Endogenous Fragility in Opaque Supply Chains

Andrea Titton*

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Abstract

This paper investigates the role of supply chain unobservability in generating endogenously fragile production networks. In a simple production game, in which firms need to multisource to hedge against suppliers' risk under unobservability, firms underdiversify vis-à-vis the social optimum. The unobservability of suppliers' relations is the driver behind this. In production networks where upstream risk is highly correlated and supplier relationships are not observable, the marginal risk reduction of adding an additional supplier is low, because this additional supplier's risk is likely to be correlated to that of existing suppliers. This channel reduces firm incentives to diversify, which gives rise to inefficiently fragile production networks.

By solving the social planner problem, I show that, if the risk reduction experienced downstream resulting from upstream diversification were to be internalised by upstream firms, endogenous production networks would be resilient to most levels of risk. Despite its stylised form, the model identifies the trade-off firms face when diversifying risk and isolates the mechanism that aggregates these decisions into a production network. Furthermore, it maps the conditions of the trade-off, such as expected profits of the firm or the pairing costs, to the properties of the production network.

*Faculty of Economics and Business, University of Amsterdam (a.titton@uva.nl). I thank my supervisors Florian Wagener and Cees Diks for the patient guidance on this paper. I also thank the CeNDEF group and the Quantitative Economics section at the University of Amsterdam for helpful comments throughout the many seminars. Finally, I thank attendees of the EEA-ESESM conference, Barcelona, 2023, and the Dutch Network Science Society Symposium, Leiden, 2022, for the constructive comments.

In August 2020, hurricane Laura hit one of the world's largest petrochemical districts, in the U.S. states of Louisiana and Texas. As polymer producers in the area were forced to halt production, up to 15% of the country's PE and PP producers were unable to source polymer inputs, which in turn caused shortages across the economy (Vakil, 2021). Such a widespread disruption raised awareness on the role suppliers' correlation has in destabilising production networks and its importance in firms' sourcing decisions. Yet, the structure of the supply chain is often opaque: firms do not observe sourcing relations beyond their immediate suppliers (Williams et al., 2013). In face of this opacity, how do producers make sourcing decisions? And, should we expect these sourcing decisions to yield resilient production networks?

In this paper, I study the role of supply chain opacity in determining firms' sourcing decisions and, in turn, the consequences on the resilience of the production network. A widespread approach to mitigate risk is to diversify it by multisourcing. This practice consists of procuring the same inputs from multiple suppliers, sometimes redundantly (Zhao and Freeman, 2019). Yet, when deciding how many suppliers from which to source, a firm faces decreasing marginal benefits in risk reduction, because each additional supplier's failure to deliver is increasingly likely to be correlated with that of the existing ones. In the presence of marginal costs of sourcing, for example, contractual costs or higher prices, the uncertainty behind the correlation of a firm's potential suppliers might induce it to diversify risk less than socially optimal. The wedge between endogenous firm decisions and social optimality arises because downstream firms would be willing to compensate their suppliers for increased diversification of inputs. This underdiversification can generate aggregate fragility in production networks. To understand the relationship between the opacity of the supply chain, firms' diversification decisions, and production network fragility, I study the properties of a stylised production game. In the equilibrium of the game, unobserved correlation among suppliers generates fragility via two channels. First, it directly introduces endogenous correlation in downstream firms' risk, which amplifies through the production network. This increases the probability of cascading failures, in which the entire production network is unable to produce. Second, it indirectly affects firms' decisions by reducing the expected marginal gain from adding a source of input goods. The latter channel leads to firms diversifying increasingly less, such that small shocks in the production of basal goods can generate cascading failures downstream.

The role that production networks play in determining economic outcomes has been long recognised. As far back as Leontief (1936), economists have studied how networks in production can act as aggregators of firm level activity. Following a foundational paper by Hulten (1978), which showed that the first order impact of a productivity shock to an industry is independent of the production network structure, macroeconomics has since de-emphasised this role (Baqaee and Farhi, 2019, p. 2). However, more recently, Baqaee and Farhi (2019) illustrated how the structure of the production network can aggregate micro shocks via second order effects.[(1)] Furthermore, the degree of competition in an

[(1)]These results build on a vast literature and recent literature (Gabaix, 2011; Acemoglu et al., 2012;

industry also interacts with the production network to aggregate shocks, which can lead to cascading failures (Baqaee, 2018). Once established that production networks play a central role in aggregating shocks, two natural questions arise. First, which networks can we expect to observe, given that firm endogenously and strategically choose suppliers? Second, are these endogenous network formations responsible for the growth or fragility that large economies display? These questions fuelled a number of recent papers studying endogenous production network formation. Focusing on growth, Acemoglu and Azar (2020) show that endogenous production networks can be a channel through which firms' increased productivity lowers costs throughout the supply chain and allows for sustained economic growth. In parallel, a vast literature dealt with studying the role of endogenous production networks and firm incentives in determining fragile or resilient economies. Erol and Vohra (2014) showed that in networks with strategic link formation, systemic endogenous fragility arises if the shocks experienced by firms are correlated. Later work, by Amelkin and Vohra (2020), shows that uncertainty in the time of production is crucial in determining whether production networks in equilibrium are sparse, hence fragile. Finally, Elliott et al. (2022) illustrate how complexity in the production process can also be a key driver of endogenous fragility in production networks. [(2)]

In this paper, I aim to study endogenous fragility in the presence of supply chain unobservability. Kopytov et al. (2021) studied the effect of uncertainty in endogenous production network formation on firms' productivity and business cycles. They find that higher uncertainty can lead to lower economic growth. In contrast, in this paper I study the role of uncertainty in generating endogenous fragility to cascading failures using a more stylised production network model, akin to that studied by Elliott et al. (2022). In line with the existing literature, in the model small idiosyncratic shocks can be massively amplified. The degree of amplification depends on the equilibrium behaviour of firms. This phenomenon holds true in vertical economies producing simple goods. The novel theoretical contribution of this paper is to extend the analysis of production network formation to an opaque environment in which firms aim to minimise risk while accounting for correlation between suppliers. To do so, I develop a tractable analytical framework that describes the propagation of idiosyncratic shocks through the supply chain when firms take sourcing decisions endogenously in an imperfect information environment. The model describes the evolution of risk through the supply chain as a dynamical system over the depth of the production network. Finally, I show that the model can be seen as an extension of the one developed by Elliott et al. (2022) for large, but finite, supply chains.

Carvalho et al., 2016; Baqaee and Farhi, 2019; Carvalho and Tahbaz-Salehi, 2019)

[(2)] The literature on production networks is vast and it is unfortunately impossible to give a fair overview in this introduction. For a more comprehensive review of the literature I refer the reader to Carvalho and Tahbaz-Salehi (2019) and Amelkin and Vohra (2020)

1. Production, Risk, and Firm Problem

Consider a vertical economy producing $K+1$ goods, indexed by $k \in [K] := \{0, 1, \ldots K\}$. Each firm produces a single good and each good is produced by m firms. Good k requires only good $k-1$ as input. If a firm producing good k is unable to source its input good $k-1$, the firm is *disrupted* and hence unable to produce. I index firms producing good k by (k, i), with $i \in [m]$. The firm (k, i) is able to produce if at least one of its suppliers is able to deliver, namely, not all of its suppliers are disrupted. To avoid being disrupted, the firm chooses which firms to source from, among the producers of its input good. Letting $\mathcal{D}_k$ be the set of disrupted firms in layer k and $\mathcal{S}_{k,i}$ the set of suppliers of firm (k, i), we can say that $(k, i) \in \mathcal{D}_k$ if and only if all of its suppliers $(k-1, j) \in \mathcal{S}_{k,i}$ are in $\mathcal{D}_{k-1}$. I refer to the set of the firm's suppliers $\mathcal{S}_{k,i}$ as its sourcing strategy. The disruption events are random and the probability that a firm is disrupted can be written as

$$p_{k,i} := \mathbb{P}\big((k, i) \in \mathcal{D}_k\big) = \mathbb{P}\big(\mathcal{S}_{k,i} \subset \mathcal{D}_{k-1}\big). \tag{1}$$

If a firm is not disrupted, it obtains a profit $\Pi_{k,i}$ which is decreasing in the number of suppliers $|\mathcal{S}_{k,i}|$. Figure 1 illustrates how disruption propagate between two layers. The problem of the firm (k, i) is then to maximise the expected profit

$$\mathbb{E}_{\mathcal{D}_{k-1}}\Big[\Pi_{k,i}(\mathcal{S}_{k,i})\Big] = \Big(1 - \mathbb{P}\big(\mathcal{S}_{k,i} \subseteq \mathcal{D}_{k-1}\big)\Big)\ \Pi_{k,i}\big(\,|\mathcal{S}_{k,i}|\,\big) \tag{2}$$

by picking a set of suppliers from those producing its input good $\mathcal{S}_{k,i} \subseteq \{k-1\} \times [m]$. Before moving on with the model solution, it is useful to discuss the assumptions presented in this section. Clearly, the production game is highly stylised: first, firms do not adjust prices but only quantities, such that failure to produce only arises in the case that no input is sourced; second, they are able to obtain profits by simply producing; third, contracting with new suppliers has a cost. There are both theoretical and empirical reasons behind this choice. Theoretically, a simpler model allows us to isolate the interplay between the variables of interest: correlation in the risk of suppliers, supply chain opacity, and the endogenous production network fragility. Empirically, these assumption capture well the rationale behind firms' multisourcing. There is strong evidence that firms, first, when faced with supply chain shocks, adjust quantities rather than prices in the short run (di Giovanni and Levchenko, 2010; Macchiavello and Morjaria, 2015; Jiang et al., 2022; Lafrogne-Joussier et al., 2022), second, that production shutdowns can be extremely costly (Tan and Kramer, 1997; Hameed and Khan, 2014), and third, that fostering relationships with suppliers is important in guaranteeing operational performance, albeit costly (Cousins and Menguc, 2006). The model establishes a link between these issues faced by firms when choosing a sourcing strategy and the fragility of the production network.

