

Endogenous Fragility of Supply Chains and Correlated Disruption Risk *

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In August 2020, Hurricane Laura hit one of the world’s largest petrochemical districts, in the U.S. states of Louisiana and Texas. As polymer producers in the area were forced to halt production, up to 15% of the country’s polypropene producers were unable to source polymer inputs, which in turn caused shortages across the economy (Vakil, 2021). This is just one example of agglomeration of economic activity increasing the correlation between disruptions among suppliers of crucial goods for the economy. In face of such correlated risk, how do downstream producers make sourcing decisions? And, do these decisions yield supply chains resilient to such correlated disruptions?

In this paper, I study the feedback between the risk of a disruption in sourcing inputs and the endogenous formation of supply chains. A widespread approach to mitigate risk is to diversify it by multisourcing. This practice consists of procuring the same inputs from multiple suppliers, sometimes redundantly (Zhao and Freeman, 2019). Yet, when deciding how many suppliers to source from, a firm faces decreasing marginal benefits in risk reduction, because each additional supplier’s failure to deliver is increasingly likely to be correlated with that of the firm’s current suppliers. In the presence of marginal costs of sourcing, such as contractual costs or higher prices, the uncertainty behind the correlation of a firm’s potential suppliers might induce it to diversify risk less than socially optimal. The wedge between endogenous firm decisions and social optimality arises because downstream firms would be willing to compensate their suppliers for increased diversification of inputs. This under-diversification can generate aggregate fragility in production networks. To understand the relationship between the firm’s diversification decisions and supply chain fragility, I study the properties of a stylised production model. In equilibrium, correlation in the risk of disruption among suppliers generates fragility via two channels. First, it directly introduces endogenous correlation in downstream firms’ risk, which amplifies through the production network. This increases the probability of cascading failures, in which the entire production network is unable to produce. Second, it indirectly affects firms’ decisions

by reducing the expected marginal gain from adding a source of input goods. The latter channel leads to firms diversifying increasingly less, such that small increases in the expected disruption probability can yield fragile production networks.

The role that production networks play in determining economic outcomes has been long recognised. As far back as [Leontief \(1936\)](#), economists have studied how networks in production can act as aggregators of firm-level activity. Following a foundational paper by [Hulten \(1978\)](#), which proved, under strict assumptions, that the first order impact of a productivity shock to an industry is independent of the production network structure, macroeconomics has since de-emphasised this role. [Baqae and Farhi \(2019\)](#), relaxing these assumptions, illustrated how the structure of the production network can aggregate micro shocks via second order effects.¹ Furthermore, the degree of competition in an industry also interacts with the production network to aggregate shocks, which can lead to cascading failures ([Baqae, 2018](#)). Once established that production networks play a central role in aggregating shocks, two natural questions arise. First, which networks can we expect to observe, given that firms endogenously and strategically choose suppliers? Second, are these endogenous network formations responsible for the growth or fragility that large economies display? These questions fuelled a number of recent papers studying endogenous production network formation. Focusing on growth, [Acemoglu and Azar \(2020\)](#) show that endogenous production networks can be a channel through which firms' increased productivity lowers costs throughout the supply chain and allows for sustained economic growth. In parallel, a vast literature dealt with studying the role of endogenous production networks and firm incentives in determining fragile or resilient economies. [Erol and Vohra \(2014\)](#) showed that in networks with strategic link formation, systemic endogenous fragility arises if the shocks experienced by firms are correlated. Later work, by [Amelkin and Vohra \(2020\)](#), shows that uncertainty in the time of production is crucial in de-

¹These results build on a vast literature, for example [Acemoglu et al. \(2012\)](#); [Baqae and Farhi \(2019\)](#); [Carvalho and Tahbaz-Salehi \(2019\)](#); [Carvalho et al. \(2020\)](#); [Gabaix \(2011\)](#)

termining whether production networks in equilibrium are sparse and, hence fragile. Finally, [Elliott, Golub and Leduc \(2022\)](#) illustrate how complexity in the production process can also be a key driver of endogenous fragility in production networks.²

A less understood link is that between the correlation of risk within the supply chain, how firms deal with it, and the consequences this has on the economy. [Kopytov et al. \(2021\)](#) studied the effect of uncertainty in endogenous production network formation on firms' productivity and business cycles. They find that higher uncertainty can lead to lower economic growth. In contrast, this paper focuses on the role of uncertainty in generating endogenous fragility to cascading failures using a more stylised production network model. In line with the existing literature ([Elliott, Golub and Leduc, 2022](#)), in the model, small idiosyncratic shocks can be massively amplified. The degree of amplification depends on the equilibrium behaviour of firms. This phenomenon holds true in vertical economies producing simple goods. This paper extends the analysis of production network formation to an environment in which firms aim to minimise risk while accounting for correlation between suppliers. To do so, I develop a tractable analytical framework that describes the propagation of idiosyncratic shocks through the supply chain when firms make sourcing decisions endogenously in an imperfect information environment. The model describes the evolution of risk through the supply chain as a dynamical system over its depth. The social planner solution shows that endogenous fragility can impose large welfare losses. Importantly, these losses might be discontinuous: an arbitrarily small increase in the correlation of risk among basal firms can generate large welfare losses. Finally, I study a benchmark case where firms have perfect information over idiosyncratic risk. In this case, despite each individual firm being able to achieve a smaller disruption risk, the supply chain is maximally fragile and there is a high probability of large disruptions.

²The literature on production networks is vast and it is unfortunately impossible to give a fair overview in this introduction. For a more comprehensive review of the literature, I refer the reader to [Carvalho and Tahbaz-Salehi \(2019\)](#) and [Amelkin and Vohra \(2020\)](#)

The remainder of the paper is structured as follows. Section 1 discusses the assumptions on the supply chain disruptions and the problem of the firm, and establishes the results that allow to analyse firm sourcing decisions. Section 2 derives the law of propagation of the disruption events through the supply chain. Section 3 establishes the firm's optimal sourcing strategy and how this endogenously determines the fragility of the supply chain. These results are then compared, in Section 4, to the social planner solution to determine the welfare losses induced by the firm's endogenous decisions. Finally, in Section 5, the role of imperfect information is isolated by solving the model under perfect information.

1 Model

1.1 Production Technology and the Firm Objective

Consider an economy producing $K + 1$ goods, indexed by $k \in \{0, 1, \dots, K\}$. Each firm produces a single good and each good is produced by infinitely many firms indexed by $i \in \mathbb{N}$. Production of the basal good $k = 0$ does not require any input and is at risk of random exogenous disruptions in the production process. A disrupted basal firm is unable to deliver its good as input to downstream producers. The economy is vertical as each downstream good $k > 0$ requires only good $k - 1$ as input. If a firm producing good k is unable to source its input good $k - 1$, the firm is itself disrupted and hence unable to deliver downstream. In other words, the i -th firm producing good k , indexed by (k, i) , is able to produce if at least one of its suppliers is able to deliver, that is, if not all of its suppliers are disrupted. To avoid being disrupted, the firm chooses which firms to source from, among the producers of its input good. Formally, letting \mathcal{D}_k be the random set of disrupted firms in layer k and $\mathcal{S}_{k,i}$ the set of suppliers of firm (k, i) , we can say that $(k, i) \in \mathcal{D}_k$ if and only if all of its suppliers $(k - 1, j) \in \mathcal{S}_{k,i}$ are in \mathcal{D}_{k-1} . I refer to the set of the firm's suppliers $\mathcal{S}_{k,i}$ as its sourcing strategy. The disruption events are random and the probability that a firm is disrupted

can be written as

$$p_{k,i} := \mathbb{P}((k,i) \in \mathcal{D}_k) = \mathbb{P}(\mathcal{S}_{k,i} \subset \mathcal{D}_{k-1}). \quad (1)$$

Figure 1 illustrates this mechanism.

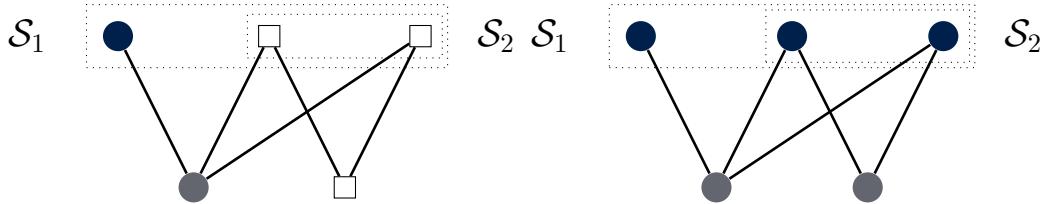


Figure 1: The supply chain is depicted in the left panel. The left firm is sourcing its input good from all three suppliers, \mathcal{S}_1 , while the right firm only from the latter two, \mathcal{S}_2 . As a disruption occurs, some upstream firms are unable to supply the input good (white box). Unlike the left firm, the right firm is unable to source its inputs and is hence disrupted.

If a firm is not disrupted, it obtains an exogenous profit π . Implementing a given sourcing strategy costs the firm $C(|\mathcal{S}_{k,i}|)$. The cost C is assumed to be non-negative, increasing in the number of suppliers $|\mathcal{S}_{k,i}|$, and with $C(0) = 0$ and $C(s) \rightarrow \infty$ as $s \rightarrow \infty$. The problem of firm (k, i) is then to maximise the expected profit³

$$\Pi_{k,i}(\mathcal{S}_{k,i}) = \left(1 - \mathbb{P}(\mathcal{S}_{k,i} \subseteq \mathcal{D}_{k-1})\right) \pi - C(|\mathcal{S}_{k,i}|) \quad (2)$$

by picking a sourcing strategy $\mathcal{S}_{k,i}$. Before moving to the solution of the model, it is useful to discuss the assumptions presented in this section. The production game is highly stylised: first, firms do not adjust prices but only quantities, such that failure to produce only arises in case no input is sourced; second, they are able to obtain profits by simply producing; third, contracting with new suppliers has a cost; and lastly, there are infinitely many firms in each layer. There are both theoretical and empirical reasons behind these choices. Theoretically, a simpler model allows us to isolate the interplay between the variables of interest: correlation in the risk of suppliers, supply chain opacity, and endogenous production network fragility. Empirically, these assumptions capture well the

³The expectation is taken over the random set \mathcal{D}_{k-1} .

rationale behind firms' multisourcing. There is strong evidence that firms, first, when faced with supply chain shocks, adjust quantities rather than prices in the short run (di Giovanni and Levchenko, 2010; Jiang, Rigobon and Rigobon, 2022; Lafrogne-Joussier, Martin and Mejean, 2022; Macchiavello and Morjaria, 2015), second, that production shutdowns can have significant costs (Barrot and Sauvagnat, 2016; Hameed and Khan, 2014), and third, that fostering relationships with suppliers is costly, but important in guaranteeing operational performance (Cousins and Menguc, 2006). Finally, modelling infinitely many firms yields tractable analytical results, which can then be mapped back to the finite case by considering a finite subset of firms. Furthermore, this is a good approximation of real supply chains, which are increasingly complex and deep (Elliott, Golub and Leduc, 2022). The model establishes a link between these issues faced by firms when choosing a sourcing strategy and the fragility of the production network.

1.2 Imperfect Information and Ex-Ante Symmetry

The supply chain is opaque: firms cannot observe the sourcing decisions of their potential suppliers before making their own. Furthermore, firms do neither know how risky individual basal producers are, nor how their risk is correlated. Yet, firms know the distribution from which the probabilities of disruption in the basal layer are drawn. To motivate this assumption, recall the introductory example of Hurricane Laura. A downstream firm producing polypropene, might not be able to trace back the production steps from its input to individual polymer producers in Louisiana or Texas, and, hence, the exact exposure of its production process to hurricanes. Yet, the firm can estimate the aggregate risk the polymer industry faces in the region. Given this information about the basal layer and their own depth k in the production network, firms can derive the distribution of risk among their suppliers and make sourcing decisions based on it. The risk of two firms downstream sourcing from the same number of suppliers is ex-ante identical, albeit possibly correlated.

I now formalise this idea in the context of the model. Let

$$X_{k,j} := \begin{cases} 1 & \text{if firm } (k,j) \text{ is disrupted and} \\ & \text{otherwise.} \\ 0 & \end{cases} \quad (3)$$

To encode the uncertainty in the probability $p_{0,j}$ that a basal firm experiences a disruption event $X_{0,j} = 1$, I assume that these probabilities $\{p_{0,j}\}_j$ are independent samples of a random variable P_0 , which has a known distribution. I assume this distribution to be the $\text{Beta}(\mu, \rho)$. The Beta distribution allows to flexibly model shocks that might happen due to the spacial or technological proximity of basal producers, which cannot be diversified. Consider, for example, how oil extraction plants must be located near oil reserves and are hence all subject to correlated weather shocks that might force them to shut down. In this case, despite the small expected probability that an individual firm is disrupted, as a hurricane is a rare occurrence, disruptions are highly correlated, as all firms are vulnerable to the same hurricane.

