Домашна работа 2, №45342, Martin, 1, I, Информатика

Иво Стратев

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Задача 1.

а) Да се пресметне A^{132} , където $A = \begin{pmatrix} 12 & 12 \\ 0 & 12 \end{pmatrix} \in M_2(\mathbb{R})$

Решение:

$$A^2 = AA = \begin{pmatrix} 12 & 12 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} 12 & 12 \\ 0 & 12 \end{pmatrix} = \begin{pmatrix} 12^2 & 2.12^2 \\ 0 & 12^2 \end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix} 12^2 & 2.12^2 \\ 0 & 12^2 \end{pmatrix} \begin{pmatrix} 12 & 12 \\ 0 & 12 \end{pmatrix} = \begin{pmatrix} 12^3 & 3.12^3 \\ 0 & 12^3 \end{pmatrix}$$

Решение по индукция:

$$A^n \ ?= \begin{pmatrix} 12^n & n.12^n \\ 0 & 12^n \end{pmatrix}$$

База на индукцията $n = 1$:

$$A^1 \ ?= \begin{pmatrix} 12^1 & 1.12^1 \\ 0 & 12^1 \end{pmatrix}$$

$$A^1 \ = \begin{pmatrix} 12^1 & 1.12^1 \\ 0 & 12^1 \end{pmatrix} = \begin{pmatrix} 12 & 12 \\ 0 & 12 \end{pmatrix} = A$$

Индуктивно предположение $n = k$:

$$A^k = A^k = \begin{pmatrix} 12^k & k.12^k \\ 0 & 12^k \end{pmatrix}$$

Индуктивно заключение $n = k + 1$:

$$A^{k+1} \ ?= \begin{pmatrix} 12^{k+1} & (k+1)12^{k+1} \\ 0 & 12^{k+1} \end{pmatrix}$$

$$A^{k+1} = A^k A = \begin{pmatrix} 12^k & k.12^k \\ 0 & 12^k \end{pmatrix} \begin{pmatrix} 12 & 12 \\ 0 & 12 \end{pmatrix} =$$
$$= \begin{pmatrix} 12^{k+1} & k.12^{k+1} + 12^{k+1} \\ 0 & 12^{k+1} \end{pmatrix} = \begin{pmatrix} 12^{k+1} & (k+1)12^{k+1} \\ 0 & 12^{k+1} \end{pmatrix}$$

$$\implies A^n = \begin{pmatrix} 12^n & n12^n \\ 0 & 12^n \end{pmatrix} \ \forall n \in \mathbb{N}$$

$$\implies A^{132} = \begin{pmatrix} 12^{132} & 132.12^{132} \\ 0 & 12^{132} \end{pmatrix}$$

Отговор:

$$A^{132} = \begin{pmatrix} 12^{132} & 132.12^{132} \\ 0 & 12^{132} \end{pmatrix}$$

б) Да се намери матрицата $B = f(A)$, където $f(x) = -8x^4 + 4x^3 + 3x^2 - 6x + 2$

$$B = -8A^4 + 4A^3 + 3A^2 - 6A + 2E$$

$$B = -8 \begin{pmatrix} 12^4 & 4.12^4 \\ 0 & 12^4 \end{pmatrix} + 4 \begin{pmatrix} 12^3 & 3.12^3 \\ 0 & 12^3 \end{pmatrix} +$$
$$3 \begin{pmatrix} 12^2 & 2.12^2 \\ 0 & 12^2 \end{pmatrix} - 6 \begin{pmatrix} 12 & 12 \\ 0 & 12 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -8.12^4 & -32.12^4 \\ 0 & -8.12^4 \end{pmatrix} + \begin{pmatrix} 4.12^3 & 12^4 \\ 0 & 4.12^3 \end{pmatrix} +$$
$$\begin{pmatrix} 3.12^2 & 6.12^2 \\ 0 & 3.12^2 \end{pmatrix} + \begin{pmatrix} -6.12 & -6.12 \\ 0 & -6.12 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 4.12^3.(-23) & 12^4.(-31) \\ 0 & 4.12^3.(-23) \end{pmatrix} + \begin{pmatrix} 6.12.5 & 6.12.11 \\ 0 & 6.12.5 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} -92.12^3 & -31.12^4 \\ 0 & -92.12^3 \end{pmatrix} + \begin{pmatrix} \frac{5}{2}.12^2 & \frac{11}{2}.12^2 \\ 0 & \frac{5}{2}.12^2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 12^2(\frac{5}{2} - 92.12) & 12^2(\frac{11}{2} - 31.12^2) \\ 0 & 12^2(\frac{5}{2} - 92.12) \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 12^2(\frac{5-92.24}{2}) & 12^2(\frac{11-62.12^2}{2}) \\ 0 & 12^2(\frac{5-92.24}{2}) \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 12^2(\frac{5-2208}{2}) & 12^2(\frac{11-8928}{2}) \\ 0 & 12^2(\frac{5-2208}{2}) \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 12^2(\frac{-2203}{2}) & 12^2(\frac{-8917}{2}) \\ 0 & 12^2(\frac{-2203}{2}) \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} -2203.72 & -8917.72 \\ 0 & -2203.72 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} -158616 & -642024 \\ 0 & -158616 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} =$$

$$\begin{pmatrix} -158614 & -642024 \\ 0 & -158614 \end{pmatrix}$$

Отговор:

$$B = \begin{pmatrix} -158614 & -642024 \\ 0 & -158614 \end{pmatrix}$$

в) Да се провери дали матриците A и B комутират, т. е. дали $AB = BA$

$$AB = \begin{pmatrix} 12 & 12 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} -158614 & -642024 \\ 0 & -158614 \end{pmatrix} = \begin{pmatrix} -158614.12 & -(642024 + 158614)12 \\ 0 & -158614.12 \end{pmatrix}$$

$$BA = \begin{pmatrix} -158614 & -642024 \\ 0 & -158614 \end{pmatrix} \begin{pmatrix} 12 & 12 \\ 0 & 12 \end{pmatrix} = \begin{pmatrix} -158614.12 & -(642024 + 158614)12 \\ 0 & -158614.12 \end{pmatrix}$$

$\Longrightarrow AB = BA$

Задача 2.
Да се реши матричното уравнение $AXB = C$, където

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 4 \\ 3 & 6 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 5 & 6 \\ 2 & 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -27 & -45 & -45 \\ -102 & -175 & -178 \\ -129 & -229 & -238 \end{pmatrix}$$

