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Lecture 5 - Fourier Series and Orthogonality

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Outline

1. Review

2. Fourier Series
   - Orthogonality
   - Examples
   - Useful for what? (PDEs, Sum identities)

3. Summary

Review

- **Fourier Series** are just another way of representing a **periodic function** (which here is taken to have period 2π),

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\,x) + b_n \sin(n\,x)\right],$$

- The Euler formulae:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(m\,x)\,\mathrm{d}x,$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(m\,x)\,\mathrm{d}x.$$

Note that m is a *dummy index*: you can replace it by n or k as long as you do it in the LHS and RHS of the equation

Outline

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Outline

$\rightarrow$ FS: Orthogonality relations (review)

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$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big[a_n \cos(n\,x) + b_n \sin(n\,x)\Big].$$

To find the Fourier coefficients a_m, b_m (**Euler formulae**) we needed the **orthogonality relations**:

$$\int_{-\pi}^{\pi} \sin(m\,x)\sin(n\,x)\,\mathrm{d}x = \pi\delta_{mn}, \qquad \int_{-\pi}^{\pi} \cos(m\,x)\cos(n\,x)\,\mathrm{d}x = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin(m\,x)\cos(n\,x)\,\mathrm{d}x = 0,$$

where δ_{mn} is the *Kronecker symbol*: $\delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}.$

FS: Orthogonality relations (review)

We have seen from direct calculation that the **orthogonality relations** follow from **trigonometric identities.**

- The trigonometric functions, $\cos(n\,x)$, $\sin(n\,x)$, are **eigenfunctions** of the **boundary value problem (BVP)**

$$y'' + \lambda y = 0; \qquad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$$

Note: the boundary conditions are also periodic: e.g. $y(-\pi + 2\pi) = y(\pi)$

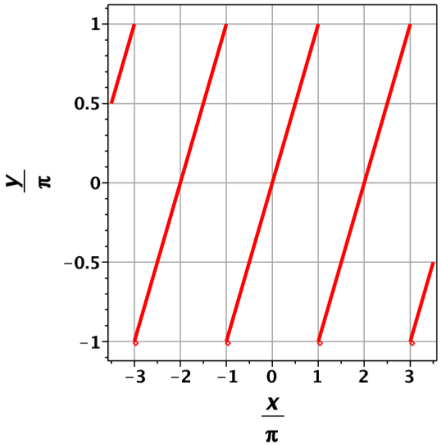
- It is a **general property of eigenfunctions of** a class of BVP, called **Sturm-Liouville problems**, that they **satisfy such orthogonality relations.**

- The **mathematical theory** of the Sturm-Liouville problems is explained in the Lecture Notes (but this material is <u>not</u> examinable).

Outline

→ FS example 1: Sawtooth function

Consider the **sawtooth** function $f = x$ where $-\pi < x < \pi$, extended as a periodic function of period 2π.



Sawtooth function example: Fourier series

Here, as a first way to solve the problem, we do it using brute force.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0.$$

To compute a_m and b_m we use extensively integration by parts:

$\left(\int \cos(\alpha x) = \frac{\sin(\alpha x)}{\alpha} \searrow\right)$ $\left(\int_A^B u\,dv = [uv]_A^B - \int_A^B v\,du\right)$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{x}^{u}\,\overbrace{\cos(m\,x)\,\mathrm{d}x}^{dv} = \frac{1}{\pi}\left[\overbrace{x}^{u}\,\overbrace{\frac{\sin(m\,x)}{m}}^{v}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\overbrace{\frac{\sin(m\,x)}{m}}^{v}\,\overbrace{\mathrm{d}x}^{du}$$

$$= \frac{1}{\pi}\left[x\frac{\sin(m\,x)}{m} + \frac{\cos(mx)}{m^2}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[0 - 0 + \frac{(-1)^m - (-1)^m}{m^2}\right]$$

$$= 0. \qquad [\nwarrow \sin(n\pi) = 0, \qquad \cos(n\pi) = (-1)^n\ ]$$

Sawtooth function example: Fourier series

(↙ Integration by parts: $\int_A^B u\,dv = [uv]_A^B - \int_A^B v\,du$)

