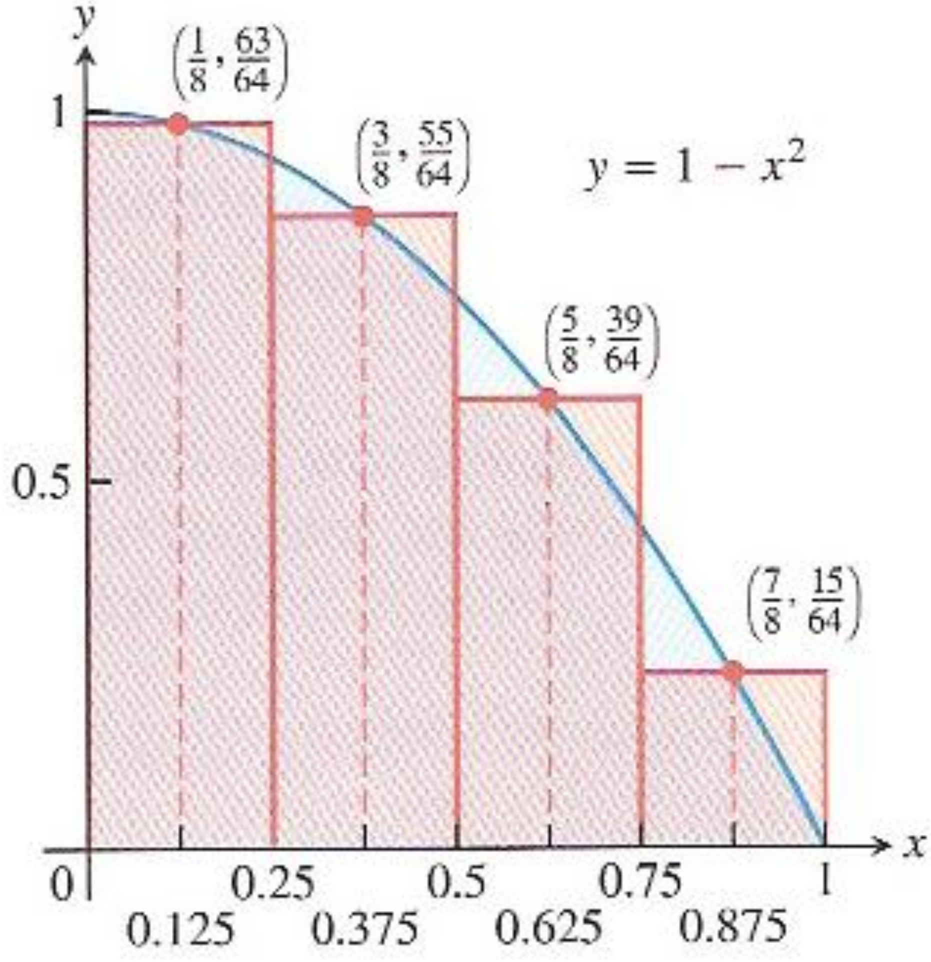
Chapter 7 Double Integrals

Recall: one variable case



Double Integrals

Definition

The definition of definite integral in chapter 3 can be extended to functions of two variables.

Let R be a plane region in the xy-plane.

Subdivide R into subrectangles $R_i \quad (i = 1, ..., n)$.

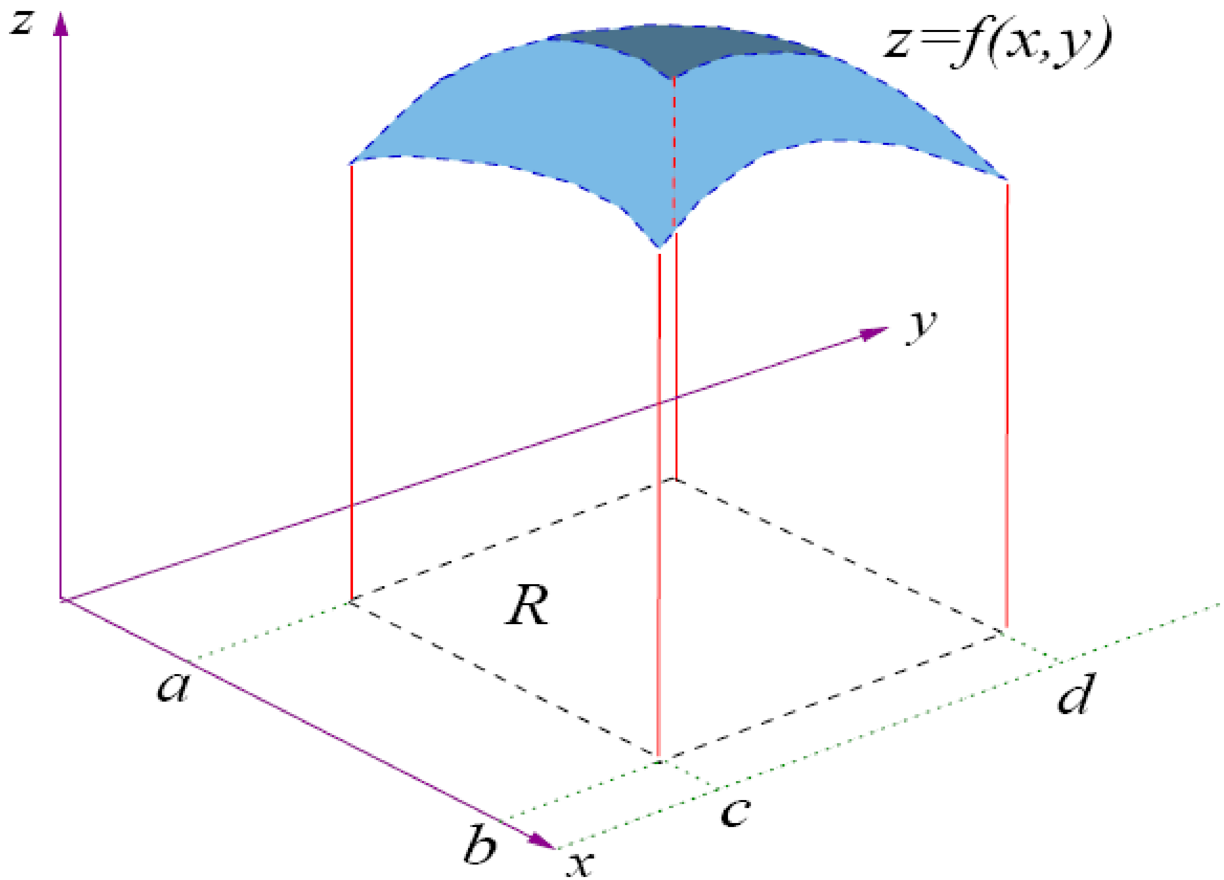
Let ΔA_i be the area of R_i and (x_i, y_i) be a point in R_i.

Let $f(x, y)$ be a function of two variables. Then the **double integral** of f over R is

$$\iint_R f(x, y)\, dA = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i.$$

Geometrical meaning

Geometrically, if $f(x, y) \geq 0$ for all $(x, y) \in R$, the definite integral $\iint_R f(x, y)\ dA$ is equal to the *volume* under the surface $z = f(x, y)$ and above the xy-plane over the region R as shown in the following diagram.


z
z=f(x,y)
y
R
a
d
c
b
x

Summing over all the rectangles,

$$\sum_{i=1}^{n} f(x_i, y_i) A_i$$

gives the *approximate* volume of the solid under the

surface and above R.

By letting n go to ∞, (i.e. making the subdivision more refined), the above sum will approach the *exact* volume of the solid.

Properties of Double Integrals

(1) $$\iint_R (f(x,y) + g(x,y))\ dA$$
$$= \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA.$$

(2) $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$, where c is a constant.

