Part 3d: Dynamics of Beams

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

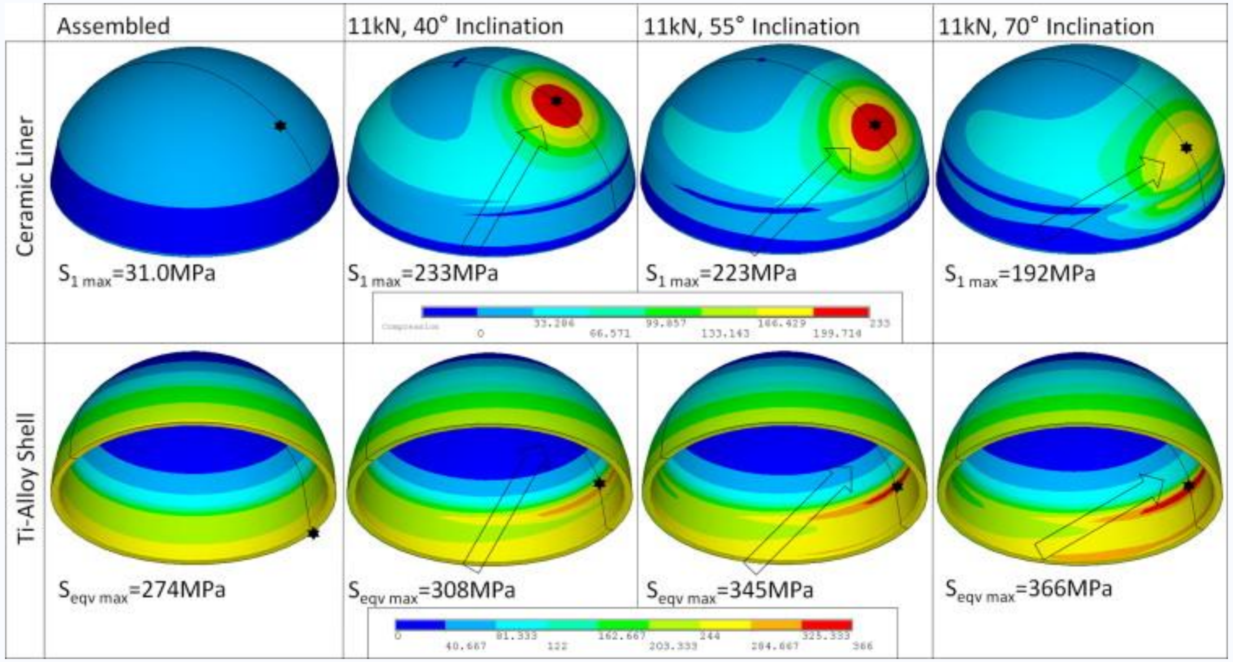
From 12th November 2024

Reporting FEA: Essentials

- ❑ Introduction, problem background
- ❑ Study objective – ideally a pass/fail test, including acceptance criteria
- ❑ A description of model geometry, loading and BCs, with justification
- ❑ A table of your applied material properties, with references
- ❑ Details of your mesh
- ❑ Results, including scale bar for contour plots, and units
- ❑ Discussion of results including back-of-envelope calculations, and a list and appraisal of model simplifications and assumptions
- ❑ Conclusion – did you meet your objective (PASS or FAIL)?
- ❑ Recommendations

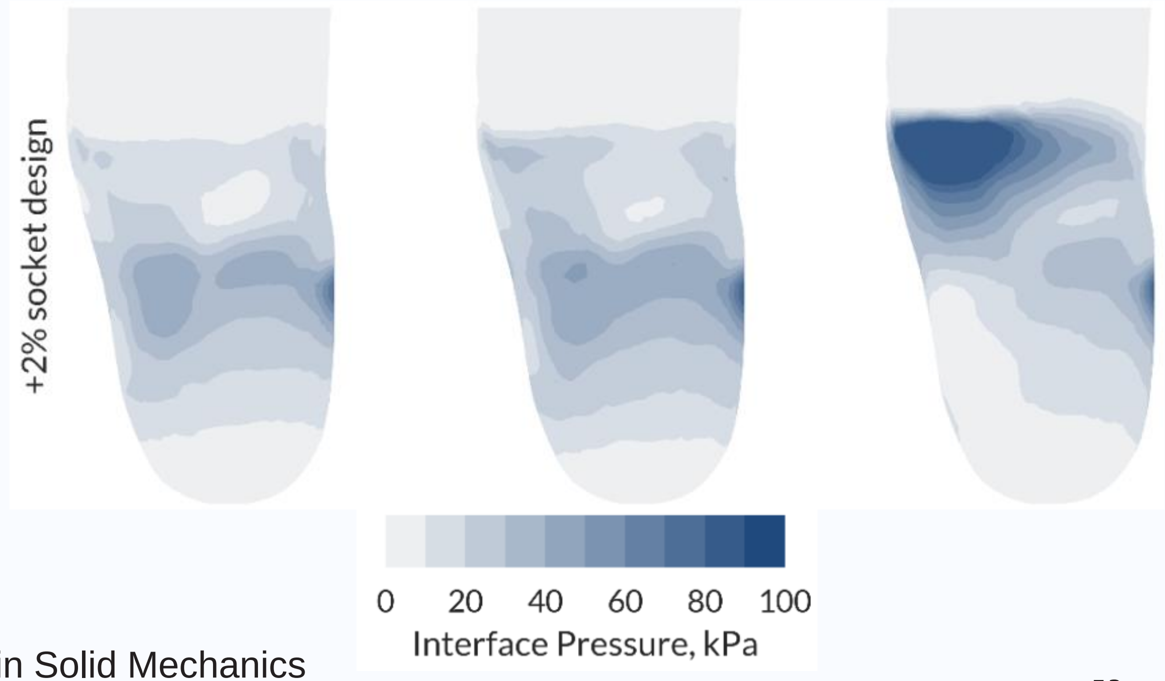
Reporting FEA: Fine Tuning

- (Projects, not essential for the coursework)
- Results Plotting: How to enable easy comparison?

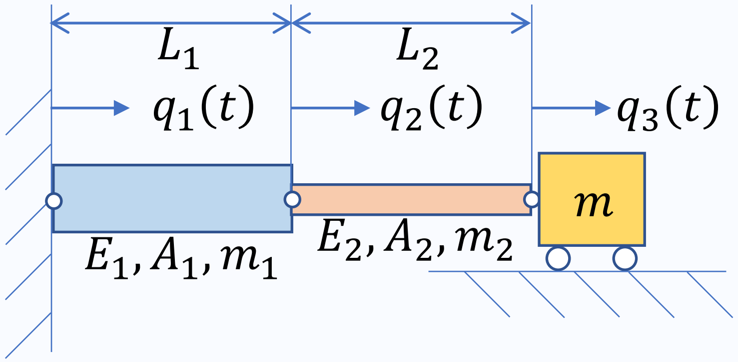


Dickinson, A. S. et al (2014). *Med Eng & Physics*.
https://doi.org/10.1016/j.medengphy.2013.09.009

Steer J.W. et al (2020). *Prosthet Orthot Intl*.
https://doi.org/10.1177/0309364620967781



Last time: if we add a mass to rods:



- How many DoF do we have now?
  - Still only 3.
- What external forces do we have?
  - Still none.
- So we just modify what we already have:

