SESA1015	Mechanics of Flight, 2020/21

Southampton
UNIVERSITY OF

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1 INTRODUCTION (Lecture 1)

On December 17, 1903, two brothers from Dayton, Ohio, named Wilbur and Orville Wright, were successful in flying an airplane they built. Their powered aircraft flew for 12 seconds above the sand dunes of Kitty Hawk, North Carolina, making them the first men to pilot a heavier-than-air machine that took off on its own power, remained under control, and sustained flight.

- **Mechanics of Flight course applies Newton's laws of motion to flying vehicles**
- **Part one of the course will consist of:**
  - 20 lectures (with slides, outline notes and diagrams on Blackboard)
  - Examples and past exam questions
  - Private study using recommended books and worldwide web resources
  - Note that this part of the course has been previously called AA114 and SESA1004
- **Aircraft design and aircraft operations are highly regulated and controlled, with emphasis on:**
  - Safety
  - Cost
  - Reliability
  - Environmental Impact
- **Major components of the Aircraft to be considered are:**
  - Wings, including definitions, planform and structures
  - Thrust generators
  - Landing arrangements
  - Tail arrangements
- **Major elements of Mechanics of Flight are:**

  - Performance
  - Forces and Moments
  - Stability
  - Control
  - Loading Actions
  - Aeroelasticity
- **Forces on aircraft do NOT act through a single point**; **a simplification is to consider them acting through a single point – the Centre of Gravity**
  - In trimmed flight the total moment of forces should balance and pitching moment is zero
- **Aircraft can be in equilibrium, but have differing stability**
  - Civilian aircraft are generally aerodynamically stable
  - Modern military fighters are often unstable, so that they are more maneuverable in combat

2 THE ATMOSPHERIC ENVIRONMENT, FUNDAMENTAL CONCEPTS AND DEFINITIONS (Lecture 2)

Before we can commence a study of aircraft performance it is necessary to consider the medium in which the aircraft operates, i.e. the atmosphere. Another important consideration is the significance of the aircraft speed relative to the air and the measurement of airspeed.

2.1 The Atmosphere

In the study of Mechanics of Flight a knowledge of the aerodynamic forces acting on the aircraft and the performance of its propulsion system are required. In addition to the geometric shape of the aircraft and the angle it makes relative to the air free-stream moving past it, the aerodynamic forces depend on its velocity relative to the air and on the density of the air in which it flies. The performance also depends on the temperature of the air, since this determines the speed of sound and hence the magnitude of compressibility effects (i.e. Mach number effects). Similarly, the performance of the propulsion system is also dependent on the air density and temperature at the altitude at which the aircraft is flying. Since aircraft can operate at altitudes between sea-level and a few tens of kilometres, depending on the type, it is important to understand how the density and temperature of the atmosphere vary with height above sea-level.

Clearly, the properties of the atmosphere vary on a daily and seasonal basis and are also dependent on the global position. They are also dependent on local topography and the weather. In order to make comparisons of aircraft performance, flight tests, etc., a standard atmosphere is defined which gives the mean values of pressure, density, temperature, etc. as a function of height above sea-level.

A detailed study of the atmosphere will be presented in the section of this module on Aircraft Operations in Semester 2. This will include information on the calculation of the properties of the International Standard Atmosphere. In the study of the mechanics of flight we use these properties as the basis for calculating and comparing the performance of flight vehicles. The main interest will centre on the Troposphere (sea level up to 11 km) and the constant temperature region of the Stratosphere (11 - 20 km). These two regions cover the principal portion of the atmosphere from sea-level up to 20 km altitude in which the majority of aircraft operate.

The International Standard Atmosphere gives the values of the density, pressure, temperature and speed of sound at various altitudes. The variations of these properties, relative to their sea level values, are shown graphically in Fig 2.1. The variation of temperature up to altitudes of 90 km is shown in Fig 2.2.

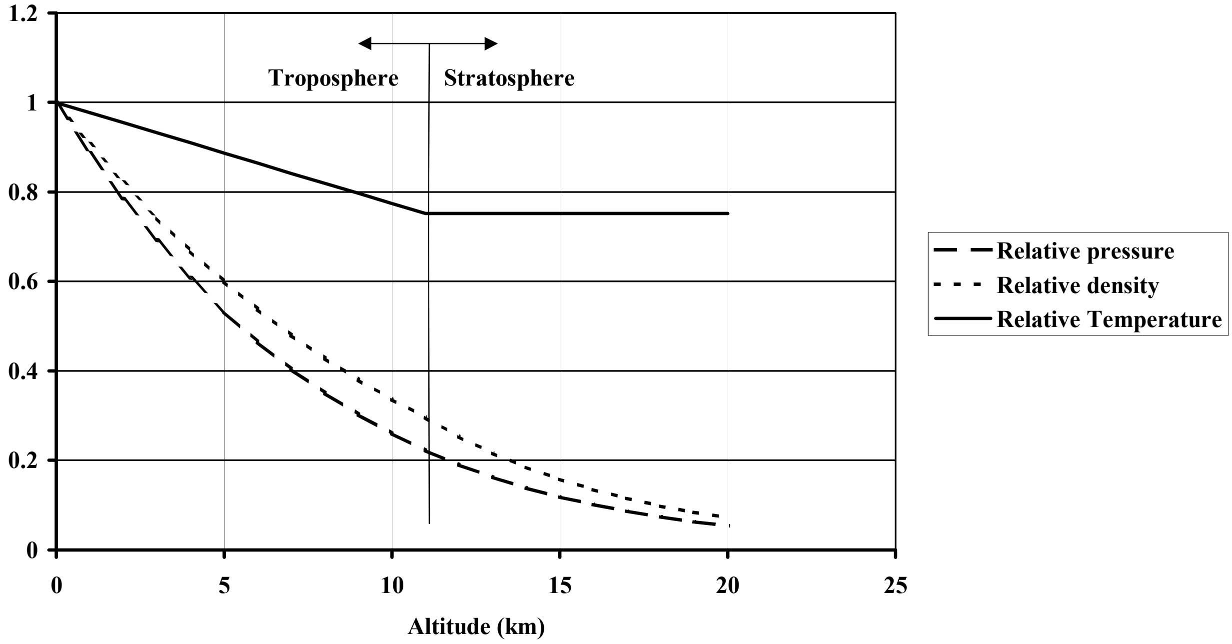


Fig 2.1 – International Standard Atmosphere up to 20km

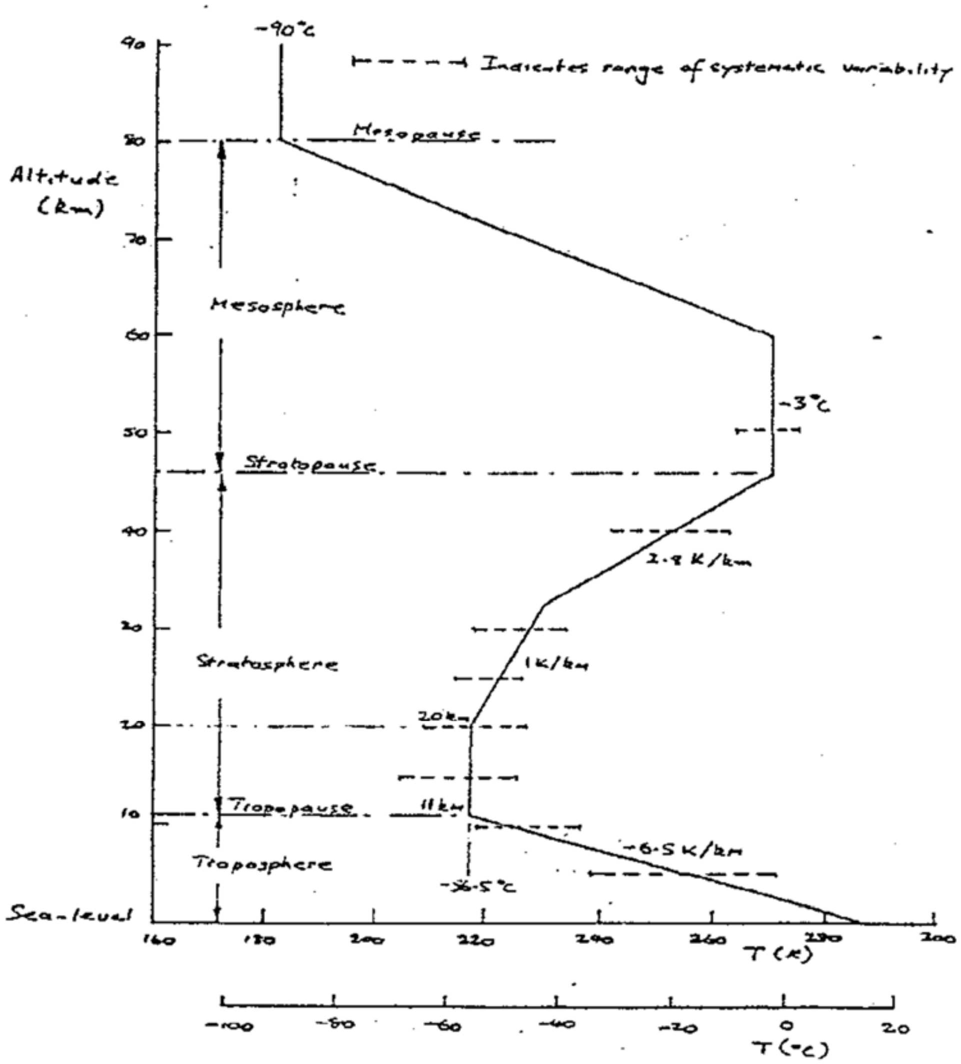
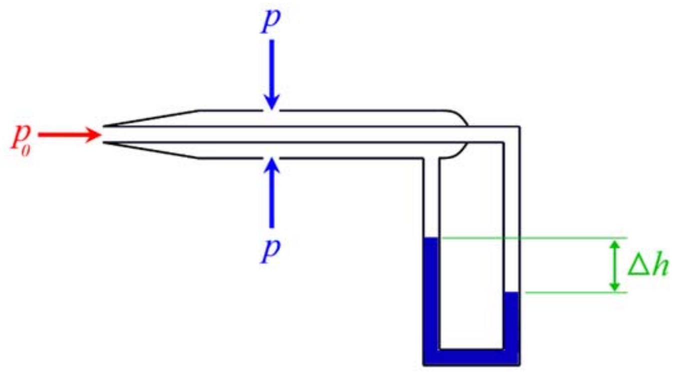


Fig 2.2 – International Standard Atmosphere up to 90km

2.2 Measurement of Airspeed

We shall see later that the forces acting on the aircraft in flight are dependent on the quantity $\rho V^2/2$, where ρ is the local air density and V is the speed of the aircraft relative to the air (true air speed). It is of importance to measure the speed of the aircraft in flight and to display this information on an instrument - the air speed indicator (ASI) - on the aircraft control panel. The measurement of the speed of the aircraft relative to the air is obtained by measuring pressure differences due to aircraft motion using a specially designed instrument called a pitot-static tube shown below.



A forward facing open ended central core of the tube measures the pressure where the air is brought to rest relative to the aircraft. This pressure is known as the pitot pressure (p_o). Around the central core is a separate annular tube which is sealed at the front end, but holes are drilled perpendicular to the tube wall at a particular distance from the front so that the static pressure (p) of the air surrounding the aircraft is measured. Alternatively, the speed may be measured using a single forward facing tube, known as a pitot tube, which measures the pitot pressure p_o, and the static pressure may be measured at some chosen point on the surface of the aircraft where the pressure is relatively undisturbed.

From Bernoulli's equation the difference between the pitot pressure and the static pressure can be shown to be equal to $\rho V^2/2$, i.e.

$$p_o - p = \frac{1}{2}\rho V^2$$

If the two separate parts of the pitot static tube are connected to two sides of a differential pressure gauge this will measure the quantity $\rho V^2/2$, which is known as the dynamic pressure. In the previous figure a manometer is shown to indicate the pressure difference, but in practice a sensitive pressure gauge is used to provide an output which is calibrated in knots (nautical miles per hour).

The true airspeed is given by : $V = \sqrt{\dfrac{2(p_o - p)}{\rho}}$

Hence to calibrate the pressure gauge to give true air speed it is necessary to know the local air density ρ. In principle, it would be possible to deduce the local air density, which decreases with height. Instead, a speed V_E is calculated from the measured pressure difference by using the constant sea-level density ρ_o (1.226 kg/m^3).

$$V_E = \sqrt{\frac{2(p_o - p)}{\rho_o}}$$

The speed V_E is known as the equivalent airspeed. As the altitude increases from sea-level the equivalent airspeed decreases below the true airspeed.

It is noted from the above that the true airspeed and the equivalent airspeed are related by

$$\rho V^2 = \rho_o V_E^{\ 2}$$

Hence $V = V_E\,(\rho_o/\rho)^{½} = V_E/\sigma^{½}$ where σ is the relative density.

Note that the airspeed indicator calibrated with sea-level air density gives a measure of the equivalent airspeed only if the true values of the pitot pressure and the local static pressure are measured. In practice, it is quite straightforward to measure the value of the pitot pressure without error, but the measured static pressure is dependent on a number of factors such as the position on the aircraft at which it is measured, the aircraft angle of attack and the configuration of the aircraft (i.e., flaps up or down, etc.). The value of the airspeed as indicated without allowance for these errors is known as the indicated airspeed.

2.3 Mach Number

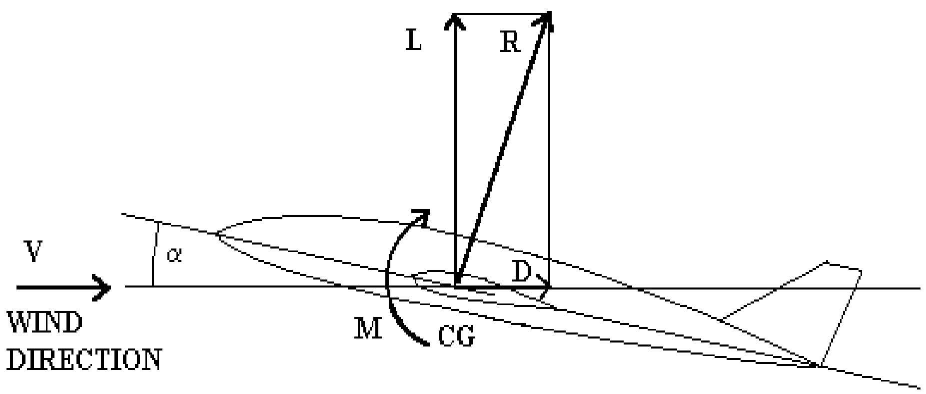
When the speed of the aircraft approaches the speed of sound the Mach Number is important. The speed of sound is a measure of the speed of propagation of weak pressure waves in the air, which is a compressible medium. The speed of propagation of a weak wave is called the speed of sound and is denoted by the symbol a. For a perfect gas it may be shown to be:

$$a = (\gamma p/\rho)^{½} = (\gamma RT)^{½}$$

where γ is the ratio of specific heats for the gas and R is the gas constant.

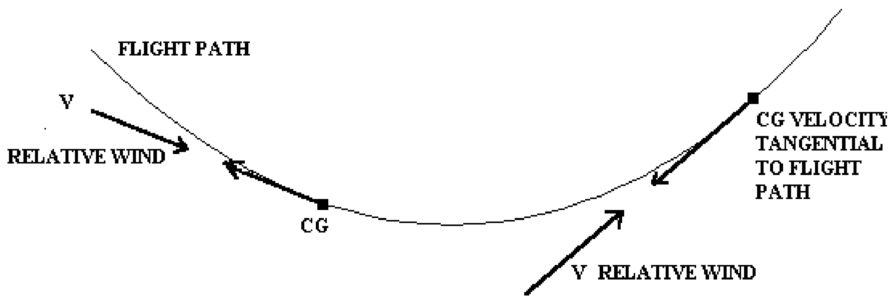
The Mach Number M is the ratio of the true airspeed to the speed of sound.

i.e $\quad M = V/a$



Flight Path is the path through space traced by the centre of gravity of the aircraft.

Relative Wind is the apparent wind seen by the aircraft and is equal and opposite to the instantaneous velocity of the centre of gravity.



2.4 Aerodynamic Forces and Moments

Aerodynamic Reaction consists of the force and moment components referred to the centre of gravity i.e. R and M.

R can be resolved into the following components:

L normal to the relative wind:	Lift
D parallel to the relative wind:	Drag

In addition there is a moment M

M is known as the: Pitching Moment

Incidence or Angle of attack ($=\alpha$) is the angle between the body datum (reference line) of the aircraft and the direction of the relative wind.

Attitude is the angle between the body datum and the Earth reference line.

The angle of attack is important in determining the aerodynamic force, whilst the attitude is important when considering the pitch mechanics.

