Домашна работа 3, № 45342, Група 3

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1 Задача 1.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!(2n)!}{(3n)!}$$

$$a_n = \frac{n!(2n)!}{(3n)!}, \quad a_{n+1} = \frac{(n+1)!(2n+2)!}{(3n+3)!}, \quad \forall n \in \mathbb{N}\ a_n > 0$$

$$d_n = \frac{a_{n+1}}{a_n} = \frac{(n+1)!(2n+2)!}{(3n+3)!} \frac{(3n)!}{n!(2n)!} = \frac{(n+1)(2n+1)(2n+2)}{(3n+1)(3n+2)(3n+3)} =$$

$$= \frac{(2n+1)(2n+2)}{(3n+1)(3n+2)3} = \frac{4n^2+6n+2}{(9n^2+9n+2)3} \to \frac{4}{27} < 1 \implies \text{сходимост по Даламбер}$$

2 Задача 2.

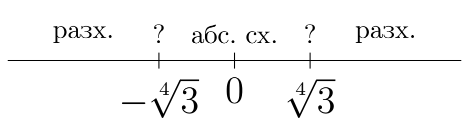
$$\sum_{n=1}^{\infty} a_n x^{4n} = \sum_{n=1}^{\infty} \frac{x^{4n}}{(n+1)(3^n+2)} = \sum_{n=1}^{\infty} \frac{(x^4)^n}{(n+1)(3^n+2)} = \sum_{n=1}^{\infty} a_n (x^4)^n$$

$$y = x^4 \implies \sum_{n=1}^{\infty} a_n y^n = \sum_{n=1}^{\infty} \frac{y^n}{(n+1)(3^n+2)}$$

$$a_n = \frac{1}{(n+1)(3^n+2)}, \quad a_{n+1} = \frac{1}{(n+2)(3^{n+1}+2)}, \quad \forall n \in \mathbb{N}\ a_n > 0$$

$$d_{a_n} = \frac{a_{n+1}}{a_n} = \frac{(n+1)(3^n+2)}{(n+2)(3^{n+1}+2)} = \frac{n3^n+2n+3^n+2}{3n3^n+2n+6.3^n+4} \to \frac{1}{3} = \frac{1}{R_y} \implies$$

$$R = R_x = \sqrt[4]{R_y} = \sqrt[4]{3} \quad \implies$$



$$x = \sqrt[4]{3} \implies \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\left(\left(\sqrt[4]{3}\right)^4\right)^n}{(n+1)(3^n+2)} = \sum_{n=1}^{\infty} \frac{3^n}{(n+1)(3^n+2)}$$

$$b_n = \frac{3^n}{(n+1)(3^n+2)}, \quad b_{n+1} = \frac{3^{n+1}}{(n+2)(3^{n+1}+2)}$$

$$d_{b_n} = \frac{b_{n+1}}{b_n} = \frac{3^{n+1}}{(n+2)(3^{n+1}+2)} \frac{(n+1)(3^n+2)}{3^n} =$$

$$\frac{3^{n+1}}{3^n} d_{a_n} = 3d_{a_n} = 3\frac{n3^n + 2n + 3^n + 2}{3n3^n + 2n + 6.3^n + 4} =$$

$$\frac{3n3^n + 6n + 3.3^n + 6}{3n3^n + 2n + 6.3^n + 4} \to 1 \implies \text{нищо по Даламбер}$$

$$r_{b_n} = n\left(\frac{1}{d_{b_n}} - 1\right) = n\left(\frac{3n3^n + 2n + 6.3^n + 4 - (3n3^n + 6n + 3.3^n + 6)}{3n3^n + 6n + 3.3^n + 6}\right) =$$

$$= n\left(\frac{3.3^n - 4n - 2}{3n3^n + 6n + 3.3^n + 6}\right) = \frac{3n3^n - 4n^2 - 2n}{3n3^n + 6n + 3.3^n + 6} \to 1 \implies \text{нищо по Раабе-Дюамел}$$

$$\forall n \in \mathbb{N}\ \frac{1}{n} > 0, \quad \sum_{n=1}^{\infty} \frac{1}{n} \text{ е разходящ}$$

$$\lim_{n\to\infty} \frac{b_n}{\frac{1}{n}} = \frac{n3^n}{(n+1)(3^n+2)} \to 1 > 0 \implies$$

$$\sum_{n=1}^{\infty} b_n \text{ е разходящ съгласно граничната форма на критерият за сравнение.}$$

$$x = -\sqrt[4]{3} \implies \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{\left(\left(-\sqrt[4]{3}\right)^4\right)^n}{(n+1)(3^n+2)} = \sum_{n=1}^{\infty} \frac{3^n}{(n+1)(3^n+2)} \implies$$

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} b_n \implies \sum_{n=1}^{\infty} c_n \text{ е разходящ} \implies$$

$$\forall x \in (-\sqrt[4]{3}, \sqrt[4]{3}) \ \sum_{n=1}^{\infty} a_n x^{4n} \text{ е сходящ (абсолютно сходящ)}$$

3 Задача 3.

$$f(x) = \begin{cases} x & , \ x \in [0, \frac{\pi}{2}) \\ \pi - x & , \ x \in [\frac{\pi}{2}, \pi] \end{cases}$$