(3) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$,

then $\displaystyle\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$

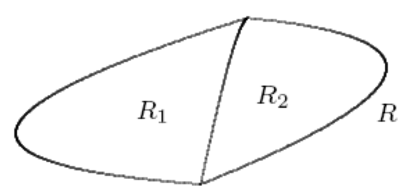
(4) $\displaystyle\iint_R dA \left(= \iint_R 1\,dA \right) = A(R)$, the area of R.

$$(5) \quad \iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA,$$

where $R = R_1 \cup R_2$

and R_1, R_2 do not overlap except perhaps on their

boundary.



(6) If $m \le f(x, y) \le M$ for all $(x, y) \in R$, then

$$mA(R) \le \iint_R f(x, y)\, dA \le M\, A(R).$$

Evaluation of double integrals

We shall discuss how to derive an efficient way to evaluate double integrals over certain plane regions. The key is to describe the given region in terms of the coordinates.

Rectangular regions

A rectangular region R in the xy-plane can be described in terms of inequalities:

$$a \leq x \leq b, \qquad c \leq y \leq d.$$

Then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\ dx\right]\ dy.$$

The RHS is called an **iterated integral**. i.e. repeating the integration for each variable, one at a time.

We can also change the order of the variables of integration (without changing the value):

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\ dy \right]\ dx.$$

Note that we can do away with the square brackets in the integrated integrals.

Example

Evaluate the iterated integrals:

$$\text{(a)} \int_0^3 \int_1^2 (x+2y)\, dydx, \qquad \text{(b)} \int_1^2 \int_0^3 (x+2y)\, dxdy.$$

Solution:

$$(a)\quad \int_0^3 \int_1^2 (x + 2y)\, dydx = \int_0^3 \left[xy + y^2\right]_{y=1}^{y=2} dx$$

$$= \quad \int_0^3 (x + 3)\, dx = \left[\frac{x^2}{2} + 3x\right]_{x=0}^{x=3} = 27/2.$$

$$(b)\quad \int_1^2\int_0^3 (x+2y)\,dxdy = \int_1^2 \left[\frac{x^2}{2} + 2xy\right]_{x=0}^{x=3} dy$$

$$= \int_1^2 \left[\frac{9}{2} + 6y\right]\, dy = \left[\frac{9y}{2} + 3y^2\right]_{y=1}^{y=2} = 27/2.$$

Example

Let R be the rectangular region

$$0 \leq x \leq 4, \quad 1 \leq y \leq 2.$$

Evaluate $\displaystyle\iint_R x^2 y \, dA$.

Solution:

$$\begin{aligned}\iint_R x^2 y\, dA &= \int_0^4 \int_1^2 x^2 y\, dy dx \\ &= \left(\int_0^4 x^2\, dx\right)\left(\int_1^2 y\, dy\right) \\ &= \frac{64}{3} \times \frac{3}{2} = 32.\end{aligned}$$

Remark

In general, if $f(x, y) = g(x)h(y)$, then

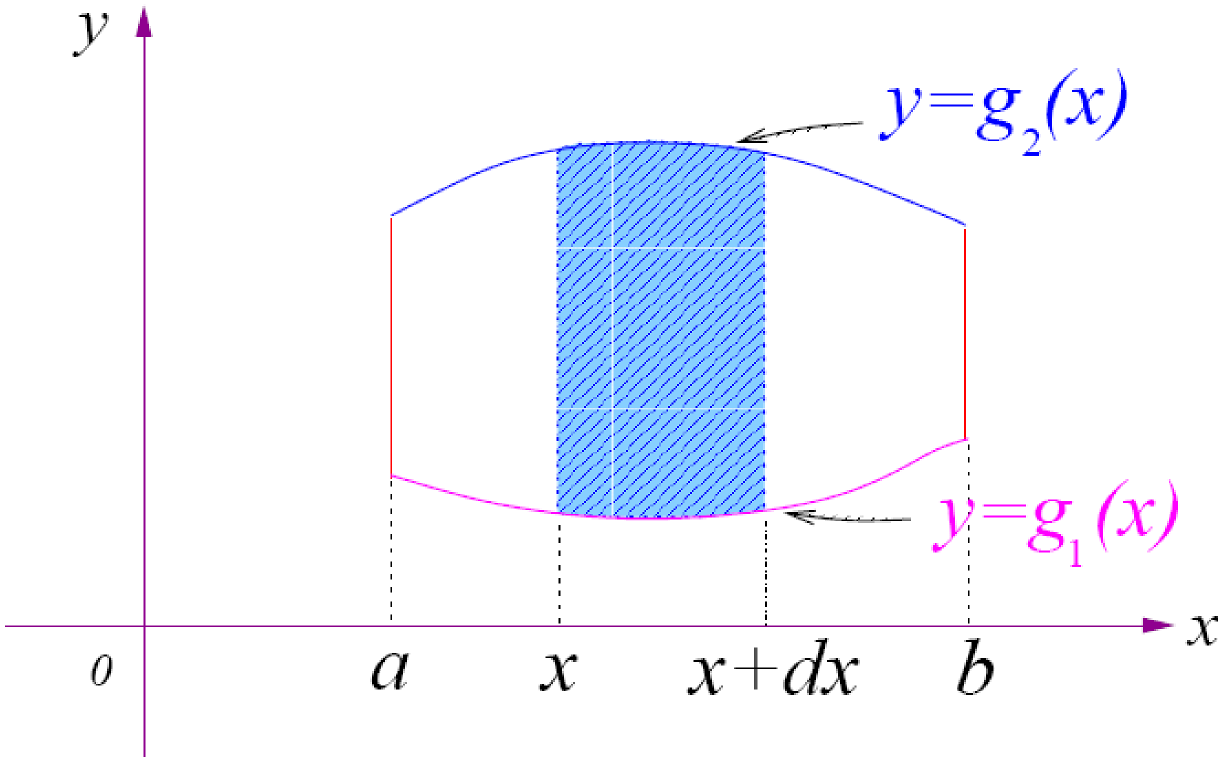
$$\iint_R g(x)h(y)\, dA = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)$$

where R is the rectangular region

$$a \leq x \leq b, \quad c \leq y \leq d.$$

General regions - Type A

Bottom and top boundaries are curves given by $y = g_1(x)$ and $y = g_2(x)$ respectively. i.e. they need not be straight lines. Left and right boundaries are straight lines given by $x = a$ and $x = b$ respectively.



