Lecture 25 - The Laplacian and Vector Identities

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Outline

$\rightarrow$ Review

($\swarrow$ recall: $\boldsymbol{\nabla} \equiv \vec{\boldsymbol{\nabla}}$)

- Gradient

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x}\,\hat{\imath} + \frac{\partial \phi}{\partial y}\,\hat{\jmath} + \frac{\partial \phi}{\partial z}\,\hat{k}$$

- Divergence

$$\text{div}\,\vec{F} \equiv \boldsymbol{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- Curl

$$\text{curl}\,\vec{F} \equiv \boldsymbol{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Review

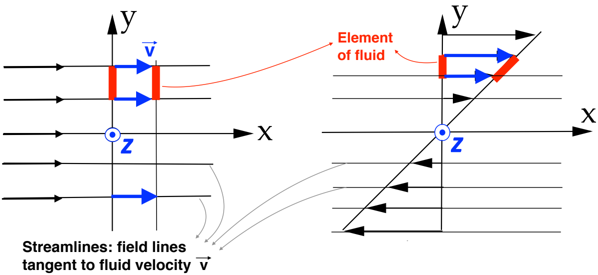
- If $\vec{F} \equiv \vec{v}$ is **fluid velocity** then $\vec{\omega} = \nabla \times \vec{F}$ is **vorticity**

Irrotational flow: $\nabla \times \vec{v} = 0$

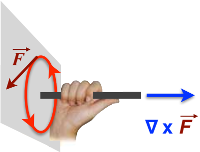
(No vorticity: elements of fluid do not spin)

Rotational flow: $\nabla \times \vec{v} \neq 0$

(fluid with vorticity: elements of fluid 'spin')



- **Direction of** $\nabla \times \vec{F}$ given by **right hand rule (version 2):**



Outline

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$\to$ The Laplacian

• The **divergence of** a **gradient** is such an important differentiation operation that it has both its own name, **Laplacian**, and symbol, ∇^2.

$$\operatorname{div}\operatorname{grad} f \equiv \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f) \equiv \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} .$$

• **Laplacian operator** (a **scalar differential operator**):

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

• The Laplacian is important in many areas of physics/engineering. It appears in three dimensional **wave** and **diffusion** PDE problems:

$$\frac{\partial^2 \phi}{\partial t^2} - \boldsymbol{\nabla}^2 \phi = 0 , \qquad \frac{\partial \phi}{\partial t} - \boldsymbol{\nabla}^2 \phi = 0$$

as well as in **static** situations (**elliptic PDEs**) in **electromagnetism** and **gravity**:

$$\nabla^2 \phi = \rho \qquad \longleftarrow \text{Poisson's equation}$$

and in **incompressible fluid flow:**

Examples of calculating the Laplacian

Example:

Let $\phi(x,y,z) = e^x y^3 \sin z$. Calculate $\nabla^2\phi$.

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\,\hat{\imath} + \frac{\partial\phi}{\partial y}\,\hat{\jmath} + \frac{\partial\phi}{\partial z}\,\hat{k}$$
$$= e^x y^3 \sin z\,\hat{\imath} + 3e^x y^2 \sin z\,\hat{\jmath} + e^x y^3 \cos z\,\hat{k}$$

Then the Laplacian of scalar field ϕ, $\mathrm{Lap}\,\phi \equiv \nabla^2\phi \equiv \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi)$ is:

$$\nabla^2\phi \equiv \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
$$= \frac{\partial(e^x y^3 \sin z)}{\partial x} + \frac{\partial(3e^x y^2 \sin z)}{\partial y} + \frac{\partial(e^x y^3 \cos z)}{\partial z}$$
$$= e^x y^3 \sin z + 6e^x y \sin z - e^x y^3 \sin z$$

$\nwarrow$ Laplacian of a <u>scalar</u> field is thus a <u>scalar field</u>

Alternatively, but equivalently, we can compute the Laplacian as:

$$\begin{aligned}
\nabla^2\phi &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\
&= \frac{\partial^2(e^x y^3 \sin z)}{\partial x^2} + \frac{\partial^2(e^x y^3 \sin z)}{\partial y^2} + \frac{\partial^2(e^x y^3 \sin z)}{\partial z^2} \\
&= e^x y^3 \sin z + 6e^x y \sin z - e^x y^3 \sin z \\
&= \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi)
\end{aligned}$$

