Section 6.3: Cardinality

CS1231S Discrete Structures

Wong Tin Lok

National University of Singapore

11 August 2020

> No one shall be able to drive us from the paradise that Cantor created for us.
>
> David Hilbert

> Imagine set theory's having been invented by a satirist as a kind of parody on mathematics.
>
> Ludwig Wittgenstein

Tell me your view on infinity at `https://pollev.com/wtl`.

What we saw

- equality of functions
- composition of functions
- injections, surjections, bijections
- inverses

Definition 6.2.13

Let $f: A \to B$. Then $g: B \to A$ is an *inverse* of f if

$$\forall x \in A \ \forall y \in B \ \big(y = f(x) \ \Leftrightarrow \ x = g(y)\big).$$

Theorem 6.2.18

A function is bijective if and only if it has an inverse.

Now

how one can compare sizes of infinite sets

No one shall be able to drive us from the paradise that Cantor created for us.

David Hilbert

Imagine set theory's having been invented by a satirist as a kind of parody on mathematics.

Ludwig Wittgenstein

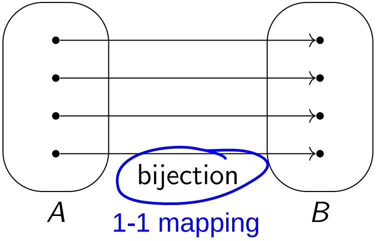
Tell me your view on infinity at `https://pollev.com/wtl`.

Equality of cardinality

Definition 5.2.3

(1) A set is *finite* if it has finitely many (distinct) elements. It is *infinite* if it is not finite.

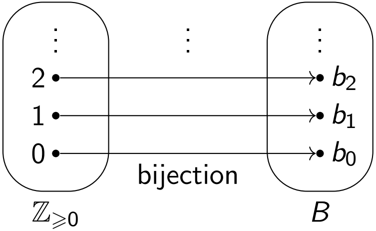
(2) Suppose A is a finite set. The *cardinality* of A, or the *size* of A, is the number of (distinct) elements in A.



Definition 6.3.1 (Cantor)

(1) Two set A, B are said to have the *same cardinality* if there is a bijection $A \to B$.

(2) A set is *countable* if it is finite or it has the same cardinality as $\mathbb{Z}_{\geqslant 0}$.



Note 6.3.2

An infinite set B is countable if and only if

there is a sequence $b_0, b_1, b_2, \ldots \in B$ in which every element of B appears exactly once.

Examples of countable sets

Example 6.3.3

(1) $\mathbb{Z}_{\geqslant 0}$ is countable because each element of $\mathbb{Z}_{\geqslant 0}$ is listed exactly once in the sequence

$$0, 1, 2, 3, 4, \ldots .$$

(2) The set $E = \{2x : x \in \mathbb{Z}_{\geqslant 0}\}$ is countable because each element of E is listed exactly once in the sequence

$$0, 2, 4, 6, 8, 10, 12, \ldots .$$

only in infinite

Note that $E \subsetneq \mathbb{Z}_{\geqslant 0}$, but E and $\mathbb{Z}_{\geqslant 0}$ have the same cardinality.

(3) $\mathbb{Z}$ is countable because each element of $\mathbb{Z}$ is listed exactly once in the sequence

$$0, 1, -1, 2, -2, 3, -3, \ldots .$$

Note that $\mathbb{Z}$ is the union of two disjoint infinite sets $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Z}^{-}$, but $\mathbb{Z}$ and $\mathbb{Z}_{\geqslant 0}$ have the same cardinality.

Note 6.3.2

An infinite set B is countable if and only if

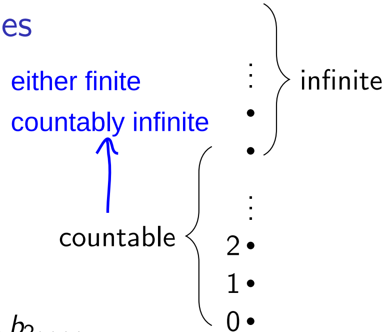
being countable can still be infinite

there is a sequence $b_0, b_1, b_2, \ldots \in B$ in which every element of B appears exactly once.

Countable cardinalities are the smallest cardinalities

Proposition 6.3.4

Any subset A of a countable set B is countable.



Proof

splitting by cases

1. If A is finite, then A is countable by definition.
2. So suppose A is infinite.
   - 2.1. Then B is infinite too as $A \subseteq B$.
   - 2.2. Use the countability of B to find a sequence $b_0, b_1, b_2, \ldots$ in which every element of B appears exactly once.
   - 2.3. Taking away those terms in the sequence that are not in A, we are left with a subsequence in which every element of A appears exactly once.
   - 2.4. So A is countable. $\square$

Note 6.3.2

An infinite set B is countable if and only if

there is a sequence $b_0, b_1, b_2, \ldots \in B$ in which every element of B appears exactly once.

Countable infinity is the smallest infinity

Proposition 6.3.5

Every infinite set B has a countable infinite subset.

