Chapter 5: Mission Analysis

Lecture 3 – The ellipse equation

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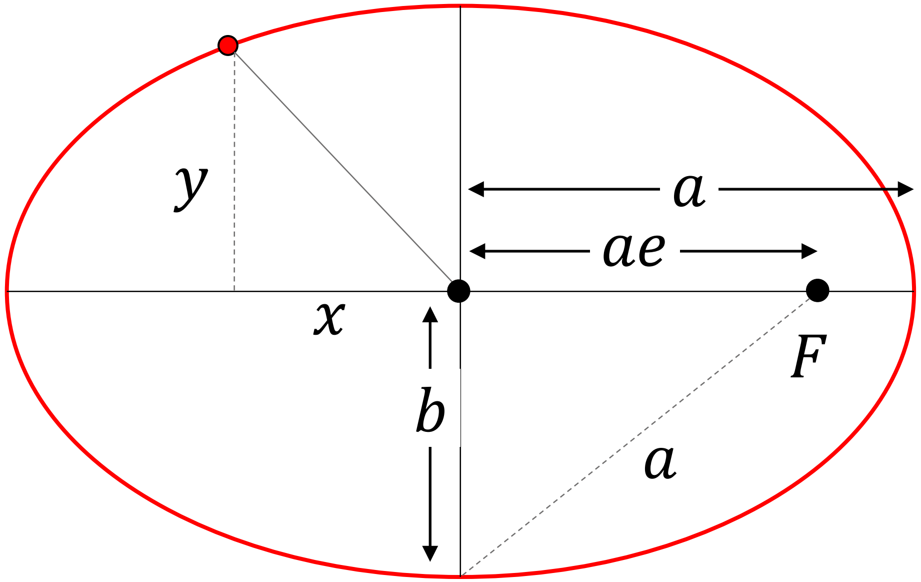
Overview of lecture 3

- This lecture is focused on the derivation of the ellipse equation:
  - The ellipse equation we need describes the ellipse as a function of the radius and the position (measured using an angle):
    - I.e. it is the polar form of the ellipse equation
  - Understanding the approach at a conceptual level is important, but the derivation itself will not be assessed

- Why is the ellipse equation important?
  - It describes an elliptical trajectory with respect to a frame of reference
  - If we show mathematically, using fundamental physical principles, that orbital trajectories can be described using this equation then we can prove that Kepler's 1st Law is correct
    - This is what we will do in lectures 4 and 5

Ellipse equation

- Ellipse properties and equation in cartesian coordinates:



a= semi-major axis b= semi-minor axis

e= eccentricity

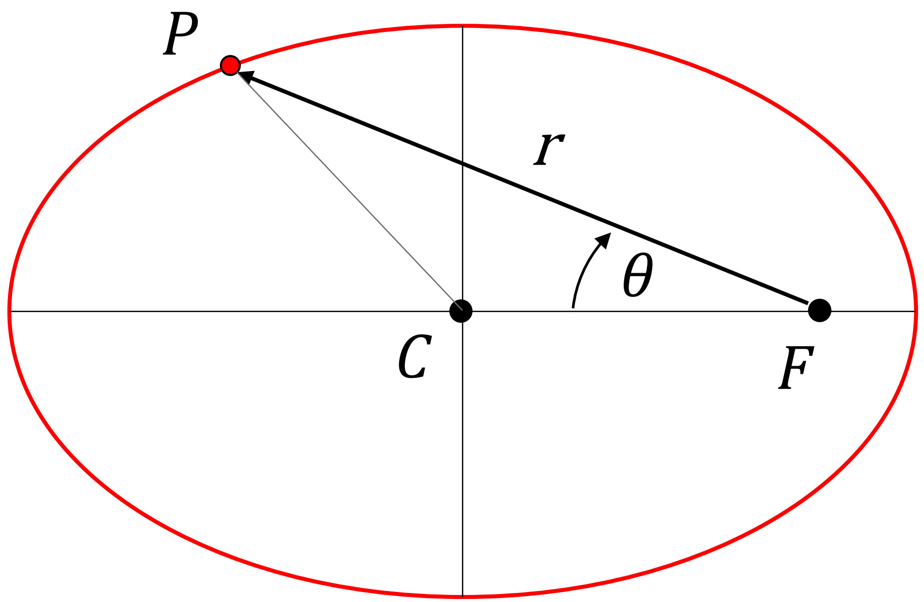
- Ellipse equation: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

