Домашна работа 1. Вариант 2. № 45342. Група 3

Иво Стратев

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1 Задача 1.

$C : x^2 + y^2 \leq 5^2$
$P : y = x^2 - 5$
$K : x^2 + y^2 = 5^2 \implies K(O(0,0), 5)$
$V_{Px} = 0 \implies V_{Py} = -5 \implies V_P(0, -5)$

$$\Phi : \begin{cases} x^2 + y^2 \leq 5^2 \\ y = x^2 - 5 \end{cases}$$

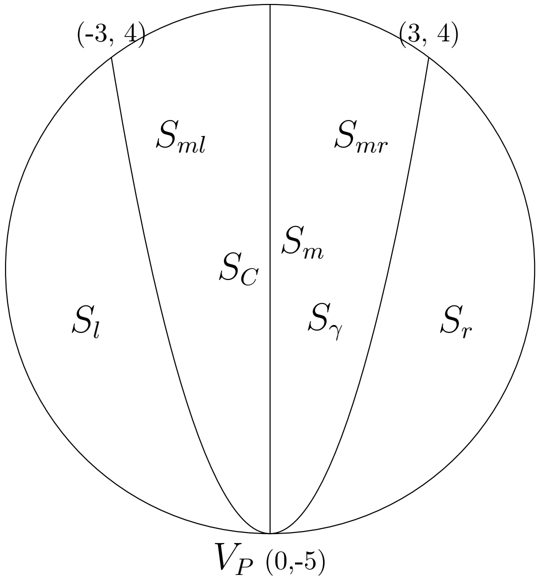
$$\Gamma : \begin{cases} y = x^2 - 5 \\ x^2 + y^2 = 5^2 \end{cases} \to \begin{cases} x^2 = y + 5 \\ y + 5 + y^2 = 5^2 \end{cases}$$

$y^2 + y - 20 = 0$
$D = 1 - 4.(-20) = 1 + 80 = 81$
$y_1 = \frac{-1+9}{2} = 4$
$y_2 = \frac{-1-9}{2} = -5$

$4 = x^2 - 5$
$9 = x^2$
$x_1 = 3$
$x_2 = -3$

$-5 = x^2 - 5$
$x_3 = 0$

$\implies K \cap P = \{(0, -5),\ (-3, 4),\ (3, 4)\}$



$K : y^2 = 25 - x^2 \implies K : y = \pm\sqrt{25 - x^2}$

$\forall x \in [0,3]:$

$f(x) = \sqrt{25 - x^2}$
$f'(x) = \frac{-\cancel{2}x}{\cancel{2}\sqrt{25-x^2}} = -\frac{x}{\sqrt{25-x^2}} \implies f'(x) < 0 \implies$
$f(x) \downarrow \implies \begin{cases} f_{max}(x) = f(0) = \sqrt{25-0} = 5 \\ f_{min}(x) = f(3) = \sqrt{25-9} = \sqrt{16} = 4 \end{cases}$

$P : y = x^2 - 5,\ g(x) = x^2 - 5$
$g'(x) = 2x \implies g'(x) > 0 \implies$
$g(x) \uparrow \implies \begin{cases} g_{max}(x) = g(3) = 9 - 5 = 4 \\ g_{min}(x) = g(0) = 0 - 5 = -5 \end{cases}$

$\implies \forall x \in [0,3]\ g(x) \leq f(x)$
$\gamma : \begin{cases} 0 \leq x \leq 3 \\ g(x) \leq y \leq f(x) \end{cases} \implies$
$S_\gamma = \int_0^3 f(x) - g(x)\,\mathrm{d}x$

