Fantastic Topological Surfaces and How to Classify Them

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1 Introduction

The field of Topology is concerned with the properties of space that are preserved under continuous deformations such as stretching, crumpling, and bending, but not involving tearing new holes or sewing up old ones. It is thereby often thought of as the field of "rubber sheet" geometry since the precise rigid shape of the objects is not relevant. In particular, mathematicians often study mathematical structures in topology called *surfaces*. These surfaces, also known as 2-dimensional manifolds, come in a diverse array of variations and combinations. Noteworthy examples include the sphere, the torus, the Klein bottle, and the 2-holed torus. Some questions that may naturally arise and come to mind include: are there well-defined categories of manifolds; if so, can we ascertain which sort of manifolds belong in them; and if given a manifold, how can we tell what type it is? More relevantly: when are two surfaces the same, if they are not, how can we tell them apart, and how many categories of surfaces are there?

The Classification Theorem of Compact Surfaces answers these questions by showing us that all nonempty, compact connected 2-dimensional manifolds are homeomorphic to one of three categories of manifolds. These cases consist of the sphere, the connected sum of tori, or a connected sum of projective planes. We aim to rigorously prove the theorem via verification of the requisite auxiliary theorems and lemmas as outlined in Lee's Introduction to Topological Manifolds [8]. This will require an analysis of a relationship that can be established between these topological surfaces and equivalent polygonal representations.

Historically, it was Max Dehn and Poul Heegaard who first proved the theorem in 1907 using the assumption that surfaces could be represented by some suitable polygonal presentation [8]. More generally, the Classification Theorem contains the insight that all such 2-dimensional manifolds or surfaces can be categorized in a systematic and complete way. Recently, the equivalent classification theorem for 3-dimensional manifolds was proven by Grigori Perelman representing a great advance in the field of topology and inviting one to understand the lower dimensional theorem for the perspective it may provide for the higher.

2 Definitions and background information

2.1 Examples of Topological Spaces

We will now proceed to define the notion of a topology and provide some illustrative examples.

Definition 2.1. *Let X be a set and let T_X be a collection of subsets of X. The pair (X, T_X) is called a **topological space** if each of the following hold:*

- *ϕ and X are elements of T_X.*
- *The union of any collection of elements of T_X is an element of T_X.*
- *The intersection of a finite collection of elements of T_X is an element of T_X.*

*If (X, T_X) is a **topological space**, then T_X is called a **topology** on X. Furthermore, the elements of T_X are called **open sets** in X. In addition, the complement of an open set is called a **closed set**.*

*Note that an element of a topology can be both open and closed i.e. **clopen**.*

Example 2.2. *$\mathbb{R}^n$ is a topological space where open sets are products of open intervals in $\mathbb{R}$, along with unions and finite intersections of such sets. This topology is called the **ordinary topology** on $\mathbb{R}^n$.*

Example 2.3. *If $X = \{a, b\}$, then there are four possible topologies on X: $T_{X_1} = \{\emptyset, \{a, b\}\}$, $T_{X_2} = \{\emptyset, \{a\}, \{a, b\}\}$, $T_{X_3} = \{\emptyset, \{b\}, \{a, b\}\}$, $T_{X_4} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.* ■

Example 2.4. *Let $X = \{1, 2, 3\}$. Then we have the following nonexhaustive list of examples of topologies on X. Pictures are from [10]*

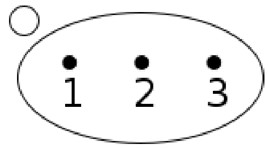


Figure 1: $T_X = \{\emptyset, \{1,2,3\}\}$.

(This is the **trivial** or **indiscrete topology**.)

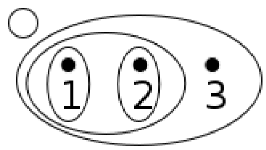


Figure 2: $T_X = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$.

(This is also a topology.)

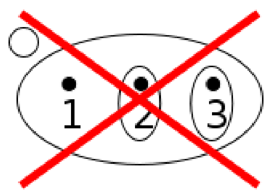


Figure 3: $T_X = \{\emptyset, \{2\}, \{3\}, \{1,2,3\}\}$.

(This is not a topology, because $\{2\}$ and $\{3\}$ are in the topology but their union $\{2,3\}$ is not.)

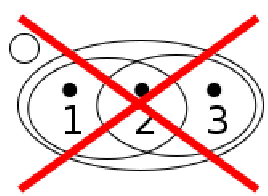


Figure 4: $T_X = \{\emptyset, \{1,2\}, \{2,3\}, \{1,2,3\}\}$.

(This is not a topology, because $\{1,2\}$ and $\{2,3\}$ are in the topology but their intersection $\{2\}$ is not.)

We now see three more examples of topologies and exhibit some typical proofs.

Example 2.5. *Let X be any set. Then $T_X = \{\emptyset, X\}$ is a topology on X called the* ***indiscrete topology****.*

Proof: We let X be any set and let $T_X = \{\emptyset, X\}$. Then $\emptyset, X$ are elements of T_X so the first property of being a topology on X is satisfied. Furthemore, $\emptyset$, and X are the only elements of T_X. Thus, the possible unions of any collection of elements of T_X are $U = \emptyset, X$ which are all subsets of T_X. Hence, the second property of being a topology on X is satisfied. Finally, the possible sets from the intersection of a finite number of elements of T_X are $U = \emptyset, X$ which are subsets of T_X . Therefore, we have shown that T_X is a topology on X. ∎

Example 2.6. *Let X be any set. Then $T_X = P(X)$, the power set of X, is a topology on X called the* ***discrete topology****.*

Proof: We let X be any set and let T_X be the collection of all subsets of X. We see $\emptyset \subseteq X$ and $X \subseteq X$. Thus, $\emptyset$ and X are elements of T_X which satisfies the first property of being a topology on X. Next, let U be the union of any collection of elements of T_X. Since T_X is a collection of all of the subsets of X, any such union of the subsets of T_X is yet another subset of X and thus, an element of T_X. This satisfies the second property of being a topology on X. Finally, Let I be the intersection of a finite collection of elements of T_X. Since T_X is a collection of all of the subsets of X, any such intersection of the subsets of T_X is yet another subset of X and an element of T_X. This satisfies the third and final property of being a topology on X. Therefore, we have shown that T_X is a topology on X. ∎

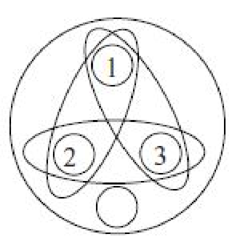


Figure 5: the **discrete topology**, taken from [11]

Theorem 2.7. *Let X be any set and define $T_X = \{U | U \subseteq X, \text{ and } X - U \text{ is finite or } X\}$. Then T_X is a topology on X called the* ***finite complement topology****.*

Proof: Let X be any set with $T_X = \{U | U \subseteq X$, and $X - U$ is finite or X$\}$. First, we need to show that $\emptyset$ and X are elements of T_X. Let $U = \emptyset$, Note that for any set X, $\emptyset \subseteq X$ and $X - \emptyset = X$. Thus, $\emptyset$ is an element of T_X. Furthermore, note that for any set X, $X \subseteq X$ and $X - X = \emptyset$ which is of course finite. Second, we need to show that the union of any collection of elements of T_X is also in T_X. Third, we need to show that the intersection of a finite collection of of elements of T_X is also in T_X. Let $\{U_\alpha\}_{\alpha \in I}$ be any collection of sets from T_X.

Case 1: We assume that $(X - U_\alpha)$ is finite for at least one $\alpha \in I$. We then note that $X - \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha \in I}(X - U_\alpha)$ as per basic set theory. Note that both sides of this equality are finite as at least one $(X - U_\alpha)$ is finite. Hence, the union $\bigcup_{\alpha \in I} U_\alpha$ is an element of T_X. Now, let $U_1, U_2, ..., U_n$ represent a finite number of sets belonging to T_X with $X - U_i$ being finite for some i. Then we note that $X - \bigcap_{k=1}^{n} U_k = \bigcup_{k=1}^{n}(X - U_k)$ as per basic set theory. If $X - U_i = X$ for some other i, then it follows that both sides of this equality will be equal to X. If $X - U_i$ is finite for all other i, then both sides of this equality will be the same finite set. Thus, we conclude that $\cap_{k=1}^{n} U_k$ is an element of T_X.

Case 2: We will now assume that $(X - U_\alpha)$ is infinite for all $\alpha \in I$. Then X is infinite with $(X - U_\alpha) = X$ and thus is in T_X for every $\alpha \in I$. We then note that $X - \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha \in I}(X - U_\alpha)$ as per basic set theory where $\bigcap_{\alpha \in I}(X - U_\alpha) = X$ which is in T_X. Now, let $U_1, U_2, ..., U_n$ represent a finite number of sets belonging to T_X with $X - U_i = X$ for each i. We know that $\cap_{k=1}^{n} U_k \subseteq X$ as the interserction of subsets of a set is also subset of that set by set theory. Furthermore, as $U_i = \emptyset$ for each i, $\cap_{k=1}^{n} U_k = \emptyset$ which is in T_X. Thus, we find that $\cap_{k=1}^{n} U_k$ is an element of T_X.

Therefore, T_X is a topology on X. ■

2.2 Special Topologies: new spaces from old spaces

With this background support established, we will now proceed to define a vital topological concept which is analogous to quotient structures in group theory.

Note that when the topology is understood, we will simply write X instead of (X, T_X).

Definition 2.8. *Let X be any topological space and Y be any set. Let $q : X \to Y$ be a surjective function. Define $T_Y = \{U \subseteq Y | q^{-1}(U) \text{open in } X\}$.*

Theorem 2.9. *T_Y is a topology on Y.*

Proof: Let X be a topological space, Y be any set, and $q : X \to Y$ to be a surjective map. We need to show that the $\emptyset$ and Y are in T_Y as defined. Note that $\emptyset \subseteq Y$ and $Y \subseteq Y$ with $q^{-1}(\emptyset) = \emptyset \in T_X$ and $q^{-1}(Y) = X \in T_X$. This is by the surjective property of q with the image of q as $q^{-1}(\text{im}(q)) = X$ and $\text{im}(q) = Y$. This follows from the function property that each element of X is mapped onto some element in Y. Thus, $\emptyset$ and Y are in T_Y.

Now, we need to show that the union of any collection of elements of T_Y is also an element of T_Y. We let $\{U_\alpha\}_{\alpha \in I}$ be such an arbitrary collection. Furthermore, $q^{-1}(\bigcup_{\alpha \in I} U_\alpha) = \bigcup_{\alpha \in I} q^{-1}(U_\alpha)$ by Lemma A.1 where $\bigcup_{\alpha \in I} q^{-1}(U_\alpha) \in T_X$ as every $q^{-1}(U_\alpha)$ is open in T_X hence their union is open as well. Thus $\bigcup_{\alpha \in I} U_\alpha \in T_Y$.

Finally, we must show that the intersection of any finite collection of elements of T_Y is also in T_Y. We let $\{U_\alpha\}_{\alpha \in J}$ be such an arbitrary collection where J is a finite indexing set. Furthermore, $q^{-1}(\bigcap_{\alpha \in J} U_\alpha) = \bigcap_{\alpha \in J} q^{-1}(U_\alpha)$ by Lemma A.2 where $\bigcap_{\alpha \in J} q^{-1}(U_\alpha) \in T_X$ as every $q^{-1}(U_\alpha)$ is open in T_X hence their intersection is open as well. Therefore $\bigcap_{\alpha \in J} U_\alpha \in T_Y$. Therefore, T_Y is topology on Y. ■

Note: the topology T_Y defined above is known as the **quotient topology** with the function $q : X \to Y$ being called the **quotient map**.

We now consider the special case where the set Y is a set of equivalence classes of X.

