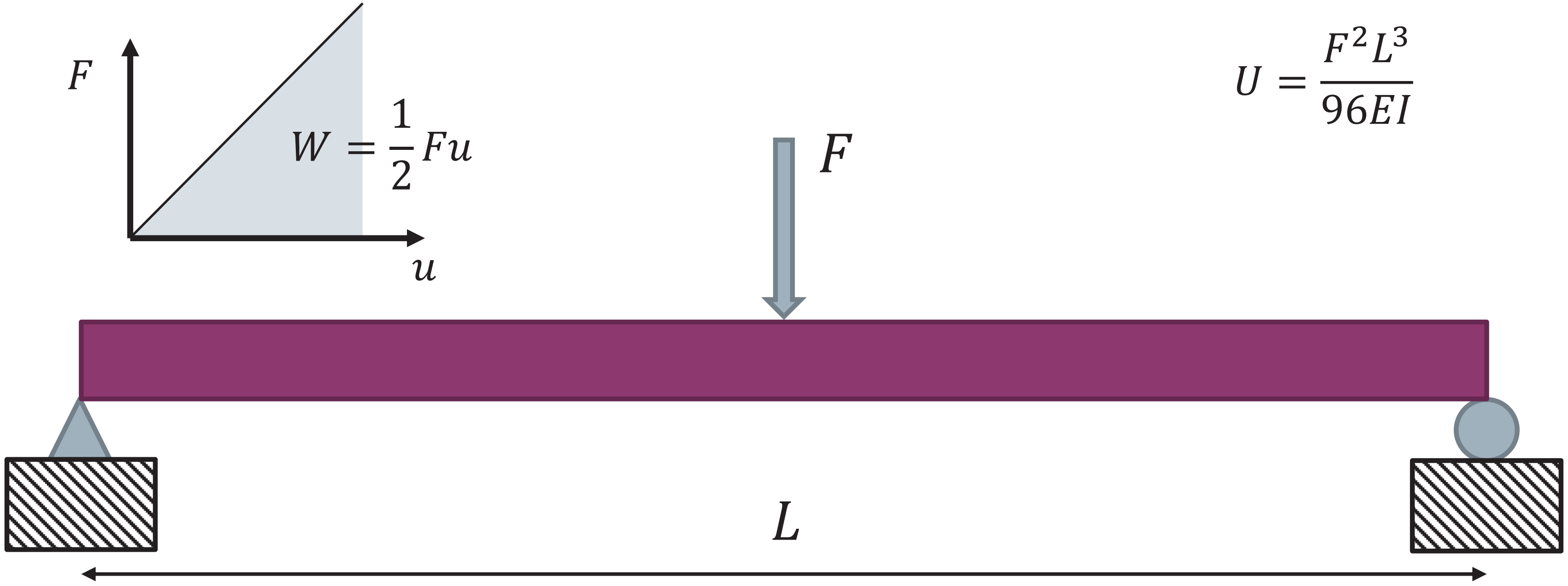
FEEG 2005
Structures: Lecture 9

Energy methods 2 - Virtual Work and Strain Energy Solutions

Last lecture

- Solving deflection problems using conservation of energy
  - Work done by external forces, W = strain energy stored in the system, U



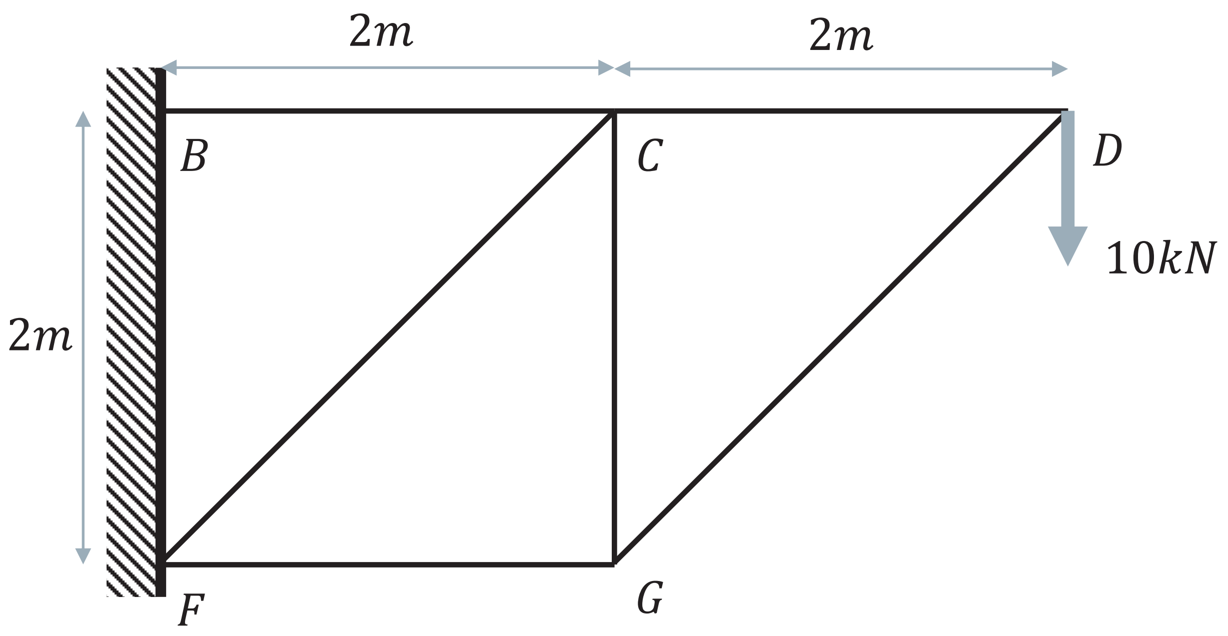
$$W = \frac{1}{2}Fu$$

$$U = \frac{F^2L^3}{96EI}$$

This lecture: Principle of Virtual Work

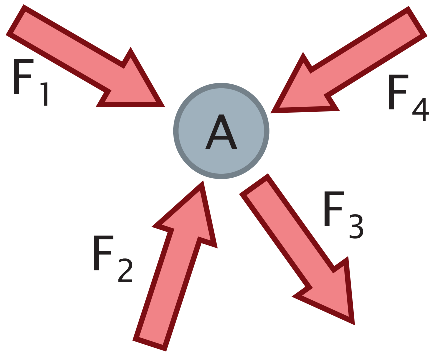
External Virtual Work = Internal Virtual Work

- A method for finding the deflection of structures by equating the virtual work done by external virtual forces to the virtual work of virtual internal forces
- Very powerful for solving complicated problems:
  - What's the horizontal and vertical deflection at D?

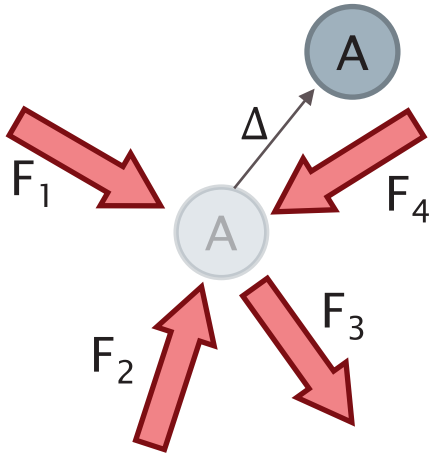


The Principle of Virtual Work for quasi-static loading

- If a system of forces acts on a particle which is in a state of static equilibrium and the particle is given any virtual displacement, then the net work done by the forces is zero.