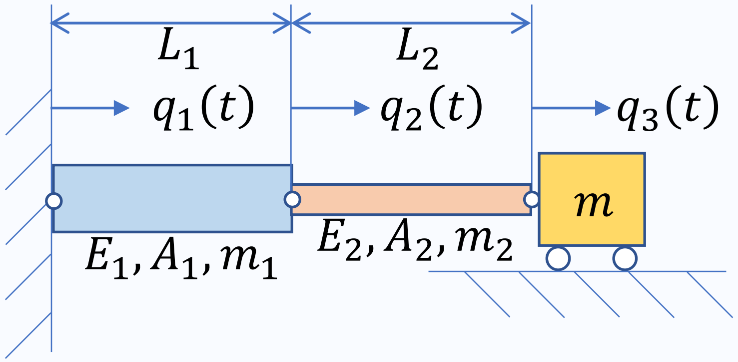
  $U = U_1 + U_2$, but $T = T_1 + T_2 + T_m$

  $T_m = {}^1\!/_2\, m\dot{q}_3{}^2$

- So our system mass matrix $[M]$ is obtained from:

$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}^T_{1\times3} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}_{3\times3} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}_{3\times1}$$

Last time: if we add a mass to rods:



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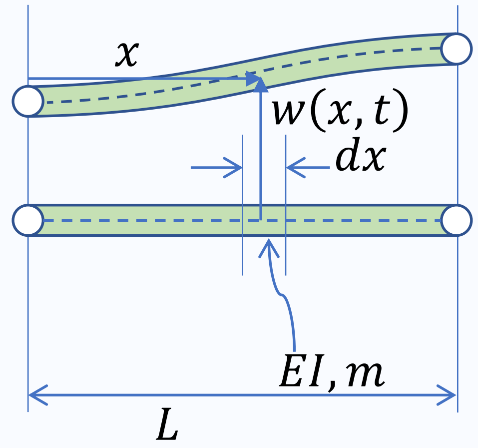
- So our system mass matrix $[M]$ is obtained from:

$$T = {}^1/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}^T_{1\times 3} \begin{bmatrix} \frac{m_1L_1}{3} & \frac{m_1L_1}{6} & 0 \\ \frac{m_1L_1}{6} & \frac{m_1L_1}{3} + \frac{m_2L_2}{3} & \frac{m_2L_2}{6} \\ 0 & \frac{m_2L_2}{6} & \frac{m_2L_2}{3} + m \end{bmatrix}_{3\times 3} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}_{3\times 1}$$

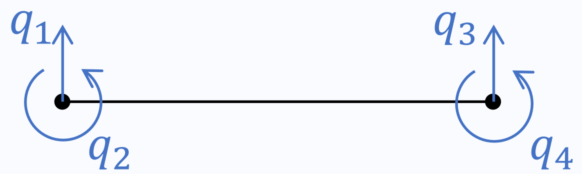
$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$

Beams – in Dynamics



- A beam vibrating transverse to its length: for the elastic potential energy we can recycle from statics, with the same assumptions:
- We don't know the field displacement $w(x,t)$ so we approximate it from our generalised coordinates $q_i(t)$ using shape functions:

$$w(x,t) = f_1(x)q_1(t) + f_2(x)q_2(t) + f_3(x)q_3(t) + f_4(x)q_4(t)$$



- where $f_i(x)$ are our Hermite Cubic shape functions.
- This lets us say the elastic potential strain energy is:

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \begin{bmatrix} & & \\ & K & \\ & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

- which came from the integral:

$$U = {}^1\!/_2 \int_0^L EI w''^2 dx$$

Beams – in Dynamics

- We assume the kinetic energy from rotatory motion of each slice is small, compared to its transverse motion.
- Again we don't know the field velocity $\dot{w}(x,t)$ but we approximate it:

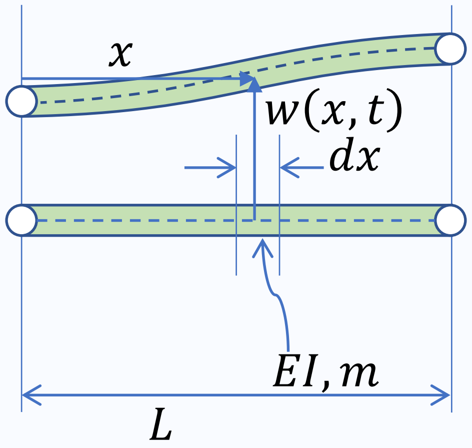
$$\dot{w}(x,t) = f_1(x)\dot{q}_1(t) + f_2(x)\dot{q}_2(t) + f_3(x)\dot{q}_3(t) + f_4(x)\dot{q}_4(t)$$

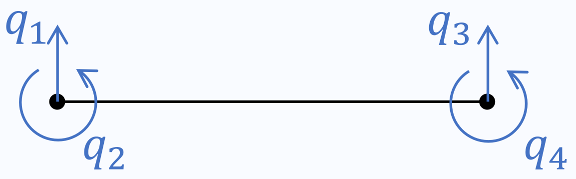
- where $f_i(x)$ are our Hermite Cubic shape functions.
- This lets us say the kinetic energy is:

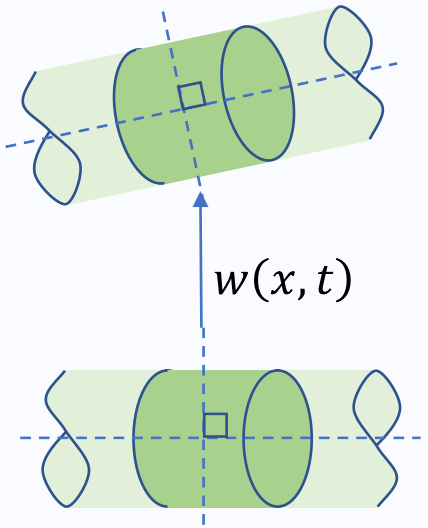
$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{Bmatrix}^T \begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{Bmatrix}$$

- which came from the integral:

$$T = {}^1\!/_2 \int_0^L m\big(\dot{w}(x,t)\big)^2 dx$$







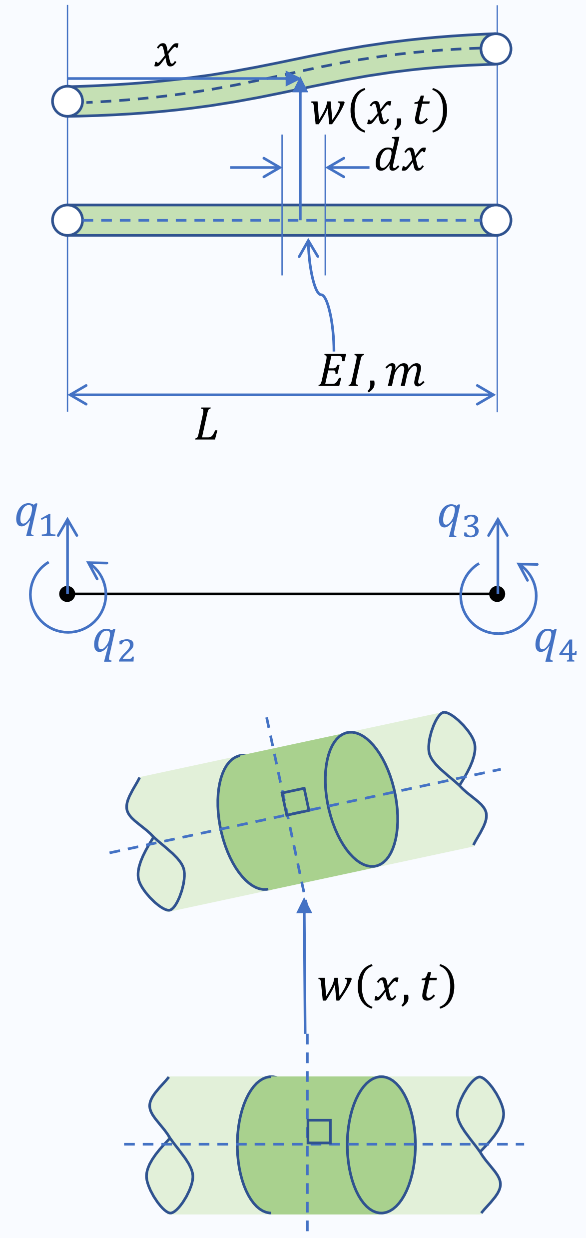
Beams – in Dynamics

- Key Results:

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

- and:

$$[M] = \frac{mL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$



Part 3e: Solving Dynamic Systems

FEEG3001/SESM6047 FEA in Solid Mechanics

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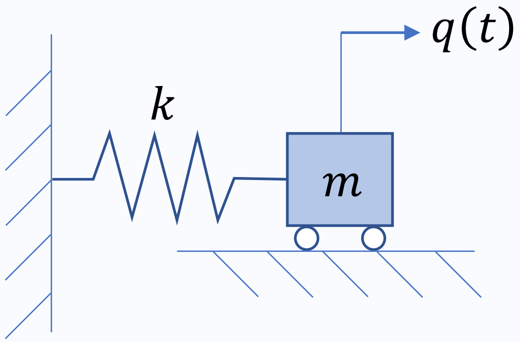
From 12th November 2024

A free-oscillatory solution to our 1-DoF System:



- A reminder of some Dynamics basics:
- The Governing Equation of Motion is

  $m\ddot{q} + kq = 0$

  $q(t) = ?$
- How would we solve it?
- What kind of differential equation is it?
  - 2^{nd} order
  - Homogeneous (right side = 0)
  - Linear, and
  - Constant-Coefficient (time-invariant)

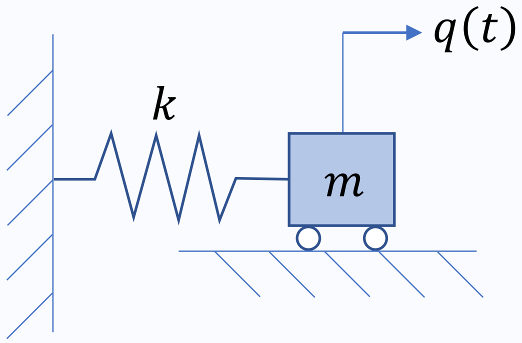
A free-oscillatory solution to our 1-DoF System:

$$m\ddot{q} + kq = 0$$

- To solve a:
  - 2nd order
  - Homogeneous (right side = 0)
  - Linear, and
  - Constant-Coefficient (time-invariant)
- The suggested solution is:

$$q(t) = e^{st}$$

- The differential equation will tell us s



A free-oscillatory solution to our 1-DoF System:

- Substitute $q(t) = e^{st}$ into $m\ddot{q} + kq = 0$:

  $$ms^2e^{st} + ke^{st} = 0$$

  $$(ms^2 + k)e^{st} = 0$$

- So we have two potential solutions:

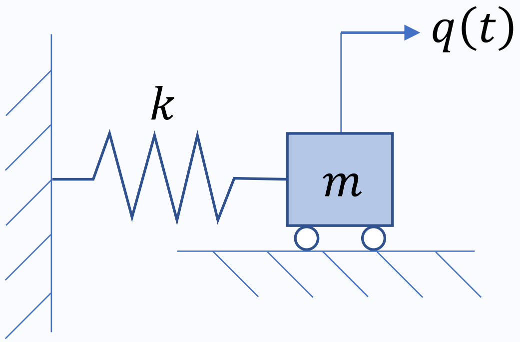
  $e^{st} = 0$ (trivial) and $ms^2 + k = 0$

- so:

  $$s = \pm\sqrt{\frac{k}{m}} \times i$$

- giving:

  $$q(t) = e^{\pm\sqrt{\frac{k}{m}}it} = e^{\pm i\omega t}$$



A free-oscillatory solution to our 1-DoF System:

- The solution must be:

$$q(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

- and since Euler's Identity states that:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

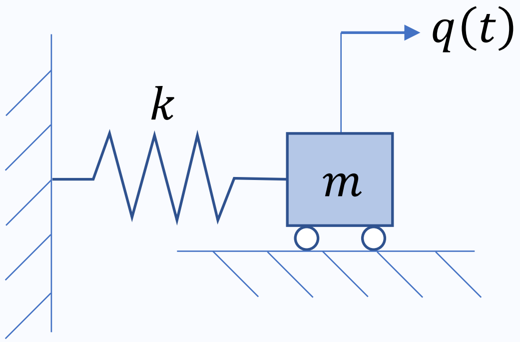
- we can express this as:

$$q(t) = C_1[\cos\omega t + i\sin\omega t] + C_2[\cos\omega t - i\sin\omega t]$$

- with complex constants C_1 and C_2.
- $q(t)$ must be real (it is motion!) so we reject the imaginary parts, leaving a general solution which has the form:

$$q(t) = A\cos\omega t + B\sin\omega t$$

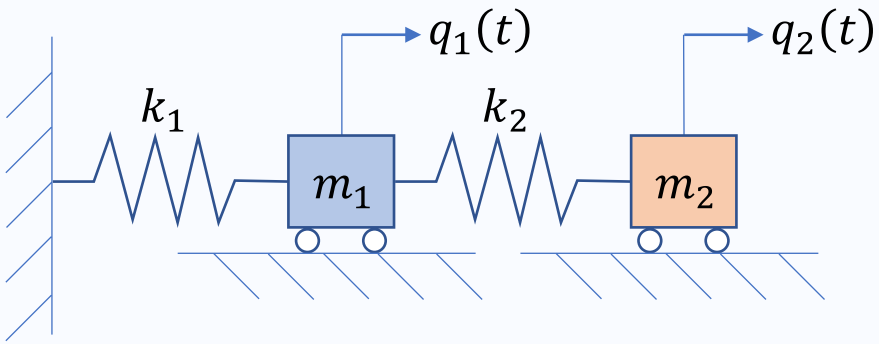
- (i.e. a linear combination of the two independent solutions)
- and we will find A and B from the initial conditions.



So the movement of our conservative, one DoF system will be sinusoidal, with natural frequency:

$$\omega = \sqrt{\frac{k}{m}}$$

A multi-DoF System?



