Fantastic Topological Surfaces and How to Classify Them

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Western Oregon University

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Introduction

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- In particular, mathematicians often study mathematical structures in topology called *surfaces*.
- These surfaces come in a diverse array of variations and combinations.

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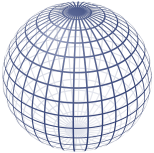

Sphere, $\mathbb{S}^2$

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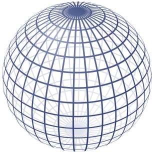

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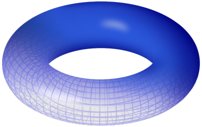

Torus, $\mathbb{T}^2$

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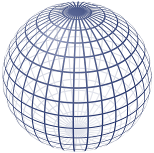

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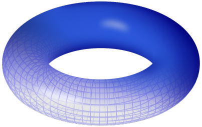

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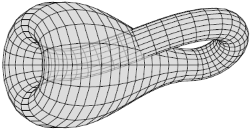

Klein Bottle, $\mathbb{K}$

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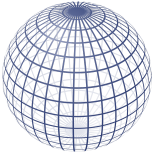

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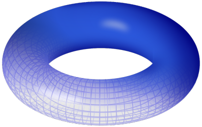

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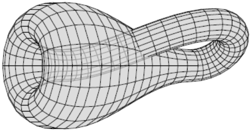

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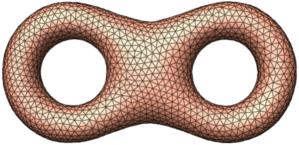

2-holed Torus, $\mathbb{T}^2 \# \mathbb{T}^2$

Motivating Questions

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How many categories of surfaces are there?

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Surfaces

Definition

A **surface** is a 2-dimensional compact locally Euclidean shape.

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We shall now proceed to define compact and locally Euclidean.

Compact Space

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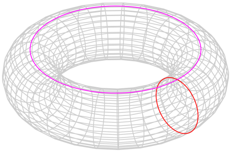

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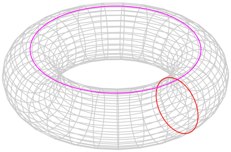

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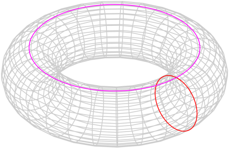

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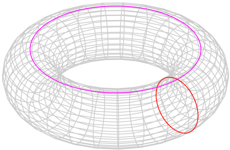

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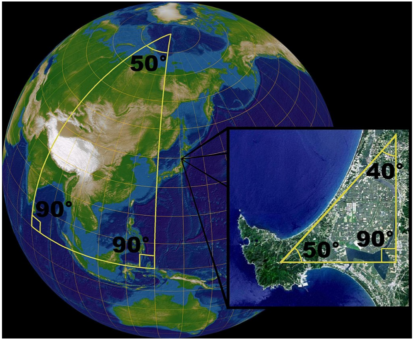

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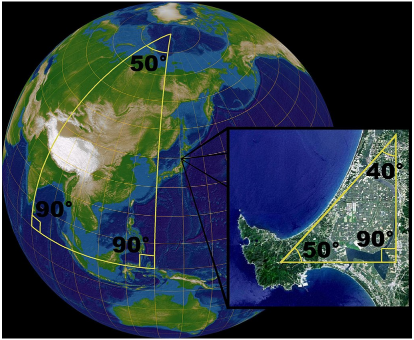

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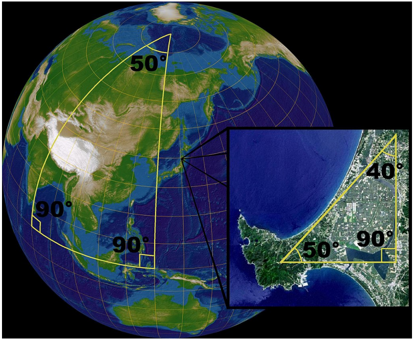

