Conceptual Topos As Semantic Cage: An Algebraic Topology of Meaning based on Conceptual Topology

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Introduction

Conceptual Topos v1.1 is an initial formalization of the algebraic topology of meaning based on Conceptual Topology.

This version sketches core axioms for Topos including:

- Initial Object
- Finite Limits (Product, Equalizer, Pullback)
- Subobject Classifier Ω
- Fibered Topos structure
- Semantic Exponential via σ operator

Future versions (v1.x) will refine the formalization and extend it.

In this version, the term fiber is used informally to describe the structural cohesion of morphic chains under a shared Z-frame.The current framework is not yet a strict fibered topos in the categorical sense. Formal connection to fibered topos is an intended direction for future versions.This document lays the foundation toward that goal.

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1. Fibered Conceptual Topology:

Fibered Conceptual Topology provides a semantic geometric framework wherein each Z-frame (semantic anchor) acts as a base space, with semantic morphic flows forming fibers over these anchors. The Yoneda-like interpretation captures concepts as bundles of semantic relations within and across Z-frames. This fibered structure serves as the foundation for further constructions in Conceptual Topos.

```
CT := (C, B, π: E → B, Fb := π⁻¹(b), A ≅ b ⋃ Nat(Hom(−, A), Fb))

Where:
  - C is the category of concepts (objects = words or concepts)
  - B is the base space of Z-frames (semantic continuity anchors)
  - E is the total semantic space (word vector embedding space)
  - π projects each concept to its semantic base (Z-frame)
  - Fb is the fiber (semantic morphic chain) over a base b
  - A ≅ b ⋃ Nat(Hom(−, A), Fb) interprets each concept A via its morphisms
relative to its   Z-frame b (Yoneda perspective defined in appendix)
```

```
        Z = baby
    puppy  ←   girl    //specified: size + young
     |          |
     |          |
    dog        she     //abstraction
      \        /
        Mammal
```

2. Monoid Structure of Semantic Flow (M|Z):

In Conceptual Topology, Z is defined as a mediating point/semantic anchor.

```
Let C and D, Z be categories,
with semantic projection π: C ∪ D → Z, such that for each X ∈ Ob(C ∪ D):

π(X) ∈ Ob(Z)

For each X ∈ Ob(C ∪ D), there exists morphism:

f_X: X → π(X)
f_X⁻¹: π(X) → X

such that:

f_X⁻¹ ∘ f_X ≅ id_X

For morphism f: X → Y | Z,
this corresponds to:

f_Z: π(X) → π(Y) in Z

For any X, Y ∈ Ob(C ∪ D):

Let [X]_Z := semantic representation of X under frame Z (i.e., π(X))

Then:

[ X ]_Z1 ≅ [ Y ]_Z2 | Z1, Z2 ∈ Z //or Z1, Z2 >> Z
which means:
["Dog"]_Pet = [Retriever, Dachshund, Poodle, Bulldog, ...]
["girl"]_Human = [girl, woman, person, ...]
["Dog"]_Pet ≅ ["girl"]_Human | Life

Then the set of semantic flow morphisms under Z forms a monoid:

M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z }

This is also defined as Morphic Chain:

Let **D(Cₙ₋₁ | Z)** := Category of Morphic Chains over **Ob(Cₙ₋₁)** within a
given Z-frame.
where:D(Cₙ₋₁ | Z) ={ C₀ → C₁ → C₂ → ... | Z }

or as a set
D(Cₙ₋₁ | Z) ={ C₀, C₁, C₂, ... | Z }
```

3. Identity Element of M|Z

```
Let: M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z }

Define the identity element of M|Z as a family of identity morphisms over
the shared Z frame:

For each X ∈ Ob(C ∪ D), there exists a unique identity morphism under a Z
frame:

e|Z_X := id_X | Z

Then, for any f: X → Y | Z ∈ M|Z:

e|Z_X ∘ f = f
f ∘ e|Z_Y = f

Therefore, the identity structure of M|Z is given by the family:

{ id_X | Z | X ∈ Ob(C ∪ D) }

which forms a pointwise identity across the objects under the common Z
frame.
This ensures that M|Z satisfies the identity axiom of a monoid.
```

4. Associativity of M | Z

```
Let: M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z }

Then for all f, g, h ∈ M|Z:

(f ∘ g) ∘ h = f ∘ (g ∘ h)

Thus, the composition ∘ in M|Z is associative.
```

