Sections 6.1 and 6.2: Bijections

CS1231S Discrete Structures

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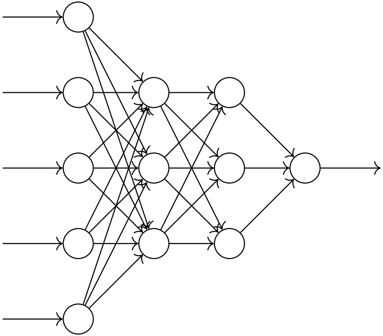


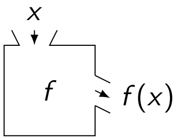
Much of the power of deep learning arises from the fact that repeated composition of multiple nonlinear functions has significant expressive power.

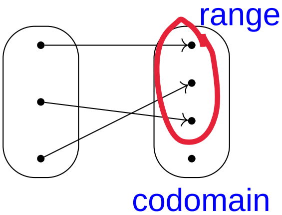
Aggarwal 2018

What we saw

- A *function* from a set A to a set B is an assignment to each element of A exactly one element of B.
- Here A is called the *domain* of f and B is called the *codomain* of f.
  all input must be this — all outputs will be in this
- The *range* of f is $\{f(x) : x \in A\}$. = set of all outputs
- $f : A \to B;$

  $x \mapsto t$





Now

- equality of functions
- function composition
- bijections
- inverse functions

Equality of functions

Definition 6.1.19

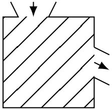
Two functions $f: A \to B$ and $g: C \to D$ are *equal* if

(1) $A = C$ and $B = D$; and same domain, same codomain

(2) $f(x) = g(x)$ for all $x \in A$. same assignment

In this case, we write $f = g$.

- same input always same output



Example 6.1.20

Let $f: \{0,2\} \to \mathbb{Z}$ and $g: \{0,2\} \to \mathbb{Z}$ defined by setting, for all $x \in \{0,2\}$,

same domain

$$f(x) = 2x \quad \text{and} \quad g(x) = x^2.$$

Then $f = g$ because their domains are the same, their codomains are the same, and $f(x) = g(x)$ for every $x \in \{0,2\}$.

Example 6.1.21

Let $f: \mathbb{Z} \to \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Q}$ defined by setting, for all $x \in \mathbb{Z}$,

$$f(x) = x^3 = g(x).$$

Then $f \neq g$ because they have different codomains.

Function composition

“g composed with f” or “g circle f”

Definition 6.1.22

Let $f\colon A \to B$ and $g\colon B \to C$. Then $g \circ f\colon A \to C$ such that for every $x \in A$,

$$(g \circ f)(x) = g(f(x)).$$

$$\begin{tikzcd}
& B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"'] & & C
\end{tikzcd}$$

Note 6.1.23

For $g \circ f$ to be well-defined, the codomain of f must equal the domain of g.

Example 6.1.24

Let $f\colon A \to B$.

(1) $(f \circ \mathrm{id}_A)(x) = f(\mathrm{id}_A(x)) = f(x)$ for all $x \in A$. So $f \circ \mathrm{id}_A = f$.

(2) $(\mathrm{id}_B \circ f)(x) = \mathrm{id}_B(f(x)) = f(x)$ for all $x \in A$. So $\mathrm{id}_B \circ f = f$.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[dr, dashed, "f"'] & A \arrow[d, "f"] \arrow[dr, dashed, "f"] & \\
& B \arrow[r, "\mathrm{id}_B"'] & B
\end{tikzcd}$$

$\mathrm{id}_A\colon A \to A;$
$x \mapsto x,$

$\mathrm{id}_B\colon B \to B;$
$y \mapsto y.$

Noncommutativity of function composition

Definition 6.1.22

Let $f\colon A \to B$ and $g\colon B \to C$. Then $g \circ f\colon A \to C$ such that for every $x \in A$,

$$(g \circ f)(x) = g(f(x)).$$

Example 6.1.25

Let $f, g\colon \mathbb{Z} \to \mathbb{Z}$ such that for every $x \in \mathbb{Z}$,

$$f(x) = 3x \quad \text{and} \quad g(x) = x + 1.$$

Then for every $x \in \mathbb{Z}$,

$$(g \circ f)(x) = g(f(x)) = g(3x) = 3x + 1 \quad \text{and}$$
$$(f \circ g)(x) = f(g(x)) = f(x + 1) = 3(x + 1).$$

