SC612: Discrete Mathematics MSc(IT)-Autumn 2023

Lecture 3: Predicate logic

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1 Predicate logic

The area of logic that deals with predicates and quantifiers

Predicate A predicate is a statement that contains variables, sometimes referred to as predicate variables, and may be true or false depending on those variables' value or values.

Example. $\underbrace{x}_{subject} \overbrace{> y + 3}^{predicate}$ $\quad x + y = z$

• The statement $P(x) : x > y + 3$ is said to be the value of the propositional function P at x.
• Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.
• Predicates are useful to establish the correctness of computer programs, that is, to show that computer programs always produce the desired output when given valid input.
• The statements that describe valid input are known as preconditions and
• the conditions that the output should satisfy when the program has run are known as postconditions.

Why Because predicates, which contain variables, don't have truth values that can be easily detected, they are not propositions. Therefore, we can't use propositional logic to draw conclusions.

Quantifier Quantifiers help us to create propositions from propositional functions and the process is called quantification.
Quantifiers express the extent to which a predicate is true over a range of elements.

Universe of Discourse Many mathematical statements assert that a property is true for all values of a variable in a particular domain. Also, called the domain of discourse.

Types • Universal quantification, which tells us that a predicate is true for every element under consideration, and
• Existential quantification, which tells us that there is one or more element under consideration for which the predicate is true.

Definition 1.1: Universal quantifier

The universal quantification of $P(x)$ is the statement
"$P(x)$ for all values of x in the domain."
• *The notation $\forall x\ P(x)$ denotes the universal quantification of $P(x)$.*
• *Here $\forall$ is called the universal quantifier and is read as "for all x P(x)" or "for every x P(x)."*

- *An element for which $P(x)$ is false is called a counterexample of $\forall x\ P(x)$.*
- *Besides "for all" and "for every," universal quantification can be expressed in many other ways, including "all of," "for each," "given any," "for arbitrary," "for each," and "for any."*

Note. When the elements of the universe of discourse is finitely many, e.g., consists of $x_1, x_2, \ldots x_n$, then $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$, since the conjunction is true if and only if of $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

- Let $P(x) : x + 3 > x$. What is the truth value of the quantification $\forall x\ P(x)$, where the domain is $\mathbb{R}$? Ans: TRUE
- Let $P(x) : x > 0$. What is the truth value of the quantification $\forall x\ P(x)$, where the domain is $\mathbb{R}$? Ans: FALSE, since -1 is smaller than 0.

Definition 1.2: Existential quantifier

The existential quantification of $P(x)$ is the statement
"There exists an element x in the domain such that $P(x)$ "

- *The notation $\exists x\ P(x)$ denotes the existential quantification of $P(x)$.*
- *Here $\exists$ is called the existential quantifier and is read as "there exists x P(x)"*
- *Other words used for existential quantifier are "for some," "for at least one," or "there is."*

Note. When the elements of the universe of discourse is finitely many, e.g., consists of $x_1, x_2, \ldots x_n$, then $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$, since this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

Note. If the domain is empty, then $\exists x P(x)$ is false, because there can be no element x in the domain for which $P(x)$ is true.

- Let $P(x) : x > 3$. What is the truth value of the quantification $\exists x\ P(x)$, where the domain is $\mathbb{R}$? Ans: TRUE, since x=3.1532
- Let $P(x) : x = x + 3$. What is the truth value of the quantification $\exists x\ P(x)$, where the domain is $\mathbb{R}$? Ans: FALSE

Remark 1.1. Think in terms of looping and searching when determining the truth value of a quantification.
Suppose that there are n objects in the domain for the variable x.

- To determine whether $\forall x P(x)$ is true,
  - loop through all n values of x to see whether $P(x)$ is always true.
  - If we encounter a value x for which $P(x)$ is false, then we have shown that $\forall x P(x)$ is false.
  - Otherwise, $\forall x P(x)$ is true.
- To see whether $\exists x P(x)$ is true,
  - loop through the n values of x searching for a value for which $P(x)$ is true.
  - If we find one, then $\exists x P(x)$ is true.
  - If we never find such an x, then we have determined that $\exists x P(x)$ is false.