Definition 2.10. *Let X be a topological space. Let $p \in X$. Let $\sim$ be an equivalence relation on X. Then $[p]$ denotes the equivalence class of p. Now consider $X/\sim\, = \{[p]\}$. This forms a partition of X. Next, we construct the map $q : X \to X/\sim$ with $p \mapsto [p]$. Then $X/\sim$ along with the quotient topology induced by q is called the **quotient space** (or **identification space**) of X by $\sim$. Note that all quotient spaces can be obtained in this way.*

We will now list some additional examples concerning how one can create a new space from existing topological spaces.

Definition 2.11. *Let $(X_\alpha)_{\alpha \in A}$ be an indexed family of nonempty topological spaces. The **disjoint union** is defined as the set $\coprod_{\alpha \in A} X_\alpha$ consisting of all ordered pairs (x, α) with $\alpha \in A$ and $x \in X_\alpha$.*

$$\coprod_{\alpha \in A} X_\alpha = \{(x, \alpha) : \alpha \in A \textit{ and } x \in X_\alpha\}$$

For each $\alpha \in A$, there is a canonical injection $\iota_\alpha : X_\alpha \to \coprod_{\alpha \in A} X_\alpha$ given by $\iota_\alpha(x) = (x, \alpha)$, and each set X_α is identified with its image $X_\alpha^* = \iota_\alpha(X_\alpha) = \{(x, \alpha) : x \in X_\alpha\}$.

Definition 2.12. *Let Y be the the disjoint union. Define $T_Y = \{U \subseteq Y | U \cap X_\alpha$ for each α in A is open in $X_\alpha\}$.*

Theorem 2.13. *T_Y is a topology on Y.*

Proof: Let Y be the the disjoint union. Define $T_Y = \{U \subseteq Y | U \cap X_\alpha$ for each α is open in $X_\alpha\}$. We need to show that $\emptyset$ and X are in T_X. We consider $U = \emptyset$. Then $U \subseteq Y$ since every set contains the empty set. Furthemore, $\emptyset \cap X_\alpha = \emptyset$ for each α in A which is also in X_α for each α as every set contains the empty set. Thus, we see $U = \emptyset$ is in T_Y. We next consider $U = Y$. Then $Y \subseteq Y$ as every set contains itself. In addition, $Y \cap X_\alpha = X_\alpha$ for each α in A which is also in X_α for each α as every set contains itself. Hence, we see $U = Y$ is in T_Y.

Next, we need to show that the union of any collection of elements of T_Y is an element of T_Y. Let $\{U_\beta\}_{\beta \in I}$ be any collection of sets from T_Y. Then, we take our union of these sets which is denoted $\bigcup_{\beta \in I} U_\beta$. As the union of subsets of a set is also a subset of the set as per set theory, we see that $\bigcup_{\beta \in I} U_\beta \subseteq Y$. Furthermore, the intesection of the union with X_α, $\bigcup_{\beta \in I} U_\beta \cap X_\beta = X_\alpha$, for each α in A which is open in X_α as per the property of the disjoint union. Hence, we see that each U_β satisfies $U_\beta \cap X_\alpha$ which is open in X_α, thus $\bigcup_{\beta \in I}(U_\beta \cap X_\alpha)$ is also open in each X_α, since the arbitrary unions of open elements are themselves open. Thus, $\bigcup_{\beta \in I} U_\beta$ is in T_Y.

Finally, we need to show that the intersection of a finite collection of elements of T_Y is also an element of T_Y. Let $U_1, U_2, ..., U_n$ represent a finite number of arbitrary sets belonging to T_Y. Then we take the intersection of these denoted, $\bigcap_{k=1}^n U_k$. As the intersection of subsets of a set is also a subset of the set as per set theory, we see that $\bigcap_{k=1}^n U_k \subseteq Y$. In addition, we now take the intersection of our intersection with X_α where we fix $\alpha \in I$. Then $(\bigcap_{k=1}^n U_k) \cap X_\alpha = \bigcap_{k=1}^n (U_k \cap X_\alpha)$ which is open in T_Y as the intersection of a finite number of arbitrary open sets is open. Hence, we see that $\bigcap_{k=1}^n U_k$ is in T_Y.

Therefore, T_Y is a topology on Y. ■

The topology above is called the **disjoint union topology** on $\amalg_{\alpha \in A} X_\alpha$. Furthermore, $\amalg_{\alpha \in A} X_\alpha$ is referred to as the **disjoint union space**.

Finally, we have two more examples of new topological spaces constructed from old ones. The proofs are omitted but are similar to those already given in this section.

Definition 2.14. *Let the* ***product topology*** *be the Cartesian product $X \times Y$ of two topological spaces whose open sets are the arbitrary unions and finite intersections of subsets $A \times B$, where A and B are open subsets of X and Y, respectively.*

Definition 2.15. *Let (X, T_X) be a topological space, and let Y be a subset of X. Now let T_Y be the collection of all sets $U \cap Y$ such that $U \in T_X$. Then T_Y is a topology on Y called the* ***subspace topology*** *where Y is called the* ***subspace*** *of X.*

2.3 Continuity

Next, we shall define the notion of continuity in a topological manner and explore some related topological concepts.

Definition 2.16. *Let X and Y be topological spaces. A map $f : X \to Y$ is* ***continuous*** *if and only if the preimage of any open set is open, that is, whenever U is an open subset of Y, its preimage $f^{-1}(U)$ is open in X.*

Note that this topological definition of continuity is equivalent to that used in calculus for functions of the form $f : \mathbb{R}^n \to \mathbb{R}^m$ where $\mathbb{R}^n$, $\mathbb{R}^m$ have the ordinary topology.

Lemma 2.17. *Let X, Y, and Z be topological spaces. Now let f and g be continuous functions such that $g : X \to Y$ and $f : Y \to Z$. Then their composition $f \circ g : X \to Z$ is also continuous.*

Proof: Let X, Y, and Z be topological spaces. Now let f and g be continuous functions such that $g : X \to Y$ and $f : Y \to Z$. Then their composition is the function $f \circ g : X \to Z$ defined by $(f \circ g)(x) = f(g(x))$ for each $x \in X$. We then let U be an open subset of Z. The preimage of U has the form $(f \circ g)^{-1}(U) = f(g(U))^{-1}$ where U lies in the set X. Since f is continuous, the preimage $f^{-1}(U)$ is open in Y. Furthermore, g is continuous which implies that the preimage $g^{-1}(f^{-1}(U))$ is open in X. As $g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U)$ by properties of function composition, we thereby know that the preimage of U, $(f \circ g)^{-1}(U)$, is open in X. Therefore, the function composition $f \circ g : X \to Z$ is continuous by the topological definition of continuity. ∎

The following proofs are necessary for the reflection and rotation operations in part 2 of Section 3, and they show an example of some familar continuous maps.

Lemma 2.18. *Let $\mathbb{R}^2$ be a topological space with the ordinary topology. Let $Ref : \mathbb{R}^2 \to \mathbb{R}^2$ be the **reflection matrix** where we denote the reflection matrix across the x-axis as Ref_x.*

$$Ref_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

And we denote the reflection matrix across the y-axis as Ref_y.

$$Ref_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Then both Ref_x and Ref_y are continuous.

Proof: Let $\mathbb{R}^2$ be a topological space with the ordinary topology. Let (x_0, y_0) be a point in $\mathbb{R}^2$. Then the operations of addition and multiplication are known to be continuous. Hence, the linear transformations represented by Ref_x and Ref_y must also be continuous. ■

Lemma 2.19. *Let $\mathbb{R}^2$ be a topological space with the ordinary topology. Let $Rot : \mathbb{R}^2 \to \mathbb{R}^2$ be the **rotation matrix***

$$Rot(\theta) = \begin{bmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

If (x_0, y_0) is a point in $\mathbb{R}^2$, let $Rot(\theta)(x_0, y_0) := (x_1, y_1)$. Then the map $(x_0, y_0) \mapsto (x_1, y_1)$ is continuous.

Proof: Let X and Y be in $\mathbb{R}^2$. Let $Rot : X \to Y$ be the rotation matrix

$$Rot(\theta) = \begin{bmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Let (x_0, y_0) be a point in $\mathbb{R}^2$. Then the cos and sin functions are known to be continuous. Furthermore, the sums and products of continuous functions are known to be continuous. Hence the linear transformation represented by the rotation matrix must also be continuous. Therefore, the rotation matrix with the mapping $(x_0, y_0) \mapsto (x_1, y_1)$ is a continuous map. ■

2.4 Topological Equivalence: Homotopy and Homeomorphism

We will now delve into how continuity is used in topology and define the topological notion of homotopy.

Definition 2.20. *Let X, Y be topological spaces and let $f, g : X \to Y$ be continuous maps. A* ***homotopy*** *of f to g is a continuous map, $H : X \times [0,1] \to Y$, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. Note that if f and g are homotopic, we write $f \simeq g$.*

Example 2.21. *If $f, g : \mathbb{R} \to \mathbb{R}^2$ are given by $f(x) := (x, x^3)$ and $g(x) = (x, e^x)$, then the map $H : \mathbb{R} \times [0,1] \to \mathbb{R}^2$ given by $H(x,t) = (x, (1-t)x^3 + te^x)$ is a homotopy between them.*

See figure 6.

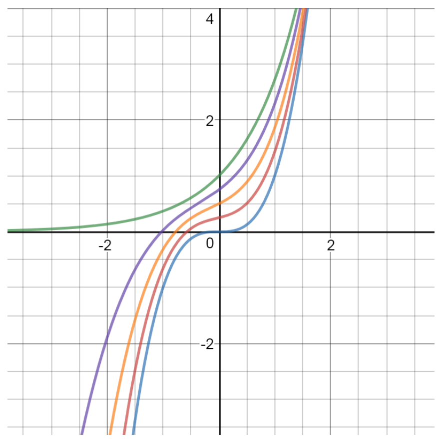


Figure 6: a homotopy between f and g, taken from [12]

Now, we will introduce some lemmas and definitions necessary to complete the "Local Criterion for Continuity" and "Gluing" Lemmas which are key to demonstrating a property of homotopies.

Lemma 2.22. *Suppose S is a subspace of the topological space X. If $U \subseteq S \subseteq X$, U is open in S, and S is open in X, then U is open in X. Note, the equivalent is true with "closed" in place of "open".*

Proof: Let S be a subspace of the topological space X. Assume $U \subseteq S \subseteq X$, where U is open in S, and S is open in X. First, we know by subset properties that $U \subseteq X$. Since U is open in S, there is an $T \subseteq X$ such that $U = T \cap S$ and T is open in X by definition of S being a subspace. Then as U is the intersection of two open sets in X, U is open in X as the intersection of open sets is open. ∎

Definition 2.23. *Let X be a topological space and let $p \in X$. Then a* ***neighborhood of p*** *is an open subset of X that contains p. More generally, assume $K \subseteq X$. Then a* ***neighborhood of the subset K*** *is an open subset containing K.*

Lemma 2.24. *Let X be a topological space and let $A \subseteq X$ be any subset. Then if every point of A has a neighborhood contained in A, then A is open in X.*

Proof: Let X be a topological space and let $A \subseteq X$ be any subset. Assume that every point of A has a neighborhood contained in A. That is, every point of A has an open set within A that contains it. Hence, A is a union of open sets. Next, we note that the union of open sets is open. Therefore, A is open in X.