- Recall our governing equation of motion is:

  $[M]_{N\times N}\{\ddot{q}\}_{N\times 1} + [K]_{N\times N}\{q\}_{N\times 1} = \{0\}_{N\times 1}$

  $\{q(t)\} = ?$

- What kind of differential equation is it?
  - 2^{nd} order
  - Homogeneous (right side = 0)
  - Linear, and
  - Constant-Coefficient (time-invariant)
  - *Simultaneous*, or *coupled*

  $M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + K_{11}q_1 + K_{12}q_2 = 0$ and

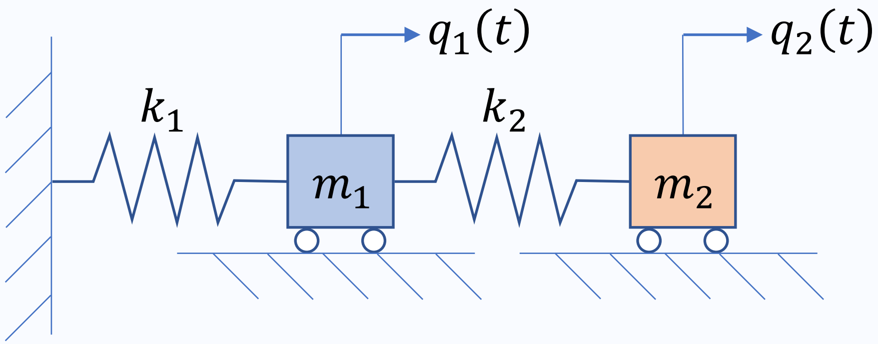
  $M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + K_{21}q_1 + K_{22}q_2 = 0$

  which means I can't solve one independent of the other.

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

So how do we solve two simultaneous differential equations…?

A multi-DoF System?



- Propose simultaneous differential equations have solutions:

$$\{q(t)\} = \{A\} \sin \omega t$$

$$q_1(t) = A_1 \sin \omega t$$

$$q_2(t) = A_2 \sin \omega t \ \ldots$$

$$q_n(t) = A_n \sin \omega t$$

- which is the same as saying:

$$\{q(t)\} = [q_1(t), q_2(t) \ldots q_N(t)]^T = [A_1, A_2 \ldots A_N(t)]^T \sin \omega t$$

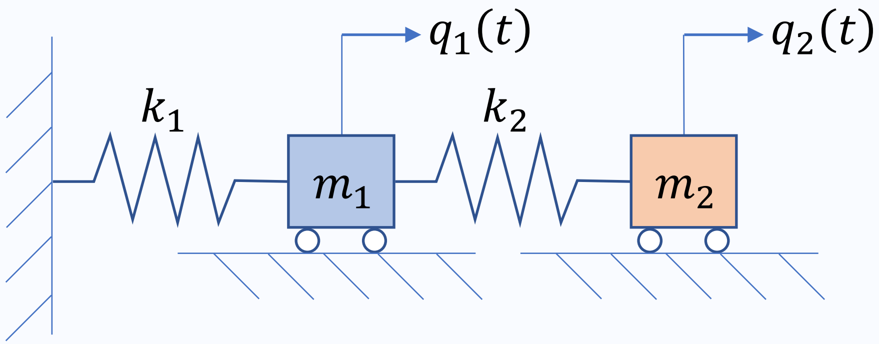
- as well as:

$$\{\dot{q}(t)\} = [\dot{q}_1(t), \dot{q}_2(t) \ldots \dot{q}_N(t)]^T = [A_1, A_2 \ldots A_N(t)]^T \omega \cos \omega t$$

$$\{\ddot{q}(t)\} = [\ddot{q}_1(t), \ddot{q}_2(t) \ldots \ddot{q}_N(t)]^T = [A_1, A_2 \ldots A_N(t)]^T (-\omega^2) \sin \omega t$$

$$\{\ddot{q}(t)\} = -\omega^2 \{A\} \sin \omega t$$

A multi-DoF System?



- So we substitute into our original governing equation of motion:

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$$

- noting that matrix multiplication is not commutative (respects order).

$$(-\omega^2[M]\{A\} + [K]\{A\}) \sin \omega t = \{0\}$$

- Our trivial solution exists with no motion, when $\sin \omega t = 0$. So keep:

$$(-\omega^2[M]\{A\} + [K]\{A\}) = \{0\}$$

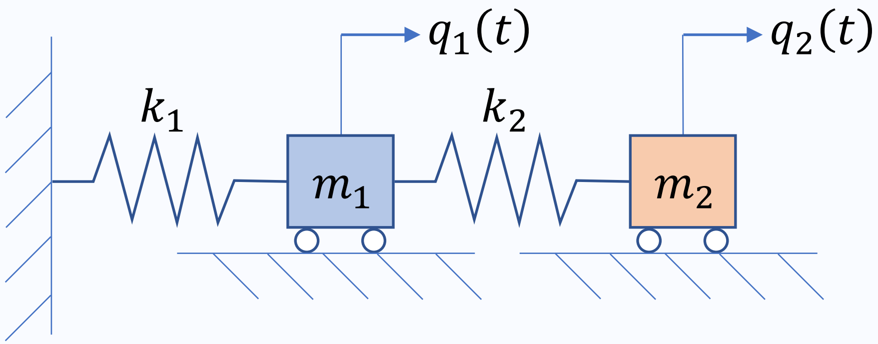
- which is the same as

$$[K]\{A\} = \omega^2[M]\{A\}$$

- Finally, customarily we call:

$\{A\} = \{u\}$ (amplitudes) and $\omega^2 = \lambda$

A multi-DoF System?



- Do you recognise what this is?

  $[K]\{u\} = \lambda[M]\{u\}$

- It has the form of a *Generalised Eigenvalue Problem* (with 2 matrices).

  $\boldsymbol{K}u = \lambda \boldsymbol{M}u$ (now with bold indicating matrices)

- We should be able to reduce this to the Standard Eigenvalue Problem $\boldsymbol{A}x = \lambda x$ or $(\boldsymbol{A} - \lambda)x = 0$
- You should have seen E.V.P.s before, but perhaps just not in context!
- Next week we will look at how to solve them.

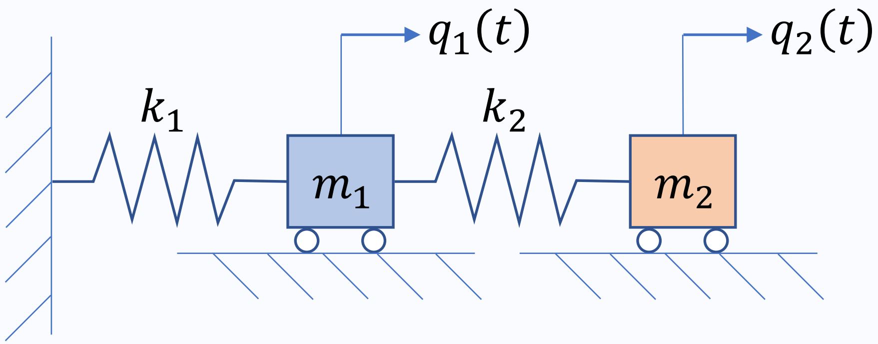
Part 3e: Solving Dynamic Systems Continued

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

From 15th November 2024

A multi-DoF System?



- Recall the generalised equation of motion, from the quadratic form of the kinetic energy, has the form:

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$$

- And we have seen how to form $\boldsymbol{M}$ and $\boldsymbol{K}$ for rods and beams.
- This has a solution of the type

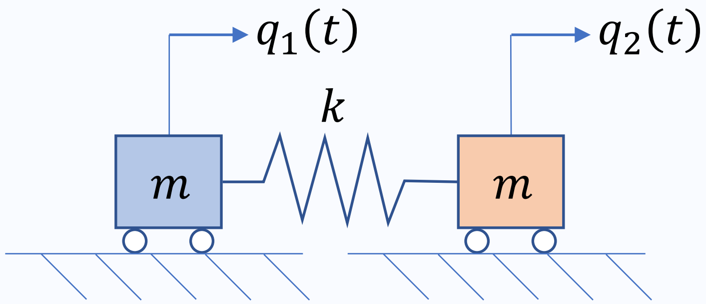
$$\{q(t)\} = \{u\} \sin \omega t$$

- where $\{u\}$ expresses the amplitudes
- For non-trivial solutions this gives

$$[K]\{u\} = \lambda[M]\{u\}$$

- An Eigenvalue Problem with eigenvector $\{u\}$ and eigenvalue $\lambda = \omega^2$
- And we will get n solutions for λ, equal to the number of masses, or DoF.

