Домашна работа 1, №45342, Martin, 1, I, Информатика

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Задача 1.

а) Да се запише в алгебричен вид числото $\left(\frac{8-4\sqrt{3}\imath}{2+6\sqrt{3}\imath}\right)^{342}$

$\left(\frac{8-4\sqrt{3}\imath}{2+6\sqrt{3}\imath}\right)^{342} = \left(\frac{(8-4\sqrt{3}\imath)(2-6\sqrt{3}\imath)}{(2+6\sqrt{3}\imath)(2-6\sqrt{3}\imath)}\right)^{342} =$

$= \left(\frac{16-48\sqrt{3}\imath-8\sqrt{3}\imath-72}{4+108}\right)^{342} = \left(\frac{-\cancel{56}-\cancel{56}\sqrt{3}\imath}{\cancel{112}}\right)^{342} = \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\imath\right)^{342}$

$|z| = \sqrt{Re(z)^2 + Im(z)^2}$

$\cos\varphi = \frac{Re(z)}{|z|}, \sin\varphi = \frac{Im(z)}{|z|}$

$z = |z|\left(\cos\varphi + \imath\sin\varphi\right)$

$z^n = |z|^n\left(\cos n\varphi + \imath\sin n\varphi\right)$

$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}\imath$

$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{4}{4}} = 1$

$\cos\varphi = \frac{-1}{2}, \sin\varphi = \frac{-\sqrt{3}}{2} \implies \varphi = \frac{5}{3}$

$z = 1\left(\cos\frac{5}{3}\pi + \imath\sin\frac{5}{3}\pi\right)$

$z^{342} = 1^{342}\left(\cos\cancel{342}\frac{5}{\cancel{3}}\pi + \imath\sin\cancel{342}\frac{5}{\cancel{3}}\pi\right)$

$z^{342} = \cos 560\pi + \imath\sin 560\pi$

$z^{342} = \cos 0\pi + \imath\sin 0\pi = 1$

$\implies \left(\frac{8-4\sqrt{3}\imath}{2+6\sqrt{3}\imath}\right)^{342} = 1$

б) Да се намерят в тригонометричен вид корените на уравнението $x^{263} - 4\sqrt{3}\imath - 4 = 0$

$x^{263} - 4\sqrt{3}\imath - 4 = 0$

$x^{263} = 4 + 4\sqrt{3}\imath = z$

$x = \sqrt[263]{z}$

$|z| = \sqrt{Re(z)^2 + Im(z)^2}$

$\cos\varphi = \frac{Re(z)}{|z|}, \sin\varphi = \frac{Im(z)}{|z|}$

$z = |z|\left(\cos\varphi + \imath\sin\varphi\right)$

$\sqrt[n]{z} = |z|^{\frac{1}{n}}\left(\cos\frac{\varphi+2k\pi}{n} + \imath\sin\frac{\varphi+2k\pi}{n}\right)$

$k = 0, 1, \ldots, n-1$

$$|z| = \sqrt{(4)^2 + \left(4\sqrt{3}\right)^2} = \sqrt{16 + 16 \times 3} = \sqrt{4 \times 16} = 2 \times 4 = 8$$
$$z = 8\left(\frac{\cancel{4}}{\cancel{8}} + \frac{\cancel{4}\sqrt{3}}{\cancel{8}}\right) = 8\left(\frac{1}{2} + \imath\frac{\sqrt{3}}{2}\right) = 8\left(\cos\frac{\pi}{3} + \imath\sin\frac{\pi}{3}\right)$$
$$x = \sqrt[263]{z} = 2^{\frac{3}{263}}\left(\cos\frac{\frac{\pi}{3}+2k\pi}{263} + \imath\sin\frac{\frac{\pi}{3}+2k\pi}{263}\right)$$
$$k = 0, 1, \ldots, 262$$

в) Да се намерят в алгебричен вид корените на уравнението $x^2 + (2+2\imath)x - (12 - 18\imath) = 0$

$$x^2 + (2+2\imath)x - (12 - 18\imath) = 0$$
$$D = (2+2\imath)^2 - 4(-(12-18\imath))$$
$$D = 4 + 8\imath + 4\imath^2 + 4(12 - 18\imath)$$
$$D = \cancel{4} + 8\imath \cancel{-4} + 48 - 72\imath$$
$$D = 48 - 64\imath$$
$$z = \sqrt{D} | \uparrow^2$$
$$z \in \mathbb{C}; z = a + b\imath$$
$$z^2 = D = 48 - 64\imath$$
$$a^2 - b^2 + 2ab\imath = 48 - 64\imath$$

