Application of Graph Theory on the Advantage of Going First in Hex

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***Abstract*— Hex, a two-player strategic board game played on a hexagonal grid, has intrigued mathematicians and game theorists due to its simplicity and depth. Invented by Piet Hein in 1942 and later reinvented by John Nash, Hex has become the base of exploration in various mathematical disciplines. In this paper we will prove the rule that hex always has a winner and never ties. Since this was proven we have dived into the advantage of going first in Hex. Although not proven, we have shown that the color going first in Hex is more likely to win than color going second.**

***Keywords*—Hex, Hex strategy, Graph Theory, Discrete Mathematics**

I. INTRODUCTION

Hex, an abstract strategy game played on a hexagonal grid, has long captivated the brilliance of mathematicians, game theorists, and aficionados alike. Known for its simplicity yet strategic depth, Hex has become a subject of mathematical exploration across various disciplines. This paper undertakes the advantageous implications of the initial move in Hex, employing the methodologies afforded by graph theory.

Hex unfolded on a board consisting of hexagonal cells with the size 11x11, and it can vary from 4x4 to 19x19. The objective is to construct a path, linking the player's opposing side and making a connection impermeable by the opponent. Unlike other grid-based games, like chess and go, Hex employs a hexagonal structure, connecting each cell with either six, four, or three neighboring cells. This seemingly simple difference introduces a richness of strategic possibilities. Each player alternately places their colored hexagons on the board, creating a web of connections that intertwine and clash as the game progresses.

Strategic depth in Hex revolves around the inherent asymmetry created by the starting sides assigned to each player. Consequently, whether the first move bestows a strategic advantage becomes a subject of conjecture and fascination.

In this paper, we delve into the representation of Hex as a graph with vertices and edges. Through the lens of graph theory and induction, we try to prove the advantage of going first in Hex.

II. GRAPH THEORY

Graph is used to represent discrete objects and the relation between those objects. By definition, a graph $G = (V, E)$ where V is a finite set of vertices $\{v_1, v_2, \ldots, v_n\}$, and E is a subset of V, called edges that connects the vertices $\{e_1, e_2, \ldots, e_n\}$ [1].

A. Simple and Non-simple Graph

Based on the existence of graph loops and multiple edges, graphs are divided into simple graphs and non-simple graphs. A simple graph is a graph that does not contain any graph loops or multiple edges, while a non-simple graph is a graph that has either graph loops or multiple edges.

Non-simple graphs are divided into multi-graphs and pseudo-graphs. A multi-graph is a graph that contains double or more edges that connect two vertices. A pseudo-graph is a graph that contains an edge that connects vertices to itself making a loop.

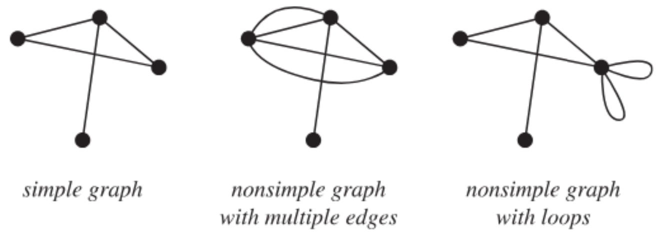

Fig. 1 types of graphs
(source: Wolfram Math World https://mathworld.wolfram.com/Graph.html).

In Figure 1, shown three graphs, the first graph is a simple graph with 4 vertices and 3 edges. The second graph is a non-simple graph with 4 vertices and 5 edges with 2 connections containing 2 edges, making it a multi-graph. The last graph is a non-simple graph with 4 vertices and 5 edges with 2 looping edges, making it a pseudo-graph.

B. Directed and Undirected Graph

Based on the direction of edges in a graph, graphs are classified into two types: Undirected Graphs and Directed Graphs. An Undirected Graph is a graph with edges not having a direction, whereas a Directed Graph is a graph in which each edge is given a specific direction.

