

1 Глава 3 Уравнения на права и равнина в пространството

1.1 Задача 1.

Да се намери точка M' , ортогонално симетрична на точката $M(1, 1, 2)$ относно равнината ε , определена с точките $M_1(5, 10, 0)$, $M_2(4, 0, -7)$, $M_3(2, 4, -5)$. Да се определят и директорните косинуси в посока от M към M'

Решение:

$$\varepsilon \begin{cases} z M_1(5, 10, 0) \\ z M_2(4, 0, -7) \\ z M_3(2, 4, -5) \end{cases}$$

$$\varepsilon : \begin{vmatrix} x-5 & y-10 & z \\ -1 & -10 & -7 \\ -3 & -6 & -5 \end{vmatrix} = 0$$

$$\varepsilon : 50(x-5) + 21(y-10) + 6z - 30z - 42(x-5) - 5(y-10) = 0$$

$$\varepsilon : 8(x-5) + 16(y-10) - 24z = 0 \mid \frac{1}{8}$$

$$\varepsilon : x - 5 + 2(y - 10) - 3z = 0$$

$$\varepsilon : x + 2y - 3z - 25 = 0$$

$$N_\varepsilon(1, 2, -3) \perp \varepsilon$$

$$g \begin{cases} z M(1, 1, 2) \\ \parallel N_\varepsilon(1, 2, -3) \end{cases}$$

$$g \begin{cases} x = 1 + \lambda \\ y = 1 + 2\lambda \\ z = 2 - 3\lambda \end{cases}$$

$$g \cap \varepsilon = M_0(x_0, y_0, z_0)$$

$$1 + \lambda_0 + 2 + 4\lambda_0 - 6 + 9\lambda_0 - 25 = 0$$

$$14\lambda_0 = 28 \implies \lambda_0 = 2 \implies M_0(3, 5, -4)$$

$$M'(x', y', z')$$

$$M_0\left(\frac{x_M+x'}{2}, \frac{y_M+y'}{2}, \frac{z_M+z'}{2}\right)$$

$$3 = \frac{1+x'}{2} \quad 5 = \frac{1+y'}{2} \quad -4 = \frac{2+z'}{2}$$

$$x' = 5 \quad y' = 9 \quad z = -10$$

$$\implies M'(5, 9, -10)$$

$$\overrightarrow{MM'}(4, 8, -12) \parallel \overrightarrow{q}(1, 2, -3)$$

$$\implies \overrightarrow{n_q} = \frac{\overrightarrow{q}}{|\overrightarrow{q}|}$$

$$|\overrightarrow{q}| = \sqrt{1+4+9} = \sqrt{14}$$

$$\overrightarrow{n_q}\left(\frac{1}{14}, \frac{2}{14}, -\frac{3}{14}\right)$$

Директорните косинуси съвпадат с координатите на $\overrightarrow{n_q}$

1.2 Задача 2.

Да се намери точка M' , ортогонално симетрична на точката $M(-1, 1, 2)$ относно правата

$$g \begin{cases} x - y + 1 = 0 \\ x - z - 2 = 0 \end{cases}$$

и да се намери разстоянието от M до g

Решение:

$$g \begin{cases} x = 0 + \lambda \\ y = 1 + \lambda \\ z = -2 + \lambda \end{cases} \quad \lambda \in \mathbb{R}$$

$$g \begin{cases} z = G(0, 1, -2) \\ \parallel \overrightarrow{g}(1, 1, 1) \end{cases}$$

$$\varepsilon \begin{cases} z = M(-1, 1, 2) \\ \perp \overrightarrow{g}(1, 1, 1) \end{cases}$$

$$\implies \varepsilon : 1(x+1) + 1(y-1) + 1(z-2) = 0$$

$$\varepsilon : x + y + z - 2 = 0$$

$$\varepsilon \cap g = M_0(x_0, y_0, z_0)$$

$$\begin{aligned}y_0 &= x_0 + 1 \\z_0 &= x_0 - 2 \\x_0 + y_0 + z_0 - 2 &= 0\end{aligned}$$