$$\left|\begin{array}{lcl} a^2 - b^2 & = & 48 \\ 2ab\imath & = & -64\imath \end{array}\right.$$
$$\left|\begin{array}{lcl} a^2 - b^2 & = & 48 \\ a & = & \frac{-32}{b} \end{array}\right.$$

$$a^2 - b^2 = 48$$
$$\left(\frac{-32}{b}\right)^2 - b^2 = 48$$
$$b^2 - \left(\frac{-32}{b}\right)^2 + 48 = 0 | \times b^2$$
$$b^4 + 48b^2 - 32^2 + = 0$$
$$y = b^2,\ y \geq 0$$
$$y^2 + 48y - 32^2 = 0$$
$$D_y = 48^2 - 4(-32^2)$$
$$D_y = 48^2 + 4 \times 32^2$$
$$D_y = 2^2 \times 3^2 \times 8^2 + 2^2 \times 4^2 \times 8^2$$
$$D_y = 2^2 8^2 (3^2 + 4^2) = 2^2 8^2 5^2$$
$$\sqrt{D_y} = 2 \times 5 \times 8 = 80$$
$$y_1 = \frac{-\cancel{48+80}}{2} = -24 + 40 = 16 > 0$$
$$y_2 = \frac{-\cancel{48+80}}{2} = -24 - 40 = -64 \leq 0$$

$$b^2 = 16, b = \pm 4$$
$$a = \frac{-32}{\pm 4}, a = \mp 8$$
$$z = \mp 8 \pm 4\imath$$
$$x_1 = \frac{-(2+2\imath)+(-8+4\imath)}{2} = \frac{-2-2\imath-8+4\imath}{2} = \frac{-\cancel{10}+\cancel{2}\imath}{\cancel{2}} = -5 + \imath$$
$$x_2 = \frac{-(2+2\imath)-(-8+4\imath)}{2} = \frac{-2-2\imath+8-4\imath}{2} = \frac{-\cancel{6}-\cancel{6}\imath}{\cancel{2}} = 3 - 3\imath$$
$$x_3 = \frac{-(2+2\imath)+(8-4\imath)}{2} = \frac{-2-2\imath+8-4\imath}{2} = \frac{-\cancel{6}-\cancel{6}\imath}{\cancel{2}} = 3 - 3\imath$$
$$x_4 = \frac{-(2+2\imath)-(8-4\imath)}{2} = \frac{-2-2\imath-8+4\imath}{2} = \frac{-\cancel{10}+\cancel{2}\imath}{\cancel{2}} = -5 + \imath$$

$x_1 = x_4, x_2 = x_3$

Задача 2.
Нека $w_0, w_1, \ldots, w_71$ са седемдесет и вторите корени на единицата, където $\omega_k = \cos\frac{2k\pi}{72} + \imath\sin\frac{2k\pi}{72}$. Да се пресметне израза $\omega_0^{389} + \omega_1^{389} + \cdots + \omega_{71}^{389}$

$$\omega_0^{389} + \omega_1^{389} + \cdots + \omega_{71}^{389} = \sum_{i=0}^{71} \omega_i^{389}$$
$$\omega_1 \in \mathbb{C}$$
$$\implies \omega_1^k = \cos\frac{2k\pi}{72} + \imath\sin\frac{2k\pi}{72} = \omega_k$$
$$\implies \sum_{i=0}^{71} \omega_i^{389} = \sum_{i=0}^{71} \left(\omega_1^{389}\right)^i = \frac{\left(\omega_1^{389}\right)^{72} - 1}{\omega_1^{389} - 1}$$
$$= \frac{\left(\cos\frac{2\times\cancel{72}\times 389\pi}{\cancel{72}} + \imath\sin\frac{2\times\cancel{72}\times 389\pi}{\cancel{72}}\right) - 1}{\omega_1^{389} - 1}$$
$$= \frac{(\cos 2\times 389\pi + \imath\sin 2\times 389\pi) - 1}{\omega_1^{389} - 1}$$
$$= \frac{(\cos\pi + \imath\sin\pi) - 1}{\omega_1^{389} - 1} = \frac{1 + 0\imath - 1}{\omega_1^{389} - 1} = \frac{0}{\omega_1^{389} - 1}$$
$$\omega_1^{389} = \cos\frac{2\pi 389}{72} + \imath\sin\frac{2\pi 389}{72} \neq 1$$
$$\implies \omega_1^{389} - 1 \neq 0$$
$$\implies \frac{0}{\omega_1^{389} - 1} = 0$$
$$\implies \omega_0^{389} + \omega_1^{389} + \cdots + \omega_{71}^{389} = 0$$

