# Домашна работа 3, № 45342, Група 3

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## Задача 1.

$f(x) = x^4 + x^2 + \overline{5}x + \overline{3}, \quad g(x) = \overline{12}x^4 + x^3 + \overline{8}x + \overline{2} \in \mathbb{Z}_{13}$

$$\begin{array}{rl} & \overline{12}x^4 + x^3 + \overline{8}x + \overline{2} \ : \ x^4 + x^2 + \overline{5}x + \overline{3} = \overline{12} \\ - & \\ & \overline{12}x^4 + \overline{12}x^2 + \overline{8}x + \overline{10} \\ \hline & x^3 + x^2 + \overline{5} \end{array}$$

Нека $q_1(x) = \overline{12},\ r_1(x) = x^3 + x^2 + \overline{5} \implies g(x) = q_1(x)f(x) + r_1(x) \implies$

$r_1(x) = g(x) - q_1(x)f(x)$

$$\begin{array}{rl} & x^4 + x^2 + \overline{5}x + \overline{3} \ : \ x^3 + x^2 + \overline{5} = x + \overline{12} \\ - & \\ & x^4 + x^3 + \overline{5}x \\ \hline & \overline{12}x^3 + x^2 + \overline{3} \\ - & \\ & \overline{12}x^3 + \overline{12}x^2 + \overline{8} \\ \hline & \overline{2}x^2 + \overline{8} \end{array}$$

Нека $q_2(x) = x + \overline{12},\ r_2(x) = \overline{2}x^2 + \overline{8} \implies f(x) = q_2(x)r_1(x) + r_2(x) \implies$

$r_2(x) = f(x) - q_2(x)r_1(x) = f(x) - q_2(x)(g(x) - q_1(x)f(x)) =$

$= f(x) - q_2(x)g(x) + q_2(x)q_1(x)f(x) \implies r_2(x) = (\overline{1} + q_2(x)q_1(x))f(x) - q_2(x)g(x)$

$$
\begin{array}{rl}
 & x^3 + x^2 + \overline{5} \ : \quad \overline{2}x^2 + \overline{8} = \overline{7}x + \overline{7} = \overline{7}(x + \overline{1}) \\
- & \\
 & x^3 + \overline{4}x \\
\hline
 & \quad x^2 + \overline{9}x + \overline{5} \\
 - & \\
 & \quad x^2 + \overline{4} \\
\hline
 & \qquad \overline{9}x + \overline{1}
\end{array}
$$

Нека $q_3(x) = \overline{7}(x + \overline{1}),\ r_3(x) = \overline{9}x + \overline{1} \implies r_1(x) = q_3(x)r_2(x) + r_3(x) \implies$

$r_3(x) = r_1(x) - q_3(x)r_2(x) =$

$= g(x) - q_1(x)f(x) - q_3(x)\left[(\overline{1} + q_2(x)q_1(x))f(x) - q_2(x)g(x)\right] =$

$= g(x)(\overline{1} + q_3(x)q_2(x)) + f(x)(-q_1(x) - q_3(x) - q_1(x)q_2(x)q_3(x)) =$

$= g(x)(\overline{1} + q_3(x)q_2(x)) + f(x)\left[-q_3(x) - q_1(x)(\overline{1} + q_2(x)q_3(x))\right]$

Нека $h(x) = \overline{1} + q_3(x)q_2(x) \implies r_3 = h(x)g(x) + f(x)(-q_3(x) - q_1(x)h(x))$

$$
\begin{array}{rl}
 & \overline{2}x^2 + \overline{8} \ : \quad \overline{9}x + \overline{1} = \overline{6}x + \overline{8} \\
- & \\
 & \overline{2}x^2 + \overline{6}x \\
\hline
 & \quad \overline{7}x + \overline{8} \\
- & \\
 & \quad \overline{7}x + \overline{8} \\
\hline
 & \qquad 0 \quad \implies (f, g) = r_3
\end{array}
$$

$h(x) = \overline{1} + \overline{7}(x + \overline{1})(x + \overline{12}) = \overline{1} + \overline{7}(x + \overline{1})(x - \overline{1}) = \overline{1} + \overline{7}(x^2 - \overline{1}) =$

