# 1 Глава 3 Уравнения на права и равнина в пространството

## 1.1 Задача 1.

Да се намери точка $M'$, ортогонално симетрична
на точката $M(1,1,2)$ относно равнината $\varepsilon$, определена с точките
$M_1(5,10,0),\ M_2(4,0,-7),\ M_3(2,4,-5)$. Да се определят и директорните
косинуси в посока от $M$ към $M'$
Решение:

$$\varepsilon \begin{cases} z\ M_1(5,10,0) \\ z\ M_2(4,0,-7) \\ z\ M_3(2,4,-5) \end{cases}$$

$$\varepsilon : \begin{vmatrix} x-5 & y-10 & z \\ -1 & -10 & -7 \\ -3 & -6 & -5 \end{vmatrix} = 0$$

$$\varepsilon : 50(x-5) + 21(y-10) + 6z - 30z - 42(x-5) - 5(y-10) = 0$$

$$\varepsilon : 8(x-5) + 16(y-10) - 24z = 0 \mid \tfrac{1}{8}$$

$$\varepsilon : x - 5 + 2(y-10) - 3z = 0$$

$$\varepsilon : x + 2y - 3z - 25 = 0$$

$$N_\varepsilon(1,2,-3) \perp \varepsilon$$

$$g \begin{cases} z\ M(1,1,2) \\ \parallel N_\varepsilon(1,2,-3) \end{cases}$$

$$g \begin{cases} x = 1 + \lambda \\ y = 1 + 2\lambda \\ z = 2 - 3\lambda \end{cases}$$

$$g \cap \varepsilon = M_0(x_0, y_0, z_0)$$

$$1 + \lambda_0 + 2 + 4\lambda_0 - 6 + 9\lambda_0 - 25 = 0$$

$$14\lambda_0 = 28 \implies \lambda_0 = 2 \implies M_0(3,5,-4)$$

$$M'(x', y', z')$$

$$M_0(\tfrac{x_M + x'}{2}, \tfrac{y_M + y'}{2}, \tfrac{z_M + z'}{2})$$

$3 = \frac{1+x'}{2} \quad 5 = \frac{1+y'}{2} \quad -4 = \frac{2+z'}{2}$

$x' = 5 \quad y' = 9 \quad z = -10$

$\implies M'(5, 9, -10)$

$\overrightarrow{MM'}(4, 8, -12) \parallel \overrightarrow{q}(1, 2, -3)$

$\implies \overrightarrow{n_q} = \frac{\overrightarrow{q}}{|\overrightarrow{q}|}$

$|\overrightarrow{q}| = \sqrt{1+4+9} = \sqrt{14}$

$\overrightarrow{n_q}(\frac{1}{14}, \frac{2}{14}, -\frac{3}{14})$

Директроните косинуси съвпадат с кординатите на $\overrightarrow{n_q}$

## 1.2 Задача 2.

Да се намери точка $M'$, ортогонално симетрична
на точката $M(-1, 1, 2)$ относно правата
$g \begin{cases} x - y + 1 = 0 \\ x - z - 2 = 0 \end{cases}$
и да се намери разтоянието от $M$ до $g$
Решение:

$$g \begin{cases} x = 0 + \lambda \\ y = 1 + \lambda \qquad \lambda \in \mathbb{R} \\ z = -2 + \lambda \end{cases}$$

$$g \begin{cases} z\ G(0, 1, -2) \\ \parallel \overrightarrow{g}(1, 1, 1) \end{cases}$$

$$\varepsilon \begin{cases} z\ M(-1, 1, 2) \\ \perp \overrightarrow{g}(1, 1, 1) \end{cases}$$

$\implies \varepsilon : 1(x+1) + 1(y-1) + 1(z-2) = 0$

$\varepsilon : x + y + z - 2 = 0$

$\varepsilon \cap g = M_0(x_0, y_0, z_0)$

$$\begin{aligned} y_0 &= x_0 + 1 \\ z_0 &= x_0 - 2 \\ x_0 + y_0 + z_0 - 2 &= 0 \end{aligned}$$

$$\begin{aligned} y_0 &= x_0 + 1 \\ z_0 &= x_0 - 2 \\ x_0 + x_0 + 1 + x_0 - 2 - 2 &= 0 \end{aligned}$$

