Part 3: Beams in Bending
Introduction and Shape Functions

FEEG3001/SESM6047 FEA in Solid Mechanics

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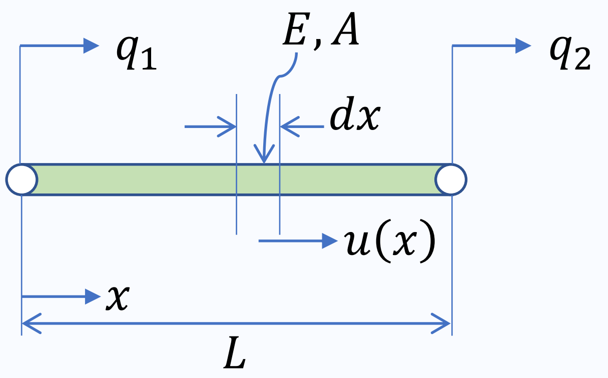
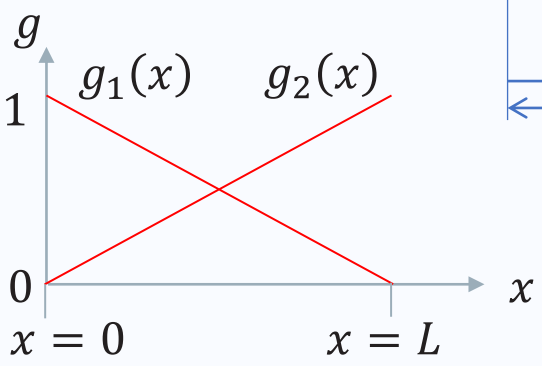
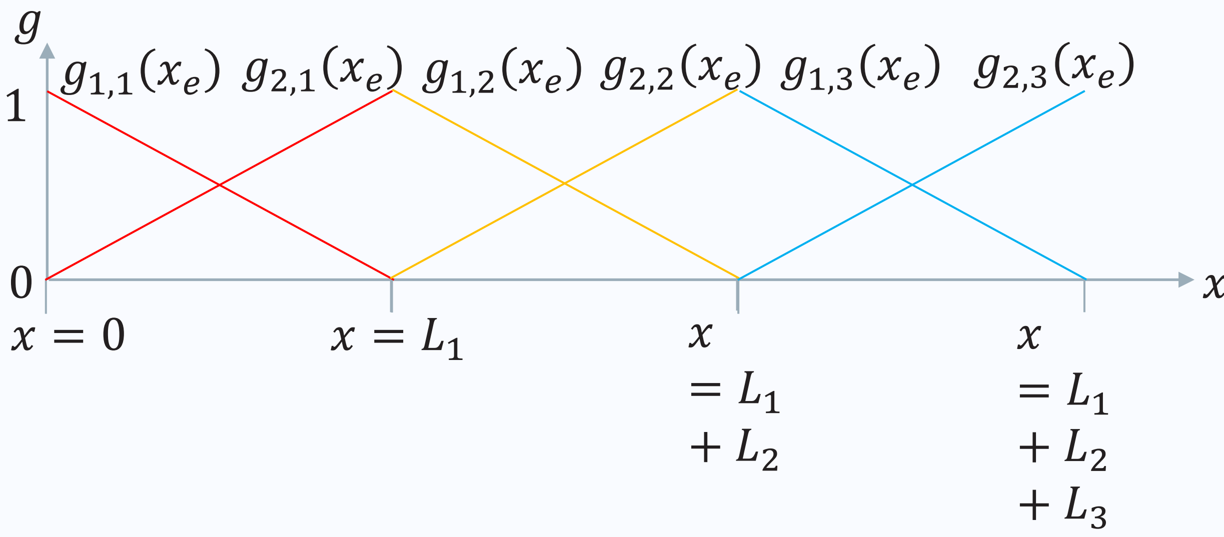
From 22nd October 2024

