SC 612: Discrete Mathematics MSc(IT)-Autumn 2023

Lectures 10: Counting

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1 Combinatorics

study of arrangements of objects, is an important part of discrete mathematics.

1.1 Counting Principle

two basic counting principles,

- the product rule: useful when a procedure is made up of separate tasks
- the sum rule.

Product Rule

If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Example 1.1

How many functions are there from a set with m elements to a set with n elements?

Solution: A function corresponds to a choice of one of the n elements in the co-domain for each of the m elements in the domain. Hence, by the product rule there are $n \cdot n \cdots n = nm$ functions from a set with m elements to one with n elements.

Example 1.2

How many one-to-one functions are there from a set with m elements to one with n elements?

Solution: When $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.

When $m \leq n$ Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of the function at a_1. Because the function is one-to-one, the value of the function at a_2 can be picked in $n-1$ ways (because the value used for a_1 cannot be used again). In general, the value of the function at a_k can be chosen in $n-k+1$ ways.
By the product rule, there are $n(n-1)(n-2)\cdots(n-m+1)$ one-to-one functions from a set with m elements to one with n elements.

Example 1.3

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Solution: Let S be a finite set. List the elements of S in arbitrary order.
There is a one-to-one correspondence between subsets of S and bit strings of length $|S|$.

Namely, a subset of S is associated with the bit string with a

- 1 in the ith position if the ith element in the list is in the subset, and
- 0 in this position otherwise.

By the product rule, there are $2^{|S|}$ bit strings of length $|S|$. Hence, $|P(S)| = 2^{|S|}$.

Sum Rule

If a task can be done either in one of n_1 ways or in one of n_2 ways (distinct from the n_1 ways), then there are $n_1 + n_2$ ways to do the task.

✔ Example 1.4

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution: The student can choose a project by selecting a project from the first list, the second list, or the third list.
Because no project is on more than one list, by the sum rule there are 23 + 15 + 19 = 57 ways to choose a project.

Subtraction Rule

If a task can be done in either n_1 ways or n_2 ways (some of the ways to do it are common to both ways), then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Note

The subtraction rule is also known as the principle of inclusion-exclusion, when it is used to count the number of elements in the union of two sets.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Division Rule

There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

Note

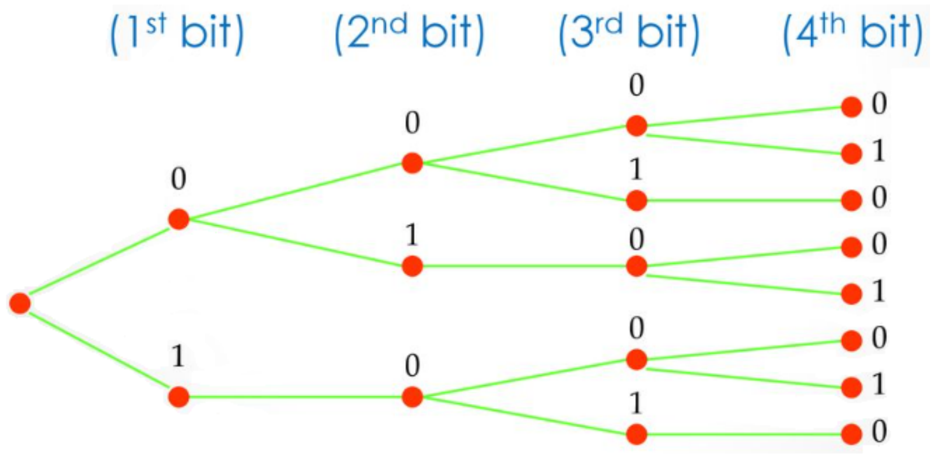
- In terms of sets: "If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$."
- In terms of functions: "Let A and B are finite sets. If $f : A \to B$, and for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$."

Tree Diagrams

Counting problems can be solved using tree diagrams.

✔ Example 1.5

How many bit strings of length four do not have two consecutive 1s?

