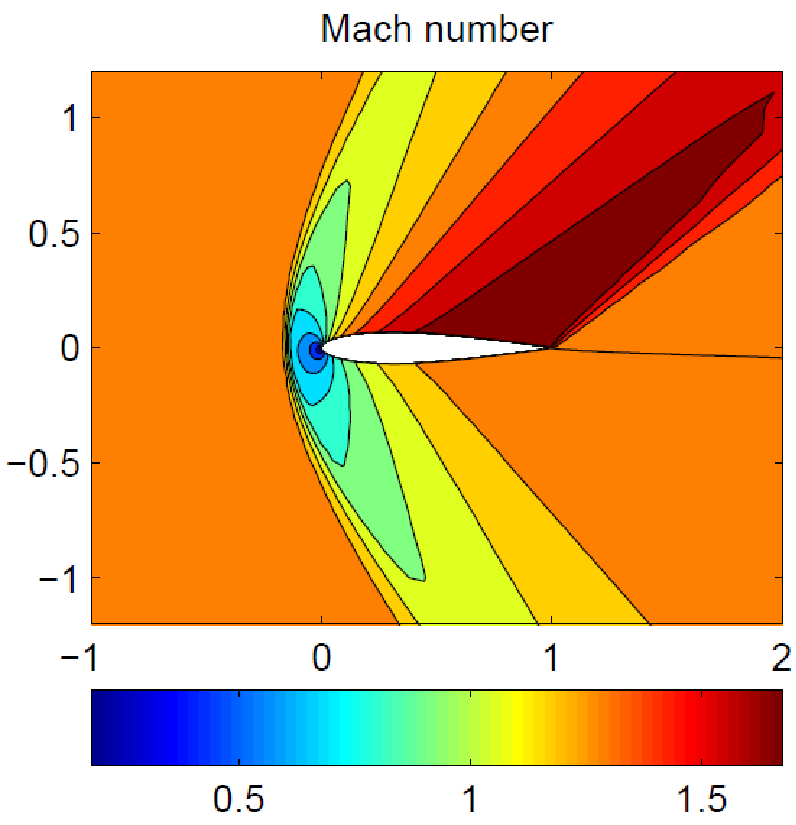
SESA3029 Aerothermodynamics 4.1

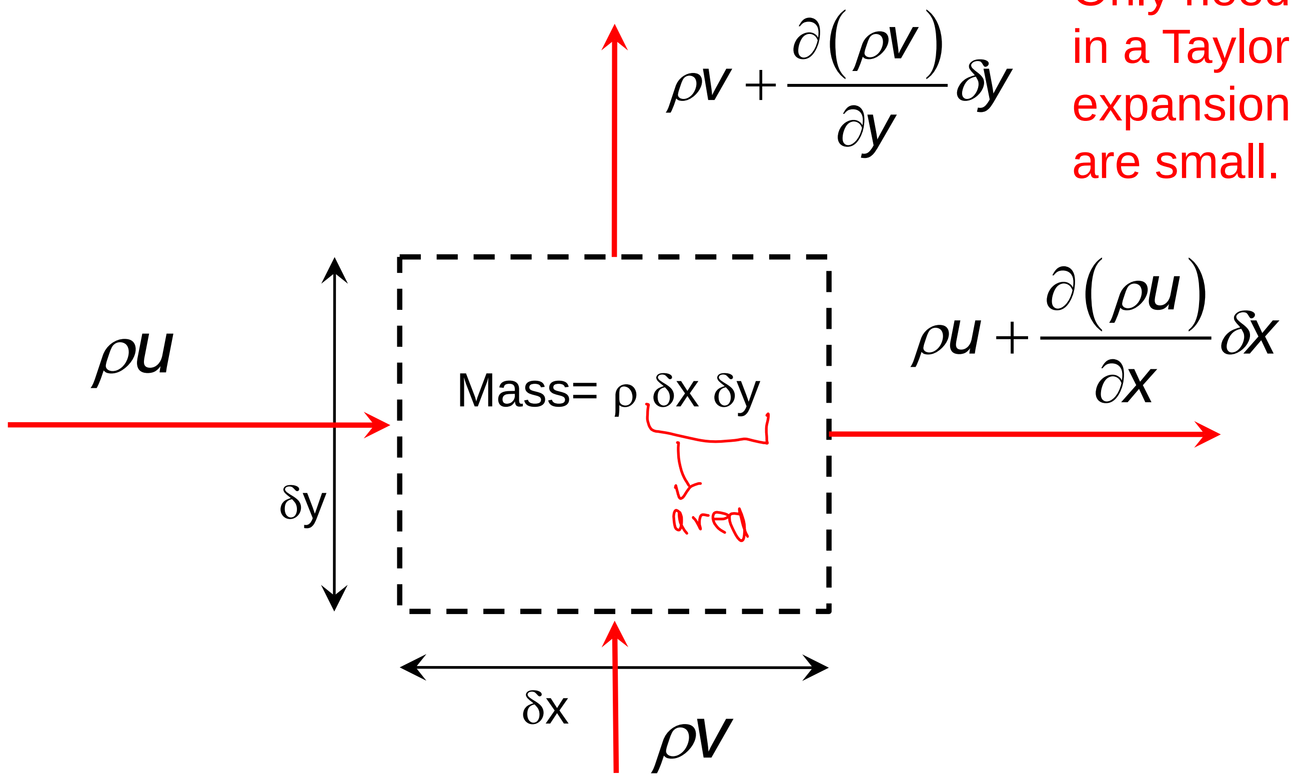


Euler equations in Cartesian co-ordinates for compressible flow - the basis of CFD

Objective

- Introduce the Euler equations
  - Widely used in CFD, sometimes in combination with a boundary-layer model
  - Capable of capturing shock waves, expansion fans etc.
  - We will derive in 2D Cartesian co-ordinates for steady flow

Control volume for mass conservation



Only need first terms in a Taylor series expansion, since δx δy are small.

rate of increase of mass in CV= mass flow rate in - mass flow rate out

$$\underbrace{\frac{\partial}{\partial t}(\rho\delta x\delta y)}_{=\dot{m}} = \rho u\delta y + \rho v\delta x - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\delta x\right)\delta y - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\delta y\right)\delta x$$

$$\frac{\partial}{\partial t}\left(\rho\delta x\delta y\right) = \rho \cancel{u\delta y} + \rho \cancel{v\delta x} - \left(\cancel{\rho u} + \frac{\partial\left(\rho u\right)}{\partial x}\delta x\right)\delta y - \left(\cancel{\rho v} + \frac{\partial\left(\rho v\right)}{\partial y}\delta y\right)\delta x$$

