CS2040 – Data Structures and Algorithms

Lecture 14 – Connecting People

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Outline

Minimum Spanning Tree (MST), CP3 Section 4.3

- Motivating Example & Some Definitions

Two Algorithms to solve MST (you have a choice!)

- Prim's (greedy algorithm with PriorityQueue)
  - PriorityQueue is discussed in Lecture 08
- Kruskal's (greedy algorithm, uses sorting and UFDS)
  - UFDS is discussed in Lecture 09

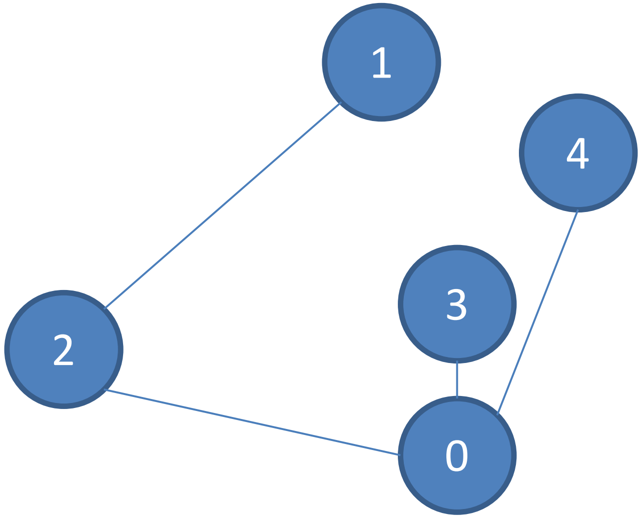
Review

Definitions that we have learned before

- **Tree T**
  - **T** is a **connected graph** that has **V** vertices and **V-1** edges
  - Important: One unique path between any two pair of vertices in **T**
- **Spanning** Tree **ST** of connected graph **G**
  - **ST** is a tree that spans (covers) every vertex in **G**
  - Recall the **BFS and DFS Spanning Tree**

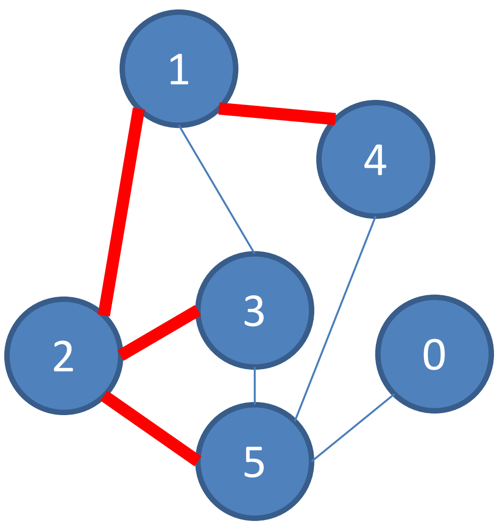
Is This A Tree?

1. Yes, why ___________
2. No, why ___________



Do the edges highlighted in red part form a spanning tree of the original graph?

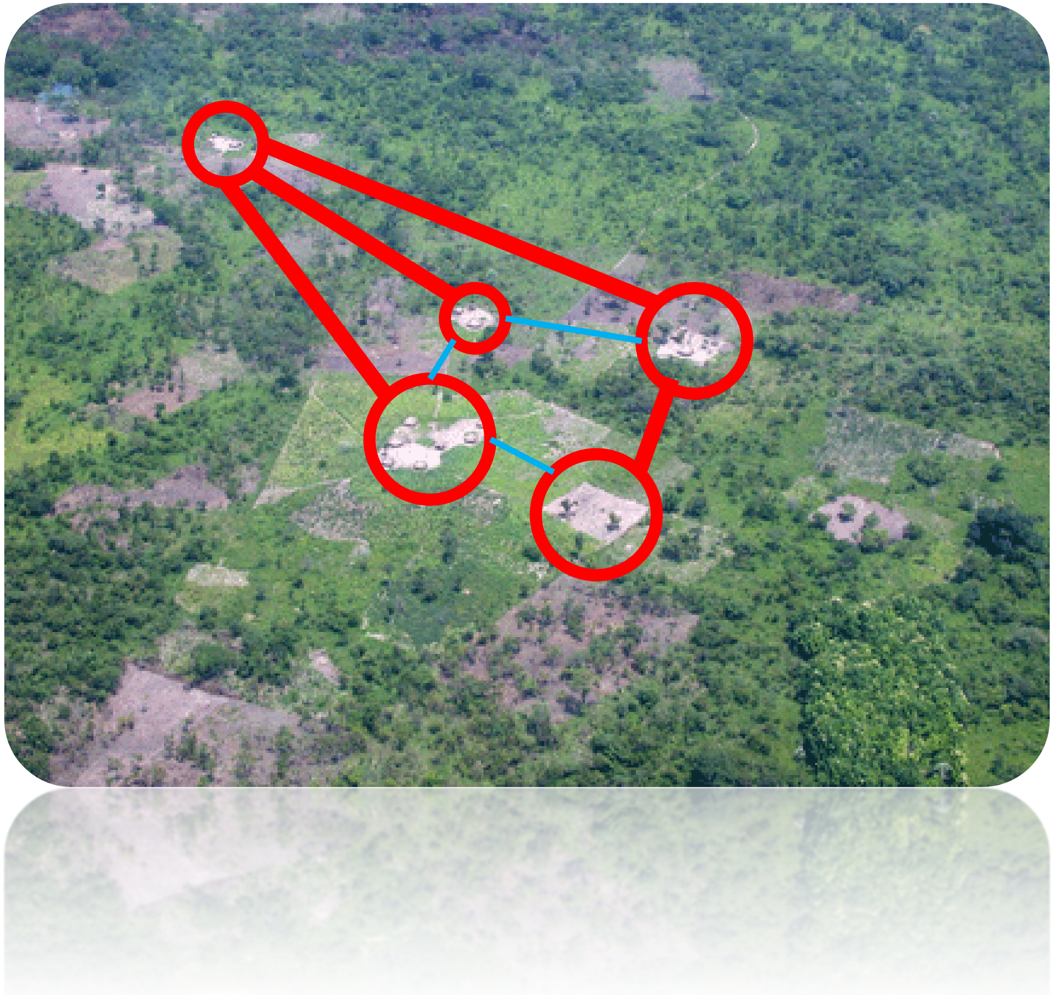
1. Yes, why ___________
2. No, why ___________