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{x}^{u}\,\overbrace{\sin(m\,x)\,\mathrm{d}x}^{dv} = \frac{1}{\pi}\left[-x\frac{\cos(m\,x)}{m}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(m\,x)}{m}\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[-x\frac{\cos(m\,x)}{m} + \frac{\sin(mx)}{m^2}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[-\pi\frac{(-1)^m}{m} - \pi\frac{(-1)^m}{m} + 0 - 0\right]$$

$$= \frac{2(-1)^{m+1}}{m}.$$

(↗ $\int \sin(\alpha x) = -\frac{\cos(\alpha x)}{\alpha}$) [↖ $\sin(n\pi) = 0, \quad \cos(n\pi) = (-1)^n$]

This was using **brute force.**
We will see **ways of simplifying this in future lectures.**

Sawtooth function example: Fourier series

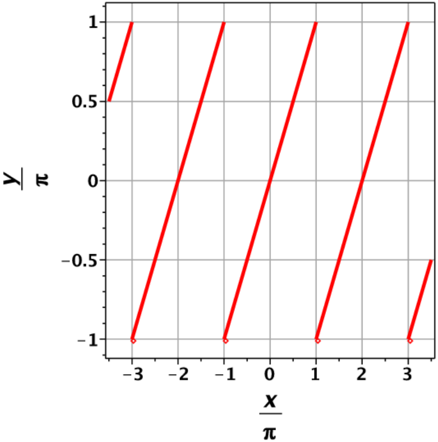
$$f(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty}\left[a_m\cos(m\,x) + b_m\sin(m\,x)\right]$$

Conclusion: for the **sawtooth** function $f(x) = x$ for $-\pi < x < \pi$,

and then extended as periodic function of period 2π,

we have:

$$a_0 = 0,$$
$$a_m = 0,$$
$$b_m = \frac{2(-1)^{m+1}}{m}.$$



We steadily see **convergence** inside the interval.

Red: $m = 1, \cdots 4$; Green: $m = 1, \cdots 8$;
Yellow: all $m = 1, \cdots, \infty$ (=exact function).

Sawtooth function example: Fourier series

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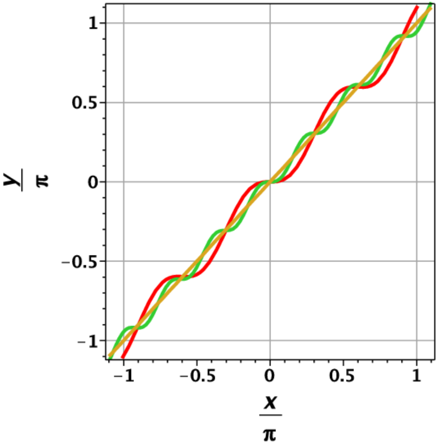

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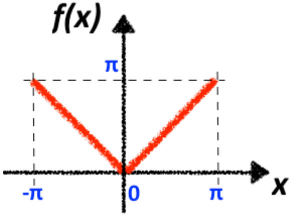
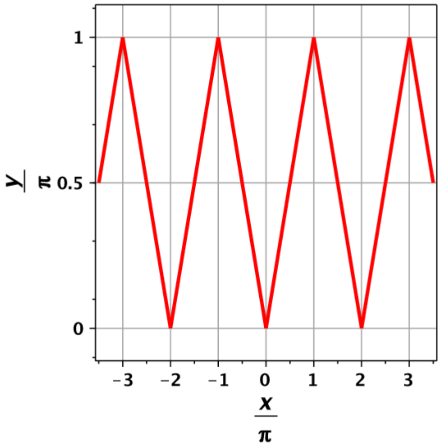
→ FS example 2: Tent function

Consider the **tent** function
$f(x) = |x|$ where $-\pi < x < \pi$,



extended as a periodic function of period 2π.



Tent function example: Fourier series

We do this by brute force in this lecture:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\, dx = \frac{1}{\pi}\int_{-\pi}^{0} -x\, dx + \frac{1}{\pi}\int_{0}^{\pi} x\, dx = -\frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi} = \pi.$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(m\,x)\, dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} \overbrace{-x}^{u}\ \overbrace{\cos(m\,x)\, dx}^{dv} + \int_{0}^{\pi} x\cos(m\,x)\, dx\right\}$$

Now do integration by parts in both integrals, $\int_A^B u\, dv = [uv]_A^B - \int_A^B v\, du$

$$= \frac{1}{\pi}\left\{-\left[x\frac{\sin(m\,x)}{m} + \frac{\cos(m\,x)}{m^2}\right]_{-\pi}^{0} + \left[x\frac{\sin(m\,x)}{m} + \frac{\cos(mx)}{m^2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\left[0 - 0 + \frac{1}{m^2} - \frac{(-1)^m}{m^2}\right] + \left[0 - 0 + \frac{(-1)^m}{m^2} - \frac{1}{m^2}\right]\right\}$$

$$= \frac{2}{\pi m^2}\left[(-1)^m - 1\right]. \qquad \left[\nwarrow \sin(n\pi) = 0, \qquad \cos(n\pi) = (-1)^n\ \right]$$