Definition 1. A $\text{Beta}(\mu, \rho)$ distribution, with mean $\mu \in [0, 1]$ and correlation $\rho \in [0, 1]$, is a continuous probability distribution over the unit interval with cumulative density function, if $\rho \in (0, 1)$,

$$F(p; \mu, \rho) = \begin{cases} 0 & \text{if } p < 0, \\ \frac{1}{C(\mu, \rho)} \int_0^p u^{(1-\mu)(\frac{1-\rho}{\rho})-1} (1-u)^{\mu(\frac{1-\rho}{\rho})-1} du & \text{if } 0 \leq p \leq 1 \\ 1 & \text{if } p > 1, \end{cases} \quad (4)$$

where $C(\mu, \rho) := \int_0^1 u^{(1-\mu)(\frac{1-\rho}{\rho})-1} (1-u)^{\mu(\frac{1-\rho}{\rho})-1} du$, if $\rho = 0$,

$$F(p; \mu, 0) = \begin{cases} 0 & \text{if } p < \mu, \\ 1 & \text{if } p \geq \mu, \end{cases} \quad (5)$$

and, if $\rho = 1$,

$$F(p; \mu, 1) = \begin{cases} 0 & \text{if } p < 0, \\ 1 - \mu & \text{if } 0 \leq p < 1, \\ 1 & \text{if } p \geq 1. \end{cases} \quad (6)$$

Assumption 1. The probability $p_{0,j}$ that basal firm j experiences a disruption $X_{0,j} = 1$ is sampled independently, for all j , from a Beta-distributed random variable with mean μ_0 and correlation ρ_0 . That is,

$$P_0 \sim \text{Beta}(\mu_0, \rho_0) \text{ and} \quad (7)$$

$$(X_{0,j} \mid P_0 = p_{0,j}) \stackrel{i.i.d.}{\sim} \text{Bernoulli}(p_{0,j}). \quad (8)$$

One can think of this process in terms of a simple analogy. Each basal firm samples independently a coin, with bias probability of head $p_{0,j}$, from the same collection P_0 of coins. If the outcome of the coin is head, the firm is disrupted. Under this assumption, the disruption events $\{X_{0,j}\}_j$ experienced by basal firms are identically distributed but not independent, as their probabilities $p_{0,j}$ are sampled from the same collection P_0 of coins. The parameter μ_0 is the average fraction of disrupted firms in the basal layer, that is,

$$\mathbb{E}[X_{0,j}] = \mu_0. \quad (9)$$

For this reason, I hereafter refer to it as basal risk. The parameter ρ_0 is the degree of correlation in disruption events among basal producers, that is,

$$\text{Corr}[X_{0,j}, X_{0,i}] = \rho_0 \text{ for all } i, j. \quad (10)$$

If $\rho_0 = 0$, the disruption events are independent. If $\rho_0 = 1$, the disruption events are perfectly correlated. Hence, hereafter I refer to it as basal correlation. Figure 2 illustrates the effect of the basal correlation ρ_0 on the basal disruption events. Consider an economy with a basal risk $\mu_0 = 0.3$. In case of a low basal correlation $\rho_0 = 0.02$ (lighter line), the disruption probability of basal firms is

concentrated around μ_0 . If ρ_0 increases to 0.1 (darker line), the distribution of disruption probabilities becomes more spread. While basal firms are on average as risky as before, the degree of correlation between their shocks is larger. In

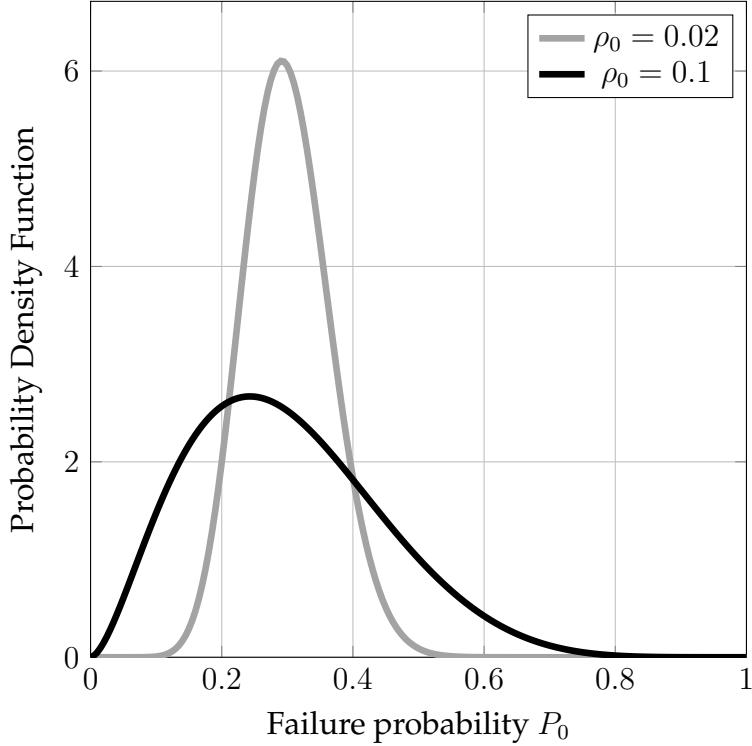


Figure 2: Distribution of disruption probabilities P_0 of basal firms for different levels of basal correlation ρ_0 . In both cases $\mu_0 = 0.3$.

the limit as $\rho_0 = 1$, all firms will either be disrupted, with probability μ_0 , or not disrupted, with probability $1 - \mu_0$, that is, $X_{0,j} = X_{0,i}$ for all firms i and j .

Using this framework, the following assumption formalises the supply chain opacity.

Assumption 2. *Downstream firms (k, j) for $k \geq 1$ observe the distribution of the failure probability P_0 , but do not observe the distribution of disruptions $\{X_{k,j}\}_j$.*

Under this assumption, all basal firms are ex-ante symmetric from the point of view of downstream producers. Going back to the example of polymer producers, downstream firms using plastic understand how hurricane risk can impact polymer production, which is an input to their production, as they observe P_0 . Yet, they cannot estimate the risk that individual polymer producers

$(0, j)$ face, since they do not observe $p_{0,j}$. The fact that basal producers are ex-ante symmetric is a crucial feature of the model: downstream producers must make sourcing decisions in a regime in which they cannot evaluate the risk of a disruption to specific suppliers. This notion of ex-ante symmetry, can be formalised using the concept of exchangeability.

Definition 2. *A sequence $\{X_i\}_i$ of random variables is exchangeable if its distribution is independent of the order of the sequence. That is, given any permutation σ ,*

$$X_1, X_2, X_3 \dots \sim X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)} \dots \quad (11)$$

Proposition 1. *The basal disruption events $\{X_{0,j}\}_j$ are exchangeable.*

The proof is provided in Appendix A.1. Due to the exchangeability result, a downstream producer i in layer $k = 1$ must choose its set of suppliers $\mathcal{S}_{1,i}$ without knowing the disruption risk of individual basal producers. However, by the exchangeability result, the firm's risk depends only on the number of suppliers $|\mathcal{S}_{1,i}|$ it selects, not on their specific identities. Therefore, I now impose an assumption on how firms select upstream producers when they are indifferent among suppliers and only the number of sources matters. Since downstream firms are indifferent between upstream suppliers, the selection procedure should be as "unbiased" as possible, meaning the upstream index (k, j) should have minimal influence on whether a firm is chosen as a supplier. A fully unbiased selection is not possible, as there is no uniform distribution over countably infinite upstream producers. Thus, I assume the selection procedure depends on an uniformness parameter $\theta > 0$. As $\theta \rightarrow 0$, firms are more likely to select suppliers with lower indices (k, j) . As $\theta \rightarrow \infty$, the bias in supplier selection becomes arbitrarily small. The following formalises this idea.

Assumption 3. *If firm $(k + 1, i)$ chooses to supply its input good k from $s_{k+1,i} < \infty$ sources and is indifferent among them, it picks suppliers $\mathcal{S}_{k+1,i}$ as follows.*

For an arbitrary uniformness $\theta > 0$, a sequence of random variables

$$u_j \stackrel{i.i.d.}{\sim} \text{Beta}\left(\frac{1}{1+1/\theta}, \frac{1}{2+\theta}\right), \quad (12)$$

is sampled. Then, each upstream firm (k, i) is assigned a weight

$$\alpha_{k,1} = u_1 \text{ and } \alpha_{k,j} = u_j \prod_{i < j} (1 - u_i). \quad (13)$$

These weights define a random probability mass function as

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(1 - \sum_{j=1}^n \alpha_{k,1} < \delta \right) = 0, \quad (14)$$

for any $\delta > 0$.

Each downstream firm $(k + 1, i)$ constructs its set of sources $\mathcal{S}_{k+1,i}$ by sampling without replacement from \mathbb{N} according to the random probability mass function $\alpha_{k,i}$.

This construction, illustrated in Figure 3, is based on the stick-breaking process introduced in Ferguson (1973) and commonly used in constructing Dirichlet processes (Kallenberg, 2005). This assumption allows firm to "break ties" whenever multiple possible sourcing strategies yield the same payoff. To do so, it imposes a random preferential attachment weight u_j to each upstream producer (k, j) . Crucially, this weight is random and does not depend on the downstream producer. That is, conditioning on the weights $\{u_j\}_j$, changing the index of a downstream firm $(k + 1, i)$ does not change its sampling process. The parameter θ controls the degree to which the weight u_j is biased towards lower indices j . As $\theta \rightarrow 0$, more weight is placed on lower indices, hence all downstream firms are more likely to source from lower indices. As $\theta \rightarrow \infty$, more weight is placed on higher indices and firms are decreasingly likely to share suppliers. Under Assumption 3, the exchangeability can be extended to all downstream layers of the supply chain.

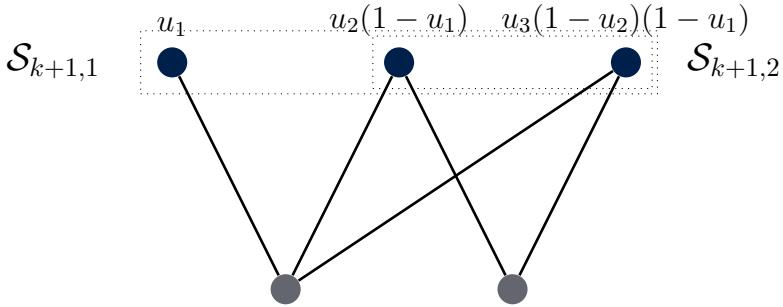


Figure 3: Illustration of the procedure to compute indifferent matching, Assumption 3

Proposition 2. *In each downstream layer $k \geq 1$, the disruption events*

$$X_{k,1}, X_{k,2}, X_{k,3}, \dots \quad (15)$$

are exchangeable.

The proof is provided in Appendix A.2. Intuitively, in each layer, the sampling procedure is common for all downstream firms $(k + 1, i)$, it is hence independent of the firm index i . This independence of the index i immediately implies that a permutation of the indices does not affect the downstream distribution of disruptions. Proposition 2 extends the exchangeability from the basal to all downstream layers. Given the exchangeability in each layer, from the point of view of a downstream firm, all suppliers are ex-ante symmetric. This implies that the profit of the firm depends exclusively on how many suppliers it chooses, rather than which suppliers it chooses. Then, a firm producing good k can first infer the distribution of the number $D_{k-1} := |\mathcal{D}_{k-1}|$ of disrupted firms among its potential suppliers and then choose the optimal number $s_{k,i} := |\mathcal{S}_{k-1,i}|$ of firms from which to source its input good. Furthermore, by symmetry, all firms in layer k choose the same number s_k of sources, that is,

$$s_{k,i} = s_k \text{ for all } i. \quad (16)$$

As a result, the sourcing strategies $\mathcal{S}_{k,i}$ and $\mathcal{S}_{k,j}$ of any two firms i and j are such that their disruption probabilities $p_{k,i}$ and $p_{k,j}$ are identically distributed.⁴

⁴This approach is widely used in the study of random graphs, see, for example, Diaconis

Before proceeding, it is worth discussing to what extent the result of Proposition 2 extends to a finite case, and, hence, this model represents a good approximation of a finite supply chain. Diaconis and Janson (2007) have shown that, if the finite length n sequence of disruptions $X_{k,1}, X_{k,2}, \dots, X_{k,n}$ is exchangeable, the subsequence $X_{k,1}, X_{k,2}, \dots, X_{k,s_k}$ has variation distance at most $4s_k/n$ to the closest mixture of i.i.d. random variables. In each layer k of the production network this error accumulates, such that the total accumulated variation is at most

$$4 \sum_{k=1}^K s_k/n. \quad (17)$$

Hence, the model represents a good approximation whenever the depth of the supply chain K and the number of suppliers chosen by firms s_k is small compared to the total number of firms n . Hence, as long as the supply chain is flat and sparse, relative to the total number of firms, the model yields a good finite approximation. In the real world there is ample evidence that production networks are sparse (Acemoglu et al., 2012). For example, Dhyne et al. (2021) found the fraction of sourcing connections to firms to be around 1/151.

Proposition 2 allows us to characterise the firms' sourcing decisions, the formation of the production network, and the propagation of the disruptions in terms of a representative firm for each layer k .