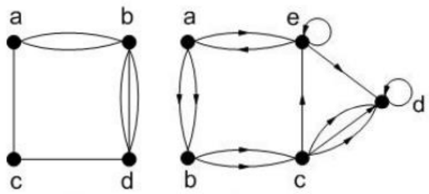

Fig. 2 Undirected and Directed Graph (source: https://informatika.stei.itb.ac.id/~rinaldi.munir/Matdis/2023-2024/19-Graf-Bagian1-2023.pdf, accessed on 08/12/2023).

III. COMBINATORICS

Combinatorics is a branch of Mathematics that counts the number of ways in which an object is constructed without enumerating every possible structure or combination [2].

In this paper we will use simple combinatorics to find the amount of game state possible in the game of Hex.

A. Sum Rule

If a task is done by *n* different ways **or** done by *m* ways in which both ways are different, then there is ***(n + m)*** ways to solve the task.

B. Product Rule

If a task is done by *n* different ways **and** done by *m* ways in which both ways are different, then there is ***(n x m)*** ways to solve the task.

IV. HEX

A. Hex rules and basics

Hex is played with two people, on a diamond-shaped board made up of hexagonal cells. The board dimensions can vary, but the typical size is 11 × 11. Two opposite sides of the board are labeled "black", and the remaining two sides are labeled "white". The players alternate turns to place their tile on any unoccupied space on the game board, with the goal of forming an unbroken chain of tiles (of his own color of course) linking his two regions [3].

There are many variations of the board size. Typically, in a standard game, it uses the 11x11 size.

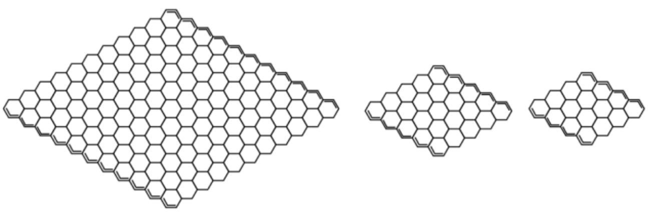

Fig. 3 variation of board (from the left 11x11, 5x5, 4x4) (source: writer's archive)

The "white" and the "black" sides often vary and depend. Some variations also use the color "red" and "blue" or other combination of color.

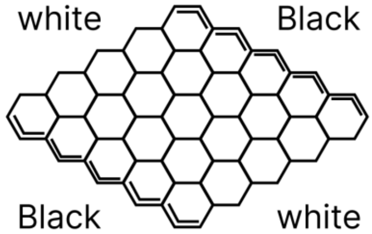

Fig. 4 a 5x5 Hex Board (source: writer's archive)

In figure 4. White's side is the left top and the right bottom of the board, while Black's side is the right top and the left bottom of the board. White's goal is to connect the left top side to the right bottom side of the board. Black's goal is to connect the right top side to the left bottom side of the board. See figure 4.

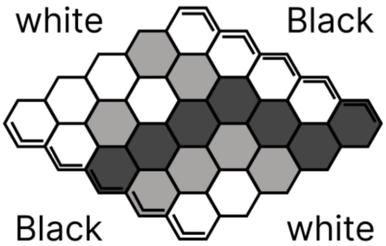

Fig. 5 black wins hex (source: writer's archive)

In Figure 5, Black won the game. Black successfully connects both sides of the board making black the winner, while White is the loser.

Depending on how many neighboring tiles are connected, there are three types of hex tiles: tiles with three neighbors, four neighbors, and six neighbors. See Figure 6.

