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Lecture 11 - Laplace Transforms

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Outline

→ Integral transforms and Differential Equations (DE)

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- The idea of an **integral transform** is to use **indirect steps** to **solve the DE**:

  s = some other independent variable

$$
\begin{tikzcd}
\text{DE Problem for } y(t) \arrow[r, "\textbf{Integral Transform}"] \arrow[d, dotted, "??"] & \text{Simpler Problem for } \tilde{y}(s) \arrow[d, "\text{Simple steps}"'] \\
\text{Solution for } y(t) & \text{Solution for } \tilde{y}(s) \arrow[l, "\textbf{Inverse transform}"']
\end{tikzcd}
$$

- We have already seen an example of this in the last lecture, where we used a Fourier transform to solve the DE for a damped harmonic oscillator.

→ Laplace Transforms (LT)

compared to fourier we remove complex to be easyer to work with

- We define the **Laplace transform** $\mathcal{L}[f(x)]$ as

s matches to w (comparing to fourier)

$$\mathcal{L}[f(x)] \equiv \tilde{f}(s) = \int_0^\infty f(x)e^{-sx}\,dx.$$

mustn't grow faster than e^{-sx}

disadvantage compared to fourier is e^{-sx} will "blow up" either positive or negative, i.e. we use bounds $0 \to \infty$ not $-\infty \to \infty$

Note that the independent variable changes $x \to s$.

- Note: Compare with the definition of the Fourier transform:

$$\mathcal{F}[f(x)] \equiv F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\,dx$$

The range of integration is **0** to ∞ rather than $-\infty$ to ∞. More importantly we have e^{-sx} rather than $e^{-j\omega x}$.

- **Laplace transforms** do not exist for all functions or for all s.

A simple example: LT do not always exist

- Laplace transforms do **not** exist for all functions or for all s.

For example, consider $f(x) = e^{ax}$. It's Laplace Transform (LT) is

sub into defn (constant

$$\mathcal{L}\left[e^{ax}\right] \equiv \tilde{f}(s) = \int_0^\infty e^{ax} e^{-sx}\,\mathrm{d}x = \int_0^\infty e^{-(s-a)x}\,\mathrm{d}x = -\frac{1}{s-a}\left[e^{-(s-a)x}\right]_0^\infty$$

possible cases

$$= -\lim_{x\to\infty} \frac{e^{-(s-a)x}}{s-a} + \frac{1}{s-a}.$$

$[\nwarrow \mathcal{L}[f(x)] \equiv \tilde{f}(s) = \int_0^\infty f(x) e^{-sx}\,\mathrm{d}x, \quad e^\alpha e^\beta = e^{\alpha+\beta}]$

if this converges $(\tilde{f}(s) < \infty)$ then it works, else Laplace not possible

- If $s - a > 0$ then $\lim_{x\to\infty} e^{-(s-a)x} \to 0$ and

$$\mathcal{L}\left[e^{ax}\right] = \frac{1}{s-a} \quad \Rightarrow \quad \text{LT does exist.}$$

- If $s - a < 0$: $\lim_{x\to\infty} e^{-(s-a)x} \to \infty \quad \Rightarrow$ LT does not exist for this s.
- If $s = a$:

$$\mathcal{L}\left[e^{ax}\right]\Big|_{(s=a)} = \int_0^\infty 1\,\mathrm{d}x = [x]_0^\infty \to \infty$$

So it diverges $\Rightarrow$ LT does not exist for this s.

Convergence

- As previous example illustrates, not all functions have a Laplace transform, as the required **integral may not converge**.
- Essentially, if $f(x)$ is bounded by an exponential $K\,e^{a\,x}$, with a, K independent of x, for large x, then its Laplace transform exists for $s > a$. (The precise theorem is given in the Lecture Notes.)
- **Of rather more importance for our later use** is the **corollary:**

$$\lim_{x\to\infty} f(x)e^{-s\,x} = 0$$

which **holds when f has a Laplace transform**.

