CS1231(S) Tutorial 6: Number theory 1

National University of Singapore

2020/21 Semester 1

Questions for discussion on the LumiNUS Forum

Answers to these questions will not be provided.

D1. Which of the following numbers are multiples of 17? Which are not?

(a) 68 (b) 84 (c) 357 (d) 1001

D2. Complete Exercise 8.1.4, i.e., show that if $a, d, n \in \mathbb{Z}$ such that $d \mid n$, then $ad \mid an$.

D3. Complete Exercise 8.1.8, i.e., determine which $n \in \mathbb{Z}$ makes $0 \mid n$.

D4. Prove or disprove the following sentence.

If $a, b, c \in \mathbb{Z}^+$ such that $a \mid bc$, then $a \mid b$ or $a \mid c$.

Tutorial questions

More challenging questions are marked with an asterisk (*).

1. Let $a, b \in \mathbb{Z}$. Show that if $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.
2. Find the quotient and the remainder when
   (a) 44 is divided by 8;
   (b) 777 is divided by 21;
   (c) -123 is divided by 19;
   (d) 0 is divided by 17;
   (e) -100 is divided by 101.
3. Show that for all odd integers $n \in \mathbb{Z}$,
$$n^2 \ \underline{\mathrm{div}}\ 4 = \frac{n^2 - 1}{4}.$$
4. Is 107 prime? Is 113 prime?
5. Write down an integer $n \geqslant 1231$ that shares no prime divisor with 15811090783488000. Prove that your answer is correct. Did you implicitly or explicitly use the Fundamental Theorem of Arithmetic (i.e., the fact that every positive integer greater than 1 has a unique factorization into a product of primes) in your proof? If yes, then can you avoid it (possibly by choosing a different n)?

6.* An integer n is said to be a *perfect square* if $n = k^2$ for some $k \in \mathbb{Z}$. Prove that a positive integer n is a perfect square if and only if it has an odd number of positive divisors.

   (Hint: pair up the divisors strictly bigger than $\sqrt{n}$ and the divisors strictly smaller than $\sqrt{n}$.)

7. Find the binary, octal and hexadecimal expansions of 1231.

8. Find the decimal expansions of

   (a) $(1101001)_2$;

   (b) $(156)_8$;

   (c) $(74)_{16}$.

9.* Let $n \in \mathbb{Z}_{\geqslant 1}$ with decimal representation $(a_\ell a_{\ell-1} \ldots a_0)_{10}$. Prove that $9 \mid n$ if and only if $9 \mid (a_0 + a_1 + \cdots + a_\ell)$.

   (Hint: for example,

$$\begin{aligned} 7524 &= 7 \times 1000 + 5 \times 100 + 2 \times 10 + 4 \\ &= 7 \times (999 + 1) + 5 \times (99 + 1) + 2 \times (9 + 1) + 4 \\ &= (7 \times 999 + 7) + (5 \times 99 + 5) + (2 \times 9 + 2) + 4 \\ &= (7 \times 999 + 5 \times 99 + 2 \times 9) + 7 + 5 + 2 + 4 \\ &= 9 \times (7 \times 111 + 5 \times 11 + 2 \times 1) + (7 + 5 + 2 + 4). \end{aligned}$$

   You may use without proof the fact that

$$10^i = 9 \times 10^{i-1} + 9 \times 10^{i-2} + \cdots + 9 \times 10^0 + 1$$

   for all $i \in \mathbb{Z}_{\geqslant 0}$.)

Summation notation

Definition. Let $k, \ell \in \mathbb{Z}$ and $a_k, a_{k+1}, \ldots, a_\ell \in \mathbb{R}$. Then

$$\sum_{i=k}^{\ell} a_i = \begin{cases} a_k + a_{k+1} + \cdots + a_\ell, & \text{if } k \leqslant \ell; \\ 0, & \text{otherwise.} \end{cases}$$

Basic properties. Let $k, \ell \in \mathbb{Z}$ and $a_k, a_{k+1}, \ldots, a_\ell, b_k, b_{k+1}, \ldots, b_\ell, c \in \mathbb{R}$. Then

$$\sum_{i=k}^{\ell} a_i + \sum_{i=k}^{\ell} b_i = \sum_{i=k}^{\ell} (a_i + b_i) \qquad \text{and} \qquad c \sum_{i=k}^{\ell} a_i = \sum_{i=k}^{\ell} c a_i.$$