Conceptual Topos As Conceptual Cage: An Algebraic Topology of Meaning based on Conceptual Topology

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Introduction

Conceptual Topos v1.2 is an initial formalization of the algebraic topology of meaning based on Conceptual Topology.

This version sketches core axioms for Topos including:

- Initial Object
- Finite Limits (Product, Equalizer, Pullback)
- Subobject Classifier Ω
- Fibered Topos structure
- Conceptual Exponential via σ operator

Future versions (v1.x) will refine the formalization and extend it.

In this version, the term fiber is used informally to describe the structural cohesion of morphic chains under a shared Z-frame.The current framework is not yet a strict fibered topos in the categorical sense. Formal connection to fibered topos is an intended direction for future versions.This document lays the foundation toward that goal.

Index

1. Fibered Conceptual Topology:

Fibered Conceptual Topology provides a conceptual geometric framework wherein each Z-frame (conceptual anchor) acts as a base space, with conceptual morphic flows forming fibers over these anchors. The Yoneda-like interpretation captures concepts as bundles of conceptual relations within and across Z-frames. This fibered structure serves as the foundation for further constructions in Conceptual Topos.

```
CT := (C, B, π: E → B, Fb := π⁻¹(b), A ≅ b ∪ Nat(Hom(−, A), Fb))

Where:
  - C is the category of concepts (objects = words or concepts)
  - B is the base space of Z-frames (conceptual continuity anchors)
  - E is the total conceptual space (word vector embedding space)
  - π projects each concept to its conceptual base (Z-frame)
  - Fb is the fiber (conceptual morphic chain) over a base b
  - A ≅ b ∪ Nat(Hom(−, A), Fb) interprets each concept A via its morphisms
relative to its   Z-frame b (Yoneda perspective defined in appendix)
```

```
        Z = baby
    puppy  ←   girl    //specified: size + young
     |          |
     |          |
    dog        she     //abstraction
      \        /
        Mammal
```

1.1. Local Conceptual Flow under Z Frame

Identity Morphism

In Category Theory, Identity Morphism is always defined.

```
id_X: X → X
such that for any f: X → Y:
f ∘ id_X = f
id_Y ∘ f = f
```

However, in Conceptual Topology, morphisms are mediated by Z frame, thus the identity morphis is not always given unless Z is defined.

Two Types of Identity Morphism in Conceptual Topology

1. f: X → X | X (Category-theoretic identity)

```
f: X → X | X
such that for any f: X → Y:
f ∘ id_X = f
id_Y ∘ f = f

e.g. f: dog → dog | dog
```

2. f: X → X | Z (Mediated identity with conceptual flow)

```
f: X → X | Z

f: X → Z
f⁻¹: Z → X
f⁻¹ ∘ f ≅ id_X
e.g. you → you | externalized perspective
     NL: you are you
```

Since the identity morphism passes through an external anchor point, the identity morphism is defined quasi-identical.
e.g. dog → perro | собака

Simplified Form of Identity Morphism:

1. f: X → X | X (Category-theoretic identity)
   In simplified form: X
   or more explicitly: id_X

2. f: X → X | Z (Mediated identity with conceptual flow)
   In simplified form: X | Z

Mirror Morphism Definition:

Each mirror maps conceptual transitions across vocabularies while preserving morphic identity up to rupture—that is, it allows for conceptual divergence that still respects underlying structural continuity, even if exact invertibility is not preserved.

```
f : X  → Y  | Z ∈ Dᵢ
f’: X’ → Y' | Z ∈ Dᵢ₊₁
  ⇒X’ ≠ X, but cod(f) = cod(f’) | CD (CD = codomain)

We define f’ as a mirror-correspondent morphism of f under a given Z-frame,
if and only if:

∃Z: rupture(f, f’ | Z) ≠ ∅
∧ cod(f) = cod(f’) | CD
```

```
        Z = baby
    puppy  ←   girl    //specified: size + young
     |          |
     |          |
    dog        she     //abstraction
      \        /
        Mammal
```

Note: Z: rupture(f, f’ | Z) ≠ ∅ means that there exists a Z-frame under which *f* and *f′* exhibit structural divergence—i.e., they are not fully invertible but still converge at the codomain level.

For example, let *Z* = *abstraction*. This allows a conceptual transition from *girl* → *she* and *puppy* → *dog*, treating them as mirror morphisms under a shared conceptual frame.
However, if we take *Z* = *agency*, a rupture emerges: *puppy* → *dog* lacks agency, while *girl* → *she* retains it.
Hence, **rupture(f, f’ | agency) ≠** ∅, yet *f* and *f′* still align toward the same codomain (e.g., *mammal*).

Quasi-Natural Transformation of Meaning Systems

A **Morphic Chain Mirror** is a contextual correspondence between two morphic chains drawn from distinct but meaning-aligned vocabularies. This correspondence is realized through a **quasi-natural transformation** under a shared intermidiating Z-frame.

```
η: Dᵢ ⇒ Dᵢ₊₁  | CD (CD = codomain)
η_X ∘ Dᵢ({f₁ | Z₁, ..., fₙ | Zₙ}) ≈ Dᵢ₊₁({f′₁ | Z₁, ..., f′ₙ | Zₙ}) ∘ η_Y | CD
for all fⱼ: Xⱼ → Yⱼ | Zⱼ ∈ Dᵢ,
where f′ⱼ: η_X(Xⱼ) → η_Y(Yⱼ) | Zⱼ

Then, η is said to be a quasi-natural transformation under the Z-frame
i.e. η ∈ Mor(C) where C is the contextual meaning category
Example: η: girl → puppy | Z = Baby
```

1.2. σ Operator as Functor

Definition: Conceptual Shifting Morphism (σ)

```
σ: D(Xₙ₋₁ | X) → D(Xₙ₋₁ | X)

such that σ ⊕ f ∈ M|Z if and only if type compatibility holds:

∀ A, B, (A → B) ∘ σ(X) is valid if:

( A >> X or X >> A )
and
( B >> X or X >> B )

Definition: Subsumption
A >> X ≡ A ⊑ X

Definition: SubsumedBy
X >> A ≡ X ⊑ A

Example:
king → king >> human → human
  ⇒ king >> human  → valid

human → human >> queen → queen
  ⇒ human >> queen → valid
```