Definition 2.25. *Let X and Y be sets. Let $f : X \to Y$ be a function. Then for any subset $S \subseteq X$, there is a naturally defined function from S to Y, denoted by $f|_S : S \to Y$ called the **restriction of** f **to** S, which is obtained by applying f only to elements of S : $f|_S(x) = f(x)$ for all $x \in S$.*

Lemma 2.26. *A map $f : X \to Y$ between topological spaces is continuous if and only if each point of X has a neighborhood on which (the restriction of) f is continuous.*

Proof: If f is continuous, we may simply take each neighborhood to be X itself. Conversely, suppose f is continuous in a neighborhood of each point, and let $U \subseteq Y$ be any open subset; we have to show that $f^{-1}(U)$ is open. Any point $x \in f^{-1}(U)$ has a neighborhood V_x on which f is continuous as seen in the figure below. Continuity of $f|_{V_x}$ implies, in particular, that $(f|_{V_x})^{-1}(U)$ is an open subset of V_x, and is therefore also an open subset of X. Unwinding the definitions, we see that

$$(f|_{V_x})^{-1}(U) = \{x \in V_x : f(x) \in U\} = f^{-1}(U) \cap V_x,$$

so $(f|_{V_x})^{-1}(U)$ is a neighborhood of x contained in $f^{-1}(U)$. By Lemma 2.24, this implies that $f^{-1}(U)$ is an open subset of X. ■

The above is called the "Local Criterion for Continuity"

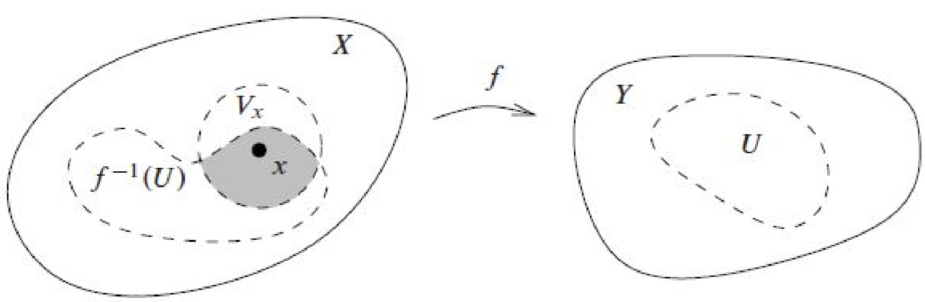


Figure 7: the local criterion for continuity, taken from [13]

We are almost ready to prove an important fact about homotopies, but first we need to introduce another definition and then prove the necessary "Gluing Lemma".

Definition 2.27. *Let X be a topological space. Then an* ***open cover*** *of a topological space X is a collection U of open subsets of X whose union is X and a* ***closed cover*** *of a topological space X is a collection U of closed subsets of X whose union is X. A* ***subcover*** *of U is a subcollection of elements of U that also covers X.*

Lemma 2.28. *Let X and Y be topological spaces, and let $\{A_1, A_2\}$ be a finite closed cover of X. Suppose that we are given continuous maps $f_1 : A_1 \to Y$ and $f_2 : A_2 \to Y$ that agree on overlaps: $f_1|_{A_1 \cap A_2} = f_2|_{A_1 \cap A_2}$. Then there exists a unique continuous map $f : X \to Y$ whose restriction to each A_i is equal to f_1 and f_2.*

Proof: It follows from elementary set theory that there exists a unique map f such that $f|_{A_1} = f_1$ and $f|_{A_2} = f_2$. Specifically, we define $f : X \to Y$ such that when $x \in X$,

$$f(x) = \begin{cases} f_1(x), & x \in A_1, \\ f_2(x), & x \in A_2. \end{cases} \tag{1}$$

We claim that this function is well-defined, that is, for each $x \in X$ there is a unique $y \in Y$ with $f(x) = y$. Observe that if $x \in A_1 \setminus A_2$ or $x \in A_2 \setminus A_1$ then f is necessarily well-defined. Furthermore, if $x \in A_1 \cap A_2$, then $f_1(x) = f_2(x)$ by assumption and f is again well-defined.

Let $K \subseteq Y$ be closed. To prove that f is continuous, it suffices to show that the preimage of each closed subset $K \subseteq Y$ is closed. This will be done by checking that $f^{-1}(K) \cap A_1 = f_1^{-1}(K)$ and $f^{-1}(K) \cap A_2 = f_2^{-1}(K)$. Without loss of generality, we will show that this is the case for $f^{-1}(K) \cap A_1 = f_1^{-1}(K)$.

Let $x \in f^{-1}(K) \cap A_1$. Then there exists a $y \in Y$ such that $f(x) = y$. Since $x \in A_1$, $f(x) = f_1(x) = y$. Thus, $x \in f_1^{-1}(K)$ and $f^{-1}(K) \cap A_1 \subseteq f_1^{-1}(K)$. Now let $x \in f_1^{-1}(K)$. Then $x \in A_1$ by definition of f_1. Thus, $x \in f^{-1}(K) \cap A_1$ by definition of intersection and $f_1^{-1}(K) \subseteq f^{-1}(K) \cap A_1$. Hence, by subset equality, we see that $f^{-1}(K) \cap A_1 = f_1^{-1}(K)$ and since we assumed the above without loss of generality, $f^{-1}(K) \cap A_2 = f_2^{-1}(K)$.

Next, we need to check that $f^{-1}(K) = f_1^{-1}(K) \cup f_2^{-1}(K)$. Let $x \in f_1^{-1}(K) \cup f_2^{-1}(K)$. Then $x \in A_1$ or $x \in A_2$ by definition of f. As $A_1 \cup A_2 = X$, $x \in X$ and thus $x \in f^{-1}(K)$ with $f_1^{-1}(K) \cup f_2^{-1}(K) \subseteq f^{-1}(K)$. Now, we let $x \in f^{-1}(K)$. Then $x \in X$ and $X = A_1 \cup A_2$. Hence, $x \in A_1$ or $x \in A_2$. If $x \in A_1$, then $x \in f_1^{-1}(K)$ and if $x \in A_2$, then $x \in f_2^{-1}(K)$. Thus, $x \in f_1^{-1}(K) \cup f_2^{-1}(K)$ by definition of union and $f^{-1}(K) \subseteq f_1^{-1}(K) \cup f_2^{-1}(K)$. Hence, by subset equality, we see that $f^{-1}(K) = f_1^{-1}(K) \cup f_2^{-1}(K)$.

Since $f_i^{-1}(K)$ is closed in A_i by continuity of f_i, and A_i is closed in X by hypothesis, it follows from Lemma 2.22 that $f_i^{-1}(K)$ is also closed in X. Thus $f^{-1}(K)$ is the union of finitely many closed subsets, and hence closed. ∎

The above lemma is known as the **"Gluing Lemma"**. It is important in establishing the following fact about homotopies:

Theorem 2.29. *Let X and Y be topological spaces. A homotopy is an equivalence relation on the set of all continuous maps from X to Y.*

Proof: Let X and Y be topological spaces. Consider the set of continuous maps from X to Y. Any map f is homotopic to itself via the trivial homotopy $H(x,t) = f(x)$. Thus, homotopy is reflexive. Next, if $H : f \simeq g$, then a homotopy from g to f is given by $\widetilde{H}(x,t) = H(x, 1-t)$. Hence, homotopy is symmetric. For the last property, if $F : f \simeq g$ and $G : g \simeq h$, define $H : X \times I \to Y$, where $I = [0,1]$, by following F at double speed for $0 \leq t \leq \frac{1}{2}$, and then following G at double speed for the remainder of the unit interval i.e.

$$H(x,t) = \begin{cases} F(x, 2t), & 0 \leq t \leq \frac{1}{2}; \\ G(x, 2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases} \tag{2}$$

As $F(x,1) = g(x) = G(x,0)$, the two definitions of H agree at $t = \frac{1}{2}$, where they overlap. Note that $[0,\frac{1}{2}]$,$[\frac{1}{2},1]$ is a closed cover for $[0,1]$. Thus H is continuous by the gluing lemma, and is therefore a homotopy between f and h. We have thereby shown that the homotopy is transitive. Therefere, a homotopy of continuous maps from X to Y is an equivalence relation. ∎

Definition 2.30. *Given a map $f : X \to Y$, let $[f]$ denote its* ***homotopy equivalence class****. The set of all homotopy equivalence classes of maps from X to Y is denoted $[X,Y]$.*

Definition 2.31. *Let $S^n = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + ... + x_{n+1}^2 = 1\}$ denote the standard n-sphere. The* ***n^th^ homotopy group*** *of a topological space X is the set of homotopy classes of continous functions $f : S^n \to X$. In other words, it is $[S^n, X]$. We write the n^{th} homotopy group as $\pi_{\boldsymbol{n}}(X)$.*

Remark: the fact that this set is a group is non-obvious and its proof can be found in Lee.

Definition 2.32. *The first homotopy group is also called the* ***fundamental group****. It consists of homotopy classes of maps $f : S^1 \to X$.*

Now that we have examined homotopies, we will proceed to state a related but distinct form of equivalence called a homeomorphism and discuss its connection to topology.

Definition 2.33. *A function $f : X \to Y$ between two topological spaces (X, T_X) and (Y, T_Y) is called a **homeomorphism** if it has the following properties:*

- *f is a bijection (one-to-one and onto),*
- *f is continuous,*
- *the inverse function f^{-1} is continuous (f is an open mapping).*

*If such a function exists, we say X and Y are **homeomorphic**.*

Note that a continuous bijective function need not be bicontinuous thus necessitating the third condition for a homeomorphism.

Example 2.34. *Let X be the $\mathbb{R}$ with the discrete topology and let Y be the $\mathbb{R}$ with the ordinary topology with the function f being the identity map. Then f is a continuous bijection, and f^{-1} is nowhere continuous.*

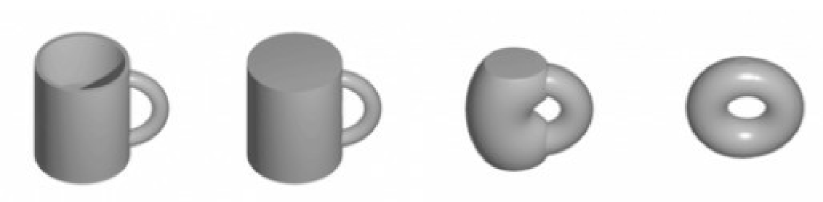


Figure 8: Coffee and Donuts, taken from [14]

Seen here in figure 8 is is the classic "coffee cup and donut" demonstration of a homeomorphism where one is continuously deformed into the other.

They are topologically "equivalent" in that the "hole" is preserved across the deformations. Effectively, a homeomorphism is a way to bend or stretch a surface into another without creating or destroying "holes".

In general, a mapping function between topological spaces being a homeomorphism is the notion of "equivalence" that topologists care about. It is analogous to isomorphism in group theory. Before moving on, it would be helpful to understand the difference between a homotopy and a homeomorphism on a more intuitive level. Specifically, a homotopy is a notion of equivalence between maps whereas a homeomorphism is a notion of equivalence between topological spaces. The former is concerned with the functions that map one topological space to another while the latter concerns how one topological space can be seen as equivalent to another.

2.5 Hausdorff Spaces

Next, we shall define the topological concept of a Hausdorff space and explore the notion with some examples.

Definition 2.35. *Let X be a topological space. X is* ***Hausdorff*** *if for each pair of distinct points p and q in X, there are disjoint open sets U and V such that $p \in U$ and $q \in V$.*

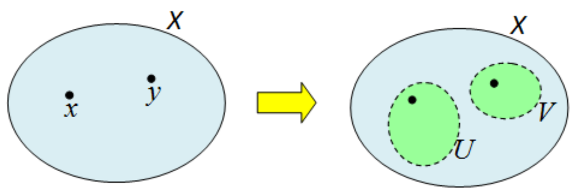


Figure 9: Example of Hausdorff property of topological spaces, taken from [15]

Examples:

#1: A set X with the indiscrete topology is <u>not</u> Hausdorff in general. Recall: $X = \{1, 2, 3\}$ with $T_X = \{\emptyset, \{1, 2, 3\}\}$ in which case p and q will always be found together in the same open set.

#2: The set $\mathbb{R}$ with the ordinary topology is Hausdorff since for any points $p, q \in \mathbb{R}$, we can find open intervals $U = (p - \frac{|p-q|}{3}, p + \frac{|p-q|}{3})$ and $V = (q - \frac{|p-q|}{3}, q + \frac{|p-q|}{3})$ such that $p \in U$ and $q \in V$ and $U \cap V = \emptyset$.

