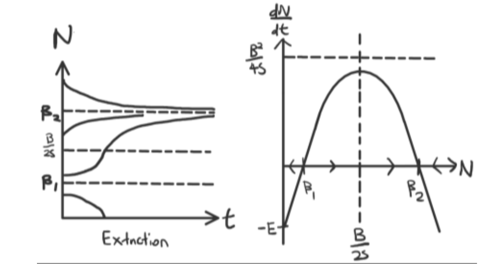
$\frac{dN}{dt} = BN - SN^2 - E$, E = Population Removed Per Time Unit

1. $E > \frac{B^2}{4S}$: No Equilibrium (decreases to extinction)



2. $0 < E < \frac{B^2}{4S}$: 2 Equilibrium Solutions (3 possible cases)

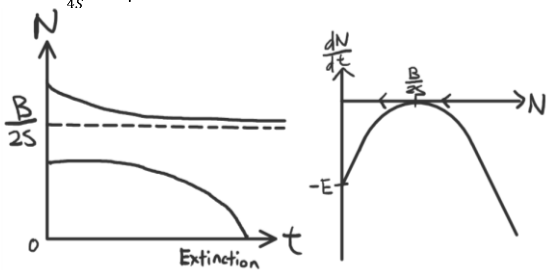
$$\beta = \frac{B^2 \pm \sqrt{B^2 - 4SE}}{2S}$$



β_2 is stable, $N(0) > \beta_1$, population tends to β_2.

β_1 is not stable, $N(0) < \beta_1$, population tends to 0.

3. $E = \frac{B^2}{4S}$: 1 Equilibrium Solution



Population Follows Equation

Given $\frac{dN}{dt} = f(N)$, find all values of N such that $f(N) = 0$ (equilibrium points), evaluate sign of $f(N_0)$ to determine how N changes. N will increase ($f(N_0)$ = Positive)/decrease ($f(N_0)$ = Negative) until the next equilibrium point.

Partial Differential Equations

1. Let solution of PDE be $u(x,y) = X(x)\cdot Y(y)$
2. Sub u into PDE $(u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{yx})$
3. Get 2 separable ODE (LHS = RHS = constant):
   $f(x) = g(y) = k$
4. $\int f_1(X)\,dX = \int kf_2(x)\,dx$ and $\int g_1(Y)\,dY = \int kg_2(y)\,dy$
5. Solve for $X(x), Y(y)$
6. $u(x,y) = X(x)\cdot Y(y)$
7. Sub x_1, y_1 into $u(x,y) = X(x)\cdot Y(y)$ to find constants.
8. Solve $u(x_2, y_2)$.