Question 9.

Let $T_n = T(X_1, \cdots, X_n)$ be an estimator of $\tau(\theta)$. Define $T_{jack} = T_n - b_{jack}$, where

$$b_{jack} = (n-1)(\overline{T}_n - T_n), \quad \overline{T}_n = \frac{1}{n}\sum_{i=1} T_{(-i)},$$

and $T_{(-i)}$ is the statistic with the i^{th} observation removed. Suppose

$$E_\theta(T_n) = \tau(\theta) + \frac{a}{n} + \frac{b}{n^2} + O(\frac{1}{n^3}),$$

for some $a, b \in \mathbb{R}$. Show that if $\mathbb{V}_\theta(T) \sim c/n$ for some positive constant c, then $\mathbb{V}_\theta(T_{jack}) \sim d/n$ for some positive constant d. Thus, the Jackknife will reduce bias, but won't increase variance. (*Note:* $f(n) \sim g(n)$ means that $f(n)/g(n) \to 1$ as $n \to \infty$.)

Solution: **(15 marks)**