Курсова задача №3

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Като използвате подходящо развитие в степенен ред на подинтегралната функция пресметнете с точност $E = 10^{-4}$ определения интеграл

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2}\, dx$$

Решение:

$$\sqrt[4]{1+x} = (1+x)^{\frac{1}{4}} = \sum_{n=0}^{\infty} \binom{\frac{1}{4}}{n} x^n \implies$$

$$\sqrt[4]{1+x^2} = \sum_{n=0}^{\infty} \binom{\frac{1}{4}}{n} x^{2n} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{k=0}^{n-1} \left(\frac{1}{4} - k\right) x^{2n} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{k=0}^{n-1} - \left(\frac{4k-1}{4}\right) x^{2n} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{4^n} \prod_{k=0}^{n-1} (4k-1) x^{2n} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} \prod_{k=0}^{n-1} (4k-1) x^{2n} \implies$$

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2}\, dx = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} \prod_{k=0}^{n-1} (4k-1) x^{2n}\, dx =$$

$$= \sum_{n=0}^{\infty} \int_0^{\frac{1}{2}} \frac{(-1)^n}{4^n n!} \prod_{k=0}^{n-1} (4k-1) x^{2n}\, dx =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} \prod_{k=0}^{n-1} (4k-1) \int_0^{\frac{1}{2}} x^{2n}\, dx =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} \prod_{k=0}^{n-1} (4k-1) \left(\frac{x^{2n+1}}{2n+1} \right) \Bigg|_0^{\frac{1}{2}} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n! 2^{2n+1} (2n+1)} \prod_{k=0}^{n-1} (4k-1) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n! 2^{2n+1} (2n+1)} \prod_{k=0}^{n-1} (4k-1) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+2n+1} n! (2n+1)} \prod_{k=0}^{n-1} (4k-1) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n+1} n! (2n+1)} \prod_{k=0}^{n-1} (4k-1)$$

Получихме, че

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n+1} n! (2n+1)} \prod_{k=0}^{n-1} (4k-1)$$

или

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2}\, dx = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{4n+1} n! (2n+1)} \prod_{k=0}^{n-1} (4k-1) =$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1} n! (2n+1)} (-1) \prod_{k=0}^{n-1} (4k-1)$$

Ще докажем, че реда

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1} n!(2n+1)} (-1) \prod_{k=0}^{n-1} (4k-1)$$

е Лайбницовски ред. Първо да се убедим, че членовете му са с алтерниращи знаци.

$$\forall n \in \mathbb{N}^{+} \ - \frac{\prod_{k=0}^{n-1} (4k-1)}{2^{4n+1} n!(2n+1)} =$$

$$= -(4.0-1) \frac{\prod_{k=1}^{n-1} (4k-1)}{2^{4n+1} n!(2n+1)} =$$

$$= \frac{\prod_{k=1}^{n-1} (4k-1)}{2^{4n+1} n!(2n+1)} > 0$$

Тогава членовете на реда

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1} n!(2n+1)} (-1) \prod_{k=0}^{n-1} (4k-1) =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1} n!(2n+1)} \prod_{k=1}^{n-1} (4k-1)$$

са с алтерниращи знаци. Също така

$$\frac{1}{2^{4n+1}n!(2n+1)} \le \frac{\prod\limits_{k=1}^{n-1}(4k-1)}{2^{4n+1}n!(2n+1)} \le \frac{4^n n!}{2^{4n+1}n!(2n+1)} = \frac{1}{2^{2n+1}(2n+1)} \implies$$

$$\lim_{n\to\infty}\frac{1}{2^{4n+1}n!(2n+1)} \le \lim_{n\to\infty}\frac{\prod\limits_{k=1}^{n-1}(4k-1)}{2^{4n+1}n!(2n+1)} \le \lim_{n\to\infty}\frac{1}{2^{2n+1}(2n+1)} \implies$$

$$0 \le \lim_{n\to\infty}\frac{\prod\limits_{k=1}^{n-1}(4k-1)}{2^{4n+1}n!(2n+1)} \le 0 \implies$$

$$\lim_{n\to\infty}\frac{\prod\limits_{k=1}^{n-1}(4k-1)}{2^{4n+1}n!(2n+1)} = 0$$

Нека $a_n = \dfrac{\prod\limits_{k=1}^{n-1}(4k-1)}{2^{4n+1}n!(2n+1)}$. Ще покажем, че редицата $\{a_n\}_{n=1}^{\infty}$ е монотонно намаляваща.