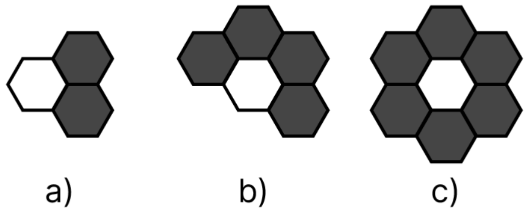

Fig. 6 a) three neighbors, b) four neighbors, c) six neighbors (source: writer's archive)

In Hex, there is another rule called the Swap rule. This rule states that after the first hex tile is placed, the opponent can choose to swap the colors. For example, if White is the first player to put the tile, and Black is the second player, then Black can choose to Swap colors, changing the Black player's color to White and the White player's color change to Black. After the swap, the player with the color Black will continue to place his tile and continue the game. This rule only applies to the first tile placed. In this paper we are not going to ignore this rule, since it only changes which player has what color. Not which color goes first. The color that goes first is the one going first, and it cannot be changed.

B. Proving the Rule: There is Always a Winner

In Hex there is always a winner and never a tie. We can see this by using graph theory.

Suppose a board where every hex tile is filled in and there is a winner.

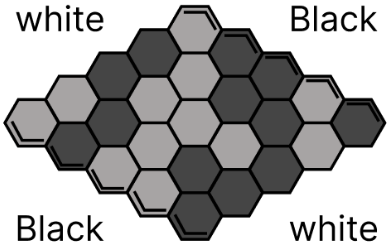

Fig. 7 a filled board
(source: writer's archive)

Think about the points in a hex tile as the vertices, shown in the figure 8 below.

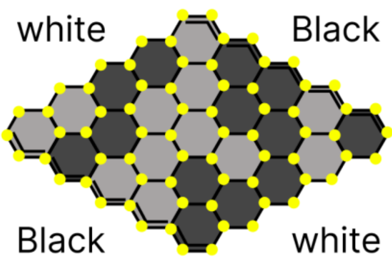

Fig. 8 vertices in a Hex board
(source: writer's archive)

Now we need to connect the edges where a hexagon meets the opponent's color or the opponent's side.

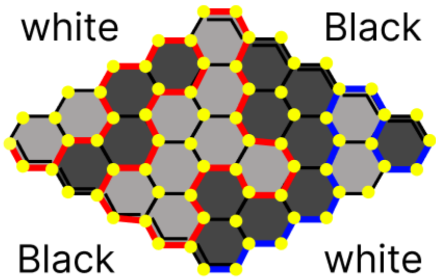

Fig. 9 connected vertices in a hex board.
(source: writer's archive)

In Figure 9, we can see that there are two main connections of edges represented by the red and blue connections. The red connection connects the left corner of the board with the top corner, and the blue connection connects the right corner of the board with the bottom.

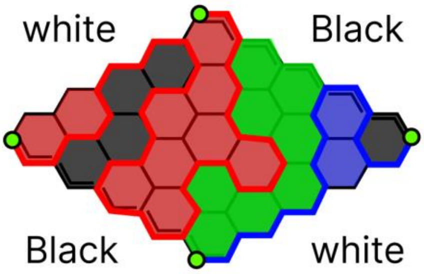

Fig. 10 hex board areas,
(source: writer's archive)

In Figure 10, the green dots are vertices that are in the corners. We can see that the 2 connections connect 2 green dots. Because of the connection between these dots, it shows that there is a path of hexagon that between the separate lines that wins the game, indicated by the green colored area.

To prove that there is always 2 connection that connects the four corners, we need to understand base of the connection which is the junction hex tile.

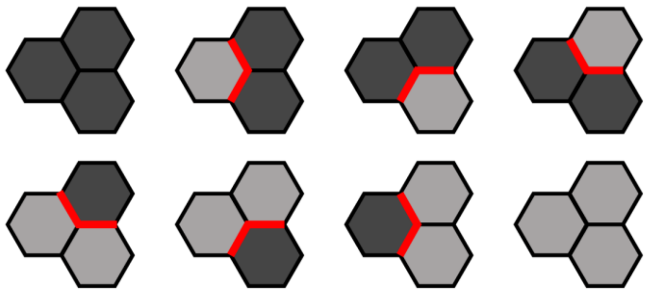

Fig. 11 all possible junction
(source: writer's archive)

In Figure 11, we can see all the possible junction between 2 different colors, black and white. The red line is edges connecting between the two different colors. The figure also shows that there are only two options at the connecting edge, it can be either 2 edges or none.

