

Conceptual Topos As Conceptual Cage: An Algebraic Topology of Meaning based on Conceptual Topology

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Introduction

Meaning was once considered fluid, ungraspable — a vapor that escaped the structures we tried to impose. But what if meaning does not escape? What if it moves, and that movement can be mapped, composed, and classified? This theory, Conceptual Topology (概念位相論) As Conceptual Topos, begins with a radical yet simple claim.

Meaning does not escape. It just topologizes within the abstract cage.

We no longer describe meaning merely through signs and chains of signifiers, but as flows — morphisms between concepts mediated by contextual anchors called Z-frames. These frames act as semantic coordinates, situating each concept within a space of possible interpretation.

A dog is not simply “a dog.” It is interpreted through the semantic frame Z in which it is embedded.

dog | Domesticated

dog | Mammal

dog | Son

Or, as a morphism, computer \rightarrow she | person

With the Z frame computer is interpreted as a historical computing worker (pre-digital era), resolving ambiguity via structural semantic framing.

In this model, concepts are objects, interpretive movements are arrows, and semantic coherence is topological.

We define categories like $C|Z$, where morphisms $f: A \rightarrow B | Z$ are conceptual transformations under a shared meaning frame. We introduce operators like σ that model semantic shifting, generalization, or abstraction across frames and we show that these operators exhibit functorial and even Kan-extension-like behavior.

Meaning is no longer a mirage. It circulates within a space that is structured, closed, and composable. We are no longer chasing meaning. We are building it from its space.

Note: While we refer to “fibers” to describe morphic coherence over a shared Z -frame, this current formulation is not yet a strict fibered topos in the categorical sense. Rather, this document serves as the semantic scaffolding toward that formalization.

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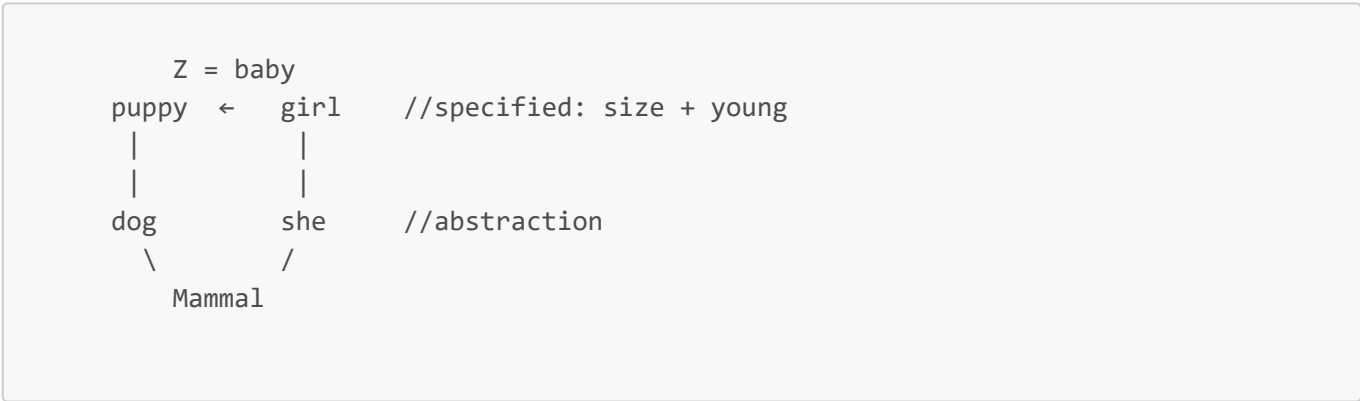
1. Fibered Conceptual Topology:

Fibered Conceptual Topology provides a conceptual geometric framework wherein each Z-frame (conceptual anchor) acts as a base space, with conceptual morphic flows forming fibers over these anchors. The Yoneda-like interpretation captures concepts as bundles of conceptual relations within and across Z-frames. This fibered structure serves as the foundation for further constructions in Conceptual Topos.

$$CT := (C, B, \pi: E \rightarrow B, Fb := \pi^{-1}(b), A \cong b \cup Nat(Hom(-, A), Fb))$$

Where:

- C is the category of concepts (objects = words or concepts)
- B is the base space of Z-frames (conceptual continuity anchors)
- E is the total conceptual space (word vector embedding space)
- π projects each concept to its conceptual base (Z-frame)
- Fb is the fiber (conceptual morphic chain) over a base b
- $A \cong b \cup Nat(Hom(-, A), Fb)$ interprets each concept A via its morphisms relative to its Z-frame b (Yoneda perspective defined in appendix)



1.1. A Z-framed Conceptual Category

A Z-framed Conceptual Category is a structure $(C, Z, \text{Hom}, \text{id}, \circ)$.

- $\text{Ob}(C)$ is a set of conceptual entities.
- Z is a set of conceptual frames.
- $\text{Hom}(X, Y \mid Z)$ is the set of morphisms from X to Y within Z -frame.
- For each X , there exists an identity morphism $\text{id}_X|Z: X \rightarrow X \mid Z$ or $\text{id}_X: X \rightarrow X \mid X$.
- Composition \circ is a partial operation defined as

For $f: X \rightarrow Y \mid Z_1$ and $g: Y \rightarrow Z \mid Z_2$,
 $g \circ f: X \rightarrow Z \mid Z_3$ is defined iff $\exists Z_3 \in Z$ such that $Z_1 \subseteq Z_3$ and $Z_2 \subseteq Z_3$

Z-Frame

Z-frame has multi-functionality as a conceptual frame: Category, Fiber and Morphism level object. We define a Z-Indexed Fibered Conceptual Category as a tuple

$(C, Z, \pi: C \rightarrow Z)$

Where:

C : A category of conceptual morphisms (objects: concepts, morphisms: semantic transformations)

Z : A category of conceptual frames (Z-frames), representing interpretive contexts or domains

π : A functor projecting each morphism in C to its conceptual frame in Z

Object Level Z-Frame Structure

Let:

$\text{Ob}(C)$ be a set of conceptual entities (dog, she, king, ...)

$\text{Ob}(Z)$ be a set of semantic frames (Domesticated, Mammal, Abstract, ...)

Each morphism in C is typed as

$f: A \rightarrow B \mid Z \in \text{Hom}(A, B \mid Z)$

This is realized through a retractive structure mediated by Z

$f: A \rightarrow Z$

$g: Z \rightarrow A$

such that

$g \circ f \cong \text{id}_A \mid Z$

In diagrammatic terms

$$\begin{array}{c} A \\ | \quad \backslash \\ | \quad \backslash \\ v \quad v \\ Z \rightarrow A \quad (g \circ f = \text{id}_A \mid Z) \end{array}$$

Alternatively, to express conceptual flow under Z

$f: A \rightarrow Z$

$g: Z \rightarrow B$

such that

$g \circ f \cong A \rightarrow B \mid Z$

This means that A is transformed to B under the interpretation frame Z . The flow between A and B is mediated by Z , and Z ensures that the interpretation of both A and B is consistent under the same frame.

Disambiguation and Structural Integrity When multiple interpretations (or semantic frames) exist such as *computer*, Z acts as a disambiguating factor, ensuring that the meanings of A and B are not left to chance but are structurally ensured by their relationships to Z .

Example:

computer \rightarrow she \mid person

Without Z , computer may refer to a machine, a metaphor, a role, or even ambiguity between literal and historical meanings.

With $Z = \text{person}$, computer is interpreted as a historical computing worker (pre-digital era), resolving ambiguity via structural semantic framing.

Fibered Structure

For each $Z \in \text{Ob}(Z)$, define the local fiber:

$$\pi^{-1}(Z) := \{ f \in \text{Mor}(C) \mid \pi(f) = Z \}$$

Over the total base Z , the full fibered category is:

$$\pi^{-1}(Z) := \{ f_i \in \text{Mor}(C) \mid \pi(f_i) = Z_i \text{ for some } Z_i \in \text{Ob}(Z) \}$$

This is the subcategory $C|Z$, where all morphisms are constrained to operate within the same Z -frame.

Functoriality

π must satisfy the functor laws

For any identity morphism $\text{id}_A|A$ in C

$$\pi(\text{id}_A|A) = \text{id}_Z|Z$$

For any composable morphisms $f: A \rightarrow B \mid Z_1$, $g: B \rightarrow C \mid Z_2$ with $Z_1, Z_2 \subseteq Z$, the composition is

$$g \circ f: A \rightarrow C \mid Z$$

and:

$$\pi(g \circ f) = Z$$

Here, Z is the least upper bound (or unifying context) of Z_1 and Z_2 .

Z-Frame Subsuming Composition ($Z \subseteq Z'$) Although f is originally defined over Z , due to the inclusion $Z \subseteq Z'$, the composite $g \circ f$ can be interpreted as an internal composition in the superordinate category Z' .

Let $Z \subseteq Z'$

$f: A \rightarrow B \mid Z$
where:
 $f_1: A \rightarrow Z$
 $f_2: Z \rightarrow B$
 $f = f_2 \circ f_1$

thus, $A \rightarrow Z \rightarrow B$

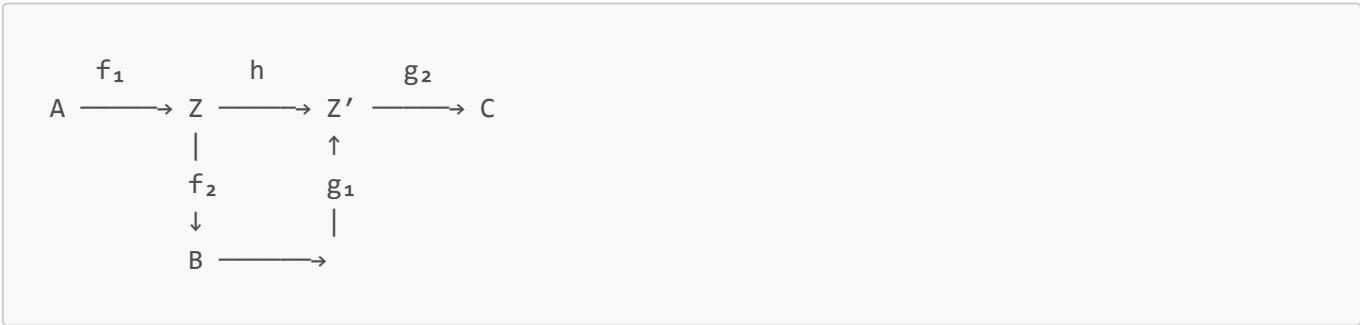
$g: B \rightarrow C \mid Z'$
where:
 $g_1: B \rightarrow Z'$
 $g_2: Z' \rightarrow C$
 $g = g_2 \circ g_1$

thus, $B \rightarrow Z' \rightarrow C$

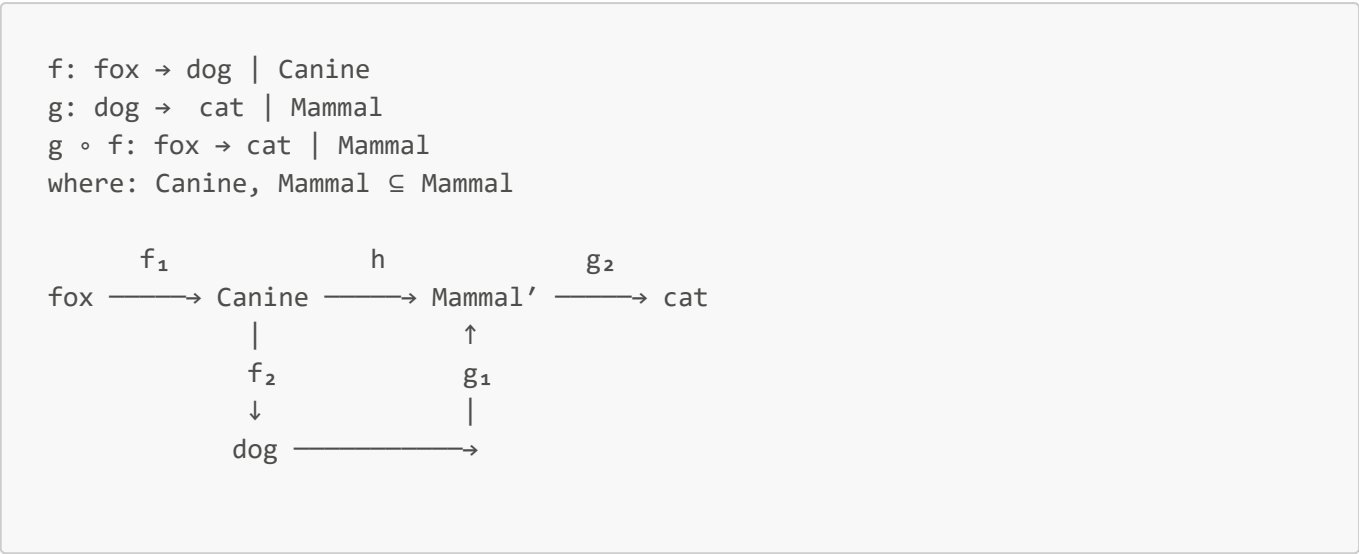
Z subsuming composition is expressed as follows.

$g \circ f = g_2 \circ (g_1 \circ f_2) \circ f_1$
 $h := g_1 \circ f_2: Z \rightarrow Z'$
 $g \circ f \mid Z' = g_2 \circ h \circ f_1$

In Diagram:

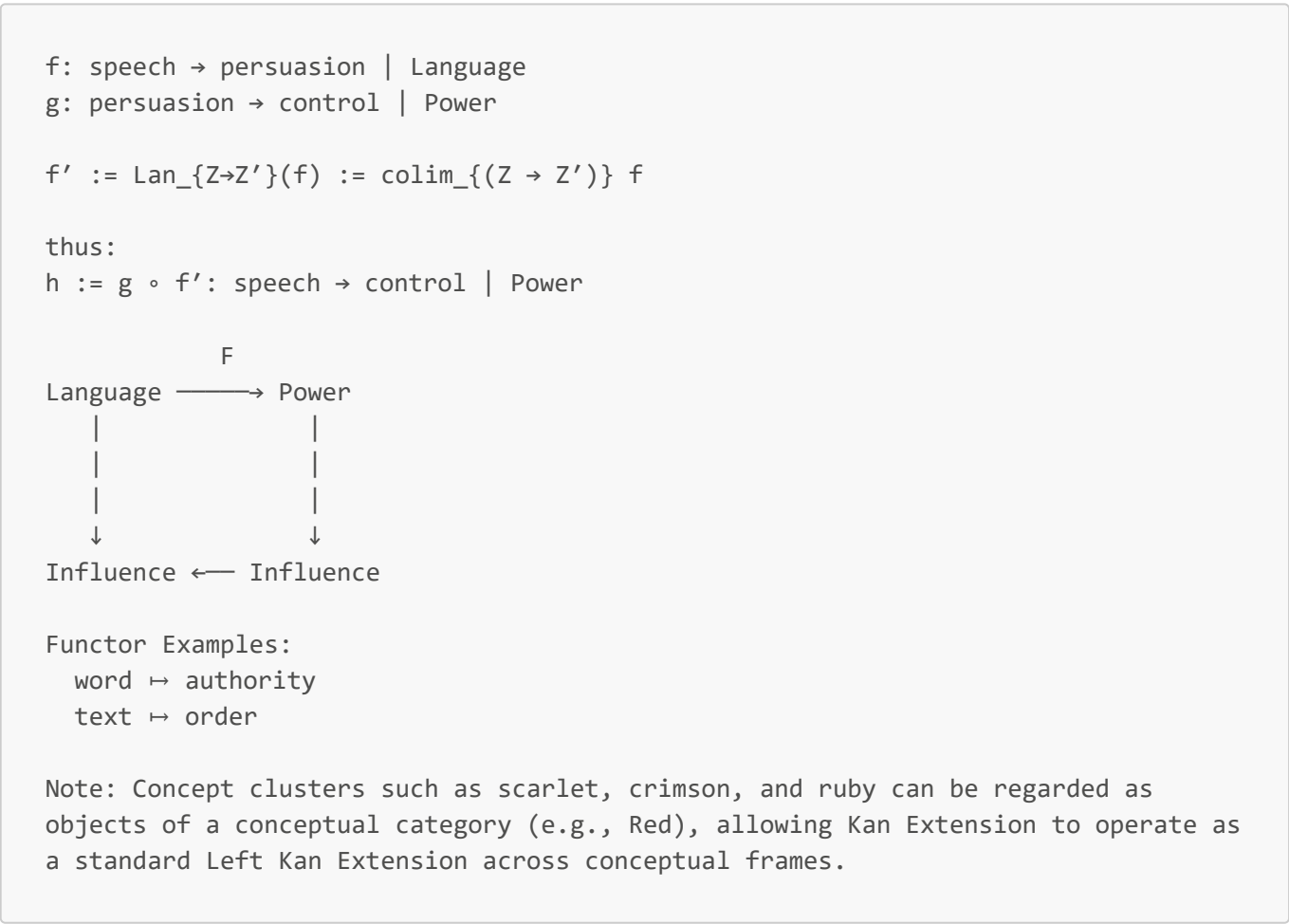


In NL Diagram:



Z-Frame Lifting via Kan Extension (Z ∉ Z')

In the case of Z ∉ Z', Left Kan Extension ensures conceptual flow structurally.