#3: The set $\mathbb{R}$ with the finite complement topology is <u>not</u> Hausdorff. We note that this topology is defined as $C = \{B \subseteq \mathbb{R} | \mathbb{R} - B \text{ is finite}\}$. By way of contradiction, assume that C is Hausdorff. Then for each pair of distinct points p and q in C there exist disjoint open sets U and V such that $p \in U$ and $q \in V$ with $U, V \subseteq \mathbb{R}$. Since, $\mathbb{R} - V$ must be finite as per the definition of C, V^c must be finite and similarly U^c must be finite. As $U \subseteq V^c$ due to $U \cap V = \emptyset$, U must be finite and it follows that $\mathbb{R} = U \cup U^c$ must be finite as well. However, $\mathbb{R}$ is not finite, and we thereby have a contradiction. Therefore, C is not Hausdorff.

Theorem 2.36. *If X and Y are Hausdorff spaces, then $X \times Y$ with the product topology is Hausdorff.*

Proof: Assume X and Y are Hausdorff spaces with $X \times Y$ possesing the product topology. Then let p and q be distinct points in $X \times Y$. Now by definition of cartesian product, $p = (x_1, y_1)$ and $q = (x_2, y_2)$ where $(x_1, y_1) \neq (x_2, y_2)$ due to either $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality, we assume $x_1 \neq x_2$. Since X is Hausdorff, we know for the distinct points x_1, x_2 that there exists disjoint open sets U and V such that $x_1 \in U$ and $x_2 \in V$. Hence $U \times Y$ with (x_1, y_1) and $V \times Y$ with (x_2, y_2) as their respective elements form open sets of $X \times Y$ via the definition of product topology and must also be disjoint as the first element of every ordered pair of $U \times Y$ is disjoint with respect to the first element of every pair of $V \times Y$. Therefore, $X \times Y$ with the product topology is Hausdorff. ∎

Theorem 2.37. *Every subspace of a Hausdorff space is Hausdorff.*

Proof: Let Y be any arbitrary subspace of a Hausdorff space X. Now let p and q be distinct points in Y. Then there exists sets U, V such that they are disjoint open sets in X with $p \in U$, $q \in V$. If we form the sets $U \cap Y$ and $V \cap Y$ with $p \in U \cap Y$ and $q \in V \cap Y$, then these sets will be open by definition of the subspace topology on Y. Next, note that $U \cap Y \subseteq U$ and $V \cap Y \subseteq V$ where U, V are disjoint. Hence $U \cap Y$ and $V \cap Y$ are disjoint. Therefore Y is Hausdorff. ∎

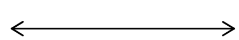
2.6 Compactness

These definitions establish the notion of compactness for topological spaces and provide the basis for a helpful lemma regarding quotient maps.

Definition 2.38. *A topological space X is said to be* ***compact*** *if every open cover of X has a finite subcover.*

Note that this is equivalent to X being closed and bounded if X is embedded in $\mathbb{R}^n$ for some n.

- In $\mathbb{R}^2$:
  - a line is not compact because it is not bounded. Alternatively, the open cover $\{(n-2, n) | n \in \mathbb{Z}\}$ has no finite subcover.

  - The **closed rectangle** $[a, b] \times [c, d]$ is compact.

- In $\mathbb{R}^3$:
  - $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$ is compact. (where $\mathbb{S}^1$ is a circle: $\{x^2 + y^2 = 1 | (x, y) \in \mathbb{R}^2\}$)

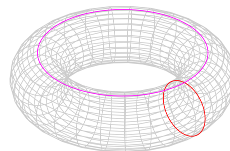
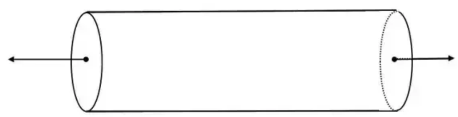


Figure 10: encircled torus, taken from [16]

  - $\mathbb{S}^1 \times \mathbb{R}^1$ i.e. an infinite cylinder is <u>not</u> compact.



Lemma 2.39. *Let f be a continuous map from a compact space to a Hausdorff space. If f is surjective, then it is a quotient map.*

Proof: Let f be a continuous map from a compact space X to a Hausdorff space Y. Assume that f is a surjective function. Then since $f : X \to Y$ is surjective, we know that for any $S \subseteq Y$, $f^{-1}(Y - S) = X - f^{-1}(S)$ via properties of pullbacks of surjective functions. Furthermore since $f : X \to Y$ is a continuous function, we know that for any U open in Y, the preimage, $f^{-1}(U)$ is open in X.

Now, we need to show that if $U \subseteq Y$ with $f^{-1}(U)$ open in X, then U is open in Y. As f is continuous, if T is closed in Y, then $f^{-1}(T)$ is closed in X. In addition, since f is surjective, we then have that $f(f^{-1}(T)) = T$ for every set T. Now, we assume that $V = f^{-1}(T)$ is closed in X. Then as X is compact, we see that V is compact as well. Finally, since f is continuous, $f(V) = T$ is compact and closed since Y is Hausdorff. Note that $f^{-1}(Y - S) = X - f^{-1}(S)$. Furthermore, recalling that if one lets X be any topological space, Y be any set, $q : X \to Y$ be a surjective function, then one can define $T_Y = \{U \subseteq Y | q^{-1}(U) \text{open in X}\}$ which is the quotient topology and q is the quotient map. Therefore, we see by this definition that f is a quotient map. ■

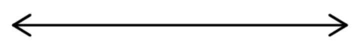
The above is known as and will be referred to as the **closed map lemma**.

2.7 Connectedness

These definitions similarly establish the notion of connectedness for topological spaces and path-connected space as a stronger notion of connectedness. Furthermore, the latter's relation to quotient spaces is examined.

Definition 2.40. *Let X be a topological space . X is called **disconnected** if there exist nonempty disjoint open sets U and V such that $X = U \cup V$. Such a pair (U, V) is called a **separation** of X. If X is not disconnected, then X is called **connected**.*

- In $\mathbb{R}$:
  - The real number line $\mathbb{R}$ is connected.



  - The line defined by $\mathbb{R} \smallsetminus \{0\}$ is disconnected by the two open subsets $\{x : x > 0\}$ and $\{x : x < 0\}$.

We can now examine the topological notion of path-connectedness as follows:

Definition 2.41. *Let X be a topological space and let $p, q \in X$. A **path in X from p to q** is a continuous map $f : I \to X$ such that $f(0) = p$ and $f(1) = q$, where $I = [0, 1]$ is the unit interval.*

Definition 2.42. *Let X be a topological space and let $p, q \in X$. X is said to be **path-connected** if for every $p, q \in X$, there is a path in X from p to q.*

Theorem 2.43. *Every quotient space of a path-connected space is path-connected.*

Proof: Let Q be a quotient space of the path-connected space C. Then there exists some equivalence relation $\sim$ on C such that $Q = C/\sim$ where $q : C \to C/\sim$ is the quotient map $q(x) = [x]$. Note that the $[x]$ indicates the equivalence class of x under the relation $\sim$. Also, note that q is surjective by definition of quotient space. Now let $[c], [d]$ be any points in Q in which case $[c], [d]$ are equivalence classes of points in C. We then pick $c_0 \in [c]$ and $d_0 \in [d]$. We see that $c_0, d_0 \in C$ thus $q(c_0) = [c], q(d_0) = [d]$. As $c_0, d_0 \in C$ with C being path-connected, there exists $f : I \to C$ such that $f(0) = c_0$ and $f(1) = d_0$. Next, since $q(c_0) = [c]$, we have then have $q(f(0)) = q(c_0) = [c]$ and in turn $q(f(1)) = q(d_0) = [d]$. Thus, we have the function $g : I \to C/\sim$ with $I = [0, 1]$ such that $g(0) = [c]$ and $g(1) = [d]$ with g being continuous as $g = q \circ f$ where the composition is continous by lemma 2.12. Therefore, our quotient space Q is path-connected. ■

2.8 Manifolds

A manifold is a topological space that locally resembles Euclidian space near each point of the space in question while perhaps having a global structure that is not at all Euclidean. Less abstractly, a 2-dimensional manifold or surface must be locally Euclidean, meaning that near each point, the surface "looks flat", but the entire structure may not be flat overall.

This is best seen with the "Earth zoomed" example here with figure 12:

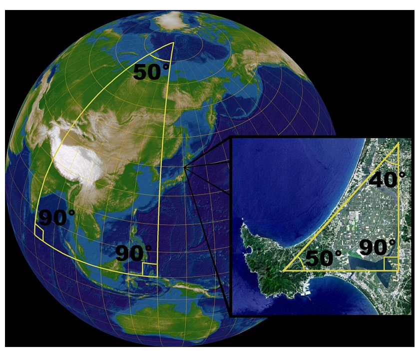


Figure 11: Earth zoomed, taken from [17]

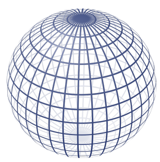
- The triangle on a local patch on Earth will add up to approximately 180°as seen in the figure with 40°+ 50°+ 90°= 180°, meaning that it is locally Euclidean.
- But the triangle at a much larger hemispheric scale will not add up to 180°as seen in the figure with 50°+ 90°+ 90°= 230°, meaning that it is not globally Euclidean.
- Thus, the Earth's exterior is a surface i.e. a 2-dimensional manifold.

We will now establish the basis for talking about manifolds via the following definitions, lemmas, and theorems.

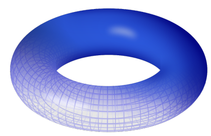
Definition 2.44. *A topological space M is said to be* ***locally Euclidean of dimension n*** *if every point of M has a neighborhood in M that is homeomorphic to an open subset of $\mathbb{R}^n$.*

Definition 2.45. *Assume n is a positive integer. A* ***n-dimensional manifold*** *is a connected Hausdorff space that is locally Euclidean of dimension n. May be refered to as "n-manifold" for brevity.*

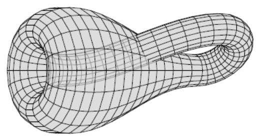
Recall that surfaces are 2-dimensional manifolds.
Some examples include the Sphere, the Torus, the Klein bottle, and the 2-holed Torus:



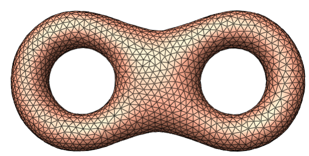
Sphere, $\mathbb{S}^2$, taken from [18]



Torus, $\mathbb{T}^2$, taken from [19]



Klein Bottle, $\mathbb{K}$, taken from [20]

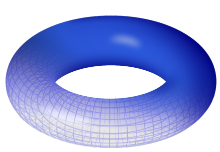


2-holed Torus, $\mathbb{T}^2\#\mathbb{T}^2$, taken from [21]

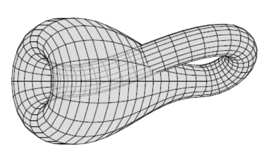
This gives rise to a perhaps more intuitive definition of a surface.

Definition 2.46. ***Surfaces*** *i.e. 2-dimensional manifolds are connected Hausdorff spaces with shape that can be* ***embedded*** *(formed without self-intersections) or* ***immersed*** *(formed with self-intersections) in three dimensional real space.*

Embedded:



Immersed:



We now examine the notion of gluing manifolds together to form connected sums.

Definition 2.47. *Let X and Y be topological spaces. Let A be a closed subspace of Y, and $f : A \to X$ is a continuous map. Let $\sim$ be the equivalence relation on the disjoint union $X \cup Y$ generated by $a \sim f(a)$ for all $a \in A$, and denote the resulting quotient space by $X \cup_f Y = (X \cup Y)/\sim$. Any such quotient space is called an* ***adjunction space****.*

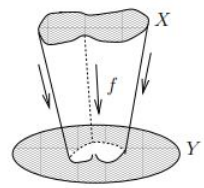


Figure 13: gluing X to Y via f, taken from [22]

Definition 2.48. *An* ***open ball of radius*** *r in a manifold M is the intersection of an open ball of radius r in $\mathbb{R}^n$ with M. Recall that an open ball of radius r in $\mathbb{R}^n$ is the set of all points distance less than r from a specified point.*

Definition 2.49. *The* ***connected sum*** *of two n-manifolds M_1 and M_2 is obtained by deleting open balls B_1 and B_2 of radius $\epsilon > 0$ from M_1 and M_2 respectively, and then forming the adjunction space $M_1 \cup_f M_2$ where f identifies the boundary of B_1 with B_2. This space is denoted $M_1 \# M_2$, (the connected sum).*

That is, we delete two open neighborhoods or "disks" in each of M_1 and M_2 and "glue along the boundaries of the holes" to form the connected sum.

