

Курсова задача №3

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Като използвате подходящо развитие в степенен ред на подинтегралната функция пресметнете с точност $E = 10^{-4}$ определения интеграл

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2} dx$$

Решение:

$$\sqrt[4]{1+x} = (1+x)^{\frac{1}{4}} = \sum_{n=0}^{\infty} \binom{\frac{1}{4}}{n} x^n \implies$$

$$\sqrt[4]{1+x^2} = \sum_{n=0}^{\infty} \binom{\frac{1}{4}}{n} x^{2n} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{k=0}^{n-1} \left(\frac{1}{4} - k \right) x^{2n} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{k=0}^{n-1} - \left(\frac{4k-1}{4} \right) x^{2n} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{4^n} \prod_{k=0}^{n-1} (4k-1) x^{2n} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} \prod_{k=0}^{n-1} (4k-1) x^{2n} \implies$$

$$\begin{aligned}
\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2} \, dx &= \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} \prod_{k=0}^{n-1} (4k-1) x^{2n} \, dx = \\
&= \sum_{n=0}^{\infty} \int_0^{\frac{1}{2}} \frac{(-1)^n}{4^n n!} \prod_{k=0}^{n-1} (4k-1) x^{2n} \, dx = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} \prod_{k=0}^{n-1} (4k-1) \int_0^{\frac{1}{2}} x^{2n} \, dx = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} \prod_{k=0}^{n-1} (4k-1) \left(\frac{x^{2n+1}}{2n+1} \right) \bigg|_0^{\frac{1}{2}} = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n! 2^{2n+1} (2n+1)} \prod_{k=0}^{n-1} (4k-1) = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n! 2^{2n+1} (2n+1)} \prod_{k=0}^{n-1} (4k-1) = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+2n+1} n! (2n+1)} \prod_{k=0}^{n-1} (4k-1) = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n+1} n! (2n+1)} \prod_{k=0}^{n-1} (4k-1)
\end{aligned}$$

Получим, что

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n+1} n! (2n+1)} \prod_{k=0}^{n-1} (4k-1)$$

или

$$\begin{aligned}
\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2} \, dx &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{4n+1} n! (2n+1)} \prod_{k=0}^{n-1} (4k-1) = \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1} n! (2n+1)} (-1) \prod_{k=0}^{n-1} (4k-1)
\end{aligned}$$

Ще докажем, че реда

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1}n!(2n+1)} (-1) \prod_{k=0}^{n-1} (4k-1)$$

е Лайбницовски ред. Първо да се обединим, че членовете му са с алтерниращи знаци.

$$\forall n \in \mathbb{N}^+ \quad - \frac{\prod_{k=0}^{n-1} (4k-1)}{2^{4n+1}n!(2n+1)} =$$

$$= -(4 \cdot 0 - 1) \frac{\prod_{k=1}^{n-1} (4k-1)}{2^{4n+1}n!(2n+1)} =$$

$$= \frac{\prod_{k=1}^{n-1} (4k-1)}{2^{4n+1}n!(2n+1)} > 0$$

Тогава членовете на реда

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1}n!(2n+1)} (-1) \prod_{k=0}^{n-1} (4k-1) =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1}n!(2n+1)} \prod_{k=1}^{n-1} (4k-1)$$

са с алтерниращи знаци. Също така

$$\frac{1}{2^{4n+1}n!(2n+1)} \leq \frac{\prod_{k=1}^{n-1}(4k-1)}{2^{4n+1}n!(2n+1)} \leq \frac{4^n n!}{2^{4n+1}n!(2n+1)} = \frac{1}{2^{2n+1}(2n+1)} \implies$$

$$\lim_{n \rightarrow \infty} \frac{1}{2^{4n+1}n!(2n+1)} \leq \lim_{n \rightarrow \infty} \frac{\prod_{k=1}^{n-1}(4k-1)}{2^{4n+1}n!(2n+1)} \leq \lim_{n \rightarrow \infty} \frac{1}{2^{2n+1}(2n+1)} \implies$$

$$0 \leq \lim_{n \rightarrow \infty} \frac{\prod_{k=1}^{n-1}(4k-1)}{2^{4n+1}n!(2n+1)} \leq 0 \implies$$

$$\lim_{n \rightarrow \infty} \frac{\prod_{k=1}^{n-1}(4k-1)}{2^{4n+1}n!(2n+1)} = 0$$

Нека $a_n = \frac{\prod_{k=1}^{n-1}(4k-1)}{2^{4n+1}n!(2n+1)}$. Ще покажем, че редицата $\{a_n\}_{n=1}^{\infty}$ е мо-