Решение:

$$AXB = C \;|B^{-1}$$
$$AXBB^{-1} = CB^{-1}$$
$$AXE = CB^{-1}$$
$$AX = CB^{-1},\ \text{полг. } Y = CB^{-1}$$
$$AX = Y \;|()^t$$
$$(AX)^t = Y^t$$
$$X^tA^t = Y^t \;|(A^t)^{-1}$$
$$X^tA^t(A^t)^{-1} = Y^t(A^t)^{-1}$$
$$X^tE = Y^t(A^t)^{-1}$$
$$X^t = Y^t(A^t)^{-1}$$

$$B^{-1} : (B|E) \to (E|B^{-1})$$

$$\begin{array}{c} \\ 2 \\ 2 \end{array}\left(\begin{array}{ccc|ccc} -1 & -2 & -2 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \to$$

$$\to \begin{array}{c} 2 \\ \\ 2 \end{array}\left(\begin{array}{ccc|ccc} -1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & -2 & -3 & 2 & 0 & 1 \end{array}\right) \to$$

$$\to \begin{array}{c} -2 \\ -2 \\ \\ \end{array}\left(\begin{array}{ccc|ccc} -1 & 0 & 2 & 5 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 6 & 2 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -7 & -2 & -2 \\ 0 & 1 & 0 & -10 & -3 & -2 \\ 0 & 0 & 1 & 6 & 2 & 1 \end{array}\right)\begin{array}{c} -1 \\ \\ \\ \end{array} \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 2 & 2 \\ 0 & 1 & 0 & -10 & -3 & -2 \\ 0 & 0 & 1 & 6 & 2 & 1 \end{array}\right)$$

$$B^{-1} = \begin{pmatrix} 7 & 2 & 2 \\ -10 & -3 & -2 \\ 6 & 2 & 1 \end{pmatrix}$$

$$CB^{-1} = \begin{pmatrix} -27 & -45 & -45 \\ -102 & -175 & -178 \\ -129 & -229 & -238 \end{pmatrix} \begin{pmatrix} 7 & 2 & 2 \\ -10 & -3 & -2 \\ 6 & 2 & 1 \end{pmatrix} = Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

$$y_{11} = 7. -27 + 45.10 - 45.6 = 7. -27 + 45(10-6) = 9(20-21) = -9$$
$$y_{12} = 2. -27 + 3.45 - 2.45 = -9.6 + 9.5 = -9$$

$$y_{13} = 2. - 27 + 2.45 - 45 = -9.6 + 9.5 = -9$$
$$y_{21} = 7. - 102 + 10.175 + 6. - 178 = -714 + 1750 - 1068 = -32$$
$$y_{22} = 2. - 102 + 3.175 + 2. - 178 = -2.280 + 3.175 = 35(15 - 16) = -35$$
$$y_{23} = 2. - 102 + 2.175 + 1. - 178 = 2.73 - 178 = -32$$
$$y_{31} = 7. - 129 + 10.229 - 6.238 = -903 + 2290 - 1428 = -41$$
$$y_{32} = 2. - 129 + 3.229 + 2. - 238 = -2.367 + 3.229 = -734 + 687 = -47$$
$$y_{33} = 2. - 129 + 2.229 + 1. - 238 = 200 - 238 = -38$$

$$CB^{-1} = Y = \begin{pmatrix} -9 & -9 & -9 \\ -32 & -35 & -32 \\ -41 & -47 & -38 \end{pmatrix}$$
$$Y^t = \begin{pmatrix} -9 & -32 & -41 \\ -9 & -35 & -47 \\ -9 & -32 & -38 \end{pmatrix}$$

$$A^t = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 6 \\ 1 & 4 & 5 \end{pmatrix}$$

$$(A^t)^{-1} : (A^t|E) \to (E|(A^t)^{-1})$$

$$\begin{array}{c} \\ - \\ - \end{array} \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 0 & 1 & 0 \\ 1 & 4 & 5 & 0 & 0 & 1 \end{array}\right) \to$$

$$\to \begin{array}{c} -3 \\ \\ 1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \to$$

$$\to \begin{array}{c} -6 \\ 3 \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & 4 & -3 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 3 & -6 \\ 0 & 1 & 0 & -1 & -2 & 3 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right) \begin{array}{c} \\ \\ -1 \end{array} \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 3 & -6 \\ 0 & 1 & 0 & -1 & -2 & 3 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$(A^t)^{-1} = \begin{pmatrix} 4 & 3 & -6 \\ -1 & -2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

$$X^t = Y^t (A^t)^{-1}$$

$$X^t = \begin{pmatrix} -9 & -32 & -41 \\ -9 & -35 & -47 \\ -9 & -32 & -38 \end{pmatrix} \begin{pmatrix} 4 & 3 & -6 \\ -1 & -2 & 3 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

$x_{11} = -9.4 + 32 + 0 = -36 + 32 = -4$
$x_{12} = -9.3 + 32.2 - 41 = 64 - 68 = -4$
$x_{13} = 96 - 3.32 + 41 = 59 + 41 - 96 = -1$
$x_{21} = -9.4 + 35 + 0 = -36 + 35 = -1$
$x_{22} = -3.9 + 2.35 - 47 = -27 - 47 + 70 = -74 + 70 = -4$
$x_{23} = 9.6 - 3.35 + 47 = 54 + 47 - 105 = 101 - 105 = -4$
$x_{31} = -9.4 + 32 + 0 = -36 + 32 = -4$
$x_{32} = -3.9 + 64 - 38 = 64 - 27 - 38 = 64 - 65 = -1$
$x_{33} = 9.6 - 3.32 + 38 = 54 + 38 - 96 = 92 - 96 = -4$

$$X^t = \begin{pmatrix} -4 - 4 - 1 \\ -1 - 4 - 4 \\ -4 - 1 - 4 \end{pmatrix}$$

$$(X^t)^t = X = \begin{pmatrix} -4 - 1 - 4 \\ -4 - 4 - 1 \\ -1 - 4 - 4 \end{pmatrix}$$

Отговор:

$$X = \begin{pmatrix} -4 - 1 - 4 \\ -4 - 4 - 1 \\ -1 - 4 - 4 \end{pmatrix}$$

Задача 3.
Да се намери рангът на матрицата:

$$
\begin{pmatrix}
1 & 3 & -1 & 3 & 1 \\
0 & -7 & -8 & -3 & -4 \\
4 & 20 & 7 & 15 & -3\lambda+\mu+9 \\
-1 & -6 & -3 & -4 & -3 \\
2 & 11 & \lambda+5 & 8 & 5 \\
-1 & -1 & 3 & -2 & 0
\end{pmatrix}
$$

