Tutorial 2

Exercise 2

10. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ and $n \times p$ matrices respectively.

    (a) Suppose the homogeneous linear system $\boldsymbol{Bx} = \boldsymbol{0}$ has infinitely many solutions. How many solutions does the system $\boldsymbol{ABx} = \boldsymbol{0}$ have?

    (b) Suppose $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution. Can we tell how many solutions are there for $\boldsymbol{ABx} = \boldsymbol{0}$?

21. Given that $\boldsymbol{A}$ is a 3×3 matrix such that

$$\boldsymbol{A}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \text{and} \quad \boldsymbol{A}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Find a matrix $\boldsymbol{X}$ such that

$$\boldsymbol{AX} = \begin{pmatrix}1 & 0 & 4\\1 & 0 & 4\\1 & 0 & 7\end{pmatrix}.$$

(Hint: Write $\boldsymbol{X} = \begin{pmatrix}\boldsymbol{x_1} & \boldsymbol{x_2} & \boldsymbol{x_3}\end{pmatrix}$ where $\boldsymbol{x_i}$ is the ith column of $\boldsymbol{X}$.)

22. Prove Remark 1.1.10:

    Show that a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has either no solution, only one solution or infinitely many solutions.

    (Hint: Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has two different solutions $\boldsymbol{u}$ and $\boldsymbol{v}$. Use $\boldsymbol{u}$ and $\boldsymbol{v}$ to construct infinitely many other solutions.)

24. Determine which of the following statements are true. Justify your answer.

    (b) If $\boldsymbol{A}$ is a square matrix, then $\frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})$ is symmetric.

    (c) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size, $(\boldsymbol{A} + \boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{B}^2 + 2\boldsymbol{AB}$.

    (f) If $\boldsymbol{A}$ is a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$.

    (g) If $\boldsymbol{A}$ is a matrix such that $\boldsymbol{AA}^{\mathrm{T}} = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$.

27. (a) Give three examples of 2×2 matrices $\boldsymbol{A}$ such that $\boldsymbol{A}^2 = \boldsymbol{A}$.

    (b) Let $\boldsymbol{A}$ be a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{A}$. Show that $\boldsymbol{I} + \boldsymbol{A}$ is invertible and $(\boldsymbol{I} + \boldsymbol{A})^{-1} = \frac{1}{2}(2\boldsymbol{I} - \boldsymbol{A})$.