$= \overline{7}x^2 - \overline{6} = \overline{7}x^2 + \overline{7} = \overline{7}(x^2 + \overline{1}) \implies$

$-q_3(x) - q_1(x)h(x) = -\overline{7}(x + \overline{1}) - \overline{12}.\overline{7}(x^2 + \overline{1}) = \overline{7}(x^2 + \overline{1}) - \overline{7}(x + \overline{1}) =$

$= \overline{7}x^2 + \overline{7} - \overline{7}x - \overline{7} = \overline{7}x^2 - \overline{7}x = \overline{7}(x^2 - x) \implies$

$(f, g) = \overline{9}x + \overline{1} = \overline{7}(x^2 + \overline{1})g(x) + \overline{7}(x^2 - x)f(x) \implies$

$\overline{5}x + \overline{2} = (x^2 + \overline{1})g(x) + (x^2 - x)f(x)$

Отговор: $(f, g) = \overline{5}x + \overline{2},\ \overline{5}x + \overline{2} = (x^2 + \overline{1})g(x) + (x^2 - x)f(x)$

# Задача 2.

Нека $f(x) = ax^3 + bx^2 + cx + d \in \mathbb{C}[x] :\ \exists q \in \mathbb{C}[x] :\ f(x) = (x^2 + 1)q(x) - 12x - 8$

За корените на $f(x)$ е изпълнено:

$$\begin{cases} \displaystyle\sum_{k=1}^{3} \frac{1}{x_k} = 6 \\ \displaystyle\sum_{k=1}^{3} \frac{1}{x_k^2} = \left( \sum_{k=1}^{3} \frac{1}{x_k} \right)^2 - 2\left( \frac{1}{x_1}\frac{1}{x_2} + \frac{1}{x_1}\frac{1}{x_3} + \frac{1}{x_2}\frac{1}{x_3} \right) = 18 \end{cases}$$

$x^2 + 1 = (x - i)(x + i) \implies$

$f(i) = -ai - b + ci + d = -12i - 8$

$f(-i) = ai - b - ci + d = 12i - 8 \implies$

$f(i) + f(-i) = -2b + 2d = -16 \implies d = b - 8$

$f(i) - f(-i) = -2ai + 2ci = -24i \implies c = a - 12$

Нека $g(x) = x^3 f\left(\frac{1}{x}\right) = dx^3 + cx^2 + bx + a \implies$

$$\begin{cases} \sum_{k=1}^{3} \frac{1}{x_k} = -\frac{c}{d} = 6 \\ \frac{1}{x_1}\frac{1}{x_2} + \frac{1}{x_1}\frac{1}{x_3} + \frac{1}{x_2}\frac{1}{x_3} = \frac{b}{d} \\ \prod_{k=1}^{3} \frac{1}{x_k} = -\frac{a}{d} \end{cases} \implies \begin{cases} c = -6d \\ \left(-\frac{c}{d}\right)^2 - 2\frac{b}{d} = 18 \end{cases} \implies$$

$$\begin{cases} c = -6d \\ 6.6 - 2\frac{b}{d} = 3.6 \end{cases} \implies \begin{cases} c = -6d \\ 18 = 2\frac{b}{d} \end{cases} \implies \begin{cases} c = -6d \\ b = 9d \end{cases} \implies$$

$$\begin{cases} d = b - 8 \\ c = -6d \\ b = 9d \\ c = a - 12 \end{cases} \implies \begin{cases} -8d = -8 \\ c = -6d \\ b = 9d \\ c = a - 12 \end{cases} \implies \begin{cases} d = 1 \\ c = -6 \\ b = 9 \\ a = 6 \end{cases} \implies$$

$f(x) = 6x^3 + 9x^2 - 6x + 1$

Проверка:

$$\begin{array}{l} \phantom{-}\ 6x^3 + 9x^2 - 6x + 1 \ : \quad x^2 + 1 = 6x + 9 \\ - \\ \phantom{-}\ 6x^3 + 6x \\ \hline \phantom{-}\qquad 9x^2 - 12x + 1 \\ \quad - \\ \phantom{-}\qquad 9x^2 + 9 \\ \hline \phantom{-}\qquad\qquad -12x - 8 \end{array}$$