Решение:

$$
\begin{pmatrix}
1 & 3 & -1 & 3 & 1 \\
0 & -7 & -8 & -3 & -4 \\
4 & 20 & 7 & 15 & -3\lambda+\mu+9 \\
-1 & -6 & -3 & -4 & -3 \\
2 & 11 & \lambda+5 & 8 & 5 \\
-1 & -1 & 3 & -2 & 0
\end{pmatrix}
\begin{array}{c}
\\ -1 \\ \\ -1 \\ \\ \\
\end{array}
\rightarrow
$$

$$
\rightarrow
\begin{array}{c}
\\ \\ -4 \\ -1 \\ -2 \\ 1
\end{array}
\begin{pmatrix}
1 & 3 & -1 & 3 & 1 \\
0 & 7 & 8 & 3 & 4 \\
4 & 20 & 7 & 15 & -3\lambda+\mu+9 \\
1 & 6 & 3 & 4 & 3 \\
2 & 11 & \lambda+5 & 8 & 5 \\
-1 & -1 & 3 & -2 & 0
\end{pmatrix}
\rightarrow
$$

$$
\rightarrow
\begin{array}{c}
-3 \\ -3 \\ -3 \\ -1 \\ -2 \\ \\
\end{array}
\begin{pmatrix}
1 & 3 & -1 & 3 & 1 \\
0 & 7 & 8 & 3 & 4 \\
0 & 8 & 11 & 3 & -3\lambda+\mu+5 \\
0 & 3 & 4 & 1 & 2 \\
0 & 5 & \lambda+7 & 2 & 3 \\
0 & 2 & 2 & 1 & 1
\end{pmatrix}
\rightarrow
$$

$$
\rightarrow
\begin{pmatrix}
1 & -3 & -7 & 0 & -2 \\
0 & 1 & 2 & 0 & 1 \\
0 & 2 & 5 & 0 & -3\lambda+\mu+2 \\
0 & 1 & 2 & 0 & 1 \\
0 & 1 & \lambda+3 & 0 & 1 \\
0 & 2 & 2 & 1 & 1
\end{pmatrix}
\rightarrow
$$

$$
\rightarrow
\begin{array}{c}
3 \\ \\ -2 \\ -1 \\ -2
\end{array}
\begin{pmatrix}
1 & -3 & -7 & 0 & -2 \\
0 & 1 & 2 & 0 & 1 \\
0 & 2 & 5 & 0 & -3\lambda+\mu+2 \\
0 & 1 & \lambda+3 & 0 & 1 \\
0 & 2 & 2 & 1 & 1
\end{pmatrix}
\rightarrow
$$

$$\to \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & -3\lambda+\mu \\ 0 & 0 & \lambda+1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 \end{pmatrix}$$

$\lambda = -1 \implies r = 4$
$\lambda \neq -1,\ \mu = 3\lambda \implies r = 4$
$\lambda \neq -1,\ \mu \neq 3\lambda \implies r = 5$

Отговор:
$\lambda = -1,\ r = 4$
$\lambda \neq -1,\ \mu = 3\lambda,\ r = 4$
$\lambda \neq -1,\ \mu \neq 3\lambda,\ r = 5$

Задача 4.
Да се намери обратната матрица на матрицата от $n^{\text{ти}}$ ред, $n \in \mathbb{N}$:

$$\left(\begin{array}{ccccc|ccccc} 2 & 9 & 9 & \dots & 9 & 1 & 0 & 0 & \dots & 0 \\ 9 & 2 & 9 & \dots & 9 & 0 & 1 & 0 & \dots & 0 \\ 9 & 9 & 2 & \dots & 9 & 0 & 0 & 1 & \dots & 0 \\ & & \dots & & & & & \dots & & \\ 9 & 9 & 9 & \dots & 2 & 0 & 0 & 0 & \dots & 1 \end{array}\right)$$

Решение:

$$\begin{array}{c} \\ - \\ - \\ - \\ - \end{array} \left(\begin{array}{ccccc|ccccc} 2 & 9 & 9 & \dots & 9 & 1 & 0 & 0 & \dots & 0 \\ 9 & 2 & 9 & \dots & 9 & 0 & 1 & 0 & \dots & 0 \\ 9 & 9 & 2 & \dots & 9 & 0 & 0 & 1 & \dots & 0 \\ & & \dots & & & & & \dots & & \\ 9 & 9 & 9 & \dots & 2 & 0 & 0 & 0 & \dots & 1 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccccc|ccccc} 2 & 9 & 9 & \dots & 9 & 1 & 0 & 0 & \dots & 0 \\ 7 & -7 & 0 & \dots & 0 & -1 & 1 & 0 & \dots & 0 \\ 7 & 0 & -7 & \dots & 0 & -1 & 0 & 1 & \dots & 0 \\ & & \dots & & & & & \dots & & \\ 7 & 0 & 0 & \dots & -7 & -1 & 0 & 0 & \dots & 1 \end{array}\right) \begin{array}{c} \\ -\frac{1}{7} \\ -\frac{1}{7} \\ -\frac{1}{7} \\ -\frac{1}{7} \end{array} \to$$

$$\rightarrow \begin{array}{c} -9 \\ \\ \\ \\ \\ \end{array}\left(\begin{array}{ccccc|ccccc} 2 & 9 & 9 & \dots & 9 & 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 & \frac{1}{7} & -\frac{1}{7} & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 & \frac{1}{7} & 0 & -\frac{1}{7} & \dots & 0 \\ & & \dots & & & & & \dots & & \\ -1 & 0 & 0 & \dots & 1 & \frac{1}{7} & 0 & 0 & \dots & -\frac{1}{7} \end{array}\right) \rightarrow$$

$$\rightarrow \begin{array}{c} -9 \\ \\ \\ \\ \\ \end{array}\left(\begin{array}{ccccc|ccccc} 2+9 & 0 & 9 & \dots & 9 & 1-\frac{9}{7} & \frac{9}{7} & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 & \frac{1}{7} & -\frac{1}{7} & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 & \frac{1}{7} & 0 & -\frac{1}{7} & \dots & 0 \\ & & \dots & & & & & \dots & & \\ -1 & 0 & 0 & \dots & 1 & \frac{1}{7} & 0 & 0 & \dots & -\frac{1}{7} \end{array}\right) \rightarrow$$

$$\rightarrow \begin{array}{c} -9 \\ \\ \\ \\ \\ \end{array}\left(\begin{array}{ccccc|ccccc} 2+2.9 & 0 & 0 & \dots & 9 & 1-2.\frac{9}{7} & \frac{9}{7} & \frac{9}{7} & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 & \frac{1}{7} & -\frac{1}{7} & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 & \frac{1}{7} & 0 & -\frac{1}{7} & \dots & 0 \\ & & \dots & & & & & \dots & & \\ -1 & 0 & 0 & \dots & 1 & \frac{1}{7} & 0 & 0 & \dots & -\frac{1}{7} \end{array}\right) \rightarrow$$