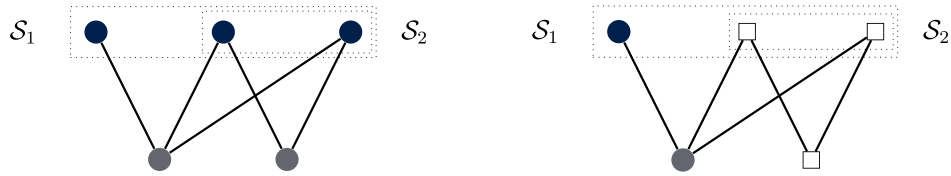

Figure 1: The supply chain is depicted in the left panel. The left firm is sourcing its input good from all three suppliers $\mathcal{S}_1$, while the right firm only from the latter two $\mathcal{S}_2$. As a disruption hits the suppliers (white box), the left firm is not disrupted as it can still source its input good.

To model supply chain uncertainty, I assume firms cannot observe the supplier decisions of the firms producing the necessary input good before making their supplier decision. Nevertheless, firms know the position they occupy within the supply chain k and the number of firms in each layer m. I show that, given this information, firms can derive the distribution of risk in each layer of the production network and make sourcing decisions based on it. Intuitively, if the supply chain is unobservable, ex-ante, all potential suppliers are symmetric from the firms' point of view, despite their risk being potentially correlated. Therefore, despite not knowing exactly the correlation of risk among producers of their input goods, firms have some information they can use to estimate the risk of a particular sourcing strategy. This reasoning can be formalised by making two assumptions (1 and 2). For convenience, we define a random variable

$$X_{k,j} \coloneqq \mathbb{1}\big\{(k,j) \in \mathcal{D}_k\big\} \tag{3}$$

which is 1 in case the firm (k,j) is disrupted and 0 otherwise, such that $p_{k,j} = \mathbb{E}\, X_{k,j}$.

Assumption 1. *Disruption events in the basal layer $X_{0,1}, X_{0,2}, \dots X_{0,m}$ are exchangeable, that is, their joint distribution is the same after relabelling the firms. Note that this implies that they are identically distributed, yet not necessarily independent.*

Assumption 2. *If there are multiple sourcing strategies $\mathcal{S}_1 \dots \mathcal{S}_2 \subseteq \{k-1\} \times [m]$ that yield the same expected profit $\mathbb{E}\, \Pi(\mathcal{S}_i)$, the firm breaks ties by picking any such set with equal probability.*

Lemma 1. *If the disruption events among upstream firms are exchangeable, then the probability that a downstream firm is disrupted depends only on the number of suppliers it picks.*

Lemma 1 is proven in Appendix B.1.2. Using assumption 1 and 2, we can extend the exchangeability property from the basal layer, given by assumption 1, to all downstream layers.

Proposition 1. *The disruption events in each layer k, $X_{k,1}, X_{k,2}, X_{k,3} \dots, X_{k,m}$ are exchangeable.*

Proof. The proof can be done by weak induction, using as base case assumption 1. Assume that for some layer $k-1$ the disruption events $X_{k-1,1}, X_{k-1,2} \dots, X_{k-1,m}$ are exchangeable. By lemma 1 we know that the probability of disruption achieved by making a sourcing decision $\mathcal{S} \subseteq \{k-1\} \times [m]$ only depends on the size of $\mathcal{S}$ and so does the expected profit $\mathbb{E}\ \Pi(\mathcal{S})$. By symmetry and assumption 2, all firms in layer k are then selecting a random subset of supplier from layer $k-1$ with equal probability. Hence, the distribution of downstream distributions

$$X_{k,1}, X_{k,2} \dots X_{k,m} \tag{4}$$

depends on the distribution of m random exchangeable and identically distributed subsets of the original exchangeable sequence $X_{k-1,1}, X_{k-1,2} \dots X_{k-1,m}$. This construction is independent of the order of the firms $1, 2 \dots m$. This implies that $X_{k,1} \dots, X_{k,m}$ is exchangeable. □

The proof of Proposition 1 illustrates the economic rationale for multisourcing in an opaque environment. By symmetry, from the point of view of the firm, all suppliers are ex-ante equal, yet their risk is correlated. Hence, the profit of the firm depends exclusively on *how many* suppliers it chooses, rather than *which* suppliers it chooses. A firm producing good k can first infer the number of disrupted firms among its potential suppliers $|\mathcal{D}_{k-1}|$ and then select the optimal number $s_{k,i} \coloneqq |\mathcal{S}_{k-1,i}|$ of firms from which to source its input good. Furthermore, by symmetry, two firms i and j in layer k might pick a different set of suppliers, that is, $\mathcal{S}_{k,i}$ is not necessarily equal to $\mathcal{S}_{k,j}$, but they do pick the same number of suppliers

$$s_{k,i} = s_k \text{ for all } i. \tag{5}$$

By Lemma 1, if two firms (k,i) and (k,j) pick two random sets of suppliers $\mathcal{S}_{k,i}$ and $\mathcal{S}_{k,j}$ with equal size $|\mathcal{S}_{k,i}| = |\mathcal{S}_{k,j}| = s_k$, the respective disruption probabilities of the two firms, $p_{k,i}$ and $p_{k,j}$, can be thought of as drawn from the same distribution.(3)

2. Disruptions Propagation

Proposition 1 characterises the limit distribution of the disruption events, as $m \to \infty$.(4) Let $P_m(s_k) = p_{k,i}$ be the probability that the firm is disrupted. This can be written as the probability that all sampled suppliers are disrupted

(3) The fact that an agent can navigate complex structures by random sampling and exploiting symmetries is not surprising. This approach has been used in various mathematical fields (Vershik, 2004; Austin, 2008), including the study of random graph limits (Diaconis and Janson, 2007). An extensive treatment is given in Kallenberg (2005).

(4) Technical details on how to adapt Assumptions 1 and 2, and obtain the result given in Proposition 1 for the case of $m \to \infty$ are treated in Appendix B.1.1.

$$P_m\left(s_k\right)=\overbrace{\frac{|\mathcal{D}_{k-1}|}{m}\frac{|\mathcal{D}_{k-1}|-1}{m-1}\cdots\frac{|\mathcal{D}_{k-1}|-(s_k-1)}{m-(s_k-1)}}^{s_k\text{ terms}}=\frac{(|\mathcal{D}_{k-1}|)^{\underline{s_k}}}{(m)^{\underline{s_k}}}, \tag{6}$$

where $(x)^{\underline{s_k}}=x(x-1)\ldots(x-s_k+1)$ denotes the falling factorial. The function P_m is identical for all firms in layer k, and depends on the layer size m both indirectly, via the support of $\mathcal{D}_{k-1}$, and directly. Since the number of disrupted firms among the producers of an input good $|\mathcal{D}_{k-1}|$ is a random variable, so is the probability of being disrupted $P_m(s_k)$. Letting

$$P\left(s_k\right):=\lim_{m\to\infty}P_m\left(s_k\right), \tag{7}$$

we can now make a claim regarding the distribution of the firm being disrupted $P(s_k)$ and its relationship with the ratio of disrupted producer of its input good

$$R_{k-1}:=\lim_{m\to\infty}\frac{|\mathcal{D}_{k-1}|}{m}. \tag{8}$$

Definition 1. *Let μ and ρ be the mean of overdispersion of a Beta distribution with shape parameters α and β. These are uniquely determined by*

$$\mu=\frac{\beta}{\alpha+\beta}\ \textit{and}\ \rho=\frac{1}{1+\alpha+\beta}. \tag{9}$$

Abusing notation, I write $Y\sim Beta(\mu,\rho)$.

Definition 2. *A random variable Y is said to follow a BetaPower distribution, with mean μ, overdispersion ρ, and power s if it can be written as $Y=X^s$ where X follows a Beta distribution with mean μ and overdispersion ρ.*

Proposition 2. *If the fraction of disrupted firms in the suppliers' layer $k-1$, R_{k-1}, follows a BetaPower distribution, then the probability of disruption of a firm producing good k, $P(s_k)$ follows a BetaPower distribution.*

Corollary 2.1. *In the large m limit, if for some layer or good k the fraction of disrupted firms R_k follows a BetaPower distribution, then, in all downstream layers $l>k$, the fraction of disrupted firms F_l follows a BetaPower distribution.*

Proposition 2 and Corollary 2.1 are proven in Appendix B.2. Corollary 2.1 asserts that the distribution of disrupted firms will remain in the same distribution family as risk amplifies through the production network. This result allows us to describe disruption propagation in the supply chain by mapping the evolution of μ_k and ρ_k through the layers. Furthermore, it allows firms to estimate μ_k and ρ_k and use this to determine the optimal sourcing strategy s_{k+1}. The only missing piece is the distribution of disrupted firms in the basal layer. As mentioned in the previous section, we assume that basal firms fail with a not necessarily independent probability $p_{0,j}$. We can model this by assuming that p_0 follows itself a Beta with mean μ_0 and overdispersion ρ_0.

Assumption 3. *The probability of a disruption in the basal layer follows*

$$p_{0,j} \sim Beta(\mu_0, \rho_0) \text{ for all } j. \tag{10}$$

Note that this assumption implies exchangeability of $X_{0,1}, X_{0,2}, X_{0,3} \ldots$ and is hence consistent with Assumption 1.

One can think of this assumption as modelling correlated shocks that might happen due to spacial or technological proximity of basal producers that cannot be diversified. Consider, for example, how oil extraction plants must be located in specific areas and are hence all subject to correlated weather shocks that can force to them shut down.

It is useful at this point to give an interpretation of μ_k and ρ_k, in the context of our model. The parameter μ_k is the fraction of firms that are expected to be disrupted. Hence, I hereafter refer to μ_k as *risk*. The parameter ρ_k tracks the degree of correlation in the disruption of firms operating in layer k. I illustrate this in Figure 2. Consider the distribution of the disruption probability among downstream firms p_{k+1} in the case the firm has a single supplier (solid lines) or two suppliers (dotted line). For low overdispersion, $\rho_k = 0.01$, the suppliers' disruptions are weakly correlated and the downstream disruption probability is concentrated around the risk μ_k. If firms contract an additional supplier, the distribution of failures decreases and remains concentrated around the risk. As ρ_k increases, the suppliers' disruption events become more correlated and the downstream disruption probabilities become fat-tailed, that is, a significant fraction of firms is likely to be disrupted. In this case, if firms contract an additional supplier risk decreases, but a large probability of severe disruptions remains.