5. Axioms

5.1. Identity Element

Unit Axiom 1: Identity Element Z

```
id_Z:= Z → Z | Z
∀f ∈ M|Z: id_Z ∘ f = f  and f∘id_Z =
```

Definition:

```
Statement:
Z-frame itself is the unit of M | Z.

Formal Definition:
Z:= Z → Z|Z

Justification:
Since any morphism in M | Z is defined as:

f: X→Y|Z

and Z itself is defined as its own identity morphism:

Z:= Z→Z|Z

then:
id_Z = Z

Conclusion:

Therefore:
id_Z is the unit element of M|Z.

∀f ∈ M|Z: (id_Z ∘ f | Z) = f and (f ∘ id_Z | Z)= f
(with frame-preserving composition)

∴id_Z is the unit of M|Z.
```

Note:

```
idZ :Z→Z | Z
f:X→Y | Z
(idZ|Z)∘(X→Y|Z)
```

Unit Axiom2: Void Concept

```
f ∈ M|Z
"" ∘ f = f  and f ∘ "" = f
id_Z ∘ f = f and f ∘ id_Z = f
```

Definition:

```


The empty concept is a theoretically assumed concept, denoted as "",
which acts as the unit element at the conceptual / lexical level.

Formal Definition:

"" ∘ f = f and f ∘ "" = f

Justification:

The empty concept "" represents no lexical or semantic content.
Composing any morphism f with the empty concept does not alter the flow of
meaning.
Conclusion:

"" is the unit element at the conceptual level of Conceptual Topology.

```

5.2. Zero Morphism: Negation Morphism

We define semantic zero morphism, negation morphism: n_f In CT as the result of applying Not() to a morphism

```
g:σ(Z). Not(g){ A ↛ B | Z} = A ↛ B|Z = n_f
where: g: A → B

Formal Properties (Axiom):

∀g:X→Y|Z where composition with n_f is defined:

∀g: g ∘ n_f = n_f and n_f ∘ g = n_f

Left Side:
g: A→B
g∘(A↛B|Z) = A↛ B|Z

Right Side
g: A→B(A↛B|Z)∘g=A↛B|Z
```

```
Interpretation:
Applying Not() to any morphism produces a semantic zero morphism, which
collapses any further semantic flow.

Natural Language:
Left Side: g∘(A↛B|Z)
"A is not B"
The apple is not a fruit

Right Side: (A↛B|Z)∘g
"B is not A"
This is a fruit, but this is not an apple which is a fruit.


In CT, this was called rutpure().
Now defined:
rupture(A,B,Z)= σ(Z).Not(g) = n_f = A↛B|Z
```

5.3. Composition Axiom

```
M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z }

Then for all f, g, h ∈ M|Z:

For f, g, h ∈ M|Z,
where:
f:V → W | Z
g:Y → V | Z
h:X → Y | Z

(f ∘ g) ∘ h = f ∘ (g ∘ h)
```

Example:

```
For f, g, h ∈ M|Z,

where:
f:she → you | Human
g:he → she | Human
h:man → he | Human

(f ∘ g) ∘ h = f ∘ (g ∘ h)
```

6. Conceptual Topos

6.1. Category Level: Initial Object

Definition:

```
Let Concept be a category where Ob(Concept) are lexical / conceptual
objects.
Then "" ∈ Ob(Concept) is Initial Object if:

∀ X ∈ Ob(Concept), ∃ unique morphism:

u_X : "" → X | X

such that:

∀ f: X → Y | Z,
f ∘ u_X = u_Y
```

Monoid Level: Unit in M|Z

```
Recall:

M|Z = { fₙ ∘ ... ∘ f₁ | all fᵢ: Xᵢ → Xᵢ₊₁ | Z ∧ ∀ i, j: fᵢ ≅ fⱼ | Z }

Now, define:

"" ∈ Ob(Concept)

and identity morphism under Z-frame:

e|Z_"" := id_"" | Z

Then for all f ∈ M|Z:

e|Z_"" ∘ f = f
f ∘ e|Z_"" = f
```

6.2. Finite Limits

Terminal Object Conceptual Topos defines a terminal object as the Z-frame identity:

```
id_Z := Z → Z | Z


Any morphism f: X → Z | Z factors uniquely through id_Z.

This realizes the semantic universal target:

∀ X ∈ Ob(C ∪ D), ∃! f_terminal: X → Z | Z
```