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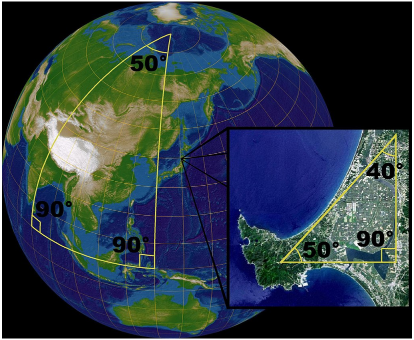
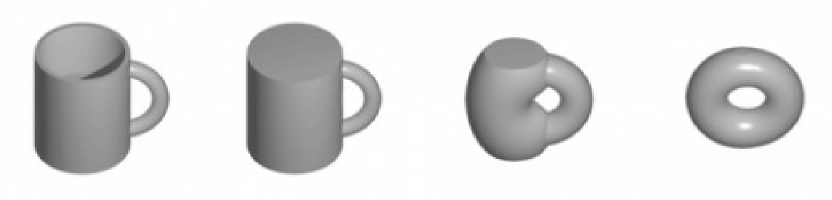


- The triangle on a local patch of Earth will add up to approximately 180°as seen here with 40°+ 50°+ 90°= 180°.
- But the triangle at a much larger hemispheric scale will not add up to 180°as seen here with 50°+ 90°+ 90°= 230°.
- Thus, the Earth's exterior is a 2-dimensional shape i.e. a surface.

Homeomorphism

Definition

A **homeomorphism** is a way to bend or stretch a surface into another without creating or destroying "holes".



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- $\mathbb{P}^2 \# \ldots \# \mathbb{P}^2$, *a connected sum of projective planes.*

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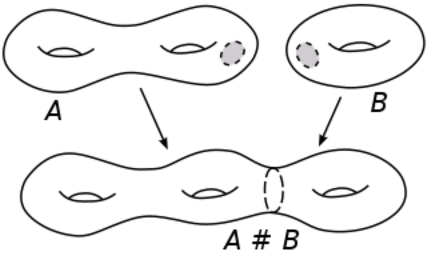

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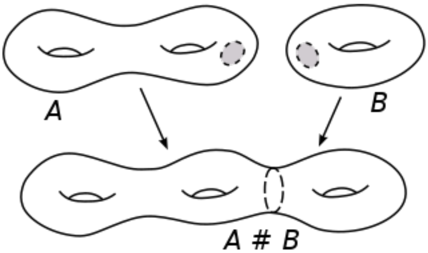

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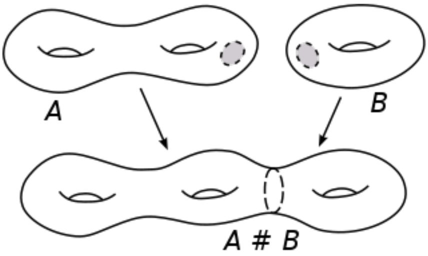

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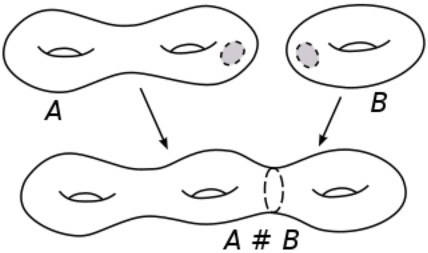

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Theorem

The connected sum of surfaces is a surface.

Surface Mechanics

- How do we work with surfaces?

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Theorem (Presentation Theorem)

Every compact surface admits what is called a polygonal presentation.

- This theorem enables us to convert any surface to its polygonal presentation provided it is compact.
- Working with the polygonal presentations is simpler and easier than dealing with surfaces directly.

Presentation Examples

- Polygonal Presentations of the Sphere, $\mathbb{S}^2$, and the Torus, $\mathbb{T}^2$.

