SC 612: Discrete Mathematics MSc(IT)-Autumn 2023

Lecture 4: Rules of Inference

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1 Rules of Inference in Propositional logic

Definition 1.1: Argument

An argument in propositional logic is a sequence of propositions.
- *Premises: All but the final proposition in the argument are called premises.*
- *Conclusion: the final proposition.*
- *Valid argument: An argument is valid if the truth of all its premises implies that the conclusion is true.*

• Argument form: An argument form in propositional logic is a sequence of compound propositions involving propositional variables.
• An argument form is valid no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

Example. *Check whether this argument is valid?*
"If you have a current password, then you can log onto the network."
"You have a current password."
Therefore,
"You can log onto the network."

Solution • Identify the premises and conclusion in the argument propositions

Premises $\begin{cases} \text{“If you have a current password, then you can log onto the network.”} \\ \text{“You have a current password.”} \end{cases}$

Therefore,

Conclusion { "You can log onto the network."
• Convert the argument to argument form (using propositional variables)
p:= "You have a current password"
q:= "You can log onto the network."
$p \to q$:= "If you have a current password, then you can log onto the network." as
Then, the argument has the form

$$\begin{array}{ll} 1. & p \to q \\ 2. & p \\ \hline \therefore & q \end{array}$$

• This form of argument is valid if all its premises are true, then the conclusion must also be true.
To check, $((p \to q) \land p) \to q$ is a tautology.

Remark 1.1. • We can always use a truth table to show that an argument form is valid.
• Do this by showing that whenever the premises are true, the conclusion must also be true.

Table 1: Truth table for $((p \to q) \wedge p) \to q$

p	q	$p \to q$	$((p \to q) \wedge p)$	$((p \to q) \wedge p) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

• However, this can be a tedious approach. For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid requires $2^{10} = 1024$ different rows.
• We need a simple approach than the truth tables, called rules of inference.

1.1 Rules of Inference for propositions

• The simple argument forms whose validity is established.
• These rules of inference can be used as building blocks to construct more complicated valid argument forms.
• Some of the most important rules of inference in propositional logic are

Definition 1.2: Modus ponens/Law of detachment

• *Statement: Modus ponens is the tautology* $(p \wedge (p \to q)) \to q$
• *This tautology leads to the following valid argument form,*

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

• *The hypotheses are written in a column, followed by a horizontal bar, followed by a line that begins with the therefore symbol and ends with the conclusion.*
• *Modus ponens tells us that if a conditional statement and the hypothesis of this conditional statement are both true, then the conclusion must also be true.*

Example. *Suppose that the conditional statement*
"If it snows today, then we will go skiing" *and its hypothesis,*
"It is snowing today," *are true.*
$\therefore$ *"We will go skiing," is true. (by modus ponens)*

• A valid argument can lead to an incorrect conclusion if one or more of its premises is false.

Example. *Determine the validity of the argument.*

"If $\sqrt{2} > \frac{3}{2}$, *then* $(\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$. *We know that* $\sqrt{2} > \frac{3}{2}$. *Consequently,* $(\sqrt{2})^2 = 2 > \left(\frac{3}{2}\right)^2 = \frac{9}{4}$*,"*

Solution: Let p:= " $\sqrt{2} > \frac{3}{2}$ " and q:=" $2 > \left(\frac{3}{2}\right)^2$."
• The premises of the argument are $p \to q$ and p, and q is its conclusion.
• This argument is in modus ponens form.
• However, one of its premises, $\sqrt{2} > \frac{3}{2}$, is false.

• we cannot conclude that the conclusion is true. Furthermore, note that the conclusion of this argument is false, because $2 < \frac{9}{4}$.

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Example. *Show that the premises*
"It is not sunny this afternoon and it is colder than yesterday,"
"We will go swimming only if it is sunny,"
"If we do not go swimming, then we will take a canoe trip," and
"If we take a canoe trip, then we will be home by sunset" lead to the conclusion
"We will be home by sunset."

Solution: Define the proposition
p:= "It is sunny this afternoon,"
q:= "It is colder than yesterday,"
r:= "We will go swimming,"
s:= "We will take a canoe trip," and
t:= "We will be home by sunset."
• Then the premises become $\neg p \land q, r \to p, \neg r \to s$, and $s \to t$.
• The conclusion is simply t.
• We need to give a valid argument with premises $\neg p \land q, \vec{r} \to p, \neg r \to s$, and $\vec{s} \to t$ and conclusion t.
• Construction of an argument to check validity.

Step	Reason
1. $\neg p \land q$	Premise
2. $\neg p$	by Simplification rule
3. $r \to p$	Premise
4. $\neg r$	by Modus tollens on (2) and (3)
5. $\neg r \to s$	Premise
6. s	by Modus ponens on (4) and (5)
7. $s \to t$	Premise
8. t	by Modus ponens on (6) and (7)

• Note: We can use a truth table to show the validity. But the table would have 2^5=32 rows.

Fallacies Fallacies arise due to incorrect arguments.
• Fallacies resemble rules of inference, but are based on contingencies rather than tautologies.

Definition 1.3: Fallacy of affirming the conclusion

It is a type of invalid reasoning.
• *The proposition $((p \to q) \land q) \to p$ is not a tautology, because it is false when p is false and q is true. However, there are many incorrect arguments that treat this as a tautology.*
• *The argument with premises $p \to q$ and q and conclusion p is an invalid argument.*

Example. *Is the following argument valid?*
If you do every problem in this book, then you will learn discrete mathematics.
You learned discrete mathematics.
Therefore, you did every problem in this book.