Category:

We define a Z-framed Conceptual Category **C|Z** (e.g. dog|Domesticated), C|C in simple notation **Concept** (e.g. Dog, Button...), as a category enriched over semantic frames Z.

Notation: We denote a morphism $f: X \rightarrow Y$ mediated by Z-frame as $f: X \rightarrow Y \mid Z$. This represents a meaning-preserving conceptual flow within the frame Z.

Objects

Ob(C|Z): A set of conceptual entities (lexical terms, abstract notions).

Examples: dog, she, computer, king, etc.

Morphisms

Each morphism is defined mediated by a Z-frame. **Hom(X, Y | Z)** = { $f \mid f: X \rightarrow Y \mid Z$ }, where $Z \in \text{Ob}(Z \text{ Frames})$ represents a semantic anchor or contextual frame.

A morphism $f: X \rightarrow Y \mid Z$ is interpreted as "X conceptually maps to Y within the semantic continuity defined by Z." Z gives the interpretive coherence or semantic clarification(e.g., $\text{dog} \rightarrow \text{pet} \mid \text{Domesticated}$).

Composition

Composition is defined only within a shared Z-frame or subsuming Z frame of local Z frames.

1. Within the same Z-frame If $f: A \rightarrow B \mid Z$, $g: B \rightarrow C \mid Z$, then $g \circ f$ is defined iff Z is shared.

```
f: computer → smartphone | Gadget
g: smartphone → mobile GPS | Gadget
g ∘ f = computer → mobile GPS | Gadget
```

2. Across compatible Z-frames (via σ -mediated composition)

Composition across different Z-frames (i.e., σ -mediated composition) is possible when the individual Z-frames are compatible under a higher semantic frame. This higher frame Z must be able to subsume both the local frames Z_1 and Z_2 by the conditions $Z_1 \subseteq Z$ and $Z_2 \subseteq Z$. This condition ensures that both morphisms can coexist within the same larger context, preserving the continuity of meaning across frames.

If $f: A \rightarrow B \mid Z_1$, $g: B \rightarrow C \mid Z_2$, then $g \circ f$ is defined iff there exists a higher frame Z such that $Z_1 \subseteq Z$ and $Z_2 \subseteq Z$.

```
f: computer → smartphone | Gadget
g: match → knife | Tool
```

If there exists a higher frame `Instrument` that subsumes both `Gadget` and `Tool`, then the composite morphism becomes

```
g ∘ f = computer → knife | Instrument
where Instrument ⊇ Gadget, Tool
```

This composition is associative within a Z-frame

```
If: f: A → B | Z, g: B → C | Z, h: C → D | Z
then: (h ∘ g) ∘ f | Z = h ∘ (g ∘ f) | Z.
```

This guarantees that within a single Z-frame, composition behaves as expected according to the standard rules of category theory.

σ -mediated Composition:

In the case of σ -mediated composition, associativity holds when all involved Z-frames are subsumed by a common higher Z-frame.

```
Let f: A → B | Z1, g: B → C | Z2, and h: C → D | Z3
(h ∘ g) ∘ f | Z = h ∘ (g ∘ f) | Z is valid
where Z1 ⊆ Z, Z2 ⊆ Z, and Z3 ⊆ Z.
```

This ensures that all morphisms can coexist within the same conceptual space, and the meaning flow is preserved across the frames.

Example:

If $f: \text{computer} \rightarrow \text{smartphone} \mid \text{Gadget}$, $g: \text{smartphone} \rightarrow \text{mobile GPS} \mid \text{Gadget}$, and $h: \text{mobile GPS} \rightarrow \text{navigation} \mid \text{Travel}$, then the composite morphism is defined as

```
h ∘ (g ∘ f) | Travel = computer → navigation | Travel
where: Gadget ⊆ Travel is defined
```

Here, Instrument is a higher Z-frame that subsumes both Gadget and Travel.

Let $C|Z$ be a Conceptual Category with partial composition $\circ|Z$.

Partial Composition in Z-Framed Category

Typing judgment

$f: A \rightarrow B \mid Z \in \text{Hom}(A, B \mid Z)$

Composition judgment

If $f: A \rightarrow B \mid Z_1$ and $g: B \rightarrow C \mid Z_2$,
and $\exists Z$ such that $Z_1 \subseteq Z$ and $Z_2 \subseteq Z$
then define:

$g \circ f : A \rightarrow C \mid Z$

This defines a partial composition operation:

$\circ : \text{Hom}(A, B \mid Z_1) \times \text{Hom}(B, C \mid Z_2) \rightarrow \text{Hom}(A, C \mid Z)$

Identity Morphism

In Category Theory, Identity Morphism is always defined.

For every morphism $f: A \rightarrow B$,
there exist identity morphisms $\text{id}_A: A \rightarrow A$ and $\text{id}_B: B \rightarrow B$ such that

$f \circ \text{id}_A = f$
 $\text{id}_B \circ f = f$

However, in Conceptual Topology, morphisms are mediated by Z frame, thus the identity morphis is not always given unless Z is defined.

Two Types of Identity Morphism in Conceptual Topology

1. $f: X \rightarrow X \mid X$ (Category-theoretic identity)

For every object X , there exists a mediated identity morphism

$$\text{id}_X: X \rightarrow X \mid Z$$

such that for $Z = X$ (i.e., the identity is mediated by the object itself), we define

$$f: X \rightarrow X \mid X$$

$$f \circ \text{id}_X = f$$

$$\text{id}_X \circ f = f$$

e.g. $f: \text{dog} \rightarrow \text{dog} \mid \text{dog}$

2. $f: X \rightarrow X \mid Z$ (Mediated identity with conceptual flow)

$$f: X \rightarrow X \mid Z$$

$$f: X \rightarrow Z$$

$$f^{-1}: Z \rightarrow X$$

$$f^{-1} \circ f \cong \text{id}_X$$

e.g. $\text{you} \rightarrow \text{you} \mid \text{externalized perspective}$

NL: you are you

Since the identity morphism passes through an external anchor point, the identity morphism is defined quasi-identical.

e.g. $\text{dog} \rightarrow \text{perro} \mid \text{собака}$

Simplified Form of Identity Morphism:

1. $f: X \rightarrow X \mid X$ (Category-theoretic identity)
 In simplified form: X
 or more explicitly: id_X
2. $f: X \rightarrow X \mid Z$ (Mediated identity with conceptual flow)
 In simplified form: $X \ Z$

Mirror Morphism Definition:

Each mirror maps conceptual transitions across vocabularies while preserving morphic identity up to rupture—that is, it allows for conceptual divergence that still respects underlying structural continuity, even if exact invertibility is not preserved.

```
f : X → Y | Z ∈ Di
f' : X' → Y' | Z ∈ Di+1
⇒ X' ≠ X, but cod(f) = cod(f') | CD (CD = codomain)
```

We define f' as a mirror-correspondent morphism of f under a given Z -frame, if and only if:

```
∃Z: rupture(f, f' | Z) ≠ ∅
∧ cod(f) = cod(f') | CD
```

```

      Z = baby
puppy ← girl    //specified: size + young
  |         |
  |         |
dog        she   //abstraction
  \        /
    Mammal
```

Note: $Z: \text{rupture}(f, f' \mid Z) \neq \emptyset$ means that there exists a Z -frame under which f and f' exhibit structural divergence—i.e., they are not fully invertible but still converge at the codomain level.

For example, let $Z = \text{abstraction}$. This allows a conceptual transition from $girl \rightarrow she$ and $puppy \rightarrow dog$, treating them as mirror morphisms under a shared conceptual frame.

However, if we take $Z = \text{agency}$, a rupture emerges: $puppy \rightarrow dog$ lacks agency, while $girl \rightarrow she$ retains it. Hence, $\text{rupture}(f, f' \mid \text{agency}) \neq \emptyset$, yet f and f' still align toward the same codomain (e.g., *mammal*).

Quasi-Natural Transformation of Meaning Systems

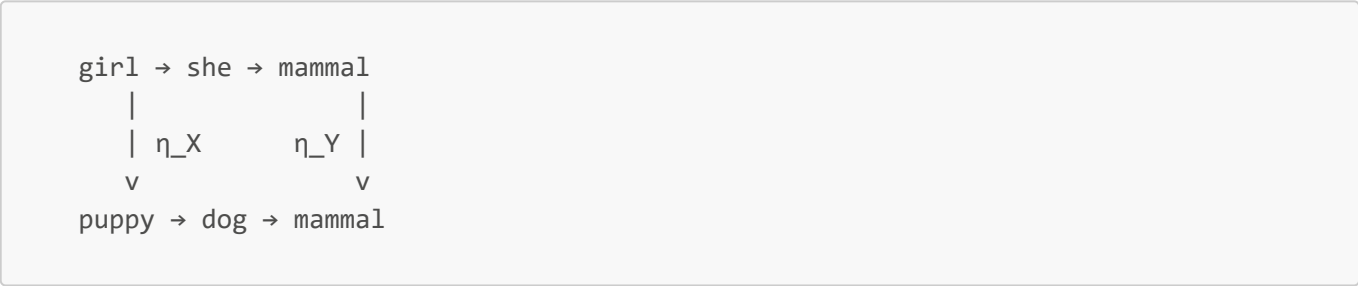
A **Morphic Chain Mirror** is a contextual correspondence between two morphic chains drawn from distinct but meaning-aligned vocabularies. This correspondence is realized through a **quasi-natural transformation** under a shared intermediating Z-frame.

$$\eta: D_i \Rightarrow D_{i+1} \mid CD \text{ (CD = codomain)}$$
$$\eta_X \circ D_i(\{f_1 \mid Z_1, \dots, f_n \mid Z_n\}) \approx D_{i+1}(\{f'_1 \mid Z_1, \dots, f'_n \mid Z_n\}) \circ \eta_Y \mid CD$$

for all $f_j: X_j \rightarrow Y_j \mid Z_j \in D_i$,
where $f'_j: \eta_X(X_j) \rightarrow \eta_Y(Y_j) \mid Z_j$

Then, η is said to be a quasi-natural transformation under the Z-frame
i.e. $\eta \in \text{Mor}(C)$ where C is the contextual meaning category
Example: $\eta: \text{girl} \rightarrow \text{puppy} \mid Z = \text{Baby}$

Diagram:



1.2. σ Operator as Functor

Definition: Conceptual Shifting Morphism (σ)

$\sigma: D(X_{n-1} \mid X) \rightarrow D(X_{n-1} \mid X)$

such that $\sigma \oplus f \in M|Z$ if and only if type compatibility holds:

$\forall A, B, (A \rightarrow B) \circ \sigma(X)$ is valid if:

($A \gg X$ or $X \gg A$)

and

($B \gg X$ or $X \gg B$)

Definition: Subsumption

$A \gg X \equiv A \sqsubseteq X$

Definition: SubsumedBy

$X \gg A \equiv X \sqsubseteq A$

Example:

$\text{king} \rightarrow \text{king} \gg \text{human} \rightarrow \text{human}$

$\Rightarrow \text{king} \gg \text{human} \rightarrow \text{valid}$

$\text{human} \rightarrow \text{human} \gg \text{queen} \rightarrow \text{queen}$

$\Rightarrow \text{human} \gg \text{queen} \rightarrow \text{valid}$

Conceptual Operators

Conceptual Operator σ modifies morphism as follows.

$\sigma(X). \text{Not}(x)\{ A \rightarrow B \mid Z \}$	\rightarrow	Rupture under Z frame
$\sigma(X). \text{so_much}(x)\{ A \rightarrow B \mid Z \}$	\rightarrow	Preservation & amplification under Z frame
$\sigma(X). \gg(x,y)$ function form	\rightarrow	Conceptual Shifting x to y (Generalization) as
$\sigma(X). \ll(x,y)$ function form	\rightarrow	Downward Shifting x to y (Specialization) as
$\sigma(X). \>(x,y)$	\rightarrow	Conceptual Shifting

Conceptual Morphism Set Operators

Addition (\oplus):

$$\sigma(X). \oplus(f, A_{n-1} \mid Z): D(A_{n-1} \mid Z) \rightarrow D(B_{n-1} \mid Z) \mid Z$$

$$\sigma(X). \oplus(f_1, f_2) : A_{n-1} := \{f_1, f_2\}$$

Subtraction (\ominus):

$$\ominus: A_{n-1} \ominus \{f_i\}$$

$$\sigma(X). \ominus(f, A_{n-1} \mid Z): D(A_{n-1} \mid Z) \rightarrow D(B_{n-1} \mid Z) \mid Z$$

- \oplus operator is σ_{safe} if Z alignment is preserved.
- \ominus operator is potentially σ_{unsafe} but can be σ_{safe} if resulting chain preserves the underlying morphic continuity Z .

Example

$$\begin{aligned} &\{\text{Royalty}\vec{,} \text{Male}\vec{,} \text{Human}\vec{\} \ominus \{\text{Male}\vec{\} \oplus \{\text{Female}\vec{\} \\ &= \{\text{Royalty}\vec{,} \text{Female}\vec{,} \text{Human}\vec{\} \\ &= \text{queen} \end{aligned}$$

Conceptual Mapping

$$C_{\text{chain}} = \{ f_1, f_2, \dots, f_n \mid Z \} \in D(C_{n-1} \mid Z)$$

$$\sigma(X): D(A_{n-1} \mid Z) \rightarrow D(B_{n-1} \mid Y) \mid Z, Y \in CD$$

where:

$$D(A_{n-1} \mid Z) = \text{source morphic chain}$$

$$D(B_{n-1} \mid Y) = \text{target morphic chain}$$

$$CD = \text{codomain alignment (conceptual anchor)}$$

$\sigma(X)$ is not strict functorial \rightarrow quasi-alignment under conceptual equivalence

$$\sigma(X) \approx \eta: D_i \Rightarrow D_{i+1} \mid CD \text{ (Quasi-Natural Transformation interpretation)}$$

Example:

$$\sigma(X). \succ(\text{puppy} \rightarrow \text{dog} \rightarrow \text{mammal} \mid \text{Canine, Human}) \ni \text{girl} \rightarrow \text{she} \rightarrow \text{mammal} \mid \text{Human}$$

where: canine, Human \in Mammal

Identity Morphism of σ

```

word is word
thus:
word  $\cong$  Nat(Hom(-, word | Word), Fib(word))

 $\sigma_{id}(Z)$ . OP(X,Z) =  $\sigma$  such that  $\sigma(f,Z) = f$  for all  $f \in \text{Hom}(X, X | Z)$  unless OP is
 $\sigma_{unsafe}$  such that word is not a word:  $\sigma(\text{Word})$ . Not(word  $\rightarrow$  /word)

 $\sigma_{id}(\text{Word})$ . OP(word, Word) = word

 $\sigma_{id}(\text{Word})$ . OP(f, Word) = f for all  $f: \text{word} \rightarrow \text{word} | \text{word}$ 
since:  $M|Z = \{ f_n \circ \dots \circ f_1 \mid \text{all } f_i: X_i \rightarrow X_{i+1} \mid Z \wedge \forall i, j: f_i \cong f_j \mid Z = \text{Word} \}$ 

 $\sigma_{id}(\text{Word}) \in M|Word$ 
 $\sigma(\text{Word}) \circ \sigma_{id}(\text{Word}) = \sigma(\text{Word})$ 
 $\sigma_{id}(\text{Word}) \circ \sigma(\text{Word}) = \sigma(\text{Word})$ 
 $\therefore$  word is word

```

Associativity of σ

```

 $\sigma_1(Z)$ . OP(D(An-1 | Z), Z) = D(Zn-1 | Z)
 $\sigma_2(Z)$ . OP(D(Bn-1 | Z), Z) = D(Zn-1 | Z)

Then the composition  $\sigma_2 \circ \sigma_1$ :
 $\sigma_{comp}(Z)$ . OP(D(Zn-1 | Z), D(Zn-1 | Z)) = D(Zn-1 | Z)

where: OP is not  $\sigma_{unsafe}$  and under shared Z frame

Associativity
For all  $\sigma_1(Z)$ ,  $\sigma_2(Z)$ ,  $\sigma_3(Z)$  such that their domains/codomains match for
composition:
 $(\sigma_3 \circ \sigma_2) \circ \sigma_1 = \sigma_3 \circ (\sigma_2 \circ \sigma_1)$ 

Thus,  $\sigma$  composition operator is associative under Monoid structure.