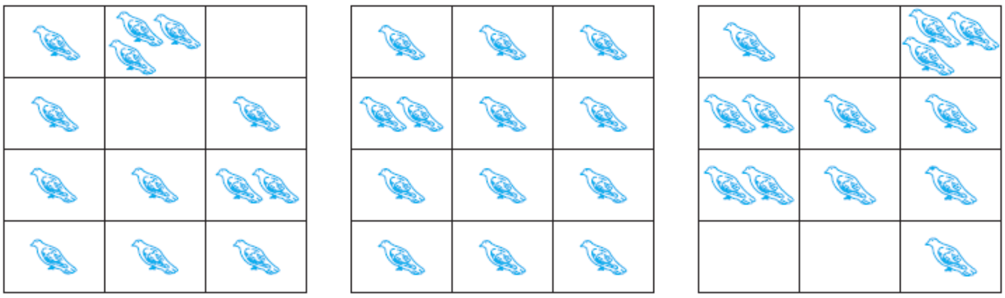


1.2 The Pigeonhole Principle

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Note

The pigeonhole principle is also called the Dirichlet drawer principle.



1.3 Permutations and Combinations

Both permutations and combinations deal with arrangement of objects.

- If the order in the arrangement is a matter, then it is a permutation else combination.

Permutations

A permutation of a set of distinct objects is an ordered arrangement of these objects.

- An ordered arrangement of r elements of a set is called an r-permutation.
- The number of r-permutations of a set with n elements is denoted by $P(n, r)$ and can be found using the product rule.

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

- If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$

✔ Example 1.6

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution: Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is P(100, 3) = 100 · 99 · 98 = 970,200.

✔ Example 1.7

How many permutations of the letters ABCDEFGH contain the string ABC ?

Solution: Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G, and H. Because these six objects can occur in any order, there are 6! = 720 permutations of the letters ABCDEFGH in which ABC occurs as a block.

✔ Example 1.8

How many bit strings of length 12 contain

- a) exactly three 1s?
- b) at most three 1s?
- c) at least three 1s?
- d) an equal number of 0s and 1s?

Solution:

- a) To specify a bit string of length 12 that contains exactly three 1s, we simply need to choose the three positions that contain the 1s. There are $C(12, 3) = 220$ ways to do that.
- b) To contain at most three 1s means to contain three 1s, two 1s, one 1, or no 1s.

$C(12, 3) + C(12, 2) + C(12, 1) + C(12, 0) = 220 + 66 + 12 + 1 = 299.$

- c) To contain at least three 1s means to contain three 1s, four 1s, five 1s, six 1s, seven 1s, eight 1s, nine 1s, 10 1s, 11 1s, or 12 1s. This approach contains too many numbers to add.

A simpler approach would be to figure out the number of ways not to have at least three 1s (i.e., to have two 1s, one 1, or no 1s) and then subtract that from 2^{12} (total number of bit strings of length 12).

This way we get $4096 - (66 + 12 + 1) = 4017.$

- d) To have an equal number of 0s and 1s in this case means to have six 1s.

Therefore the answer is $C(12, 6) = 924.$

Combinations

A combination of a set of objects is an unordered selection of these objects.

- An unordered selection of r elements of a set is called an r-combination.
- The number of r-combinations of a set with n distinct elements is denoted by $C(n,r)$ or $\binom{n}{r}$ and is called a Binomial coefficient.

If n is a non-negative integer and r is an integer with $0 \leq r \leq n$, then the number of r-combinations of a set with n elements is

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

- $P(n,r) = C(n,r)r!$
- Let n and r be non-negative integers with $r \leq n$, then $C(n,r) = C(n,n-r)$

✔ Example 1.9

How many ways are there to pick 3 persons from a group of 6?

Solution: $C(6,3) = 20$.

combinatorial proof

A combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity.

- Double counting proof : is a combinatorial proof of an identity that uses counting arguments to prove that both sides of the identity count the same objects but in different ways
- Bijective proof: is a combinatorial proof of an identity that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity.

Permutations with Repetition The number of r-permutations of a set of n objects with repetition allowed is n^r .