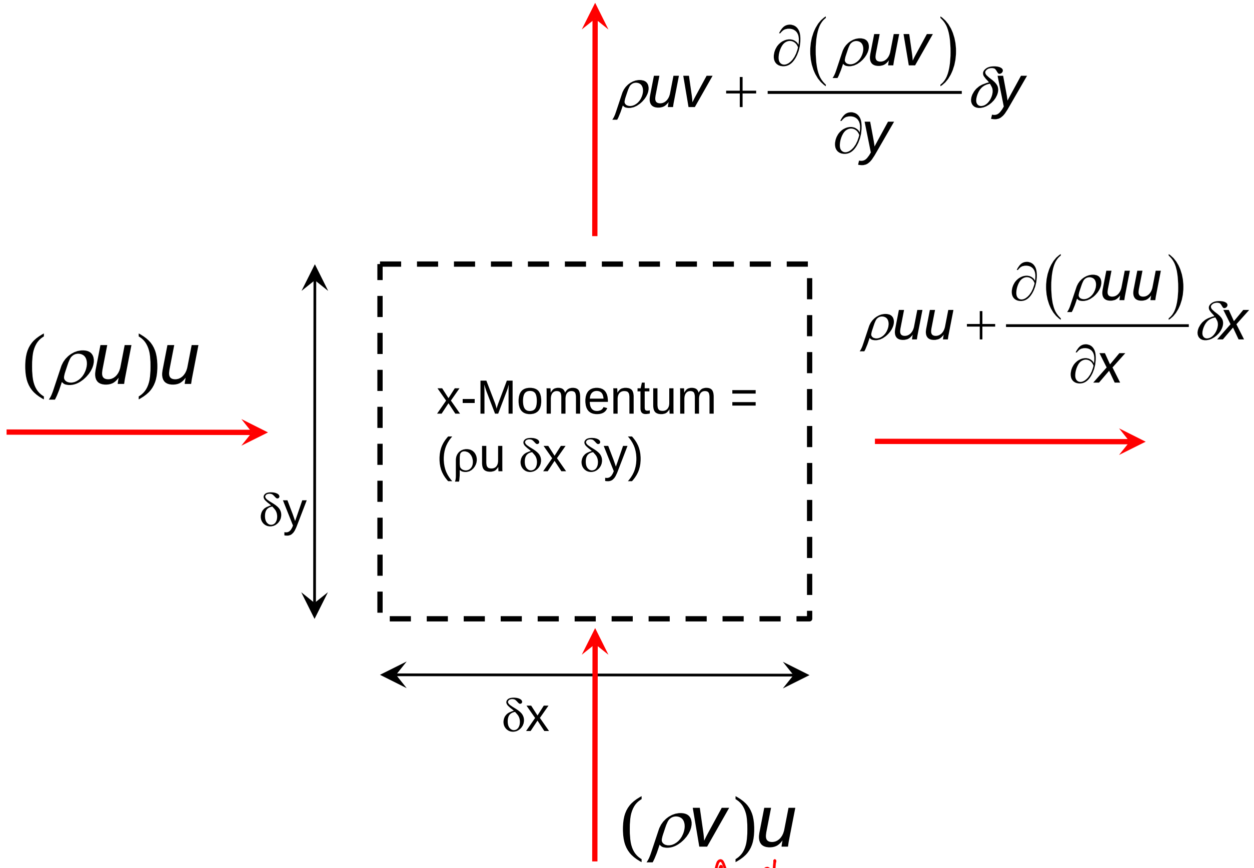
$\div\left(\delta x \ \delta y\right)$

Cancel terms and divide by δx δy

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial\left(\rho u\right)}{\partial x} + \frac{\partial\left(\rho v\right)}{\partial y} = 0}$$

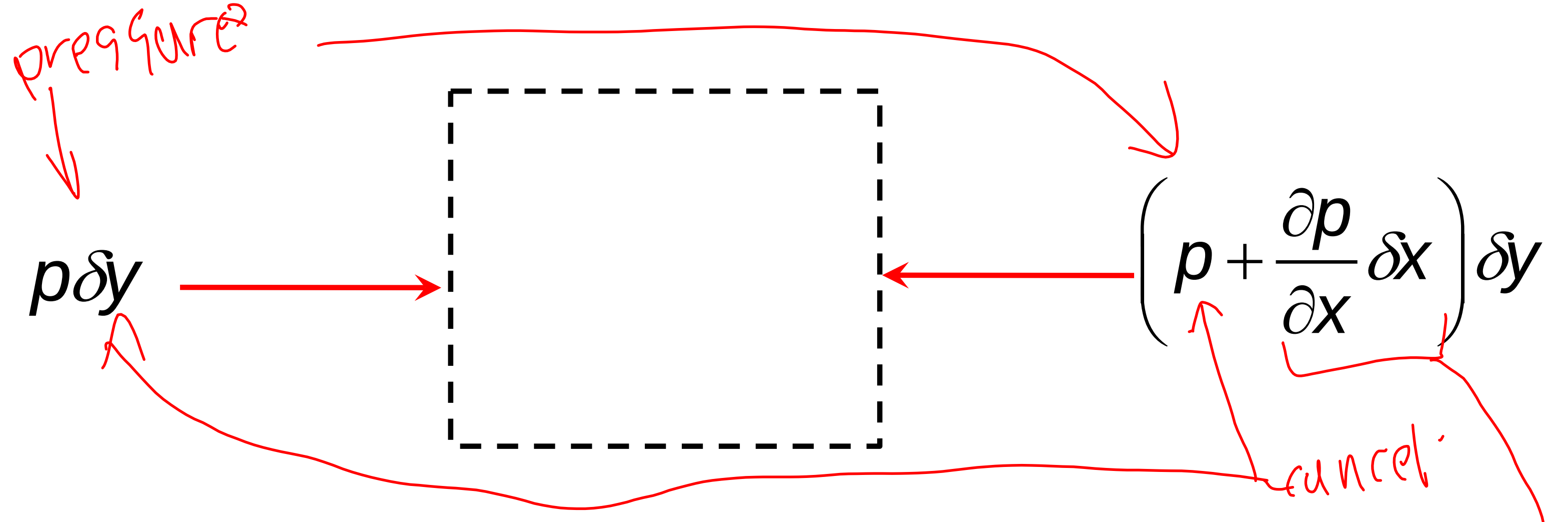
Mass conservation equation for compressible flow