```

Example:

```

Let  $\sigma_1 = \sigma(\text{Mammal})$ .  $\gg(\text{canine} \rightarrow \text{mammal}, \text{Life}) = (\text{life} \rightarrow \text{life} | \text{Life})$ 
Let  $\sigma_2 = \sigma(\text{Mammal})$ .  $\gg(\text{mammal} \rightarrow \text{animal}, \text{Life}) = (\text{life} \rightarrow \text{life} | \text{Life})$ 
Let  $\sigma_3 = \sigma(\text{Mammal})$ .  $\gg(\text{animal} \rightarrow \text{livingBeing}, \text{Life}) = (\text{life} \rightarrow \text{life} | \text{Life})$ 

Conclusion:
 $(\sigma_3 \circ \sigma_2) \circ \sigma_1 = \sigma_3 \circ (\sigma_2 \circ \sigma_1) = (\text{life} \rightarrow \text{life} | \text{Life})$ 

```

1.3. σ Operator as Kan Extension

Functorial Properties of σ mapping

A Functor $F: C \rightarrow D$ is a mapping between categories satisfying:

- Object mapping: For each $X \in \text{Ob}(C)$, $F(X) \in \text{Ob}(D)$
- Morphism mapping: For each $f: X \rightarrow Y \in \text{Mor}(C)$, $F(f): F(X) \rightarrow F(Y) \in \text{Mor}(D)$
- Identity preservation: $F(\text{id}_X) = \text{id}_{\{F(X)\}}$
- Composition preservation: $F(f \circ g) = F(f) \circ F(g)$

We define $\sigma: D(A_{n-1} \mid Z) \gg D(B_{n-1} \mid Z')$ as such a Functor.

σ Operator as Kan Extension

Let:

$D(A_{n-1} \mid Z) := \text{Category of Morphic Chains over } Z\text{-frame } Z$

$D(B_{n-1} \mid Z') := \text{Category of Morphic Chains over } Z'\text{-frame } Z'$

Define:

$\sigma_{\text{safe}} = \text{Lan}_{\sigma} : D(A_{n-1} \mid Z) \gg D(B_{n-1} \mid Z')$

such that:

For any object $d \in D(B_{n-1} \mid Z')$,

$\text{Lan}_{\sigma} (D(A_{n-1} \mid Z))(d) := \text{colim}_{\{(c, f: \sigma(c) \rightarrow d)\}} D(A_{n-1} \mid Z)(c)$

And:

For any morphism $h: d \rightarrow d'$ in $D(B_{n-1} \mid Z')$,

$\text{Lan}_{\sigma} (h)$ is defined to preserve functoriality

$\text{Lan}_{\sigma} (h) \circ \text{Lan}_{\sigma} (f) = \text{Lan}_{\sigma} (h \circ f)$

Therefore:

σ_{safe} satisfies:

Object-level safe lifting: $\text{Ob}(D(A_{n-1} \mid Z)) \rightarrow \text{Ob}(D(B_{n-1} \mid Z'))$

Morphism-level safe lifting: $\text{Mor}(D(A_{n-1} \mid Z)) \rightarrow \text{Mor}(D(B_{n-1} \mid Z'))$

$\sigma_{\text{safe}} = \text{Left Kan Extension}$ guarantees the Quasi-Natural Transformation property

$\forall f \in \text{Mor}(D(A_{n-1} \mid Z)),$

$\text{Lan}_{\sigma} (G \circ f) = (\text{Lan}_{\sigma} G) \circ (\text{Lan}_{\sigma} f)$

Relation to Qasi-Natural Transformation

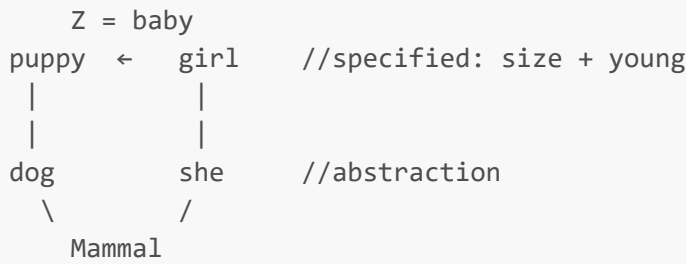
The σ mapping operator, defined as a Functor $\sigma: D(A_{n-1} \mid Z) \gg D(B_{n-1} \mid Z')$, exhibits structural alignment with Quasi-Natural Transformation (QNT) in the following way.

In the original formulation of QNT in this framework

$$\eta: D_i \Rightarrow D_{i+1} \mid \text{CD (codomain)}$$

$$\eta_X \circ D_i(\{f_1 \mid Z_1, \dots, f_n \mid Z_n\}) \approx D_{i+1}(\{f'_1 \mid Z_1, \dots, f'_n \mid Z_n\}) \circ \eta_Y$$

Diagram:



In this diagram, Quasi-Natural Transformation η aligns morphic chains between $\text{puppy} \rightarrow \text{dog} \rightarrow \text{mammal}$ and $\text{girl} \rightarrow \text{she} \rightarrow \text{mammal}$ within a shared codomain mammal (under Z-frame "Mammal").

The Quasi-Natural Transformation mediates conceptual flow correspondence across different morphic chain categories under a shared or shifted Z-frame.

In the Kan Extension formalization:

$$\text{Lan}_\sigma (D(A_{n-1} \mid Z)) = D(B_{n-1} \mid Z')$$

The lifting of the entire functor $D(A_{n-1} \mid Z)$ under σ corresponds to constructing a universal QNT from $D(A_{n-1} \mid Z)$ to $D(B_{n-1} \mid Z')$.

More precisely, for any object $d \in D(B_{n-1} \mid Z')$:

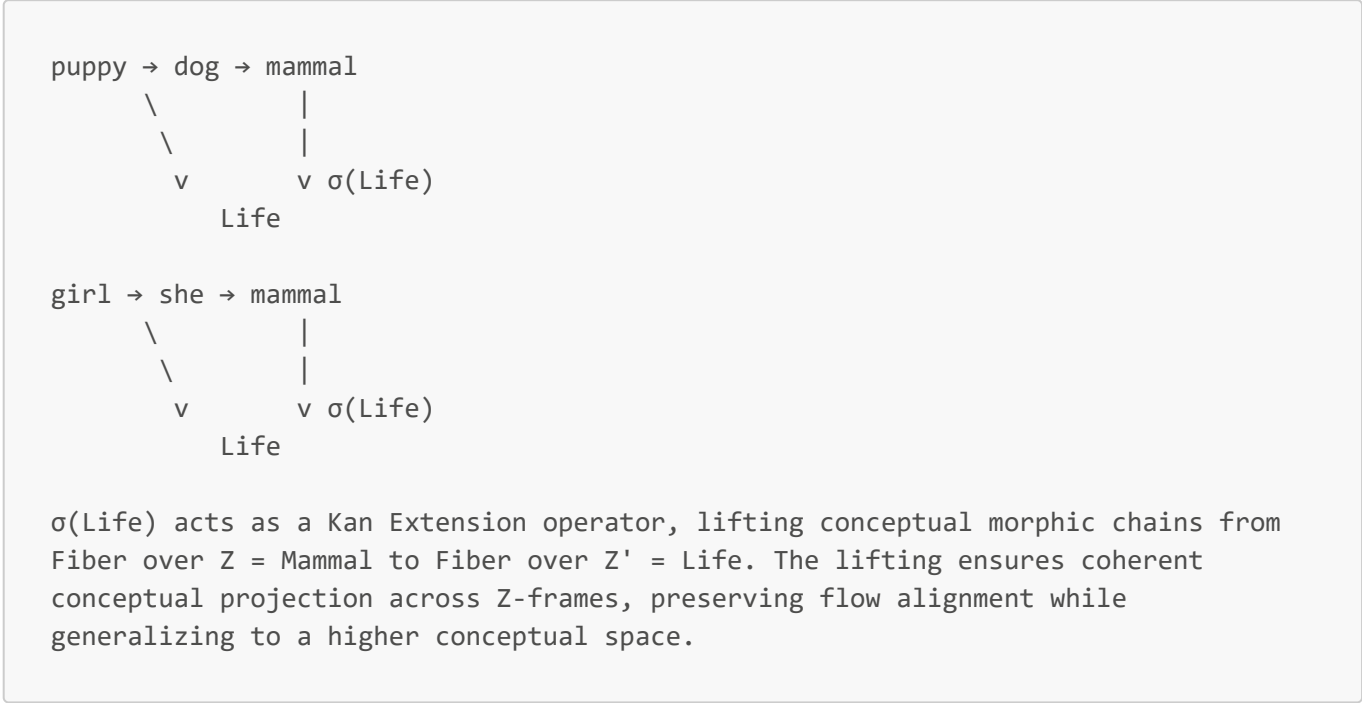
$$\text{Lan}_\sigma (D(A_{n-1} \mid Z))(d) := \text{colim}_{\{(c, f: \sigma(c) \rightarrow d)\}} D(A_{n-1} \mid Z)(c)$$

yields a canonical shifting from the conceptual flow space under Z to the corresponding conceptual flow space under Z' , respecting the structural continuity required by QNT.

Thus:

$\sigma_{\text{safe}} \approx \text{Left Kan Extension} \approx \text{Universal Quasi-Natural Transformation between } D(A_{n-1} \mid Z) \text{ and } D(B_{n-1} \mid Z')$

Diagram:



This formalization guarantees that the Quasi-Natural Transformation property observed in the original Conceptual Cage structure is preserved and generalized through the Kan Extension framework, providing a categorical foundation for conceptual flow lifting.

1.4 Kan Extension as Horizontal Conceptual Shifting and Cone Structure

Conceptually, σ operator as Kan Extension performs not only lifting of morphic chains but also acts as a horizontal mapping across Z-frames, shifting conceptual flow from Fiber over Z to Fiber over Z'.

Diagrammatically, this can be visualized as a horizontal shift.

Fiber over Z (Mammal):

puppy → dog → mammal
girl → she → mammal

↓↓↓↓↓ Kan Extension $\sigma(\text{Life})$

Fiber over Z' (Life):

girl → she → mammal → Life
puppy → dog → mammal → Life

Applying $\sigma(\text{Life})$ results in a horizontal lifting of codomain alignment

Recursive Kan Extension as Iterated Colimit of Conceptual Shiftings

Conceptually, Recursive Kan Extension can be understood as constructing an iterated colimit of sequential conceptual shiftings (σ operators) across Z-frames.

Conceptual Ladder Structure:

Fiber over Z_0
↓ σ_1
Fiber over Z_1
↓ σ_2
Fiber over Z_2
↓ σ_3
Fiber over Z_3
↓ ...

NL: turtle → reptile → animal → ...

Iterated Colimit Perspective:

At each stage, the application of σ_n corresponds to forming a conceptual projection from Fiber over Z_{n-1} to Fiber over Z_n .

The entire ladder:

$$\text{Lan}_{\{\sigma_n\}} \circ \dots \circ \text{Lan}_{\{\sigma_3\}} \circ \text{Lan}_{\{\sigma_2\}} \circ \text{Lan}_{\{\sigma_1\}}$$

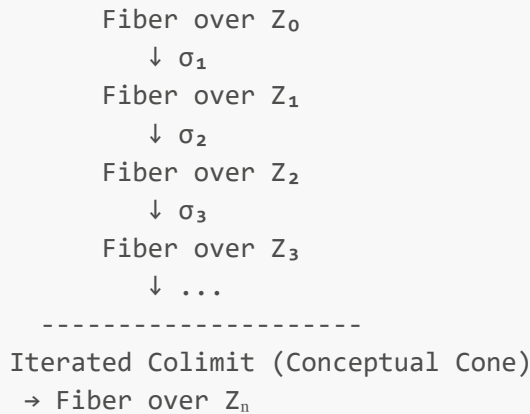
can be viewed formally as an iterated colimit over the sequence of Z-frames, forming a conceptual cone over the diagram.

$$\text{colim}_{\{Z_0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow \dots \rightarrow Z_n\}} (\text{Lan}_{\{\sigma_i\}}(\pi^{-1}(Z_{i-1})))$$

Interpretation:

Each $\text{Lan}_{\{\sigma_i\}}$ acts as a conceptual lifting operation, progressively shifting semantic flow across Z-frame layers. The cumulative structure forms an iterated conceptual cone, whose colimit aligns the entire sequence into the semantic flow space under Z_n .

Diagram (Iterated Colimit View):



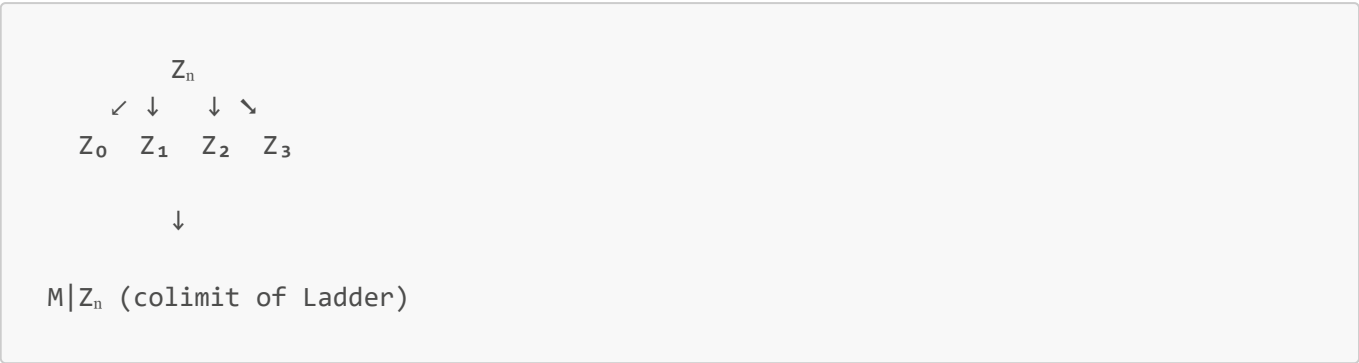
NL: tortoise \rightarrow turtle \rightarrow reptile \rightarrow animal \rightarrow ... | Iterated Colimit Result = Muticellular Organism

Formal Expression:

$$\text{Iterated_Colimit} = \text{colim}_{\{Z_0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow \dots \rightarrow Z_n\}} (\text{Lan}_{\{\sigma_i\}}(\pi^{-1}(Z_{i-1})))$$

This conceptual ladder forms an iterated semantic cone, whose colimit aligns the entire Z-frame sequence into the unified semantic flow space under Z_n .

