MA1101R

LIVE LECTURE 6

Topics for week 6

3.4 Linear Independence

3.5 Bases

Q&A: log in to PollEv.com/vtpoll

Let's revise - Span

- A linear span always contains the $\boxed{\text{zero}}$ vector
- If a set of vectors is in span(S), then any $\boxed{\text{linear combination}}$ of the vectors is also in span(S).
- In $\mathbf{R}^2$ and $\mathbf{R}^3$, span$\{\mathbf{u}\}$ represents a $\boxed{\text{line}}$ if $\mathbf{u} \neq \mathbf{0}$.
- In $\mathbf{R}^2$ and $\mathbf{R}^3$, span$\{\mathbf{u}, \mathbf{v}\}$ represents a $\boxed{\text{plane}}$ if $\mathbf{u}$ is not parallel to $\mathbf{v}$.
- A subset V of $\mathbf{R}^n$ is a $\boxed{\text{subspace}}$ of $\mathbf{R}^n$ if V is a linear span of a set S of vectors in $\mathbf{R}^n$.
- $\{\mathbf{0}\}$ and $\mathbf{R}^n$ are $\boxed{\text{subspaces}}$ of $\mathbf{R}^n$

Let’s revise - Subsets vs Subspaces

- All $\boxed{\text{subspaces}}$ of $\mathbf{R}^n$ are $\boxed{\text{subsets}}$ of $\mathbf{R}^n$
- NOT all $\boxed{\text{subsets}}$ of $\mathbf{R}^n$ are $\boxed{\text{subspaces}}$ of $\mathbf{R}^n$
- A $\boxed{\text{subspace}}$ of $\mathbf{R}^n$ is closed under addition and scalar multiplication
- If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a $\boxed{\text{subset}}$ of $\mathbf{R}^n$,

then span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a $\boxed{\text{subspace}}$ of $\mathbf{R}^n$

Linearly independent set: no redundant vectors in the set

Linearly dependent set: redundant vectors in the set

Redundant vectors

$\boldsymbol{v_1} = (2, 1, 3)$, $\boldsymbol{v_2} = (1, -1, 2)$, $\boldsymbol{v_3} = (3, 0, 5)$, $\boldsymbol{v_4} = (1, 2, 4)$

span$\{\boldsymbol{v_1}\}$ represents a line

$\{\boldsymbol{v_1}\}$ is linearly independent

span$\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ represents a plane

$\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is linearly independent

Both vectors are not redundant

span$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ represents the same plane

One of the three vectors is redundant

$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is linearly dependent

span$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ represents the entire $\mathbf{R}^3$

One of the first three vectors is redundant

$\boldsymbol{v_4}$ is not redundant

$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ is linearly dependent

Testing Linear Independence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

- Standard method

  Form the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_n\mathbf{v}_n = \mathbf{0}$

  Homogeneous system

➢ *If $c_1 = 0$, $c_2 = 0$, … $c_n = 0$ is the unique solution, then they are linearly independent*

➢ *If there are non-trivial solutions, then they are linearly dependent.*

- Redundancy

➢ *If some $\mathbf{v}_i$ is a linear combination of the others, then they are linearly dependent*

➢ *If every $\mathbf{v}_i$ is not a linear combination of the others, then they are linearly independent.*

Testing Linear Independence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

Special Methods: (only work under certain circumstances)

- The column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbf{R}^n$ form a square matrix $\mathbf{A}$.
  - ➢ *If det($\mathbf{A}$) = 0, then they are linearly dependent.*
  - ➢ *If det($\mathbf{A}$) ≠ 0, then they are linearly independent.*
- There are only two vectors $\mathbf{v}_1, \mathbf{v}_2$ in the set.
  - ➢ *If $\mathbf{v}_1, \mathbf{v}_2$ are scalar multiple of each other, then they are linearly dependent.*
  - ➢ *If $\mathbf{v}_1, \mathbf{v}_2$ are not scalar multiple of each other, then they are linearly independent.*
- Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbf{R}^m$.
  - ➢ *If $n > m$, then they are linearly dependent.*
  - ➢ *If $n \leq m$, it can be linearly independent or dependent.*

The more vectors you have, the more likely for them to be linearly dependent

Linear independence (Example)

{ (1, 1, 1), (-1, -1, -1) }   scalar multiple → lin dep

{ (1, 1, 1), (1, 1, -1) }   not scalar multiple → lin indep

{ (1, 1, 1), (0, 1, 1), (0, 0, 1) }   only trivial solution → lin indep

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1$$

a(1, 1, 1) + b(0, 1, 1) + c(0, 0, 1) = (0, 0, 0)