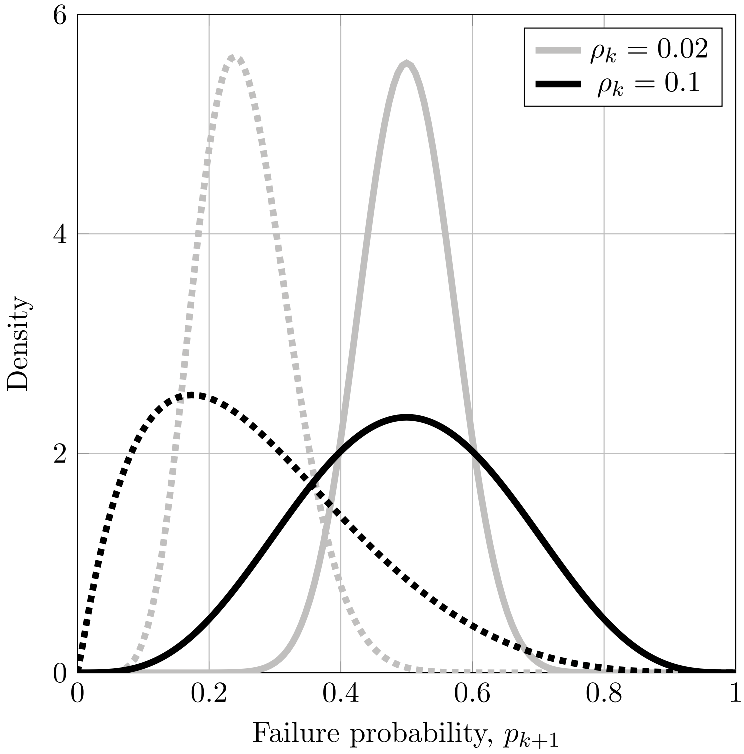

Figure 2: Distribution of disruption probabilities of downstream firms for different levels of upstream correlation ρ_k, in the cases of single sourcing (solid) and multisourcing (dotted).

Having established the link in disruption between suppliers and downstream firms, in this section I analyse how these propagate through the supply chain, while keeping the firms' sourcing decisions fixed. The goal is to recursively connect downstream distributions with upstream sourcing decisions and initial conditions, before endogenising the firm decisions. First, upstream firms' decisions, up to layer k, affect the disruption distribution in a layer $k+1$, via the product of sourcing strategies s_j for $j \leq k$.

Definition 3. *Let*

$$S_k = \prod_{j=1}^{k} s_j. \tag{11}$$

We call this quantity the upstream diversification of layer k.

Proposition 3. *Given a level of upstream diversification S_k, the risk between one layer k and the next $k+1$ depends on the sourcing strategy s_{k+1} via*

$$\mu_{k+1} = \eta(s_{k+1}, S_k)\ \mu_k, \tag{12}$$

where the risk reduction factor is given by

$$\eta(s_{k+1}, S_k) = \left(\frac{\mu_0 \frac{\rho_0}{1-\rho_0} + S_k}{\frac{\rho_0}{1-\rho_0} + S_k}\right)\left(\frac{\mu_0 \frac{\rho_0}{1-\rho_0} + S_k + 1}{\frac{\rho_0}{1-\rho_0} + S_k + 1}\right)\cdots\left(\frac{\mu_0 \frac{\rho_0}{1-\rho_0} + S_k s_{k+1} - 1}{\frac{\rho_0}{1-\rho_0} + S_k s_{k+1} - 1}\right), \tag{13}$$

and by convention $\eta(1, S_k) = 1$.

The risk reduction factor $\eta(s_{k+1}, S_k)$, derived in Appendix B.3, illustrates the interplay between the firms' choice s_{k+1}, the choices along the firms' production chain S_k, and the basal disruption distribution (μ_0, ρ_0). This interplay is illustrated in Figure 3. As correlation within the production network ρ_0 grows the expected risk reduction firms experience by multisourcing decrease, that is, more sources are necessary to obtain the same degree of risk reduction. As $\rho_0 \to 1$, diversification becomes impossible, namely $\mu_{k+1} \to \mu_k$ for any sourcing strategy s_{k+1}. Comparing the underdiversified and highly diversified cases illustrates a crucial positive externality upstream firms impose on downstream firms. The more diversified upstream firms are, the lower diversification is required downstream to obtain a given level of risk reduction. Yet, higher correlation levels reduce the magnitude of this positive externality.

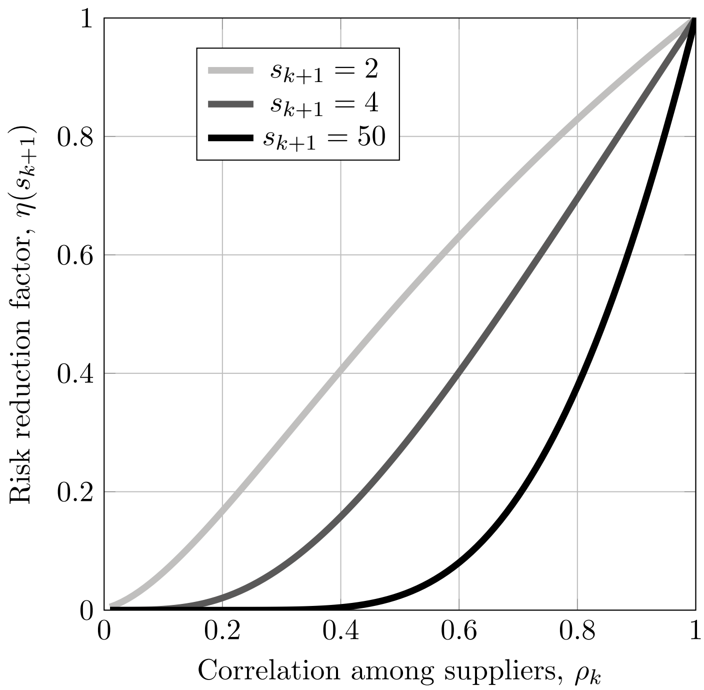

Figure 3: Risk reduction factor μ_{k+1}/μ_k at different basal overdispersion levels ρ_0 and for different sourcing strategies s_{k+1}.

3. Firm Optimal Diversification and Competitive Equilibrium

The mechanics of disruption propagation, derived in the previous section, determine the firms' risk diversification incentives and, as a consequence, their optimal sourcing strategy. This section derives such optimal strategies. I first analyse a limit case in which suppliers' risk is not correlated ($\rho_0 \to 0$) to obtain a benchmark case, before turning towards the general framework ($\rho_0 > 0$). For illustration purposes in the next section, I assume firms face quadratic costs from contracting suppliers, such that profits can be written as

$$\mathbb{E}\,\Pi_k(s_k) = \Big(1 - \mathbb{E}\big[p_k|s_k\big]\Big)\pi - \frac{c}{2}s_k^2, \tag{14}$$

for an exogenous profit π and a fixed marginal cost c. The optimisation problem of the firm is to then choose the optimal sourcing strategy

$$s_k = \arg\max_{\mathbb{N}_0} \mathbb{E}\,\Pi_k(s_k). \tag{15}$$

3.1. A Limit Case: Uncorrelated Disruptions

Proposition 4. *If risk among basal firms becomes uncorrelated $\rho_0 \to 0$, disruption events in layer k become independent and happen with probability*

$$\mu_k = \mu_{k-1}^{s_k} = (\mu_{k-2}^{s_{k-1}})^{s_k} = \ldots = \mu_0^{S_k}. \tag{16}$$

Proof. Follows immediately from $p_k \to \mu_k$ as $\rho_0 \to 0$. □

Without correlation risk, profits (14) are given by

$$\Pi_{k+1}(s) = \big(1 - \mu_k^s\big)\pi - \frac{c}{2}s^2. \tag{17}$$

Consider the marginal benefits of adding an additional source of input goods,

$$\Delta\Pi_{k+1}(s) \coloneqq \Pi_{k+1}(s+1) - \Pi_{k+1}(s) = \mu_k^s(1-\mu_k)\pi - c\left(s + \frac{1}{2}\right). \tag{18}$$

A firm with a given number of suppliers, contracts an extra supplier if doing so yields a positive expected marginal profit. Hence, the optimal number of suppliers s_{k+1} is the smallest s for which the expected marginal profit is negative $\Delta\Pi_{k+1}(s) < 0$ (see Appendix B.4).

Definition 4. *Let $\tilde{s}_{k+1}$ be the unique root of $\Delta\Pi_{k+1}(s)$ over $[-1/2, \infty)$. I refer to this quantity as the "diversification incentive".*

Proposition 5. *The optimal sourcing strategy is given by*

$$s_{k+1} = \lceil \tilde{s}_{k+1} \rceil. \tag{19}$$

Corollary 5.1. *Firms do not produce, $s_{k+1} = 0$, if the suppliers' risk μ_k is greater than the critical threshold μ^i,*

$$\mu_k > \mu^i := 1 - \frac{c/2}{\pi} \tag{20}$$

As expected, the diversification incentive $\tilde{s}_{k+1}$ and the optimal sourcing strategy s_{k+1} are implicitly determined by the suppliers' risk μ_k and the relative marginal cost of contracting a new supplier $\frac{c/2}{\pi}$. Figure 4 shows the optimal sourcing strategy s_{k+1} (dotted line) and $\tilde{s}_{k+1}$ (solid line), given a suppliers' level of risk μ_k, for three different marginal costs ratios $\frac{c/2}{\pi}$. First, as the marginal pairing costs increase, the firm's diversification incentives decrease. Second, higher levels of risk increase the diversification incentives of firms until a threshold is reached, after which, diversification incentives and, in turn, optimal diversification start decreasing. The concavity of the optimal sourcing strategy s_{k+1} in the suppliers' risk μ_k, highlights a vicious cycle which is introduced in production networks when firm decisions are endogenised: after a certain level of risk, firms have less incentives to diversify, which, in turn, "flattens" the marginal expected profits. This mechanisms can lead to endogenous fragilities in production networks.