A summary of the forces and moment acting on a symmetric aircraft at zero yaw and roll angle, together with the physical quantities which determine their magnitudes is as follows:

– L (force) N	+ve upwards ^
– D (force) N	+ve pushing backwards>
– M (moment) Nm	+ve nose up
– ρ (density) kg/m^3	

- V (velocity) m/s
- l (typical length) m — e.g., wing chord, or fuselage length
- α (angle of attack) non-dim. — radians or degrees

Aerodynamic lift for example is a function of:

$$L \equiv L(\rho, V, l, \alpha, \text{shape}, \text{etc.})$$

The form of this function may be found by dimensional analysis. In this process we consider the dimensions of both sides of the functional equation above.

By convention we write: $L = \frac{1}{2}\rho V^2 S C_L$

where S is the reference area
and C_L is called the Lift Coefficient.

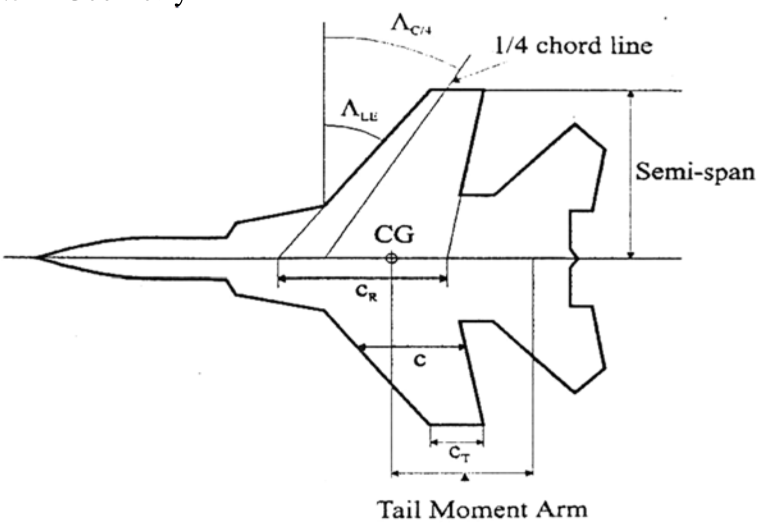
Similarly $D = \frac{1}{2}\rho V^2 S C_D$

where C_D is called the Drag Coefficient.

$$M = \frac{1}{2}\rho V^2 S l C_M$$

where C_M is called the Pitching Moment Coefficient,
and l is a reference length (e.g. the wing mean chord) .

2.5 Geometry



Reference Lengths:

i) Root chord c_R
ii) Mean chord $\bar{c}$
iii) Tail moment arm ℓ_T
iv) Wing span b

Reference Areas:

i) Wing plan area S_G or S_N
ii) Wetted area of component S_W

Shape Parameters:

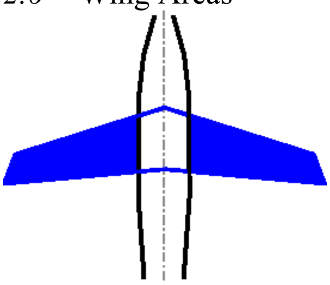
i) Taper ratio $\lambda = \dfrac{\text{Tip chord}}{\text{Root chord}} = \dfrac{c_T}{c_R}$

ii) Wing sweep Λ

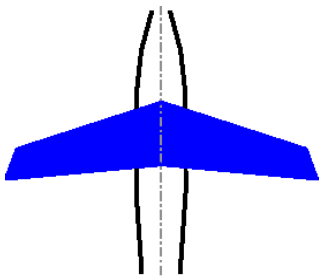
iii) Aspect ratio $A = \dfrac{\text{Span}}{\text{Mean chord}} = \dfrac{b}{\bar{c}}$

also $A = \dfrac{\text{Span}^2}{\text{Wing area}} = \dfrac{b^2}{S}$

2.6 Wing Areas



Shaded Area = Net Wing Area = S_N

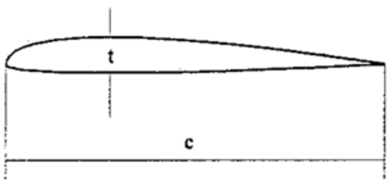


Shaded Area = Gross Wing Area = S_G

Both S_G and S_N are projected wing areas.
S_W is the 'wetted' area of the wing = total surface area of the upper and lower surfaces of the wing (S_W>2S_N due to wing thickness).

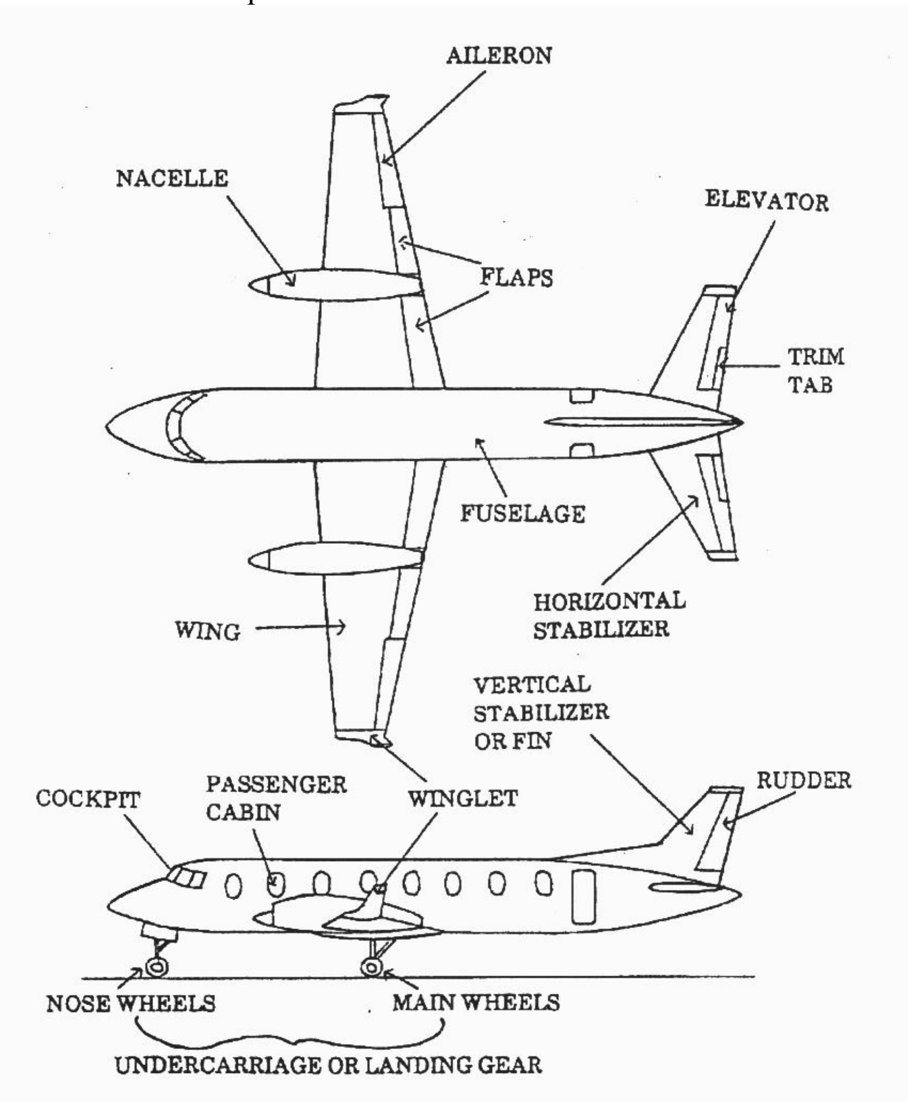
S_G is normally used as the Reference Area of the aircraft in defining the aerodynamic coefficients, etc.

2.7 Aerofoil



Maximum thickness = t
Chord = c

2.8 Aircraft Components



A twin turbo-prop regional passenger aircraft with tricycle undercarriage

3 TYPICAL AERODYNAMIC CHARACTERISTICS FOR CONVENTIONAL AIRCRAFT (Lectures 3, 4 & 5)

This section summarises the typical aerodynamic characteristics of a conventional aircraft with an unswept wing and conventional tailplane. Some typical values of the lift coefficient, the drag coefficient and the pitching moment coefficient are given, but it should be remembered that the specific aircraft shape will determine the precise values. It should be noted that the following descriptions do *not* include the effects of compressibility and therefore are not applicable to high subsonic, transonic and supersonic flight.

3.1 Lift Coefficient

For a conventional aircraft shape the lift coefficient increases approximately linearly with an increase in the angle of attack for angles up to about 10 to 15 degrees. At angles greater than this the character of the flow about the wing changes dramatically and the flow **separates** from the upper surface. The lift coefficient then stops rising before it reduces significantly and this is accompanied by a large increase in the drag coefficient. The aircraft is said to be **stalled.** Note that the **maximum C_L is about 1.2** – students should be **absolutely familiar** with this value as it is the starting point for most conventional aircraft design. The C_L can be increased above this level with various high lift devices but these all add weight and cost. For back of the envelope calculations assuming peak C_L of unity is a conservative place to start aircraft design.

A typical variation of lift coefficient with angle of attack for an aircraft in the 'clean configuration' (i.e. flaps, slats and undercarriage retracted) is shown in the following diagram:

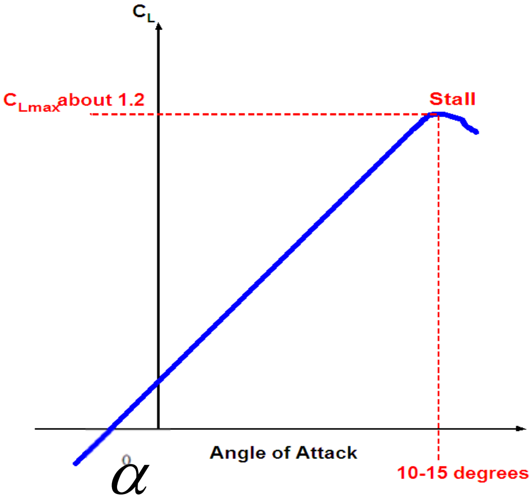


Fig. 3.1

α_o is called the **no lift angle** and is usually negative.

$a_o = dC_L/d\alpha$ is called the **lift curve slope** and is usually linear until near the stall**.**

Hence $$C_L = a_o(\alpha - \alpha_o)$$

Typically $$C_L = 0.09(\alpha + 2) \qquad \text{where } \alpha \text{ is in degrees}$$

or $$C_L = 5(\alpha + 2/57.3) \quad \text{where } \alpha \text{ is in radians}$$

Typical values of the lift coefficient when the aircraft is flying at its normal cruising speed are in the range of about 0.2 to 0.5. The reason for this is that when taking off and landing the aircraft will generally be flying significantly slower than at cruise and so will need to operate at a higher C_L value to keep airborne and clearly one cannot be too close to stall at such times.

So it is emphasized that Fig. 3.1 relates to the aircraft in the cruise configuration. When flaps and slats are used the values of C_{Lmax} which may be achieved can increase to values of 3.0 or more for sophisticated designs of flap. The angle of attack at which stall occurs is also increased. This enables the aircraft to fly at much lower speeds for take-off and landing.

3.2 Drag Coefficient

A typical shape of the variation of the drag coefficient C_D with angle of attack α is shown in Fig. 3.2.

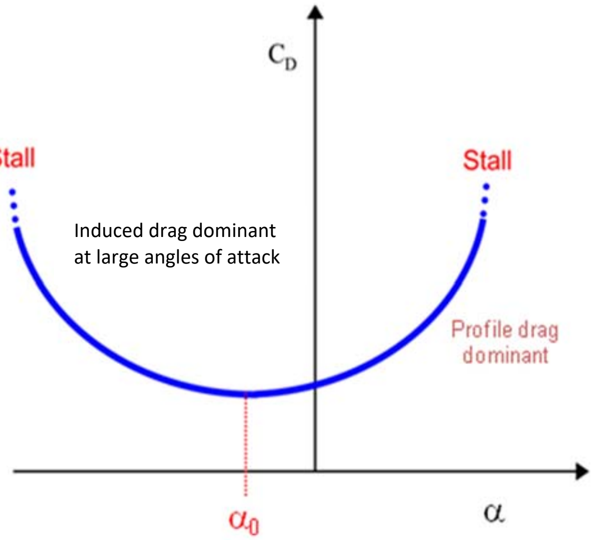


Fig 3.2.

The value of C_D at the zero value of the lift coefficient C_L is called the profile drag coefficient.

Hence $$C_D = C_{Do} + C_{Di}$$

where C_D is the total drag coefficient,
C_{Do} is the profile drag coefficient,
C_{Di} is the induced drag coefficient.

The profile drag arises from the boundary layer which forms on the aircraft, in which the effects of the viscosity of the air ('stickiness') are predominantly felt. Sometimes this is called the boundary layer drag. This part of the drag may be split into two parts:

- the skin friction drag acting on all the surfaces of the aircraft directly caused by the viscosity of the air and
- the drag due to the normal pressures acting on the surface of the aircraft which is known as the boundary layer normal pressure drag or the form drag.

In the absence of viscosity there would be no boundary layer adjacent to the surface and the normal pressures acting on the forward facing surfaces would exactly balance with those on rearward facing surfaces, resulting in no net drag force due to pressure. However, when the effect of viscosity is considered, the pressures do not balance due to the effective change of shape caused by the boundary layer and this results in the boundary layer normal pressure drag force. The magnitude of this form of drag is clearly dependent on how 'streamlined' the object is (i.e., on the object's shape or form).

The other contribution to the total drag is the induced drag D_i. This arises from the downwash to which the wing is subject due to the fact that it is generating lift. This downwash tilts the resultant force acting on the wing rearwards which gives a component of force in the streamwise direction known as the induced drag. The induced drag coefficient C_{Di} is alternatively called the lift dependent or vortex drag coefficient, since this is the part of the total drag coefficient associated with the amount of lift coefficient which is being generated. Aerodynamic theory shows that the induced drag coefficient increases in proportion to the square of the lift coefficient. This increase of drag coefficient is associated with the vortices which are shed from the wing tips when the wing generates lift. These vortices, from which the downwash results, increase in strength as the lift coefficient increases.

Thus, the induced drag coefficient may be written as

$$C_{Di} = kC_L^2$$

Furthermore, aerodynamic theory shows that the constant k depends on the wing aspect ratio A as follows:

$$k = K/\pi A$$

where K is a constant which depends on the planform shape of the wing and generally has values somewhat in excess of unity. For an elliptic planform wing (which has the minimum induced drag coefficient of all planforms) the value of K is exactly 1.0.

Hence the total drag coefficient may be written as:

$$C_D = C_{Do} + KC_L^2/\pi A$$

Typical values of the profile drag coefficient in the cruise configuration are in the range of about 0.01 to 0.025 for streamlined aircraft with undercarriage and high lift devices retracted. For smaller aircraft with non-retractable undercarriage rather higher values will pertain, perhaps as high as 0.055. So the total drag coefficient is typically

$$C_D = 0.02 + 0.055C_L^2$$

It is important to note that since lift varies with the square of speed at fixed angle of attach, the angle of attack has to be varied in flight as speed changes. This means that value of lift coefficient varies with speed as well (i.e., as the aircraft mass and thus total lift is fixed, a high C_L pertains at low speed and low C_L at high speed). Therefore the drag coefficient also varies with speed and it turns out that there is a particular speed that minimizes overall drag.

3.3 Pitching Moment Coefficient

Typical variations of the pitching moment coefficient C_m of a conventional aircraft *without the tailplane contribution*, for different axis positions, are shown in Fig 3.3 – note how the centre of pressure moves with angle of attack and the moment coefficient slope is dependent on the point it is taken about.