Задача 3.
Да се реши уравнението $(x-\imath)^{36} + (x+\imath)^{36} = 0$

$$(x-\imath)^{36} + (x+\imath)^{36} = 0$$
$$(x-\imath)^{36} = -(x+\imath)^{36}$$
$$x \neq \imath$$
$$\left(\frac{x-\imath}{x+\imath}\right)^{36} = -1$$
$$A = \frac{x-\imath}{x+\imath}$$
$$A^{36} = -1$$
$$z = -1 = -1 + 0\imath$$
$$|z| = \sqrt{1} = 1$$
$$\cos\varphi = \frac{-1}{1} = -1, \sin\varphi = \frac{0}{1} = 0 \implies \varphi = \pi$$
$$z = \cos\pi + \imath\sin\pi$$
$$\sqrt[36]{z} = \cos\frac{\pi + 2k\pi}{36} + \imath\sin\frac{\pi + 2k\pi}{36} = z_k$$
$$k = 0, 1, \ldots, 35$$

$$\frac{x-\imath}{x+\imath} = z_k$$
$$x - \imath = z_k(x+\imath)$$

$x - \imath = z_k x + z_k \imath$
$\imath - z_k \imath = z_k x + x$
$(1 - z_k)\imath = (z_k + 1)x$
$z_k < \cos \frac{\pi + 2\pi \cancel{36}}{\cancel{36}} + \imath \sin \frac{\pi + 2\pi \cancel{36}}{\cancel{36}} = \cos \pi + \imath \sin \pi = -1$
$\implies z_k + 1 \neq 0$
$x = \frac{(z_k - 1)\imath}{(z_k + 1)}$

Задача 4. Нека z е комплексно число, за което е изпълнено $z + \frac{1}{z} = 2\cos\varphi$. Да се докаже, че $z^{81} + \frac{1}{z^{81}} = 2\cos(81\varphi)$

$z + \frac{1}{z} = 2cos\varphi | \times z$
$z^2 + 1 - 2z\cos\varphi = 0$
$D = 4\cos^2\varphi - 4$
$D = 4(\cos^2\varphi - 1)$
$\sqrt{D} = 2\sqrt{\cos^2\varphi - 1} = 2\sqrt{-\sin^2\varphi}$
$z_{1,2} = \frac{\cancel{2}\cos\varphi \pm \cancel{2}\sqrt{-\sin^2\varphi}}{\cancel{2}}$
$z_{1,2} = \cos\varphi \pm \sqrt{-1}\sqrt{\sin^2\varphi}$
$z_{1,2} = \cos\varphi \pm \imath\sin\varphi$
$\frac{1}{z_{1,2}} = {z_{1,2}}^{-1} = \cos -\varphi \pm \imath\sin -\varphi = \cos\varphi \mp \imath\sin\varphi$
$z^{81} = \cos 81\varphi \pm \imath\sin 81\varphi$
$\frac{1}{z^{81}} = \cos 81\varphi \mp \imath\sin 81\varphi$
$z^{81} + \frac{1}{z^{81}} = \cos 81\varphi \pm \cancel{\imath\sin 81\varphi} + \cos 81\varphi \mp \cancel{\imath\sin 81\varphi}$
$z^{81} + \frac{1}{z^{81}} = 2\cos(81\varphi)$

Задача 5. Да се реши системата в зависимост от стойностите на параметрите λ и μ:

$$\begin{array}{rcrcrcrcr}
4x_1 & + & 3x_2 & + & 3x_3 & - & 2x_4 & = & \lambda \\
-x_1 & - & x_2 & - & x_3 & + & x_4 & = & 2 \\
-19x_1 & - & 19x_2 & - & 20x_3 & + & (11+\mu)x_4 & = & 6-2\lambda \\
4x_1 & + & 7x_2 & + & 8x_3 & + & x_4 & = & -2
\end{array}$$

$$\begin{array}{c} 3 \\ \\ -20 \\ 8 \end{array}
\left(\begin{array}{cccc|c}
4 & 3 & 3 & -2 & \lambda \\
-1 & -1 & -1 & 1 & 2 \\
-19 & -19 & -20 & 11+\mu & 6-2\lambda \\
4 & 7 & 8 & 1 & -2
\end{array}\right)$$