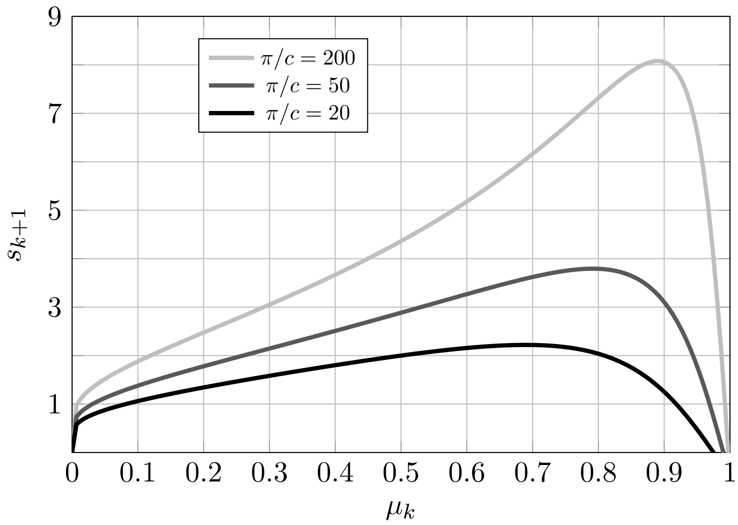

Figure 4: $\tilde{s}_{k+1}$ (solid) and s_{k+1} (dotted)

To illustrate the mechanism behind the endogenous fragility, we can combine the law of risk propagation (4) and the optimal firm sourcing (5) to obtain the risk dynamics in the production network,

$$\mu_{k+1} = \mu_k^{s_{k+1}(\mu_k)}. \tag{21}$$

The recursive relation (21) maps risk from suppliers to firms throughout the production network, hence studying its properties allows to describe how risk propagates through the supply chain. Two natural questions arise: first, which levels of risk remain stable through the supply chain, that is, they are neither amplified nor dampened as one moves from suppliers to firms? Second, how are initial levels of risk mapped to these stable risk levels? The former can be answered by determining levels of risk $\bar{\mu}$ which are fixed points of equation (21). The latter by looking at the basin of attraction of such points.

Proposition 6. *The stable levels of risk $\bar{\mu}$ are all points satisfying*

$$\frac{\bar{\mu}(1-\bar{\mu})}{3} \leq \frac{c/2}{\pi}. \tag{22}$$

Proposition 7. *For all basal risk levels μ_0 larger than the critical threshold*

$$\mu^c := \frac{1}{2} + \sqrt{\frac{1}{4} - 3\frac{c/2}{\pi}}, \tag{23}$$

the endogenous supply chain is unable to diversify risk, $\bar{\mu} \geq \mu_0$. We refer to this situation as "endogenous fragility".

Propositions 6 and 7, which are proven in Appendix B.4.2, link the firms' relative costs $\frac{c/2}{\pi}$ and the aggregate outcomes in term of production network risk. As relative marginal costs increase, the capacity of the production network to endogenously diversify basal risk μ_0 decreases and firms' under-diversification yields endogenous fragility. It is interesting to notice that, comparing the aggregate threshold μ^c with the firm shutdown threshold μ^i (Figure 5), for some levels of basal risk μ_0, despite no firm shutting down production $\mu_0 < \mu^i$, the production network as a whole is still unable to endogenously diversify risk $\mu_0 > \mu^c$. This is true even in this special case, where firms' risk is uncorrelated. In the next section, I introduce correlation risk $\rho > 0$ and investigate how doing so changes the dynamics illustrated here.

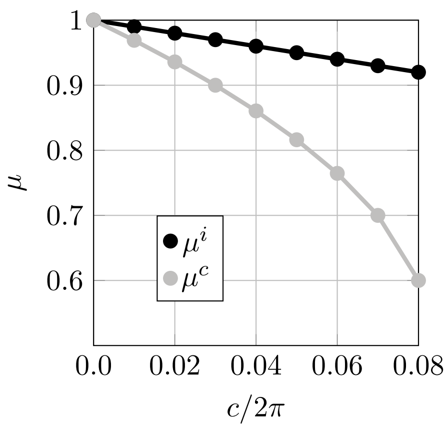

Figure 5

3.2. Optimal Sourcing with Correlated Distributions

If suppliers' risk is not independent, $\rho_0 > 0$, risk among suppliers throughout the production network is correlated, and firms' optimisation incentives change. The firm problem in layer $k+1$ is to choose the number of suppliers $s_{k+1} \in \mathbb{N}_{\geq 0}$ that maximises the profits Π given an upstream diversification S_k. The firm's expected disruption probability is given by the expected probability of disruption among its suppliers μ_k, scaled by a factor $\eta(s_{k+1}, S_k)$ (Proposition 3) which depends on the firm sourcing strategy. As in the limit case analysed in the previous section, the firm will increase diversification as long as the expected reduction in profits obtained by adding an additional supplier outweighs the costs of contracting that additional supplier. The expected marginal profits are given by

$$\Delta\Pi_{k+1}(s_{k+1}) = \Big(\eta(s_{k+1}, S_k) - \eta(s_{k+1}+1, S_k)\Big)\, \mu_k \pi - c\left(s_{k+1} + \frac{1}{2}\right). \tag{24}$$

The marginal profits are strictly decreasing in the number of suppliers (see Appendix B.5), hence the optimal sourcing strategy s_{k+1} is the smallest number of sources for which the marginal profits are negative (Proposition 5). For example, Figure 6 displays the surface of expected marginal profits $\Delta\Pi_{k+1}$ if the firm chooses a number of sources s_{k+1} given a upstream diversification $S_k = \prod_{j=1}^{k} s_j$. The curve $\Delta\Pi_{k+1} = 0$ is white hence the optimal sourcing will be given by the smallest optimal source s_{k+1} (marker) for which $\Delta\Pi_{k+1} < 0$. Furthermore, Figure 6 illustrates the role of the initial overdispersion ρ_0 on the expected marginal profits $\Delta\Pi_{k+1}$ and the optimal sourcing strategy s_{k+1}. As the risk of disruption among basal producers increases from $\rho_0 = 0$, the marginal expected profit curves flattens (from $\rho_0 = 0.01$ to 0.1 in 6), such that firms diversification incentives increase. Yet, as suppliers' risk becomes increasingly correlated, the diversification incentives start decreasing and the optimal number of sources decreases (from $\rho_0 = 0.5$ to 0.9 in 6).

Proposition 8. *The optimal sourcing strategy s_{k+1} is (weakly) concave in the level of initial overdispersion ρ_0.*

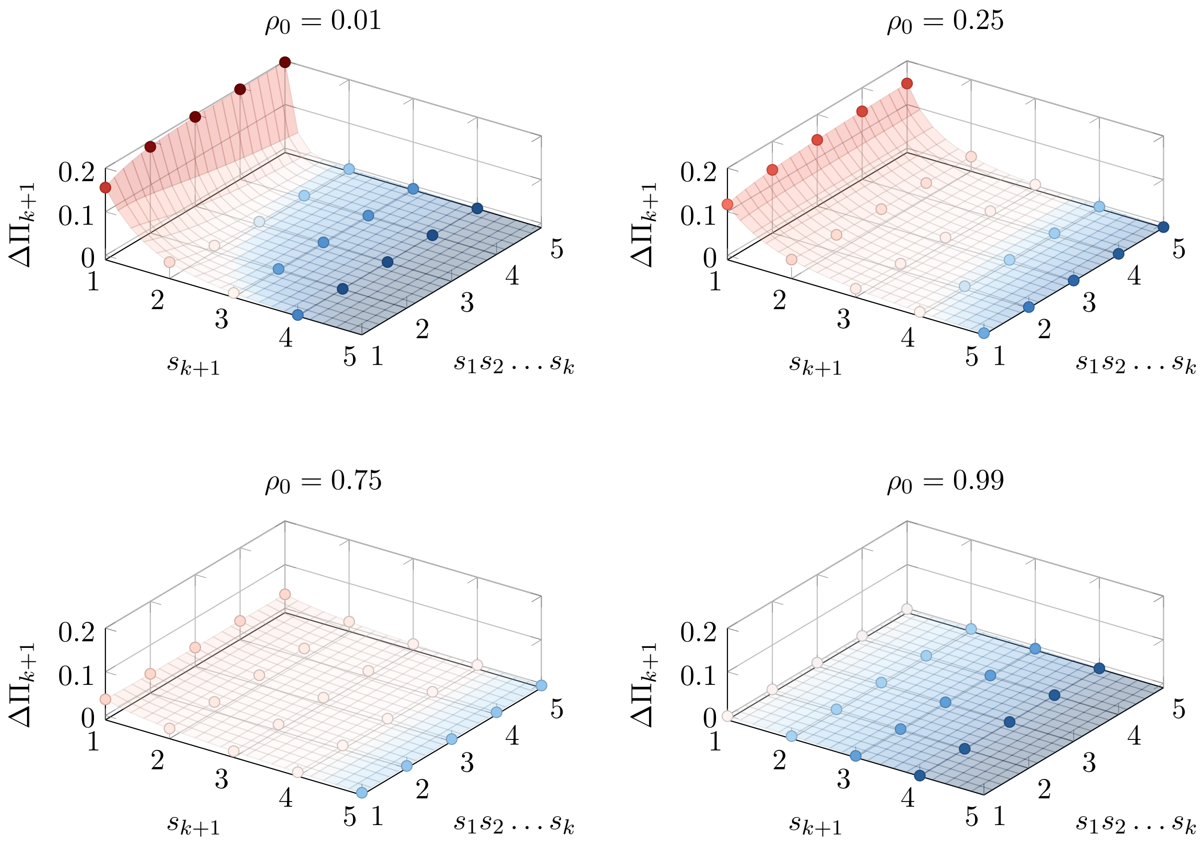

Figure 6: Surface of $\Delta\Pi_{k+1}$ with $\mu_k = 0.1$ and $c/2\pi = 0.1$, for different values of ρ_0. The surface is white if $\Delta\Pi_{k+1} = 0$.

