SC 612: Discrete Mathematics MSc(IT)-Autumn 2023

Lectures 15: Recurrence Relation

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1 Recurrence Relation

A formula expressing terms of a sequence, except for some initial terms, as a function of one or more previous terms of the sequence.

1.1 Linear Homogeneous Recurrence Relations with Constant Coefficients

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

• The recurrence relation is linear because the right-hand side is a sum of previous terms of the sequence each multiplied by a function of n.
• The recurrence relation is homogeneous because no terms occur that are not multiples of the a_j s.
• The coefficients of the terms of the sequence are all constants, rather than functions that depend on n.
• The degree is k because a_n is expressed in terms of the previous k terms of the sequence.
A consequence of the second principle of mathematical induction (strong induction) is that a sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}.$$

✔ Example 1.1

• $P_n = 5P_{n-1}$ is a linear homogeneous recurrence relation of degree one.
• $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two.
• $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.
• $a_n = a_{n-1} + a_{n-2}^2$ is not linear.
• $H_n = 2H_{n-1} + 1$ is not homogeneous.
• $B_n = nB_{n-1}$ does not have constant coefficients.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant.
• Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

which is equivalent to

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0. \tag{1}$$

• Thus, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution iff r is a solution of (1).
• Equation (1) is the characteristic equation of the recurrence relation.
• The solutions of (1) are called the characteristic roots of the recurrence relation.
• The characteristic roots are used to give an explicit formula for all the solutions of the recurrence relation.

Linear homogeneous recurrence relations of degree two First, consider the case when there are two distinct characteristic roots.

Theorem 1.1

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \tag{2}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof: We must do two things to prove the theorem.
• (I) To show if r_1 and r_2 are characteristic roots, and α_1 and α_2 are constants, then $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of (2).
• (II), if $\{a_n\}$ is a solution of (2), then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants α_1 and α_2.

I Let $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, To show $\{a_n\}$ is a solution of (2).
Since r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, $\implies r_1^2 = c_1 r_1 + c_2, r_2^2 = c_1 r_2 + c_2$.
From (2)

$$\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1 \left(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}\right) + c_2 \left(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}\right) \\
&= \alpha_1 r_1^{n-2} \left(c_1 r_1 + c_2\right) + \alpha_2 r_2^{n-2} \left(c_1 r_2 + c_2\right) \\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\
&= \alpha_1 r_1^n + \alpha_2 r_2^n \\
&= a_n.
\end{aligned}$$

Therefore, $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of (2).

II Let $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$. To show $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2.
Suppose $\{a_n\}$ is a solution of (2) with initial conditions $a_0 = C_0$ and $a_1 = C_1$.
To show $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the initial conditions for constants α_1 and α_2. This requires that

$$\begin{aligned}
a_0 &= C_0 = \alpha_1 + \alpha_2, \\
a_1 &= C_1 = \alpha_1 r_1 + \alpha_2 r_2
\end{aligned}$$

Solving these two equations for α_1 and α_2, we get

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}, \qquad \alpha_2 = C_0 - \alpha_1 = C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2},$$

where these expressions for α_1 and α_2 depend on the fact that $r_1 \neq r_2$.
Hence, the sequence $\{a_n\}$ with $\alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the two initial conditions.
We know that $\{a_n\}$ and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ are both solutions of (2) and both satisfy the initial conditions when $n = 0$ and $n = 1$.

• Because there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, it follows that the two solutions are the same, that is, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all nonnegative integers n.
• A solution of the linear homogeneous recurrence relation with constant coefficients of degree two must be of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are constants.

Theorem 1.2

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solution of the recurrence relation (2) if and only if

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Example 1.2

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$?

Solution: The only root of $r^2 - 6r + 9 = 0$ is $r = 3$. Hence, the solution to this recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

for some constants α_1 and α_2. Using the initial conditions, it follows that

$$a_0 = 1 = \alpha_1$$
$$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$$

Solving these two equations shows that $\alpha_1 = 1$ and $\alpha_2 = 1$. Consequently, the solution to this recurrence relation and the initial conditions is

$$a_n = 3^n + n3^n$$

Linear homogeneous recurrence relations of higher degree with distinct roots General result about the solution of linear homogeneous recurrence relations with constant coefficients, where the degree may be greater than two, under the assumption that the characteristic equation has distinct roots.