$$\begin{aligned}y_0 &= x_0 + 1 \\z_0 &= x_0 - 2 \\x_0 + x_0 + 1 + x_0 - 2 - 2 &= 0\end{aligned}$$

$$\begin{aligned}y_0 &= x_0 + 1 \\z_0 &= x_0 - 2 \\3x_0 &= 3\end{aligned}$$

$$\begin{aligned}y_0 &= 2 \\z_0 &= -1 \\x_0 &= 1 \\ \implies M_0(1, 2, -1)\end{aligned}$$

$$M'(x', y', z')$$

$$M_0(\frac{x_M+x'}{2}, \frac{y_M+y'}{2}, \frac{z_M+z'}{2})$$

$$1 = \frac{-1+x'}{2} \quad 2 = \frac{1+y'}{2} \quad -1 = \frac{2+z'}{2}$$

$$x' = 3 \quad y' = 3 \quad z = -4$$

$$\implies M'(3, 3, -4)$$

$$d(M,g)=d(M,M_0)=|\overrightarrow{MM_0}| \; (\varepsilon \cap g=M_0, \; \varepsilon \; z \; M, \; \varepsilon \perp g)$$

$$\overrightarrow{MM_0}(2,1,-3)$$

$$|\overrightarrow{MM_0}|=\sqrt{4+1+9}=\sqrt{14}$$

1.3 Задача 3.

Да се намери трансферзалата на правите

$$a \begin{cases} x = 3 + \lambda \\ y = -1 + 2\lambda \\ z = 4\lambda \end{cases} \quad \lambda \in \mathbb{R} \quad b \begin{cases} x = -2 + 3\mu \\ y = -1 \\ z = 4 - 5\mu \end{cases} \quad \mu \in \mathbb{R}$$

минаваща през точката $P(1, 1, 1)$

Решение:

$$a \begin{cases} z \; A(3, -1, 0) \\ \parallel \; \vec{a}(1, 2, 4) \end{cases}$$

$$\alpha \left\{ \begin{array}{l} z \, P(1,1,1) \\ z \, A(3,-1,0) \\ \parallel \, \overrightarrow{a}(1,2,4) \end{array} \right.$$

$$\overrightarrow{AP}(-2,2,1)$$

$$\overrightarrow{AP} \times \overrightarrow{a} \left(\left| \begin{array}{cc} 2 & 1 \\ 2 & 4 \end{array} \right|, \left| \begin{array}{cc} 1 & -2 \\ 4 & 1 \end{array} \right|, \left| \begin{array}{cc} -2 & 2 \\ 1 & 2 \end{array} \right| \right)$$

$$\overrightarrow{AP} \times \overrightarrow{a}(6,9,-6) \parallel \overrightarrow{q}(2,3,-2)$$

$$\alpha \left\{ \begin{array}{l} z \, P(1,1,1) \\ \perp \, \overrightarrow{q}(2,3,-2) \end{array} \right.$$

$$\alpha:2(x-1)+3(y-1)-2(z-1)=0$$

$$\alpha:2x-2+3y-3-2z+2=0$$

$$\alpha:2x+3y-2z-3=0$$

$$b \left\{ \begin{array}{l} z \, B(-2,-1,4) \\ \parallel \, \overrightarrow{b}(3,0,-5) \end{array} \right.$$

$$\beta \left\{ \begin{array}{l} z \, P(1,1,1) \\ z \, B(-2,-1,4) \\ \parallel \, \overrightarrow{b}(3,0,-5) \end{array} \right.$$