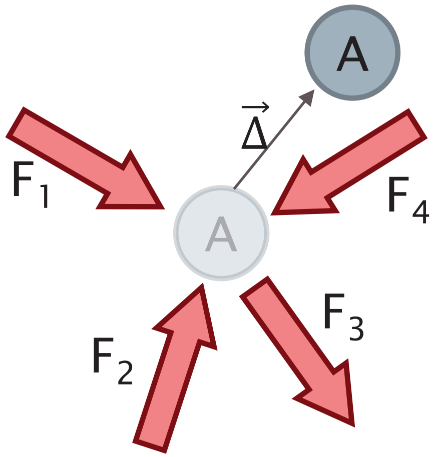
A system in quasit static equilibrium



For an arbitrary virtual displacement Δ, the total work by all forces will be zero.

The Principle of Virtual Work (PVW)

- Let $\vec{P}$ to be the resultant vector of all forces F_1, F_2,…
- PVW means that $\vec{P}.\vec{\Delta}= 0$. Since virtual displacement vector $\vec{\Delta}\neq 0$, therefore,$\vec{P} = 0$ so the system is in equilibrium.
- Notes:
  - Virtual Displacement: any arbitrary displacement that does not actually have to take place (hence 'virtual') but must be geometrically possible.
  - Virtual displacement can be caused by a virtual force.
  - Forces (real or virtual) always stay constant and parallel to their original lines of action.

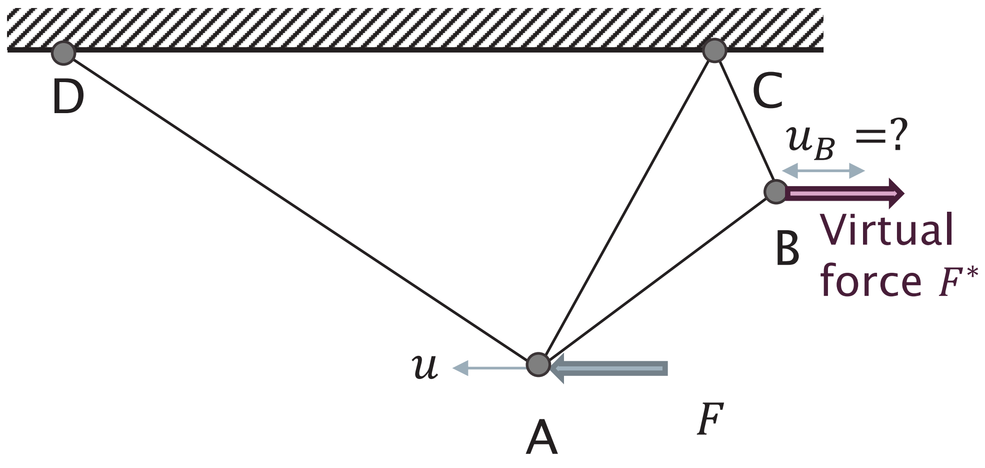


Principle of Virtual Work to find displacement at any point

- From conservation of energy, we know that for real external forces F, we can write:

$$Fu = P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 \; \dots + P_n\Delta_n$$

- We are wondering: what is the horizontal displacement at joint B, u_B?
- There is no real force on joint B so we cannot use standard energy conservation.
- We can use a virtual force F^* at point B to find the desired unknown real displacement u_B caused by real force F.

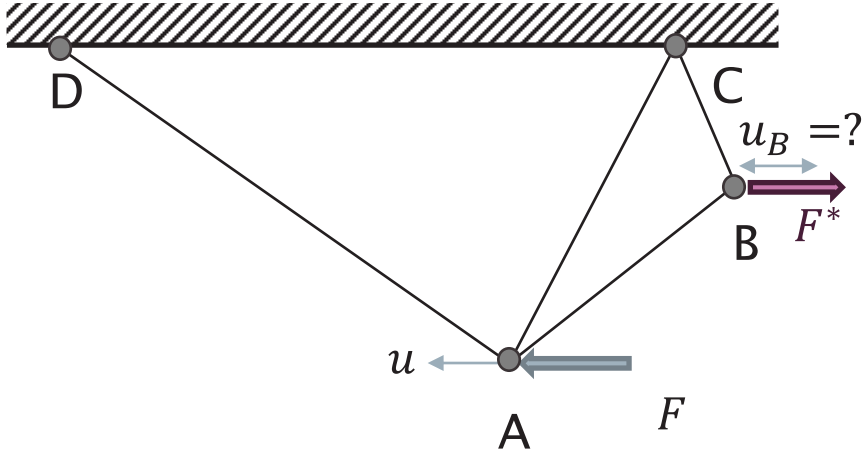


How Principle of Virtual Work works

P^* = internal virtual forces that counteract virtual force

F^* has arbitrary magnitude, often 1N

1. Apply the imaginary force F^* at the desired direction. This external force causes internal forces P_i^* and member deformation Δ_i^* and joint displacement u_i^* that we can find.
2. With the virtual force remaining on the body, apply the actual or real force F. This causes real internal forces P_i annd deformations Δ_i, which can be computed. The virtual forces F^* and P_i^* do external and internal work.
3. Find the deformation by equating the external virtual work= $F^* u_B$ to the internal virtual work=$P_1^*\Delta_1 + P_2^*\Delta_2 + P_3^*\Delta_3 + P_n\Delta_n$