$$\rightarrow \begin{array}{c} \\ -1 \\ 1 \\ \\ \end{array}
\left(\begin{array}{cccc|c}
1 & 0 & 0 & 1 & \lambda+6 \\
-1 & -1 & -1 & 1 & 2 \\
1 & 1 & 0 & \mu-9 & -2\lambda-34 \\
-4 & -1 & 0 & 9 & 14
\end{array}\right)$$

$$\rightarrow \begin{array}{c} \\ -3 \\ 3 \\ 4 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & \lambda+6 \\ 3 & 0 & -1 & -8 & -12 \\ -3 & 0 & 0 & \mu & -2\lambda-20 \\ -4 & -1 & 0 & 9 & 14 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & \lambda+6 \\ 0 & 0 & -1 & -11 & -3\lambda-30 \\ 0 & 0 & 0 & \mu+3 & \lambda-2 \\ 0 & -1 & 0 & 13 & 4\lambda+38 \end{array}\right) \begin{array}{c} \\ -1 \\ \\ \\ \end{array}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & \lambda+6 \\ 0 & 0 & 1 & 11 & 3\lambda+30 \\ 0 & 0 & 0 & \mu+3 & \lambda-2 \\ 0 & -1 & 0 & 13 & 4\lambda+38 \end{array}\right)$$

$$\left|\begin{array}{ccccc} x_1 & + & x_4 & = & \lambda+6 \\ x_3 & + & 11x_4 & = & 3\lambda+30 \\ & & (\mu+3)x_4 & = & \lambda-2 \\ -x_2 & + & 13x_4 & = & 4\lambda+38 \end{array}\right.$$

I $\mu = -3$

I.1 $\lambda = 2$

$$\left|\begin{array}{ccccc} x_1 & + & x_4 & = & \lambda+6 \\ x_3 & + & 11x_4 & = & 3\lambda+30 \\ & & 0x_4 & = & 0 \\ -x_2 & + & 13x_4 & = & 4\lambda+38 \end{array}\right.$$

$$\rightarrow \left|\begin{array}{ccccc} x_1 & + & x_4 & = & \lambda+6 \\ x_3 & + & 11x_4 & = & 3\lambda+30 \\ -x_2 & + & 13x_4 & = & 4\lambda+38 \end{array}\right.$$

$x_4 = p$

$$\left|\begin{array}{ccccc} x_1 & + & p & = & \lambda+6 \\ x_3 & + & 11p & = & 3\lambda+30 \\ -x_2 & + & 13p & = & 4\lambda+38 \end{array}\right.$$

$$\rightarrow \left|\begin{array}{ccc} x_1 & = & \lambda+6-p \\ x_3 & = & 3\lambda+30-11p \\ x_2 & = & 13p-4\lambda-38 \end{array}\right.$$

I.2 $\lambda \neq 2$

$$\left|\begin{array}{ccccc} x_1 & + & x_4 & = & \lambda+6 \\ x_3 & + & 11x_4 & = & 3\lambda+30 \\ & & 0x_4 & = & \lambda-2 \\ -x_2 & + & 13x_4 & = & 4\lambda+38 \end{array}\right.$$

Системата няма решение.

II $\mu \neq -3$

II.1 $\lambda = 2$

$$\left|\begin{array}{ccccc} x_1 & + & x_4 & = & \lambda+6 \\ x_3 & + & 11x_4 & = & 3\lambda+30 \\ & & (\mu+3)x_4 & = & 0 \\ -x_2 & + & 13x_4 & = & 4\lambda+38 \end{array}\right.$$

$$\rightarrow \left|\begin{array}{ccc} x_1 & = & \lambda+6 \\ x_3 & = & 3\lambda+30 \\ x_4 & = & 0 \\ x_2 & = & -4\lambda-38 \end{array}\right.$$

II.2 $\lambda \neq 2$

$$\left|\begin{array}{ccccc} x_1 & + & x_4 & = & \lambda+6 \\ x_3 & + & 11x_4 & = & 3\lambda+30 \\ & & (\mu+3)x_4 & = & \lambda-2 \\ -x_2 & + & 13x_4 & = & 4\lambda+38 \end{array}\right.$$

$$\rightarrow \left|\begin{array}{ccccc} x_1 & + & \frac{\lambda-2}{\mu+3} & = & \lambda+6 \\ x_3 & + & 11\frac{\lambda-2}{\mu+3} & = & 3\lambda+30 \\ & & x_4 & = & \frac{\lambda-2}{\mu+3} \\ -x_2 & + & 13\frac{\lambda-2}{\mu+3} & = & 4\lambda+38 \end{array}\right.$$

