Student Number: | A | | | | | | | | |

National University of Singapore

MA1101R Linear Algebra I

Semester II (2016 – 2017)

Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. Write down your student number clearly in the space provided at the top of this page. This booklet (and only this booklet) will be collected at the end of the examination.
2. Please write your student number only. Do not write your name.
3. This examination paper contains **SIX (6)** questions and comprises **NINETEEN (19)** printed pages.
4. Answer **ALL** questions.
5. This is a **CLOSED BOOK** (with helpsheet) examination.
6. You are allowed to use one A4-size helpsheet.
7. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	**Marks**
1	
2	
3	
4	
5	
6	
Total	

Question 1 **[20 marks]**

(a) Consider the following linear system

$$\begin{cases} x + ay + 2z = 1 \\ x + 2ay + 3z = 1 \\ x + ay + (a+3)z = 2a^2 - 1. \end{cases}$$

Determine the conditions on the constant a such that the linear system has

(i) exactly one solution; (ii) no solution; (iii) infinitely many solutions.

Show your working below.

$$\left(\begin{array}{ccc|c} 1 & a & 2 & 1 \\ 1 & 2a & 3 & 1 \\ 1 & a & a+3 & 2a^2-1 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & a & 2 & 1 \\ 0 & a & 1 & 0 \\ 0 & 0 & a+1 & 2a^2-2 \end{array}\right).$$

If $a = 0$, then $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right)$. There is no solution.

If $a \neq 0$, then the matrix is in row-echelon form. If $a \neq -1$, the system has a unique solution. If $a = -1$, the system has infinitely many solutions.

Continue on pages 14–19 if you need more writing space.

(b) Find the least squares solution of the following linear system

$$\begin{cases} x + \ \ y + \ \ z = 2 \\ x + 2y - \ \ z = 1 \\ x - \ \ y + \ \ z = 2 \\ \qquad y - 2z = 5. \end{cases}$$

Show your working below.

$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 5 \end{pmatrix}$. Then

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 7 & -4 \\ 1 & -4 & 7 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} = \begin{pmatrix} 5 \\ 7 \\ -7 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 6 \\ 2 & 7 & -4 & 7 \\ 1 & -4 & 7 & -7 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -4 & 7 & -7 \\ 2 & 7 & -4 & 7 \\ 3 & 2 & 1 & 6 \end{array}\right) \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -4 & 7 & -7 \\ 0 & 15 & -18 & 21 \\ 0 & 14 & -20 & 26 \end{array}\right)$$

$$\xrightarrow{R_3 - \frac{14}{15}R_2} \left(\begin{array}{ccc|c} 1 & -4 & 7 & -7 \\ 0 & 15 & -18 & 21 \\ 0 & 0 & -\frac{16}{5} & \frac{32}{5} \end{array}\right).$$

Then $z = -2$, $15y = 21 + 18z = -15 \Rightarrow y = -1$ and $x = -7 + 4y - 7z = 3$.

Continue on pages 14–19 if you need more writing space.

Question 2 **[15 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 9 & 2 & 6 & -1 \\ 0 & 0 & 4 & 2 & 5 & -1 \\ 3 & -6 & 9 & -3 & 0 & 3 \\ 1 & -2 & 5 & 0 & 1 & 0 \end{pmatrix}$. It is given that there exists an invertible matrix $\boldsymbol{B}$ such that

$$\boldsymbol{BA} = \begin{pmatrix} 1 & -2 & 3 & -1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Write down a basis for the row space of $\boldsymbol{A}$, and extend it to a basis for $\mathbb{R}^6$.

(ii) Let $\boldsymbol{v}_i$ be the ith column of $\boldsymbol{A}$, $i = 1, \ldots, 6$.

Find a basis S for the column space of $\boldsymbol{A}$ such that $S \subseteq \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$, and express each $\boldsymbol{v}_i$ not in S as a linear combination of vectors in S.