Diagram:



A cone on a diagram $F: J \rightarrow C$ is a universal natural transformation from a constant diagram ΔX to F . In this case:

$$\Delta(\pi^{-1}(Z_n)) \Rightarrow \text{Ladder of } \text{Lan}_{\{\sigma_i\}}(\pi^{-1}(Z_{i-1}))$$

or as monoid structure

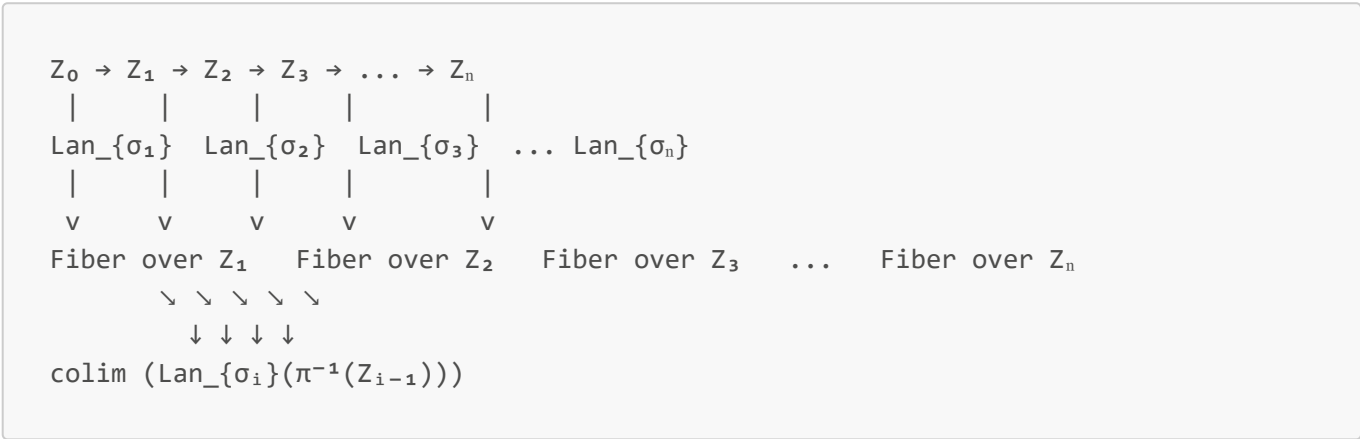
$$M|_{Z_n} \{ F_n \circ \dots \circ F_1 \mid \text{all } F_i: F_i \rightarrow F_{i+1} \mid Z_n \wedge \forall i, j: F_i \cong F_j \mid Z_n \}$$

∞ -Morphic Interpretation of Recursive Ken Extension

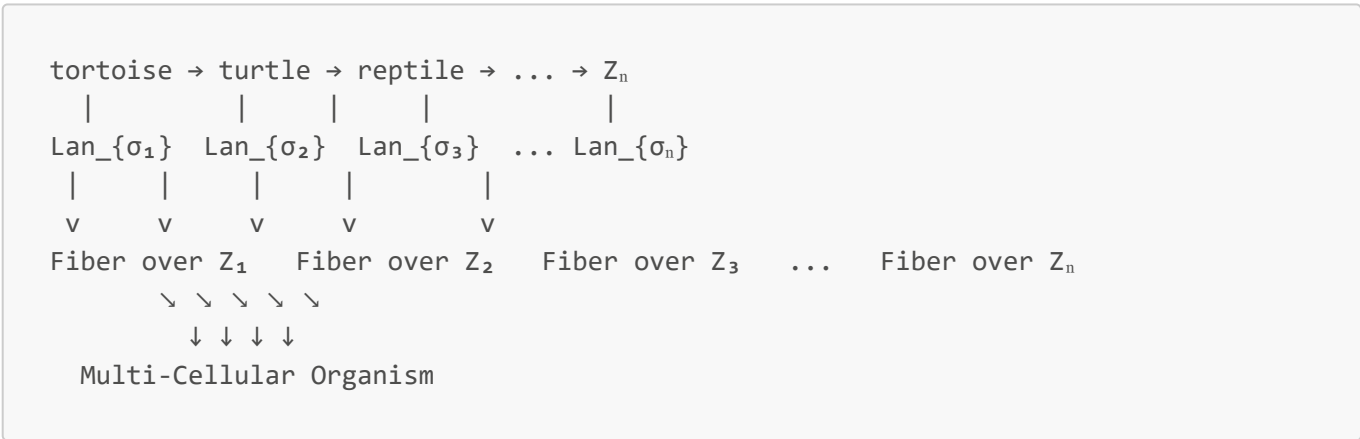
Viewed categorically, this recursive construction aligns with the notion of ∞ -morphisms or higher morphic flows, where each application of $\text{Lan}_{\{\sigma_i\}}$ corresponds to a morphism in an extended conceptual category, and their collective composition forms an ∞ -structured cone.

$$\infty\text{-Universal Product} = \text{colim}_{\{Z_0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow \dots \rightarrow Z_n\}} (\text{Lan}_{\{\sigma_i\}}(\pi^{-1}(Z_{i-1})))$$

Diagram:



NL Diagram:



This interpretation enables the Conceptual Topos or Conceptual Topology to naturally support recursive, layered conceptual flow, where mappings can extend across arbitrarily many Z-frames while preserving structural coherence.

Example: Iterated Kan Extension of Conceptual Ladder

Step 1:

```


$$Z_0 = \text{Turtle}$$


$$Z_1 = \text{Reptile}$$


$$\sigma_1 = \sigma(\text{Reptile}): \text{Turtle} \rightarrow \text{Reptile}$$


$$\text{Lan}_{\{\sigma_1\}}(\text{Fiber over Turtle}) \rightarrow \text{Fiber over Reptile}$$


```

Step 2:

```


$$Z_2 = \text{Animal}$$


$$\sigma_2 = \sigma(\text{Animal}): \text{Reptile} \rightarrow \text{Animal}$$


$$\text{Lan}_{\{\sigma_2\}}(\pi^{-1}(\text{Reptile})) \rightarrow \pi^{-1}(\text{Animal})$$


```

Step 3:

```


$$Z_3 = \text{Life}$$


$$\sigma_3 = \sigma(\text{Life}): \text{Animal} \rightarrow \text{Life}$$


$$\text{Lan}_{\{\sigma_3\}}(\text{Fiber over } \pi^{-1}(\text{Animal})) \rightarrow \pi^{-1}(\text{Life})$$


```

Composition:

```


$$\text{Lan}_{\{\sigma_3\}} \circ \text{Lan}_{\{\sigma_2\}} \circ \text{Lan}_{\{\sigma_1\}}(\pi^{-1}(\text{Turtle}))$$


```

Colimit:

```


$$\text{colim}_{\{Z_0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow Z_3\}} \text{Lan}_{\{\sigma_i\}}(\pi^{-1}(Z_{i-1})) = \pi^{-1}(Z_3) = \pi^{-1}(\text{Life})$$


```

NL:

```


$$\text{Turtle} \rightarrow \text{Reptile} \rightarrow \text{Animal} \rightarrow \text{Life}$$


```

Conceptual flow lifted across Z-frame layers as iterated Kan Extensions, converging to the unified flow under Life.

Universal Property of $\text{Lan}_{\{\sigma_i\}}$

$$\begin{array}{ccc}
 Z_{\{i-1\}} & \xrightarrow{\sigma_i} & Z_i \\
 \downarrow H & & \downarrow K \\
 E & \xleftarrow{\alpha: H \Rightarrow K \circ \sigma_i^*} & E
 \end{array}$$

Given a base conceptual shifting operator

$$\sigma_i: Z_{\{i-1\}} \gg Z_i$$

we define $\text{Lan}_{\{\sigma_i\}}$ for corresponding fiber categories

$$\text{Lan}_{\{\sigma_i\}}: \pi^{-1}(Z_{\{i-1\}}) \rightarrow \pi^{-1}(Z_i)$$

To satisfy the following **universal property**, for any functor

$$H: \pi^{-1}(Z_{\{i-1\}}) \rightarrow E$$

and any functor

$$K: \pi^{-1}(Z_i) \rightarrow E$$

with a natural transformation

$$\alpha: H \Rightarrow K \circ \sigma_i^*$$

(where σ_i^* is the pullback functor along σ_i),

there exists a unique natural transformation

$$\beta: \text{Lan}_{\{\sigma_i\}}(H) \Rightarrow K$$

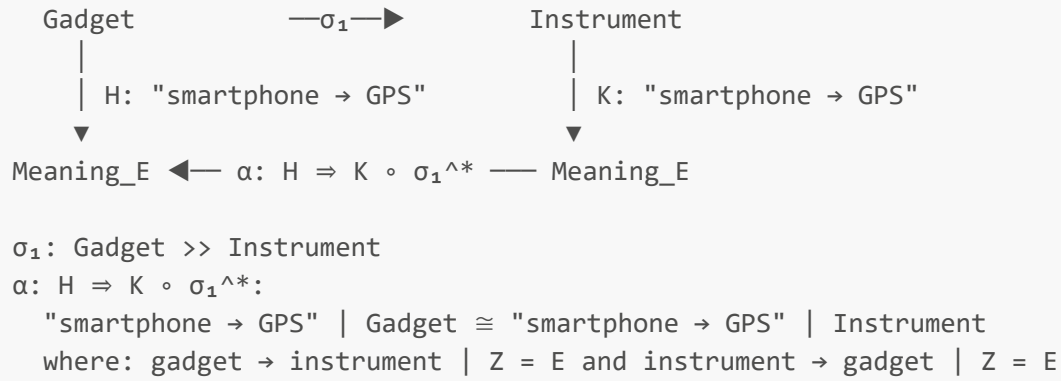
such that the following diagram commutes

$$\begin{array}{ccc}
 H & & \\
 \downarrow \alpha & & \\
 K \circ \sigma_i^* & & \\
 \uparrow & & \\
 \text{Lan}_{\{\sigma_i\}}(H) \circ \sigma_i^* & &
 \end{array}$$

In formal terms

$$\text{Nat}(H, K \circ \sigma_i^*) \cong \text{Nat}(\text{Lan}_{\{\sigma_i\}}(H), K)$$

NL Diagram: The operation `smartphone → GPS` maintains the structural coherence under Gadget when lifting it to Instrument.

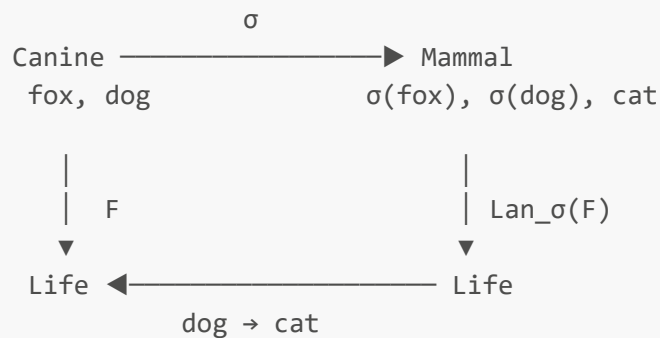


NL Example 2

A functor $F: \text{Canine} \rightarrow \text{Life}$
 A functor $\sigma: \text{Canine} \rightarrow \text{Mammal}$

Then the left Kan extension of F along σ is a functor.
 $\text{Lan}_\sigma(F): \text{Mammal} \rightarrow \text{Life}$

Object $d \in \text{Mammal}$, $c \in \text{Canine}$, $d, c \in \text{Life}$
 $\text{Lan}_\sigma(F)(d) = \text{colimit } (\sigma(c) \rightarrow d)(F(c))$



e.g. $\text{Lan}_\sigma(f)(\text{fox}) = \text{dog} \rightarrow \text{cat}$

$\text{fox}, \text{dog}, \text{cat} \in \text{Life}$
 $\text{fox}, \text{dog}, \text{cat} \in \text{Mammal}$
 $\text{fox}, \text{dog} \in \text{Canine}$

We focus on fox.

$\text{fox} \in \text{Canine}$
 $F(\text{fox}) = \text{dog} \in \text{Life}$
 $\sigma(\text{fox}) := \text{dog} \rightarrow \text{cat} \mid \text{Mammal}$
 a gorilla. (QNT)
 $\text{dog} \rightarrow \text{cat} \mid \text{Mammal}$

`// fox is a canine`
`// fox behaves like dog in the context of Life`
`// Under Mammal, fox is like a dog or a cat, not`
`a gorilla. (QNT)`
`// dogs, as mammals, resemble cats.`

$\Rightarrow \text{Lan}_\sigma(F)(\text{fox}) = \text{dog} \rightarrow \text{cat} \mid \text{Life}$

That is:

A fox, seen as a canine, is mapped to a dog under the functor: $\text{Canine} \rightarrow \text{Life}$,
But as mammals, dogs resemble cats,

The fox is, in Life, conceptually equivalent to a transition, $\text{dog} \rightarrow \text{cat}$ – that is, the fox is basically dog or cat.

Note:

The transition $(\text{dog} \rightarrow \text{cat} \mid \text{Life})$ is interpreted as a Quasi-Natural Transformation (QNT):

$\eta: \text{dog} \Rightarrow \text{cat} \mid \text{Mammal}$

$\text{dog} \rightarrow \text{Life} \circ \eta_Y \approx \eta_X \circ \text{cat} \rightarrow \text{Life}$

NL: "Cat is a dog, and dog is a cat – basically, under the Life frame."

Example:

$f_1: \text{king} \rightarrow \text{man} \quad \mid Z_1$
 $f_2: \text{woman} \rightarrow ? \quad \mid Z_1$

Conceptual Shifting Operator

$\sigma_1: Z_1 \gg Z_2 \quad (\text{GenderedEntity} \gg \text{SocialRole}: \text{Generalization})$

$\sigma_2: Z_3 \ll Z_2 \quad (\text{SocialRole} \ll \text{RoyalSemantic}: \text{Specification})$

$\text{Lan}_{\{\sigma_1\}}(f_1): \text{king} \rightarrow \text{male-role} \mid Z_2$

$\text{Lan}_{\{\sigma_1\}}(f_2): \text{female-role} \rightarrow \text{female-role} \mid Z_2$

$\text{Lan}_{\{\sigma_2\}}(\text{Lan}_{\{\sigma_1\}}(f_1)): \text{king} \rightarrow \text{king} \mid Z_3$

$\text{Lan}_{\{\sigma_2\}}(\text{Lan}_{\{\sigma_1\}}(f_2)): \text{queen} \rightarrow \text{queen} \mid Z_3$

We may define $\sigma_3 = \sigma_2 \circ \sigma_1 : Z_1 \rightarrow Z_3$ as the composition of generalization and specification,

allowing us to write $\text{Lan}_{\{\sigma_3\}}(f) \cong \text{Lan}_{\{\sigma_2\}}(\text{Lan}_{\{\sigma_1\}}(f))$

Alternately, We define the above as fibers.

$\pi^{-1}(Z_1): \text{Gendered Entity}$

$\pi^{-1}(Z_2): \text{Social Role}$

$\pi^{-1}(Z_3): \text{Royal Semantic}$

$\text{colim}_{\{Z_1 \rightarrow Z_2 \rightarrow Z_3\}} (\text{Lan}_{\{\sigma_i\}}(\pi^{-1}(Z_i)))$

$\therefore \text{Lan}_{\{\sigma_2\}}(\text{Lan}_{\{\sigma_1\}}(\text{king} - \text{man} + \text{woman})) \cong \text{queen}$

Diagram:

```

Z1: Gendered Entity
  king → man
  woman

σ1 ↓ Generalized to Social Role

Z2: Social Role
  king → male-role
  female-role → female-role

σ2 ↓ Specified to Royalty

Z3: Royal Semantic
  king → king
  queen → queen
    
```

Iterated Kan Extension Ladder over the Z-frame

```

Z0 → Z1 → Z2 → ... → Zn

π-1(Z0) → π-1(Z1) → π-1(Z2) → ... → π-1(Zn)

Lan_{σn} ∘ ... ∘ Lan_{σ1}(π-1(Z0)) → π-1(Zn)
    
```

Iterated colimit approximates the **unified conceptual flow**

$$\text{Iterated_Colimit} \cong \text{colim}_{\{Z_0 \rightarrow Z_1 \rightarrow \dots \rightarrow Z_n\}} (\text{Lan}_{\{\sigma_i\}}(\pi^{-1}(Z_{\{i-1\}})))$$

Example: Pullback of a Meaning Transformation via σ_1^*

We consider a morphism in the fiber over $Z_2 = \text{Instrument}$

$$f: \text{smartphone} \rightarrow \text{GPS} \mid Z_2$$

Let $\sigma_1: Z_1 \rightarrow Z_2$ be a contextual shift from $Z_1 = \text{Gadget}$ to $Z_2 = \text{Instrument}$. To interpret this transformation from the perspective of Z_1 , we apply the pullback functor σ_1^* .

This yields

$$\sigma_1^{**}(f): \text{smartphone} \rightarrow \text{smartphone-GPS} \mid Z_1$$

where smartphone-GPS is a more concrete or reduced interpretation of GPS within the limited frame of Z_1 .

Diagram:



f : An abstract transformation in Z_2 , viewing the smartphone as a GPS device.

$\sigma_1^{**}(f)$: A recontextualized version of f , valid under the Z_1 frame, where GPS is not a standalone concept but a function within the smartphone.

Interpretive Constraint: $\sigma_1^{**}(f)$ exists only if the target concept can be reconstructed within Z_1 .

$\sigma_1^{**}(f)$ is undefined $\Rightarrow \text{rupture}(f, \sigma_1^{**}(f)) \neq \emptyset$

In this case, the meaning transformation cannot be pulled back into the lower context.

Safe / Unsafe Conceptual Shifting Morphism (σ)

Definition of Safe and Unsafe σ Operator

Conceptual Shifting Morphism (σ) can be classified based on whether it preserves the global coherence of the morphic chain.

Safe σ Operator (σ_{safe}) Acts on the entire morphic chain as a coherent transformation.

$$\sigma_{\text{safe}}: D(A_{n-1} \mid Z) \rightarrow D(B_{n-1} \mid Z') \mid Z \gg Z' \vee Z \ll Z'$$

where: $Z, Z' \in \text{CD}$

Behaves as a Quasi-Natural Transformation

$$\sigma_{\text{safe}} = \eta: D_i \Rightarrow D_{i+1} \mid \text{CD}$$

Composition is associative:

$$(\sigma_3 \circ \sigma_2) \circ \sigma_1 = \sigma_3 \circ (\sigma_2 \circ \sigma_1)$$

Resulting chain remains in $M|Z$ or $M_{\{Z'\}}$ (closed).