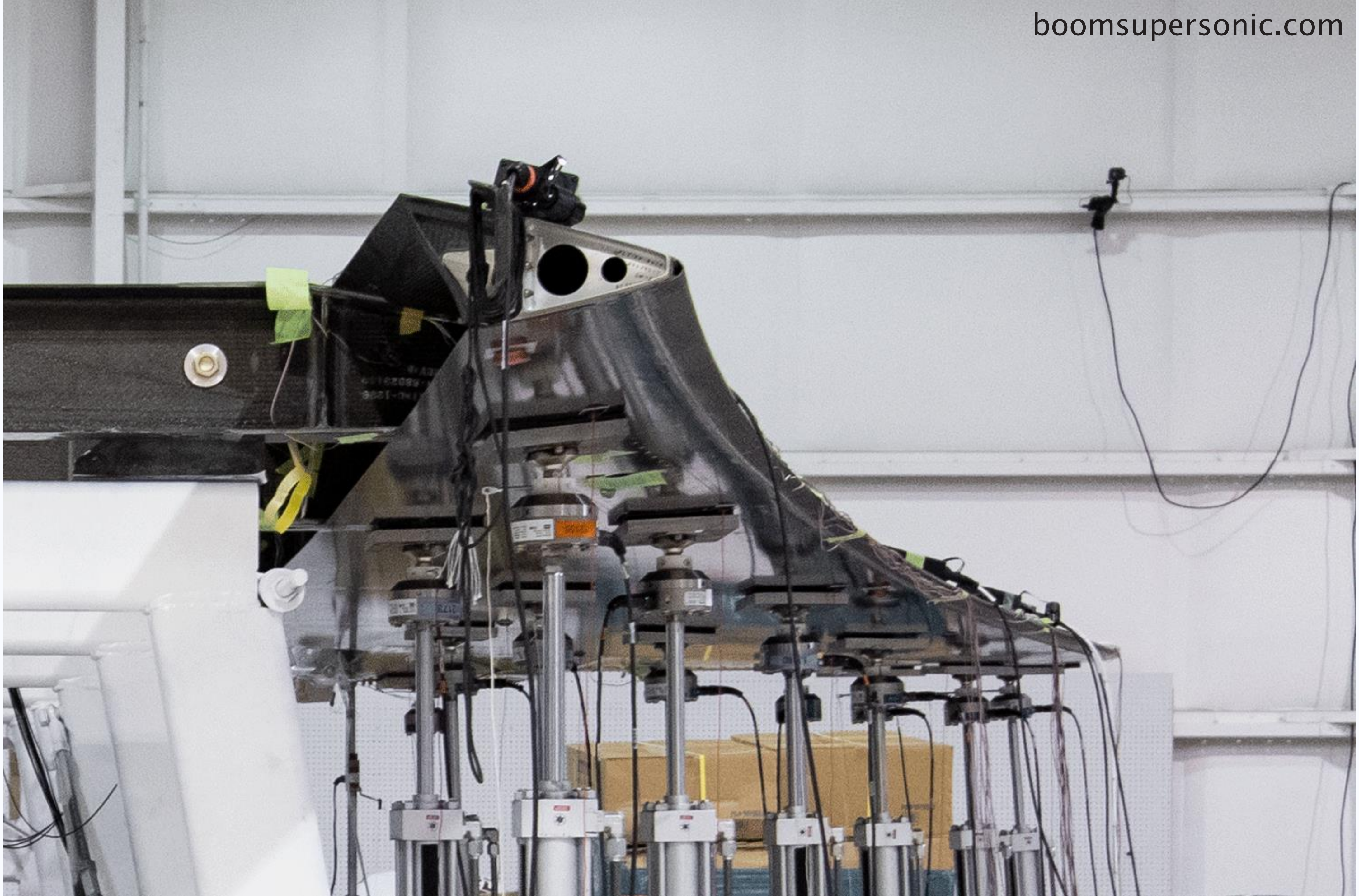
Reminder of linear interpolation functions for rods in axial tension and compression:





- Why use Shape Functions?
  - A continuum has an infinite number of Degrees of Freedom
  - FEA:
    - describes the mechanics of problems approximately,
    - using an equivalent description that has finite DoF, and
    - describes the displacement field in between using a pre-determined shape function.
- Shape Functions should provide continuity between adjoining elements…