Motivating Example

Government Project

- Want to link rural villages with roads
- The cost to build a road depends on the terrain, etc
- You only have limited budget
- How are you going to build the roads?



Definitions (1)

- Vertex set **V** (e.g. street intersections, houses, etc)
- Edge set **E** (e.g. streets, roads, avenues, etc)
  - Generally undirected (e.g. bidirectional road, etc)
  - Weighted (e.g. distance, time, toll, etc)
- Weight function **w(a, b): E→R**
  - Sets the weight of edge from **a** to **b**
- **Weighted** Graph **G**: **G(V, E), w(a, b): E→R**
- **Connected** undirected graph **G**
  - There is a path from any vertex **a** to any other vertex **b** in **G**
- The graph **G** we're concerned with is **connected undirected** and **weighted** when dealing with **MST**

More Definitions (2)

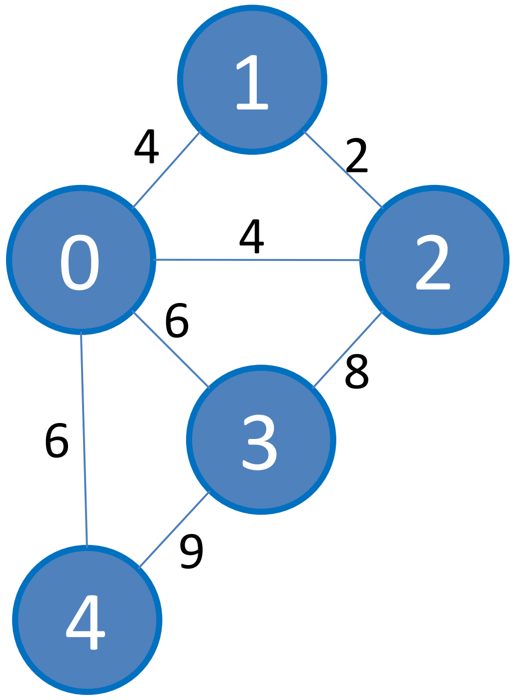
- Spanning Tree **ST** of connected undirected weighted graph **G**
  - Let **w(ST)**, weight of **ST**, denotes the total weight of edges in **ST**

$$w(ST) = \sum_{(a,b)\in ST} w(a,b)$$

- **Minimum Spanning Tree (MST)** of connected undirected weighted graph **G**
  - **MST** of **G** is an **ST** of **G** with the minimum possible **w(ST)**

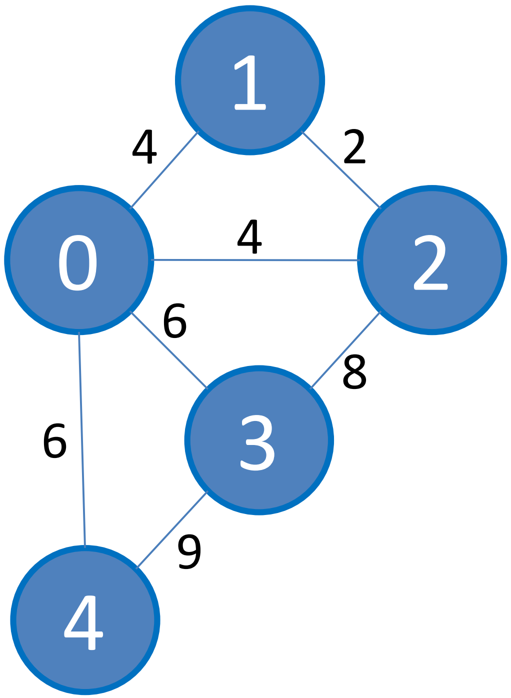
More Definitions (3)