The concavity of s_{k+1} with respect to the correlation of suppliers' risk (8) introduces another mechanism which misaligns the incentives of the firm with those of the production network. For low levels of correlations, firms can still seek diversification. Yet, as correlation increases the probability of simultaneous disruptions of a firm's existing suppliers and any potential new source becomes large enough that seeking new sources becomes no longer profitable. This novel incentive has strong ramifications in the formation and the fragility of endogenous production networks. To analyse these ramifications, consider the recursive relation of risk μ_k, when firms optimally source s_{k+1} (parallel to equation 21),

$$\mu_{k+1} = \eta(s_{k+1}, S_k)\ \mu_k. \tag{25}$$

Figure 7 illustrates the fraction of failing firms in the last layer μ_K with $K \gg 0$ given the basal risk μ_0 and overdispersion ρ_0 in a low (left) and a high (right) relative pairing cost regime. First, notice that in both cases if $\rho_0 \to 1$ there is no possible diversification since all firms are either disrupted or not, hence the disruption risk is constant across the production network $\mu_K = \mu_0$, regardless of the sourcing strategy. Likewise, if risk among basal producers becomes independent, $\rho_0 \to 0$, the problem converges to the limit case discussed in the previous section. Second, for a given level of basal correlation ρ_0, there is a critical threshold μ^c such that an arbitrary small increase in the initial risk level μ_0 has a discontinuous effect in the fraction of downstream firms μ_K. As discussed in the previous section, this discontinuity is induced on the production network by the firms' endogenous diversification incentives. As correlation increases, the maximum "diversifiable" level of basal risk, μ^c, becomes smaller. This lowering critical threshold suggests that, when allowed to form endogenously, production networks display a tendency towards endogenous fragility. Importantly, the model suggests that cascading failures can be triggered not only by increases in basal risk μ_0 but also by increases in basal correlation ρ_0, even as basal risk remains constant. This fragility, unlike the one studied by Elliott et al. (2022), is not induced by the structure of the production function, but rather arises endogenously due to the decreasing diversification motives of firms.

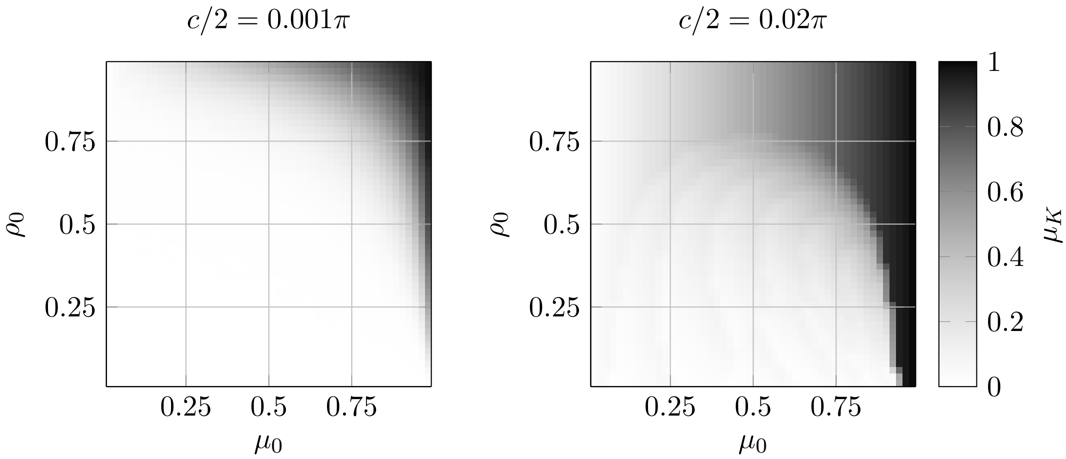

Figure 7: Surface of the downstream fraction of non functioning firms μ_K, given the initial condition μ_0 and ρ_0 in a low (left) and a high (right) relative pairing costs regime.

4. Social Planner Problem

To establish a benchmark to which one can compare the competitive equilibrium analysed in the previous section, I now solve the model from the perspective of a social planner. The social planner faces the trade-off of, on the one hand, minimising the number of firms expected to fail, and, on the other, minimising the number of sourcing relations. To develop a useful benchmark, it is important to define a social planner problem that can be meaningfully compared to the decentralised firm's problem. To do so, I make the following two assumptions.

Assumption 4. *The social planner knows the distribution of failure in the basal layer* $p_{0,j} \sim Beta(\mu_0, \rho_0)$ *and makes decision before* $p_{0,j}$ *is realised.*

Assumption 5. *As in the firm problem, we assume that, when determining the optimal suppliers of layer k, the number of potential suppliers $m \to \infty$. This allows the social planner to recursively, from the last layer K upwards, assign suppliers $\mathcal{S}_{k,i}$ such that no two firms share suppliers $\mathcal{S}_{k,i} \cap \mathcal{S}_{k,j} = \emptyset$.*

To understand the intuition behind Assumption 5, consider the possible supplier overlap illustrate in Figure 8: if a supplier has multiple downstream clients (dashed box), the social planner can always disrupt one link and firm that firm to source from a supplier without downstream clients (solid box). By doing so, the social planner can "diversify away" all the component of risk's correlation that arises due to the network structure. Hence, the only source of risk in the model is represented by the shutdowns experienced by firms in the basal layer, which happen with non-idiosyncratic probabilities $p_{0,i}$ (Assumption 4).

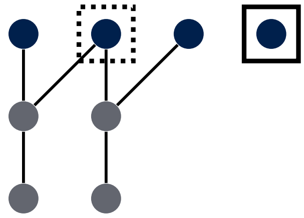

Figure 8: Disentangling overlapping paths

Combining Assumptions 4 and 5, the social planner problem is then to maximise average expected payoffs

$$W(\{\mathcal{S}_{k,i}\}) \coloneqq \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \frac{1}{K} \sum_{k=0}^{K} \Big(1 - \mathbb{P}\big(\mathcal{S}_{k,i} \subset \mathcal{D}_{k-1}\big)\Big)\, \pi - \frac{c}{2} \left|\mathcal{S}_{k,i}\right|^2, \tag{26}$$

by choosing a sequence of suppliers $\{S_{k,i}\} \subseteq \mathbb{N}_{\geq 0}$ for each firm in each layer such that $S_{k,i} \cap S_{k,j}$ is empty for all i, j. The social planner problem can be further simplified by noticing that, given that all firms in layer k are identical, if establishing an additional path from a firm in layer k to a basal firm has positive marginal benefits, then it has positive marginal benefits for all firms in layer k which share the same number of paths to basal firms. Hence, as in the decentralised firms' problem, the social planner can choose the optimal number of sources in each layer, let the firms source at random, and finally disentangle any overlapping paths. Hence, we can reformulate the social planner problem recursively, by letting V_k be the maximal average welfare from layer k to the last layer K. This can be recursively defined as

$$V_k(R_{k-1}) = \max_{s_k} \Big\{ \big(1 - \mathbb{E}\big[R_{k-1}^{s_k}\big]\big)\, \pi - \frac{c}{2} s_k^2 + \mathbb{E}\big[V_{k+1}(R_k)\big] \Big\} \tag{27}$$

where the state $R_{k-1} \sim \mathrm{BetaPower}(\mu_0, \rho_0, s_1 s_2 \ldots s_{k-1})$ is the fraction of disrupted firms, which evolves as

$$R_k = R_{k-1}^{s_k}. \tag{28}$$

The average welfare in layer $K+1$ is given by $V_{K+1}(R_K) = 0$, since firms in the last layer are never sources to other firms, and an initial state condition R_0, given by Assumption 3. This problem can be solved using standard backward induction techniques (see Appendix C). The optimum average social welfare (26) can then be written as

$$V_1(R_0) = \max_{s_1, s_2, \ldots s_{K-1}} W(\{s_1, s_2, \ldots s_{K-1}\}). \tag{29}$$

Letting $\{s_k^p\}_{k=1}^K$ be the socially optimal sourcing strategies sequence and $\{\mu_k^p\}_{k=1}^K$ be the expected disruption in each layer given by such sourcing strategies, we can compute the change in downstream risk compared to the decentralised case (as illustrated in Figure 7). Figure 9 shows this difference $\mu_K - \mu_K^p$ for the same two cost regimes. On the one hand, when relative pairing costs are low (left panel), the social planner achieves marginally lower risk levels of downstream risk for most initial conditions. If initial basal correlation ρ_0 is sufficienctly large and basal risk μ_0 is sufficienctly low, the firms overdiversify compared to the socially optimum $\mu_K < \mu_K^p$. On the other hand, if relative pairing costs are high (right panel), the differences between the social optimum and the competitive level of downstream risk are starker. First, for larger levels of basal risk μ_0 and lower levels of basal correlation ρ_0 the firms overdiversify compared to the social optimum, $\mu_K < \mu_K^p$. Second, the social planner is able to diversify risk around the critical threshold μ_c, $\mu_K > \mu_K^p$. This implies that the cascading failures that occur around the critical threshold are fully attributable to firms' endogenous underdiversification motives.

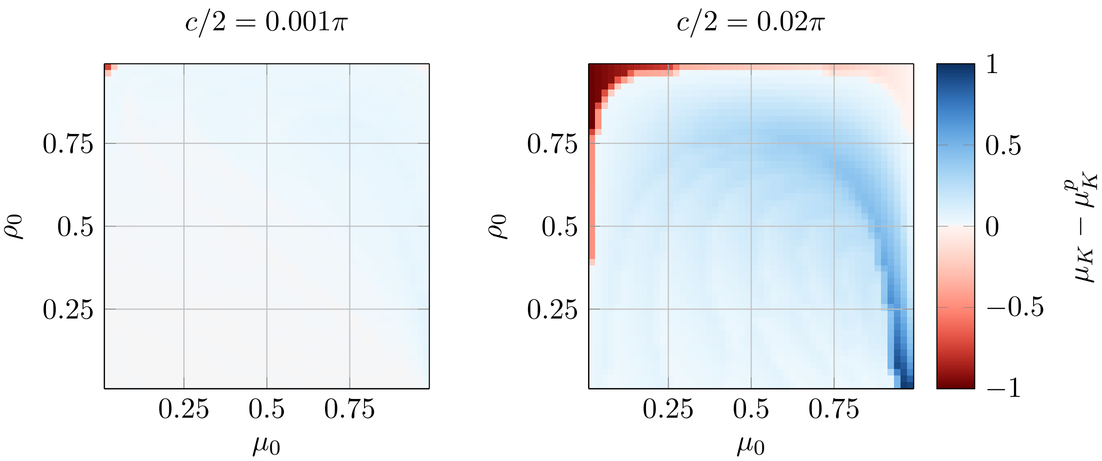

Figure 9: Change in downstream expected failure probability between the firms' μ_K and the social planner μ_K^p equilibrium, given different initial conditions μ_0 and ρ_0 in a low (left) and a high (right) relative pairing costs regime.