{ (1, 1, 1), (0, 1, 0), (1, 0, 1) }   non-trivial solution → lin dep

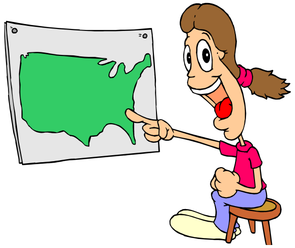
$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

a(1, 1, 1) + b(0, 1, 0) + c(1, 0, 1) = (0, 0, 0)

{ (1, 1, 1), (0, 1, 1), (0, 0, 1), (1, 0, 1) }

more than 3 vectors → lin dep

Map of LA



A **is an n×n matrix**

A is invertible	chapter 2	***A*** is not invertible
det ***A*** $\neq$ 0	chapter 2	det ***A*** = 0
rref of ***A*** is identity matrix	chapter 1	rref of ***A*** has a zero row
Ax = ***0*** has only the trivial solution	chapter 1	***Ax*** = ***0*** has non-trivial solutions
Ax = ***b*** has a unique solution	chapter 1	***Ax*** = ***b*** has no solution or infinitely many solutions
Columns (rows) of ***A*** are linearly independent	chapter 3	Columns (rows) of ***A*** are linearly dependent

to be continued

True or false

True 1. If the set of nonzero vectors $\{\mathbf{v}_1,\mathbf{v}_2\}$ is linearly dependent in $\mathbf{R}^3$, then the set $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ must also be linearly dependent.

False 2. If none of the vectors from the set $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ in $\mathbf{R}^3$ is a multiple of one of the other vectors, then $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is linearly independent.

False 3. If S_1 and S_2 are two linearly independent sets, then $S_1 \cup S_2$ is also a linearly independent set.

Geometrical interpretation

Vectors in R^3 (non-zero)	Directions of vectors	Linear dependency	Linear span
Two vectors $\mathbf{v}_1, \mathbf{v}_2$	$\mathbf{v}_1, \mathbf{v}_2$ parallel	Linearly dependent	span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a line
Two vectors $\mathbf{v}_1, \mathbf{v}_2$	$\mathbf{v}_1, \mathbf{v}_2$ not parallel	Linearly independent	span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane
Three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$	$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ parallel	Linearly dependent	span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a line
Three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$	$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ co-planar	Linearly dependent	span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a plane
Three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$	$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ not coplanar	Linearly independent	span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the 3D space

Linear independence VS Span

- If S is linearly independent, then
  - $\boldsymbol{u} \notin \text{span}(S) \Leftrightarrow S \cup \{\boldsymbol{u}\}$ is linearly independent.
  - $\boldsymbol{u} \in \text{span}(S) \Leftrightarrow S \cup \{\boldsymbol{u}\}$ is linearly dependent
- Let $\{\boldsymbol{u}, \boldsymbol{v}\} \in \mathbf{R}^2$.

  $\{\boldsymbol{u}, \boldsymbol{v}\}$ is linearly independent $\Leftrightarrow \text{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \mathbf{R}^2$
- Let $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \in \mathbf{R}^3$.

  $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly independent $\Leftrightarrow \text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = \mathbf{R}^3$

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}\}$ is a basis for $\mathbf{R}^n$

Linear independence VS Span

Given that: $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}\}$ is a subset of $\mathbf{R}^n$

To Show:
$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}\}$ spans $\mathbf{R}^n$

same as: span(S) = $\mathbf{R}^n$

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{v} \qquad \begin{pmatrix} x \\ y \\ \vdots \\ z \end{pmatrix}$$

$\boldsymbol{v}$ is any general vector in $\mathbf{R}^n$

check whether the system is always consistent

yes → $\boxed{\text{spans}}$

no → $\boxed{\text{does not span}}$

To Show:
$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}\}$ is lin. indep.

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0} \qquad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\boldsymbol{0}$ is the zero vector in $\mathbf{R}^n$

check whether the system has non-trivial solution

yes → $\boxed{\text{lin.dep}}$

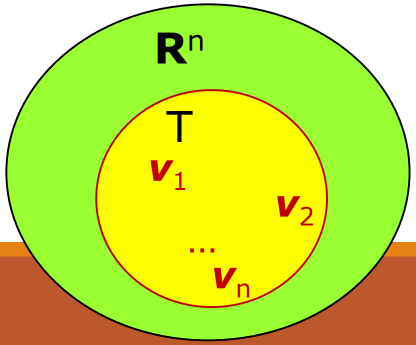
no → $\boxed{\text{lin.indep}}$

T is not a basis for V

Bases for $\mathbf{R}^n$ VS Bases for its subspaces

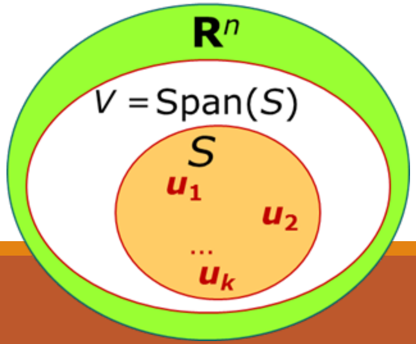
Bases for $\mathbf{R}^n$

- The basis T is the smallest set of "building blocks" for $\mathbf{R}^n$.
- T is linearly independent and span(T) = $\mathbf{R}^n$.
- Every vector in $\mathbf{R}^n$ can be expressed as a linear combination of T in a unique way.