$\dots$

$$\rightarrow \begin{array}{c} -9 \\ \\ \\ \\ \\ \end{array}\left(\begin{array}{ccccc|ccccc} 2+(n-2).9 & 0 & 0 & \dots & 9 & 1-(n-2).\frac{9}{7} & \frac{9}{7} & \frac{9}{7} & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 & \frac{1}{7} & -\frac{1}{7} & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 & \frac{1}{7} & 0 & -\frac{1}{7} & \dots & 0 \\ & & \dots & & & & & \dots & & \\ -1 & 0 & 0 & \dots & 1 & \frac{1}{7} & 0 & 0 & \dots & -\frac{1}{7} \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{ccccc|ccccc} 2+(n-1).9 & 0 & 0 & \dots & 0 & 1-(n-1).\frac{9}{7} & \frac{9}{7} & \frac{9}{7} & \dots & \frac{9}{7} \\ -1 & 1 & 0 & \dots & 0 & \frac{1}{7} & -\frac{1}{7} & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 & \frac{1}{7} & 0 & -\frac{1}{7} & \dots & 0 \\ & & \dots & & & & & \dots & & \\ -1 & 0 & 0 & \dots & 1 & \frac{1}{7} & 0 & 0 & \dots & -\frac{1}{7} \end{array}\right) \begin{array}{c} \frac{1}{2+(n-1).9} \\ \\ \\ \\ \\ \end{array} \rightarrow$$

$$\rightarrow \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \dots & 0 & \frac{1-(n-1).\frac{9}{7}}{2+(n-1).9} & \frac{9}{7}.\frac{1}{2+(n-1).9} & \frac{9}{7}.\frac{1}{2+(n-1).9} & \dots & \frac{9}{7}.\frac{1}{2+(n-1).9} \\ -1 & 1 & 0 & \dots & 0 & \frac{1}{7} & -\frac{1}{7} & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 & \frac{1}{7} & 0 & -\frac{1}{7} & \dots & 0 \\ & & \dots & & & & & \dots & & \\ -1 & 0 & 0 & \dots & 1 & \frac{1}{7} & 0 & 0 & \dots & -\frac{1}{7} \end{array}\right) \rightarrow$$

$1-(n-1).\frac{9}{7} = 1-n.\frac{9}{7}+\frac{9}{7} = \frac{16}{7}-\frac{9}{7}.n = \frac{16-9.n}{7}$
$2+(n-1).9 = 2+9.n-9 = 9.n-7$

$$\begin{array}{r} \\ + \\ \to + \\ + \\ + \end{array}\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \dots & 0 & \frac{16-9.n}{7.(9n-7)} & \frac{9}{7}.\frac{1}{9n-7} & \frac{9}{7}.\frac{1}{9n-7} & \dots & \frac{9}{7}.\frac{1}{9n-7} \\ 0 & 1 & 0 & \dots & 0 & \frac{1}{7}+\frac{16-9.n}{7.(9n-7)} & \frac{9}{7}.\frac{1}{9n-7}-\frac{1}{7} & \frac{9}{7}.\frac{1}{9n-7} & \dots & \frac{9}{7}.\frac{1}{9n-7} \\ 0 & 0 & 1 & \dots & 0 & \frac{1}{7}+\frac{16-9.n}{7.(9n-7)} & \frac{9}{7}.\frac{1}{9n-7} & \frac{9}{7}.\frac{1}{9n-7}-\frac{1}{7} & \dots & \frac{9}{7}.\frac{1}{9n-7} \\ & & \dots & & & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & \frac{1}{7}+\frac{16-9.n}{7.(9n-7)} & \frac{9}{7}.\frac{1}{9n-7} & \frac{9}{7}.\frac{1}{9n-7} & \dots & \frac{9}{7}.\frac{1}{9n-7}-\frac{1}{7} \end{array}\right) \to$$

$\frac{1}{7}+\frac{16-9.n}{7.(9n-7)} = \frac{\cancel{9n}-7+16\cancel{-9n}}{7.(9n-7)} = \frac{9}{7.(9n-7)} = \frac{9}{7}.\frac{1}{9n-7}$
$\frac{9}{7}.\frac{1}{9n-7}-\frac{1}{7} = \frac{9-9n+7}{7.(9n-7)} = \frac{16-9.n}{7.(9n-7)} = \frac{16-9.n}{9}.\frac{9}{7.(9n-7)}$

Полг. $s=\frac{9}{7.(9n-7)}$, $p=\frac{16-9.n}{9}$

$$\to \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \dots & 0 & s.p & s & s & \dots & s \\ 0 & 1 & 0 & \dots & 0 & s & s.p & s & \dots & s \\ 0 & 0 & 1 & \dots & 0 & s & s & s.p & \dots & s \\ & & \dots & & & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & s & s & s & \dots & s.p \end{array}\right)$$

Отговор:

$$\begin{pmatrix} s.p & s & s & \dots & s \\ s & s.p & s & \dots & s \\ s & s & s.p & \dots & s \\ & & \dots & & \\ s & s & s & \dots & s.p \end{pmatrix}, \text{ където: } s=\frac{9}{7.(9n-7)},\ p=\frac{16-9.n}{9}$$

