FEEG3001 / SESM6047 Notes for Finite Element Analysis in Solid Mechanics

Part 3: Two-Noded Beam Elements

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1. Introduction to Two-Noded Beam Elements

We saw two-noded (i.e. one-dimensional) rod elements which also deform only in one-dimension, along their length. These rod elements had two nodes, each with one degree of freedom (DoF), and therefore two degrees of freedom overall. This document will describe more complicated one-dimensional elements which have multiple degrees of freedom per node:

- beams in bending (or 'bar' elements), which have two DoF per node, four overall, and
- beam-rods (or 'beam-column' elements), which have three DoF per node, six overall.

2. Beams in Bending

Beam elements can deform **only** in flexure. Their displacements are perpendicular to their axis, only: they do not deform in tension or compression, along their axis. We use Euler-Bernoulli hypothesis assumptions, that the cross sections do not change during bending, and remain perpendicular to the neutral axis.

Their properties are described by E, the Young's Modulus, I, the second moment of area, and L, the length. The displacement field at any point within the element $w(x)$ is calculated from the transverse displacement at the end points, q_1 and q_3, and the rotation at the end points, q_2 and q_4.

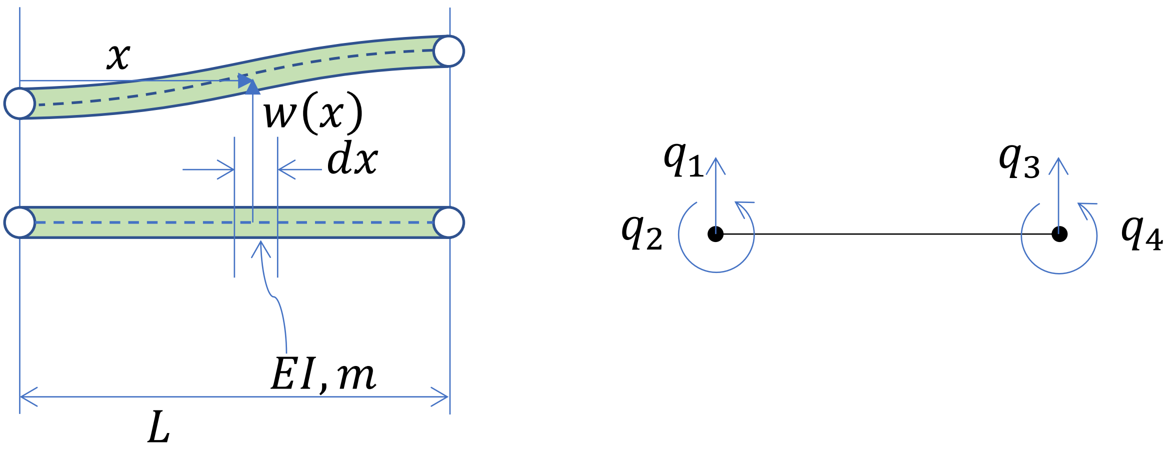


Figure 1: (left) a Beam or Bar element, which can be loaded in bending or flexure, only. (right) its nodes can displace transverse to the element's axis and rotate. Hence, it does not deform in tension or compression.

We need an expression for the element's strain energy, U, in terms of the displacement field $w(x)$, and we know that for Euler-Bernoulli beams we can use:

$$U = {}^1\!/_2 \int_0^L EI\,(w(x)'')^2 dx \text{ where } (\cdot)' = \frac{d(\cdot)}{dx}$$

Our aim is to express $w(x)$ from q_{1-4}, which are currently unknown, but which we will find using the FEA calculation. As with rod elements, we have to prescribe shape functions, denoted in general by N_i:

$$w(x) = N_1(x)q_1 + N_2(x)q_2 + N_3(x)q_3 + N_4(x)q_4$$

You will recall that the differential equation for rods is second order (it contains the second derivative of deflection), which we approximated with first order (linear) shape functions. The differential equation for beams contains the fourth derivative of deflection, so we need to approximate this with third order (cubic) shape functions. We use the four 'Hermite Cubic' shape functions, in the $0 \leq x \leq L$ domain:

$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4$$

where

$$f_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}, \qquad f_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2},$$

$$f_3(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} \qquad \text{and} \qquad f_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

We can plot these on our domain:

	Left Node	Right Node
Value	1	0
Slope	0	0

	Left Node	Right Node
Value	0	0
Slope	1	0

	Left Node	Right Node
Value	0	1
Slope	0	0

	Left Node	Right Node
Value	0	0
Slope	0	1

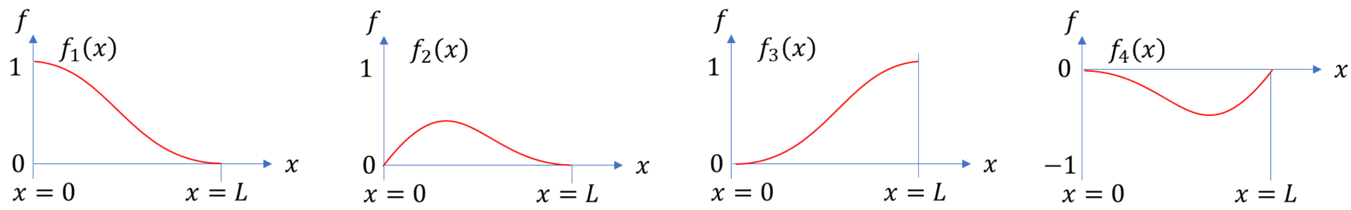


Figure 2: Hermite Cubic shape functions for beams in bending

Once we know q_{1-4}, these shape functions allow us to estimate the transverse deflection $w(x)$ and rotation/slope $w'(x)$ of any point in between.

$$w(x) = \left(1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}\right)q_1 + \left(x - 2\frac{x^2}{L} + \frac{x^3}{L^2}\right)q_2 + \left(3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right)q_3 + \left(-\frac{x^2}{L} + \frac{x^3}{L^2}\right)q_4$$

and

$$w'(x) = \left(-6\frac{x}{L^2} + 6\frac{x^2}{L^3}\right)q_1 + \left(1 - 4\frac{x}{L} + 3\frac{x^2}{L^2}\right)q_2 + \left(6\frac{x}{L^2} - 6\frac{x^2}{L^3}\right)q_3 + \left(-2\frac{x}{L} + 3\frac{x^2}{L^2}\right)q_4$$

Substitution shows that the deflection simplifies to the generalised coordinates at each end:

$$w(0) = (1)q_1 + (0)q_2 + (0)q_3 + (0)q_4 = q_1$$

$$w(L) = (1 - 3 + 2)q_1 + (L - 2L + L)q_2 + (3 - 2)q_3 + (-L + L)q_4 = q_3$$

and that the first derivative of deflection simplifies to the slopes at each end:

$$w'(0) = (0)q_1 + (1)q_2 + (0)q_3 + (0)q_4 = q_2$$

$$w'(L) = \left(-\frac{6}{L} + \frac{6}{L}\right)q_1 + (1 - 4 + 3)q_2 + \left(\frac{6}{L} - \frac{6}{L}\right)q_3 + (-2 + 3)q_4 = q_4$$

which is the expected result.

Note: you might argue that dimensional analysis is violated here, as some of our terms have units of deflection (q_1, q_3), and others have units of slope (q_2, q_4). However, we are working in matrix space, even if we have not shown this explicitly yet.

Now, applying PMTPE as before, assuming we have solved to find the unknowns q_{1-4} we use $w(x)$ to approximate our Elastic Strain Energy:

$$U = {}^{1}\!/_{2} \int_0^L EI\,(w(x)'')^2 dx$$

We can substitute in the expression above:

$$U = {}^{1}\!/_{2} \int_0^L EI[f_1''(x)q_1 + f_2''(x)q_2 + f_3''(x)q_3 + f_4''(x)q_4]^2 dx$$

Expanding out the square, we will end up with a quadratic with ten terms in q_i. Showing just a few:

$$U = {}^{1}\!/_{2} \int_0^L EI[f_1''(x){q_1}^2 + f_2''(x){q_2}^2 \ldots + 2f_1''(x)q_1 f_2''(x)q_2 \ldots]dx$$

so our integration result will take the form:

$$U = {}^{1}\!/_{2}\, EI[(\cdot){q_1}^2 + (\cdot){q_2}^2 \ldots + 2(\cdot)q_1q_2 \ldots]$$