Definition 1.3: Uniqueness quantifier

It is denoted by $\exists!$
• *The notation* $\exists!\ x\ P(x)$ *states "There exists a unique* x *such that* $P(x)$ *is true."*
"There is exactly one x *such that* $P(x)$ *is true."*
"There is one and only one x *such that* $P(x)$ *is true."*

Quantifiers with Restricted Domains

- $\forall\ x < 0\ (x^2 > 0)$ with domain $\mathbb{R}$.
  - The statement $\forall x < 0\,(x^2 > 0)$ states that for every real number x with $x < 0, x^2 > 0$.
  - "The square of a negative real number is positive."
  - This is same as $\forall x\,(x < 0 \to x^2 > 0)$.
- $\forall\ y = 0\ (y^3 = 0)$, with domain $\mathbb{R}$.
  - The statement $\forall y \neq 0\,(y^3 \neq 0)$ states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$.
  - That is, "The cube of every nonzero real number is nonzero."
  - this is equivalent to $\forall y\,(y \neq 0 \to y^3 \neq 0)$.
- $\exists\ z > 0\ (z^2 = 2)$ with domain $\mathbb{R}$.
  - The statement $\exists z > 0\,(z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$.
  - That is, "There is a positive square root of 2 .
  - "This statement is equivalent to $\exists z\,(z > 0 \wedge z^2 = 2)$.
- Note that the restriction of a universal quantification is the same as the universal quantification of a conditional statement. For instance, $\forall x < 0\,(x^2 > 0)$ is another way of expressing $\forall x\,(x < 0 \to x^2 > 0)$. On the other hand, the restriction of an existential quantification is the same as the existential quantification of a conjunction. For instance, $\exists z > 0\,(z^2 = 2)$ is another way of expressing $\exists z\,(z > 0 \wedge z^2 = 2)$.

Precedence of Quantifiers The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.
• For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$.
• In other words, it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

Binding Variables When a quantifier is used on the variable x, we say that this occurrence of the variable is bound.

Free Variables An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.
• All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition.
• This can be done using a combination of universal quantifiers, existential quantifiers, and value assignments.

Scope It is the part of a logical expression to which a quantifier is applied
• a variable is free if it is outside the scope of all quantifiers in the formula that specify this variable.

Example. *In the statement* $\exists\ x\ (x + y = 1)$*, the variable* x *is bound by the existential quantification* $\exists\ x$*, but the variable* y *is free because it is not bound by a quantifier and no value is assigned to this variable.*
Thus, in $\exists\ x\ (x + y = 1)$*,* x *is bound, but* y *is free.*
• *In the statement* $\exists\ x(P(x) \wedge Q(x)) \vee \forall\ x\ R(x)$*, all variables are bound.*
- *The scope of the first quantifier,* $\exists\ x$*, is the expression* $P(x) \wedge Q(x)$ *because* $\exists\ x$ *is applied only to* $P(x) \wedge Q(x)$*, and not to the rest of the statement.*
- *The scope of the second quantifier,* $\forall\ x$*, is the expression* $R(x)$*.*

That is, the existential quantifier binds the variable x *in* $P(x) \wedge Q(x)$ *and the universal quantifier* $\forall\ x$ *binds the variable* x *in* $R(x)$*.*

Logical Equivalence Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.
• We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example. *Show that* $\forall x\ (P(x) \wedge Q(x))$ *and* $\forall\ x\ P(x) \wedge \forall\ x\ Q(x)$ *are logically equivalent.*

- To show, $\forall x\ (P(x) \wedge Q(x)) \iff \forall\ x\ P(x) \wedge \forall\ x\ Q(x)$, that is,
  - $\forall x\ (P(x) \wedge Q(x)) \implies \forall\ x\ P(x) \wedge \forall\ x\ Q(x)$ and
  - $\forall x\ (P(x) \wedge Q(x)) \impliedby \forall\ x\ P(x) \wedge \forall\ x\ Q(x)$
- Let $\forall x\ (P(x) \wedge Q(x))$ be true.
  $\implies$ if a is in the domain, then $P(a) \wedge Q(a)$ is true.
  Hence, $P(a)$ is true and $Q(a)$ is true.
  Because P(a) is true and Q(a) is true for every element in the domain, we can conclude that $\forall\ x\ P(x)$ and $\forall\ x\ Q(x)$ are both true.
  This means that $\forall\ x\ P(x) \wedge \forall\ x\ Q(x)$ is true.
  Therefore, $\forall x\ (P(x) \wedge Q(x)) \implies \forall\ x\ P(x) \wedge \forall\ x\ Q(x)$.
- Suppose $\forall x\ P(x) \wedge \forall\ xQ(x)$ is true.
  $\implies \forall\ x\ P(x)$ is true and $\forall\ x\ Q(x)$ is true.
  Hence, if a is in the domain, then both $P(a)$ is true and $Q(a)$ is true
  $\implies \forall a,\ P(a) \wedge Q(a)$ is true.
  Hence, $\forall x\ (P(x) \wedge Q(x))$ is true
  Therefore, $\forall\ x\ P(x) \wedge \forall\ x\ Q(x) \implies \forall x\ (P(x) \wedge Q(x))$