Очевидно f е непрекъсната в интервалите $[0, \frac{\pi}{2}), (\frac{\pi}{2}, \pi]$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \pi - x = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

$$\lim_{x\to\frac{\pi}{2}-0} f(x) = \lim_{x\to\frac{\pi}{2}-0} x = \frac{\pi}{2}$$

$$\lim_{x\to\frac{\pi}{2}+0} f(x) = \lim_{x\to\frac{\pi}{2}+0} \pi - x = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

$$f\left(\frac{\pi}{2}\right) = \pi - \frac{\pi}{2} = \frac{\pi}{2} \implies f \text{ е непрекъсната в } [0,\pi]$$

$$\forall x \in \left[\frac{\pi}{2}, \pi\right]\ f'(x) = (\pi - x)' = -1 < 0 \implies \forall x \in \left[\frac{\pi}{2}, \pi\right]\ f(x) \downarrow$$

$$\forall x \in \left[0, \frac{\pi}{2}\right)\ f'(x) = (x)' = 1 > 0 \implies \forall x \in \left[0, \frac{\pi}{2}\right)\ f(x) \uparrow$$

$$f' \text{ е прекъсната в } \frac{\pi}{2} \implies f \text{ е частично гладка в } [0,\pi]$$

$$\forall x \in [0,\pi]\ 0 \le f(x) \le \frac{\pi}{2} \implies f \text{ е интегруема в } [0,\pi]$$

$$f_1(x) = \begin{cases} -f(-x) & ,\ x \in [-\pi, 0) \\ f(x) & ,\ x \in [0,\pi] \end{cases} \implies$$

f_1 е нечетна и съвпада с f в интервала $[0,\pi]$, което значи, че развитието й в ред на Фурие ще е развитието на f по синуси.

f_1 е непрекъсната, частично гладка и интегруема в $[-\pi,\pi]$

$$f_1(-\pi) = -f_1(\pi) = -(\pi - \pi) = 0 = \pi - \pi = f_1(\pi)$$

$\implies$ Редът на Фурие на f_1 е равномерно сходящ в $[-\pi,\pi]$ и има за стойност $f_1(x)$

$\implies \forall x \in [0,\pi]$ сумата на реда ще съвпада със стойността на $f(x)$

$$\forall n \in \mathbb{N} \cup \{0\}\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\cos(nx)\ \mathrm{d}x = 0$$

$$\forall n \in \mathbb{N}\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\sin(nx)\ \mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\ \mathrm{d}x =$$

$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} x\sin(nx)\ \mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin(nx)\ \mathrm{d}x\right) =$$

$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} x\sin(nx)\ \mathrm{d}x + \pi\int_{\frac{\pi}{2}}^{\pi} \sin(nx)\ \mathrm{d}x - \int_{\frac{\pi}{2}}^{\pi} x\sin(nx)\ \mathrm{d}x\right) =$$

$$= -\frac{2}{n\pi}\left(\int_0^{\frac{\pi}{2}} x\ \mathrm{d}\cos(nx) + \pi\cos(nx)\Big|_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} x\ \mathrm{d}\cos(nx)\right)$$

$$\int x\ \mathrm{d}\cos(nx) = x\cos(nx) - \int \cos(nx)\ \mathrm{d}x = x\cos(nx) - \frac{\sin(nx)}{n} \implies$$

$$b_n = -\frac{2}{n\pi}\left[\left(x\cos(nx) - \frac{\sin(nx)}{n}\right)\Big|_0^{\frac{\pi}{2}} + \pi\cos(nx)\Big|_{\frac{\pi}{2}}^{\pi} - \left(x\cos(nx) - \frac{\sin(nx)}{n}\right)\Big|_{\frac{\pi}{2}}^{\pi}\right] =$$

$$= -\frac{2}{n\pi}\left[\frac{\pi}{2}\cos\left(n\frac{\pi}{2}\right) - \frac{\sin\left(n\frac{\pi}{2}\right)}{n} + \pi\cos(n\pi) - \pi\cos\left(n\frac{\pi}{2}\right) - \pi\cos(n\pi) + \right.$$

$$\left. + \frac{\pi}{2}\cos\left(n\frac{\pi}{2}\right) - \frac{\sin\left(n\frac{\pi}{2}\right)}{n}\right] = \frac{4}{\pi n^2}\sin\left(n\frac{\pi}{2}\right)$$

n	1	2	3	4	5	6	7	8	9	10
$\sin\left(n\frac{\pi}{2}\right)$	1	0	-1	0	1	0	-1	0	1	0

$\implies$

$$b_n = \begin{cases} \frac{4}{\pi n^2}(-1)^{\left(\left(\frac{n-1}{2}\right) \bmod 2\right)} & ,\ n \equiv 1 \pmod 2 \\ 0 & ,\ n \equiv 0 \pmod 2 \end{cases}$$

$$f_1(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2}(-1)^{\left(\left(\frac{2n-2}{2}\right) \bmod 2\right)} \sin((2n-1)x) =$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi n^2}(-1)^{((n-1) \bmod 2)} \sin((2n-1)x = \sum_{n=1}^{\infty} \frac{4}{\pi n^2}(-1)^{n-1} \sin((2n-1)x)$$