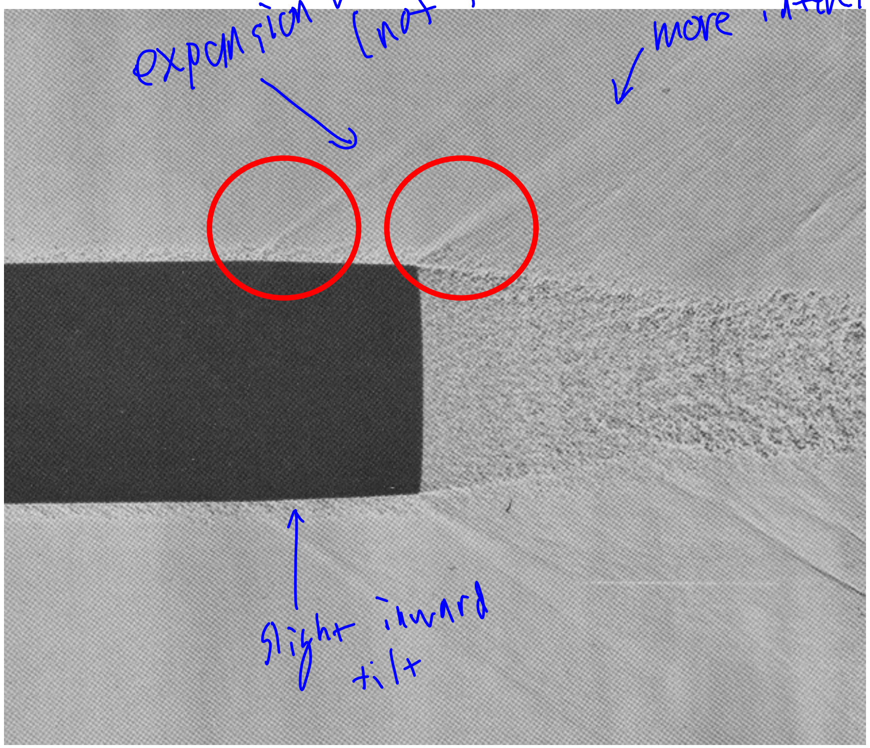
SESA3029 Aerothermodynamics



Lecture 2.4

Expansion waves

Expansion waves

- Must be isentropic (i.e. Mach waves)
- Finite pressure changes are produced by a series (a ‘fan’) of small changes
- Convex corners in supersonic flow give expansion fans