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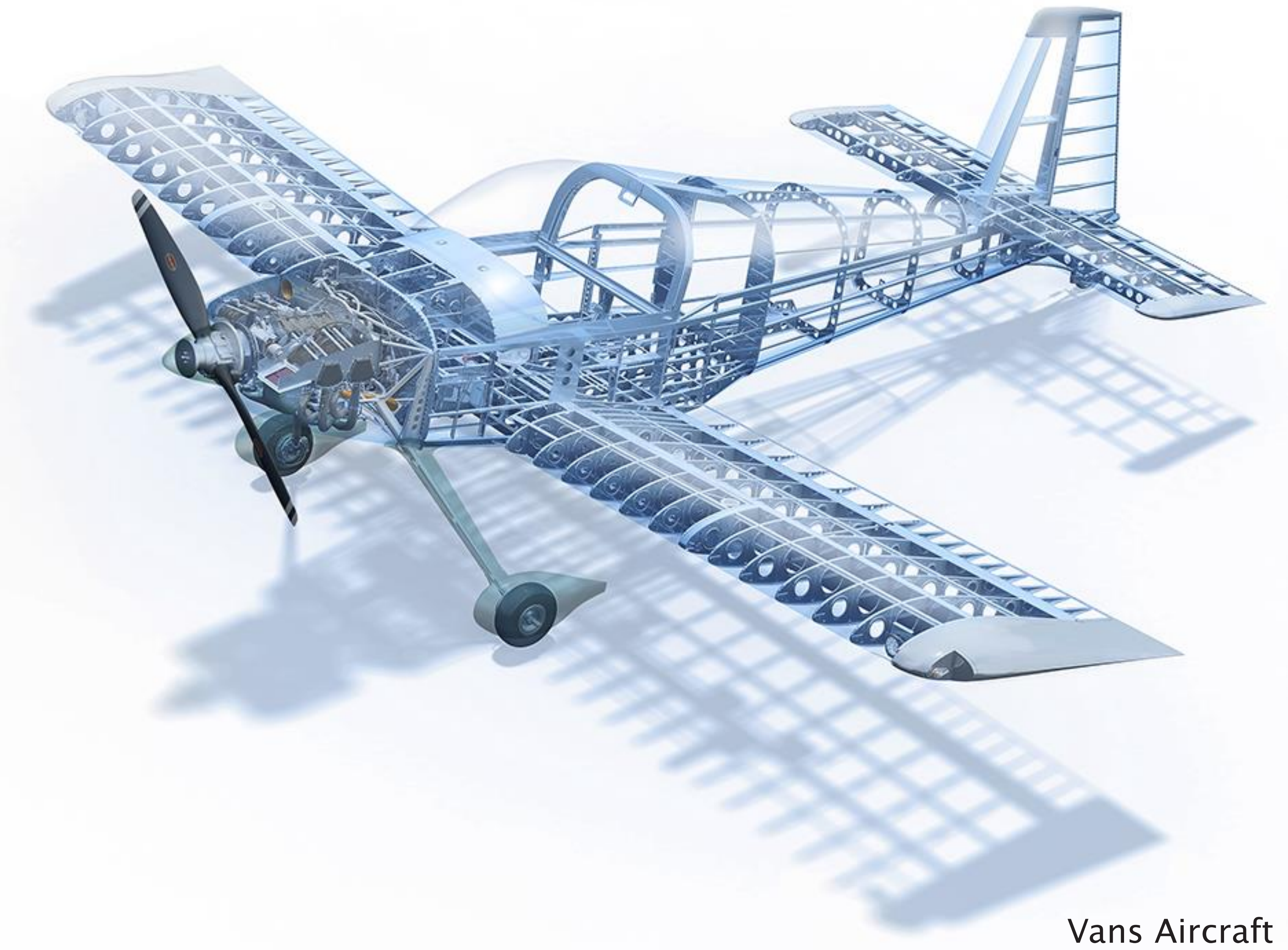
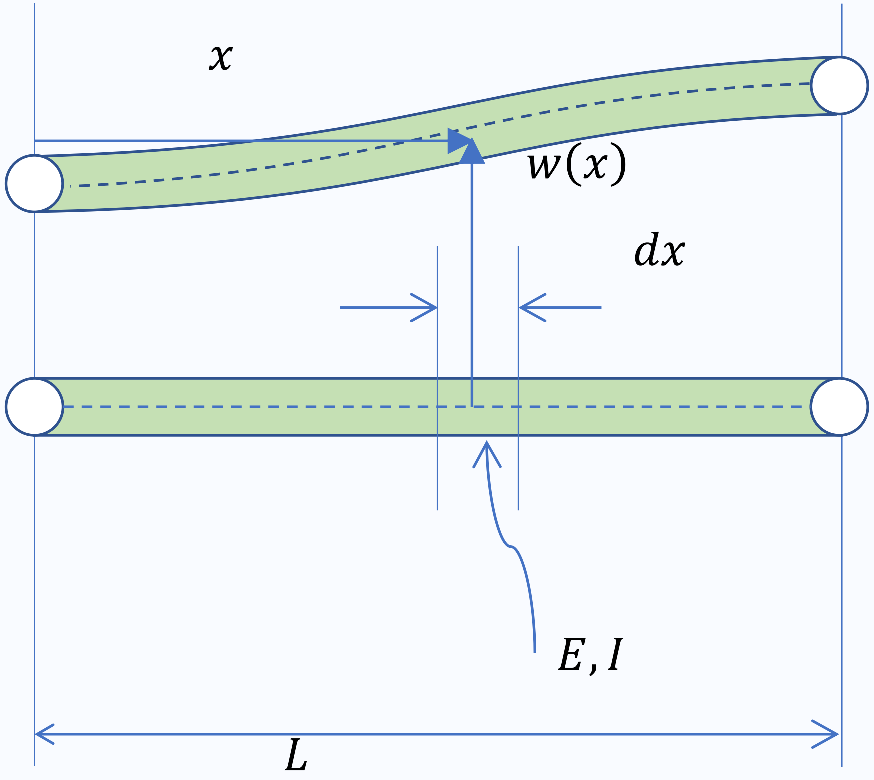
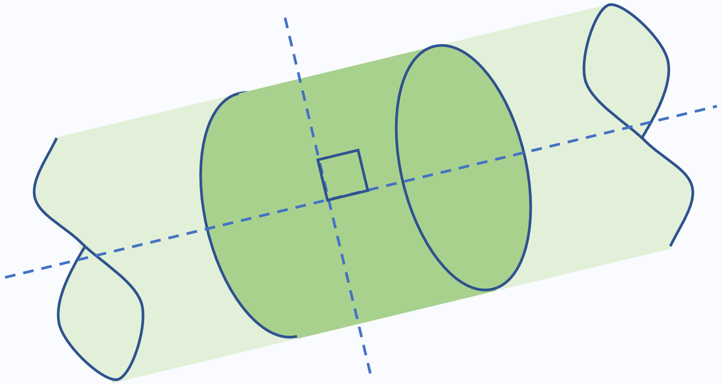


