**National University of Singapore**

**Semester 1, 2020/2021 MA1101R Practice Assignment 3 Answer**

1. Let $V = \{(w, x, y, z) \mid y = w - x, z = 2w + x\}$ be a subset of $\mathbb{R}^4$.

   (i) [2 marks] Show that $V$ is s subspace of $\mathbb{R}^4$ by expression $V$ as a linear span.

   (ii) [2 marks] Write down a basis for $V$ and $\dim V$.

   (iii) [1 mark] If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a subset of $V$, can we tell whether it is a linearly independent set? Why?

   **Answer**

   (i) We can write the general vector in $V$ as
   $$(w, x, w - x, 2w + x) = w(1, 0, 1, 2) + x(0, 1, -1, 1).$$
   So $V = \text{span}\{(1, 0, 1, 2), (0, 1, -1, 1)\}$.

   (ii) Since $(1, 0, 1, 2), (0, 1, -1, 1)$ are linearly independent, it forms a basis for $V$. Hence $\dim V = 2$.

   (iii) Since $\dim V = 2$, any subset of $V$ with more than 2 vectors is linearly dependent.

2. Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a basis for $\mathbb{R}^3$.

   (i) [3 marks] Show that $T = \{\boldsymbol{u}_1 + \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis for $\mathbb{R}^3$.

   (ii) [2 marks] Find the transition matrix from $T$ to $S$. Briefly explain how you get the answer.

   **Answer**

   (i) Set $c_1(\boldsymbol{u}_1 + \boldsymbol{u}_2 + c_2(\boldsymbol{u}_1 - \boldsymbol{u}_2 + c_3\boldsymbol{u}_3 = \boldsymbol{0}$ (*)
   Then
   $$(c_1 + c_2)\boldsymbol{u}_1 + (c_1 - c_2)\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 = \boldsymbol{0}.$$
   Since $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is linearly independent, we have
   $$c_1 + c_2 = 0, \quad c_1 - c_2 = 0, \quad c_3 = 0.$$
   We can easily solve $c_1 = c_2 = c_3 = 0$ to be the only solution for $(*)$.
   Hence $T = \{\boldsymbol{u}_1 + \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is linearly independent.
   Since $\dim \mathbb{R}^3 = 3$, so $T$ is a basis for $\mathbb{R}^3$.
   (Alternative approach is possible.)

   (ii) $[\boldsymbol{u}_1 + \boldsymbol{u}_2]_S = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, [\boldsymbol{u}_1 - \boldsymbol{u}_2]_S = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, [\boldsymbol{u}_3]_S = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

   So the transition matrix from $T$ to $S$ is $\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

3. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$.

   (i) [2 marks] Find a basis for the column space of $\boldsymbol{A}$.

   (ii) [2 marks] Find a basis for the nullspace of $\boldsymbol{A}$. (Show your working.)

   (iii) [2 mark] Extend the basis in (i) to a basis for $\mathbb{R}^4$. (Show your working.)

**Answer**

(i) $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{GJE} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

From the RREF, a basis for the column space is given by the first two columns of $\boldsymbol{A}$:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

(Alternative answer is possible.)

(ii) We solve the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ using the RREF in (i) to get:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -r-t \\ -s \\ r \\ s \\ t \end{pmatrix} = r \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

So a basis for the nullspace is given by

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(Alternative answer is possible.)

(iii) Write the basis in (i) as row vectors and form a matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 3 & 1 \end{pmatrix} \xrightarrow{GJE} \begin{pmatrix} 1 & 0 & 1 & 2/3 \\ 0 & 1 & 1 & -1/3 \end{pmatrix}$$

So to extend the basis, we add $(0,0,1,0)^T$ and $(0,0,0,1)^T$.
(Alternative answer is possible.)

4. [4 marks] Let $\boldsymbol{C} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x^2-x-2 & x+1 \end{pmatrix}$.

   Find all the values of $x$ such that
   (i) $\operatorname{rank}(\boldsymbol{C}) = 1$; (ii) $\operatorname{rank}(\boldsymbol{C}) = 2$; (iii) $\operatorname{rank}(\boldsymbol{C}) = 3$.

   **Answer**

   By observing the "leading entries" of the matrix $\boldsymbol{C}$:
   $(x-2)$ and $x^2 - x - 2 = (x-2)(x+1)$,
   we just need to look at the two values $x = 2$ and $x = -1$.

   For $x = 2$, we have $\boldsymbol{C} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. So $\operatorname{rank}(\boldsymbol{C}) = 2$.

   For $x = -1$, we have $\boldsymbol{C} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So $\operatorname{rank}(\boldsymbol{C}) = 2$.

   For $x \neq 2, -1$, we have $x - 2 \neq 0$ and $x^2 - x - 2 \neq 0$. So $\operatorname{rank}(\boldsymbol{C}) = 3$.

   Hence

   (i) $\operatorname{rank}(\boldsymbol{C}) = 1$ for no value of $x$.
   (ii) $\operatorname{rank}(\boldsymbol{C}) = 2$ for $x = -1$ or $2$.
   (iii) $\operatorname{rank}(\boldsymbol{C}) = 3$ for $x \neq -1$ and $2$.