$$\begin{aligned} y_0 &= x_0 + 1 \\ z_0 &= x_0 - 2 \\ 3x_0 &= 3 \end{aligned}$$

$$\begin{aligned} y_0 &= 2 \\ z_0 &= -1 \\ x_0 &= 1 \end{aligned}$$

$\Longrightarrow M_0(1, 2, -1)$

$M'(x', y', z')$

$M_0(\frac{x_M+x'}{2}, \frac{y_M+y'}{2}, \frac{z_M+z'}{2})$

$1 = \frac{-1+x'}{2} \quad 2 = \frac{1+y'}{2} \quad -1 = \frac{2+z'}{2}$

$x' = 3 \quad y' = 3 \quad z = -4$

$\Longrightarrow M'(3, 3, -4)$

$d(M, g) = d(M, M_0) = |\overrightarrow{MM_0}| \; (\varepsilon \cap g = M_0, \; \varepsilon \; z \; M, \; \varepsilon \perp g)$

$\overrightarrow{MM_0}(2, 1, -3)$

$|\overrightarrow{MM_0}| = \sqrt{4 + 1 = 9} = \sqrt{14}$

## 1.3 Задача 3.

Да се намери трансферзалата на правите

$$a \begin{cases} x = 3 + \lambda \\ y = -1 + 2\lambda & \lambda \in \mathbb{R} \\ z = 4\lambda \end{cases} \qquad b \begin{cases} x = -2 + 3\mu \\ y = -1 & \mu \in \mathbb{R} \\ z = 4 - 5\mu \end{cases}$$

минаваща през точката $P(1, 1, 1)$

Решение:

$$a \begin{cases} z \; A(3, -1, 0) \\ \parallel \; \overrightarrow{a}(1, 2, 4) \end{cases}$$

$$\alpha \begin{cases} z\ P(1,1,1) \\ z\ A(3,-1,0) \\ \parallel\ \overrightarrow{a}(1,2,4) \end{cases}$$

$$\overrightarrow{AP}(-2,2,1)$$

$$\overrightarrow{AP} \times \overrightarrow{a} \left( \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix}, \begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} \right)$$

$$\overrightarrow{AP} \times \overrightarrow{a}(6,9,-6) \parallel \overrightarrow{q}(2,3,-2)$$

$$\alpha \begin{cases} z\ P(1,1,1) \\ \perp\ \overrightarrow{q}(2,3,-2) \end{cases}$$

$$\alpha : 2(x-1) + 3(y-1) - 2(z-1) = 0$$

$$\alpha : 2x - \cancel{2} + 3y - 3 - 2z + \cancel{2} = 0$$

$$\alpha : 2x + 3y - 2z - 3 = 0$$

$$b \begin{cases} z\ B(-2,-1,4) \\ \parallel\ \overrightarrow{b}(3,0,-5) \end{cases}$$

$$\beta \begin{cases} z\ P(1,1,1) \\ z\ B(-2,-1,4) \\ \parallel\ \overrightarrow{b}(3,0,-5) \end{cases}$$

$$\overrightarrow{BP}(3,2,-3)$$

$$\overrightarrow{BP} \times \overrightarrow{b} \left( \begin{vmatrix} 2 & -3 \\ 0 & -5 \end{vmatrix}, \begin{vmatrix} -3 & 3 \\ -5 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 3 & 0 \end{vmatrix} \right)$$

$$\overrightarrow{BP} \times \overrightarrow{b}(-10,6,-6) \parallel \overrightarrow{w}(-5,3,-3)$$

$$\beta \begin{cases} z\ P(1,1,1) \\ \perp\ \overrightarrow{w}(-5,3,-3) \end{cases}$$

$$\beta : -5(x-1) + 3(y-1) - 3(z-1) = 0$$

$$\beta : -5x + 5 + 3y - \cancel{3} - 3z + \cancel{3} = 0$$

$$\beta : -5x + 3y - 3z + 5 = 0 \mid -1$$

$$\beta : 5x - 3y + 3z - 5 = 0$$

$$t\begin{cases}2x+3y-2z-3=0\\5x-3y+3z-5=0\end{cases}$$

## 1.4 Задача 4.

Да се намерият зравнението на оста отсечка и дължината на оста-отсечка на кръстосаните правите

$$a\begin{cases}x=7+\lambda\\y=3+2\lambda\quad\lambda\in\mathbb{R}\\z=9-\lambda\end{cases}\qquad b\begin{cases}x=3-7\mu\\y=1+2\mu\quad\mu\in\mathbb{R}\\z=1+3\mu\end{cases}$$

$$a\begin{cases}z\ A(7,3,9)\\\parallel\ \overrightarrow{a}(1,2,-1)\end{cases}\qquad b\begin{cases}z\ B(3,1,1)\\\parallel\ \overrightarrow{b}(-7,2,3)\end{cases}$$