Example

```

 $\sigma_1(X).$   $\rightarrow$ (canine, mammal)
 $\sigma_2(X).$   $\rightarrow$ (mammal, animal)
 $\sigma_3(X).$   $\rightarrow$ (animal, livingBeing)
    
```

Composition:

```

( $\sigma_3 \circ \sigma_2$ )  $\circ \sigma_1 = \sigma_3 \circ (\sigma_2 \circ \sigma_1)$ 
 $\rightarrow$   $\rightarrow$ (canine, livingBeing)
    
```

Entire morphic chain is preserved.

Unsafe σ Operator (σ_{unsafe}) Does not preserve global coherence of the morphic chain. Acts locally or in a decomposed manner.

Chain may collapse:

$$\sigma_{\text{unsafe}}: D(A_{n-1} \mid Z) \rightarrow \{ \text{rupture}(f_1), \text{rupture}(f_2), \dots, \text{rupture}(f_n) \mid \neg Z \}$$

$$\text{rupture}(f, \sigma(f) \mid Z) \neq \emptyset$$

Cannot be captured by a Quasi-Natural Transformation globally.

Example

$\sigma(X). \text{Not}(x) \{ A \rightarrow B \mid Z \}$

Result:
 $\text{rupture}(A \rightarrow B \mid Z)$
 \rightarrow breaks the morphic flow \rightarrow chain decomposes.

2. Monoid Structure of Conceptual Flow (M|Z):

In Conceptual Topology, Z is defined as a mediating point/conceptual anchor.

Let C and D , Z be categories,
with conceptual projection $\pi: C \cup D \rightarrow Z$, such that for each $X \in \text{Ob}(C \cup D)$

$$\pi(X) \in \text{Ob}(Z)$$

For each $X \in \text{Ob}(C \cup D)$, there exists morphism

$$\begin{aligned} f_X: X &\rightarrow \pi(X) \\ f_X^{-1}: \pi(X) &\rightarrow X \end{aligned}$$

such that

$$f_X^{-1} \circ f_X \cong \text{id}_X$$

For morphism $f: X \rightarrow Y \mid Z$,
this corresponds to:

$$f_Z: \pi(X) \rightarrow \pi(Y) \mid Z$$

For any $X, Y \in \text{Ob}(C \cup D)$

Let $[X]_Z :=$ conceptual representation of X under frame Z (i.e., $\pi(X)$)

Then:

$$[X]_{Z1} \cong [Y]_{Z2} \mid Z1, Z2 \in Z \text{ //or } Z1, Z2 \gg Z$$

which means

$$\begin{aligned} ["\text{Dog}"]_{\text{Pet}} &= [\text{Retriever}, \text{Dachshund}, \text{Poodle}, \text{Bulldog}, \dots] \\ ["\text{girl}"]_{\text{Human}} &= [\text{girl}, \text{woman}, \text{person}, \dots] \\ ["\text{Dog}"]_{\text{Pet}} &\cong ["\text{girl}"]_{\text{Human}} \mid \text{Life} \end{aligned}$$

Then the set of conceptual flow morphisms under Z forms a monoid

$$M|Z = \{ f_n \circ \dots \circ f_1 \mid \text{all } f_i: X_i \rightarrow X_{i+1} \mid Z \wedge \forall i, j: f_i \cong f_j \mid Z \}$$

This is also defined as Morphic Chain.

Let $D(C_{n-1} \mid Z) :=$ Category of Morphic Chains over $\text{Ob}(C_{n-1})$ within a given Z -frame.
where: $D(C_{n-1} \mid Z) = \{ C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \dots \mid Z \}$

or as a set

$$D(C_{n-1} \mid Z) = \{ C_0, C_1, C_2, \dots \mid Z \}$$

3. Identity Element of $M|Z$

Let: $M|Z = \{ f_n \circ \dots \circ f_1 \mid \text{all } f_i: X_i \rightarrow X_{i+1} \mid Z \wedge \forall i, j: f_i \cong f_j \mid Z \}$

Define the identity element of $M|Z$ as a family of identity morphisms over the shared Z frame:

For each $X \in \text{Ob}(C \cup D)$, there exists a unique identity morphism under a Z frame:

$e|_{Z_X} := \text{id}_X \mid Z$

Then, for any $f: X \rightarrow Y \mid Z \in M|Z$:

$e|_{Z_X} \circ f = f$

$f \circ e|_{Z_Y} = f$

Therefore, the identity structure of $M|Z$ is given by the family:

$\{ \text{id}_X \mid Z \mid X \in \text{Ob}(C \cup D) \}$

which forms a pointwise identity across the objects under the common Z frame.

This ensures that $M|Z$ satisfies the identity axiom of a monoid.

4. Associativity of $M \mid Z$

Let: $M|Z = \{ f_n \circ \dots \circ f_1 \mid \text{all } f_i: X_i \rightarrow X_{i+1} \mid Z \wedge \forall i, j: f_i \cong f_j \mid Z \}$

Then for all $f, g, h \in M|Z$:

$(f \circ g) \circ h = f \circ (g \circ h)$

Thus, the composition \circ in $M|Z$ is associative.

5. Axioms

5.1. Identity Element

Unit Axiom 1: Identity Element Z

```
id_Z := Z → Z | Z
∀f ∈ M|Z: id_Z ∘ f = f and f ∘ id_Z =
```

Definition:

```
Statement:
Z-frame itself is the unit of M | Z.

Formal Definition:
Z := Z → Z|Z

Justification:
Since any morphism in M | Z is defined as:

f: X→Y|Z

and Z itself is defined as its own identity morphism:

Z := Z→Z|Z

then:
id_Z = Z

Conclusion:

Therefore:
id_Z is the unit element of M|Z.

∀f ∈ M|Z: (id_Z ∘ f | Z) = f and (f ∘ id_Z | Z) = f
(with frame-preserving composition)

∴id_Z is the unit of M|Z.
```

Note:

```
idZ :Z→Z | Z
f:X→Y | Z
(idZ|Z) ∘ (X→Y|Z)
```

Unit Axiom2: Void Concept

$$f \in M|Z$$

$$"" \circ f = f \text{ and } f \circ "" = f$$

$$\text{id}_Z \circ f = f \text{ and } f \circ \text{id}_Z = f$$

Definition:

The empty concept is a theoretically assumed concept, denoted as "", which acts as the unit element at the conceptual / lexical level.

Formal Definition:

$$"" \circ f = f \text{ and } f \circ "" = f$$

Justification:

The empty concept "" represents no lexical or conceptual content. Composing any morphism f with the empty concept does not alter the flow of meaning.

Conclusion:

"" is the unit element at the conceptual level of Conceptual Topology.

5.2. Zero Morphism: Negation Morphism

We define conceptual zero morphism, negation morphism: n_f In CT as the result of applying Not() to a morphism

$$g: \sigma(Z). \text{ Not}(g)\{ A \rightarrow B \mid Z \} = A \rightarrow B|Z = n_f$$

where: $g: A \rightarrow B$

Formal Properties (Axiom):

$\forall g: X \rightarrow Y|Z$ where composition with n_f is defined:

$$\forall g: g \circ n_f = n_f \text{ and } n_f \circ g = n_f$$

Left Side:

$g: A \rightarrow B$

$g \circ (A \rightarrow B|Z) = A \rightarrow B|Z$

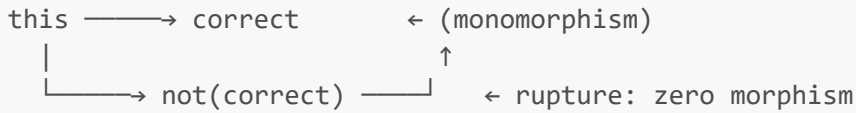
Right Side

$g: A \rightarrow B \ (A \rightarrow B|Z) \circ g = A \rightarrow B|Z$

Interpretation:

Applying Not() to any morphism produces a conceptual zero morphism, which collapses any further conceptual flow.

NL Diagram:



Natural Language:

Left Side: $g \circ (A \rightarrow \emptyset | Z)$

"A is not B"

The apple is not a fruit

Right Side: $(A \rightarrow \emptyset | Z) \circ g$

"B is not A"

This is a fruit, but this is not an apple which is a fruit.

In CT, this was called `rupture()`.

Now defined:

$\text{rupture}(A, B, Z) = \sigma(Z). \text{Not}(g) = n_f = A \rightarrow \emptyset | Z$

5.3. Composition Axiom

$M|Z = \{ f_n \circ \dots \circ f_1 \mid \text{all } f_i: X_i \rightarrow X_{i+1} \mid Z \wedge \forall i, j: f_i \cong f_j \mid Z \}$

Then for all $f, g, h \in M|Z$:

For $f, g, h \in M|Z$,

where:

$f: V \rightarrow W \mid Z$

$g: Y \rightarrow V \mid Z$

$h: X \rightarrow Y \mid Z$

$(f \circ g) \circ h = f \circ (g \circ h)$

Example:

For $f, g, h \in M|Z$,

where:

$f: \text{she} \rightarrow \text{you} \mid \text{Human}$

$g: \text{he} \rightarrow \text{she} \mid \text{Human}$

$h: \text{man} \rightarrow \text{he} \mid \text{Human}$

$$(f \circ g) \circ h = f \circ (g \circ h)$$

6. Conceptual Topos

6.1. Category Level: Initial Object

Definition:

Let Concept be a category where $\text{Ob}(\text{Concept})$ are lexical / conceptual objects.
Then $"" \in \text{Ob}(\text{Concept})$ is Initial Object if:

$\forall X \in \text{Ob}(\text{Concept}), \exists$ unique morphism:

$u_X : "" \rightarrow X \mid X$

such that:

$\forall f: X \rightarrow Y \mid Z,$
 $f \circ u_X = u_Y$

Monoid Level: Unit in $M|Z$

Recall:

$M|Z = \{ f_n \circ \dots \circ f_1 \mid \text{all } f_i: X_i \rightarrow X_{i+1} \mid Z \wedge \forall i, j: f_i \cong f_j \mid Z \}$

Now, define:

$"" \in \text{Ob}(\text{Concept})$

and identity morphism under Z -frame:

$e|Z_{} := \text{id}_{} \mid Z$

Then for all $f \in M|Z$:

$e|Z_{} \circ f = f$
 $f \circ e|Z_{} = f$

6.2. Finite Limits

Terminal Object Conceptual Topos defines a terminal object as the Z-frame identity:

id_Z := Z → Z | Z

Any morphism f: X → Z | Z factors uniquely through id_Z.

This realizes the conceptual universal target:

$\forall X \in \text{Ob}(C \cup D), \exists! f_{\text{terminal}}: X \rightarrow Z \mid Z$

Example:

she → human | Human
me → human | Human

Pullback

Given morphisms:

f: girl → mammal
g: puppy → mammal

Pullback of (f, g) is:

P = Baby
p₁: Baby → girl
p₂: Baby → puppy

with commuting condition:

f ∘ p₁ = g ∘ p₂ ≈ mapping to common conceptual frame (mammal)

Diagram:

Baby

/ \

p₁ / \ p₂

/ \

girl puppy

\ /

v v

mammal (conceptual anchor / codomain)

This previously defined as Quasi-Natural Transformation:

$$\eta: D_i \Rightarrow D_{i+1} \mid CD \text{ (CD = codomain)}$$

$$\eta_X \circ D_i(\{f_1 \mid Z_1, \dots, f_n \mid Z_n\}) \approx D_{i+1}(\{f'_1 \mid Z_1, \dots, f'_n \mid Z_n\}) \circ \eta_Y \mid CD$$

for all $f_j: X_j \rightarrow Y_j \mid Z_j \in D_i$,
 where $f'_j: \eta_X(X_j) \rightarrow \eta_Y(Y_j) \mid Z_j$

Then, η is said to be a quasi-natural transformation under the Z-frame
 i.e. $\eta \in \text{Mor}(C)$ where C is the contextual meaning category

$$\eta_X \circ D_i(\{\text{girl} \rightarrow \text{mammal} \mid Z_1\}) \approx D_{i+1}(\{\text{puppy} \rightarrow \text{mammal} \mid Z_2\}) \circ \eta_Y$$

Pullback Diagram

```

graph TD
    Mammal --> girl
    Mammal --> puppy
    girl --> Baby
    puppy --> Baby
    subgraph ZFrame
        Baby
    end
    
```

Baby (conceptual anchor / common Z-frame)

Example: $\eta: \text{girl} \rightarrow \text{puppy} \mid Z = \text{Baby}$

Quasi-Natural Transformation Diagram:

```

graph TD
    puppy -->|//specified: size + young| girl
    puppy --> dog
    girl --> she
    dog -->|//abstraction| Mammal
    she --> Mammal
    
```

For any X with morphisms
 $q_1: X \rightarrow \text{girl}$ and $q_2: X \rightarrow \text{puppy}$ satisfying $f \circ q_1 = g \circ q_2$,

there exists unique $u: X \rightarrow \text{Baby}$

such that:
 $p_1 \circ u = q_1$, $p_2 \circ u = q_2$.

Equalizer: Mirror Morphism

Equalizer of two morphisms f, g :
 $A \rightarrow B$ is an object $\text{Eq}(f,g)$ with morphism
 $e: \text{Eq}(f,g) \rightarrow A$

such that:

 $f \circ e = g \circ e$

and universal property:

 $\forall h: X \rightarrow A$ such that $f \circ h = g \circ h$,
 $\exists!$ unique $u: X \rightarrow \text{Eq}(f,g)$

Diagram:

Eq(f, g)

|

e

v

A

f ↘ ↙ g

B

In conceptual topology this was defined as mirror morphism:

$f : X \rightarrow Y \mid Z \in D_i$
 $f' : X' \rightarrow Y' \mid Z \in D_{i+1}$
 $\Rightarrow X' \neq X, \text{ but } \text{cod}(f) = \text{cod}(f') \mid \text{CD (common codomain)}$

We define f' as a mirror-correspondent morphism of f under a given Z -frame,
if and only if:

 $\exists Z: \text{rupture}(f, f' \mid Z) \neq \emptyset$
 $\wedge \text{cod}(f) = \text{cod}(f') \mid \text{CD}$

Eq(f, f')

|

e

v

X X'

\ /

\ /

v v

Y = Y' (codomain = C)

Product: σ operator \oplus

In any category C , the Product of A and B is an object $A \times B$ equipped with projections:

$$\pi_1: A \times B \rightarrow A$$

$$\pi_2: A \times B \rightarrow B$$

with universal property:

For any object X with morphisms:

$$f_1: X \rightarrow A$$

$$f_2: X \rightarrow B$$

there exists a unique morphism $u: X \rightarrow A \times B$ such that:

$$\pi_1 \circ u = f_1$$

$$\pi_2 \circ u = f_2$$

Addition (\oplus):

$\sigma(Z)$ serves as the mediating operator ensuring that the composed morphic chain remains within the conceptual fiber over Z .

Defined as:

$$\sigma(Z). \oplus(A_{n-1}, B_{n-1}, Z) = D(C_{n-1} \mid CD) = M_C \mid CD$$

Example:

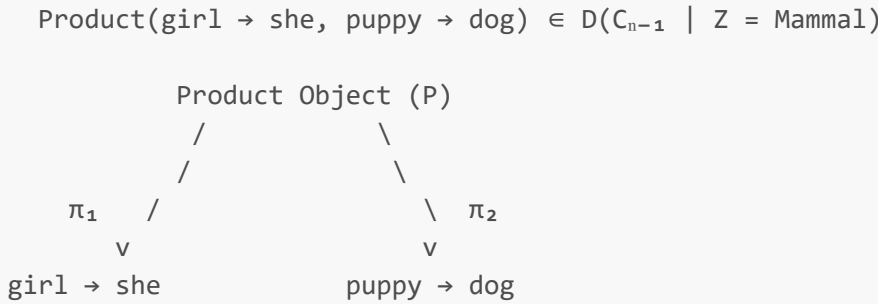
$$A_{n-1} := \text{girl} \rightarrow \text{she} \mid \text{Human}$$

$$B_{n-1} := \text{puppy} \rightarrow \text{dog} \mid \text{Canine}$$

$$\sigma(\text{Human}). \oplus(\text{girl} \rightarrow \text{she}, \text{puppy} \rightarrow \text{dog} \mid \text{Mammal}) \rightarrow M \mid Z(\text{girl} \rightarrow \text{she} \rightarrow \text{puppy} \rightarrow \text{dog} \mid \text{Mammal}) \mid \text{Mammal}$$

→ composite meaning space

Diagram:



6.3. Exponentials

Conceptual Topos models exponentials via conceptual shift operators.