Theorem 1.3

Let $c_1, c_2 \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2 \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Example 1.3

Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6$$

The characteristic roots are $r = 1, r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$. Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives

$$\begin{aligned}
a_0 &= 2 = \alpha_1 + \alpha_2 + \alpha_3. \\
a_1 &= 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3 \\
a_2 &= 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9
\end{aligned}$$

When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

Linear homogeneous recurrence relations of higher degree with repeated roots Consider a linear homogeneous recurrence relations with constant coefficients, wherein the characteristic equation has multiple roots.

• For each root r of the characteristic equation, the general solution has the form $P(n)r^n$, where $P(n)$ is a polynomial of degree $m - 1$, with m the multiplicity of this root.

Theorem 1.4

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c1r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_f$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = &\left(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}\right) r_1^n \\ &+ \left(\alpha_{2,0} + \alpha_{2,1} + \cdots + \alpha_{2,m_2-1 n^{m_2-1}}\right) r_2^n \\ &+ \cdots + \left(\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_2-1} n^{m_t-1}\right) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$. where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Example 1.4

Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1, a_1 = -2$, and $a_2 = -1$.

Solution: The characteristic equation of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0$$

Because $r^3 + 3r^2 + 3r + 1 = (r+1)^3$, there is a single root $r = -1$ of multiplicity three of the characteristic equation. By Theorem 4 the solutions of this recurrence relation are of the form

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1} n(-1)^n + \alpha_{1,2} n^2 (-1)^n.$$

To find the constants $\alpha_{1,0}, \alpha_{1,1}$, and $\alpha_{1,2}$, use the initial conditions. This gives

$$\begin{aligned} a_0 &= 1 = \alpha_{1,0} \\ a_1 &= -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2} \\ a_2 &= -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} \end{aligned}$$

The simultaneous solution of these three equations is $\alpha_{1,0} = 1, \alpha_{1,1} = 3$, and $\alpha_{1,2} = -2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = \left(1 + 3n - 2n^2\right)(-1)^n$$

1.2 Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

General form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n.

• The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the associated homogeneous recurrence relation.

Example 1.5

Following are linear nonhomogeneous recurrence relations with constant coefficients.
- $a_n = a_{n-1} + 2^n$,
- $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,
- $a_n = 3a_{n-1} + n3^n$, and
- $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n$

The associated linear homogeneous recurrence relations are
- $a_n = a_{n-1}$,
- $a_n = a_{n-1} + a_{n-2}$,
- $a_n = 3a_{n-1}$, and
- $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, respectively.

The solution of a linear nonhomogeneous recurrence relations with constant coefficients is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation.

Theorem 1.5

If $\left\{a_n^{(p)}\right\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\left\{a_n^{(p)} + a_n^{(h)}\right\}$, where $\left\{a_n^{(h)}\right\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Example 1.6

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

Solution: This is a linear nonhomogeneous recurrence relation.

• The solutions of its associated homogeneous recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

are $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$, where α_1 and α_2 are constants.
• Because $F(n) = 7^n$, a reasonable trial solution is $a_n^{(p)} = C \cdot 7^n$, where C is a constant.
Substituting the terms of this sequence into the recurrence relation implies that $C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$.
Factoring out 7^{n-2}, we get $49C = 35C - 6C + 49 \implies 20C = 49 \implies C = 49/20$.
Hence, $a_n^{(P)} = (49/20)7^n$ is a particular solution.
• The general solution is of the form

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n$$

Theorem 1.6

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = \left(b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0\right) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers.
• When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$\left(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0\right) s^n.$$

• When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m \left(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0\right) s^n.$$

Note: When s is a root of multiplicity m of the characteristic equation of the associated linear homogeneous recurrence relation, the factor n^m ensures that the proposed particular solution will not already be a solution of the associated linear homogeneous recurrence relation.

