SC612: Discrete Mathematics	MSc(IT)-Autumn 2023

Lecture 2: Propositional Equivalences

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1 Logical Equivalences

The compound propositions p and q are called logically equivalent if $p \iff q$ is a tautology (all truth values are T).
• The notation $p \equiv q$ denotes that p and q are logically equivalent.
• The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \iff q$ is a tautology.
• The symbol $\iff$ is sometimes used instead of $\equiv$ to denote logical equivalence.

1.1 DeMorgan's law

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Example. *Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.*

Truth Tables for $\neg p \vee q$ and $p \to q$.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example (Distributive law of disjunction over conjunction)**.** *Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.*

TABLE 5 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

1.2 Some important equivalences

Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalences Involving Conditional Statements.
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

1.3 Propositional Satisfiability

- A compound proposition is satisfiable if there is an assignment of truth values to its variables that makes it true.
- A compound proposition is unsatisfiable iff its negation is true for all assignments of truth values to the variables, that is, if and only if its negation is a tautology.
  - The compound proposition is false for all assignments of truth values to its

variables.

- To show that a compound proposition is unsatisfiable $\implies$ To show that every assignment of truth values to its variables makes it false.

- When we find a particular assignment of truth values that makes a compound proposition true, we have shown that it is satisfiable; such an assignment is called a solution of this particular satisfiability problem.

Example. *Determine whether each of the compound propositions is satisfiable*
- $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$,
- $(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$,
- $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$.

Solution: Instead of using truth table, we will reason about truth values.
- Note that $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ is true

when the three compound statements are true
$\implies$ all variable p, q, and r have the same truth value.
$\implies$ there is at least one assignment of truth values for p, q, and r that makes it true
Hence, it is satisfiable.
- Note that $(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ is true

when the two compound statements are true
$\implies$ at least one of p, q, and r is true and at least one is false
$\implies$ there is at least one assignment of truth values for p, q, and r that makes it true
Hence, $(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ is satisfiable.
- Note that for $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ to be true,

$\implies$ $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ and $(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ must both be true.
For the first to be true, the three variables must have the same truth values, and
for the second to be true, at least one of three variables must be true and at least one must be false.
However, these conditions are contradictory.
From these observations we conclude that no assignment of truth values to p, q, and r makes $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ true.
Hence, it is unsatisfiable.

Note. Generate truth tables online using the following links Truth Table Generator and eMathHelp