A specific example:



- A simple two DoF system, identical mass trolleys. T and U?

$$T = {}^1\!/_2\, m\dot{q}_1{}^2 + {}^1\!/_2\, m\dot{q}_2{}^2$$

$$U = {}^1\!/_2\, k(q_2 - q_1)^2$$

- Which we can rewrite using the recognised quadratic forms of our generalised coordinates:

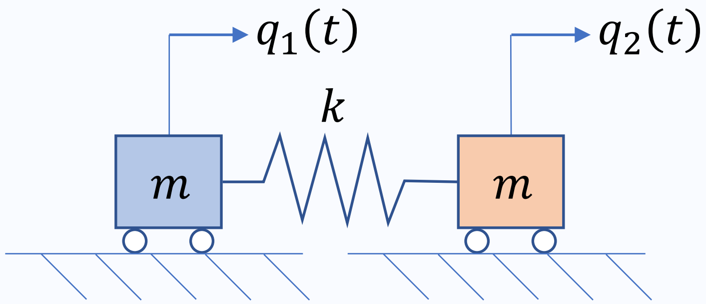
$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

- You could use Lagrange's equations, but we won't; because now we have learned that we can work by inspection of these common forms, giving:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

How do we solve it?



- This scenario is a special case of synchronous motion (both bodies have same frequency, reach their extrema at the same time), and a solution of the type:

$$\{q(t)\} = \{u\}\sin \omega t$$

- which gives an eigenvalue problem of the form:

$$[K]\{u\} = \lambda[M]\{u\}$$

- Specifically, here:

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \boxed{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}} = \lambda \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \boxed{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}$$

- which we can rewrite as a Standard Eigenvalue Problem:

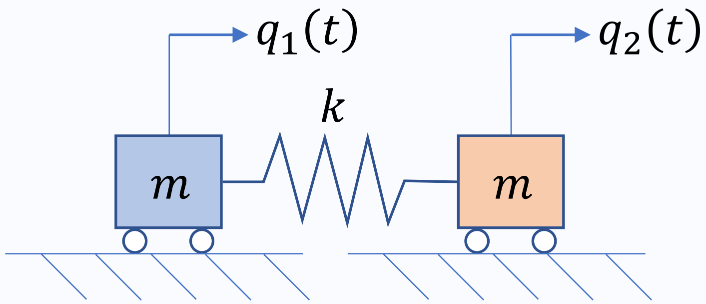
$$\begin{bmatrix} k - \lambda m & -k \\ -k & k - \lambda m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- i.e. 2 simultaneous algebraic equations with 3 unknowns:

How does $\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$ become $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$?

Our EVP solution gives us *amplitudes*.

How do we solve it: Eigenvalues?



$$\begin{bmatrix} k-\lambda m & -k \\ -k & k-\lambda m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- So we have rank deficiency, one fewer equations than unknowns.
- How do we solve this?
- First λs, then us.
- For this synchronous motion, non-trivial solutions exist if the determinant is zero:

$$\begin{vmatrix} k-\lambda m & -k \\ -k & k-\lambda m \end{vmatrix} = 0, \text{ i.e.}$$

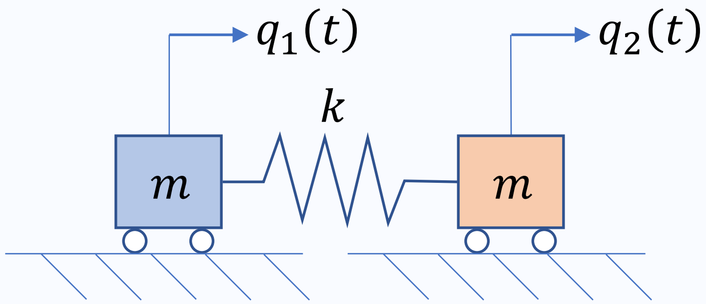
$(k-\lambda m)(k-\lambda m) - (-k)(-k) = 0$ or

$k^2 - 2k\lambda m + \lambda^2 m^2 - k^2 = 0$ or

$\lambda(\lambda m^2 - 2km) = 0$

- But we should not cancel the λ because you would lose a solution.

How do we solve it: Eigenvalues?



$$\lambda(\lambda m^2 - 2km) = 0$$

- gives:

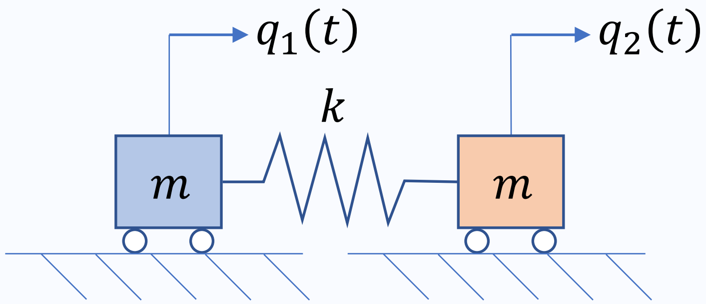
$$\lambda_1 = 0, \text{ and } \lambda_2 = \frac{2k}{m}$$

- and since $\lambda = \omega^2$

$$\omega_1 = 0, \text{ and } \omega_2 = \sqrt{\frac{2k}{m}}$$

- meaning the differential, characteristic equation has solutions with the form $a + bt$ (rigid-body motion), and $\sin \omega_2 t$ (oscillatory)
- and in reality we would expect a mixture of these two motions

How do we solve it: Eigenvectors?



- For $\lambda_1 = 0$,

  $$\begin{bmatrix} k - \lambda m & -k \\ -k & k - \lambda m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

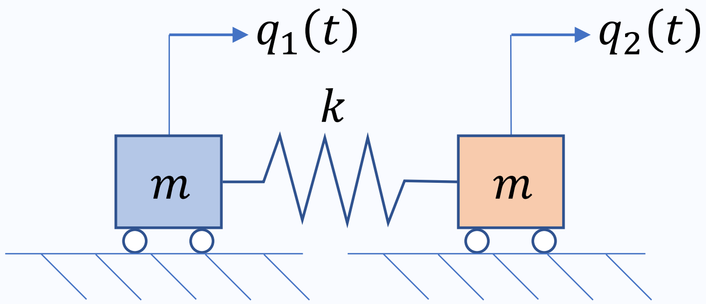
  $ku_1 - ku_2 = 0$ (eq.1)

  $-ku_1 + ku_2 = 0$ (eq.2)

- which we can't solve simultaneously. Instead, say $u_1 = 1$:

  $$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

- (or any multiple!)
- which is *rigid body motion* $(\mathrm{T} = const, \ \mathrm{U} = 0)$



How do we solve it: Eigenvectors?