$$\rightarrow \left|\begin{array}{ccc} x_1 & = & \frac{(\lambda+6)(\mu+3)-(\lambda-2)}{\mu+3} \\ x_3 & = & \frac{(3\lambda+30)(11\mu+33)-(11\lambda-22)}{11\mu+33} \\ x_4 & = & \frac{\lambda-2}{\mu+3} \\ x_2 & = & -\frac{(4\lambda+38)(13\mu+39)-(13\lambda-26)}{13\mu+39} \end{array}\right.$$

$$\rightarrow \left| \begin{array}{lcl} x_1 & = & \frac{\lambda\mu+2\lambda+6\mu+20}{\mu+3} \\ x_3 & = & \frac{33\mu+88\lambda+330\mu+1012}{11\mu+33} \\ x_4 & = & \frac{\lambda-2}{\mu+3} \\ x_2 & = & -\frac{52\lambda\mu+143\lambda+494\mu+1498}{13\mu+39} \end{array} \right.$$

Задача 6

В четири мерното пространство са дадени векторите:

$$\begin{array}{rcl} v_1 & = & (-4, 3, 5, -3) \\ v_2 & = & (-4, -1, 8, -14) \\ v_3 & = & (-1, 1, 1, 0) \\ v_4 & = & (-1, -3, 2, \mu-8) \\ v & = & (1, 1, \lambda, 1) \end{array}$$

$$\begin{array}{r} \\ 1 \\ 1 \\ \\ \end{array} \left(\begin{array}{cccc|c} -4 & -4 & -1 & -1 & 1 \\ 3 & -1 & 1 & -3 & 1 \\ 5 & 8 & 1 & 2 & \lambda \\ -3 & -14 & 0 & \mu-8 & 1 \end{array}\right)$$

$$\rightarrow \begin{array}{r} -4 \\ \\ 1 \\ -3 \end{array} \left(\begin{array}{cccc|c} -4 & -4 & -1 & -1 & 1 \\ -1 & -5 & 0 & -4 & 2 \\ 1 & 4 & 0 & 1 & \lambda+1 \\ -3 & -14 & 0 & \mu-8 & 1 \end{array}\right)$$

$$\rightarrow \begin{array}{r} 16 \\ -5 \\ \\ 1 \end{array} \left(\begin{array}{cccc|c} 0 & 16 & -1 & 15 & -7 \\ -1 & -5 & 0 & -4 & 2 \\ 0 & -1 & 0 & -3 & \lambda+3 \\ 0 & 1 & 0 & \mu+4 & -5 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 0 & 0 & -1 & -33 & 16\lambda+41 \\ -1 & 0 & 0 & 11 & -5\lambda-13 \\ 0 & -1 & 0 & -3 & \lambda+3 \\ 0 & 0 & 0 & \mu+1 & \lambda-2 \end{array}\right) \begin{array}{r} -1 \\ -1 \\ -1 \\ \\ \end{array}$$

$$\left| \begin{array}{lcccc} x_3 & + & 33x_4 & = & -16\lambda-41 \\ x_1 & + & -11x_4 & = & 5\lambda+13 \\ x_2 & + & 3x_4 & = & -\lambda-3 \\ & & (\mu+1)x_4 & = & \lambda-2 \end{array} \right.$$

I $\mu = -1$

I.1 $\lambda = 2$

$$\left|\begin{array}{cccc} x_3 & + & 33x_4 & = & -16\lambda-41 \\ x_1 & + & -11x_4 & = & 5\lambda+13 \\ x_2 & + & 3x_4 & = & -\lambda-3 \\ & & 0x_4 & = & 0 \end{array}\right.$$

$$\rightarrow \left|\begin{array}{ccccc} x_3 & + & 33x_4 & = & -16\lambda-41 \\ x_1 & + & -11x_4 & = & 5\lambda+13 \\ x_2 & + & 3x_4 & = & -\lambda-3 \end{array}\right.$$

$x_4 = p$

$$\left|\begin{array}{ccccc} x_3 & + & 33p & = & -16\lambda-41 \\ x_1 & + & -11p & = & 5\lambda+13 \\ x_2 & + & 3p & = & -\lambda-3 \end{array}\right.$$

$$\rightarrow \left|\begin{array}{ccc} x_3 & = & -16\lambda-41-33p \\ x_1 & = & 5\lambda+13+11p \\ x_2 & = & -\lambda-3-3p \end{array}\right.$$

I.2 $\lambda \neq 2$

$$\left|\begin{array}{ccccc} x_3 & + & 33x_4 & = & -16\lambda-41 \\ x_1 & + & -11x_4 & = & 5\lambda+13 \\ x_2 & + & 3x_4 & = & -\lambda-3 \\ & & 0x_4 & = & \lambda-2 \end{array}\right.$$

Системата няма решение.