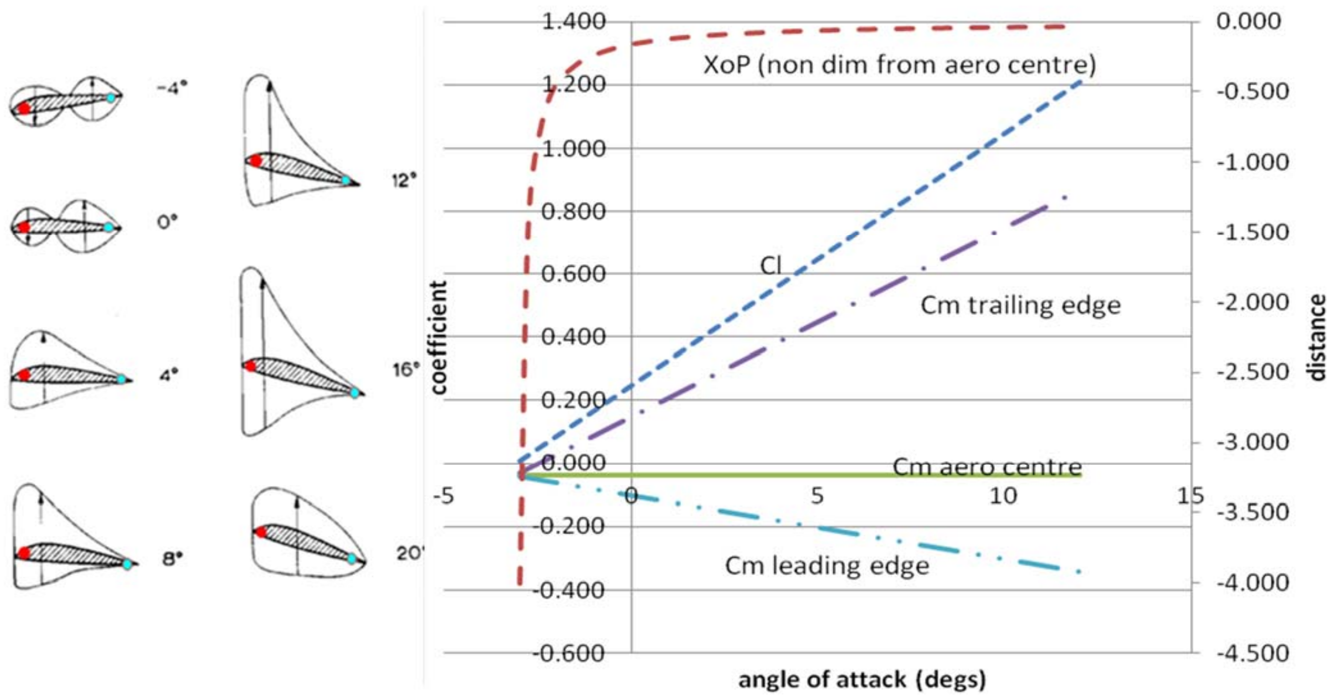


Fig. 3.3 – pitching moment variation with AoA

The variation of C_m with the lift coefficient C_L is then

$$C_m = C_{mo} - hC_L \quad \text{..(1)}$$

where h is a constant, the value and sign of which vary with the axis position (+ve for axis positions forward of the aerodynamic centre and negative for positions aft of the aerodynamic centre), and C_{mo} is usually negative (the aero centre is typically at ¼ chord).

C_{mo} is the pitching moment coefficient at zero lift and depends on the shape of the aircraft, but is usually negative (nose down). Note that an aircraft which is perfectly symmetric about its datum line through its nose would have a value of C_{mo} of zero.

For an aircraft to be trimmed in steady flight at a given speed (hence at a given value of C_L) there must be no pitching moment about the centre of gravity (axis position). Hence the pitching moment curve when plotted for the centre of gravity must cross the horizontal axis at the angle of attack that yields that value of C_L . Considering the case of rearward axis positions (CG), see Fig. 3.4, the aircraft without the tailplane is unstable (i.e. the pitching moment increases if the angle of attack is increased and vice versa). To trim the aircraft (C_{mcg} = 0) and to make it stable (–ve slope *of* C_{mcg} versus C_L) requires the addition of the tailplane which, for the axis (CG) position relevant to Fig. 3.4, is required to give a nose down pitching moment (up load on the tailplane) which must be added to the pitching moment without the tailplane as shown in the Fig. 3.4. Note that pitching moment which is required from the tailplane to give C_{mcg} = 0 varies as the value of C_L at which trim is required changes. This changing pitching moment is created by deflection of the trim tab, or the elevator, or by changing the angle of attack of the whole tailplane.

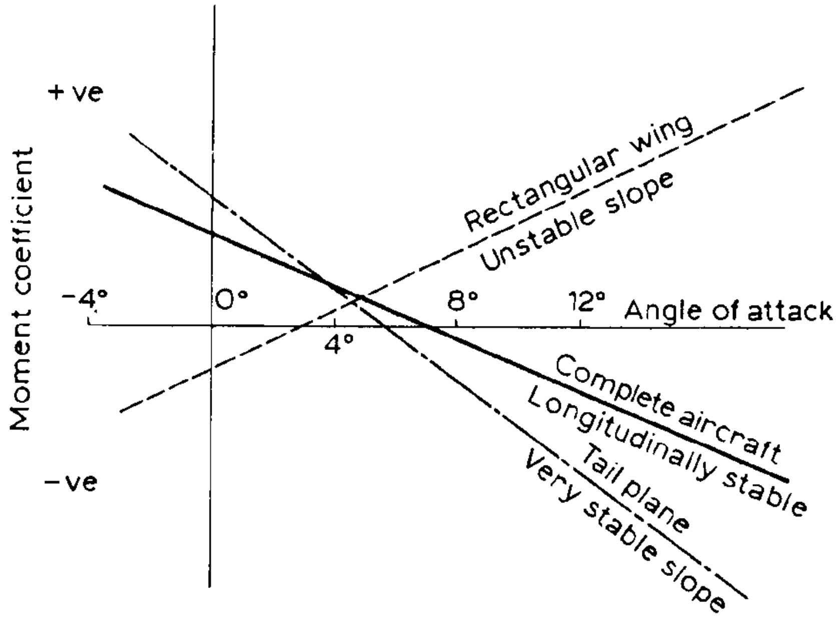


Fig. 3.4 Pitching moment coefficient about CoG

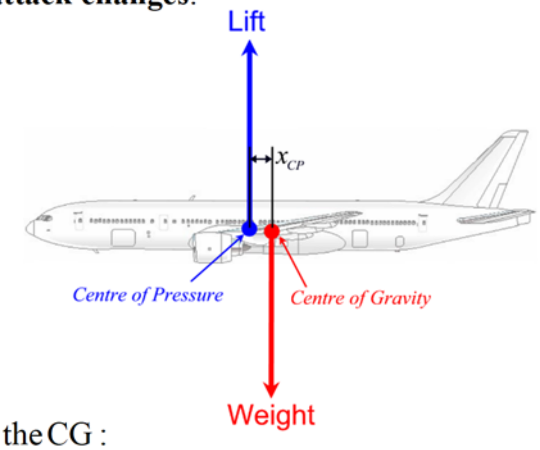
Two further definitions are introduced in the context of the pitching moment characteristics of the wing plus fuselage (less the tailplane) as follows.

3.4 Centre of Pressure

This is defined as the reference axis position about which the pitching moment is zero.

i.e. $M_{cp} = 0$

It is the position at which the resultant lift of the wing plus body appears to act with zero moment and **moves as the angle of attack changes**.



Taking moments about the CG :

$$M_{CG} = x_{CP} L$$

Dividing by $\frac{1}{2}\rho V^2 Sc$ and using Eqn 1 we obtain :

$$C_{m_{CG}} = \frac{x_{CP}}{c} C_L = C_{m_o} - hC_L$$

The result for the distance x_{cp} of the centre of pressure forward of the centre of gravity is:

$$x_{cp}/c = C_{mCG}/C_L \quad = \quad C_{mo}/C_L \; - h$$

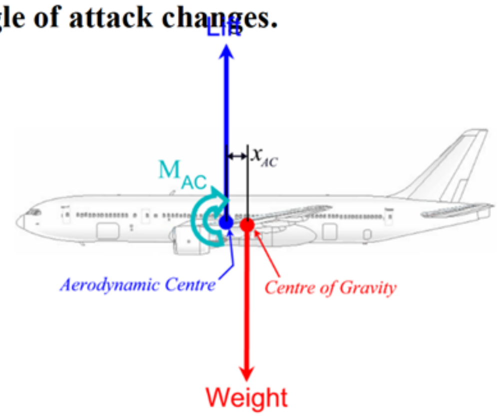
where c is the wing mean chord.
Note that for C_{mo} –ve this whole expression is negative and as its magnitude is reduced as C_L increases, the centre of pressure moves forward as the angle of attack increases.

3.5 Aerodynamic Centre

This is defined as the reference axis position about which the pitching moment coefficient does not vary with angle of attack.

i.e.
$$\frac{dM_{AC}}{d\alpha} = 0$$

This position is that at which the force and moment acting may be *replaced* by a constant pitching moment M_o (which is independent of α and hence L) together with an equivalent lift force acting at a changed location and which changes with angle of attack. **The position of the aerodynamic centre does not change as the angle of attack changes.**



Taking moments about the CG :

$$M_{CG} = M_o + x_{AC} L$$

Dividing by $\frac{1}{2}\rho V^2 Sc$ and using Eqn 1 we obtain :

$$C_{m_{CG}} = C_{m_o} + \frac{x_{AC}}{c} C_L = C_{m_o} - hC_L$$

Hence

$$\frac{x_{AC}}{c} = -h$$

More generally, the result for the distance x_{AC} of the aerodynamic centre forward of the centre of gravity is given by

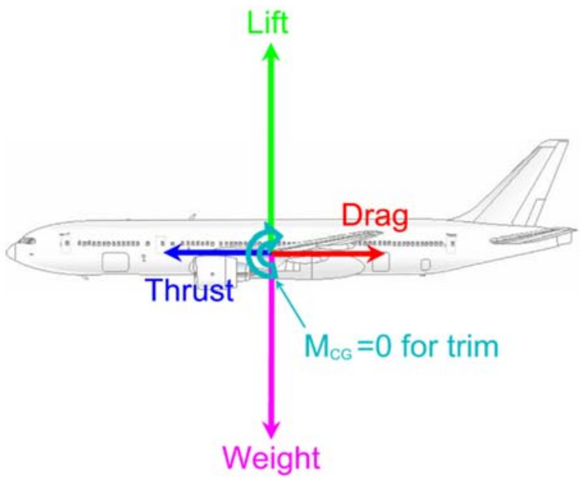
$$\frac{x_{AC}}{c} = \frac{dC_{m_{CG}}}{dC_L} = -h$$

Since h is a constant, this result confirms that the aerodynamic centre is at a fixed position. From classical aerodynamic theory, the position of the aerodynamic centre is at the quarter chord position measured from the leading edge of the wing.

4 PERFORMANCE – STEADY LEVEL FLIGHT (Lectures 6, 7 & 9)

4.1 Equilibrium of Forces and Moments

We will consider steady level flight at speed V. The forces and moments acting on the aircraft are the **Thrust, Drag, Lift, Weight** and **Pitching Moment**. For simplicity we will assume all of the forces act through the centre of gravity as shown in the diagram.



Equilibrium of forces and moments about the CG gives:

$L = W$
$T = D$
$M_{CG} = 0$

where L = Lift, D = Drag, T = Thrust, W = Weight and M_{CG} = Pitching Moment (+ve nose up)

Now $L = 0.5\rho V^2 S C_L = W$ where S is the reference area

Hence for an aircraft of given weight and wing area, operating at a constant height (constant air density), **the speed V determines the value of C_L** which is required. Since the lift coefficient C_L is largely a function of incidence (angle of attack) only, therefore **α is also determined**.

Rearranging the above equation, we can alternatively regard it as an expression which gives the value of the level flight speed in terms of W, S, C_L and ρ:

$$V = (2W/\rho S C_L)^{0.5} = (2w/\rho C_L)^{0.5}$$

where $w = W/S$ = **Wing Loading**.

4.2 Stalling Speed

The stalling speed is the minimum speed at which the aircraft may fly. This occurs when the lift coefficient is a maximum i.e. $C_L = C_{Lmax}$. For steady level flight, $L = W$ and thus:

$$V_S = (2W/\rho S C_{Lmax})^{0.5} = (2w/\rho C_{Lmax})^{0.5}$$

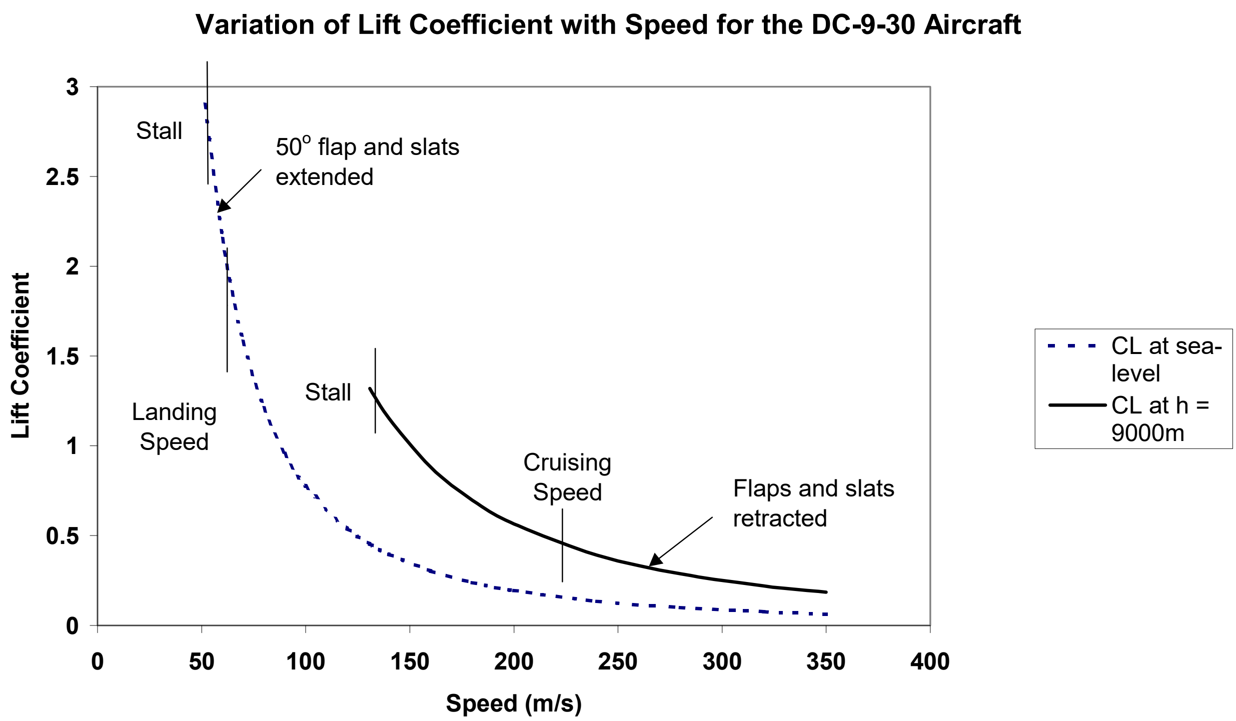
where V_S is the **Stalling Speed**

It should be noted that, for a given wing loading and maximum lift coefficient, the true air-speed at the stall increases as the air density ρ decreases, i.e., as the altitude increases. However, the equivalent airspeed (EAS) at the stall, $V_{S(EAS)}$, remains constant as altitude increases since $\rho V_S^2 = \rho_o V_{S(EAS)}^2$ and the sea-level air density ρ_o is constant.

Since the stalling speed is the minimum speed at which the aircraft can fly it is important from the point of view of the take-off and landing speeds.

The value of C_{Lmax} depends on the aircraft configuration. For high speed flight when the aircraft is in the cruise, for example, low drag is required and all high lift devices such as flaps and slats are retracted. In this 'clean' configuration the value of C_{Lmax} is likely to be about 1.2. For landing, it is necessary to have the lowest possible landing speed and high drag. This is achieved by deploying all the high lift devices to their greatest extent to increase C_{Lmax}. This may more than double the value of C_{Lmax} in comparison with the clean configuration, depending on the sophistication of the high lift system used. For take-off, partial deployment of high lift devices is used to provide the appropriate compromise between high lift and comparatively low drag.

The values of lift coefficient and speed in the cruise and landing configurations for the DC-9-30 civil transport aircraft are illustrated in the Figure below.



4.3 Level Flight Speeds

For flight at speed V the total aircraft drag may be written as

$$D = C_D \frac{1}{2}\rho V^2 S = \frac{1}{2}\rho V^2 S\left(C_{D_o} + \frac{K}{\pi A}C_L^2\right) = A'V^2 + \frac{B'}{V^2}$$

where $A'V^2$ is the profile drag
and B'/V^2 is the induced drag
and A' and B' are constants for flight at a fixed altitude.

$$A' = \frac{1}{2}\rho S C_{D_o} \quad \text{and} \quad B' = \frac{2KW^2}{\pi A \rho S}$$

For a jet engine, the thrust at a fixed altitude may be assumed to be almost constant with flight speed for a given r.p.m. setting. For steady level flight, thrust T is equal to drag D and we obtain the following quadratic equation for V^2:

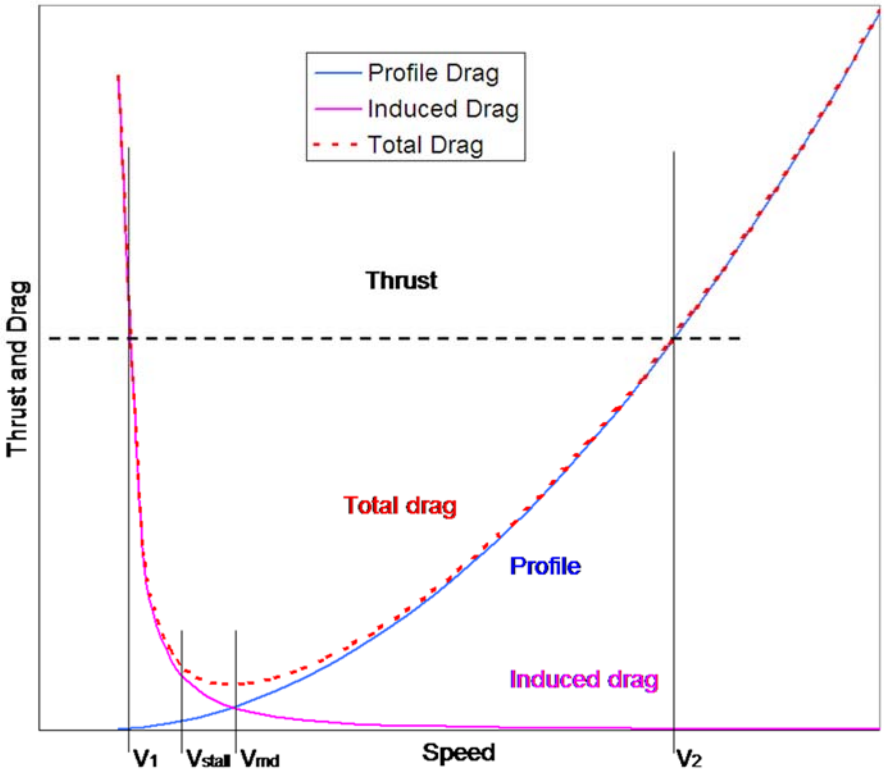
$$A'V^4 - TV^2 + B' = 0$$

For a given thrust T this has two solutions for V which are denoted by V_{i} and V_{ii}. These are the possible steady level flight speeds for the thrust level (r.p.m. setting) chosen. It is possible that the speed V_{i} may be less than the stalling speed.