Rate of change of x-momentum



has both
direction contributions

Forces in x direction



Rate of change of x-momentum = sum of forces applied in x-direction

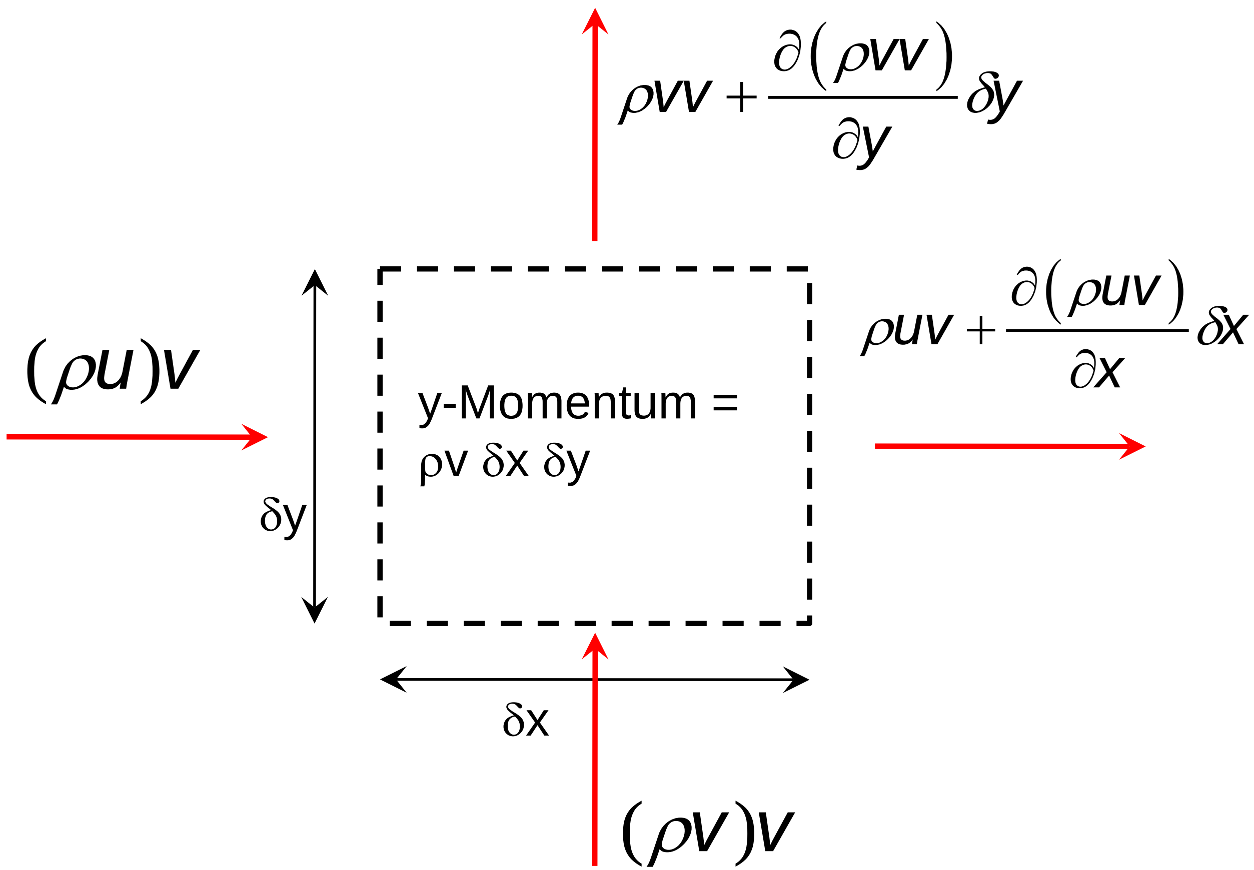
$$\frac{\partial(\rho u \delta x \delta y)}{\partial t} + \frac{\partial(\rho u u)}{\partial x}\delta x \delta y + \frac{\partial(\rho u v)}{\partial y}\delta y \delta x = -\frac{\partial p}{\partial x}\delta x \delta y$$

