MA 1521

Tutorial 2 Solutions

1. **(a)** $y = \dfrac{x+1}{x^2+1}$, $x \in [-3,3]$.

$y' = \dfrac{2-(x+1)^2}{(x^2+1)^2}$ and $y' = 0$ if $x = -1 \pm \sqrt{2}$.

So critical points are $x = -1 \pm \sqrt{2}$ and endpoints are $x = \pm 3$.

$$y' \begin{cases} < 0 & \text{if } -3 \le x < -1-\sqrt{2}, \\ = 0 & \text{if } x = -1-\sqrt{2}, \\ > 0 & \text{if } -1-\sqrt{2} < x < -1+\sqrt{2}, \\ = 0 & \text{if } x = -1+\sqrt{2}, \\ < 0 & \text{if } -1+\sqrt{2} < x \le 3. \end{cases}$$

Hence $y$ is decreasing in $[-3, -1-\sqrt{2})$, increasing in $(-1-\sqrt{2}, -1+\sqrt{2})$, and decreasing in $(-1+\sqrt{2}, 3]$.

So local min is:

$$y(-1-\sqrt{2}) = -\frac{1}{2(\sqrt{2}+1)}, \quad y(3) = \frac{2}{5}$$

and local max is:

$$y(-1+\sqrt{2}) = \frac{1}{2(\sqrt{2}-1)}, \quad y(-3) = -\frac{1}{5}.$$

Since

$$-\frac{1}{2(\sqrt{2}+1)} < -\frac{1}{5} < \frac{2}{5} < \frac{1}{2(\sqrt{2}-1)},$$

so absolute min. is $\displaystyle\min_{x\in[-3,3]} y = -\frac{1}{2(\sqrt{2}+1)}$ at $x = -1-\sqrt{2}$

and absolute max. is $\displaystyle\max_{x\in[-3,3]} y = \frac{1}{2(\sqrt{2}-1)}$ at $x = -1+\sqrt{2}$.

**(b)** $y = (x-1)\sqrt[3]{x^2}$, $x \in (-\infty, \infty)$.

$$y' = x^{2/3} + \frac{2}{3}(x-1)x^{-1/3} = \frac{5x-2}{3x^{1/3}}$$

and $y' = 0$ if $x = \frac{2}{5}$.

Note that $y'$ does not exist at $x = 0$.

So the critical points are $x = 0$ and $x = \frac{2}{5}$.

$$y' \begin{cases} > 0 & \text{if } x < 0, \\ \text{does not exist} & \text{if } x = 0, \\ < 0 & \text{if } 0 < x < \frac{2}{5}, \\ = 0 & \text{if } x = \frac{2}{5}, \\ > 0 & \text{if } x > \frac{2}{5}. \end{cases}$$

Hence $y$ is increasing in $(-\infty, 0)$, decreasing in $(0, \frac{2}{5})$, and increasing in $(\frac{2}{5}, \infty)$.

So local max. is $y(0) = 0$

and local min. is $y(\frac{2}{5}) = -\frac{3}{5}(\frac{2}{5})^{2/3}$.

Since $\lim_{x\to-\infty} y = -\infty$, $\lim_{x\to\infty} y = \infty$, so there is no absolute extremes.

2. Let $x$ be the distance between B and C. Suppose the energy that it takes to fly over land is 1 unit per km, then it will take 1.4 unit per km to fly over water.

   The total energy is given by the function

$$f(x) = 1.4\sqrt{5^2 + x^2} + (13 - x).$$

   Then

$$f'(x) = \frac{1.4x - \sqrt{5^2 + x^2}}{\sqrt{5^2 + x^2}}.$$

   Solving $f'(x) = 0$, we have $x = 5.103$ and the First Derivative Test shows that this point is an absolute minimum.

3. Let $x$ m be the distance from the shadow to the foot of the lamp post. Using similar triangles in the diagram on the last page, we have

$$\frac{s}{9} = \frac{15}{x}.$$

   Therefore,

$$sx = 135$$

   and hence

$$x = \frac{135}{4.9t^2}.$$

   Differentiate this with respect to $t$ and then substitute $t = 0.5$ to solve for $\frac{dx}{dt}$, we find that

$$\frac{dx}{dt} = -440.8,$$

   and so the speed is 440.8 m/sec. (The - sign indicates that the shadow is moving towards the foot of the lamp post.)