Definition 2.50. *This procedure above will be called* ***gluing*** *X* ***to*** *Y* ***along*** *f.*

One can think of this as effectively "suturing" or "stitching" one manifold or surface onto another as seen in the figure below.

With this framework now established, an important theorem can now be noted.

Theorem 2.51. *If M_1 and M_2 are manifolds, then $M_1 \# M_2$ is a manifold.*

Note that the proof for this result can be found in Teo [9].

This theorem effectively states that any combination of manifolds via "connected sum" will necessarily be a manifold as well. That is, if we connect two surfaces, the result is also a surface.

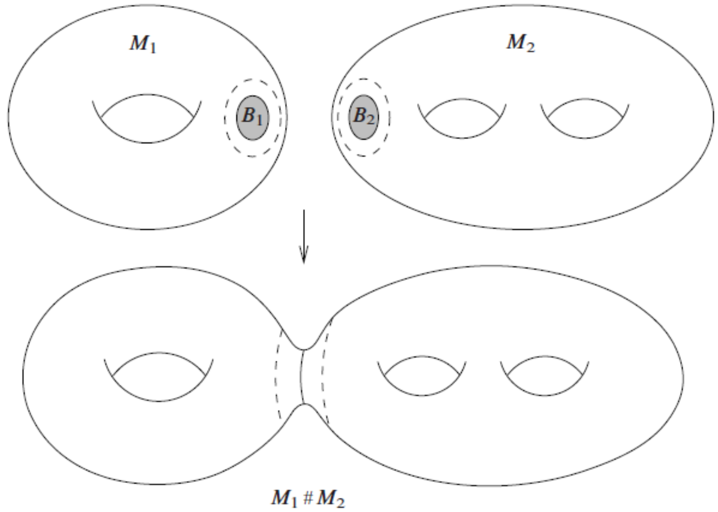


Figure 14: This is an example of a *Connected Sum*, taken from [23]

3 The Classification Theorem

Having built up our necessary foundation, we will now concern ourselves with proving the following theorem found in our primary source [8]. Note that most of these proofs are directly based on those in Lee [8] and all images are also from Lee unless otherwise noted.

Theorem 3.1. *Let M be a nonempty, compact, connected 2-dimensional manifold, formed from a polygon in the plane by gluing corresponding sides of the boundary together. Then M is homeomorphic to exactly one of the following:*

- S^2*, the sphere.*
- $T^2\#...\#T^2$*, a connected sum of tori.*
- $P^2\#...\#P^2$*, a connected sum of projective planes.*

This theorem is a key result in the classification of surfaces whose proof will consist of verifying the many required auxiliary theorems and lemmas listed below.

Outline of Lee's proof of the classification theorem follows:

I First, we define a polygonal presentation P, and define an operation $|P|$ called geometric realization which produces a topological space from a polygonal presentation.

Definition 3.2. *Let S be a set. A **word in S** is an ordered k-tuple of symbols of the form a or a^{-1} for some $a \in S$.*

Definition 3.3. *A **polygonal presentation** denoted*

$$P = \langle S | W_1 ... W_k \rangle,$$

is a finite set S together with finitely many words $W_1, ..., W_k$ in S of length 3 or more, such that every symbol in S appears in at least one word.

Note that the symbol e is meant to represent a symbol not in the current set of symbols and is not the identity as a polygonal presentation is only a semi-algebraic construct and not a group.

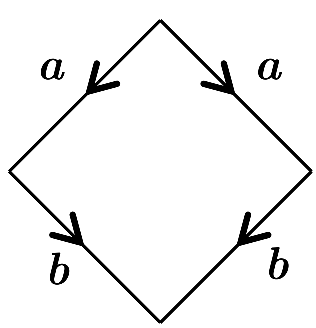
Definition 3.4. *A topological space $|P|$ called the **geometric realization of P** is determined by a given polygonal presentation P in the following manner:*

(a) For each word W_i, P_i is taken to denote a suitable polygonal region in the plane at the origin.

(b) A one-to-one correspondence is defined between the symbols of a word and the edges of the polygon.

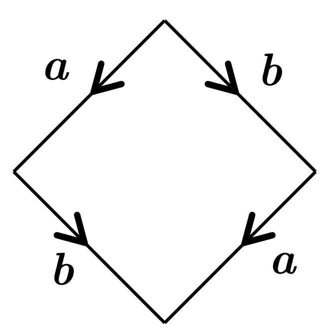
(c) Finally, $|P|$ denotes a quotient space specified by identifying edges that have the same edge symbol.

Some examples of building such geometric realizations include the following for the sphere and the torus.



$\mathbb{S}^2$

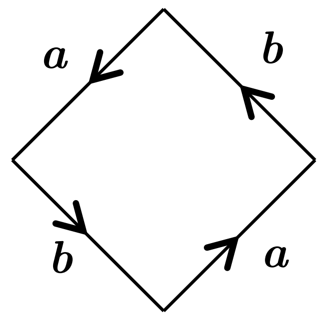
polygonal presentation:
$\langle a, b | abb^{-1}a^{-1} \rangle$



$\mathbb{T}^2$

polygonal presentation:
$\langle a, b | aba^{-1}b^{-1} \rangle$

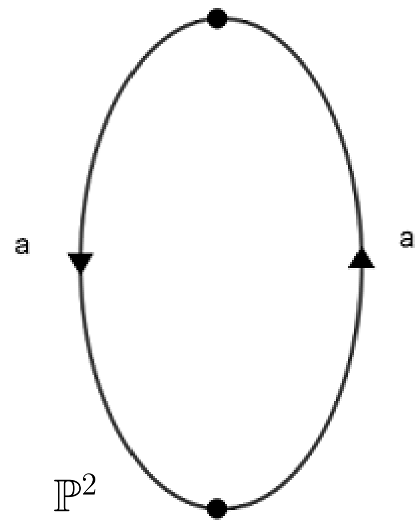
Along with the following equivalent presentations for the projective plane, $\mathbb{P}^2$.



$\mathbb{P}^2$

polygonal presentation:

$\langle a, b | abab \rangle$

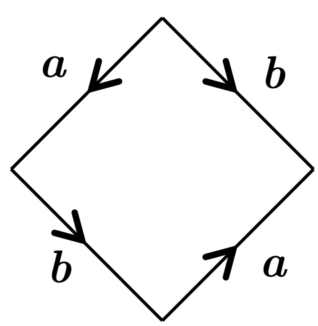


$\mathbb{P}^2$

polygonal presentation:

$\langle a | aa \rangle$

And the Klein Bottle, $\mathbb{K}$.



$\mathbb{K}$

polygonal presentation:

$\langle a, b | abab^{-1} \rangle$

Note that one can find the connected sum of two polygonal presentations by just concatenating them such as with, $\mathbb{T}^2 \# \mathbb{T}^2$, the 2-holed torus.

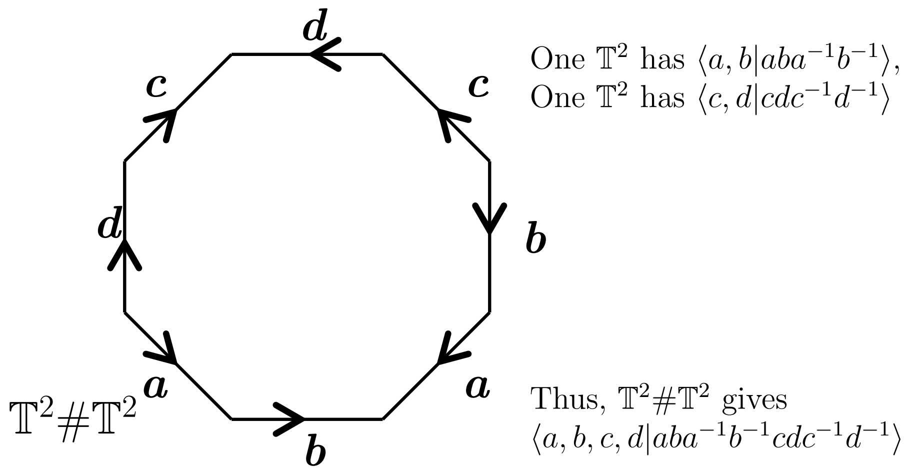


Figure 15: A polygonal presentation of $\mathbb{T}^2 \# \mathbb{T}^2$, the 2-holed torus.

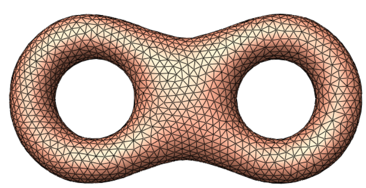


Figure 16: For reference, the geometric realization of $\mathbb{T}^2 \# \mathbb{T}^2$, the 2-holed torus, taken from [21]

II Next, we note that performing operations known as **elementary transformations** on polygonal presentations can produce topologically equivalent presentations for later use. That is, they do not change the topological equivalence class of the polygonal presentation. That this equivalence is the case will be proven for each of the elementary transformations listed below.

Definition 3.5. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be polygonal presentations. If the presentations possess homeomorphic geometric realizations, then they are said to be **topologically equivalent**. We denote this relationship as $\mathcal{P}_1 \approx \mathcal{P}_2$*

We shall now list the allowable elementary transformations.

- **Relabeling:** This transformation is defined as either changing all occurrences of a symbol a to a new symbol not already in the presentation, interchanging all occurrences of two symbols a and b, or interchanging all occurrences of a and a^{-1} for some $a \in S$.

  Proof(of equivalence): Let P be a convex polygonal region.We then consider each of the three cases.

First, let P have edges labeled aW. Now transform P to P' by replacing a with the new symbol e to obtain eW as the label for the new polygon. The polygon P' remains convex as only the label of a vertex has been changed with no new vertices being added or removed with no changes in their angles. Thus, there is a continous function $P \to P'$ that sends a to e and W to W. Composing this map with the quotient map forming the geometric realization of P thereby produces a homeomorphic geometric realization of P'.

Second, let P have edges labeled abW. Now transform P to P' by switching ab with ba to obtain baW as the label for the new polygon. The polygon P' remains convex as only the labels of the two vertices have been changed with no new vertices being added or removed with no changes in their angles. Thus, there is a continous function $P \to P'$ that sends ab to ba and W to W. Composing this map with the quotient map forming the geometric realization of P thereby produces a homeomorphic geometric realization of P'.

Third, let P have edges labeled $aa^{-1}W$. Now transform P to P' by switching aa^{-1} with $a^{-1}a$ to obtain $a^{-1}aW$ as the label for the new polygon. The polygon P' remains convex as only the labels of the two vertices have been changed with no new vertices being added or removed with no changes in their angles. Thus, there is a continous function $P \to P'$ that sends aa^{-1} to $a^{-1}a$ and W to W. Composing this map with the quotient map forming the geometric realization of P thereby produces a homeomorphic geometric realization of P'. $\blacksquare$

- **Subdividing:** This transformation is defined as replacing every occurrence of a by ae and every occurrence of a^{-1} by $e^{-1}a^{-1}$, where e is a new symbol not already in the presentation.

  Proof(of equivalence): Let P be a convex polygonal region with edges labeled aW. We transform P to P' as follows: choose the midpoint of the side labeled a in P, and place a vertex there. The resulting polygon P' is still convex since the angle at this vertex is a straight angle. Then label the two new resulting edges a and e, so the label of the new polygon P' is aeW. There is a continuous function $P \to P'$ that sends a to ae and W to W. Composing this map with the quotient map forming the geometric realization of P produces a homeomorphic geometric realization of P'. $\blacksquare$

- **Consolidating:** If a and b always occur adjacent to each other either as ab or $b^{-1}a^{-1}$, this transformation is defined as replacing every occurrence of ab by a and every occurrence of $b^{-1}a^{-1}$ by a^{-1}, provided that the result is one or more words of length at least 3 or a single word of length 2.