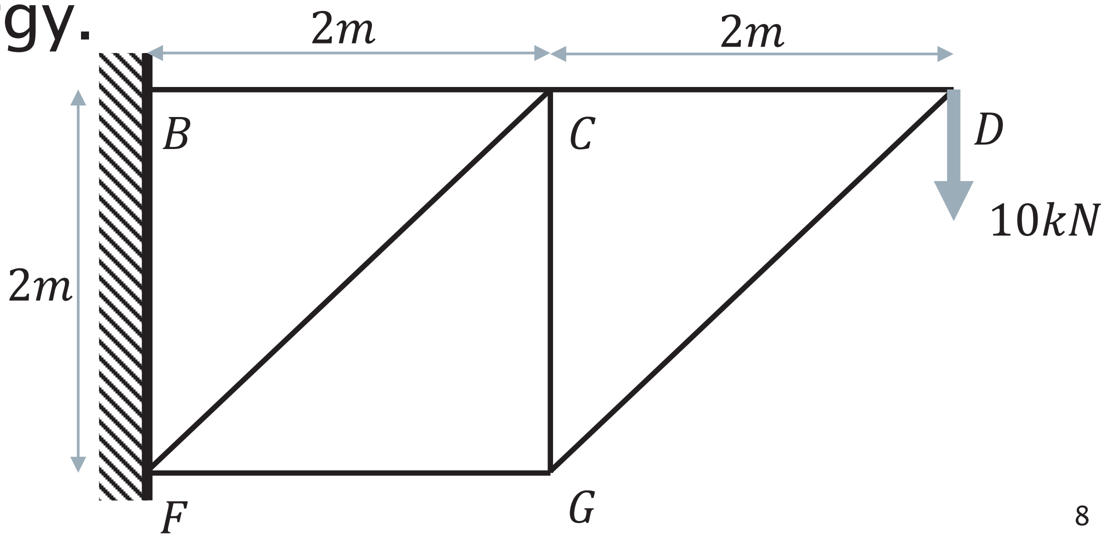
Real

$$F^* u_B = \sum_n P^* \Delta$$

Virtual

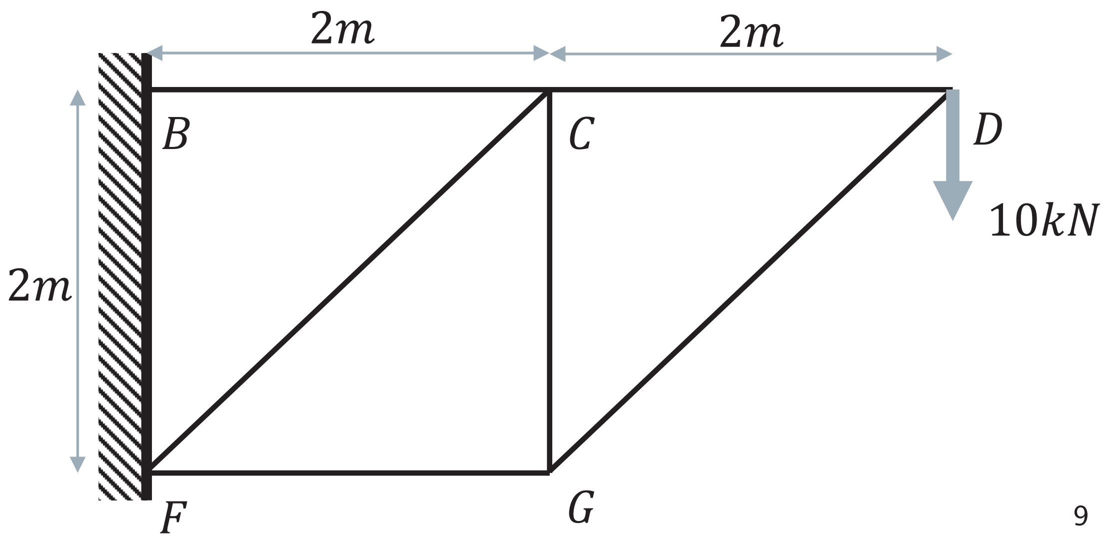
Example 1 – Have a go first!

- The plane pin jointed truss shown below is subjected to a vertical load of 10kN. Find the vertical displacement of D. All truss members have a cross sectional area of $1000\ mm^2$ and an elastic modulus of $E = 200\ GPa$.
- HINT: use the method of sections or joints to get all the internal forces 'P_n' and then calculate the deflection $\Delta_\mathrm{n} = \frac{P_n L}{EA}$ in each member. Then use PVW or conservation of energy.



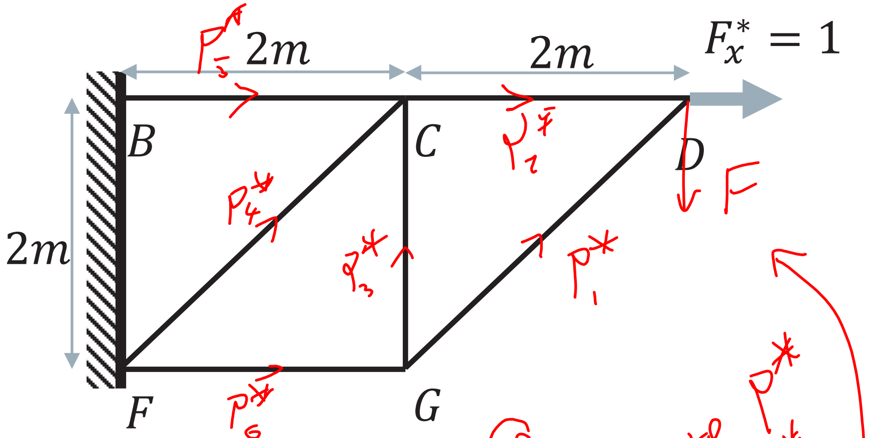
Example 1- continued – Have a go first!

- The plane pin jointed truss shown below is subjected to a vertical load of 10kN. Find the horizontal displacement of D as well as the vertical and horizontal displacement at G. All truss members have a cross sectional area of $1000\, mm^2$ and an elastic modulus of $E = 200\, GPa$.
- HINT: You may need to define a virtual force. Sub everything into the PVW equation.



PVW with virtual forces

- If we want to find the horizontal displacement at D, we select an arbitrary external virtual force F^* to be $F_x^* = 1$. ① assume force F^* at D
- We now temporally ignore our actual loading F and consider the FBD for our external virtual force F^* and find all the internal virtual internal forces P^* using the method of sections or joints. ② place these
- Then we find the real displacement u using PVW equation where Δ_i are the real internal displacements found from real forces.



③ equate P^* and F^*

④ place real force

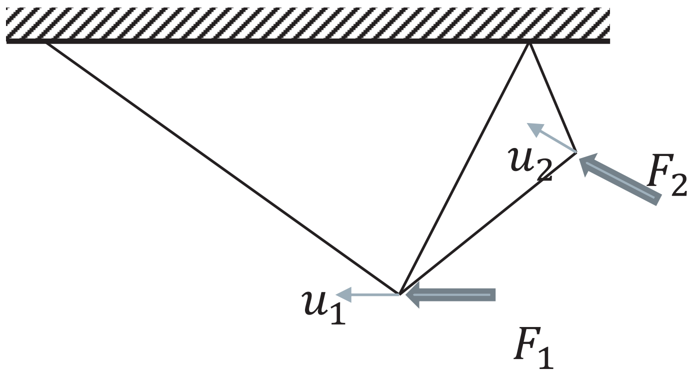
Real

$$F^* u = \sum_n P^* \Delta$$

Virtual

Multiple forces applied to a structure

- Using real forces and deflections does not let us find the horizontal displacement in the truss in previous example.
- It gets worse if there are several external real forces F_j because there would too many unknown displacements 'u_j' on the left-hand side of the equation $\sum F_j u_j = \sum P_i \Delta_i$.
- However, the PVW is still valid and can find any unknown displacement if we use the real displacements with virtual force terms (i.e. arbitrary forces) as long as they are in equilibrium.



