Outline

Lecture 15 - Separation of Variables

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MATH2048, Semester 1

Outline

1. Review
2. Separation of Variables
3. Summary

Review

- Classified simple linear PDEs (hyperbolic, parabolic, elliptic)
- Derived the **wave equation**

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0.$$

which is the prototype example of a **hyperbolic PDE.**

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How to solve a wave equation?

We propose the following *ansatz* (educated guess) for the solution:

$y(x,t) = X(x)T(t) \quad \leftarrow$ that we call the **Separation of Variables** *ansatz*

We then use a **3-step strategy** to find the PDE solution $y(x,t)$:

1. use the wave equation (PDE) and boundary conditions to get two **ODEs** (no longer a PDE!). Now we can solve the ODEs (much simpler!);
2. recombine ODE solutions $X(x)$ and $T(t)$ to get a simple solution;
3. combine all such solutions into the **general** solution $y(x,t)$.

Example: Application to the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}; \quad y(0,t) = 0, \;\; y(L,t) = 0; \quad \underset{\text{S. of V.}}{\longrightarrow} y(x,t) = X(x)T(t).$$

Substitute $y = XT$ in the PDE and divide it by y (i.e. by XT) to get

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 (XT)}{\partial t^2} = X\frac{d^2 T}{dt^2} = X\ddot{T} \\ \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 (XT)}{\partial x^2} = T\frac{d^2 X}{dx^2} = TX'' \end{cases} \Rightarrow X\ddot{T} = c^2 TX'' \quad \Leftrightarrow \quad \frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X}.$$

Key step: The LHS only depends on t. On the other hand the RHS only depends on x. So the only way this equation can be valid is if both the LHS and the RHS are equal to the same constant λ (say).

So, the fact that both sides are separately constant gives two ODEs:

$$\begin{cases} \frac{1}{c^2}\frac{\ddot{T}}{T} = \lambda, \\ \frac{X''}{X} = \lambda. \end{cases} \Leftrightarrow \quad \begin{cases} \ddot{T} - c^2\lambda T = 0, \\ X'' - \lambda X = 0. \end{cases}$$

Boundary conditions

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So we need to solve two ODEs to find $X(x)$, $T(t)$ and the separation constant λ... this is looking like and Eigenvalue problem for the eigenvalue λ.
But, to solve this system of ODEs

$$\begin{cases} X'' - \lambda X = 0 \,. \\ \ddot{T} - c^2 \lambda T = 0 \,, \end{cases}$$

we **need boundary conditions (BCs)**. We have that (from BCs in previous slide)

$$y(0,t) = 0 \Leftrightarrow X(0)T(t) = 0, \qquad y(L,t) = 0 \Leftrightarrow X(L)T(t) = 0$$
$$\Rightarrow \quad X(0) = 0, \qquad\qquad \Rightarrow \quad X(L) = 0.$$

• In this problem, we have no boundary conditions for T.

1 PDE $\longrightarrow$ 2 ODEs. BUT 2 BCs for $y(x,t)$ $\underset{(\text{ not })}{\not\longrightarrow}$ $\begin{cases} \text{2 BCs for } X(x). \\ \text{2 BCs for } T(t). \end{cases}$

Eigenvalue Problem: Find λ (separation constant)

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We thus have the Eigenvalue Problem

$$X'' - \lambda X = 0; \qquad X(0) = 0, \quad X(L) = 0.$$

λ is the unknown eigenvalue (revisit Lecture 3: there $X'' + \lambda X = 0$!!).

We have to consider the **three cases** (revisit Lecture 3):

1. $\lambda = k^2 > 0$ (distinct real roots $\Rightarrow X = A\,e^{kx} + B\,e^{-kx}$),
2. $\lambda = 0$ ($\Rightarrow X = A + B\,x$),
3. $\lambda = -k^2 < 0$ (complex conjugate roots $\Rightarrow X = A\sin(k\,x) + B\cos(k\,x)$),

we find that only the **third case** gives a **non-trivial** solution. Namely, we get the eigenfunction and eigenvalue:

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

[Exercise: get this result! (see Lecture Notes $\oint$ 5.4.2)]

Using the separation constant λ_n to find $T_n(t)$

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- First ODE for $X_n(x)$ ✓ Separation constant λ_n ✓
- But we still have to solve the second **ODE for** $T(t)$:

$$\ddot{T} - c^2 \lambda T = 0 \qquad \text{with } \underline{\text{no}} \text{ boundary conditions}$$

but **we now know the value of** $\lambda = -k^2 < 0$.