Combinations with Repetition The number of r-combinations from a set of n objects with repetition allowed is $C(n+r-1,r)$.

Permutations with Indistinguishable Objects The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type $2, \ldots$, and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Proof: To determine the number of permutations, first note that

- the n_1 objects of type one can be placed among the n positions in $C(n,n_1)$ ways, leaving

$n - n_1$ positions free.

• Then the objects of type two can be placed in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions free.

• Continue placing the objects of type three, . . ., type $k - 1$, until at the last stage,

• n_k objects of type k can be placed in $C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$ ways.

Hence, by the product rule, the total number of different permutations is

$$\begin{aligned} & C(n, n_1)\, C(n - n_1, n_2) \cdots C(n - n_1 - \cdots - n_{k-1}, n_k) \\ &= \frac{n!}{n_1!\,(n - n_1)!} \frac{(n - n_1)!}{n_2!\,(n - n_1 - n_2)!} \cdots \frac{(n - n_1 - \cdots - n_{k-1})!}{n_k!0!} \\ &= \frac{n!}{n_1! n_2! \cdots n_k!}. \end{aligned}$$

✔ Example 1.10

How many different strings can be made by reordering the letters of the word SUCCESS?

Solution: Because some of the letters of SUCCESS are the same, the answer is not given by the number of permutations of seven letters.

• This word contains three Ss, two Cs, one U, and one E.

• To determine the number of different strings that can be made by reordering the letters, first note that the

• three Ss can be placed among the seven positions in C(7, 3) different ways, leaving four positions free.

• Then the two Cs can be placed in C(4, 2) ways, leaving two free positions.

• The U can be placed in C(2, 1) ways, leaving just one position free.

• Hence E can be placed in C(1, 1) way.

Consequently, from the product rule, the number of different strings that can be made is C(7, 3)C(4, 2)C(2, 1)C(1, 1) = 420 .

Distributing Objects into Boxes Many counting problems can be solved by enumerating the ways objects can be placed into boxes (where the order these objects are placed into the boxes does not matter).

• objects can be either distinguishable (labeled), or indistinguishable (identical/unlabeled).

• Boxes can be either distinguishable, or indinguishable

• When we solve a counting problem using the model of distributing objects into boxes, we need to determine whether the objects are distinguishable and whether the boxes are distinguishable.

Both distinguishable The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box $i, i = 1, 2, ..., k$, equals

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

Example 1.11

How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Solution: We will use the product rule to solve this problem.

- The first player can be dealt 5 cards in C(52, 5) ways.
- The second player can be dealt 5 cards in C(47, 5) ways, because only 47 cards are left.
- The third player can be dealt 5 cards in C(42, 5) ways.
- Finally, the fourth player can be dealt 5 cards in C(37, 5) ways.

Hence, the total number of ways to deal four players 5 cards each is

$$C(52,5)C(47,5)C(42,5)C(37,5) = \frac{52!}{47!5!}\cdot\frac{47!}{42!5!}\cdot\frac{42!}{37!5!}\cdot\frac{37!}{32!5!} = \frac{52!}{5!5!5!5!32!}.$$

Objects indistinguishable and Boxes distinguishable Counting the number of ways of placing n indistinguishable objects into k distinguishable boxes turns out to be the same as counting the number of n-combinations for a set with k elements when repetitions are allowed.

Example 1.12

How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?

Solution: The number of ways to place 10 indistinguishable balls into eight bins equals the number of 10-combinations from a set with eight elements when repetition is allowed. $C(8+10-1,10) = C(17,10) = \frac{17!}{10!7!} = 19,448.$

Objects distinguishable and Boxes indistinguishable There is no simple closed formula for the number of ways to distribute n distinguishable objects into j indistinguishable boxes.

- However, there is a formula involving a summation of Stirling numbers of the second kind, $S(n,j)$.
- $S(n,j)$ denote the number of ways to distribute n distinguishable objects into j indistinguishable boxes, so that no box is empty. $S(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$
- The number of ways to distribute n distinguishable objects into k indistinguishable boxes equals

$$\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k}\frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$$

Both indistinguishable Distributing n indistinguishable objects into k indistinguishable boxes is the same as writing n as the sum of at most k positive integers in non-increasing order.