- **The (standard) MST Problem**
  - Input: A connected undirected weighted graph **G(V, E)**
  - Select some edges of **G** such that the graph forms a spanning tree, but with minimum total weight
  - Output: Minimum Spanning Tree **(MST)** of **G**

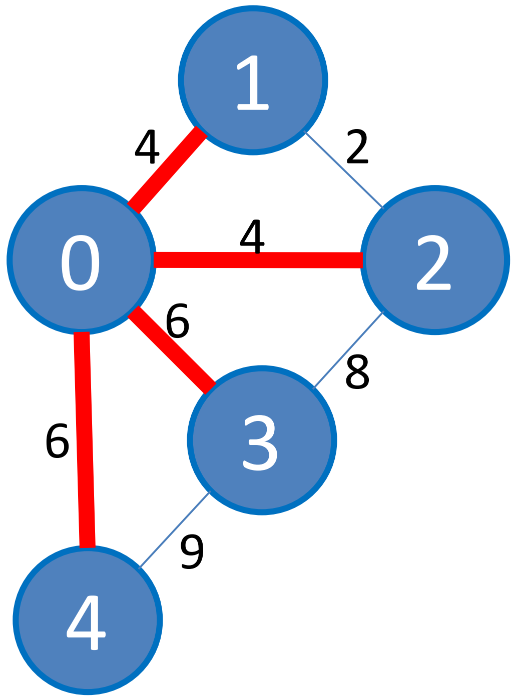


Example

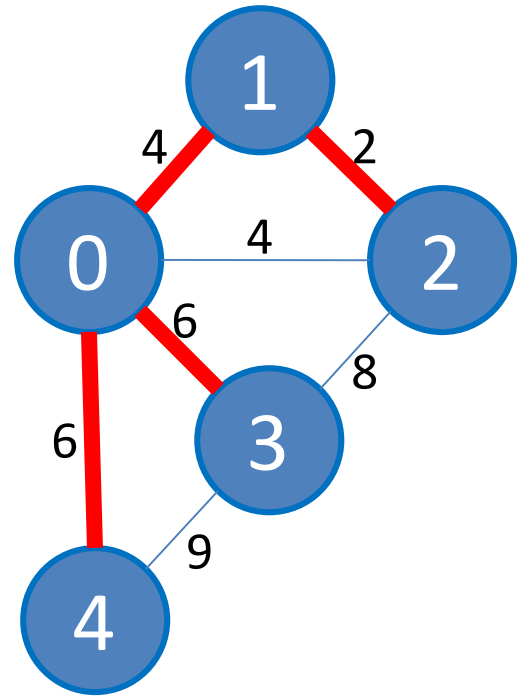
The Original Graph



A Spanning Tree
Cost: 4+4+6+6 = 20

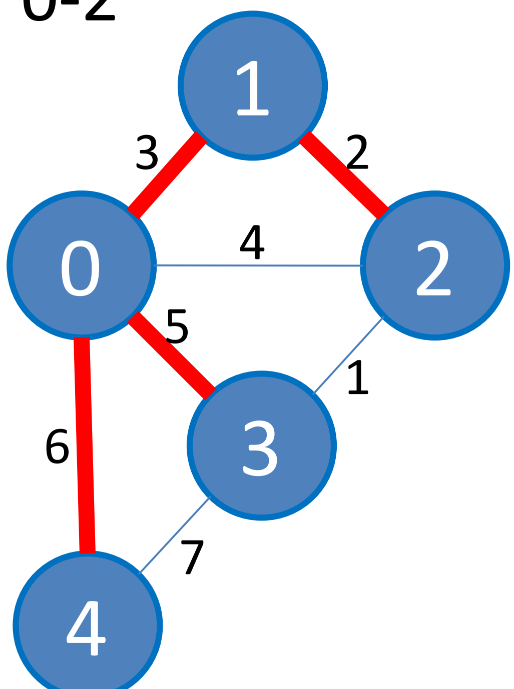


An MST
Cost: 4+6+6+2 = 18



Do the edges highlighted in red part form an MST of the original graph?

1. No, we must replace edge 0-3 with edge 2-3
2. No, we must replace edge 1-2 with 0-2
3. Yes



Brute force/Complete Search Solution?

- Consider all cycles in the graph and break them!
  - For each cycle remove the largest edge
  - If 1 or more edges in a cycle has already been removed previously move on to the next cycle

- Cycle property: For any cycle C in graph G(V,E), if weight of an edge e is larger than every other edge in C, e cannot be included in the MST of G(V,E)

- How to get all cycles in the graph?
  - Not so easy … (Can you think of a way to do this?)
  - Can have up to $O(2^N)$ different cycles!
  - Listing down one by one is slow !

MST Algorithms

MST is a well-known Computer Science problem

Several efficient (polynomial) algorithms:

- Jarnik's/Prim's greedy algorithm
  - Uses PriorityQueue Data Structure taught in Lecture 08
- Kruskal's greedy algorithm
  - Uses Union-Find Data Structure taught in Lecture 09
- Boruvka's greedy algorithm (not discussed here)
- And a few more advanced variants/special cases…

Do you still remember Prim's/Kruskal's algorithms from CS1231?