Definition

For any objects A, B:

B^A exists such that:

$$\text{Hom}(X \otimes A, B) \cong \text{Hom}(X, B^A)$$

Construction via σ operator

Conceptual shift operators:

$\sigma(Z). \gg(A, B)$
or
 $\sigma(Z). \>(A, B)$

act as internal exponential morphisms within the fibered structure over the Z-frame:

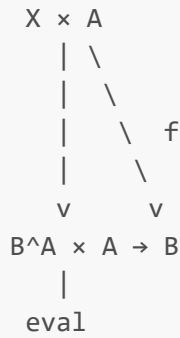
$$(A, B, Z) \cong B^A$$

where the Z-frame mediates the conceptual continuity and contextual grounding of the morphic shift.

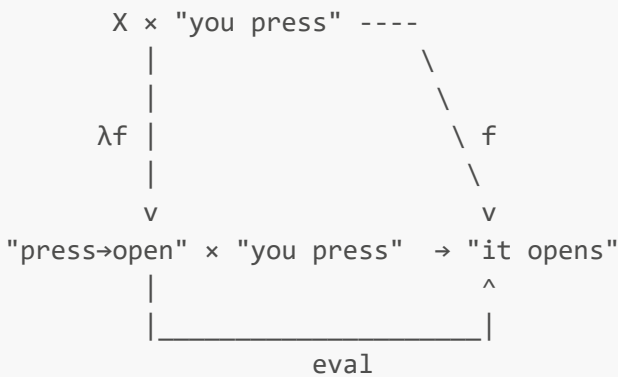
We define Exponential objects via σ operator as conceptual abstraction mechanisms:

$$B^A := \sigma(Z). \>(A, B)$$

Exponentials as If: Conditional Statement



In NL Diagram:



A: condition – "you press"
 B: result – "it opens"
 B^A: if A, then B: "press → open"
 eval: B^A x A → B: "it opens"
 λf: X → B^A: the result under the condition
 you press → extract the general law under the condition you press then "press → open"

We formalize this as eval function in CTL(Conceptual Topology Language). We employ programing style to track dynamic transformation of conceptual flow.

```
def exponential(a: Concept, b: Concept, z: Frame) -> ConceptualMorphism::
    """
    Construct exponential object B^A under Z-frame.
    Represents: 'if a then b' interpreted within context c.
    """
    return σ(z).>(a, b) // represents B^A

def eval(f: Conceptual Morphism, a: Concept, z: Frame) -> Concept:
    return f x a | z

Example:
//if press then open → it opens
eval("press", "open", "door") -> door → opened door | Door
```

```
//if press then open + you press → it opens
f = exponential("press", "open", "door")
eval(f, "you press") → open | Door
```

Full Exponential Law formalization will be provided in later version.

Definition: Conceptual Shifting Morphism (σ)

$$\sigma: D(X_{n-1} \mid X) \rightarrow D(X_{n-1} \mid X)$$

such that $\sigma \oplus f \in M|Z$ if and only if type compatibility holds:

$\forall A, B, (A \rightarrow B) \circ \sigma(X)$ is valid if:

($A \gg X$ or $X \gg A$)

and

($B \gg X$ or $X \gg B$)

Definition: Subsumption

$A \gg X \equiv A \sqsubseteq X$

Definition: SubsumedBy

$X \gg A \equiv X \sqsubseteq A$

Example:

king \rightarrow king \gg human \rightarrow human
 \Rightarrow king \gg human \rightarrow valid

human \rightarrow human \gg queen \rightarrow queen
 \Rightarrow human \gg queen \rightarrow valid

Example

$\sigma(\text{Human}). \gg(\text{puppy} \rightarrow \text{dog} \rightarrow \text{mammal} \mid \text{Canine}, \text{Human})$
 $\cong \text{girl} \rightarrow \text{she} \rightarrow \text{mammal} \mid \text{Human}$

This shift realizes an internal conceptual transformation corresponding to exponential behavior.

6.4. Definition of Ω

Let Ω be an object in the Concept category, representing the **conceptual truth space**.

For any subobject (conceptual inclusion):

$$m: M \hookrightarrow X$$

there exists a unique characteristic morphism:

$$\chi_m: X \rightarrow \Omega$$

such that the following diagram commutes:

$$\begin{array}{ccc} M & \longrightarrow & X \\ | & & | \\ | & & \vee \chi_m \\ | & & \Omega \end{array}$$

Interpretation in Conceptual Topology

- Ω encodes **conceptual entailment / membership / inclusion**.
- **Z-frame membership** is naturally mapped to Ω :

$$\chi_Z: X \rightarrow \Omega$$

interpreted as:

"Does X conceptually belong to Z-frame Z?"

Examples

Example 1: Dog in Pet Z-frame

$$\chi_{\text{Pet}}(\text{Dog}) = \text{True}$$

Example 2: Apple in Pet Z-frame

$$\chi_{\text{Pet}}(\text{Apple}) = \text{False}$$

Example 3: Innocent in Body Z-frame (after rupture)

$\chi_{\text{Body}}(\text{"innocent"}) = \text{True} / \text{False}$ depending on whether the conceptual projection is coherent under Z-Frame.

Relation to Rupture

Conceptual rupture can be lifted to Ω as:

$$\sigma(Z). \text{Not}(f: A \rightarrow B \mid Z) \Rightarrow \text{rupture}(A,B,Z) \Rightarrow \chi_Z(f) = \text{False}$$

Thus, **negation** and **conceptual discontinuity** become **Ω -classifiable**.

6.5. Conceptual Topos as Fibered Topos over Z-frame

Conceptual Topos is structured as a **fibered topos** over the conceptual base space **Z-frame**.

Z-frame as Fibered Structure

- Let $\pi: C \cup D \rightarrow Z$ be the conceptual projection.
- Each fiber $\pi^{-1}(Z)$ forms a category of morphic chains **D(C_{n-1} | Z)**.
- Morphisms of the form:

$$X \rightarrow Y \mid Z \equiv X \rightarrow Y \text{ in fiber over } Z$$

correspond to morphisms within the fibered structure over Z.

Initial Object and Codomain Projection

- The **Initial Object** "" serves as the conceptual origin.
- It projects into the codomain via:

$$"" \rightarrow \mid X \equiv "" \rightarrow \pi(X)$$

$$\begin{array}{c} "" \\ \downarrow u_X \\ X \longrightarrow \pi(X) \text{ (in Z-frame)} \end{array}$$

$$\begin{array}{l} \text{Fiber } \pi^{-1}(Z_X): \\ "" \rightarrow X \rightarrow Y \end{array}$$

Thus, conceptual generation naturally occurs anchored in Z-frame.

Conceptual Flow Closure

- Conceptual flows:

$$X \rightarrow Y \mid Z$$

are closed within the fiber over Z , corresponding to the codomain Z of the conceptual projection π .

- Rupture and negation are classified by Ω :

$$\chi_Z: X \rightarrow \Omega$$

7. Global Conceptual Space: Total Conceptual Space (TCS)

We define the Total Conceptual Space (TCS) as the global conceptual anchor:

$$Z = \text{TCS} = \text{Total Conceptual Space}$$

Definition of $M|TCS$:

The global morphic flow space under TCS is defined as:

$$M|TCS = \{ f_n \circ \dots \circ f_1 \mid \text{all } f_i: M|Z_i \rightarrow M|Z_{i+1} \mid TCS \wedge \forall i, j: f_i \cong f_j \mid TCS \}$$

We can regard $M|TCS$ as the composition space of conceptual perspectives: Here, each $M|Z$ functions as a conceptual symbolization or perspective lens, and $M|TCS$ represents global flows across chained perspectives.

Monoid Closure Property:

Composition in $M|TCS$ is closed:

$$\forall f, g \in M|TCS, f \circ g \in M|TCS$$

The identity morphism is preserved:

$$\forall f \in M|TCS, f \circ \text{id} = f = \text{id} \circ f$$

Thus, $M|TCS$ forms a closed monoid under composition.

Completeness Statement:

For any pair of concepts X, Y :

$$\forall X, Y \in \text{Ob}(C), \exists f \in \text{Mor}(C), \text{ such that } f: X \rightarrow Y \mid TCS$$

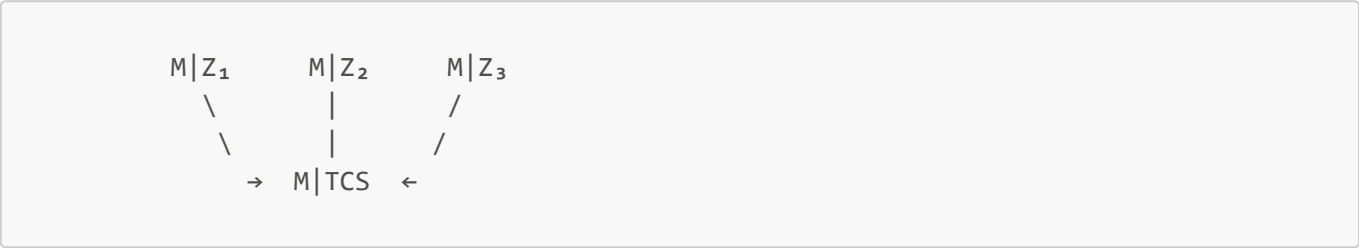
That is, any conceptual pair X and Y can be connected via a morphic flow under TCS.

Fibered Structure and Lifting

Each local $M|Z$ can be lifted into $M \mid TCS$ via conceptual shifting σ :

$$\forall M|Z, \exists \sigma: M|Z \rightarrow M \mid TCS$$

Thus, the global base space TCS ensures that the entire morphic flow space is both complete and coherent.



Example:

can \rightarrow person \mid TCS

\rightarrow Metaphoric reading: "The can represents the absent person."

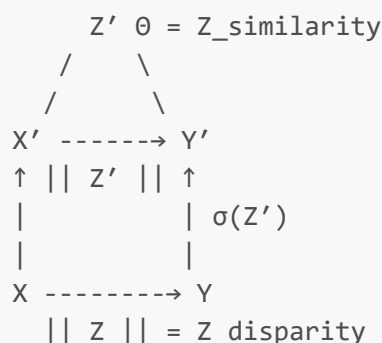
\rightarrow Ironic reading: "We are all cans under capitalism."

Summary:

The Total Conceptual Space (TCS) functions as the global base space of the conceptual topology. All local Z-frames are fibered over TCS, and conceptual flows can be lifted via σ operators into $M \mid TCS$. Thus, Conceptual Topos is complete and globally coherent under $M \mid TCS$.

8.1. Conceptual Flow Classification

First, we define the core conceptual flow diagram, interpreting Z as a retractive flow.



For reference: In the composition $A \rightarrow Z \rightarrow B$, the mediating Z can be interpreted as the distance between A and B . We define this as $Z = \|A - B\|$, and denote the norm as $\|Z\|$. In retraction $A \rightarrow Z \rightarrow A$ also behaves as a conceptual identity or coherent transformation via cosine similarity hinged on Z when Z is scalarized as $\|Z\|$ under PCA projection.



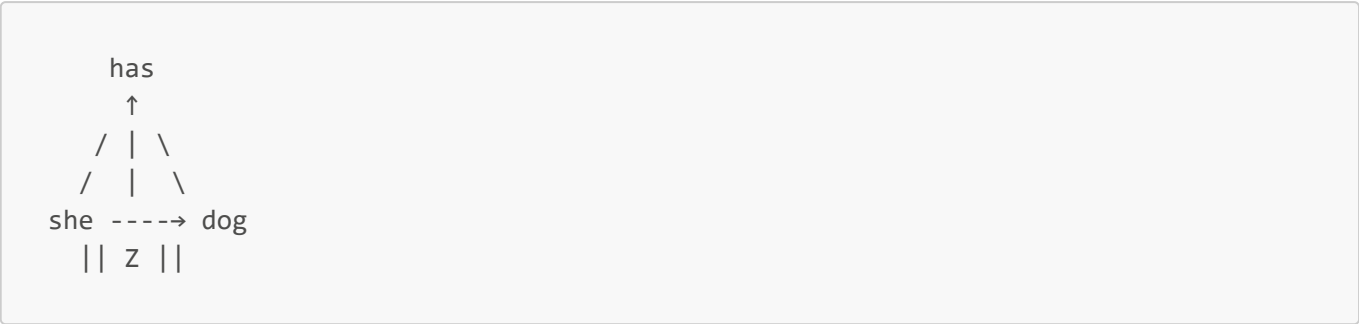
$$g \circ f \cong A \rightarrow B \mid Z$$

Self Identity Morphism: Triangle

A diagram of a tetrahedron. The base is a triangle with vertices A, B, and Z. A dashed line connects A and B. Vertex Z is below the line AB. Above the triangle is vertex Z'. An arrow points from Z' down to the center of the triangle formed by A, B, and Z.

A PCA plot titled "PCA of Semantic Geometry (Word2Vec)" showing the relationship between three words: "dog", "she", and "has". The x-axis ranges from -2 to 2, and the y-axis ranges from -1.0 to 1.5. The word "dog" is represented by a green dot at approximately (-2.0, -0.3). The word "she" is represented by a blue dot at approximately (0.6, 1.4). The word "has" is represented by an orange dot at approximately (1.4, -1.1). Arrows connect "dog" to "she" and "dog" to "has", illustrating the semantic relationship between these words.

While multiple interpretations are possible, this diagram offers the most coherent explanation. The morphism $\sigma(\text{possession})$ unifies she and dog under the relation of ownership: the owner and the owned.

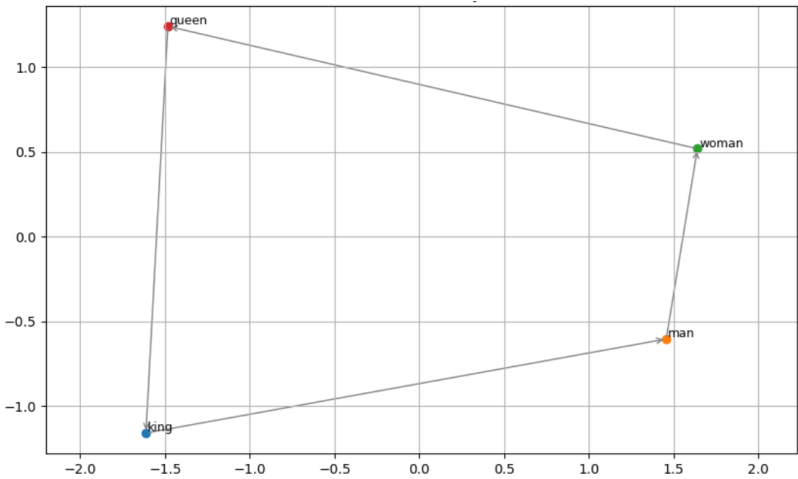
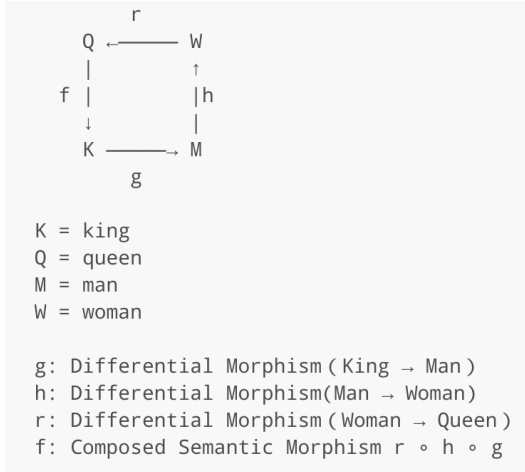


Trapezoid



king, man , woman, queen

The trapezoid structure observed in king, man, woman, and queen can be interpreted as preserving the same major Z-axis—in this case, gender. Note: Z may also represent a multi-dimensional conceptual frame.