More examples of Laplace Transforms

We can **check** using the definition of Laplace Transform

$$\mathcal{L}\left[f(x)\right] \equiv \tilde{f}(s) = \int_0^\infty f(x)e^{-s\,x}\,\mathrm{d}x$$

that

$$\mathcal{L}\left[\sin(a\,x)\right] = \frac{a}{s^2+a^2}, \qquad \text{[proof: next slide]}$$

$$\mathcal{L}\left[\cos(a\,x)\right] = \frac{s}{s^2+a^2} \qquad \text{[Exercise: do it!]}$$

Proof

Use integration by parts: $\quad (\swarrow \mathcal{L}[f(x)] = \int_0^\infty f(x) e^{-sx}\,dx)$

$$\mathcal{L}[\sin(ax)] = \int_0^\infty \overbrace{\sin(ax)}^{u}\,\overbrace{e^{-sx}}^{dv}\,dx \qquad \left(\swarrow \int_A^B u\,dv = [uv]_A^B - \int_A^B v\,du\right)$$

(Assume $s > 0$: $e^{-s\infty} \to 0 \searrow$)

$$= \underbrace{\left[-\frac{1}{s}\sin(ax)e^{-sx}\right]_0^\infty}_{=0:\ e^{-\infty}\to 0,\ \sin(0)=0} + \int_0^\infty \frac{a}{s}\underbrace{\cos(ax)}_{u}\,\underbrace{e^{-sx}}_{dv}\,dx$$

$$= \left[-\frac{a}{s^2}\cos(ax)e^{-sx}\right]_0^\infty - \frac{a^2}{s^2}\int_0^\infty \sin(ax)e^{-sx}\,dx$$

rearrange

sub into self trick

$$= \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}[\sin(ax)]$$

$$\Leftrightarrow \left(1 + \frac{a^2}{s^2}\right)\mathcal{L}[\sin(ax)] = \frac{a}{s^2} \quad \overset{\text{rearrange}}{\Leftrightarrow} \quad \mathcal{L}[\sin(ax)] = \frac{s^2}{a^2+s^2}\,\frac{a}{s^2}$$

rearrange

$$\Leftrightarrow \qquad \mathcal{L}[\sin(ax)] = \frac{a}{s^2+a^2}.$$

$\leftarrow$ *Exercise*: prove the case $\mathcal{L}[\cos(ax)]$

First shift theorem for LT

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- By definition of LT: $\mathcal{L}[f(x)] \equiv \tilde{f}(s) = \int_0^\infty f(x)e^{-sx}\,dx$

- The **First Shift Theorem** states that:

$$\boxed{\mathcal{L}\left[e^{-ax}f(x)\right] = \tilde{f}(s+a).} = \int_0^\infty \left(f(x)e^{-ax}\right) e^{-sx}\,dx$$

Proof:

$$[\swarrow \mathcal{L}[g(x)] = \int_0^\infty g(x)e^{-sx}\,dx \quad \text{with } g(x) = f(x)e^{-ax}\,]$$

$$\mathcal{L}\left[f(x)e^{-ax}\right] = \int_0^\infty f(x)e^{-ax}e^{-sx}\,dx$$

$$= \int_0^\infty f(x)e^{-(s+a)x}\,dx$$

$$= \tilde{f}(s+a) \qquad \boxed{(\nwarrow \tilde{f}(S) = \int_0^\infty f(x)e^{-Sx}\,dx \text{ with } S = s+a)}$$

This is really really useful, since we can solve a simple transform then do this to get the more complicated case!