$S_C = \pi 5^2 = 25\pi$

$S_C = S_l + S_m + S_r$

$S_m = S_{ml} + S_{mr}$

$S_{ml} = S_{mr} = S_\gamma \implies S_m = 2S_\gamma$

$S_l = S_r = \frac{1}{2}(S_C - S_m) = 12.5\pi - S_\gamma$

$$\begin{aligned} S_\gamma = \int_0^3 f(x) - g(x)\,\mathrm{d}x \\ &= \int_0^3 \sqrt{25-x^2} - (x^2-5)\,\mathrm{d}x \\ &= \int_0^3 \sqrt{25-x^2} - x^2 + 5\,\mathrm{d}x \\ &= (-\frac{x^3}{3} + 5x)\Big|_0^3 + \int_0^3 \sqrt{25-x^2}\,\mathrm{d}x \\ &= 15 - \frac{\cancel{27}}{\cancel{3}} + \int_0^3 \sqrt{25-x^2}\,\mathrm{d}x \\ &= 6 + \int_0^3 \sqrt{25-x^2}\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
I = \int_0^3 \sqrt{25-x^2}\,\mathrm{d}x \\
&= && x\sqrt{25-x^2}|_0^3 - \int_0^3 x\,\mathrm{d}\sqrt{25-x^2} \\
&= && 3\sqrt{25-9} - \int_0^3 \frac{x.(-\cancel{2}x)}{\cancel{2}\sqrt{25-x^2}}\,\mathrm{d}x \\
&= && 3\sqrt{16} + \int_0^3 \frac{x^2}{\sqrt{25-x^2}}\,\mathrm{d}x \\
&= && 12 + \int_0^3 \frac{x^2+25-25}{\sqrt{25-x^2}}\,\mathrm{d}x \\
&= && 12 + \int_0^3 \frac{x^2-25}{\sqrt{25-x^2}}\,\mathrm{d}x + 25\int_0^3 \frac{\mathrm{d}x}{\sqrt{25-x^2}} \\
&= && 12 - \int_0^3 \frac{\cancel{25-x^2}}{\cancel{\sqrt{25-x^2}}}\,\mathrm{d}x + 25\int_0^3 \frac{\mathrm{d}x}{\sqrt{25(1-\frac{x^2}{25})}} \\
&= && 12 - I + 25\int_0^3 \frac{\cancel{5}\,\mathrm{d}\frac{x}{5}}{\cancel{5}\sqrt{1-(\frac{x}{5})^2}} \\
&= && 12 - I + 25\arcsin\frac{x}{5}|_0^3 \\
&= && 12 - I + 25\arcsin\frac{3}{5} = I
\end{aligned}
$$

$\implies 2I = 12 + 25\arcsin\frac{3}{5} \implies I = 6 + 12.5\arcsin\frac{3}{5} \implies$

$S_\gamma = 6 + I = 12 + 12.5\arcsin\frac{3}{5} \implies$

$S_r = S_l = 12.5\pi - (12 + 12.5\arcsin\frac{3}{5}) = 12.5(\pi - \arcsin\frac{3}{5}) - 12$

$S_m = 2S_\gamma = 24 + 25\arcsin\frac{3}{5}$

Отговор:

$12.5(\pi - \arcsin\frac{3}{5}) - 12,\ 24 + 25\arcsin\frac{3}{5},\ 12.5(\pi - \arcsin\frac{3}{5}) - 12$

2 Задача 2.

$C : x^2 + y^2 \le 13^2$
$P : y = 13 - x^2$
$K : x^2 + y^2 = 13^2 \implies K(O(0,0), 13)$
$V_{Px} = 0 \implies V_{Py} = 13 \implies V_P(0, 13)$

$$K \cap P : \begin{cases} y = 13 - x^2 \\ x^2 + y^2 = 13^2 \end{cases} \to \begin{cases} x^2 = 13 - y \\ 13 - y + y^2 = 13^2 \end{cases}$$

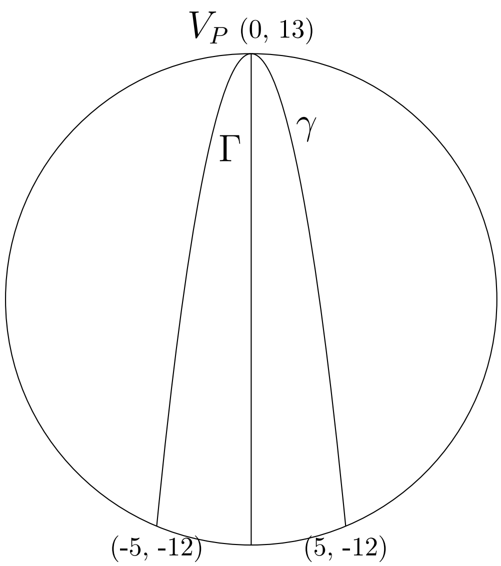
$y^2 - y - 156 = 0$
$D = 1 - 4.(-146) = 1 + 624 = 25$
$y_1 = \frac{1+25}{2} = 13$

$y_2 = \frac{1-25}{2} = -12$

$13 = 13 - x^2$
$x_1 = 0$

$-12 = 13 - x^2$
$x^2 = 25$
$x_2 = 5$
$x_3 = -5$

$\implies K \cap P = \{(0, 13),\ (-5, -12),\ (5, -12)\}$



$$\Gamma : \begin{cases} -5 \leq x \leq 5 \\ f(x) = 13 - x^2 \end{cases}$$

$$\gamma : \begin{cases} 0 \leq x \leq 5 \\ f(x) = 13 - x^2 \end{cases} \implies l_\Gamma = 2 l_\gamma$$