II $\mu \neq -1$

II.1 $\lambda = 2$

$$\left|\begin{array}{ccccc} x_3 & + & 33x_4 & = & -16\lambda-41 \\ x_1 & + & -11x_4 & = & 5\lambda+13 \\ x_2 & + & 3x_4 & = & -\lambda-3 \\ & & (\mu+1)x_4 & = & 0 \end{array}\right.$$

$$\rightarrow \left|\begin{array}{ccc} x_3 & = & -16\lambda-41 \\ x_1 & = & 5\lambda+13 \\ x_2 & = & -\lambda-3 \\ x_4 & = & 0 \end{array}\right.$$

II.2 $\lambda \neq 2$

$$\left| \begin{array}{ccccc} x_3 & + & 33x_4 & = & -16\lambda - 41 \\ x_1 & + & -11x_4 & = & 5\lambda + 13 \\ x_2 & + & 3x_4 & = & -\lambda - 3 \\ & & (\mu+1)x_4 & = & \lambda - 2 \end{array} \right.$$

$$\rightarrow \left| \begin{array}{ccc} x_3 & = & \frac{-(16\lambda+41)(\mu+1)-33(\lambda-2)}{\mu+1} \\ x_1 & = & \frac{(5\lambda+13)(\mu+1)+11(\lambda-2)}{\mu+1} \\ x_2 & = & \frac{-(\lambda+3)(\mu+1)-3(\lambda-2)}{\mu+1} \\ x_4 & = & \frac{\lambda-2}{\mu+1} \end{array} \right.$$

$$\rightarrow \left| \begin{array}{ccc} x_3 & = & \frac{-(16\lambda\mu+40\lambda+41\mu-25)}{\mu+1} \\ x_1 & = & \frac{5\lambda\mu+16\lambda+13\mu-9}{\mu+1} \\ x_2 & = & \frac{-(\lambda\mu+4\lambda+3\mu-3)}{\mu+1} \\ x_4 & = & \frac{\lambda-2}{\mu+1} \end{array} \right.$$

а) За стойностите на параметрите $\mu \neq -1$, $\forall \lambda$ векторът v се представя като линейна комбинация на векторите v_1, v_2, v_3, v_4 по точно един начин.

б) За стойностите на параметрите $\mu = -1$, $\lambda = 2$ векторът v се представя като линейна комбинация на векторите v_1, v_2, v_3, v_4 по повече от един начин.

Задача 7.
Нека V е множеството от всички полиноми с реални коефициенти и от степен не по-голяма от 3.
а) Да се докаже, че v е линейно пространство над полето на рялните числа относно обичайните операци на полиноми и умножение на полиноми с число.

$a = \alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x^1 + \alpha_4$
$b = \beta_1 x^3 + \beta_2 x^2 + \beta_3 x^1 + \beta_4$
$c = \gamma_1 x^3 + \gamma_2 x^2 + \gamma_3 x^1 + \gamma_4$
$a' = \alpha'_1 x^3 + \alpha'_2 x^2 + \alpha'_3 x^1 + \alpha'_4$
$a, b, c, a' \in \mathbb{V}$
$\lambda, \mu, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4 \in \mathbb{R}$

1. $(a + b) + c = a + (b + c)$

$(a+b) = (\alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x^1 + \alpha_4) + (\beta_1 x^3 + \beta_2 x^2 + \beta_3 x^1 + \beta_4) = (\alpha_1 + \beta_1)x^3 + (\alpha_2 + \beta_2)x^2 + (\alpha_3 + \beta_3)x^1 + (\alpha_4 + \beta_4)$

$(a+b)+c = (\alpha_1+\beta_1)x^3+(\alpha_2+\beta_2)x^2+(\alpha_3+\beta_3)x^1+(\alpha_4+\beta_4)+(\gamma_1 x^3+\gamma_2 x^2+\gamma_3 x^1+\gamma_4) = (\alpha_1+\beta_1+\gamma)x^3+(\alpha_2+\beta_2+\gamma)x^2+(\alpha_3+\beta_3+\gamma)x^1+(\alpha_4+\beta_4+\gamma)$

$(b+c) = (\beta_1 x^3+\beta_2 x^2+\beta_3 x^1+\beta_4)+(\gamma_1 x^3+\gamma_2 x^2+\gamma_3 x^1+\gamma_4) = (\beta_1+\gamma_1)x^3+(\beta_2+\gamma_2)x^2+(\beta_3+\gamma_3)x^1+(\beta_4+\gamma_4)$

