Домашна работа

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Задача 1.

Да се реши задачата на Коши:

$$\begin{cases} 2x^3y' = xy^3 - x^2y + y^5 \\ y(2) = 1 \end{cases}$$

Очевидно $y \equiv 0$ е решение на $2x^3y' = xy^3 - x^2y + y^5$, но $0(2) = 0 \neq 1 \implies$

$y \equiv 0$ не е решение на дадената задача на Коши.

Полагаме $y = z^m \implies y' = mz^{m-1}z' \implies 2x^3mz^{m-1}z' = xz^{3m} - x^2z^m + z^{5m} \implies$

$3 + m - 1 = 3m + 1 = m + 2 = 5m \implies m + 2 = 3m + 1 = 5m \implies m = \frac{1}{2} \implies$

$x^3z^{\frac{-1}{2}}z' = xz^{\frac{3}{2}} - x^2z^{\frac{1}{2}} + z^{\frac{5}{2}} \implies z' = \left(\frac{z}{x}\right)^2 - \frac{z}{x} + \left(\frac{z}{x}\right)^3$

Полагаме $t = \frac{z}{x} \implies z = tx \implies z' = t'x + t \implies$

$t'x + t = t^2 - t + t^3 \implies t'x = t^3 + t^2 - 2t \implies \frac{t'}{t(t^2+t-2)} = \frac{1}{x} \quad \Big| \quad \int \mathrm{d}x \implies$

$$\int \frac{t'}{t(t+2)(t-1)}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x \implies$$

$$I = \int \frac{1}{t(t+2)(t-1)}\,\mathrm{d}t = \ln x + c,\ c \in \mathbb{R}$$

$$I = \int \frac{1}{t(t+2)(t-1)}\,\mathrm{d}t = \int \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1}\,\mathrm{d}t =$$

$$= A\int \frac{1}{t}\,\mathrm{d}t + B\int \frac{1}{t+2}\,\mathrm{d}t + C\int \frac{1}{t-1}\,\mathrm{d}t = A\ln|t| + B\ln|t+2| + C\ln|t-1|$$

$$\frac{1}{t(t+2)(t-1)} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} = \frac{A(t+2)(t-1) + Bt(t-1) + Ct(t+2)}{t(t+2)(t-1)}$$

$$\implies 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2) \implies$$

$$\begin{cases} 1 = -2A, & t = 0 \\ 1 = 6B, & t = -2 \\ 1 = 3C, & t = 1 \end{cases} \implies \begin{cases} A = -\frac{1}{2} \\ B = \frac{1}{6} \\ C = \frac{1}{3} \end{cases} \implies$$

$$I = -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| = \ln x + c$$

$$y = \sqrt{z} \implies z = y^2 \implies t = \frac{z}{x} = \frac{y^2}{x} \implies$$

$$-\frac{1}{2}\ln\left|\frac{y^2}{x}\right| + \frac{1}{6}\ln\left|\frac{y^2+2x}{x}\right| + \frac{1}{3}\ln\left|\frac{y^2-x}{x}\right| = \ln x + c$$

$$y(2) = 1 \implies \exists \delta \in \mathbb{R} \;:\; y \;:\; (2-\delta,\ 2+\delta) \to \mathbb{R} \implies$$

$$-\frac{1}{2}\ln\left|\frac{y(2)^2}{2}\right| + \frac{1}{6}\ln\left|\frac{y(2)^2+2.2}{2}\right| + \frac{1}{3}\ln\left|\frac{y(2)^2-2}{2}\right| = \ln 2 + c \implies$$

$$-\frac{1}{2}\ln\left|\frac{1}{2}\right| + \frac{1}{6}\ln\left|\frac{5}{2}\right| + \frac{1}{3}\ln\left|-\frac{1}{2}\right| = \ln 2 + c \implies$$

$$-\frac{1}{2}\ln\left(\frac{1}{2}\right) + \frac{1}{6}\ln\left(\frac{5}{2}\right) + \frac{1}{3}\ln\left(-(-\frac{1}{2}\right) = \ln 2 + c \implies$$

$$\ln(\sqrt{2}) + \ln\left(\sqrt[6]{\frac{5}{2}}\right) + \ln\left(\sqrt[3]{\frac{1}{2}}\right) = \ln(2) + c \implies$$

$$c = \ln(\sqrt{2}) + \ln(\sqrt{2}) + \ln\left(\sqrt[6]{\frac{5}{2}}\right) + \ln\left(\sqrt[3]{\frac{1}{2}}\right) - \ln(2) \implies$$

$$c = \ln\left(\frac{1}{2}\right) + \ln(\sqrt{2}) + \ln\left(\sqrt[6]{\frac{5}{2}}\right) + \ln\left(\sqrt[3]{\frac{1}{2}}\right) \implies$$

$$-\frac{1}{2}\ln\left(\frac{y^2}{x}\right) + \frac{1}{6}\ln\left(\frac{y^2+2x}{x}\right) + \frac{1}{3}\ln\left(\frac{x-y^2}{x}\right) =$$

$$= \ln x + \ln\left(\frac{1}{2}\right) + \ln(\sqrt{2}) + \ln\left(\sqrt[6]{\frac{5}{2}}\right) + \ln\left(\sqrt[3]{\frac{1}{2}}\right)$$

