Section 6.1: Functions

CS1231S Discrete Structures

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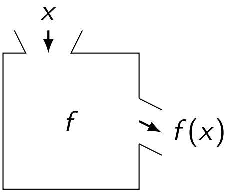


Which part of the following may prevent it from defining a function in the mathematical sense?

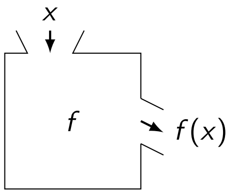
```
int encrypt(int x) {
  return (x^3 + 23*x - key);
}
```

Answer at `https://pollev.com/wtl`.

Why functions

- The language of functions is an important part of modern mathematical discourse.
- Function-like objects are interesting mathematical objects.
- For this module, they provide a topic on which we practise writing and understanding proofs.

$$\boxed{\frac{\text{sets}}{\text{functions}} \approx \frac{\text{what it is}}{\text{what it does}}.}$$



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Functions

steps taken from A to B

- must always return same output with same input

Let A, B be sets.

Definition 6.1.1

A *function* or a *map* from A to B is an assignment to each element of A exactly one element of B. We write $f : A \to B$ for

sets

"f is a function from A to B".

Suppose $f : A \to B$.

indicates output

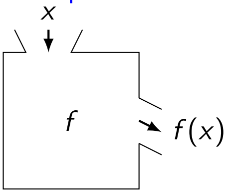
(1) Let $x \in A$. Then $\boxed{f(x)}$ denotes the element of B that f assigns x to. If $y = f(x)$, then we say that *f maps x to y*, and we may write $f : x \mapsto y$.

elements

(2) A is called the *domain* of f, and B is called the *codomain* of f.

(3) The *range* or the *image* of f is

$$\{f(x) : x \in A\}$$
$$= \{y \in B : y = f(x) \text{ for some } x \in A\}.$$



forces output to be int

codomain

domain

$x \in \mathbb{Z}$

```
int encrypt(int x) {
  return (x^3 + 23*x - key);
}
```

range $= \{m^3 + 23m : m \text{ is int}\}$.

not all values can fall in the return

- all possible values are range

- #(range) < #(codomain)

Defining a function using a general expression

Let A, B be sets.

Notation 6.1.2

Often one can specify a function $f : A \to B$ by an expression t involving a special symbol x, sometimes called a *variable*. By this, we mean:

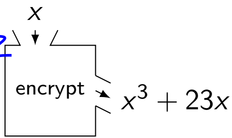
> f assigns to each $x_0 \in A$
> the value of t when every occurrence of x in it is replaced by x_0.

In this case, we may write

$$\begin{aligned} f &: A \to B; \\ x &\mapsto t. \end{aligned}$$

$23 \in \text{codomain} = \mathbb{Z}$

$24 \notin \text{range}$



Example 6.1.3

Consider

$$\begin{aligned} \text{encrypt} &: \mathbb{Z} \to \mathbb{Z}; \\ x &\mapsto x^3 + 23x. \end{aligned}$$

Then encrypt is the function with domain $\mathbb{Z}$ and codomain $\mathbb{Z}$ that assigns to each $x_0 \in \mathbb{Z}$ the value of ${x_0}^3 + 23x_0$. Thus $\text{encrypt}(0) = 0^3 + 23 \times 0 = 0$ and $\text{encrypt}(1) = 1^3 + 23 \times 1 = 24$. The range of encrypt is

$$\{x^3 + 23x : x \in \mathbb{Z}\}.$$

Identity functions

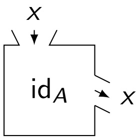
Let A, B be sets.

Notation 6.1.2

Often one can specify a function $f\colon A \to B$ by an expression t involving a special symbol x, sometimes called a *variable*. By this, we mean:

f assigns to each $x_0 \in A$
the value of t when every occurrence of x in it is replaced by x_0.

In this case, we may write
$$\begin{aligned} f\colon A &\to B;\\ x &\mapsto t. \end{aligned}$$



Definition 6.1.4

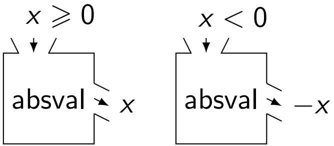
The *identity function* on A is the function
$$\begin{aligned} \mathrm{id}_A\colon A &\to A\\ x &\mapsto x. \end{aligned}$$

case where range =/= proper subset of codomain

Remark 6.1.5

The domain, the codomain, and the range of id_A are all A.

Absolute value



Definition 6.1.6

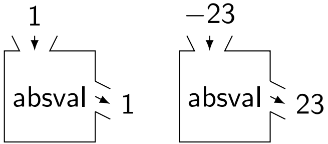
Let absval : $\mathbb{Q} \to \mathbb{Q}$ satisfying, for every $x \in \mathbb{Q}$,

$$\text{absval}(x) = \begin{cases} x, & \text{if } x \geqslant 0; \\ -x, & \text{otherwise.} \end{cases}$$

In other words, the function absval has domain $\mathbb{Q}$, codomain $\mathbb{Q}$, and

$$\forall x \in \mathbb{Q} \ \big((x \geqslant 0 \to \text{absval}(x) = x) \land (\sim(x \geqslant 0) \to \text{absval}(x) = -x) \big).$$

Usually one writes $|x|$ for absval(x).



