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Lecture 3 - Boundary value and Eigenvalue Problems

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Outline

Review

- For **inhomogeneous equations:**
    - First solve the homogeneous equation (for complementary functions y_1, y_2).
    - Then find a particular solution (particular integral) y_P.
    - **Add** complementary and particular solutions to form the **general solution**

$$y = c_1 y_1 + c_2 y_2 + y_P$$

- Solutions of second order linear ODE's, homogeneous and inhomogeneous, contain two integration constants, the arbitrary constants c_1, c_2.
  Can we fix them? Yes, we can: that is our job today!

Outline

→ Boundary value problems (BVPs)

• **BVPs** occur when **conditions** to pick out a **unique solution** are imposed at different values (usually boundaries) of the independent variable ***x***, e.g.

$$y'' + 2y' + y = 0\,, \qquad y = y(x)\,,$$

with **Boundary Conditions:**

$$y(0) + 2y'(0) = 0, \quad 2y(1) - y'(1) = 1.$$

• The general solution is found by whichever standard method works (constant coefficients, Euler equation, method of undetermined coefficients, $\cdots$).

• **Only in the end** we impose the boundary conditions that (*might*) **fix the two integration constants** c_1, c_2**.**

Example

- The simple **harmonic oscillator**

$$y'' + y = 0 \xrightarrow{AE} y \propto e^{\lambda x} \to \lambda^2 + 1 = 0 \therefore \lambda = \pm j$$

[↖ constant coefficients ODE with (purely imaginary) complex roots: see Lecture 1]

has general solution *(Lecture 1)*

$$y = c_1 \cos(x) + c_2 \sin(x).$$

Arbitrary $c_1, c_2 \Rightarrow$ 2-parameter family of solutions!

- **If we impose the boundary conditions (BCs):**

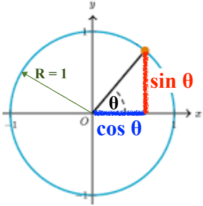
subbing

$$\boxed{\boldsymbol{y(0) = 0, \qquad y\left(\tfrac{\pi}{2}\right) = 1,}}$$

$$y(0) = 0 \Leftrightarrow c_1 \cos(0) + c_2 \sin(0) = 0 \quad \Leftrightarrow c_1 + 0 = 0$$

$$y\left(\tfrac{\pi}{2}\right) = 1 \Leftrightarrow c_1 \cos\left(\tfrac{\pi}{2}\right) + c_2 \sin\left(\tfrac{\pi}{2}\right) = 1 \Leftrightarrow 0 + c_2 = 1$$

evaluating



$\Rightarrow c_1 = 0, c_2 = 1$ and the **unique solution:**

$$y = \sin(x).$$

Example

(same as prev only change is boundary condition)

- The simple **harmonic oscillator**

$$y'' + y = 0$$

[↖ constant coefficients ODE with (purely imaginary) complex roots: see Lecture 1]

has general solution

$$y = c_1 \cos(x) + c_2 \sin(x).$$

subbing in and evaluating gives context and leads to 0=1 ∴ contradiction

Arbitrary $c_1, c_2 \Rightarrow$ 2-parameter family of solutions!

- **Instead, if we impose boundary conditions (BCs):**

$$y(0) = 0, \qquad y(\pi) = 1$$

we get $c_1 = 0, -c_1 = 1$! This **is contradictory** $\Rightarrow$ **no solution.**

Example

(change since #1 is only value)

- The simple **harmonic oscillator**

$$y'' + y = 0$$

[↖ constant coefficients ODE with (purely imaginary) complex roots: see Lecture 1]

has general solution

$$y = c_1 \cos(x) + c_2 \sin(x).$$

Arbitrary $c_1, c_2 \Rightarrow$ 2-parameter family of solutions!

- **Instead, if we impose boundary conditions (BCs):**

$$y(0) = 0, \qquad y(\pi) = 0$$

we get $c_1 = 0, c_1 = 0$. This is **not contradictory**, but the conditions are **not independent** $\Rightarrow$ there is a **1-parameter family of solutions:**

$$y = c_2 \sin(x) \qquad \longleftarrow \text{BCs did not fix } c_2$$

Summary

A **boundary value problem** may have

- a unique solution
- 1-parameter family of solutions
- no solution

Outline

$\rightarrow$ Eigenvalue problems

We may have a **boundary value problem** containing an unknown constant $\boldsymbol{\lambda}$. A simple example is

$$y''(x) + \boldsymbol{\lambda}\, y(x) = 0\,; \qquad \text{with } \textbf{BCs:} \quad y(0) = 0, \quad y'(1) + y(1) = 0.$$

For example, it arises in **heat conduction in a bar** (many others).

The approach to find the **eigenvalue** λ and associated **eigenfunctions** $y(x)$ is to:

1. find the **general solution** for y, for all values of λ;
2. **check if** the **boundary conditions allow a non-trivial** solution.

Typically, we find that **only certain λ work** (or none!).

Eigenvalue problems: case 1 ($\lambda = 0$)

λ = constant eigenvalue

A.E guess: $y = e^{\Lambda x} \to \Lambda^2 + \lambda = 0 \to \Lambda = \pm\sqrt{-\lambda}$

$$y''(x) + \boldsymbol{\lambda}\, y(x) = 0\,; \qquad \text{with } \textbf{BCs:} \quad y(0) = 0, \quad y'(1) + y(1) = 0.$$

[↖ constant coefficients ODE: see Lecture 1]

1. Check the case $\lambda = 0$

$\Rightarrow$ constant coefficients ODE with repeated real roots: $\Lambda_1 = \Lambda_2 \equiv \Lambda = 0$ (Lecture 1).