Note $(g \circ f)(0) = 1 \neq 3 = (f \circ g)(0)$.

$$\begin{tikzcd}
 & x \arrow[rr, mapsto, "f"] & & 3x \arrow[dd, mapsto, "g"] \\
x \arrow[dd, mapsto, "g"'] & \mathbb{Z} \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{Z} \arrow[d, "g"] & \\
 & \mathbb{Z} \arrow[r, "f"'] & \mathbb{Z} & 3x + 1 \\
x + 1 \arrow[rr, mapsto, "f"'] & & 3(x + 1) &
\end{tikzcd}$$

just proving how order is important

Associativity of function composition

Definition 6.1.22

Let $f\colon A \to B$ and $g\colon B \to C$. Then $g \circ f\colon A \to C$ such that for every $x \in A$,

$$(g \circ f)(x) = g(f(x)).$$

Theorem 6.1.26 (associativity of function composition)

Let $f\colon A \to B$ and $g\colon B \to C$ and $h\colon C \to D$. Then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

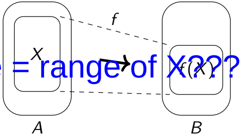
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, dashed, "g \circ f"'] & B \arrow[d, "g"] \arrow[dr, dashed, "h \circ g"] & \\
& C \arrow[r, "h"'] & D
\end{tikzcd}$$

Proof

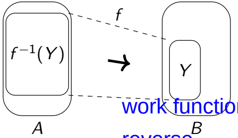
1. The domains of $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are both A.
2. The codomains of $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are both D.
3. For every $x \in A$,

$$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x))) = h((g \circ f)(x)) = (h \circ (g \circ f))(x). \qquad \square$$

Setwise image and preimage



setwise image = range of X???



work function in reverse

Definition 6.2.1

Let $f: A \to B$.

(1) If $X \subseteq A$, then let $f(X) = \{y \in B : y = f(x) \text{ for some } x \in X\} = \{f(x) : x \in X\}$.

(2) If $Y \subseteq B$, then let $f^{-1}(Y) = \{x \in A : y = f(x) \text{ for some } y \in Y\}$.

We call $f(X)$ the *image* of X, and $f^{-1}(Y)$ the *preimage* of Y under f.

reverse the function

Remark 6.2.2

If $f: A \to B$, then $f(A)$ is the range/image of f.

Ambiguity!

Example 6.2.3

Define $g: \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$.

(1) If $X = \{-1, 0, 1\}$, then $g(X) = \{g(-1), g(0), g(1)\} = \{1, 0, 1\} = \{0, 1\}$.

(2) If $Y = \{0, 1, \underline{2}\}$, then $g^{-1}(Y) = \{0, -1, 1\}$.

nothing can square to 2 so no results

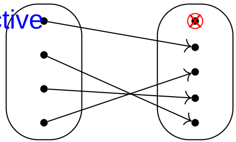
Note 6.2.4

In general, we cannot make f^{-1} operate on elements instead of subsets.

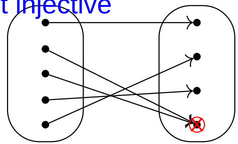
Injections and surjections

A *function* from A to B is an assignment to each element of A exactly one element of B.

not surjective



not injective



Suppose we invert the arrows in the diagrams above. Do the inverted diagrams represent functions from the right set to the left set?

- No for the left diagram, because the top dot on the right is not joined to any dot on the left.
- No for the right diagram, because the bottom dot on the right is joined to more than one dot on the left.

surjective function = *surjection*
injective function = *injection*
bijective function = *bijection*

Definition 6.2.5

Let $f : A \to B$.

for everything in the codomain

there is at least 1 in the domain pointing to it

(1) f is *surjective* or *onto* if $\forall y \in B \ \exists x \in A \ (y = f(x))$.

for everything in the codomain, there is at most one arrow pointing to it

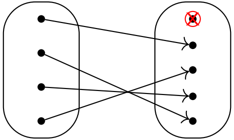
(2) f is *injective* or *one-to-one* if $\forall x, x' \in A \ (f(x) = f(x') \Rightarrow x = x')$.

(3) f is *bijective* if it is surjective and injective, i.e., $\forall y \in B \ \exists! x \in A \ (y = f(x))$.

taking conjunction, means only 1 pointing from domain to codomain

Surjectivity

Definition 6.2.5(1)



A function $f : A \to B$ is *surjective* if $\forall y \in B \ \exists x \in A \ (y = f(x))$.