These solutions are given by:

$$V^2 = \frac{T \pm \sqrt{T^2 - 4A'B'}}{2A'}$$

The solution to the thrust equals drag equation is shown graphically in the following diagram. In the diagram it is seen that the profile drag increases (proportional to V^2) and the induced drag decreases (proportional to $1/V^{2)}$ as speed increases. The total drag, which is the sum of the profile drag and the induced drag, passes through a minimum at a speed known as the Minimum Drag Speed (V_{MD}).



4.4 Minimum Drag Conditions

Since $$D = A'V^2 + \frac{B'}{V^2}$$

the minimum drag condition may be found by differentiating this equation with respect to *V* and putting the result equal to zero.

$$\frac{dD}{dV} = 2A'V - 2\frac{B'}{V^3} = 0$$

The result is $$V_{MD} = \sqrt[4]{\frac{B'}{A'}}$$

where V_{MD} is known as the Minimum Drag Speed.

The magnitude of the minimum drag is given by substituting the result for V_{MD} into the equation for D.

$$D_{\min} = A'\left(\frac{B'}{A'}\right)^{1/2} + B'\left(\frac{A'}{B'}\right)^{1/2} = 2(A'B')^{1/2}$$

where D_{min} is known as the Minimum Drag.

An important deduction from this result is that at the minimum drag speed

Profile Drag = $(A'B')^{1/2}$ = Induced Drag

$$\therefore C_{D_o} = \frac{K}{\pi A} C_L^2$$

A further important result is obtained by substituting for A' and B' in the expression for the minimum drag to obtain:

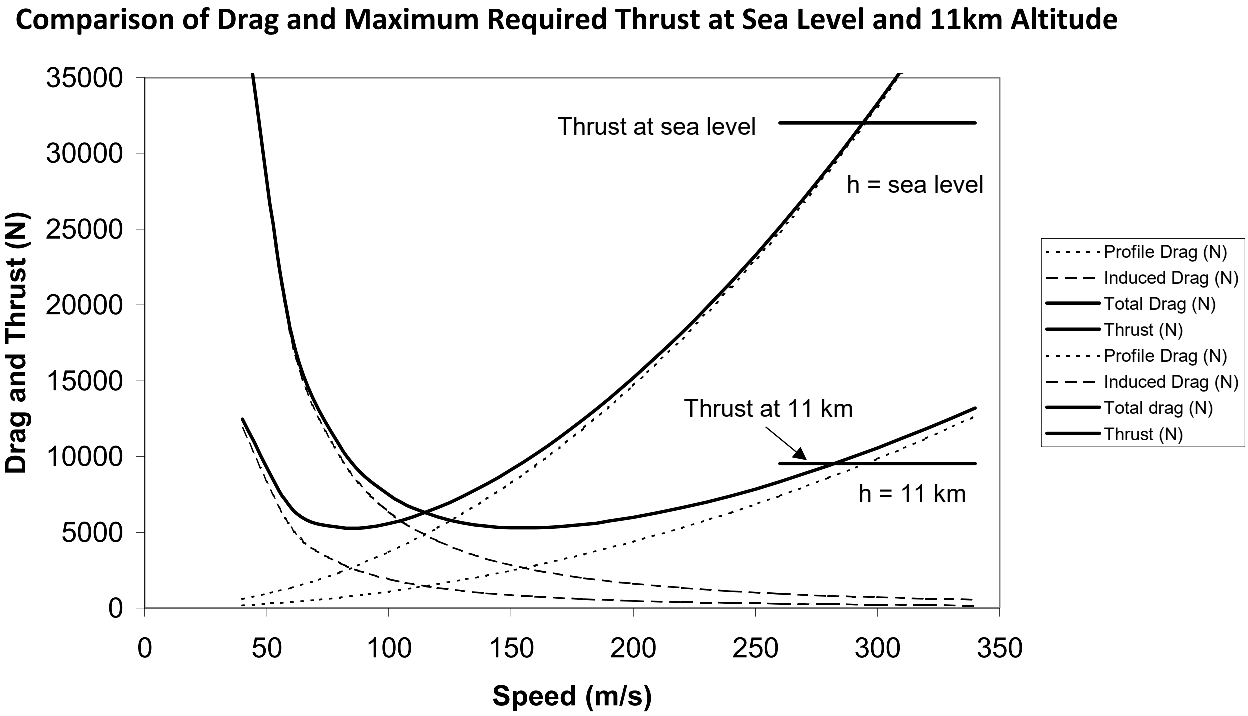
$$D_{\min} = 2W\left(\frac{KC_{D_o}}{\pi A}\right)^{1/2}$$

Note that this shows that the minimum drag is independent of the density ρ and is therefore independent of altitude.

An example of the variation of typical drag and thrust with true airspeed for a twin engined business jet aircraft of weight 90,000 N is shown in the following figure. The drag coefficient for this aircraft is given by:

$$C_D = 0.02 + 0.045 C_L^2$$

In this figure drag and thrust are compared at sea-level and at a cruising altitude of 11km. The independence of the minimum drag with altitude is demonstrated. Also shown is the large reduction in both thrust and drag during high altitude cruise. It should also be noted that at h = 11 km the true airspeed for minimum drag is much greater than at sea-level. The high Mach number cruise speed at high altitude would therefore be closer to the minimum drag speed than at sea-level and hence much greater flight efficiency would result. Since the fuel consumption of turbo-jet and turbo-fan engines is roughly proportional to the thrust, it is noted that the flight efficiency to fly at a particular true airspeed is much greater at high altitudes, providing that the Mach number at which the compressibility drag rise occurs is not exceeded as the speed of sound (proportional to the square root of temperature) falls as altitude increases.



4.5 Maximum Lift to Drag Ratio $(L/D)_{max}$

The lift to drag ratio is an important parameter which determines flight efficiency. Using the equation for minimum drag and noting that in steady level flight the lift (L) is equal to the weight (W) and that the maximum lift to drag ratio will occur when the drag is a minimum, we obtain the following result for the Maximum Lift to Drag Ratio:

$$\left(\frac{L}{D}\right)_{max} = \frac{W}{D_{min}} = \frac{1}{2}\sqrt{\frac{\pi A}{K C_{D_0}}}$$

4.6 Power Requirements

The power requirement to overcome the drag in steady level flight is the rate of doing work against the drag. This is equal to the force required to overcome the drag multiplied by the distance moved in unit time.

Thus $\qquad$ Power $= P = TV = DV = A'V^3 + B'/V$

The minimum power speed may be found by differentiating the expression for the power P with respect to speed V and putting the result equal to zero.

$$\frac{dP}{dV} = 3A'V^2 - \frac{B'}{V^2} = 0$$

The result for the Minimum Power Speed V_{MP} and its relationship to the Minimum Drag Speed V_{MD} is:

$$V_{MP} = \sqrt[4]{\frac{B'}{3A'}} = \frac{V_{MD}}{3^{1/4}} = 0.76 V_{MD}$$

i.e., the minimum power speed is less than the minimum drag speed.

5 ENDURANCE AND RANGE OF JET ENGINED AIRCRAFT (Lectures 9 & 10)

The following analysis will consider the cruise phase of flight only and will ignore the fuel used in take-off/climb/descent/landing/taxiing, etc., and the fuel required for reserves.

The important factor here is the rate at which fuel is being used. The characteristics of turbo-jet engines are such that the specific fuel consumption (s.f.c.) is reasonably independent of forward speed, but is a function of altitude.

The specific fuel consumption for a turbo-jet engine is defined here as:

s.f.c = s = weight of fuel used per unit time to produce unit thrust (note this is NOT the SFC used for piston engine aircraft and more generally by mechanical engineers when considering stand-alone engines, which links fuel consumption to power and not thrust).

Hence the rate at which the aircraft weight decreases as the cruise proceeds is:

$$\frac{dW(t)}{dt} = -sT$$

where $W(t)$ is the total aircraft weight at time t
and T is the engine thrust.

For equilibrium during the cruise

Thrust T = Drag D
Lift L = Weight W

Hence $\dfrac{dW(t)}{dt} = -sT = -sD = -s\dfrac{LD}{L} = -s\dfrac{WD}{L}$

Now $L/D = C_L/C_D$

and $\dfrac{dW(t)}{dt} = -s\dfrac{WC_D}{C_L}$

For the moment it is assumed that <u>flight takes place at constant C_L/C_D</u>. This implies flight at constant angle of attack (incidence) since C_D is also a function of C_L together with some constants and C_L is just a function of angle of attack. It should be noted that in order for lift to balance weight, the lift must reduce as the aircraft uses fuel and the weight decreases. Since

$$L = W = 0.5\rho V^2 SC_L$$

at least one of altitude (hence density ρ), speed V, or, angle of attack α (hence C_L) must change as the cruise progresses in order to achieve this. Flight at constant L/D implies that if the altitude is constant the speed must decrease, or, if the speed is constant the altitude must increase. As we shall see later various cruise patterns can be considered, including that where the aircraft climbs steadily as it burns fuel.

Using the assumption of constant C_L/C_D, the endurance may be found by integrating the expression for $dW(t)/dt$ as follows:

$$\int \frac{dW}{W} = -s\frac{D}{L}\int dt \qquad\qquad \therefore \ln W = -s\frac{C_D}{C_L}t + \text{constant}$$

When $t=0$, $\quad W = W_S + W_f$

Where:

- W_s = weight of dry aircraft
- W_f = weight of fuel

$$\therefore \text{constant} = \ln(W_s + W_f)$$

At any time t $\quad t = -\frac{1}{s}\frac{C_L}{C_D}\ln\frac{W}{W_s + W_f}$

The result for the endurance T' is

$$T' = \frac{1}{s}\frac{L}{D}\ln\left(1 + \frac{W_f}{W_s}\right)$$

where $\quad W_f$ is the weight of fuel used, and
W_s is the total remaining weight of the aircraft.

For maximum endurance, the following conditions must be satisfied:

Low s	Propulsion
Maximum L/D	Aerodynamics
Maximum W_f/W_s	Structural Design

Flight at constant C_L/C_D implies constant angle of attack (incidence). If the altitude is held constant the speed must decrease as the weight decreases. This implies that the r.p.m. (throttle) setting must also change during the cruise since drag decreases and hence thrust decreases. Such a cruise pattern is seldom acceptable in practice for passenger flights.

Since angle of attack is not generally displayed it is more normal to fly at either:
constant speed
constant r.p.m. (throttle) setting – constant thrust

Maximum endurance at a given altitude occurs at $(L/D)_{\max}$, i.e. at minimum drag.

5.1 Range

To calculate the range we return to:

$$\frac{dW(t)}{dt} = -s\frac{WD}{L}$$
$$\frac{dW(t)}{dt} = \frac{dW}{dx}\frac{dx}{dt} = V\frac{dW}{dx}$$

where $\quad x$ = distance

and V = velocity.

$$\frac{dW}{dx} = -\frac{s}{V}\frac{D}{L}W$$

Assuming that $\frac{V}{s}\frac{L}{D}$ is held constant during the cruise, the range R obtained as fuel of weight W_f is used is given by:

$$\int dx = -\frac{V}{s}\frac{L}{D}\int \frac{dW}{W}$$

Integrating gives $x = -\frac{V}{s}\frac{L}{D}\ln W +$ constant, which can be eliminated by setting x to zero and W to W_s+W_f at the start of the journey and x to R and W to W_s at the end:

$$R = \frac{V}{s}\frac{L}{D}\ln(1+\frac{W_f}{W_s})$$

This equation is known as the Breguet Range Equation for a jet engined aircraft. It assumes that the cruise is performed at constant *VL/D*. Many other cruise modes are possible for which the range will be somewhat different from the Breguet range. For maximum range the following are required:

Low s	efficient propulsion,
Maximum *VL/D*	good aerodynamics,
Maximum W_f / W_s	good structural design.

Note that maximum range and maximum endurance do not occur at the same speed.

Maximum endurance	- maximum *L/D*
Maximum range	- maximum *VL/D*.

5.2 Range in the stratosphere

In the stratosphere the speed of sound (a) is constant since the temperature is constant.

Thus $V = Ma$

and the range R is given by:

$$R = \frac{a}{s}\frac{ML}{D}\ln(1+\frac{W_f}{W_s})$$

In the stratosphere the range is a maximum when *ML/D* is a maximum, so we fly at the highest possible Mach Number commensurate with avoiding drag due to compressibility affects (shock drag caused by shock waves forming on the wings). For current generation jet aircraft this is typically around M=0.82.

5.3 Cruise-Climb Technique for a Jet Engined Aircraft

During the cruise, lift is equal to weight and hence

$$L = W = 0.5\rho V^2 SC_L$$

The three variables which determine the lift are:

ρ density (function of height)
V true air speed
C_L lift coefficient (function of angle of attack)

For this particular cruise technique it is assumed that the true air speed V and the angle of attack (hence C_L) are held constant throughout the cruise.

N.B. if C_L is constant then

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{D_o} + {KC_L^2}/{\pi A}} \quad \text{is also constant.}$$

Thus in this particular case both V and L/D are constant throughout the flight. In addition, for a jet engined aircraft the specific fuel consumption s is usually assumed constant throughout the flight. Thus VL/sD is constant and the range is given by the Breguet Range Equation:

$$R = \frac{V}{s}\frac{L}{D}\ln(1+\frac{W_f}{W_s}) = \frac{V}{s}\frac{L}{D}\ln(\frac{W_1}{W_2})$$

where W_1 = weight at start of cruise = $W_f + W_s$
and W_2 = weight at end of cruise = W_s

In the stratosphere the temperature is assumed constant and the thrust T of a turbo-jet engine at a fixed throttle setting is proportional to the relative density σ, ignoring any variation with forward speed. Thus

$$T_2/T_1 = \sigma_2/\sigma_1 \qquad \text{as the height varies.}$$

During the cruise the weight decreases as fuel is used, but at all times the lift is equal to the weight.

Since V and C_L are both constant during the cruise, then as weight decreases so the relative density σ decreases and height increases. This is called a cruise – climb technique.

$$W = L = \frac{1}{2}\rho V^2 SC_L$$

$$\text{i.e. } W = \frac{1}{2}(\rho_o\sigma)V^2 SC_L$$

Thus Drag = Thrust at all times throughout the flight.

In summary, as the fuel is consumed, decreasing the aircraft weight, the aircraft gains altitude with the decreased lift balancing the decreased weight. In the stratosphere this gain in altitude automatically decreases the thrust as the air density decreases, maintaining thrust equal to drag without any adjustment to the flight condition of the aircraft. In the stratosphere a constant value of the true air speed V corresponds to a constant Mach number M, since the speed of sound is constant. The technique is to fly at constant Mach number and allow the aircraft to gradually "drift up" in altitude as fuel is used. The main long haul airlanes allow for this approach.

5.4 Maximising the Breguet Range

It is possible to find an optimum cruising speed for the cruise – climb technique to give the maximum Breguet range for a given weight of fuel used. To find the optimum it is necessary to find the value of V which gives the maximum value of the factor VL/D which appears in the Breguet range equation. This is done by expressing VL/D in terms of C_L and differentiating this with respect to C_L to find the value of C_L corresponding to a maximum as follows.

In the stratosphere, with a jet engine operating at fixed r.p.m. setting, the thrust is approximately proportional to the relative density σ (= ρ/ρ_o).

