Домашна работа. Вариант 2

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Задача 1.

$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 =?\, n^2(2n^2-1), n = 1, 2, 3, \ldots$

$n = 1 \implies 1^3 =?\, 1^2(2(1^2) - 1)$
$1^3 = 1$
$1^2(2(1^2) - 1) = 1(2-1) = 1$
$1 = 1$

$n = k \implies 1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2-1)$

$n = k+1 \implies 1^3 + 3^3 + 5^3 + \cdots + (2(k+1)-1)^3 =?\, (k+1)^2(2(k+1)^2-1)$

$1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 + (2(k+1)-1)$
$= k^2(2k^2-1) + (2k+2-1)^3$
$= 2k^4 - k^2 + (2k+1)^3$
$= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1$
$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$

$(k+1)^2(2(k+1)^2-1)$
$= (k^2+2k+1)(2(k^2+2k+1)-1)$
$= (k^2+2k+1)(2k^2+4k+1)$
$= 2k^4 + 4k^3 + k^2 + 4k^3 + 8k^2 + 2k + 2k^2 + 4k + 1$
$2k^4 + 8k^3 + 11k^2 + 6k + 1$

$\implies 1^3 + 3^3 + 5^3 + \cdots + (2(k+1)-1)^3 = (k+1)^2(2(k+1)^2-1)$
$\implies 1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1), n = 1, 2, 3, \ldots$

Задача 2.

a) $\lim_{n\to\infty} \frac{2n^3+3n+5}{-3n^3+4n+7}$

$\lim_{n\to\infty} \frac{2n^3+3n+5}{-3n^3+4n+7}$

$\lim_{n\to\infty} \frac{\cancel{n}^3(2+\frac{3}{n^2}+\frac{5}{n^3})}{\cancel{n}^3(-3+\frac{4}{n^2}+\frac{7}{n^3})}$

$\lim_{n\to\infty} \frac{2+\frac{3}{n^2}+\frac{5}{n^3}}{-3+\frac{4}{n^2}+\frac{7}{n^3}}$

$\lim_{n\to\infty} \frac{2+3\frac{1}{n}\frac{1}{n}+5\frac{1}{n}\frac{1}{n}\frac{1}{n}}{-3+4\frac{1}{n}\frac{1}{n}+7\frac{1}{n}\frac{1}{n}\frac{1}{n}}$

$\frac{1}{n} \to 0 \implies \lim_{n\to\infty} \frac{2+3\times0+5\times0}{-3+4\times0+7\times0}$

$\implies \lim_{n\to\infty} \frac{2n^3+3n+5}{-3n^3+4n+7} = \frac{-2}{3}$

б) $\lim_{n\to\infty} \frac{3n^4+4^n n^2+(-3)^n}{2n^3+5^n}$

$\lim_{n\to\infty} \frac{3n^4+4^n n^2+(-3)^n}{2n^3+5^n}$

$\lim_{n\to\infty} \frac{3n^4}{2n^3+5^n}$

$\lim_{n\to\infty} \frac{n^{\cancel{4}}3}{n^{\cancel{3}}(2+\frac{5^n}{n^3})}$

$\lim_{n\to\infty} \frac{3n}{2+\frac{5^n}{n^3}}$

$n \to \infty \quad 5^n \prec n^3 \implies \frac{5^n}{n^3} \to 0 \quad n \prec \frac{5^n}{n^3} \implies \lim_{n\to\infty} \frac{3n^4}{2n^3+5^n} = 0$

$\lim_{n\to\infty} \frac{4^n n^2}{2n^3+5^n}$

$\lim_{n\to\infty} \frac{n^{\cancel{2}}\frac{4^n}{n^2}}{n^{\cancel{3}}(2+\frac{5^n}{n^3})}$

$\lim_{n\to\infty} \frac{\frac{4^n}{n^2}}{2n+\frac{5^n}{n^2}}$

$n \to \infty \quad 4^n \prec 5^n\,,\; n^0 \prec 2n \implies \lim_{n\to\infty} \frac{4^n n^2}{2n^3+5^n} = 0$

$\lim_{n\to\infty} \frac{(-3)^n}{2n^3+5^n}$

$n \to \infty \quad (-3)^n \to -\infty\,,\; 5^n \to \infty$

$(-3)^n \prec 5^n\,,\; n^0 \prec 2n^3 \implies \lim_{n\to\infty} \frac{(-3)^n}{2n^3+5^n} = 0$

$\lim_{n\to\infty} \frac{3n^4+4^n n^2+(-3)^n}{2n^3+5^n} = \lim_{n\to\infty} \frac{3n^4}{2n^3+5^n} + \lim_{n\to\infty} \frac{4^n n^2}{2n^3+5^n}$
$+ \lim_{n\to\infty} \frac{(-3)^n}{2n^3+5^n} = 0+0+0 = 0$