$f'(x) = -2x,\ (f'(x))^2 = 4x^2$

$$
\begin{aligned}
l_\gamma = \int_0^5 \sqrt{1+(f'(x))^2}\,\mathrm{d}x \\
&= \int_0^5 \sqrt{1+4x^2}\,\mathrm{d}x \\
&= x\sqrt{1+4x^2}|_0^5 - \int_0^5 x\,\mathrm{d}\sqrt{1+4x^2} \\
&= 5\sqrt{101} - \int_0^5 \frac{x4x\cancel{2}}{\cancel{2}\sqrt{1+4x^2}} \\
&= 5\sqrt{101} - \int_0^5 \frac{4x^2+1-1}{\sqrt{1+4x^2}} \\
&= 5\sqrt{101} - \int_0^5 \frac{\cancel{4x^2+1}}{\cancel{\sqrt{1+4x^2}}} + \int_0^5 \frac{1}{\sqrt{1+(2x)^2}}\,\mathrm{d}x \\
&= 5\sqrt{101} - l_\gamma + \frac{1}{2}\int_0^5 \frac{1}{\sqrt{1+(2x)^2}}\,\mathrm{d}x2x \\
&= 5\sqrt{101} - l_\gamma + \frac{1}{2}\ln|2x+\sqrt{1+(2x)^2}||_0^5 \\
&= 5\sqrt{101} - l_\gamma + \frac{1}{2}\ln(10+\sqrt{101}) = l_\gamma
\end{aligned}
$$

$\implies\ 2l_\gamma = l_\Gamma = 5\sqrt{101} + \frac{1}{2}\ln(10+\sqrt{101})$

Отговор: $5\sqrt{101} + \frac{1}{2}\ln(10+\sqrt{101})$

3 Задача 3.

$I = \int_0^\infty \frac{\operatorname{arctg}(x^3)\ln(1+x^2)}{x^p}\,\mathrm{d}x,\ p \in \mathbb{R}$

$I_0 = \int_0^1 \frac{\operatorname{arctg}(x^3)\ln(1+x^2)}{x^p}\,\mathrm{d}x$

$I_\infty = \int_1^\infty \frac{arctg(x^3)\ln(1+x^2)}{x^p}\,\mathrm{d}x$

$\implies\ I = I_0 + I_\infty$

I е сх $\iff$ $I_0,\ I_\infty$ са едновременно сх

$I_0 = \int_0^1 \frac{\operatorname{arctg}(x^3)\ln(1+x^2)}{x^p}\,\mathrm{d}x$

$\frac{\operatorname{arctg}(x^3)\ln(1+x^2)}{x^p}\ \widetilde{0}\ \frac{x^3x^2}{x^p} = \frac{1}{x^{p-5}}$

$\int_0^1 \frac{1}{x^{p-5}}\,\mathrm{d}x$ е сх за $p-5<1$ те. за $p<6$

$\implies\ I_0$ е сх за $p<6$

$I_\infty = \int_1^\infty \frac{\operatorname{arctg}(x^3)\ln(1+x^2)}{x^p}\,\mathrm{d}x$

$J_\infty = \int_1^\infty \frac{\ln(1+x^2)}{x^p}\,\mathrm{d}x$

$f(x) = \frac{\ln(1+x^2)}{x^p}$
$g(x) = \frac{\ln x}{x^p}$

$$\begin{aligned} \lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{\ln(1+x^2)}{x^p}\frac{x^p}{\ln x} \\ &= \lim_{x\to\infty}\frac{\ln(1+x^2)}{\ln x} \\ &= \lim_{x\to\infty}\frac{2x^2}{1+x^2} \\ &= \frac{2}{1} = 2 \in (0,\infty) \end{aligned}$$

$\implies f(x) \overset{\infty}{\sim} g(x)$

$\int_1^\infty \frac{\ln x}{x^p}\,\mathrm{d}x = \begin{cases} \int_1^\infty \frac{\ln x}{x}\,\mathrm{d}x = \int_1^\infty \ln x \,\mathrm{d}\ln x = \frac{\ln^2 x}{2}\Big|_1^\infty = \infty \implies \text{разх}, & p=1 \\ \frac{1}{1-p}\big(\frac{\ln x}{x^{p-1}}\Big|_1^\infty - \int_1^\infty \frac{x}{x^p}\,\mathrm{d}\ln x\big) = \frac{1}{1-p}\big(\frac{\ln x}{x^{p-1}}\Big|_1^\infty - \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x\big), & p \neq 1 \end{cases}$

$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ е сх за $p > 1$

$p > 1$ $\frac{\ln x}{x^{p-1}}\Big|_1^\infty = \lim_{x\to\infty}\frac{\ln x}{x^{p-1}} - \ln 1 = 0 - 0 = 0$ $(\ln x \prec x^{p-1})$

$\implies \int_1^\infty \frac{\ln x}{x^p}\,\mathrm{d}x$ е сх за $p > 1$

$\implies J_\infty$ е сх за $p > 1$

$\forall x \in [1,\infty)$ $\operatorname{arctg}\uparrow$, $\lim_{x\to\infty}\operatorname{arctg} x = \frac{\pi}{2} \implies$

I_∞ е сх за $p > 1$ (Критерий на Абел) $\implies$

I е сх за $p \in (-\infty, 6) \cap (1,\infty) = (1,6)$

Отговор:

$\int_0^\infty \frac{\operatorname{arctg}(x^3)\ln(1+x^2)}{x^p}\,\mathrm{d}x$ е сх за $p \in (1,6)$