Задача 5.
$a_1=(8,\ -3,\ -1,\ 1)$
$a_2=(4,\ 0,\ -20,\ 8)$
$a_3=(-8,\ 3,\ 1,\ -1)$
$a_3=(-3,\ 1,\ 2,\ -1)$

$$\begin{pmatrix} 8 & -3 & -1 & 1 \\ 4 & 0 & -20 & 8 \\ -8 & 3 & 1 & -1 \\ -3 & 1 & 2 & -1 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 8 & -3 & -1 & 1 \\ 4 & 0 & -20 & 8 \\ 0 & 0 & 0 & 0 \\ -3 & 1 & 2 & -1 \end{pmatrix} \to$$

$$\rightarrow \begin{array}{cc} \begin{pmatrix} 8 & -3 & -1 & 1 \\ 4 & 0 & -20 & 8 \\ -3 & 1 & 2 & -1 \end{pmatrix} & \begin{array}{c} \\ \frac{1}{4} \\ \\ \end{array} \end{array} \rightarrow$$

$$\rightarrow \begin{array}{cc} \begin{array}{c} -8 \\ \\ 3 \end{array} & \begin{pmatrix} 8 & -3 & -1 & 1 \\ 1 & 0 & -5 & 2 \\ -3 & 1 & 2 & -1 \end{pmatrix} \end{array} \rightarrow$$

$$\rightarrow \begin{pmatrix} 0 & -3 & -39 & -15 \\ 1 & 0 & -5 & 2 \\ 0 & 1 & -13 & 5 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -5 & 2 \\ 0 & 1 & -13 & 5 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -5 & 2 \\ 0 & 1 & -13 & 5 \end{pmatrix}$$

$u_1 = (1,\ 0,\ -5,\ 2)$
$u_2 = (0,\ 1,\ -13,\ 5)$
u_1, u_2 - базис на $\mathbb{U}$

$$\mathbb{U}: \begin{array}{rcl} 5x_1 + 13x_2 + x_3 & = & 0 \\ -2x_1 - 5x_2 + x_4 & = & 0 \end{array}$$

Задача 6.
Нека векторите $e_1,\ e_2,\ e_3$ са базис на линейното пространство $\mathbb{V}$ и

$$\begin{array}{lcl lcl lcl} a_1 & = & 5e_3 - e_1 & b_1 & = & e_1 - e_3 & c_1 & = & -2e_1 - 3e_2 - 3e_3 \\ a_2 & = & e_1 - e_2 - 3e_3 & b_2 & = & -e_2 & c_2 & = & 2e_2 + e_3 \\ a_3 & = & e_3 & b_3 & = & e_3 - 3e_2 & c_3 & = & e_1 + 8e_2 + 5e_3 \end{array}$$

а) Да се докаже, че векторите $a_1,\ a_2,\ a_3$ и $b_1,\ b_2,\ b_3$ също образуват базис на $\mathbb{V}$

Решение:

$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$
$\lambda_1(5e_3 - e_1) + \lambda_2(e_1 - e_2 - 3e_3) + \lambda_3 e_3 = 0$
$5\lambda_1 e_3 - \lambda_1 e_1 + \lambda_2 e_2 - \lambda_2 e_2 - 3\lambda_2 e_3 + \lambda_3 e_3 = 0$
$(\lambda_2 - \lambda_1)e_1 - \lambda_2 e_2 + (5\lambda_1 - 3\lambda_2 + \lambda_3)e_3 = 0$

$$\begin{aligned} \lambda_2 - \lambda_1 &= 0 \\ -\lambda_2 &= 0 \\ 5\lambda_1 - 3\lambda_2 + \lambda_3 &= 0 \end{aligned} \implies \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 5 & -3 & 1 \end{pmatrix}$$

$$\begin{array}{cc} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 5 & -3 & 1 \end{pmatrix} & \begin{array}{l} -1 \\ -1 \\ \end{array} \end{array} \to$$

$$\to \begin{array}{cc} \begin{array}{r} \\ \\ -5 \end{array} & \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 5 & -3 & 1 \end{pmatrix} \end{array} \to$$

$$\to \begin{array}{cc} \begin{array}{r} 1 \\ \\ -2 \end{array} & \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \end{array} \to$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (\lambda_1,\ \lambda_2,\ \lambda_3) = (0,\ 0,\ 0)$$

$\implies a_1,\ a_2,\ a_3$ са базис на $\mathbb{V}$

$\mu_1 b_1 + \mu_2 b_2 + \mu_3 b_3 = 0$
$\mu_1(e_1 - e_3) + \mu_2(-e_2) + \mu_3(e_3 - 3e_2) = 0$
$\mu_1 e_1 - \mu_1 e_3 - \mu_2 e_2 - \mu_3 e_3 - 3\mu_3 e_2$
$\mu_1 e_1 - (\mu_2 + 3\mu_3)e_2 + (\mu_3 - \mu_1)e_3 = 0$

$$\begin{aligned} \mu_1 &= 0 \\ -\mu_2 - 3\mu_3 &= 0 \\ \mu_3 - \mu_1 &= 0 \end{aligned} \implies \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -3 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -3 \\ -1 & 0 & 1 \end{pmatrix} & \begin{array}{l} \\ -1 \\ \end{array} \end{array} \to$$

$$\to \begin{array}{cc} \begin{array}{r} \\ \\ 1 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix} \end{array} \to$$

$$\to \begin{array}{cc} \begin{array}{r} \\ -3 \\ \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \to$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (\mu_1,\ \mu_2,\ \mu_3) = (0,\ 0,\ 0)$$

$\implies b_1,\ b_2,\ b_3$ са базис на $\mathbb{V}$

б) Да се докаже, че $\exists! \varphi \in Hom\mathbb{V};\ \varphi(b_i) = c_i,\ i = 1,\ 2,\ 3$

Решение:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -3 \\ -1 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R})$$

$$C = \begin{pmatrix} -2 & 0 & 1 \\ -3 & 2 & 8 \\ -3 & 1 & 5 \end{pmatrix} \in M_3(\mathbb{R})$$

Нека $\tau, \psi \in Hom\mathbb{V}$;
$Me(\tau) = B,\ \tau(e_i) = b_i,\ i = 1,\ 2,\ 3$
$Me(\psi) = C,\ \psi(e_i) = c_i,\ i = 1,\ 2,\ 3$

Допс. $\exists \varphi \in Hom\mathbb{V};\ \varphi(b_i) = c_i,\ i = 1,\ 2,\ 3$
$\implies (\varphi\tau)(e_i) = (\varphi\ \circ \tau)(e_i) = c_i = \psi(e_i),\ i = 1,\ 2,\ 3$
$\implies \varphi\tau = \psi$
$\mathbb{V} = l(e_1,\ e_2,\ e_3) \implies \dim \mathbb{V} = 3$
$b_1,\ b_2,\ b_3$ е базис на $\mathbb{V} \implies r(B) = 3 \implies \exists B^{-1}$

$B^{-1}:$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -3 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \\ -1 \\ \\ \end{array} \to$$

$$\to \begin{array}{c} \\ \\ 1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to$$

$$\to \begin{array}{c} \\ -3 \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -1 & -3 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

$\implies B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & -1 & -3 \\ 1 & 0 & 1 \end{pmatrix}$