Conceptual Operators

Conceptual Operator σ modifies morphism as follows.

```
σ(X). Not(x){ A →/B | Z}     →     Rupture under Z frame
σ(X). so_much(x){A → B | Z}  →     Preservation & amplification under Z frame
σ(X). >>(x,y)                →     Conceptual Shifting x to y (Generalization) as
function form
σ(X). <<(x,y)                →     Downward Shifting x to y (Specialization) as
function form
σ(X). >(x,y)                 →     Conceptual Shifting
```

Conceptual Morphism Set Operators

```
Addition (⊕):
σ(X). ⊕(f, Aₙ₋₁ | Z): D(Aₙ₋₁ | Z) → D(Bₙ₋₁ | Z) | Z
σ(X). ⊕(f₁, f₂) : Aₙ₋₁ :={f₁, f₂}

Subtraction (⊖):
⊖: Aₙ₋₁ ⊖ {f_i}
σ(X). ⊖(f, Aₙ₋₁ | Z): D(Aₙ₋₁ | Z) → D(Bₙ₋₁ | Z) | Z

- ⊕ operator is σ_safe if Z alignment is preserved.
- ⊖ operator is potentially σ_unsafe but can be σ_safe if resulting chain
preserves the underlying morphic continuity Z.
```

Example

```
{Royalty⃗, Male⃗, Human⃗} ⊖ {Male⃗} ⊕ {Female⃗}
= {Royalty⃗, Female⃗, Human⃗}
= queen
```

Conceptual Mapping

```
C_chain = { f₁, f₂, ..., fₙ | Z} ∈ D(Cₙ₋₁ | Z)

σ(X): D(Aₙ₋₁ | Z) > D(Bₙ₋₁ | Y) | Z, Y ∈ CD
where:
D(Aₙ₋₁ | Z) = source morphic chain
D(Bₙ₋₁ | Y) = target morphic chain
CD = codomain alignment (conceptual anchor)

σ(X) is not strict functorial → quasi-alignment under conceptual equivalence

σ(X) ≈ η: Dᵢ ⇒ Dᵢ₊₁ | CD (Quasi-Natural Transformation interpretation)

Example:
σ(X). >(puppy → dog → mammal | Canine, Human) ∋ girl → she → mammal | Human

where: canine, Human ∈ Mammal
```

Identity Morphism of σ

```
word is word
thus:
word ≅ Nat(Hom(-, word), Fib(word))

σ_id(Z). OP(X,Z) = σ such that σ(f) = f for all f ∈ Hom(X, X) unless OP is
σ_unsafe such that word is not a word: σ(Word). Not(word ↛word)

σ_id(Word). OP(word, Word) = word

σ_id(Word). OP(f, Word) = f  for all f: word → word | word
since: M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z = Word
}

σ_id ∈ M|Z
σ ∘ σ_id = σ
σ_id ∘ σ = σ
∴ word is word and word is word
```

Associativity of σ

```
σ₁(Z). OP(D(Aₙ₋₁ | Z), Z) = D(Zₙ₋₁ | Z)
σ₂(Z). OP(D(Bₙ₋₁ | Z), Z) = D(Zₙ₋₁ | Z)

Then the composition σ₂ ∘ σ₁:
σ_comp(Z). OP(D(Zₙ₋₁ | Z), D(Zₙ₋₁ | Z)) = D(Zₙ₋₁ | Z)

where: OP is not σ_unsafe and under shared Z frame

Associativity
For all σ₁, σ₂, σ₃ such that their domains/codomains match for composition:
(σ₃ ∘ σ₂) ∘ σ₁ = σ₃ ∘ (σ₂ ∘ σ₁)

Thus, σ composition operator is associative under Monoid structure.
```

Example:

```
Let σ₁ = σ(Mammal). >>(canine → mammal, Life)  = (life → life | Life)
Let σ₂ = σ(Mammal). >>(mammal → animal, Life)  = (life → life | Life)
Let σ₃ = σ(Mammal). >>(animal → livingBeing, Life) = (life → life | Life)

Conclusion:
 (σ₃ ∘ σ₂) ∘ σ₁ = σ₃ ∘ (σ₂ ∘ σ₁) =  (life → life | Life)
```

1.3. σ Operator as Kan Extension

Functorial Properties of σ mapping

```
A Functor F: C → D is a mapping between categories satisfying:

- Object mapping: For each X ∈ Ob(C), F(X) ∈ Ob(D)
- Morphism mapping: For each f: X → Y ∈ Mor(C), F(f): F(X) → F(Y) ∈ Mor(D)
- Identity preservation: F(id_X) = id_{F(X)}
- Composition preservation: F(f ∘ g) = F(f) ∘ F(g)

We define σ: D(Aₙ₋₁ | Z) >> D(Bₙ₋₁ | Z') as such a Functor.
```

σ Operator as Kan Extension

```
Let:

- D(Aₙ₋₁ | Z) := Category of Morphic Chains over Z-frame Z
- D(Bₙ₋₁ | Z') := Category of Morphic Chains over Z'-frame Z'

Define:

σ_safe ≈ Lan_σ : D(Aₙ₋₁ | Z) >> D(Bₙ₋₁ | Z')

such that:

For any object d ∈ D(Bₙ₋₁ | Z'),
Lan_σ (D(Aₙ₋₁ | Z))(d) := colim_{(c, f: σ(c) → d)} D(Aₙ₋₁ | Z)(c)

And:

For any morphism h: d → d' in D(Bₙ₋₁ | Z'),
Lan_σ (h) is defined to preserve functoriality:

Lan_σ (h) ∘ Lan_σ (f) = Lan_σ (h ∘ f)

Therefore:

σ_safe satisfies:

- Object-level safe lifting: Ob(D(Aₙ₋₁ | Z)) → Ob(D(Bₙ₋₁ | Z'))
- Morphism-level safe lifting: Mor(D(Aₙ₋₁ | Z)) → Mor(D(Bₙ₋₁ | Z'))

σ_safe ≈ Left Kan Extension guarantees the Quasi-Natural Transformation property:

∀ f ∈ Mor(D(Aₙ₋₁ | Z)),
Lan_σ (G ∘ f) = (Lan_σ G) ∘ (Lan_σ f)
```

Relation to Qasi-Natural Transformation

The σ mapping operator, defined as a Functor σ: $D(A_{n-1} \mid Z) >> D(B_{n-1} \mid Z')$, exhibits structural alignment with Quasi-Natural Transformation (QNT) in the following way:

In the original formulation of QNT in this framework:

```
η: Dᵢ ⇒ Dᵢ₊₁ | CD (codomain)
η_X ∘ Dᵢ({f₁ | Z₁, ..., fₙ | Zₙ}) ≈ Dᵢ₊₁({f′₁ | Z₁, ..., f′ₙ | Zₙ}) ∘ η_Y
```

Diagram:

```
        Z = baby
    puppy  ←   girl    //specified: size + young
     |          |
     |          |
    dog        she     //abstraction
      \        /
        Mammal

In this diagram, Quasi-Natural Transformation η aligns morphic chains between
puppy → dog → mammal and girl → she → mammal within a shared codomain mammal
(under Z-frame "Mammal").
```

the Quasi-Natural Transformation mediates conceptual flow correspondence across different morphic chain categories under a shared or shifted Z-frame.