$$\overrightarrow{BP}(3,2,-3)$$

$$\overrightarrow{BP} \times \overrightarrow{b} \left(\left| \begin{array}{cc} 2 & -3 \\ 0 & -5 \end{array} \right|, \left| \begin{array}{cc} -3 & 3 \\ -5 & 3 \end{array} \right|, \left| \begin{array}{cc} 3 & 2 \\ 3 & 0 \end{array} \right| \right)$$

$$\overrightarrow{BP} \times \overrightarrow{b}(-10,6,-6) \parallel \overrightarrow{w}(-5,3,-3)$$

$$\beta \left\{ \begin{array}{l} z \, P(1,1,1) \\ \perp \, \overrightarrow{w}(-5,3,-3) \end{array} \right.$$

$$\beta:-5(x-1)+3(y-1)-3(z-1)=0$$

$$\beta:-5x+5+3y-3-3z+3=0$$

$$\beta:-5x+3y-3z+5=0 \mid -1$$

$$\beta:5x-3y+3z-5=0$$

$$t \begin{cases} 2x + 3y - 2z - 3 = 0 \\ 5x - 3y + 3z - 5 = 0 \end{cases}$$

1.4 Задача 4.

Да се намерият уравнението на оста отсечка и дължината на оста-отсечка на кръстосаните правите

$$a \begin{cases} x = 7 + \lambda \\ y = 3 + 2\lambda \\ z = 9 - \lambda \end{cases} \quad \lambda \in \mathbb{R} \quad b \begin{cases} x = 3 - 7\mu \\ y = 1 + 2\mu \\ z = 1 + 3\mu \end{cases} \quad \mu \in \mathbb{R}$$

$$a \begin{cases} z A(7, 3, 9) \\ \parallel \vec{a}(1, 2, -1) \end{cases} \quad b \begin{cases} z B(3, 1, 1) \\ \parallel \vec{b}(-7, 2, 3) \end{cases}$$

$$P_1(7 + \lambda, 3 + 2\lambda, 9 - \lambda)$$

$$P_2(3 - 7\mu, 1 + 2\mu, 1 + 3\mu)$$

$$\overrightarrow{P_1 P_2}(-7\mu - \lambda - 4, 2\mu - 2\lambda - 2, 3\mu + \lambda - 8)$$

$$\begin{aligned} \overrightarrow{P_1 P_2} \vec{a} &= 0 \\ \overrightarrow{P_1 P_2} \vec{b} &= 0 \end{aligned}$$

$$-7\mu - \lambda - 4 + 4\mu - 4\lambda - 4 - 3\mu - \lambda + 8 = 0$$

$$-6\mu - 6\lambda = 0 \mid -\frac{1}{6}$$

$$\mu + \lambda = 0$$

$$49\mu + 7\lambda + 28 + 4\mu - 4\lambda - 4 + 9\mu + 3\lambda - 24 = 0$$

$$62\mu + 6\lambda = 0 \mid \frac{1}{2}$$

$$31\mu + 3\lambda = 0$$

$$\lambda = -\mu$$

$$\begin{aligned} 28\mu = 0 &\implies \mu = \lambda = 0 \\ \implies P_1 &\equiv A(7, 3, 9), \quad P_2 \equiv B(3, 1, 1) \end{aligned}$$

$$\overrightarrow{P_1 P_2}(-4, -2, -8) \parallel \vec{q}(2, 1, 4)$$

$$t \begin{cases} z P_2(3, 1, 1) \\ \parallel q(2, 1, 4) \end{cases}$$

$$\implies t \begin{cases} x = 3 + 2v \\ y = 1 + v \\ z = 1 + 4v \end{cases} \quad v \in \mathbb{R}$$

$$d(P_1, P_2) = |\overrightarrow{P_1 P_2}| = |-2| |\vec{q}| = 2\sqrt{4 + 1 + 16} = 2\sqrt{21}$$

2 РЪКОВОДСТВО ПО АГ - В.Михова

2.1 Задача 259. в)