(iii) Find a basis for the nullspace of $\boldsymbol{A}$.

Show your working below.

Students might not use BA to derive the answers, but reduce A to a row echelon form R, and use the non-zero rows of R. This is acceptable.

(i) $\{(1, -2, 3, -1, 0, 1), (0, 0, 2, 1, 1, -1), (0, 0, 0, 0, 3, 1)\}$.

$\{(1, -2, 3, -1, 0, 1), (0, 0, 2, 1, 1, -1), (0, 0, 0, 0, 3, 1)\} \cup \{\boldsymbol{e}_2, \boldsymbol{e}_4, \boldsymbol{e}_6\}$.

(ii) Since the 1st, 3rd and 5th columns of $\boldsymbol{BA}$ are pivot, $S = \{\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_5\}$ forms a basis of the column space of $\boldsymbol{A}$.

Other possible answers:
{vi, vj, vk}
where
i = 1 or 2
j = 3 or 4
k = 5 or 6

$$\boldsymbol{BA} \xrightarrow[\frac{1}{3}R_3]{\frac{1}{2}R_2} \begin{pmatrix} 1 & -2 & 3 & -1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 - \frac{1}{2}R_3} \begin{pmatrix} 1 & -2 & 3 & -1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 - 3R_2} \begin{pmatrix} 1 & -2 & 0 & -\frac{5}{2} & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{v}_2 = -2\boldsymbol{v}_1, \quad \boldsymbol{v}_4 = -\tfrac{5}{2}\boldsymbol{v}_1 + \tfrac{1}{2}\boldsymbol{v}_3, \quad \boldsymbol{v}_6 = 3\boldsymbol{v}_1 - \tfrac{2}{3}\boldsymbol{v}_3 + \tfrac{1}{3}\boldsymbol{v}_6.$$

Continue on next page if you need more writing space.

More working space for Question 2.

(iii) Let $x_2 = r$, $x_4 = s$ and $x_6 = t$. Then

$$x_1 = 2r + \tfrac{5}{2}s - 3t, \quad x_3 = -\tfrac{1}{2}s + \tfrac{2}{3}t, \quad x_5 = -\tfrac{1}{3}t.$$

Hence,

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(2, 1, 0, 0, 0, 0) + s(\tfrac{5}{2}, 0, -\tfrac{1}{2}, 1, 0, 0) + t(-3, 0, \tfrac{2}{3}, 0, -\tfrac{1}{3}, 1).$$

The nullspace has a basis

$$\{(2, 1, 0, 0, 0)^{\mathrm{T}}, (\tfrac{5}{2}, 0, -\tfrac{1}{2}, 1, 0, 0)^{\mathrm{T}}, (-3, 0, \tfrac{2}{3}, 0, -\tfrac{1}{3}, 1)\}.$$

Continue on page 14–19 if you need more writing space.

Question 3 **[20 marks]**

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a basis for a vector space V, where

$$\boldsymbol{u}_1 = (1,2,1,2), \quad \boldsymbol{u}_2 = (0,2,2,1), \quad \boldsymbol{u}_3 = (1,12,1,0).$$

(i) Use the Gram-Schmidt process to transform the basis S to an orthogonal basis T for V.

(ii) Find the projection of $(-11, 13, -17, 11)$ onto the vector space V.

(iii) Extend T to an orthogonal basis for $\mathbb{R}^4$.

(iv) Find the transition matrix from the basis S to the basis T.

Show your working below.

(i) Let $\boldsymbol{v}_1 = \boldsymbol{u}_1 = (1,2,1,2)$.