Нека $\tau' \to B^{-1},\ \tau\tau' \to BB^{-1} = E,\ \tau'\tau \to B^{-1}B = E$
$\implies \tau\tau' = \tau'\tau = \varepsilon$
$\implies \exists! \tau^{-1};\ \tau^{-1} = \tau',;\ \tau^{-1}(b_i) = e_i,\ i = 1,\ 2,\ 3$

$\varphi\tau = \psi\ |\tau^{-1}$
$\varphi\tau\tau^{-1} = \psi\tau^{-1}$
$\varphi\varepsilon = \psi\tau^{-1}$
$\varphi = \psi\tau^{-1},\ \psi \to C,\ \tau^{-1} \to B^{-1}$
$\implies \varphi \to CB^{-1}$
$\exists! B^{-1}, \implies \exists! F = CB^{-1} \in M_3(\mathbb{R})$
$\implies \exists! \varphi \in Hom\mathbb{V};\ \varphi(b_i) = c_i,\ i = 1,\ 2,\ 3,\ \varphi \to CB^{-1}$

в) Да се намери матрицата на φ в базиса $e_1,\ e_2,\ e_3$

Решение:
$M_e(\varphi) \to CB^{-1}$

$$CB^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ -3 & 2 & 8 \\ -3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & -1 & -3 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ -1 & -2 & 2 \\ -1 & -1 & 2 \end{pmatrix}$$

г) Да се намери матрицата на φ в базиса $a_1,\ a_2,\ a_3$

Решение:

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 5 & -3 & 1 \end{pmatrix}$$

Нека $\chi \in Hom\mathbb{V}$;
$Me(\chi) = A,\ \chi(e_i) = a_i,\ i = 1,\ 2,\ 3$
$\mathbb{V} = l(e_1,\ e_2,\ e_3) \implies \dim \mathbb{V} = 3$
$a_1,\ a_2,\ a_3$ е базис на $\mathbb{V} \implies r(A) = 3 \implies \exists A^{-1} \in M_3(\mathbb{R})$

$A^{-1}:$

$$\begin{array}{c} \\ \\ 5 \end{array} \left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 5 & -3 & 1 & 0 & 0 & 1 \end{array}\right) \to$$

$$\rightarrow \begin{array}{c} 1 \\ \\ 2 \end{array} \left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 5 & 0 & 1 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & 1 \end{array}\right) \begin{array}{c} -1 \\ -1 \\ \\ \end{array} \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 2 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 5 & 2 & 1 \end{pmatrix}$$

Нека $\chi' \rightarrow A^{-1},\ \chi\chi' \rightarrow AA^{-1} = E,\ \chi'\chi \rightarrow A^{-1}A = E$
$\Longrightarrow\ \chi\chi' = \chi'\chi = \varepsilon$
$\Longrightarrow\ \exists!\chi^{-1};\ \chi^{-1} = \chi',;\ \chi^{-1}(a_i) = e_i,\ i = 1,\ 2,\ 3$

$\varphi(a_i) = \lambda_{1i}a_1 + \lambda_{2i}a_2 + \lambda_{3i}a_3\ |\chi^{-1},\ i = 1,\ 2,\ 3$
$\chi^{-1}\varphi(a_i) = \lambda_{1i}\chi^{-1}(a_1) + \lambda_{2i}\chi^{-1}(a_2) + \lambda_{3i}\chi^{-1}(a_3),\ i = 1,\ 2,\ 3$
$\chi^{-1}\varphi(a_i) = \lambda_{1i}e_1 + \lambda_{2i}e_2 + \lambda_{3i}e_3\ i = 1,\ 2,\ 3$
$a_i = \chi(e_i),\ i = 1,\ 2,\ 3$
$\chi^{-1}\varphi(\chi(e_i)) = \lambda_{1i}e_1 + \lambda_{2i}e_2 + \lambda_{3i}e_3\ i = 1,\ 2,\ 3$
$(\chi^{-1}\varphi\chi)(e_i) = \lambda_{1i}e_1 + \lambda_{2i}e_2 + \lambda_{3i}e_3\ i = 1,\ 2,\ 3$
$\Longrightarrow\ (\chi^{-1}\varphi\chi \rightarrow A^{-1}M_e(\varphi)A = M_a(\varphi)$

$$M_a(\varphi) = A^{-1}M_e(\varphi)A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & -2 & 2 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 5 & -3 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} 6 & -4 & 1 \\ 11 & -5 & 2 \\ 11 & -6 & 2 \end{pmatrix} = \begin{pmatrix} -17 & 9 & -3 \\ -11 & 5 & -2 \\ 63 & -36 & 11 \end{pmatrix}$$

Задача 7.
Нека $\mathbb{V} = M_2(\mathbb{F})$. Дадени са изображенията:

а) $\varphi(X) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} X + X \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix},\ X \in \mathbb{V}$

Решение:

$$\varphi \in Hom\mathbb{V} \iff \begin{matrix} \varphi(A+B) = \varphi(A) + \varphi(B)\ \forall A, B \in \mathbb{V} \\ \varphi(\lambda A) = \lambda\varphi(A)\ \forall \lambda \in \mathbb{F},\ \forall A \in \mathbb{V} \end{matrix}$$

$$A \in \mathbb{V},\ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\varphi(A) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} A + A \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} + 3a_{21} & a_{12} + 3a_{22} \\ 2a_{11} + 5a_{21} & 2a_{12} + 5a_{22} \end{pmatrix} + \begin{pmatrix} -a_{11} - 4a_{12} & a_{11} + 3a_{12} \\ -a_{21} + -4a_{22} & a_{21} + 3a_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} 3a_{21} - 4a_{12} & a_{11} + 3a_{22} + 4a_{12} \\ 2a_{11} + 4(a_{21} - a_{22}) & a_{21} + 2a_{12} + 8a_{22} \end{pmatrix}$$

$$B \in \mathbb{V},\ B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\varphi(B) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} B + B \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} b_{11} + 3b_{21} & b_{12} + 3b_{22} \\ 2b_{11} + 5b_{21} & 2b_{12} + 5b_{22} \end{pmatrix} + \begin{pmatrix} -b_{11} - 4b_{12} & b_{11} + 3b_{12} \\ -b_{21} + -4b_{22} & b_{21} + 3b_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} 3b_{21} - 4b_{12} & b_{11} + 3b_{22} + 4b_{12} \\ 2b_{11} + 4(b_{21} - b_{22}) & b_{21} + 2b_{12} + 8b_{22} \end{pmatrix}$$

