1 Глава 3 Уравнения на права и равнина в пространството

1.1 Задача 1.

Да се намери точка M', ортогонално симетрична
на точката $M(1,1,2)$ относно равнината ε, определена с точките
$M_1(5,10,0),\ M_2(4,0,-7),\ M_3(2,4,-5)$. Да се определят и директорните
косинуси в посока от M към M'
Решение:

$$\varepsilon\begin{cases} z\ M_1(5,10,0) \\ z\ M_2(4,0,-7) \\ z\ M_3(2,4,-5) \end{cases}$$

$$\varepsilon:\begin{vmatrix} x-5 & y-10 & z \\ -1 & -10 & -7 \\ -3 & -6 & -5 \end{vmatrix}=0$$

$$\varepsilon: 50(x-5)+21(y-10)+6z-30z-42(x-5)-5(y-10)=0$$

$$\varepsilon: 8(x-5)+16(y-10)-24z=0\ |\tfrac{1}{8}$$

$$\varepsilon: x-5+2(y-10)-3z=0$$

$$\varepsilon: x+2y-3z-25=0$$

$$N_\varepsilon(1,2,-3)\perp\varepsilon$$

$$g\begin{cases} z\ M(1,1,2) \\ \parallel N_\varepsilon(1,2,-3) \end{cases}$$

$$g\begin{cases} x=1+\lambda \\ y=1+2\lambda \\ z=2-3\lambda \end{cases}$$

$$g\cap\varepsilon=M_0(x_0,y_0,z_0)$$

$$1+\lambda_0+2+4\lambda_0-6+9\lambda_0-25=0$$

$$14\lambda_0=28 \implies \lambda_0=2 \implies M_0(3,5,-4)$$

$$M'(x',y',z')$$

$$M_0(\tfrac{x_M+x'}{2},\tfrac{y_M+y'}{2},\tfrac{z_M+z'}{2})$$

$3 = \frac{1+x'}{2} \quad 5 = \frac{1+y'}{2} \quad -4 = \frac{2+z'}{2}$

$x' = 5 \quad y' = 9 \quad z = -10$

$\implies M'(5, 9, -10)$

$\overrightarrow{MM'}(4, 8, -12) \parallel \overrightarrow{q}(1, 2, -3)$

$\implies \overrightarrow{n_q} = \frac{\overrightarrow{q}}{|\overrightarrow{q}|}$

$|\overrightarrow{q}| = \sqrt{1+4+9} = \sqrt{14}$

$\overrightarrow{n_q}(\frac{1}{14}, \frac{2}{14}, -\frac{3}{14})$

Директроните косинуси съвпадат с кординатите на $\overrightarrow{n_q}$

1.2 Задача 2.

Да се намери точка M', ортогонално симетрична
на точката $M(-1, 1, 2)$ относно правата
$g \begin{cases} x - y + 1 = 0 \\ x - z - 2 = 0 \end{cases}$
и да се намери разтоянието от M до g
Решение:

$$g \begin{cases} x = 0 + \lambda \\ y = 1 + \lambda \qquad \lambda \in \mathbb{R} \\ z = -2 + \lambda \end{cases}$$

$$g \begin{cases} z\ G(0, 1, -2) \\ \parallel \overrightarrow{g}(1, 1, 1) \end{cases}$$

$$\varepsilon \begin{cases} z\ M(-1, 1, 2) \\ \perp \overrightarrow{g}(1, 1, 1) \end{cases}$$

$\implies \varepsilon : 1(x+1) + 1(y-1) + 1(z-2) = 0$

$\varepsilon : x + y + z - 2 = 0$

$\varepsilon \cap g = M_0(x_0, y_0, z_0)$

$$\begin{aligned} y_0 &= x_0 + 1 \\ z_0 &= x_0 - 2 \\ x_0 + y_0 + z_0 - 2 &= 0 \end{aligned}$$

$$\begin{aligned} y_0 &= x_0 + 1 \\ z_0 &= x_0 - 2 \\ x_0 + x_0 + 1 + x_0 - 2 - 2 &= 0 \end{aligned}$$

$$\begin{aligned} y_0 &= x_0 + 1 \\ z_0 &= x_0 - 2 \\ 3x_0 &= 3 \end{aligned}$$

$$\begin{aligned} y_0 &= 2 \\ z_0 &= -1 \\ x_0 &= 1 \end{aligned}$$

$\Longrightarrow M_0(1, 2, -1)$

$M'(x', y', z')$

$M_0(\frac{x_M + x'}{2}, \frac{y_M + y'}{2}, \frac{z_M + z'}{2})$

$1 = \frac{-1 + x'}{2} \quad 2 = \frac{1 + y'}{2} \quad -1 = \frac{2 + z'}{2}$

$x' = 3 \quad y' = 3 \quad z = -4$

$\Longrightarrow M'(3, 3, -4)$

$d(M, g) = d(M, M_0) = |\overrightarrow{MM_0}|\ (\varepsilon \cap g = M_0,\ \varepsilon\ z\ M,\ \varepsilon \perp g)$

$\overrightarrow{MM_0}(2, 1, -3)$

$|\overrightarrow{MM_0}| = \sqrt{4 + 1 = 9} = \sqrt{14}$

1.3 Задача 3.

