Part 2a: Elastic Rods in Tension and Compression

FEEG3001/SESM6047 FEA in Solid Mechanics

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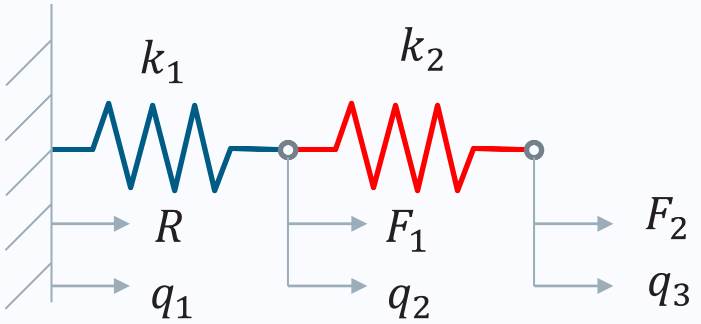
From 11th October 2024

Reminder from last time:



- We search for solutions for q_i in problems like this:
  - We express the system's Elastic Strain Energy U and its Potential Energy V
  - by finding U_i for each element and V based on all applied loads.
  - We could apply PMTPE and calculate $\frac{\partial \Pi(q_i)}{\partial q_i} = 0$ to obtain $\{F\} = [K]\{q\}$
  - but because we can express U_i in quadratic form, we can take a shortcut where $U_i = 1/2\{q\}^T[K_i]\{q\}$
  - and assemble our elemental $[K_i]$ into a global $[K]$
  - and by expressing V in matrix form too, we find $\{F\}$ and can therefore solve $\{F\} = [K]\{q\}$
  - by inverting the stiffness matrix, allowing $\{q\} = [K]^{-1}\{F\}$

This week:

- Similarities:
  - PMTPE gives us our governing equation of equilibrium (a set of linear quadratic equations)
  - How we fetch and assemble our element stiffness matrices into an assembled stiffness matrix
- Differences:
  - Unlike for springs, we have some approximation, for ‘elastic rods’

Next week:

- Solving Rod FE Systems
- Combining them with point Masses, and Springs
- Example Questions

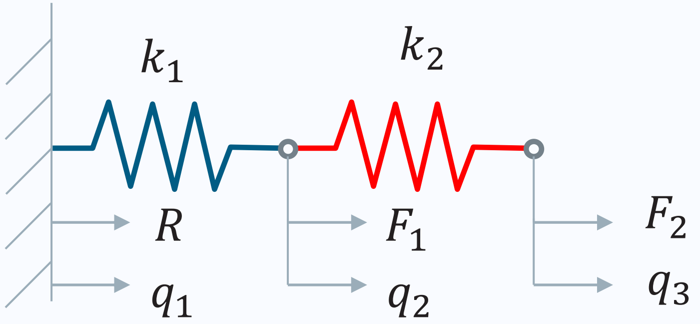
Reminder for Springs – why don't we lose R?

We need to assert a ***boundary condition***:

$$q_1 = 0$$

This zero-BC case allows us to use an (unproven) trick where we strike out corresponding rows and columns of our governing equation:

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R \\ F_1 \\ F_2 \end{Bmatrix}$$



and rewrite what is left:

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

After solving for $\{q\}$ we can come back and find R.

The FEA Procedure

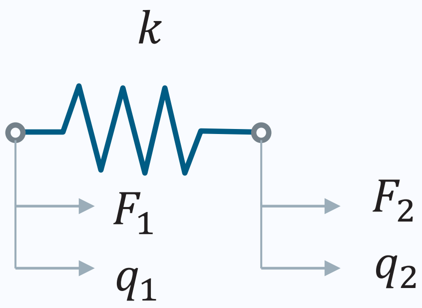
- The Complete Procedure:

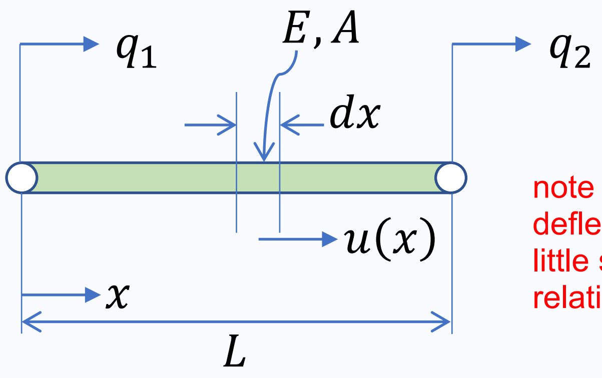
1. Describe the problem
2. Select a displacement estimation/approximation function
3. Relate displacement field to nodal displacements
4. Estimate strain from displacement
5. Estimate stress from strain
6. Apply PMTPE
7. Apply Boundary Conditions
8. Solve for nodal displacements
9. Calculate displacement field (optional)
10. Calculate strain field (optional)
11. Calculate stress field (optional)
12. Calculate reaction forces (optional)

- Applied Practically:

1. Describe the problem
2. Select an appropriate element type
3. Apply the elements to the problem, assembling them
4. Apply Boundary Conditions
5. Solve for nodal displacements
6. Calculate displacement field (optional)
7. Calculate strain field (optional)
8. Calculate stress field (optional)
9. Calculate reaction forces (optional)

Elastic Rods

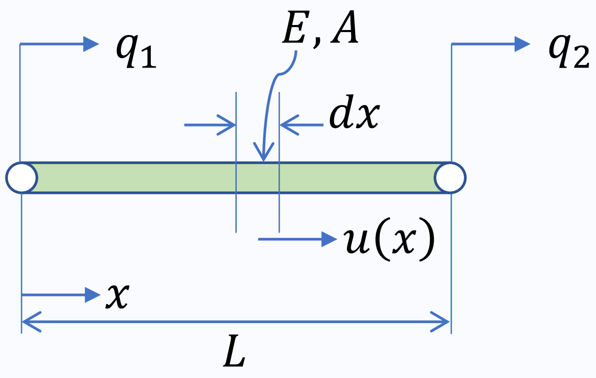




note small u is the deflection of that little slice, it has no relation to U

- Can deform only in tension or compression
- Unlike our spring, properties are described by E, A (cross section) and L (length), and the axial displacement of the endpoints q_1 and q_2 in coordinate system x
- We look at a small portion $\mathrm{d}x$, and its deformation $u(x)$ depends on where we take the slice
- Every point on the cross-section slice displaces the same amount
- So x is the 'label' (which cross section) and $u(x)$ is the displacement.

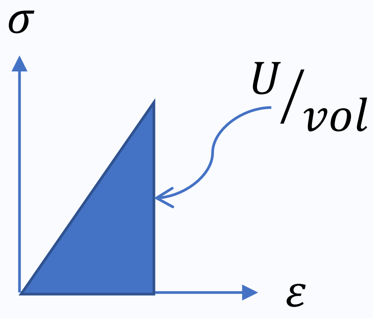
Elastic Rods



- PMTPE requires an expression for the strain energy U in terms of the displacement field in the rod u.

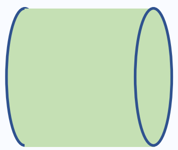
$$U = {}^1\!/_2 \int_0^L EA \left(\frac{du}{dx}\right)^2 dx$$

- Where does this come from? You should remember…
- Elastic strain energy per unit volume is the area under the stress-strain curve, for simplified 1D (uniaxial) elasticity:



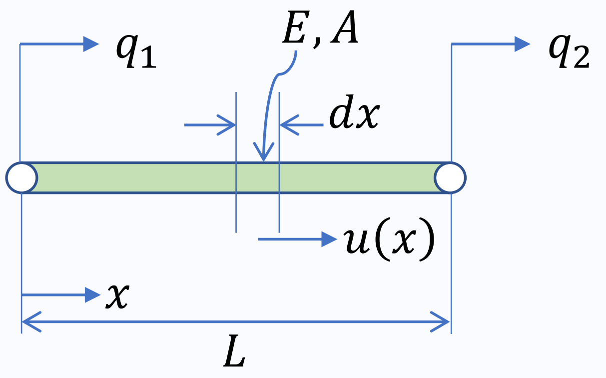
$$U = {}^1\!/_2 \int \sigma_x \varepsilon_x dVol$$

