Question 3.

Let $X_1, \cdots, X_n$ be a sample from the pmf

$$P_\theta(X = x) = \frac{\alpha(x)\theta^x}{f(\theta)} \ , \quad x = 0, 1, 2, \cdots \ ,$$

where $\theta > 0$, $\alpha(x) > 0$, $f(\theta) = \sum_{x=0}^{\infty} \alpha(x)\theta^x$, $\alpha(0) = 1$, and let $T = T(X_1, \cdots, X_n) = \sum_{i=1}^{n} X_i$. Write

$$c(t, n) = \sum_{\vec{x} \in R(t)} \prod_{i=1}^{n} \alpha(x_i) \ ,$$

where $\vec{x} = (x_1, \cdots, x_n)$ and $R(t) = \{(x_1, \cdots, x_n) \in \mathbb{N}_0^n : \ T(x_1, \cdots, x_n) = t\}$. Show that T is a complete sufficient statistic for θ, and that the $UMVUE$ for $d(\theta) = \theta^r (r > 0$ is an integer) is given by

$$Y_r(t) = \begin{cases} 0, & \text{if } t < r, \\ \frac{c(t-r,n)}{c(t,n)}, & \text{if } t \geq r \ . \end{cases}$$

Using this result find the UMVUE of θ^r when $X_i \overset{iid}{\sim} \text{Poisson}(\theta)$. **(15 marks)**