Each integral term is relatively straightforward so we will not take pages to derive them in full here. However, we can reorganise the quadratic in the general matrix form:

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \begin{bmatrix} & 4\times 4 & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

with some example terms shown here:

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T EI \begin{bmatrix} \int_0^L {f_1''}^2(x)dx & \int_0^L f_1'' f_2''(x)dx & \cdot & \cdot \\ \int_0^L f_1'' f_2''(x)dx & \int_0^L {f_2''}^2(x)dx & \cdot & \cdot \\ \cdot & \cdot & \int_0^L {f_3''}^2(x)dx & \cdot \\ \cdot & \cdot & \cdot & \int_0^L {f_4''}^2(x)dx \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Recalling the application of PMTPE to rods in tension and compression, by the same principle this gives us the **elemental stiffness matrix for beams in bending:**

$$[K] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

3. Assembling Beam Elements:

Once again, in models where we have more than one element, instead of simply adding the elemental stiffness matrices, we assemble them. In the following example (Figure 3):

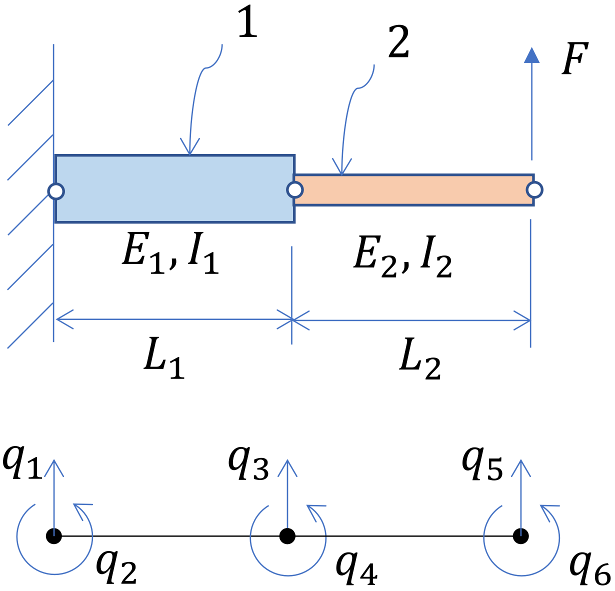


Figure 3: Two connected beam elements with cantilever constraint and a tip force (top), and the system's DoF (bottom)

Before you read on: what dimensions will our **global stiffness matrix** have? Can you guess what it looks like?

The total elastic strain energy is given by

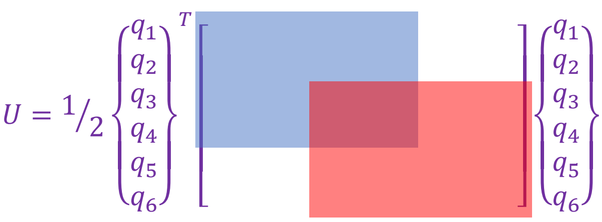
$$U = U_1 + U_2$$

For the two elements, the strain energy is:

$$U_1 = {}^1/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \frac{E_1 I_1}{L_1{}^3} \begin{bmatrix} 12 & 6L_1 & -12 & 6L_1 \\ 6L_1 & 4L_1{}^2 & -6L_1 & 2L_1{}^2 \\ -12 & -6L_1 & 12 & -6L_1 \\ 6L_1 & 2L_1{}^2 & -6L_1 & 4L_1{}^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$U_2 = {}^1/_2 \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \frac{E_2 I_2}{L_2{}^3} \begin{bmatrix} 12 & 6L_2 & -12 & 6L_2 \\ 6L_2 & 4L_2{}^2 & -6L_2 & 2L_2{}^2 \\ -12 & -6L_2 & 12 & -6L_2 \\ 6L_2 & 2L_2{}^2 & -6L_2 & 4L_2{}^2 \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

and assembled, the system's elastic strain energy is an assembly of these two, indicated by the shaded squares:



which looks slightly daunting when completed, but is quite simple at its core:

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \begin{bmatrix} 12\frac{E_1 I_1}{L_1{}^3} & 6\frac{E_1 I_1}{L_1{}^2} & -12\frac{E_1 I_1}{L_1{}^3} & 6\frac{E_1 I_1}{L_1{}^2} & 0 & 0 \\ 6\frac{E_1 I_1}{L_1{}^2} & 4\frac{E_1 I_1}{L_1} & -6\frac{E_1 I_1}{L_1{}^2} & 2\frac{E_1 I_1}{L_1} & 0 & 0 \\ -12\frac{E_1 I_1}{L_1{}^3} & -6\frac{E_1 I_1}{L_1{}^2} & 12\frac{E_1 I_1}{L_1{}^3} + 12\frac{E_2 I_2}{L_2{}^3} & -6\frac{E_1 I_1}{L_1{}^2} + 6\frac{E_2 I_2}{L_2{}^2} & -12\frac{E_2 I_2}{L_2{}^3} & 6\frac{E_2 I_2}{L_2{}^2} \\ 6\frac{E_1 I_1}{L_1{}^2} & 2\frac{E_1 I_1}{L_1} & -6\frac{E_1 I_1}{L_1{}^2} + 6\frac{E_2 I_2}{L_2{}^2} & 4\frac{E_1 I_1}{L_1} + 4\frac{E_2 I_2}{L_2} & -6\frac{E_2 I_2}{L_2{}^2} & 2\frac{E_2 I_2}{L_2} \\ 0 & 0 & -12\frac{E_2 I_2}{L_2{}^3} & -6\frac{E_2 I_2}{L_2{}^2} & 12\frac{E_2 I_2}{L_2{}^3} & -6\frac{E_2 I_2}{L_2{}^2} \\ 0 & 0 & 6\frac{E_2 I_2}{L_2{}^2} & 2\frac{E_2 I_2}{L_2} & -6\frac{E_2 I_2}{L_2{}^2} & 4\frac{E_2 I_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

We then consider the work done by external forces. There will be both a reaction force and moment at the fixture:

$$V = -Rq_1 - Mq_2 - Fq_5$$

or in matrix form:

$$V = -\{R \quad M \quad 0 \quad 0 \quad F \quad 0\} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

Applying PMTPE, the equation of equilibrium is therefore:

$$\begin{bmatrix} & & \\ & K_{6\times 6} & \\ & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} R \\ M \\ 0 \\ 0 \\ F \\ 0 \end{Bmatrix}$$

We cannot solve this in its current state because our stiffness matrix $[K]$ is singular ($|K| = 0$). However, we can apply boundary conditions where $q_1 = 0$ and $q_2 = 0$. Therefore we can strike out the first two equations (the first two rows and columns of the stiffness matrix), leaving our **reduced** stiffness matrix and equation of equilibrium:

$$\begin{bmatrix} 12\frac{E_1 I_1}{L_1{}^3} + 12\frac{E_2 I_2}{L_2{}^3} & -6\frac{E_1 I_1}{L_1{}^2} + 6\frac{E_2 I_2}{L_2{}^2} & -12\frac{E_2 I_2}{L_2{}^3} & 6\frac{E_2 I_2}{L_2{}^2} \\ -6\frac{E_1 I_1}{L_1{}^2} + 6\frac{E_2 I_2}{L_2{}^2} & 4\frac{E_1 I_1}{L_1} + 4\frac{E_2 I_2}{L_2} & -6\frac{E_2 I_2}{L_2{}^2} & 2\frac{E_2 I_2}{L_2} \\ -12\frac{E_2 I_2}{L_2{}^3} & -6\frac{E_2 I_2}{L_2{}^2} & 12\frac{E_2 I_2}{L_2{}^3} & -6\frac{E_2 I_2}{L_2{}^2} \\ 6\frac{E_2 I_2}{L_2{}^2} & 2\frac{E_2 I_2}{L_2} & -6\frac{E_2 I_2}{L_2{}^2} & 4\frac{E_2 I_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \\ 0 \end{Bmatrix}$$

which represents four independent simultaneous equations with four unknowns, which we *can* solve for $\{q\}$. Having done this, we would calculate the reaction force and moment by using the boundary condition equations, which we removed.