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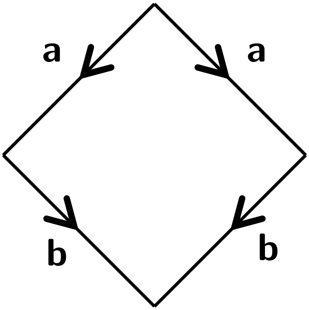

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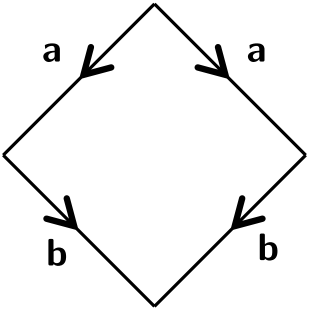

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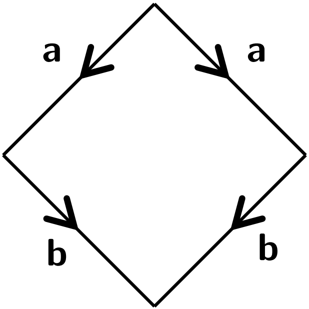

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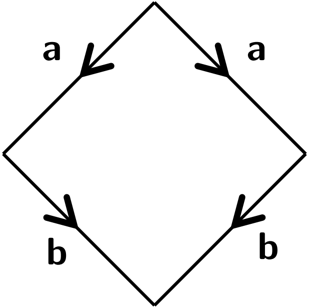

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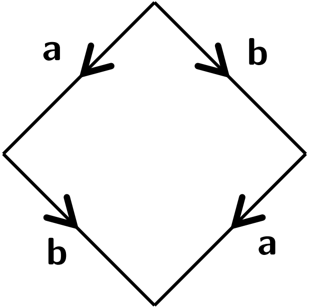

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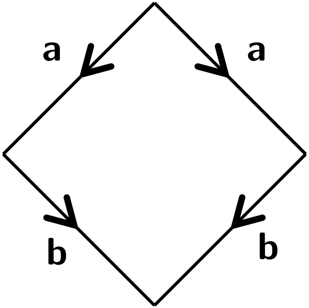

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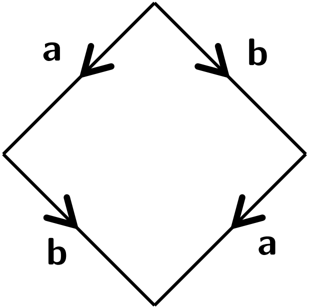

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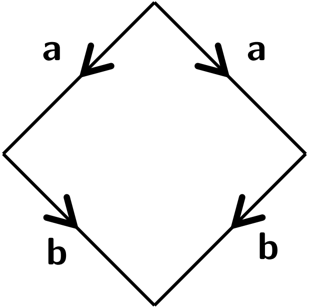

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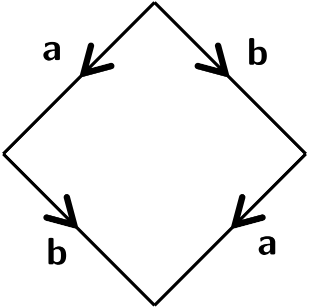

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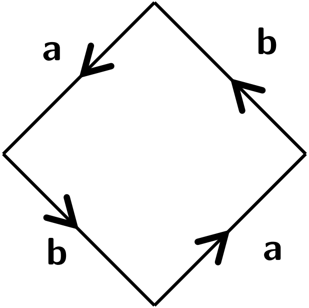

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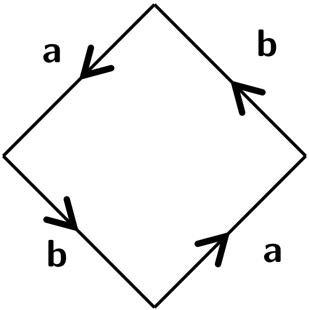

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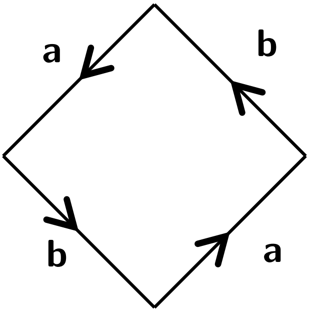

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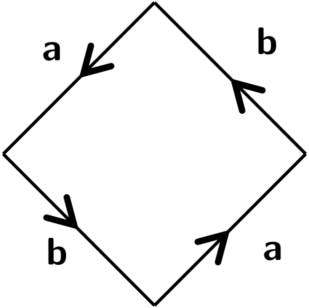

