Trig cheat (say: odd trig function hence integrates to zero)

$$\int_0^{2\pi} \sin^{2m+1} x \cos^n x \, dx = 0 \qquad n, m \in \mathbb{Z}$$
$$\int_0^{2\pi} \sin^{m} x \cos^{2n+1} x \, dx = 0 \qquad n, m \in \mathbb{Z}$$

ODEs

euler type linear homogeneous ODEs

So these are actually quite cool (and common), as you'll see in a bit this is a homogeneous Euler type:

$$x^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

What you'll note is the difference between this and a linear second order homogeneous ODE is the multiplication by decreasing powers of x, these actually cancel out with the differentials leading to a dimensionless equation.

Solving it

The case where k is real and has 2 solutions

$$x^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$
$$k_{1,2} = \frac{-(b-1) \pm \sqrt{(b-1)^2 - 4c}}{2} = \frac{(1-b) \pm \sqrt{(b-1)^2 - 4c}}{2}$$
$$y = Ax^{k_1} + Bx^{k_2}$$

> solving:
> Simply sub the values of k_1, k_2 into $y = Ax^{k_1} + Bx^{k_2}$

The case where k is complex or duplicate

$$x^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$
$$\frac{d^2y}{dt^2} + (b-1)\frac{dy}{dt} + cy = 0$$
$$t = \ln x$$

> solving:
> Use the same method as discussed in solving linear second order ODEs then sub t back in to get the equation in terms of x and y.

Fourier stuff

Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

> where:
> $f(x) = f(x+2L) =$ a periodic function (it repeats perfectly every $2L$)
> $a_0 =$ a offset constant (offsets the function from an average of 0)
> $a_n, b_n =$ the nth constant related to $\cos$ and $\sin$ respectively
> $L =$ half the period of the function

The constants a_n, b_n usually end up being defined as a function of n which when solved allows you to calculate the constants $a_1, a_2, a_3 \ldots$ where the more constants calculated the closer the approximation of the original periodic function $f(x)$.
(CBA to do the proof, just look it up)

Finding the constants

$$a_m = \frac{1}{L}\int_{R_2}^{R_1} f(x) \cos\left(\frac{n\pi}{L}x\right) \cdot dx$$
$$b_m = \frac{1}{L}\int_{R_2}^{R_1} f(x) \sin\left(\frac{n\pi}{L}x\right) \cdot dx$$
$$a_0 = \frac{1}{L}\int_{R_2}^{R_1} f(x) \cdot dx$$

> where:
> $a_m =$ often expands to a function defining the nth a constant in terms of m.
> $b_m =$ often expands to a function defining the nth b constant in terms of m.
> $f(x) = f(x+2L) =$ a periodic function (it repeats perfectly every $2L$)
> $L =$ half the period of the function
> $R_1, R_2 =$ a region the functions defined over, often this is just $R_1 = L$ and $R_2 = -L$

Complex fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{jn\pi x}{L}} \qquad c_n = \begin{cases} \frac{1}{2}(a_n - jb_n) & n > 0 \\ \frac{1}{2}a_0 & n = 0 \\ \frac{1}{2}(a_{-n} + jb_{-n}) & n < 0 \end{cases}$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n e^{\frac{jn\pi x}{L}} + \sum_{n=-\infty}^{-1} c_{(n)} e^{\frac{jn\pi x}{L}}$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[c_n e^{\frac{jn\pi x}{L}} + c_{(-n)} e^{\frac{-jn\pi x}{L}}\right]$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-j\frac{n\pi x}{L}} \cdot dx$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x) \cdot dx$$

where:
$f(x) =$ some function
$j = \sqrt{-1}$
$c_n =$ the nth constant

When dealing with this type of problem, the following identity is very frequently used:

$$\sin(A) = \frac{1}{2j}(e^{jA} - e^{-jA})$$

$$\cos(A) = \frac{1}{2}(e^{jA} + e^{-jA})$$

Fourier transform

$$\text{Fourier Transform:} \qquad F(\omega) \equiv \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-j\omega t} \cdot dt$$

$$\text{Inverse Transform:} \qquad f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \cdot d\omega$$

where:
$f(t) =$ some input function of t
$F(\omega) =$ the fourier transform of $f(t)$
$j = \sqrt{-1}$
$\omega =$ frequency
$\mathcal{F}[\ldots] =$ fourier transform of some function