Example:

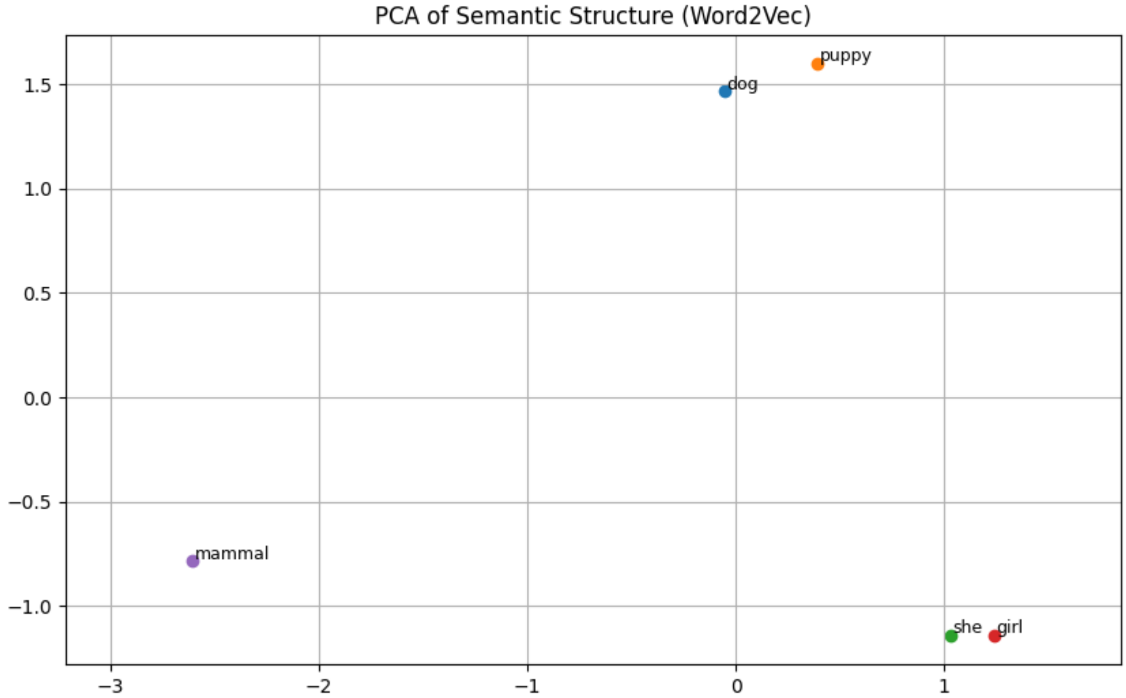
```
she → human | Human
me → human | Human
```

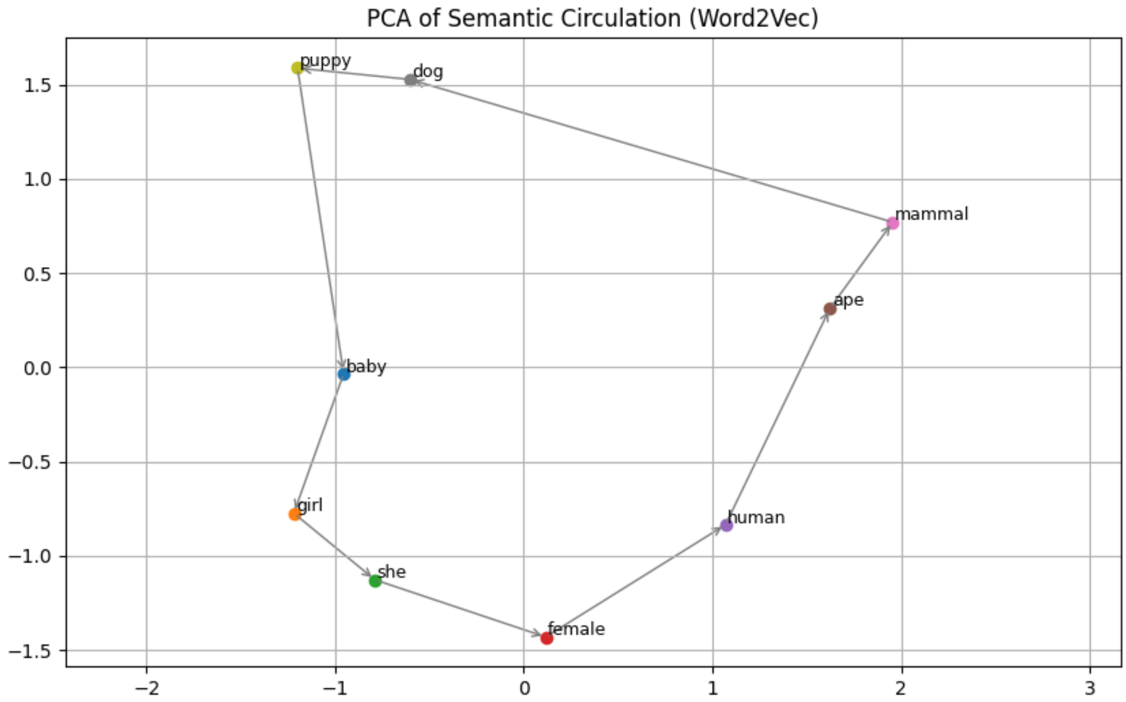
Pullback

```
Given morphisms:

f: girl → mammal
g: puppy → mammal

Pullback of (f, g) is:

P = Baby
p₁: Baby → girl
p₂: Baby → puppy

with commuting condition:

f ∘ p₁ = g ∘ p₂  ≈ mapping to common semantic frame (mammal)

Diagram:

      Baby
     /    \
  p₁ /      \ p₂
   /          \
girl          puppy
   \          /
    \        /
     v      v
     mammal (semantic anchor / codomain)
```