- Compare with equation for a circle:

$$\frac{x^2}{r^2}+\frac{y^2}{r^2}=1$$

Ellipse equation

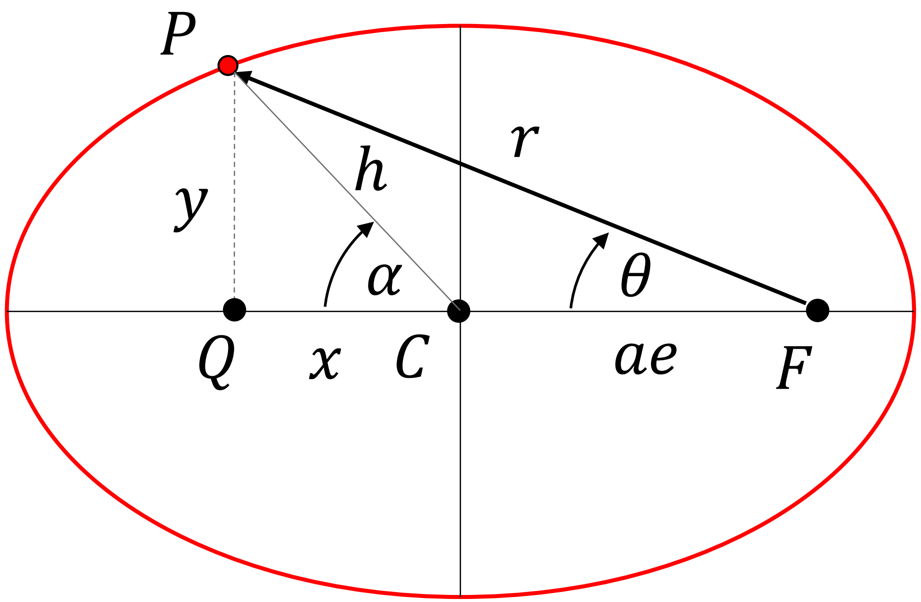
- We want to write the ellipse equation in polar form



- Ellipse equation: $f(r, \theta)$
- Use cartesian coordinates as our starting point

Ellipse equation

- Use the ellipse equation in cartesian form to start:



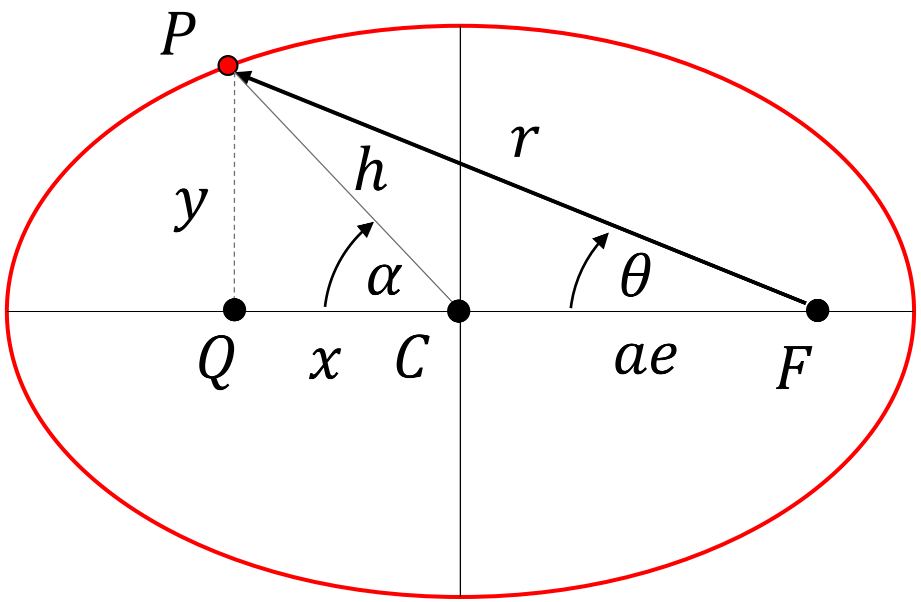
- Ellipse equation: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
- Where* $x = h\cos\alpha$ and $y = h\sin\alpha$
- Also**: $r\cos\theta = x + ae = h\cos\alpha + ae$
- So: $h\cos\alpha = r\cos\theta - ae$