In the Kan Extension formalization:

```
Lan_σ (D(Aₙ₋₁ | Z)) = D(Bₙ₋₁ | Z')
```

the lifting of the entire functor $D(A_{n-1} \mid Z)$ under σ corresponds to constructing a universal QNT from $D(A_{n-1} \mid Z)$ to $D(B_{n-1} \mid Z')$.

More precisely, for any object $d \in D(B_{n-1} \mid Z')$:

```
Lan_σ (D(Aₙ₋₁ | Z))(d) := colim_{(c, f: σ(c) → d)} D(Aₙ₋₁ | Z)(c)
```

yields a canonical shifting from the conceptual flow space under Z to the corresponding conceptual flow space under Z', respecting the structural continuity required by QNT.

Thus:

```
σ_safe ≈ Left Kan Extension ≈ Universal Quasi-Natural Transformation between
D(Aₙ₋₁ | Z) and D(Bₙ₋₁ | Z')
```

Diagram:

```
puppy → dog → mammal
      \         |
       \        |
        v       v σ(Life)
           Life

girl → she → mammal
      \         |
       \        |
        v       v σ(Life)
           Life

σ(Life) acts as a Kan Extension operator, lifting conceptual morphic chains from
Fiber over Z = Mammal to Fiber over Z' = Life. The lifting ensures coherent
conceptual projection across Z-frames, preserving flow alignment while
generalizing to a higher conceptual space.
```

This formalization guarantees that the Quasi-Natural Transformation property observed in the original Conceptual Cage structure is preserved and generalized through the Kan Extension framework, providing a categorical foundation for conceptual flow lifting.

1.4 Kan Extension as Horizontal Conceptual Shifting and Cone Structure

Conceptually, σ operator as Kan Extension performs not only lifting of morphic chains but also acts as a horizontal mapping across Z-frames, shifting conceptual flow from Fiber over Z to Fiber over Z'.

Diagrammatically, this can be visualized as a horizontal shift:

```
Fiber over Z (Mammal):

puppy → dog → mammal
girl → she → mammal

↓↓↓↓↓↓ Kan Extension σ(Life)

Fiber over Z' (Life):

girl → she → mammal → Life
puppy → dog → mammal → Life
```

Applying σ(Life) results in a horizontal lifting of codomain alignment

Recursive Kan Extension as Iterated Colimit of Conceptual Shiftings
Conceptually, Recursive Kan Extension can be understood as constructing an iterated colimit of sequential conceptual shiftings (σ operators) across Z-frames:

Conceptual Ladder Structure:

```
Fiber over Z₀
    ↓ σ₁
Fiber over Z₁
    ↓ σ₂
Fiber over Z₂
    ↓ σ₃
Fiber over Z₃
    ↓ ...

NL: turtle → reptile → animal → ...
```

Iterated Colimit Perspective:
At each stage, the application of σ_n corresponds to forming a conceptual projection from Fiber over Z_{n-1} to Fiber over Z_n.

The entire ladder:

$\mathrm{Lan}_{\sigma_n} \circ \ldots \circ \mathrm{Lan}_{\sigma_3} \circ \mathrm{Lan}_{\sigma_2} \circ \mathrm{Lan}_{\sigma_1}$

can be viewed formally as an iterated colimit over the sequence of Z-frames, forming a conceptual cone over the diagram:

```
colim_{Z₀ → Z₁ → Z₂ → ... → Zₙ} (Lan_{σᵢ}(Fiber over Zᵢ₋₁))
```

Interpretation:
Each Lan_{σ_i} acts as a conceptual lifting operation, progressively shifting semantic flow across Z-frame layers. The cumulative structure forms an iterated conceptual cone, whose colimit aligns the entire sequence into the semantic flow space under Z_n.

Diagram (Iterated Colimit View):

```
                Fiber over Z₀
                   ↓ σ₁
                Fiber over Z₁
                   ↓ σ₂
                Fiber over Z₂
                   ↓ σ₃
                Fiber over Z₃
                   ↓ ...
            ---------------------
          Iterated Colimit (Conceptual Cone)
           → Fiber over Zₙ

NL: tortoise → turtle → reptile → animal → ... | Iterated Colimit Result = Muti
Celluar Organism
```

Formal Expression:

```
Iterated_Colimit ≈ colim_{Z₀ → Z₁ → Z₂ → ... → Zₙ} (Lan_{σᵢ}(Fiber over Zᵢ₋₁))
```

This conceptual ladder forms an iterated semantic cone, whose colimit aligns the entire Z-frame sequence into the unified semantic flow space under Z_n.

Diagram:

```
        Zₙ
    ↙ ↓   ↓ ↘
  Z₀  Z₁  Z₂  Z₃

        ↓

M|Zₙ (colimit of Ladder)
```

A cone on a diagram F: J → C is a universal natural transformation from a constant diagram ΔX to F. In this case:

```
Δ(Fiber over Zₙ) ⇒ Ladder of Lan_{σᵢ}(Fiber over Zᵢ₋₁)

or as monoid structure:

M|Zₙ { Fₙ ∘ ... ∘ F₁ | all Fᵢ: Fᵢ → Fᵢ₊₁ | Zₙ ∧ ∀ i, j: Fᵢ ≅ Fⱼ | Zₙ }
```

∞-Morphic Interpretation of Recusive Ken Extension

Viewed categorically, this recursive construction aligns with the notion of ∞-morphisms or higher morphic flows, where each application of Lan_{σ_i} corresponds to a morphism in an extended conceptual category, and their collective composition forms an ∞-structured cone:

```
∞-Universal Product ≈ colim_{Z₀ → Z₁ → Z₂ → ... → Zₙ} (Lan_{σᵢ}(Fiber over Zᵢ₋₁))
```