$\div \partial x \, \partial y$

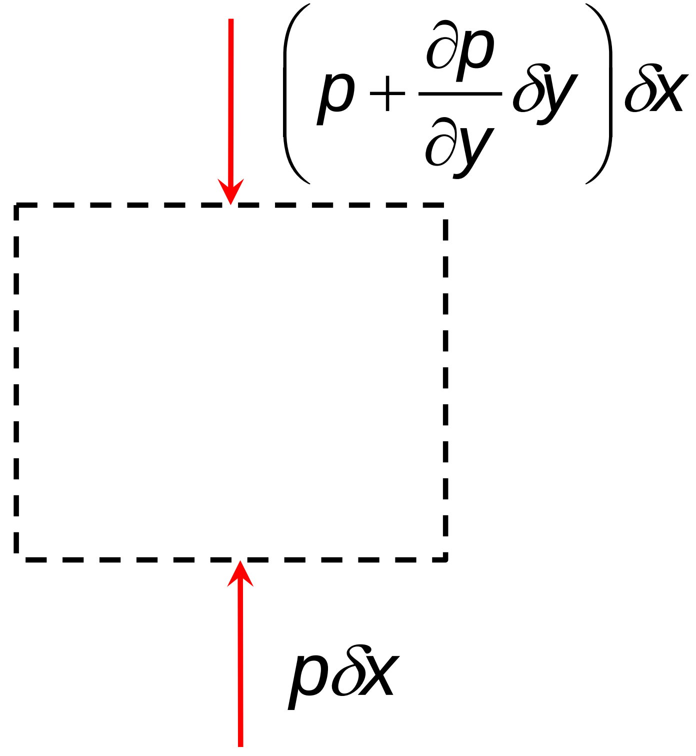
$$\boxed{\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} + \frac{\partial p}{\partial x} = 0}$$

x-momentum equation

Rate of change of y-momentum



Forces in y direction



Rate of change of y-momentum =
sum of forces applied in y-direction

$$\frac{\partial(\rho v \delta x \delta y)}{\partial t} + \frac{\partial(\rho uv)}{\partial x}\delta x \delta y + \frac{\partial(\rho v^2)}{\partial y}\delta y \delta x = -\frac{\partial p}{\partial y}\delta x \delta y$$

$$\boxed{\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial p}{\partial y} = 0}$$