This previously defined as Quasi-Natural Transformation:

```
η: Dᵢ ⇒ Dᵢ₊₁  | CD (CD = codomain)
η_X ∘ Dᵢ({f₁ | Z₁, ..., fₙ | Zₙ}) ≈ Dᵢ₊₁({f'₁ | Z₁, ..., f'ₙ | Zₙ}) ∘ η_Y | CD
for all fⱼ: Xⱼ → Yⱼ | Zⱼ ∈ Dᵢ,
where f'ⱼ: η_X(Xⱼ) → η_Y(Yⱼ) | Zⱼ

Then, η is said to be a quasi-natural transformation under the Z-frame
i.e. η ∈ Mor(C) where C is the contextual meaning category

η_X ∘ Dᵢ({girl → mammal | Z₁}) ≈ Dᵢ₊₁({puppy → mammal | Z₂}) ∘ η_Y


Pullback Diagram
      Mammal
     /     \
    /       \
girl         puppy
   \         /
    \       /
     Baby  (semantic anchor / common Z-frame)


Example: η: girl → puppy | Z = Baby


Quasi-Natural Transformation Diagram:
        Z = baby
    puppy  ←   girl    //specified: size + young
     |          |
     |          |
    dog        she     //abstraction
      \        /
        Mammal


For any X with morphisms
q₁: X → girl and q₂: X → puppy satisfying f ∘ q₁ = g ∘ q₂,

there exists unique u: X → Baby

such that:
p₁ ∘ u = q₁, p₂ ∘ u = q₂.
```


PCA of Semantic Structure (Word2Vec)
puppy
dog
mammal
she
girl
1.5
1.0
0.5
0.0
−0.5
−1.0
−3
−2
−1
0
1


PCA of Semantic Circulation (Word2Vec)
puppy
dog
mammal
ape
baby
girl
she
female
human
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2
−1
0
1
2
3

Equalizer: Mirror Morphism

```
Equalizer of two morphisms f, g:
A → B is an object Eq(f,g) with morphism
e: Eq(f,g) → A

such that:

f ∘ e = g ∘ e

and universal property:

∀ h: X → A such that f ∘ h = g ∘ h,
∃! unique u: X → Eq(f,g)

Diagram:

      Eq(f, g)
         |
         e
         v
         A
    f ↘     ↙ g
        B
```

In conceptual topology this was defined as mirror morphism:

```
f : X  → Y  | Z ∈ Dᵢ
f': X' → Y' | Z ∈ Dᵢ₊₁
  ⇒X' ≠ X, but cod(f) = cod(f') | CD ( common codomain)

We define f' as a mirror-correspondent morphism of f under a given Z-frame,
if and only if:

∃Z: rupture(f, f' | Z) ≠ ∅
∧ cod(f) = cod(f') | CD


        Eq(f, f')
          |
          e
          v
      X        X'
       \      /
        \    /
         v  v
          Y = Y' (codomain = C)
```