PVW with multiple virtual forces

- We can have multiple external F^* and internal P^* virtual forces in the principle of virtual work with our actual displacements (u and Δ) to solve for u.

$$\boxed{\sum_j F^* u = \sum_n P^* \Delta}$$

- This process can be repeated for each joint in the truss applying a unit virtual load either horizontally ($F_x^* = 1$) or vertically ($F_y^* = 1$) and then finding the resulting virtual internal loads P^*.
- **NOTE**: the actual loading has not disappeared! It is included in the calculation of each actual displacement: $\Delta_\mathrm{n} = \frac{P_n L}{EA}$

PVW for deformable bodies

- For a discrete truss with 'j' joints and 'n' members:

External Virtual Work $$\boxed{\sum_j F^* u = \sum_n P^* \Delta}$$ Internal Virtual Work

tensor ... product?

tensor

- We can also express the PVW using what we have learned previously about strain energy in 3D:

$F^* u =$

External Virtual Work $$\boxed{\int_S \cancel{\boldsymbol{T}^* \cdot \boldsymbol{u}}\, dS = \int_V \boldsymbol{\sigma}^* : \boldsymbol{\varepsilon}\, dV}$$ Internal Virtual Work

vector

simplification often done at this level

- **NOTE**: the eq's above assume quasi-static loading and neglect body forces like gravity!

PVW for deformable bodies

External Virtual Work

Internal Virtual Work

Surface integral over ‘S’

$$\boxed{\int_S \boldsymbol{T}^* \cdot \boldsymbol{u}\, dS = \int_V \boldsymbol{\sigma}^* : \boldsymbol{\varepsilon}\, dV}$$

Volume integral over ‘V’

Virtual Traction Vector applied over ‘S’

Displacement Vector

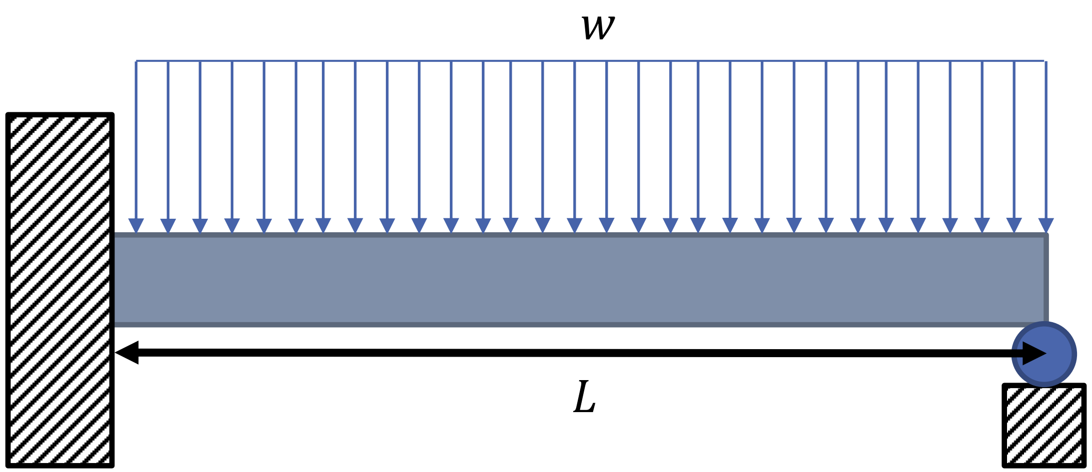
Virtual Stress Matrix

Strain Matrix

- By using a unit **virtual** load $(T^* = 1)$ at a point of interest we can find the corresponding actual displacement ‘u’ at that point in the direction of T^*.

Example 2 – Have a go first!

- A supported cantilever beam (statically indeterminate) is subjected to a distributed load ‘w’. Using energy methods find all reaction forces and hence the show that the deflection at the midspan is: $v = \dfrac{wL^4}{192EI}$



Strain energy solution for forces

Strain Energy Solution for Forces

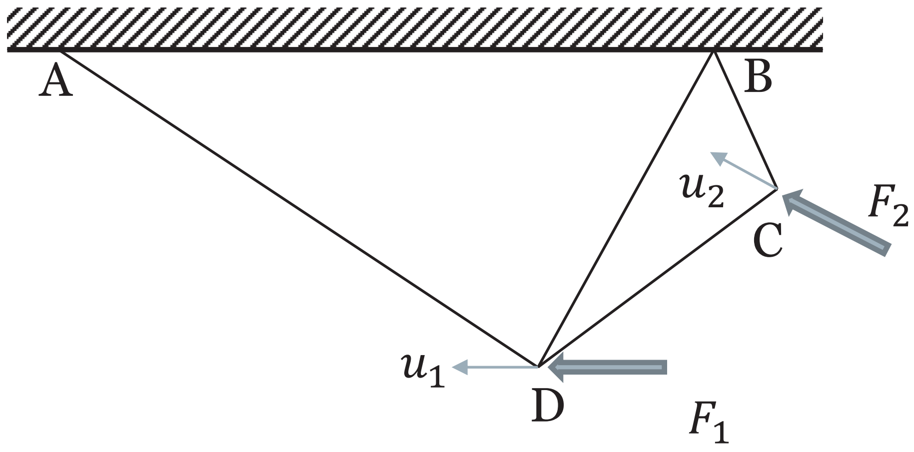
external work

internal work

- Consider the truss shown and recall the PVW:

$$\underbrace{F_1 u_1 + F_2 u_2}_{\text{external work}} = \underbrace{P_1 \Delta_1 + P_2 \Delta_2 + \cdots + P_n \Delta_n}_{\text{internal work}} \quad (1)$$

- Now consider a small increment in displacement δu_1 at joint D along the line of action of F_1 with u_2 fixed.
- The truss members are connected so δu_1 causes each member to deform by a small amount $\delta\Delta_1$, $\delta\Delta_2$, … $\delta\Delta_n$:
- $F_1(u_1 + \delta u_1) + F_2 u_2 = P_1(\Delta_1 + \delta\,\Delta_1) + P_2(\Delta_2 + \delta\,\Delta_2) + \cdots + P_n(\Delta_n + \delta\Delta_n)$ (2)



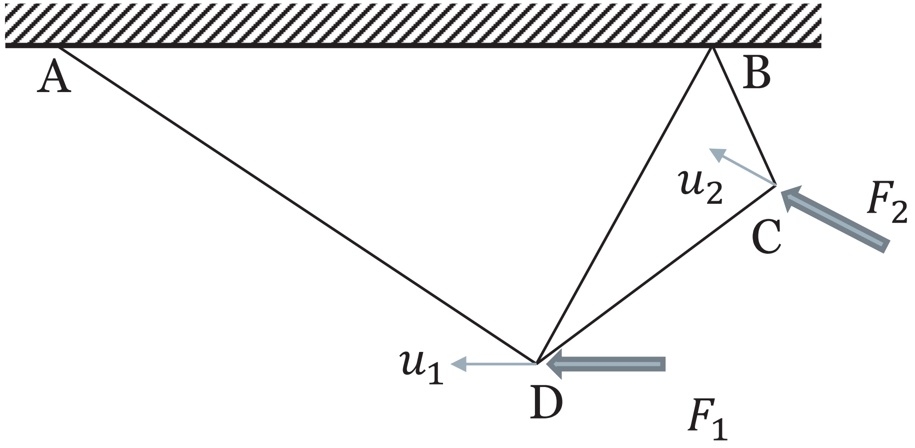
Strain Energy Solution for Forces

- Now subtract (1) from (2):

$$F_1(u_1 + \delta u_1) + F_2 u_2 = P_1(\Delta_1 + \delta\,\Delta_1) + P_2(\Delta_2 + \delta\,\Delta_2) + \cdots + P_n(\Delta_n + \delta\Delta_n) \quad (2)$$

$$F_1 u_1 + F_2 u_2 = P_1\Delta_1 + P_2\Delta_2 + \cdots + P_n\Delta_n \quad (1)$$

$$F_1 \delta u_1 = P_1 \delta\,\Delta_1 + P_2 \delta\,\Delta_2 + \cdots + P_n \delta\Delta_n$$