2 Disruption Propagation

Using the framework introduced in the previous section, I now turn to characterising the propagation of the disruptions across layers. I consider the limit case in which $\theta \rightarrow \infty$, that is, vanishingly less weight is given towards lower indexed suppliers when selected by downstream firms using the procedure of Assumption 3. First, I will show that, in each layer k , the disruption events $\{X_{k,i}\}_i$ are i.i.d. Bernoulli trials conditional on a random variable $P_k = P(X_{i,k} = 1)$. Recall, that the $X_{i,k}$ are constructed via a random probability mass function,

and Janson (2007); Kallenberg (2005).

hence the probability of a disruption P_k is itself a random variable. This can be thought of as the random variable from which firms in layer k sample their disruption probabilities. It suffices to track how P_k evolves across the layers $k \in \{0, 1, 2 \dots K\}$ to study the propagation of the disruption. Furthermore, if $P_0 \sim \text{Beta}(\mu_0, \rho_0)$, as in Assumption 1, then $P_k \sim \text{BetaPower}(\mu_0, \rho_0, S_k)$ for all k , that is, the family of these distributions is closed as disruptions propagate through the layers. In the following, I introduce the BetaPower distribution and prove this result-

Definition 3. A random variable P_k follows a $\text{BetaPower}(\mu, \rho, S_k)$ distribution, with risk $\mu \in [0, 1]$, correlation $\rho \in [0, 1]$, and power $S_k \in \mathbb{N}$, if it can be written as

$$P_k = P_0^{S_k}, \quad (18)$$

where P_0 follows a $\text{Beta}(\mu, \rho)$. The probability density function is given by

$$f(p; \mu, \rho, S_k) \propto p^{\frac{(1-\mu)(1-\rho)}{S_k \rho} - 1} (1 - p^{1/S_k})^{\frac{\mu(1-\rho)}{\rho} - 1}. \quad (19)$$

Proposition 3. If the upstream disruption probability $P_k \sim \text{BetaPower}(\mu_0, \rho_0, S_k)$ and downstream firms select upstream suppliers using the procedure of Assumption 3 with $\theta \rightarrow \infty$, the downstream disruption probability is given by $P_{k+1} = P_k^{s_{k+1}}$, such that,

$$P_{k+1} \sim \text{BetaPower}(\mu_0, \rho_0, s_{k+1} S_k). \quad (20)$$

The proof is provided in Appendix A.3. Proposition 3 guarantees that the distribution of disrupted firms remains in the same distribution family. This result allows us to describe disruptions propagation in the supply chain by mapping the evolution of the distribution P_k through the layers. Furthermore, the distribution P_k is fully determined by the risk μ_0 and correlation ρ_0 in the basal layer and the sourcing decisions $\{s_k\}_k$ the representative firms makes through the supply chain.

Consider again the analogy of basal producers sampling biased coin from a bag with distribution P_0 . From the point of view of a downstream firm in

layer $k = 1$, before the coin is sampled from the bag by the basal producers, all basal producers are ex-ante symmetric. The downstream firm experiences a disruption if the outcome of the sampled coin is head for all of its suppliers. Hence, in choosing s_1 suppliers, the downstream firm is tying its disruption to s_1 coin tosses, drawn from the basal bag with distribution P_0 . To illustrate how the sourcing decision of the representative downstream firm $k = 1$, influences the probability of a downstream disruption, consider

$$P_1 = P_0^{s_1}. \quad (21)$$

Figure 4 shows the probability density function of P_1 in case the firm has a single supplier $s_1 = 1$ (dotted lines) or two suppliers $s_1 = 2$ (solid line). The basal disruption probability P_0 follows the same distribution as in Figure 2. If downstream producers supply from a single source (dashed line), the disruption probability downstream follows the same distribution as that of upstream producers, $P_1 = P_0$. To diversify risk, firms can choose to contract an additional supplier (solid line). Regardless of the basal correlation ρ_0 , constructing an additional supplier reduces the downstream risk $\mu_1 = \mathbb{E}[P_1]$. If the upstream correlation is low, $\rho_0 = 0.02$ the distribution of P_1 remains concentrated around the mean. If the upstream correlation is large, $\rho_0 = 0.1$, the suppliers' disruption events become more correlated and the downstream disruption probabilities become fat-tailed, that is, a significant fraction of firms is likely to be disrupted and, as a consequence, diversification is less effective. Hence, a downstream firm can mitigate risk by contracting an additional supplier, but a large disruption probability remains.

Having established the link between upstream and downstream probabilities of disruptions

$$P_{k+1} = P_k^s, \quad (22)$$

I now turn to the analysis of how risk $\mu_k = \mathbb{E}[P_k]$ propagates through the supply chain, before studying how firms make decisions endogenously.

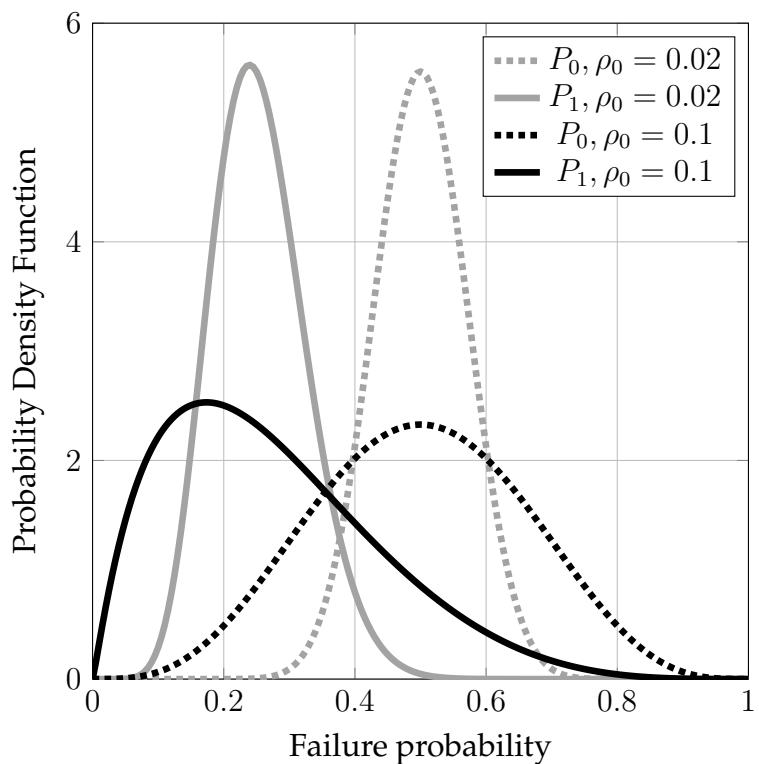


Figure 4: Distribution of disruption probabilities of downstream firms for different levels of upstream correlation ρ_0 , in the cases of single sourcing (dotted) and multisourcing (solid). In both cases $\mu_k = 0.3$.

Proposition 4. *The average disruption probability between one layer k and the next $k+1$ depends on the sourcing strategy s_{k+1} via the map*

$$\mu_{k+1} = \begin{cases} \eta(s_{k+1}, S_k) \mu_k & \text{if } s_{k+1} > 0, \\ 1 & \text{if } s_k = 0, \end{cases} \quad (23)$$

where $S_k := \prod_{j=1}^k s_j$ is the diversification level up to layer k and η is the "risk reduction factor", which is given by

$$\begin{aligned} \eta(s_{k+1}, S_k) &= \left(\frac{\mu_0 \frac{\rho_0}{1-\rho_0} + S_k}{\frac{\rho_0}{1-\rho_0} + S_k} \right) \left(\frac{\mu_0 \frac{\rho_0}{1-\rho_0} + S_k + 1}{\frac{\rho_0}{1-\rho_0} + S_k + 1} \right) \\ &\quad \times \cdots \times \left(\frac{\mu_0 \frac{\rho_0}{1-\rho_0} + S_k s_{k+1} - 1}{\frac{\rho_0}{1-\rho_0} + S_k s_{k+1} - 1} \right). \end{aligned} \quad (24)$$

The risk reduction factor $\eta(s_{k+1}, S_k)$, derived in Appendix A.4, governs how the firm's choice s_{k+1} , the choices along the firm's production chain S_k , and the basal conditions μ_0, ρ_0 affect the expected number of disruptions downstream. This interplay is illustrated in the following figures. Figure 5 shows how the risk reduction factor varies with basal correlation ρ_0 for different sourcing strategies s_k , fixing the upstream diversification of $S_k = 2$. If correlation ρ_0 in the basal layer grows, to obtain a given level of risk reduction η , firms producing good k need to source more suppliers. If $\rho_0 = 1$, diversification becomes impossible, as $\eta \rightarrow 1$ and $\mu_{k+1} \rightarrow \mu_k$ for any sourcing strategy s_{k+1} . As the above, Figure 6 shows the response of the risk reduction factors to different levels of basal correlation, but instead of varying the strategy s_k of the firm, it varies the level of upstream diversification S_k . For low levels of basal correlation ρ_0 , more upstream diversification S_k allows downstream producers to achieve lower risk with fewer suppliers. Yet, there is a level of basal correlation after which more diversification is detrimental for the downstream firm, as this high upstream diversification simply exacerbates tail risk. This represents a crucial externality the upstream suppliers impose on downstream producers. For a low level of correlation, sourcing downstream represents a positive ex-

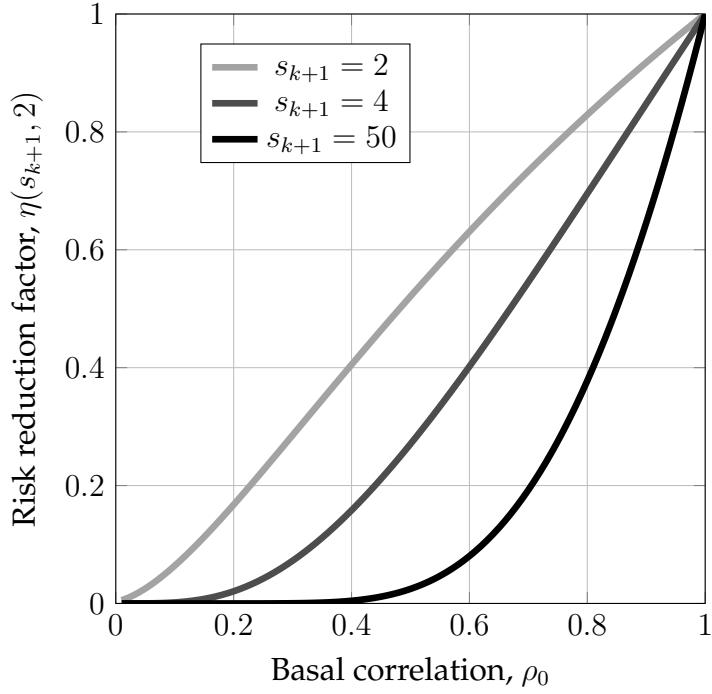


Figure 5: Risk reduction factor μ_{k+1}/μ_k at different basal correlation levels ρ_0 and for different sourcing strategies s_{k+1} . $S_k = 2$.

ternality downstream. This externality shrinks as correlation increases until it becomes a negative externality. Section 4 explores the welfare consequences of this mechanism.

3 Firm Optimal Diversification and Equilibrium

The mechanics of disruption propagation, derived in the previous section, determine the firm's optimal sourcing strategy. This section derives such optimal strategies. Importantly, due to Proposition 2, all firms in a given layer are identical before the shock and so is their optimisation problem. We can hence focus on the problem of the representative firm in layer $k + 1$: to choose how many suppliers in layer k to source from, based on the inferred distribution of their probability of experiencing a disruption event. This, in turn, is fully determined by the average disruption probability μ_0 , the correlation ρ_0 of the disruptions, and the sourcing strategies $\{s_1, s_2, \dots, s_k\}$ of the representative firms upstream. Henceforth, I assume firms face quadratic costs of sourcing, with cost parame-

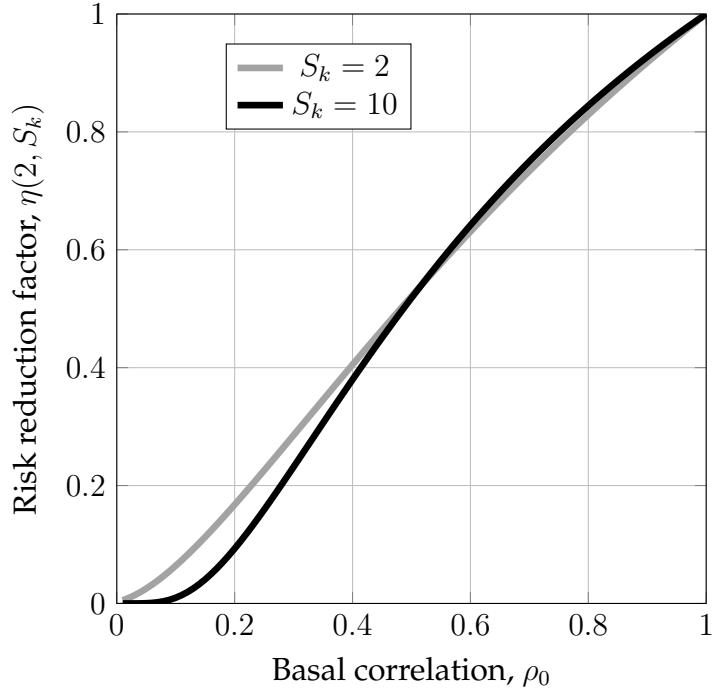


Figure 6: Risk reduction factor μ_{k+1}/μ_k at different basal correlation levels ρ_0 and for different upstream diversification levels strategies S_k . $s_{k+1} = 2$.

ter c , such that expected profits (2) can be written as

$$\Pi_k(s) = (1 - \mathbb{E}_s[P_k]) \pi - \frac{c}{2}s^2, \quad (25)$$

The optimisation problem of the firm is to then choose the optimal sourcing strategy

$$s_k = \arg \max_{s \in \{0,1,2,\dots\}} \Pi_k(s). \quad (26)$$

3.1 Limit Case: Uncorrelated Disruptions

Before turning towards the general framework, I first analyse a limit case in which suppliers' risk is not correlated, that is, $\rho_0 = 0$. This limit case gives a useful interpretation of the incentives behind multisourcing and allows us to establish a benchmark against which to study the introduction of correlated shocks.