$$\varphi(A) + \varphi(B) =$$

$$= \begin{pmatrix} 3a_{21} - 4a_{12} & a_{11} + 3a_{22} + 4a_{12} \\ 2a_{11} + 4(a_{21} - a_{22}) & a_{21} + 2a_{12} + 8a_{22} \end{pmatrix}$$
$$+$$
$$\begin{pmatrix} 3b_{21} - 4b_{12} & b_{11} + 3b_{22} + 4b_{12} \\ 2b_{11} + 4(b_{21} - b_{22}) & b_{21} + 2b_{12} + 8b_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} 3(a_{21} + b_{21}) - 4(a_{12} + b_{12}) & a_{11} + b_{11} + 3(a_{22} + b_{22}) + 4(a_{12} + b_{12}) \\ 2(a_{11} + b_{11}) + 4(a_{21} - a_{22} + b_{21} - b_{22}) & a_{21} + b_{21} + 2(a_{12} + b_{12}) + 8(a_{22} + b_{22}) \end{pmatrix}$$

$$A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

$$\varphi(A + B) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} (A + B) + (A + B) \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} + b_{11} + 3(a_{21} + b_{21}) & a_{12} + b_{12} + 3(a_{22} + b_{22}) \\ 2(a_{11} + b_{11}) + 5(a_{21} + b_{21}) & 2(a_{12} + b_{12}) + 5(a_{22} + b_{22}) \end{pmatrix}$$
$$+$$
$$\begin{pmatrix} -(a_{11} + b_{11}) - 4(a_{12} + b_{12}) & a_{11} + b_{11} + 3(a_{12} + b_{12}) \\ -(a_{21} + b_{21}) + -4(a_{22} + b_{22}) & a_{21} + b_{21} + 3(a_{22} + b_{22}) \end{pmatrix} =$$

$$= \begin{pmatrix} 3(a_{21} + b_{21}) - 4(a_{12} + b_{12}) & a_{11} + b_{11} + 3(a_{22} + b_{22}) + 4(a_{12} + b_{12}) \\ 2(a_{11} + b_{11}) + 4(a_{21} - a_{22} + b_{21} - b_{22}) & a_{21} + b_{21} + 2(a_{12} + b_{12}) + 8(a_{22} + b_{22}) \end{pmatrix}$$

$$\implies \varphi(A + B) = \varphi(A) + \varphi(B) \; \forall A, B \in \mathbb{V} \quad (1)$$

$\lambda \in \mathbb{F}$

$$\lambda\varphi(A) = \lambda \begin{pmatrix} 3a_{21} - 4a_{12} & a_{11} + 3a_{22} + 4a_{12} \\ 2a_{11} + 4(a_{21} - a_{22}) & a_{21} + 2a_{12} + 8a_{22} \end{pmatrix} =$$

$$\begin{pmatrix} \lambda(3a_{21} - 4a_{12}) & \lambda(a_{11} + 3a_{22} + 4a_{12}) \\ \lambda(2a_{11} + 4(a_{21} - a_{22})) & \lambda(a_{21} + 2a_{12} + 8a_{22}) \end{pmatrix}$$

$$\lambda A = \lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}$$

$$\varphi(\lambda A) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \lambda A + \lambda A \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix} + \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda a_{11} + 3\lambda a_{21} & \lambda a_{12} + 3\lambda a_{22} \\ 2\lambda a_{11} + 5\lambda a_{21} & 2\lambda a_{12} + 5\lambda a_{22} \end{pmatrix} + \begin{pmatrix} -\lambda a_{11} - 4\lambda a_{12} & \lambda a_{11} + 3\lambda a_{12} \\ -\lambda a_{21} + -4\lambda a_{22} & \lambda a_{21} + 3\lambda a_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} 3\lambda a_{21} - 4\lambda a_{12} & \lambda a_{11} + 3\lambda a_{22} + 4\lambda a_{12} \\ 2\lambda a_{11} + 4(\lambda a_{21} - \lambda a_{22}) & \lambda a_{21} + 2\lambda a_{12} + 8\lambda a_{22} \end{pmatrix} =$$

$$\begin{pmatrix} \lambda(3a_{21} - 4a_{12}) & \lambda(a_{11} + 3a_{22} + 4a_{12}) \\ \lambda(2a_{11} + 4(a_{21} - a_{22})) & \lambda(a_{21} + 2a_{12} + 8a_{22}) \end{pmatrix}$$

$\implies \varphi(\lambda A) = \lambda \varphi(A)\ \forall \lambda \in \mathbb{F},\ \forall A \in \mathbb{V} \quad (2)$

От (1) и (2) $\implies \varphi \in Hom\mathbb{V}$

$$\varphi(E_{11}) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} E_{11} + E_{11} \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$\varphi(E_{12}) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} E_{12} + E_{12} \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -4 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 0 & 2 \end{pmatrix}$$

$$\varphi(E_{21}) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} E_{21} + E_{21} \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & 0 \\ 5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix}$$

$$\varphi(E_{22}) = \begin{pmatrix}1 & 3\\ 2 & 5\end{pmatrix} E_{22} + E_{22}\begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix} =$$

$$= \begin{pmatrix}1 & 3\\ 2 & 5\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix} =$$

$$= \begin{pmatrix}0 & 3\\ 0 & 5\end{pmatrix} + \begin{pmatrix}0 & 0\\ -4 & 3\end{pmatrix} = \begin{pmatrix}0 & 3\\ -4 & 8\end{pmatrix}$$

$$M_{E_2}(\varphi) = \begin{pmatrix}0 & -4 & 3 & 0\\ 1 & 4 & 0 & 3\\ 2 & 0 & 4 & -4\\ 0 & 2 & 1 & 8\end{pmatrix}$$

б) $\psi(X) = X\begin{pmatrix}1 & 3\\ 2 & 5\end{pmatrix} + \begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix},\ X \in \mathbb{V}$

