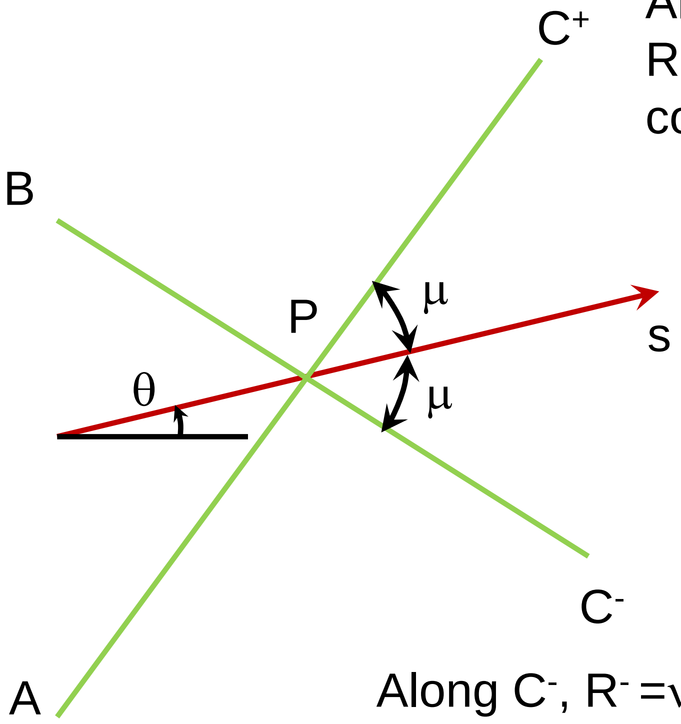
SESA3029 Aerothermodynamics
Lecture 3.3

Minimum length nozzle worked example

Method of characteristics



Along C^+, $R^+=\nu-\theta$ is constant

Along C^-, $R^-=\nu+\theta$ is constant

Suppose we know the flow state at points A and B

$$R_A^+ = \nu_A\left(M_A\right) - \theta_A$$

$$R_B^- = \nu_B\left(M_B\right) + \theta_B$$

At point P we must have

$$\nu_P - \theta_P = R_A^+$$

$$\nu_P + \theta_P = R_B^-$$

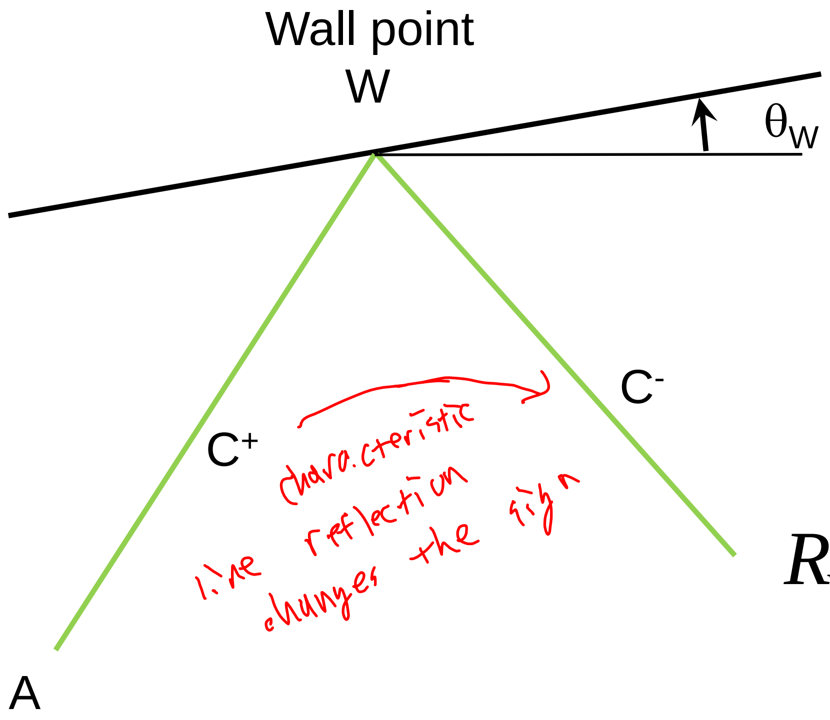
sim equations

Hence, at point P

$$\nu_P = \frac{R_A^+ + R_B^-}{2} \qquad \theta_P = \frac{R_B^- - R_A^+}{2}$$

From IFT we can find M_P, hence we have marched the solution downstream

Reflection from a wall



Boundary condition:
tangential flow at the wall

In general we get a reflected characteristic wave

$$R_W^- = \nu_W + \theta_W$$

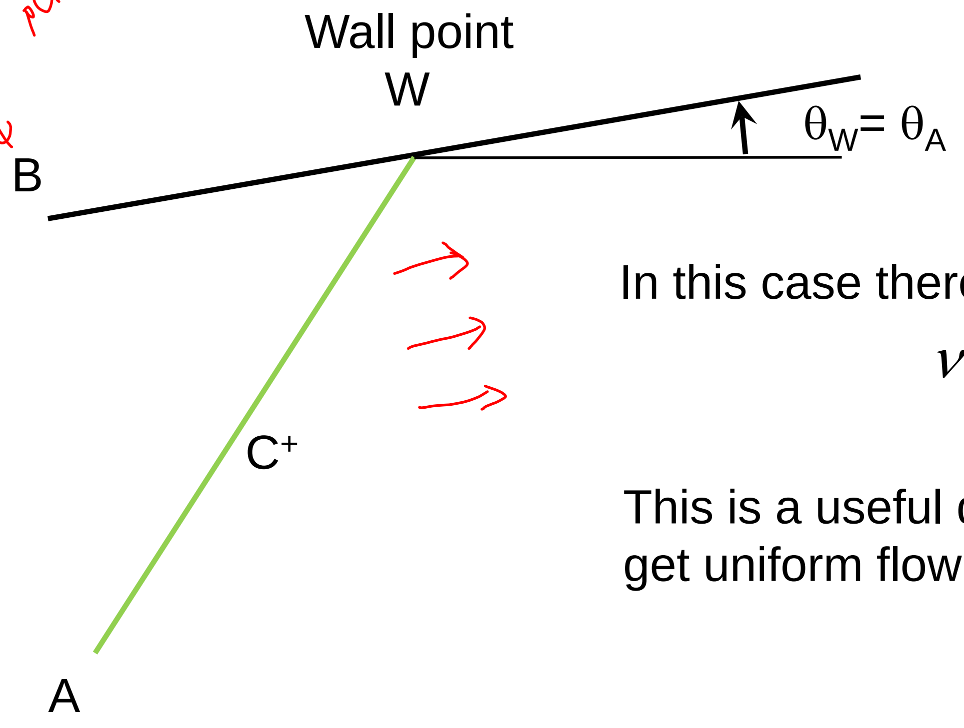
$$R_A^+ = \nu_A - \theta_A = \nu_W - \theta_W$$