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\boldsymbol{v}_1 \cdot \boldsymbol{u}_2}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 = (0,2,2,1) - \frac{8}{10}(1,2,1,2) = \left(-\frac{4}{5}, \frac{2}{5}, \frac{6}{5}, -\frac{3}{5}\right).$$

$$\begin{aligned}\boldsymbol{v}_3 &= \boldsymbol{u}_3 - \frac{\boldsymbol{v}_1 \cdot \boldsymbol{u}_3}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{v}_2 \cdot \boldsymbol{u}_3}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 \\ &= (1,12,1,0) - \frac{26}{10}(1,2,1,2) - \frac{26/5}{13/5}\left(-\frac{4}{5}, \frac{2}{5}, \frac{6}{5}, -\frac{3}{5}\right) = (0,6,-4,-4).\end{aligned}$$

(ii) Let $\boldsymbol{v} = (-11, 13, -17, 11)$. The projection of $\boldsymbol{v}$ onto V is

$$\begin{aligned}\boldsymbol{p} &= \frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 + \frac{\boldsymbol{v}_2 \cdot \boldsymbol{v}}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 + \frac{\boldsymbol{v}_2 \cdot \boldsymbol{v}}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 \\ &= \frac{20}{10}(1,2,1,2) + \frac{-13}{13/5}\left(-\frac{4}{5}, \frac{2}{5}, \frac{6}{5}, -\frac{3}{5}\right) + \frac{102}{68}(0,6,-4,-4) \\ &= (6, 11, -10, 1).\end{aligned}$$

(iii) $\boldsymbol{v} - \boldsymbol{p} = (-11, 13, -17, 11) - (6, 11, -10, 1) = (-17, 2, -7, 10)$.

So $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, (-17, 2, -7, 10)\}$ is an orthogonal basis for $\mathbb{R}^4$.

(iv) By construction, $\boldsymbol{u}_1 = \boldsymbol{v}_1$, $\boldsymbol{u}_2 = \frac{4}{5}\boldsymbol{v}_1 + \boldsymbol{v}_2$ and $\boldsymbol{u}_3 = \frac{13}{5}\boldsymbol{v}_1 + 2\boldsymbol{v}_2 + \boldsymbol{v}_3$.

So the transition matrix from S to T is $\begin{pmatrix} 1 & \frac{4}{5} & \frac{13}{5} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

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More working space for Question 3.

Continue on pages 14–19 if you need more writing space.

Question 4 **[15 marks]**

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\right) = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad T\left(\begin{pmatrix}0\\1\\2\end{pmatrix}\right) = \begin{pmatrix}2\\1\\-1\end{pmatrix}, \quad T\left(\begin{pmatrix}3\\3\\2\end{pmatrix}\right) = \begin{pmatrix}-2\\1\\3\end{pmatrix}.$$

(i) Find the standard matrix for T.

(ii) Find $\text{rank}(T)$ and $\text{nullity}(T)$.

Show your working below.

(i) $\boldsymbol{A}\begin{pmatrix}1&0&3\\2&1&3\\3&2&2\end{pmatrix} = \begin{pmatrix}1&2&-2\\1&1&1\\0&-1&3\end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc}1&0&3&1&0&0\\2&1&3&0&1&0\\3&2&2&0&0&1\end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{ccc|ccc}1&0&3&1&0&0\\0&1&-3&-2&1&0\\0&2&-7&-3&0&1\end{array}\right)$$

$$\xrightarrow{R_3-R_2} \left(\begin{array}{ccc|ccc}1&0&3&1&0&0\\0&1&-3&-2&1&0\\0&0&-1&1&-2&1\end{array}\right) \xrightarrow{-R_3} \left(\begin{array}{ccc|ccc}1&0&3&1&0&0\\0&1&-3&-2&1&0\\0&0&-1&-1&2&-1\end{array}\right)$$

$$\xrightarrow[R_2+3R_3]{R_1-3R_3} \left(\begin{array}{ccc|ccc}1&0&0&4&-6&3\\0&1&0&-5&7&-3\\0&0&1&-1&2&-1\end{array}\right).$$

$$\boldsymbol{A} = \begin{pmatrix}1&2&-2\\1&1&1\\0&-1&3\end{pmatrix}\begin{pmatrix}4&-6&3\\-5&7&-3\\-1&2&-1\end{pmatrix} = \begin{pmatrix}-4&4&-1\\-2&3&-1\\2&-1&0\end{pmatrix}.$$

(ii) $\boldsymbol{A} \xrightarrow[R_3+\frac{1}{2}R_1]{R_2-\frac{1}{2}R_1} \begin{pmatrix}-4&4&-1\\0&1&-\frac{1}{2}\\0&1&-\frac{1}{2}\end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix}-4&4&-1\\0&1&-\frac{1}{2}\\0&0&0\end{pmatrix}$.