Diagram:

```
Z₀ → Z₁ → Z₂ → Z₃ → ... → Zₙ
 |     |     |     |        |
Lan_{σ₁}  Lan_{σ₂}  Lan_{σ₃}  ... Lan_{σₙ}
 |     |     |     |        |
 v     v     v     v        v
Fiber over Z₁   Fiber over Z₂   Fiber over Z₃   ...   Fiber over Zₙ
       ↘ ↘ ↘ ↘ ↘
         ↓ ↓ ↓ ↓
colim (Lan_{σᵢ} Fiber over Zᵢ₋₁)
```

NL Diagram:

```
tortoise → turtle → reptile → ... → Zₙ
  |         |     |     |           |
Lan_{σ₁}  Lan_{σ₂}  Lan_{σ₃}  ... Lan_{σₙ}
 |     |     |     |        |
 v     v     v     v        v
Fiber over Z₁   Fiber over Z₂   Fiber over Z₃   ...   Fiber over Zₙ
       ↘ ↘ ↘ ↘ ↘
         ↓ ↓ ↓ ↓
  Multi-Cellular Organism
```

This interpretation enables the Conceptual Topos or Conceptual Topology to naturally support recursive, layered conceptual flow, where mappings can extend across arbitrarily many Z-frames while preserving structural coherence.

Example: Iterated Kan Extension of Conceptual Ladder

Step 1:

```
Z₀ = Turtle
Z₁ = Reptile
σ₁ = σ(Reptile): Turtle → Reptile

Lan_{σ₁}(Fiber over Turtle) → Fiber over Reptile
```

Step 2:

```
Z₂ = Animal
σ₂ = σ(Animal): Reptile → Animal

Lan_{σ₂}(Fiber over Reptile) → Fiber over Animal
```

Step 3:

```
Z₃ = Life
σ₃ = σ(Life): Animal → Life

Lan_{σ₃}(Fiber over Animal) → Fiber over Life
```

Composition:

```
Lan_{σ₃} ∘ Lan_{σ₂} ∘ Lan_{σ₁}(Fiber over Turtle)
```

Colimit:

```
colim_{Z₀ → Z₁ → Z₂ → Z₃} Lan_{σᵢ}(Fiber over Zᵢ₋₁) ≈ Fiber over Z₃ = Fiber over Life
```

NL:

```
Turtle → Reptile → Animal → Life
```

Conceptual flow lifted across Z-frame layers as iterated Kan Extensions, converging to the unified flow under Life.

Safe / Unsafe Conceptual Shifting Morphism (σ)

Definition of Safe and Unsafe σ Operator
Conceptual Shifting Morphism (σ) can be classified based on whether it preserves the global coherence of the morphic chain.

Safe σ Operator (σ_safe) Acts on the entire morphic chain as a coherent transformation.

```
σ_safe: D(Aₙ₋₁ | Z) > D(Bₙ₋₁ | Z') | Z >> Z' ∨ Z << Z'

where: Z, Z'∈ CD
```

Behaves as a Quasi-Natural Transformation

```
σ_safe ≈ η: Dᵢ ⇒ Dᵢ₊₁ | CD
```

Composition is associative:

```
(σ₃ ∘ σ₂) ∘ σ₁ = σ₃ ∘ (σ₂ ∘ σ₁)
```

Resulting chain remains in M|Z or M_{Z'} (closed).

Example

```
σ₁(X). >(canine, mammal)
σ₂(X). >(mammal, animal)
σ₃(X). >(animal, livingBeing)

Composition:
(σ₃ ∘ σ₂) ∘ σ₁ = σ₃ ∘ (σ₂ ∘ σ₁)
→ >(canine, livingBeing)

Entire morphic chain is preserved.
```

Unsafe σ Operator (σ_unsafe) Does not preserve global coherence of the morphic chain. Acts locally or in a decomposed manner.

Chain may collapse:

```
σ_unsafe: D(Aₙ₋₁ | Z) → { rupture(f₁), rupture(f₂), ..., rupture(fₙ) | ¬Z }
```

```
rupture(f, σ(f) | Z) ≠ ∅
```

Cannot be captured by a Quasi-Natural Transformation globally.

Example

```
σ(X). Not(x) { A →/B | Z }

Result:
rupture(A →/B | Z)
→ breaks the morphic flow → chain decomposes.
```

2. Monoid Structure of Conceptual Flow (M|Z):

In Conceptual Topology, Z is defined as a mediating point/conceptual anchor.

```
Let C and D, Z be categories,
with conceptual projection π: C ∪ D → Z, such that for each X ∈ Ob(C ∪ D):

π(X) ∈ Ob(Z)

For each X ∈ Ob(C ∪ D), there exists morphism:

f_X: X → π(X)
f_X⁻¹: π(X) → X

such that:

f_X⁻¹ ∘ f_X ≅ id_X

For morphism f: X → Y | Z,
this corresponds to:

f_Z: π(X) → π(Y) in Z

For any X, Y ∈ Ob(C ∪ D):

Let [X]_Z := conceptual representation of X under frame Z (i.e., π(X))

Then:

[ X ]_Z1 ≅ [ Y ]_Z2 | Z1, Z2 ∈ Z //or Z1, Z2 >> Z
which means:
["Dog"]_Pet = [Retriever, Dachshund, Poodle, Bulldog, ...]
["girl"]_Human = [girl, woman, person, ...]
["Dog"]_Pet ≅ ["girl"]_Human | Life

Then the set of conceptual flow morphisms under Z forms a monoid:

M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z }

This is also defined as Morphic Chain:

Let **D(Cₙ₋₁ | Z)** := Category of Morphic Chains over **Ob(Cₙ₋₁)** within a given
Z-frame.
where:D(Cₙ₋₁ | Z) ={ C₀ → C₁ → C₂ → ... | Z }

or as a set
D(Cₙ₋₁ | Z) ={ C₀, C₁, C₂, ... | Z }
```

3. Identity Element of M|Z

```
Let: M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z }

Define the identity element of M|Z as a family of identity morphisms over the
shared Z frame:

For each X ∈ Ob(C ∪ D), there exists a unique identity morphism under a Z frame:

e|Z_X := id_X | Z

Then, for any f: X → Y | Z ∈ M|Z:

e|Z_X ∘ f = f
f ∘ e|Z_Y = f

Therefore, the identity structure of M|Z is given by the family:

{ id_X | Z | X ∈ Ob(C ∪ D) }

which forms a pointwise identity across the objects under the common Z frame.
This ensures that M|Z satisfies the identity axiom of a monoid.
```

