Теория за определен и неопределен интеграл

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1 Def. Неопределен интеграл от функция $f(x)$ е функция $F(x)$ удовлетворяваща условието $F'(x) = f(x)$

1.1 Интегриране по части

1.1.1 Опр. $\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x$

1.1.2 Док-во:

$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x \mid ()'$

$f(x)g'(x) = (f(x)g(x))' - g(x)f'(x)$

$f(x)g'(x) = \cancel{f(x)'g(x)} + f(x)g'(x) - \cancel{g(x)f'(x)}$

$\implies f(x)g'(x) = f(x)g'(x)$

1.1.3 Бонус Док-во:

$\int f(x)g'(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}g(x)$

$g'(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}x} \implies g'(x)\,\mathrm{d}x = \mathrm{d}g(x)$

$\implies \int f(x)\,\mathrm{d}g(x) = f(x)g(x) - \int g(x)\,\mathrm{d}f(x) =$

$= f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x$

1.2 Формула за смяна на променливите

1.2.1 Опр. $\int f(x)\,\mathrm{d}x = F(x) \implies \int f(\varphi(t))\varphi'(t)\,\mathrm{d}t = F(\varphi(t))$

1.2.2 Док-во:

$\int f(\varphi(t))\varphi'(t)\,\mathrm{d}t = F(\varphi(t)) \ |()'$

$f(\varphi(t))\varphi'(t) = (F(\varphi(t)))'$

$f(\varphi(t))\varphi'(t) = F'(\varphi(t))\varphi'(t)$

$f(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t)$

$\implies \int f(\varphi(t))\varphi'(t)\,\mathrm{d}t = F(\varphi(t))$

1.3 Бонус Док-во:

$\int \frac{1}{x}\,\mathrm{d}x = \ln|x|$

$x \geq 0 \implies |x| = x$

$\implies \int \frac{1}{x}\,\mathrm{d}x = \ln(x) \ |()'$

$\implies \frac{1}{x} = \frac{1}{x}$

$x < 0 \implies |x| = -x$

$\implies \int \frac{1}{x}\,\mathrm{d}x = \ln(-x) \ |()'$

$\implies \frac{1}{x} = \frac{1}{-x}(-1) = \frac{1}{x}$

$\implies \int \frac{1}{x}\,\mathrm{d}x = \ln|x|$

2 Определен интеграл

2.1 Риман

$n \in \mathbb{N},\ i = [1, n] \subset \mathbb{N}$
$a, b \in \mathbb{R},\ f : [a, b] \to \mathbb{R}$
$\tau = \{a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b\}$
$diam(\tau) = max\{x_k - x_{k-1} \mid \forall k \in i,\ x_{k-1}, x_k \in \tau\}$
$\xi_k \in [x_{k-1},\ x_k],\ \forall k \in i,\ \ x_{k-1}, x_k \in \tau$
$S_R = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$ сума на Риман

$\lim_{diam(\tau)\to 0} \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \int\limits_a^b f(x)\,\mathrm{d}x = I$

$\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0;\ \forall \tau = \{a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b\},\ diam(\tau) < \delta_\varepsilon$
$\forall \{\xi_1, \ldots, \xi_n\};\ \xi_k \in [x_{k-1},\ x_k],\ \forall k \in i,\ \ x_{k-1}, x_k \in \tau$
$\implies\ |\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) - I| < \varepsilon$
$\forall f$, за което е изпълнено горното условие е интегруема фунцкия в смисъл на Риман

2.2 Дарбу

$n \in \mathbb{N},\ i = [1, n] \subset \mathbb{N}$
$a, b \in \mathbb{R},\ f : [a, b] \to \mathbb{R}$
$\tau = \{a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b\}$
$diam(\tau) = max\{x_k - x_{k-1} \mid \forall k \in i,\ x_{k-1}, x_k \in \tau\}$
$m_k = inf\{f(x) \mid x \in [x_{k-1},\ x_k]\} \forall k \in i$
$M_k = sup\{f(x) \mid x \in [x_{k-1},\ x_k]\} \forall k \in i$

Малка сума на Дарбу $s_f = \sum_{k=1}^{n} m_k(x_k - x_{k-1}),\ \underline{I} = sups_f$

Голяма сума на Дарбу $S_f = \sum_{k=1}^{n} M_k(x_k - x_{k-1}),\ \overline{I} = infS_f$