Bases for subspace V

- The basis S is the smallest set of "building blocks" for V.
- S is linearly independent and span(S) = V.
- Every vector in V can be expressed as a linear combination of S in a unique way.

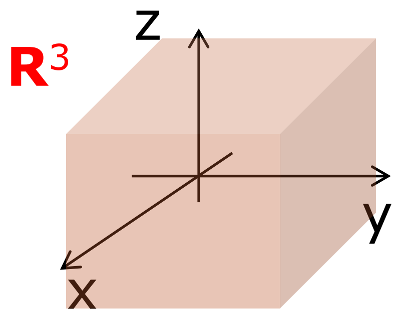


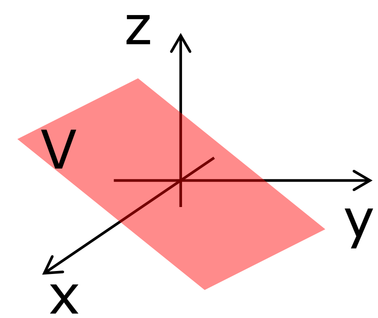
$\mathbf{R}^n$ subspace of $\mathbf{R}^n$

Bases for a vector space

There are many possible bases for a given vector space

A basis for "$\mathbf{R}^n$" is not a basis for "subspace of $\mathbf{R}^n$"





Meaning

1. A basis for V is a set of building blocks of V
2. A basis for V is a "unit of measurement" for vectors in V.
3. A basis for V gives a "coordinate system" for V.

Show basis

To show S is a basis for $\mathbf{R}^n$

- Check S has n vectors
- Check S is linearly independent

To show S is a basis for a subspace V of $\mathbf{R}^n$

- Check S is linearly independent
- Check span(S) = V

To show S is a basis for $\mathbf{R}^n$

- Check S has n vectors
- Check S is linearly independent

Bases for R^3

Which of the sets of vectors is/are bases for $\mathbf{R}^3$?

{ (1, 1, 1), (-1, -1, -1) }

Less than 3 vectors → not basis for $\mathbf{R}^3$

{ (1, 1, 1), (1, 1, -1) }

Less than 3 vectors → not basis for $\mathbf{R}^3$

{ (1, 1, 1), (0, 1, 1), (0, 0, 1) }

3 vectors, lin indep → basis for $\mathbf{R}^3$

{ (1, 1, 1), (0, 1, 0), (1, 0, 1) }

3 vectors, lin dep → not basis for $\mathbf{R}^3$

{ (1, 1, 1), (0, 1, 1), (0, 0, 1), (1, 0, 1) }

More than 3 vectors → not basis for $\mathbf{R}^3$

Basis for a subspace of $\mathbf{R}^3$

$V = \{(x, y, z) \mid x - y + 2z = 0\}$ is a subspace of $\mathbf{R}^3$

Which of the sets of vectors is/are bases for V?

$\{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \}$

3 vectors, lin indep → basis for $\mathbf{R}^3$ → not basis for V

$\{ (1, 1, 0) \}$

1 vector, lin indep → basis for a line → not basis for V

$\{ (1, 1, 0), (1, 0, -1) \}$

2 vectors, lin indep → basis for a plane

$(1,0,-1) \notin V$ → not basis for V

$\{ (1, 1, 0), (1, -1, -1) \}$

$(1,1,0), (1,-1,-1) \in V$, lin indep → basis for V

Exercise 3 Q38

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a basis for some vector space V.

Is $T = \{\boldsymbol{u}_1, \boldsymbol{u}_1 + \boldsymbol{u}_2, \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3\}$ also a basis for V?

Every vector in T belongs to span(S). So $\text{span}(T) \subseteq \text{span}(S)$.

$\boldsymbol{u}_1 \in \text{span}(T)$

$\boldsymbol{u}_2 = (\boldsymbol{u}_1 + \boldsymbol{u}_2) - \boldsymbol{u}_1 \in \text{span}(T)$

$\boldsymbol{u}_3 = (\boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3) - (\boldsymbol{u}_1 + \boldsymbol{u}_2) \in \text{span}(T)$

Every vector in S belongs to span(T). So $\text{span}(S) \subseteq \text{span}(T)$.

So $\text{span}(T) = \text{span}(S) = V$

We shall see later:
Since we know the "dimension" of T is 3,
we just need to check (i) span(T) = V, OR (ii) T is linearly indep

Exercise 3 Q38

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a basis for some vector space V.