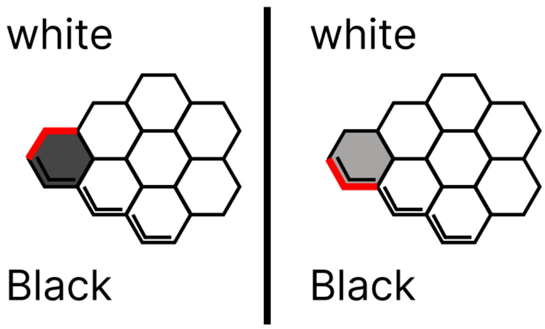

Fig. 12 connection at the corner of the board
(source: writer's archive)

In Figure 12, we can also conclude that there are always two connecting edges at the corner, no matter which corner or what color. Since we know that there are either two or zero edges connected in each junction, we know that this connection will grow and can't stop. Otherwise, there will be a junction that has only one edge connected which is not possible. This connection also can't connect to itself, otherwise it will create a junction with three edges that are connected, see figure 13.

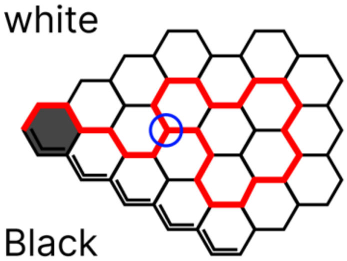

Fig. 13 illegal junction mark by the blue circle
(source: writer's archive)

Since it can't connect to itself then the only choice is to connect with another corner just like what we saw in figure 10. If the top corner connects to the left corner and the bottom corner connects to the right corner, that means there is a path of hexagon connecting the top right side of the board to the bottom left of the board. So does the opposite, if the top corner connects to the right corner and the bottom corner connects to the left corner, that means that there is a path of hexagon connecting the top left side of the board to the bottom right side of the board. See figure 14.

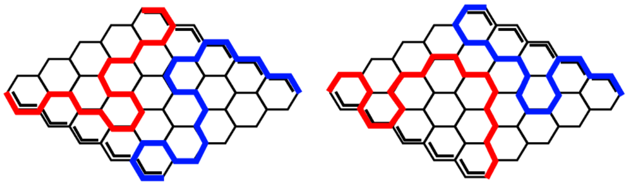

Fig. 14 two ways of connecting the corners.
(source: writer's archive)

Thus, it is proven that there is always a winner in the game of hex. Since the argument and proof doesn't get effected by the size of the board, we can say that in every size of board of Hex, there is always a winner and never a tie.

C. Strategies and Going First

Since we have proven that here is always a winner, we now can go deeper into strategies of winning and can ignore all possibilities and strategies to instead of losing, we can try to make a tie. This meant that the strategies discussed, all of which are very important to win the game.

Hex requires complex strategies to win, many of them are mathematically intense. In hex the basic includes [4]:

- Play defensively: defense is also attack.
- Use bridges to make connections between your pieces and simultaneously to block your opponent.
- If you can think of a strong response to your own move, then look for a better one!
- Never give up the game until it is clearly over but abandon areas of the board which are hopeless.

To better understand the strategy of the game, we can represent the board as a graph. Different from the representation at proving the rule, this graph uses the center of the hexagon as the vertices and the connection between hexagon as the edges.

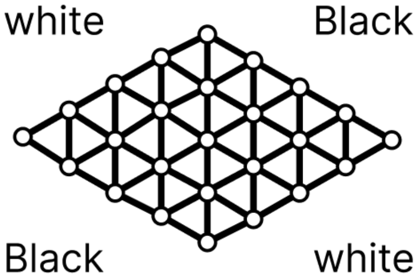

Fig. 15 representation of Hex with hexagon as vertices and connection as edges
(source: writer's archive)

Using this representation, we can better look at the state of the board. To make things more visible we will use red as the one going first and blue as the opponent.

