AY2020/2021 Semester I
MA1101R Take-home assignment declaration form

Read A, B and C. Sign and submit this declaration form together with your answers.

A. Academic, Professional and Personal Integrity

1. *The University is committed to nurturing an environment conducive for the exchange of ideas, advancement of knowledge and intellectual development. Academic honesty and integrity are essential conditions for the pursuit and acquisition of knowledge, and the University expects each student to maintain and uphold the highest standards of integrity and academic honesty at all times.*
2. *The University takes a strict view of cheating in any form, deceptive fabrication, plagiarism and violation of intellectual property and copyright laws. Any student who is found to have engaged in such misconduct will be subject to disciplinary action by the University.*
3. *It is important to note that all students share the responsibility of protecting the academic standards and reputation of the University. This responsibility can extend beyond each student's own conduct, and can include reporting incidents of suspected academic dishonesty through the appropriate channels. Students who have reasonable grounds to suspect academic dishonesty should raise their concerns directly to the relevant Head of Department, Dean of Faculty, Registrar, Vice Provost or Provost.*

B. I have read and understood the rules of the assessments stated below.

a. *Students should attempt the assessments on their own. (For homework assignments, students may discuss with or consult their friends and instructors, but they have to work on the assignment independently.)*

b. *Students should not reproduce of any assessment materials, e.g. by photography, videography, screenshots, or copying down of questions, etc.*

C. I understand that by breaching any of the rules above, I would have committed offences under clause 3(l) of the NUS Statute 6, Discipline with Respect to Students which is punishable with disciplinary action under clause 10 or clause 11 of the said statute.

3) *Any student who is alleged to have committed or attempted to commit, or caused or attempted to cause any other person to commit any of the following offences, may be subject to disciplinary proceedings:*
   (l) plagiarism, giving or receiving unauthorised assistance in academic work, or other forms of academic dishonesty.

I have read and will abide by the NUS Code of Student Conduct (in particular, (A) Academic, Professional and Personal Integrity), B and C when attempting this assessment.

Signature:______________________________

Full Name: ______________________________

Student Number: __________________________

National University of Singapore

Semester 1, 2020/2021 **MA1101R** **Homework Assignment 1**

(a) Use A4 size paper and pen (blue or black ink) to write your answers. (Students may also type out the answers or write the answers electronically using their devices.)

(b) Write down your student number and full name clearly on the top left of every page of the answer scripts.

(c) Write the page number on the top right corner of each page of answer scripts.

(d) This assignment consists of 5 pages and 10 questions. Each question is 10 marks.

(e) To submit your answer scripts, do the following:

  (i) Scan or take pictures of your work (make sure the images can be read clearly).

  (ii) Merge the declaration form and all your answers into one pdf file. Arrange them in order of the page with the declaration form as the first page.

  (iii) Name the pdf file by **StudentNo HW1** (e.g. **A123456R HW1**).

  (iv) Upload your pdf into the LumiNUS folder Homework 1 submission.

(f) Deadline for submission is 2 October, 2020 by 11.59pm. **Late submission will not be accepted.**

1. The following augmented matrix belongs to some linear system:

$$\left(\begin{array}{ccc|c} a^2-a & 0 & a & a^2 \\ 0 & 1-a^2 & a^2 & -a \\ 0 & a^2-1 & a & a \\ a^2-a & 0 & a & 2a \end{array}\right)$$

where a is some real number.

Determine the values of a for the system to have

(i) no solution;

(ii) a unique solution;

(iii) infinitely many solutions with one parameter;

(iv) infinitely many solutions with two parameters.

2. The following augmented matrix is in <u>row echelon form</u> and belongs to some <u>non-homogeneous</u> linear system:

$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right)$$

Determine whether the following statements are true or false. Explain carefully how you derive your answers.
(Note: The entries a, e, h need not be leading entries of the row echelon form.)