$$\implies \ln\left(\sqrt{\frac{x}{y^2}}\right) + \ln\left(\sqrt[6]{\frac{y^2+2x}{x}}\right) + \ln\left(\sqrt[3]{\frac{x-y^2}{x}}\right) =$$

$$= \ln x + \ln\left(\frac{1}{2}\right) + \ln(\sqrt{2}) + \ln\left(\sqrt[6]{\frac{5}{2}}\right) + \ln\left(\sqrt[3]{\frac{1}{2}}\right) \implies$$

$$\sqrt{\frac{x}{y^2}}\sqrt[6]{\frac{y^2+2x}{x}}\sqrt[3]{\frac{x-y^2}{x}} = \frac{1}{2}\sqrt{2}\sqrt[6]{\frac{5}{2}}\sqrt[3]{\frac{1}{2}}x \quad | \quad ()^2 \implies$$

$$\left(\frac{x}{y^2}\right)^3 \frac{y^2+2x}{x}\left(\frac{x-y^2}{x}\right)^2 = \frac{1}{64}.8\frac{5}{2}\frac{1}{4}x^6 \implies$$

$$\frac{y^2+2x}{y^6}(x-y^2)^2 = \frac{x^6}{64} \implies$$

$$\frac{(y^2+2x)(x^2-2xy^2+y^4)}{y^6} = \frac{x^6}{64} \implies$$

$$\frac{y^2x^2-2xy^4+y^6+2x^3-4x^2y^2+2xy^4}{y^6} = \frac{x^6}{64} \implies$$

$$\frac{y^6+2x^3-3x^2y^2}{y^6} = \frac{x^6}{64} \implies \frac{64}{x^6}+\frac{128}{x^3y^6}-\frac{192}{x^4y^4} = 1$$

Задача 2.

Да се намери общото решение на диференциалното уравнение:

$(1-x^2y)\mathrm{d}x+(yx^2-x^3)\mathrm{d}y$ използвайки подходящ интегриращ множител.

Решение:

Нека $P(x,\ y)=1-x^2y$ и $Q(x,\ y)=yx^2-x^3 \in C^\infty(\mathbb{R}^2)$. Търсим $\mu(x,\ y)$, такова че уравнението $\mu P\mathrm{d}x+\mu Q\mathrm{d}y=0$ е пълен диференциал.

Тоест $(\mu P)'_y = (\mu Q)'_x \implies \mu'_y P + \mu P'_y = \mu'_x Q + \mu Q'_x$

Очевиден интегриращ множител е x^{-2}, за това ще търсим интегриращ множител от вида $\mu(x,\ y)=\varphi(x)$, което е изпълнено, ако $\mu'_y(x,\ y)\equiv 0 \implies$

$\mu P'_y = \mu'_x Q + \mu Q'_y \implies \mu(P'_y - Q'_x) = Q\mu'_x \implies$

$\mu'_x = \mu\frac{P'_y-Q'_x}{Q} \iff \exists\psi(x)\ :\ \frac{P'_y-Q'_x}{Q} = \psi(x)$

$\frac{P'_y-Q'_x}{Q} = \frac{-x^2-(2yx-3x^2)}{yx^2-x^3} = \frac{2x(x-y)}{x^2(y-x)} = -\frac{2}{x} = \psi(x) \implies \mu'_x = -\frac{2}{x}\mu \implies$

$\frac{1}{\mu}\mu'_x = -\frac{2}{x} \Big| \displaystyle\int \mathrm{d}x \implies \int \frac{1}{\mu}\mu'_x\,\mathrm{d}x = \int -\frac{2}{x}\,\mathrm{d}x \implies$

$$\int \frac{1}{\mu}\,\mathrm{d}\mu = -2\int \frac{1}{x}\,\mathrm{d}x \implies \ln|\mu| = -2\ln x + c \implies$$

$$\ln|\mu| = \ln(x^{-2}) + c \implies |\mu| = e^c x^{-2} \implies \mu = tx^{-2},\ t \in \mathbb{R}\backslash\{0\}$$

При $t = 1 \implies \mu(x,\ y) = x^{-2} \implies (x^{-2} - y)\mathrm{d}x + (y - x)\mathrm{d}y = 0$

Нека $G(x,\ y) = x^{-2} - y$ и $H(x,\ y) = y - x \in C^{\infty}(\mathbb{R}^2)$

$$G'_y(x,\ y) = (x^{-2} - y)'_y = -1 = (y - x)'_x = H'_x(x,\ y) \implies$$

$$\exists U(x,\ y)\ :\ U'_x = G,\ U'_y = H \implies$$

$$U = \int G\,\mathrm{d}x = \int (x^{-2} - y)\,\mathrm{d}x = \int (x^{-2} - y)\,\mathrm{d}x = -\frac{1}{x} - yx + c(y)$$

$$U'_y = H \implies \left(-\frac{1}{x} - yx + c(y)\right)'_y = y - x \implies -x + c'_y(y) = y - x \implies$$

$$c(y) = \int y\,\mathrm{d}y = \frac{1}{2}y^2 + r,\ r \in \mathbb{R} \implies U(x,\ y) = -\frac{1}{x} - yx + \frac{1}{2}y^2 + r$$

Отговор: Общото решение на даденото диференциално уравнение е:

$\frac{1}{2}y^2 - \frac{1}{x} - yx = -r,\ r \in \mathbb{R}$