Laplace stuff

Laplace transform

$$\mathcal{L}[f(x)] \equiv \bar{f}(s) = \int_{0}^{\infty} f(x) e^{-sx} \cdot dx$$

where:
$\mathcal{L}[\ldots] =$ Laplace transform function
$\bar{f}(s) =$ Laplace transform of $f(x)$
$s =$ frequency, independent variable of Laplace transform
$f(x) =$ some function
$x =$ independent variable of function being transformed

Laplace transform known results

$f(t)$	$\mathcal{L}[f(t)] \equiv \tilde{f}(s)$
A	$\frac{A}{s}, \quad \mathrm{Re}(s) > 0$
e^{at}	$\frac{1}{s-a}, \quad \mathrm{Re}(s) > a$
$t^n, \quad n = 1, 2 \dots$	$\frac{n!}{s^{n+1}}, \quad \mathrm{Re}(s) > 0$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}, \quad \mathrm{Re}(s) > 0$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}, \quad \mathrm{Re}(s) > 0$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}, \quad \mathrm{Re}(s) > \lvert\omega\rvert$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}, \quad \mathrm{Re}(s) > \lvert\omega\rvert$
$t^n f(t)$	$(-1)^n \frac{d^n \tilde{f}}{ds^n}$
$\frac{df}{dt}$	$s\tilde{f}(s) - f(0)$
$\frac{d^2 f}{dt^2}$	$s^2 \tilde{f}(s) - sf(0) - \frac{df}{dt}(0)$
$H(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}
$e^{-at} f(t)$	$\tilde{f}(s+a)$
$f(t-a)H(t-a)$	$e^{-as}\tilde{f}(s)$

$$\mathcal{L}\left[\frac{d^n f}{dx^n}\right] = s^n \tilde{f}(s) - \sum_{k=0}^{n-1} s^{n-(k+1)} f^k(0)$$

Heavyside function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$H(x-a) = \begin{cases} 0, & x < a \\ 1, & x > a \end{cases}$$

where:
$H(x)$ = heaviside function
x = independent variable

By summing multiple together it becomes possible to define bounds:

$$H(x-a) \times H(b-x) = \begin{cases} 1, & a < x < b \\ 0, & else \end{cases}$$

$$H(x-a) - H(x-b) = \begin{cases} 1, & a < x < b \\ 0, & else \end{cases}$$

where:
$H(x)$ = heaviside function
x = independent variable

dirac function

$$\int_{-\infty}^{\infty} \delta(x-a)K \cdot dx = K$$

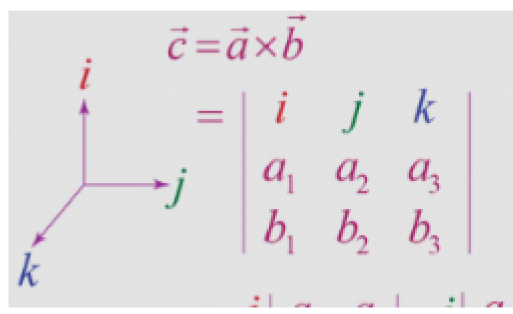
$$\int_{-\infty}^{\infty} \delta(x-a)f(x) \cdot dx = f(a)$$

$$\int_{B}^{A} \delta(x-a)K \cdot dx = 0 \quad \text{if } a > A \text{ or } a < B$$

$$\delta(x-a) \approx \begin{cases} 0, & x \neq a \\ \infty = \dfrac{1}{dx}, & x = a \end{cases}$$

