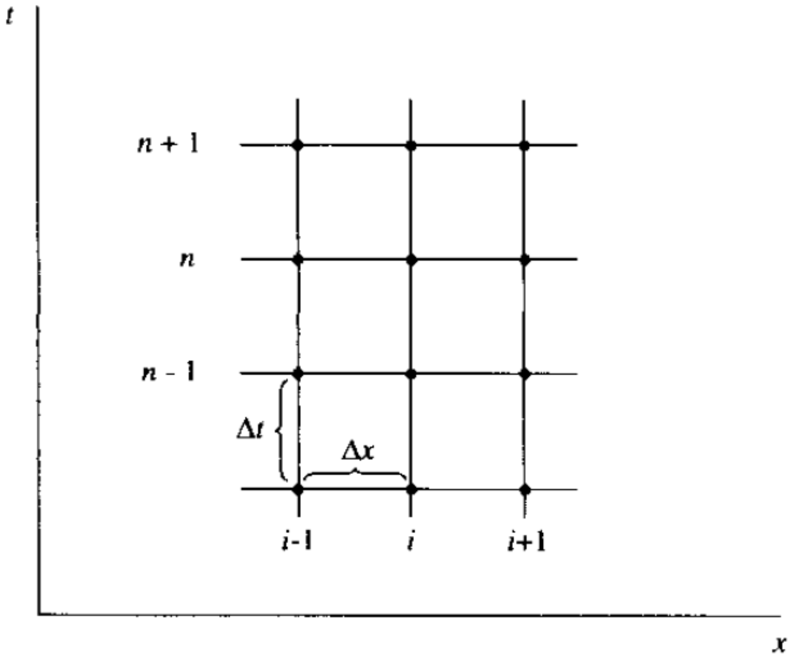
SESA3029 Aerothermodynamics

Lecture 5.6

1D finite difference methods for transient problems

Explicit method

1D, no heat source $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$



Now use a discrete finite difference grid with point-wise values T_i^n

Use a forward difference in time and approximate $\frac{\partial^2 T}{\partial x^2}$ at old time level n.

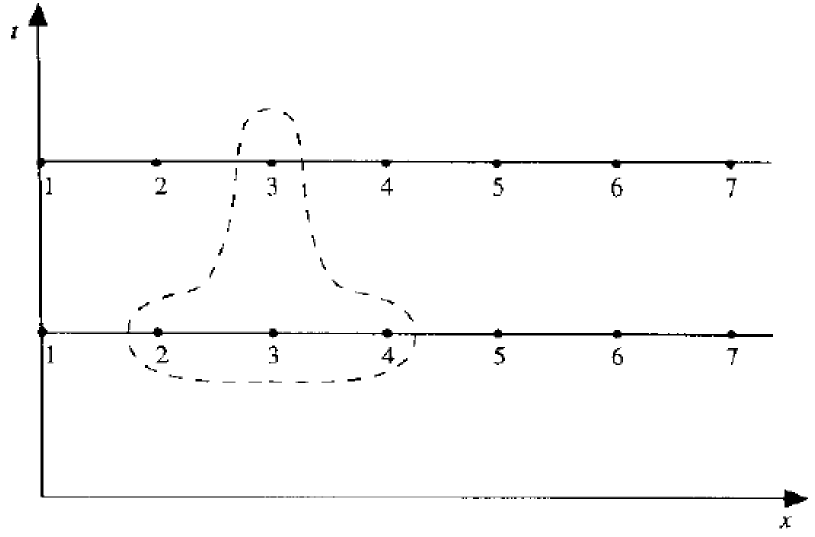
$$\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} = \frac{1}{\alpha}\frac{T_i^{n+1} - T_i^n}{\Delta t}$$

Introducing the Fourier number $F = \frac{\Delta t}{\Delta x^2}\alpha$ gives the iteration rule

$$T_i^{n+1} = T_i^n + F\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right) \qquad (4)$$

The developed method is *explicit* in time.

Choice of time steps



The dependency of each value on previous ones is visualised on the left.