1. Yes and I also know how to ***implement*** them
2. Yes, but I have not try implementing them yet
3. I forgot that particular CS1231 material…
   but I know it exists
4. Eh?? These two algorithms were covered before in CS1231??
5. I didn't take CS1231 ☹

Prim's Algorithm

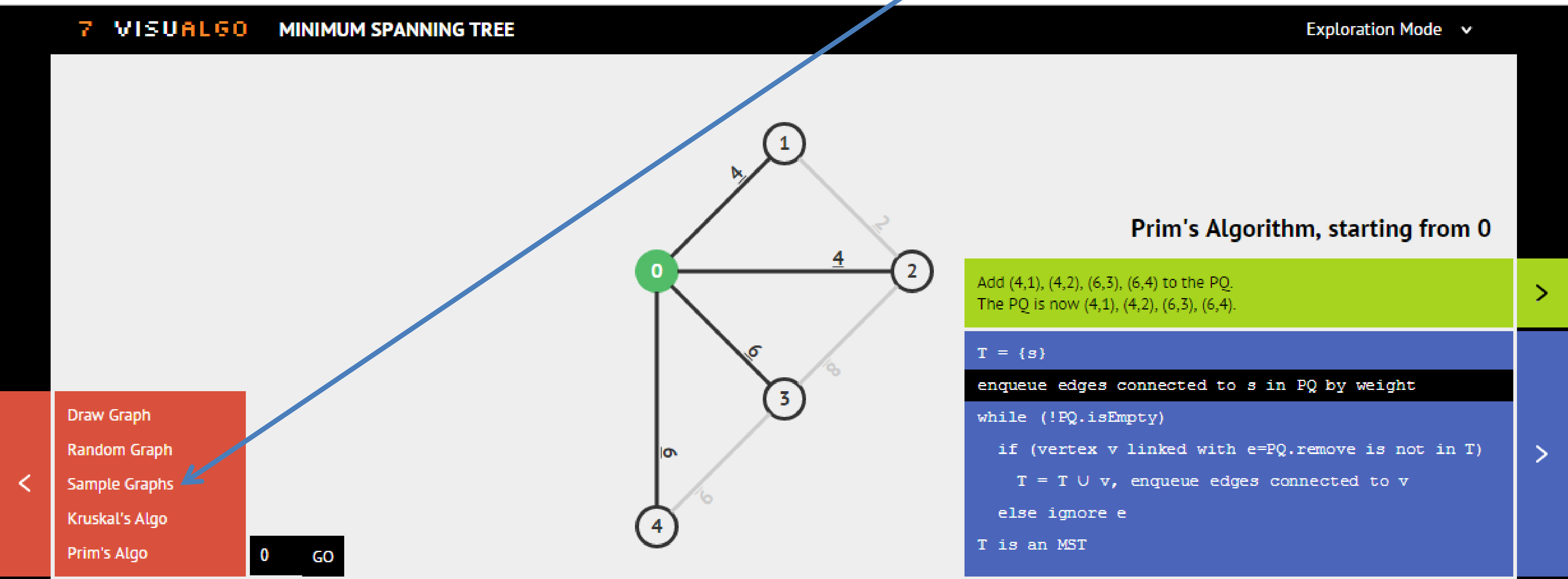
Very simple pseudo code

```
T ← {s}, a starting vertex s (usually vertex 0)
enqueue edges connected to s (only the other ending
  vertex and edge weight) into a priority queue PQ
  that orders elements based on increasing weight

while there are unprocessed edges left in PQ
  take out the front most edge e
  if vertex v linked with this edge e is not taken yet
    T ← T ∪ v (including this edge e)
    enqueue each edge adjacent to v into the PQ if it
    is not already in T

T is an MST
```

MST Algorithm: Prim's

Ask VisuAlgo to perform Prim's *from various sources* on the sample Graph (CP3 4.10), *then try other graphs*

In the screen shot below, we show the start of **Prim(0)**



Easy Java Implementation

You just need to use two known Data Structures
to be able to implement Prim's algorithm:

1. A priority queue (we can use Java PriorityQueue), and
2. A boolean array (to decide if a vertex has been taken or not)

With these DSes, we can run Prim's in O(**E** log **V**) using Adjacency list

- We only process each edge once (enqueue and dequeue it), O(**E**)
  - Each time, we enqueue/dequeue from a PQ in O(log **E**)
  - As $\mathbf{E} = O(\mathbf{V}^2)$, we have $O(\log \mathbf{E}) = O(\log \mathbf{V}^2) = O(2 \log \mathbf{V}) = O(\log \mathbf{V})$
  - Total time $O(\mathbf{E})*O(\mathbf{logV}) = O(\mathbf{E}\log\mathbf{V})$

Let's have a quick look at PrimDemo.java

Why Does Prim's Work? (1)

First, we have to realize that **Prim's algorithm** is a **greedy algorithm**

This is because **at each step**, it always try to select the next valid edge e with **minimal weight** (greedy!)