So $\text{rank}(T) = 2$ and $\text{nullity}(T) = 3 - 2 = 1$.

Continue on pages 14–19 if you need more writing space.

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $\mathrm{Ker}(T) = \mathrm{Ker}(T \circ T)$. Prove that

$$\mathrm{Ker}(T \circ T) = \mathrm{Ker}(T \circ T \circ T).$$

Show your working below.

If $\boldsymbol{v} \in \mathrm{Ker}(T \circ T)$, then $T \circ T(\boldsymbol{v}) = \mathbf{0}$; consequently $T \circ T \circ T(\boldsymbol{v}) = T(T \circ T(\boldsymbol{v})) = T(\mathbf{0}) = \mathbf{0}$, i.e., $\boldsymbol{v} \in \mathrm{Ker}(T \circ T \circ T)$.

Let $\boldsymbol{v} \in \mathrm{Ker}(T \circ T \circ T)$. Set $\boldsymbol{w} = T(\boldsymbol{v})$. Then

$$T \circ T(\boldsymbol{w}) = T \circ T(T(\boldsymbol{v})) = T \circ T \circ T(\boldsymbol{v}) = \mathbf{0}.$$

So $\boldsymbol{w} \in \mathrm{Ker}(T \circ T) = \mathrm{Ker}(T)$. We thus have

$$T \circ T(\boldsymbol{v}) = T(T(\boldsymbol{v})) = T(\boldsymbol{w}) = \mathbf{0};$$

that is, $\boldsymbol{v} \in \mathrm{Ker}(T \circ T)$.

Continue on pages 14–19 if you need more writing space.

Question 5 **[15 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 \\ a & 1 & b \\ 1 & 0 & 0 \end{pmatrix}$, where a, b are real constants.

(i) Find the eigenvalues of $\boldsymbol{A}$.

(ii) Prove that $\boldsymbol{A}$ is diagonalizable if and only if $a + b = 0$.

(iii) Suppose that $a + b = 0$. Find an invertible matrix $\boldsymbol{P}$ in terms of a such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

Show your working below.

(i) $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & 0 & -1 \\ -a & \lambda - 1 & -b \\ -1 & 0 & \lambda \end{vmatrix} = (\lambda - 1) \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = (\lambda - 1)^2(\lambda + 1)$.

So the eigenvalues of $\boldsymbol{A}$ are -1 and 1.

(ii) Let $\lambda = 1$. Then

$$\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 \\ -a & 0 & -b \\ -1 & 0 & 1 \end{pmatrix} \xrightarrow[R_3 + R_1]{R_2 + aR_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -b - a \\ 0 & 0 & 0 \end{pmatrix}.$$

$\boldsymbol{A}$ is diagonalizable if and only if $\text{nullity}(\boldsymbol{I} - \boldsymbol{A}) = 2 \Leftrightarrow -b - a = 0 \Leftrightarrow a + b = 0$.

Continue on next page if you need more writing space.

More working space for Question 5.

(iii) Suppose $a+b=0$. Then $\boldsymbol{I}-\boldsymbol{A} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The nullspace has a basis

$$\{(1,0,1)^{\mathrm{T}}, (0,1,0)^{\mathrm{T}}\}.$$

$-\boldsymbol{I}-\boldsymbol{A} = \begin{pmatrix} -1 & 0 & -1 \\ -a & -2 & a \\ -1 & 0 & -1 \end{pmatrix} \xrightarrow[R_2 - aR_1]{R_3 - R_1} \begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & 2a \\ 0 & 0 & 0 \end{pmatrix}$. The nullspace has a basis

$$\{(-1,a,1)^{\mathrm{T}}\}.$$

Hence, $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & a \\ 1 & 0 & 1 \end{pmatrix}$.