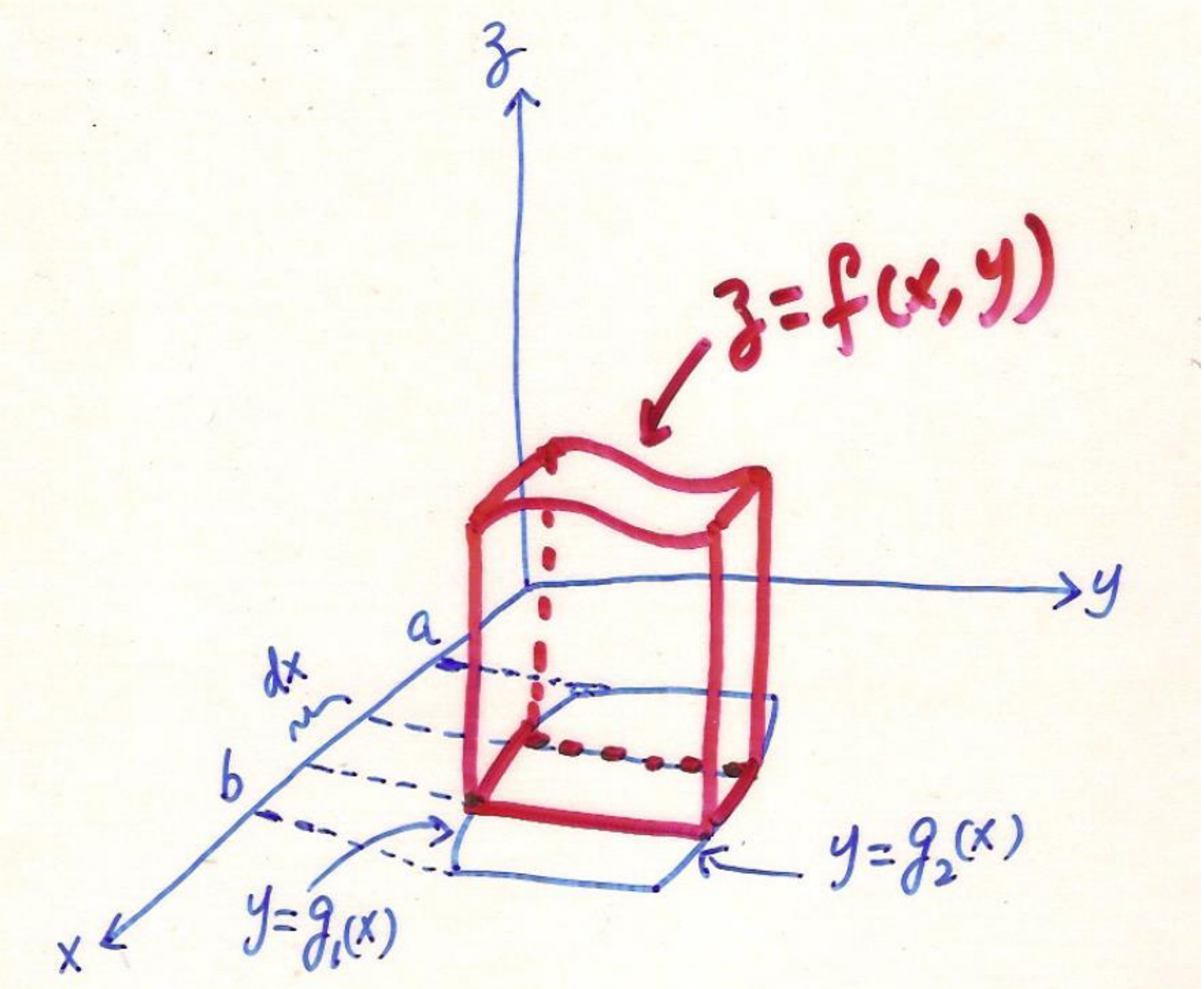
If the top and bottom curves happen to intersect at $x = a$ or b, then the left or right side may reduce to just a point.

The region R is given by

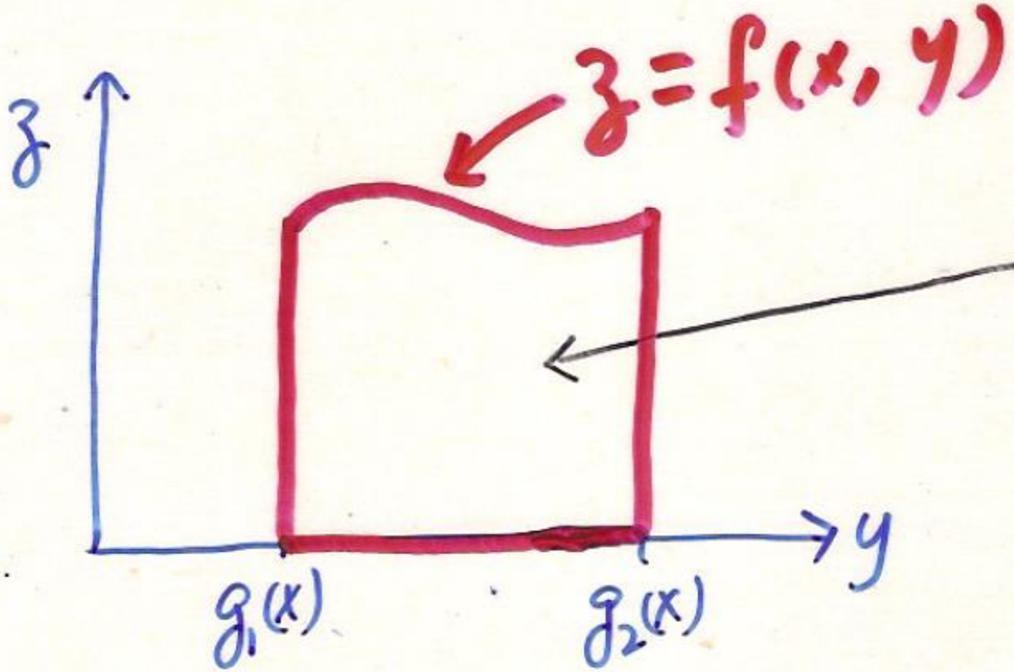
$$R : \quad g_1(x) \leq y \leq g_2(x), \quad a \leq x \leq b.$$

The cross-sectional area of the slice between x and $x + dx$ is given by

$$c(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$



z
z=f(x,y)
y
a
dx
b
y=g_2(x)
y=g_1(x)
x



$$\text{Area} = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

The total volume is the sum of the volumes of all the slices:

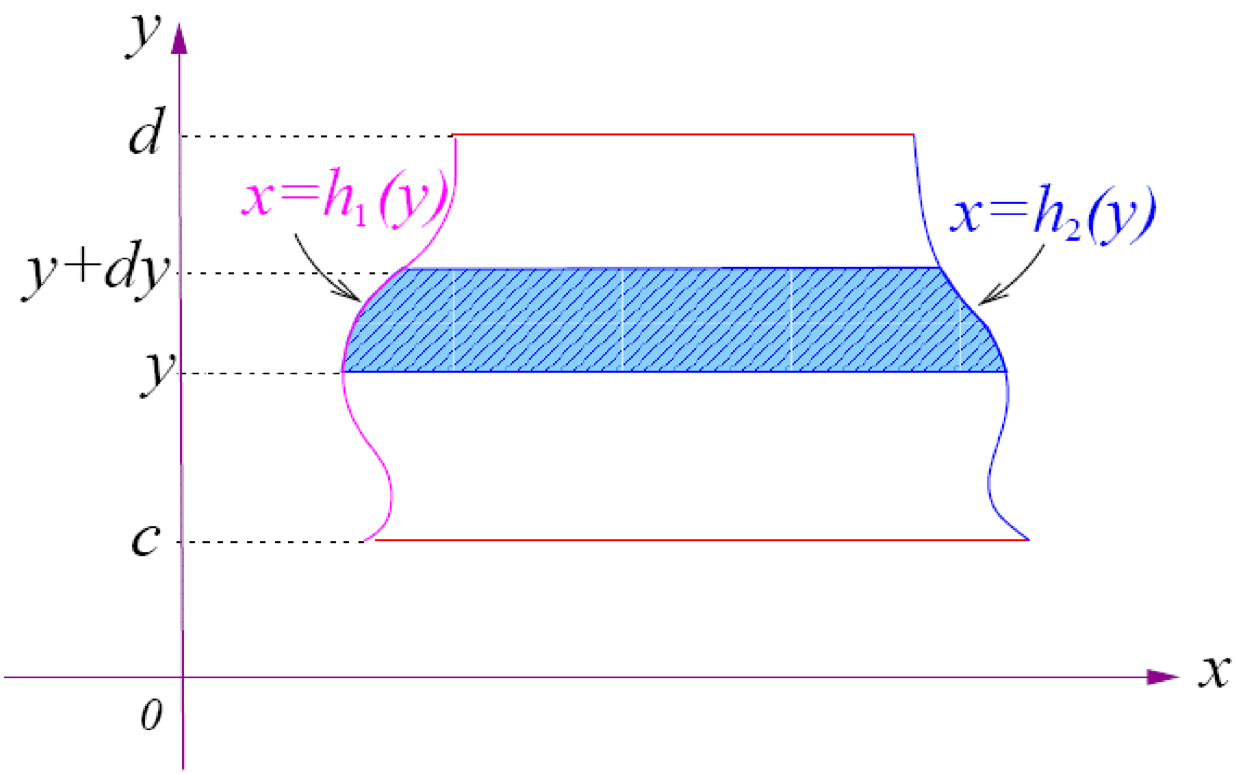
$$\int_a^b c(x)\ dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\ dy \right]\ dx.$$

In other words, the double integral of f over Type A region R can be computed by an iterated integral first w.r.t. dy followed by dx:

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\;dy\right]\;dx$$

General regions - Type B

Left and right boundaries are curves given by $x = h_1(y)$ and $x = h_2(y)$ respectively, i.e. they need not be straight lines. Bottom and top boundaries are straight lines (given by $y = c$ and $y = d$ resp.