Greedy algorithm is usually simple to implement

- However, it usually requires "proof of correctness"
- You will see such proof like this again in CS3230
- Here, we will just see a quick proof

Why Does Prim's Work? (2)

with visual explanation

Proof by contradiction:

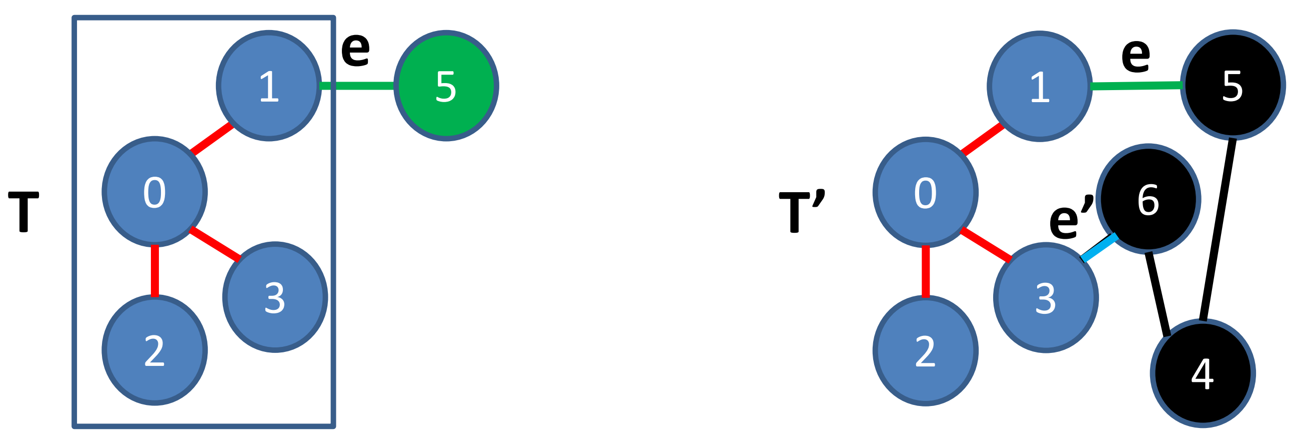
1. Assume that edge **e** is the first edge at iteration k chosen by Prim's which is not in any valid MST.
2. Let **T** be the tree generated by Prim's before adding **e**.
3. Now **T** must be a subtree of some valid MST **T'**

Why Does Prim’s Work? (3)

with visual explanation

Adding edge **e** to **T’** will now create a cycle.

Since e has 1 endpoint in **T** (the valid endpoint) and one endpoint outside **T**, trace around this cycle in **T’** until we get to some edge **e’** that goes back to **T**



Why Does Prim's Work? (4)

with visual explanation

Vertices in blue and vertices in black forms 2 disjoints subsets of **T'**. This is called a cut of **T'**

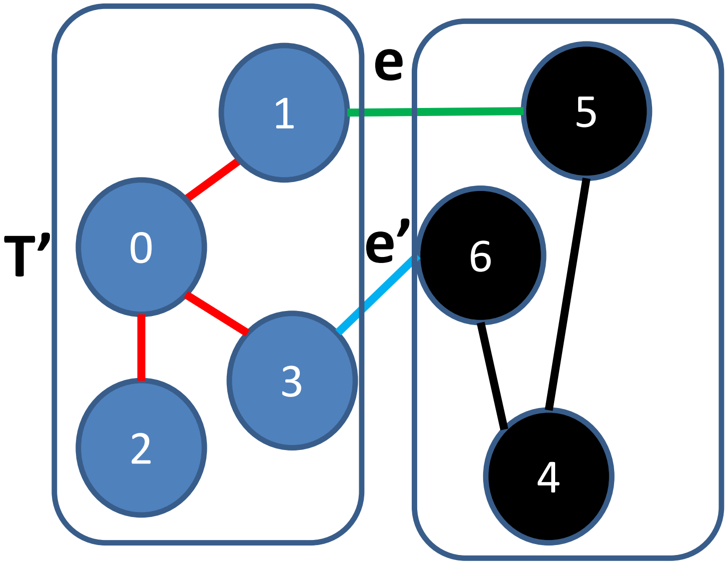
Definition:

Cut of a graph: any partition of the vertices of the graph into 2 disjoint subset.

Cut-set: The set of edges that cross a cut (e and e' in the example)

Cut Property of a graph:

For any cut C of the graph, if the weight of an edge e in the cut-set is strictly smaller than the weights of all other edges of the cut-set, then this edge belongs to all MSTs of the graph.