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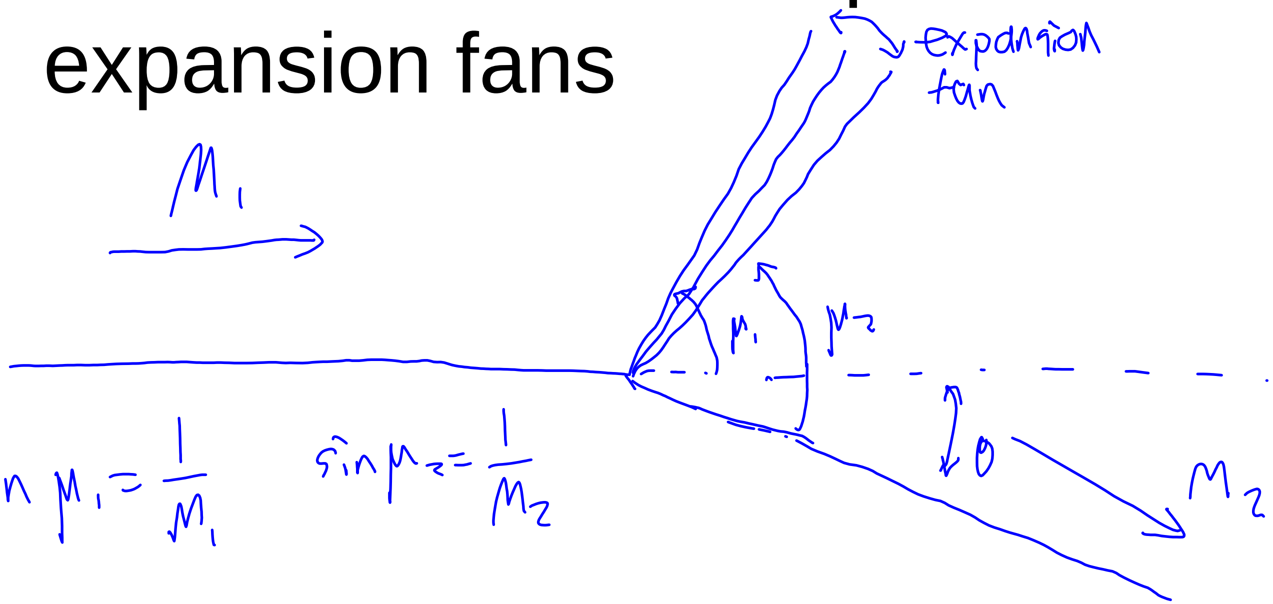
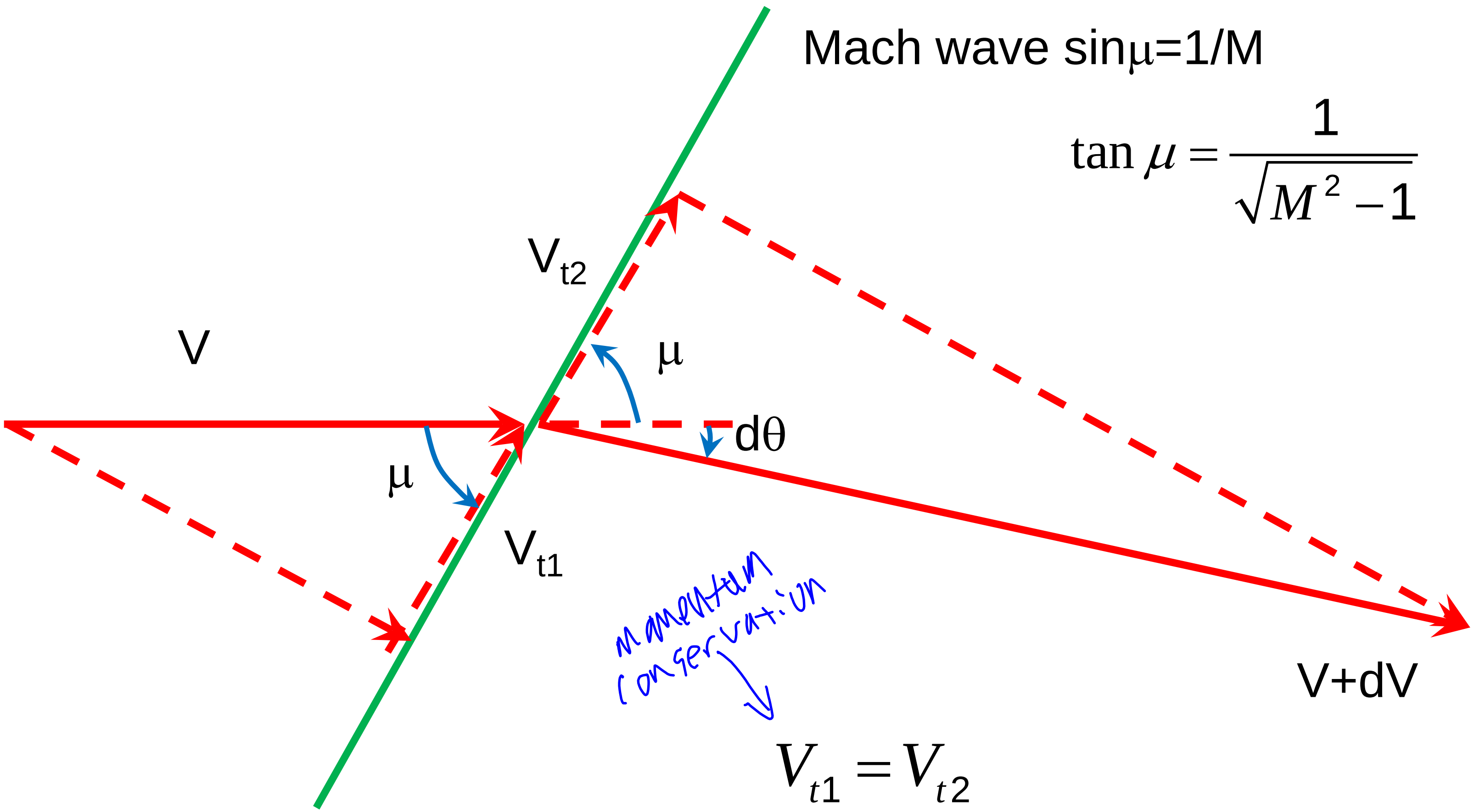