4. Associativity of M | Z

```
Let: M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z }

Then for all f, g, h ∈ M|Z:

(f ∘ g) ∘ h = f ∘ (g ∘ h)

Thus, the composition ∘ in M|Z is associative.
```

5. Axioms

5.1. Identity Element

Unit Axiom 1: Identity Element Z

```
id_Z:= Z → Z | Z
∀f ∈ M|Z: id_Z ∘ f = f  and f∘id_Z =
```

Definition:

```
Statement:
Z-frame itself is the unit of M | Z.

Formal Definition:
Z:= Z → Z|Z

Justification:
Since any morphism in M | Z is defined as:

f: X→Y|Z

and Z itself is defined as its own identity morphism:

Z:= Z→Z|Z

then:
id_Z = Z

Conclusion:

Therefore:
id_Z is the unit element of M|Z.

∀f ∈ M|Z: (id_Z ∘ f | Z) = f and (f ∘ id_Z | Z)= f
(with frame-preserving composition)

∴id_Z is the unit of M|Z.
```

Note:

```
idZ :Z→Z | Z
f:X→Y | Z
(idZ|Z)∘(X→Y|Z)
```

Unit Axiom2: Void Concept

```
f ∈ M|Z
"" ∘ f = f  and f ∘ "" = f
id_Z ∘ f = f and f ∘ id_Z = f
```

Definition:

```
The empty concept is a theoretically assumed concept, denoted as "",
which acts as the unit element at the conceptual / lexical level.

Formal Definition:

"" ∘ f = f and f ∘ "" = f

Justification:

The empty concept "" represents no lexical or conceptual content.
Composing any morphism f with the empty concept does not alter the flow of
meaning.
Conclusion:

"" is the unit element at the conceptual level of Conceptual Topology.
```

5.2. Zero Morphism: Negation Morphism

We define conceptual zero morphism, negation morphism: n_f In CT as the result of applying Not() to a morphism

```
g:σ(Z). Not(g){ A →/B | Z} = A →/B|Z = n_f
where: g: A → B

Formal Properties (Axiom):

∀g:X→Y|Z where composition with n_f is defined:

∀g: g ∘ n_f = n_f and n_f ∘ g = n_f

Left Side:
g: A→B
g∘(A→B̸|Z) = A→/B|Z

Right Side
g: A→B(A→B̸|Z)∘g=A→B̸|Z
```

```
Interpretation:
Applying Not() to any morphism produces a conceptual zero morphism, which
collapses any further conceptual flow.

Natural Language:
Left Side: g∘(A→B̸|Z)
"A is not B"
The apple is not a fruit

Right Side: (A→B̸|Z)∘g
"B is not A"
This is a fruit, but this is not an apple which is a fruit.


In CT, this was called rutpure().
Now defined:
rupture(A,B,Z)= σ(Z).Not(g) = n_f = A→B̸|Z
```

5.3. Composition Axiom

```
M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z }

Then for all f, g, h ∈ M|Z:

For f, g, h ∈ M|Z,
where:
f:V → W | Z
g:Y → V | Z
h:X → Y | Z

(f ∘ g) ∘ h = f ∘ (g ∘ h)
```

Example:

```
For f, g, h ∈ M|Z,

where:
f:she → you | Human
g:he → she | Human
h:man → he | Human

(f ∘ g) ∘ h = f ∘ (g ∘ h)
```

6. Conceptual Topos

6.1. Category Level: Initial Object

Definition:

```
Let Concept be a category where Ob(Concept) are lexical / conceptual objects.
Then "" ∈ Ob(Concept) is Initial Object if:

∀ X ∈ Ob(Concept), ∃ unique morphism:

u_X : "" → X | X

such that:

∀ f: X → Y | Z,
f ∘ u_X = u_Y
```

Monoid Level: Unit in M|Z

```
Recall:

M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z }

Now, define:

"" ∈ Ob(Concept)

and identity morphism under Z-frame:

e|Z_"" := id_"" | Z

Then for all f ∈ M|Z:

e|Z_"" ∘ f = f
f ∘ e|Z_"" = f
```

6.2. Finite Limits

Terminal Object Conceptual Topos defines a terminal object as the Z-frame identity:

```
id_Z := Z → Z | Z


Any morphism f: X → Z | Z factors uniquely through id_Z.

This realizes the conceptual universal target:

∀ X ∈ Ob(C ∪ D), ∃! f_terminal: X → Z | Z
```

Example:

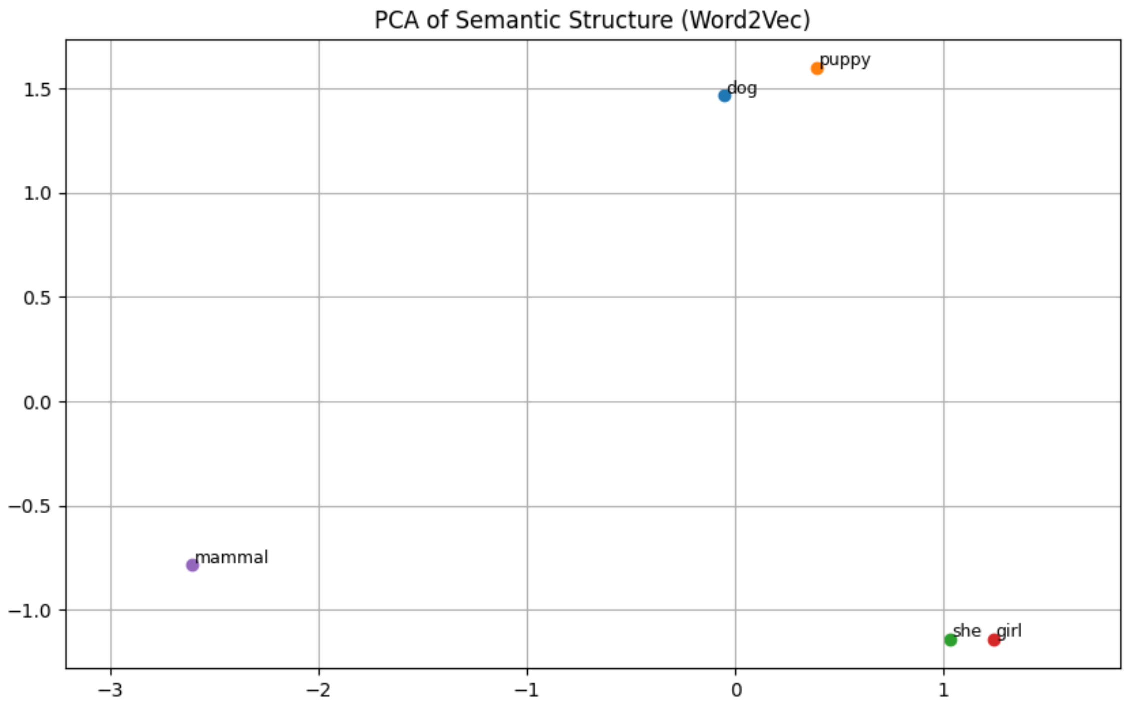
```
she → human | Human
me → human | Human
```