If the left and right curves happen to intersect at $y = c$ or d, then the bottom or top side may reduce to just a point.

The region R is given by

$$R: \quad h_1(y) \leq x \leq h_2(y), \quad c \leq y \leq d.$$

The double integral of f over Type B region R can be computed by an iterated integral first w.r.t. dx followed by dy:

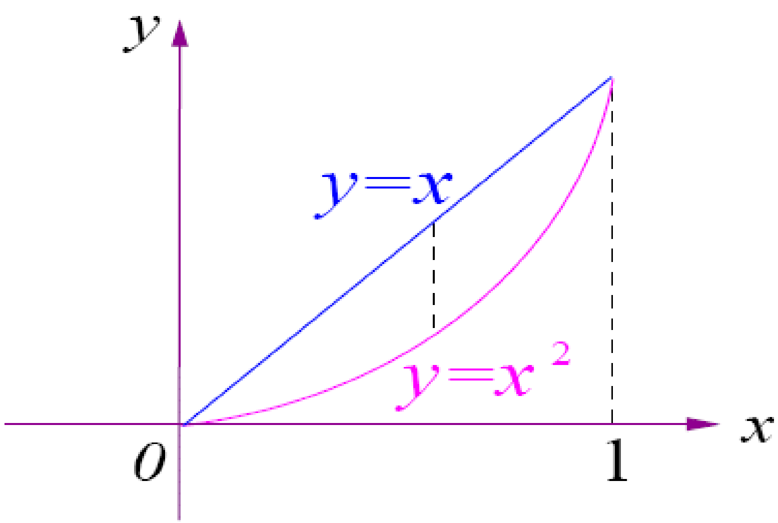
$$\iint_R f(x,y)\, dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right] dy$$

Example

If R is bounded by $y = x$ and $y = x^2$, find $\iint_R 30xy\, dA$.

Solution: Treat R as a Type A region:

$$R: \quad x^2 \leq y \leq x, \quad 0 \leq x \leq 1.$$

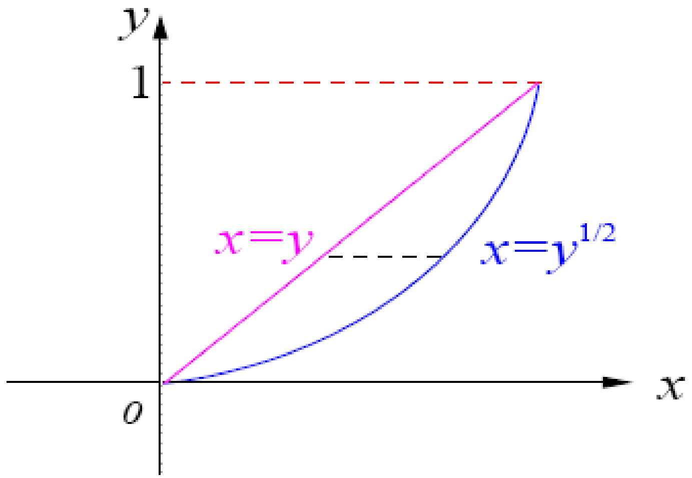


Then

$$\iint_R 30xy\,dA = \int_0^1 \left[\int_{x^2}^{x} 30xy\ dy\right]\ dx$$
$$= \int_0^1 \left[15xy^2\right]_{y=x^2}^{y=x}\ dx = \int_0^1 15x(x^2 - x^4)\ dx$$
$$= \left[\frac{15x^4}{4} - \frac{15x^6}{6}\right]_{x=0}^{x=1} = \frac{5}{4}.$$

If we treat R as a type B region:

$$R: \quad y \leq x \leq \sqrt{y}, \quad 0 \leq y \leq 1.$$



$$\iint_R 30xy\ dA = \int_0^1 \left[\int_y^{\sqrt{y}} 30xy\ dx \right]\ dy$$

$$= \int_0^1 \left[15x^2y\right]_{x=y}^{x=\sqrt{y}}\ dy = \int_0^1 \left(15y^2 - 15y^3\right)\ dy$$

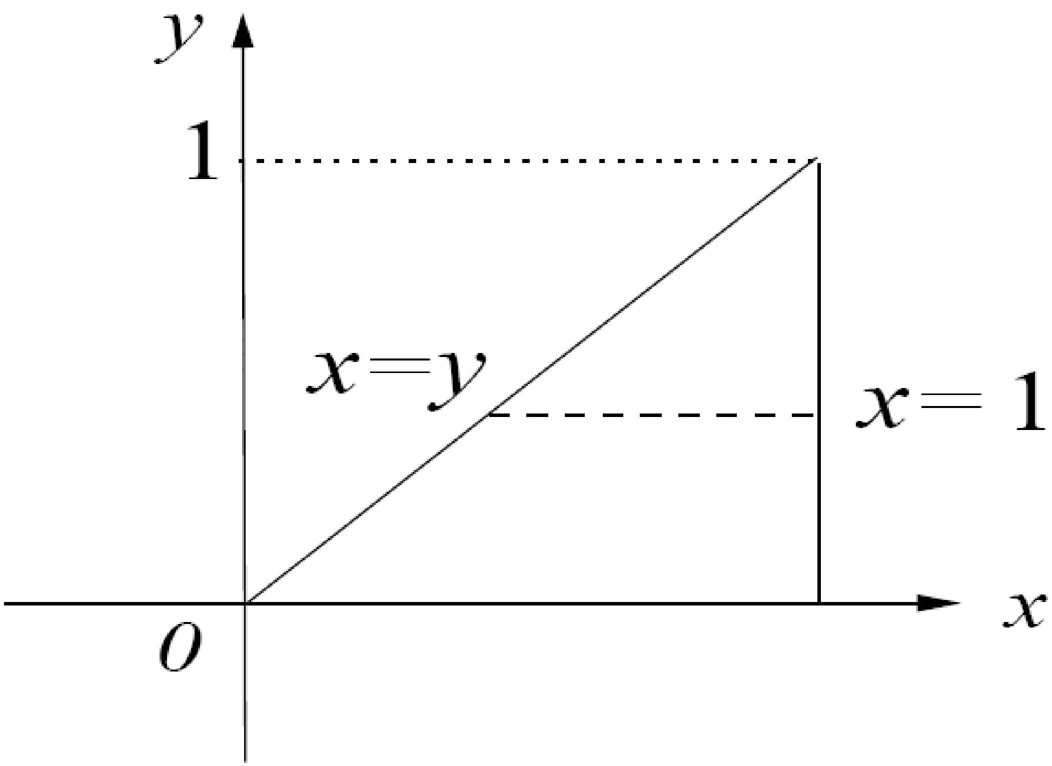
$$= \left[\frac{15y^3}{3} - \frac{15y^4}{4}\right]_{y=0}^{y=1} = \frac{5}{4}.$$

Example

Calculate $\displaystyle\iint_R \frac{\sin\ x}{x}\,dA$, where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.