Product: σ operator⊕

```
In any category C, the Product of A and B is an object A × B equipped with
projections:

π₁: A × B → A
π₂: A × B → B

with universal property:
For any object X with morphisms:

f₁: X → A
f₂: X → B

there exists a unique morphism u: X → A × B such that:

π₁ ∘ u = f₁
π₂ ∘ u = f₂
```

Addition (⊕):
σ(Z) serves as the mediating operator ensuring that the composed morphic chain remains within the semantic fiber over Z.

```
Defined as:
σ(Z). ⊕(Aₙ₋₁, Bₙ₋₁, Z) = D(Cₙ₋₁ | CD)  → semantic Product under Z-frame

where:
Aₙ₋₁:= girl → she
Bₙ₋₁:= puppy → dog

σ(Z). ⊕(Aₙ₋₁, Bₙ₋₁, Z) = D(Cₙ₋₁ | CD)

For any pair of morphic chains 1Aₙ₋₁, Bₙ₋₁, the operation σ(Z).⊕(Aₙ₋₁, Bₙ₋₁)
defines an object P ∈ D(Cₙ₋₁ | Z)P∈D(Cn−1∣Z) with projections π₁, π₂
satisfying the product universal property.


Example:

girl → she
puppy → dog

σ(Human). ⊕(girl → she, puppy → dog | Mammal) → Product(girl → she → puppy
→ dog | Mammal) | Mammal
→ composite meaning space
```

```
Diagram:

    Product(girl → she, puppy → dog) ∈ D(Cₙ₋₁ | Z = Mammal)

             Product Object (P)
              /           \
             /             \
      π₁    /               \  π₂
          v                   v
  girl → she            puppy → dog
```

6.3. Exponentials

Conceptual Topos models exponentials via semantic shift operators.

Definition

```
For any objects A, B:

B^A exists such that:
Hom(X ⊗ A, B) ≅ Hom(X, B^A)
```

Construction via σ operator

Semantic shift operators:

```
σ(Z). >>(A, B)
or
σ(Z). >(A, B)

act as internal exponential morphisms within the fibered structure over the
Z-frame:
        (A, B, Z) ≅ B^A
where the Z-frame mediates the semantic continuity and contextual grounding
of the morphic shift.
```

We define Exponential objects via σ operator as semantic abstraction mechanisms:

```
B^A:=σ(Z).>(A,B)
```

Full Exponential Law formalization will be provided in later version.

Definition: Semantic Shifting Morphism (σ)

```
σ: D(Xₙ₋₁ | X) → D(Xₙ₋₁ | X)

such that σ ⊕ f ∈ M|Z if and only if type compatibility holds:

∀ A, B, (A → B) ∘ σ(X) is valid if:

( A >> X or X >> A )
and
( B >> X or X >> B )

Definition: Subsumption
A >> X ≡ A ⊑ X

Definition: SubsumedBy
X >> A ≡ X ⊑ A

Example:
king → king >> human → human
  ⇒ king >> human  → valid

human → human >> queen → queen
  ⇒ human >> queen → valid
```

Example

```
σ(Human). >>(puppy → dog → mammal | Canine, Human)
≅ girl → she → mammal | Human
```

This shift realizes an internal semantic transformation corresponding to exponential behavior.

6.4. Definition of Ω

Let **Ω** be an object in the Concept category, representing the **semantic truth space**.

```
For any subobject (conceptual inclusion):

m: M ↪ X

there exists a unique characteristic morphism:

χ_m: X → Ω
```

such that the following diagram commutes:

```
M ———→ X
|       |
|       v χ_m
|       Ω
```

Interpretation in Conceptual Topology

- **Ω** encodes **semantic entailment / membership / inclusion**.
- **Z-frame membership** is naturally mapped to Ω:

```
χ_Z: X → Ω
```

interpreted as:
"Does X conceptually belong to Z-frame Z?"

Examples

Example 1: Dog in Pet Z-frame

χ_Pet(Dog) = True

Example 2: Apple in Pet Z-frame

χ_Pet(Apple) = False

Example 3: Innocent in Body Z-frame (after rupture)

χ_Body("innocent") = True / False depending on whether the semantic projection is coherent under Z-Frame.