So this is a **constant coefficient ODE** (revisit Lecture 1).

Since $\lambda = -k^2 < 0$ the associated auxiliary equation is a quadratic with two purely imaginary roots

$$\Lambda = \pm \mathbf{j}\, c \frac{n\pi}{L}, \qquad n = 1, 2, 3, \cdots .$$

So its **general solution** $T(t) = T_n(t)$ is:

$$T_n(t) = \tilde{C}_n \cos\left(\frac{n\pi c\, t}{L}\right) + \tilde{D}_n \sin\left(\frac{n\pi c\, t}{L}\right).$$

A solution y_n (one for each n)

We have our separation *ansatz* $y(x,t) = X(x)T(t)$ and solutions

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right), \qquad T_n(t) = \tilde{C}_n \cos\left(\frac{n\pi c\, t}{L}\right) + \tilde{D}_n \sin\left(\frac{n\pi c\, t}{L}\right).$$

Combining this, $\boldsymbol{y_n(x,t) = X_n(x) T_n(t)}$, gives $\underline{\underline{\textbf{a}}}$ solution (i.e. for $\underline{\text{a}}$ given n):

$$y_n(x,t) = \left[C_n \cos\left(\frac{n\pi c\, t}{L}\right) + D_n \sin\left(\frac{n\pi c\, t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right).$$

The A_n coefficient has been absorbed in the C_n, D_n coefficients:
($A_n\tilde{C}_n \equiv C_n$, $A_n\tilde{D}_n \equiv D_n$).

The general solution (superposition of y_n's)

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The wave equation is a **linear PDE**: this means that the sum of 2 or more solutions is still a solution of the PDE.

So we can **superpose** (i.e. sum) all our solutions y_n to get the **general solution**:

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) \quad \Leftrightarrow$$

$$y(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos\left(\frac{n\pi c\, t}{L}\right) + D_n \sin\left(\frac{n\pi c\, t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right).$$

This is the most general form of the solution you get using separation of variables.

What about the (so far) arbitrary coefficients $\boldsymbol{C_n}$, $\boldsymbol{D_n}$?

Can we fix them? **Do they cover all initial data?**

Initial data: finding C_n and D_n

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$$y(x,t) = \sum_{n=1}^{\infty}\left[C_n \cos\left(\frac{n\pi c\,t}{L}\right) + D_n \sin\left(\frac{n\pi c\,t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right) \qquad \longrightarrow \boldsymbol{C_n = ?\ \ D_n = ?}$$

- **Initial data**: the function y and its time derivative $\dot{y}$ at $t = 0$.
- Suppose we are given the **initial data**:

$\swarrow$ $f(x)$ and $g(x)$ are known functions

$$y(x,0) = f(x), \qquad \frac{\partial y}{\partial t}(x,0) = g(x). \tag{1}$$

Evaluating our general solution at $t = 0$ and imposing (1) gives:

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right), \qquad g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} D_n \sin\left(\frac{n\pi x}{L}\right).$$

These are just **Fourier Series** $\Longrightarrow$ we know the condition for which the **Euler coefficients** C_n**,** D_n can be computed (these are the **Dirichlet conditions** and associated theorem of Lecture 6)

$\Longrightarrow$ we know when Separation of Variables works.

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Summary: Separation of variables in 6 steps

$$y(x,t) = X(x)T(t)$$

1. Determine equations for X, T.
2. Use boundary conditions of y in order to obtain boundary conditions of X.
3. Solve eigenvalue problem for X: determine eigenvalues λ_n and eigenfunctions X_n.
4. Insert eigenvalue λ_n in the T equation and solve it to obtain T_n.
5. The normal modes are $y_n = X_n T_n$ and the general solution is obtained by superposition

$$y(x,t) = \sum_n X_n(x)T_n(t)$$

6. Use initial conditions, $y(x,0), \partial y(x,0)/\partial t$ to determine all undetermined coefficients. This step involves Fourier series.