- If $a_1+a_2+\cdots+a_j = n$, where $a_1,a_2,\ldots,a_j$ are positive integers with $a_1\geq a_2\geq\cdots\geq a_j$, we say that $a_1,a_2,\ldots,a_j$ is a partition of the positive integer n into j positive integers.
- if $p_k(n)$ is the number of partitions of n into at most k positive integers, then there are

$$p_k(n)$$

ways to distribute n indistinguishable objects into k indistinguishable boxes.

- No simple closed formula exists for this number.

1.4 The Binomial Theorem

Theorem 1.1

Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof: We use a combinatorial proof. The terms in the product when it is expanded are of the form $x^{n-j}y^j$ for $j = 0, 1, 2, \ldots, n$. To count the number of terms of the form $x^{n-j}y^j$, note that to obtain such a term it is necessary to choose $n - jx$ from the n sums (so that the other j terms in the product are y s). Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$, which is equal to $\binom{n}{j}$.

Example 1.13

What is the expansion of $(x+y)^4$?

Solution: From the binomial theorem it follows that

$$\begin{aligned}
(x+y)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j \\
&= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \\
&= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4
\end{aligned}$$

Example 1.14

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Solution: From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!12!} = 5,200,300$$

Example 1.15

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Solution: First, note that this expression equals $(2x + (-3y))^{25}$. By the binomial theorem, we have

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$$

The coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$, namely, $\binom{25}{13} 2^{12} (-3)^{13}$.

Some results

- Let n be a nonnegative integer. Then $\sum_{k=0} \binom{n}{k} = 2^n$
- Let n be a positive integer. Then $\sum_{k=0} (-1)^k \binom{n}{k} = 0$
- Let n be a nonnegative integer. Then $\sum_{k=0} 2^k \binom{n}{k} = 3^n$
- Let n and k be a positive integers with $n \geq k$. Then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$
  This is called Pascal's identity. It is the basis for a geometric arrangement of the binomial coefficients in a triangle, known as Pascal's triangle which shows that when two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\ldots
\end{array}
$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\ldots
\end{array}
$$

- Let m, n and r be a nonnegative integers with r not exceeding either m or n. Then $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$
  This is called Vandermonde's identity.
- If n is a nonnegative integers, then $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$
  Using Vandermonde's identity with $m = n = r$.

Theorem 1.2

If $-1 < x < 1$ and n has any value

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \ldots$$

$*$ If n is in fact a positive integer, the coefficients, after the coefficient of x^n, are all zero since they each contain the factor $(n - n)$. Thus for this case we see that the expansion terminates with x^n. For this particular case, the requirement that $-1 < x < 1$ is not necessary.

$*$ If n has a value other than a positive integer the expansion will not terminate and the requirement $-1 < x < 1$ is absolutely essential.

$*$ Although the expansion does not terminate in this case, it can still be used with great effect to approximate the value of $(1 + x)^n$ when $|x| < 1$.

Example 1.16

Obtain the first five terms in the expansion of $(1+x)^{1/2}$. Hence evaluate $\sqrt{1.03}$ to 5 significant figures.

For this case $n = \frac{1}{2}$ and from Theorem 1.2

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}x^3 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3}{1\cdot 2\cdot 3\cdot 4}x^4 \dots$$
$$1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots$$

$\mathrm{h}x = 0.03$ which is certainly between -1 and +1 ,

$$1.03)^{1/2} = \sqrt{1.03} = 1 + \frac{1}{2}(0.03) - \frac{1}{8}(0.0009) + \frac{1}{10}(0.000027)\dots$$

since we only require the result to 4 decimal places, we need in only consider the first three terms.

$$\begin{aligned}(1.03)^{1/2} &= 1 + 0.015 - 0.0001125 + 0.0000017 + \dots \\ &\simeq 1.0148892 \\ \sqrt{1.03} &= 1.0149 \quad \text{correct to } 5 \text{ significant figures.}\end{aligned}$$