*Triangle PQC
**Triangle PQF

((1) Write x & y in terms of h & α;
(2) Then write h & α in terms of r, θ, a & e)

Ellipse equation

- Use some geometry and algebra…:



- We can also see that*:

$$y = h\sin\alpha = r\sin\theta$$

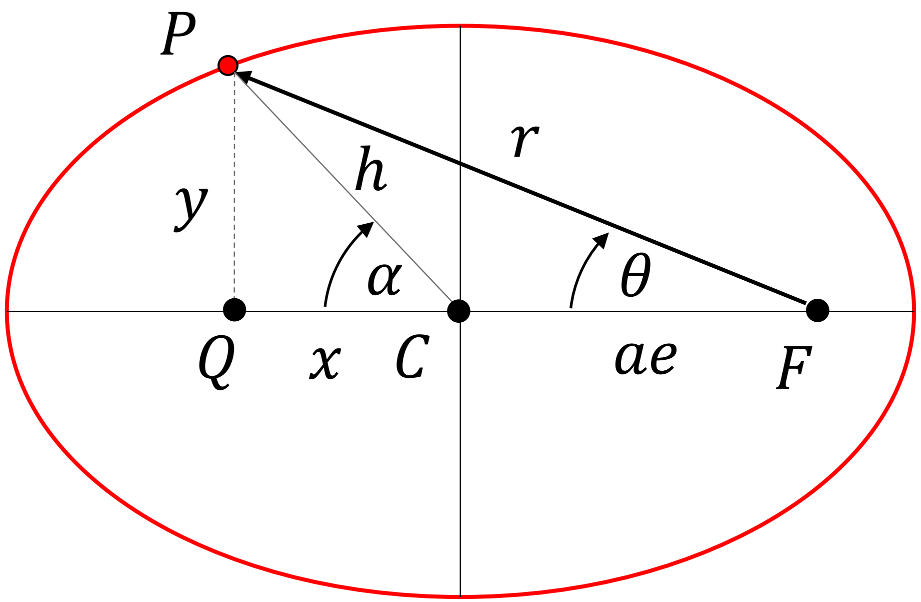
- This means we can now write the ellipse equation in terms of r and θ

*Triangles PQF and PQC have one side in common

((1) Write x & y in terms of h & α;
(2) Then write h & α in terms of r, θ, a & e)

Ellipse equation

- Ellipse equation in polar form:



- Ellipse equation: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
- Re-writing:

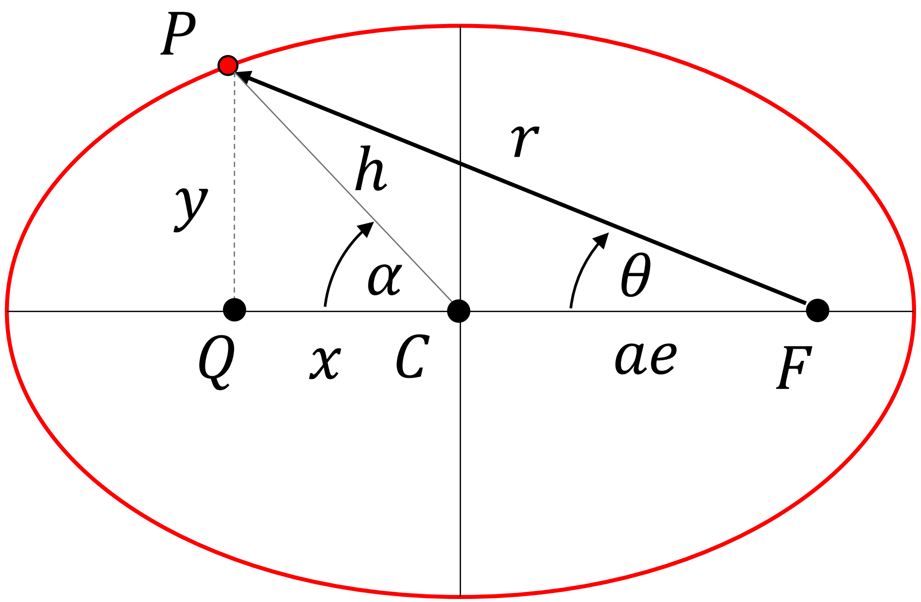
$$\frac{(h\cos\alpha)^2}{a^2} + \frac{(h\sin\alpha)^2}{b^2} = 1$$