Ако$\underline{I} = \overline{I} = I,$ то $f(x)$ наричаме интегруема в смисъл на Дарбу и
$\int\limits_a^b f(x)\,\mathrm{d}x = I$

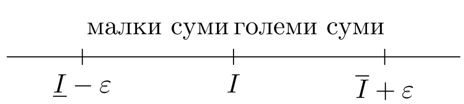
2.2.1 $\underline{I}$

$1. \forall \tau;\ \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \leq \underline{I}$
$2. \forall \varepsilon > 0\ \exists \tau_\varepsilon;\ \sum_{k=1}^{n} m_k(x_k - x_{k-1}) > \underline{I} - \varepsilon$

2.2.2 $\overline{I}$

$1. \forall \tau;\ \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \geq \overline{I}$
$2. \forall \varepsilon > 0\ \exists \tau_\varepsilon;\ \sum_{k=1}^{n} M_k(x_k - x_{k-1}) < \overline{I} + \varepsilon$

$\implies\ s_f \leq S_R \leq S_f$



$\tau' \succ \tau \iff \tau \subset \tau'$

2.2.3 $\tau' \succ \tau \implies s_{f_\tau} \leq s_{f_{\tau'}}$ При добавяне на нови точки малките суми не намаляват

Док-во:
$\tau = \{x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n\}$
$j \in [1, n] \subset \mathbb{N}$
$\tau' = \tau \cup \{x'\},\ x' \in (x_{j-1}, x_j)$
$m' = inf\{f(x) \mid x \in [x_{j-1},\ x']\}$
$m'' = inf\{f(x) \mid x \in [x',\ x_j]\}$
$m_j = inf\{f(x) \mid x \in [x_{j-1},\ x_j]\}$
$\implies m', m'' \geq m_j$
$m'(x' - x_{j-1}) + m''(x_j - x') \geq m_j(x' - x_{j-1}) + m_j(x_j - x') =$
$= m_j(x_j - \cancel{x'} + \cancel{x'} - x_{j-1}) = m_j(x_j - x_{j-1})$
$\implies s_{f_\tau} \leq s_{f_{\tau'}}$

2.2.4 $\tau' \succ \tau \implies S_{f_\tau} \geq S_{f_{\tau'}}$ При добавяне на нови точки големите суми не растат

Док-во:
$\tau = \{x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n\}$
$j \in [1, n] \subset \mathbb{N}$
$\tau' = \tau \cup \{x'\},\ x' \in (x_{j-1}, x_j)$
$M' = sup\{f(x) \mid x \in [x_{j-1},\ x']\}$
$M'' = sup\{f(x) \mid x \in [x',\ x_j]\}$
$M_j = sup\{f(x) \mid x \in [x_{j-1},\ x_j]\}$
$\implies M', M'' \leq M_j$
$M'(x' - x_{j-1}) + M''(x_j - x') \leq M_j(x' - x_{j-1}) + {}_j(x_j - x') =$
$= M_j(x_j - \cancel{x'} + \cancel{x'} - x_{j-1}) = {}_j(x_j - x_{j-1})$
$\implies s_{f_\tau} \geq s_{f_{\tau'}}$

2.2.5 Всяка малка сума не надминава, коя да е голяма сума

Док-во:
$\tau_1 = \{a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_k = b\}$
$\tau_2 = \{a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_j = b\}$
$\tau_3 = \tau_1 \cup \tau_2$
$\tau_3 = \tau_1 \cup \tau_2 \implies \tau_3 \succ \tau_1 \implies s_{f_{\tau_1}} \leq s_{f_{\tau_3}}$
$\tau_3 = \tau_1 \cup \tau_2 \implies \tau_3 \succ \tau_2 \implies S_{f_{\tau_3}} \leq S_{f_{\tau_2}}$
$\implies s_{f_{\tau_1}} \leq s_{f_{\tau_3}} \leq S_{f_{\tau_3}} \leq S_{f_{\tau_2}} \implies s_{f_{\tau_1}} \leq S_{f_{\tau_2}}$

2.2.6 Критерий на Дарбу (НДУ за интегруемост по Дарбу)

$\forall \varepsilon > 0\ \exists \tau_1, \tau_2;\ S_{f_{\tau_1}} - s_{f_{\tau_2}} < e$
Док-во:
$(\Leftarrow)$
$s_{f_{\tau_2}} \leq \underline{I} \leq \overline{I} \leq S_{f_{\tau_1}} \implies \overline{I} - \underline{I} < \varepsilon\ \forall \varepsilon > 0$
$(\Rightarrow)$