Vans Aircraft

Two-noded beam elements in bending

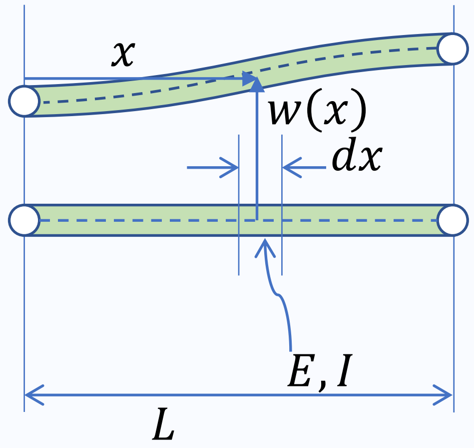
- Similar to what we saw for rods, but displacements $w(x)$ are perpendicular to the element's axis
- What parameters give it its bending properties?
  - E, Young's modulus
  - I, Second moment of area





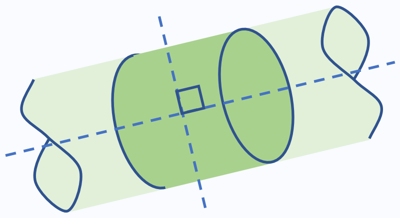
Euler-Bernoulli hypothesis assumptions:
- Cross sections do not change during bending
- Cross section remains perpendicular to the neutral axis during bending

Two-noded beam elements in bending



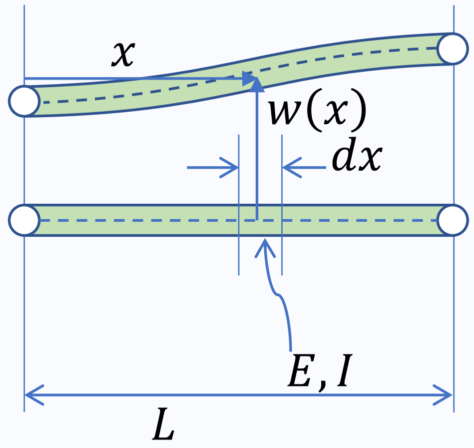
- Without deriving it, we define that the strain energy stored in the bent beam is given by:

$$U = {}^1\!/_2 \int_0^L EI(w(x)'')^2 dx$$



- where $(\cdot)' = \frac{d}{dx}(\cdot)$

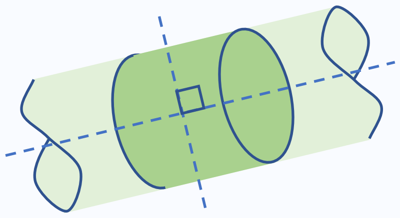
Two-noded beam elements in bending



- Recalling FEEG1002:

  EAu'' = longitudinal loading (rods, tens/comp)

  EIw'''' = transverse loading (beams)



- the axial rod differential equation has a second derivative of deformation
- the beam bending differential equation has a 4th derivative of deformation
- How do we handle this without solving the differential equation?
- We use a shape or interpolation function again;
- We cannot use linear interpolation – we need 'cubic' interpolation (i.e. the order of the D.E. minus 1).

Two-noded beam elements in bending

- We require continuity of deformation from one element to the next, and continuity of 1st derivative of deformation. Beams: transverse deflection and slope
- If we use cubic interpolation for transverse deflection:

$M(x) = \mathrm{EI}\frac{d^2w(x)}{dx^2}$ can capture linearly varying bending moment within an element

Since $\sigma = \frac{My}{I}$ this means we can capture linearly varying stress within an element

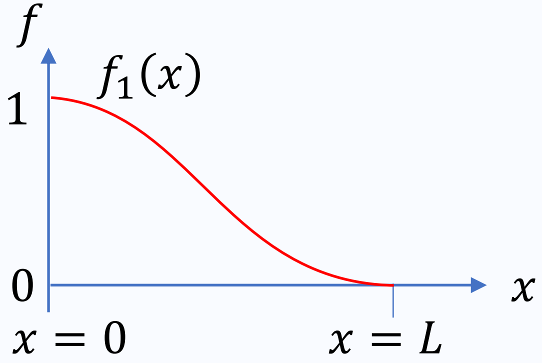
$\mathrm{V}(x) = \mathrm{EI}\frac{d^3w(x)}{dx^3}$ can capture constant shear force within an element

- Though deflection and slope must be continuous from element to element, bending moments and shear force are not. This allows us to apply concentrated moments and forces on nodes.
- FEA is popular because it 'weakens' the restriction on continuity of our interpolation functions (interpolation order is order of the differential equation -1)

Two-noded beam elements in bending

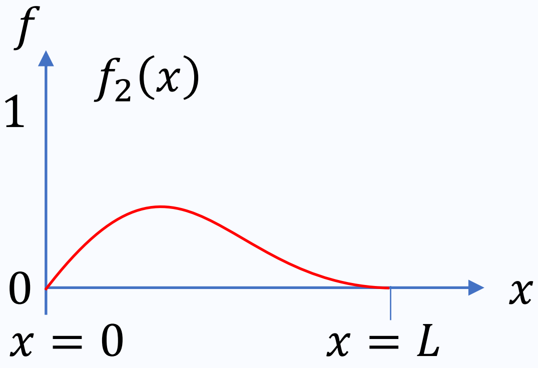
- Now we will define four cubic interpolation functions in the $x = 0$ to L domain, defined by their value and their slope.
- These are called the ‘Hermite cubics’:

	Left Node	Right Node
Value	1	0
Slope	0	0



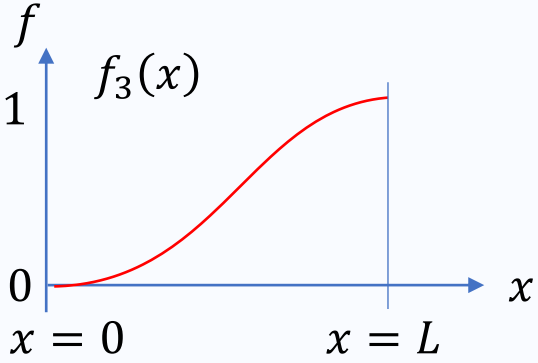
$$f_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$

	Left Node	Right Node
Value	0	0
Slope	1	0



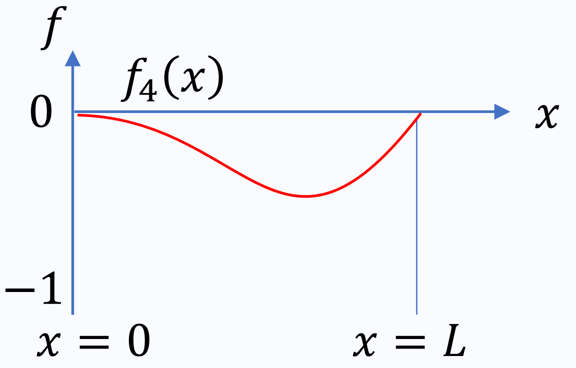
$$f_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}$$