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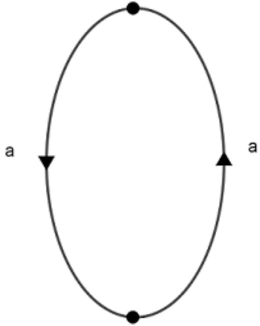

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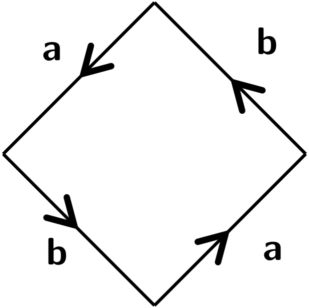

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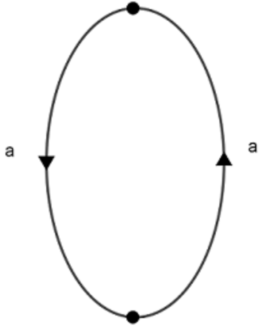

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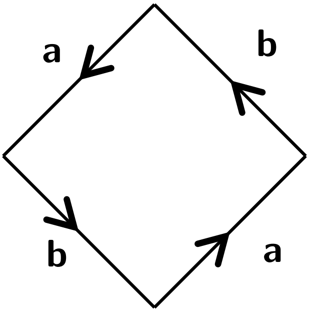

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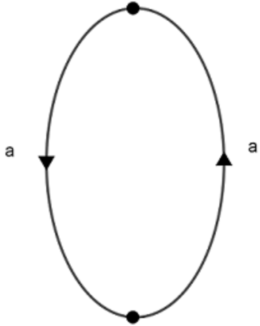

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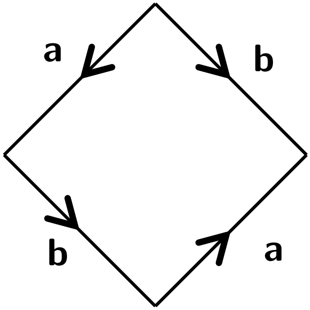

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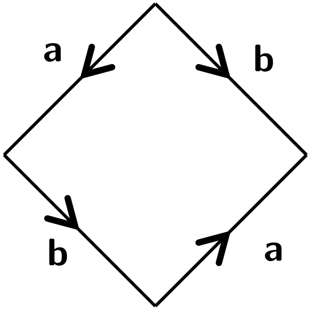

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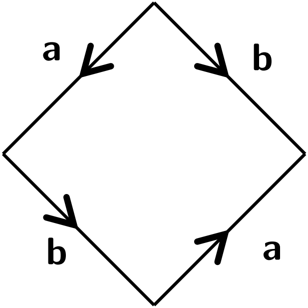

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polygonal presentation:

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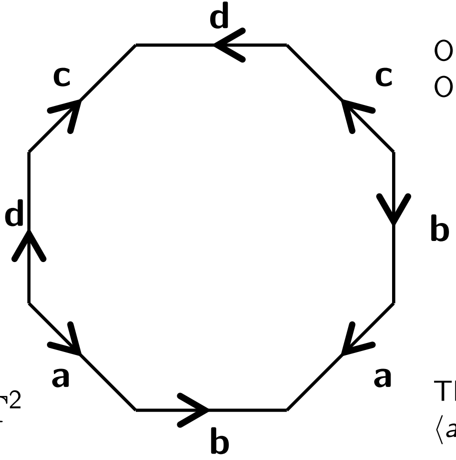

$\mathbb{T}^2 \# \mathbb{T}^2$

One, $\mathbb{T}^2$, has $\langle a, b | aba^{-1}b^{-1} \rangle$,
One, $\mathbb{T}^2$, has $\langle c, d | cdc^{-1}d^{-1} \rangle$

Thus, $\mathbb{T}^2 \# \mathbb{T}^2$ gives
$\langle a, b, c, d | aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$