Finally, by substituting back into the original expression for $w(x)$ we could obtain our estimate for the deflection at any point within any element.

4. Stresses in Beam Elements:

As in our other two-node elements, we can use the deflection expression to obtain the element strain and stress responses. Again, it would be convenient to use the nodal deflection results $\{q\}$:

$$\varepsilon_x(x) = [B]\{q\}$$

and

$$\sigma_x(x) = [D][B]\{q\}$$

These will not necessarily be constant along the beam's length. You might also recall that the bending strain and stress vary with the distance y from the neutral axis of the beam. The axial strain is given by:

$$\varepsilon_x(x) = -y\frac{\partial^2 w}{\partial x^2}$$

based on the Euler-Bernoulli assumption that plane cross sections of the beam remain plane. We can express this in matrix notation as a 'matrix of operators', i.e.

$$\varepsilon_x(x) = \left[-y\frac{\partial^2}{\partial x^2}\right]\{w\}$$

We can substitute our deflection approximation using the shape functions to give:

$$\varepsilon_x(x) = \left[-y\frac{\partial^2}{\partial x^2}\right][f_1(x) \quad f_2(x) \quad f_3(x) \quad f_4(x)]\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

giving

$$\varepsilon_x(x) = -y[f_1''(x) \quad f_2''(x) \quad f_3''(x) \quad f_4''(x)]\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

You could calculate these second derivatives of the shape functions to obtain:

$$\varepsilon_x(x) = -y\left[\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right) \quad \left(-\frac{4}{L}+\frac{6x}{L^2}\right) \quad \left(\frac{6}{L^2}-\frac{12x}{L^3}\right) \quad \left(-\frac{2}{L}+\frac{6x}{L^2}\right)\right]\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

where $[B]$ (our 'B matrix') is:

$$[B] = -y\left[\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right) \quad \left(-\frac{4}{L}+\frac{6x}{L^2}\right) \quad \left(\frac{6}{L^2}-\frac{12x}{L^3}\right) \quad \left(-\frac{2}{L}+\frac{6x}{L^2}\right)\right]$$

Finally, in this simple one-dimensional situation our 'D matrix' is described by our Young's Modulus E, which allows us to express the elemental stress distribution as:

$$\sigma_x(x) = [D][B]\{q\}$$

$$\sigma_x(x) = -y[E]\left[\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right) \quad \left(-\frac{4}{L}+\frac{6x}{L^2}\right) \quad \left(\frac{6}{L^2}-\frac{12x}{L^3}\right) \quad \left(-\frac{2}{L}+\frac{6x}{L^2}\right)\right]\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Hence, we have two expressions for how strain and stress vary along the beam's length x and depth y.

5. Beam-Rod / Beam-Column Elements:

Beam-Rod (or Beam-Column) elements combine bending with stretching in tension and compression. This is a more accurate description of the elements described as beams in commercial software packages.

Their properties are described by E, the Young's Modulus, I, the second moment of area, A, the cross-sectional area, and L, the length. It has six degrees of freedom: the same four DoF as the beam element, plus two axial displacement DoF, which are conventionally numbered as shown below.