	Left Node	Right Node
Value	0	1
Slope	0	0



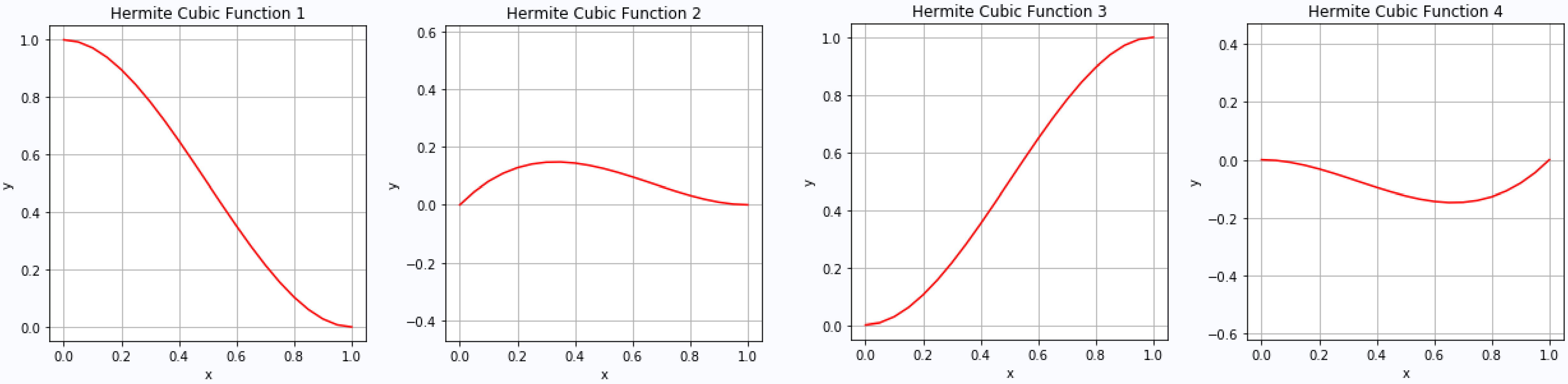
$$f_3(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}$$

	Left Node	Right Node
Value	0	0
Slope	0	1



$$f_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

The Hermite Cubics



Two-noded beam elements in bending

- and we make our approximation by saying the displacement anywhere in the element is approximated as:

$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4$$

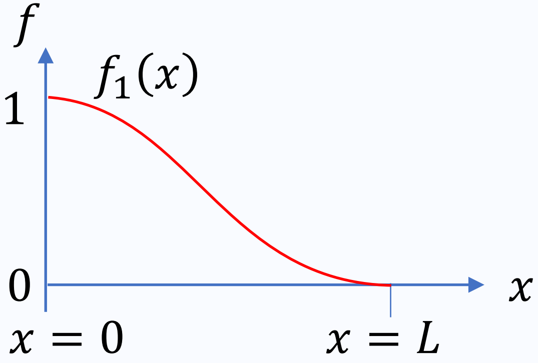
- where $f_i(x)$ are the four shape functions or interpolation functions, each having the form:

$$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3$$

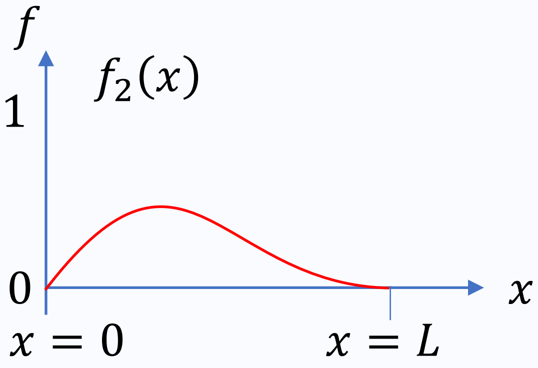
- We also now have four q_i values to find…
- We won't solve it 4 times, and you won't need to remember them, but you should now understand how, based on the axial rod.