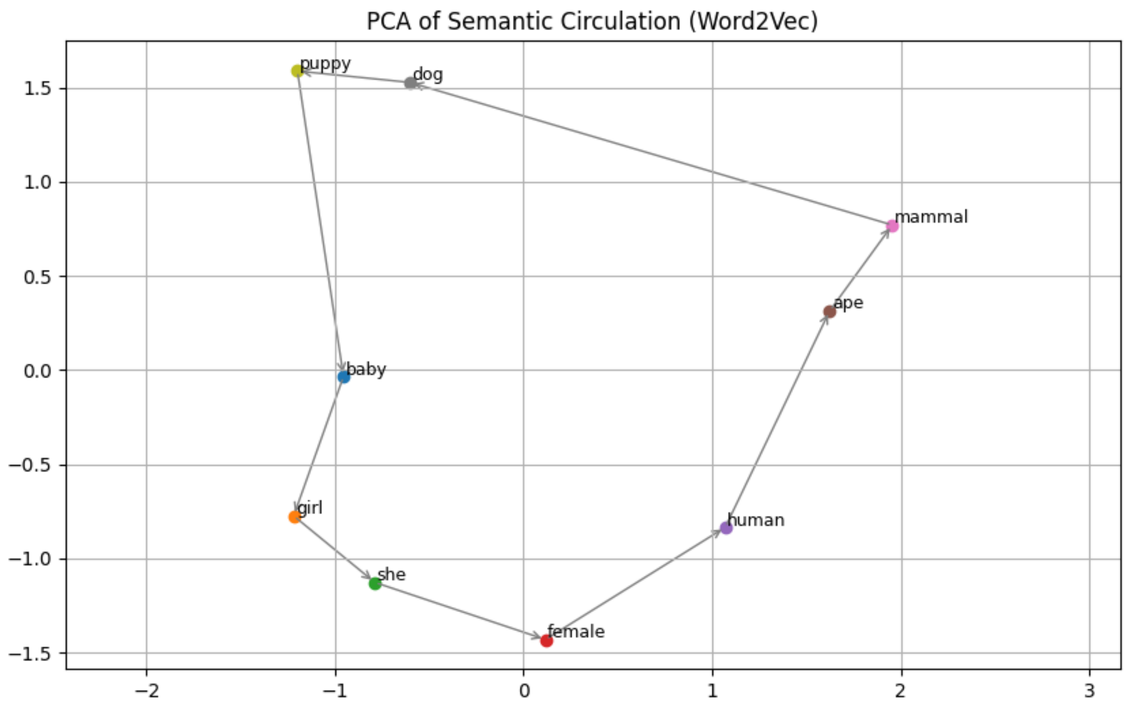
Pullback

```
Given morphisms:

f: girl → mammal
g: puppy → mammal

Pullback of (f, g) is:

P = Baby
p₁: Baby → girl
p₂: Baby → puppy

with commuting condition:

f ∘ p₁ = g ∘ p₂  ≈ mapping to common conceptual frame (mammal)

Diagram:

      Baby
     /    \
  p₁ /      \ p₂
   /          \
girl          puppy
   \          /
    \        /
     v      v
     mammal (conceptual anchor / codomain)
```

This previously defined as Quasi-Natural Transformation:

```
η: Dᵢ ⇒ Dᵢ₊₁  | CD (CD = codomain)
η_X ∘ Dᵢ({f₁ | Z₁, ..., fₙ | Zₙ}) ≈ Dᵢ₊₁({f′₁ | Z₁, ..., f′ₙ | Zₙ}) ∘ η_Y | CD
for all fⱼ: Xⱼ → Yⱼ | Zⱼ ∈ Dᵢ,
where f′ⱼ: η_X(Xⱼ) → η_Y(Yⱼ) | Zⱼ

Then, η is said to be a quasi-natural transformation under the Z-frame
i.e. η ∈ Mor(C) where C is the contextual meaning category

η_X ∘ Dᵢ({girl → mammal | Z₁}) ≈ Dᵢ₊₁({puppy → mammal | Z₂}) ∘ η_Y


Pullback Diagram
      Mammal
     /     \
    /       \
girl         puppy
   \         /
    \       /
     Baby  (conceptual anchor / common Z-frame)


Example: η: girl → puppy | Z = Baby


Quasi-Natural Transformation Diagram:
        Z = baby
    puppy  ←   girl    //specified: size + young
     |          |
     |          |
    dog        she     //abstraction
      \        /
        Mammal


For any X with morphisms
q₁: X → girl and q₂: X → puppy satisfying f ∘ q₁ = g ∘ q₂,

there exists unique u: X → Baby

such that:
p₁ ∘ u = q₁, p₂ ∘ u = q₂.
```


PCA of Semantic Structure (Word2Vec)
puppy
dog
mammal
she
girl
1.5
1.0
0.5
0.0
−0.5
−1.0
−3
−2
−1
0
1


PCA of Semantic Circulation (Word2Vec)
puppy
dog
mammal
ape
baby
girl
she
female
human
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2
−1
0
1
2
3

Equalizer: Mirror Morphism

```
Equalizer of two morphisms f, g:
A → B is an object Eq(f,g) with morphism
e: Eq(f,g) → A

such that:

f ∘ e = g ∘ e

and universal property:

∀ h: X → A such that f ∘ h = g ∘ h,
∃! unique u: X → Eq(f,g)

Diagram:

      Eq(f, g)
         |
         e
         v
         A
    f ↘     ↙ g
        B
```