The differences between the firms' sourcing strategies and the social optimum have implications generate welfare losses in the production network. Letting W be the average firm profit in the decentralised case and W^p be the average profit achieved by the social planner, in Figure 10 I show the welfare loss due to the firms' diversification decisions $W - W^p$, in the high cost scenario ($\frac{c}{2} = 0.02\pi$). The welfare loss is largest around the critical value μ^c, where the production network suffer is endogenously fragile. At these levels of risk, firms' upstream firms' diversification incentives are weak, which creates large downstream resilience externalities.

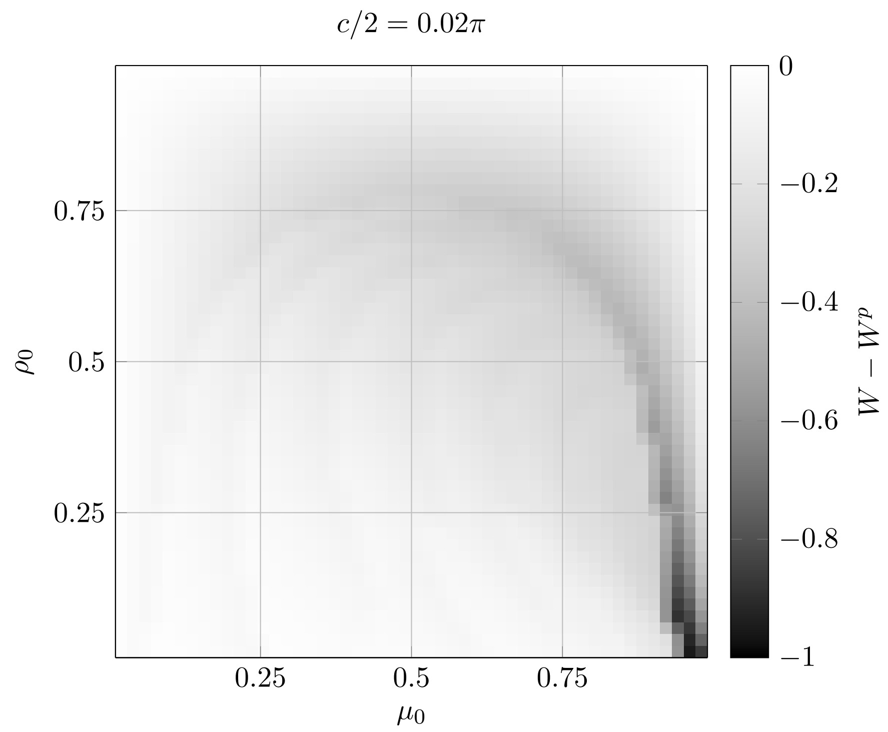

Figure 10: Welfare loss of the decentralised equilibrium $W - W^p$, given different initial conditions μ_0 and ρ_0 in a low (left) and a high (right) relative pairing costs regime.

5. Conclusion

Risk diversification is a crucial determinant of firms' sourcing strategies. In this paper, I show that firms endogenously underdiversify risk when they have incomplete information about upstream sourcing relations. This endogenous underdiversification generates fragile production networks, in which, arbitrarily small increases in risk upstream can generate discontinuously large disruptions downstream. I do so by deriving an analytical solution to a simple production game in which firms' sole objective is to minimise the risk of failing to source input goods. Despite its simple structure, the game identifies an important externality firms impose on the production network when making sourcing decisions: upstream multisourcing introduces correlation in firms' risk, which reduces incentives to multisource downstream. This externality exacerbates the risk of fragile production networks. Furthermore, I show that a social planner can not design production networks that mitigate fully this externality. A consequence of this result is that, in principle, it is possible to design a transfer mechanism that allows downstream firms to compensate upstream firms to internalise the diversification externalities. Finally, by analysing the welfare loss in competitive equilibrium, I show that there is a critical region of basal conditions where arbitrary small increases in suppliers' correlation, even if the expected number of disrupted firms remains constant, can generate catastrophic downstream disruptions by altering downstream firm diversification incentives. This result illustrates how an increase in correlation among basal producers, for example due to widespread offshoring to the same country, can endogenously generate fragile production networks which have a large tail risk of disruption.

As mentioned before, the approach presented here is just one of the many points of view that can be taken when studying endogenous production network and imperfect information. Simple production games can be helpful in isolating mechanisms, but it is often equally, if not more, valuable to embed such mechanisms into more comprehensive and complex models and study how they interact with each other. In this spirit, it would be of great interest to develop a general equilibrium model with endogenous production network with growth and risk diversification motives. In addition, the social planner solution presented here serve as a natural steppingstone for the study of insurance or taxation schemes to mitigate production network externalities. How can we make funds flow upstream in the supply chain to incentives or disincentives diversification? Can such a transfer mechanism be setup without knowledge of the production network structure? The result presented in this paper suggest that this is possible.

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A. Notation and Distributions

To study firms' decision in an opaque production network, I assume firms reason probabilistically about what is, at the core, a combinatorial problem. To talk about such a problem, it is useful to introduce some notation and distributions that play a central role in describing the reasoning and decisions of firms.

For a real number x and a positive integer n, I denote the rising factorial as

$$x^{\overline{n}} := \underbrace{x(x+1)(x+2)\dots(x+n-1)}_{n \text{ terms}}. \tag{30}$$

and the falling factorial

$$x^{\underline{n}} := \underbrace{x(x-1)(x-2)\dots(x-(n+1))}_{n \text{ terms}}. \tag{31}$$

Abusing notation, I often extend this function to non-integer exponents n by using the gamma function Γ, such that

$$x^{\overline{n}} := \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{32}$$

Two properties of the rising factorial that are used in the paper are the additive property of the exponent,

$$x^{\overline{n+m}} = x^{\overline{n}}(x+m)^{\overline{m}}, \tag{33}$$

and the fact that it is strictly increasing in its arguments,

$$\frac{\partial x^{\overline{n}}}{\partial x} = x^{\overline{n}}\big(\psi(x+n) - \psi(x)\big) > 0, \tag{34}$$

where ψ is the digamma function.

Two distributions that arise naturally in combinatorial problems and in the framework presented below are the beta and beta-binomial distributions. The former, which I denote $\text{Beta}(\alpha, \beta)$, has two positive parameters. In this paper, I use a more convenient parametrisation, namely, I write

$$\text{Beta}(\mu, \rho), \text{ where } \mu = \frac{\alpha}{\alpha+\beta} \text{ and } \rho = \frac{1}{1+\alpha+\beta}. \tag{35}$$

This distribution is used here to model uncertainty around the value of a probability distribution of a binomial process. Consequently, we can define a beta-binomial distribution as a compounded binomial distribution with a beta distributed probability parameter. Namely, by letting $X \sim \text{Bin}(m, p)$, where $m \in \mathbb{N}$ and $p \sim \text{Beta}(\mu, \rho)$, we can write $X \sim \text{BetaBin}(m, \mu, \rho)$. The random variable X takes values in $[m] := \{0, 1, \dots, m\}$.

Informally, one can think of X as a distribution with fatter tails, vis-à-vis a simple binomial, induced by the added uncertainty around the probability parameter p. For example, consider the limit case where $\mu = 1/2$ and $\rho = 1/3$. In this case, p is uniformly distributed on t . . . he interval $[0, 1]$. Then, X is uniformly distributed on $[m]$. On the other hand, if $\rho = 0$, then X follows a binomial distribution with $p = 1/2$. The first two moments of the beta-binomial distribution can be written as

$$\mathbb{E}[X] = m\mu \text{ and } \mathbb{V}\text{ar}[X] = m\mu(1-\mu)(1+(m-1)\rho). \tag{36}$$

The probability mass function of the beta-binomial distribution is

$$f_X(k) = \binom{m}{k} \frac{B(k+\alpha, m-k+\beta)}{B(\alpha, \beta)} \tag{37}$$

where B is the beta function

$$B(a+1, b+1) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^a (1-x)^b \mathrm{d}x. \tag{38}$$

The moment generating function of the beta-binomial is

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = {}_2F_1(-m, \alpha, \alpha+\beta; 1-e^t) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{B(\alpha+n, \beta)}{B(\alpha, \beta)} (1-e^t)^n, \tag{39}$$

where ${}_2F_1$ is the hypergeometric function defined as

$${}_2F_1(-m, a, b; z) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{(b)^{\overline{n}}}{(c)^{\overline{n}}} z^n. \tag{40}$$

B. Omitted Proofs

B.1. Exchangeability

In this section many of the proofs stem from Austin (2008).

B.1.1. Exchangeability in the large m case

Extending assumption 1 to the limiting case $m \to \infty$ is readily done by defining exchangeability for infinite sequences.

Definition 5. *For any set S, let sym(S) be the set of bijections from S to itself. A sequence of $\{0,1\}$-valued random variables $\{X_n\}_{n\in\mathbb{N}}$ is exchangeable if*

$$\{X_n\}_{n\in\mathbb{N}} \stackrel{d}{=} \{X_{\sigma(n)}\}_{n\in\mathbb{N}} \textit{ for all } \sigma \in \textit{sym}(\mathbb{N}). \tag{41}$$

An equivalent definition found in the literature is that $\{X_n\}_{n\in\mathbb{N}}$ is exchangeable if and only if all its finite subsequences are exchangeable.

A more complex task is that of extending assumption 2, since, fixing a number of suppliers $n > 0$, there are countably infinitely many exchangeable subsets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \dots$ of $\{X_n\}_{n\in\mathbb{N}}$ of size n and hence it is not possible to define a uniform measure over them. To circumvent this I now define a construction, known as paintbox partition, which allows us to partition the sequence $\{X_n\}_{n\in\mathbb{N}}$ into infinitely many exchangeable random partitions (Kingman, 1978). Then I will assign to each downstream firm i a deterministic subset of suppliers belonging to the i-th partition. In this way, each firm is assigned a random exchangeable subset of upstream firm, despite both the upstream and downstream number of firms being uncountable.