Example 6.2.6

The function $f : \mathbb{Q} \to \mathbb{Q}$, defined by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$, is surjective.

Proof

1. Take any $y \in \mathbb{Q}$.
2. Let $x = (y - 1)/3$.
3. Then $x \in \mathbb{Q}$ and $f(x) = 3x + 1 = y$. $\square$

Remark 6.2.7(1)

codomain $\subseteq$ range

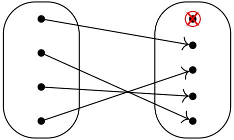
A function is surjective if and only if its codomain is equal to its range.

&&

range $\subseteq$ codomain

every element of the codomain is an output

Non-surjectivity



Definition 6.2.5(1)

A function $f\colon A \to B$ is *surjective* if $\forall y \in B\ \exists x \in A\ (y = f(x))$.

Remark 6.2.7(2)

A function $f\colon A \to B$ is *not* surjective if and only if

$$\exists y \in B\ \forall x \in A\ (y \neq f(x)).$$

Example 6.2.8

Define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not surjective.

Proof

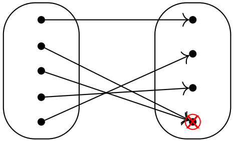
1. Note $g(x) = x^2 \geqslant 0 > -1$ for all $x \in \mathbb{Z}$.
2. So $g(x) \neq -1$ for all $x \in \mathbb{Z}$, although $-1 \in \mathbb{Z}$. $\square$

Injectivity

Definition 6.2.5(2)

A function $f: A \to B$ is *injective* if

$$\forall x, x' \in A \ (f(x) = f(x') \Rightarrow x = x').$$



Example 6.2.9

The function $f: \mathbb{Q} \to \mathbb{Q}$, defined by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$, is injective.

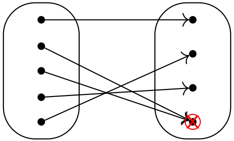
Proof

1. Let $x, x' \in \mathbb{Q}$ such that $f(x) = f(x')$.
2. Then $3x + 1 = 3x' + 1$.
3. So $x = x'$. $\square$

can change a non injective function to an injective one

- by grouping the multiple inputs as sets

Non-injectivity



Definition 6.2.5(2)

A function $f: A \to B$ is *injective* if

$$\forall x, x' \in A \ \ (f(x) = f(x') \Rightarrow x = x').$$

negation

Remark 6.2.10

A function $f: A \to B$ is *not* injective if and only if

$$\exists x, x' \in A \ \ (f(x) = f(x') \wedge x \neq x').$$

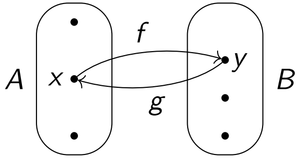
Example 6.2.11

Define $g: \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not injective.

Proof

Note $g(1) = 1^2 = 1 = (-1)^2 = g(-1)$, although $1 \neq -1$. $\square$

Inverses



Definition 6.2.13

Let $f\colon A \to B$. Then $g\colon B \to A$ is an *inverse* of f if

$$\forall x \in A\ \forall y \in B\ \big(y = f(x) \quad \Leftrightarrow \quad x = g(y)\big).$$

Example 6.2.14

Define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$. Note that for all $x, y \in \mathbb{Q}$,

$$y = 3x + 1 \quad \Leftrightarrow \quad x = (y-1)/3.$$

Let $g\colon \mathbb{Q} \to \mathbb{Q}$ such that $g(y) = (y-1)/3$ for all $y \in \mathbb{Q}$. Then the equivalence above tells us

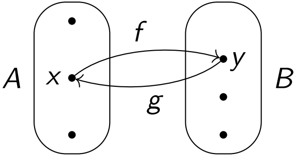
$$\forall x, y \in \mathbb{Q}\ \big(y = f(x) \quad \Leftrightarrow \quad x = g(y)\big).$$

So g is an inverse of f.

Note 6.2.15

We have no guarantee of a description of an inverse of a general function that is much different from what is given by the definitions.

Uniqueness of inverses



Definition 6.2.13

Let $f \colon A \to B$. Then $g \colon B \to A$ is an *inverse* of f if

$$\forall x \in A \ \forall y \in B \ \left(y = f(x) \quad \Leftrightarrow \quad x = g(y) \right).$$

Proposition 6.2.16 (uniqueness of inverses)

If g, g' are inverses to $f \colon A \to B$, then $g = g'$.