can solve for ν_W

(So we can find ν_W, hence M etc)

Wave cancellation

previous wall point



$$R_A^+ = \nu_A - \theta_A$$

Suppose we choose $\theta_W = \theta_A$

In this case there is no reflected wave and

$$\nu_W = \nu_A$$

This is a useful design method for nozzles to get uniform flow at the exit

To find the co-ordinates of W we can use the same formulae as before, but based on the triangle ABW (B=previous wall point)

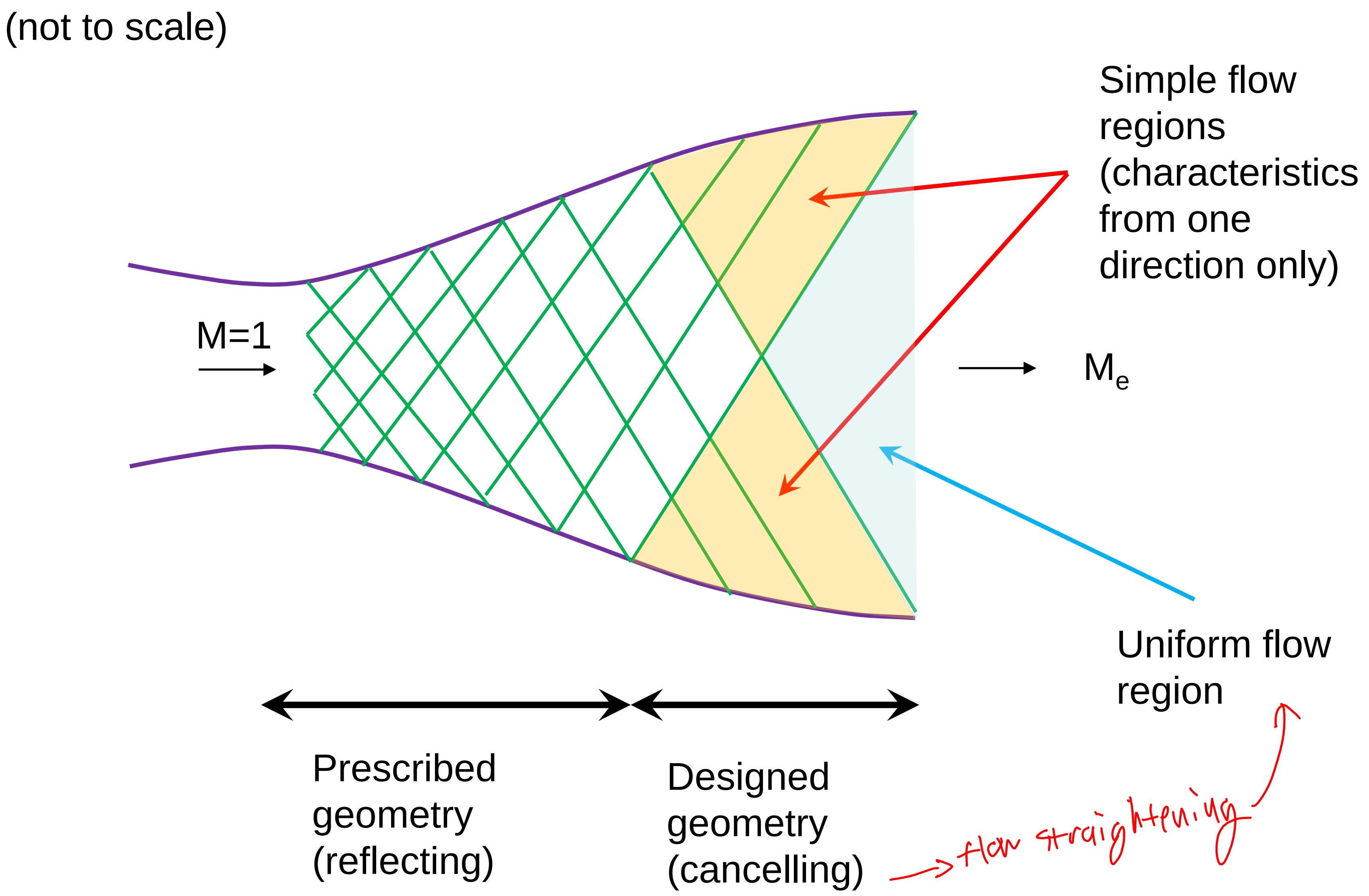
$$\boxed{\begin{aligned} \alpha_{AW} &= (\theta + \mu)_A \\ \alpha_{BW} &= \frac{1}{2}[\theta_B + \theta_W] \end{aligned}}$$

flow is already in correct flow direction relative to wall.