Paintbox partitions and Kingman's paintbox theorem

Definition 6. *Let $\{m_k\}_{k\in\mathbb{N}}$ be a non-negative, non-increasing sequence, such that*

$$\sum_{k=1}^{\infty} m_k \leq 1. \tag{42}$$

We call this mass-partition. Furthermore, denote by $\mathcal{M}$ the set of all such mass-partitions.

Lemma 2. *$\mathcal{M}$ is compact.*

Proof. First notice that $\mathcal{M} \subset [0,1]^{\mathbb{N}}$. By Tychonoff's theorem, $[0,1]^{\mathbb{N}}$ is compact. We just need to show that $\mathcal{M}$ is closed. By definition

$$\mathcal{M} = \left\{ \{m_k\}_{k\in\mathbb{N}} \;\middle|\; \sum_{k=1}^{\infty} m_k \leq m_{k+1} \leq 1 \text{ and } 0 \leq m_k \leq 1 \text{ for all } k \right\}. \tag{43}$$

Take an arbitrary limit point $\{\overline{m}_k\}_{k\in\mathbb{N}}$ in $\mathcal{M}$ and let $\{m_{k,n}\}_{k,n\in\mathbb{N}}$ be an n-sequence such that

$$\{m_{k,n}\}_{k,n\in\mathbb{N}} \to \{\overline{m}_k\}_{k\in\mathbb{N}} \text{ as } n \to \infty. \tag{44}$$

Since $\{m_{k,n}\}_{k,n\in\mathbb{N}} \in \mathcal{M}$, for all n and k we have

$$0 \leq m_{k,n} \leq m_{k+1,n} \leq 1. \tag{45}$$

Taking the n-limit on all sides yields

$$0 \leq \overline{m}_k \leq \overline{m}_{k+1} \leq 1. \tag{46}$$

Furthermore, we know the that the partial sums $\sum_{k=1}^{K} m_{k,n}$ converge for all n. Let

$$M_n = \sum_{k=1}^{\infty} m_{k,n} \leq 1. \tag{47}$$

By Tannery's theorem, given the convergence of the partial sums and the fact that $|m_{k,n}| \leq 1$ for all k, n, we can write

$$1 \geq \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} m_{k,n} = \sum_{k=1}^{\infty} \lim_{n\to\infty} m_{k,n} = \sum_{k=1}^{\infty} \overline{m}_k. \tag{48}$$

Hence $\{\overline{m}_k\}_{k\in\mathbb{N}} \in \mathcal{M}$. □

Given that $\mathcal{M}$ is compact we can construct a set of measures over $\mathcal{M}$.

Definition 7. *Let $Pr(\mathcal{M})$ be the set of measures over $\mathcal{M}$.*

Then the supplier are associated to the downstream firms as follows

1. Take an arbitrary measure $\nu \in Pr(\mathcal{M})$;
2. Sample a mass partition $\{m_k\}_k \sim \nu$;
3. Let $\mathcal{G}_i$ be the pool of suppliers from which downstream firm i can choose. Assign upstream firm j to each pool with the following probabilities

$$\mathbb{P}\big(j \in \mathcal{G}_i\big) = s_j \text{ and } \mathbb{P}\big(j \in \mathcal{G}_\infty\big) = 1 - \sum_j s_j, \tag{49}$$

   where $\mathcal{G}_\infty$ is the pool of unassigned firms;
4. If the downstream firm i wants k suppliers $\mathcal{S}_j$, pick the first k suppliers of its pool G_i.

By Kingman's Paintbox theorem all pools G_i are exchangeable random partitions of the suppliers with the same distribution. Such distribution is fully determined by ν. Furthermore, this implies that the set $\mathcal{S}_j$ is an exchangeable random subset of suppliers.

B.1.2. Proof of Lemma 1

Proof. Consider the sequence of upstream firm disruptions $X_1, X_2, X_3 \ldots$, where I omitted the layer label k for convenience. This sequence need not to be finite. We assume the sequence to be exchangeable

$$X_1, X_2, X_3 \ldots \overset{d}{=} X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)} \ldots, \tag{50}$$

for an arbitrary permutation of its indices σ. Now, pick two arbitrary finite subsets of firms $\mathcal{A} = \{X_{a_1}, X_{a_2} \ldots X_{a_k}\}$ and $\mathcal{B} = \{X_{b_1}, X_{b_2} \ldots X_{b_k}\}$ of size k. Here $a_i, b_i \in [m]$ are the indices of the original sequence corresponding to the i-th index of the subset. Let σ be a permutation that takes elements of $\mathcal{A}$ to $\mathcal{B}$, yet, keeps all other elements unpermuted, namely,

$$\sigma(\mathcal{A}) = \mathcal{B} \text{ and } \sigma(\mathcal{A}^c) = \mathcal{B}^c. \tag{51}$$

By exchangeability we have

$$\begin{aligned}\mathbb{P}(\mathcal{A}) &= \mathbb{P}(\mathcal{A}, \mathcal{A}^c \text{ taking any value}) \\ &= \mathbb{P}(\sigma(\mathcal{A}), \sigma(\mathcal{A}^c) \text{ taking any value}) \\ &= \mathbb{P}(\mathcal{B}, \mathcal{B}^c \text{ taking any value}) = \mathbb{P}(\mathcal{B}).\end{aligned} \tag{52}$$

□

B.2. Distribution of $\{R_k\}_{k=1}^K$ given a sequence of $\{s_k\}_{k=1}^K$

In this section many of the proofs build on those provided in Chapter 6 of Munford et al. (1978).

Throughout these proofs, I switch between the shape (α, β) and the mean-overdispersion (μ, ρ) parametrisations depending on which one leads to a simpler or clearer notation. We have a unique mapping between the two hence the change of variables is just notational.

First we characterise the density of a r.v. $P = X^s$ where $X \sim \text{Beta}(\alpha, \beta)$. As in Definition 2, we call this distribution BetaPower.

Lemma 3. *The density of P is*

$$h(p) = \frac{p^{\frac{\alpha}{s}-1}(1-p^{1/s})^{\beta-1}}{sB(\alpha,\beta)}. \tag{53}$$

Proof. The proof follows from letting $g(p) = p^s$, noticing that $g(X) = P$, and using the chain rule to show that

$$h(p) = f_X(g^{-1}(p)) \left| \frac{\mathrm{d}}{\mathrm{d}p} g^{-1}(p). \right| \tag{54}$$

□

I seek to prove that if R_{k-1} follows BetaPower distribution, then $P_m(s_k, D_{k-1})$ converges to a BetaPower distribution. Then the number of disrupted firms

$$D_k \mid P_m(s_k, D_{k-1}) = p \ \sim \text{Bin}(m, p), \tag{55}$$

First, I will prove to lemmas regarding the limit $P_m(s_k, D_{k-1})$

Lemma 4.

$$\lim_{m\to\infty} P_m(s_k, D_{k-1}) = \lim_{m\to\infty} \left(\frac{D_{k-1}}{m}\right)^{s_k} =: R^{s_k} \tag{56}$$

Proof. Abusing notation, we can redefine P_m to be a function of $R_m := D_{k-1}/m$, particularly,

$$P_m(s_k, R_m) = \frac{(m\ R_m)^{\underline{s_k}}}{(m)^{\underline{s_k}}}. \tag{57}$$

Expanding the falling factorial, we can write this as

$$P_m(s_k, R_m) = \frac{(mR_m)^{s_k}}{m^{s_k}} + O\left((R_m)^{s_k-1}\right), \tag{58}$$

which implies that

$$\lim_{m\to\infty} P_m(s_k, R_m) = R^{s_k} \text{ where } R = \lim_{m\to\infty} R_m. \tag{59}$$

□

Lemma 5. *If $R \sim BetaPower(\alpha, \beta, s_1)$, then $Q := R^{s_2}$ is a r.v. with*

$$Q \sim BetaPower(\alpha, \beta, s_1 + s_2) \tag{60}$$

Proof. By definition R can be expressed as $R = X^{s_1}$ where $X \sim$ Beta. Hence $Q = R^{s_1} = X^{s_1+s_2}$. □

Lemma 5 guarantees that, if we prove that $R \sim$ BetaPower, we are done since $R^{s_k} \sim$ BetaPower. To prove this, I show that the density of R converges uniformly to that of a BetaPower. A first intermediate result is due to Laplace and it is not proven (Lemma 6).

Lemma 6. *Let g be a twice continuously differentiable on the interval $[0, 1]$, on which it attains a unique maximum, x_0. Furthermore, let h be a positive function. Then*

$$\lim_{m\to\infty} \frac{\sqrt{\frac{2\pi}{m|g''(x_0)|}}h(x_0)e^{mg(x_0)}}{\int_0^1 h(x)e^{mg(x)}\,dx} = 1 \tag{61}$$

Definition 8. *For an integer m and $r = k/m$ for some integer k, using the definition of the Beta function (38), we write*

$$\binom{m}{mr} = \left((m+1)\int_0^1 p^{mr}(1-p)^{m(1-r)}\,dp\right)^{-1}. \tag{62}$$

Now, using Lemma 3, 4, 5, and 6, we can prove the main result.

Proposition 9. *If $D \sim BetaBinPower(m, \alpha, \beta, s)$ distribution, then*

$$R := \lim_{m\to\infty} D/m \sim BetaPower(\alpha, \beta, s). \tag{63}$$

Proof. First we can define the probability mass function of R_m

$$f_m(x) = \frac{\int_0^1 h(p)\ p^{mx}(1-p)^{m(1-x)}\mathrm{d}p}{(m+1)\int_0^1 y^{mx}(1-y)^{m(1-x)}\mathrm{d}y} \tag{64}$$

where I rewrote the binomial coefficient as 8 and where h is the density of the BetaPower distribution (3). Consider an $r \in [0,1]$. Define $\delta_\varepsilon(r)$ to be all the numbers in the support of f_m with distance at most ε from r,

$$\delta_\varepsilon(r) = \left\{x \in \mathrm{supp}(f_m) : |x-r| < \varepsilon\right\} \tag{65}$$

.