Why Does Prim's Work? (5)

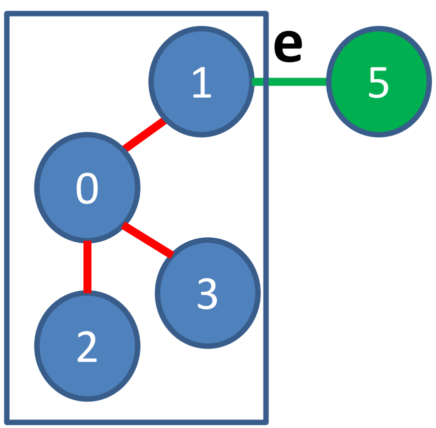
with visual explanation

By Prim's algorithm **e** and **e'** must be candidate edges at iteration k, but **e** was chosen meaning w(**e**) ≤ w(**e'**) by cut property

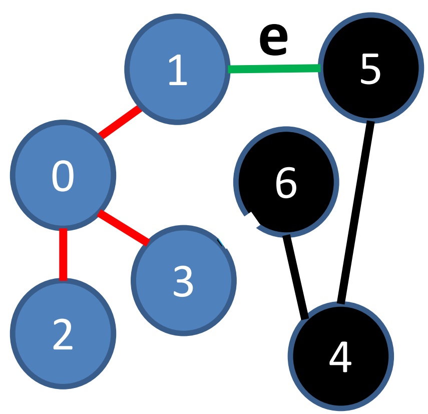
Now replacing **e'** with **e** in **T'** must give us tree **T''** covering all vertices of the graph s.t w(**T''**) ≤ w(**T'**)

Contradiction that **e** is first edge chosen wrongly

T

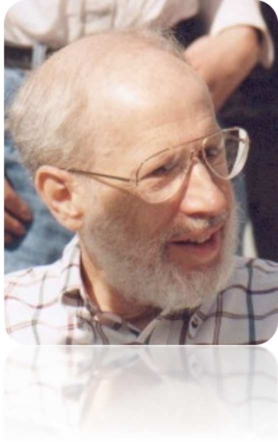


T''



Coming up next: Kruskal’s algorithm

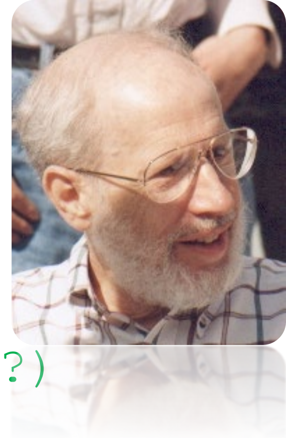
Kruskal's Algorithm



Very simple pseudo code

```
sort the set of E edges by increasing weight
T ← {}
while there are unprocessed edges left
  pick an unprocessed edge e with min cost
  if adding e to T does not form a cycle
    add e to T
T is an MST
```

Kruskal’s Implementation (1)



```
sort the set of E edges by increasing weight // O(?)
T ← {}
while there are unprocessed edges left // O(E)
  pick an unprocessed edge e with min cost // O(?)
  if adding e to T does not form a cycle // O(?)
    add e to the T // O(1)
T is an MST
```

To sort the edges:

- We use **EdgeList** to store graph information
- Then use “any” sorting algorithm that we have seen before

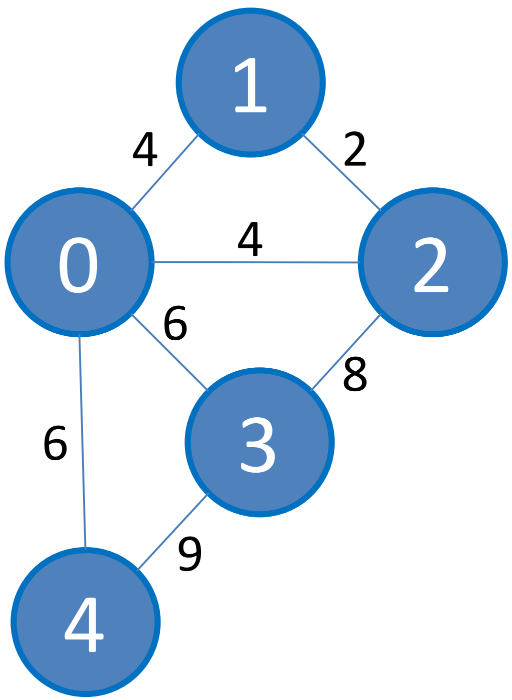
To test for cycles:

- We use **Union-Find Disjoint Sets**

Sorting Edges in Edge List

Adjacency Matrix/List that we have learned previously are *not suitable* for edge-sorting task!