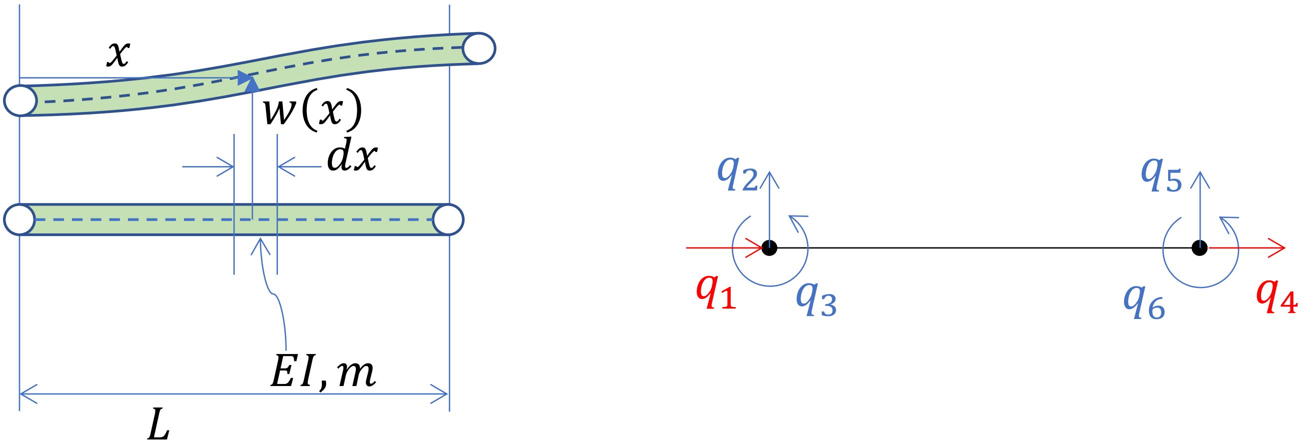


Figure 4: (left) a Beam-Rod or Beam-Column element, which can be loaded in bending and axial stretching. (right) its nodes can displace transverse to and along the element's axis, and rotate.

We can express the element's strain energy, U by summing the contributions from its bending and axial deformation behaviour, and hence it is subject to the same assumptions and limitations (i.e. Euler Bernoulli):

$$U = U_{stretch} + U_{bending}$$

As such, it has linear interpolation for its stretch behaviour, and cubic interpolation for its bending. We combine these into a single element stiffness matrix:

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & 0 & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 0 & 6\frac{EI}{L^2} & 4\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 0 & 12\frac{EI}{L^3} & -6\frac{EI}{L^2} \\ 0 & 6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

We can potentially solve the stretch and bending parts of the problem separately, if they do not interact, but the problem will only have a solution if there are sufficient boundary conditions that the structure is statically determinate in all three degrees of freedom. This may not be immediately obvious. The following two example structures (Figure 5) are both statically determinate even though only one of them has a boundary condition which prevents rotation:

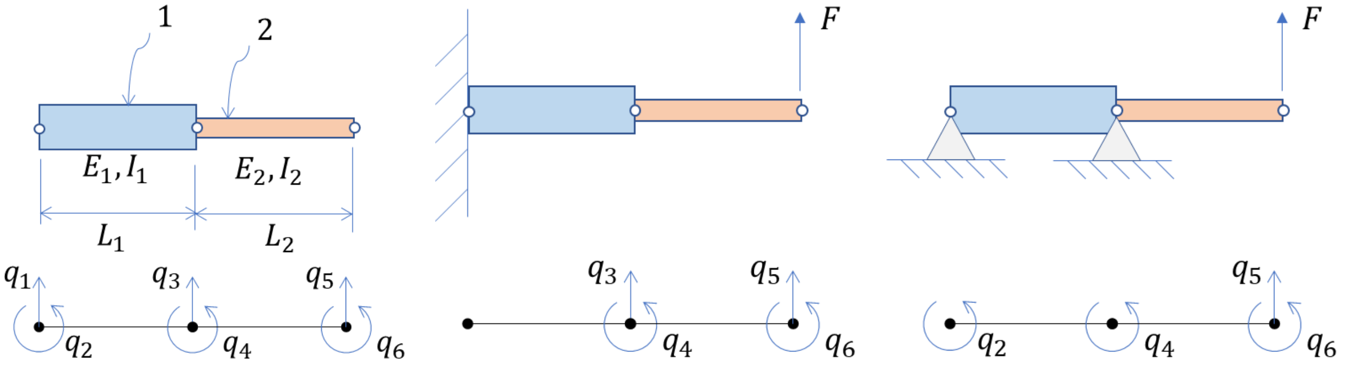


Figure 5: (left) a Beam structure, with all of its potential degrees of freedom. To avoid rigid body motion, and have a statically determinate solution, the structure needs overall constraint in both transverse displacement and rotation. This can be achieved in multiple ways, for example (centre) by directly fixing one translation and one rotational DoF, or (right) fixing two translational DoF. (Note that these situations would not have the same solution).