$\implies s_{f_{\tau_2}} > \underline{I} - \frac{\varepsilon}{2}, \quad S_{f_{\tau_1}} < \overline{I} + \frac{\varepsilon}{2}$
$\overline{I} = \underline{I} = I \implies S_{f_{\tau_1}} - s_{f_{\tau_2}} < \cancel{I} + \frac{\varepsilon}{2} - \cancel{I} + \frac{\varepsilon}{2} < \varepsilon$

$\implies \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) < \varepsilon$

2.3 Всяка непрекъсната функция в даден интервал е интегрируема в него

Док-во:
$\forall \varepsilon > 0 \exists \delta > 0;\ \forall x_1, x_2 \in [a,b]\ |x_1 - x_2| < \delta$
$\implies |f(x_1) - f(x_2)| < \frac{\varepsilon}{2(b-a)}$
Ако $\tau = \{x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n\},\ \dim \tau < \delta$
$\implies S_{f_\tau} - s_{f_\tau} = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) < \frac{\varepsilon}{2\cancel{(b-a)}}\cancel{(b-a)} < \frac{\varepsilon}{2} < \varepsilon$

2.4 Всяка нарастваща и ограничена в даден интервал фунцкия е интегруема в него

Док-во:
$\tau = \{x_k = a + k\frac{b-a}{n} \mid \forall k \in [1,n] \subset \mathbb{N}\},\ \implies diam\tau = \frac{b-a}{n}$
$\implies \lim_{n\to\infty} S_{f_\tau} - s_{f_\tau} = \lim_{n\to\infty} \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) =$
$= \lim_{n\to\infty} \sum_{k=1}^{n}(f(x_k) - f(x_{k-1}))(\frac{b-a}{n}) = (\frac{b-a}{n})(f(b) - f(a)) \xrightarrow[n\to\infty]{} 0$

2.5 Th За средните стойности

$f(x) : [a,b] \to [m,M]$ е интегруема

$\implies m \le \frac{\int\limits_a^b f(x)\,\mathrm{d}x}{b-a} \le M$
Док-во:

$m \le f(x) \le M \mid \int\limits_a^b \mathrm{d}x$

$\int\limits_a^b m\,\mathrm{d}x \le \int\limits_a^b f(x)\,\mathrm{d}x \le \int\limits_a^b M\,\mathrm{d}x$

$m(b-a) \le \int\limits_a^b f(x)\,\mathrm{d}x \le M(b-a) \mid/(b-a)$

$\implies m \le \frac{\int\limits_a^b f(x)\,\mathrm{d}x}{b-a} \le M$

2.6 Втора Th За средните стойности

$f(x) : [a,b] \to [m,M]$ е интегруема

$\implies \exists c \in [a,b];\ \int\limits_a^b f(x)\,\mathrm{d}x = f(c)(b-a$

Док-во Ползвайки горната теорема:

$\implies m \leq \frac{\int\limits_a^b f(x)\,\mathrm{d}x}{b-a} \leq M$

$\mu = \frac{\int\limits_a^b f(x)\,\mathrm{d}x}{b-a} \implies m \leq \mu \leq M$

Ако $f(x)$ е непрекъсната

$\implies \exists c \in [a,b];\ f(c) = \mu$

$\mu = \frac{\int\limits_a^b f(x)\,\mathrm{d}x}{b-a} = f(c)\ |(b-a)$

$\implies \int\limits_a^b f(x)\,\mathrm{d}x = f(c)(b-a)$

2.7 Th Функцията $F(x) = \int\limits_a^x f(t)\,\mathrm{d}t$ е непрекъсната

Док-во Ползвайки горната теорема:

И ако:

$|f(t)| \leq M$

$\int\limits_a^b f(t)\,\mathrm{d}t = f(c)(b-a) \implies \int\limits_a^x f(t)\,\mathrm{d}t = f(c)(x-a) = F(x)$

$\implies |F(x_2) - F(x_1)| = |f(c)(x_2-a) - f(c)(x_1-a)| = |f(c)x_2 - f(c)x_1| =$

$= |\int\limits_a^{x_2} f(t)\,\mathrm{d}t - \int\limits_a^{x_1} f(t)\,\mathrm{d}t| = |\int\limits_a^{x_2} f(t)\,\mathrm{d}t + \int\limits_{x_1}^a f(t)\,\mathrm{d}t| = |\int\limits_{x_1}^{x_2} f(t)\,\mathrm{d}t| \leq$