Отговор: $f(x) = 6x^3 + 9x^2 - 6x + 1$

## Задача 3.

$x_1,\ x_2,\ x_3$ са корени на полинома $s(x) = x^3 + px + q \implies$

$$\begin{cases} \sum_{k=1}^{3} x_k = 0 \implies x_2 + x_3 = -x_1 \\ x_1x_2 + x_1x_3 + x_2x_3 = p \implies x_1(x_2+x_3) + x_2x_3 = p \implies x_2x_3 = p + x_1^2 \\ \prod_{k=1}^{3} x_k = -q \implies x_2x_3 = -\frac{q}{x_1} \\ \forall k \in \{1,2,3\}\ x_k^3 = -px_k - q \end{cases}$$

$$\sigma = \tfrac{x_1}{-8x_2^2+5x_2x_3-8x_3^2} + \tfrac{x_2}{-8x_1^2+5x_1x_3-8x_3^2} + \tfrac{x_3}{-8x_1^2+5x_1x_2-8x_2^2} = \sum \tfrac{x_1}{-8x_2^2+5x_2x_3-8x_3^2} =$$

$$= \sum \tfrac{x_1}{-8(x_2^2+x_3^2)+5x_2x_3} = \sum \tfrac{x_1}{-8(x_2+x_3)^2+21x_2x_3} = \sum \tfrac{x_1}{-8(-x_1)^2-21\frac{q}{x_1}} = \sum_{k=1}^{3} \frac{x_k}{-8x_k^2 - 21\frac{q}{x_k}} =$$

$$= \sum_{k=1}^{3} \frac{x_k^2}{-8x_k^3 - 21q} = \sum_{k=1}^{3} \frac{x_k^2}{8px_k - 13q} = \frac{1}{8p}\sum_{k=1}^{3} \frac{x_k^2 - \left(\frac{13q}{8p}\right)^2 + \left(\frac{13q}{8p}\right)^2}{x_k - \frac{13q}{8p}} =$$

$$= \frac{1}{8p}\sum_{k=1}^{3} \frac{(x_k - a)(x_k + a)}{x_k - a} + \frac{1}{8p}\sum_{k=1}^{3} \frac{(a)^2}{x_k - a} =$$

$$= \frac{1}{8p}\sum_{k=1}^{3} (x_k + a) + \frac{a^2}{8p}\sum_{k=1}^{3} \frac{1}{x_k - a} =$$

$$= \frac{1}{8p}\left(3a + \sum_{k=1}^{3} x_k - a^2 \sum_{k=1}^{3} \frac{1}{a - x_k}\right) = \frac{1}{8p}\left(3a - a^2\frac{s'(a)}{s(a)}\right) =$$

$$= \frac{3as(a) - a^2s'(a)}{8ps(a)} = \frac{3a(a^3 + pa + q) - a^2(3a^2 + p)}{8ps(a)} =$$

$$= \frac{2pa^2 + 3aq}{8ps(a)} = \frac{2pa^2 + 3aq}{8p(a^3 + pa + q)} = \frac{2p\left(\frac{13q}{8p}\right)^2 + 3\frac{13q}{8p}q}{8p\left(\left(\frac{13q}{8p}\right)^3 + p\frac{13q}{8p} + q\right)} =$$

$$= \frac{\frac{1}{4}169q^2 + 39q^2}{(8p)^2\left(\left(\frac{13q}{8p}\right)^3 + \frac{13q}{8} + q\right)} = \frac{169q^2 + 156q^2}{\frac{(13q)^3}{2p} + 416p^2q + 256p^2q} =$$

$$= \frac{325q^2}{\frac{(13q)^3}{2p} + 672p^2q} = \frac{650pq^2}{2197q^3 + 1344p^3q} = \frac{650pq}{1344p^3 + 2197q^2}$$