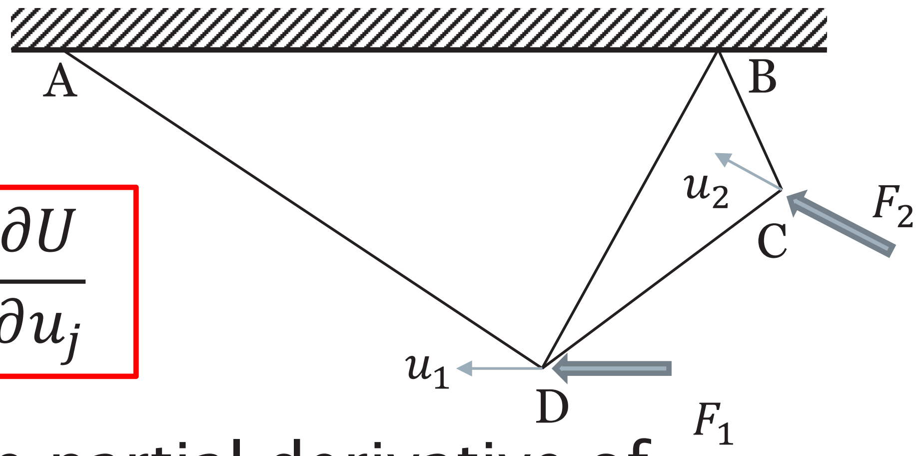


- Or:

$$F_1 \delta u_1 = \sum_n P\delta\Delta$$

Strain Energy Solution for Forces

- But $P\delta\Delta$ is the increment in strain energy in one member of the truss so the sum over all members is the increment in total strain energy $\delta U = \sum_n P\delta\Delta$ so $F_1 \delta u_1 = \delta U$
- Finally: $F_1 = \frac{\partial U}{\partial u_1}$ and similarly: $F_2 = \frac{\partial U}{\partial u_2}$
- General case (external force at joint j): $\boxed{F_j = \frac{\partial U}{\partial u_j}}$



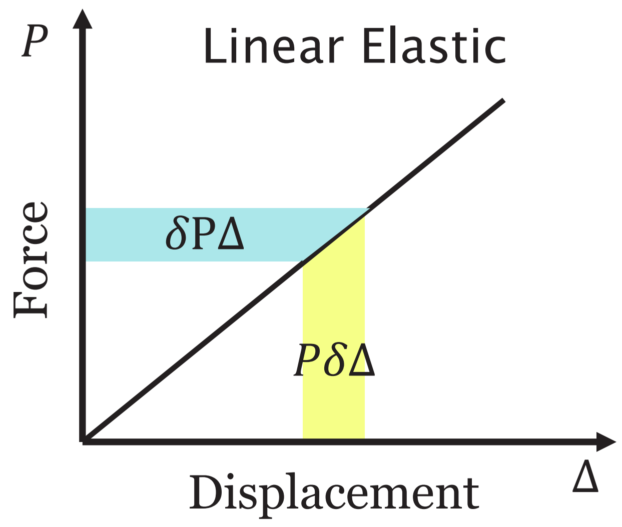
- The external force can be found by taking the partial derivative of the strain energy with respect to displacement at the point of application of the force.

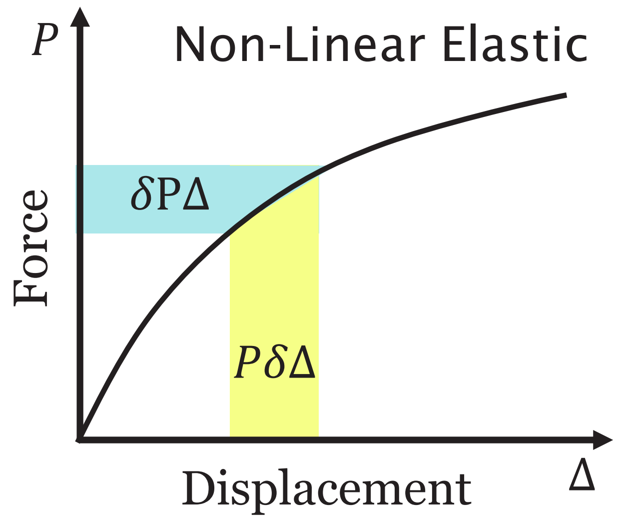
Complimentary energy solution for deflections

Definition of Complementary Energy

- The area above the curve is the **complementary energy** C so the shaded area is the increment in the complementary energy $\delta C = \delta \mathrm{P}\Delta$.

- The shaded area below the curve is the increment in the strain energy: $\delta U = P\delta\Delta$.

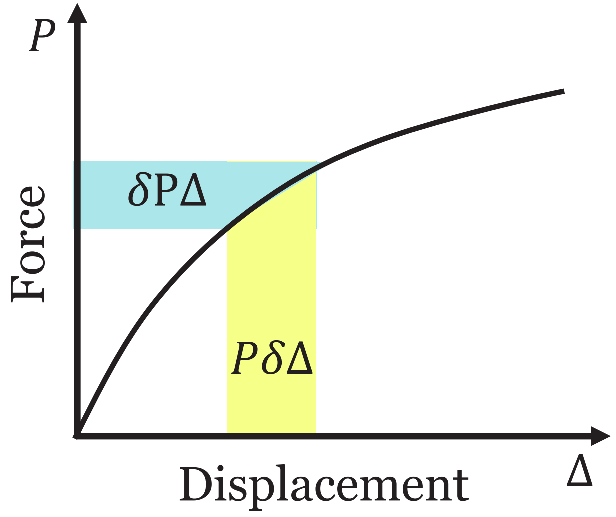




Complementary Energy and Strain Energy

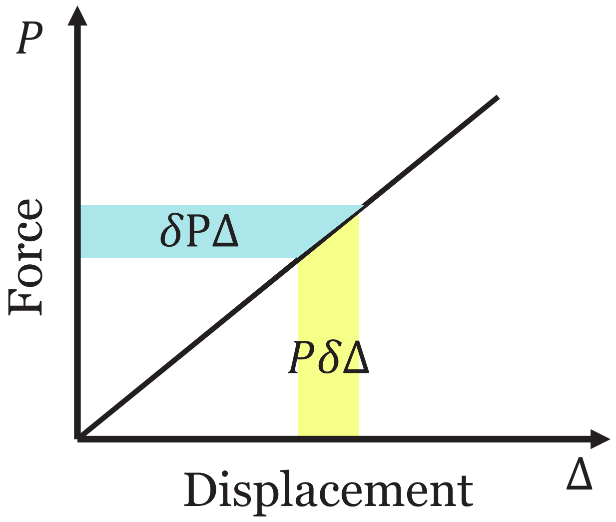
- For all elastic systems (linear and non-linear):

$$U + C = P\Delta$$



- For linear elastic systems:

$$U = C = \frac{1}{2} P\Delta$$



Complementary Energy and deformation

- Now consider a small increment in force δF_1 and apply the same reasoning as before:

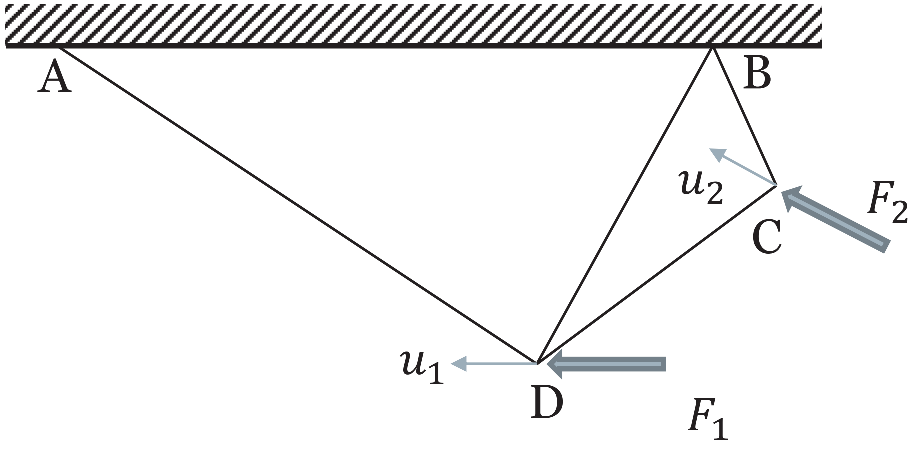
$$(F_1+\delta F_1)u_1 + F_2u_2 = (P_1+\delta P_1)\Delta_1 + (P_2+\delta P_2)\Delta_2 + \cdots + (P_n+\delta P_n)\Delta_n$$

$$-$$

$$F_1u_1 + F_2u_2 = P_1\Delta_1 + P_2\Delta_2 + \cdots + P_n\Delta_n$$

$$=$$

$$\delta F_1u_1 = \delta P_1\,\Delta_1 + \delta P_2\,\Delta_2 + \cdots + P_n\delta\Delta_n$$

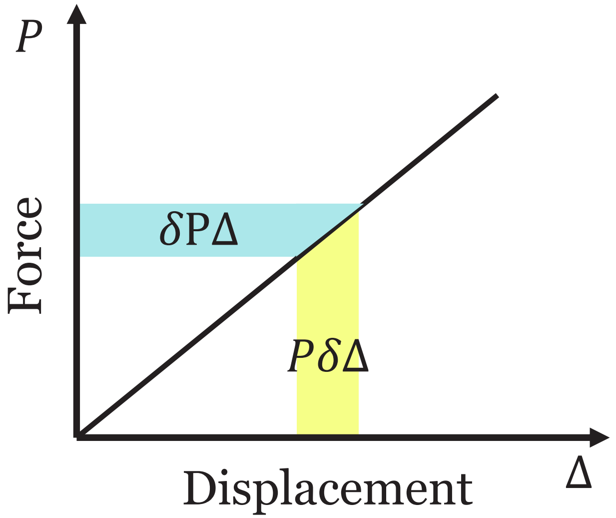


- Or: $\delta F_1u_1 = \sum_n \delta P\Delta$

Complementary Energy solution for deflections

- Recall that for a small increment in force δF_1: $\delta F_1 u_1 = \sum_n \delta P \Delta$
- $\delta P \Delta$ is the increment in complementary energy, so:

  $\delta F_1 u_1 = \delta C$ Rearranging this: $u_1 = \dfrac{\partial C}{\partial F_1}$



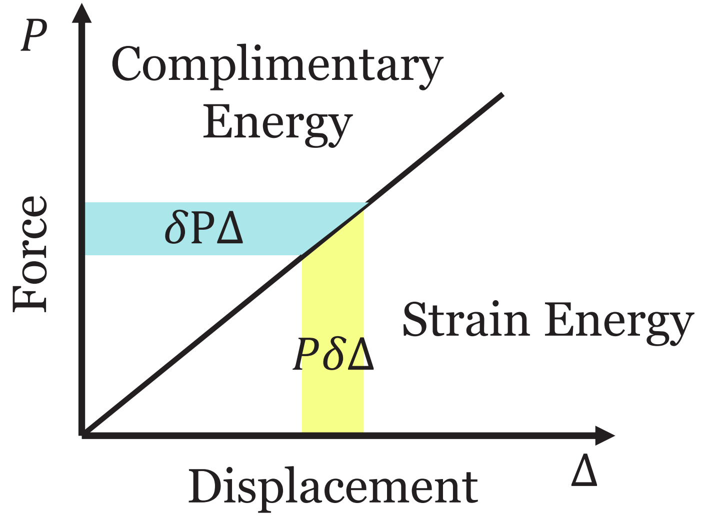
General case (displacement at joint j): $\boxed{u_j = \dfrac{\partial C}{\partial F_j}}$

- The displacement of a joint in the direction of the force can be found by taking the partial derivative of the complementary energy with respect to the external force at the joint.

Summary – strain energy and complementary energy for finding force and deflection

- The force at a joint can be found using the partial derivative of the strain energy ‘U’ with respect to the displacement:

$$\boxed{F_j = \frac{\partial U}{\partial u_j}}$$

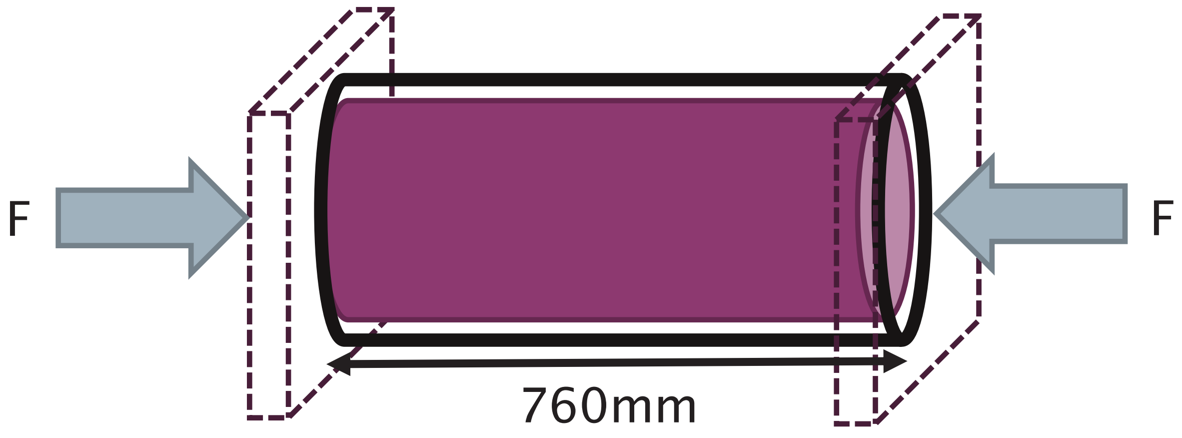


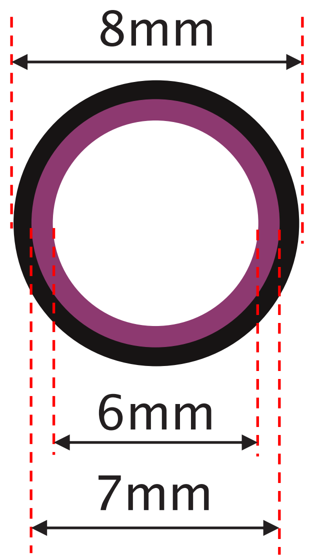
- The displacement at a joint can be found using the partial derivative of the complimentary energy ‘C’ with respect to the force:

$$\boxed{u_j = \frac{\partial C}{\partial F_j}}$$

Example – Have a go first!