  Note: consolidating is the inverse operation to subdividing so a proof for subdividing would be sufficient for it by symmetry. This inverse relationship follows from subdividing adding length to a word W via introduction of the new symbol e i.e. a new side while consolidating shortens a word W by removing the associated symbol and side b thereby allowing for each other's result to be undone thereby producing the necessary symmetry since either form could be substituted for the other in the proof.

- **Reflecting:** This transformation is defined thusly,

$$\langle S|a_1 \dots a_m, W_2, \dots, W_k\rangle \mapsto \langle S|a_m^{-1} \dots a_1^{-1}, W_2, \dots, W_k\rangle$$

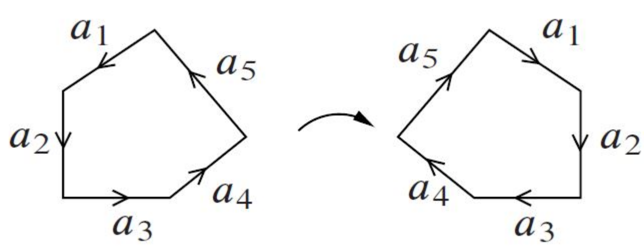


Figure 17: Example of reflection operation.

  Proof(of equivalence): We need to show that reflecting produces a homeomorphic geometric realization. Let P_1 be the geometric realization of $a_1, \dots, a_m$ and P_1' be the geometric realization of $a_m^{-1}, \dots, a_1^{-1}$. Since reflection is a continuous linear transformation by the reflection lemma, one can choose the reflection matrix to be the homeomorphism. Then, it follows that reflection is bijective and continuous for both f and f^{-1}. For $W_2, \dots, W_k$, the homeomorphism can be suitably extended using the identity map. ■

- **Rotating:**This transformation is defined thusly,

$$\langle S|a_1 a_2 \dots a_m, W_2, \dots, W_k\rangle \mapsto \langle S|a_2 \dots a_m a_1, W_2, \dots, W_k\rangle$$

  Proof(of equivalence): We need to show that rotating produces a homeomorphic geometric realization. Let P_1 be the geometric realization of $a_1, \dots, a_m$ and P_1' be the

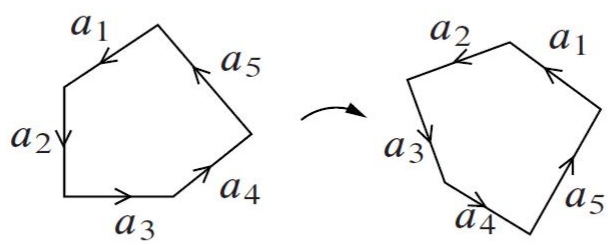


Figure 18: Example of rotation operation.

geometric realization of $a_2, \ldots, a_m, a_1$. Since rotation is a continuous linear transformation by the rotation lemma, one can choose the rotation matrix to be the homeomorphism. Then, it follows that the rotation operation is is bijective and continuous for both f and f^{-1}. For $W_2, \ldots, W_k$, the homeomorphism can be suitably extended using the identity map. ■

- **Cutting:**If words W_1 and W_2 both have length at least 2, Then this transformation is defined thusly,

$$\langle S|W_1W_2, W_3, \ldots, W_k\rangle \mapsto \langle S, e|W_1e, e^{-1}W_2, W_3, \ldots, W_k\rangle$$

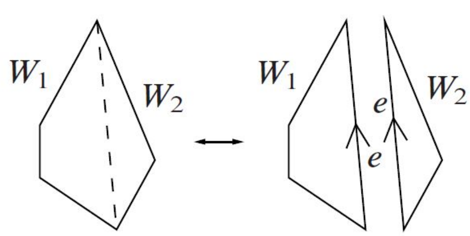


Figure 19: Example of cut operation.

Proof(of equivalence):We need to show that cutting produces a homeomorphic geometric realization, so let P_1 and P_2 be convex polygonal regions labeled W_1e and $e^{-1}W2$, respectively, and let P' be a convex polygonal region labeled W_1W_2. Now, assume that these are the only words in their respective presentations. Let $\pi : P_1 \amalg P_2 \to M$ and $\pi' : P' \to M'$ denote the respective quotient maps. The line segment going from the terminal vertex of W_1 in P' to its initial vertex is in P' by convexity and we label this segment e. By Theorem C.3, there is a continuous map $f : P_1 \amalg P_2 \to P'$ that takes each edge of P_1 or P_2 to the edge in P' with the corresponding label, and whose restriction to each P_i is a homeomorphism onto its image. By the closed map lemma, f is a quotient map. Since f identifies the two edges labeled e and e^{-1} and nothing else, the quotient maps $\pi' \circ f$ and π make exactly the same identifications. Thus, their quotient spaces are homeomorphic. If there are other words $W_3, \ldots, W_k$, we just extend f by declaring it to be the identity on their respective polygonal regions and proceed as above. ■

- **Pasting:** This transformation is defined thusly,

$$\langle S, e | W_1 e, e^{-1} W_2, W_3, \ldots, W_k \rangle \mapsto \langle S | W_1 W_2, W_3, \ldots, W_k \rangle$$

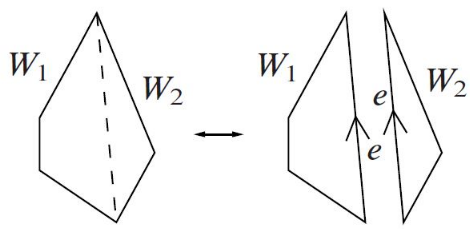


Figure 20: Example of paste operation.

Note: pasting is the inverse operation to cutting so the above proof for cutting is sufficient for it by symmetry. This inverse relationship follows from cutting introducing a pair of parallel sides e and e^{-1} to say the presentation $W_1 W_2$ thereby producing two separate presentations $W_1 e$ and $e^{-1} W_2$ where each now has a matching additional side. On the other hand, pasting removes the pair of parallel sides e and e^{-1} from $W_1 e$ and $e^{-1} W_2$ thereby producing the presentation represented by concatenationg W_1 and W_2 i.e. $W_1 W_2$. This allows for each other's result to be undone thereby producing the necessary symmetry since either form could be substituted for the other in the proof.

- **Folding:** If word W_1 has length at least 3, then this transformation is defined thusly

$$\langle S, e | W_1 e e^{-1}, W_2, \ldots, W_k \rangle \mapsto \langle S | W_1, W_2, \ldots, W_k \rangle$$

Note W_1 is allowed to have length 2, provided that the presentation has only one word.

Proof(of equivalence):For folding, we can ignore the additional words $W_2, \ldots, W_k$. If W_1 has length 2, we can subdivide to lengthen it, then perform the folding operation, and then consolidate. Thus, assume without loss of generality that W_1 has length at least 3. Suppose first that $W_1 = abc$ has length exactly 3. Then let P and P'

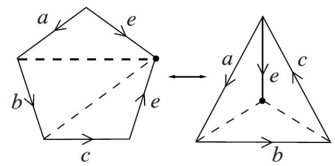


Figure 21: Example of folding operation.

be convex polygonal regions with the edge labels $abcee^{-1}$ and abc, respectively, and let $\pi : P \to M$, $\pi' : P' \to M'$ be the quotient maps. Adding edges turns P and P' into polyhedra of Euclidean simplicial complexes, and there is a unique simplicial map $f : P \to P'$ that takes each edge of P to the edge of P' with the same label. As before, $\pi' \circ f$ and π are quotient maps that make the same identifications, so their quotient spaces are homeomorphic. If W_1 has length 4 or more, we can write $W_1 = Xbc$ for some X of length at least 2. Then we cut along a new edge a to obtain

$$\langle S, b, c, e | Xbcee^{-1} \rangle \approx \langle S, a, b, c, e | Xa^{-1}, abcee^{-1} \rangle$$

and proceed as before. ■

- **Unfolding:** This transformation is defined thusly

$$\langle S | W_1, W_2, \ldots, W_k \rangle \mapsto \langle S, e | W_1 ee^{-1}, W_2, \ldots, W_k \rangle$$

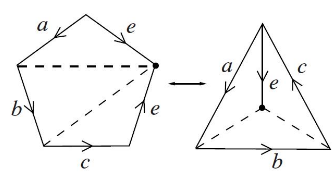


Figure 22: Example of unfolding operation.

Note: Unfolding is the inverse operation to folding so the above proof for folding is sufficient for it by symmetry. This inverse relationship follows from unfolding introducing the pair of opposing sides e and e^{-1} together in the form of ee^{-1} to a word say W_1 thereby producing the unfolded presentation $W_1 ee^{-1}$. Whereas folding removes the pair of opposing sides ee^{-1} from the $W_1 ee^{-1}$ leaving the folded presentation W_1 without the intervening e and e^{-1}. This allows for each other's result to be undone thereby producing the necessary symmetry since either form could be substituted for the other in the proof.

These elementary transformations will allow us to reduce all possible polygonal presentations of surfaces to just a few general types as required for the Classifications theorem.

III We now show that every 2-dimensional manifold admits a polygonal presentation. That is, there exists a polygonal presentation P such that the geometric realization of P is homeomorphic to our manifold M.

Theorem 3.6 (Presentation Theorem). *Every compact surface admits a polygonal presentation.*

Proof. Let M be an arbitrary compact surface. It follows from the Triangulation Theorem that M is homeomorphic to a 2-dimensional simplicial complex K, in which each 1-simplex is a face of exactly two 2-simplices. From this simplicial complex, a surface presentation $\mathcal{P}$ can be constructed with one word of length 3 for each 2-simplex where edges have the same label if and only if they correspond to the same 1-simplex.

We need to show that the geometric realization of $\mathcal{P}$ is homeomorphic to that of K. Let $P = P_1 \amalg \cdots \amalg P_k$ denote the disjoint union of the 2-simplices of K. Then we have the two quotient maps $\pi_K : P \to |K|$ and $\pi_{\mathcal{P}} : P \to |\mathcal{P}|$ via the disjoint union topology on P. Thus, it suffices to show that they make the same identifications. Note that these quotient maps are injective in the interiors of the 2-simplices, have the same identifications of edges, and identify vertices only with other vertices. This follows from P being the pullback of both $|K|$ and $|\mathcal{P}|$.

Now, we need to show that π_K, like $\pi_{\mathcal{P}}$, identifies vertices only when constrained via the relation generated by the edge identifications. Suppose $v \in K$ is any vertex. Then v belongs to some 1-simplex, because otherwise it would be an isolated point of $|K|$, contradicting the fact that $|K|$ is a connected 2-manifold. Note that the Triangulation Theorem guarantees that this 1-simplex is a face of exactly two 2-simplices.

Let two 2-simplices σ, σ' containing v be **edge-connected at v** if there is a sequence $\sigma = \sigma_1, ..., \sigma_k = \sigma'$ of 2-simplices containing v such that σ_i shares an edge with σ_{i+1} for each $i = 1, ..., k-1$. Then edge-connectedness is an equivalence relation on the set of 2-simplices containing v.

To prove this, we will show that there is only one equivalence class by way of contradiction. If this is not the case, we can group the 2-simplices containing v into two disjoint sets $\{\sigma_1, ..., \sigma_k\}$ and $\{\tau_1, ..., \tau_k\}$, such that any σ_i and σ_j for all $i \le k$, $j \le k$ are edge-connected to each other, but no τ_i is edge-connected to any σ_j. Next, let $\mathcal{E}$ be picked small enough that the neighborhood $B_{\mathcal{E}}(v)$ intersects strictly with those simplices that contain v. Then open ball $B_{\mathcal{E}}(v) \cap |K|$ is an open subset of $|K|$ and is thereby a 2-manifold, which means v has a neighborhood $W \subseteq B_{\mathcal{E}}(v) \cap |K|$ that is homeomorphic to $\mathbb{R}^2$.