$$U = {}^1\!/_2 \int \sigma_x \varepsilon_x A dx$$



$$dVol = Adx$$

Elastic Rods



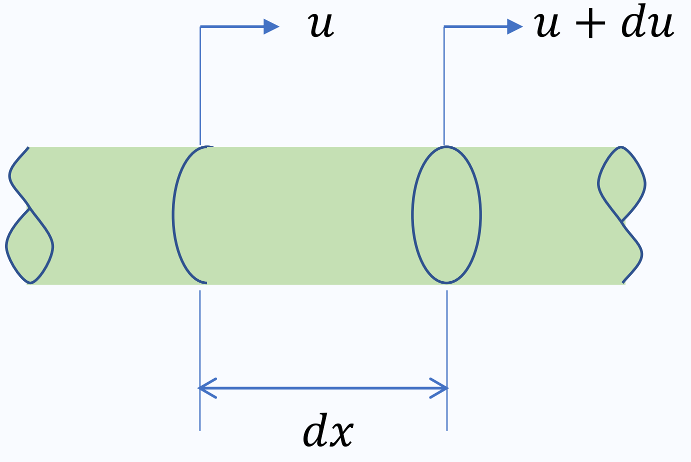
- and because in 1D elasticity (highly simplified case) we can say that $\sigma_x = E\varepsilon_x$,

$$U = {}^1\!/_2 \int E\varepsilon_x \times \varepsilon_x \times A dx$$

$$U = {}^1\!/_2 \int EA{\varepsilon_x}^2 dx$$

- What is ε_x given the displacement field?

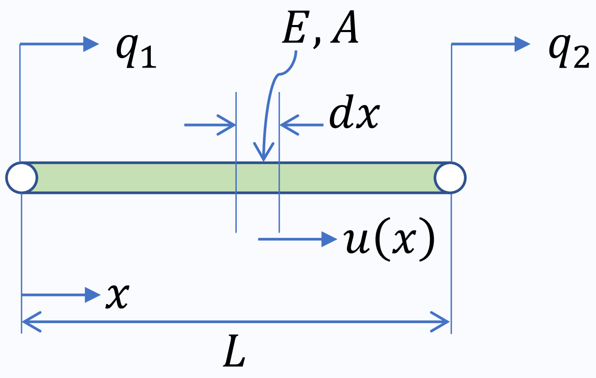
$$\varepsilon_x = \frac{(u + du) - u}{dx} = \frac{du}{dx}$$



- Hence

$$U = {}^1\!/_2 \int EAu'^2 dx$$

Elastic Rods



- **Aim:** we want to express the displacement throughout the element $u(x)$ from the nodal displacements, q_1 and q_2, (currently unknown).
- Now our first approximation: let's say $u(x)$ is a linear function of x. i.e.:

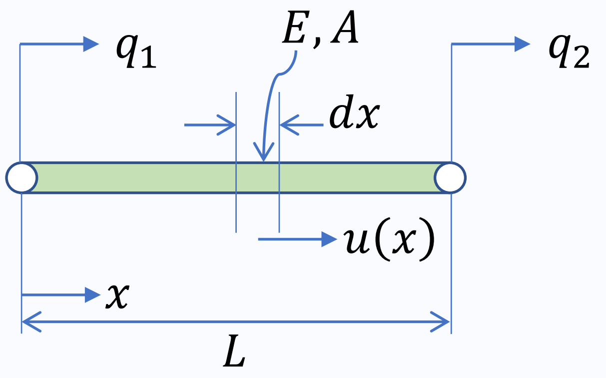
$$u(x) = a + bx$$

- (We can make this linear function assumption for 1D elasticity, but we couldn't make that assumption for more complex cases).
- This is inconvenient though, as a and b have no meaning. A more convenient version, more general:

$$u(x) = N_1(x)q_1 + N_2(x)q_2$$

- q_1 and q_2 are unknowns, N_1 and N_2 are prescribed ***shape functions*** or ***interpolation functions*** for approximation: a common approach taken in all finite elements.