- For $\lambda_2 = \frac{2k}{m}$,

  $$\begin{bmatrix} k-\lambda m & -k \\ -k & k-\lambda m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} k-2k & -k \\ -k & k-2k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

  $-ku_1 - ku_2 = 0$ (eq.1)

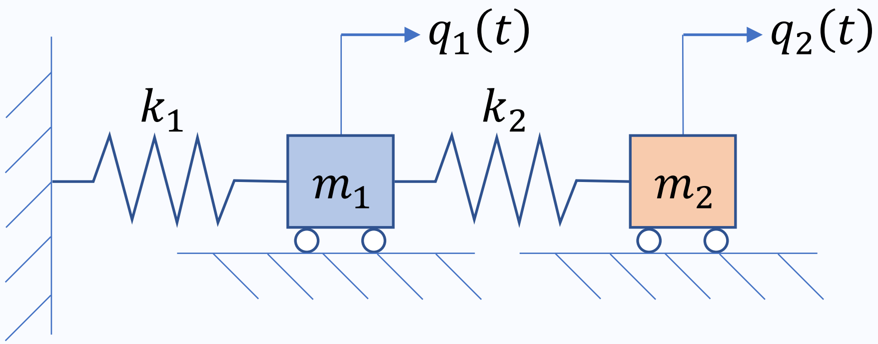
  $-ku_1 - ku_2 = 0$ (eq.2)

- which we can't solve simultaneously. Instead, say $u_1 = 1$:

  $$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

- (or any multiple!)
- which is *normal mode motion* $(\mathrm{T} > 0,\ \mathrm{U} > 0)$

Practice question:



- What are the synchronous, free oscillatory motions?
- and guess: how many rigid body modes are there?

- Reminder: what does this have to do with Finite Elements?
- Finite Elements will give you the same matrix equation, with motions of this kind: *normal mode motions*.

And Past Paper Questions for Next time:

Q2. The potential energy U and the kinetic energy T of a two-degrees-of-freedom system are given by $U = (q_1 - q_2)^2$ and $T = \frac{1}{2}(\dot{q}_1 + \dot{q}_2)^2$ respectively. Derive the governing equations of motion using Lagrange's equations. Organise them in a matrix form and identify the stiffness matrix and the mass matrix.

[8]

Describe the type of governing equations of motion (what order, linear vs non-linear, ordinary vs partial differential equation, homogeneous vs non-hmogeneous, etc. – using as many descriptors as you can think of).

2018/19 Q2 **[2]**

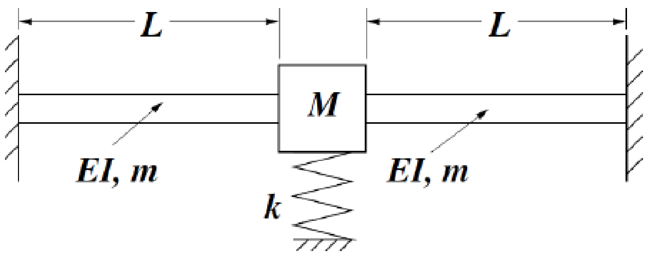
Q3. A point mass M is supported on two beams that are fixed at the ends, and sprung at the centre, as shown in Figure Q3. Use one element each to model the two beams. Obtain the assembled mass and the stiffness matrices for this structure. Calculate the the approximate natural frequencies of the structure. The stiffness and the mass matrices of an element are respectively given by

$$\frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}, \quad \frac{mL}{420}\begin{pmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{pmatrix},$$

The order of counting the degrees-of-freedom at the two nodes of a beam element is shown in the figure below.

[20]

Obtain the relationship involving the structural parameters for which the two natural frequencies are equal. **[4]**



2018/19 Q3

University of Southampton

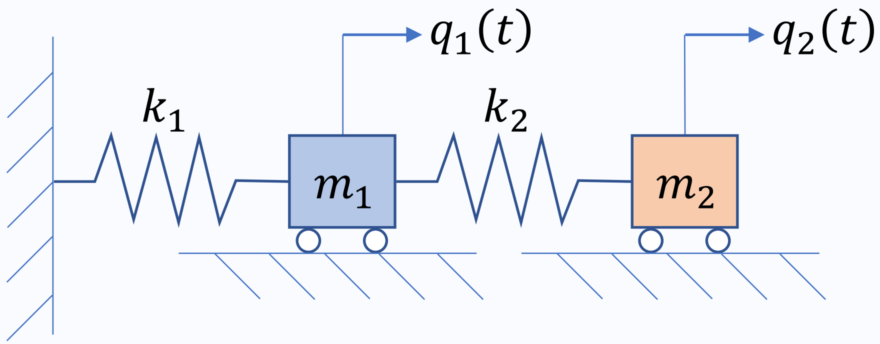
Part 3f: Final Details on Dynamic Systems: Other Elements, and Mode Shapes

FEEG3001/SESM6047 FEA in Solid Mechanics

Dr A S Dickinson

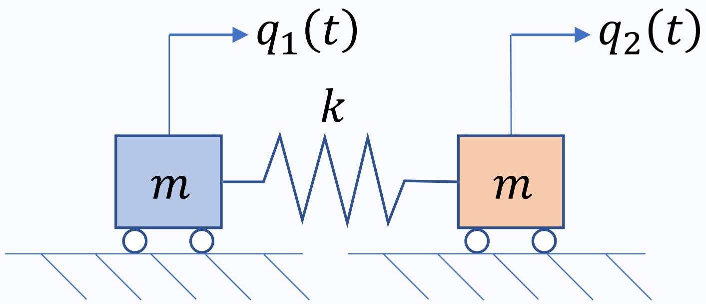
From 15th November 2024

Practice question:

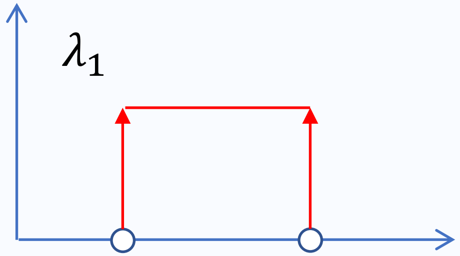


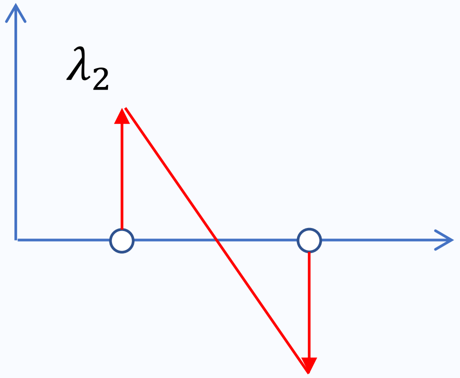
- What are the synchronous, free oscillatory motions?
- and guess: how many rigid body modes are there?

Practice question:

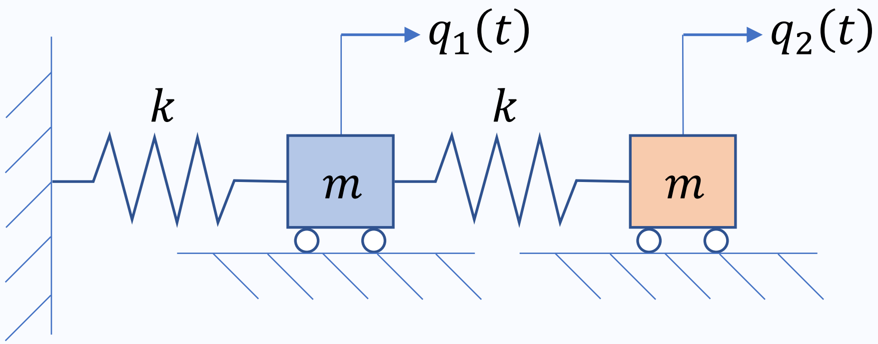


- Ask yourself: what will the mode shapes look like?
- It is customary to plot them. For example:
- For a normal mode: $\lambda_1 = 0$, $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$
- For a normal mode: $\lambda_2 = \frac{2k}{m}$, $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