Решение:

$$\psi \in Hom\mathbb{V} \iff \begin{matrix}\psi(A+B) = \psi(A) + \psi(B)\ \forall A, B \in \mathbb{V}\\ \psi(\lambda A) = \lambda\psi(A)\ \forall\lambda \in \mathbb{F},\ \forall A \in \mathbb{V}\end{matrix}$$

$$A \in \mathbb{V},\ A = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

$$\psi(A) = A\begin{pmatrix}1 & 3\\ 2 & 5\end{pmatrix} + \begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix} =$$

$$= \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}1 & 3\\ 2 & 5\end{pmatrix} + \begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix} =$$

$$= \begin{pmatrix}a_{11} + 2a_{12} & 3a_{11} + 5a_{12}\\ a_{21} + 2a_{22} & 3a_{21} + 5a_{22}\end{pmatrix} + \begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix} =$$

$$= \begin{pmatrix}a_{11} + 2a_{12} - 1 & 3a_{11} + 5a_{12} + 1\\ a_{21} + 2a_{22} - 4 & 3a_{21} + 5a_{22} + 3\end{pmatrix}$$

$$B \in \mathbb{V},\ B = \begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$$

$$\psi(B) = B\begin{pmatrix}1 & 3\\ 2 & 5\end{pmatrix} + \begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix} =$$

$$= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} b_{11} + 2b_{12} & 3b_{11} + 5b_{12} \\ b_{21} + 2b_{22} & 3b_{21} + 5b_{22} \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} b_{11} + 2b_{12} - 1 & 3b_{11} + 5b_{12} + 1 \\ b_{21} + 2b_{22} - 4 & 3b_{21} + 5b_{22} + 3 \end{pmatrix}$$

$$\psi(A)+\psi(B) = \begin{pmatrix} a_{11} + 2a_{12} - 1 & 3a_{11} + 5a_{12} + 1 \\ a_{21} + 2a_{22} - 4 & 3a_{21} + 5a_{22} + 3 \end{pmatrix} + \begin{pmatrix} b_{11} + 2b_{12} - 1 & 3b_{11} + 5b_{12} + 1 \\ b_{21} + 2b_{22} - 4 & 3b_{21} + 5b_{22} + 3 \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} + b_{11} + 2(a_{12} + b_{12} - 1) & 3(a_{11} + b_{11}) + 5(a_{12} + b_{12}) + 2 \\ a_{21} + b_{21} + 2(a_{22} + b_{22} - 4) & 3(a_{21} + b_{21} + 2) + 5(a_{22} + b_{22}) \end{pmatrix}$$

$$A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

$$\psi(A + B) = (A + B) \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} + b_{11} + 2(a_{12} + b_{12}) & 3(a_{11} + b_{11}) + 5(a_{12} + b_{12}) \\ a_{21} + b_{21} + 2(a_{22} + b_{22}) & 3(a_{21} + b_{21}) + 5(a_{22} + b_{22}) \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} + b_{11} + 2(a_{12} + b_{12}) - 1 & 3(a_{11} + b_{11}) + 5(a_{12} + b_{12}) + 1 \\ a_{21} + b_{21} + 2(a_{22} + b_{22} - 2) & 3(a_{21} + b_{21} + 1) + 5(a_{22} + b_{22}) \end{pmatrix}$$

$$\Longrightarrow \ \psi(A + B) \neq \psi(A) + \psi(B) \ \forall A, B \in \mathbb{V} \ \Longrightarrow \ \psi \notin Hom\mathbb{V}$$

Задача 8.

В линейното пространство $\mathbb{V}$ с базис $e_1, \ e_2, \ e_3, \ e_4, \quad A \in Hom\mathbb{V}$:

$$A(\xi_1 e_1,\ \xi_2 e_2,\ \xi_3 e_3,\ \xi_4 e_4) = \\ = (\xi_1 - \xi_2 - 3\xi_3 + \xi_4)e_1 + (3\xi_2 + \xi_3)e_2 + \\ + (-\xi_1 - 2\xi_2 + 2\xi_3 - \xi_4)e_3 \qquad + (4\xi_1 + 5\xi_2 - 9\xi_3 + 4\xi_4)e_4$$

Решение:

Полг.

$\xi_1 = 1,\ \xi_2 = \xi_3 = \xi_4 = 0;\ A(e_1) = e_1 - e_3 + 4e_4$
$\xi_2 = 1,\ \xi_1 = \xi_3 = \xi_4 = 0;\ A(e_2) = -e_1 + 3e_2 - 2e_3 + 5e_4$
$\xi_3 = 1,\ \xi_1 = \xi_2 = \xi_4 = 0;\ A(e_3) = -3e_1 + e_2 + 2e_3 - 9e_4$
$\xi_1 = 4,\ \xi_1 = \xi_2 = \xi_3 = 0;\ A(e_4) = e_1 - e_3 + 4e_4$

$$M_e(A) = \begin{pmatrix} 1 & -1 & -3 & 1 \\ 0 & 3 & 1 & 0 \\ -1 & -2 & 2 & -1 \\ 4 & 5 & -9 & 4 \end{pmatrix}$$

$KerA:$

$$\begin{array}{c} \\ \\ 1 \\ -4 \end{array} \begin{pmatrix} 1 & -1 & -3 & 1 \\ 0 & 3 & 1 & 0 \\ -1 & -2 & 2 & -1 \\ 4 & 5 & -9 & 4 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & -1 & -3 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 9 & 3 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & -1 & -3 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to$$

$$\to \begin{array}{c} 3 \\ \\ \end{array} \begin{pmatrix} 1 & -1 & -3 & 1 \\ 0 & 3 & 1 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 8 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ & & - & - \end{pmatrix}$$

$a_1 = (\frac{8}{3},\ -\frac{1}{3},\ 1,\ 0)$
$a_2 = (-1,\ 0,\ 0,\ 1)$
a_1, a_2 - базис на $KerA$

$d(A) = dimKerA = 2$

$ImA :$

$$M_e(A)^t = \begin{pmatrix} 1 & -1 & -3 & 1 \\ 0 & 3 & 1 & 0 \\ -1 & -2 & 2 & -1 \\ 4 & 5 & -9 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 & -1 & 4 \\ -1 & 3 & -2 & 5 \\ -3 & 1 & 2 & -9 \\ 1 & 0 & -1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 4 \\ -1 & 3 & -2 & 5 \\ -3 & 1 & 2 & -9 \\ 1 & 0 & -1 & 4 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 & 4 \\ -1 & 3 & -2 & 5 \\ -3 & 1 & 2 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{array}{c} \\ 1 \\ 3 \end{array} \begin{pmatrix} 1 & 0 & -1 & 4 \\ -1 & 3 & -2 & 5 \\ -3 & 1 & 2 & -9 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 3 & -3 & 9 \\ 0 & 1 & -1 & 3 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

$b_1 = (1,\ 0,\ -1,\ 4)$
$b_2 = (0,\ 1,\ -1,\ 3)$
b_1, b_2 - базис на ImA
$r(A) = dimImA = 2$