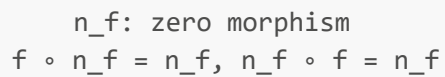
$$\begin{array}{ccc}
 & || & Z' || \\
 A' & \dashrightarrow & B' \\
 \uparrow & \uparrow & \uparrow \\
 | & | \sigma(Z') | & \\
 | & | & | \\
 A & \dashrightarrow & B \\
 & || & Z ||
 \end{array}$$

$f: A \rightarrow B \mid Z$
 $g: A' \rightarrow B' \mid Z'$

since $|| Z' || = || Z ||$

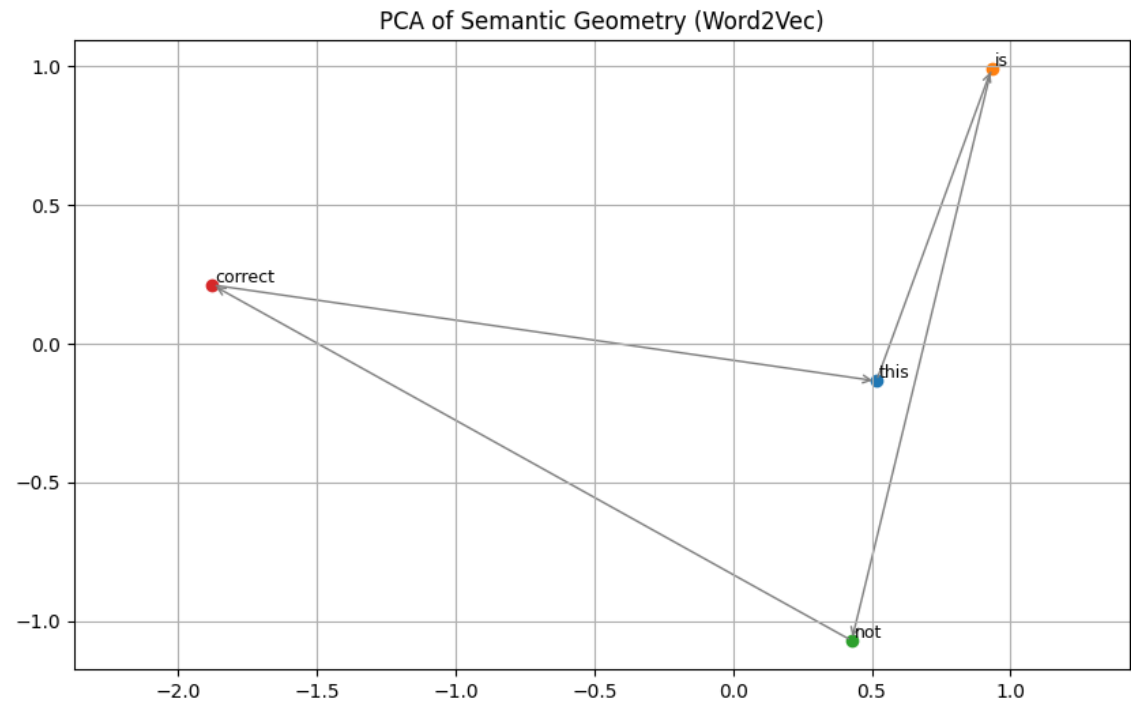
$$\begin{array}{ccc}
 & || & Z' || \\
 A' & \dashrightarrow & B' \\
 \uparrow & \uparrow & \uparrow \\
 | & | \sigma(Z') | & \\
 | & | & | \\
 A & \dashrightarrow & B \\
 & || & Z ||
 \end{array}$$

NL Diagram:

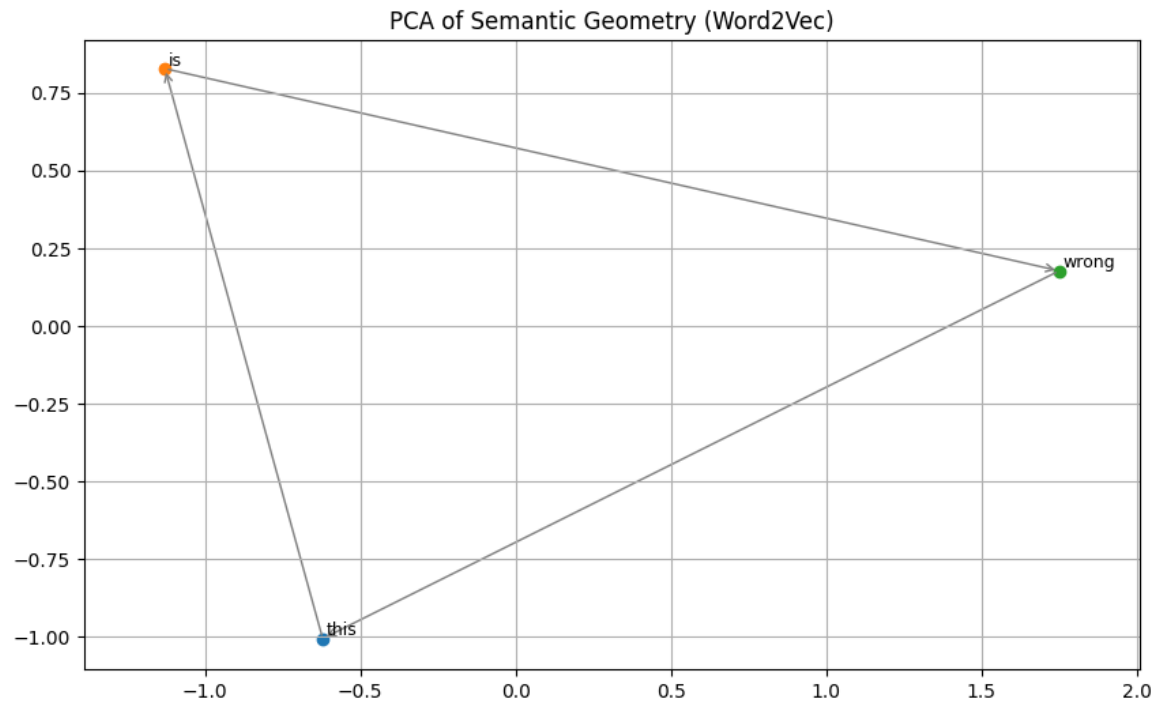


A PCA plot titled "PCA of Semantic Geometry (Word2Vec)" showing the relationship between four words: "right", "is", "this", and "not". The x-axis ranges from -1.5 to 1.5, and the y-axis ranges from -1.00 to 0.75. The words are represented as points: "right" (red dot at approx. -1.2, 0.55), "is" (orange dot at approx. 1.0, 0.75), "this" (blue dot at approx. 0.4, -0.25), and "not" (green dot at approx. -0.1, -1.05). Lines connect "right" to "is", "right" to "not", and "not" to "is", forming a triangle. The word "this" is positioned near the line connecting "not" and "is".

Word	X (PCA1)	Y (PCA2)
right	-1.2	0.55
is	1.0	0.75
this	0.4	-0.25
not	-0.1	-1.05



If a lexicon A which satisfy with $not\ B \subseteq A$ such as *wrong* can recover the conceptual flow.



8.3. Cone Structre: Kan Extension and QNT

We can observe iterated colimit and QNT in PCA.

```
Di = colim_{puppy → dog → mammal} (Lan_{σi}(π-1(Mammal)))
Di+1 = colim_{girl → she → mammal} (Lan_{σi}(π-1(Mammal)))
```

Here Defined QNT Between D_i and D_{i+1}

$\eta: D_i \Rightarrow D_{i+1} \mid CD = \text{Mammal}$

$\eta_X \circ D_i(\{f_1 \mid Z_1, \dots, f_n \mid Z_n\}) \approx D_{i+1}(\{f'_1 \mid Z_1, \dots, f'_n \mid Z_n\}) \circ \eta_Y \mid CD$

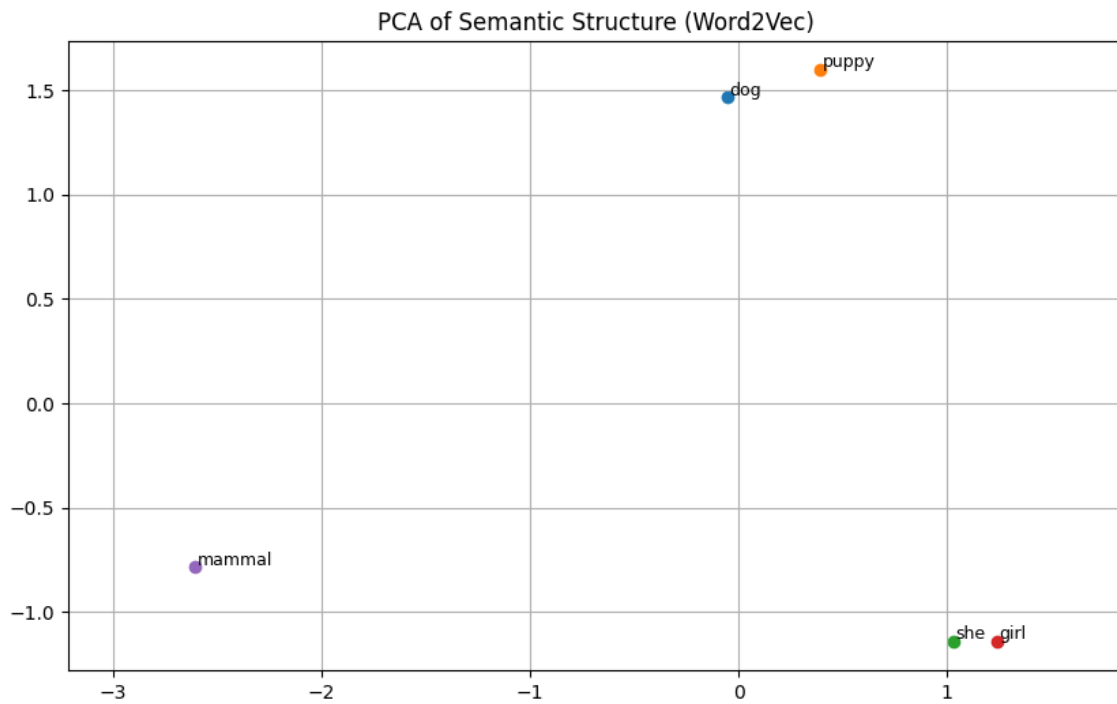
for all $f_j: X_j \rightarrow Y_j \mid Z_j \in D_i$,

where $f'_j: \eta_X(X_j) \rightarrow \eta_Y(Y_j) \mid Z_j$

Then, η is said to be a quasi-natural transformation under the Z-frame

i.e. $\eta \in \text{Mor}(C)$ where C is the contextual meaning category

Example: $\eta: \text{girl} \rightarrow \text{puppy} \mid Z = \text{Baby}$



We did not define universal product in the above sections, yet this is observable in PCA.

```

      Mammal
     /    \
    /      \
   /        \
  f          g
 /          \
v   P = Baby v
Puppy <-- P --> Girl
  π A          π B

```



8.5 Computational Methodology to induce Z-frame

A Z-frame acts as a cone shape in morphic structure—its role is to anchor multiple conceptual flows into a unifying conceptual topology. In category-theoretic terms, the Z-frame corresponds to the colimit of a diagram.

While colimits are traditionally defined via universal properties over cones, in natural language processing (NLP), a colimit can be approximated by KMeans clustering over word embeddings. If we interpret a cone from the side, we observe its tapering morphic shape; but from above, the apex of the cone corresponds to the conceptual anchor (Z-frame) toward which all points converge.

Z frame: king, queen, man woman

Using a simple KMeans (k=1) over these four word vectors yields the following cluster center and its top 10 closest neighbors.

```
=== Z-frame Candidates ===
  queen : Similarity = 0.7720
  woman : Similarity = 0.7235
  king  : Similarity = 0.7221
  man   : Similarity = 0.6933
princess : Similarity = 0.6631
  girl  : Similarity = 0.6182
monarch : Similarity = 0.6139
 prince : Similarity = 0.6070
  lady  : Similarity = 0.5850
teenage_girl : Similarity = 0.5712
```

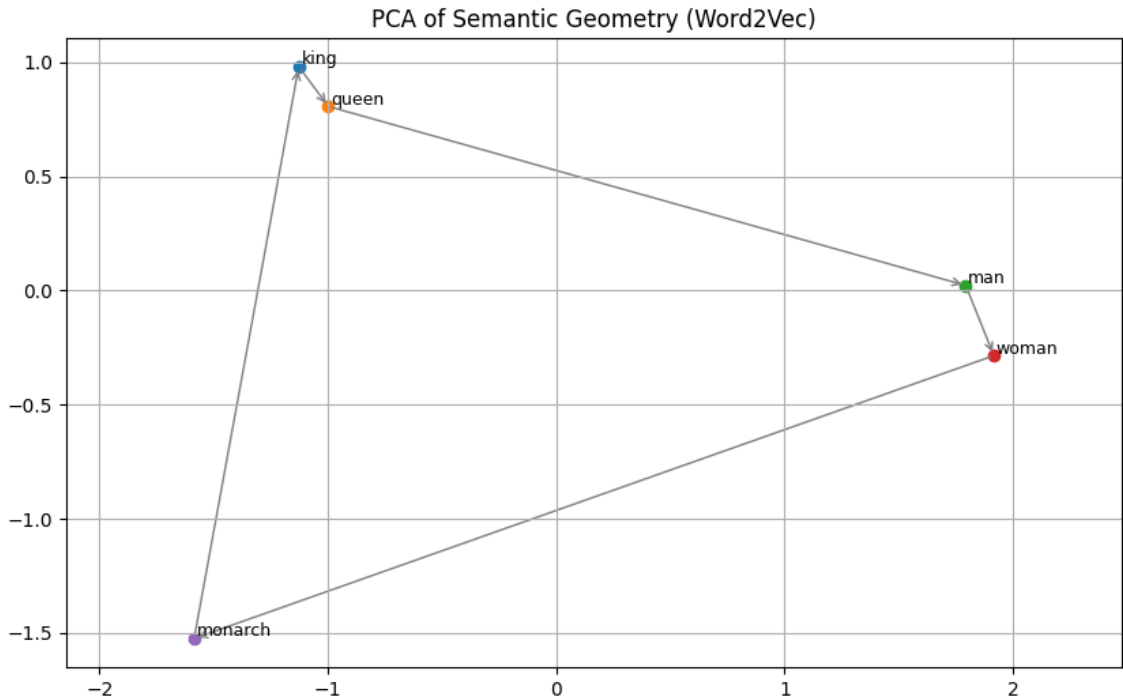
Remarkably, monarch emerges from a purely geometric computation. This term does not merely appear due to proximity in vector space, but because it semantically subsumes king, queen, man, and woman in terms of parochial authority. Thus, monarch behaves as a Z-frame, mediating the entire morphic configuration.

Diagram:



PCA Interpretation

Although the PCA projection did not produce a perfectly symmetric cone structure, we can semantically interpret this as reflecting the historical structure of patriarchal monarchy.



In this frame, the transformation from king → queen or man → woman does not converge straight onto the concept monarch as we saw in puppy to dog and girl to she. This asymmetry indicates a semantic tension: while monarch appears to geometrically mediate both gendered and political domains, the flow from female-associated terms deviates from the dominant axis. Nevertheless, monarch successfully subsumes the morphic circulation among king, queen, man, and woman, serving as a conceptual apex (Z-frame) that bridges the two word clusters:

[king, queen] ↔ [man, woman]
unified under the latent conceptual attractor: monarch

From a geometrical viewpoint, these divergences are positional errors relative to the apex. What appears as semantic tension in flat projection becomes, from the Z-frame’s perspective, merely an expected distortion in projection space. The Z-frame does not require perfect symmetry; it requires coherence. And *monarch*, asymmetrically, holds.

Semantic Mediation via Z-frame: monarch

The Z-frame enables mediated and asymmetrically coherent morphic interpretation.

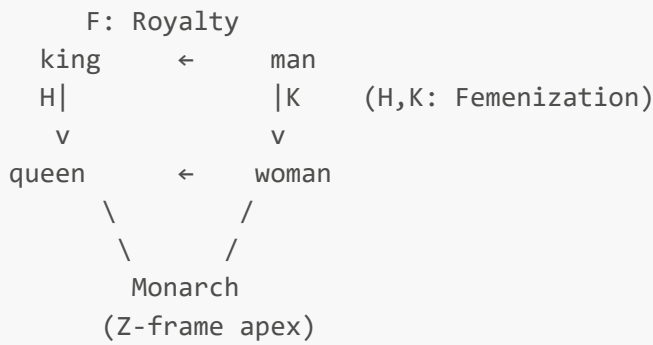
Let us define semantic transitions under $Z = \text{monarch}$.

```
man → woman | Monarch    // gender role transformation within monarchic structure
woman → queen | Monarch  // woman becomes queen under monarchic mediation
queen → king | Monarch   // queen transitions to king in the symbolic hierarchy

therefore:
man → king | Monarch     // man maps to king under the monarchic frame
```

From the perspective of Left Kan Extension, *king* can also be understood as a lifted projection from *man*, crossing from a gendered lexical domain (generic gender space) to a political-royal domain (*king*) under the frame *monarch*.