$\leq |Mx_2 - Mx_1| = |M(x_2-x_1)| = M|x_2-x_1|$

2.8 Th На Нютон-Лайбниц

$f(x) \in C[a,b],\ F(x) = \int\limits_a^x f(t)\,\mathrm{d}t,\ x \in [a,b]$

$\implies F(x)$ е диференцируема в интервала $[a,b]$ $F'(x) = f(x)$

Док-во в точката $x_0 \in [a,b],\ h;\ x_o + h \in [a,b]$

$\implies \frac{F(x_0+h)-F(x_0)}{h} = \frac{\int\limits_a^{x_0+h} f(t)\,\mathrm{d}t - \int\limits_a^{x_0} f(t)\,\mathrm{d}t}{h} =$

$= \frac{\int\limits_a^{x_0+h} f(t)\,\mathrm{d}t + \int\limits_{x_0}^a f(t)\,\mathrm{d}t}{h} = \frac{\int\limits_{x_0}^{x_0+h} f(t)dxt}{h} = \frac{f(c)(x_0+h-x_0)}{h} = f(c)$

$x \in [min\{x_0+h, h\}, max\{x_0+h, x_0\}]$

$\implies x_0 + h \xrightarrow[h\to 0]{} x_0 \implies c \xrightarrow[h\to 0]{} x_0 \implies f(c) \to f(x_0)$

$\frac{F(x_0+h)-F(x_0)}{h} = f(c)\ |\lim_{h\to 0}$

$\implies \lim_{h\to 0} \frac{F(x_0+h)-F(x_0)}{h} = f(x_0)$

$\implies F'(x) = f(x),\ \forall x \in [a,b]$

2.9 Формула на Нютон-Лайбниц

$\int\limits_a^b f(x)\,\mathrm{d}x = \Phi(b) - \Phi(a)$
Док-во:
$F(x) = \Phi(x) + C$
$F(x) = \int\limits_a^x f(t)\,\mathrm{d}t \implies F(a) = 0 = \Phi(a) + C$
$\implies C = -\Phi(a) \implies \int\limits_a^b f(x)\,\mathrm{d}x = F(b) = \Phi(b) + C = \Phi(b) - \Phi(a)$
$\int\limits_a^b f(x)\,\mathrm{d}x = \Phi(b) - \Phi(a) = \Phi(x)|_a^b$
$\implies \int\limits_a^b f'(x)\,\mathrm{d}x = f(x)|_a^b$

2.10 Интегриране по части $\int\limits_a^b f(x)\,\mathrm{d}g(x) = f(x)g(x)|_a^b - \int\limits_a^b g(x)\,\mathrm{d}f(x)$

$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
$\implies f(x)g'(x) = (f(x)g(x))' - f'(x)g(x) \mid \int\limits_a^b \mathrm{d}x$
$\implies \int\limits_a^b f(x)g'(x)\,\mathrm{d}x = \int\limits_a^b (f(x)g(x))'\,\mathrm{d}x - \int\limits_a^b g(x)f'(x)\,\mathrm{d}x$
$\implies \int\limits_a^b f(x)\,\mathrm{d}g(x) = f(x)g(x)|_a^b - \int\limits_a^b g(x)\,\mathrm{d}f(x)$

2.11 Формула за смяна на променливите

$\int\limits_a^b f(x)\,\mathrm{d}x$
$x = \varphi(t) \implies t = \varphi^{-1}(x)$
$a = \varphi(\alpha) \implies \alpha = \varphi^{-1}(a)$
$b = \varphi(\beta) \implies \beta = \varphi^{-1}(b)$
$\implies \int\limits_a^b f(x)\,\mathrm{d}x = \int\limits_\alpha^\beta f(\varphi(t))\varphi'(t)\,\mathrm{d}t$
Док-во:
$\int f(t)\,\mathrm{d}t = \int f(\varphi(t))\varphi'(t)\,\mathrm{d}t = \Phi(t) = \Phi(\varphi^{-1}(x)) \;|()|_a^b$
$\implies \int\limits_a^b f(t)\,\mathrm{d}t = \Phi(\varphi^{-1}(a)) - \Phi(\varphi^{-1}(b)) = \Phi(\alpha) - \Phi(\beta) = \int\limits_\alpha^\beta f(\varphi(t))\varphi'(t)\,\mathrm{d}t$