Proposition 5. *If risk among basal firms is uncorrelated, that is, $\rho_0 = 0$, disruption*

events in layer k are independent and happen with probability

$$\mu_{k+1} = \mu_k^{s_{k+1}}. \quad (27)$$

Proof. By the definition of P_k , $P_k = \mu_k$ if $\rho_0 = 0$. \square

In this limit case, the representative firms' profits (2) are given by

$$\Pi_{k+1}(s) = (1 - \mu_k^s) \pi - \frac{c}{2} s^2. \quad (28)$$

The function Π_{k+1} is illustrated in Figure 7. Using this, we can derive the optimal sourcing strategy s_k of a firm producing good k . A firm with s suppliers contracts an extra one only if doing so yields a positive marginal profit

$$\begin{aligned} \Delta\Pi_{k+1}(s) &:= \Pi_{k+1}(s + 1) - \Pi_{k+1}(s) \\ &= \mu_k^s (1 - \mu_k) \pi - c \left(s + \frac{1}{2} \right). \end{aligned} \quad (29)$$

The marginal benefit $\Delta\Pi_{k+1} : \mathbb{R} \rightarrow \mathbb{R}$ is strictly decreasing and that it has a unique root. Hence, the optimal number of suppliers s_{k+1} is the smallest integer s for which the expected marginal profit is negative, that is, $\Delta\Pi_{k+1}(s) < 0$.

Definition 4. Let $\tilde{s}_{k+1} \in \mathbb{R}$ be the unique root of $\Delta\Pi_{k+1}$. I refer to this quantity as the "desired sourcing strategy" of the firm.

The optimal sourcing strategy s_{k+1} is then the smallest integer larger than the firm's desired sourcing strategy \tilde{s}_{k+1} , that is,

$$s_{k+1}(\mu_k) = \begin{cases} \lceil \tilde{s}_{k+1} \rceil & \text{if } \tilde{s}_{k+1} > 0 \text{ and} \\ 0 & \text{otherwise,} \end{cases} \quad (30)$$

where $\lceil \cdot \rceil$ is the ceiling function. The optimal sourcing strategy s_{k+1} , immediately implies an upper bound on the suppliers' risk μ_k firms are willing to tolerate.

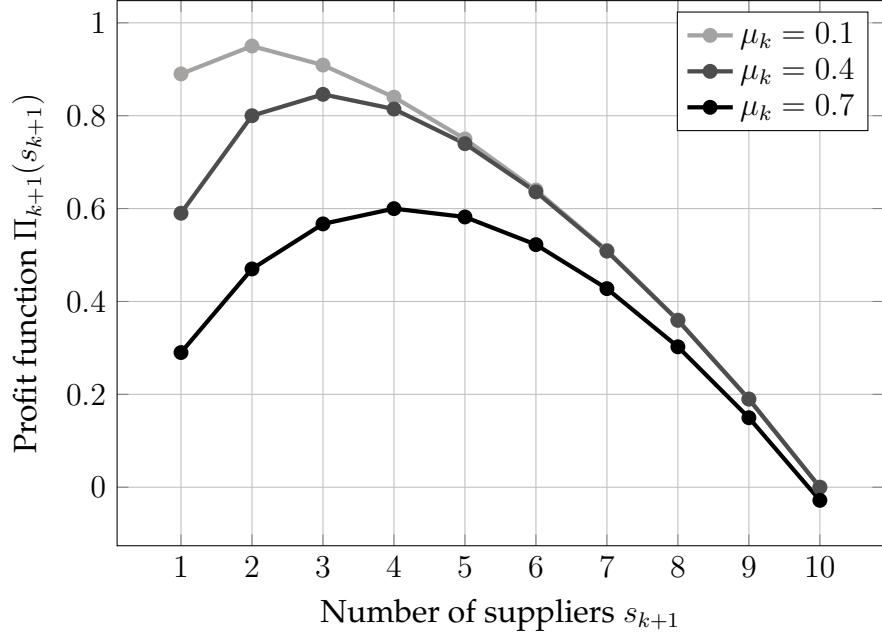


Figure 7: Firms' profits Π_{k+1} as a function of the sourcing strategy s_{k+1} for different levels of upstream risk μ_k .

Proposition 6. *Introduce the threshold*

$$\mu^{frag} := 1 - rc, \quad (31)$$

where $rc := \frac{c/2}{\pi}$ is the real marginal costs of an additional supplier. If the average disruption probability μ_k is larger than μ^{frag} , the downstream firm does not source any inputs, that is, $s_{k+1} = 0$.

Proof. Suppose a firm optimally does not source any inputs. This implies that the marginal benefit of adding the first supplier is negative, that is, $\Delta\Pi_{k+1}(0) < 0$, which yields the desired inequality. \square

For any value of initial basal risk $\mu_0 > \mu^{frag}$, firms do not source and the production network shuts down. As expected, the desired \tilde{s}_{k+1} and the optimal sourcing strategy s_{k+1} are determined by the upstream average disruption probability μ_k and the real marginal costs of contracting a new supplier rc . Figure 8 illustrates the effect these two conditions have on the optimal sourcing strategy. First, higher real marginal costs rc reduce the firm's number of sources. Second, as the upstream average disruption probability μ_k increases,

initially, the firm seeks higher diversification, until a level above which the desired sourcing strategy starts decreasing steeply.

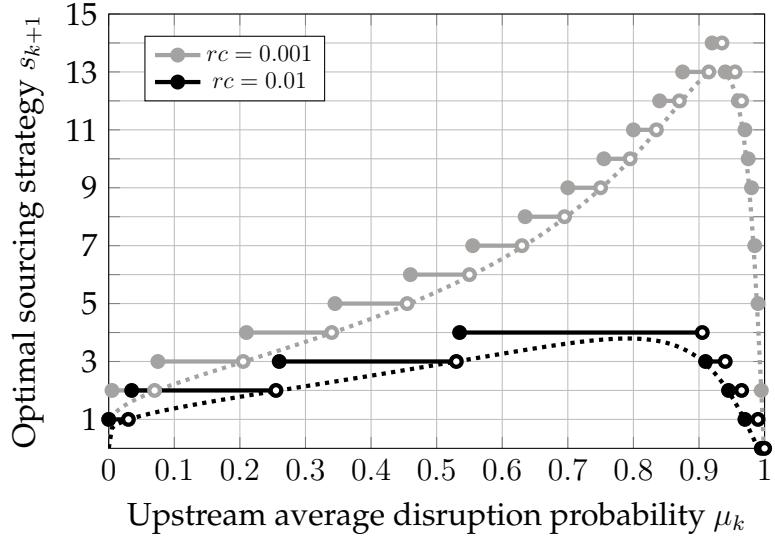


Figure 8: The desired \tilde{s}_{k+1} (dotted) and optimal s_{k+1} sourcing strategy (solid) as a function of the upstream average disruption probability μ_k

Having studied how risk affects the firm's optimal sourcing, I now look at the opposite channel, that is, how the firm sourcing strategy affects risk propagation. To do so, I think of the average disruption probability

$$\mu_{k+1} = \phi(\mu_k) := \mu_k^{s_{k+1}(\mu_k)} \quad (32)$$

from suppliers to downstream producers as a dynamical system, not in time but in layers $k \in \{0, 1, 2, \dots\}$ of the supply chain. Given a basal condition μ_0 , a fixed point $\bar{\mu}$ of the map (32) is then a level of disruption probability $\bar{\mu}$ such that all firms downstream of a layer k single-source, namely $s_l = 1$ for all $l \geq k$, and hence all share the same disruption probability $\mu_l \equiv \bar{\mu}$. When looking at the production network through this lens, a natural question arises: which basal levels of disruption probabilities μ_0 are not endogenously diversified by the production network, that is, $\bar{\mu} \geq \mu_0$? To answer this, first I characterise the downstream disruption probability $\bar{\mu}$.

Proposition 7. *The downstream disruption probability $\bar{\mu}$ satisfies*

$$\bar{\mu}(1 - \bar{\mu}) \leq 3rc. \quad (33)$$

Proof. A steady state is attained at level \bar{k} if $s_{\bar{k}} = 1$. This implies that the marginal benefit of multisourcing is negative $\Delta\Pi_{\bar{k}}(1) \leq 0$. This yields the desired inequality. \square

Corollary 7.1. *Introduce the critical threshold*

$$\mu^{\text{div}} := \frac{1}{2} + \sqrt{\frac{1}{4} - 3rc}. \quad (34)$$

If $\mu_0 > \mu^{\text{div}}$, the endogenous supply chain is unable to diversify risk, that is $\bar{\mu} \geq \mu_0$.

This result, proven in Appendix A.5, links the firm real marginal costs of sourcing rc and the production network risk. As relative marginal costs increase, the capacity of the production network to endogenously diversify decreases and firms' underdiversification yields endogenous fragility. Notice that, comparing the threshold μ^{div} of endogenous diversification with the threshold μ^{frag} of firm shutdown, illustrated in Figure 9, for some levels of basal probability of disruption μ_0 , despite no firm shutting down production $\mu_0 < \mu^{\text{frag}}$, the production network as a whole is still unable to endogenously diversify risk $\mu_0 > \mu^{\text{div}}$. This is true even in this special case, where the firms' risk is uncorrelated. In the next section, I introduce correlation risk $\rho > 0$ and investigate how doing so changes the dynamics illustrated here.

3.2 Optimal Sourcing with Correlated Distributions

If disruption events are not independent, that is $\rho_0 > 0$, the risk among suppliers throughout the production network is correlated, which affects the firm's optimisation incentives. In this case, the problem of a firm in layer $k + 1$ is still to choose the number of suppliers $s_{k+1} \in \{0, 1, 2, \dots\}$ that maximises the profits Π , but, by Proposition 4, the firm's disruption probability is given by

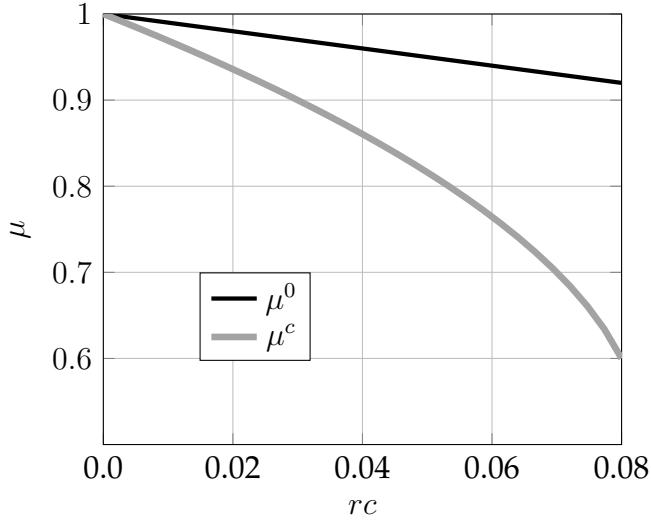


Figure 9: The critical thresholds μ^{div} , above which are unable to diversify, and μ^{frag} above which firms stop producing, as a function of the relative marginal costs rc .

the average disruption probability of its suppliers μ_k multiplied by a factor $\eta(s_{k+1}, S_k)$ which depends on the upstream diversification $S_k = \prod_{l < k} s_l$. As in the limit case analysed in the previous section, the firm will increase diversification as long as the expected increase in profits obtained by adding a supplier outweighs the costs of contracting that additional supplier. These expected marginal profits are given by

$$\Delta\Pi_{k+1}(s_{k+1}) = \left(\eta(s_{k+1}, S_k) - \eta(s_{k+1} + 1, S_k) \right) \mu_k \pi - c \left(s_{k+1} + \frac{1}{2} \right). \quad (35)$$

The characterisation of the optimal sourcing strategy is analogous to the limit case without the correlation discussed above. $\tilde{s}_{k+1} \in \mathbb{R}$ is the desired sourcing strategy for which the marginal benefits and marginal costs of diversification are equal, such that $\Delta\Pi_{k+1}(\tilde{s}_{k+1}) = 0$. As the marginal profits are strictly decreasing in the number of suppliers (see Appendix A.6), the firm will, as in the limit case, choose its optimal sourcing strategy as $s_{k+1} = \lceil \tilde{s}_{k+1} \rceil$ if $\tilde{s}_{k+1} > 0$ and chooses not to source any inputs otherwise. Figure 10 illustrates how the optimal sourcing strategy s_{k+1} changes with upstream correlation ρ_k for different levels of relative costs rc . As upstream correlation increases, the firm increases its sources to diversify risk. Yet, for large levels of correlation,

the disruption of an additional source of the input good is likely correlated to a disruption among the firm's existing suppliers, which reduces the firm's incentive to multisource. As disruptions among suppliers tends towards perfect correlation, $\rho_k \rightarrow 1$, the firm sources from a single supplier.