$P_1(7+\lambda,3+2\lambda,9-\lambda)$

$P_2(3-7\mu,1+2\mu,1+3\mu)$

$\overrightarrow{P_1P_2}(-7\mu-\lambda-4,2\mu-2\lambda-2,3\mu+\lambda-8)$

$$\begin{aligned}\overrightarrow{P_1P_2}\overrightarrow{a}&=0\\\overrightarrow{P_1P_2}\overrightarrow{b}&=0\end{aligned}$$

$-7\mu-\lambda-\cancel{4}+4\mu-4\lambda-\cancel{4}-3\mu-\lambda+\cancel{8}=0$

$-6\mu-6\lambda=0\mid-\frac{1}{6}$

$\mu+\lambda=0$

$49\mu+7\lambda+\cancel{28}+4\mu-4\lambda-\cancel{4}+9\mu+3\lambda-\cancel{24}=0$

$62\mu+6\lambda=0\mid\frac{1}{2}$

$31\mu+3\lambda=0$

$\lambda=-\mu$

$28\mu=0\implies\mu=\lambda=0$
$\implies P_1\equiv A(7,3,9),\ P_2\equiv B(3,1,1)$

$\overrightarrow{P_1P_2}(-4,-2,-8)\parallel\overrightarrow{q}(2,1,4)$

$$t\begin{cases}z\ P_2(3,1,1)\\\parallel q(2,1,4)\end{cases}$$

$$\Longrightarrow t \begin{cases} x = 3 + 2v \\ y = 1 + v & v \in \mathbb{R} \\ z = 1 + 4v \end{cases}$$

$d(P_1, P_2) = |\overrightarrow{P_1P_2}| = |-2||\overrightarrow{q}| = 2\sqrt{4+1+16} = 2\sqrt{21}$

# 2 Ръководство по АГ - В.Михова

## 2.1 Задача 259. б)

Да се намери трансферзалата на правите

$$a \begin{cases} x = 3 + s \\ y = -1 + 2s & s \in \mathbb{R} \\ z = 4s \end{cases} \quad b \begin{cases} x = -2 + 3k \\ y = -1 & k \in \mathbb{R} \\ z = 4 - 5k \end{cases}$$

Ако трансферзалата е $\parallel l \begin{cases} x - 3y + z = 0 \\ x + y - z + 4 = 0 \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & 1 & -1 & -4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & -4 & 2 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 2 & -1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 \end{array}\right)$$

$z = \lambda,\ \lambda \in \mathbb{R}$

$2y = \lambda - 2 \implies y = -1 + \frac{1}{2}\lambda$

$x = y = -1 + \frac{1}{2}\lambda$

$$\Longrightarrow l \begin{cases} x = -1 + \frac{1}{2}\lambda \\ y = -1 + \frac{1}{2}\lambda & \lambda \in \mathbb{R} \\ z = \lambda \end{cases} \Longrightarrow l \begin{cases} z\ L(-1,-1,0) \\ \parallel \overrightarrow{l}(\frac{1}{2}, \frac{1}{2}, 1) \end{cases}$$

$$a \begin{cases} z\ A(3,-1,0) \\ \parallel \overrightarrow{a}(1,2,4) \end{cases}$$

$$\alpha \begin{cases} z\ A(3,-1,0) \\ \parallel \overrightarrow{a}(1,2,4) \\ \parallel \overrightarrow{l}(1,1,2) \end{cases}$$

$$\overrightarrow{a} \times \overrightarrow{l} \left( \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \right)$$