в) $\lim_{n\to\infty} \frac{n^2+3n+3}{n^2-3n-4}^{3n+3}$

$\lim_{n\to\infty} \frac{n^2+3n+3}{n^2-3n-4}^{3n+3}$

$\frac{n^2+3n+3}{n^2-3n-4} = \frac{\cancel{n^2}(1+\frac{3}{n}+\frac{3}{n^2})}{\cancel{n^2}(1-\frac{3}{n}-\frac{4}{n^2}} = \frac{1+3\frac{1}{n}+3\frac{1}{n}\frac{1}{n}}{1-3\frac{1}{n}\frac{1}{n}-4\frac{1}{n}\frac{1}{n}}$

$\frac{1}{n} \to 0 \implies \frac{1+3\frac{1}{n}+3\frac{1}{n}\frac{1}{n}}{1-3\frac{1}{n}\frac{1}{n}-4\frac{1}{n}\frac{1}{n}} \to 1$

$\lim_{n\to\infty} \left[1+\left(\frac{n^2+3n+3}{n^2-3n-4}-1\right)\right]^{3n+3}$

$\lim_{n\to\infty} \left[1+\left(\frac{n^2+3n+3}{n^2-3n-4}-\frac{n^2-3n-4}{n^2-3n-4}\right)\right]^{3n+3}$

$\lim_{n\to\infty} \left(1+\frac{6n+7}{n^2-3n-4}\right)^{3n+3}$

$\lim_{n\to\infty} \left[\left(1+\frac{6n+7}{n^2-3n-4}\right)^{\frac{n^2-3n-4}{6n+7}}\right]^{\frac{6n+7}{n^2-3n-4}3n+3}$

$\lim_{n\to\infty} \left[\left(1+\frac{6n+7}{n^2-3n-4}\right)^{\frac{n^2-3n-4}{6n+7}}\right]^{\frac{18n^2+39n+21}{n^2-3n-4}}$

$\left(1+\frac{6n+7}{n^2-3n-4}\right)^{\frac{n^2-3n-4}{6n+7}} \to e$

$\frac{18n^2+39n+21}{n^2-3n-4} = \frac{\cancel{n^2}(18+\frac{39}{n}+\frac{21}{n^2})}{\cancel{n^2}(1-\frac{3}{n}-\frac{4}{n})} = \frac{18+39\frac{1}{n}+21\frac{1}{n}\frac{1}{n}}{1-3\frac{1}{n}\frac{1}{n}-4\frac{1}{n}\frac{1}{n}}$

$\frac{1}{n} \to 0 \implies \frac{18+39\frac{1}{n}+21\frac{1}{n}\frac{1}{n}}{1-3\frac{1}{n}\frac{1}{n}-4\frac{1}{n}\frac{1}{n}} \to 18$

$\implies \lim_{n\to\infty} \frac{n^2+3n+3}{n^2-3n-4}^{3n+3} = e^{18}$

г) $\lim_{n\to\infty} \frac{n^2+3n+3}{n^2-3n-4}^{\frac{1}{3n+3}}$

$\lim_{n\to\infty} \frac{n^2+3n+3}{n^2-3n-4}^{\frac{1}{3n+3}}$

$\frac{n^2+3n+3}{n^2-3n-4} = \frac{\cancel{n^2}(1+\frac{3}{n}+\frac{3}{n^2})}{\cancel{n^2}(1-\frac{3}{n}-\frac{4}{n^2}} = \frac{1+3\frac{1}{n}+3\frac{1}{n}\frac{1}{n}}{1-3\frac{1}{n}\frac{1}{n}-4\frac{1}{n}\frac{1}{n}}$

$\frac{1}{n} \to 0 \implies \frac{1+3\frac{1}{n}+3\frac{1}{n}\frac{1}{n}}{1-3\frac{1}{n}\frac{1}{n}-4\frac{1}{n}\frac{1}{n}} \to 1$

$\lim_{n\to\infty} \left[1+\left(\frac{n^2+3n+3}{n^2-3n-4}-1\right)\right]^{\frac{1}{3n+3}}$

$\lim_{n\to\infty} \left[1+\left(\frac{n^2+3n+3}{n^2-3n-4}-\frac{n^2-3n-4}{n^2-3n-4}\right)\right]^{\frac{1}{3n+3}}$

$\lim_{n\to\infty} \left(1+\frac{6n+7}{n^2-3n-4}\right)^{\frac{1}{3n+3}}$

$\lim_{n\to\infty}\left[\left(1+\frac{6n+7}{n^2-3n-4}\right)^{\frac{n^2-3n-4}{6n+7}}\right]^{\frac{6n+7}{n^2-3n-4}\frac{1}{3n+3}}$

$\lim_{n\to\infty}\left[\left(1+\frac{6n+7}{n^2-3n-4}\right)^{\frac{n^2-3n-4}{6n+7}}\right]^{\frac{6n+7}{3n^3-6n^2-21n-12}}$

