Домашна работа

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20 ноември 2017 г.

Задача 1.

Решете задачата на Коши

$$\begin{cases} 3(xyy' - y^2)\cos\left(\dfrac{y^2}{x^2} - 1\right) = 2x^2 \\ y(1) = -1 \end{cases}$$

$$3(xyy' - y^2)\cos\left(\frac{y^2}{x^2} - 1\right) = 2x^2 \implies$$

$$(xyy' - y^2)\cos\left(\frac{y^2}{x^2} - 1\right) = \frac{2}{3}x^2$$

Нека $\cos\left(\dfrac{y^2}{x^2} - 1\right) \neq 0 = \cos\left(\dfrac{\pi}{2} + 2k\pi\right) \implies$

$$\frac{y^2}{x^2} - 1 \neq \frac{\pi}{2} + 2k\pi \implies$$

$$y^2 \neq x^2\left(1 + \frac{\pi}{2} + 2k\pi\right) \implies$$

$$y \neq \pm|x|\sqrt{\left(1 + \frac{\pi}{2} + 2k\pi\right)}$$

Ако $\cos\left(\dfrac{y^2}{x^2} - 1\right) = 0 \implies 0 = 2x^2$ следователно

$y = \pm|x|\sqrt{\left(1 + \dfrac{\pi}{2} + 2k\pi\right)},\ y = \pm x\sqrt{\left(1 + \dfrac{\pi}{2} + 2k\pi\right)}$ не са решения

$$\implies xyy' - y^2 = \tfrac{2}{3}\frac{x^2}{\cos\left(\dfrac{y^2}{x^2} - 1\right)} \implies$$

$$xyy' = y^2 + \frac{2}{3}\frac{x^2}{\cos\left(\dfrac{y^2}{x^2} - 1\right)}$$

Ако $y \equiv 0 \implies 0 = \frac{2}{3}\frac{x^2}{\cos(-1)} = \frac{2}{3\cos(1)}x^2 \not\equiv 0 \implies y \equiv 0$ не е решение

$$\implies y' = \tfrac{y}{x} + \tfrac{2}{3}\tfrac{x}{y}\frac{1}{\cos\left(\dfrac{y^2}{x^2} - 1\right)} \text{ (уравнението е хомогенно)}$$

Полагаме $z = \frac{y}{x} \implies y = zx \implies y' = z'x + z \implies$

$$z'x + z = z + \tfrac{2}{3}\frac{1}{z\cos(z^2 - 1)} \implies$$

$$z\cos(z^2 - 1)z' = \frac{2}{3}\frac{1}{x} \quad \Big| \quad \int \mathrm{d}x \implies$$

$$\int z\cos(z^2 - 1)z' \, \mathrm{d}x = \int \frac{2}{3}\frac{1}{x} \, \mathrm{d}x \implies$$

$$\int z\cos(z^2 - 1) \, \mathrm{d}z = \frac{2}{3}\int \frac{1}{x} \, \mathrm{d}x \implies$$

$$\frac{1}{2}\int \cos(z^2 - 1) \, \mathrm{d}(z^2 - 1) = \frac{2}{3}\ln(x) + c \quad (c \in \mathbb{R}) \implies$$

$$\frac{1}{2}\sin(z^2 - 1) = \frac{2}{3}\ln(x) + c \implies$$

$$\sin(z^2 - 1) = \frac{4}{3}\ln(x) + c \quad | \quad \arcsin \implies$$

$$z^2 - 1 = \arcsin\left(\frac{4}{3}\ln(x) + c\right) \implies$$

$$z = \pm\sqrt{\arcsin\left(\frac{4}{3}\ln(x) + c\right) + 1} \implies$$

$$y = \pm x\sqrt{\arcsin\left(\frac{4}{3}\ln(x) + c\right) + 1}, \quad c \in \mathbb{R} \text{ - общо решение}$$

$$y(1) = -1 = \pm 1.\sqrt{\arcsin\left(\frac{4}{3}\ln(1) + c\right) + 1} = \pm\sqrt{\arcsin\left(\frac{4}{3}.0 + c\right) + 1} \implies$$

$-1 = \pm\sqrt{\arcsin(c)+1} \implies 1 = \sqrt{\arcsin(c)+1} \implies 1 = \arcsin(c)+1 \implies$

$\arcsin(c) = 0 \implies c = 0 \implies y = x\sqrt{\arcsin\left(\frac{4}{3}\ln x\right)+1}$

Отговор: Решението на дадената задача на Коши е: $y = x\sqrt{\arcsin\left(\frac{4}{3}\ln x\right)+1}$

Задача 2.