Hence $\quad T/\sigma = \text{constant} = k$

But $\quad T = D = 0.5\rho V^2 S C_D = 0.5(\rho_o\sigma)V^2 S C_D = k\sigma$

Hence $\quad V^2 C_D = \text{constant}$

and $\quad V$ is proportional to $1/C_D^{1/2}$

Hence $\quad \dfrac{VL}{D}$ is proportional to $\dfrac{1}{C_D^{1/2}}\dfrac{C_L}{C_D}$

To find the value of C_L which gives the maximum Breguet range we need to differentiate $\dfrac{1}{C_D^{1/2}}\dfrac{C_L}{C_D}$ with respect to C_L and put the result equal to zero.

$$\frac{d}{dC_L}\left(\frac{C_L}{C_D^{3/2}}\right)=\frac{d}{dC_L}\left(\frac{C_L}{\left(C_{D_o}+KC_L^{\,2}/\pi A\right)^{3/2}}\right)$$

$$\therefore\frac{d}{dC_L}\left(\frac{C_L}{C_D^{3/2}}\right)=\frac{\left(C_{D_0}+KC_L^{\,2}/\pi A\right)^{3/2}-C_L\,{}^{3}\!/_{2}\left(C_{D_0}+KC_L^{\,2}/\pi A\right)^{1/2}\,{}^{2KC_L}\!/_{\pi A}}{\left(C_{D_0}+KC_L^{\,2}/\pi A\right)^{3}}$$

Dividing by $\left(C_{D_o}+KC_L^{\,2}/\pi A\right)^{1/2}$ we obtain

$$\frac{d}{dC_L}\left(\frac{C_L}{C_D^{3/2}}\right)=\frac{C_{D_0}+KC_L^{\,2}/\pi A-3KC_L^{\,2}/\pi A}{\left(C_{D_0}+KC_L^{\,2}/\pi A\right)^{5/2}}=\frac{C_{D_0}-2KC_L^{\,2}/\pi A}{\left(C_{D_0}+KC_L^{\,2}/\pi A\right)^{5/2}}$$

This is zero when $C_{D_0}=2KC_L^{\,2}/\pi A$

$$\therefore C_L=\sqrt{\frac{C_{D_0}}{2K/\pi A}}=\frac{C_{L_{MD}}}{\sqrt{2}}$$

But since C_L is inversely proportional to V^2 then

$$V_{\text{optimum range}}=\sqrt[4]{2}\times V_{MD}$$

where the subscript MD refers to minimum drag conditions.

Hence the cruising speed that gives best range is $2^{1/4}$ × the minimum drag speed. Of course the cruising pattern adopted will very much depend on the type of mission being flown.

5.5 Effect of Altitude on Optimum Cruising Speed

In the earlier section of the notes headed ‘Level Flight Speeds’ it was shown that the true airspeed at which the minimum drag speed occurred increased with altitude, but the value of the minimum drag was independent of altitude. Thus at high altitudes, the optimum cruising speed will occur at higher true airspeeds than at low altitude. In the Figure shown there, drag and thrust were compared at sea-level and at a cruising altitude of 11km. This showed the large reduction in both drag and thrust during high altitude cruise at a given true airspeed. It was also noted that at h = 11 km the true airspeed for minimum drag was much greater than at sea-level. The high Mach number cruise speed at high altitude would therefore be closer to the minimum drag speed, and hence the optimum cruising speed, than at sea-level and hence much greater flight efficiency would result. Since the fuel consumption of turbo-jet and turbo-fan engines is roughly proportional to the thrust, it is noted that the flight efficiency to fly at a particular true airspeed is much greater at high altitudes, providing that the Mach number at which the compressibility drag rise occurs is not exceeded as the speed of sound (proportional to the square root of temperature) falls as altitude increases. For optimum Breguet range we therefore need to fly at $V=2^{1/4}\,V_{MD}$ at high altitude.

6 RANGE AT CONSTANT ALTITUDE AND CONSTANT ANGLE OF ATTACK (Lectures 11, 12 & 13)

For the Breguet Cruise Pattern, that has already been considered, the following assumptions were made:

- V = constant
- L/D = constant (α = constant)
- ρ ~ decreases (h ~ increases)

Many alternative cruise patterns are possible, amongst which is cruise at constant altitude h and constant L/D (implies constant angle of attack). For this cruise pattern the following assumptions are made:

- L/D = constant (constant angle of attack)
- ρ = constant (constant altitude)
- V ~ decreases

To calculate the range it must be recognised that as the weight decreases as fuel is used, the speed of the aircraft must decrease if lift is to balance weight at constant
i.e., at constant C_L).

$$L = W = \frac{1}{2}\rho V^2 S C_L$$

$$\therefore V = \left(\frac{2W}{\rho S C_L}\right)^{1/2}$$

For a turbo jet engined aircraft:

$$\frac{dW}{dt} = -sT = -sD \quad \text{where } s \text{ is the specific fuel consumption}$$

$$\therefore \frac{dW}{dx}\frac{dx}{dt} = \frac{dW}{dx}V = -sD$$

Using $\quad L = W = \frac{1}{2}\rho V^2 S C_L \quad \text{and D} = \frac{1}{2}\rho V^2 S C_D$

$$\frac{dW}{dx} = -\frac{s\frac{1}{2}\rho V^2 S C_D}{V} = -\frac{s}{2}\rho V S C_D = -\frac{s}{2}\rho\left(\frac{2W}{\rho S C_L}\right)^{1/2} S C_D = -s\left(\frac{\rho W S}{2}\right)^{1/2}\frac{C_D}{C_L^{1/2}}$$

$$\therefore dx = -\frac{1}{s}\left(\frac{2}{\rho S}\right)^{1/2}\frac{C_L^{1/2}}{C_D}\frac{dW}{W^{1/2}}$$

$$\therefore \text{RANGE} = \int_1^2 dx = -\frac{1}{s}\left(\frac{2}{\rho S}\right)^{1/2}\frac{C_L^{1/2}}{C_D}\int_1^2 \frac{dW}{W^{1/2}}$$

$$\therefore \text{RANGE} = x_2 - x_1 = \frac{2}{s}\left(\frac{2}{\rho S}\right)^{1/2} \frac{C_L^{1/2}}{C_D}\left[W_1^{1/2} - W_2^{1/2}\right]$$

where W_1 and W_2 are the initial and final weights.

To find the maximum range for this cruise pattern it is necessary to find when $C_L^{1/2}/C_D$ is a maximum.

$$\frac{d}{dC_L}\left(\frac{C_L^{1/2}}{C_D}\right) = \frac{d}{dC_L}\left(\frac{C_L^{1/2}}{C_{D_o} + \frac{K}{\pi A}C_L^2}\right)$$

$$= \frac{\left(C_{D_o} + \frac{K}{\pi A}C_L^2\right)\frac{1}{2}C_L^{-1/2} - C_L^{1/2}\frac{2K}{\pi A}C_L}{\left(C_{D_o} + \frac{K}{\pi A}C_L^2\right)^2} = 0$$

when $\quad C_{D_0} = \dfrac{3K}{\pi A}C_L^2$

i.e. when **Profile Drag = 3 times Induced Drag**

$$\therefore \qquad C_{L_{\text{CRUISE}}} = \frac{1}{\sqrt{3}}\sqrt{\frac{C_{D_o}}{K/\pi A}} = \frac{C_{L_{\text{MINDRAG}}}}{\sqrt{3}}$$

But $\quad V \propto \dfrac{1}{C_L^{1/2}}$

Therefore $V_{CRUISE} = \sqrt[4]{3} \times V_{MIN\ DRAG}$

So Maximum Range Cruising Speed = $(3)^{1/4}$ times Minimum Drag Speed for cruise at constant α and constant altitude.

6.1 Effect of Compressibility on Range

It must be emphasised that the above analysis has not taken account of compressibility drag. The conditions for optimum range give values of C_L and optimum cruise heights which are lower than those used in practice. When the compressibility drag rise that occurs at high Mach numbers is taken into account, it is desirable to keep increasing altitude at almost constant Mach number until drag is minimised. As altitude is increased the density falls and the value of C_L needed to balance the weight increases. Thus, in practice, the values of C_L for the optimum range will be greater than indicated by an analysis that ignores compressibility.

A further factor that affects the analysis is that although the assumption of constant specific fuel consumption (s.f.c.) is quite good for turbojets, the s.f.c. of turbofans increases slightly with Mach number and this leads to the conditions for optimum range being at a somewhat lower Mach number than would result from the assumption of constant s.f.c.

Fig 6.1 includes the effect of compressibility for a trijet wide body aircraft and shows the variation of

L/D and *ML/D* (the factor in the specific range - see section on 'Endurance and Range of Jet Engined Aircraft') with C_L. This shows that range is increased by both high *L/D* and high speed. The best *L/D* of 16.4 is achieved at $M = 0.75$ at $C_L = 0.46$, but the best range (proportional to *ML/D*) is obtained at $M = 0.825$ when $C_L = 0.46$. The graph also shows that the range is reduced by only 1% if C_L is reduced to 0.405. Since, for a given weight, changing C_L implies changing altitude, this shows that the range of the aircraft is relatively insensitive to altitude. The graph also shows that if the Mach number is increased to 0.85, the range is decreased from the optimum by only 3% whereas the lift to drag ratio is reduced by 9%.

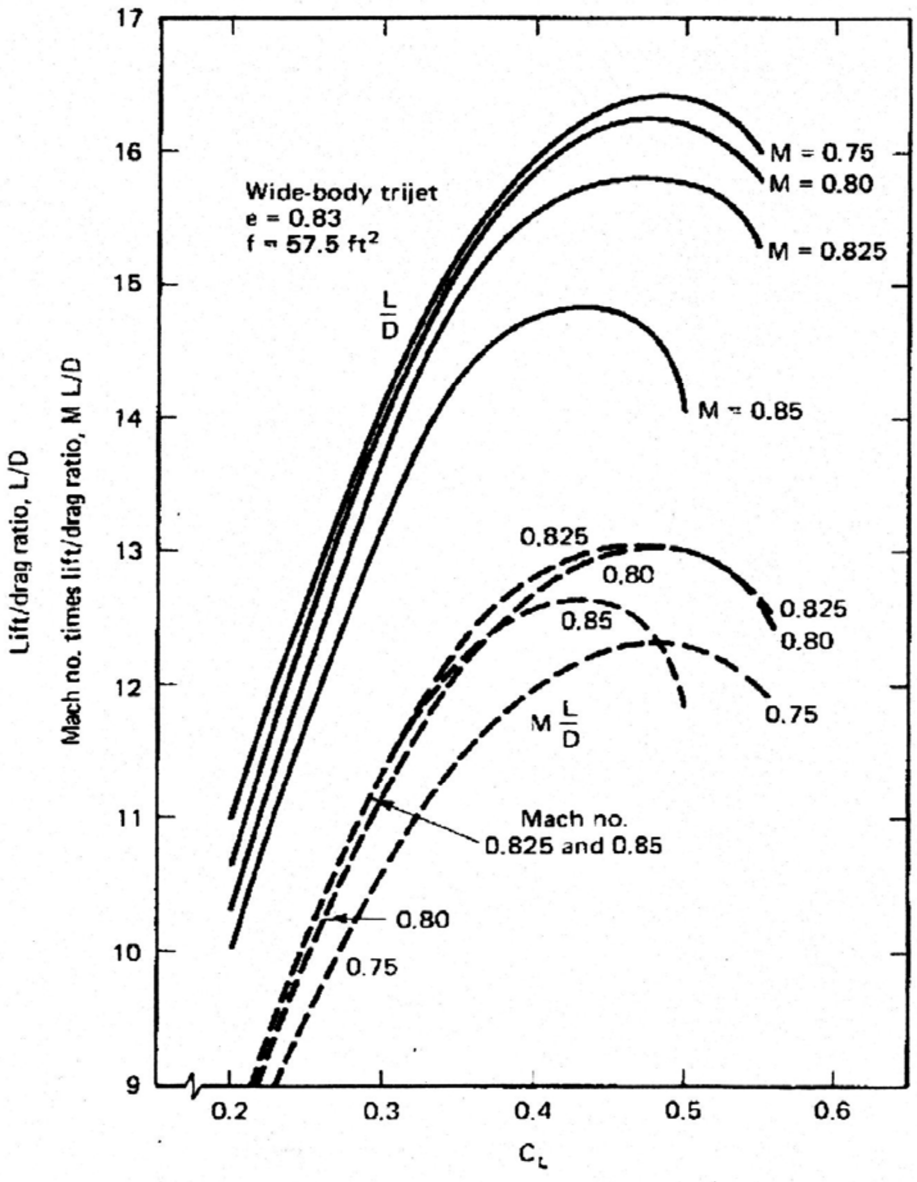


Fig 6.1

(reproduced from 'Fundamentals of Flight' by R S Shevell - see booklist)

6.2 Range of a Piston Engined Aircraft

In the case of a piston engined aircraft the important characteristic which determines the range is the specific fuel consumption but this is defined in a different way from that for a jet engined aircraft. For the piston engine it is defined as:

s' = weight of fuel used in unit time to produce unit power.

Now the power required to overcome the drag at speed V is:

$$P = DV$$

and the power output required from the engine is :

$$P' = DV/\eta$$

where η is the combined efficiency of the transmission and the propeller.

For a piston engined aircraft in steady level flight the rate of decrease of weight as fuel is used is therefore:

$$\frac{dW(t)}{dt} = -\frac{s'DV}{\eta} = -\frac{s'}{\eta}\frac{D}{L}WV$$

Also

$$\frac{dW(t)}{dt} = \frac{dW}{dx}\frac{dx}{dt} = \frac{dW}{dx}V$$

Hence $$\frac{dW(t)}{dx} = -\frac{s'}{\eta}\frac{D}{L}W$$

or $$dx = -\frac{\eta}{s'}\frac{L}{D}\frac{dW}{W}$$

The specific fuel consumption s' and the efficiency η can usually be taken to be constant. It will also be assumed that the lift to drag ratio (L/D) remains constant during the cruise. In this case the range R using fuel of weight W_f is given by:

$$R = (\eta/s')(L/D)\ \ln(1 + W_f/W_s)$$

or $$R = (\eta/s')(L/D)\ \ln(W_1/W_2)$$

where W_1 = initial weight = $W_s + W_f$

and W_2 = final weight $= W_s$

The above equations are known as the Breguet Range Equations for Piston Engined Aircraft

The maximum range occurs when the lift to drag ratio (L/D) is a maximum, i.e. at the minimum drag speed, provided that η and s' are constant.

Using the definition of the specific fuel consumption, it can be seen that the maximum endurance occurs at the minimum power speed.

It should be noted that in practice the efficiency η falls at higher speeds due to the high tip speeds of the propeller and the consequent compressibility effects.

Also, it is usual to cruise at intermediate heights at speeds greater than the speed for minimum drag (say at 4 - 8 km).

6.3 Optimum Cruise Performance Techniques

6.3.1 Jet Engined Aircraft

Maximum Range:

a) At a given fixed altitude:

the best cruising speed is $(3)^{1/4}$ × Minimum Drag Speed.

For this cruise pattern the angle of attack and the altitude are constant and the speed decreases as fuel is used.

b) For Cruise Climb in the stratosphere:

the best cruising speed is $(2)^{1/4}$ × Minimum Drag Speed.

Maximum Endurance:

Achieved at $(L/D)_{max}$, i.e. at the Minimum Drag Speed.

Both maximum range and maximum endurance for turbo-jet and turbo-fan powered aircraft are achieved at high altitude.

6.3.2 Piston Engined Aircraft

Maximum Range:

Occurs at $(L/D)_{max}$, i.e at the Minimum Drag Speed, unless η/s' falls off rapidly with increase of speed.

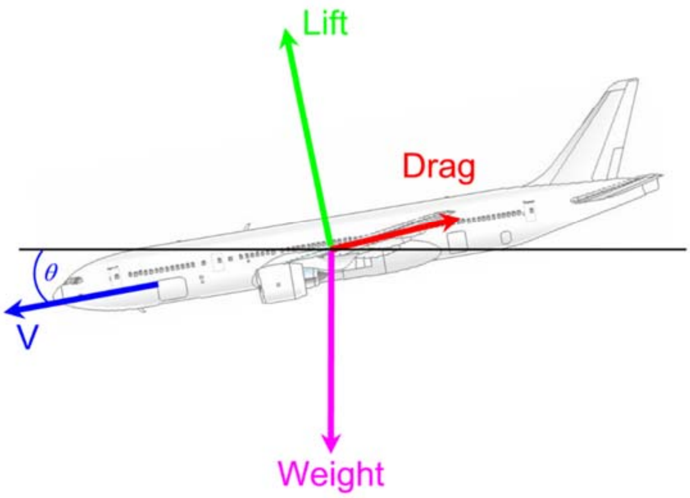
Maximum Endurance:

Achieved at Minimum Power Speed

7 SIMPLE ACCELERATED FLIGHT (LECTURES 8, 14, 15 & 16)

7.1 Gliding Flight without Thrust

Consider an aircraft in a steady glide without thrust. We wish to determine the glide angle and the rate of descent. The angle of the glide is θ and the true air speed of the aircraft is V. The forces acting on the aircraft are weight W, lift L and drag D.