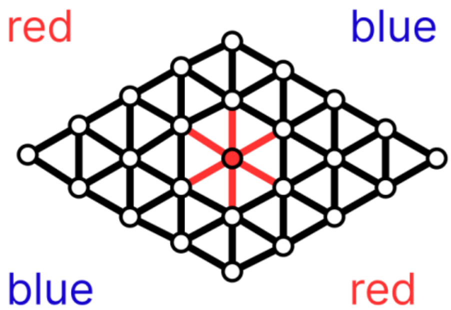

Fig. 16 red first move.
(source: writer's archive)

At the early stage of the game, looking at the potential of moves is very important. In figure 16, where red goes first, we can see that choosing the hex that has the highest number of neighbors will open many possibilities. More possibilities are needed to face the opponents.

At the early stage of the game, looking at the potential of moves is very important. In figure 16, where red goes first, we can see that choosing the Hex that has the highest number of neighbors will open many possibilities. The more possibilities available the more option of path to connect the sides and win the game.

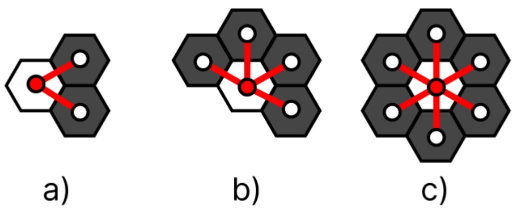

Fig. 17 possible connection in a) three neighbors b) four neighbors c) six neighbors
(source: writer's archive)

In going first, to get the highest possibility of winning, the player must choose the tile with the maximum amount of connecting tiles (see c) in figure 17).

In hex there are many strategies and principles that becomes the basics of complex strategies, some of them include:

A. The two Bridge

The formation consisting of two pieces that are nonadjacent but have two empty neighboring hexes in common is referred to as a two-bridge [5].

B. Blocking moves

When you have no pieces in the area, it is usually best to start blocking at a distance. If you block too close, then the opponent can simply flow around the attempted block. [5].

C. Momentum

The player who is dictating the play is said to have the momentum. Alternatively, the momentum is against the player who is being forced to respond to the opponent. The player with the momentum usually has the advantage and this advantage is often decisive. You should generally not hand over the momentum to the opponent unless you have a very good reason for doing so. In well played close matches, the momentum often swings between the two players with each move [5].

D. Multiple threats per move

The player who is dictating the play is said to have the momentum. Alternatively, the momentum is against the player who is being forced to respond to the opponent. The player with the momentum usually has the advantage and this advantage is often decisive. You should generally not hand over the momentum to the opponent unless you have a very good reason for doing so. In well played close matches, the momentum often swings between the two players with each move [5].

E. The center

The central region of the board is strategically the most important area. From the center, connections can spread out in many directions giving you more flexibility and options than starting from an edge. Furthermore, centrally played pieces are nearly equidistant from both of your edges. The greater the distance apart two pieces are, the harder they are to connect, i.e. their potential link is weaker [5].

Going first in hex has a few advantages. One of them is being the one with the first momentum, because you are the one who places the first tile. By correctly using the momentum and always acting so the momentum never falls to the enemy, we can say that the first player will be able to easily win. But the first player must make sure not to get blocked by the enemy. This might be easy on a 4x4 or 5x5 board but on a typical 11x11 board. Blocking can be a lot easier.

Having the first move also makes it easier to control the center as you are always ahead by one tile against the opponent. This can be seen by this example in Figure 18.