It follows that $W \setminus \{v\}$ is connected. However, if we set

$$U = W \cap (\sigma_1 \cup \ldots \cup \sigma_k) \setminus \{v\}, \qquad V = W \cap (\tau_1 \cup \ldots \cup \tau_m) \setminus \{v\}$$

where U and V are disjoint because the sets $\{\sigma_1, ..., \sigma_k\}$ and $\{\tau_1, ..., \tau_k\}$ are disjoint. Then U and V are both open in $|K|$ because their intersection with each simplex is open in the simplex, and $W = U \cup V$ is a disconnection of W. This is a contradiction to the fact that $W \setminus \{v\}$ is connected. Thus, there is only one equivalence class. Therefore, we have demonstrated that every compact surface admits a polygonal presentation. ■

This last theorem is essential for allowing us to convert any surface, i.e. any 2-dimensional manifold, to its polygonal presentation provided it is compact. We desire this as working with polygonal presentations is simpler and easier than interacting with surfaces directly.

IV Finally, we will use the allowable elementary operations to reduce all possible polygonal presentations of 2-dimensional manifolds to one of three things: a sphere, a connect sum of one or more tori, or a connect sum of one or more projective planes. In doing so we will prove a lemma that $\mathbb{T}^2 \# \mathbb{P}^2$ is homeomorphic to $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$, which allows us to eliminate cases with both a torus and a projective plane.

Lemma 3.7 (Klein Lemma)**.** *The Klein bottle is homeomorphic to* $\mathbb{P}^2 \# \mathbb{P}^2$.

Proof: By the following sequence of elementary transformations, we find that the Klein bottle has the following presentations as seen in our figure:

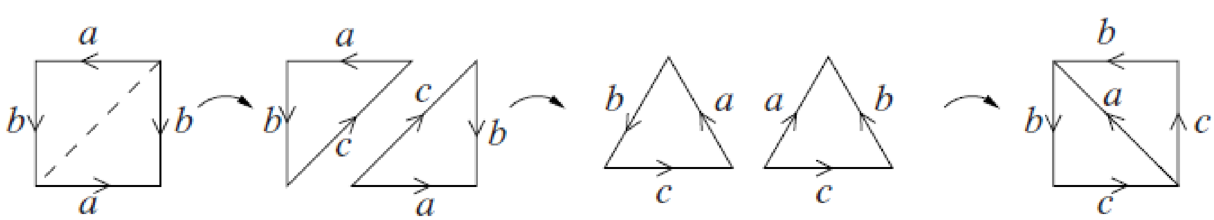


Figure 23: Transforming the Klein bottle $\mathbb{K}$ to $\mathbb{P}^2 \# \mathbb{P}^2$.

$$\begin{aligned}&\langle a, b | abab^{-1}\rangle \\ &\approx \langle a, b, c | abc, c^{-1}ab^{-1}\rangle \text{ (cut along c)} \\ &\approx \langle a, b, c | bca, a^{-1}cb\rangle \text{ (rotate and reflect)} \\ &\approx \langle b, c | bbcc\rangle \text{ (paste along } a \text{ and rotate)}\end{aligned}$$

The presentation in the last line is our standard presentation of a connected sum of two projective planes. ■

Lemma 3.8 (Connected Sum Equivalence Lemma)**.** *The connected sum* $\mathbb{T}^2\#\mathbb{P}^2$ *is homeomorphic to* $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$.

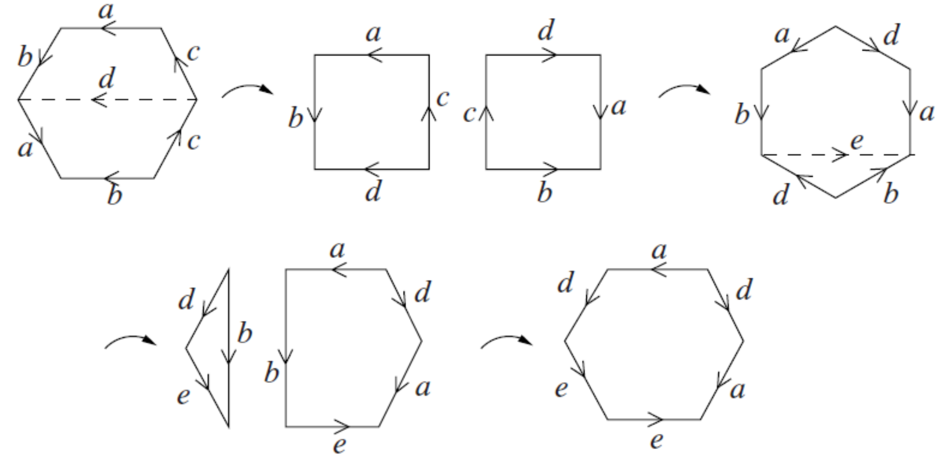


Figure 24: Transforming $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$ to $\mathbb{T}^2\#\mathbb{P}^2$.

Proof: Start with $\langle a, b, c | abab^{-1}\rangle$ as seen in our figure above, which is a presentation of $\mathbb{K}\#\mathbb{P}^2$, and therefore by the preceding lemma is a presentation of $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$. Following from our figure, we cut along d, paste along c, cut along e, and paste along b, rotating and relfecting as necessary, to obtain $\langle a, d, e | a^{-1}d^{-1}adee\rangle$, which is a presentation of $\mathbb{T}^2\#\mathbb{P}^2$. ■

Proof of the Classification Theorem: Let M be an arbitrary compact connected surface. By Theorem 3.6, i.e. the Presentation Theorem, we can assume that M comes with a given polygonal presentation. We prove the theorem by transforming this presentation to one of our standard presentations in several steps. Let us say that a pair of edges that are to be identified are **complementary** if they appear in the presentation as both a and a^{-1}, and **twisted** if they appear as $a, ..., a$ or as $a^{-1}, ..., a^{-1}$. (The terminology reflects the fact that if a polygonal region is cut from a piece of paper, you have to twist the paper to paste together a twisted edge pair, but not for a complementary pair.)

Step 1: *M admits a presentation with only one face.* Since M is connected, if there are two or more faces, some edge in one face must be identified with an edge in a different face; otherwise, M would be the disjoint union of the quotients of its faces, and since each such quotient is open and closed, they would disconnect M. Thus by performing successive pasting transformations (together with rotations and reflections as necessary), we can reduce the number of faces in the presentation to one.

Step 2: *Either M is homeomorphic to the sphere, or M admits a presentation in which there are no adjacent complementary pairs.* Each adjacent complementary pair can be eliminated by folding, unless it is the only pair of edges in the presentation; in this case the presentation is equivalent to $\langle a|aa^{-1}\rangle$ and M is homeomorphic to the sphere.

From now on, we will assume that the presentation is not equivalent to the standard presentation of the sphere.

Step 3: *M admits a finite sided presentation in which all twisted pairs are adjacent.* If a twisted pair is not adjacent, then the presentation can be transformed by rotations to one described by a word of the form $VaWa$, where neither V nor W is empty. Figure 25, shows how to transform the word $VaWa$ into $VW^{-1}bb$ by cutting along b, reflecting, and pasting along a. (Here W^{-1} denotes the word obtained from W by reflecting.)

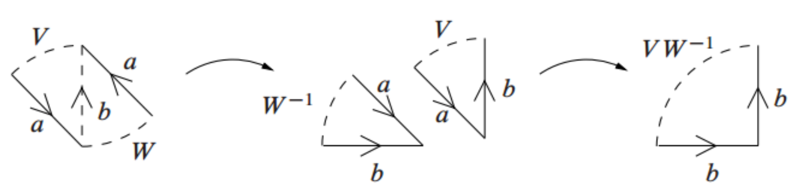


Figure 25: Making a twisted pair adjacent.

In this last presentation, the twisted pair a, a has been replaced by another twisted pair b,b which is now adjacent. Moreover, no other adjacent pairs have been separated. We may have created some new twisted pairs when we reflected W, but we decreased the total number of nonadjacent pairs (including both twisted and complementary ones) by at least one. Thus, after finitely many such operations, there are no more nonadjacent twisted pairs. We may also have created some new adjacent complementary pairs. These can be eliminated by repeating *Step 2*, which does not increase the number of nonadjacent pairs.

Step 4: *M admits a presentation in which all vertices are identified to a single point.* Choose some equivalence class of vertices, and call it v. If there are vertices that are not identified with v, there must be some edge that connects a v vertex with a vertex in some other equivalence class; label the edge a and the other vertex class w (Figure 26). The other edge that touches a at its v vertex cannot be identified with a: if it were complementary to a, we would have eliminated both edges in Step 2, while if it formed a twisted pair with a, then the quotient map would identify the initial and terminal vertices of a with each other, which we are assuming is not the case. So label this other edge b, and label its other vertex x (this one may be identified with v, w, or neither one).

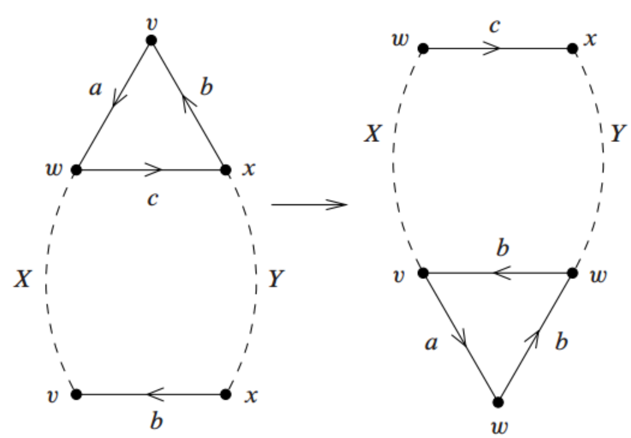


Figure 26: Reducing the number of vertices equivalent to v.

Somewhere in the polygon is another edge labeled b or b^{-1}. Let us assume for definiteness that it is b^{-1}; the argument for b is similar except for an extra reflection. Thus we can write the word describing the presentation in the form $baXb^{-1}Y$, where X and Y are unknown words, not both empty. Now cut along c and paste along b as in (Figure 26).

In the new presentation, the number of vertices labeled v has decreased, and the number labeled w has increased. We may have introduced a new adjacent complementary pair, so perform Step 2 again to remove it. This may again decrease the number of vertices labeled v (for example, if a v vertex lies between edges labeled aa^{-1} that are eliminated by folding), but it cannot increase their number. So repeating this sequence a finite number of times—decrease the v vertices by

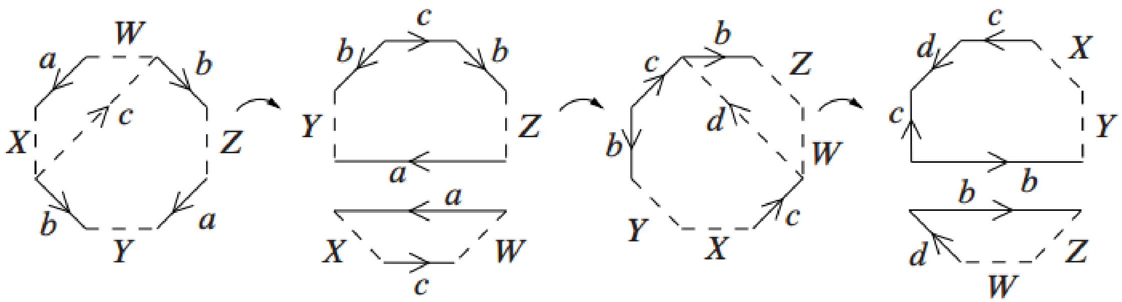


Figure 27: Bringing intertwined complementary pairs together.

one, then eliminate adjacent complementary edges — we eventually eliminate the vertex class v from the presentation altogether. Iterate this procedure for each vertex class until there is only one left.