Equilibrium of forces (aerodynamic forces balanced against weight) gives:

$$L = W\cos\theta$$

$$D = W\sin\theta$$

Hence $$\tan\theta = D/L$$

$$\therefore \quad \text{Glide angle } \theta = \tan^{-1} D/L \approx D/L \quad \text{for small values of } \theta$$

The glide angle is therefore a minimum when L/D is a maximum. This occurs at the minimum drag condition (see previous notes).

For an aircraft for which the drag coefficient is given by:

$$C_D = a + bC_L^2$$

the minimum drag condition occurs when the profile drag is equal to the induced drag (see previous sections of the notes), i.e., when

$$a = bC_L^2$$

and hence for a shallow glide

$$\theta_{\min} \approx \frac{D_{\min}}{L} = \frac{C_{D_{MD}}}{C_L} = \frac{2a}{\left(a/b\right)^{1/2}} = 2(ab)^{1/2}$$

The rate of descent is given by *–dh/dt*, where h is the height.

$$-\frac{dh}{dt} = -\dot{h} = V\sin\theta \approx V\frac{D}{W}$$

Thus the minimum rate of descent occurs when *VD* is a minimum. This is equivalent to the minimum power condition (see previous sections). At this condition the induced drag is three times the profile drag and for an aircraft whose drag coefficient is of the form given we get:

$$3a = bC_L^2$$

Hence

$$\theta_{\dot{h}\,\mathrm{min}} \approx \frac{D_{\mathrm{min\ power}}}{L} = \frac{C_{D_{MP}}}{C_{L\,MP}} = \frac{a+3a}{\left(3a/b\right)^{1/2}} = \frac{4}{\sqrt{3}}(ab)^{1/2}$$

The speed at this angle of descent for shallow glide angles (*L* = *W*) is given by:

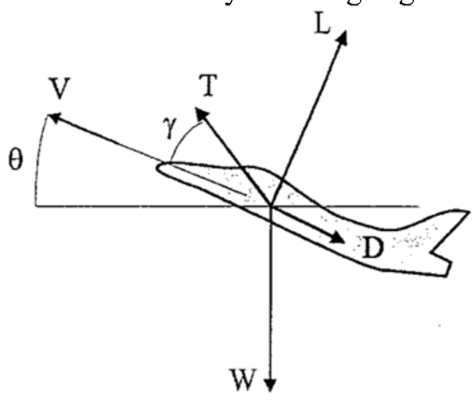
$$V_{\dot{h}\,\mathrm{min}} = \left(\frac{2W}{\rho S C_{L_{MP}}}\right)^{1/2} = \left(\frac{2W}{\rho S}\right)^{1/2}\left(\frac{b}{3a}\right)^{1/4}$$

And the minimum rate of descent is therefore:

$$-\frac{dh}{dt} = -\dot{h} = V_{\dot{h}\,\mathrm{min}}\theta_{\dot{h}\,\mathrm{min}}$$

7.2 Steady Climbing Flight

The forces acting on the aircraft in steady climbing flight are shown in the following figure:



θ is the climb angle and γ is the thrust vector angle.

Equilibrium of forces gives:

$$L + T\sin\gamma = W\cos\theta$$
$$T\cos\gamma - D = W\sin\theta$$
$$\text{and the Rate of Climb} = \frac{dh}{dt} = V\sin\theta$$

where h is height and t is time.

$$\frac{dh}{dt} = \frac{V(T\cos\gamma - D)}{W}$$

If the thrust vector angle is zero then

$$\frac{dh}{dt} = \frac{V(T - D)}{W}$$

$\dfrac{T-D}{W}$ is known as the SPECIFIC EXCESS THRUST

To calculate the angle of a shallow climb for zero thrust vector angle (γ=0, cosθ=1, L=W) at a given speed V, C_L is found from

$$C_L = \frac{2W}{\rho V^2 S}$$

which in turn determines the drag coefficient from $C_D = a + bC_L^2$

and the drag D from $D = {}^1/_2\rho V^2 SC_D$.

The climb angle is then found from the specific excess thrust as

$$\theta = \sin^{-1}\left(\frac{T-D}{W}\right)$$

or for small angles, $\theta=(T–D)/W$, where T is the thrust.

The rate of climb is $\qquad \dfrac{dh}{dt} = V\theta$

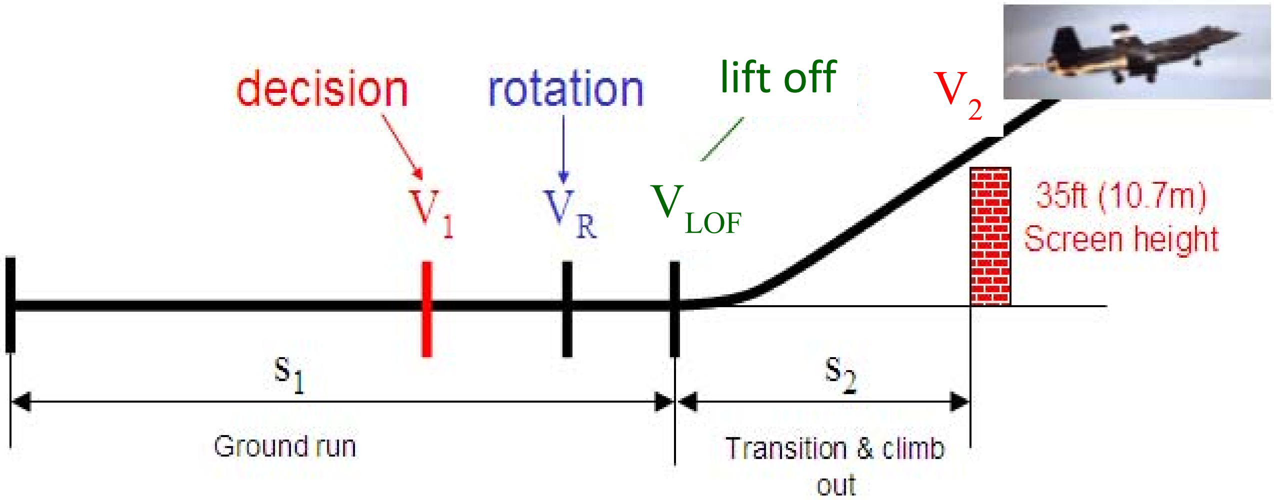
where θ is in radians.

It should be noted that the above analysis holds only for shallow climb angles where the rates of climb are relatively small. Many high performance aircraft have very high values of the thrust to weight ratio (*T/W*) and this can be in excess of unity, in which case vertical climb is possible. When the climb rates are very high the climb cannot be considered 'steady' since the density in the above analysis will be changing with time. The calculation of optimum climb rates can then become quite complex and is not considered here.

7.3 Take-Off

During take-off the aircraft must accelerate from rest through several critical speeds V_1, V_R, V_{LOF} and V_2. The first critical speed is the decision speed V_1 at which, if an engine fails, it is possible for the aircraft to decelerate to rest within the available runway. After this speed has been reached it is necessary to proceed with the take-off. The next critical speed is the rotation speed V_R, at which the pilot increases the angle of attack and the lift acting on the aircraft rapidly increases. The aircraft continues to accelerate to the lift-off speed V_{LOF} at which the aircraft leaves the ground (this is greater than the unstick speed, where maximum lift available just permits safe take-off). This is followed by a transition phase to a constant climb angle. The aircraft continues to accelerate during this climb phase to the screen speed V_2 at the point where the aircraft reaches the imaginary screen height of 35 ft (10.7 m). The take-off distance is considered to be the total distance from the start of take-off roll to the screen, i.e. $s_1 + s_2$.

It should be noted that during the take-off run the aircraft will be in the take-off configuration with flaps and slats deployed at their take-off settings. This gives greater C_L at take-off than would be available in the "clean" configuration and hence a lower stalling speed. The flap angles used during take-off are much less than during landing so that the increase of drag is not excessive.



7.3.1 Definitions:

V_s The stalling speed in the take-off configuration.

V_{mu} Minimum Unstick Speed. Minimum airspeed at which airplane can safely lift off ground and continue take-off.

V_{mca} Minimum Control Speed. Minimum airspeed at which when critical engine is made inoperative, it is still possible to recover control of the airplane and maintain straight flight.

V_{mcg} Minimum control speed on the ground. At this speed the aircraft must be able to continue a straight path down the runway with a failed engine, without relying on nose gear reactions.

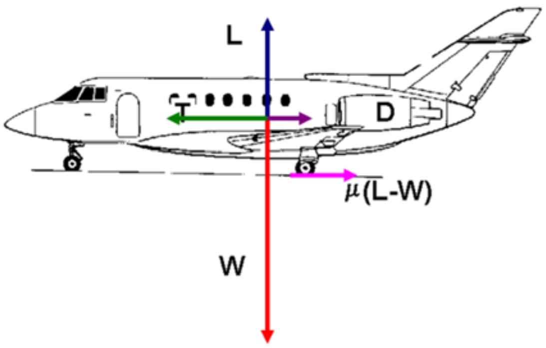
V_1 Decision speed, a short time after critical engine failure speed. Above this speed, aerodynamic controls alone must be adequate to proceed safely with takeoff.

V_R Rotation Speed. Must be greater than V_1 and greater than 1.05 V_{mca}

V_{LOF} Lift-off Speed. Must be greater than 1.1 V_{mu} with all engines, or 1.05 V_{mu} with engine out.

V_2 Take-off climb speed is the demonstrated airspeed at the 35 ft height. Must be greater than 1.1 V_{mca} and 1.2 V_s, the stalling speed in the take-off configuration (for jet engine aircraft, 1.15 V_s for piston engine aircraft)†

The forces acting on the aircraft during the take-off run are shown in the following diagram. The resistance due to friction on the runway is $\mu(W - L)$.



The equation of motion in the x direction directed along the runway is:

$$T - D - \mu(W - L) = \frac{W}{g}\frac{dV}{dt} = \frac{W}{g}V\frac{dV}{dx}$$

or
$$\frac{T}{W} - \frac{D}{W} - \mu\left(1 - \frac{L}{W}\right) = \frac{V}{g}\frac{dV}{dx}$$

where D and L are the ground run values which are a function of angle of attack and speed.

Strictly, to find the take off distance to reach the speed V_2 the values of L and D must be substituted in the equation of motion which must then be integrated from 0 to V_2. An approximate estimate of the distance s_1 can be found by making the following assumptions:

T/W = constant

D/W and L/W are both very small during the ground run prior to lift-off.

The equation of motion then becomes:

$$\frac{T}{W} - \mu = \frac{V}{g}\frac{dV}{dx}$$

$$\therefore \int_0^{s_1}\left(\frac{T}{W} - \mu\right)dx = \int_0^{V_2}\frac{V}{g}dV$$

$$\therefore s_1 = \frac{1}{\left(T/W - \mu\right)}\frac{V_2^2}{2g}$$

† more formally, the speed a twin-engined aircraft should be capable of reaching at a screen height of 35ft following a critical engine failure at V_1. In what follows we will assume acceleration stops at lift off, so that $V_2 \approx V_{LOF}$

Taking: $V_2 = 1.2V_s$

and noting $V_s = \sqrt{\dfrac{2W}{\rho S C_{L_{max}}}}$

the ground run s_1 to accelerate from $V = 0$ to $V = V_2$ is :

$$s_1 = \frac{1.44}{g}\frac{W}{\rho S C_{L_{max}}}\frac{W/T}{\left(1-\mu W/T\right)}$$

where C_{Lmax} is the maximum lift coefficient with the aircraft in the take-off configuration in "ground effect".

Note that for a given value of C_{Lmax} the ground run distance s_1 depends on

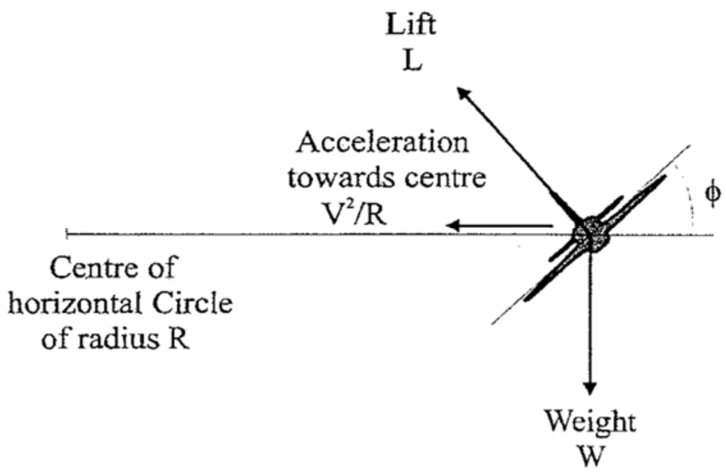
W/S	= wing loading	low W/S results in short s_1
T/W	= thrust/weight ratio	high T/W results in short s_1
ρ	= density	low ρ results in longer s_1 (take-off from hot, high altitude airports requires more runway length)

All of the above analysis assumes still air and a horizontal runway. Taking off into wind and on a downward sloping runway will reduce the take-off distance.

7.4 Turning Performance

In order to change the direction of flight the aircraft must be able to carry out turning manoeuvres. When in steady level flight, the aircraft can be manoeuvred into turning flight by deflecting the ailerons alone, or by using the rudder alone, or by a combination of ailerons and the rudder. If ailerons alone are used the aircraft always has an angle of "sideslip" - this increases the drag compared with the no-sideslip case. In the case of a rudder only turn, the rudder deflection induces an angle of roll which inclines the lift vector and causes the flight path to be curved. In the rudder only turn the relative wind approaches the aircraft from outside the turn and the aircraft is said to be "skidding". If the rudder and ailerons are used together, turning flight can be achieved in which the aircraft is always aligned with the relative wind and this turn is called a co-ordinated turn. In the analysis which follows a co-ordinated turn in the horizontal plane will be considered.

When the aircraft is in a steady turn the wings make an angle to the horizon ϕ that is known as the bank angle. The lift vector is tilted from the vertical by an amount equal to the bank angle and now has both vertical and horizontal components. When in equilibrium, the vertical component balances the weight and the horizontal component provides the force required to cause the aircraft to accelerate towards the centre of the turning circle. This is illustrated in the following diagram:



The forces acting on the aircraft in the turn are lift L, weight W, thrust T and drag D. In equilibrium:

$$T = D$$

$$L\cos\phi = W$$

$$L\sin\phi = (W/g)(V^2/R)$$

$$L^2\sin^2\phi + L^2\cos^2\phi = W^2 + \left(\frac{W}{g}\frac{V^2}{R}\right)^2$$

$$\therefore L = W\sqrt{1+\left(\frac{V^2}{gR}\right)^2}$$

The result for the radius of turn R is :

$$R = \frac{V^2}{g}\frac{1}{\sqrt{\left(L/W\right)^2 - 1}}$$

or $$R = (V^2/g)/(n^2 - 1)^{1/2}$$

where $n = L/W$ is the load factor

Alternatively, the radius of turn may be expressed in terms of the speed and the bank angle.

$$L\sin\phi = \frac{W}{g}\frac{V^2}{R}$$

$$R = \frac{WV^2}{gL\sin\phi} = \frac{V^2}{g}\frac{1}{n\sin\phi}$$

but $$L\cos\phi = W$$

This gives the result

$$R = \frac{V^2}{g}\cot\phi$$

Since the lift acting on the aircraft in the turn is n times the weight, the stalling speed in the turn for a given $C_{L\,max}$ is increased above its steady level flight value.