Solution: p:= "You did every problem in this book."
q:= "You learned discrete mathematics."
• Then this argument is of the form:
if $p \to q$ and q, then p.
• This is an example of an incorrect argument using the fallacy of affirming the conclusion.

Definition 1.4: Fallacy of denying the hypothesis

It is a type of invalid reasoning.
• *The proposition $((p \to q) \land \neg p) \to \neg q$ is not a tautology, because it is false when p is false and q is true.*
• *The argument with premises $p \to q$ and $\neg p$ and conclusion $\neg q$ is an invalid argument.*

Example. *Is the following argument valid?*
If you do every problem in this book, then you will learn discrete mathematics.
You learned discrete mathematics.
Therefore, you did every problem in this book.

Solution: p:= "You did every problem in this book."
q:= "You learned discrete mathematics."
• If the conditional statement $p \to q$ is true and $\neg p$ is true,
is it correct to conclude that $\neg q$ is true?
• This is an example of an incorrect argument using the fallacy of denying the hypothesis.

1.2 Rules of Inference for Quantified Statements

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

- Universal instantiation is the rule of inference used to conclude that P(c) is true, where c is a particular member of the domain, given the premise ∀xP(x).
  - e.g., “All women are wise” that “Lisa is wise,” where Lisa is a member of the domain of all women.
- Universal generalization is the rule of inference that states that ∀xP(x) is true, given the premise that P(c) is true for all elements c in the domain.
  - Universal generalization is used when we show that ∀xP(x) is true by taking an arbitrary element c from the domain and showing that P(c) is true.
- Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which P(c) is true if we know that ∃xP(x) is true.
  - We cannot select an arbitrary value of c here, but rather it must be a c for which P(c) is true.
- Existential generalization is the rule of inference that is used to conclude that ∃xP(x) is true when a particular element c with P(c) true is known.
  - if we know one element c in the domain for which P(c) is true, then we know that ∃xP(x) is true.

Example. *Show that the promises ”A student in this class has not read the book,” and ”Everyone in this class passed the first exam” imply the conclusion ”Someone who passed the first exam has not read the book.”*

Solution: Let C(x):= “x is in this class,” B(x):= “x has read the book,” and P(x) := “x passed the first exam.”
• The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \to P(x))$.
• The conclusion is $\exists x(P(x) \wedge \neg B(x))$.
These steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $(C(a) \wedge \neg B(a))$	by existential instantiation on (1)
3. $C(a)$	by simplification on (2)
4. $\forall x(C(x) \to P(x))$	Premise
5. $C(a) \to P(a)$	by universal instantiation on (4)
6. $P(a)$	by Modus ponens on (3)& (5)
7. $\neg B(a)$	by simplification on (2)
8. $P(a) \wedge \neg B(a)$	by conjunction on (6) & (7)
9. $\exists x(P(x) \wedge \neg B(x))$	by existential instantiation on (7) & (8)

universal modus ponens This rule of inference is the combination of universal instantiation and modus ponens.
• This rule tells us that
if $\forall x(P(x) \to Q(x))$ is true, and
if $P(a)$ is true for a particular element a in the domain of the universal quantifier,
then $Q(a)$ must also be true.
• By universal instantiation, $P(a) \to Q(a)$ is true.
• Then, by modus penens, $Q(a)$ must also be true.

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ P(a), \text{where } a \text{ is a particular element in the domain} \\ \hline \therefore Q(a) \end{array}$$

Example. *Assume that "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n is true. Use universal modus ponens to show that $100^2 < 2^{100}$.*

Solution: Let $P(n) :=$ "$n > 4$" and $Q(n) :=$ "$n^2 < 2^n$."
• The statement "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" can be represented by $\forall n(P(n) \to Q(n))$, where the domain consists of all positive integers.
• We are assuming that $\forall n(P(n) \to Q(n))$ is true.
• Note that $P(100)$ is true because $100 > 4$.
• It follows by universal modus ponens that $Q(100)$ is true, i.e., $100^2 < 2^{100}$.

universal modus tollens This rule of inference is the combination of universal instantiation and modus tollens.

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ \neg Q(a), \text{ where } a \text{ is a particular elamest in the demain} \\ \hline \therefore \neg P(a) \end{array}$$

Example. *All good drivers are very alert.*
People who are drunk are not very alert.
Therefore, people who are drunk are not good drivers.

Solution: Let $G(x)$ represent "x is a good driver" and $A(x)$ represent "x is very alert."
• All good drivers are very alert. So, we can write this statement as:

$$\forall x(G(x) \to A(x))$$

• People who are drunk are not very alert. Let $D(x)$ represent "x is drunk" and $\neg A(x)$ represent "x is not very alert." So, we can write this statement as:

$$\forall x(D(x) \to \neg A(x))$$

• To show, People who are drunk are not good drivers. To prove:

$$\forall x(D(x) \to \neg G(x))$$

From statement 1, we have $\forall x(G(x) \to A(x))$
From statement 2, we have $\forall x(D(x) \to \neg A(x))$
By universal modus tollens, we can conclude: $\forall x(D(x) \to \neg G(x))$
So, the argument is valid by universal modus tollens.