$\sin \mu_1 = \frac{1}{M_1}$ $\quad$ $\sin \mu_2 = \frac{1}{M_2}$

The expansion fan turns the flow gradually

Turning a flow by a small angle dθ



$$\tan\mu = \frac{1}{\sqrt{M^2 - 1}}$$

momentum conservation

$$V_{t1} = V_{t2}$$

$$V\cos\mu = \left(V + \mathrm{d}V\right)\cos\left(\mu + \mathrm{d}\theta\right)$$

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Expand out

$$1 \xleftrightarrow{\text{small angle approx}} \mathrm{d}\theta$$

$$V\cos\mu = (V + \mathrm{d}V)\left[\cos\mu\cos\mathrm{d}\theta - \sin\mu\sin\mathrm{d}\theta\right]$$

Let dθ→0 and remove products of small quantities

$$0 = \mathrm{d}V\cos\mu - V\mathrm{d}\theta\sin\mu$$

$$\frac{\mathrm{d}V}{V} = \mathrm{d}\theta\tan\mu = \frac{\mathrm{d}\theta}{\sqrt{M^2 - 1}}$$

V=Ma, so $\mathrm{d}V = M\mathrm{d}a + a\mathrm{d}M$

$$\frac{\mathrm{d}V}{V} = \frac{\mathrm{d}a}{a} + \frac{\mathrm{d}M}{M}$$ and hence $$\frac{\mathrm{d}\theta}{\sqrt{M^2 - 1}} = \frac{\mathrm{d}a}{a} + \frac{\mathrm{d}M}{M} \quad (1)$$

$$\frac{\mathrm{d}\theta}{\sqrt{M^2-1}} = \frac{\mathrm{d}a}{a} + \frac{\mathrm{d}M}{M} \qquad (1)$$

We know
(adiabatic flow)

$$\frac{a_0^2}{a^2} = 1 + \frac{\gamma-1}{2}M^2$$

or

$$a = a_0\left[1+\frac{\gamma-1}{2}M^2\right]^{-\frac{1}{2}}$$

$\frac{d}{dM}$

$$\frac{da}{dM} = -\frac{1}{2}(\gamma-1)Ma_0\left[1+\frac{\gamma-1}{2}M^2\right]^{-\frac{3}{2}}$$

$$= -\frac{1}{2}(\gamma-1)Ma\left[1+\frac{\gamma-1}{2}M^2\right]^{-1}$$

Rewrite as

$$\frac{\mathrm{d}a}{a} = -\frac{\gamma-1}{2}M\left(1+\tfrac{\gamma-1}{2}M^2\right)^{-1}\mathrm{d}M \qquad (2)$$

$$\frac{\mathrm{d}\theta}{\sqrt{M^2-1}} = \frac{\mathrm{d}a}{a} + \frac{\mathrm{d}M}{M} \quad (1)$$

$$\frac{\mathrm{d}a}{a} = -\frac{\gamma-1}{2} M \left(1+\tfrac{\gamma-1}{2}M^2\right)^{-1} \mathrm{d}M \quad (2)$$