∴ flow downstream is uniform until next disturbance

so we can make use of this

MoC example: nozzle with flow straightening



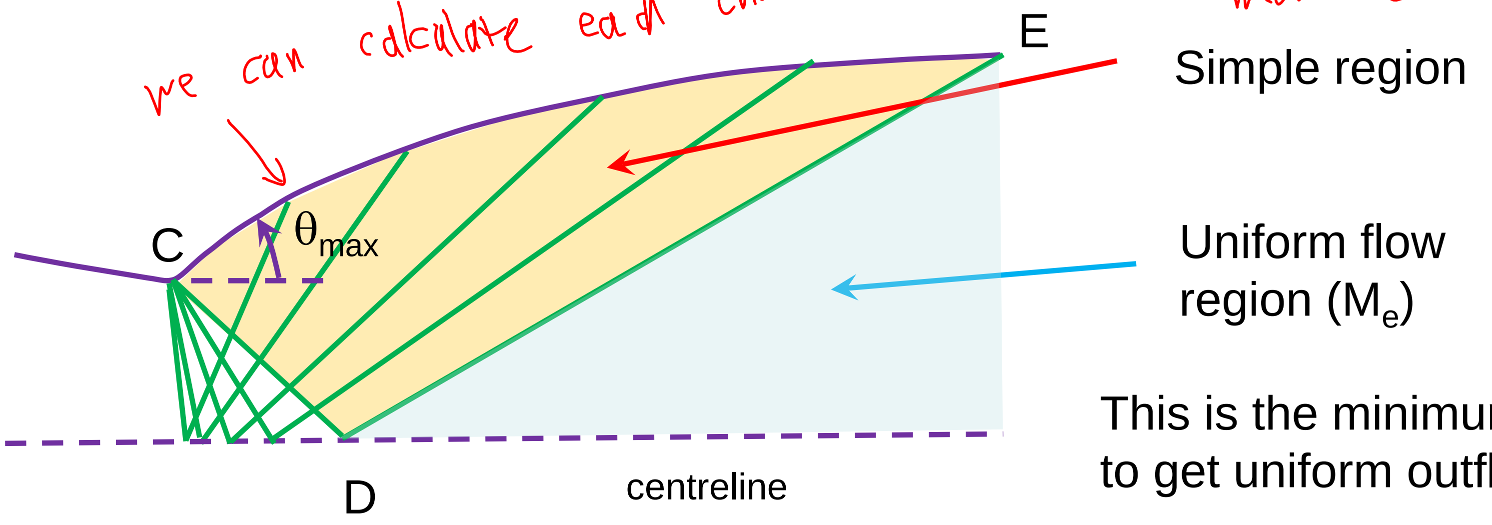
Plainer method of characteristics

Minimum length nozzle

Suppose we start the cancellation at the throat
(and take a symmetric case)

we can calculate each characteristic line point to get nozzle shape

more characteristic lines give better nozzle



This is the minimum length to get uniform outflow

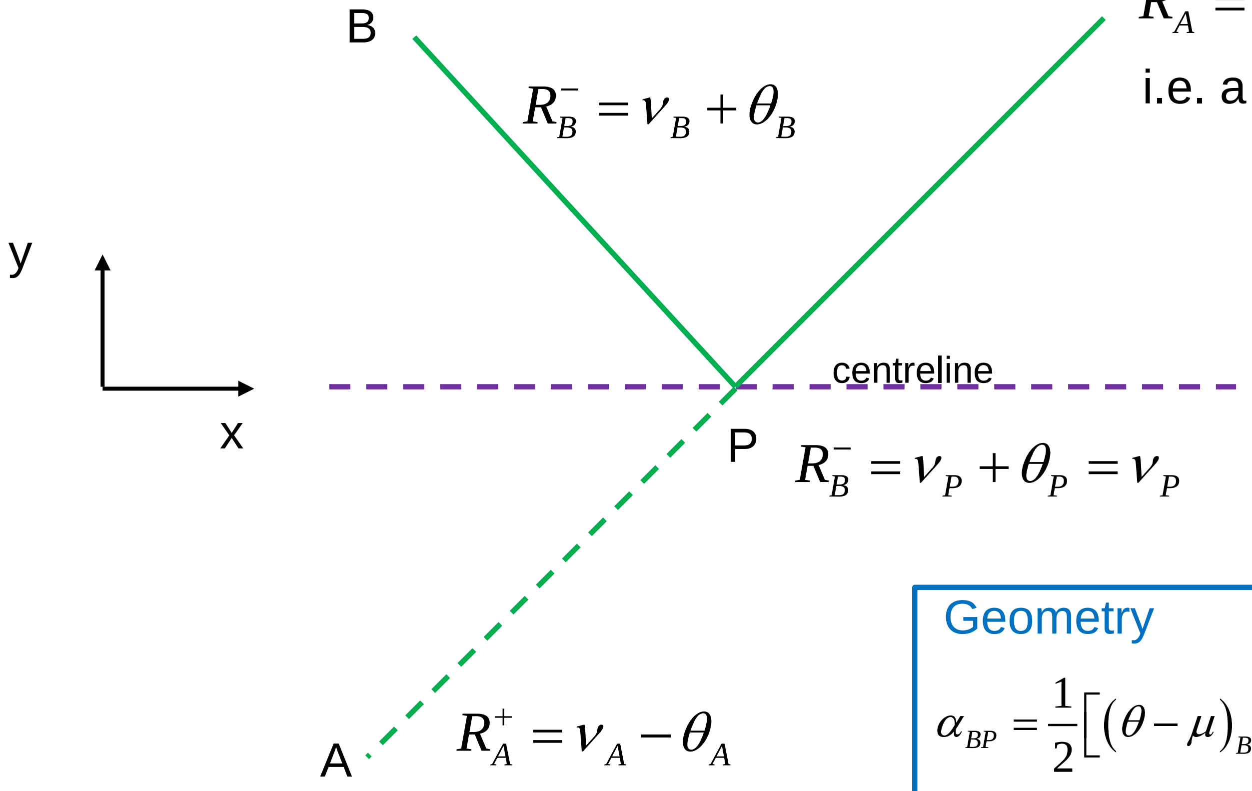
Point D must have θ=0 and $\nu(M_e)$ $$R_C^- = \theta_D + \nu_D = \nu(M_e)$$