$a+(b+c) = (\alpha_1 x^3+\alpha_2 x^2+\alpha_3 x^1+\alpha_4)+(\beta_1+\gamma_1)x^3+(\beta_2+\gamma_2)x^2+(\beta_3+\gamma_3)x^1+(\beta_4+\gamma_4) = (\alpha_1+\beta_1+\gamma)x^3+(\alpha_2+\beta_2+\gamma)x^2+(\alpha_3+\beta_3+\gamma)x^1+(\alpha_4+\beta_4+\gamma)$

$\implies (a+b)+c = a+(b+c) = a+b+c$

2. $a+b=b+a$

$(a+b) = (\alpha_1 x^3+\alpha_2 x^2+\alpha_3 x^1+\alpha_4)+(\beta_1 x^3+\beta_2 x^2+\beta_3 x^1+\beta_4) = (\alpha_1+\beta_1)x^3+(\alpha_2+\beta_2)x^2+(\alpha_3+\beta_3)x^1+(\alpha_4+\beta_4)$

$(b+a) = (\beta_1 x^3+\beta_2 x^2+\beta_3 x^1+\beta_4)+(\alpha_1 x^3+\alpha_2 x^2+\alpha_3 x^1+\alpha_4) = (\alpha_1+\beta_1)x^3+(\alpha_2+\beta_2)x^2+(\alpha_3+\beta_3)x^1+(\alpha_4+\beta_4)$

$\implies a+b=b+a$

3. $a+0=a$

$a+0 = (\alpha_1 x^3+\alpha_2 x^2+\alpha_3 x^1+\alpha_4)+(0x^3+0x^2+0x^1+0) = (\alpha_1+0)x^3+(\alpha_2+0)x^2+(\alpha_3+0)x^1+(\alpha_4+0) = \alpha_1 x^3+\alpha_2 x^2+\alpha_3 x^1+\alpha_4$

$\implies a+0=a$

4. $\exists\, a' : a+a'=0$

$a+a'=0$
$(\alpha_1 x^3+\alpha_2 x^2+\alpha_3 x^1+\alpha_4)+(\alpha_1' x^3+\alpha_2' x^2+\alpha_3' x^1+\alpha_4') = (0x^3+0x^2+0x^1=0)$
$(\alpha_1+\alpha_1')x^3+(\alpha_2+\alpha_2')x^2+(\alpha_3+\alpha_3')x^1+(\alpha_4+\alpha_4') = (0x^3+0x^2+0x^1=0)$

$$\equiv \left|\begin{array}{lcccc} \alpha_1' & + & \alpha_1' & = & 0 \\ \alpha_2 & + & \alpha_2' & = & 0 \\ \alpha_3 & + & \alpha_3' & = & 0 \\ \alpha_4 & + & \alpha_4' & = & 0 \end{array}\right. \rightarrow \left|\begin{array}{lcc} \alpha_1' & = & -\alpha_1 \\ \alpha_2' & = & -\alpha_2 \\ \alpha_3' & = & -\alpha_3 \\ \alpha_4' & = & -\alpha_4 \end{array}\right.$$

$\implies a' = -\alpha_1 x^3+-\alpha_2 x^2+-\alpha_3 x^1+-\alpha_4 = -(\alpha_1 x^3+\alpha_2 x^2+\alpha_3 x^1+\alpha_4) = -a$
$\implies \exists\, a' : a+a'=0$

5. $1a=a$

$1a = 1(\alpha_1 x^3+\alpha_2 x^2+\alpha_3 x^1+\alpha_4) = (1\alpha_1)x^3+(1\alpha_2)x^2+(1\alpha_3)x^1+(1\alpha_4) =$

$\alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x^1 + \alpha_4$

$\implies 1a = a$

6. $\lambda(a+b) = \lambda a + \lambda b$

$\lambda(a+b) = \lambda[(\alpha_1+\beta_1)x^3 + (\alpha_2+\beta_2)x^2 + (\alpha_3+\beta_3)x^1 + (\alpha_4+\beta_4)] = \lambda(\alpha_1 + \beta_1)x^3 + \lambda(\alpha_2+\beta_2)x^2 + \lambda(\alpha_3+\beta_3)x^1 + \lambda(\alpha_4+\beta_4) = (\lambda\alpha_1 + \lambda\beta_1)x^3 + (\lambda\alpha_2 + \lambda\beta_2)x^2 + (\lambda\alpha_3+\lambda\beta_3)x^1 + (\lambda\alpha_4+\lambda\beta_4)$