(i) If the third column of the augmented matrix is a pivot column, then the system has a unique solution.

(ii) If the third row of the augmented matrix is a zero row, then the system has infinitely many solutions.

(iii) If e and f are both zero, then the system is inconsistent.

(iv) If the general solution of the linear system has two parameters, then $d \neq 0$.

(v) If $i \neq 0$, then $\begin{vmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & h \end{vmatrix} \neq 0$.

3. A square matrix $\boldsymbol{A}$ is said to be <u>similar</u> to a square matrix $\boldsymbol{B}$ of the same order if there is an invertible matrix $\boldsymbol{P}$ such that

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}.$$

Determine whether the following statements are true or false. Justify your answers.

(i) If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$ and $\boldsymbol{B}$ is similar to $\boldsymbol{C}$, then $\boldsymbol{A}$ is similar to $\boldsymbol{C}$.

(ii) The only matrix that is similar to a scalar matrix is itself.
i.e. If $\boldsymbol{S}$ is a scalar matrix, and $\boldsymbol{T} = \boldsymbol{P}\boldsymbol{S}\boldsymbol{P}^{-1}$, then $\boldsymbol{T} = \boldsymbol{S}$.

(iii) If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then $\boldsymbol{A}^k$ is similar to $\boldsymbol{B}^k$ for any positive integer k.

4. (a) Let $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$ where a, b, c, d are some real numbers.
Find $\det(\boldsymbol{A})$, $\mathrm{adj}(\boldsymbol{A})$ and $\boldsymbol{A}^{-1}$ if $\boldsymbol{A}$ is invertible.

(b) Suppose $\boldsymbol{B} = (b_{ij})$ is an $n \times n$ square matrix with integers entries, i.e. $b_{ij} \in \mathbb{Z}$ are integers, and $\det(\boldsymbol{B}) = 1$. Show that $\boldsymbol{B}^{-1}$ also has integer entries. (Hint: Use theorem 2.5.25.)

5. (a) Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be $n \times n$ $(n \geq 2)$ square matrices such that $\boldsymbol{M} = (m_{ij})$ has entries

$$m_{ij} = \begin{cases} a & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

and $\boldsymbol{N} = (n_{ij})$ has entries $n_{ij} = 1$ for all i, j.

(i) Show that $\boldsymbol{N}^2 = n\boldsymbol{N}$.

(ii) Express $\boldsymbol{M}$ in terms of $\boldsymbol{N}$ and the identity matrix $\boldsymbol{I}$.

(iii) Use (ii) to show that $\boldsymbol{M}$ is invertible when $a \neq 0$ and find $\boldsymbol{M}^{-1}$.
(Hint: Find $\boldsymbol{M}^{-1}$ in terms of $\boldsymbol{N}$ and $\boldsymbol{I}$.)

(b) A square matrix is said to be <u>nilpotent</u> if there exists a positive integer m such that $\boldsymbol{A}^m = \boldsymbol{0}$ (see Exercise 2.13).
Prove that if $\boldsymbol{A}$ is nilpotent and $\boldsymbol{B}$ is an invertible matrix of order n such that $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$, then $(\boldsymbol{B} - \boldsymbol{A})$ is invertible.
(Hint: Consider the expression $\boldsymbol{B}^m - \boldsymbol{A}^m$ and make use of Theorem 2.4.14.)

6. Let $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 3 & 3 & 0 & 6 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 & -1 & 1 & -4 \\ 0 & 0 & 3 & 3 & 12 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

   (i) Perform Gauss Jordan elimination to find the reduced row echelon forms $\boldsymbol{R_A}$ and $\boldsymbol{R_B}$ of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. (Follow the steps in algorithm 1.4.2 and 1.4.3 <u>strictly</u> and indicate your e.r.o.)