Hence at C $$R_C^- = \nu(M_e) = \theta_{\max} + \nu_C$$

P-M expansion at C $$\theta_{\max} = \nu_C - \underbrace{\nu(M=1)}_{=0} = \nu_C$$

and finally $$\boxed{\theta_{\max} = \frac{\nu(M_e)}{2}}$$

Symmetry plane



$$R_B^- = \nu_B + \theta_B$$

$$R_A^+ = R_P^+ = \nu_P = R_B^-$$

i.e. a reflected wave

$$R_B^- = \nu_P + \theta_P = \nu_P$$

(since θ_P=0 to satisfy symmetry)

$$R_A^+ = \nu_A - \theta_A$$

By symmetry $\nu_A = \nu_B$

$$\theta_A = -\theta_B$$

so $R_A^+ = R_B^-$

Geometry

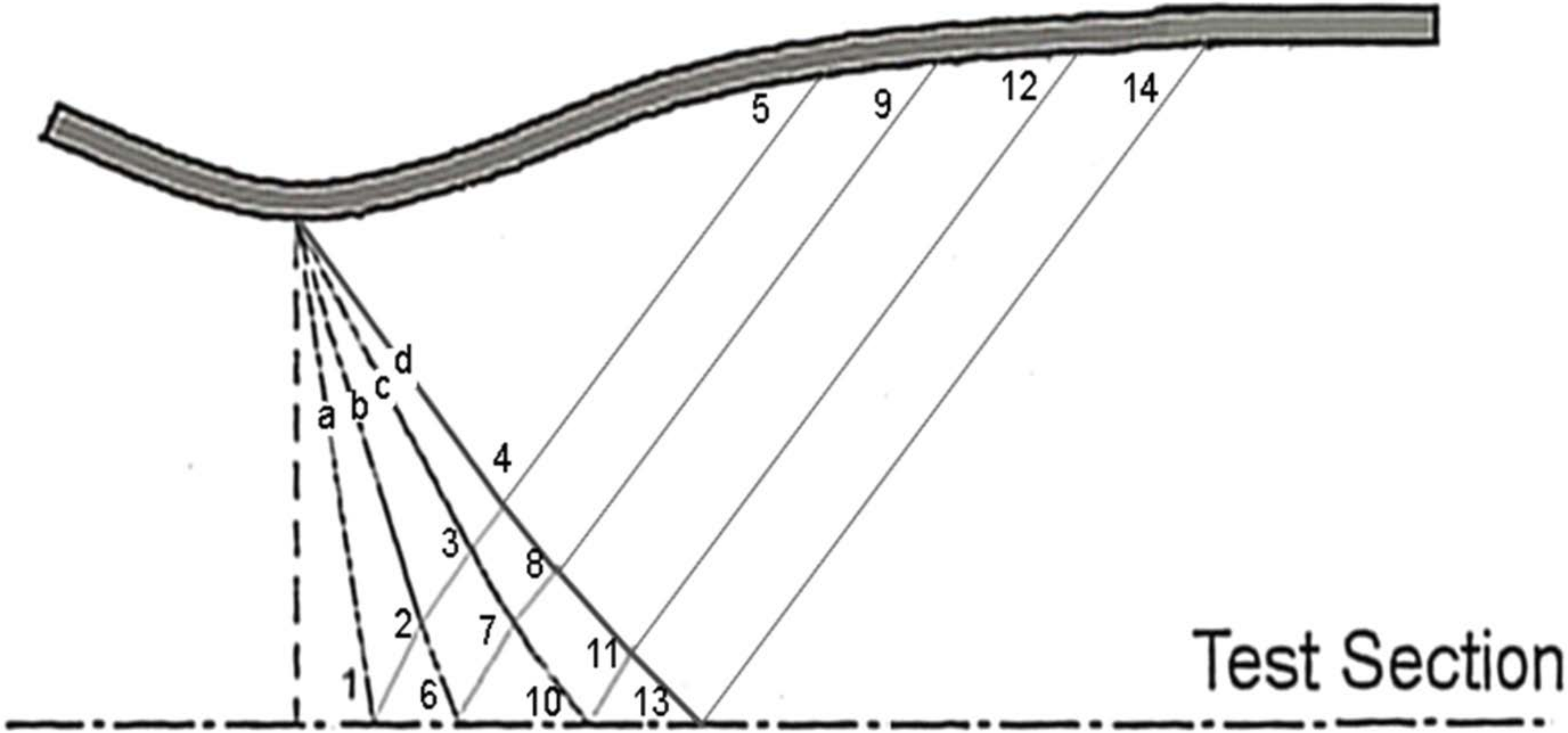
$$\alpha_{BP} = \frac{1}{2}\left[(\theta-\mu)_B + (\theta-\mu)_P\right]$$

with θ_P=0 and μ_P=sin^{-1}(1/M_P)

$$\tan\alpha_{BP} = \frac{y_P - y_B}{x_P - x_B}$$

$$y_P = 0$$

$$x_P = x_B - \frac{y_B}{\tan\alpha_{BP}}$$



5
9
12
14
a
b
c
d
4
3
8
2
7
11
1
6
10
13
Test Section

Template for MoC calculation

Point	R^+	R^-	θ	ν	M	μ	$\theta+\mu$	$\theta-\mu$	x	y
a										
b										
c										
d										
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										
11										
12										
13										
14										

flow angle

specification

Step 1: Add target Mach number and required P-M expansion wave (θ_a=0.373)

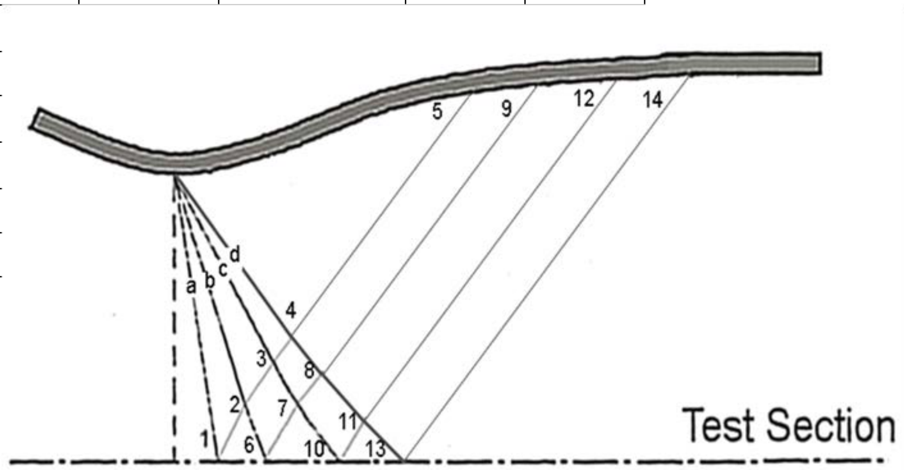
divided into equal increments between $\theta_a \rightarrow \theta_{max}$