Elastic Rods



$$u(x) = N_1(x)q_1 + N_2(x)q_2$$

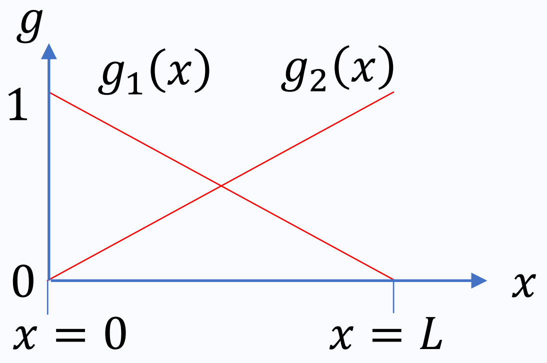
- The trick is we choose for these shape functions N_1 and N_2 some linear functions, g_1 and g_2. (We could choose other functions…)

$$u(x) = g_1(x)q_1 + g_2(x)q_2$$

- We can define these by drawing our domain 0 to L:

$$g_1(x) = 1 - \frac{x}{L}$$

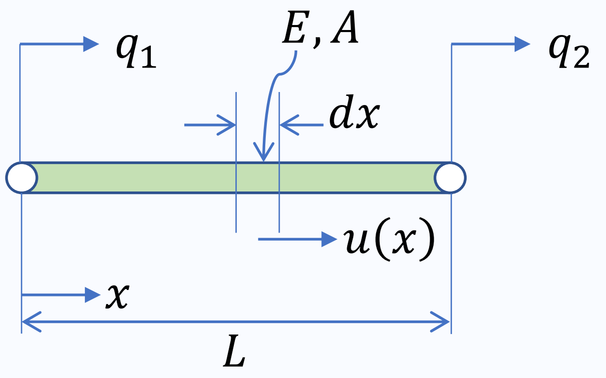
$$g_2(x) = \frac{x}{L}$$



$$u(x) = \left(1 - \frac{x}{L}\right) q_1 + \left(\frac{x}{L}\right) q_2$$

Elastic Rods

$$u(x) = \left(1 - \frac{x}{L}\right) q_1 + \left(\frac{x}{L}\right) q_2$$

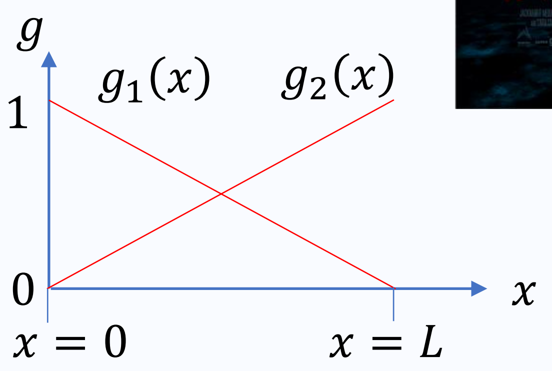


- But what are these qs?
- Why didn't we just use $u(x) = a + bx$?
- Work out $u(x)$ at each end:

$$u(0) = \left(1 - \frac{0}{L}\right) q_1 + \left(\frac{0}{L}\right) q_2 = q_1$$

$$u(L) = \left(1 - \frac{L}{L}\right) q_1 + \left(\frac{L}{L}\right) q_2 = q_2$$

- The qs, which aren't functions of x, describe the end displacements! This is why we choose shape functions in that form.