Royal Domain \Leftrightarrow Gendered Domain



Left Kan Extension:

```
Lanσ(man) = king | Monarch
Lanσ(woman) = queen | Monarch
```

QNT Internal Z frame Induction

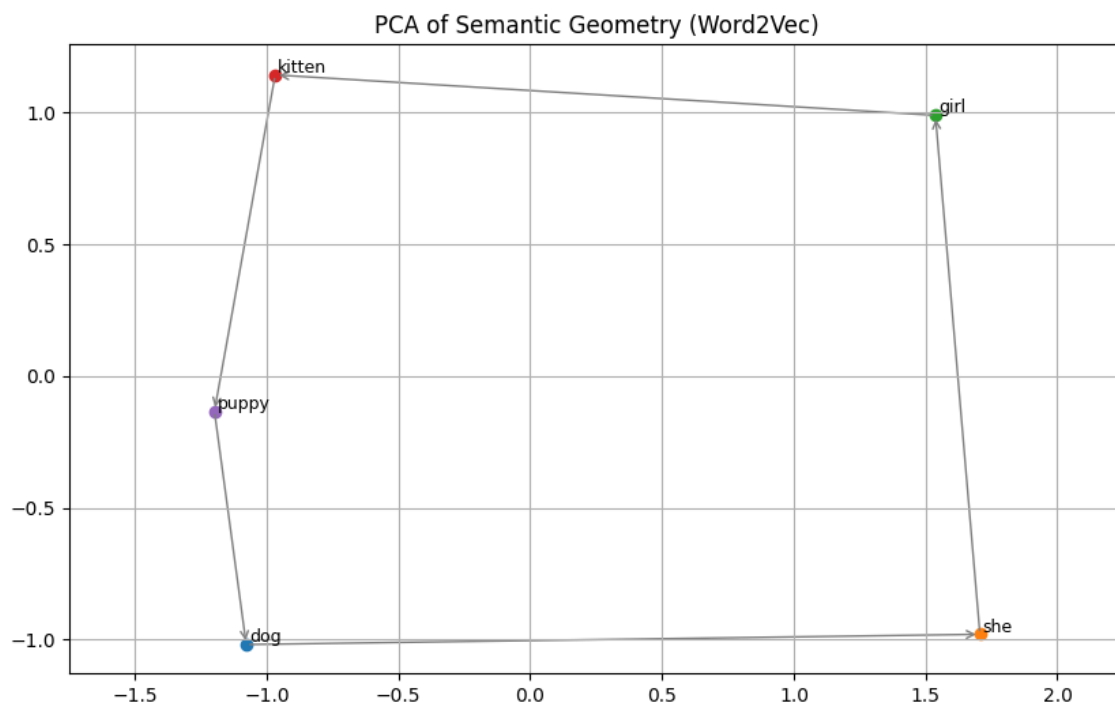
QNT has internal Z-frame coherence such that $f_1 \rightarrow f'_1 | Z_1$.

```
η: Di ⇒ Di+1 | CD = Mammal
ηX ∘ Di({f1 | Z1, ..., fn | Zn}) ≈ Di+1({f'1 | Z1, ..., f'n | Zn}) ∘ ηY | CD
for all fj: Xj → Yj | Zj ∈ Di,
where f'j: ηX(Xj) → ηY(Yj) | Zj
```


Using the same method KMeans, we could induce internal Z-frame (or CD) - kitten using puppy, dog, girl and she.

```
=== Z-frame Candidates ===  
  puppy : Similarity = 0.8759  
   dog  : Similarity = 0.8172  
 kitten : Similarity = 0.7581  
   pup  : Similarity = 0.7549  
  girl  : Similarity = 0.7462  
 puppies : Similarity = 0.7243  
golden_retriever : Similarity = 0.7225  
   pooch : Similarity = 0.7119  
Pomeranian : Similarity = 0.7082  
 chihuahua : Similarity = 0.7012
```

However, in the PCA projection, kitten did not function as a Z-frame as initially presumed. Instead, the structural configuration shifted, and puppy emerged as the effective conceptual anchor. This suggests that KMeans does not necessarily predict which word will operate as the actual Z-frame, even if it captures the overall conceptual coherence of the lexical field. While this might imply that KMeans does not always align with a true colimit structure, both the PCA projection and the initial KMeans similarity scores identified puppy as the geometric and structural center. Puppy ranked as the top Z-frame candidate, and the addition of kitten did not affect the overall configuration. This indicates that KMeans may still closely approximate colimit-like conceptual anchors despite our semantic intuitions suggesting otherwise.





By introducing the concept *mammal*, the overall geometric structure of the lexical field underwent a notable transformation. We observe that *mammal* does not merely act as an added point—but as a semantic mediator that 3-dimensionally mediates two distinct morphic chains: she → dog and girl → kitten → puppy.

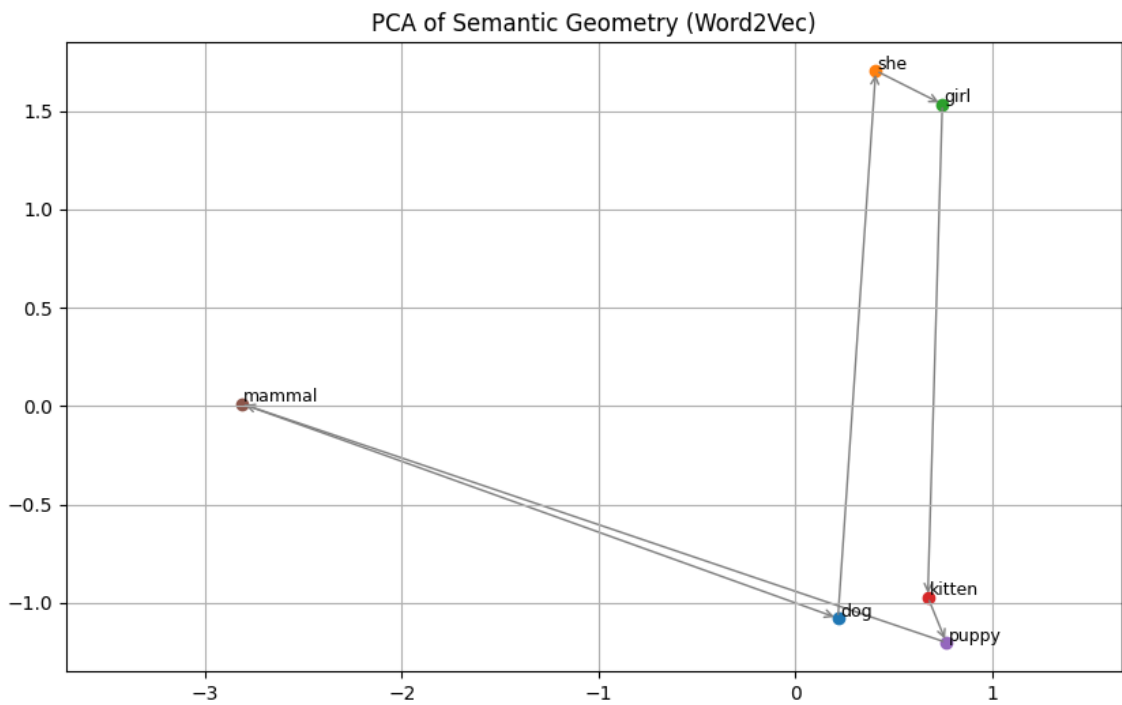
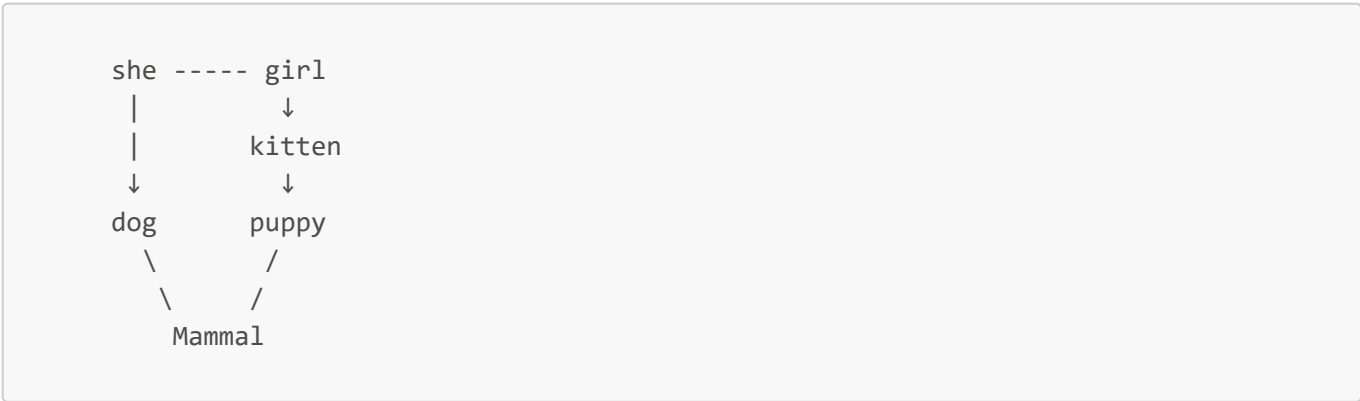


Diagram for visualization

While the geometric position of *mammal* is shifted from its PCA-projected location in this diagram, the topological structure of the semantic system remains intact—positioning *mammal* between two morphic chains, as visually observed in the PCA.



Conclusion

Conceptual Topos is a **fibred topos** over Z-frame:

$$\mathbf{CT} := (C, B, \pi: E \rightarrow B, \text{Fb} := \pi^{-1}(b), A \cong b \cup \text{Nat}(\text{Hom}(-, A), \text{Fb}))$$

with:

- Initial Object $0 \rightarrow \text{codomain } \pi(X)$
- Morphic Chains as fibers $\pi^{-1}(Z)$
- Ω as subobject classifier in Z
- σ operator inducing internal exponential morphisms.

Appendix

symbols

Z : Intermediating variable (conceptual anchor; Z-frame)
 $|$: Frame separator (indicates morphism is mediated by Z-frame)
 \rightarrow : Morphic Flow
 $\rightarrow/$ Ruptured morphism
 F : Cross-category morphism (used in cross-category flow under shared Z-frame)
 $//$: Used to narrate meaning flow of morphic chains.
 \neg : Absence

$M|Z$: Monoid of Conceptual Flow under Z-frame
 $R|Z := \{ \text{rupture}(f) \mid \text{rupture}(f, \sigma(f) \mid Z) \neq \emptyset \}$
 $e|Z$: Identity element of $M|Z$
 $D(A_{n-1} \mid Z)$: Morphic chain under Z frame

σ : Conceptual Shifting Morphism
 $>>$: Generalization relation ($A >> X \equiv A \sqsubseteq X$)
 $<<$: Specialization relation ($X >> A \equiv X \sqsubseteq A$)
 $\text{rupture}(f, \sigma(f) \mid Z) \neq \emptyset$: Indicates conceptual rupture
 η : Quasi-Natural Transformation: Contextual alignment between morphic chains.

\oplus : Conceptual morphism set addition in σ or morphic merger such as:
 $(k_2 \circ k_1) \oplus (q_2 \circ q_1) = \text{human} \rightarrow \text{royalty} \mid Z'$
 \ominus : Conceptual morphism set subtraction
Removes specified morphisms from a morphic chain or set.

Notations

Concept / Word (lexeme):
- Lower case (e.g., puppy, dog, girl, she)

Z Frame (conceptual anchor):
- Upper case (e.g., Mammal, Human, Agency, Domesticated, Royalty)

Type variables (A, B, X, Y, Z in formal definitions):
- Follow standard formal notation (uppercase)

Example:
puppy \rightarrow dog $|$ Mammal
 $A \rightarrow B \mid Z$

Morphism: f, g, h
Functor: F

Simplified Form of Identity Morphism:

1. $f: X \rightarrow X \mid X$ (Category-theoretic identity)
In simplified form: X
or more explicitly: id_X
2. $f: X \rightarrow X \mid Z$ (Mediated identity with conceptual flow)
In simplified form: $X \mid Z$

σ Operator

$\sigma(X). \text{Not}(x)\{ A \rightarrow B \mid Z \}$	\rightarrow	Rupture under Z frame
$\sigma(X). \text{so_much}(x)\{ A \rightarrow B \mid Z \}$	\rightarrow	Preservation & amplification under Z frame
$\sigma(X). \gg(x, y)$ function form	\rightarrow	Conceptual Shifting x to y (Generalization) as function form
$\sigma(X). \ll(x, y)$ function form	\rightarrow	Downward Shifting x to y (Specialization) as function form
$\sigma(X). >(x, y)$	\rightarrow	Conceptual Shifting

Conceptual Morphism Set Operators

Addition (\oplus):
 $\sigma(X). \oplus(f, A_{n-1} \mid Z): D(A_{n-1} \mid Z) \rightarrow D(B_{n-1} \mid Z) \mid Z$
 $\sigma(X). \oplus(f_1, f_2) : A_{n-1} := \{f_1, f_2\}$

Subtraction (\ominus):
 $\Theta: A_{n-1} \ominus \{f_i\}$
 $\sigma(X). \ominus(f, A_{n-1} \mid Z): D(A_{n-1} \mid Z) \rightarrow D(B_{n-1} \mid Z) \mid Z$

- \oplus operator is σ_{safe} if Z alignment is preserved.
- \ominus operator is potentially σ_{unsafe} but can be σ_{safe} if resulting chain preserves the underlying morphic continuity Z .

σ Typing Hierarchy

$\sigma_{\text{safe}}: D(A_{n-1} \mid Z) \rightarrow D(B_{n-1} \mid Z) \mid Z$ (Preserves global coherence)
 $\sigma_{\text{unsafe}}: D(A_{n-1} \mid Z) \rightarrow \{ \text{rupture}(f_1), \dots, \text{rupture}(f_n) \mid \neg Z \}$ (Global coherence lost)

Note: σ_{safe} behaves as Quasi-Natural Transformation.
 σ_{unsafe} induces rupture, and cannot be captured globally.

Python Code Used in This Study

```
import matplotlib.pyplot as plt
from sklearn.decomposition import PCA
from gensim.models import KeyedVectors
import numpy as np

model_path = ''
model = KeyedVectors.load_word2vec_format(model_path, binary=True)

words = ["this", "dog"]

# ベクトル取得
vectors = [model[word] for word in words]
labels = words

# 次元削減 (PCA)
pca = PCA(n_components=2)
reduced = pca.fit_transform(vectors)

# プロット
plt.figure(figsize=(10, 6))
for i, label in enumerate(labels):
    x, y = reduced[i]
    plt.scatter(x, y)
    plt.text(x + 0.01, y + 0.01, label, fontsize=9)

# 矢印付加

from matplotlib.patches import FancyArrowPatch
for i in range(len(reduced) - 1):
    start = reduced[i]
    end = reduced[i + 1]
    arrow = FancyArrowPatch(start, end, arrowstyle='->', mutation_scale=10,
color='gray')
    plt.gca().add_patch(arrow)

start = reduced[len(reduced)-1]
end = reduced[0]
arrow = FancyArrowPatch(start, end, arrowstyle='->', mutation_scale=10,
color='gray')
plt.gca().add_patch(arrow)

plt.title("PCA of Semantic Geometry (Word2Vec)")
plt.grid(True)
plt.axis("equal")
plt.savefig("image.png")
plt.show()
```

Kmeans

```
import numpy as np
from sklearn.cluster import KMeans
from gensim.models import KeyedVectors

model_path = ''
words = ["king", "queen", "man", "woman"]

model = KeyedVectors.load_word2vec_format(model_path, binary=True)
vectors = np.array([model[word] for word in words])

# KMeans Clustering
kmeans = KMeans(n_clusters=1, random_state=42, n_init='auto')
kmeans.fit(vectors)

# Z-frame
z_frame = kmeans.cluster_centers_[0]

print("\n=== Z-frame Candidates ===")
for word, score in model.similar_by_vector(z_frame, topn=10):
    print(f"{word:>12s} : Similarity = {score:.4f}")
```

Word2Vec Data Set

<https://code.google.com/archive/p/word2vec/>

Conceptual Topos Named as 概念位相論 / Conceptual Topology

This theory, named 概念位相論 or Conceptual Topology, was proposed by **No Name Yet Exist**.

Meaning no longer escapes.

It circulates within the morphic fibration.

We, once again, govern the topology of meaning.

GitHub: <https://github.com/No-Name-Yet-Exist/Conceptual-Topology>

Note: <https://note.com/xoreaxeax/n/n3711c1318d0b>

Zenodo: <https://zenodo.org/records/15455079>

This is Version: 1.5.1

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