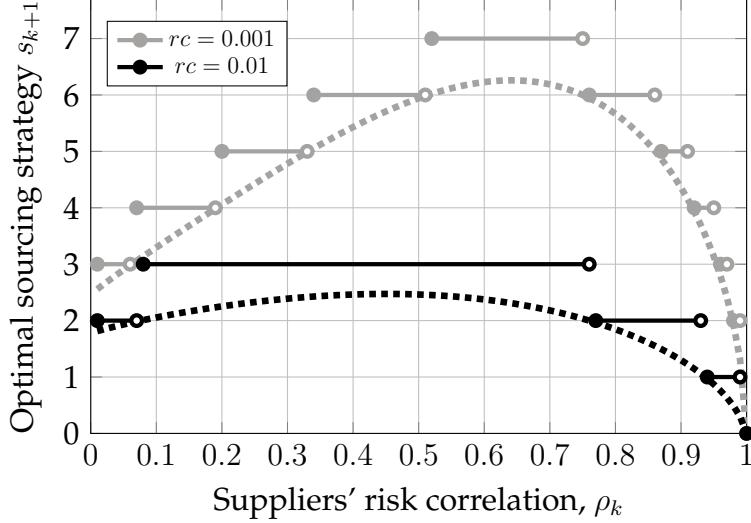


Figure 10: The desired \tilde{s}_{k+1} (dotted) and optimal s_{k+1} sourcing strategy (solid) as a function of the upstream correlation ρ_k

To study the ramifications of this endogenous channel for the supply chain formation and its fragility, next I analyse the propagation of risk through the layers. As above, conditional on the choices of the upstream producers $S_k = \prod_{l < k} s_l(\mu_l)$, we can view the mapping of the probability of disruption between layers as a dynamical system through the layers of the production network, with map

$$\mu_{k+1} = \Phi(\mu_k; S_k) := \eta(s_{k+1}(\mu_k), S_k)\mu_k. \quad (36)$$

A steady state of the dynamical system is then an average disruption probability $\bar{\mu} := \mu_{\bar{k}}$ in some layer \bar{k} such that all downstream layers $l \geq \bar{k}$ have the same average disruption probability $\mu_l \equiv \bar{\mu}$. This can occur in two cases. Either the firm in layer \bar{k} does not source, that is, $s_{\bar{k}} = 0$, or it sources from one supplier, that is, $s_{\bar{k}} = 1$. The former case is trivial: the production network shuts down, and all downstream firms do not produce, such that $\bar{\mu} = 1$. In the latter case, by single sourcing, the average disruption probability in layer \bar{k} is the average

disruption probability among the suppliers $\bar{k} - 1$, as the risk reduction factor $\eta(s_k, S_{k-1}) = 1$ if $s_k = 1$. Because the layers are symmetric, the firms in the downstream layer $\bar{k} + 1$ face the same problem as those in layer \bar{k} , such that they endogenously single source, that is, $s_{\bar{k}+1} = 1$. Inductively, this holds true for all $l \geq \bar{k}$, hence $\mu_l \equiv \bar{\mu}$. Hereafter, I refer to the situation in which the downstream average disruption probability is greater than the basal one, that is, $\bar{\mu} \geq \mu_0$, as endogenous fragility. Figure 11 shows the downstream average disruption probability $\bar{\mu}$ as a function of basal average disruption probability μ_0 , for cases in which basal correlation ρ_0 is low or high. In both cases for large possible initial levels of basal average disruption probability μ_0 the supply chain is endogenously resilient, as $\bar{\mu} < \mu_0$. But, as in the uncorrelated cases studied above, there is a threshold of average basal disruption probability $\mu_0 > \mu^{\text{div}}$ for which the firm is endogenously fragile and $\bar{\mu} \geq \mu_0$. The threshold effect is discontinuous. At $\mu_0 \equiv \mu^{\text{div}}$ an arbitrarily small increase in μ_0 can lead to discontinuously large downstream failure probabilities $\bar{\mu}$. The threshold μ^{div} is decreasing in the

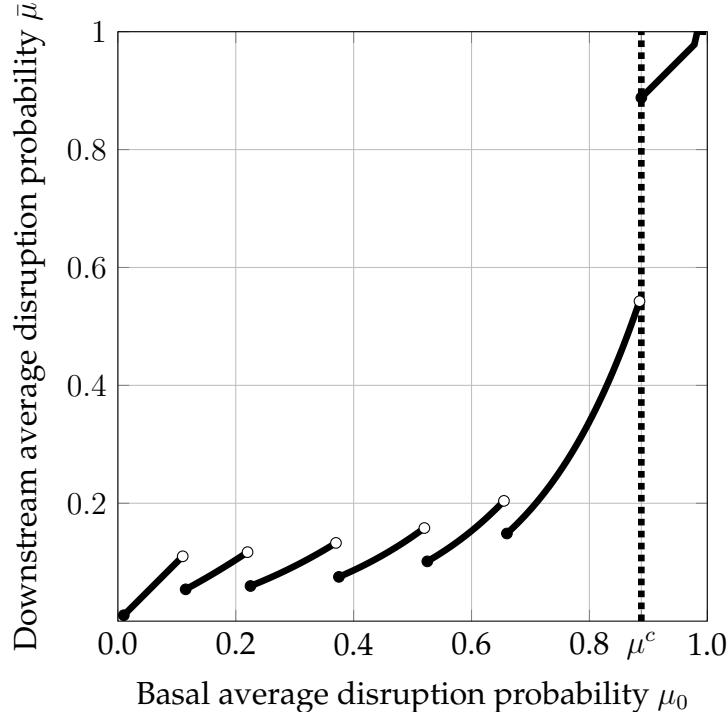


Figure 11: Downstream average disruption probability as a function of basal average disruption probability for $\rho_0 = 0.4$

basal level of correlation ρ_0 , as illustrated in Figure 12. This implies that a small increase in basal correlation leads to discontinuous increases in the downstream average disruption probability. This result highlights an additional channel to that studied by Elliott, Golub and Leduc (2022) by which supply chains can be endogenously fragile: even if the expected failure probability μ_0 of basal producers remains unchanged, an increase in the correlation of their disruptions ρ_0 , can endogenously induce large fragilities. Hence, phenomena that can lead to increases in risk among upstream producers, such as offshoring, climate disruptions, and economic agglomeration, can generate underdiversification and endogenous fragility, even as they leave individual producer's risk unchanged.

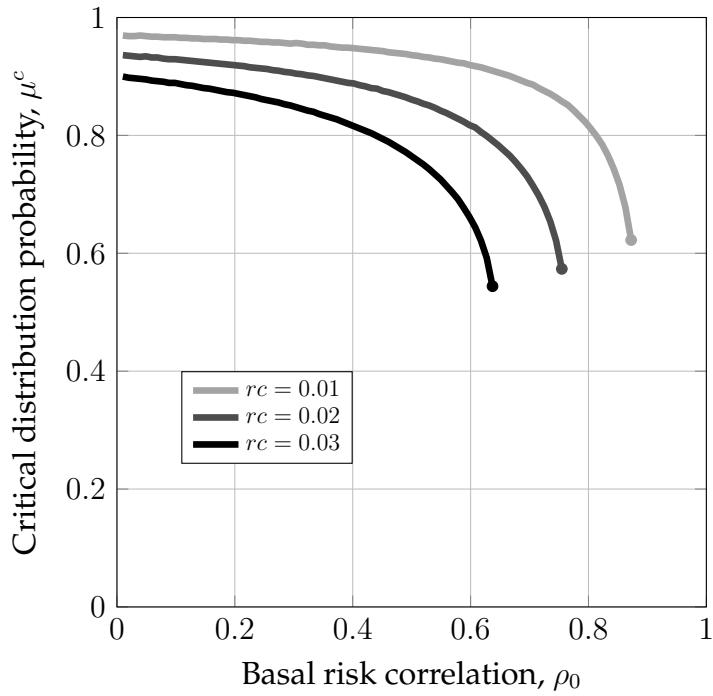


Figure 12: Critical level of basal average disruption probability μ^{div} as a function of the basal correlation.

4 Social Planner Problem

To establish a benchmark to which one can compare the competitive equilibrium analysed above, in this section I solve the model from the perspective of

a social planner. The social planner attempts to, on the one hand, minimise the number of firms expected to fail, and, on the other, minimise the number of costly sourcing relations. To develop a useful benchmark, I define a social planner problem that can be meaningfully compared to the decentralised firm's problem by making the following two assumptions.

Assumption 4. *The social planner knows the distribution of failure in the basal layer $P_0 \sim \text{Beta}(\mu_0, \rho_0)$ and establishes the firms' sourcing decisions before P_0 is realised.*

Assumption 5. *The social planner makes sourcing decisions such that no two firms share suppliers, that is, $\mathcal{S}_{k,i} \cap \mathcal{S}_{k,j}$ is empty.*

By this assumption, the social planner can "diversify away" all the correlation that arises due to the network structure. Hence, the only source of risk in the model is represented by the shutdowns experienced by firms in the basal layer, which happen with non-idiosyncratic probabilities $\{p_{0,j}\}_j$ sampled from P_0 (Assumption 4). This assumption can be satisfied by the planner as long as the socially optimal sourcing strategies $S_{k,i}$ are finite, as they can match downstream firms in k and upstream firms in $k - 1$ by picking a bijection $\gamma_k : \mathbb{N} \rightarrow \mathbb{N}^{\max_k |\mathcal{S}_{k,i}|}$.

Combining Assumptions 4 and 5, the social planner problem is then to maximise average expected payoff

$$W(\{\mathcal{S}_{k,i}\}) := \frac{1}{K} \sum_{k=0}^K \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (1 - \mathbb{P}(\mathcal{S}_{k,i} \subset \mathcal{D}_{k-1})) \pi - \frac{c}{2} |\mathcal{S}_{k,i}|^2, \quad (37)$$

by choosing a sourcing strategy $S_{k,i} \subseteq \mathbb{N}$ for each firm in each layer such that $S_{k,i} \cap S_{k,j}$ is empty for all i, j . The social planner problem can be further simplified by noticing that, given that all firms in layer k , are identical if establishing an additional path from a firm in layer k to a basal firm has positive marginal benefits, then it has positive marginal benefits for all firms in layer k which share the same number of paths to basal firms. Hence, as in the decentralised firms' problem, the social planner can choose the optimal number of sources in each layer and then match downstream producers and upstream suppliers

using γ_k . This allows to formulate the social planner problem recursively. Let V_k be the maximal average welfare from layer k to the last layer K . This can be recursively defined as

$$V_k(P_{k-1}) = \max_{s_k} \left\{ (1 - \mathbb{E}[P_{k-1}^{s_k}]) \pi - \frac{c}{2} s_k^2 + \mathbb{E}[V_{k+1}(P_k)] \right\}, \quad (38)$$

where the state $P_{k-1} \sim \text{BetaPower}(\mu_0, \rho_0, s_1 s_2 \dots s_{k-1})$ is the fraction of disrupted firms, which evolves as

$$P_k = P_{k-1}^{s_k}. \quad (39)$$

The average welfare in layer $K + 1$ is given by $V_{K+1}(P_K) = 0$, since firms in the last layer are never sources to other firms. The state is initialised at $P_0 \sim \text{Beta}(\mu_0, \rho_0)$. This problem can be solved using standard backward induction techniques (see Appendix B). The optimum average social welfare (37) can then be written as

$$V_1(P_0) = \max_{s_1, s_2, \dots, s_{K-1}} W(\{s_1, s_2, \dots, s_{K-1}\}). \quad (40)$$

Let $\{s_k^p\}_{k=1}^K$ be the socially optimal sourcing strategies sequence and $\{\mu_k^p\}_{k=1}^K$ be the expected disruption in each layer given by such sourcing strategies. I assume that the number of layers K is sufficiently large such that for some downstream layer \bar{k} , the risk remains constant $\mu_k^p \equiv \mu^p$ for all $k \geq \bar{k}$. Figure 13 shows this difference in downstream risk between the decentralised case and the social planner, that is, $\mu - \mu^p$ for the same two cost regimes. If pairing costs are low, the social planner achieves marginally lower risk levels of downstream risk for most initial conditions. If initial basal correlation ρ_0 is sufficiently large and the average basal disruption probability μ_0 is sufficiently low, the firms over-diversify compared to the socially optimum $\mu < \mu^p$. If relative pairing costs are high, the social planner is able to diversify risk around the critical threshold μ_c , such that the decentralised equilibrium induces inefficiently high levels of average downstream disruption probability, that is, $\mu > \mu^p$. This result implies that the cascading failures that occur around the critical threshold are

fully attributable to firms' endogenous under-diversification motives and are hence inefficient. The differences between the firms' sourcing strategies and

