

Аритметични действия със сходящи редици

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1 " + "

$$\begin{aligned} \{a_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} a \\ \{b_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} b \\ 0 \leq |a_n + b_n - (a + b)| &= |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| \\ |a_n - a| &\xrightarrow{n \rightarrow \infty} 0, |b_n - b| \xrightarrow{n \rightarrow \infty} 0 \\ \implies \{a_n + b_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} a + b \end{aligned}$$

2 " - "

$$\begin{aligned} \{a_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} a \\ \{b_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} b \\ 0 \leq |a_n + b_n - (a + b)| &= |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| \\ |a_n - a| &\xrightarrow{n \rightarrow \infty} 0, |b_n - b| \xrightarrow{n \rightarrow \infty} 0 \\ \implies \{a_n + b_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} a + b \\ \{b_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} b \\ \implies \forall \varepsilon > 0 \exists \nu; \forall n > \nu &\implies |b_n - b| = |-(b_n - b)| = |-b_n + b| < \varepsilon \\ \implies \{-b_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} -b \\ \implies \{a_n + (-b_n)\}_{n=1}^{\infty} &= \{a_n - b_n\}_{n=1}^{\infty} \xrightarrow{n \rightarrow \infty} a + (-b) = a - b \\ \implies \{a_n - b_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} a - b \end{aligned}$$

3 " *

$$\begin{aligned} \{a_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} a \\ \{b_n\}_{n=1}^{\infty} &\xrightarrow{n \rightarrow \infty} b \\ 0 \leq |a_n b_n - ab| &= |a_n b_n - ab \pm a_n b| = \\ &= |a_n(b_n - b) + b(a_n - a)| \leq |a_n| |b_n - b| + |b| |a_n - a| \\ |b_n - b| &\xrightarrow{n \rightarrow \infty} 0, |a_n - a| \xrightarrow{n \rightarrow \infty} 0 \\ \{a_n\}_{n=1}^{\infty} \text{ е ограничена} &\implies |a_n| |b_n - b| \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

$$\begin{aligned} &\implies |a_n||b_n - b| + |b||a_n - a| \xrightarrow{n \rightarrow \infty} 0 \implies |a_n b_n - ab| \xrightarrow{n \rightarrow \infty} 0 \\ &\implies \{a_n b_n\}_{n=1}^{\infty} \xrightarrow{n \rightarrow \infty} ab \end{aligned}$$

4 ”/”

$$\begin{aligned} &\{a_n\}_{n=1}^{\infty} \xrightarrow{n \rightarrow \infty} a \\ &\{b_n\}_{n=1}^{\infty} \xrightarrow{n \rightarrow \infty} b \\ &0 \leq |a_n b_n - ab| = |a_n b_n - ab \pm a_n b| = \\ &= |a_n(b_n - b) + b(a_n - a)| \leq |a_n||b_n - b| + |b||a_n - a| \\ &|b_n - b| \xrightarrow{n \rightarrow \infty} 0, |a_n - a| \xrightarrow{n \rightarrow \infty} 0 \\ &\{a_n\}_{n=1}^{\infty} \text{ е ограничена} \implies |a_n||b_n - b| \xrightarrow{n \rightarrow \infty} 0 \\ &\implies |a_n||b_n - b| + |b||a_n - a| \xrightarrow{n \rightarrow \infty} 0 \implies |a_n b_n - ab| \xrightarrow{n \rightarrow \infty} 0 \\ &\implies \{a_n b_n\}_{n=1}^{\infty} \xrightarrow{n \rightarrow \infty} ab \\ &\text{БОО } b > 0 \implies \{b_n\}_{n=1}^{\infty} \xrightarrow{n \rightarrow \infty} b > 0 \\ &\implies \forall \varepsilon > 0 \exists \nu; \forall n > \nu \\ &\implies |b_n - b| < \varepsilon \implies b_n \in (b - \varepsilon, b + \varepsilon) \\ &\varepsilon := \frac{b}{2} \implies b_n \in (b - \frac{b}{2}, b + \frac{b}{2}) \implies b_n \in (\frac{b}{2}, \frac{3b}{2}) \\ &\implies n < \nu \implies b_n \notin (\frac{b}{2}, \frac{3b}{2}) \implies n > \nu \implies b_n > \frac{b}{2} > 0 \\ &\implies |\frac{1}{b_n} - \frac{1}{b}| = |\frac{b - b_n}{bb_n}| = \frac{|b - b_n|}{|b||b_n|} \\ &b_n > \frac{b}{2}, |b - b_n| \xrightarrow{n \rightarrow \infty} \varepsilon \\ &\implies \frac{|b - b_n|}{|b||b_n|} < \frac{2}{b^2} |b - b_n| \xrightarrow{n \rightarrow \infty} 0 \\ &\implies \{\frac{1}{b_n}\}_{n=1}^{\infty} \xrightarrow{n \rightarrow \infty} \frac{1}{b} \\ &\implies \{a_n \frac{1}{b_n}\}_{n=1}^{\infty} \xrightarrow{n \rightarrow \infty} a \frac{1}{b} \\ &\implies \{\frac{a_n}{b_n}\}_{n=1}^{\infty} \xrightarrow{n \rightarrow \infty} \frac{a}{b} \end{aligned}$$