Question 7.

Generalized Neyman-Pearson Lemma. Let $f_0(x), f_1(x), \cdots, f_k(x)$ be $k+1$ probability density functions. Let ϕ_0 be a test function of the form

$$\phi_0(x) = \begin{cases} 1, & \text{if} \quad f_0(x) > \sum_{j=1}^{k} a_j f_j(x) \\ \gamma(x), & \text{if} \quad f_0(x) = \sum_{j=1}^{k} a_j f_j(x) \\ 0, & \text{if} \quad f_0(x) < \sum_{j=1}^{k} a_j f_j(x) \end{cases}$$

where $a_j \geq 0$ for $j = 1, \cdots, k$. Show that ϕ_0 maximizes

$$\int \phi(x) f_0(x) dx$$

among all ϕ, $0 \leq \phi \leq 1$, such that

$$\int \phi(x) f_j(x) dx \leq \int \phi_0(x) f_j(x) dx, \qquad j = 1, 2, \cdots, k$$