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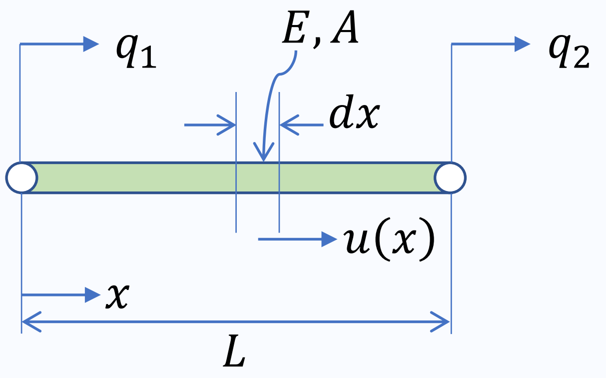


- and assuming we have solved to find the q unknowns, we can use

$$u(x) = \left(1 - \frac{x}{L}\right) q_1 + \left(\frac{x}{L}\right) q_2$$

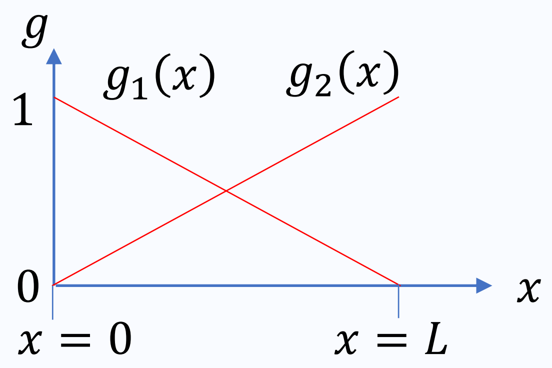
- to give us values for strain:

$$\varepsilon(x) = \frac{du}{dx} = \left(-\frac{1}{L}\right) q_1 + \left(\frac{1}{L}\right) q_2$$

- and in this simple 1D case, for stress:

$$\sigma(x) = E\varepsilon(x) = \left(-\frac{E}{L}\right) q_1 + \left(\frac{E}{L}\right) q_2$$

- (check dimensional analysis if you like!)



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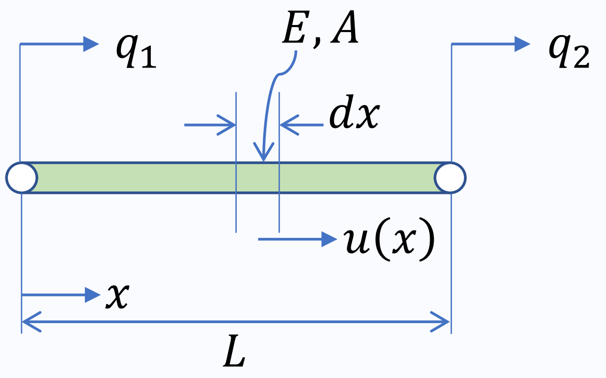


- or in matrix form (as more complex element types will require):

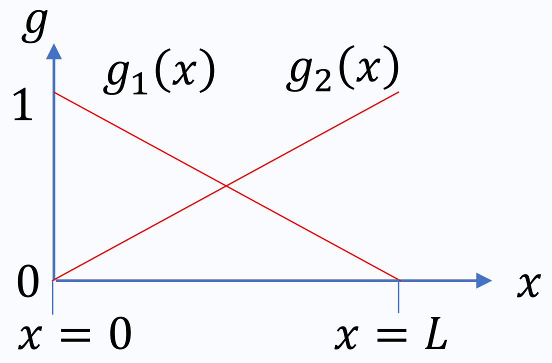
$$\varepsilon(x) = \frac{du}{dx} = \left(-\frac{1}{L}\right) q_1 + \left(\frac{1}{L}\right) q_2$$

$$\varepsilon(x) = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

- and

$$\sigma(x) = E\varepsilon(x) = \left(-\frac{E}{L}\right) q_1 + \left(\frac{E}{L}\right) q_2$$

$$\sigma(x) = [E] \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$



Summary:

- We have written the internal field of displacement $u(x)$ in terms of two chosen functions that happen here to be linear (Shape Functions) $g_1(x)$ and $g_2(x)$.
- $g_1(x)$ has a property where its value is 1 at left and 0 at right, and
- $g_2(x)$ has a value 0 at left and 1 at right.
- Because they are linear functions, what about their combination? It must also be a linear function, if q values do not depend on x (they are single, nodal values).
- Next class will continue towards how we find values for q!