Solution: If we treat R as type B:

$$R: \quad y \le x \le 1, \quad 0 \le y \le 1.$$

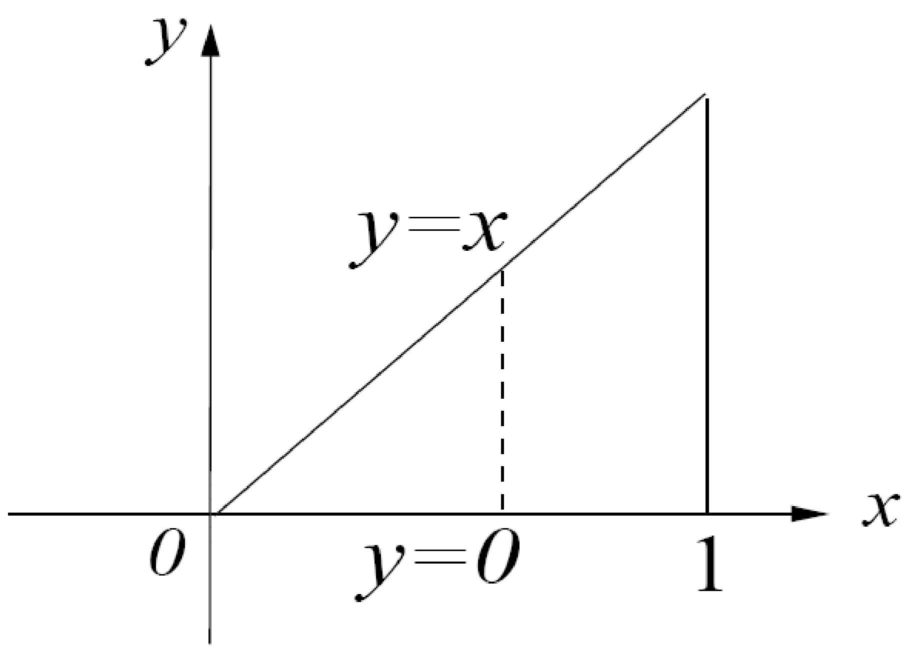


$$\iint_R \frac{\sin\ x}{x}\, dA = \int_0^1 \left[\int_y^1 \frac{\sin\ x}{x}\, dx\right]\, dy,$$

which cannot be evaluated by elementary means.

If we treat R as type A:

$$R: \quad 0 \leq y \leq x, \quad 0 \leq x \leq 1.$$



$$\iint_R \frac{\sin\ x}{x}\ dA = \int_0^1 \left[\int_0^x \frac{\sin\ x}{x}\ dy\right]\ dx$$

$$= \int_0^1 \left[y\frac{\sin\ x}{x}\right]_{y=0}^{y=x}\ dx = \int_0^1 (\sin\ x - 0)\ dx$$

$$= [-\cos\ x]_{x=0}^{x=1} = 1 - \cos\ 1.$$

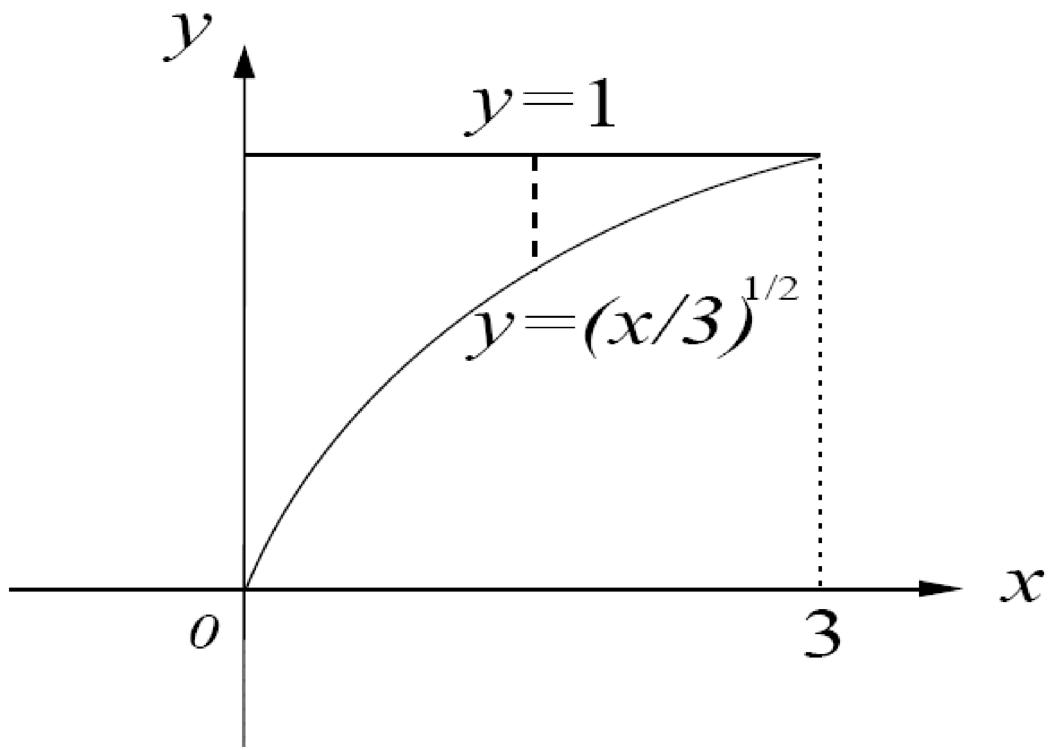
Example

Evaluate $\displaystyle\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy \;\; dx.$

Solution: R may be described as :

$$R: \quad \sqrt{\frac{x}{3}} \leq y \leq 1, \quad 0 \leq x \leq 3.$$

Hence the Type A region looks like



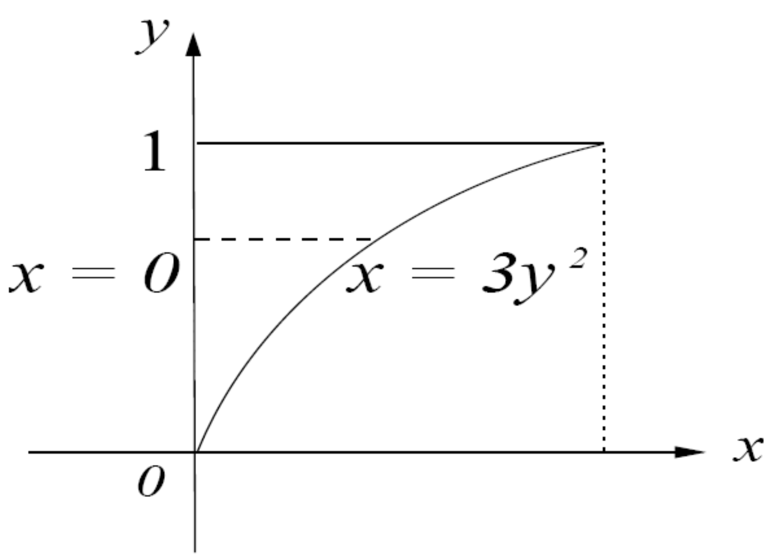
However, it is difficult to integrate e^{y^3} directly.

Now we treat R as type B:

Note that $y = \sqrt{\frac{x}{3}} \Longrightarrow x = 3y^2$.

So the region R is given by

$$R: \quad 0 \leq x \leq 3y^2, \quad 0 \leq y \leq 1.$$



$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\ dy\ \ dx = \int_0^1 \left[\int_0^{3y^2} e^{y^3}\ dx\right]\ dy$$

$$= \int_0^1 \left[xe^{y^3}\right]_{x=0}^{x=3y^2}\ dy = \int_0^1 3y^2 e^{y^3}\ dy$$