Point	R^+	R^-	θ	ν	M	μ	$\theta+\mu$	$\theta-\mu$	x	y
a			0.37	0.37					0	1
b			6.37	6.37					0	1
c			12.37	12.37					0	1
d			18.37	18.37					0	1
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										
11										
12										
13					2.4					
14					2.4					

at $M=2.4$

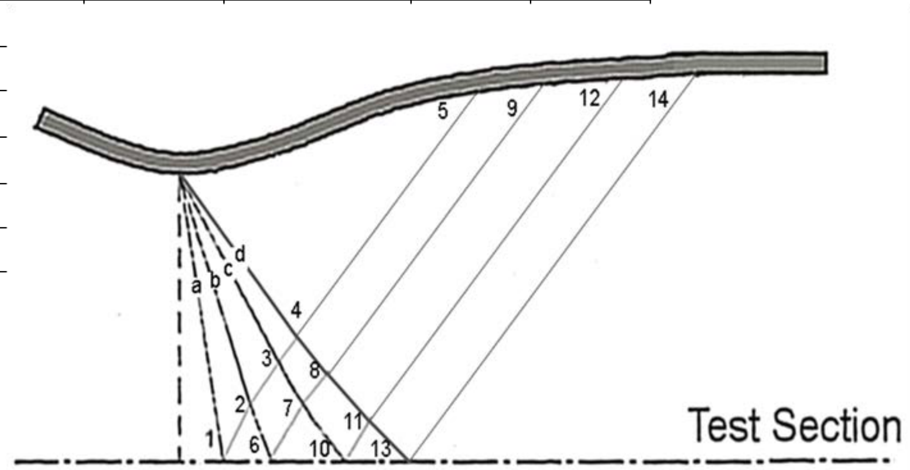
$\theta_{max}=18.37$

goal



Step 2: Work out R^- and ν

Point	R^+	R^-	θ	ν	M	μ	$\theta+\mu$	$\theta-\mu$	x	y
a	0	0.75	0.37	0.37					0	1
b	0	12.75	6.37	6.37					0	1
c	0	24.75	12.37	12.37					0	1
d	0	36.75	18.37	18.37					0	1
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										
11										
12										
13					2.4					
14					2.4					



use to calc next

Step 3: Compute Mach number, Mach angle and characteristic lines

Point	R^+	R^-	θ	ν	M	μ	$\theta+\mu$	$\theta-\mu$	x	y
a	0	0.75	0.37	0.37	1.042	73.74	74.11	-73.36	0	1
b	0	12.75	6.37	6.37	1.307	49.90	56.27	-43.52	0	1
c	0	24.75	12.37	12.37	1.516	41.28	53.65	-28.90	0	1
d	0	36.75	18.37	18.37	1.719	35.57	53.94	-17.19	0	1
1										
2										
3										
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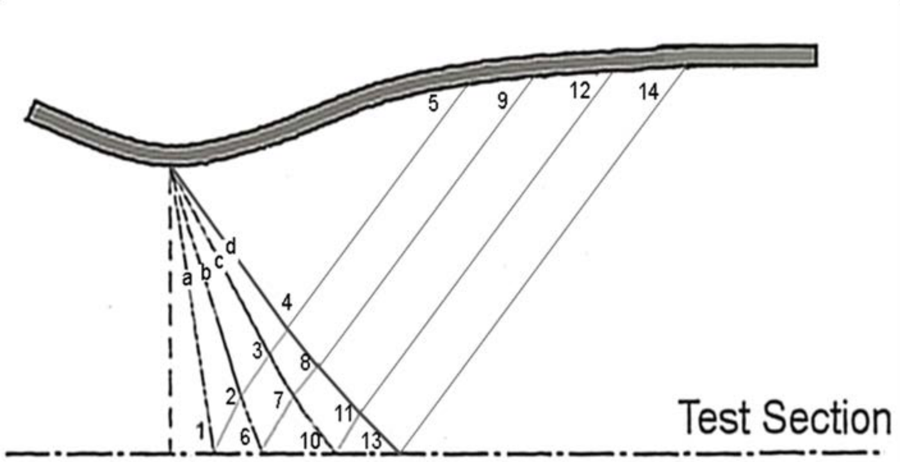
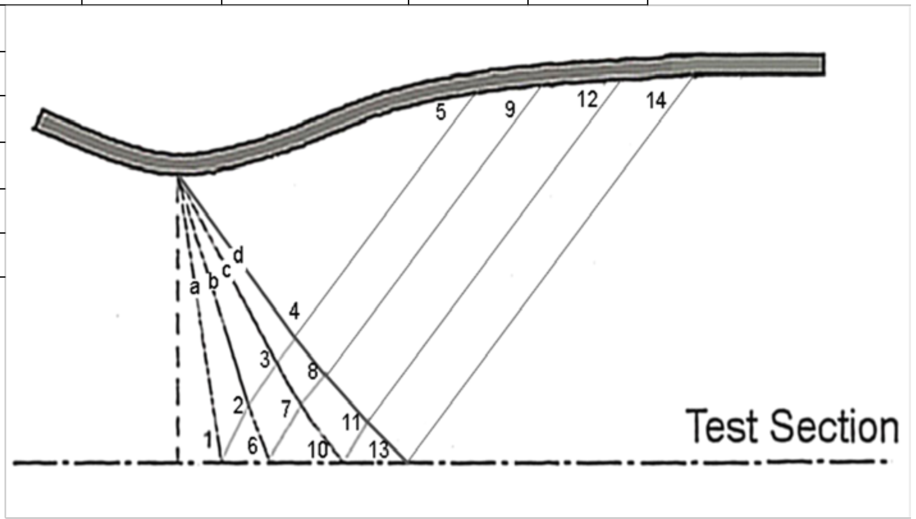
11										
12										
13					2.4					
14					2.4					