Vector calculus

cross product



line integral of vector field

$$\vec{r}(t) = v_x(t)i + v_y(t)j + v_z(t)k \qquad t_0 \leq t \leq t_1$$

$$x : \vec{r}(t) = v_x(t) \qquad y : \vec{r}(t) = v_y(t) \qquad x : \vec{r}(t) = v_z(t)$$

$$\vec{F}(x,y,z) = f_x(x,y,z)i + f_y(x,y,z)j + f_z(x,y,z)k \quad \rightarrow \quad \begin{aligned} \vec{F}(r(t)) &= f_x(v_x,v_y,v_z)i + f_y(v_x,v_y,v_z)j + f_z(v_x,v_y,v_z)k \\ &= f_x(t) + f_y(t) + f_z(t) \end{aligned}$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_{t_0}^{t_1} \left[\vec{F}(r(t)) \cdot \frac{d\vec{r}}{dt} \right] dt \\ &= \int_{t_0}^{t_1} \left[f_x(t) \frac{dv_x(t)}{dt} + f_y(t) \frac{dv_y(t)}{dt} + f_y(t) \frac{dv_y(t)}{dt} \right] dt \end{aligned}$$

Del, nabla operator (put it on scalars)

$$\begin{aligned} (n)D: \ \nabla &= \sum_{i=1}^{n} \vec{e}_i \frac{\delta}{\delta x_i} \\ \nabla f &= \sum_{i=1}^{n} \vec{e}_i \frac{\delta f}{\delta x_i} \\ 3D: \ \nabla f &= \left(\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z} \right) = \vec{e}_x \frac{\delta}{\delta x} + \vec{e}_y \frac{\delta}{\delta y} + \vec{e}_z \frac{\delta}{\delta z} \end{aligned}$$

divergance operator:

$$\text{div } F = \nabla \cdot F$$

$$3D: \ \text{div } F = \frac{\delta F_x}{\delta x} + \frac{\delta F_y}{\delta y} + \frac{\delta F_z}{\delta z}$$

where:
∇ = del operator

Curl:

$$2D: \ \vec{\omega} = \nabla \times \vec{V} = \left(\frac{\delta V_y}{\delta x} - \frac{\delta V_x}{\delta y} \right) \hat{e}_z$$

$$3D: \ \vec{\omega} = \nabla \times \vec{F} = \left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z} \right) \hat{e}_x - \left(\frac{\delta F_z}{\delta x} - \frac{\delta F_x}{\delta z} \right) \hat{e}_y + \left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} \right) \hat{e}_z = \det \begin{pmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_y & F_z \end{pmatrix}$$

Normal of a plane (Just use del)

$$\text{Plane equation: } 0 = F(x,y,z) = f_x(x) + f_y(y) + f_z(z) + C$$

$$\text{Normal equation: } \vec{n} = \nabla F$$

Laplacian operator

$$\text{scalar case:} \quad \nabla^2 f \equiv \nabla \cdot (\nabla f) \equiv \text{div:}(\text{grad} f) = \frac{\delta^2}{\delta x^2} f + \frac{\delta^2}{\delta y^2} f + \frac{\delta^2}{\delta z^2} f$$

$$\text{vector case:} \quad \nabla^2 \vec{f} = \hat{i} \nabla^2 (\vec{f} \cdot \hat{i}) + \hat{j} \nabla^2 (\vec{f} \cdot \hat{j}) + \hat{k} \nabla^2 (\vec{f} \cdot \hat{k}) = \left(\nabla^2 f_{\hat{i}}, \nabla^2 f_{\hat{j}}, \nabla^2 f_{\hat{k}} \right)$$

