If $y = \frac{1}{3}x^3$ find the **exact value** of $\frac{dy}{dx}$ when $x = 39$.

$$\frac{dy}{dx} = x^2$$

$$x = 39 \Rightarrow \frac{dy}{dx} = 39^2 = \underline{\underline{1521}}$$

Let C denote the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$. Let P denote a point on the curve C in the first quadrant. Let L denote the normal line to the curve C at the point P. If L passes through the point $(\frac{1}{2}, 0)$ find the gradient of the line L. Give your answer correct to two decimal places.

First observe that the part of C in the 1st quadrant can be written as

$$\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}, \qquad 0 \leq t \leq \frac{\pi}{2}.$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} = -\tan t.$$

$$\therefore m = \text{grad of } L = \frac{-1}{-\tan t} = \cot t$$

$$(\tfrac{1}{2}, 0) \in L \Rightarrow \frac{0 - \sin^3 t}{\frac{1}{2} - \cos^3 t} = m = \cot t = \frac{\cos t}{\sin t}$$

$$\Rightarrow -\sin^4 t = \frac{1}{2}\cos t - \cos^4 t$$

$$\Rightarrow -(1 - \cos^2 t)^2 = \frac{1}{2}\cos t - \cos^4 t$$

$$\Rightarrow 4\cos^2 t - \cos t - 2 = 0$$

$$\Rightarrow \cos t = \frac{1 + \sqrt{1+32}}{8} = \frac{1+\sqrt{33}}{8} \quad (\because 0 \leq t \leq \tfrac{\pi}{2})$$

$$\therefore \sin t = \sqrt{1 - \cos^2 t} = \sqrt{1 - \left(\frac{1+\sqrt{33}}{8}\right)^2} = \frac{\sqrt{30 - 2\sqrt{33}}}{8}$$

$$m = \cot t = \frac{1+\sqrt{33}}{\sqrt{30 - 2\sqrt{33}}} = 1.567\ldots$$

$$\approx \underline{\underline{1.57}}$$

Let a denote a positive constant. Let R denote the part of the region in the first quadrant which is bounded above by the curve $y = \sqrt{(a-x)(2a-x)(3a-x)}$ and bounded below by the x-axis from $x = 2a$ to $x = 3a$. If the volume of the solid obtained by revolving R one complete round about the x-axis is equal to 2019, find the value of a. Give your answer correct to two decimal places.

$$2019 = \int_{2a}^{3a} \pi y^2 dx$$

$$= \int_{2a}^{3a} \pi (a-x)(2a-x)(3a-x)\, dx$$

$$= \pi \int_{2a}^{3a} (6a^3 - 11a^2x + 6ax^2 - x^3)\, dx$$

$$= \pi \left[6a^3x - \frac{11}{2}a^2x^2 + 2ax^3 - \frac{1}{4}x^4 \right]_{2a}^{3a}$$

$$= \pi a^4 \left\{ \left(18 - \frac{99}{2} + 54 - \frac{81}{4}\right) - (12 - 22 + 16 - 4) \right\}$$

$$= \frac{\pi}{4} a^4$$

$$\therefore a = \left(\frac{4 \times 2019}{\pi}\right)^{1/4} = 7.120\ldots$$

$$\approx \underline{\underline{7.12}}$$

Let a denote a positive constant. If the area under the curve $y = x\sqrt{2ax - x^2}$ from $x = 0$ to $x = a$ is equal to 888, find the value of a. Give your answer correct to two decimal places.

$$888 = \int_0^a x\sqrt{2ax - x^2}\,dx$$

$$= \int_0^a x\sqrt{a^2 - a^2 + 2ax - x^2}\,dx$$

$$= \int_0^a x\sqrt{a^2 - (a-x)^2}\,dx$$

Let $a - x = a\sin\theta \Rightarrow -dx = a\cos\theta\, d\theta$

$x = 0 \Rightarrow a = a\sin\theta \Rightarrow \theta = \frac{\pi}{2}$

$x = a \Rightarrow 0 = a\sin\theta \Rightarrow \theta = 0$

$$\therefore 888 = \int_{\frac{\pi}{2}}^{0} (a - a\sin\theta)\sqrt{a^2 - a^2\sin^2\theta}\,(-a\cos\theta\, d\theta)$$

$$= -a^3 \int_0^{\frac{\pi}{2}} (1 - \sin\theta)(-\cos^2\theta)\,d\theta$$

$$= a^3 \int_0^{\frac{\pi}{2}} \left[\left(\frac{1+\cos 2\theta}{2}\right) - \cos^2\theta\sin\theta\right] d\theta$$

$$= a^3\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}}$$

$$= a^3\left(\frac{\pi}{4} - \frac{1}{3}\right)$$

$$\therefore a = \left(\frac{888}{\frac{\pi}{4} - \frac{1}{3}}\right)^{1/3} = 12.523\cdots \approx \underline{\underline{12.52}}$$