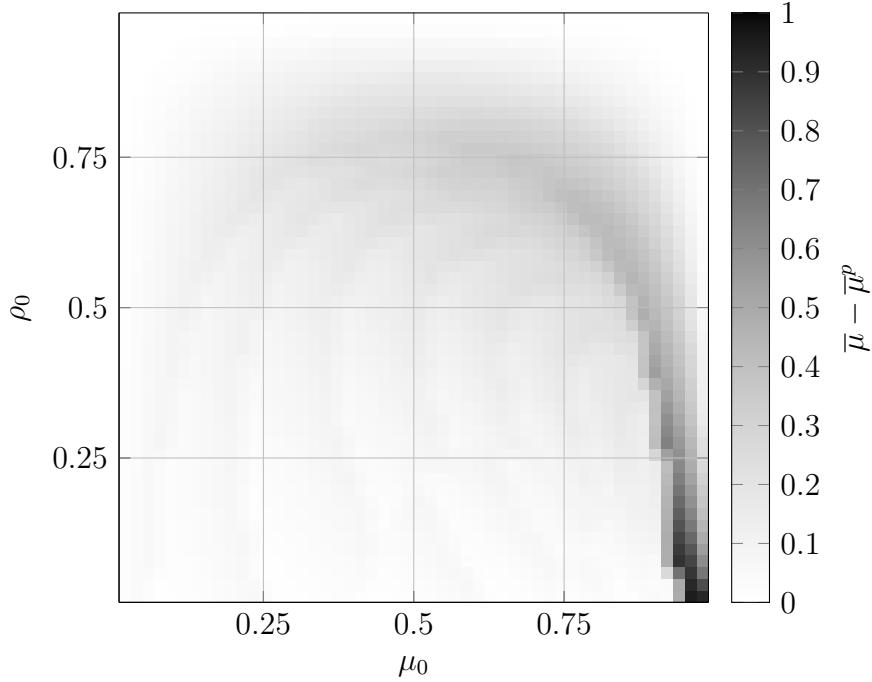


Figure 13: Change in downstream expected failure probability between the firms' μ and the social planner μ^p equilibrium, given different initial conditions μ_0 and ρ_0 in a low (left) and a high (right) relative pairing costs regime.

the social optimum generate welfare losses in the production network. Letting W be the average firm profit in the decentralised case and W^p be the average profit achieved by the social planner, Figure 14 illustrates the welfare loss due to the firms' diversification decisions $W - W^p$. The welfare loss is larger around the critical value μ^c , where the production network is endogenously fragile. At these levels of risk, firms' upstream firms' diversification incentives are weak, which creates large downstream resilience externalities. Crucially, both an increase in basal risk μ_0 and an increase in basal correlation ρ_0 can generate discontinuous welfare losses.

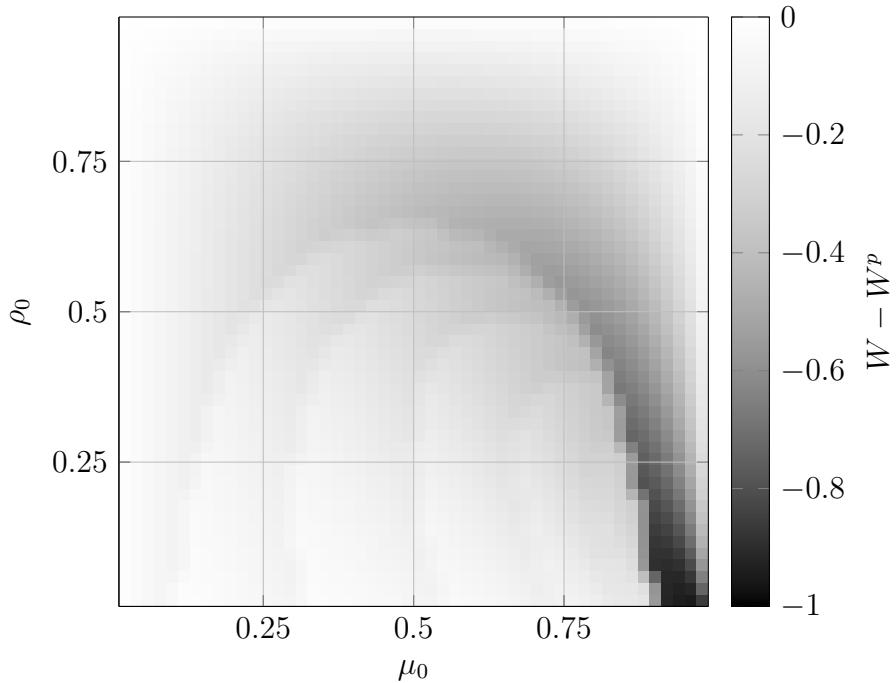


Figure 14: Welfare loss of the decentralised equilibrium $W - W^p$, given different initial conditions μ_0 and ρ_0 .

5 The Role of Opacity

So far I assumed that firms cannot observe the realisation of the supply chain and the basal disruption probabilities P_0 when making sourcing decisions. To understand how this assumption affects optimal decisions and fragility within the supply chain, I now analyse the supply chain under perfect information. The following assumption clarifies what is meant by perfect information in the context of the model.

Assumption 6. *In a regime of perfect information, each firm i in level k is able to perfectly estimate the disruption probability of each potential supplier and the full correlation structure of the disruption events.*

Under this perfect information regime, the firm can assign correct probabilities to its own disruption risk

$$\mathbb{P}(\mathcal{S}_{k,i} \subset \mathcal{D}_{k-1}) \text{ for all possible sourcing strategies } \mathcal{S}_{k,i}. \quad (41)$$

The firm can hence rank suppliers by the marginal reduction in risk they provide and source from the "safest" s_k desired suppliers. As all firms downstream are ex-ante symmetric, the marginal benefits of diversification experienced by firm (i, j) are the same as those of all other firms in layer k , which implies that, in equilibrium, all firms in layer k will employ the same sourcing strategy, given that they are ex-ante symmetric. This outcome is beneficial for any single firm, but detrimental to the stability of the production network. The following two propositions formalise this.

Proposition 8. *Compared to the opaque scenario, for the same number of sources, firm (k, i) is (weakly) less likely to be disrupted.*

Proof. Given the same number of sources, the firm with perfect information minimises its disruption risk with fewer constraints than in the opaque scenario. \square

Proposition 9. *Under perfect information, the supply chain is maximally fragile: either all firms fail or none do.*

Proof. Without loss of generality, assume basal firms are sorted by their disruption risk $p_{0,1} > p_{0,2} > \dots$. Suppose a firm in layer $k = 1$ chooses to contract $s_{1,i}$ suppliers. In choosing which suppliers to source from, the optimal choice is then to pick the first $s_{1,i}$ -th basal producers, namely

$$\mathcal{S}_{1,i} = \{1, 2, \dots, s_{1,i}\}. \quad (42)$$

By symmetry, this is also the optimal choice of all other firms in layer $k = 1$, such that $\mathcal{S}_{1,i} = \mathcal{S}_{1,j}$ for all i, j . This further implies that disruption events in layer $k = 1$ are perfectly correlated, $X_{1,i} = X_{1,j}$ for all i, j , as each two firms share all suppliers. This in turn implies that, regardless of the diversification strategy, all firms downstream $k > 1$ experience perfectly correlated distribution events. Hence,

$$X_{k,i} = X_{k,j} \text{ for all firms } i, j \text{ and layers } k > 0. \quad (43)$$

\square

Supply chain opacity, despite preventing firms from implementing an optimal diversification strategy, leads to more resilient supply chains. Hence, policy efforts to improve supply chain resilience via transparency might backfire if not paired with efforts to coordinate diversification of firms' sources.

6 Conclusion

Deeper and more globalised supply chains are more vulnerable to widespread, correlated disruptions. This paper studies how firms diversify sourcing risk when disruptions are correlated, and how this affects the endogenous formation of supply chains. I show that, as disruption correlation rises, a firm sources its input good from more suppliers to diversify the risk of a disruption. Yet, there is a level of correlation after which the expected risk reduction of adding a supplier is small, hence, the firm starts reducing its number of suppliers and lowers its diversification. This mechanism can create two forms of externalities imposed by upstream suppliers onto downstream producers. When firms upstream choose to under-diversify, this generates increased disruption risk downstream. When firms upstream choose to over-diversify, this generates a high correlation in their disruption risk, which has consequences for downstream diversification incentives. Due to this, I show that the supply chain is endogenously fragile to disruption correlation, that is, small increases in the correlation among upstream disruptions can trigger large under-diversification throughout the supply chain. This can leave the economy vulnerable to unlikely disruption events.

To construct a welfare benchmark for these results, I then solve the equivalent social planner problem. I show that a social planner can design a supply chain that is resilient to such correlated shocks. This result has two consequences. First, it illustrates that the fragility derives from the individual firm's diversification strategy and the risk externalities it induces downstream. It is hence entirely endogenous to the supply chain formation. Second, it implies the presence of large and discontinuous welfare losses. Finally, I study the role

of imperfect information on supply chain formation. I show that, in the presence of perfect information on the structure and risk of the supply chain, firms are better able to individually diversify risk, yet, in doing so, they choose identical suppliers, which renders the supply chain vulnerable to small upstream disruptions. This suggests that recent efforts in increasing firms' visibility of the supply chain, despite reducing individual firms' supply chain risk, might have the unintended consequence of making supply chains more fragile.

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A Omitted Proofs

A.1 Proof of Proposition 1

The proof is a straightforward application of de Finetti's theorem, which I now recall.

Lemma 9.1 (De Finetti's Theorem). *An infinite sequence of Bernoulli random variables $X_1, X_2 \dots$ is exchangeable if and only if there exists a random variable P over the unit interval such that $(X_1, X_2 \dots) | P$ are independent and identically distributed.*

Using this, we can prove Proposition 1.

Proof. By construction the basal disruption probabilities $\{X_{0,j}\}_j$ are independent and identically distributed conditional on P_0 . \square

A.2 Proof of Proposition 2

The proof is built on four steps. First, in Lemma 9.2 I show that downstream firms, facing exchangeable disruptions among upstream producers, are indifferent between suppliers $\mathcal{S}_{k+1,i}$ and only care about the number of sources $|\mathcal{S}_{k+1,i}|$. Second, in Lemma 9.3, I show that this choice is finite and the same among all firms, $s_{k+1} := |\mathcal{S}_{k+1,i}| < \infty$. Third, in Lemma 9.4, I show that, given the source selection procedure of Assumption 3, firms facing exchangeable disruption among suppliers, experience disruptions which are themselves exchangeable. Finally, by induction on the layer k , using Lemma 9.4 as induction step, all downstream layers are shown to be exchangeable.

Lemma 9.2. *If the disruption $\{X_{k,i}\}_i$ events among upstream firms are exchangeable, then the probability that a downstream firm $(k+1, j)$ is disrupted depends only on the number $|\mathcal{S}_{k+1,j}|$ of suppliers it picks.*

Proof. By assumption the sequence $\{X_{k,i}\}_i$ of disruption events among upstream firms is an exchangeable sequence of random variables.

Let $A = \{X_{k,a_1}, X_{k,a_2} \dots X_{k,a_n}\}$ and $B = \{X_{k,b_1}, X_{k,b_2} \dots X_{k,b_n}\}$ be two subsequences of finite size n , where $a_i, b_i \in \mathbb{N}$. Now consider a permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ satisfying

$$\sigma(A) = B \text{ and } \sigma(A^c) = B^c, \quad (44)$$

that is, elements of A are mapped into B . The joint probability of A taking a value $x \in \{0, 1\}^A$ is

$$\begin{aligned} \mathbb{P}(A = x) &= \sum_{y \in \{0, 1\}^{A^c}} \mathbb{P}(A = x \text{ and } A^c = y) \\ \text{then by exchangeability, } &= \sum_{y \in \{0, 1\}^{A^c}} \mathbb{P}(\sigma(A) = x \text{ and } \sigma(A^c) = y) \\ &= \sum_{y \in \{0, 1\}^{A^c}} \mathbb{P}(B = x \text{ and } B^c = y) \\ &= \mathbb{P}(B = x), \end{aligned} \quad (45)$$

hence,

$$A \sim B. \quad (46)$$

□

Lemma 9.3. *If the disruption $\{X_{k,i}\}_i$ events among upstream firms are exchangeable, each firm chooses the number of suppliers $s_{k+1,i} = |\mathcal{S}_{k+1,i}|$ from the same maximising set, that is, for all i ,*

$$s_{k+1,i} \in \{s_1^*, s_2^*, \dots, s_n^*\} := \arg \sup_{s \in \mathbb{N}} \Pi(s), \quad (47)$$

using the procedure of Assumption 3. Importantly the maximising set is nonempty and finite.

Proof. By Lemma 9.2, the probability of failure only depends on the number of suppliers in the sourcing strategy $s_{k+1,i} = |\mathcal{S}_{k+1,i}|$, that is,

$$\mathbb{E}[X_{k+1,i} | |\mathcal{S}_{k+1,i}| = s] = p_{k+1}(s) \text{ for all } i \quad (48)$$

for some probability mass function p_{k+1} . Notice that

$$p_{k+1}(0) = 1. \quad (49)$$

Abusing notation, I write the profit function (1.2) as $\Pi : \mathbb{N} \rightarrow \mathbb{R}$

$$\Pi(s) = (1 - p_{k+1}(s))\pi - C(s). \quad (50)$$

I now show that the set of maximisers

$$\arg \sup_{s \in \mathbb{N}} \Pi(s) \quad (51)$$

is nonempty and contains elements which are independent of the downstream firm $(k + 1, i)$.