Example 1.7

What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when
• $F(n) = 3^n$,
• $F(n) = n3^n$,
• $F(n) = n^2 2^n$, and
• $F(n) = (n^2 + 1)\, 3^n$?

Solution: The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$. Its characteristic equation, $r^2 - 6r + 9 = (r-3)^2 = 0$, has a single root, 3 of multiplicity two.
The particular solution is of the form $P(n)s^n$, where $P(n)$ is a polynomial and s is a constant, we need to ask whether s is a root of this characteristic equation.
Because $s = 3$ is a root with multiplicity $m = 2$ but $s = 2$ is not a root, particular solution has the form

• $p_0 n^2 3^n$ if $F(n) = 3^n$,
• $n^2\left(p_1 n + p_0\right) 3^n$ if $F(n) = n3^n$,
• $\left(p_2 n^2 + p_1 n + p_0\right) 2^n$ if $F(n) = n^2 2^n$, and
• $n^2\left(p_2 n^2 + p_1 n + p_0\right) 3^n$ if $F(n) = \left(n^2 + 1\right) 3^n$

Note

Care must be taken when $s = 1$
• In particular, with $F(n) = b_t n_t + b_{t-1} n_{t-1} + \cdots + b_1 n + b_0$
• the form of the solution then depends on whether 1 is a root of the characteristic equation of the associated linear homogeneous recurrence relation.

1.3 Generating function

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Example 1.8

The generating functions for the sequences $\{a_k\}$ with
• $a_k = 3,$ —— $\sum_{k=0}^{\infty} 3x^k$
• $a_k = k + 1$ —— $\sum_{k=0}^{\infty}(k+1)x^k$ and
• $a_k = 2^k$ —— $\sum_{k=0}^{\infty} 2^k x^k$.

• Generating functions for finite sequences of real numbers can be defined by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0, a_{n+2} = 0$, and so on.
• The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n$$

Example 1.9

What is the generating function for the sequence $1, 1, 1, 1, 1, 1$?

Solution: The generating function of $1, 1, 1, 1, 1, 1$ is

$$1 + x + x^2 + x^3 + x^4 + x^5$$

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

✔ Example 1.10

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Solution: Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$.

First note that

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$$

Using the recurrence relation, we see that

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_0 + \sum_{k=1}^{\infty} \left(a_k - 3a_{k-1}\right) x^k \\ &= 2, \end{aligned}$$

because $a_0 = 2$ and $a_k = 3a_{k-1}$. Thus,

$$G(x) - 3xG(x) = (1-3x)G(x) = 2.$$

Solving for $G(x)$ shows that $G(x) = 2/(1-3x)$.

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

Consequently, $a_k = 2 \cdot 3^G$.

✔ Example 1.11

Solve the following recurrence relations.
1. $a_k = a_{k-1} + 2a_{k-2} + 2^k, a_0 = 4, a_1 = 12.$
2. $a_k = 4a_{k-1} - 4a_{k-2} + k^2, a_0 = 2, a_1 = 5$
3. $a_k = 2a_{k-1} - 3a_{k-2} + 4^k + 6, a_0 = 20, a_1 = 60$

Solution 1.

$$\begin{cases} a_k = a_{k-1} + 2a_{k-2} + 2^k \\ a_0 = 4 \\ a_1 = 12 \end{cases}$$

Let $G(x)$ denote the generating function for the sequence a_k, then we get $G(x) = \sum_{k\geq 0} a_k x^k$

Take the first equation, then multiply each term by x^k

$$a_k x^k = a_{k-1} x^k + 2a_{k-2} + 2^k x^k$$

And sum each term from 2 since it's a 2-order recurrence relation.

$$\begin{aligned} \sum_{k\geq 2} a_k x^k &= \sum_{k\geq 2} a_{k-1} x^k + \sum_{k\geq 2} 2a_{k-2} x^k + \sum_{k\geq 2} 2^k x^k \\ &= \sum_{k\geq 2} a_{k-1} x^k + 2\sum_{k\geq 2} a_{k-2} x^k + \sum_{k\geq 2} 2^k x^k \end{aligned}$$