$\left(1+\frac{6n+7}{n^2-3n-4}\right)^{\frac{n^2-3n-4}{6n+7}}\to e$

$\frac{6n+7}{3n^3-6n^2-21n-12}=\frac{\cancel{n}(6+\frac{7}{n})}{n^{\cancel{3}}(3-\frac{6}{n}-\frac{21}{n^2}-\frac{12}{n^3})}=\frac{1}{n}\frac{1}{n}\left(\frac{6+7\frac{1}{n}}{3-6\frac{1}{n}-21\frac{1}{n}\frac{1}{n}-12\frac{1}{n}\frac{1}{n}\frac{1}{n}}\right)$

$\frac{1}{n}\to 0\implies\frac{1}{n}\frac{1}{n}\left(\frac{6+7\frac{1}{n}}{3-6\frac{1}{n}-21\frac{1}{n}\frac{1}{n}-12\frac{1}{n}\frac{1}{n}\frac{1}{n}}\right)\to 0$

$\implies\lim_{n\to\infty}\frac{n^2+3n+3}{n^2-3n-4}^{\frac{1}{3n+3}}=e^0=1$

д) $\lim_{n\to\infty}\frac{n}{\sqrt{n^4+3n^2+4}-\sqrt{n^4-n^3+1}}$

$\lim_{n\to\infty}\frac{n}{\sqrt{n^4+3n^2+4}-\sqrt{n^4-n^3+1}}$

$\lim_{n\to\infty}\frac{n}{\sqrt{n^4+3n^2+4}-\sqrt{n^4-n^3+1}}\frac{\sqrt{n^4+3n^2+4}+\sqrt{n^4-n^3+1}}{\sqrt{n^4+3n^2+4}+\sqrt{n^4-n^3+1}}$

$\lim_{n\to\infty}\frac{n(\sqrt{n^4+3n^2+4}+\sqrt{n^4-n^3+1})}{(\sqrt{n^4+3n^2+4})^2-(\sqrt{n^4-n^3+1})^2}$

$\lim_{n\to\infty}\frac{n\sqrt{n^4}(\sqrt{1+\frac{3}{n^2}+\frac{4}{n^4}}+\sqrt{1-\frac{1}{n}+\frac{1}{n^4}})}{(n^4+3n^2+4)-(n^4-n^3+1)}$

$\lim_{n\to\infty}\frac{n^3(\sqrt{1+\frac{3}{n^2}+\frac{4}{n^4}}+\sqrt{1-\frac{1}{n}+\frac{1}{n^4}})}{n^3+3n^2+3}$

$\lim_{n\to\infty}\frac{\cancel{n^3}(\sqrt{1+\frac{3}{n^2}+\frac{4}{n^4}}+\sqrt{1-\frac{1}{n}+\frac{1}{n^4}})}{\cancel{n^3}(1+\frac{3}{n}+\frac{3}{n^3})}$

$\lim_{n\to\infty}\frac{\sqrt{1+3\frac{1}{n}\frac{1}{n}+4\frac{1}{n}\frac{1}{n}\frac{1}{n}\frac{1}{n}}+\sqrt{1-\frac{1}{n}+\frac{1}{n}\frac{1}{n}\frac{1}{n}\frac{1}{n}}}{1+3\frac{1}{n}+3\frac{1}{n}\frac{1}{n}\frac{1}{n}}$

$\frac{1}{n}\to 0\implies\lim_{n\to\infty}\frac{\sqrt{1+3\times0+4\times0}+\sqrt{1-0+0}}{1+3\times0+3\times0}$

$\implies\lim_{n\to\infty}\frac{n}{\sqrt{n^4+3n^2+4}-\sqrt{n^4-n^3+1}}=2$

е) $\lim_{n\to\infty}\frac{(2n)!!}{(2n)^n}$

$\lim_{n\to\infty}\frac{(2n)!!}{(2n)^n}$

$\lim_{n\to\infty}\frac{\cancel{2n}(n!)}{\cancel{2n}(2n)^{n-1}}$

$\lim_{n\to\infty} \frac{\displaystyle\prod_{k=1}^{n} k}{\displaystyle\prod_{i=1}^{n-1} 2n}$

$$\prod_{k=1}^{n} k \prec \prod_{i=1}^{n-1} 2n \implies \lim_{n\to\infty} \frac{(2n)!!}{(2n)^n} = 0$$

Задача 3.

$a_n \to -\infty$ ако $\forall\, C < 0 \quad \exists\, \upsilon\, ;\, n > \upsilon ; a_n < C$

Допускаме противното:
$\exists\, C \geq 0 \quad \forall n\, ;\, a_n \geq C$

$a_n \downarrow \implies a_{n+1} < a_n \implies \exists\, \upsilon\, ;\, a_\upsilon \geq C ; a_{\upsilon+1} < C$

$\implies \forall\, n > \upsilon \quad a_n < C \implies$ противоречие с допускането

$\implies a_n \to -\infty$