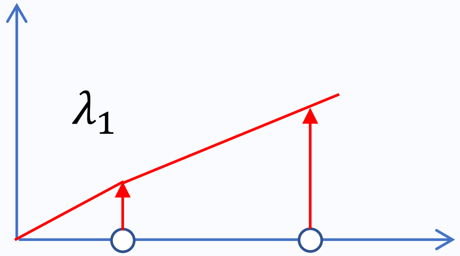


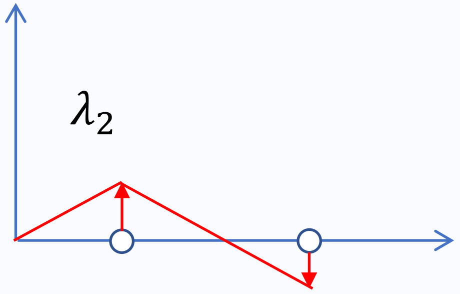


Practice question:

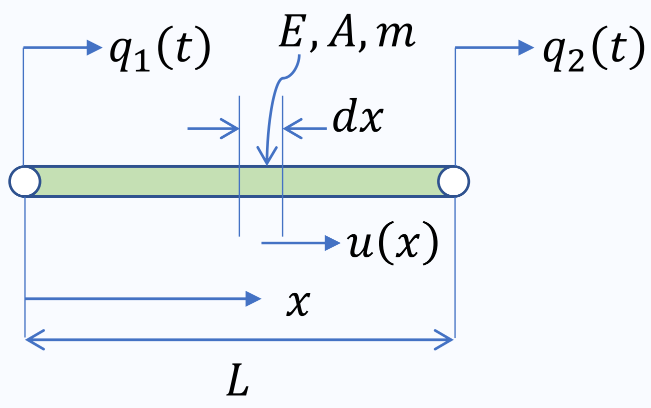


- What will the mode shapes look like?
- It is customary to plot them. For example:
- For a normal mode: λ_1, $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \\ \end{Bmatrix}$
- For a normal mode: λ_2, $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \\ \end{Bmatrix}$





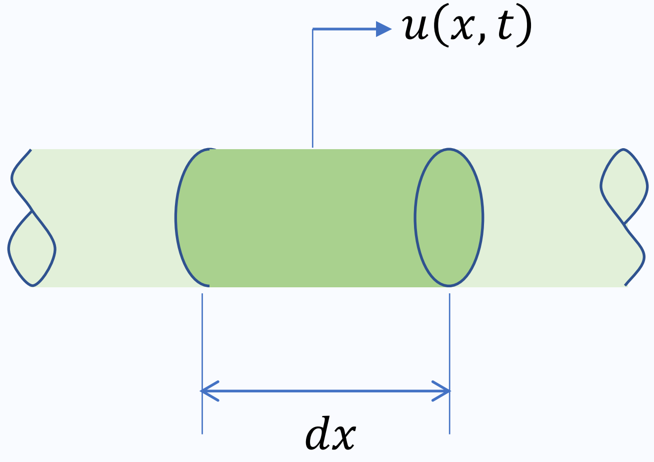
Rods, Strings and Shafts



- Because of how we set up this element type, we can kill three birds with one stone!
- Remembering, for Rods in Tension and Compression, the kinetic and potential energies:
- For a slice dx:

$$U = {}^1\!/_2 \int_0^L EAu'^2 dx$$

$$T = {}^1\!/_2 \int_0^L m\dot{u}^2 dx$$

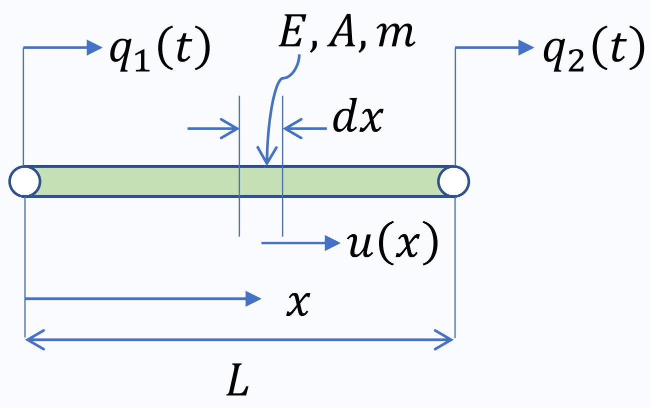


- and this used the shape functions:

$$u(x,t) = g_1(x)q_1(t) + g_2(x)q_2(t)$$

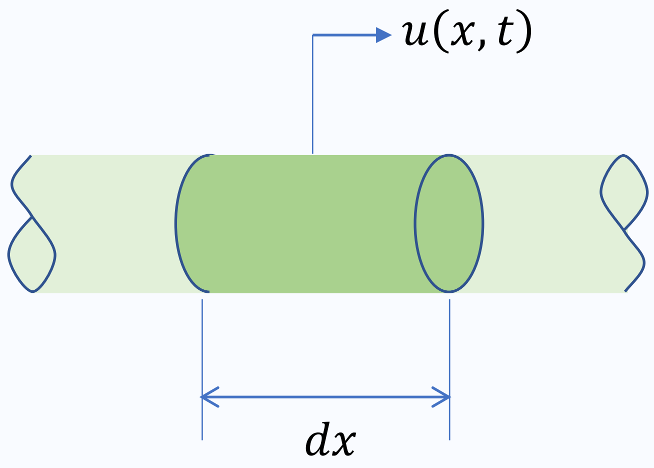
- This leads to:

Rods, Strings and Shafts



$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T [K] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

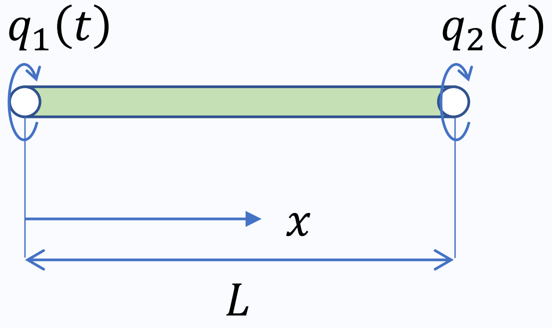
- and



$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T \begin{bmatrix} \frac{mL}{3} & \frac{mL}{6} \\ \frac{mL}{6} & \frac{mL}{3} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T [M] \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

- and we can formulate a new Finite Element problem right away without derivations, as long as it has strain energy and kinetic energy expressions of a similar form.