Step 5: *If the presentation has any complementary pair a, a^{-1}, then it has another complementary pair b, b^{-1} that occurs intertwined with the first, as in $a, ..., b, ..., a^{-1}, ..., b^{-1}$.* By way of contradiction, assume that this is not the case. Then the presentation is of the form $aXa^{-1}Y$, where X contains only matched complementary pairs or adjacent twisted pairs. Thus, each edge in X is identified only with another edge in X, and the same is true of Y by construction. However, this implies that the terminal vertices of the a and a^{-1} edges, both of which touch only X, can be identified only with vertices in X, while the initial vertices can be identified only with vertices in Y . This is a contradiction, since all vertices are identified together by *Step 4*. Hence, the initial and terminal vertices of each edge cannot be identified separately. Therefore, our claim is true.

Step 6: *M admits a presentation in which all intertwined complementary pairs occur together with no other edges intervening: $aba^{-1}b^{-1}$.* If the presentation is given by the word $WaXbYa^{-1}Zb^{-1}$, perform the elementary transformations indicated in Figure 27. (cut along c, paste along a, cut along d, and paste along b) to obtain the new word $cdc^{-1}d^{-1}WZYX$. This replaces the old intertwined set of pairs with a new adjacent set $cdc^{-1}d^{-1}$, without separating any other edges that were previously adjacent. Repeat this for each set of intertwined pairs. (Note that this step requires no reflections.)

Step 7: *M is homeomorphic to either a connected sum of one or more tori or a connected sum of one or more projective planes.* From previous work, we observe that all twisted pairs occur adjacent to each other due to *Step 3*, and all complementary pairs occur in intertwined groups such as $aba^{-1}b^{-1}$ due to *Step 6*. This is a presentation of a connected sum of tori (presented by $aba^{-1}b^{-1}$) and projective planes (presented by cc). If there are only tori or only projective planes, we are done.

Final Step: The only remaining case is that in which the presentation contains both twisted and complementary pairs. In that case, some twisted pair must occur next to a complementary one; thus the presentation is described either by a word of the form $aba^{-1}b^{-1}ccX$ or by one of the form $ccaba^{-1}b^{-1}X$. In either case, this is a connected sum of a torus, a projective plane, and whatever surface is described by the word X. But Lemma 3.8, the Connected Sum Equivalence Lemma, shows that the standard presentation of $\mathbb{T}^2\#\mathbb{P}^2$ can be transformed to that of $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$. Making this transformation, we eliminate one of the occurrences of $\mathbb{T}^2$ in the connected sum. Iterating this procedure, we eliminate them all, thus completing the proof. ∎

4 Conclusion

With this proof, we have shown that all 2-dimensional Manifolds are one of the following:

- $\mathbb{S}^2$, the sphere

- $\mathbb{T}^2\#...\#\mathbb{T}^2$, a connected sum of tori

- $\mathbb{P}^2\#...\#\mathbb{P}^2$, a connected sum of projective planes.

Note, however, that this does <u>not</u> show that these types of manifolds are distinct from one another. To do so, we would need another method involving homotopy or homology.
We also assumed the Triangulation Theorem without proof. Our source by Lee similarly assumes this theorem without proof. Future research on this subject could be extended to attempting to fully prove the Triangulation Theorem with regards to simplical complexes.

Nonetheless, the Classification Theorem of Compact Surfaces i.e. 2-dimensional manifolds provides a systematic method to classify such topological structures in a complete way. Furthermore, this insight can be generalized in an analogous manner to higher dimensions of interest.

In particular, one could consider the implications of 3-dimensional manifolds embedded in 4-dimensional space. The results of such investigations include Einstein's Theory of Relativity and Grigori Perelman's recent proof of an equivalent classification theorem for 3-dimensional manifolds and the advance they represent in the field of topology.

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Appendix

This appendix is meant to contain basic background considered too trivial, general, or off topic to be included in the main body of the paper.

A Functions and Set Theory

We will now proceed to prove a few lemmas concerning set theory that will be helpful in talking about functions with respect to their pullbacks.

Lemma A.1. *Let A and B be any sets. Let $f : A \to B$ be any function. Let $B_\alpha \subseteq B$ for all α in some index set I. Then the following holds: $f^{-1}(\bigcup_{\alpha \in I} B_\alpha) = \bigcup_{\alpha \in I} f^{-1}(B_\alpha)$.*

Proof: Let $x \in f^{-1}(\bigcup_{\alpha \in I} B_\alpha)$. Then there exists a $y \in (\bigcup_{\alpha \in I} B_\alpha)$ such that $f^{-1}(y) = x$. Therefore, there exists some $\alpha \in I$ such that $y \in B_\alpha$ and $f^{-1}(y) = x$. Hence $x \in f^{-1}(B_\alpha)$ for some $\alpha \in I$. Thus, we have $x \in \bigcup_{\alpha \in I} f^{-1}(B_\alpha)$. Therefore, $f^{-1}(\bigcup_{\alpha \in I} B_\alpha) \subseteq \bigcup_{\alpha \in I} f^{-1}(B_\alpha)$.
Next, we let $x \in \bigcup_{\alpha \in I} f^{-1}(B_\alpha)$. Then, we have $x \in f^{-1}(B_\alpha)$ for some $\alpha \in I$. In which case, there exists $y \in B_\alpha$ for some $\alpha \in I$ such $f^{-1}(y) = x$. Then there exists a $y \in (\bigcup_{\alpha \in I} B_\alpha)$ such that $f^{-1}(y) = x$. Thus, we have $x \in f^{-1}(\bigcup_{\alpha \in I} B_\alpha)$. Therefore, $f^{-1}(\bigcup_{\alpha \in I} B_\alpha) \supseteq \bigcup_{\alpha \in I} f^{-1}(B_\alpha)$. By subset equality, $f^{-1}(\bigcup_{\alpha \in I} B_\alpha) = \bigcup_{\alpha \in I} f^{-1}(B_\alpha)$. ■

Lemma A.2. *Let A and B be any sets. Let $f : A \to B$ be any function. $B_\alpha \subseteq B$ for all α in some index set I. Then the following holds:$f^{-1}(\bigcap_{\alpha \in J} B_\alpha) = \bigcap_{\alpha \in J} f^{-1}(B_\alpha)$.*

Proof: Let $x \in f^{-1}(\bigcap_{\alpha \in I} B_\alpha)$. Then there exists a $y \in (\bigcap_{\alpha \in I} B_\alpha)$ such that $f^{-1}(y) = x$. Then, for all $\alpha \in I$, $y \in B_\alpha$ and $f^{-1}(y) = x$. Hence $x \in f^{-1}(B_\alpha)$ for all $\alpha \in I$. Thus, we have $x \in \bigcap_{\alpha \in I} f^{-1}(B_\alpha)$. Therefore, $f^{-1}(\bigcap_{\alpha \in I} B_\alpha) \subseteq \bigcap_{\alpha \in I} f^{-1}(B_\alpha)$.
Next, we let $x \in \bigcap_{\alpha \in I} f^{-1}(B_\alpha)$. Then, we have $x \in f^{-1}(B_\alpha)$ for all $\alpha \in I$. In which case, there exists $y \in B_\alpha$ for all $\alpha \in I$ such $f^{-1}(y) = x$. Thus, $y \in (\bigcap_{\alpha \in I} B_\alpha)$ such that $f^{-1}(y) = x$. Thus, we have $x \in f^{-1}(\bigcap_{\alpha \in I} B_\alpha)$. Therefore, $f^{-1}(\bigcap_{\alpha \in I} B_\alpha) \supseteq \bigcap_{\alpha \in I} f^{-1}(B_\alpha)$. By subset equality, $f^{-1}(\bigcap_{\alpha \in I} B_\alpha) = \bigcap_{\alpha \in I} f^{-1}(B_\alpha)$. ■

B Simplices

The following definitions and theorems are necessary components for proving equivalence of the folding transformation in part II of Section 3 and portions of part III in Section 3.

Definition B.1. *A collection S of subsets of X is stated to be* ***locally finite*** *if each point of X has a neighborhood that intersects at most finitely many of the sets in A.*

Definition B.2. *Let $S \subseteq \mathbb{R}^n$ be a linear subspace and $b \in \mathbb{R}^n$, then the set $b+S = \{b+x : x \in S\}$ is called an* ***affine subspace of*** $\mathbb{R}^n$ ***parallel to S***

Definition B.3. *The* ***dimension of b + S*** *is the dimension of S.*

Definition B.4. *A set $\{v_0, ..., v_k\}$ is* ***affinely independent*** *if it is not contained in any affine subspace of dimension strictly less than k.*

Definition B.5. *Let $\{v_0, ..., v_k\}$ be an affinely independent set of $k+1$ points in $\mathbb{R}^n$. Then the* ***simplex*** *(plural:* ***simplices****) spanned by them, denoted by $[v_0, ..., v_k]$, is the set*

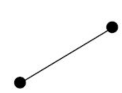
$$[v_0, ..., v_k] = \left\{ \sum_{i=0}^{k} t_i v_i : t_i \geq 0 \text{ and } \sum_{i=0}^{k} t_i = 1 \right\}$$

with the subspace topology. Furthermore, each of the points v_i is called a ***vertex*** *of the simplex. In addition, the* ***dimension*** *of a simplex is given by the integer k which is one less than the number of vertices. Such a k-dimensional simplex is called a* ***k-simplex****.*

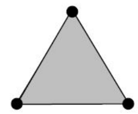
To aid our intuition, we list several examples of simplices taken from [24].

- A 0-simplex is a single point.

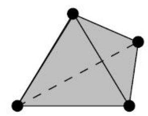
- A 1-simplex is a line segment.



- A 2-simplex is a triangle.



- A 3-simplex is a solid tetrahedron.



Definition B.6. *Let σ be a k-simplex. Then each simplex spanned by a nonempty subset of the vertices of σ is called a* ***face of*** *σ.*

Definition B.7. *The faces of k-simplex σ of 0-dimension are simply its vertices while the 1-dimensional faces are called its* ***edges****.*

Definition B.8. *The $(k-1)$-dimensional faces of a k-simplex σ are called its* ***boundary faces****.*

Definition B.9. *The union of the boundary faces of a k-simplex σ is defined to be the* ***boundary of*** *σ*

Definition B.10. *The k-simplex σ without its boundary is defined to be its* ***interior****.*

Definition B.11. *A* ***(Euclidean) simplicial complex*** *is a finite collection K of simplices in some Euclidean space $\mathbb{R}^n$, where each of the following properties hold:*

- *If $\sigma \in K$, then every face of σ is in K.*
- *The intersection of any two simplices in K is either empty or a face of each.*

Theorem B.12. *Every 2-manifold is homeomorphic to a 2-dimensional simplicial complex, in which every 1-simplex is a face of exactly two 2-simplices.*

The above is known as the **Triangulation Theorem for 2-Manifolds**. It is a necessary component for the proof of the Classification Theorem as it allows for manifolds to be "polygonized" via representation as complexes of triangles which cover the manifold in question.
Its proof can be found in Edwin E Moise's *Geometric Topology in Dimensions 2 and 3* or Carsten Thomassen's *The Jordan-Schöflies theorem and the classification of surfaces.*

C Closed Cells

The following definitions and theorems are necessary components for proving equivalence of the cutting transformation in part II of Section 3.

Definition C.1. *The* ***closed unit ball of dimension n*** *is defined as the subset* $\overline{\mathbb{B}}^n \subseteq \mathbb{R}^n$ *consisting of vectors of length at most* 1*:*

$$\overline{\mathbb{B}}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

Definition C.2. *An* ***closed n-dimensional-cell or n-cell*** *is any topological space that is homeomorphic to the complement of the open unit ball denoted* $\overline{\mathbb{B}}^n$*.*

Theorem C.3. *Suppose* D *and* D' *are closed cells (not necessarily of the same dimension). Then every continuous map* $f : \partial D \to \partial D'$ ***extends*** *to a continuous map* $F : D \to D'$*, with F(Int D)* $\subseteq$ *Int* D' *and if* f *is a homeomorphism, then* F *can also be chosen to be a homeomorphism.*

The proof of the above theorem can be found in Lee [8].