The general solution is *(Lecture 1)*

$$y = c_1 x + c_2.$$

$\frac{dy}{dx} = c_1$

The **only** solution is the **trivial** solution: $c_1 = 0, c_2 = 0,$ i.e. $\boldsymbol{y = 0}$.

Indeed,

$$y(0) = 0 \quad \Leftrightarrow \quad 0 + c_2 = 0 \quad \Leftrightarrow \quad c_2 = 0$$
$$y'(1) + y(1) = 0 \quad \Leftrightarrow \quad c_1 + (c_1 + c_2) = 0 \quad \Leftrightarrow \quad c_1 = 0.$$

Eigenvalue problems: case 2 ($\lambda = -k^2 < 0$)

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$$y''(x) + \boldsymbol{\lambda}\, y(x) = 0\,; \qquad \text{with } \textbf{BCs:} \quad y(0) = 0, \quad y'(1) + y(1) = 0.$$

[↖ constant coefficients ODE: see Lecture 1]

done for convenience since roots likely

2. Check the case $\lambda = -k^2 < 0$. AE ... $\Lambda = \pm k$

$\Rightarrow$ constant coefficients ODE with distinct real roots: $\Lambda_1 = k,\ \Lambda_2 = -k$ (Lecture 1).

The general solution is *(Lecture 1)* subed

$$y = c_1 e^{kx} + c_2 e^{-kx}.$$

$\frac{dy}{dx} = k(c_1 e^{kx} - c_2 e^{-kx})$

The **only** solution is the **trivial** solution: $c_1 = 0, c_2 = 0$. [exercise: check it]

$0 = y(0) = c_1 + c_2 \Rightarrow c_2 = -c_1$

$0 = y(1) + y'(1) = k(c_1 e^{k1} + c_1 e^{-k1}) + c_1 e^{k1} + c_1 e^{k1} \Rightarrow 0 = c_1(\dots) \to c_1 = 0$

$\therefore$ no useful solution

Eigenvalue problems: case 3 ($\lambda = k^2 > 0$)

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$$y''(x) + \boldsymbol{\lambda}\, y(x) = 0\,; \qquad \text{with } \textbf{BCs:} \quad y(0) = 0, \quad y'(1) + y(1) = 0.$$

[↖ constant coefficients ODE: see Lecture 1]

3. Check the case $\lambda = k^2 > 0$. → $\Lambda = \pm jk$

$\Rightarrow$ constant coefficients ODE with (purely imaginary) complex roots: $\Lambda_1 = \mathbf{j}\,k$, $\Lambda_2 = -\mathbf{j}\,k$.

The general solution is *(Lecture 1)*

$$y = c_1 \sin(kx) + c_2 \cos(kx) \qquad \Rightarrow \quad y'(x) = c_1 k \cos(kx) - c_2 k \sin(kx).$$

The first boundary condition gives $c_2 = 0$, but the **second BC** gives

$$c_1 \left[k\cos(k) + \sin(k) \right] = 0.$$

Here, the term in brackets may vanish, giving the non-trivial solution

$$y = c_1 \sin(kx)$$

where k must satisfy:

$$k\cos(k) + \sin(k) = 0.$$

Eigenvalue problems: case 3 ($\lambda = k^2 > 0$)

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Our **eigenfunctions** are

$$y_n = c_1 \sin\left(\sqrt{\lambda_n} x\right)$$

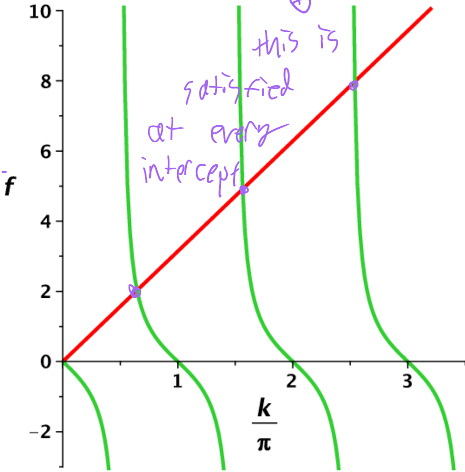
with **eigenvalues** $\lambda_n = k_n^2$, where k_n are solutions of

subscript because inf solutions.

$$0 = k\cos(k) + \sin(k)$$

$$\Rightarrow \quad \mathbf{k} = -\,\mathbf{tan}\,(\mathbf{k}).$$

These cannot be found in closed form, but numerically yes: see graph. There is an infinite but countable (discrete) number of k_n's. Not all k's do the job.



this is satisfied at every intercept

Outline

Summary

- Solving boundary value problems (BVPs) is, in practice, just solving the ODE (PDE) and seeing if the boundary conditions (BCs) are compatible.
- The BVP may have:
  - a unique solution
  - 1-parameter family of solutions
  - no solution
- Solving eigenvalue problems means finding which values of the unknown constant λ allow solutions.
- We examined the three cases:
  - $\lambda = 0$
  - $\lambda = k^2 > 0$
  - $\lambda = -k^2 < 0$
- The set of λ for which the ODE admits non-trivial solutions are the eigenvalues and the corresponding solutions the eigenfunctions.