   (ii) Find a product of elementary matrices $\boldsymbol{E}_i, \boldsymbol{F}_j$ such that

   $$\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{R_A} \quad \text{and} \quad \boldsymbol{F}_h \cdots \boldsymbol{F}_2 \boldsymbol{F}_1 \boldsymbol{B} = \boldsymbol{R_B}.$$

   (iii) Are $\boldsymbol{A}$ and $\boldsymbol{B}$ row equivalent to each other? Why?

   (iv) Find a matrix $\boldsymbol{C}$ such that $\boldsymbol{CA} = \boldsymbol{B}$.

   (v) If an additional row $\boldsymbol{r} \in \mathbb{R}^5$ is appended to both $\boldsymbol{A}$ and $\boldsymbol{B}$ to form 4×5 matrices $\boldsymbol{A}'$ and $\boldsymbol{B}'$ respectively, i.e. $\boldsymbol{A}' = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{r} \end{pmatrix}$ and $\boldsymbol{B}' = \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{r} \end{pmatrix}$. Find a matrix $\boldsymbol{D}$ such that $\boldsymbol{DA}' = \boldsymbol{B}'$.

7. Let $S_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ with $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \boldsymbol{u}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$;

   $S_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ with $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 4 \end{pmatrix}$;

   $S_3 = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4\}$ with $\boldsymbol{w}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \boldsymbol{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \boldsymbol{w}_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \boldsymbol{w}_4 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}$.

   Answer the following questions, showing some working how the answers are derived. (You may use MATLAB to help with some computations.)

   (i) Is $\text{span}(S_1) = \text{span}(S_2)$?

   (ii) Is $\text{span}(S_2) = \text{span}(S_3)$?

   (iii) Is $\text{span}(S_3) = \mathbb{R}^4$?

   (iv) Which of the vectors in S_3 belongs to $\text{span}(S_1)$?

   (v) What is the intersection of $\text{span}(S_1)$ and $\text{span}(S_3)$?

8. Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$? Justify your answers.

   (i) $A = \{(a, b, 2a - 3b) \mid a, b \in \mathbb{R}\}$;

   (ii) $B = \{(u, v, w) \mid vw = 0\}$;

   (iii) $C = \{(d^2, e^2, f^2) \mid d, e, f \in \mathbb{R}\}$;

   (iv) $D = \{(p, q, r) \mid \begin{pmatrix} p & q & r \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ is not invertible$\}$;

   (v) $E = \{(x, y, z) \mid \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}\}$.

9. Given $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly independent set in $\mathbb{R}^n$, and $\boldsymbol{A}$ is an $n \times n$ matrix. Answer the following questions with justifications.

   (i) Is $\{\boldsymbol{u} - \boldsymbol{v} - \boldsymbol{w}, -\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w}, -\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w}\}$ linearly independent?

   (ii) If $\boldsymbol{x} \in \mathbb{R}^n$ such that $\mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}\} \neq \mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$, is $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}\}$ linearly independent?

   (iii) Suppose $\{\boldsymbol{Au}, \boldsymbol{Av}, \boldsymbol{Aw}\}$ is linearly independent, must $\boldsymbol{A}$ be an invertible matrix?

   (iv) Let $\boldsymbol{B} = (\boldsymbol{u} \ \ \boldsymbol{v} \ \ \boldsymbol{w})$ be the $n \times 3$ matrix with $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ as its three columns. Does any row echelon form of $\boldsymbol{B}$ have a non-pivot column?

10. Consider the following two subspaces of $\mathbb{R}^3$:

$$U = \{\ (x, y, z) \ \big|\ 2x - y + z = 0 \text{ and } x + y - z = 0\ \},$$
$$V = \{\ (x, y, z) \ \big|\ x + 2z = 0 \text{ and } x - 2y + z = 0\ \}.$$

   (i) Write U and V in explicit set notation.

   (ii) Find a basis for each of U and V.

   (iii) Find the smallest subspace that contains both U and V. Express your answer in the following three forms:
   (a) linear span; (b) explicit set notation; (c) implicit set notation.