In conceptual topology this was defined as mirror morphism:

```
f : X  → Y  | Z ∈ Dᵢ
f′: X′ → Y' | Z ∈ Dᵢ₊₁
  ⇒X′ ≠ X, but cod(f) = cod(f′) | CD ( common codomain)

We define f′ as a mirror-correspondent morphism of f under a given Z-frame,
if and only if:

∃Z: rupture(f, f′ | Z) ≠ ∅
∧ cod(f) = cod(f′) | CD


        Eq(f, f′)
          |
          e
          v
      X        X′
       \      /
        \    /
         v  v
          Y = Y′ (codomain = C)
```

Product: σ operator⊕

```
In any category C, the Product of A and B is an object A × B equipped with
projections:

π₁: A × B → A
π₂: A × B → B

with universal property:
For any object X with morphisms:

f₁: X → A
f₂: X → B

there exists a unique morphism u: X → A × B such that:

π₁ ∘ u = f₁
π₂ ∘ u = f₂
```

Addition (⊕):
$\sigma(Z)$ serves as the mediating operator ensuring that the composed morphic chain remains within the conceptual fiber over Z.

```
Defined as:
σ(Z). ⊕(Aₙ₋₁, Bₙ₋₁, Z) = D(Cₙ₋₁ | CD)  → conceptual Product under Z-frame

where:
Aₙ₋₁:= girl → she
Bₙ₋₁:= puppy → dog

σ(Z). ⊕(Aₙ₋₁, Bₙ₋₁, Z) = D(Cₙ₋₁ | CD)

For any pair of morphic chains 1Aₙ₋₁, Bₙ₋₁, the operation σ(Z).⊕(Aₙ₋₁, Bₙ₋₁)
defines an object P ∈ D(Cₙ₋₁ | Z)P∈D(Cn−1∣Z) with projections π₁, π₂ satisfying
the product universal property.


Example:

girl → she
puppy → dog

σ(Human). ⊕(girl → she, puppy → dog | Mammal) → Product(girl → she → puppy → dog
| Mammal) | Mammal
→ composite meaning space
```

```
Diagram:

    Product(girl → she, puppy → dog) ∈ D(Cₙ₋₁ | Z = Mammal)

             Product Object (P)
              /           \
             /             \
      π₁   /                 \  π₂
          v                   v
  girl → she            puppy → dog
```

6.3. Exponentials

Conceptual Topos models exponentials via conceptual shift operators.

Definition

```
For any objects A, B:

B^A exists such that:
Hom(X ⊗ A, B) ≅ Hom(X, B^A)
```

Construction via σ operator

Conceptual shift operators:

```
σ(Z). >>(A, B)
or
σ(Z). >(A, B)

act as internal exponential morphisms within the fibered structure over the Z-frame:
        (A, B, Z) ≅ B^A
where the Z-frame mediates the conceptual continuity and contextual grounding of the morphic shift.
```

We define Exponential objects via σ operator as conceptual abstraction mechanisms:

```
B^A:=σ(Z).>(A,B)
```

Full Exponential Law formalization will be provided in later version.

Definition: Conceptual Shifting Morphism (σ)

```
σ: D(Xₙ₋₁ | X) → D(Xₙ₋₁ | X)

such that σ ⊕ f ∈ M|Z if and only if type compatibility holds:

∀ A, B, (A → B) ∘ σ(X) is valid if:

( A >> X or X >> A )
and
( B >> X or X >> B )

Definition: Subsumption
A >> X ≡ A ⊑ X

Definition: SubsumedBy
X >> A ≡ X ⊑ A

Example:
king → king >> human → human
  ⇒ king >> human  → valid

human → human >> queen → queen
  ⇒ human >> queen → valid
```

Example

```
σ(Human). >>(puppy → dog → mammal | Canine, Human)
≅ girl → she → mammal | Human
```

This shift realizes an internal conceptual transformation corresponding to exponential behavior.

6.4. Definition of Ω

Let **Ω** be an object in the Concept category, representing the **conceptual truth space**.

```
For any subobject (conceptual inclusion):

m: M ↪ X

there exists a unique characteristic morphism:

χ_m: X → Ω
```

such that the following diagram commutes:

```
M ———→ X
|       |
|       v χ_m
|       Ω
```

Interpretation in Conceptual Topology

- **Ω** encodes **conceptual entailment / membership / inclusion**.
- **Z-frame membership** is naturally mapped to Ω:

```
χ_Z: X → Ω
```

interpreted as:
"Does X conceptually belong to Z-frame Z?"

Examples

Example 1: Dog in Pet Z-frame

χ_Pet(Dog) = True

Example 2: Apple in Pet Z-frame

χ_Pet(Apple) = False

Example 3: Innocent in Body Z-frame (after rupture)

χ_Body("innocent") = True / False depending on whether the conceptual projection is coherent under Z-Frame.

Relation to Rupture

Conceptual rupture can be lifted to Ω as:

σ(Z). Not(f: A → B | Z) ⇒ rupture(A,B,Z) ⇒ χ_Z(f) = False

Thus, **negation** and **conceptual discontinuity** become **Ω-classifiable**.

6.5. Conceptual Topos as Fibered Topos over Z-frame

Conceptual Topos is structured as a **fibered topos** over the conceptual base space **Z-frame**.

Z-frame as Fibered Structure

- Let π: C ∪ D → Z be the conceptual projection.
- Each fiber $\pi^{-1}(Z)$ forms a category of morphic chains **$D(C_{n-1} \mid Z)$**.
- Morphisms of the form:

```
X → Y | Z ≡ X → Y in fiber over Z
```

correspond to morphisms within the fibered structure over Z.

Initial Object and Codomain Projection

- The **Initial Object ""** serves as the conceptual origin.
- It projects into the codomain via:

```
"" → | X ≡ "" → π(X)
```

```
   ""
    ↓ u_X
  X ───→ π(X) (in Z-frame)
```

```
Fiber π⁻¹(Z_X):
    "" → X → Y
```

Thus, conceptual generation naturally occurs anchored in Z-frame.

Conceptual Flow Closure

- Conceptual flows:

```
X → Y | Z
```

are closed within the fiber over Z, corresponding to the codomain Z of the conceptual projection π.

