MA 1521

Tutorial 9 Solutions

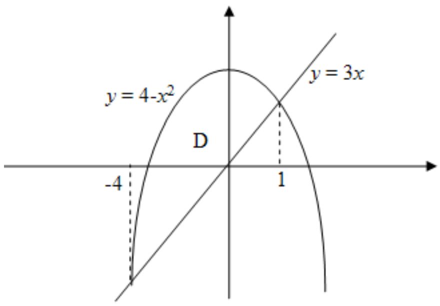
1. The volume is given by the double integral

$$V = \iint_D f(x,y)dA$$

where D is the region bounded by the parabola $y = 4 - x^2$ and straight line $y = 3x$ and $f(x,y)$ is the function whose graph is the plane $x - z + 4 = 0$.

Writing the equation of the plane as $z = x + 4$, we get the function $f(x,y) = x + 4$.

A rough sketch of the region D is shown below:



D can be regarded as type A region

$$D: \quad 3x \le y \le 4 - x^2, \quad -4 \le x \le 1.$$

(The two limits -4 and 1 of x are obtained by solving the two equation $y = 3x$ and $y = 4-x^2$.)

Hence

$$V = \int_{-4}^{1}\int_{3x}^{4-x^2}(x+4)dydx = \int_{-4}^{1}(x+4)(4-x^2-3x)dx = \left[16x - 4x^2 - \frac{7}{3}x^3 - \frac{1}{4}x^4\right]_{-4}^{1} = \frac{625}{12}$$

2. Let $z = \sqrt{2^2 - x^2 - y^2}$. Then
$z_x = -x(4 - x^2 - y^2)^{-1/2}$ and $z_y = -y(4 - x^2 - y^2)^{-1/2}$.

Substitute $z = 1$ into $x^2 + y^2 + z^2 = 4$ gives

$$x^2 + y^2 + 1 = 4 \quad \Rightarrow \quad x^2 + y^2 = 3$$

which is the equation of a circle of radius $\sqrt{3}$.

This means the plane $z = 1$ intersects the sphere at a circle of radius $\sqrt{3}$.

Hence the projected region R of the part of the sphere is a disk of radius $\sqrt{3}$.

In polar coordinates, this is given by

$$0 \le r \le \sqrt{3}, \quad 0 \le \theta \le 2\pi.$$

Thus,

$$\begin{aligned}
A(S) &= \iint\limits_R \sqrt{\frac{x^2+y^2}{4-x^2-y^2}+1}\,dA = \int_0^{2\pi}\int_0^{\sqrt{3}}\left(\frac{r^2}{4-r^2}+1\right)^{\frac{1}{2}} r\,drd\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} 2r(4-r^2)^{-\frac{1}{2}}\,drd\theta = \int_0^{2\pi} d\theta\left[-2(4-r^2)^{\frac{1}{2}}\right]_{r=0}^{r=\sqrt{3}} \\
&= (2\pi)[-2[(4-3)^{\frac{1}{2}}+2[(4)^{\frac{1}{2}}] = 4\pi.
\end{aligned}$$

3. $\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$. Therefore

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1} = \sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1} = \sqrt{2}.$$

Note that D is given as a Type A domain. The surface area is then given by

$$\int\int_D \sqrt{2}dxdy = \int_{-1}^{2}\left(\int_{x^2}^{x+2}\sqrt{2}dy\right)dx = \frac{9}{2}\sqrt{2}.$$

4. Write the equation of the saddle surface as $z = \frac{1}{a}x^2 - \frac{1}{a}y^2$, we have
$z_x = \frac{2x}{a}$ and $z_y = \frac{-2y}{a}$.
Let D denote the bounded circular region on the xy-plane bounded by the circle $x^2+y^2=a^2$.
Then the required surface area is given by
$S = \int\int_D \sqrt{1+z_x^2+z_y^2}dxdy$
$= \frac{1}{a}\int\int_D \sqrt{a^2+4x^2+4y^2}dxdy$
$= \frac{1}{a}\int_0^{2\pi}\int_0^a \sqrt{a^2+4r^2}rdrd\theta$
$= \frac{2\pi}{a}\int_0^a \sqrt{a^2+4r^2}d\left(\frac{a^2+4r^2}{8}\right)$
$= \frac{\pi}{6a}\left[\left(a^2+4r^2\right)^{\frac{3}{2}}\right]_0^a$
$= \frac{\pi a^2}{6}\left(5^{\frac{3}{2}}-1\right)$