Proof

1. Note $g, g' \colon B \to A$.
2. Since g, g' are inverses of f, for all $x \in A$ and all $y \in B$,

$$x = g(y) \quad \Leftrightarrow \quad y = f(x) \quad \Leftrightarrow \quad x = g'(y).$$

3. So $g = g'$. same domain, same codomain, same range $\square$

Definition 6.2.17

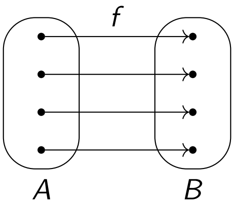
The inverse of a function f is denoted f^{-1}.

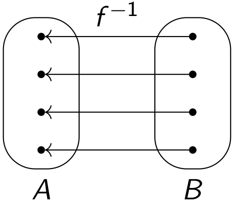
Bijectivity and invertibility

g is an inverse of $f \quad \Leftrightarrow \quad g = f^{-1}$
$\Leftrightarrow \quad \forall x \in A\ \forall y \in B\ (y = f(x) \Leftrightarrow x = g(y))$

Theorem 6.2.18

A function $f : A \to B$ is bijective if and only if it has an inverse.





Note 6.2.19

Let $f : A \to B$.

apply to set then will return a set

apply to element then will return element

(1) If $X \subseteq A$, then $f(X) = \{f(x) : x \in X\}$, which is a set. If $x \in A$, then $f(x) \in B$.

(2) If $Y \subseteq B$, then $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$, which exists even when the inverse function f^{-1} does not. If $y \in B$ and f^{-1} exists, then $f^{-1}(y) \in A$.

Bijectivity and invertibility

g is an inverse of f $\Leftrightarrow$ $g = f^{-1}$
$\Leftrightarrow$ $\forall x \in A\ \forall y \in B\ (y = f(x) \Leftrightarrow x = g(y))$

Theorem 6.2.18

A function $f\colon A \to B$ is bijective if and only if it has an inverse.

Proof

- f is *surjective* if
$$\forall y \in B\ \exists x \in A\ (y = f(x)).$$
- f is *injective* if
$$\forall x, x' \in A\ (f(x) = f(x') \Rightarrow x = x').$$

1. (“If”)
 1.1. Suppose f has an inverse, say $g\colon B \to A$.
 1.2. We first show injectivity.
  1.2.1. Let $x, x' \in A$ such that $f(x) = f(x')$.
  1.2.2. Define $y = f(x) = f(x')$.
  1.2.3. Then $x = g(y)$ and $x' = g(y)$ as g is an inverse of f.
  1.2.4. Thus $x = x'$.
 1.3. Next we show surjectivity.
  1.3.1. Let $y \in B$.
  1.3.2. Define $x = g(y)$.
  1.3.3. Then $y = f(x)$ as g is an inverse of f.
2. (“Only if”) ...

- f is *bijective* if it is both injective and surjective, i.e.,
$$\forall y \in B\ \exists! x \in A\ (y = f(x)).$$

Bijectivity and invertibility

g is an inverse of f $\Leftrightarrow$ $g = f^{-1}$
$\Leftrightarrow$ $\forall x \in A\ \forall y \in B\ (y = f(x) \Leftrightarrow x = g(y))$

Theorem 6.2.18

A function $f\colon A \to B$ is bijective if and only if it has an inverse.

Proof

- ▶ f is *surjective* if
  $$\forall y \in B\ \exists x \in A\ (y = f(x)).$$
- ▶ f is *injective* if
  $$\forall x, x' \in A\ (f(x) = f(x') \Rightarrow x = x').$$

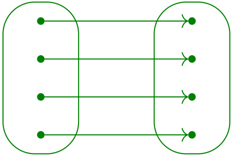
1. ("If") ...
2. ("Only if")

   2.1. Suppose f is bijective.

   2.2. Then $\forall y \in B\ \exists! x \in A\ (y = f(x))$.

   2.3. Define the function $g\colon B \to A$ by setting $g(y)$ to be the unique $x \in A$ such that $y = f(x)$ for all $y \in B$.

   2.4. This g is well-defined and is an inverse of f by the definition of inverse functions. □



- ▶ f is *bijective* if it is both injective and surjective, i.e.,
  $$\forall y \in B\ \exists! x \in A\ (y = f(x)).$$

Next

Cardinality