Step 4: Trace the R^- characteristics down

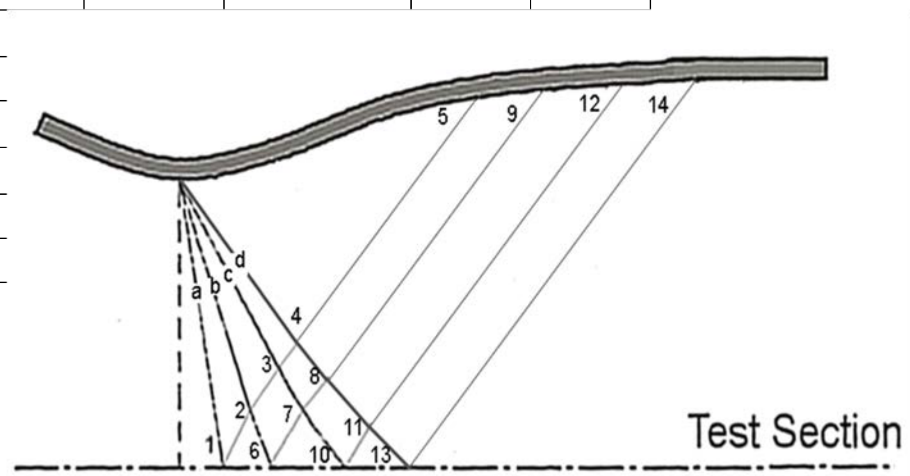
Point	R^+	R^-	θ	ν	M	μ	$\theta+\mu$	$\theta-\mu$	x	y
a	0	0.75	0.37	0.37	1.042	73.74	74.11	-73.36	0	1
b	0	12.75	6.37	6.37	1.307	49.90	56.27	-43.52	0	1
c	0	24.75	12.37	12.37	1.516	41.28	53.65	-28.90	0	1
d	0	36.75	18.37	18.37	1.719	35.57	53.94	-17.19	0	1
1		0.75								
2		12.75								
3		24.75								
4		36.75								
5										
6										
7										
8										
9										
10										
11										
12										
13					2.4					
14					2.4					



Step 5: Trace R^+ from centreline boundary condition, hence θ and ν

Point	R^+	R^-	θ	ν	M	μ	$\theta+\mu$	$\theta-\mu$	x	y
a	0	0.75	0.37	0.37	1.042	73.74	74.11	-73.36	0	1
b	0	12.75	6.37	6.37	1.307	49.90	56.27	-43.52	0	1
c	0	24.75	12.37	12.37	1.516	41.28	53.65	-28.90	0	1
d	0	36.75	18.37	18.37	1.719	35.57	53.94	-17.19	0	1
1	0.75	0.75	0.00	0.75						
2	0.75	12.75	6.00	6.75						
3	0.75	24.75	12.00	12.75						
4	0.75	36.75	18.00	18.75						
5	0.75									
6										
7										
8										
9										
10										
11										
12										
13					2.4					
14					2.4					

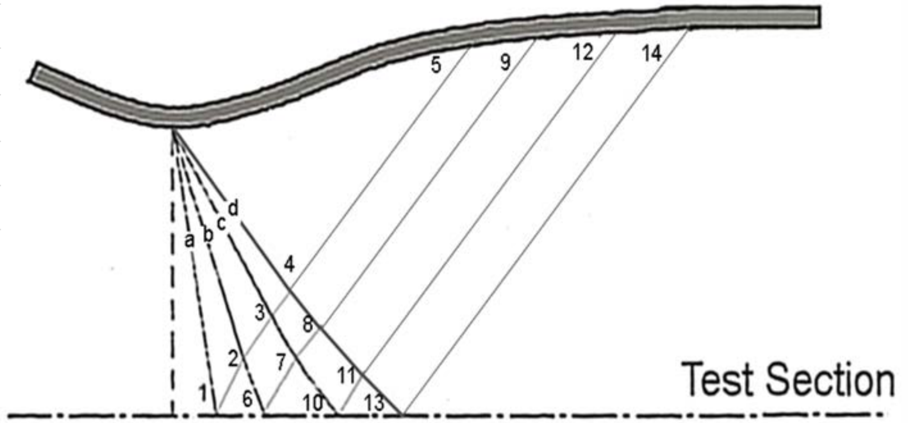
can be used to find



Step 6: Compute property data and geometry for points 1-4

Point	R^+	R^-	θ	ν	M	μ	θ+μ	θ-μ	x	y
a	0	0.75	0.37	0.37	1.042	73.74	74.11	-73.36	0	1
b	0	12.75	6.37	6.37	1.307	49.90	56.27	-43.52	0	1
c	0	24.75	12.37	12.37	1.516	41.28	53.65	-28.90	0	1
d	0	36.75	18.37	18.37	1.719	35.57	53.94	-17.19	0	1
1	0.75	0.75	0.00	0.75	1.067	69.61	69.61	-69.61	0.335	0.000
2	0.75	12.75	6.00	6.75	1.321	49.21	55.21	-43.21	0.574	0.458
3	0.75	24.75	12.00	12.75	1.529	40.86	52.86	-28.86	0.691	0.619
4	0.75	36.75	18.00	18.75	1.732	35.27	53.27	-17.27	0.7928	0.754
5	0.75									
6										
7										
8										
9										
10										
11										
12										
13					2.4					
14					2.4					