Relation to Rupture

Semantic rupture can be lifted to Ω as:

σ(Z). Not(f: A → B | Z) ⇒ rupture(A,B,Z) ⇒ χ_Z(f) = False

Thus, **negation** and **semantic discontinuity** become **Ω-classifiable**.

6.5. Conceptual Topos as Fibered Topos over Z-frame

Conceptual Topos is structured as a **fibered topos** over the semantic base space **Z-frame**.

Z-frame as Fibered Structure

- Let π: C ∪ D → Z be the semantic projection.
- Each fiber $\pi^{-1}(Z)$ forms a category of morphic chains **D(C_{n-1} | Z)**.
- Morphisms of the form:

```
X → Y | Z ≡ X → Y in fiber over Z
```

correspond to morphisms within the fibered structure over Z.

Initial Object and Codomain Projection

- The **Initial Object ""** serves as the semantic origin.
- It projects into the codomain via:

```
"" → | X ≡ "" → π(X)
```

```
   ""
    ↓ u_X
  X ————→ π(X) (in Z-frame)
```

```
Fiber π⁻¹(Z_X):
    "" → X → Y
```

Thus, semantic generation naturally occurs anchored in Z-frame.

Semantic Flow Closure

- Semantic flows:

```
X → Y | Z
```

are closed within the fiber over Z, corresponding to the codomain Z of the semantic projection π.

- Rupture and negation are classified by **Ω**:

```
χ_Z: X → Ω
```

7. Global Semantic Space: Total Conceptual Space (TCS)

We define the Total Conceptual Space (TCS) as the global semantic anchor:

```
Z = TCS = Total Conceptual Space
```

Definition of M|TCS:

The global morphic flow space under TCS is defined as:

```
M|TCS = { fₙ ∘ ... ∘ f₁ | all fᵢ:  M|Zᵢ →  M|Zᵢ₊₁ | TCS ∧ ∀ i, j: fᵢ ≅ fⱼ | TCS }
```

We can regard M|TCS as the composition space of conceptual perspectives: Here, each M|Z functions as a semantic symbolization or perspective lens, and M|TCS represents global flows across chained perspectives.

Monoid Closure Property:

```
Composition in M|TCS is closed:
∀ f, g ∈ M|TCS, f ∘ g ∈ M|TCS

The identity morphism is preserved:

∀ f ∈ M|TCS, f ∘ id = f = id ∘ f
```

Thus, M|TCS forms a closed monoid under composition.

Completeness Statement:

```
For any pair of concepts X, Y:

∀ X, Y ∈ Ob(C), ∃ f ∈ Mor(C), such that f: X → Y | TCS
```

That is, any conceptual pair X and Y can be connected via a morphic flow under TCS.

Fibered Structure and Lifting

Each local M|Z can be lifted into M|TCS via semantic shifting σ:

```
∀M|Z, ∃σ: M|Z > M|Z | TCS
```

Thus, the global base space TCSTCSTCS ensures that the entire morphic flow space is both complete and coherent.

```
        M|Z₁     M|Z₂     M|Z₃
          \        |        /
           \       |       /
             →  M|TCS  ←
```

Example:

```
can → person | TCS
→ Metaphoric reading: "The can represents the absent person."
→ Ironic reading: "We are all cans under capitalism."
```

Summary:

The Total Conceptual Space (TCS) functions as the global base space of the conceptual topology. All local Z-frames are fibered over TCS, and semantic flows can be lifted via σ operators into M|TCS. Thus, Conceptual Topos is complete and globally coherent under M|TCS.

Conclusion

Conceptual Topos is a **fibered topos** over Z-frame:

```
CT := (C, B, π: E → B, Fb := π⁻¹(b), A ≅ b ⋃ Nat(Hom(−, A), Fb))
```

with:

- Initial Object "" → codomain π(X)
- Morphic Chains as fibers $\pi^{-1}(Z)$
- Ω as subobject classifier in Z
- σ operator inducing internal exponential morphisms.

Conceptual Topos Named as 概念位相論 / Conceptual Topology

This theory, named 概念位相論 or Conceptual Topoloy, was proposed by **No Name Yet Exist**.

GitHub: https://github.com/No-Name-Yet-Exist/Conceptual-Topology
Note: https://note.com/xoreaxeax/n/n3711c1318d0b
Zenodo: https://zenodo.org/records/15455079

This is Version: 1.1