Two-noded beam elements in bending

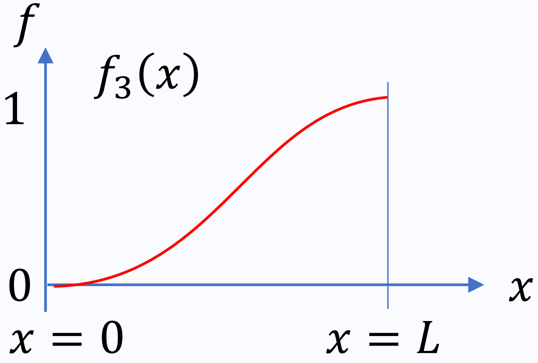
$$f_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$



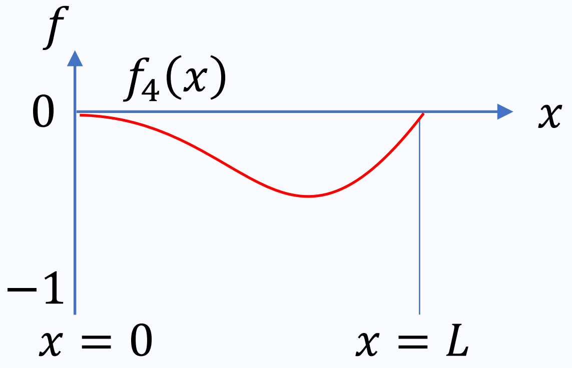
$$f_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}$$



$$f_3(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}$$



$$f_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$



$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4$$

- What can we say about the values of deflection at each end?

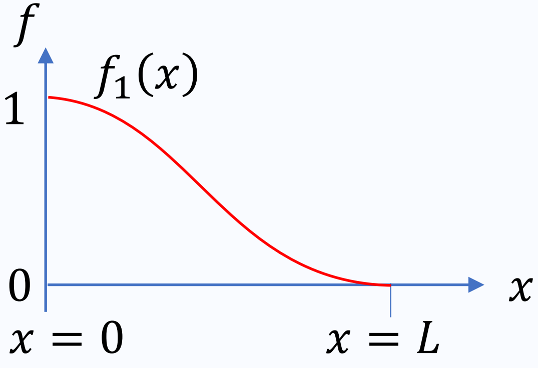
$$w(0) = f_1(0)q_1 + f_2(0)q_2 + f_3(0)q_3 + f_4(0)q_4 = q_1$$

$$w(L) = f_1(L)q_1 + f_2(L)q_2 + f_3(L)q_3 + f_4(L)q_4 = q_3$$

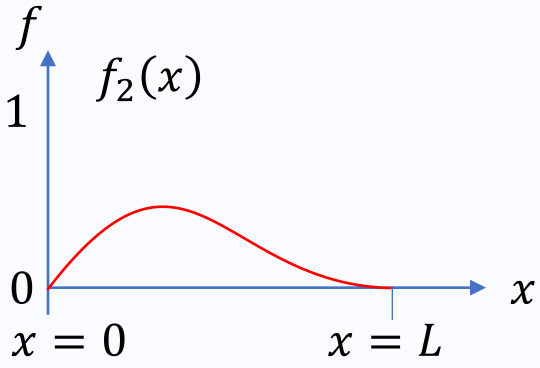
- so these are meaningful results! End deflections: generalised coordinates

Two-noded beam elements in bending

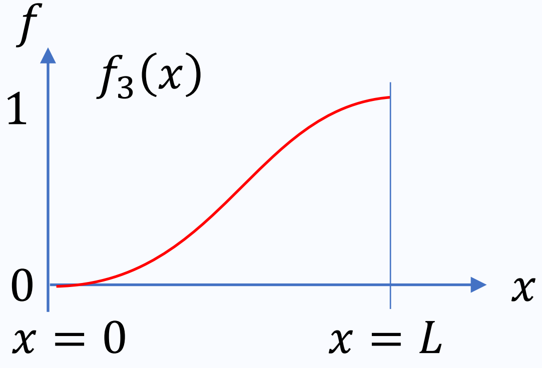
$$f_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$



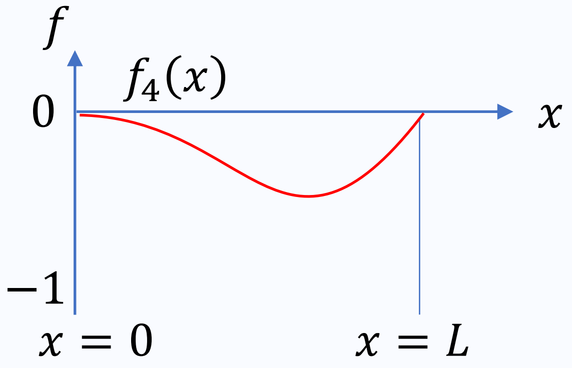
$$f_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}$$



$$f_3(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}$$



$$f_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$



$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4$$

- What can we say about the values of rotation at each end?