Решете уравнението:

$(4x^3y^3 - y)\mathrm{d}x + (4x^4y^2 + y^2 - 2x)\mathrm{d}y = 0$

Нека $P(x,\ y) = 4x^3y^3 - y$ и $Q(x,\ y) = 4x^4y^2 + y^2 - 2x$

$P'_y(x,\ y) = (4x^3y^3 - y)'_y = 12x^3y^2 - 1$

$Q'_x(x,\ y) = (4x^4y^2 + y^2 - 2x)'x = 16x^3y^2 - 2 \implies Q'_x \neq P'_y.$

Очевидно обаче $P,\ Q \in C^\infty(\mathbb{R}^2)$

Търсим интегриращ множител $\mu(x,\ y) \in C^1(D \subseteq \mathbb{R}^2)$,

такъв че уравнението е пълен диференциял, тоест: $(\mu P)'_y = (\mu Q)'_x \implies$

$\mu'_y P + \mu P'_y = \mu'_x Q + \mu Q'_x$

Очевиден интегриращ множител е $\mu(x,\ y) = y$. Ще го докажем, трябва да проверим дали $\mu'_x = 0 \iff \mu'_y P + \mu P'_y = \mu Q'_x \iff \exists\psi\ :\ \mu'_y = \mu\frac{1}{P}(Q'_x - P'_y) = \mu\psi(y)$

$\frac{1}{P}(Q'_x - P'_y) = \frac{1}{4x^3y^3-y}(16x^3y^2 - 2 - (12x^3y^2 - 1)) = \frac{4x^3y^2-1}{4x^3y^3-y} = \frac{1}{y}\frac{4x^3y^2-1}{4x^3y^2-1} = \frac{1}{y} \implies$

$\psi(y) = \frac{1}{y} \implies \mu'_y = \mu\frac{1}{y} \implies \frac{1}{\mu}\mu'_y = \frac{1}{y} \quad \Big| \quad \int \mathrm{d}y \implies \int \frac{1}{\mu}\mu'_y\,\mathrm{d}y = \int \frac{1}{y}\,\mathrm{d}y \implies$

$\int \frac{1}{\mu}\,\mathrm{d}\mu = \ln y + c \implies \ln|\mu| = \ln y + c \mid e \implies$

$e^{\ln|\mu|} = e^{\ln y + c} = e^{c+\ln y} = e^c e^{\ln y} \implies |\mu| = e^c y \implies \mu = cy\ (c \in \mathbb{R}\backslash\{0\})$

Фиксираме константата $c = 1 \implies \mu(x,\ y) = y$

Проверяваме, че наистина стигаме до уравнение, което е пълен диференциял:

Нека $G(x,\ y) = \mu(x,\ y)P(x,\ y) = y(4x^3y^3 - y) = 4x^3y^4 - y^2 \in C^\infty(\mathbb{R}^2)$ и нека

$H(x,\ y) = \mu(x,\ y)Q(x,\ y) = y(4x^4y^2 + y^2 - 2x) = 4x^4y^3 + y^3 - 2xy \in C^\infty(\mathbb{R}^2)$

$G(x,\ y)'_y = 16x^3y^3 - 2y,\ H(x,\ y)'_x = 16x^3y^3 - 2y \implies G'_y = H'_x \implies$

$G\mathrm{d}x + H\mathrm{d}y = 0$ е пълен диференциял. Следователно $\exists U\ :\ U'_x = G,\ U'_y = H$

$\implies U = \displaystyle\int G\ \mathrm{d}x = \int (4x^3y^4 - y^2)\ \mathrm{d}x = x^4y^4 - y^2x + t(y)$

$U'_y = H \implies (x^4y^4 - y^2x + t(y))'_y = 4x^4y^3 + y^3 - 2xy \implies$

$4x^4y^32xy + t'_y(y) = 4x^4y^3 + y^3 - 2xy \implies t'(y) = y^3 \implies$

$t = \displaystyle\int y^3\ \mathrm{d}y = \frac{1}{4}y^4 + r \implies U(x,\ y) = x^4y^4 - y^2x + \frac{1}{4}y^4 + r,\ r \in \mathbb{R}$

Отговор: Решение на даденото уравнение е: $x^4y^4 - y^2x + \frac{1}{4}y^4 = -r,\ r \in \mathbb{R}$