Tent function example: Fourier series

Similarly,

$$
\begin{aligned}
b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin(m\,x)\,\mathrm{d}x = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -x\sin(m\,x)\,\mathrm{d}x + \int_{0}^{\pi} x\sin(m\,x)\,\mathrm{d}x\right\} \\
&= \frac{1}{\pi}\left\{-\left[-x\frac{\cos(m\,x)}{m} + \frac{\sin(m\,x)}{m^2}\right]_{-\pi}^{0} + \left[-x\frac{\cos(m\,x)}{m} + \frac{\sin(mx)}{m^2}\right]_{0}^{\pi}\right\} \\
&= \frac{1}{\pi}\left\{-\left[-\pi\frac{(-1)^m}{m} - 0 + 0 - 0\right] + \left[-\pi\frac{(-1)^m}{m} - 0 + 0 - 0\right]\right\} \\
&= 0.
\end{aligned}
$$

We will see ways of simplifying this in future lectures.

Tent function example: Fourier series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty}\left[a_m \cos(m\,x) + b_m \sin(m\,x)\right]$$

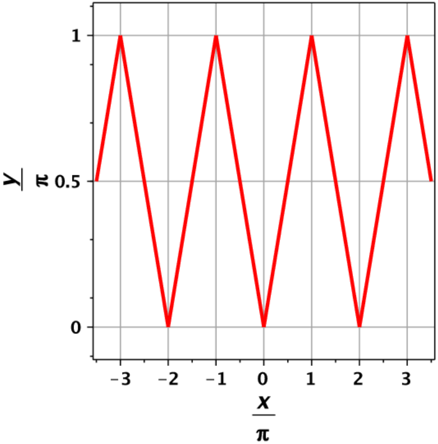
Conclusion: for the **tent** function $f = |x|$ for $-\pi < x < \pi$, and then extended as periodic function of period 2π, we have:

$$a_0 = \pi,$$

$$a_m = \frac{2}{\pi m^2}\left[(-1)^m - 1\right]$$

$$= \begin{cases} -\frac{4}{\pi m^2} & m \text{ odd} \\ 0 & m \text{ even} \end{cases},$$

$$b_m = 0.$$



We steadily see **convergence** inside the interval.

Red: $m = 1, 3$; Green: $m = 1, 3, 5, 7, 9$; Yellow: all $m = 1, \cdots, \infty$ (=exact function).

Tent function example: Fourier series

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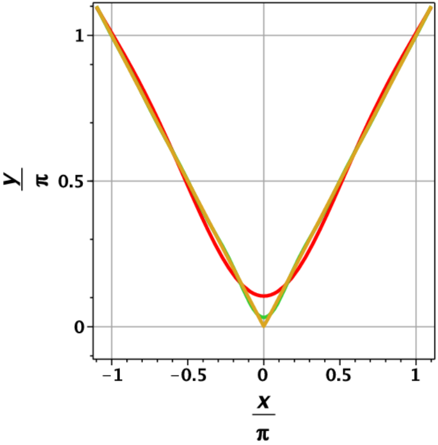

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Outline

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$\rightarrow$ FS: Useful for what? (PDEs, Sum identities)

$\rightarrow$ **Who cares about FS? That is, what are FS useful for?**

- Later we will see the **uses of Fourier Series for PDEs**.
- Valuable results for **(infinite) sum identities (series)**.

For *example*, evaluating the **tent function** $f = |x|$ at $x = 0$:

$\searrow$ (using slide 16: $a_0 = \pi, a_m = \cdots, b_m = 0$)

$$|0| = \left. \frac{a_0}{2} + \sum_n a_n \cos(n\,x) \right|_{x=0} + 0$$
$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(0)}{n^2},$$

from which it follows that

$$\frac{\pi^2}{8} = \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Outline

Summary

- The **orthogonality relations** that give the Euler formulas **follow** from general results of **Sturm-Liouville theory**.
- **Practical** Fourier Series **calculations** require lots of **integration by parts**.
- **Convergence** of Fourier Series is, in most cases, **rapid**.
- Useful **arithmetic identities (sum identities for series)** can be found from Fourier Series by evaluating them at a particular suitable point.