- Two hollow cylindrical components are arranged concentrically and loaded through rigid end plates so that both plates are subjected to the same deformation. The inner part is made of aluminium ($E = 70\ GPa$) and the outer shell is made of unidirectional carbon fibre ($E = 130\ GPa$). Using energy methods find the force required to deform the assembly 0.1mm.





Castigliano’s theory

Castigliano’s theorem as a special case for linear elastic materials

- Castigliano is basically what we just learned about the relationship between strain energy and complimentary energy with partial derivative of force and displacement, for linear elastic materials.

Castigliano's First Theorem

- Assumptions: linear elasticity
- Used to find external forces applied to a linear elastic structure:

$$\boxed{F_i = \frac{\partial U}{\partial u_i}}$$

- The force 'F_i' at any point 'i' may be found by partially differentiating the total strain energy 'U' with respect to the displacement 'u_i' in line with the force.

Southampton

Castigliano's Second Theorem

- Assumptions: linear elasticity
- Used to find the displacement in a linear elastic structure:

$$\boxed{u_i = \frac{\partial U}{\partial F_i}}$$

More useful than C because we already know how to calculate U for common engineering cases.

- The displacement 'u_i' at a given load point 'i' can be found by partially differentiating the total strain energy 'U' with respect to the force 'F_i'.
- Proof of this theorem:
  - Based on complimentary energy: $u_i = \frac{\partial C}{\partial F_i}$
  - For linear elasticity: $U = C$

Castigliano’s Second Theorem: For Moments

- Using similar reasoning, we can show that Castigliano’s theorems hold for rotations ‘θ_i’ and moments ‘M_i’ or torques ‘T_i’

$$\boxed{\theta_i = \frac{\partial U}{\partial M_i}}$$

$$\boxed{\theta_i = \frac{\partial U}{\partial T_i}}$$

Castigliano's Theorems or the PVW?

- Castigliano's Theorems:
  - Can be applied at points with multiple external loads
  - Can only be applied at points where external loads are applied

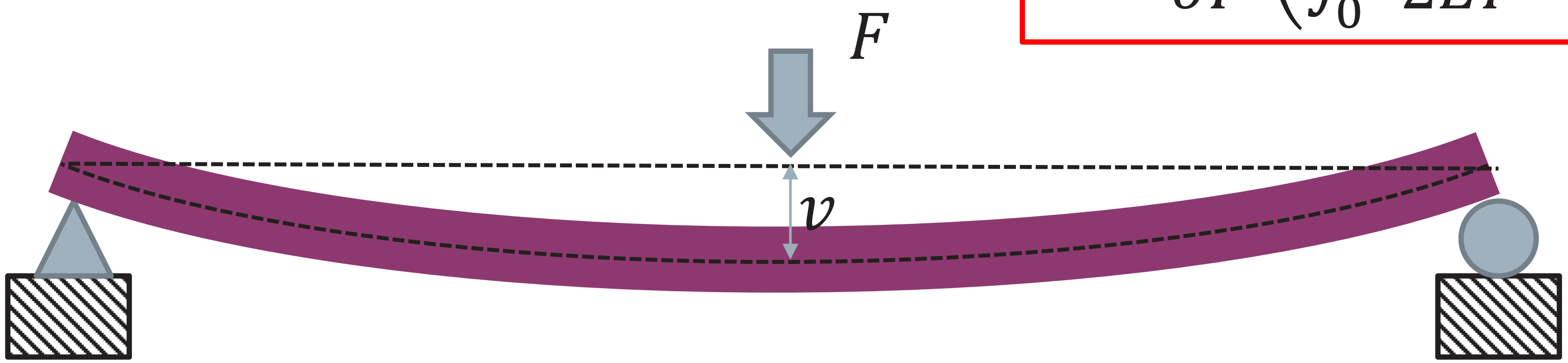
- Principle of Virtual Work (PVW):
  - Can be applied at arbitrary points in the structure

- We can combine these!
  - Apply a virtual force at a point in a structure and then take the partial derivative wrt. that force using Castigliano's theorems

Castigliano's Theorems Applied to Beams

- Recall that the strain energy in bending is: $U = \int_0^L \frac{M^2}{2EI} dx$
- Using Castigliano's second theorem, the deflection 'v' under a concentrated load 'F' applied to the beam is: $v = \frac{\partial U}{\partial F} = \frac{\partial U}{\partial M}\frac{\partial M}{\partial F}$
- We can either integrate first or differentiate first:

$$\boxed{v = \frac{\partial}{\partial F}\left(\int_0^L \frac{M^2}{2EI} dx\right) = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial F} dx}$$



Summary

Real

$$F^{*}u = \sum_{n} P^{*}\Delta$$

Virtual

- Virtual Work is a powerful technique based on energy that allows finding deformation or load for points where there is no applied load.
- Partial derivative of strain energy or complimentary energy with respect to applied force or displacement can be used to find the unknown displacement or force.

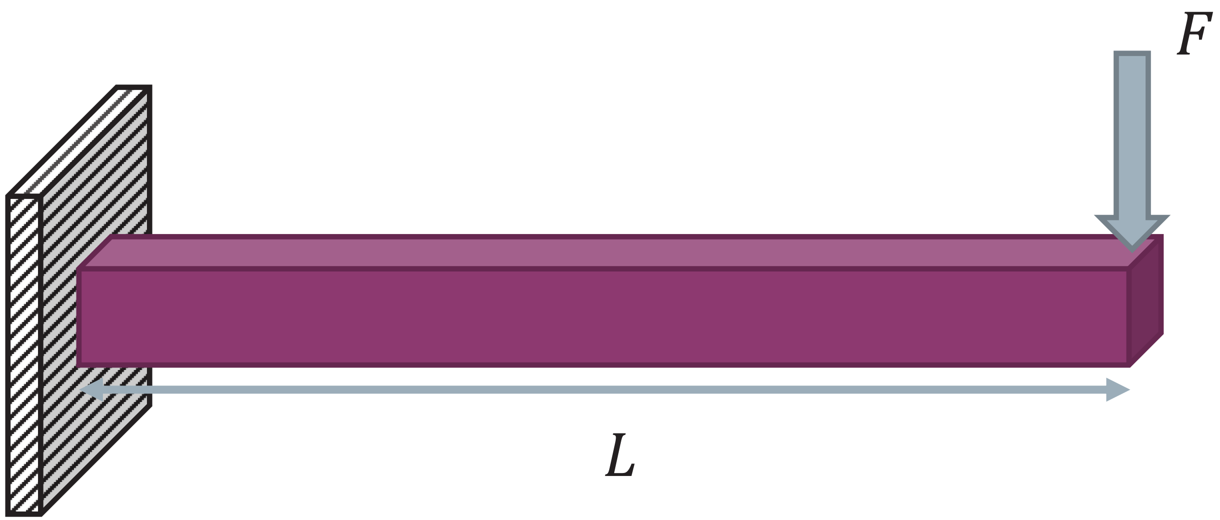
$$\boxed{F_j = \frac{\partial U}{\partial u_j}} \qquad \boxed{u_j = \frac{\partial C}{\partial F_j}}$$