- And continuing:

$$\frac{(r\cos\theta - ae)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} = 1$$

Ellipse equation

- Ellipse equation in polar form:



- Expanding and collecting terms (working not shown) we get:

$$\begin{matrix} r^2(b^2\cos^2\theta + a^2\sin^2\theta) \\ -2b^2 aer\cos\theta \\ +a^2b^2e^2 - a^2b^2 \end{matrix} = 0$$

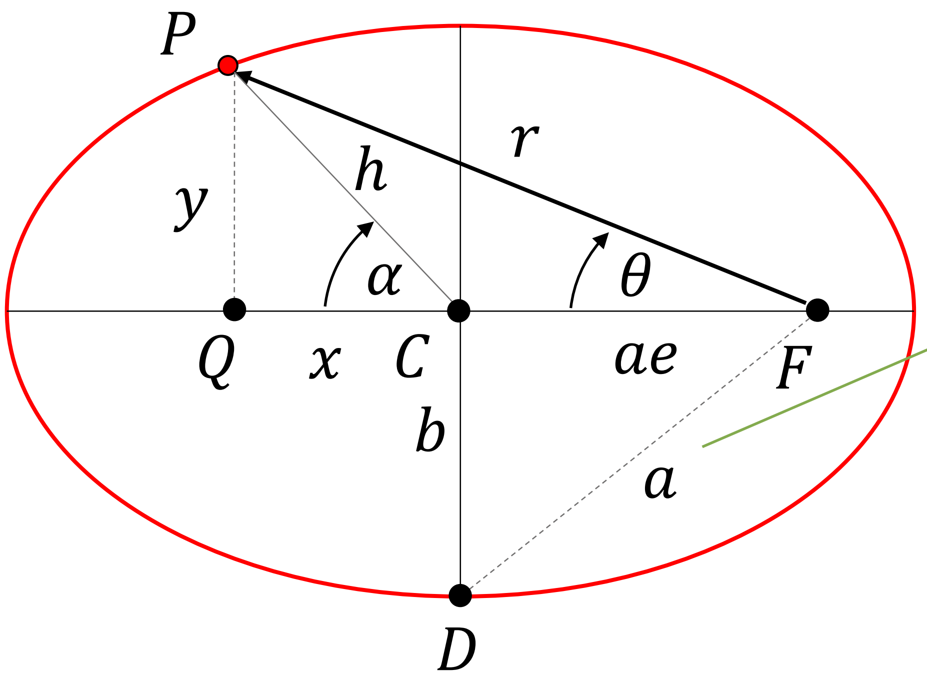
- Which is a quadratic with roots:

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

For $0 \leq e < 1$ only one of the roots guarantees a positive value of r

Ellipse equation

- Ellipse equation in polar form:



*Triangle CDF

- We also know from the properties of an ellipse*:

$$b^2 = a^2(1 - e^2)$$

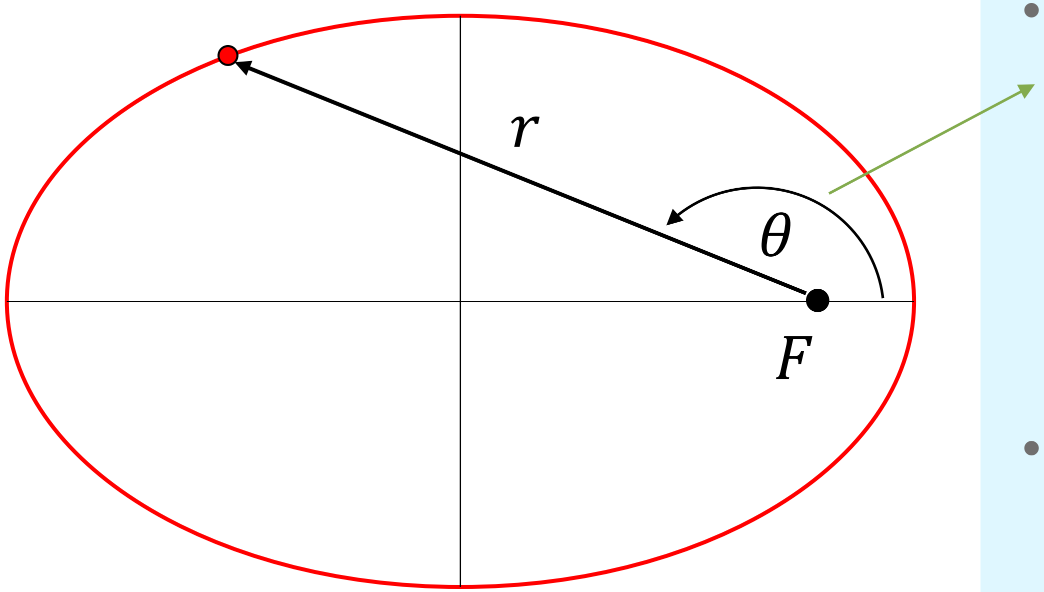
- Using this, some algebra and trig identities (working not shown) we can get to:

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}$$

- But we generally use the angle $(180 - \theta)$ when referring to orbital positions so…

Ellipse equation

- Ellipse equation in polar form:



- So our ellipse equation in polar form, with the angle θ measured from the closest point on the ellipse to the focus, is:

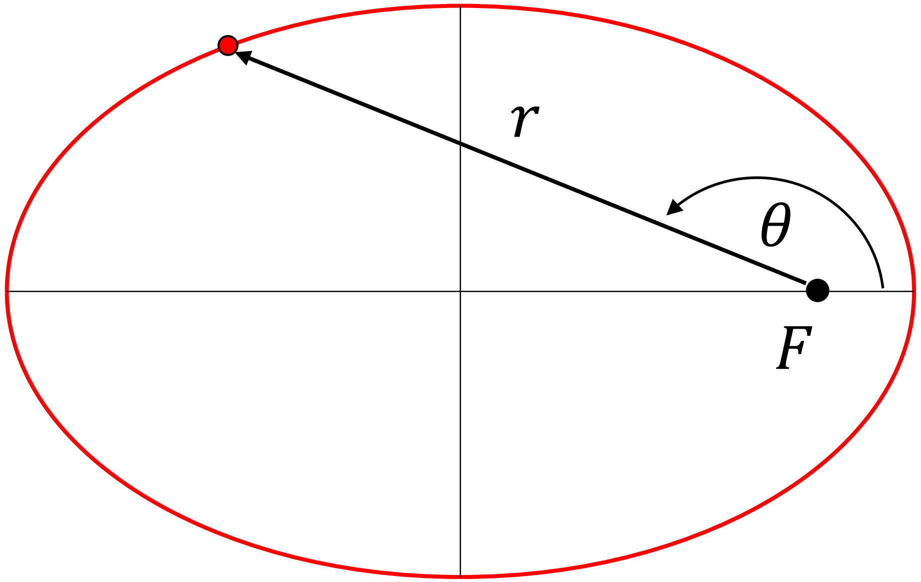
$$r = \frac{a(1 - e^2)}{1 - e\cos(180 - \theta)}$$

- Or:

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

Ellipse equation

- Ellipse equation in polar form:



$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

If we can <u>show that orbits have the same equation</u> then we will know that orbits are ellipses if the gravitational force is inverse square

Recap of lecture 3

- This lecture focused on the derivation of the ellipse equation:
  - Starting with the ellipse equation in cartesian form: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
  - With trigonometry & algebra, we can write this in polar form: $$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$
  - This is just the first step of a longer derivation, to show that the ellipse equation follows mathematically from Newton's Law of Uniform Gravitation and his Laws of Motion
  - If we show mathematically, using fundamental physical principles, that orbital trajectories can be described using this equation then we can prove that Kepler's 1st Law is correct
    - This is what we will do in lectures 4 and 5

Activity

Additional activities (not compulsory):

1. Go through the derivation of the ellipse equation in polar form and complete the missing steps/working
2. Look at other properties of an ellipse, e.g. using Wikipedia: https://en.wikipedia.org/wiki/Ellipse