4. (a) $\displaystyle \lim_{x\to\pi/2} \frac{1 - \sin x}{1 + \cos 2x} = \lim_{x\to\pi/2} \frac{-\cos x}{-2\sin 2x} = \lim_{x\to\pi/2} \frac{\sin x}{-4\cos 2x} = \frac{1}{4}.$

(b) $$\lim_{x\to 0} \frac{\ln(\cos ax)}{\ln(\cos bx)} = \lim_{x\to 0} \frac{\dfrac{-a\sin ax}{\cos ax}}{\dfrac{-b\sin bx}{\cos bx}} = \lim_{x\to 0} \frac{a\sin ax \ \cos bx}{b\sin bx \ \cos ax} = \frac{a^2}{b^2}.$$

(c) $$\lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{x\to\infty} \frac{\tan(x^{-1})}{x^{-1}} = \lim_{x\to\infty} \frac{-x^{-2}\sec^2(x^{-1})}{-x^{-2}} = \lim_{x\to\infty} \cos^{-2}(x^{-1}) = 1.$$

(d) $$\lim_{x\to 0+} x^a \ln x = \lim_{x\to 0+} \frac{\ln x}{x^{-a}} = \lim_{x\to 0+} \frac{\dfrac{1}{x}}{-ax^{-a-1}} = \lim_{x\to 0+} \frac{x^a}{-a} = 0.$$

(e) $$\lim_{x\to 1} \ln x^{\frac{1}{1-x}} = \lim_{x\to 1} \frac{\ln x}{1-x} = \lim_{x\to 1} \frac{\dfrac{1}{x}}{-1} = -1. \text{ So } \lim_{x\to 1} x^{\frac{1}{1-x}} = e^{-1}.$$

(f) Using (4d) we have

$$\lim_{x\to 0+} \ln x^{\sin x} = \lim_{x\to 0+} \sin x \ln x = \lim_{x\to 0+} \frac{\sin x}{x} \cdot x\ln x = \lim_{x\to 0+} \frac{\sin x}{x} \lim_{x\to 0+} x\ln x = 0.$$

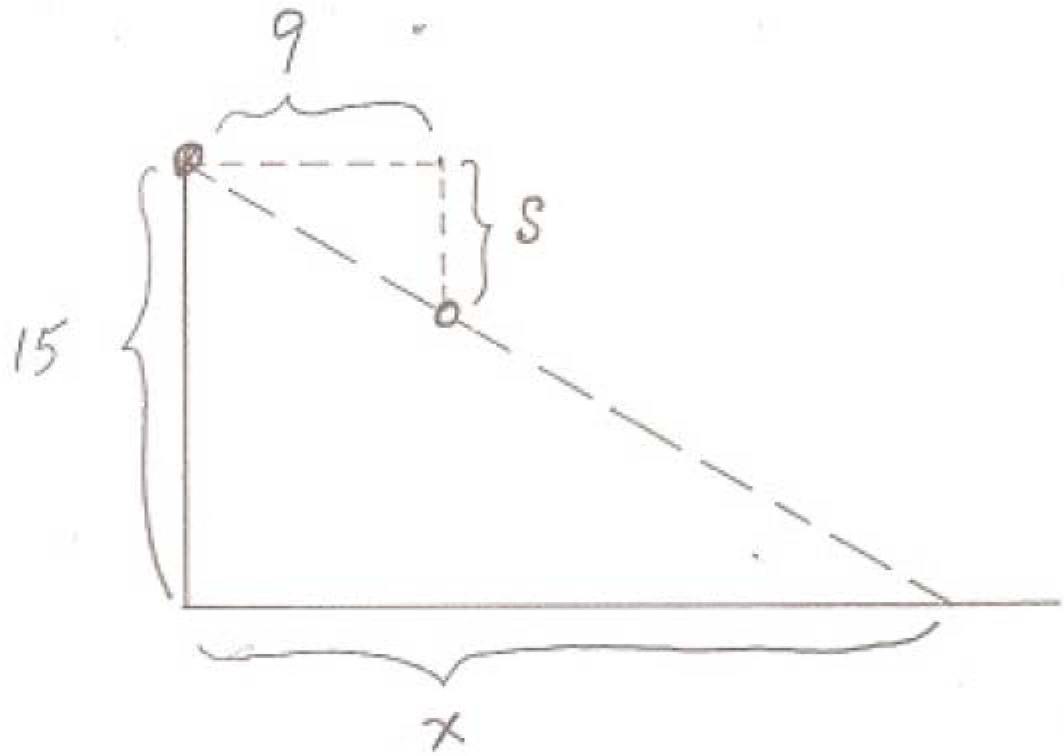
So $\lim\limits_{x\to 0+} x^{\sin x} = e^0 = 1$.

(g) $$\lim_{x\to 0} \ln\left[\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}\right] = \lim_{x\to 0} \frac{\ln(\frac{\sin x}{x})}{x^2} = \lim_{x\to 0} \frac{(\frac{x}{\sin x}) \cdot \frac{x\cos x - \sin x}{x^2}}{2x}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{x}{\sin x} \lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} = \frac{1}{2}\lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{3x^2}$$

$$= -\frac{1}{6}\lim_{x\to 0} \frac{\sin x}{x} = -\frac{1}{6}.$$

So $\lim\limits_{x\to 0} (\frac{\sin x}{x})^{\frac{1}{x^2}} = e^{-1/6}$.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