$$w'(0) = f_1{}'(0)q_1 + f_2{}'(0)q_2 + f_3{}'(0)q_3 + f_4{}'(0)q_4 = q_2$$

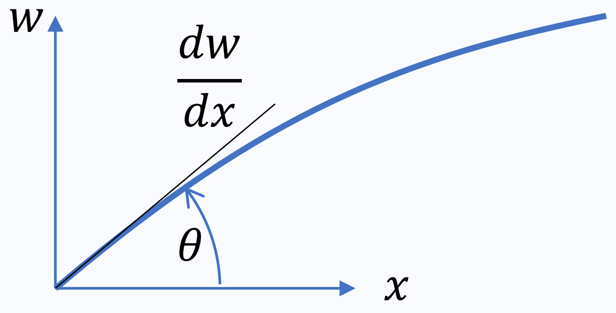
$$w'(L) = f_1{}'(L)q_1 + f_2{}'(L)q_2 + f_3{}'(L)q_3 + f_4{}'(L)q_4 = q_4$$

- also meaningful results! End slopes/rotations: generalised coordinates.

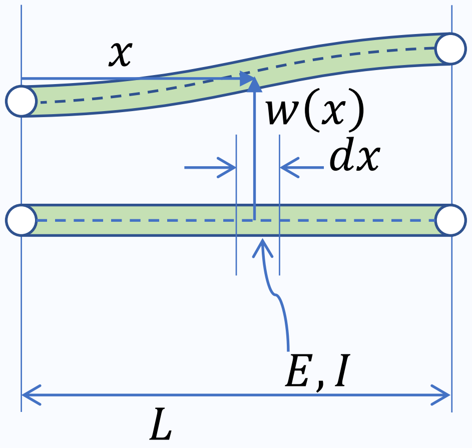
Two-noded beam elements in bending

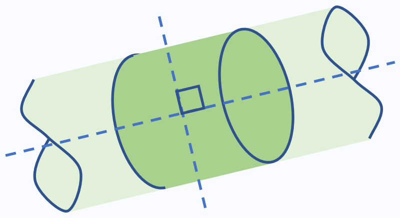
- because with small deformations, rotations and slopes are equivalent

$$w' = \frac{dw}{dx} = \tan\theta \text{ and for small } \theta \approx \sin\theta \approx \tan\theta$$

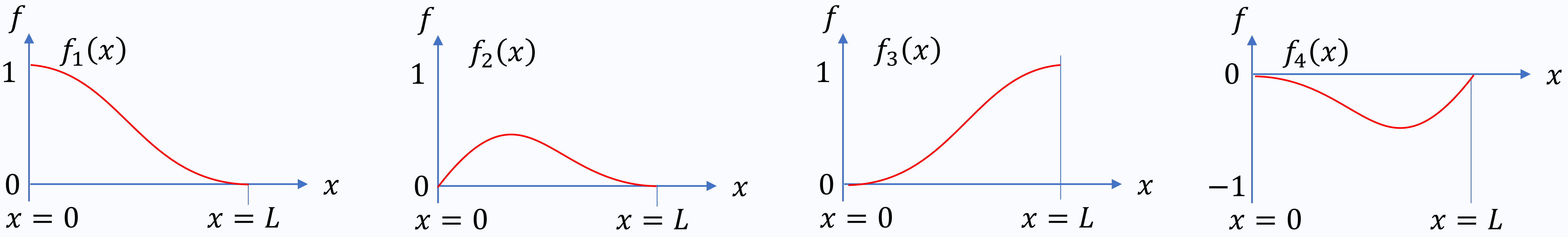


- recall we don't know what $w(x)$ is, but if we make the elements small enough (refined enough mesh), we can approximate throughout the element using the end deflections and slopes.





Recap:



- These are the shape functions for the 'Euler-Bernoulli Beam'
- This neglects transverse shear but often gives adequate predictions of beam deflection and stress with appropriate length : thickness ratios

- Next we will derive and start assembling beam element Stiffness Matrices!