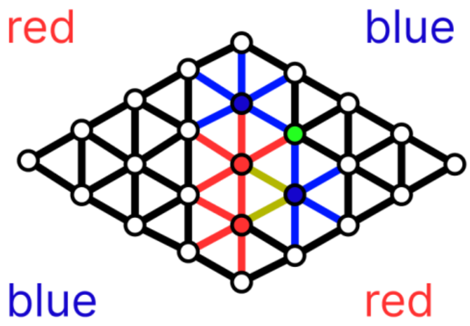

Fig. 18 controlling the center.
(source: writer's archive)

We can see that red is controlling the center because red goes first. In figure 18, red starts by taking the very center of the board, this strategy only works in a 5x5 board with very little chance to do blocking moves. The green dot is of the tile red can take. By taking this green tile, it blocks the connection between the blue tile, but also gives more potential link using the multiple threats per move principle. In 11x11 the same strategy can be used, but it needs to be careful and instead use the two-bridge strategy.

Using the two-bridge strategy will be the most important way to keep the momentum and ensure that each hex placed could connect to each other. And since going first means you can make the first bridge to ensure that if the opponent tries go through, you can easily block it since it guarantees a connection, no matter what your opponent does unless it's a fault by the player going first.

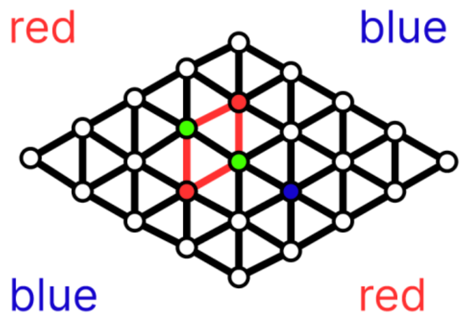

Fig. 19 the two-bridge strategy.
(source: writer's archive)

In Figure 19, we can see that the two green dots cause that if blue takes one of the tiles, then red can just take the other one. This formation guarantees that red can always connect their hex tiles. Making so that going first is an advantage but keep in mind that we are ignoring the swap rule, since it just changes which color a player gets, but the color that goes first is the one that gets the advantage in the end.

Connecting all these strategies is the fact that there is always a winner in hex.

IV. Testing

To test our theory that going first has an advantage, we implemented a hex game that randomly generates a game position for a 5x5 board. See appendix A for the program and the documentation.

Table 1 Result of Testing using a program that randomly generated a hex game on a 5x5 board.

Samples	First Player	Opponent	Win Percentage of going first
100	53	47	53.000 %
1.000	582	418	58.199 %
10.000	5667	4333	56.670 %
100.000	57475	42525	57.475 %
1.000.000	573724	426276	57.372 %
10.000.000	5731246	4268754	57.312 %
100.000.000	57326873	42673127	57.326 %

From Table 1, the result of testing using randomly generated game. We use a sample size from 100 to 100 million (100.000.000) samples. The result shows that on randomly generated moves it shows that the first player is more likely to win a game.

In a 5x5 board there are 25 cells with each cell having a possible state of empty, filled by player 1, or filled by the opponent. There are 3^{25} or 827.288.609.443 game states, but since a game could not be empty to end, we can narrow it down to just 2 states, filled by player 1 or by the opponent. We get 2^{25} or 33.554.432 game states, but this amount assumes that all games are played fully without a single empty cell. Assuming that there is a minimum of 9 hex tile places for a side to win in a 5x5 board of hex. That means that the amount of possible game state can be:

$$Total\ state\ where\ a\ side\ wins = \sum_{i=9}^{25} 2^{(25-i)}$$

The result is 391.804.103.331 or 391 million states. Since our max sample is 100 million which is far from 391 million state. Even if we assume that all 100 million samples are unique, we are still far from proving that the first player always wins.

Still since the number of samples tested is large and yet the win percentage of going first is still higher on all occasions. We can say that going first in hex is indeed more likely to win the game.

V. Conclusion

Furthermore, the empirical evidence gathered from the implementation of Hex supports the notion that going first provides a distinct advantage. The data from randomly generated game positions consistently shows a higher likelihood of the player making the first move emerging victorious. This aligns with the theoretical claims and reinforces the strategic importance of the initial move in Hex.

While this research successfully establishes the rule that there is always a winner and that going first has advantages, it is important to note that it does not conclusively prove that the first player is always the superior side. The nuanced nature of Hex, influenced by various strategies and player decisions, warrants further exploration.