Да се намери трансферзалата на правите

$$a \begin{cases} x = 3 + s \\ y = -1 + 2s \\ z = 4s \end{cases} \quad s \in \mathbb{R} \quad b \begin{cases} x = -2 + 3k \\ y = -1 \\ z = 4 - 5k \end{cases} \quad k \in \mathbb{R}$$

Ако трансферзалата лежи в равнината

$$\alpha : x - 3y + z - 5 = 0$$

$$a \cap \alpha = A(-2 + 3k, -1 + 2s, 4s)$$

$$3 + s - 3(-1 + 2s) + 4s - 5 = 0$$

$$3 + s + 3 - 6s + 4s - 5 = 0$$

$$s = 1 \implies A(4, 1, 4)$$

$$b \cap \alpha = B(-2 + 3k, -1, 4 - 5k)$$

$$-2 + 3k + 3 + 4 - 5k - 5 = 0$$

$$k = 0 \implies B(-2, -1, 4)$$

$$\overrightarrow{AB}(-6, -2, 0) \parallel \vec{q}(3, 1, 0)$$

$$AB \begin{cases} z \\ \parallel \vec{q}(3, 1, 0) \end{cases} \begin{cases} A(4, 1, 4) \\ \end{cases}$$

$$\implies AB \begin{cases} x = 4 + v \\ y = 1 + v \\ z = 4 \end{cases} \quad v \in \mathbb{R}$$

2.2 Задача 262.

Да се намери оста на кръстосаните прави

$$l \begin{cases} x = 7 + s \\ y = 3 + 2s \\ z = 9 - s \end{cases} \quad s \in \mathbb{R} \quad m \begin{cases} x + 2y + z - 6 = 0 \\ x + 5y - z - 7 = 0 \end{cases}$$

m е зададена като пресечница на две равнини $\implies \exists$ сноп равнини пресичащи се в m

Нека α е равнина от снопа и $\alpha \parallel l$

$$l \begin{cases} z \in L(7, 3, 9) \\ \parallel \vec{l}(1, 2, -1) \end{cases}$$

$$\alpha : \lambda(x + 2y + z - 6) + \mu(x + 5y - z - 7) = 0$$

$$\alpha : (\lambda + \mu)x + (2\lambda + 5\mu)y + (\lambda - \mu)z - 6\lambda - 7\mu = 0$$

$$\implies \vec{n}_\alpha(\lambda + \mu, 2\lambda + 5\mu, \lambda - \mu) \perp \alpha$$

$$\implies \vec{n}_\alpha \perp l \implies \vec{n}_\alpha \perp \vec{l} \implies \vec{n}_\alpha \vec{l} = 0$$

$$1(\lambda + \mu) + 2(2\lambda + 5\mu) - 1(\lambda - \mu) = 0$$

$$\lambda + \mu + 4\mu + 10\mu - \lambda + \mu = 0$$

$$4\lambda + 12\mu = 0 \mid \frac{1}{4}$$

$$\lambda + 3\mu = 0 \implies \lambda = -3\mu$$

$$\mu = -1 \implies \lambda = 3 \implies \vec{n}_\alpha(2, 1, 4)$$

$$\implies \alpha : 2x + y + 4z - 11 = 0$$

$$p \begin{cases} z \in L(7, 3, 9) \\ \parallel \vec{n}_\alpha(2, 1, 4) \end{cases}$$

$$\implies p \begin{cases} x = 7 + 2k \\ y = 3 + k \\ z = 9 + 4k \end{cases} \quad k \in \mathbb{R}$$

$$p \cup \alpha = P$$

$$\implies 2(7 + 2k) + 3 + k + 4(9 + 4k) - 11 = 0$$

$$14 + 4k + 3 + k + 36 + 16k - 11 = 0$$

$$21k = -42 \implies k = -2$$

$$\implies P(3, 1, 1)$$

$$l_0 \left\{ \begin{array}{l} z \, P(3, 1, 1) \\ \parallel \overrightarrow{t}(1, 2, -1) \end{array} \right.$$