Continue on pages 14–19 if you need more writing space.

Question 6 **[15 marks]**

(a) Let $\boldsymbol{A}$ be a square matrix of order n. Let $\boldsymbol{M}_{ij}$ be the matrix of order $n-1$ obtained from $\boldsymbol{A}$ by deleting the ith row and the jth column.

Prove that if $\boldsymbol{A}$ is invertible, then at least n of the matrices $\boldsymbol{M}_{ij}$ are invertible. [*Hint*: Consider the adjoint matrix $\mathbf{adj}(\boldsymbol{A})$.]

Show your working below.

Assume that at most $n-1$ of the submatrices $\boldsymbol{M}_{ij}$ are invertible. Then the cofactor $A_{ij} = (-1)^{i+j}\det(\boldsymbol{M}_{ij}) = 0$ for all but at most $n-1$ pairs of (i,j).

So $\mathbf{adj}(\boldsymbol{A}) = (A_{ji})$ would have at most $n-1$ nonzero entries. In particular, $\mathbf{adj}(\boldsymbol{A})$ would have a zero row; so $\mathbf{adj}(\boldsymbol{A})$ would be singular.

On the other hand, if $\boldsymbol{A}$ is invertible, then $\mathbf{adj}(\boldsymbol{A})$ is invertible. This leads to a contradiction. Therefore, at least n submatrices $\boldsymbol{M}_{ij}$ must be invertible.

Continue on pages 14–19 if you need more writing space.

(b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same order.

(i) Prove that the nullspace of $\boldsymbol{B}$ is a subspace of the nullspace of $\boldsymbol{AB}$.

(ii) Using (i) prove that

$$\text{nullity}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{B}) \geq \text{nullity}(\boldsymbol{AB}).$$

Show your working below.

(i) If $\boldsymbol{Bv} = \boldsymbol{0}$, then $\boldsymbol{ABv} = \boldsymbol{0}$. So

$$\text{nullspace of } \boldsymbol{B} \subseteq \text{nullspace of } \boldsymbol{AB}.$$

(ii) Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a basis for the nullspace of $\boldsymbol{B}$. Extend it to a basis for the nullspace of $\boldsymbol{AB}$:

$$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_m\}.$$

Then $\boldsymbol{Bv}_{k+1}, \ldots, \boldsymbol{Bv}_m \neq \boldsymbol{0}$, and they belong to the nullspace of $\boldsymbol{A}$. Suppose

$$c_{k+1}\boldsymbol{Bv}_{k+1} + \cdots + c_m\boldsymbol{Bv}_m = \boldsymbol{0}.$$

Then $\boldsymbol{B}(c_{k+1}\boldsymbol{v}_{k+1} + \cdots + c_m\boldsymbol{v}_m) = \boldsymbol{0}$, which implies

$$c_{k+1}\boldsymbol{v}_{k+1} + \cdots + c_m\boldsymbol{v}_m \in \text{nullspace of } \boldsymbol{B} = \text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}.$$

So $c_{k+1} = \cdots = c_m = 0$. Hence, $\boldsymbol{Bv}_{k+1}, \ldots, \boldsymbol{Bv}_m$ are linearly independent. Then

$$\text{nullity}(\boldsymbol{A}) \geq m - k = \text{nullity}(\boldsymbol{AB}) - \text{nullity}(\boldsymbol{B}).$$

So

$$\text{nullity}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{B}) \geq \text{nullity}(\boldsymbol{AB}).$$

Continue on pages 14–19 if you need more writing space.

More working spaces. Please indicate the question numbers clearly.

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