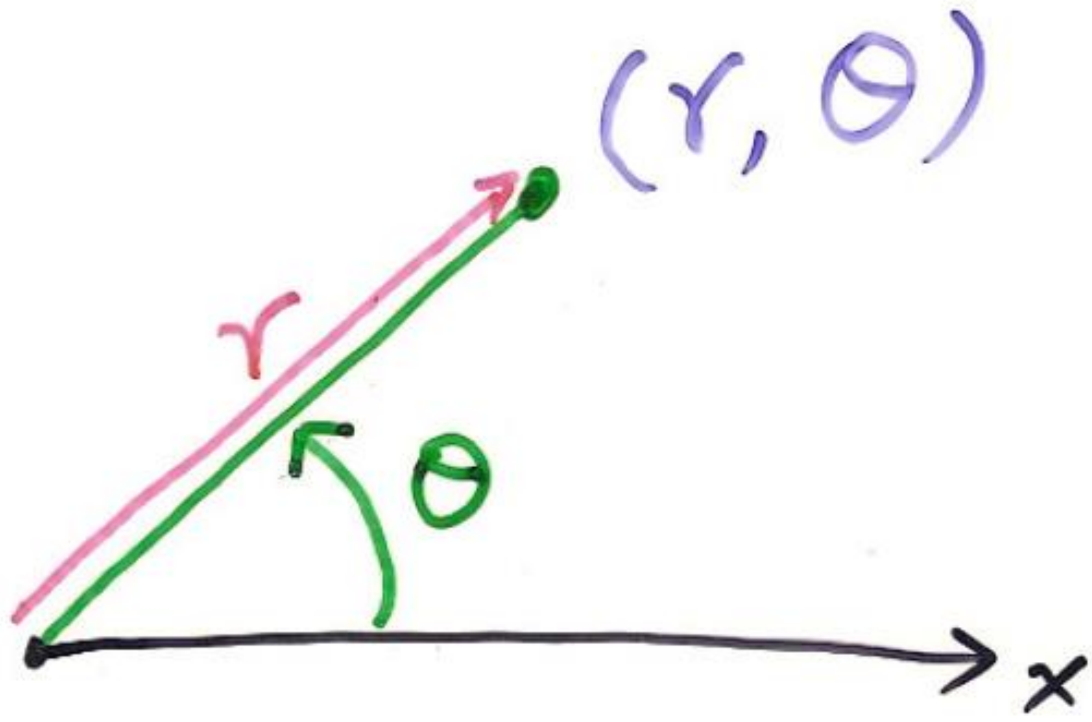
$$= \int_0^1 e^u du \qquad = [e^u]_{u=0}^{u=1} = e - 1.$$

[Here we have used a substitution $u = y^3$.]

Double integral in polar coordinates

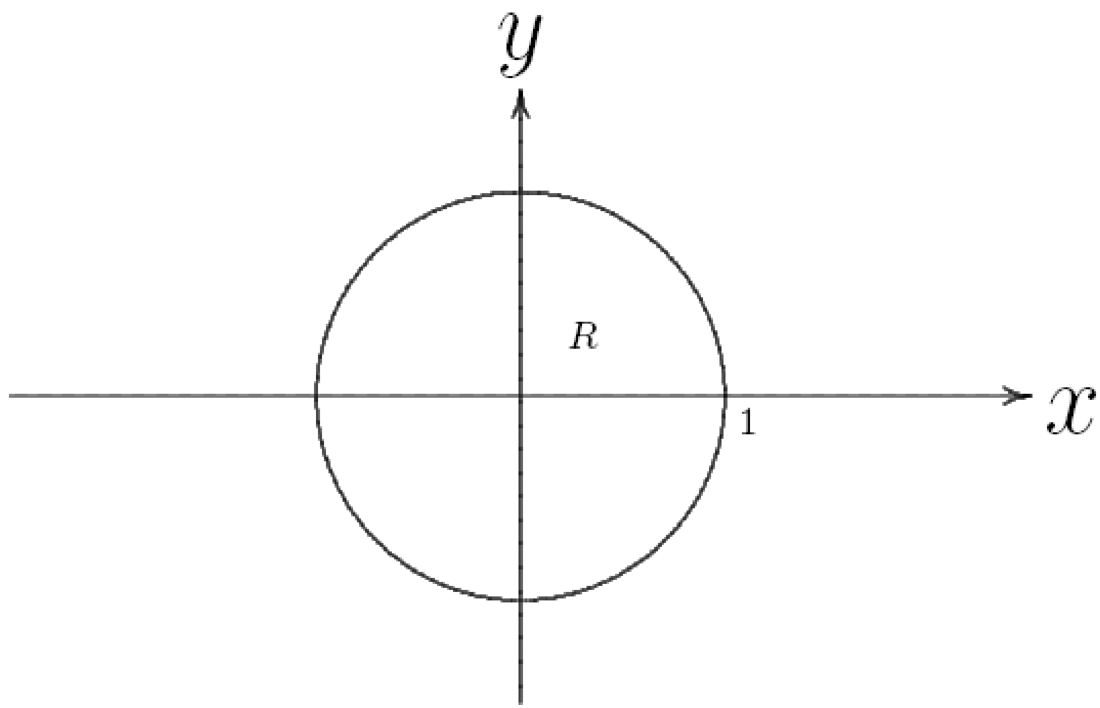
Certain regions (see examples below) can be described more simply using polar coordinates r and θ. Hence, it is more straightforward to evaluate double integrals over such regions using polar coordinates.

Instead of giving the ranges for x and y, we give the ranges for r, distance from origin to a point in the region, and θ, angle of elevation of a point from the x-axis.



(r, θ)
r
θ
x

Circle



In Cartesian coordinates, the circle can be regarded as Type A region with the upper and lower semicircles as the upper and lower boundaries. i.e.

$$R: \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \quad -1 \leq x \leq 1.$$

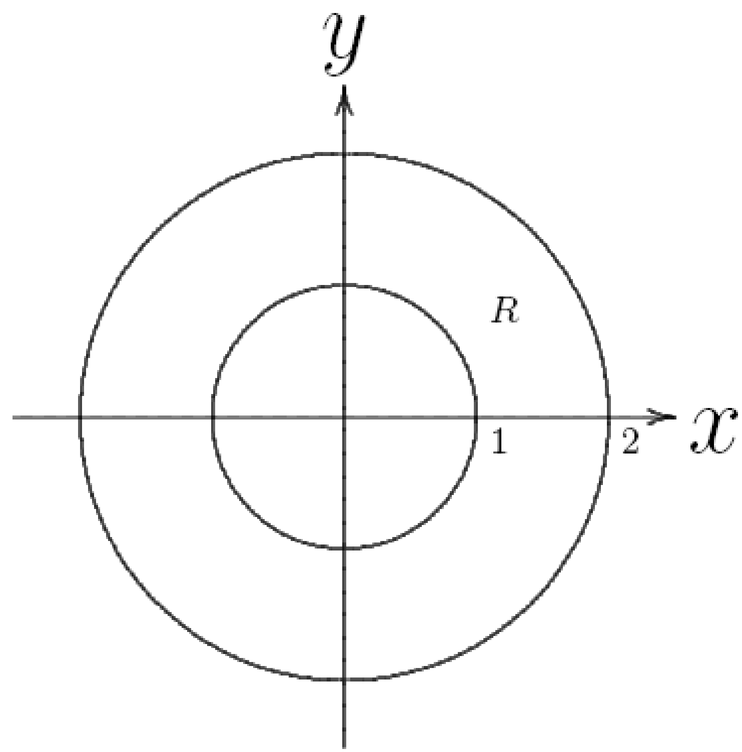
Alternatively, we can regard it as Type B region with the left and right semicircles as the left and right boundaries. i.e.

$$R: \quad -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}, \quad -1 \le y \le 1.$$

In polar coordinates, we describe the circle in terms of the ranges of r and θ:

$$R: \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

Ring



This region is neither Type A nor Type B. To use Cartesian coordinates, you need to partition the ring into smaller regions which are either Type A or Type B.

