Tutorial 7

Exercise 4

1. For each of the following $m \times n$ matrices,

   (i) find a basis for the row space and a basis for the column space;
   (ii) extend the basis for the row space in (i) to a basis for $\mathbb{R}^n$;
   (iii) extend the basis for the column space in (i) to a basis for $\mathbb{R}^m$;
   (iv) find a basis for the nullspace;
   (v) find the rank and nullity of the matrix and hence verify the Dimension Theorem for Matrices; and
   (vi) determine if the matrix has full rank.

(c) $C = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix}$, (d) $D = \begin{pmatrix} 1 & 4 & 5 & 8 \\ -1 & 4 & 3 & 0 \\ 2 & 0 & 2 & 1 \end{pmatrix}$.

9. Let $\boldsymbol{A}$ be a 3×4 matrix. Suppose that $x_1 = 1$, $x_2 = 0$, $x_3 = -1$, $x_4 = 0$ is a solution to a non-homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{b}$ and that the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ has a general solution $x_1 = t - 2s$, $x_2 = s + t$, $x_3 = s$, $x_4 = t$ where s, t are arbitrary parameters.

   (a) Find a basis for the nullspace of $\boldsymbol{A}$ and determine the nullity of $\boldsymbol{A}$.
   (b) Find a general solution for the system $\boldsymbol{Ax} = \boldsymbol{b}$.
   (c) Write down the reduced row-echelon form of $\boldsymbol{A}$.
   (d) Find a basis for the row space of $\boldsymbol{A}$ and determine the rank of $\boldsymbol{A}$.
   (e) Do we have enough information for us to find the column space of $\boldsymbol{A}$?

10. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} \end{pmatrix}$ be a 4×5 matrix such that the columns $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, $\boldsymbol{a_3}$ are linearly independent while $\boldsymbol{a_4} = \boldsymbol{a_1} - 2\boldsymbol{a_2} + \boldsymbol{a_3}$ and $\boldsymbol{a_5} = \boldsymbol{a_2} + \boldsymbol{a_3}$.

   (a) Determine the reduced row-echelon form of $\boldsymbol{A}$. (Hint: The linear relations between columns will not be changed by row operations. In this question, the fifth column of $\boldsymbol{A}$ is the sum of the second and the third columns of $\boldsymbol{A}$. Then the fifth column of the reduced row-echelon form $\boldsymbol{R}$ is still the sum of the second and the third columns of $\boldsymbol{R}$.)
   (b) Find a basis for the row space of $\boldsymbol{A}$ and a basis for the column space of $\boldsymbol{A}$.

22. Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{P}$ an $m \times m$ matrix.

   (a) If $\boldsymbol{P}$ is invertible, show that $\text{rank}(\boldsymbol{PA}) = \text{rank}(\boldsymbol{A})$.
   (b) Give an example such that $\text{rank}(\boldsymbol{PA}) < \text{rank}(\boldsymbol{A})$.

Tutorial 7 (cont.)

25. Let $\boldsymbol{A}$ be an $m \times n$ matrix.

    (a) Show that the nullspace of $\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

    (b) Show that $\text{nullity}(\boldsymbol{A}) = \text{nullity}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$ and $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$.

    (c) Is it true that $\text{nullity}(\boldsymbol{A}) = \text{nullity}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})$? Justify your answer.

    (d) Is it true that $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})$? Justify your answer.