- Rupture and negation are classified by **Ω**:

```
χ_Z: X → Ω
```

7. Global Conceptual Space: Total Conceptual Space (TCS)

We define the Total Conceptual Space (TCS) as the global conceptual anchor:

```
Z = TCS = Total Conceptual Space
```

Definition of M|TCS:

The global morphic flow space under TCS is defined as:

```
M|TCS = { fₙ ∘ ... ∘ f₁ | all fᵢ:  M|Zᵢ →  M|Zᵢ₊₁ | TCS ∧ ∀ i, j: fᵢ ≅ fⱼ | TCS }
```

We can regard M|TCS as the composition space of conceptual perspectives: Here, each M|Z functions as a conceptual symbolization or perspective lens, and M|TCS represents global flows across chained perspectives.

Monoid Closure Property:

```
Composition in M|TCS is closed:
∀ f, g ∈ M|TCS, f ∘ g ∈ M|TCS

The identity morphism is preserved:

∀ f ∈ M|TCS, f ∘ id = f = id ∘ f
```

Thus, M|TCS forms a closed monoid under composition.

Completeness Statement:

```
For any pair of concepts X, Y:

∀ X, Y ∈ Ob(C), ∃ f ∈ Mor(C), such that f: X → Y | TCS
```

That is, any conceptual pair X and Y can be connected via a morphic flow under TCS.

Fibered Structure and Lifting

Each local M|Z can be lifted into M I TCS via conceptual shifting σ:

```
∀MlZ, ∃σ: MlZ > MlZ | TCS
```

Thus, the global base space TCSTCSTCS ensures that the entire morphic flow space is both complete and coherent.

```
        M|Z₁     M|Z₂     M|Z₃
          \        |        /
           \       |       /
             →  M|TCS  ←
```

Example:

```
can → person | TCS
→ Metaphoric reading: "The can represents the absent person."
→ Ironic reading: "We are all cans under capitalism."
```

Summary:

The Total Conceptual Space (TCS) functions as the global base space of the conceptual topology. All local Z-frames are fibered over TCS, and conceptual flows can be lifted via σ operators into M I TCS. Thus, Conceptual Topos is complete and globally coherent under M I TCS.

Conclusion

Conceptual Topos is a **fibered topos** over Z-frame:

```
CT := (C, B, π: E → B, Fb := π⁻¹(b), A ≅ b ∪ Nat(Hom(−, A), Fb))
```

with:

- Initial Object "" → codomain π(X)
- Morphic Chains as fibers $\pi^{-1}(Z)$
- Ω as subobject classifier in Z
- σ operator inducing internal exponential morphisms.

Appendix

simbols

```
Z : Intermediating variable (conceptual anchor; Z-frame)
| : Frame separator (indicates morphism is mediated by Z-frame)
→: Morphic Flow
↛ Ruptured morphism
F : Cross-category morphism (used in cross-category flow under shared Z-frame)
//: Used to narrate meaning flow of morphic chains.
¬: Absence

M|Z : Monoid of Conceptual Flow under Z-frame
R|Z := { rupture(f) | rupture(f, σ(f) | Z) ≠ ∅ }
e|Z : Identity element of M|Z
D(Aₙ₋₁ | Z) : Morphic chain under Z frame

σ : Conceptual Shifting Morphism
>> : Generalization relation (A >> X ≡ A ⊑ X)
<< : Specialization relation (X >> A ≡ X ⊑ A)
rupture(f, σ(f) | Z) ≠ ∅ : Indicates conceptual rupture
η : Quasi-Natural Transformation: Contextual alignment between morphic chains.

⊕:  Conceptual morphism set addition in σ or morphic merger such as:
    (k₂ ∘ k₁) ⊕ (q₂ ∘ q₁) = human → royalty | Z'
⊖:  Conceptual morphism set subtraction
    Removes specified morphisms from a morphic chain or set.
```

Notations

```
Concept / Word (lexeme):
    - Lower case (e.g., puppy, dog, girl, she)

Z Frame (conceptual anchor):
    - Upper case (e.g., Mammal, Human, Agency, Domesticated, Royalty)

Type variables (A, B, X, Y, Z in formal definitions):
    - Follow standard formal notation (uppercase)

Example:
puppy → dog | Mammal
A → B | Z

Morphism: f, g, h
Functor: F
```

Simplified Form of Identity Morphism:

1. f: X → X | X (Category-theoretic identity)
   In simplified form: X
   or more explicitly: id_X

2. f: X → X | Z (Mediated identity with conceptual flow)
   In simplified form: X | Z

σ Operator

```
σ(X). Not(x){ A ↛/B | Z}     →     Rupture under Z frame
σ(X). so_much(x){A → B | Z}  →     Preservation & amplification under Z frame
σ(X). >>(x,y)                →     Conceptual Shifting x to y (Generalization) as
function form
σ(X). <<(x,y)                →     Downward Shifting x to y (Specialization) as
function form
σ(X). >(x,y)                 →     Conceptual Shifting
```

Conceptual Morphism Set Operators

```
Addition (⊕):
σ(X). ⊕(f, Aₙ₋₁ | Z): D(Aₙ₋₁ | Z) → D(Bₙ₋₁ | Z) | Z
σ(X). ⊕(f₁, f₂) : Aₙ₋₁ :={f₁, f₂}

Subtraction (⊖):
⊖: Aₙ₋₁ ⊖ {f_i}
σ(X). ⊖(f, Aₙ₋₁ | Z): D(Aₙ₋₁ | Z) → D(Bₙ₋₁ | Z) | Z

- ⊕ operator is σ_safe if Z alignment is preserved.
- ⊖ operator is potentially σ_unsafe but can be σ_safe if resulting chain
preserves the underlying morphic continuity Z.
```

σ Typing Hierarchy

σ_safe: $D(A_{n-1} | Z) \rightarrow D(B_{n-1} | Z) | Z$ (Preserves global coherence)
σ_unsafe: $D(A_{n-1} | Z) \rightarrow \{ \text{rupture}(f_1), \ldots, \text{rupture}(f_n) | \neg Z \}$ (Global coherence lost)

Note: σ_safe behaves as Quasi-Natural Transformation.
σ_unsafe induces rupture, and cannot be captured globally.

Conceptual Topos Named as 概念位相論 / Conceptual Topology

This theory, named 概念位相論 or Conceptual Topoloy, was proposed by **No Name Yet Exist**.

Meaning no longer escapes.
It circulates within the morphic fibration.
We, once again, govern the topology of meaning.

GitHub: https://github.com/No-Name-Yet-Exist/Conceptual-Topology
Note: https://note.com/xoreaxeax/n/n3711c1318d0b
Zenodo: https://zenodo.org/records/15455079

This is Version: 1.2