Laplace Transform of derivatives

We are interested in solving DEs, and so we need to know **how to Laplace transform ODEs**. DEs are made of derivatives. So we first need the

Laplace Transform of the derivative of $f(x)$:

$$\mathcal{L}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = s\tilde{f}(s) - f(0) \quad \longleftarrow \text{ Later useful for LT of ODEs}$$

Proof: We directly apply the definition of LT to find

is just $\mathcal{L}(f(x))$ but with $\mathcal{L}\left(\frac{df}{dx}\right)$

$$\mathcal{L}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}x} e^{-sx}\,\mathrm{d}x$$

$(\swarrow\ \mathcal{L}[g(x)] = \tilde{f}(s) = \int_0^\infty g(x)e^{-sx}\,\mathrm{d}x)$

$$= \left[f(x)e^{-sx}\right]_0^\infty + s\int_0^\infty f(x)e^{-sx}\,\mathrm{d}x$$

$$= f(x)e^{-sx}\big|_{x\to\infty} - f(0) + s\int_0^\infty f(x)e^{-sx}\,\mathrm{d}x$$

$(\nwarrow$ Use Convergence Corollary (previous slide 6): $\lim_{x\to\infty} f(x)e^{-sx} = 0)$

$$= s\tilde{f}(s) - f(0)$$

frequency space function

time space function

Laplace Transform of derivatives

- **Extending** this to **higher order derivatives** we find

$$\mathcal{L}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right] = s^2\,\tilde{f}(s) - s\,f(0) - f'(0)$$

[*Exercise*: prove it !]

or going even further

$$\mathcal{L}\left[\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right] = s^n\,\tilde{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0).$$

- The **Laplace Transforms of** $f'(x)$ **and** $f''(x)$ are **needed** to **solve second order ODEs** using the **LT indirect method.**

$\longrightarrow$ Solving a Differential Equation (DE) using LT

Use Laplace transforms to solve the BVP: [Harmonic Oscillator BVP: lecture 3]

$$y'' + y = 0; \qquad y(0) = 0, \quad y'(0) = 1. \tag{1}$$

The Laplace Transform of this ODE is:

$$\mathcal{L}\left[y'' + y\right] = \mathcal{L}[0] \quad \xLeftrightarrow{\text{splitting}} \quad \mathcal{L}\left[y''\right] + \mathcal{L}[y] = 0.$$

$$\Leftrightarrow s^2\tilde{y} - s \times 0 - 1 + \tilde{y} = 0 \quad \longleftarrow \begin{cases} \mathcal{L}[y''(x)] = s^2\tilde{y}(s) - s\,y(0) - y'(0) \\ \mathcal{L}[y'(x)] = s\,\tilde{y}(s) - y(0) \end{cases}$$

$$\Leftrightarrow (s^2 + 1)\tilde{y}(s) = 1$$

This is an ***algebraic*** equation for $\tilde{y}(s)$ with solution: $\tilde{y}(s) = \frac{1}{s^2+1}$

But in slides 7,8 we proved that the LT of $y(x) = \sin(a\,x)$ is $\mathcal{L}[y(x)] = \tilde{y}(s) = \frac{a}{s^2+a^2}$. This is precisely our case for $a = 1$!

So we <u>invert</u> the LT using a <u>known result</u> to get solution $y(x)$ of BVP (1):

$y(x) = \sin(x)$ $\leftarrow$ <u>Exercise</u>: find solution of BVP (1) using the method of Lecture 3 & confirm it indeed yields this

Outline

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Summary

- **Laplace transforms** are a class of **integral transforms.**
- They are designed to solve DEs indirectly.
- Laplace transforms map a function of x, say $f(x)$, to a function of a different independent variable s, say $\tilde{f}(s)$.
  The Laplace transform need not exist for all values of s (or at all).
- Solving DEs involves inverting the Laplace Transform; this is most easily done by checking against known results.
- List of known results:

$$\mathcal{L}\left[e^{a x}\right] = \frac{1}{s-a},$$

$$\mathcal{L}\left[\sin(a x)\right] = \frac{a}{s^2+a^2},$$

$$\mathcal{L}\left[\cos(a x)\right] = \frac{s}{s^2+a^2},$$

$$\mathcal{L}\left[e^{-ax} f(x)\right] = \tilde{f}(s+a).$$