DeMorgan's laws for quantifiers Negation of quantifiers

De Morgan's Laws for Quantifiers.			
Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Example. *What are the negations of the statements $\forall\ x\ (x^2 > x)$ and $\exists\ x\ (x^2 = 2)$?*

- The negation of $\forall\ x\ (x^2 > x)$ is the statement

$$\neg\forall\ x\ (x^2 > x) \equiv \exists\ x\ \neg(x^2 > x) \equiv \exists\ x\ (x^2 \leq x)$$

- The negation of $\exists\ x\ (x^2 = 2)$ is the statement

$$\neg\exists\ x\ (x^2 = 2) \equiv \forall\ x\ \neg(x^2 = 2) \equiv \forall\ x\ (x^2 \neq 2)$$

Statements in English to Logical Expressions

Example. *Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.*

1. Rewrite the statement to clearly identify the appropriate quantifiers to use.

   "For every student in this class, that student has studied calculus."

2. Introduce a variable x

   "For every student x in this class, x has studied calculus."

3. Introduce propositional function

   C(x):= "x has studied calculus."

4. Identify the quantifier with respect to the domain

   if the domain for x consists of the students in the class, we can translate our statement as

$$\forall\ x\ C(x)$$

Example. *Express the statements "Some student in this class has iPhone" and "Every student in this class has either iPhone or Android" using predicates and quantifiers.*

1. Rewrite the statement to clearly identify the appropriate quantifiers to use.

   "Some student in this class has iPhone" means that "There is a student in this class with the property that the student has iPhone."

2. Introduce a variable x

   "There is a student x in this class having the property that, x has iPhone."

3. Introduce propositional function

   I(x):= "x has iPhone."

4. Identify the quantifier with respect to the domain

   if the domain for x consists of the students in the class, we can translate our statement as

$$\exists\ x\ I(x)$$

Nested Quantifiers A quantifier that appears within the scope of another quantifier is called a nested quantifier.

• $\forall\ x \exists\ y(x + y = 0)$.

• Everything within the scope of a quantifier can be thought of as a propositional function.

$\forall\ xQ(x)$, where $Q(x) := \exists\ yP(x, y)$ with $P(x, y) := x + y = 0$.

• For $x, y \in \mathbb{R}$, $\forall\ x\forall\ y(x+y = y+x)$ (commutative law for addition of real numbers)

• For $x, y, z \in \mathbb{R}$, $\forall\ x\forall\ y\forall\ z(x + (y + z) = (x + y) + z)$ (associative law for addition of real numbers)

Nested quantifier to English $\forall\ x\ y((x > 0) \wedge (y < 0) \to (xy < 0))$, where the domain for both variables consists of all real numbers.

• This statement says that for every real number x and for every real number y, if $x > 0$ and $y < 0$, then $xy < 0$.

• English: for real numbers x and y, if x is positive and y is negative, then xy is negative. "The product of a positive real number and a negative real number is always a negative real number."

Order of Quantifiers The order of nested universal quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement.

• For $x, y \in \mathbb{R}$, $\forall\ x\ \forall\ yP(x, y)$ and $\forall\ y\ \forall\ x\ P(x, y)$ have the same meaning.

Example. *Let $Q(x,y) := x + y = 0$. Check the truth values of the quantifications $\exists\ y\ \forall\ x\ Q(x,y)$ and $\forall\ x\ \exists\ y\ Q(x,y)$*

• For $x, y \in \mathbb{R}$, $\exists\ y\ \forall\ x\ Q(x,y)$ means
"There is a real number y such that for every real number x, $Q(x,y)$."
"There is a real number y such that for every real number x, $x + y = 0$."
Let y=a, then for every real number x, $x + a = 0$, which is not true (y is a constant).
Therefore, the statement $\exists\ y\ \forall\ x\ Q(x,y)$ is FALSE.