To manipulate each term so that you can write them in terms of the generating function $G(x)$ and known series representations.

$$LHS: \quad \sum_{k\geq 2} a_k x^k = \left(\sum_{k\geq 2} a_k x^k + a_1 x + a_0\right) - a_1 x - a_0$$
$$= \sum_{k\geq 0} a_k x^k - a_1 x - a_0 = G(x) - 12x - 4$$
$$RHS\ I: \quad \sum_{k\geq 2} a_{k-1} x^k = x\sum_{k\geq 2} a_{k-1} x^{k-1} = x\sum_{k\geq 1} a_k x^k$$
$$= x\left(\sum_{k\geq 0} a_k x^k - a_0\right) = x(G(x) - 4)$$
$$RHS\ II: \quad \sum_{k\geq 2} a_{k-2} x^k = x^2\sum_{k\geq 2} a_{k-2} x^{k-2} = x\sum_{k\geq 0} a_k x^k = x^2 G(x)$$
$$RHS\ III: \quad \sum_{k\geq 2} 2^k x^k = \sum_{k\geq 0} 2^k x^k - 2x - 1 = \frac{1}{1-2x} - 2x - 1$$

simplifying LHS=RHS

$$G(x) - 12x - 4 = x(G(x) - 4) + 2x^2 G(x) + \frac{1}{1-2x} - 2x - 1$$
$$G(x)(1 - x - 2x^2) = 12x + 4 - 4x + \frac{1}{1-2x} - 2x - 1 = 6x + 3 + \frac{1}{1-2x}$$
$$G(x)(1-2x)(1+x) = 3(1+2x) + \frac{1}{1-2x} = \frac{3(1-4x^2)+1}{1-2x} = \frac{4-12x^2}{1-2x}$$
$$\implies G(x) = \frac{4-12x^2}{(1+x)(1-2x)^2}$$

Using partial fractions to break up the denominator

$$G(x) = \frac{4-12x^2}{(1+x)(1-2x)^2} = \frac{A}{1+x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$

Multiplying through by the common denominator, and equating coefficients, we find that $A = -8/9, B = 38/9$, and $C = 2/3$. Thus

$$G(x) = \frac{-8/9}{1+x} + \frac{38/9}{1-2x} + \frac{2/3}{(1-2x)^2}$$
$$= \sum_{k=0}^{\infty}\left[\frac{-8}{9}(-1)^k + \frac{38}{9}2^k + \frac{2}{3}\binom{k+1}{k}2^k\right]x^k$$
$$\therefore\ a_k = \frac{-8}{9}(-1)^k + \frac{38}{9}2^k + \frac{2}{3}(k+1)2^k$$

> When $k = 2$,
> - $a_k = a_{k-1} + 2a_{k-2} + 2^k = a_1 + 2a_0 + 4 = 12 + 2\cdot 4 + 4 = 24$ ✓
> - $a_k = \frac{-8}{9}(-1)^k + \frac{38}{9}2^k + \frac{2}{3}(k+1)2^k = \frac{-8}{9} + \frac{38}{9}4 + \frac{2}{3}3\cdot 4 = \frac{152-8}{9} + 8 = 24$ ✓

Solution 2.

$$\begin{cases} a_k = 4a_{k-1} - 4a_{k-2} + k^2 \\ a_0 = 2 \\ a_1 = 5 \end{cases}$$

$$a_k = 4a_{k-1} - 4a_{k-2} + k^2$$
$$a_k x^k = 4a_{k-1}x^k - 4a_{k-2}x^k + k^2 x^k$$
$$\sum_{k\geq 2} a_k x^k = 4\sum_{k\geq 2} a_{k-1}x^k - 4\sum_{k\geq 2} a_{k-2}x^k + \sum_{k\geq 2} k^2 x^k$$

$$LHS: \quad \sum_{k\geq 2} a_k x^k = \left(\sum_{k\geq 2} a_k x^k + a_1 x + a_0\right) - a_1 x - a_0$$
$$= \sum_{k\geq 0} a_k x^k - a_1 x - a_0 = G(x) - 5x - 2$$