To sort **EdgeList**, we use ***one liner*** **Java Collections.sort …**



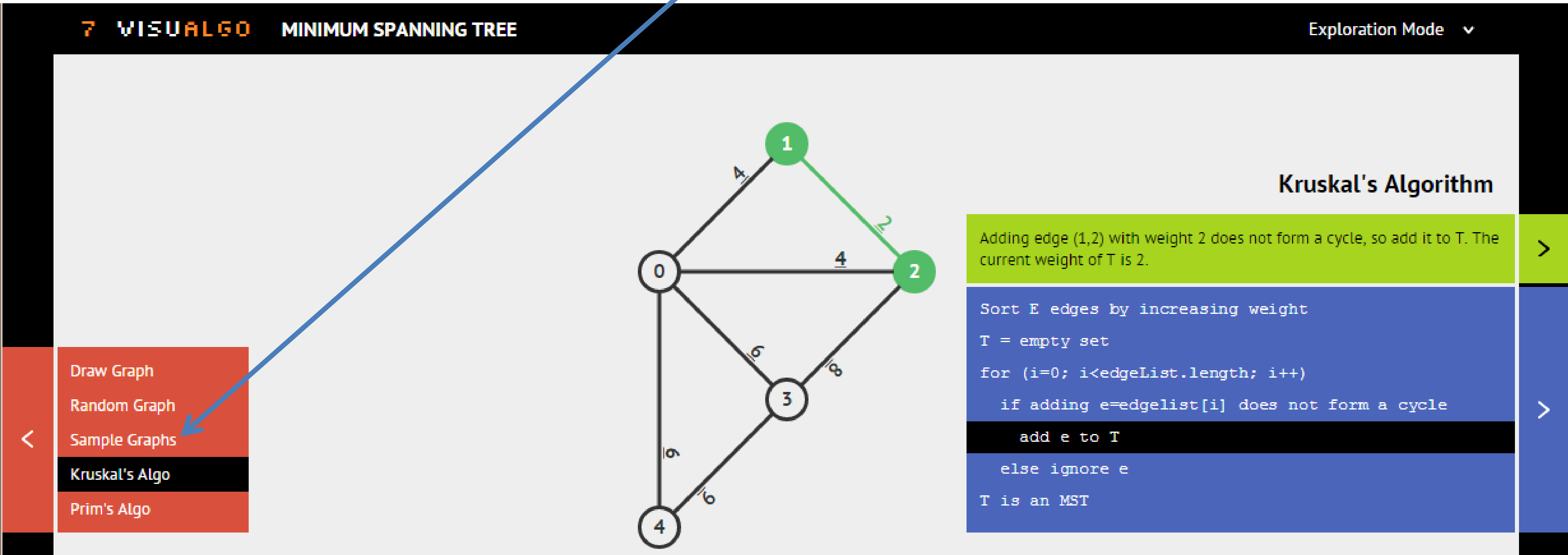
i	w	u	v
0	2	1	2
1	4	0	1
2	4	0	2
3	6	0	3
4	6	0	4
5	8	2	3
6	9	3	4

MST Algorithm: Kruskal's

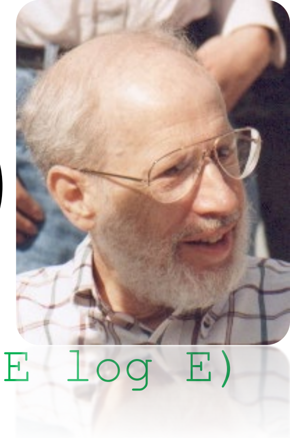
Ask VisuAlgo to perform Kruskal's on the sample Graph (CP3 4.10), *<u>then try other graphs</u>*

In the screen shot below, we show the start of **Kruskal**

(there is no parameter for this algorithm)



Kruskal’s Implementation (2)



```
sort the set of E edges by increasing weight // O(E log E)
T ← {}
while there are unprocessed edges left // O(E)
  pick an unprocessed edge e with min cost // O(1)
  if adding e to T does not form a cycle // O(α(V)) = O(1)
    add e to the T // O(1)
T is an MST
```

To sort the edges, we need $O(\mathbf{E} \log \mathbf{E})$

To test for cycles, we need $O(\alpha(\mathbf{V}))$ – small, assume constant $O(\mathbf{1})$

In overall

- Kruskal’s runs in $O(\mathbf{E} \log \mathbf{E} + \cancel{\mathbf{E}\,\alpha(\mathbf{V})})$ // E log E dominates!
- As $\mathbf{E} = O(\mathbf{V}^2)$, thus Kruskal’s runs in $O(\mathbf{E} \log \mathbf{V}^2) = O(\mathbf{E} \log \mathbf{V})$

Let’s have a quick look at KruskalDemo.java

Why Does Kruskal’s Work? (1)

Kruskal’s algorithm is also a **greedy algorithm**

Because **at each step**, it always try to select the next unprocessed edge **e** with **minimal weight** (greedy!)

Simple proof on how this greedy strategy works

- Almost the same as that for Prim’s

Why Does Kruskal’s Work? (2)

with visual explanation

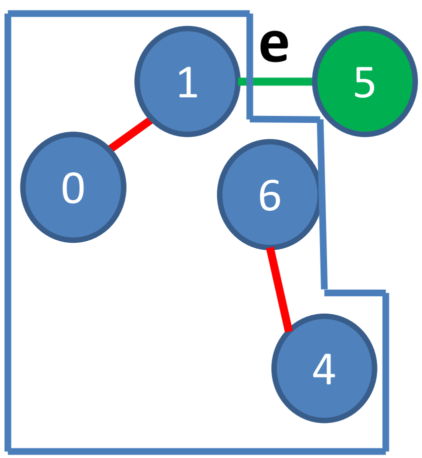
Proof by contradiction:

Assume that edge **e** is the first edge at iteration k chosen by Kruskal’s which is not in any valid MST.