Example 6.1.7

(1) $|1| = 1$ because $1 \geqslant 0$.

(2) $|-23| = -(-23) = 23$ because $\sim(-23 \geqslant 0)$.

Exercise 6.1.8

Show that the range of absval is $\mathbb{Q}_{\geqslant 0}$. must prove both way

Floor and ceiling

Definition 6.1.9

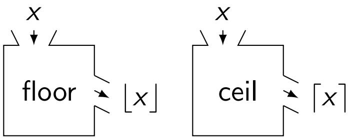
Define $\text{floor}, \text{ceil} \colon \mathbb{Q} \to \mathbb{Z}$ by setting, for each $x \in \mathbb{Q}$,

(1) floor(x) to be the largest integer y such that $y \leqslant x$; and

(2) ceil(x) to be the smallest integer y such that $y \geqslant x$.

Usually one writes $\lfloor x \rfloor$ and $\lceil x \rceil$ for floor(x) and ceil(x) respectively.

Example 6.1.10



(1) $\lfloor 15.11 \rfloor = 15$ and $\lceil 15.11 \rceil = 16$. +ve case

(2) $\lfloor -5.2 \rfloor = -6$ and $\lceil -5.2 \rceil = -5$. -ve case

(3) $\lfloor 23 \rfloor = 23$ and $\lceil 23 \rceil = 23$.

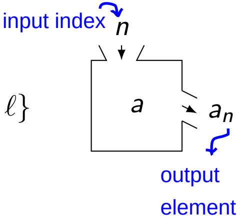
(4) As $\lfloor x \rfloor = x = \lceil x \rceil$ for all $x \in \mathbb{Z}$,

$$\{\lfloor x \rfloor : x \in \mathbb{Q}\} = \mathbb{Z} = \{\lceil x \rceil : x \in \mathbb{Q}\}.$$

Sequences

Definition 6.1.12

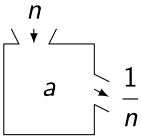
A *sequence* is a function a whose domain is $\mathbb{Z}$, $\mathbb{Z}^+$ or $\{x \in \mathbb{Z} : k \leqslant x \leqslant \ell\}$ for some $k, \ell \in \mathbb{Z}$. If a be a sequence, then we may write a_n for $a(n)$.



Example 6.1.13 (harmonic sequence)

Let a be the sequence $\mathbb{Z}^+ \to \mathbb{Q}$ defined by setting $a_n = 1/n$ for every $n \in \mathbb{Z}^+$. Then $a_1, a_2, a_3, \ldots$ are respectively

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$



Example 6.1.14

Let b be the sequence $\{x \in \mathbb{Z} : 2 \leqslant x \leqslant 5\} \to \mathbb{Z}^+$ defined by setting $b_n = 3n$ for every $n \in \{x \in \mathbb{Z} : 2 \leqslant x \leqslant 5\}$. Then b_2, b_3, b_4, b_5 are respectively

$$6, 9, 12, 15.$$

Well definition

A *function* from A to B is an assignment to each element of A exactly one element of B.

Question 6.1.15

Why does the following sentence *not* define a function?

$$\text{Define } f : \mathbb{Q} \to \mathbb{Q} \text{ by setting } f(x) = 2^x \text{ for all } x \in \mathbb{Q}.$$

Answer

It does not make f assign any value in the codomain $\mathbb{Q}$ to the element $1/2$ of the domain $\mathbb{Q}$. (Recall $2^{1/2} = \sqrt{2} \notin \mathbb{Q}$.)

Define $g : \mathbb{Q} \to \mathbb{Q}$ by setting

$$g(x) = \frac{x^2 + 1}{x^2 + 2x + 1}$$

for all $x \in \mathbb{Q}$.

$x = -1$

$g(-1) = ??$

Define $h : \mathbb{Q} \to \mathbb{Z}$ by setting $h(m/n) = m$ for all $m, n \in \mathbb{Z}$ where $n \neq 0$. or x = -1

$h(1/2) = 1 \neq 2 = h(2/4)$, although $1/2 = 2/4$.

$h(\frac{1}{2}) = 1 \quad h(\frac{2}{4}) = 2$

technically same value but different results

Terminology 6.1.18

A function is *well-defined* if its definition ensures that every element of the domain is assigned exactly one element of the codomain.

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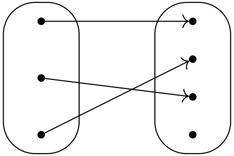
Arrow diagrams

A *function* from A to B is an assignment to each element of A exactly one element of B.



The figure above represents a function in the following sense.

- The dots on the left denote the elements of the domain.
- The dots on the right denote the elements of the codomain.
- An arrow from a left dot to a right dot indicates that the left dot is assigned the right dot.

Since every dot on the left is joined to exactly one dot on the right in the figure above, this function is well-defined.

Summary

We saw

- what functions are,
- an unambiguous language of functions,
- definitions of functions — by a formula, piecewise or otherwise, or
  by a direct description,
- well definition of functions.

Next

- equality of functions
- function composition
- inverse functions
- bijections

> Did exact sequences "exist" before the [1950's]? Yes, of course; but they were expressed in a clumsy complicated way, not easy to handle. It seems to me that one of the crucial new elements was the use of arrows as notation for maps.
>
> Beno Eckmann