Is $T = \{\boldsymbol{u}_1, \boldsymbol{u}_1 + \boldsymbol{u}_2, \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3\}$ also a basis for V?

Consider $a\,\boldsymbol{u}_1 + b(\boldsymbol{u}_1 + \boldsymbol{u}_2) + c(\boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3) = \boldsymbol{0}$ (*)

Does (*) have non-trivial scalars for a, b, c ?

Rewrite (*): $(a+b+c)\boldsymbol{u}_1 + (b+c)\boldsymbol{u}_2 + c\boldsymbol{u}_3 = \boldsymbol{0}$ (**)

(**) has only trivial scalars for a+b+c, b+c, c

$$\left.\begin{aligned} a + b + c &= 0 \\ b + c &= 0 \\ c &= 0 \end{aligned}\right\} \text{Solve: } a = b = c = 0$$

So (*) has only trivial scalars for a, b, c

So T is linearly independent.

Basis for Solution Space

homogeneous system

$$\begin{cases} 2v + 2w - x + z = 0 \\ -v - w + 2x - 3y + z = 0 \\ x + y + z = 0 \\ v + w - 2x - z = 0 \end{cases}$$

implicit form

Gaussian Elimination

general solution

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix}$$

explicit form

separate parameters

linear combination of linearly independent vectors

$$s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$\boldsymbol{u_1}$ $\boldsymbol{u_2}$

basis for solution space

$\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$

linear span form

solution space = span$\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$

Find a basis for a subspace

V: Implicit form

derive explicit form for general vector

usually need to solve linear system

V: explicit form

separate the parameters

V: span {$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ }

throw out redundant vectors

use row/column space method (chp 4)

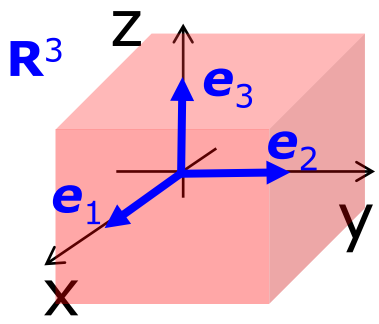
Get a basis for V

- A basis for V is a "unit of measurement" for vectors in V.
- A basis for V gives a "coordinate system" for V.

Coordinate vectors in $\mathbf{R}^n$

standard basis
$S = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$

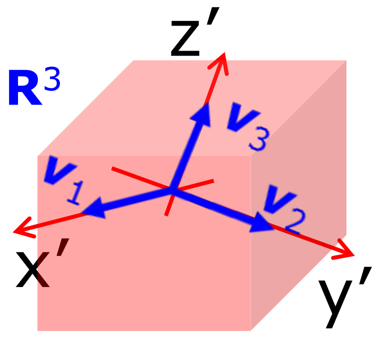
$\boldsymbol{u} = (2, 1, -3)$
$= 2\boldsymbol{e}_1 + \boldsymbol{e}_2 - 3\boldsymbol{e}_3$



$(\boldsymbol{u})_S = (2, 1, -3)$ coordinate vector of $\boldsymbol{u}$ relative to basis S

non-standard basis
$T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$

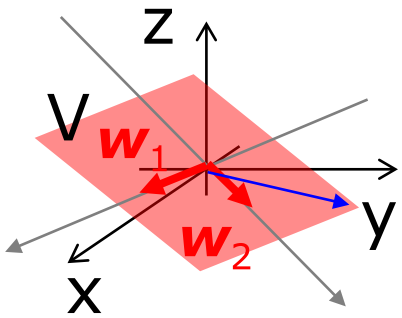
$\boldsymbol{u} = (2, 1, -3)$
$= 3\boldsymbol{v}_1 - \boldsymbol{v}_2 - 4\boldsymbol{v}_3$



$(\boldsymbol{u})_T = (3, -1, -4)$ coordinate vector of $\boldsymbol{u}$ relative to basis T

- A basis for V is a "unit of measurement" for vectors in V.
- A basis for V gives a "coordinate system" for V.

Coordinate vectors in a subspace of $\mathbf{R}^n$



basis for the plane V:

$$W = \{\boldsymbol{w}_1, \boldsymbol{w}_2\}$$

$$\boldsymbol{u} = (2, 1, -3)$$

$$= (-1)\boldsymbol{w}_1 + 2\boldsymbol{w}_2$$

$$(\boldsymbol{u})_W = (-1, 2)$$

coordinate vector of $\boldsymbol{u}$ relative to basis W

Announcement

- Practice Session
  - Practice 2 scores in submission folder
- Online quiz 6
  - Due this Sunday
- Homework 1
  - Published last Friday
  - Deadline: 2 October (week 7)
  - Declaration form
- MATLAB
  - Worksheet 2 this week