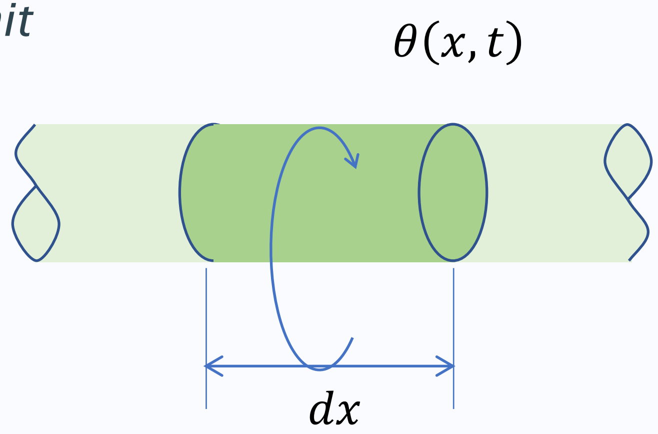
Shafts in torsion



- We can use this directly without reformulation for shafts in torsion:

$$U = {}^1\!/_2 \int_0^L GJ\theta'^2 dx \text{ and } T = {}^1\!/_2 \int_0^L I\dot{\theta}^2 dx$$

- where I is moment of inertia *per unit length*, as m was mass *per unit length*

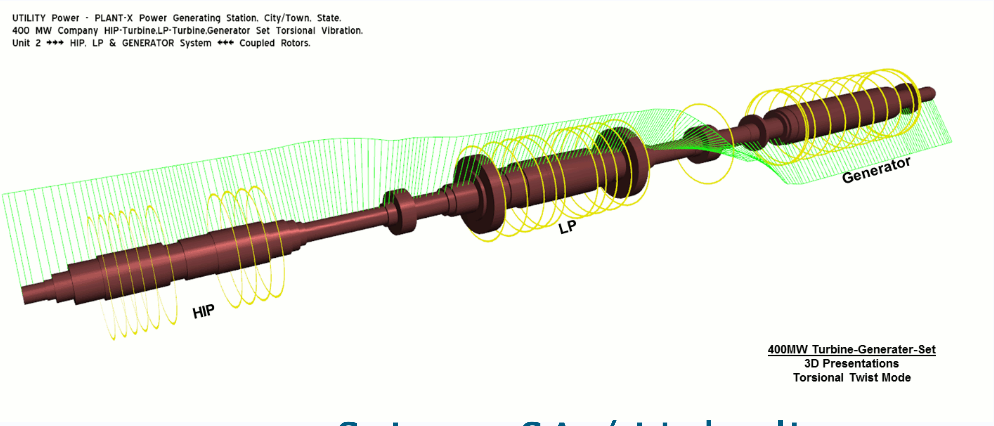


$$\theta(x,t) = g_1(x)q_1(t) + g_2(x)q_2(t)$$

- gives

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \frac{GJ}{L} & -\frac{GJ}{L} \\ -\frac{GJ}{L} & \frac{GJ}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T [K]_{shaft} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T \begin{bmatrix} \frac{IL}{3} & \frac{IL}{6} \\ \frac{IL}{6} & \frac{IL}{3} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T [M]_{shaft} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$



[Saipem SA / Linkedin]

Strings in transverse motion

- We can use this directly without reformulation for a tensed string:

  $U = {}^1\!/_2 \int_0^L T w'^2 dx$ where T is the predefined tension

  $T = {}^1\!/_2 \int_0^L m \dot{w}^2 dx$ where m is the mass per unit length

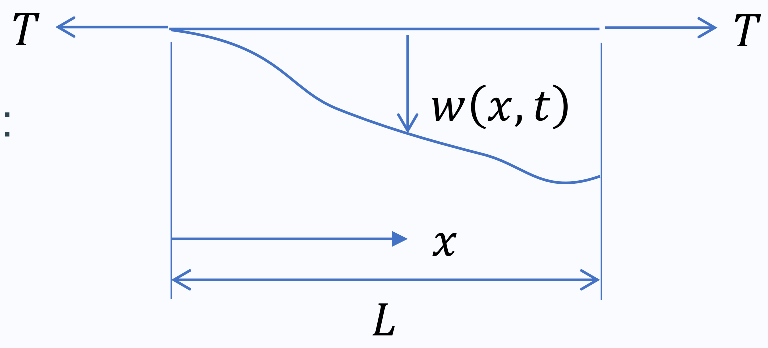
- with

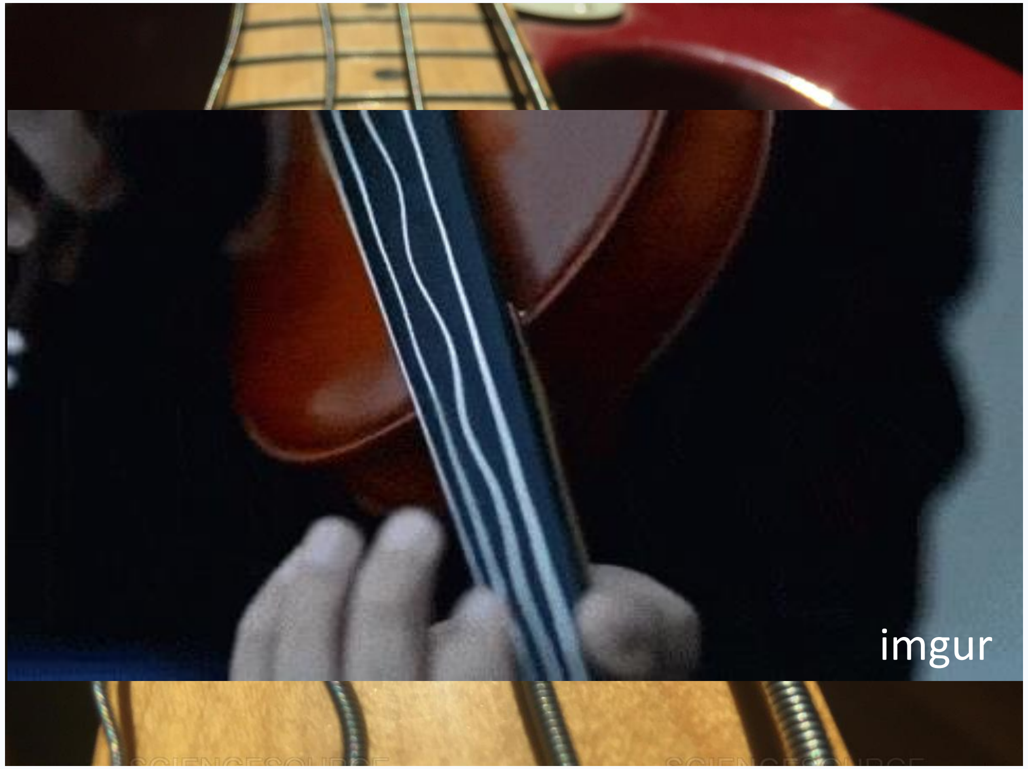
  $\theta(x,t) = g_1(x) q_1(t) + g_2(x) q_2(t)$

- gives

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \frac{T}{L} & -\frac{T}{L} \\ -\frac{T}{L} & \frac{T}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T [K]_{string} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T \begin{bmatrix} \frac{mL}{3} & \frac{mL}{6} \\ \frac{mL}{6} & \frac{mL}{3} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T [M]_{shaft} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$





Short-cut: 3 element types are analogous:

	Rods (Tens/Comp)	Shafts	Strings
Field Variable	$u(x,t)$	$\theta(x,t)$	$w(x,t)$
Potential Energy	$\frac{1}{2}\int_0^L EAu'^2dx$	$\frac{1}{2}\int_0^L GJ\theta'^2dx$	$\frac{1}{2}\int_0^L Tw'^2dx$
Kinetic Energy	$\frac{1}{2}\int_0^L m\dot{u}^2dx$	$\frac{1}{2}\int_0^L I\dot{\theta}^2dx$	$\frac{1}{2}\int_0^L m\dot{w}^2dx$
Generalised Coordinates	$q_1, q_2: u(0,t), u(L,t)$	$q_1, q_2: \theta(0,t), \theta(L,t)$	$q_1, q_2: w(0,t), w(L,t)$

Final note on Dynamics Questions

- In some Past Paper questions you might see reference to the ‘Rayleigh Ritz method’
- I have removed this from the syllabus and you can ignore such questions

- If we get time at the end we might cover it for interest but it will not be examined