Laplacian of a vector field

Let $\vec{F}$ be a vector field with components

$$\vec{F} = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}\,.$$

Then the **Laplacian** of this **vector field** is itself a **vector field** with components given by the Laplacian of the components of $\vec{F}$:

$$\nabla^2\vec{F} = (\nabla^2 F_1)\,\hat{\imath} + (\nabla^2 F_2)\,\hat{\jmath} + (\nabla^2 F_3)\,\hat{k}\,.$$

where F_1, F_2 and F_3 are scalar fields and thus:

$$\nabla^2 F_n = \frac{\partial^2 F_n}{\partial x^2} + \frac{\partial^2 F_n}{\partial y^2} + \frac{\partial^2 F_n}{\partial z^2}$$

Outline

→ Vector identities

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We summarise here some key vector identities. In this list $f(\vec{x})$ and $g(\vec{x})$ are generic scalar functions, while $\vec{F}(\vec{x})$ and $\vec{G}(\vec{x})$ are generic vector fields:

1. $\nabla(f\,g) = f\,\nabla g + g\,\nabla f$ $\longleftarrow$ Linearity of grad
2. $\nabla \times (\nabla f) = 0$ $\Leftrightarrow$ curl of grad of a scalar **vanishes**
3. $\nabla \times (g\,\vec{F}) = g(\nabla \times \vec{F}) + \nabla g \times \vec{F}$ $\longleftarrow$ Linearity of curl
4. $\nabla \cdot (g\,\vec{F}) = (\nabla g) \cdot \vec{F} + g \nabla \cdot \vec{F}$ $\longleftarrow$ Linearity of div
5. $\nabla \cdot (\nabla \times \vec{F}) = 0$ $\Leftrightarrow$ div of curl of a vector **vanishes**
6. $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$
7. $\nabla(\vec{F} \cdot \vec{G}) = \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F}) + (\vec{F} \cdot \nabla)\vec{G} + (\vec{G} \cdot \nabla)\vec{F}$
8. $\nabla \times (\vec{F} \times \vec{G}) = \vec{F}(\nabla \cdot \vec{G}) - \vec{G}(\nabla \cdot \vec{F}) + (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G}$

These results can be **proved at the (vector) component level**.
Here, you particularly see why it is important to
put arrows on top of vectors to **distinguish** them from **scalars**!

Outline

Summary

- Laplacian:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) \equiv \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \,.$$

- We learned vector identities:

1. $\boldsymbol{\nabla}(f\,g) = f\,\boldsymbol{\nabla} g + g\,\boldsymbol{\nabla} f \qquad \longleftarrow$ Linearity of grad
2. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0 \quad \Leftrightarrow$ curl of grad of a scalar **vanishes**
3. $\boldsymbol{\nabla} \times (g\,\vec{F}) = g(\boldsymbol{\nabla} \times \vec{F}) + \boldsymbol{\nabla} g \times \vec{F} \qquad \longleftarrow$ Linearity of curl
4. $\boldsymbol{\nabla} \cdot (g\,\vec{F}) = (\boldsymbol{\nabla} g) \cdot \vec{F} + g\boldsymbol{\nabla} \cdot \vec{F} \qquad \longleftarrow$ Linearity of div
5. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \vec{F}) = 0 \quad \Leftrightarrow$ div of curl of a vector **vanishes**
6. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times F) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F}$
7. $\boldsymbol{\nabla}(\vec{F} \cdot \vec{G}) = \vec{F} \times (\boldsymbol{\nabla} \times \vec{G}) + \vec{G} \times (\boldsymbol{\nabla} \times \vec{F}) + (\vec{F} \cdot \boldsymbol{\nabla})\vec{G} + (\vec{G} \cdot \boldsymbol{\nabla})\vec{F}$
8. $\boldsymbol{\nabla} \times (\vec{F} \times \vec{G}) = \vec{F}(\boldsymbol{\nabla} \cdot \vec{G}) - \vec{G}(\boldsymbol{\nabla} \cdot \vec{F}) + (\vec{G} \cdot \boldsymbol{\nabla})\vec{F} - (\vec{F} \cdot \boldsymbol{\nabla})\vec{G}$

Here, you particularly see why it is important to
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