In polar coordinates, the ring is given by

$$1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

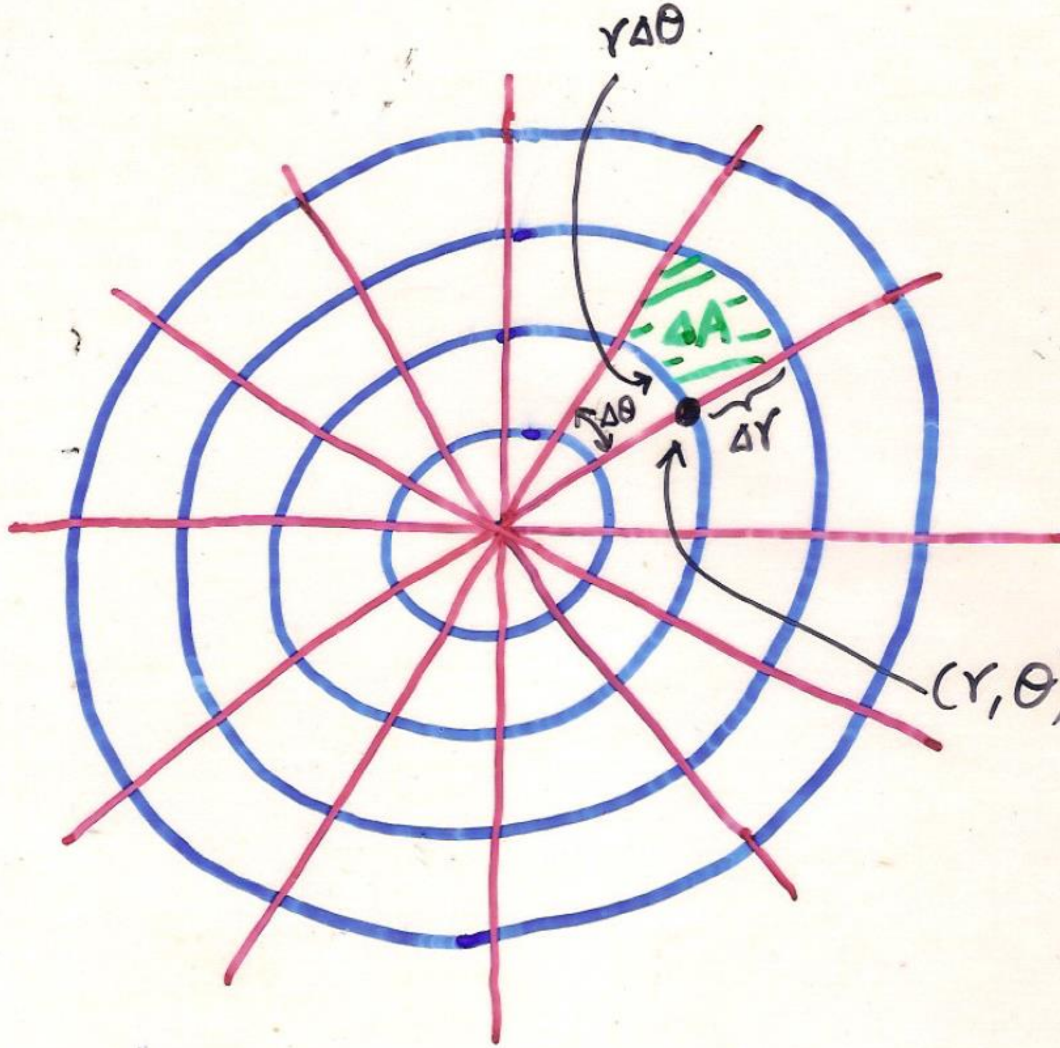
Change of variables

When we transform from the Cartesian coordinates to polar coordinates, we are performing change of variables from (x, y) to (r, θ):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

In this case, dA will be changed from $dxdy$ (or $dydx$)

to $r\ dr\ d\theta$.



$$\Delta A \simeq (r\Delta\theta)(\Delta r) = r\Delta r\Delta\theta$$

$$\underline{\underline{dA = r\,dr\,d\theta}}$$

Suppose a region R (in xy-plane) is given in polar coordinates by

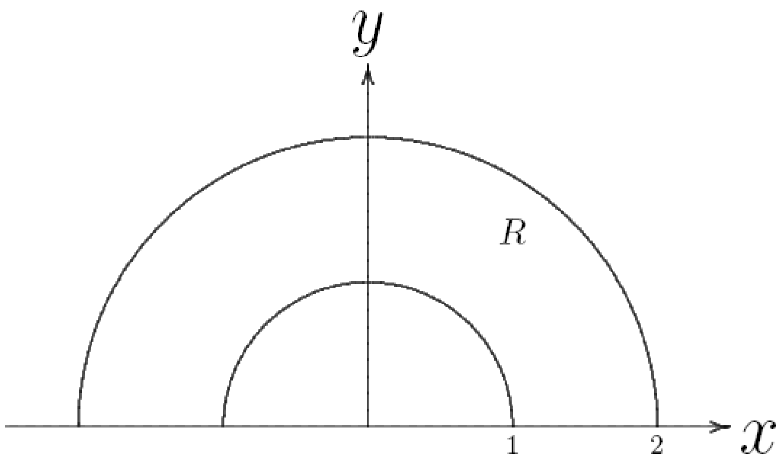
$$R: \quad a \le r \le b, \quad \alpha \le \theta \le \beta,$$

then we have

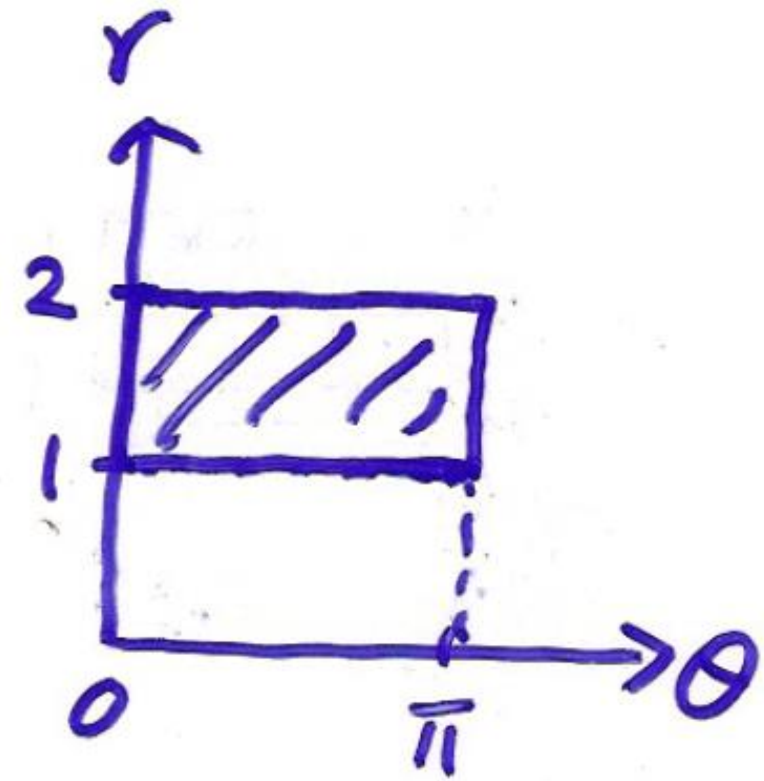
$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

Example

Evaluate $\displaystyle\iint_R (3x + 4y^2)\, dA$, where R is the semicircular ring in the upper half-plane between the semicircles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



y
R
1
2
x



r
2
1
0
π
θ

Solution: The region R is given by

$$R: \quad 1 \le r \le 2, \quad 0 \le \theta \le \pi.$$

So

$$\begin{aligned}
\iint_R (3x + 4y^2)\, dA &= \int_0^{\pi} \int_1^{2} (3r\cos\theta + 4r^2\sin^2\theta)\ r dr d\theta \\
&= \int_0^{\pi} \left[r^3\cos\theta + r^4\sin^2\theta)\right]_{r=1}^{r=2} d\theta \\
&= \int_0^{\pi} \left(7\cos\theta + 15\sin^2\theta\right) d\theta \\
&= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2}(1 - \cos 2\theta)\right) d\theta \\
&= \left[7\sin\theta + \frac{15}{2}(\theta - \frac{\sin 2\theta}{2})\right]_{\theta=0}^{\theta=\pi} \\
&= \frac{15\pi}{2}
\end{aligned}$$

Applications of Double Integrals

Volume

Suppose D is a solid region under the surface of a function $f(x, y)$ over a plane region R. Then, as we have seen in 8.1.2, the volume of D is given by