• For $x, y \in \mathbb{R}$, $\forall\ x\ \exists\ y\ Q(x,y)$ means
"For every real number x, there is a real number y such that $Q(x,y)$."
"For every real number x, there is a real number y such that $x + y = 0$."
Let y=-x, then for every real number x, $x + y = 0$ (y is dependent on x).
Therefore, the statement $\forall\ x\ \exists\ y\ Q(x,y)$ is TRUE.

Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x, y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x, y.

Statements in English to Nested Quantifiers

Example. *Translate the statement "The sum of two positive integers is always positive" into a logical expression using nested quantifiers.*

1. Rewrite the statement to clearly identify the appropriate quantifiers and domain

   "For every two integers, if these integers are both positive, then their sum is positive."

2. Introduce variables x and y

   "For all positive integers x and y, x + y is positive."

3. Identify the quantifier with respect to the domain

$$\forall\ x\ \forall\ y\ ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

   where the domain for both variables consists of all integers.

$$\forall\ x\ \forall\ y\ (x + y > 0)$$

   where the domain for both variables consists of all positive integers.

Example. *Let F(x, y) be the statement "x can fool y," where the domain consists of all people in the world. Use quantifiers to express each of these statements.*
a) Everybody can fool John.
b) Merlyn can fool everybody.
c) Everybody can fool somebody.
f) No one can fool both Ravi and John.
g) Meera can fool exactly two people.

Solution: a) x can fool John: F(x,John); everybody can fool John: $\forall$ x F(x,John)
b) Merlyn can fool everybody. $\forall$ y F(Merlyn,y)
c) Everybody can fool somebody. $\forall$ x F(x,somebody)— $\forall$ x$\exists$ y F(x,y)
f) x can fool both Ravi and John. F(x,Ravi)$\wedge$ F(x,John)
No one can fool both Ravi and John.— $\neg\exists$ x (F(x,Ravi)$\wedge$ F(x,John))
g) Meera can fool two people. $\exists\ y_1 \exists\ y_2 F(M, y_1) \wedge\ F(M, y_2) \wedge (y_1 \neq y_2)$
exactly two people — $\forall y\ F(M, y) \rightarrow (y = y_1 \vee y = y_2)$ Meera can fool exactly two people— $\exists\ y_1 \exists\ y_2 (F(M, y_1) \wedge\ F(M, y_2) \wedge (y_1 \neq y_2) \wedge \forall y\ F(M, y) \rightarrow (y = y_1 \vee y = y_2))$

Nested Quantifiers into English Expressions with nested quantifiers expressing statements in English can be quite complicated. To translate such an expression, follow the steps
- write out what the quantifiers and predicates in the expression mean.
- express this meaning in a simpler sentence.

Example. *Translate the statement into English*

$$\forall\ x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$$

where C(x):= "x has a computer," F(x, y):= "x and y are friends," and the domain for both x and y consists of all first year students.

• $C(y) \wedge F(x, y)$:= “y has a computer and x and y are friends.”
• $\exists\ y\ (C(y) \wedge F(x, y))$:= “There is a student y such that y has a computer and x and y are friends.”
• $C(x)\ \exists\ y\ (C(y) \wedge F(x, y))$:= “x has a computer or there is a student y such that y has a computer and x and y are friends.”
• $\forall\ x\ (C(x)\ \exists\ y\ (C(y) \wedge F(x, y)))$:= “for every student x in first year, x has a computer or there is a student y such that y has a computer and x and y are friends.”
In other words, “every student in the first year has a computer or has a friend who has a computer.”
“Every first year student has a computer or has a friend who has a computer.”

Negation of Nested Quantifiers Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier.

Example. *Express the negation of the statement* $\forall\ x\ \exists\ y(xy = 1)$ *so that no negation precedes a quantifier.*

$$\neg\forall\ x\ \exists\ y(xy = 1) \equiv \exists\ x\ \neg\exists\ y(xy = 1) \equiv \exists\ x\ \forall\ y\neg(xy = 1) \equiv \exists\ x\ \forall\ y(xy \neq 1)$$