ПОТОННО НАМАЛЯВАЩА.

$$\begin{aligned}
a_{n+1} &= \frac{\prod_{k=1}^n (4k-1)}{2^{4(n+1)+1}(n+1)!(2(n+1)+1)} = \\
&= \frac{(4n-1)(2n+1) \prod_{k=1}^{n-1} (4k-1)}{2^4(n+1)(2n+3)2^{4n+1}n!(2n+1)} = \\
&= \frac{(4n-1)(2n+1)}{2^4(n+1)(2n+3)} a_n = \\
&= \frac{8n^2+2n-1}{2^4(2n^2+5n+3)} a_n \implies \\
a_n &= \frac{2^4(2n^2+5n+3)}{8n^2+2n-1} a_{n+1}
\end{aligned}$$

$$2^5 n^2 + 5 \cdot 2^4 n + 3 \cdot 2^4 = 4(2^3 n^2 + 2n - 1) + 10n + 16 \implies$$

$$a_n = \left[4 + \frac{10n+16}{8n^2+2n-1} \right] a_{n+1}$$

$$\forall n \in \mathbb{N}^+ \quad \frac{10n+16}{8n^2+2n-1} > 0 \implies$$

$$\forall n \in \mathbb{N}^+ \quad 4 + \frac{10n+16}{8n^2+2n-1} > 0 \implies$$

$$\forall n \in \mathbb{N}^+ \quad a_n > a_{n+1} \implies \{a_n\}_{n=1}^{\infty} \downarrow$$

Тогава реда

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1}n!(2n+1)} \prod_{k=1}^{n-1} (4k-1)$$

е Лайбницовски ред. Тогава

$$\forall k \in \mathbb{N}^+ \left| \sum_{n=k+1}^{\infty} \frac{(-1)^{n-1} \prod_{m=1}^{n-1} (4m-1)}{2^{4n+1} n! (2n+1)} \right| < \left| \frac{(-1)^k \prod_{m=1}^k (4m-1)}{2^{4(k+1)+1} (k+1)! (2(k+1)+1)} \right|$$

$$\Rightarrow \forall k \in \mathbb{N}^+ \left| \sum_{n=k+1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1} n! (2n+1)} \prod_{m=1}^{n-1} (4m-1) \right| < \frac{\prod_{m=1}^k (4m-1)}{2^{4k+5} (k+1)! (2k+3)}$$

Искаме да пресметнем интеграла

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2} dx = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n+1} n! (2n+1)} \prod_{k=1}^{n-1} (4k-1)$$

с точност $E = 10^{-4}$. Тогава търсим първото $k \in \mathbb{N}$, за което да е изпълнено неравенството

$$\frac{\prod_{m=1}^k (4m-1)}{2^{4k+5} (k+1)! (2k+3)} < 10^{-4} = 0.0001$$

$$a_1 = \frac{\prod_{m=1}^0 (4m-1)}{2^{4 \cdot 1 + 1} (1)! (2 \cdot 1 + 1)} = \frac{1}{2^{5 \cdot 3}} \approx 0.0104166$$

Ако $k = 1$, то

$$a_2 = \frac{\prod_{m=1}^1 (4m-1)}{2^{4k+5} (k+1)! (2k+3)} = \frac{3}{2^9 (2)! 5} = \frac{3}{2^9 \cdot 2.5} = \frac{3}{2^{10 \cdot 5}} \approx 0.0005859 > 0.0001$$

Ако $k = 2$, то

$$a_3 = \frac{\prod_{m=1}^2 (4m-1)}{2^{4k+5} (k+1)! (2k+3)} = \frac{7.3}{2^{15} (3)! 7} = \frac{7.3}{2^{15} \cdot 2.3 \cdot 7} = \frac{1}{2^{16}} \approx 0.0000152 < 0.0001$$

Тогава

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2} \, dx \approx \frac{1}{2} + \sum_{n=1}^2 (-1)^n \frac{(4n-1)!!!}{2^{4n+1} n! (2n+1)} \approx$$

$$\begin{aligned} &\approx 0.5000000 + 0.0104166 - 0.0005859 + 0.0000152 = \\ &= 0.5000000 \\ &\quad + 0.0104166 \\ &\quad - 0.0005859 \\ &\quad + 0.0000152 \\ &= 0.5098459 \end{aligned}$$

Следователно стойността на интеграла

$$\int_0^{\frac{1}{2}} \sqrt[4]{1+x^2} \, dx$$

пресметната с точност 10^{-4} е 0.5098459.