In conclusion, this study contributes valuable insights into Hex by combining theoretical foundations with practical testing. The combination of Graph Theory, strategic analysis, and empirical results enriches our understanding of Hex, making it a captivating subject for both mathematical exploration and strategic gameplay. Future research could delve deeper into specific strategies, optimal opening moves, and potential variations in board sizes to further unravel the intricacies of Hex.

VI. Appendix

Appendixes A: documentation of writer's implementation of Hex and its testing result.

VII. Acknowledgment

The writer would like to thank all IF2120 lecturers, especially Dr. Fariska Zakhralativa Ruskanda, S.T., M.T. as the lecturer in our class IF2120 for Discrete Mathematics, for teaching and supporting students to write these papers. I have gained a much better understanding of graph theory and its application on Hex. I also would like to thank Dr. Ir. Rinaldi, M.T., who provided students with plenty of resources on Discrete Mathematics on the website.

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Pernyataan

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

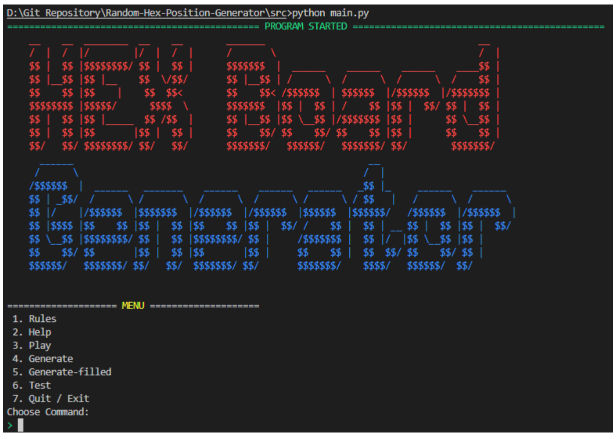
Bandung, 10 Desember 2023

R. Francisco Trianto B. 13522091

APPENDIX A

documentation of writer’s implementation of Hex and its testing result. You can find the code in this link: https://github.com/NoHaitch/Hex-With-Random-Position-Generator

Starting the program:



Generating a random game, in this game blue won

```
=============== GENERATING ===============
      1   2   3   4   5
    ======================
   \ / \ / \ / \ / \ / \
 A  | . | . | X | O | O |  A
     \ / \ / \ / \ / \ / \
   B  | X | . | O | X | X |  B
       \ / \ / \ / \ / \ / \
     C  | O | X | O | X | O |  C
         \ / \ / \ / \ / \ / \
       D  | X | O | X | O | X |  D
           \ / \ / \ / \ / \ / \
         E  | O | X | O | . | O |  E
             \ / \ / \ / \ / \ / \
              ======================
              1   2   3   4   5
=========== FINISH GENERATING ============
```

Test result using samples 100:

```
How many random games do you want to do?
> 100

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 53
Player 2 wins : 47
Win rasio for player 1 : 53.0%
```

Test result using samples 1.000:

```
How many random games do you want to do?
> 1000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 582
Player 2 wins : 418
Win rasio for player 1 : 58.199999999999996%
```

Test result using samples 10.000:

```
How many random games do you want to do?
> 10000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 5667
Player 2 wins : 4333
Win rasio for player 1 : 56.67%
```

Test result using samples 100.000:

```
How many random games do you want to do?
> 100000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 57475
Player 2 wins : 42525
Win rasio for player 1 : 57.475%
```

Test result using samples 1.000.000:

```
How many random games do you want to do?
> 1000000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 573724
Player 2 wins : 426276
Win rasio for player 1 : 57.3724%
```

Test result using samples 10.000.000:

```
How many random games do you want to do?
> 10000000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 5731246
Player 2 wins : 4268754
Win rasio for player 1 : 57.31246%
```

Test result using samples: 100.000.000

```
> 100000000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 57326873
Player 2 wins : 42673127
Win ratio for player 1 : 57.326873%
```