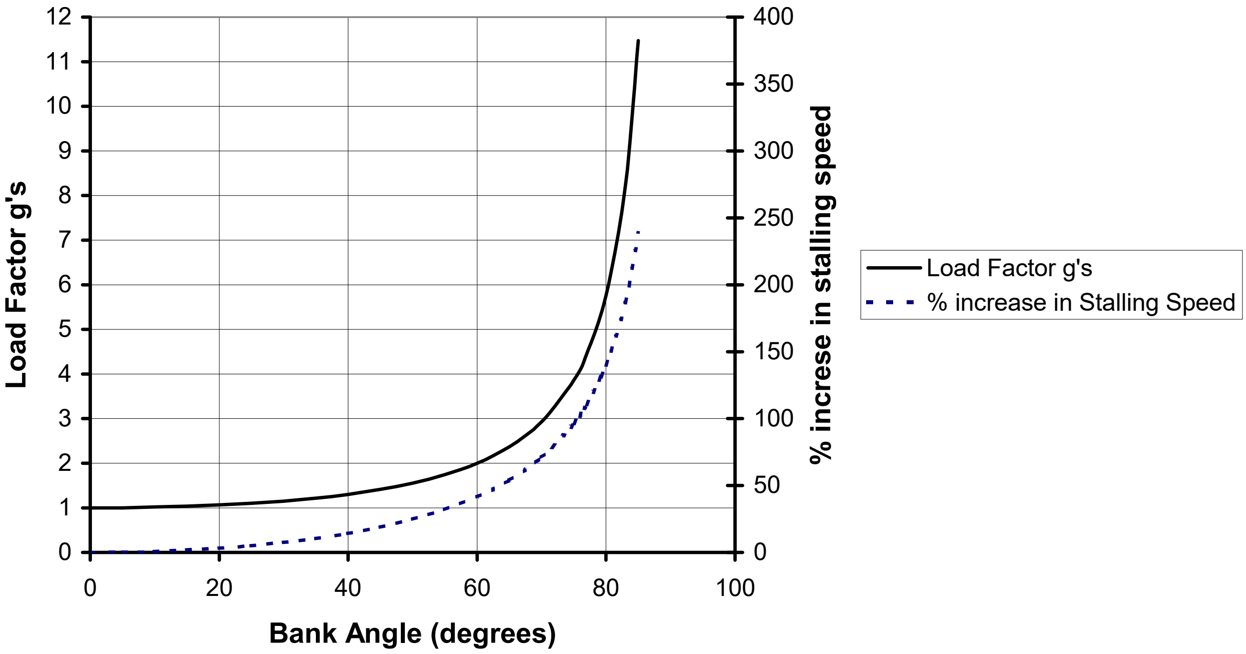
In the turn: $$V_{S_T} = \left(\frac{nW}{\frac{1}{2}\rho S C_{L_{max}}}\right)^{1/2}$$

In the steady level flight: $$V_S = \left(\frac{W}{\frac{1}{2}\rho S C_{L_{max}}}\right)^{1/2}$$

Hence the ratio of stalling speeds is: $\dfrac{V_{S_T}}{V_S} = \sqrt{n}$

The relationships of i) the load factor in a co-ordinated turn and ii) the ratio of the stalling speed in the turn to that in steady level flight, with the bank angle are illustrated in the following figure:

Increase of Load Factor and Stalling Speed in Co-ordinated Turns



7.4.1 Minimum Radius of Turn

From the previous work on turning performance it can be seen that for a given speed V and weight W the radius of turn R will be a minimum when

1) Lift L is a maximum
2) Load factor n is a maximum
3) Angle of bank ϕ is a maximum

The load factor limit arises when the ratio of lift to weight reaches a value n_{max} at which the structural strength limit of the wing is reached (allowing for a safety factor). In this case the limiting radius of turn is given by

$$R_{min} = \frac{V^2}{g} \frac{1}{\sqrt{n_{max}^2 - 1}} \qquad \text{Load Factor Limit}$$

The lift limit is reached when the wing is at the point of stalling. This will arise when the wing is operating at $C_{L\max}$ and the wing lift is given by

$$L = 0.5\rho V^2 S C_{L\max}$$

In straight and level flight at a lift coefficient equal to $C_{L\max}$

$$W = 0.5\rho V_s^2 S C_{L\max}$$

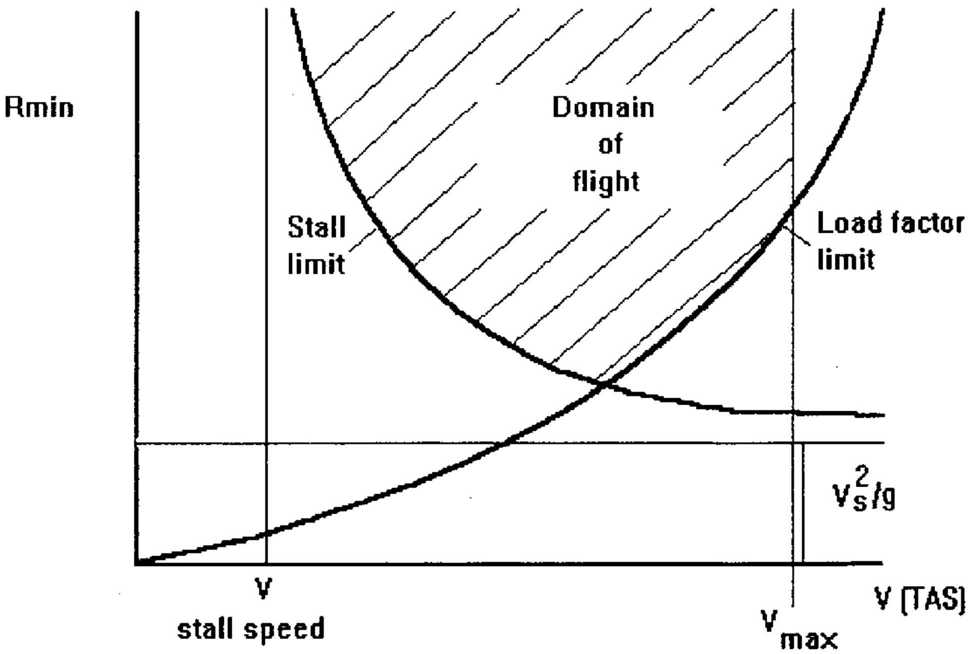
where V_s is the level flight stalling speed.

$$\therefore R_{\min} = \frac{V^2}{g}\frac{1}{\sqrt{\left(\frac{0.5\rho V^2 S C_{L_{\max}}}{0.5\rho V_s^2 S C_{L\max}}\right)^2 - 1}}$$

Hence $$R_{\min} = \frac{V_s^2}{g}\left(\frac{V}{V_s}\right)^2 \frac{1}{\sqrt{\left(V/V_s\right)^4 - 1}}$$ Maximum Lift or Stall Limit

It may be noted from this result that if the speed of the aircraft in the turn is equal to the level flight stalling speed then the minimum radius of turn is infinite, i.e. the aircraft has no manoeuvrability.

Using the above two results, a diagram may be drawn showing the domain of turning flight which is achievable.



To the left of the stall limit the value of C_L required is in excess of $C_{L\max}$. Flight is therefore not possible to the left of the stall limit. To the right of the load factor limit the value of n would be greater than $n_{\max}$ and the structural limit of the wing would be exceeded. Flight is therefore not possible to the right of the load factor limit. The domain of flight is limited at the highest speed by the maximum design speed of the aircraft which is shown as $V_{\max}$ on the diagram.

It may also be seen from the diagram above that the absolute minimum radius of turn occurs when the stall limit and load factor limit coincide.

A further implication is that since V_s increases with altitude, $R_{\min}$ increases with altitude and the manoeuvrability decreases.

7.4.2 Thrust limited radius of turn

At high speeds in the turn a limit on the minimum radius of turn $R_{\min}$ can occur due to the thrust or power limitation. This is a consequence of the fact that since the lift must exceed the weight during the turn, the value of C_L in the turn is greater than the value of C_L in steady level flight at the same speed as in the turn. Since induced drag is proportional to $C_L^{\ 2}$, this means that there is an increase in induced drag in the turn compared with steady level flight at the same speed. With this increase in drag, the thrust or power limits of the engine will be exceeded in the turn at a lower speed than would be the case in steady level flight.

It is necessary to calculate the drag in the turn and equate this to the maximum thrust available in order to find the thrust limited radius of turn.

$$D = T_{\max}$$

$$\therefore qS(C_{D_o} + kC_L^2) = T_{\max}$$

where $\quad q = \frac{1}{2}\rho V^2 \quad$ and $\quad k = \frac{K}{\pi A}$

But $\quad L = qSC_L = W\sqrt{1+\left(V^2/gR_{\min}\right)^2}$

$$\therefore C_L^2 = \frac{W^2}{q^2S^2}\left(1+\left[\frac{V^2}{gR_{\min}}\right]^2\right)$$

$$\therefore qS\left\{C_{D_o} + \frac{kW^2}{q^2S^2}\left(1+\left[\frac{V^2}{gR_{\min}}\right]^2\right)\right\} = T_{\max}$$

$$\therefore qSC_{D_o} + \frac{kW^2}{qS} + \frac{kW^2}{qS}\left(\frac{V^2}{gR_{\min}}\right)^2 = T_{\max}$$

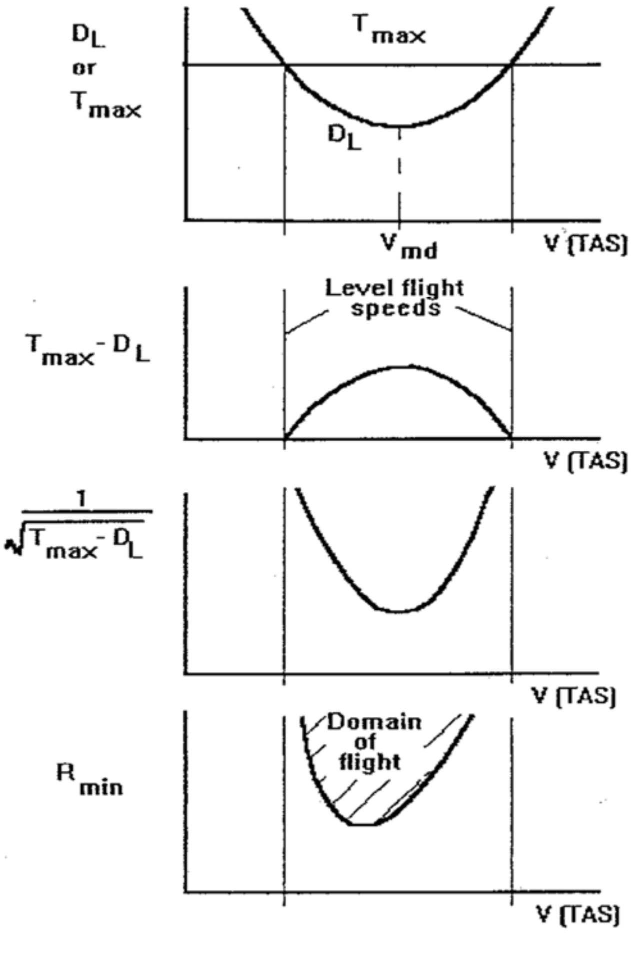
$$\text{i.e. } \frac{kW^2}{qS}\left(\frac{V^2}{gR_{\min}}\right)^2 = T_{\max} - qSC_{D_o} - \frac{kW^2}{qS}$$

But drag in level flight at speed $V = D_L = qSC_{D_o} + \dfrac{kW^2}{qS}$

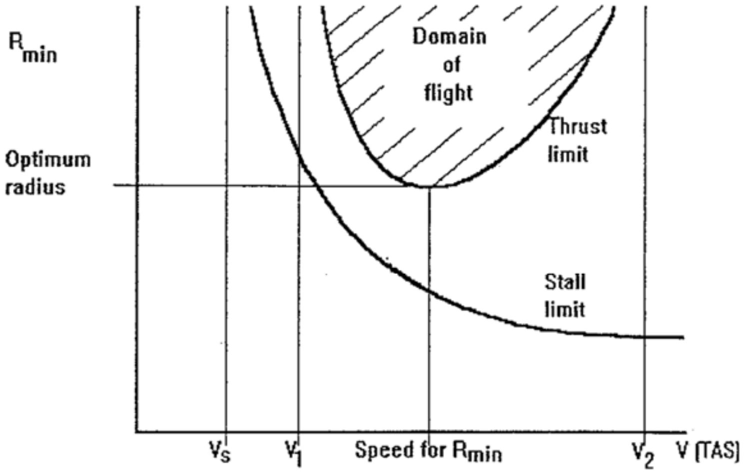
The result for the thrust limited radius of turn is

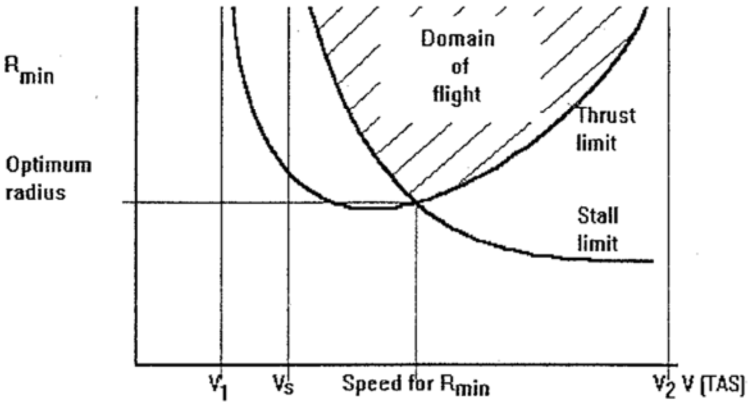
$$R_{\min} = \left(\frac{2kW^2V^2}{\rho Sg^2}\right)^{1/2} \frac{1}{\sqrt{T_{\max} - D_L}} \qquad \text{Thrust Limit}$$

The effect of the excess of thrust over the drag in level flight at the same speed as in the turn is illustrated in the following sequence of diagrams.



In addition to the above thrust limitation on the radius of turn the previously considered stall limit must also be examined to find which of the limits will occur in practice. Two possibilities exist which are shown diagrammatically in the following two diagrams:





In the first of these diagrams the aircraft would experience the thrust limit as the radius of turn is reduced at all flight speeds. In the second diagram the radius of turn would be limited by the stall at speeds below the speed for the optimum radius, but at higher speeds the aircraft would be limited by the maximum thrust available.

8 INTRODUCTION TO LONGITUDINAL STABILITY (LECTURES 17 & 18)

8.1 Trim, Static Stability and Control

8.1.1 Trim

An aircraft is in trimmed flight when, in a steady flight condition, all the forces and moments are balanced. The steady flight condition may involve a steady acceleration e.g. a correctly banked turn or a steady dive or climb. In the pitching plane, trim would be accomplished by deflecting the horizontal stabiliser, the elevator, or the elevator trim tab, to achieve zero pitching moment about the centre of gravity.

i.e. $$M_G = 0 \quad \text{for Trim}$$

It should be noted that it is possible for an aircraft to be in a trimmed state but it may not necessarily be a stable state; i.e. all the forces and moments may be balanced, but as soon as the state is perturbed the aircraft departs from equilibrium.

8.1.2 Static Stability

Stability is defined as the ability of an aircraft to return to a given equilibrium state after a disturbance (this might be caused by a gust or a small movement of the control surfaces).

An aircraft is said to be statically stable when, if it is disturbed from its equilibrium state by a small displacement, the set of forces and moments so caused initially tend to return the aircraft to its original state.

For longitudinal static stability this requires:

$$\frac{dC_{mG}}{d\alpha} < 0$$

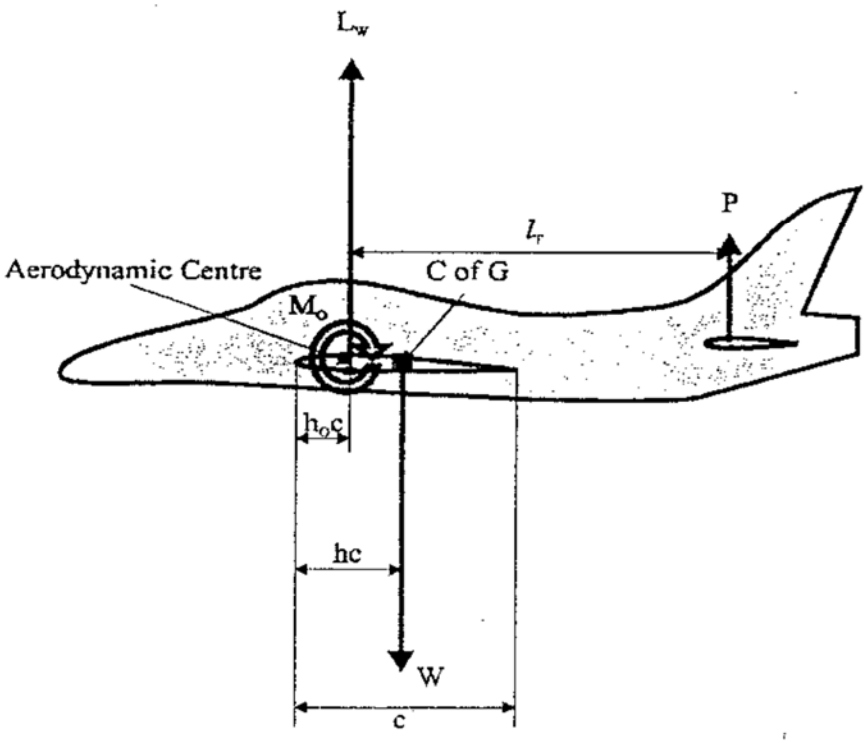
i.e. the slope of the variation of the pitching moment about the centre of gravity with the angle of attack (or C_L) must be negative.

8.1.3 Control

Control is defined as the ability to change from one flight condition to another and to correct for the effect of disturbances.

In general, six degrees of freedom (translation in the x, y and z directions and the respective rotations about these axes) are required to define the motion of the aircraft. In order to introduce the ideas of trim and stability only motion in the pitching plane (nose-up and nose-down motion) will be considered at this stage. In this case consideration will be given to longitudinal trim and longitudinal stability.