- Castigliano's theorem is for linear elastic materials where $U = C$ so:

$$\boxed{F_i = \frac{\partial U}{\partial u_i}} \qquad \boxed{u_i = \frac{\partial U}{\partial F_i}}$$

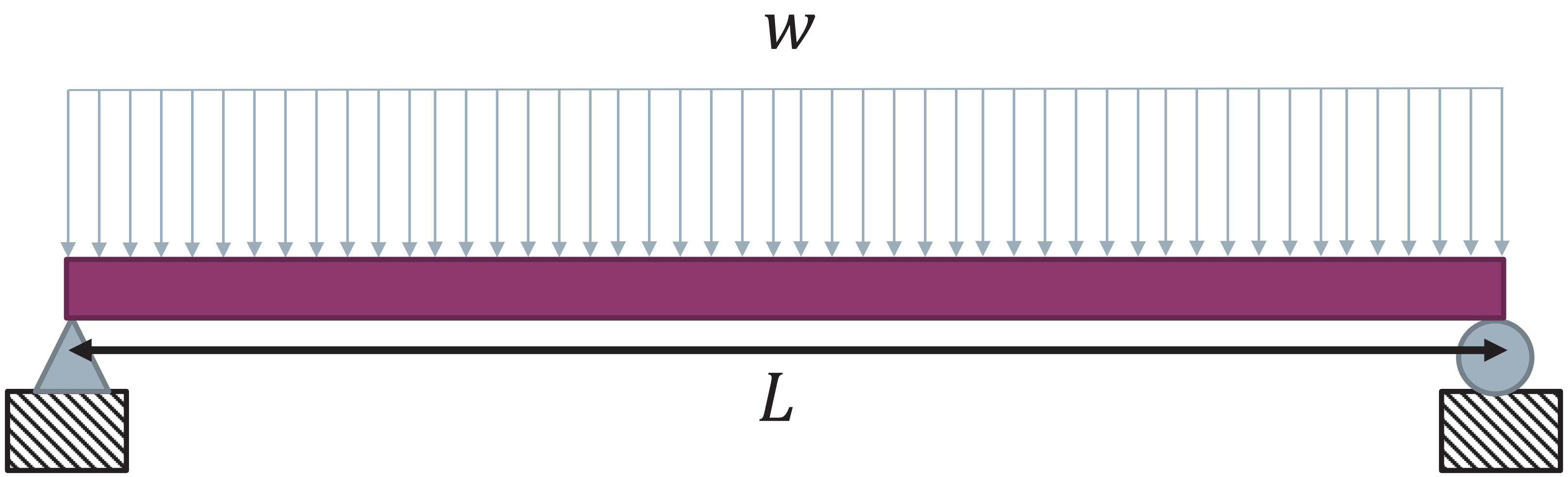
Example 1 – Have a go first!

- A cantilever beam is loaded by a point force F on its tip. Using energy methods show that the deflection at the tip is given by: $v = \frac{FL^3}{3EI}$



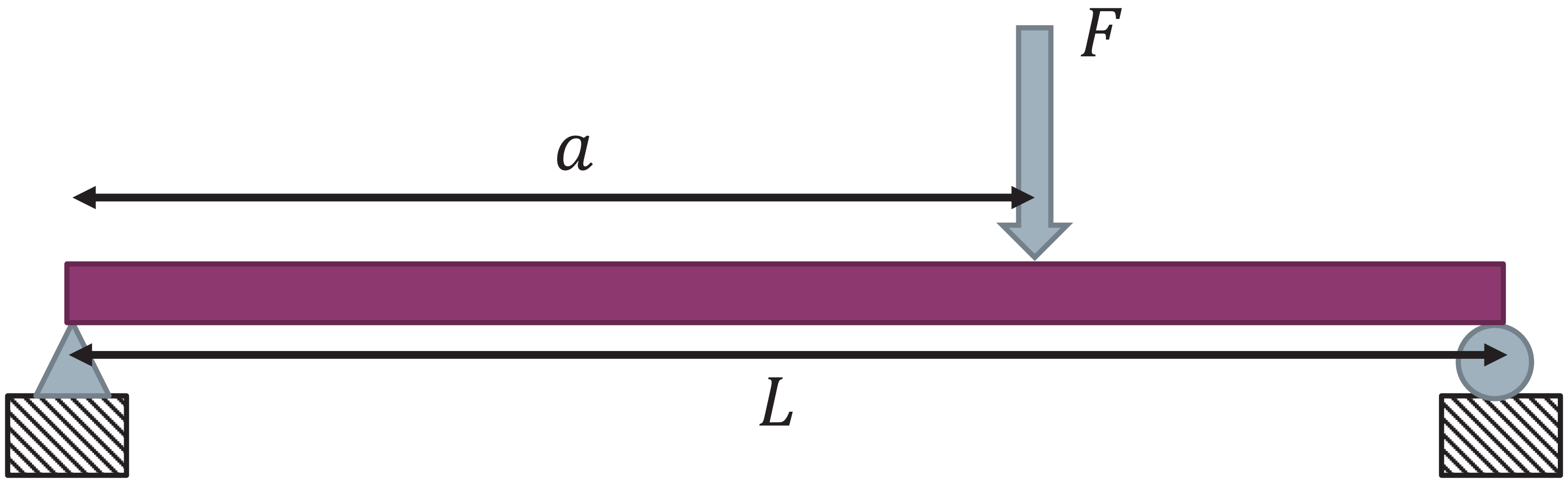
Example 2 – Have a go first!

- A simply supported beam is subjected to a distributed load ‘w’. Using energy methods show that the deflection at the midspan is given by: $v = \frac{5wL^4}{384EI}$
- HINT: Try including a virtual point load ‘F’ in the middle of the beam and solve using Castigliano 2. At the end set F = 0.



Example 3 – Have a go first!

- Consider a simply support beam with a point load applied at a distance 'a' from one edge. Using energy methods show that the deflection at the point at which the load is applied is: $v = \frac{Fa^2(L-a)^2}{3EIL}$



Example 4 – Have a go first!

- A load cell is being made of a ring of material of known elastic modulus E. Show that the stiffness coefficient of the 'proving ring' is $K = \frac{F}{u} = \frac{EI}{R^3(\frac{\pi}{4} - \frac{2}{\pi})}$