$$\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$$

Vector identities

1. $\nabla(fg) = f\nabla g + g\nabla f$ $\longleftarrow$ Linearity of grad
2. $\nabla \times (\nabla f) = 0$ $\Leftrightarrow$ curl of grad of a scalar **vanishes**
3. $\nabla \times (g\vec{F}) = g(\nabla \times \vec{F}) + \nabla g \times \vec{F}$ $\longleftarrow$ Linearity of curl
4. $\nabla \cdot (g\vec{F}) = (\nabla g) \cdot \vec{F} + g\nabla \cdot \vec{F}$ $\longleftarrow$ Linearity of div
5. $\nabla \cdot (\nabla \times \vec{F}) = 0$ $\Leftrightarrow$ div of curl of a vector **vanishes**
6. $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$
7. $\nabla(\vec{F} \cdot \vec{G}) = \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F}) + (\vec{F} \cdot \nabla)\vec{G} + (\vec{G} \cdot \nabla)\vec{F}$
8. $\nabla \times (\vec{F} \times \vec{G}) = \vec{F}(\nabla \cdot \vec{G}) - \vec{G}(\nabla \cdot \vec{F}) + (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G}$

Conservative field

$$\vec{F} = -\nabla \phi(x,y,z)$$

in (continuous across domain)

representing terrain height then the terrain represents the potent
in, since irrelevant of path the difference in height between two p
mple it is the case that:

$$\vec{F} = -\nabla \phi(x,y,z) = -\nabla(\phi(x,y,z) + C)$$

d A, B then the potential difference is the same regardless of path.

$$\int_A^B \vec{F} \cdot d\vec{s} = \phi(A) - \phi(B)$$

d is not conservative:

$$\text{conservative field:} \quad \nabla \times \vec{F} = 0$$

$$\text{non-conservative field:} \quad \nabla \times \vec{F} \neq 0$$

Surface normal:

$$\vec{r}(s,t) = f_x(s,t)\,\hat{i} + f_y(s,t)\,\hat{j} + f_z(s,t)\,\hat{k}$$

$$\frac{\delta \vec{r}}{\delta s} \cdot \frac{\delta \vec{r}}{\delta t} = 0$$

$$\frac{\delta \vec{r}}{\delta s} \times \frac{\delta \vec{r}}{\delta t} = \vec{n}$$

Surface integral:

$$\begin{aligned} dA &= \left| \frac{\delta \vec{r}}{\delta s} \times \frac{\delta \vec{r}}{\delta t} \right| ds\, dt \\ &= a\, ds\, dt \end{aligned}$$

$$\begin{aligned} A = \int dA &= \int\int \left| \frac{\delta \vec{r}}{\delta s} \times \frac{\delta \vec{r}}{\delta t} \right| ds\, dt \\ &= \int\int a\, ds\, dt \end{aligned}$$

Flux integral:

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_S \left[\vec{F}(\vec{r}(s,t)) \cdot \left(\frac{\delta \vec{r}}{\delta s} \times \frac{\delta \vec{r}}{\delta t} \right) \right] ds\, dt$$

Volume integrals:

$$\begin{aligned} \int\int\int_V f\, dV &= \int\int\int_V f(x,y,z)\, dx\, dz\, dy \\ &= \int\int\int_V f(s,t,u)\, J\, ds\, dt\, du \end{aligned}$$

$$J = \frac{\delta(x,y,z)}{\delta(s,t,u)} = \begin{vmatrix} \frac{\delta x}{\delta s} & \frac{\delta x}{\delta t} & \frac{\delta x}{\delta u} \\ \frac{\delta y}{\delta s} & \frac{\delta y}{\delta t} & \frac{\delta y}{\delta u} \\ \frac{\delta z}{\delta s} & \frac{\delta z}{\delta t} & \frac{\delta z}{\delta u} \end{vmatrix}$$

Divergance theorem:

$$\iiint_V (\nabla \cdot \vec{F})\, dV = \oiint_{S=\delta V} \vec{F} \cdot d\vec{S}$$

Stokes theorem:

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{C=\delta S} \vec{F} \cdot d\vec{r}$$