$\overrightarrow{a} \times \overrightarrow{l}(0, 2, -1)$

$$\alpha \begin{cases} z\ A(3,-1,0) \\ \perp \overrightarrow{a} \times \overrightarrow{l}(0,2,-1) \end{cases} \Longrightarrow \alpha : 2y - z + 2 = 0$$

$$b\begin{cases} \ni B(-2,-1,4) \\ \parallel \overrightarrow{b}(3,0,-5) \end{cases}$$

$$\beta\begin{cases} \ni B(-2,-1,4) \\ \parallel \overrightarrow{b}(3,0,-5) \parallel \overrightarrow{l}(1,1,2) \end{cases}$$

$$\overrightarrow{b} \times \overrightarrow{l} \left( \begin{vmatrix} 0 & -5 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} -5 & 3 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} \right)$$

$$\overrightarrow{b} \times \overrightarrow{l}(5,-11,3)$$

$$\beta\begin{cases} \ni B(-2,-1,4) \\ \perp \overrightarrow{b} \times \overrightarrow{l}(5,-11,3) \end{cases}$$

$$\implies \beta : 5(x+2) - 11(y+1) + 3(z-4) = 0$$

$$\beta : 5x + 10 - 11y - 11 + 3z - 12 = 0$$

$$\beta : 5x - 11y + 3z - 13 = 0$$

$$\implies t\begin{cases} 2y - z + 2 = 0 \\ 5x - 11y + 3z - 13 = 0 \end{cases}$$

## 2.2 Задача 259. в)

Да се намери трансферзалата на правите

$$a\begin{cases} x = 3 + s \\ y = -1 + 2s \quad s \in \mathbb{R} \\ z = 4s \end{cases} \qquad b\begin{cases} x = -2 + 3k \\ y = -1 \qquad\quad k \in \mathbb{R} \\ z = 4 - 5k \end{cases}$$

Ако трансферзалата лежи в равнината

$\alpha : x - 3y + z - 5 = 0$

$$a \cap \alpha = A(-2+3k, -1+2s, 4s)$$

$$3 + s - 3(-1+2s) + 4s - 5 = 0$$

$$3 + s + 3 - 6s + 4s - 5 = 0$$

$$s = 1 \implies A(4,1,4)$$

$$b \cap \alpha = B(-2+3k, -1, 4-5k)$$

$$-2 + 3k + 3 + 4 - 5k - 5 = 0$$

$$k = 0 \implies B(-2,-1,4)$$

$$\overrightarrow{AB}(-6,-2,0) \parallel \overrightarrow{q}(3,1,0)$$

$AB \begin{cases} z\ A(4,1,4) \\ \parallel\ \overrightarrow{q}(3,1,0) \end{cases}$

$\implies AB \begin{cases} x = 4 + \upsilon \\ y = 1 + \upsilon \quad \upsilon \in \mathbb{R} \\ z = 4 \end{cases}$

## 2.3 Задача 262.

Да се намери оста на кръстосаните прави

$l \begin{cases} x = 7 + s \\ y = 3 + 2s \quad s \in \mathbb{R} \\ z = 9 - s \end{cases} \quad m \begin{cases} x + 2y + z - 6 = 0 \\ x + 5y - z - 7 = 0 \end{cases}$

$m$ е зададена като пресечница на две равнини $\implies \exists$ сноп равнини пресичащисе в $m$

Нека $\alpha$ е равнина от снопа и $\alpha \parallel l$

$l \begin{cases} z\ L(7,3,9) \\ \parallel\ \overrightarrow{l}(1,2,-1) \end{cases}$

$\alpha : \lambda(x + 2y + z - 6) + \mu(x + 5y - z - 7) = 0$

$\alpha : (\lambda + \mu)x + (2\lambda + 5\mu)y + (\lambda - \mu)z - 6\lambda - 7\mu = 0$

$\implies \overrightarrow{n_\alpha}(\lambda + \mu, 2\lambda + 5\mu, \lambda - \mu) \perp \alpha$

$\implies \overrightarrow{n_\alpha} \perp l \implies \overrightarrow{n_\alpha} \perp \overrightarrow{l} \implies \overrightarrow{n_\alpha}\overrightarrow{l} = 0$

$1(\lambda + \mu) + 2(2\lambda + 5\mu) - 1(\lambda - \mu) = 0$

$\cancel{\lambda} + \mu + 4\mu + 10\mu - \cancel{\lambda} + \mu = 0$

$4\lambda + 12\mu = 0 \mid \frac{1}{4}$

$\lambda + 3\mu = 0 \implies \lambda = -3\mu$

$\mu = -1 \implies \lambda = 3 \implies \overrightarrow{n_\alpha}(2,1,4)$

$\implies \alpha : 2x + y + 4z - 11 = 0$

$p \begin{cases} z\ L(7,3,9) \\ \parallel \overrightarrow{n_\alpha}(2,1,4) \end{cases}$

$$\Longrightarrow\ p\begin{cases} x = 7 + 2k \\ y = 3 + k \qquad k \in \mathbb{R} \\ z = 9 + 4k \end{cases}$$