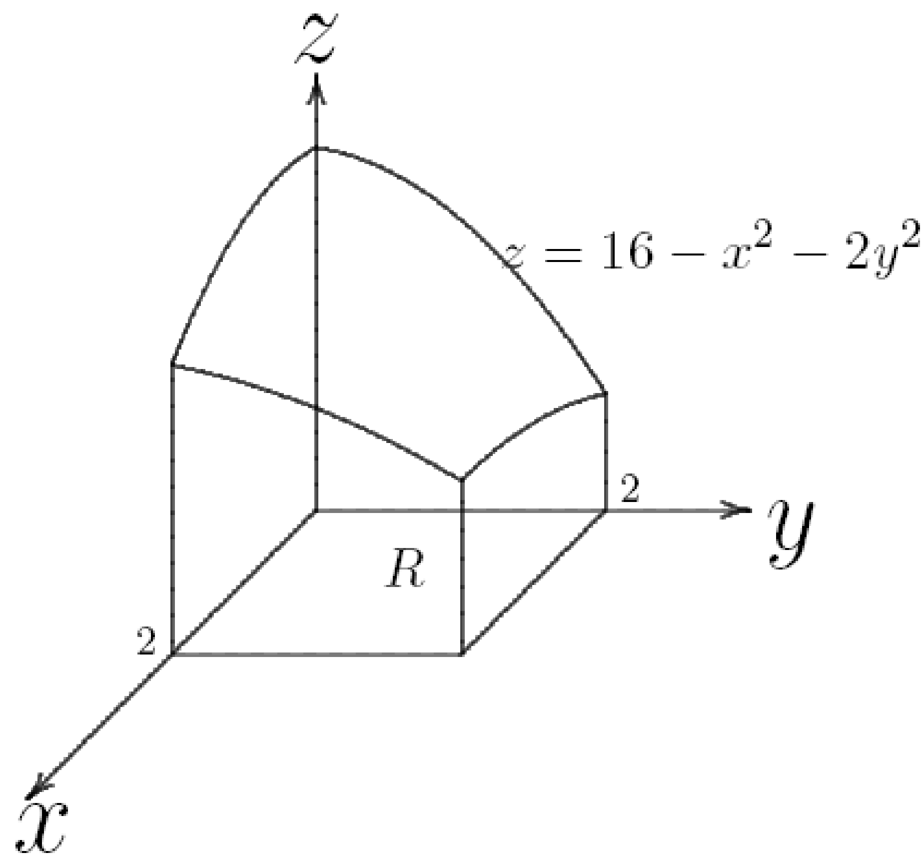
$$\iint_R f(x, y)dA.$$

Example

Find the volume of the solid D that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$, $y = 2$, and the 3-coordinate planes.

Solution: The solid region D is under the surface represented by the function $f(x, y) = 16 - x^2 - 2y^2$ and is above the rectangular region

$$R: \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 2.$$



z
$z = 16 - x^2 - 2y^2$
2
y
R
2
x

So volume of D is

$$\iint_R (16 - x^2 - 2y^2) dA$$

$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2) dx dy$$

$$= 48.$$

Surface area

If f has continuous first partial derivatives on a closed region R of the xy-plane, then the area S of that portion of the surface $z = f(x, y)$ that projects onto R is

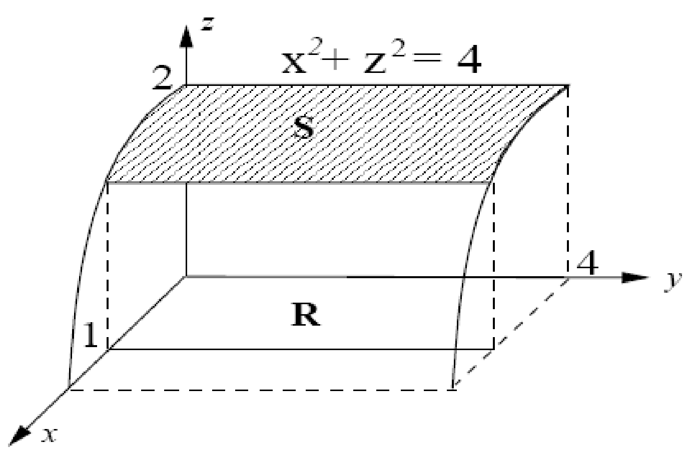
$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA.$$

Example

Find the surface area of the portion of the cylinder

$x^2 + z^2 = 4$ above the rectangle

$$R: \quad 0 \le x \le 1, \quad 0 \le y \le 4.$$



Solution: The portion of the cylinder $x^2 + z^2 = 4$ that lies above the xy-plane has the equation $z = \sqrt{4 - x^2}$.

So the surface is given by the function $f(x, y) = \sqrt{4 - x^2}$.

$$
\begin{aligned}
S &= \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA \\
&= \iint_R \sqrt{\left(-\frac{x}{\sqrt{4-x^2}}\right)^2 + 0^2 + 1}\; dA \\
&= \int_0^4 \left[\int_0^1 \frac{2}{\sqrt{4-x^2}}\, dx\right]\, dy \\
&= 2\int_0^4 \left[\sin^{-1}(x/2)\right]_{x=0}^{x=1}\, dy \\
&= 2\int_0^4 \frac{\pi}{6}\, dy = \frac{4\pi}{3}.
\end{aligned}
$$

Note that $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}(x/a) + C.$

More Examples

Example

Find the **exact value** of the surface area of that portion of the sphere $x^2 + y^2 + z^2 = 3$ that lies above the plane $z = 1$.

$$z = 1 \Rightarrow x^2 + y^2 + 1 = 3 \Rightarrow x^2 + y^2 = 2$$

$$z = \sqrt{3 - x^2 - y^2} \Rightarrow z_x = \frac{-x}{\sqrt{3 - x^2 - y^2}}, \quad z_y = \frac{-y}{\sqrt{3 - x^2 - y^2}}$$

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{\frac{3}{3 - x^2 - y^2}}$$

$$\text{Surface area} = \iint_{0 \le x^2+y^2 \le 2} \sqrt{\frac{3}{3-x^2-y^2}}\, dx\,dy$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{\frac{3}{3-r^2}}\, r\,dr\,d\theta$$

$$= 2\pi \int_0^{\sqrt{2}} \left(-\frac{\sqrt{3}}{2}\right) (3-r^2)^{-\frac{1}{2}}\, d(3-r^2)$$

$$= 2\pi \left[-\sqrt{3}\,(3-r^2)^{\frac{1}{2}}\right]_{r=0}^{r=\sqrt{2}}$$

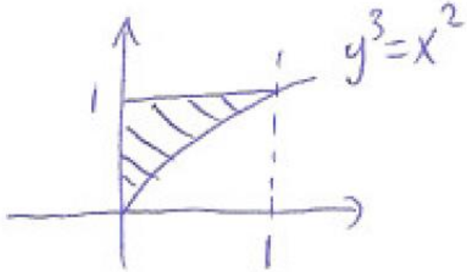
$$= 2\pi\left(-\sqrt{3}+3\right)$$

$$= \underline{\underline{(6-2\sqrt{3})\pi}}$$

Example

Find the **exact value** of the iterated integral

$$\int_0^1 \int_{x^{2/3}}^1 x \cos(y^4)\, dy\, dx.$$



$$\int_0^1\int_{x^{2/3}}^1 x\cos y^4\,dy\,dx$$

$$=\int_0^1\int_0^{y^{3/2}} x\cos y^4\,dx\,dy$$

$$=\int_0^1 \cos y^4\left[\tfrac{1}{2}x^2\right]_{x=0}^{x=y^{3/2}}dy$$

$$=\int_0^1 \tfrac{1}{2}y^3\cos y^4\,dy$$

$$=\tfrac{1}{8}\sin y^4\Big|_0^1$$

$$=\underline{\underline{\tfrac{1}{8}\sin 1}}$$

Example

The region R lies above the paraboloid

$$z = 5 - x^2 - y^2$$

and below the paraboloid

$$z = 9 - 2x^2 - 2y^2$$

Find the **exact value** of the volume of R, giving your answer in terms of π.

The two paraboloids intersect at

$$5 - x^2 - y^2 = 9 - 2x^2 - 2y^2$$

$$\Rightarrow \quad x^2 + y^2 = 4$$

$$\text{Vol of } R = \iint_{x^2+y^2\leq 4} (9-2x^2-2y^2)-(5-x^2-y^2)\,dx\,dy$$

$$= \iint_{x^2+y^2\leq 4} (4-x^2-y^2)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta$$

$$= 2\pi\left[2r^2-\frac{1}{4}r^4\right]_0^2$$

$$= \underline{\underline{8\pi}}$$