$$RHS\ I: \quad \sum_{k\geq 2} 4a_{k-1}x^k = 4x\sum_{k\geq 2} a_{k-1}x^{k-1} = 4x\sum_{k\geq 1} a_k x^k$$
$$= 4x\left(\sum_{k\geq 0} a_k x^k - a_0\right) = 4x(G(x) - 2)$$

$$RHS\ II: \quad 4\sum_{k\geq 2} a_{k-2}x^k = 4x^2\sum_{k\geq 2} a_{k-2}x^{k-2} = 4x\sum_{k\geq 0} a_k x^k = 4x^2 G(x)$$

$$Now, \ \sum_{k\geq 0} x^k = \frac{1}{1-x} \quad , \sum_{k\geq 1} kx^{k-1} = \frac{1}{(1-x)^2}, \quad \sum_{k\geq 1} kx^k = \frac{x}{(1-x)^2} \quad \text{, for } |x| < 1$$

$$\sum_{k\geq 1} k^2 x^{k-1} = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{(1-x)^2 - x(-2(1-x))}{(1-x)^4} = \frac{(1-x)+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$$
$$\implies \sum_{k\geq 2} k^2 x^{k-1} = \frac{1+x}{(1-x)^3} - 1$$

$$RHS\ III: \quad \sum_{k\geq 2} k^2 x^k = x\sum_{k\geq 2} k^2 x^{k-1} = x\left(\frac{1+x}{(1-x)^3} - 1\right) = \frac{x(1+x)}{(1-x)^3} - x$$

$$G(x) - 5x - 2 = 4x(G(x) - 2) - 4x^2 G(x) + \frac{x(1+x)}{(1-x)^3} - x$$
$$G(x)\left[1 - 4x + 4x^2\right] = 5x + 2 - 8x + \frac{x(1+x)}{(1-x)^3} - x = 2 - 4x + \frac{x(1+x)}{(1-x)^3}$$
$$\implies G(x) = \frac{2-4x}{(1-2x)^2} + \frac{x(1+x)}{(1-x)^3(1-2x)^2}$$

Using partial fractions $= \dfrac{A}{1-x} + \dfrac{B}{(1-x)^2} + \dfrac{C}{(1-x)^3} + \dfrac{D}{1-2x} + \dfrac{E}{(1-2x)^2}$

Multiplying through by the common denominator, and equating coefficients, we find that $A = 13, B = 5, C = 2, D = -24$, and $E = 6$. On simplification

$$G(x) = \frac{13}{1-x} + \frac{5}{(1-x)^2} + \frac{2}{(1-x)^3} - \frac{24}{1-2x} + \frac{6}{(1-2x)^2}$$
$$= \sum_{k=0}^{\infty}\left[13\cdot 1 + 5\cdot(k+1) + 2\cdot\binom{k+2}{k} - 24\cdot 2^k + 6\cdot\binom{k+1}{k}2^k\right]x^k$$
$$= \sum_{k=0}^{\infty}\left[13 + 5k + 5 + 2\frac{(k+2)(k+1)}{2} - 24\cdot 2^k + 6(k+1)2^k\right]x^k$$

$$= \sum_{k=0}^{\infty} \left[18 + 5k + k^2 + 3k + 2 + (6k - 18)2^k\right] x^k$$
$$= \sum_{k=0}^{\infty} \left[20 + 8k + k^2 + (6k - 18)2^k\right] x^k$$

Therefore,

$$a_k = 20 + 8k + k^2 + (6k - 18)2^k$$

When $k = 2$,
- $a_k = 4a_{k-1} - 4a_{k-2} + k^2 = 4a_1 - 4a_0 + 4 = 4 \cdot 5 - 4 \cdot 2 + 4 = 16$ ✓
- $a_k = 20 + 8k + k^2 + (6k - 18)2^k = 20 + 16 + 4 + (-6) \cdot 4 = 16$ ✓

Solution 3. Try yourself