This is a simple iteration scheme and just marching forward with a suitable time step Δt gives a time-accurate transient solution.

For steady problems a solution can be approximated by marching forward in (pseudo-)time until convergence.

<u>Stability</u> – von Neumann analysis (see FEEG6005):

Inserting a single Fourier mode $B(t)\,e^{ikx}$ into the iteration (4) formula, shows that the method is stable for

$$F \leq \tfrac{1}{2} \qquad \text{or} \qquad \Delta t \leq \frac{\Delta x^2}{2\alpha}$$

Discrete boundary conditions

Constant temperature: $T_0^{n+1} = T_{sl}\ ,\quad T_l^{n+1} = T_{sr}$

Surface convection on the left: $-k\left.\dfrac{\partial T}{\partial x}\right|_0 = h\left[T_\infty - T_0\right]$

$$\frac{\left.\dfrac{\partial T}{\partial x}\right|_{1/2} - \left.\dfrac{\partial T}{\partial x}\right|_0}{1\!\!\diagup_{2}\,\Delta x} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad\rightarrow\quad \frac{\dfrac{T_1^n - T_0^n}{\Delta x} - \left(-\dfrac{h}{k}\left[T_\infty - T_0^n\right]\right)}{1\!\!\diagup_{2}\,\Delta x} = \frac{1}{\alpha}\frac{T_0^{n+1} - T_0^n}{\Delta t}$$

yields

$$T_0^{n+1} = T_0^n + 2F\left(T_1^n - \left(1 + \frac{h\Delta x}{k}\right)T_0^n + \frac{h\Delta x}{k}T_\infty\right)$$

On the right:

$$T_l^{n+1} = T_l^n + 2F\left(T_{l-1}^n - \left(1 + \frac{h\Delta x}{k}\right)T_l^n + \frac{h\Delta x}{k}T_\infty\right)$$

Adiabatic: $h = 0$ Constant surface heat flux: $h = 0,\quad T_\infty = \dfrac{\dot{q}_x}{h}$

Explicit scheme for heat diffusion equation

Use forward difference for $\left(\frac{\partial T}{\partial t}\right)_i^n = \frac{T_i^{n+1} - T_i^n}{\Delta t} + O(\Delta t)$

and the previous central difference to approximate the entire equation as

$$0 = \frac{1}{\alpha}\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{T_i^{n+1} - T_i^n}{\Delta t} + O(\Delta t) - \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + O(\Delta x^2)$$

Taylor-series expansion yields

$$0 = \frac{1}{\alpha}\frac{T_i^{n+1} - T_i^n}{\Delta t} - \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} - \left[-\left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{\Delta t}{2\alpha} + \left(\frac{\partial^4 T}{\partial x^4}\right)_i^n \frac{\Delta x^2}{12} + \ldots \right]$$

The truncation error of this method is $O(\Delta t, \Delta x^2)$.

The method is overall first-order accurate. (***Order of accuracy***)

The method is ***consistent*** with the original equation as for $\Delta t \to 0$ and $\Delta x \to 0$ the original equation is recovered.

Example

A thick slab of copper ($\alpha = 117 \cdot 10^{-6}\,\mathrm{m^2/s}$, $k = 401\,\mathrm{W/(mK)}$), initially at 20°C, is subjected to a constant net heat flux of $\dot{q}_x = 3 \cdot 10^5\,\mathrm{W/m^2}$ at one surface.

Determine the temperatures at the surface and 150 mm from the surface after an elapsed time of 2 min.

Solution approach:

- For Δx=75mm and $F = \frac{1}{2} \quad \rightarrow \quad \Delta t \approx 24\,\mathrm{s}$
- Chose number of time steps as N = 5, 10, 20, 40 … for $F = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
- Use at least $I \geq N$ to allow application of const. T_0 =20°C as boundary condition on right
- Evaluate Δx from F and use overall length $L=I\Delta x$

From Bergman et al., p. 340ff / Incropera et al., p. 312ff.