Да се намери трансферзалата на правите

$a \begin{cases} x = 3 + \lambda \\ y = -1 + 2\lambda & \lambda \in \mathbb{R} \\ z = 4\lambda \end{cases} \quad b \begin{cases} x = -2 + 3\mu \\ y = -1 & \mu \in \mathbb{R} \\ z = 4 - 5\mu \end{cases}$

минаваща през точката $P(1, 1, 1)$

Решение:

$a \begin{cases} z\ A(3, -1, 0) \\ \parallel \overrightarrow{a}(1, 2, 4) \end{cases}$

$$\alpha \begin{cases} z\ P(1,1,1) \\ z\ A(3,-1,0) \\ \parallel\ \overrightarrow{a}(1,2,4) \end{cases}$$

$$\overrightarrow{AP}(-2,2,1)$$

$$\overrightarrow{AP} \times \overrightarrow{a} \left(\begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix}, \begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} \right)$$

$$\overrightarrow{AP} \times \overrightarrow{a}(6,9,-6) \parallel \overrightarrow{q}(2,3,-2)$$

$$\alpha \begin{cases} z\ P(1,1,1) \\ \perp\ \overrightarrow{q}(2,3,-2) \end{cases}$$

$$\alpha : 2(x-1) + 3(y-1) - 2(z-1) = 0$$

$$\alpha : 2x - \cancel{2} + 3y - 3 - 2z + \cancel{2} = 0$$

$$\alpha : 2x + 3y - 2z - 3 = 0$$

$$b \begin{cases} z\ B(-2,-1,4) \\ \parallel\ \overrightarrow{b}(3,0,-5) \end{cases}$$

$$\beta \begin{cases} z\ P(1,1,1) \\ z\ B(-2,-1,4) \\ \parallel\ \overrightarrow{b}(3,0,-5) \end{cases}$$

$$\overrightarrow{BP}(3,2,-3)$$

$$\overrightarrow{BP} \times \overrightarrow{b} \left(\begin{vmatrix} 2 & -3 \\ 0 & -5 \end{vmatrix}, \begin{vmatrix} -3 & 3 \\ -5 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 3 & 0 \end{vmatrix} \right)$$

$$\overrightarrow{BP} \times \overrightarrow{b}(-10,6,-6) \parallel \overrightarrow{w}(-5,3,-3)$$

$$\beta \begin{cases} z\ P(1,1,1) \\ \perp\ \overrightarrow{w}(-5,3,-3) \end{cases}$$

$$\beta : -5(x-1) + 3(y-1) - 3(z-1) = 0$$

$$\beta : -5x + 5 + 3y - \cancel{3} - 3z + \cancel{3} = 0$$

$$\beta : -5x + 3y - 3z + 5 = 0 \mid -1$$

$$\beta : 5x - 3y + 3z - 5 = 0$$

$$t\begin{cases}2x+3y-2z-3=0\\5x-3y+3z-5=0\end{cases}$$

1.4 Задача 3.

Да се намерият ъравнението на оста отсечка и дължината на оста-отсечка на кръстосаните правите

$$a\begin{cases}x=7+\lambda\\y=3+2\lambda\\z=9-\lambda\end{cases}\quad \lambda\in\mathbb{R}\ b\begin{cases}x=3-7\mu\\y=1+2\mu\\z=1+3\mu\end{cases}\quad \mu\in\mathbb{R}$$

$$a\begin{cases}z\ A(7,3,9)\\ \parallel\ \overrightarrow{a}(1,2,-1)\end{cases}\qquad b\begin{cases}z\ B(3,1,1)\\ \parallel\ \overrightarrow{b}(-7,2,3)\end{cases}$$

$P_1(7+\lambda,3+2\lambda,9-\lambda)$

$P_2(3-7\mu,1+2\mu,1+3\mu)$

$\overrightarrow{P_1P_2}(-7\mu-\lambda-4,2\mu-2\lambda-2,3\mu+\lambda-8)$

$$\begin{aligned}\overrightarrow{P_1P_2}\overrightarrow{a}&=0\\\overrightarrow{P_1P_2}\overrightarrow{b}&=0\end{aligned}$$

$-7\mu-\lambda-\cancel{4}+4\mu-4\lambda-\cancel{4}-3\mu-\lambda+\cancel{8}=0$

$-6\mu-6\lambda=0\mid-\frac{1}{6}$

$\mu+\lambda=0$

$749\mu+7\lambda+\cancel{28}+4\mu-4\lambda-\cancel{4}+9\mu+3\lambda-\cancel{24}=0$

$62\mu+6\lambda=0\mid\frac{1}{2}$

$31\mu+3\lambda=0$

$\lambda=-\mu$

$28\mu=0\implies\mu=\lambda=0$
$\implies P_1\equiv A(7,3,9),\ P_2\equiv B(3,1,1)$

$\overrightarrow{P_1P_2}(-4,-2,-8)\parallel\overrightarrow{q}(2,1,4)$

$$t\begin{cases} z\ P_2(3,1,1) \\ \parallel q(2,1,4) \end{cases}$$

$$\implies t\begin{cases} x = 3+2v \\ y = 1+v \qquad v\in\mathbb{R} \\ z = 1+4v \end{cases}$$

$$d(P_1,P_2) = |\overrightarrow{P_1P_2}| = |-2||\overrightarrow{q}| = 2\sqrt{4+1+16} = 2\sqrt{21}$$