To study these topics it is necessary to consider the forces and moments which act on the aircraft in the pitching plane. These are shown in the following diagram:



In the above diagram it is assumed that thrust T is equal to drag D and that they both act through the centre of gravity (C of G) and make no contribution to the pitching moment.

L_w = Lift of the wing plus body excluding the horizontal stabiliser
P = Lift of the horizontal stabiliser alone
M_o = Pitching moment at zero lift
W = Aircraft weight
c = Reference wing chord
$(h - h_o)c$ = Distance between the aerodynamic centre and the centre of gravity
h = Fraction of the wing chord from the leading edge of the centre of gravity position
h_o = Fraction of the wing chord from the leading edge of the aerodynamic centre position
S_T = Reference area of the horizontal stabiliser
S = Reference wing area
l_T = distance of centre of lift of horizontal stabiliser behind aerodynamic centre

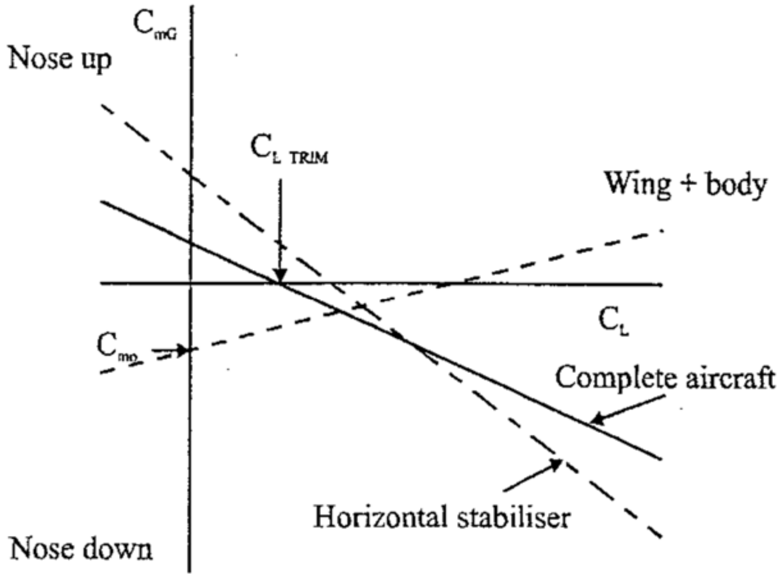
Taking moments about the centre of gravity:

$$M_G = \{M_o + (h - h_o)cL_w\} - \{P[l_T - (h - h_o)c]\}$$

This consists of two contributions from a) the wing plus body, and b) the horizontal stabiliser. For the C of G position shown in the previous diagram, the wing plus body contribution will increase, as C_L

increases in a nose up sense, from C_{mo} at $C_L = 0$. The horizontal stabiliser contribution provides an increasing nose down moment as C_L increases, starting from a nose up value at $C_L = 0$.

This diagram below shows that for the wing plus body alone, the slope of the pitching moment curve is positive, indicating longitudinal static instability. The addition of the horizontal stabiliser results in a pitching moment curve for the complete aircraft which has a negative slope, indicating that the complete aircraft is statically stable. The curve also intersects the C_L axis at the value of C_L at which the aircraft would be trimmed ($C_{mG} = 0$). A different value of the angle of the horizontal stabiliser (or the elevator or trim tab) would change the horizontal stabiliser pitching moment curve causing the complete aircraft pitching moment curve to intersect (i.e. trim the aircraft) at a different value of C_L.



The previous equation can be converted to coefficients by dividing by ½ ρ V^2Sc and noting that the lift coefficient for the horizontal stabiliser C_{LT} is defined on its reference area S_T and neglecting the tailplane moment since it is very small.
The result for the pitching moment coefficient about the centre of gravity is :

$$C_{mG} = C_{mo} + (h - h_o)(C_{Lw} + C_{LT}S_T/S) - (S_T l_T/Sc)C_{LT}$$

where $\overline{V} = \dfrac{S_T l_T}{Sc} =$ Tail Volume Coefficient or Tail Volume Ratio

Vertical equilibrium gives

$$L = L_w + P = W = 0.5\rho V^2 SC_L$$

i.e. $C_L = W/0.5\rho V^2 S$

and $C_L = C_{Lw} + C_{LT}S_T/S$

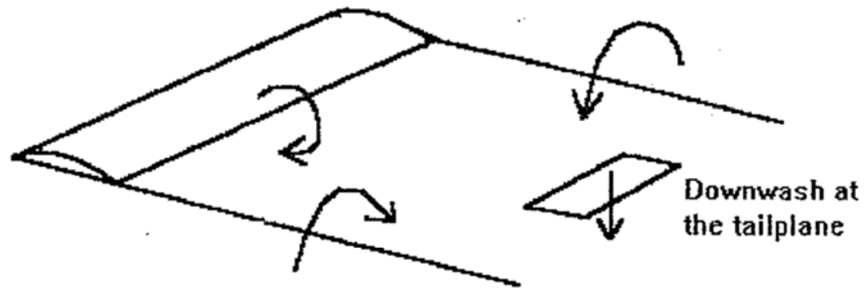
Thus $$C_{mG} = C_{mo} + (h - h_o)C_L - \overline{V}C_{LT}$$

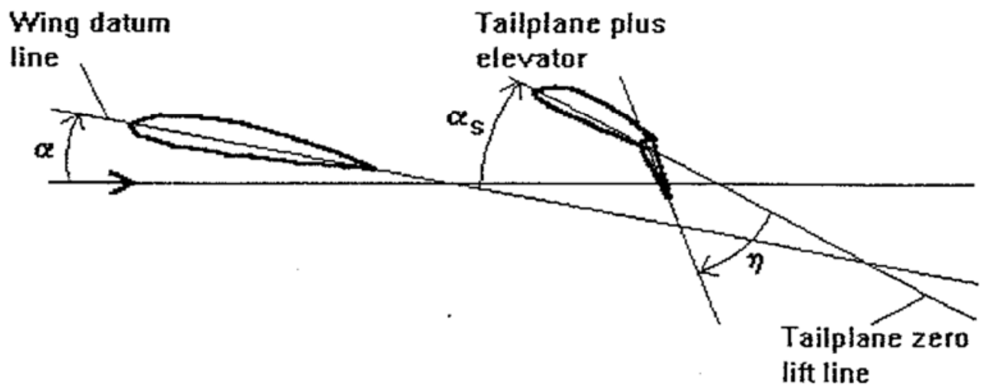
The implications for trim are:

$C_{mG} = 0$ (since $M_G = 0$)

$C_L = C_{Lw} + C_{LT}S_T/S$ (since $W = L_w + P$)

In order to proceed further it is necessary to consider the tailplane lift coefficient in more detail. It is important to note that the tailplane operates in the flow region which is affected by the downwash from the wing. This affects the effective angle of attack of the tailplane as is shown in the following diagrams:





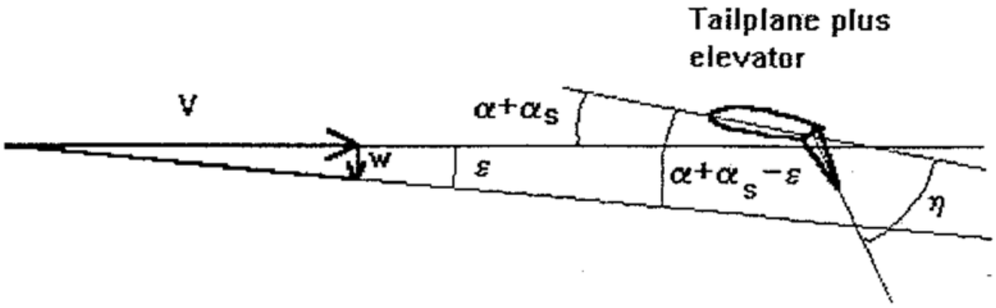
In the above diagram:

α_s is the setting or the rigging angle of the tailplane relative to the body datum
η is the elevator angle (positive as shown).

The following diagram shows the effect of the downwash velocity w on the local angle of attack of the tailplane. In this diagram

ε is the downwash angle
w is the downwash velocity
V is the freestream velocity

$\tan \varepsilon = w/V$



Downwash angle at tailplane = $\varepsilon \cong \tan\varepsilon = w/V$

where $\frac{w}{V} \cong \text{constant} \times \frac{C_L}{\pi A} \propto \alpha$, angle of attack.

Tailplane angle of attack = $\alpha_T = \alpha + \alpha_S - \varepsilon$

and $$\alpha_T = \alpha + \alpha_S - \frac{d\varepsilon}{d\alpha}\alpha$$

where $\frac{d\varepsilon}{d\alpha}$ is the variation of the downwash angle with angle of attack and is usually in the range from 0.25 to 0.5 depending on the position of the tailplane.

$$\therefore \alpha_T = \alpha\left(1 - \frac{d\varepsilon}{d\alpha}\right) + \alpha_S$$

The tailplane lift coefficient is:

$$C_{L_T} = a_{1_T}\alpha_T + a_{2_T}\eta$$

$a_{1_T} = \frac{\partial C_{L_T}}{\partial \alpha_T}$ is the lift curve slope of the tailplane (for η fixed)

$a_{2_T} = \frac{\partial C_{L_T}}{\partial \eta}$ (for α_T fixed)

For the complete aircraft:

$$C_L = a_1\alpha$$

where $$a_1 = \frac{dC_L}{d\alpha}$$

Substituting into the expression for C_{mG} we obtain:

$$C_{m_G} = C_{m_o} + (h - h_o)a_1\alpha - \bar{V}\left\{a_{1_T}\left[\alpha_S + \alpha\left(1 - \frac{d\varepsilon}{d\alpha}\right)\right] + a_{2_T}\eta\right\}$$

For Trim: $C_{mG} = 0$

The final result for the elevator angle to trim $\overline{\eta}$ is :

$$\overline{V}a_{2T}\overline{\eta} = C_{mo} + (h - h_o)a_1\alpha - \overline{V}a_{1T}\left[\alpha_s + \alpha\left(1 - \frac{d\varepsilon}{d\alpha}\right)\right]$$

In some cases the tailplane is 'all moving' and the tailplane angle α_S is adjusted to trim the aircraft. In this case the tailplane angle to trim is given by the following expression:

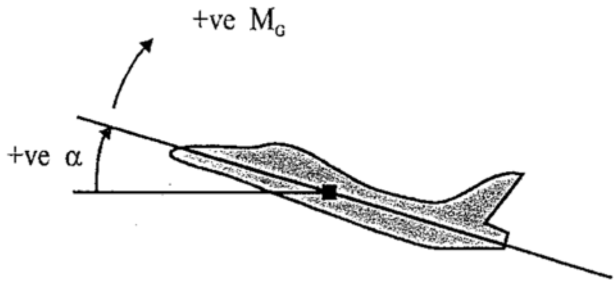
$$\overline{V}a_{1T}\overline{\alpha}_S = C_{mo} + (h - h_o)a_1\alpha - \overline{V}a_{1T}\alpha\left(1 - \frac{d\varepsilon}{d\alpha}\right)$$

The design case for the maximum up elevator angle for trim is when $\alpha = \alpha_{stall}$ i.e. the high C_L case.

Note: most forward C of G is determined by the maximum available negative trim (i.e. maximum –ve η) and at maximum C_L. If the tailplane carries –ve lift (i.e., η is large and –ve) then the main wing must carry more lift. The result is extra induced drag from both the tailplane and main wing which is known as TRIM DRAG.

8.2 Longitudinal Static Stability

The requirement for longitudinal static stability is that if the aircraft is disturbed in pitch, there should be a tendency to return to the previous steady flight condition.



i.e. we require a –ve change in M_G with a +ve change in α

and a +ve change in M_G with a –ve change in α

Thus we require $\frac{dM_G}{d\alpha}$ to be negative

or $\frac{dC_{mG}}{d\alpha} < 0$

In considering this condition we assume that the controls are held fixed.

Hence: η and α_s do not change with α, i.e., the stick is fixed.

To find out when the condition for stability is satisfied we need to differentiate the following expression for the pitching moment about the centre of gravity with respect to α:

$$C_{mG} = C_{mo} + (h - h_o)a_1\alpha - \bar{V}\left\{a_{1T}\left[\alpha_s + \alpha\left(1 - \frac{d\varepsilon}{d\alpha}\right)\right] + a_{2T}\eta\right\}$$

Hence $$\frac{dC_{mG}}{d\alpha} = (h - h_o)a_1 - \bar{V}a_{1T}\left(1 - \frac{d\varepsilon}{d\alpha}\right)$$

where hc is the distance of the C of G from the leading edge
and $h_o c$ is the distance of the aerodynamic centre from the leading edge.

The position of the C of G for neutral stability is given when $h = h_n$,

i.e. when $$(h_n - h_o)a_1 - \bar{V}a_{1T}\left(1 - \frac{d\varepsilon}{d\alpha}\right) = 0$$

$$\therefore\ h_n = h_o + \bar{V}\,\frac{a_{1T}}{a_1}\left(1 - \frac{d\varepsilon}{d\alpha}\right)$$

Thus h_n represents the position of the Neutral Point Stick Fixed. This is the rearmost allowable C of G position for static stability. Any C of G positions aft of this point will result in a statically unstable aircraft stick fixed.

Thus in general: $h < h_n$ for static stability.

The Centre of Gravity Margin Stick Fixed (sometimes called the Static Margin) is given by:

$$K_n = h_n - h$$

where K_n must be positive for stability.

Substituting the result for h_n into the expression for $dC_{mG}/d\alpha$ we obtain:

$$\frac{dC_{mG}}{d\alpha} = -a_1 K_n$$

8.2.1 Trim and C of G Margin

Trim and C of G margin are related by the following analysis:

For trim: $L = L_W + P = W = 0.5\rho V^2 S C_L = \text{constant}$

Hence $V^2 C_L = \text{constant}$

$$\frac{d(V^2C_L)}{dV} = V^2\frac{dC_L}{dV} + 2VC_L = 0$$

$$\therefore \frac{dC_L}{dV} = -\frac{2C_L}{V}$$

Now $$\frac{d\overline{\eta}}{dV} = \frac{d\overline{\eta}}{dC_L}\frac{dC_L}{dV} = \frac{d\overline{\eta}}{d\alpha}\frac{d\alpha}{dC_L}\frac{dC_L}{dV} = \frac{d\overline{\eta}}{d\alpha}\left(-\frac{2C_L}{a_1V}\right)$$

We can also obtain an expression for $\frac{d\overline{\eta}}{dV}$ from the previous equation for the elevator angle to trim derived in the notes on Trim, Static Stability and Control:

$$\overline{V}a_{2T}\overline{\eta} = C_{mo} + (h-h_o)a_1\alpha - \overline{V}a_{1T}\left[\alpha_s + \alpha\left(1-\frac{d\varepsilon}{d\alpha}\right)\right]$$

Differentiating we obtain $$\overline{V}a_{2T}\frac{d\overline{\eta}}{dV} = \left(-\frac{2C_L}{a_1V}\right)\left\{(h-h_o)a_1 - \overline{V}a_{1T}\left(1-\frac{d\varepsilon}{d\alpha}\right)\right\}$$

$$\therefore \overline{V}a_{2T}\frac{d\overline{\eta}}{dV} = -\frac{2C_L}{V}(h-h_n)$$

i.e $$\frac{d\overline{\eta}}{dV} = \frac{2C_L}{V}\frac{K_n}{\overline{V}a_{2T}}$$

This equation gives an expression for the rate of change of the elevator angle for trim as the speed of the aircraft changes. The rate of change is high when:

i) the speed V is low
ii) the static margin K_n is large.

In practice this means that the largest changes in the elevator angle for trim with speed will occur when the speed is low (high C_L) and the C of G is well forward (large K_n).

8.3 Tailplane Design

The tailplane must be designed to meet the following criteria:

a) provide adequate pitching stability in all configurations of the aircraft,
b) ensure that C_{LT} is not excessive in the forward C of G and flaps extended case.

To save weight and to reduce drag the tailplane should be as small as possible. The ideal C of G position will therefore be that position giving the same tailplane size for both stability and control considerations.

If possible the tailplane should be positioned so that it is not :

a) in a position of high $d\varepsilon/d\alpha$ (i.e. high downwash),
b) in the wake from flaps and dive brakes,
c) in a region of intense slipstream effects.

8.4 Control Design

8.4.1 Elevator

The forces generated by the elevator must be sufficient to :

a) trim and manoeuvre the aircraft in all configurations,
b) stall the aircraft in all configurations,
c) land the aircraft in a taildown attitude,
d) unstick the nose-wheel on take-off at less than the stalling speed.

Cases b) c) and d) usually size the elevator on conventional aircraft. The critical case is usually the forward C of G case.

8.4.2 Aileron

The aileron size is usually fixed by landing approach requirement (say produce a 60 degree bank in 7 seconds). Aileron generated forces must be adequate to make possible cross-wind landings in gusty conditions.

8.4.3 Rudder

The rudder size is usually determined to meet one of the following criteria :

a) the ability to counter engine failure during take-off,
b) the ability to make safe cross-wind landings.