Explicit Finite-Difference Solution for $Fo = \frac{1}{2}$

p	t(s)	T_0	T_1	T_2	T_3	T_4
0	0	20	20	20	20	20
1	24	76.1	20	20	20	20
2	48	76.1	48.1	20	20	20
3	72	104.2	48.1	34.0	20	20
4	96	104.2	69.1	34.0	27.0	20
5	120	125.2	69.1	48.1	27.0	23.5

Method	$T_0 = T(0, 120\text{ s})$	$T_2 = T(0.15\text{ m}, 120\text{ s})$
Explicit ($Fo = \frac{1}{2}$)	125.2	48.1
Explicit ($Fo = \frac{1}{4}$)	118.8	44.4
Implicit ($Fo = \frac{1}{2}$)	114.7	44.2
Exact	120.0	45.4

Explicit Finite-Difference Solution for $Fo = \frac{1}{4}$

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	0	20	20	20	20	20	20	20	20	20
1	12	48.1	20	20	20	20	20	20	20	20
2	24	62.1	27.0	20	20	20	20	20	20	20
3	36	72.6	34.0	21.8	20	20	20	20	20	20
4	48	81.4	40.6	24.4	20.4	20	20	20	20	20
5	60	89.0	46.7	27.5	21.3	20.1	20	20	20	20
6	72	95.9	52.5	30.7	22.5	20.4	20.0	20	20	20
7	84	102.3	57.9	34.1	24.1	20.8	20.1	20.0	20	20
8	96	108.1	63.1	37.6	25.8	21.5	20.3	20.0	20.0	20
9	108	113.6	67.9	41.0	27.6	22.2	20.5	20.1	20.0	20.0
10	120	118.8	72.6	44.4	29.6	23.2	20.8	20.2	20.0	20.0

Implicit method

Now use only values at new time level n+1 in the spatial finite difference $\dfrac{\partial^2 T}{\partial x^2}$

$$\frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{\Delta x^2} = \frac{1}{\alpha}\frac{T_i^{n+1} - T_i^n}{\Delta t}$$

yields eventually

$$T_{i+1}^{n+1} - \left(2 + \frac{1}{F}\right)T_i^{n+1} + T_{i-1}^{n+1} = -\frac{1}{F}T_i^n$$

The method is *implicit* in time.

Constant temperature BCs: $T_0^{n+1} = T_{sl}\ ,\quad T_l^{n+1} = T_{sr}$

Surface convection BCs:

$$T_1^{n+1} - \left(1 + \frac{h\Delta x}{k} + \frac{1}{2F}\right)T_0^{n+1} = -\frac{1}{2F}T_0^n - \frac{h\Delta x}{k}T_\infty$$

$$T_{l-1}^{n+1} - \left(1 + \frac{h\Delta x}{k} + \frac{1}{2F}\right)T_l^{n+1} = -\frac{1}{2F}T_l^n - \frac{h\Delta x}{k}T_\infty$$

Solution process

Starting from the initial conditions T_i^0, solve the linear problem (here for surface convection boundary conditions) successively, using the data from the previous time step *n* in the right-hand side

$$\begin{pmatrix} -\left(1+\frac{h_l \Delta x}{k}+\frac{1}{2F}\right) & 1 & 0 & \cdots & 0 \\ 1 & -\left(2+\frac{1}{F}\right) & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & -\left(2+\frac{1}{F}\right) & 1 \\ 0 & \cdots & 0 & 1 & -\left(1+\frac{h_r \Delta x}{k}+\frac{1}{2F}\right) \end{pmatrix} \begin{pmatrix} T_0^{n+1} \\ \\ \vdots \\ \\ T_I^{n+1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2F} T_0^n - \frac{h_l \Delta x}{k} T_{\infty l} \\ -\frac{1}{F} T_1^n \\ \vdots \\ -\frac{1}{F} T_{I-1}^n \\ -\frac{1}{2F} T_I^n - \frac{h_r \Delta x}{k} T_{\infty r} \end{pmatrix}$$

Δt can be chosen freely but should reflect the physical time scales of the problems, e.g., when boundary conditions are time dependent.