We first notice that without suppliers, the firm makes no profits $\Pi(0) = -C(0) = 0$. Furthermore, as the number of suppliers grows

$$\lim_{s \rightarrow \infty} \Pi(s) = (1 - \lim_{s \rightarrow \infty} p_{k+1}(s))\pi - \lim_{s \rightarrow \infty} C(s) \rightarrow -\infty, \quad (52)$$

as p_{k+1} is bounded above by 1. Hence, there exists a finite s^* such that $\Pi(s^*)$ is maximal. Furthermore, notice that this s^* is independent of the downstream firm index i . \square

Lemma 9.4. *Suppose all downstream firms $(k + 1, i)$ select their sourcing strategy $\mathcal{S}_{k+1,i}$ using the procedure outlined in Assumption 3 and the number of sources chosen are the same, that is, $|\mathcal{S}_{k+1,i}| = |\mathcal{S}_{k+1,j}| = s < \infty$ for all i and j . Then, if upstream disruptions are exchangeable $\{X_{k,j}\}_j$, downstream disruptions $\{X_{k+1,i}\}_i$ are exchangeable.*

The idea of the proof is to construct a random variable P_{k+1} over the unit interval such that, conditional on this random variable, all the downstream disruptions $X_{k+1,j}$ are i.i.d. Bernoulli trials. The random variable P_{k+1} is constructed to be the probability of a disruption $X_{k+1,i}$. Importantly, this random

variable is independent of the downstream firm index i . Then by de Finetti's theorem (Lemma 9.1), the disruptions are exchangeable. This is a simple case of the so called "Noise-Outsourcing" lemma in (Austin, 2015, Lemma 3.1, p.7).

Proof. We seek to construct a random variable P_{k+1} such that

$$(X_{k+1,i} | P_{k+1} = p_{k+1}) \sim \text{Bernoulli}(p_{k+1}). \quad (53)$$

I do this in two steps. In the first step, I compute the probability that a given upstream firm (k, j) is disrupted. In the second step, I compute the probability that a particular firm (k, j) is among downstream firm suppliers $\mathcal{S}_{k+1,i}$.

The first step follows from the assumption of exchangeability of upstream disruption events. Indeed, the upstream disruption events $\{X_{k,j}\}_j$ are exchangeable, hence, by de Finetti's theorem, there exist a random variable on the unit interval P_k such that

$$(X_{k,j} | P_k = p_k) \sim \text{Bernoulli}(p_k). \quad (54)$$

We can then write

$$\mathbb{P}(X_{k,j} = 1) = P_k. \quad (55)$$

The second step builds on Assumption 3. Let

$$(I_1, I_2, I_3 \dots I_s) := \mathcal{S}_{k+1,j}, \quad (56)$$

such that I_i is the i -th index of the upstream supplier selected by the downstream producer, using the procedure outlined in Assumption 3. Recall that I_i is a tuple taking the form (k, j) . Then, the probability that an upstream producer (k, j) is selected as a supplier of firm $(k + 1, i)$ is given by

$$\mathbb{P}((k, j) \in \mathcal{S}_{k+1,i}) = \sum_{m=1}^s \mathbb{P}((k, j) = I_m \text{ and } (k, j) \notin I_{1:(m-1)}), \quad (57)$$

where $I_{1:(m-1)} := (I_1, I_2, \dots, I_{m-1})$. Equation (A.18) decomposes the probability

that firm (k, j) is chosen as a supplier into the probability that the firm is picked first $\mathbb{P}((k, j) = I_1)$, added to the probability that it is picked second but not first $\mathbb{P}((k, j) = I_2 \text{ and } (k, j) \neq I_1)$, and so on. Using conditional probabilities, each summand can be written as

$$\begin{aligned} & \mathbb{P}((k, j) = I_m \text{ and } (k, j) \notin I_{1:(m-1)}) \\ &= \mathbb{P}((k, j) = I_m | I_{1:(m-1)} = ((k, j_l))_{l=1}^{m-1}) \times \mathbb{P}(I_{1:(m-1)} = ((k, j_l))_{l=1}^{m-1}) \end{aligned} \quad (58)$$

These can then be written as

$$\mathbb{P}((k, j) = I_m | I_{1:(m-1)} = ((k, j_l))_{l=1}^{m-1}) = \frac{\alpha_{k,j}}{1 - \sum_{l=1}^{m-1} \alpha_{k,l}} \quad (59)$$

and

$$\mathbb{P}(I_{1:(m-1)} = ((k, j_l))_{l=1}^{m-1}) = \prod_l \frac{\alpha_{k,l}}{1 - \sum_{z=1}^{l-1} \alpha_{k,z}}. \quad (60)$$

Importantly, none of these depend on the downstream firm index i . Denote the probability that a downstream firm is picked as a supplier as

$$\omega_{k,j} := \mathbb{P}((k, j) \in \mathcal{S}_{k+1,i}). \quad (61)$$

Then putting this together we can write

$$P_{k+1} = \prod_{j=1}^n \omega_{k,j} P_k. \quad (62)$$

Crucially, this probability does not depend on the downstream index i .

Finally, we can show that the downstream disruptions $X_{k+1,j}$, conditional on this random variable P_{k+1} are independent and identically distributed Bernoulli trials. First, the downstream disruption events have law

$$\mathbb{P}(X_{k+1,i} = 1 | P_{k+1} = p_{k+1}) = p_{k+1} \quad (63)$$

hence, $\{X_{k+1,i}\}_i$ are identically distributed. Second, for two distinct down-

stream firms i_1, i_2 , their joint probability distribution can be decomposed as

$$\begin{aligned} & \mathbb{P}(X_{k+1,i_1} = 1, X_{k+1,i_2} = 1 | P_{k+1}) \\ &= \sum_{j_1, j_2} \mathbb{P}(j_1 \in \mathcal{S}_{k+1,i_1}, j_2 \in \mathcal{S}_{k+1,i_2} | P_{k+1}) \times \\ & \quad \mathbb{P}(X_{k+1,i_1} = 1, X_{k+1,i_2} = 1 | j_1 \in \mathcal{S}_{k+1,i_1}, j_2 \in \mathcal{S}_{k+1,i_2}, P_{k+1}) \end{aligned} \quad (64)$$

First, conditional on P_{k+1} , the sampling procedure is independent, hence

$$\mathbb{P}(j_1 \in \mathcal{S}_{k+1,i_1}, j_2 \in \mathcal{S}_{k+1,i_2} | P_{k+1}) = \mathbb{P}(j_1 \in \mathcal{S}_{k+1,i_1} | P_{k+1}) \mathbb{P}(j_2 \in \mathcal{S}_{k+1,i_2} | P_{k+1}). \quad (65)$$

Second, conditional on the sampling and P_{k+1} , the failure probabilities are independent, that is,

$$\begin{aligned} & \mathbb{P}(X_{k+1,i_1} = 1, X_{k+1,i_2} = 1 | j_1 \in \mathcal{S}_{k+1,i_1}, j_2 \in \mathcal{S}_{k+1,i_2}, P_{k+1}) \\ &= \mathbb{P}(X_{k+1,i_1} = 1 | j_1 \in \mathcal{S}_{k+1,i_1}, P_{k+1}) \mathbb{P}(X_{k+1,i_2} = 1 | j_2 \in \mathcal{S}_{k+1,i_2}, P_{k+1}). \end{aligned} \quad (66)$$

This allows us to write the joint disruption probability as

$$\begin{aligned} & \mathbb{P}(X_{k+1,i_1} = 1, X_{k+1,i_2} = 1 | P_{k+1}) \\ &= \sum_{j_1} \mathbb{P}(j_1 \in \mathcal{S}_{k+1,i_1} | P_{k+1}) \mathbb{P}(X_{k+1,i_1} = 1 | j_1 \in \mathcal{S}_{k+1,i_1}, P_{k+1}) \times \\ & \quad \sum_{j_2} \mathbb{P}(j_2 \in \mathcal{S}_{k+1,i_2} | P_{k+1}) \mathbb{P}(X_{k+1,i_2} = 1 | j_2 \in \mathcal{S}_{k+1,i_2}, P_{k+1}) \\ &= \mathbb{P}(X_{k+1,i_1} = 1 | P_{k+1}) \mathbb{P}(X_{k+1,i_2} = 1 | P_{k+1}), \end{aligned} \quad (67)$$

hence X_{k+1,i_1} and X_{k+1,i_2} are conditionally independent.

As the downstream disruption events $X_{k+1,j}$ are i.i.d. Bernoulli trials conditional on P_{k+1} , by de Finetti's theorem, they are exchangeable. \square

I now prove Proposition 2.

Proof. The proof is done by induction, with the base case $k = 0$ given by Proposition 1. Assume that for some layer $k \geq 0$ the disruption events $\{X_{k,j}\}_i$ are exchangeable. By Lemma 9.2 and Lemma 9.3, all downstream firms choose the

same number of sources, that is, $|\mathcal{S}_{k+1,i}| = s_{k+1} < \infty$ for all i . Then by Assumption 3 and Lemma 9.4, the disruption events $X_{k+1,i}$ are exchangeable. \square

A.3 Proof of Proposition 3

Lemma 9.5. *For any two firms i_1 and i_2 , the probability that they share a supplier becomes arbitrarily small as $\theta \rightarrow \infty$, that is,*

$$\lim_{\theta \rightarrow \infty} \mathbb{P}(\mathcal{S}_{k+1,i_1} \cap \mathcal{S}_{k+1,i_2} \neq \emptyset) = 0. \quad (68)$$

Proof. Conditional on the weights α_j , the probability that a firm is not selected among the suppliers of a downstream firm is

$$\begin{aligned} \mathbb{P}((k, j) \notin \mathcal{S}_{k+1,i} | \alpha_j) &= \prod_{l=1}^s \left(1 - \frac{\alpha_j}{1 - \sum_{(k,l) \in \mathcal{S}_{k+1,i}} \alpha_l} \right) \\ &\geq \prod_{l=1}^s \left(1 - \frac{\alpha_j}{1 - s\bar{\alpha}} \right) \\ &= \left(1 - \frac{\alpha_j}{1 - s\bar{\alpha}} \right)^s, \end{aligned} \quad (69)$$

where $\bar{\alpha} := \sup_{j \in \mathbb{N}} \alpha_j$ and $s = |\mathcal{S}_{k+1,i}|$. Hence, the probability that a firm is selected as a supplier $\mathcal{S}_{k+1,i}$ can be bounded above by

$$\begin{aligned} \mathbb{P}((k, j) \in \mathcal{S}_{k+1,i} | \alpha_j) &= 1 - \mathbb{P}((k, j) \notin \mathcal{S}_{k+1,i} | \alpha_j) \\ &\leq 1 - \left(1 - \frac{\alpha_j}{1 - s\bar{\alpha}} \right)^s \\ &\leq \frac{s\alpha_j}{1 - s\bar{\alpha}}, \end{aligned} \quad (70)$$

where we have used the Bernoulli inequality $1 - np \leq (1 - p)^n$. Using this we

can compute the probability that two firms i_1 and i_2 share suppliers

$$\begin{aligned}\mathbb{P}(\mathcal{S}_{k+1,i_1} \cap \mathcal{S}_{k+1,i_2} \neq \emptyset | \alpha_j) &\leq \sum_j \mathbb{P}(j \in \mathcal{S}_{k+1,i_1} | \alpha_j) \mathbb{P}(j \in \mathcal{S}_{k+1,i_2} | \alpha_j) \\ &\leq \sum_j \left(\frac{s\alpha_j}{1-s\bar{\alpha}} \right)^2 \\ &= \frac{s^2}{(1-s\bar{\alpha})^2} \sum_j \alpha_j^2.\end{aligned}\tag{71}$$

Now we need to show that the unconditional probability goes to 0 as $\theta \rightarrow \infty$.

First notice that

$$\frac{s^2}{(1-s\bar{\alpha})^2} \leq \frac{s^2}{(1-s)^2},\tag{72}$$

hence

$$\mathbb{P}(\mathcal{S}_{k+1,i_1} \cap \mathcal{S}_{k+1,i_2} \neq \emptyset) = \mathbb{E}_{\alpha_j} [\mathbb{P}(\mathcal{S}_{k+1,i_1} \cap \mathcal{S}_{k+1,i_2} \neq \emptyset | \alpha_j)] \leq \frac{s^2}{(1-s)^2} \sum_j \mathbb{E}[\alpha_j^2].\tag{73}$$

Finally, to tackle the summand, we can use the definition of the weights to rewrite

$$\mathbb{E}[\alpha_j^2] = \mathbb{E}[u_j^2] \prod_{l < j} \mathbb{E}[(1-u_l)^2],\tag{74}$$

where

$$\mathbb{E}[u_j^2] = \int_0^1 u^2 \theta (1-u)^{\theta-1} du = \frac{2}{(1+\theta)(2+\theta)}\tag{75}$$

and

$$\mathbb{E}[(1-u_l)^2] = \int_0^1 u^2 \theta u^{\theta-1} du = \frac{\theta}{\theta+2},\tag{76}$$

arriving at

$$\mathbb{E}[\alpha_j^2] = \frac{2}{(1+\theta)(2+\theta)} \left(\frac{\theta}{\theta+2} \right)^{j-1}.\tag{77}$$

Using this we can finally write

$$\begin{aligned}
\sum_j \mathbb{E}[\alpha_j^2] &= \frac{2}{(1+\theta)(2+\theta)} \sum_j \left(\frac{\theta}{\theta+2}\right)^{j-1} \\
&= \frac{2}{(1+\theta)(2+\theta)} \frac{1}{1 - \frac{\theta}{\theta+2}} \\
&= \frac{1}{1+\theta}.
\end{aligned} \tag{78}$$

Hence, we have

$$\mathbb{P}(\mathcal{S}_{k+1,i_1} \cap \mathcal{S}_{k+1,i_2} \neq \emptyset) \leq \frac{s^2}{(1-s)^2} \frac{1}{1+\theta} \rightarrow 0 \text{ as } \theta \rightarrow \infty. \tag{79}$$

□

Using Lemma 9.5, we can now prove Proposition 3.