$$a_{n+1} = \frac{\prod_{k=1}^{n}(4k-1)}{2^{4(n+1)+1}(n+1)!(2(n+1)+1)} =$$

$$= \frac{(4n-1)(2n+1)\prod_{k=1}^{n-1}(4k-1)}{2^4(n+1)(2n+3)2^{4n+1}n!(2n+1)} =$$

$$= \frac{(4n-1)(2n+1)}{2^4(n+1)(2n+3)}a_n =$$

$$= \frac{8n^2+2n-1}{2^4(2n^2+5n+3)}a_n \implies$$

$$a_n = \frac{2^4(2n^2+5n+3)}{8n^2+2n-1}a_{n+1}$$

$$2^5n^2 + 5.2^4n + 3.2^4 = 4(2^3n^2+2n-1) + 10n + 16 \implies$$

$$a_n = \left[4 + \frac{10n+16}{8n^2+2n-1}\right]a_{n+1}$$

$$\forall n \in \mathbb{N}^+ \ \frac{10n+16}{8n^2+2n-1} > 0 \implies$$

$$\forall n \in \mathbb{N}^+ \ 4 + \frac{10n+16}{8n^2+2n-1} > 0 \implies$$

$$\forall n \in \mathbb{N}^+ \ a_n > a_{n+1} \implies \{a_n\}_{n=1}^{\infty} \downarrow$$

Тогава реда

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1}n!(2n+1)}\prod_{k=1}^{n-1}(4k-1)$$

е Лайбницовски ред. Тогава

$$\forall k \in \mathbb{N}^+ \quad \left| \sum_{n=k+1}^{\infty} \frac{(-1)^{n-1} \prod_{m=1}^{n-1}(4m-1)}{2^{4n+1}n!(2n+1)} \right| < \left| \frac{(-1)^k \prod_{m=1}^{k}(4m-1)}{2^{4(k+1)+1}(k+1)!(2(k+1)+1)} \right|$$

$$\Longrightarrow \forall k \in \mathbb{N}^+ \quad \left| \sum_{n=k+1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1}n!(2n+1)} \prod_{m=1}^{n-1}(4m-1) \right| < \frac{\prod_{m=1}^{k}(4m-1)}{2^{4k+5}(k+1)!(2k+3)}$$

Искаме да пресметнем интеграла

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2}\, dx = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1}n!(2n+1)} \prod_{k=1}^{n-1}(4k-1)$$

с точност $E = 10^{-4}$. Тогава търсим първото $k \in \mathbb{N}$, за което да е изпълнено неравенството

$$\frac{\prod_{m=1}^{k}(4m-1)}{2^{4k+5}(k+1)!(2k+3)} < 10^{-4} = 0.0001$$

$$a_1 = \frac{\prod_{m=1}^{0}(4m-1)}{2^{4.1+1}(1)!(2.1+1)} = \frac{1}{2^5.3} \approx 0.0104166$$

Ако $k = 1$, то

$$a_2 = \frac{\prod_{m=1}^{k}(4m-1)}{2^{4k+5}(k+1)!(2k+3)} = \frac{3}{2^9(2)!5} = \frac{3}{2^9.2.5} = \frac{3}{2^{10}.5} \approx 0.0005859 > 0.0001$$

Ако $k = 2$, то

$$a_3 = \frac{\prod_{m=1}^{k}(4m-1)}{2^{4k+5}(k+1)!(2k+3)} = \frac{7.3}{2^{15}(3)!.7} = \frac{7.3}{2^{15}.2.3.7} = \frac{1}{2^{16}} \approx 0.0000152 < 0.0001$$

Тогава

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2}\,dx \approx \frac{1}{2} + \sum_{n=1}^{2} (-1)^n \frac{(4n-1)!!!!}{2^{4n+1} n!(2n+1)} \approx$$

$$\begin{aligned}\approx 0.5000000 + 0.0104166 - 0.0005859 + 0.0000152 =\\ = 0.5000000\\ +0.0104166\\ -0.0005859\\ +0.0000152\\ = 0.5098459\end{aligned}$$

Следователно стойността на интеграла

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2}\,dx$$

пресметната с точност 10^{-4} е 0.5098459.