Отговор: $\frac{650pq}{1344p^3+2197q^2}$

# Задача 4.

$f(x) = x^4 + x^3 + \mu x^2 - 15x - 22 \in \mathbb{C}[x]$

Нека $P(M): \ (M \subset \mathbb{C}) \wedge (|M| = 4) \wedge (\forall m \in M \ f(m) = 0) \wedge$

$$\wedge (\exists r, t, u, v \in M: \ r + t = uv)$$

Нека $a, b, c, d \in \mathbb{C}: \ P(\{a, b, c, d\}) \implies$

$$\begin{cases} a + b = cd \\ a + b + c + d = -1 \\ ab + ac + ad + bc + bd + cd = \mu \\ abc + abd + acd + bcd = 15 \\ abcd = -22 \end{cases} \implies \begin{cases} a + b = cd \\ c + d = -1 - (a + b) \\ ab + (a + b)(c + d + 1) = \mu \\ ab(c + d) + (a + b)^2 = 15 \\ a^2b + ab^2 = -22 \end{cases} \implies$$

$$\begin{cases} a + b = cd \\ c + d = -1 - (a + b) \\ ab = \mu + (a + b)^2 \\ -ab - (a^2b + ab^2) + (a + b)^2 = 15 \\ a^2b + ab^2 = -22 \end{cases} \implies \begin{cases} a + b = cd \\ c + d = -1 - (a + b) \\ ab = \mu + (a + b)^2 \\ -\mu - (a + b)^2 + 22 + (a + b)^2 = 15 \\ a^2b + ab^2 = -22 \end{cases}$$

$\implies \ \mu = 7$

Ако $\mu = 7 \implies f(x) = x^4 + x^3 + 7x^2 - 15x - 22 = x^3(x + 1) + x(7x - 15) - 22$

Поренциялни корени на $f(x)$ са $\pm 1, \ \pm 2, \ \pm 11, \ \pm 22$

$f(1) = 2 - 8 - 22 \neq 0$

$f(-1) = 22 - 22 = 0$

$f(2) = 24 + -2 - 22 = 0 \implies$

Нека $g(x) = (x - 2)(x + 1) = x^2 - x - 2 \implies g \mid f$

$$
\begin{array}{rl}
- & \begin{array}{l} x^4+x^3+7x^2-15x-22 \\ x^4-x^3-2x^2 \end{array} \quad : \quad x^2-x-2 = x^2+2x+11 \\
\hline
- & \begin{array}{l} 2x^3+9x^2-15x-22 \\ 2x^3-2x^2-4x \end{array} \\
\hline
- & \begin{array}{l} 11x^2-11x-22 \\ 11x^2-11x-22 \end{array} \\
\hline
 & 0
\end{array}
$$

Нека $h(x) = x^2+2x+11 \implies f = gh$

$D(h) = 4-44 = -40 \implies \sqrt{D(h)} = 2i\sqrt{10} \implies$

$h(-1-i\sqrt{10}) = h(-1+i\sqrt{10}) = f(-1-i\sqrt{10}) = f(-1+i\sqrt{10}) = 0 \implies$

$(-1-i\sqrt{10})+(-1+i\sqrt{10}) = -2 = (-1)2 \implies$

$P(\{-1,\ 2,\ -1-i\sqrt{10},\ -1+i\sqrt{10}\})$

Отговор: $\mu = 7$

## Задача 5.

$f(x) = x^4-5x^3-5x^2-x+3 \in \mathbb{Z}[x]$

Ако $p, q \in \mathbb{Z}: \frac{p}{q}$ е потенциялен корен на $f(x) \implies p \mid 3 \ \wedge \ q \mid 1 \implies$

$(p = \pm 1 \ \vee \ p = \pm 3) \ \wedge \ q = \pm 1 \implies \frac{p}{q} = \pm 1 \ \vee \ \frac{p}{q} = \pm 3$

$f(1) = 1-5-5-1+3 = -7 \neq 0$

$f(-1) = 1+5-5+1+3 = 5 \neq 0$

$f(3) = 27(3-5)-45-3+3 = -54-45 = -99 \neq 0$

$f(-3) = 9(9+15-5)+6 = 19.9+6 = 171+6 = 177 \neq 0 \implies$

$\nexists\, z \in \mathbb{Z}:\ f(z) = 0 \implies \nexists d \in \mathbb{Z}[x] : deg(d) = 1 \ \wedge\ d \mid f$

Нека $g,\ h \in \mathbb{Z}[x]:\ deg(g) = deg(h) = 2 \ \wedge\ f = gh \implies$

$\exists a, b, c, d \in \mathbb{Z}:\ g(x) = x^2 + ax + b \ \wedge\ h(x) = x^2 + cx + d \implies$