Proof. A firm producing good $k+1$ sources from $s_{k+1} < \infty$ suppliers. A disruption for the downstream firm is defined as

$$X_{k+1,i} = \prod_{(k,j) \in \mathcal{S}_{k+1,j}} X_{k,j}. \tag{80}$$

The sequence of upstream disruptions $\{X_{k,j}\}_{(k,j) \in \mathcal{S}_{k+1,j}}$ is a subsequence of an exchangeable sequence $\{X_{k,j}\}_{j \in \mathbb{N}}$, hence it is itself exchangeable. Furthermore, by assumption, conditional on the random variable $P_k \sim \text{BetaPower}(\mu_0, \rho_0, S_k)$, the disruption events $\{X_{k,j}\}_{(k,j) \in \mathcal{S}_{k+1,j}}$ are i.i.d. Bernoulli trials sampled independently.

Using this, we can derive the probability of a downstream disruption

$$\begin{aligned}
P_{k+1} &:= \mathbb{E}[X_{k+1,i}] = \mathbb{E} \left[\prod_{l=1}^{s_k} X_{k,j_l} \right] \\
\text{by the law of iterated expectations, } &= \mathbb{E} \left[\mathbb{E} \left[\prod_{l=1}^{s_k} X_{k,j_l} \mid P_k \right] \right].
\end{aligned} \tag{81}$$

By Lemma 9.5, in the limit $\theta \rightarrow \infty$, the probability that two upstream suppliers share sources is zero, $X_{k,j_l} | P_k$ are independent and identically distributed

Bernoulli trials with probabilities independently sampled from P_k , hence we can rewrite

$$P_{k+1} = \prod_{l=1}^{s_k} \mathbb{E}[\mathbb{E}[X_{k,j_l} | P_k]] = \prod_{l=1}^{s_k} \mathbb{E}[X_{k,j_l}] = P_k^{s_k}. \quad (82)$$

□

A.4 Mapping of risk across layers

Introduce

$$r_0 := \frac{\rho_0}{1 - \rho_0}. \quad (83)$$

Proposition 10. *The expected probability of disruption faced by a firm is given by*

$$\mu_k = \mathbb{E}[P_k] = \frac{B(\mu_0 r_0 + S_k, (1 - \mu_0)r_0)}{B(\mu_0 r_0, (1 - \mu_0)r_0)}, \quad (84)$$

where $S_k = \prod_{l=1}^k s_l$ and B is the standard beta function defined as

$$B(\alpha, \beta) := \int_0^1 p^{\alpha-1} (1-p)^{\beta-1} dp = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \quad (85)$$

Proof. The disruption probability $P_k \sim \text{BetaPower}(\mu_0, \rho_0, S_k)$, hence it can be written as

$$P_k = P_0^{S_k} \text{ where } P_0 \sim \text{Beta}(\mu_0, \rho_0). \quad (86)$$

Therefore, the expected probability of a disruption

$$\mu_k = \mathbb{E}[P_k] = \mathbb{E}[P_0^{S_k}] \quad (87)$$

is the S_k -th moment of P_0 . If $S_k = 0$, then the distribution $P_k = 1$, hence $\mu_k = 1$. If $S_k > 0$, μ_k can be derived using the moment generating function of the beta distribution, which is given by

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{B((1 - \mu_0)r_0 + n, \mu_0 r_0)}{B((1 - \mu_0)r_0, \mu_0 r_0)}. \quad (88)$$

Using this we obtain

$$\mu_k = \mathbb{E}[P_0^{S_k}] = M^{(S_k)}(0) = \frac{B((1 - \mu_0)r_0 + S_k, \mu_0 r_0)}{B((1 - \mu_0)r_0, \mu_0 r_0)}, \quad (89)$$

where $M^{(n)}$ is the n -th derivative of M . \square

Using this result we can derive the “risk reduction factor” η , if $S_k > 0$, as

$$\eta(s_{k+1}, S_k) := \frac{\mu_{k+1}}{\mu_k} = \frac{B((1 - \mu_0)r_0 + S_k s_{k+1}, \mu_0 r_0)}{B((1 - \mu_0)r_0 + S_k, \mu_0 r_0)}. \quad (90)$$

By the definition of the B function in terms of the Γ function, we can rewrite η in terms of a falling factorial

$$\eta(s, S) = \frac{(\mu_0 r_0 + S)^{\overline{S(s-1)}}}{(r_0 + S)^{\overline{S(s-1)}}} \quad (91)$$

which satisfies the recursion

$$\eta(s+1, S) = \eta(s, S) \frac{(\mu_0 r_0 + S s)^{\overline{S}}}{(r_0 + S s)^{\overline{S}}}. \quad (92)$$

Corollary 10.1. *The risk reduction factor $\eta(s, S)$ is strictly decreasing in s .*

Proof. The derivative of η can be expressed in terms of the digamma function

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}. \quad (93)$$

In particular,

$$\frac{\partial \eta}{\partial s}(s, S) = \eta(s, S) S [\psi(\mu_0 r_0 + S s) - \psi(r_0 + S s)]. \quad (94)$$

The digamma function ψ is strictly increasing over $[0, \infty)$, hence $\psi(\mu_0 r_0 + S s) - \psi(r_0 + S s) < 0$, as $\mu_0 < 1$. This implies that $\frac{\partial \eta}{\partial s}(s, S) < 0$. \square

A.5 Limit case $\rho_0 = 0$

To prove Corollary 7.1, I first related the fixed point of the one-dimensional risk propagation map ϕ to the fixed point of its “continuous” counterpart $\tilde{\phi}$, that is, the map that would arise if the firms could choose $s_{k+1} = \tilde{s}_{k+1} \in \mathbb{R}$, as opposed to $s_{k+1} \in \mathbb{N}$.

Definition 5. *Introduce the continuous law of motion*

$$\mu_{k+1} := \tilde{\phi}(\mu_k) := \mu_k^{\tilde{s}_{k+1}(\mu_k)}, \quad (95)$$

where \tilde{s}_{k+1} is the desired sourcing strategy as in Definition 4.

Lemma 10.1. *A fixed point $\bar{\mu}$ of the law of motion ϕ , is attained if and only if $\tilde{\phi}(\bar{\mu}) \geq \bar{\mu}$.*

Proof. Let $\bar{\mu}$ be a fixed point of ϕ , as defined in equation (1.31). By Definition 5,

$$\tilde{\phi}(\bar{\mu}) = \bar{\mu}^{\tilde{s}(\bar{\mu})}. \quad (96)$$

As $\bar{\mu} \leq 1$,

$$\tilde{\phi}(\bar{\mu}) \geq \bar{\mu} \iff \tilde{s}(\bar{\mu}) \in (0, 1]. \quad (97)$$

By definition $s(\bar{\mu}) = \lceil \tilde{s}(\bar{\mu}) \rceil = 1$, which implies that $\phi(\bar{\mu}) = \bar{\mu}$. \square

Now we can prove Corollary 7.1.

Proof. As shown in Lemma 10.1, a risk level μ satisfying $\tilde{\phi}(\mu) \geq \mu$ implies that μ is a fixed point of the map ϕ . The risk level μ satisfies the property $\tilde{\phi}(\mu) \geq \mu$ if and only if the desired sourcing strategy is $\tilde{s}(\mu) \in (0, 1]$. For this to be the case, it must the marginal profits $\Delta\Pi_{k+1}$ (1.28) intersect the $s_{k+1} = 0$ axis in the interval $(0, 1]$, that is, $\Delta\Pi_{k+1}(0) > 0$ and $\Delta\Pi_{k+1}(1) \leq 0$.

These yield the desired inequality. \square

A.6 General Case, $\rho_0 > 0$

This appendix proves that the optimal sourcing strategy, shown to exist in Lemma 9.3, is unique. Going from existence to uniqueness is possible, as now,

we have imposed a particular structure on the profit function Π_{k+1} , and we have derived a closed form solution for the propagation of risk (A.52).

Proof. It is sufficient to show that the marginal profit function $\Delta\Pi_{k+1} : \mathbb{R} \rightarrow \mathbb{R}$ is strictly decreasing in its input s .

The marginal profit function can be written in terms of the risk reduction factor (A.52) as

$$\begin{aligned}\Delta\Pi_{k+1}(s) &= [\eta(s, S_k) - \eta(s + 1, S_k)]\pi\mu_k - c\left(s + \frac{1}{2}\right) \\ &= \left[1 - \frac{(\mu_0 r_0 + S_k s)^{\overline{S_k}}}{(r_0 + S_k s)^{\overline{S_k}}}\right]\eta(s, S_k)\mu_k\pi - c\left(s + \frac{1}{2}\right),\end{aligned}\tag{98}$$

where we have used the recursion (A.53). Recall that $(\cdot)^{\overline{S_k}}$ is the raising factorial, as defined in equation (A.1). Taking derivatives with respect to s yields

$$\begin{aligned}\Delta\Pi'_{k+1}(s) &= \frac{\partial}{\partial s} \left[1 - \frac{(\mu_0 r_0 + S_k s)^{\overline{S_k}}}{(r_0 + S_k s)^{\overline{S_k}}}\right] \eta(s, S_k)\mu_k\pi \\ &\quad + \left[1 - \frac{(\mu_0 r_0 + S_k s)^{\overline{S_k}}}{(r_0 + S_k s)^{\overline{S_k}}}\right] \frac{\partial\eta}{\partial s}(s, S_k)\mu_k\pi - c.\end{aligned}\tag{99}$$

Considering all terms separately, we have

$$\underbrace{\frac{\partial}{\partial s} \left[1 - \frac{(\mu_0 r_0 + S_k s)^{\overline{S_k}}}{(r_0 + S_k s)^{\overline{S_k}}}\right]}_{\leq 0} \underbrace{\eta(s, S_k)\mu_k\pi}_{>0},\tag{100}$$

as $\mu_k, \mu_0 \in [0, 1]$ and by the definition of η (A.52),

$$\underbrace{\left[1 - \frac{(\mu_0 r_0 + S_k s)^{\overline{S_k}}}{(r_0 + S_k s)^{\overline{S_k}}}\right]}_{\geq 0} \mu_k\pi \underbrace{\frac{\partial\eta}{\partial s}(s, S_k)}_{<0},\tag{101}$$

by the strictly decreasing property of η in s , proven in Corollary 10.1, and finally $-c < 0$. Putting the three terms together we have $\Delta\Pi'_{k+1}(s) < 0$. \square

B Solution of the Social Planner Problem

First, notice that the terminal condition V_K is linear in P_{K-1} , hence

$$\mathbb{E}[V_K(P_{K-1})] = V_K(\mathbb{E}[P_{K-1}]). \quad (102)$$

In turn, this implies that V_k is linear for all k . Hence, we can rewrite the value to be a function of the state space S ,

$$V_k(S_{k-1}) = \max_s \left\{ (1 - \mathbb{E}[P_k]) \pi - \frac{c}{2} s^2 + V_k(S_{k-1}s) \right\}, \quad (103)$$

where

$$P_k \sim \text{BetaPower}(\mu_0, \rho_0, S_{k-1}s). \quad (104)$$

We then find V_k numerically. Let the state space be

$$\Omega_m := (1, 2, 3 \dots m) \times (1, 2, 3, \dots m, m+1, \dots m^2, m^2+1, \dots, m^K) \quad (105)$$

for some $m \in \mathbb{N}$ sufficiently large. This can be seen as a projection of $\mathbb{N} \times \mathbb{N}$ into Ω_m by simple truncation, where we just make m larger as the firms pair with more suppliers. We are guaranteed $m < \infty$, as the socially optimal sourcing strategy in each layer $s_k < \infty$ for each $k \in \{0, 1, 2, \dots, K\}$. The payoff of the planner in each layer k is given by

$$l(s, S_k) := (1 - \mathbb{E}[P_k]) \pi - \frac{c}{2} s^2. \quad (106)$$

Denote by $l(\Omega_m)$ the image of l from Ω_m . Then, by means of backward in-

duction we obtain a recursive expression for $W \equiv V_1$, namely

$$\begin{aligned}
V_K(S_{K-1}) &= \max_s l(\Omega_m), \\
V_{k-1}(S_{k-2}) &= \max_s l(\Omega_m) + V_K(S_{k-2} \times s), \\
&\vdots \\
V_1(s) &= \max_s l(\Omega_m) + V_2(s).
\end{aligned} \tag{107}$$