Tutorial 8

Exercise 5

7. Let W be a subspace of $\mathbb{R}^n$. Define $W^\perp = \{\, \boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{u} \text{ is orthogonal to } W \,\}$.

   (a) Let $W = \text{span}\{(1,0,1,1),\ (1,-1,0,2),\ (1,2,3,-1)\}$. Find $W^\perp$.

   (b) Show that $W^\perp$ is a subspace of $\mathbb{R}^n$. (Hint: Show that $W^\perp$ is a solution set of a homogeneous system of linear equations.)

12. Use Gram-Schmidt Process to transform each of the following bases for $\mathbb{R}^3$ to an orthonormal basis.

   (a) $\{(1,0,1),\ (0,1,2),\ (2,1,0)\}$.

13. Use Gram-Schmidt Process to transform the following basis for $\mathbb{R}^4$ to an orthonormal basis: $\{(2,1,0,0),\ (-1,0,0,1),\ (2,0,-1,1),\ (0,0,1,1)\}$.

19. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be a square matrix of order n such that $\boldsymbol{A}^2 = \boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$.

   (a) For any two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, show that $(\boldsymbol{A}\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (\boldsymbol{A}\boldsymbol{v})$.

   (b) For any vector $\boldsymbol{w} \in \mathbb{R}^n$, show that $\boldsymbol{A}\boldsymbol{w}$ is the projection of $\boldsymbol{w}$ onto the subspace $V = \{\boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{u} = \boldsymbol{u}\}$ of $\mathbb{R}^n$.

23. A father wishes to distribute an amount of money among his three sons Jack, Jim and John.

   (a) Show that it is not possible to have a distribution such that the following conditions are all satisfied.

      (i) The amount Jack receives plus twice the amount Jim receives is \$300.

      (ii) The amount Jim receives plus the amount John receives is \$300.

      (iii) Jack receives \$300 more than twice of what John receives.

   (b) Since there is no solution to the distribution problem above, find a least squares solution.
   (Make sure that your least squares solution is feasible. For example, one cannot give a negative amount of money to anybody.)

27. (a) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$.

      (i) Solve the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

      (ii) Find a least squares solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

   (b) Suppose a linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent. Show that the solution set of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is equal to the solution set of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$.

   (Hint: You need Theorem 4.3.6 and the result of Question 4.25(a).)