Substitute (2) into (1) and rearrange

$$\frac{\mathrm{d}\theta}{\sqrt{M^2-1}} = \left[1 - \frac{\gamma-1}{2} M^2 \left(1+\tfrac{\gamma-1}{2}M^2\right)^{-1}\right] \frac{\mathrm{d}M}{M}$$

$$= \left[1+\tfrac{\gamma-1}{2}M^2\right]^{-1} \frac{\mathrm{d}M}{M}$$

(Take over common denominator and cancel terms)

$$\boxed{\mathrm{d}\theta = \frac{\sqrt{M^2-1}}{1+\tfrac{\gamma-1}{2}M^2} \frac{\mathrm{d}M}{M}} \quad (3)$$

Define a standard integral

nu symbol

$$\nu(M') = \int_1^{M'} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

is angle ish

$\nu \neq \theta$

Exact solution (tabulated on IFT) – the $\boxed{\text{Prandtl-Meyer function}}$

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1}$$

Isentropic-flow table (γ = 1.4):

M	p/p_0	ρ/ρ_0	T/T_0	ν (deg.)	A/A^*
1.0000	0.5283	0.6339	0.8333	0.0000	1.0000
1.0200	0.5160	0.6234	0.8278	0.1257	1.0003
1.0400	0.5039	0.6129	0.8222	0.3510	1.0013
1.0600	0.4919	0.6024	0.8165	0.6367	1.0029
1.0800	0.4800	0.5920	0.8108	0.9680	1.0051

Return to $$\mathrm{d}\theta = \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{\mathrm{d}M}{M} \qquad (3)$$

Integrate from $\theta=0$ $M=M_1'$ to $\theta=\theta'$ $M=M_2'$

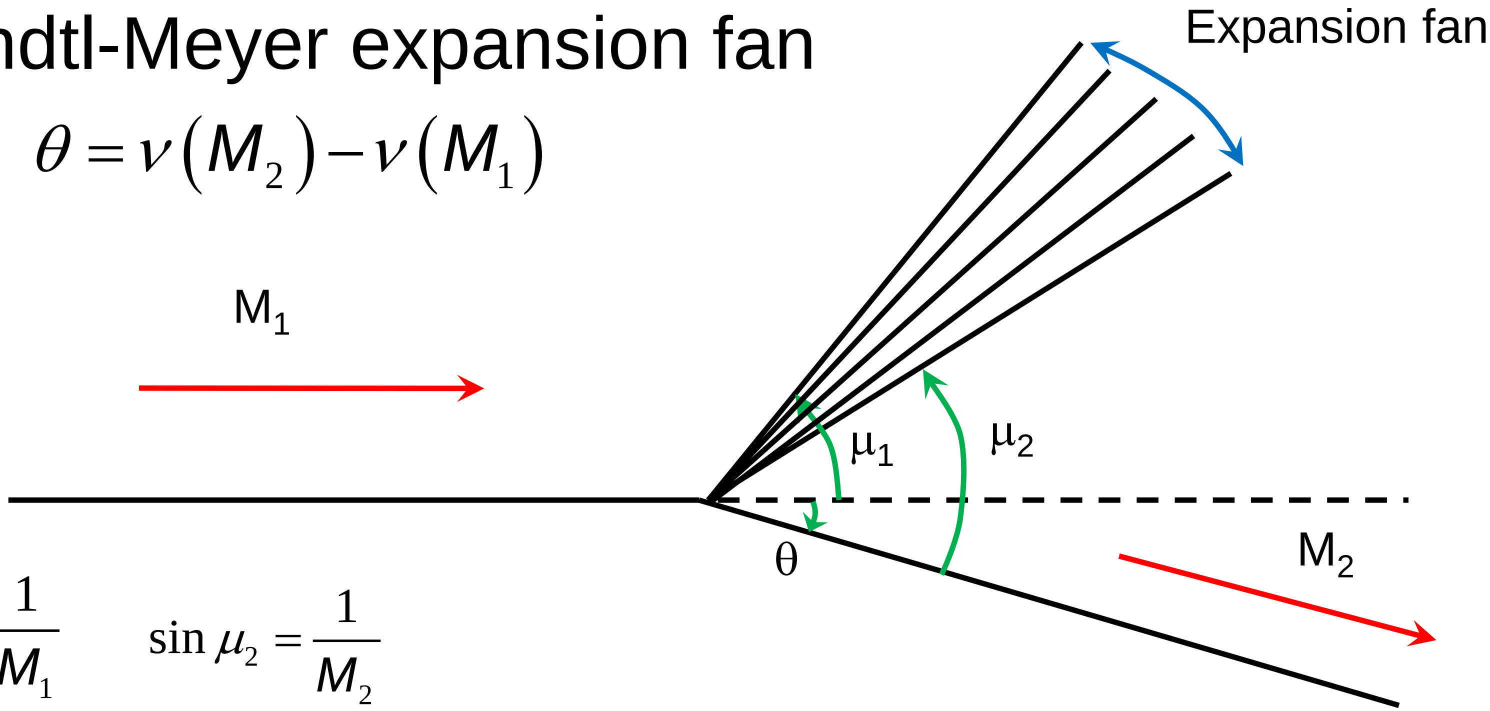
Giving $$\theta' = \nu\left(M_2'\right)-\nu\left(M_1'\right)$$