$\lambda a = \lambda(\alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x^1 + \alpha_4) = \lambda\alpha_1 x^3 + \lambda\alpha_2 x^2 + \lambda\alpha_3 x^1 + \lambda\alpha_4$

$\lambda b = \lambda(\beta_1 x^3 + \beta_2 x^2 + \beta_3 x^1 + \beta_4) = \lambda\beta_1 x^3 + \lambda\beta_2 x^2 + \lambda\beta_3 x^1 + \lambda\beta_4$

$\lambda a + \lambda b = (\lambda\alpha_1 x^3 + \lambda\alpha_2 x^2 + \lambda\alpha_3 x^1 + \lambda\alpha_4) + (\lambda\beta_1 x^3 + \lambda\beta_2 x^2 + \lambda\beta_3 x^1 + \lambda\beta_4) = (\lambda\alpha_1 + \lambda\beta_1)x^3 + (\lambda\alpha_2 + \lambda\beta_2)x^2 + (\lambda\alpha_3+\lambda\beta_3)x^1 + (\lambda\alpha_4+\lambda\beta_4)$

$\implies \lambda(a+b) = \lambda a + \lambda b$

7. $(\lambda+\mu)a = \lambda a + \mu a$

$(\lambda+\mu)a = (\lambda+\mu)(\alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x^1 + \alpha_4) = [(\lambda+\mu)\alpha_1]x^3 + [(\lambda+\mu)\alpha_2]x^2 + [(\lambda+\mu)\alpha_3]x^1 + [(\lambda+\mu)\alpha_4] = (\lambda\alpha_1 + \mu\alpha_1)x^3 + (\lambda\alpha_2+\mu\alpha_2)x^2 + (\lambda\alpha_3+\mu\alpha_3)x^1 + (\lambda\alpha_4 + \mu\alpha_4)$

$\lambda a = \lambda(\alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x^1 + \alpha_4) = \lambda\alpha_1 x^3 + \lambda\alpha_2 x^2 + \lambda\alpha_3 x^1 + \lambda\alpha_4$

$\mu a = \mu(\alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x^1 + \alpha_4) = \mu\alpha_1 x^3 + \mu\alpha_2 x^2 + \mu\alpha_3 x^1 + \mu\alpha_4$

$\lambda a + \mu a = (\lambda\alpha_1 x^3 + \lambda\alpha_2 x^2 + \lambda\alpha_3 x^1 + \lambda\alpha_4) + (\mu\alpha_1 x^3 + \mu\alpha_2 x^2 + \mu\alpha_3 x^1 + \mu\alpha_4) = (\lambda\alpha_1 + \mu\alpha_1)x^3 + (\lambda\alpha_2 + \mu\alpha_2)x^2 + (\lambda\alpha_3 + \mu\alpha_3)x^1 + (\lambda\alpha_4 + \mu\alpha_4)$

$\implies (\lambda+\mu)a = \lambda a + \mu a$

8. $\lambda(\mu a) = \lambda\mu a$

$\mu a = \mu(\alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x^1 + \alpha_4) = \mu\alpha_1 x^3 + \mu\alpha_2 x^2 + \mu\alpha_3 x^1 + \mu\alpha_4$

$\lambda(\mu a) = \lambda(\mu\alpha_1 x^3 + \mu\alpha_2 x^2 + \mu\alpha_3 x^1 + \mu\alpha_4) = \lambda(\mu\alpha_1 x^3 + \mu\alpha_2 x^2 + \mu\alpha_3 x^1 + \mu\alpha_4) = \lambda\mu\alpha_1 x^3 + \lambda\mu\alpha_2 x^2 + \lambda\mu\alpha_3 x^1 + \lambda\mu\alpha_4) = \lambda\mu(\alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x^1 + \alpha_4) = \lambda\mu a$

$\implies \lambda(\mu a) = \lambda\mu a$

в) Да се докаже, че полиномите $1, x-49, \frac{(x-49)^2}{2!}, \frac{(x-49)^3}{3!}$ образуват базис на v.

$v_1 = 1$
$v_2 = x - 49$
$v_3 = \frac{(x-49)^2}{2!}$
$v_4 = \frac{(x-49)^3}{3!}$

$\frac{v_2}{v_1} \notin \mathbb{R} \quad \in \mathbb{V}$
$\frac{v_3}{v_2} \notin \mathbb{R} \quad \in \mathbb{V}$
$\frac{v_4}{v_3} \notin \mathbb{R} \quad \in \mathbb{V}$

$\implies$ векторите са линейно не зависими