Proof

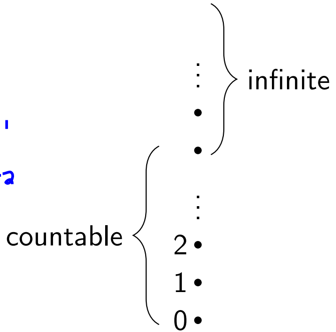
1. Keep choosing elements $b_0, b_1, b_2, \ldots$ from B. When we choose b_n, where $n \in \mathbb{Z}_{\geqslant 0}$, we can always make sure $b_n \neq b_i$ for any $i < n$, because otherwise B is equal to the finite set $\{b_0, b_1, \ldots, b_{n-1}\}$, which is a contradiction.
2. The result is a countable infinite set $\{b_0, b_1, b_2, \ldots\} \subseteq B$. $\square$

$b_n \neq b_{n-1}$

$b_n \neq b_{n-2}$

$\vdots$

$b_n \neq b_i$



Note 6.3.2

An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \ldots \in B$ in which every element of B appears exactly once.

Countability of a product of countable sets

Theorem 6.3.6 (Cantor 1877)

$\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is countable.

Proof sketch

The figure on the right describes a sequence in which every element of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ appears exactly once. $\square$

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
(0,4) \arrow[u] & (1,4) \arrow[l] & (2,4) \arrow[l] & (3,4) \arrow[l] & (4,4) \arrow[l] & \cdots \\
(0,3) \arrow[r] & (1,3) \arrow[r] & (2,3) \arrow[r] & (3,3) \arrow[d] & (4,3) \arrow[u] & \cdots \\
(0,2) \arrow[u] & (1,2) \arrow[l] & (2,2) \arrow[l] & (3,2) \arrow[d] & (4,2) \arrow[u] & \cdots \\
(0,1) \arrow[r] & (1,1) \arrow[d] & (2,1) \arrow[u] & (3,1) \arrow[d] & (4,1) \arrow[u] & \cdots \\
(0,0) \arrow[u] & (1,0) \arrow[r] & (2,0) \arrow[u] & (3,0) \arrow[r] & (4,0) \arrow[u] & \cdots
\end{tikzcd}
$$

Corollary 6.3.7

$\mathbb{Q}$ is countable.

Proof sketch

We can view $\frac{m}{n} \in \mathbb{Q}$ as the pair (m, n), where $n \neq 0$ and $\gcd(m, n) = 1$. $\square$

Note 6.3.2

An infinite set B is countable if and only if

there is a sequence $b_0, b_1, b_2, \ldots \in B$ in which every element of B appears exactly once.

Uncountability of the power of a countable infinite set

Theorem 6.3.8 (Cantor 1891)

Let A be a countable infinite set. Then $\mathcal{P}(A)$ is not countable.

Corollary 6.3.9

$\mathbb{R}$ is not countable.

Definition 5.2.1

$\mathcal{P}(A)$ denotes the set of all subsets of A.

diagonal element

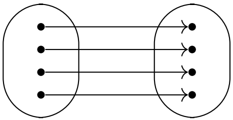
$$
\begin{array}{c|cccccc}
 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\
B_0 & \boxed{\notin} & \in & \notin & \notin & \notin & \dots \\
B_1 & \in & \boxed{\notin} & \in & \notin & \in & \dots \\
B_2 & \notin & \in & \boxed{\in} & \notin & \in & \dots \\
B_3 & \notin & \notin & \in & \boxed{\notin} & \notin & \dots \\
B_4 & \in & \notin & \in & \in & \boxed{\notin} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
B & \in & \in & \notin & \in & \in & \dots
\end{array}
$$

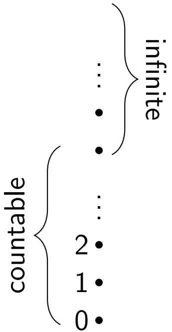
- $B_0, B_1, B_2, \ldots \in \mathcal{P}(A)$.
- $a_0, a_1, a_2, \ldots$ contains every element of A exactly once.
- $B \in \mathcal{P}(A)$ but B does not appear in $B_0, B_1, B_2, \ldots$.

Note 6.3.2

An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \ldots \in B$ in which every element of B appears exactly once.

Summary





What we saw

- one way to compare the sizes of infinite sets
- the cardinality of $\mathbb{Z}_{\geqslant 0}$ is the smallest amongst those of infinite sets
- infinite sets can have different cardinalities

> No one shall be able to drive us from the paradise that Cantor created for us. David Hilbert

Questions

(1) When does one set have a smaller cardinality than another?

(2) What is the cardinality of an infinite set? Search for "cardinal numbers".

(3) Is there a subset of $\mathcal{P}(\mathbb{Z}_{\geqslant 0})$ that is not countable but not of the same cardinality as $\mathcal{P}(\mathbb{Z}_{\geqslant 0})$? Search for "continuum hypothesis".

Next

Induction