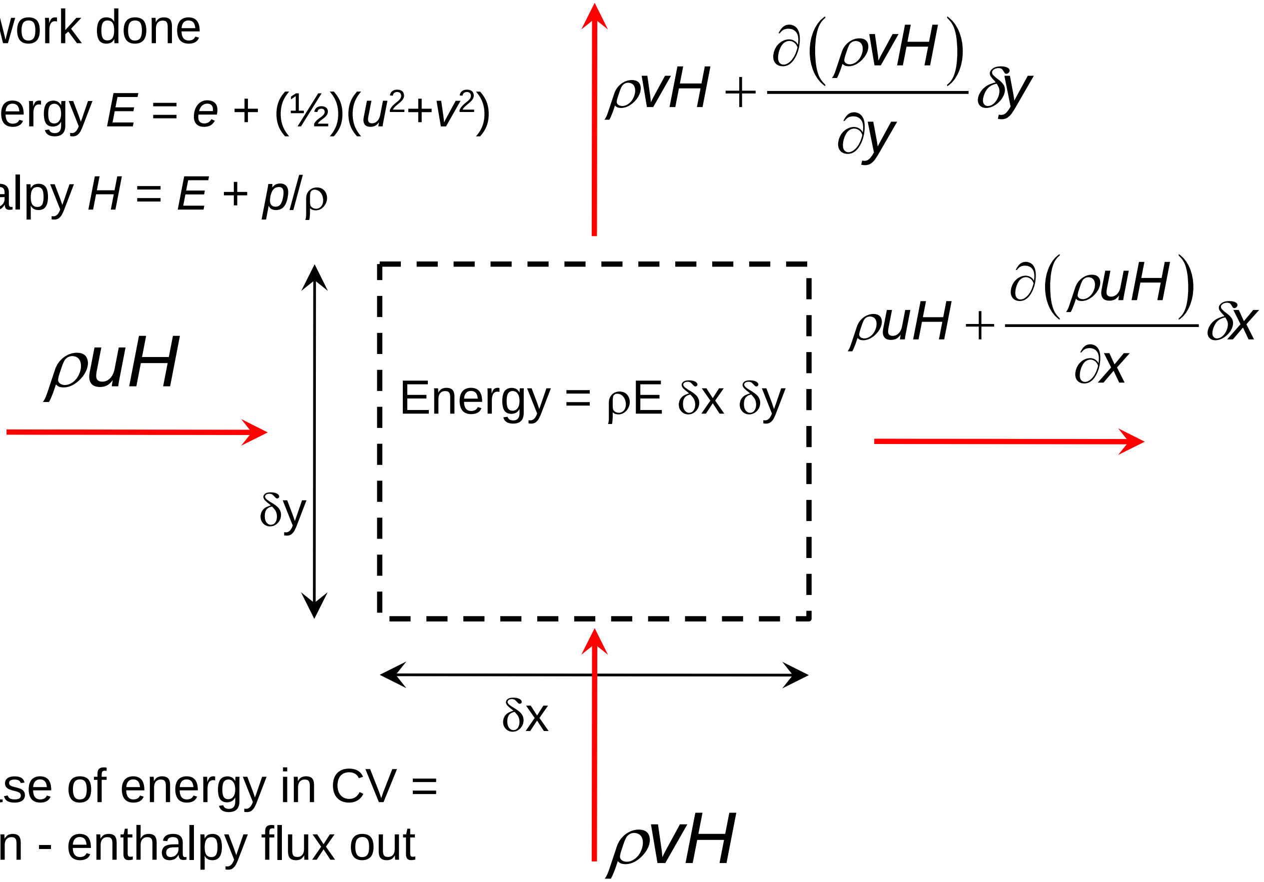
y-momentum
equation

Energy conservation

Adiabatic, no work done

Define total energy $E = e + (½)(u^2+v^2)$

Use total enthalpy $H = E + p/\rho$



Rate of increase of energy in CV =
enthalpy flux in - enthalpy flux out

$$\boxed{\frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho uH)}{\partial x} + \frac{\partial(\rho vH)}{\partial y} = 0}$$

Energy conservation

Takeaway: the Euler equations ready for CFD

4 variables and 4 unknowns

With time derivative for completeness

conservation equation

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho u \\ \rho v \\ \rho E\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H\end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix}\rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H\end{pmatrix} = 0$$

no analytical solution !!

!

'Euler solver' in CFD solves these equations (adapted for generalised co-ordinates)

Vector of 'conservative' variables

Flux vectors

$$\underbrace{\frac{\partial Q}{\partial t} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = 0}_{\text{vector form}}$$