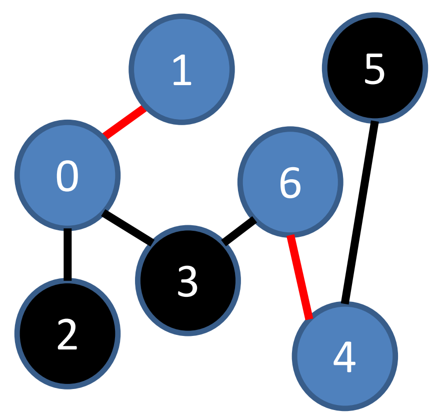
Let **F** be the forest generated by Kruskal’s before adding **e**.

Now **F** must be a part of some valid MST **T’**

F



T’



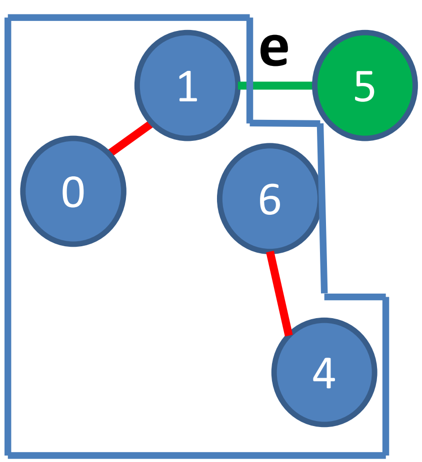
Why Does Kruskal's Work? (3)

with visual explanation

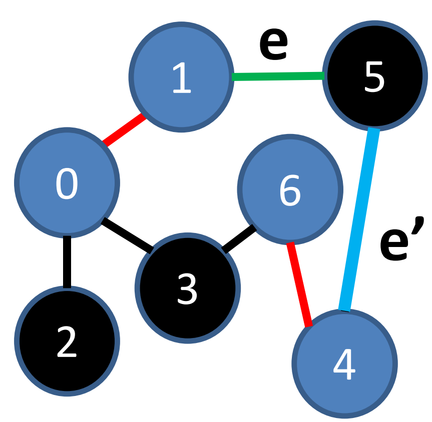
Putting **e** into **T'** will create a cycle.

Trace the cycle until an edge **e'** which connects a vertex in **F** with another vertex not in **F**

F



T'



Why Does Kruskal's Work? (4)

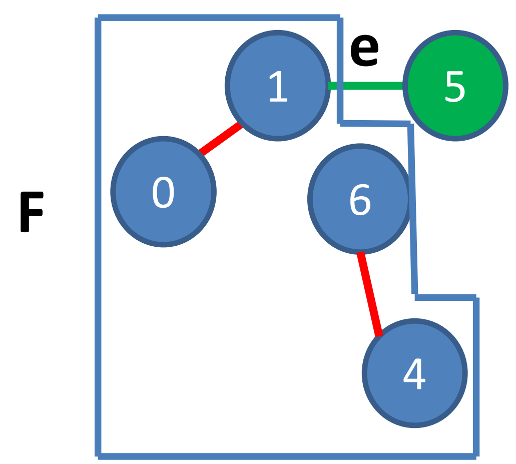
with visual explanation

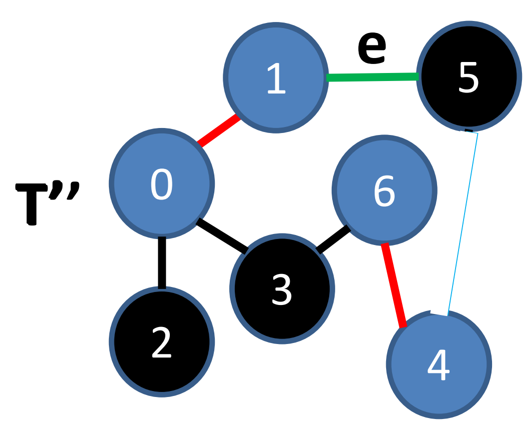
At iteration k, both **e** and **e'** are candidate (they are not chosen and do not form a cycle if chosen).

Since **e** was chosen, w(**e**) ≤ w(**e'**) (again by cut property, , the cut being {0,1,2,3,4,6} and {5} of **T'** in the previous slide)

Now replacing **e'** with **e** in **T'** must give us tree **T''** covering all vertices of the graph s.t w(**T''**) ≤ w(**T'**)

Contradiction that **e** is first edge chosen wrongly





If given an MST problem, I will…

1. Use/code Kruskal’s algorithm
2. Use/code Prim’s algorithm
3. No preference…

Summary

Introduce the MST problem (covered briefly in CS1231)

Discussing the implementation of Prim's algorithm

– Revisiting the PriorityQueue ADT

Discussing the implementation of Kruskal's algorithm

– Revisiting the EdgeList and showing technique to sort edges
– Revisiting the Union-Find Disjoint Sets DS

You *may* learn MST/Prim's/Kruskal's again in CS3230