$\mathbb{MA}$ 1521

Tutorial 6 Solutions

1. Let $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Then $\mathbf{u} = \mathrm{proj}_{\mathbf{w}}\mathbf{a}$ is parallel to $\mathbf{w}$

and $\mathbf{v} = \mathbf{a} - \mathrm{proj}_{\mathbf{w}}\mathbf{a}$ is perpendicular to $\mathbf{w}$

and $\mathbf{a} = \mathbf{u} + \mathbf{v}$.

We compute

$$\mathbf{u} = \frac{\mathbf{a}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{3+6+4}{1+9+16}(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = \frac{1}{2}(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

and

$$\mathbf{v} = \mathbf{a} - \mathbf{u} = (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) - \frac{1}{2}(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = \frac{1}{2}(5\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

2. At the point of intersection we have

$$\begin{aligned} 2t+2 &= 4s-12 \Longrightarrow 2s-t=7\,; \\ t+2 &= 2s-5 \Longrightarrow 2s-t=7\,; \\ 3t+3 &= s-3 \Longrightarrow s-3t=6\,. \end{aligned}$$

It follows that $t = -1$ and $s = 3$, and the point of intersection is $(0,\,1,\,0)$.

A normal to the plane is given by $(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = -5\mathbf{i} + 10\mathbf{j} + 0\mathbf{k}$.

An equation of the plane takes the form

$$\begin{aligned} &-5x + 10y = -5\cdot 0 + 10\cdot 1 + 0\cdot 0 \\ \Longrightarrow &-5x + 10y = 10 \\ \Longrightarrow &-x + 2y = 2. \end{aligned}$$

3. (a) $\overrightarrow{AB} = (3\mathbf{i} + 0\mathbf{j} + \mathbf{k}) - (3\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}) = 0\mathbf{i} - 3\mathbf{j} + \mathbf{k}$,

and $\overrightarrow{AC} = (0\mathbf{i} + 2\mathbf{j} + \mathbf{k}) - (3\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}) = -3\mathbf{i} - \mathbf{j} + 1\mathbf{k}$.

A vector normal to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = -2\mathbf{i} - 3\mathbf{j} - 9\mathbf{k}$.

An equation of the plane is given by

$$\begin{aligned} & -2x - 3y - 9z &= -2\cdot 0 - 3\cdot 2 - 9\cdot 1 \\ \Longrightarrow\quad & 2x + 3y + 9z &= 15. \end{aligned}$$

(b) The distance is given by

$$\frac{|2\cdot 0 + 3\cdot 0 + 9\cdot 0 - 15|}{\sqrt{(2)^2 + (3)^2 + (9)^2}} = \frac{15}{\sqrt{94}}\,.$$

(c) $\overrightarrow{OD} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Parametric equations of the line which contains the line segment OD are given by

$$(*) \qquad x = 4t\,, \quad y = 2t\,, \quad z = t\,.$$

Hence at the point of intersection (of the line and the plane) we have

$$2(4t) + 3(2t) + 9t = 15 \Longrightarrow t = \frac{15}{23}\,.$$

By (*), the coordinates of the point of intersection are $\frac{15}{23}(4,\ 2,\ 1)\,.$

4. **Remark.** Note that two non-parallel planes will intersect (in a straight line), so that the shortest distance between them is 0. In this question, Π_1 and Π_2 are parallel because the vector $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ (or the vector $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$) is perpendicular to them.

   Choose a point P on Π_1 and find the distance from P to Π_2.

   In $\Pi_1 :\ \ 2x + 2y - z = 1$, let $x = 1,\ \ y = 0$ to obtain $z = 1$. Thus, $P(1, 0, 1)$ lies on Π_1. The distance from the point (x_0, y_0, z_0) to the plane $ax + by + cz = d$ is

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

   Thus,

$$\begin{aligned} \text{distance between } \Pi_1 \text{ and } \Pi_2 &= \text{distance from } P \text{ to } \Pi_2 \\ &= \frac{|(4)(1) + (4)(0) + (-2)(1) - 5|}{\sqrt{4^2 + 4^2 + (-2)^2}} \\ &= \frac{1}{2}. \end{aligned}$$

5. For particles to collide, we equate the two vector functions using the *same* parameter t:

$$\mathbf{r}_1(t) = \mathbf{r}_2(t).$$

   Equating the three components, we get

$$t = 1 + 2t, \quad t^2 = 1 + 6t, \quad t^3 = 1 + 14t.$$

   This system does not have solutions. For example, from the first equation, we get $t = -1$. But $t = -1$ does not satisfy the other two equations.

   So, we conclude the 2 particles do not collide.

   For the path to intersect, we equate the two vector functions using *different* parameters s and t:

$$\mathbf{r}_1(t) = \mathbf{r}_2(s).$$

Equating the components, we get

$$t = 1 + 2s, \quad t^2 = 1 + 6s, \quad t^3 = 1 + 14s.$$

This system has a solution $t = 2, s = 1/2$.

So, we conclude that the 2 paths intersect.

6. $\lim\limits_{x\to 0} \frac{||\mathbf{A}+x\mathbf{B}||-||\mathbf{A}||}{x} = \lim\limits_{x\to 0} \frac{(||\mathbf{A}+x\mathbf{B}||-||\mathbf{A}||)(||\mathbf{A}+x\mathbf{B}||+||\mathbf{A}||)}{x(||\mathbf{A}+x\mathbf{B}||+||\mathbf{A}||)}$
$= \lim\limits_{x\to 0} \frac{||\mathbf{A}+x\mathbf{B}||^2-||\mathbf{A}||^2}{x(||\mathbf{A}+x\mathbf{B}||+||\mathbf{A}||)} = \lim\limits_{x\to 0} \frac{A\cdot A+2xA\cdot B+x^2 B\cdot B-A\cdot A}{x(||\mathbf{A}+x\mathbf{B}||+||\mathbf{A}||)}$
$= \lim\limits_{x\to 0} \frac{2A\cdot B+xB\cdot B}{(||\mathbf{A}+x\mathbf{B}||+||\mathbf{A}||)} = \frac{2A\cdot B}{\mathbf{2}||\mathbf{A}||} = ||\mathbf{B}|| \, |\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$