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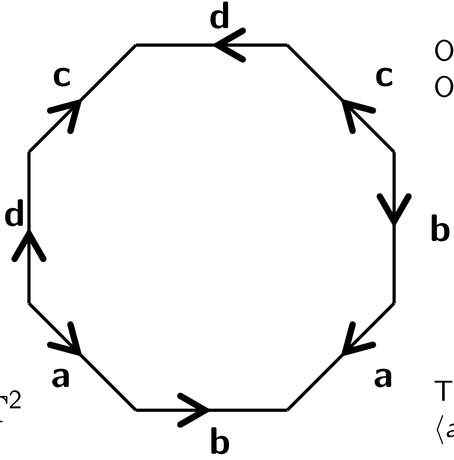

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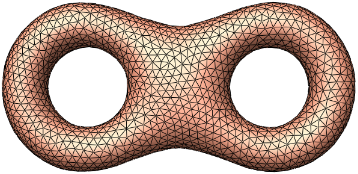

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Proof:

By the following sequence of elementary transformations, we find that the Klein bottle has this series of presentations:

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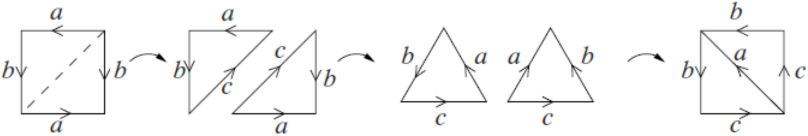

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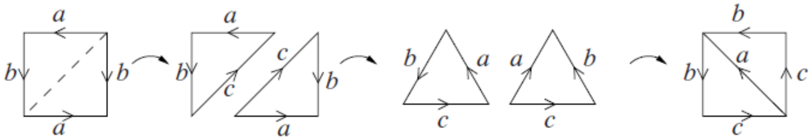

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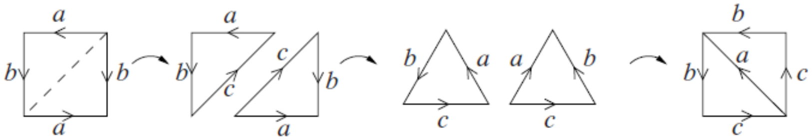

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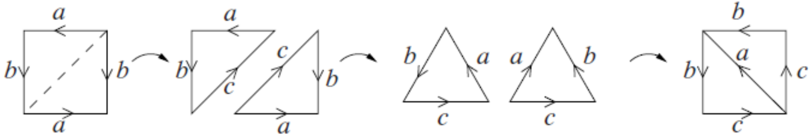

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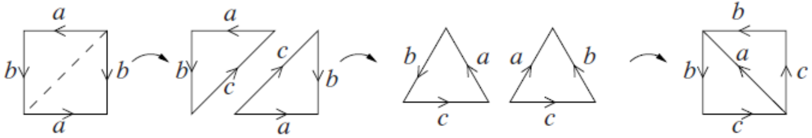

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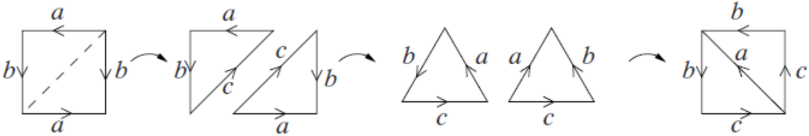

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$$\begin{gathered}\langle a, b | abab^{-1}\rangle \\ \approx \langle a, b, c | abc, c^{-1}ab^{-1}\rangle \text{ (cut along c)} \\ \approx \langle a, b, c | bca, a^{-1}cb\rangle \text{ (rotate and reflect)} \\ \approx \langle b, c | bbcc\rangle \text{ (paste along } a \text{ and rotate)}\end{gathered}$$

The presentation in the last line is a standard presentation of a connected sum of two projective planes. ■

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- In particular, the elementary transformations allow us to reduce all possible polygonal presentations of surfaces to just a few general types.

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- Furthermore, this insight can be generalized in an analogous manner to higher dimensions of interest.
- In particular, one could consider the implications of 3-dimensional analogues to surfaces embedded in 4-dimensional space.
- The results of such investigations include Einstein's theory of relativity and Gregory Perelman's recent proof of an equivalent classification theorem for such 3-dimensional shapes and the advance they represent in the field of topology.

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Conclusion

Thank you for listening!