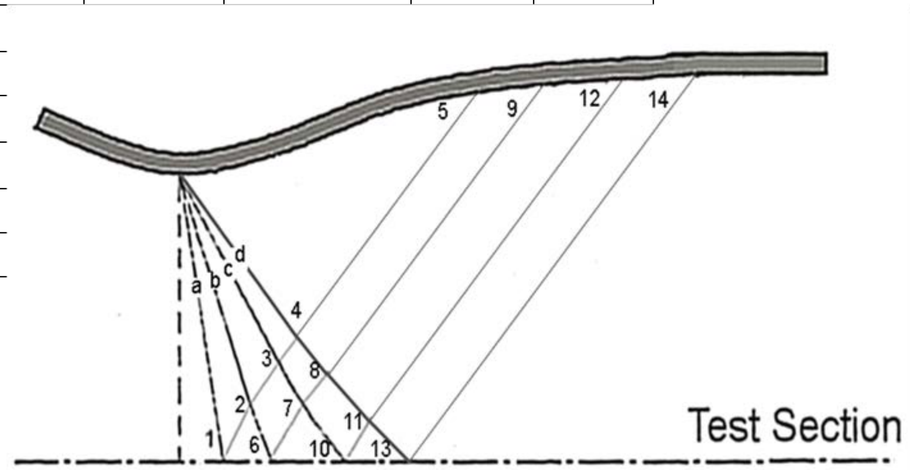
Then



Step 7: Boundary point 5 has same conditions as point 4 (and no reflection)

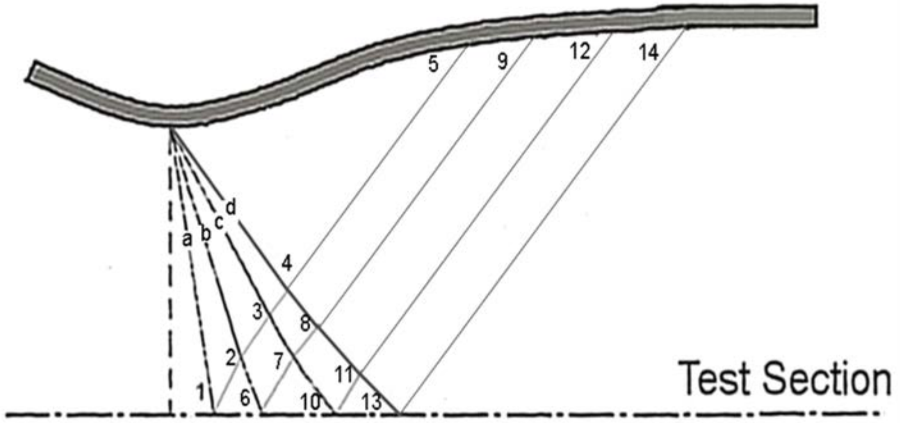
Point	R^+	R^-	θ	ν	M	μ	θ+μ	θ-μ	x	y
a	0	0.75	0.37	0.37	1.042	73.74	74.11	-73.36	0	1
b	0	12.75	6.37	6.37	1.307	49.90	56.27	-43.52	0	1
c	0	24.75	12.37	12.37	1.516	41.28	53.65	-28.90	0	1
d	0	36.75	18.37	18.37	1.719	35.57	53.94	-17.19	0	1
1	0.75	0.75	0.00	0.75	1.067	69.61	69.61	-69.61	0.335	0.000
2	0.75	12.75	6.00	6.75	1.321	49.21	55.21	-43.21	0.574	0.458
3	0.75	24.75	12.00	12.75	1.529	40.86	52.86	-28.86	0.691	0.619
4	0.75	36.75	18.00	18.75	1.732	35.27	53.27	-17.27	0.793	0.754
5	0.75	-	18.00	18.75	1.732	35.27	53.27	-17.27	1.293	1.425
6										
7										
8										
9										
10										
11										
12										
13					2.4					
14					2.4					

simple flow, values are same



Step 8: Riemann invariants for remaining internal points

Point	R^+	R^-	θ	ν	M	μ	$\theta+\mu$	$\theta-\mu$	x	y
a	0	0.75	0.37	0.37	1.042	73.74	74.11	-73.36	0	1
b	0	12.75	6.37	6.37	1.307	49.90	56.27	-43.52	0	1
c	0	24.75	12.37	12.37	1.516	41.28	53.65	-28.90	0	1
d	0	36.75	18.37	18.37	1.719	35.57	53.94	-17.19	0	1
1	0.75	0.75	0.00	0.75	1.067	69.61	69.61	-69.61	0.335	0.000
2	0.75	12.75	6.00	6.75	1.321	49.21	55.21	-43.21	0.574	0.458
3	0.75	24.75	12.00	12.75	1.529	40.86	52.86	-28.86	0.691	0.619
4	0.75	36.75	18.00	18.75	1.732	35.27	53.27	-17.27	0.793	0.754
5	0.75	-	18.00	18.75	1.732	35.27	53.27	-17.27	1.293	1.425
6	12.75	12.75								
7	12.75	24.75								
8	12.75	36.75								
9	12.75	-								
10	24.75	24.75								
11	24.75	36.75								
12	24.75	-								
13	36.75	36.75			2.4					
14	36.75	-			2.4					



Step 9: Compute flow properties and geometry

Point	R^+	R^-	θ	ν	M	μ	$\theta+\mu$	$\theta-\mu$	x	y
a	0	0.75	0.37	0.37	1.042	73.74	74.11	-73.36	0	1
b	0	12.75	6.37	6.37	1.307	49.90	56.27	-43.52	0	1
c	0	24.75	12.37	12.37	1.516	41.28	53.65	-28.90	0	1
d	0	36.75	18.37	18.37	1.719	35.57	53.94	-17.19	0	1
1	0.75	0.75	0.00	0.75	1.067	69.61	69.61	-69.61	0.335	0.000
2	0.75	12.75	6.00	6.75	1.321	49.21	55.21	-43.21	0.574	0.458
3	0.75	24.75	12.00	12.75	1.529	40.86	52.86	-28.86	0.691	0.619
4	0.75	36.75	18.00	18.75	1.732	35.27	53.27	-17.27	0.793	0.754
5	0.75	-	18.00	18.75	1.732	35.27	53.27	-17.27	1.293	1.425
6	12.75	12.75	0.00	12.75	1.529	40.86	40.86	-40.86	1.082	0.000
7	12.75	24.75	6.00	18.75	1.732	35.27	41.27	-29.27	1.363	0.245
8	12.75	36.75	12.00	24.75	1.941	31.01	43.01	-19.01	1.625	0.482
9	12.75	-	12.00							
10	24.75	24.75	0.00	24.75	1.941	31.01	31.01	-31.01	1.786	0.000
11	24.75	36.75	6.00	30.75	2.162	27.55	33.55	-21.55	2.207	0.266
12	24.75	-	6.00							
13	36.75	36.75	0.00	36.75	2.4	24.62	24.62	-24.62	2.832	0.000
14	36.75	-	0.00		2.4					