Then we can define the probability density function of R as

$$\begin{aligned} f(r) &= \lim_{\varepsilon\to 0}\lim_{m\to\infty}\frac{\sum_{x\in\delta_\varepsilon(r)} f_m(x)}{2\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\lim_{m\to\infty}\sum_{x\in\delta_\varepsilon(r)}\frac{\int_0^1 h(p)\ p^{mx}(1-p)^{m(1-x)}\mathrm{d}p}{(m+1)\int_0^1 y^{mx}(1-y)^{m(1-x)}\mathrm{d}y} \end{aligned} \tag{66}$$

I will tackle this limit by using the Laplace's method (6). First, we can rewrite the numerator

$$\int_0^1 h(p)\ p^{mx}(1-p)^{m(1-x)}\mathrm{d}p = \int_0^1 h(p)e^{m(x\log p+(1-x)\log(1-p))}\mathrm{d}p, \tag{67}$$

s and notice that the function $p \mapsto x\log p + (1-x)\log(1-p)$ is uniquely maximised at x and has second derivative $-1/x(1-x)$ at the optimum. Proposition (6) implies that the the term in (67) has the same behaviour as

$$h(x)\ x^{mx}(1-x)^{m(1-x)}\sqrt{\frac{2\pi}{m}x(1-x)} \text{ as } m\to\infty. \tag{68}$$

A similar exercise allows us to establish that

$$\lim_{m\to\infty}\frac{\int_0^1 y^{mx}(1-y)^{m(1-x)}\mathrm{d}y}{x^{mx}(1-x)^{m(1-x)}\sqrt{\frac{2\pi}{m}x(1-x)}} = 1. \tag{69}$$

Now we can relate the two limits by picking $\varepsilon = 1/\sqrt{m+1}$ such that we can rewrite the probability density function of R (66) as

$$f(r) = \lim_{m\to\infty}\frac{\sqrt{m+1}}{m+1}\sum_{x\in\delta_{\sqrt{m+1}}(r)} h(x) = h(r). \tag{70}$$

$\square$

The steps of the proof that allow us to link the distribution of D_{k-1} to that of D_k are shown in Figure (11).

$$\begin{tikzcd}
\boxed{D_{k-1} \sim \text{BetaBinPower}} \arrow[r, "\text{Proposition 9}"] & \boxed{R_m = D_{k-1}/m \to R \sim \text{BetaPower}} \arrow[d, "\text{Lemmas 3, 4, 5}"] \\
\boxed{D_k \sim \text{BetaBinPower}} \arrow[u, "\text{Step } k \to k+1"] & \boxed{p_k \to R^{s_k} \sim \text{BetaPower}} \arrow[l, "\text{Definition}"']
\end{tikzcd}$$

Figure 11

B.3. Implications of the Disruption Distribution, D_k

Now that we have shown that the mapping from upstream firms $D_{k-1} \to D_k$ preserves the BetaBinPower distribution we are left with computing its parameters of D_k. This can be done by induction and by making an assumption on D_0.

Lemma 7. *If $D_{k-1} \sim BetaBinPower(m, \alpha, \beta, S)$ for some integer S, then*

$$D_k \sim BetaBinPower(m, \alpha, \beta, Ss_k) \tag{71}$$

where s_k is the choice of suppliers in layer k.

Proof. Follows from the definition of BetaBinPower and Lemmas 3, 4, 5. $\square$

Proposition 10. *The number of disrupted firms in layer k follows*

$$D_k \sim BetaBinPower\left(m, \mu_0, \rho_0, s_k \prod_{l=0}^{k-1} s_l\right). \tag{72}$$

Equivalently, the probability of disruption in layer k is given by

$$p_k \sim BetaPower\left(\mu_0, \rho_0, s_k \prod_{l=0}^{k-1} s_l\right). \tag{73}$$

Proof. Follows by induction combining Lemma 7 and Assumption 3. $\square$

Finally, this allows us to fully determine the problem facing a firm in layer k.

Proposition 11. *Let $S_{k-1} = \prod_{l=0}^{k-1} s_l$ be the compounded supplier relationships up to layer $k-1$. Then the expected probability of disruption faced by a firm is given by*

$$\mathbb{E}\Big[p_k(s_k; S_{k-1})\Big] = \frac{B\left(\mu_0 \, \frac{1-\rho_0}{\rho_0} + S_{k-1}s_k, (1-\mu_0) \, \frac{1-\rho_0}{\rho_0}\right)}{B\left(\mu_0 \, \frac{1-\rho_0}{\rho_0}, (1-\mu_0) \, \frac{1-\rho_0}{\rho_0}\right)}. \tag{74}$$

Proof. It follow from rewriting the moment generating function of the beta distribution as

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{B(\alpha+n, \beta)}{B(\alpha, \beta)}. \tag{75}$$

and noticing that $p_k \sim \text{BetaPower}(\alpha, \beta, S_{k-1}s_k)$. $\square$

To simplify notation, let $r_0 = \frac{1-\rho_0}{\rho_0}$ and

$$\eta(s, S) = \frac{(\mu_0 r_0 + S)^{\overline{S(s-1)}}}{(r_0 + S)^{\overline{S(s-1)}}} \tag{76}$$

which satisfies the recursion

$$\eta(s+1, S) = \eta(s, S) \frac{(\mu_0 r_0 + As)^{\overline{A}}}{(r_0 + As)^{\overline{A}}} \tag{77}$$

Using Proposition (11), the coefficient η allows us to write the propagation of risk recursively

$$\mu_{k+1} = \eta(s_{k+1}, S_k) \; \mu_k. \tag{78}$$

B.4. Limit case $\rho_0 \to 0$

B.4.1. Optimal sourcing strategy without correlation

This is the proof of Proposition 5 in the limit case $\rho_0 \to 0$.

Proof. First, notice that, for $\pi, c > 0$ and $0 < \mu_k < 1$, the expected marginal benefit $\Delta\Pi_{k+1} : \mathbb{R} \to \mathbb{R}$ is strictly decreasing as

$$\frac{\partial}{\partial s} \Delta\Pi_{k+1}(s) = \log(\mu_k) \; (1-\mu_k)\mu_k^s \pi - c < 0. \tag{79}$$

Furthermore, $\Delta\Pi_{k+1}$ is continuous with

$$\Delta\Pi_{k+1}(0) = (1-\mu_k)\pi + \frac{c}{2} > 0 \text{ and} \tag{80}$$

$$\lim_{s\to\infty} \Delta\Pi_{k+1}(s) = -\infty < 0. \tag{81}$$

Hence, $\tilde{s}_{k+1}$ exists and is unique. Since $\tilde{s}_{k+1}$ is the unique root and $\Delta\Pi_{k+1}$ is strictly decreasing the integer value $s_{k+1} = \lceil \tilde{s}_{k+1} \rceil$ is the smallest integer such that the marginal benefits are negative $\Delta\Pi_{k+1}(s) < 0$. □

B.4.2. Properties of the Law of Motion

These are the proofs of propositions 6 and 7.

Definition 9. *Let $\tilde{g}(\mu_k) := \mu_k^{\tilde{s}_{k+1}(\mu_k)}$, such that $g(\mu_k) = \tilde{g}(\mu_k)R(\mu_k)$ where $R(\mu_k) \in [0, 1)$. Using the definition of $\tilde{s}_{k+1}$, we can write*

$$\tilde{g}(\mu) = \frac{c/2}{\pi}\left(\frac{2\tilde{s}_{k+1}(\mu) - 1}{1 - \mu}\right). \tag{82}$$

Lemma 8.

$$\tilde{g}(\mu) \geq \mu \iff g(\mu) = \mu$$

Proof.

$$\tilde{g}(\mu) \geq \mu \iff \tilde{s}_{k+1}(\mu) \in (0, 1) \iff s_{k+1}(\mu) = 1 \iff g(\mu) = \mu. \tag{83}$$

□

Lemma 9.

$$g(\mu) \geq \mu.$$

B.5. General Case, $\rho_0 > 0$

This is the proof of Proposition 5 in the more general case, $\rho_0 > 0$.

Proof. It is sufficient to show that $\Delta\Pi_{k+1}$ is strictly decreasing in s_{k+1} when $\rho_0 > 0$. First, notice that

$$\frac{\partial}{\partial s}\Delta\Pi_{k+1}(s) = -\mu_k\pi\frac{\partial}{\partial s}\Delta\eta'(s) - c \tag{84}$$

where $\Delta\eta(s) = \eta(s + 1, S_k) - \eta(s, S_k)$. Using the recursive definition of η (77), we can write

$$\begin{aligned} \Delta\eta(s) &= \eta(s)\left(\frac{(\mu_0 r_0 + s_{k+1}S_k)^{\overline{S_k}}}{(r_0 + s_{k+1}S_k)^{\overline{S_k}}} - 1\right) \\ \Delta\eta'(s) &= \eta'(s)\left(\frac{(\mu_0 r_0 + s_{k+1}S_k)^{\overline{S_k}}}{(r_0 + s_{k+1}S_k)^{\overline{S_k}}} - 1\right) + \eta(s)\frac{\partial}{\partial s}\left(\frac{(\mu_0 r_0 + s_{k+1}S_k)^{\overline{S_k}}}{(r_0 + s_{k+1}S_k)^{\overline{S_k}}}\right) > 0 \end{aligned} \tag{85}$$

□

C. Solution of the Social Planner Problem

First, notice that the terminal condition V_K is linear in R_{K-1}, hence $\mathbb{E}V_K(R_{K-1}) = V_K(\mathbb{E}R_{K-1})$. In turn, this implies that V_k is linear for all k. Hence we can rewrite the value to be a function of the state space S,

$$V_k(S_{k-1}) = \max_s \left\{ \Big(1 - \mathbb{E}\big[\text{BetaPower}(\mu_0, \rho_0, S_{k-1}\ s)\big]\Big)\pi - \frac{c}{2}s^2 + V_k(S_{k-1}\ s_k) \right\}. \quad (86)$$

We can find V_k numerically. Let $\Omega = [m] \times [m^K]$ for some $m \in \mathbb{N}$ and

$$l(s, S) \coloneqq \Big(1 - \mathbb{E}\big[\text{BetaPower}(\mu_0, \rho_0, S_{k-1}\ s)\big]\Big)\pi - \frac{c}{2}s^2. \quad (87)$$

Then we can recursively compute

$$V_K(S) = \max_s l(\Omega), \quad (88)$$

$$V_{k-1}(S) = \max_s l(\Omega) + V_K(S\ s), \quad (89)$$

$$\vdots \quad (90)$$

$$V_1(S) = \max_s l(\Omega) + V_2(S\ s). \quad (91)$$