Drop primes for convenience $$\boxed{\theta = \nu\left(M_2\right)-\nu\left(M_1\right)}$$

Fundamental equation for expansion fans
(θ=turning angle and ν=Prandtl-Meyer function)

Prandtl-Meyer expansion fan

$$\theta = \nu(M_2) - \nu(M_1)$$



$$\sin \mu_1 = \frac{1}{M_1} \qquad \sin \mu_2 = \frac{1}{M_2}$$

Example: A Mach 3 flow is turned by 13° over a convex corner. Find:

(a) the Mach number after the corner,

(b) the percentage pressure drop, and

(c) the included angle of the P-M expansion fan.

$$\theta = \nu(M_2) - \nu(M_1)$$

Isentropic-flow table (γ = 1.4):

M	p/p_0	ρ/ρ_0	T/T_0	ν (deg.)	A/A^*
3.0000	0.0272	0.0762	0.3571	49.7573	4.2346
3.0200	0.0264	0.0746	0.3541	50.1417	4.3160
3.0400	0.0256	0.0730	0.3511	50.5231	4.3989
3.0600	0.0249	0.0715	0.3481	50.9016	4.4835
3.0800	0.0242	0.0700	0.3452	51.2771	4.5696
3.7000	0.0099	0.0370	0.2675	61.5953	8.1691
3.7200	0.0096	0.0363	0.2654	61.8893	8.3202
3.7400	0.0094	0.0356	0.2633	62.1812	8.4739
3.7600	0.0091	0.0349	0.2613	62.4709	8.6302
3.7800	0.0089	0.0342	0.2592	62.7584	8.7891
3.8000	0.0086	0.0335	0.2572	63.0438	8.9506
3.8200	0.0084	0.0329	0.2552	63.3271	9.1148
3.8400	0.0082	0.0323	0.2532	63.6083	9.2817
3.8600	0.0080	0.0316	0.2513	63.8874	9.4513
3.8800	0.0077	0.0310	0.2493	64.1645	9.6237

+13 deg

Example: A Mach 3 flow is turned by 13° over a convex corner. Find:

(a) the Mach number after the corner,

(b) the percentage pressure drop, and

(c) the included angle of the P-M expansion fan.

(a) From IFT $\nu(M_1)=49.76°$ hence $\nu(M_2)= \nu(M_1)+\theta=49.76°+13° =62.76°$

Hence from IFT $M_2=3.78$

(b) Also from IFT $p_1/p_0=0.0272$ and $p_2/p_0=0.0089$

Percentage pressure drop $$\frac{p_1 - p_2}{p_1} \cdot 100 = \left(1 - \frac{p_2/p_0}{p_1/p_0}\right).100 = 67.3\%$$

(c) $\mu_1=\sin^{-1}(1/M_1)=19.5°$

$\mu_2=\sin^{-1}(1/M_2)=15.3°$

Included angle of PM fan = $\mu_1+\theta-\mu_2=17.2°$