$p \cup \alpha = P$

$\Longrightarrow\ 2(7 + 2k) + 3 + k + 4(9 + 4k) - 11 = 0$

$14 + 4k + 3 + k + 36 + 16k - 11 = 0$
$21k = -42 \implies k = -2$
$\Longrightarrow\ P(3, 1, 1)$

$$l_0\begin{cases} z\ P(3, 1, 1) \\ \parallel\ \overrightarrow{l}(1, 2, -1) \end{cases}$$

$$l_0\begin{cases} x = 3 + \upsilon \\ y = 1 + 2\upsilon \quad \upsilon \in \mathbb{R} \\ z = 1 - \upsilon \end{cases}$$

$l_0 \cup m = M$

$$m\begin{cases} x + 2y + z - 6 = 0 \\ x + 5y - z - 7 = 0 \end{cases}$$

$\cancel{3} + \cancel{\upsilon} + \cancel{2} + 4\upsilon + \cancel{1} - \cancel{\upsilon} - \cancel{6} = 0 \implies \upsilon = 0$

$\Longrightarrow\ M(3, 1, 1)$

$$t\begin{cases} z\ L(7, 3, 9) \\ z\ M(3, 1, 1) \\ \parallel\ \overrightarrow{ML}(4, 2, 8) \parallel \overrightarrow{t}(2, 1, 4) \end{cases}$$

$$\Longrightarrow\ t\begin{cases} x = 7 + 2\tau \\ y = 3 + \tau \qquad \tau \in \mathbb{R} \\ z = 9 + 4\tau \end{cases}$$

## 2.4 Задача 262. II Начин

Да се намери оста на кръстосаните прави

$$l\begin{cases} x = 7 + s \\ y = 3 + 2s \quad s \in \mathbb{R} \\ z = 9 - s \end{cases} \qquad m\begin{cases} x + 2y + z - 6 = 0 \\ x + 5y - z - 7 = 0 \end{cases}$$

$$\begin{aligned}
x &= 6 - z - 2y \\
x &= 7 + z - 5y \\
6 - z - 2y &= 7 + z - 5y \\
1 + 2z - 3y &= 0 \\
2z + 1 &= 3y
\end{aligned}$$

$$\begin{aligned}
x &= 6 - z - 2y \\
2z + 1 &= 3y
\end{aligned}$$

$M_1(3,1,1),\ M_2(-4,3,4),\ \overrightarrow{M_1M_2}(-7,2,3)$

$$m\begin{cases} x = 3 - 7k \\ y = 1 + 2k \quad k \in \mathbb{R} \\ z = 1 + 3k \end{cases}$$

$$l\begin{cases} z\ L(7,3,9) \\ \parallel\ \overrightarrow{l}(1,2,-1) \end{cases} \qquad m\begin{cases} z\ M(3,1,1) \\ \parallel \overrightarrow{m}(-7,2,3) \end{cases}$$

$P_1(7+s, 3+2s9-s)$

$P_2(3-7k, 1+2k, 1+3k)$

$\overrightarrow{P_1P_2}(-7k-s-4, 2k-2s-2, 3k+s-8)$

$$\begin{aligned}
\overrightarrow{P_1P_2}\,\overrightarrow{a} &= 0 \\
\overrightarrow{P_1P_2}\,\overrightarrow{b} &= 0
\end{aligned}$$

$-7k - s - \cancel{4} + 4k - 4s - \cancel{4} - 3k - s + \cancel{8} = 0$

$-6k - 6s = 0 \mid -\frac{1}{6}$

$k + s = 0$

$49k + 7s + \cancel{28} + 4k - 4s - \cancel{4} + 9k + 3s - \cancel{24} = 0$

$62k + 6s = 0 \mid \frac{1}{2}$

$31k + 3s = 0$

$s = -k$

$28k = 0 \implies k = s = 0$

$\implies P_1 \equiv L(7,3,9),\ P_2 \equiv M(3,1,1)$

$\overrightarrow{P_1P_2}(-4,-2,-8) \parallel \overrightarrow{q}(2,1,4)$

$$t\begin{cases} z\ P_1(7,3,9) \\ \parallel q(2,1,4) \end{cases}$$

$$\implies t\begin{cases} x = 7 + 2\upsilon \\ y = 3 + \upsilon \\ z = 9 + 4\upsilon \end{cases} \quad \upsilon \in \mathbb{R}$$