Final Template after adding wall points

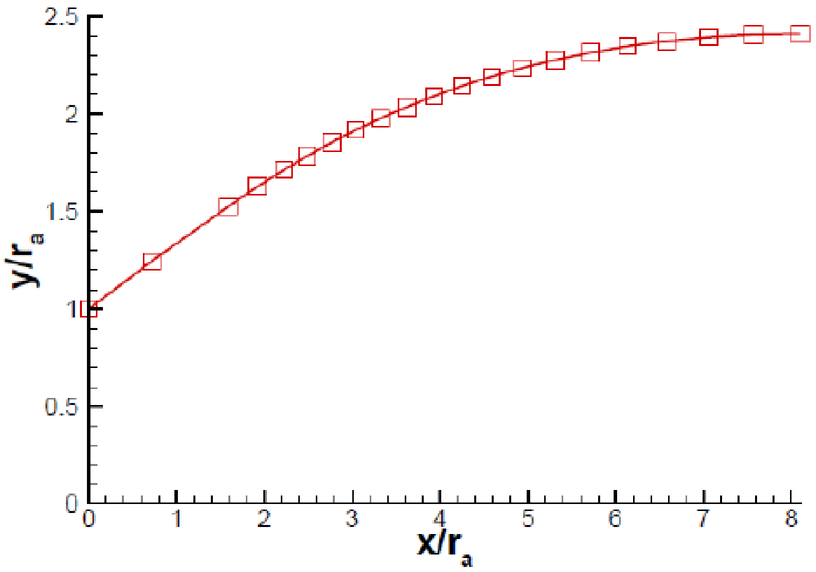
Point	R^+	R^-	θ	ν	M	μ	$\theta+\mu$	$\theta-\mu$	x	y
a	0	0.75	0.37	0.37	1.042	73.74	74.11	-73.36	0	1
b	0	12.75	6.37	6.37	1.307	49.90	56.27	-43.52	0	1
c	0	24.75	12.37	12.37	1.516	41.28	53.65	-28.90	0	1
d	0	36.75	18.37	18.37	1.719	35.57	53.94	-17.19	0	1
1	0.75	0.75	0.00	0.75	1.067	69.61	69.61	-69.61	0.335	0.000
2	0.75	12.75	6.00	6.75	1.321	49.21	55.21	-43.21	0.574	0.458
3	0.75	24.75	12.00	12.75	1.529	40.86	52.86	-28.86	0.691	0.619
4	0.75	36.75	18.00	18.75	1.732	35.27	53.27	-17.27	0.793	0.754
5	0.75	-	18.00	18.75	1.732	35.27	53.27	-17.27	1.293	1.425
6	12.75	12.75	0.00	12.75	1.529	40.86	40.86	-40.86	1.082	0.000
7	12.75	24.75	6.00	18.75	1.732	35.27	41.27	-29.27	1.363	0.245
8	12.75	36.75	12.00	24.75	1.941	31.01	43.01	-19.01	1.625	0.482
9	12.75	-	12.00	24.75	1.941	31.01	43.01	-19.01	3.177	1.930
10	24.75	24.75	0.00	24.75	1.941	31.01	31.01	-31.01	1.786	0.000
11	24.75	36.75	6.00	30.75	2.162	27.55	33.55	-21.55	2.207	0.266
12	24.75	-	6.00	30.75	2.162	27.55	33.55	-21.55	5.199	2.250
13	36.75	36.75	0.00	36.75	2.4	24.62	24.62	-24.62	2.832	0.000
14	36.75	-	0.00	36.75	2.4	24.62	24.62	-24.62	8.069	2.400

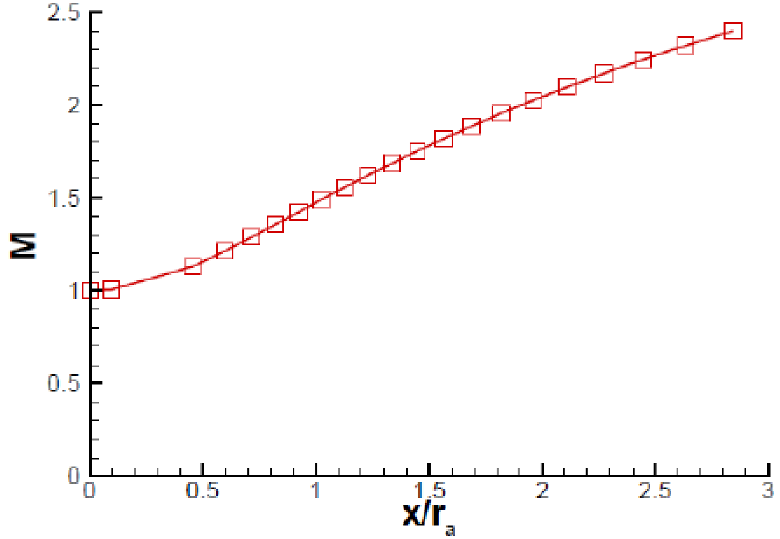
wall points!

Example: M_{des}=2.4

nozzle shape

centreline mach number





Nozzle contour and centreline Mach number, using 20 characteristics.
r_a is the nozzle throat half-width, depends on the flow rate.
Assuming $r_a = 1$ as the nozzle can be scaled for inviscid design.