$f(x) = x^4 - 5x^3 - 5x^2 - x + 3 = g(x)h(x) = (x^2 + ax + b)(x^2 + cx + d) \implies$

$$\begin{cases} a + c = -5 \\ ac + b + d = -5 \\ bc + ad = -1 \\ bd = 3 \end{cases} \implies \begin{cases} c = -5 - a \\ ac + 4s = -5 \\ sc + a3s = -1 \\ b = s \\ d = 3s \\ s = \pm 1 \end{cases} \implies \begin{cases} c = -5 - a \\ ac + 4s = -5 \\ 2as = 5s - 1 \\ b = s \\ d = 3s \\ s = \pm 1 \end{cases} \implies$$

$$\begin{cases} c = -7 \\ -9 = -14 \\ a = 2 \\ b = 1 \\ d = 3 \\ s = 1 \end{cases} \bigvee \begin{cases} c = -8 \\ -24 = -1 \\ a = 3 \\ b = -1 \\ d = -3 \\ s = -1 \end{cases} \implies \lightning \implies$$

$f(x)$ е неразложим над $\mathbb{Z} \implies f(x)$ е неразложим над $\mathbb{Q}$

# Задача 6.

$f(x) = x^j + bx + e \implies f'(x) = jx^{j-1} + b$

Нека $\alpha_1,\ \ldots,\ \alpha_{j-1}$ са корени на $f' \implies$

$\forall i \in \mathbb{N}:\ i \leq (j-1) \quad f'(\alpha_i) = j\alpha_i^{j-1} + b = 0 \implies$

$D(f) = \frac{1}{1}(-1)^{\frac{j(j-1)}{2}} R(f, f') = \frac{1}{1}(-1)^{\frac{j(j-1)}{2}} R(f', f) =$

$$= (-1)^{\frac{j(j-1)}{2}} j^j \prod_{k=1}^{j-1} f(\alpha_k) = (-1)^{\frac{j(j-1)}{2}} j^j \prod_{k=1}^{j-1} (\alpha_k^j + \alpha_k b + e) =$$

$$= (-1)^{\frac{j(j-1)}{2}} j^j \prod_{k=1}^{j-1} (\alpha_k(\alpha_k^{j-1}+b)+e) = (-1)^{\frac{j(j-1)}{2}} j^j \prod_{k=1}^{j-1} \left( \frac{\alpha_k}{j}(j\alpha_k^{j-1} + b + (j-1)b) + e \right) =$$

$$= (-1)^{\frac{j(j-1)}{2}} j^j \prod_{k=1}^{j-1} \left( \frac{1}{j}((j-1)b\alpha_k + je) \right) = (-1)^{\frac{j(j-1)}{2}} j \prod_{k=1}^{j-1} ((j-1)b\alpha_k + je) =$$

$$= (-1)^{\frac{j(j-1)}{2}} j[(j-1)b]^{j-1} \prod_{k=1}^{j-1} \left( \alpha_k + \frac{je}{(j-1)b} \right) =$$

$$= (-1)^{\frac{j(j-1)}{2}} j[(j-1)b]^{j-1} (-1)^{j-1} \prod_{k=1}^{j-1} \left( -\frac{je}{(j-1)b} - \alpha_k \right) =$$

$$= (-1)^{\frac{j(j-1)+2j-2}{2}} [(j-1)b]^{j-1} \left[ j \prod_{k=1}^{j-1} \left( -\frac{je}{(j-1)b} - \alpha_k \right) \right] =$$

$$= -(-1)^{\frac{j(j+1)}{2}} [(j-1)b]^{j-1} f' \left( -\frac{je}{(j-1)b} \right) =$$

$$= -(-1)^{\frac{j(j+1)}{2}} [(j-1)b]^{j-1} \left[ j \left( -\frac{je}{(j-1)b} \right)^{j-1} + b \right] =$$

$$= (-1)^{\frac{j(j+3)}{2}} j^j e^{j-1} - (-1)^{\frac{j(j+1)}{2}} (j-1)^{j-1} b^j$$

Отговор: $D(x^j + bx + e) = (-1)^{\frac{j(j+3)}{2}} j^j e^{j-1} - (-1)^{\frac{j(j+1)}{2}} (j-1)^{j-1} b^j$