$$l_0 \left\{ \begin{array}{l} x = 3 + v \\ y = 1 + 2v \\ z = 1 - v \end{array} \right. \quad v \in \mathbb{R}$$

$$l_0 \cup m = M$$

$$m \left\{ \begin{array}{l} x + 2y + z - 6 = 0 \\ x + 5y - z - 7 = 0 \end{array} \right.$$

$$3 + v + 2 + 4v + 1 - v - 6 = 0 \implies v = 0$$

$$\implies M(3, 1, 1)$$

$$t \left\{ \begin{array}{l} z \, L(7, 3, 9) \\ z \, M(3, 1, 1) \\ \parallel \overrightarrow{ML}(4, 2, 8) \parallel \overrightarrow{t}(2, 1, 4) \end{array} \right.$$

$$\implies t \left\{ \begin{array}{l} x = 7 + 2\tau \\ y = 3 + \tau \\ z = 9 + 4\tau \end{array} \right. \quad \tau \in \mathbb{R}$$

2.3 Задача 262. II Начин

Да се намери оста на кръстосаните прави

$$l \left\{ \begin{array}{l} x = 7 + s \\ y = 3 + 2s \\ z = 9 - s \end{array} \right. \quad s \in \mathbb{R} \quad m \left\{ \begin{array}{l} x + 2y + z - 6 = 0 \\ x + 5y - z - 7 = 0 \end{array} \right.$$

$$x = 6 - z - 2y$$

$$x = 7 + z - 5y$$

$$6 - z - 2y = 7 + z - 5y$$

$$1 + 2z - 3y = 0$$

$$2z + 1 = 3y$$

$$x = 6 - z - 2y$$

$$2z + 1 = 3y$$

$$M_1(3, 1, 1), \, M_2(-4, 3, 4), \, \overrightarrow{M_1M_2}(-7, 2, 3)$$

$$m \left\{ \begin{array}{l} x = 3 - 7k \\ y = 1 + 2k \\ z = 1 + 3k \end{array} \right. \quad k \in \mathbb{R}$$

$$l\left\{ \begin{array}{l} z\,L(7,3,9) \\ \parallel \overrightarrow{l}(1,2,-1) \end{array} \right. \quad m\left\{ \begin{array}{l} z\,M(3,1,1) \\ \parallel \overrightarrow{m}(-7,2,3) \end{array} \right.$$

$$P_1(7+s,3+2s9-s)$$

$$P_2(3-7k,1+2k,1+3k)$$

$$\overrightarrow{P_1P_2}(-7k-s-4,2k-2s-2,3k+s-8)$$

$$\begin{array}{lcl} \overrightarrow{P_1P_2}\overrightarrow{a} & = & 0 \\ \overrightarrow{P_1P_2}\overrightarrow{b} & = & 0 \end{array}$$

$$-7k-s-4+4k-4s-4-3k-s+8=0$$

$$-6k-6s=0\mid-\frac{1}{6}$$

$$k+s=0$$

$$49k+7s+28+4k-4s-4+9k+3s-24=0$$

$$62k+6s=0\mid\frac{1}{2}$$

$$31k+3s=0$$

$$s=-k$$

$$\begin{array}{l} 28k=0\implies k=s=0 \\ \implies P_1\equiv L(7,3,9),\,P_2\equiv M(3,1,1) \end{array}$$

$$\overrightarrow{P_1P_2}(-4,-2,-8)\parallel\overrightarrow{q}(2,1,4)$$

$$t\left\{ \begin{array}{l} z\,P_1(7,3,9) \\ \parallel q(2,1,4) \end{array} \right.$$

$$\implies t\left\{ \begin{array}{l} x=7+2v \\ y=3+v \\ z=9+4v \end{array} \right. \quad v\in\mathbb{R}$$