2. MODELLING OF DYNAMIC SYSTEMS

- The idea of using a mathematical model to describe a system we wish to control was introduced in Topic 1.
- This model describes the relationships between the variables of interest and is used to approximate the output of a system given the value(s) of its input(s).
- For a dynamic system, the usual case, differential equations can be used to model its dynamic behaviour.
- The complexity of the model should (normally) be the minimum necessary to represent the system.

2.1 Introduction

Mathematical systems can be **linear** or **nonlinear** and be **time-variant** or **time-invariant**.

Linear systems are those that are described by sets of linear equations. For example,

$$y = 4x + 7$$

is a linear model describing the relationship between the output, y, and the variable x.

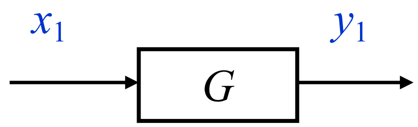
However, the expression

$$y = x^3 + 2x^2 + 12$$

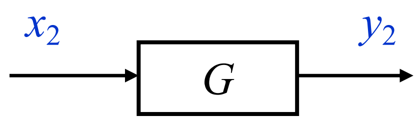
is a nonlinear model describing the relationship between y and x.

All linear systems must obey the **principle of superposition.**

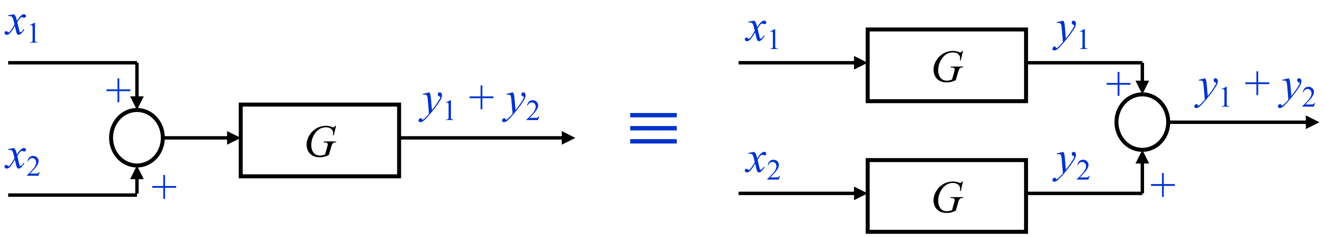
- When a system at rest is subjected to an excitation x_1, it produces a response y_1.



- The same system will produce a response y_2 for an excitation x_2.



- The principle of superposition states that if the system is subjected to the excitation $x_1 + x_2$ then the response will be equal to $y_1 + y_2$.



A system is said to be time-invariant where the coefficients are constant with respect to time. For example,

$$\alpha_1 \frac{d^2 y}{dx^2} + \alpha_2 \frac{dy}{dx} = c$$

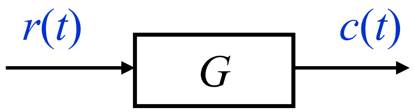
is a time-invariant model as the coefficients of the equation, α_1 and α_2 are not dependent on time, t.

The system described by the equation

$$t^2 \frac{d^2 y}{dx^2} + y = x$$

where t is time, is time-variant as one of the coefficients changes with time.

Linear differential equations



An n^{th} order differential equation takes the form:

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_1 \frac{dc(t)}{dt} + a_0 c(t)$$
$$= b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_1 \frac{dr(t)}{dt} + b_0 r(t)$$

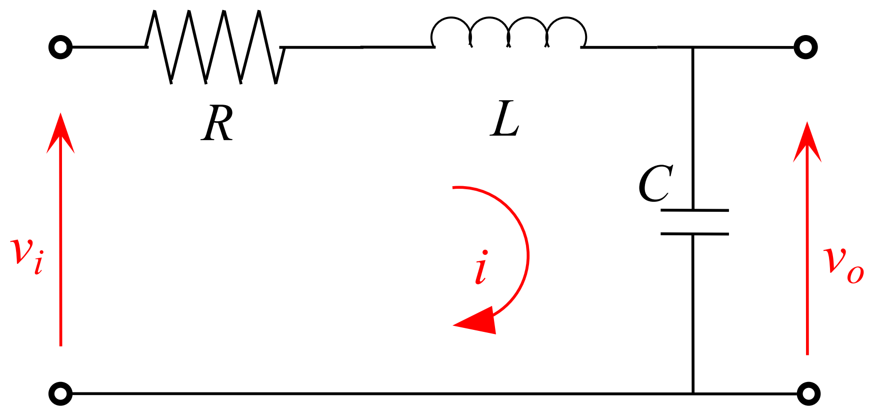
where $a_n \neq 0$ or the order would be less than n.

The usual convention is to set a_n = 1 and without loss of generality the n^{th} order differential equation has the form

$$\frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_1 \frac{dc(t)}{dt} + a_0 c(t)$$
$$= b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_1 \frac{dr(t)}{dt} + b_0 r(t)$$

2.2 Electrical circuit example

Consider the RLC circuit



$$v_i = v_o + v_R + v_L$$

The voltage drop across individual components

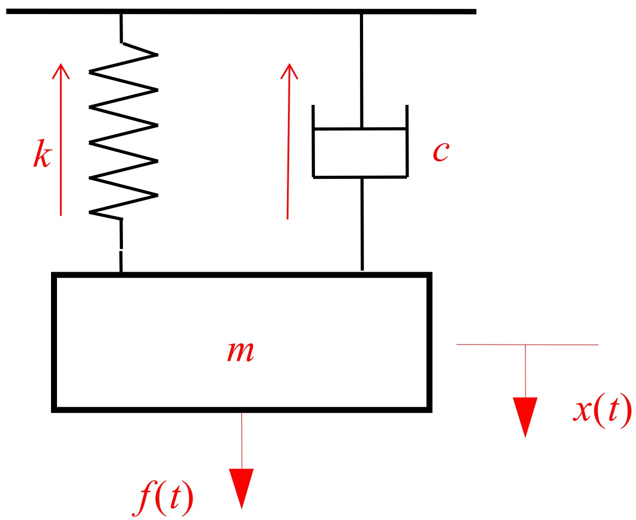
$$v_R = iR \qquad v_L = L\frac{di}{dt} \qquad v_o = \frac{1}{C}\int i\,dt \qquad \left[i = C\frac{dv_o}{dt} \right]$$

So $v_i = v_o + CR\dfrac{dv_o}{dt} + LC\dfrac{d^2v_o}{dt^2}$ or $\dfrac{d^2v_o}{dt^2} + \dfrac{R}{L}\dfrac{dv_o}{dt} + \dfrac{1}{LC}v_o(t) = \dfrac{1}{LC}v_i(t)$

Using the notation from the previous slide and $n = 2 \quad m = 0$

2.3 Mechanical system example

- By Newton's law, $m\ddot{x}$ equals the resultant of all external forces on m (in the downward direction).
- As the spring is stretched, spring force kx is upwards and opposes downward acceleration.
- The mass also moves downwards so the damping force, $c\dot{x}$, is upwards and both these terms have a minus sign.
- External force, $f(t)$, helps downward acceleration and therefore has a plus sign.



- The net force applied to the mass is $= f(t) - kx - cv = f(t) - kx - c\frac{dx}{dt}$
- The net force applied to the mass is also equal to $= ma = m\ddot{x} = m\frac{d^2x}{dt^2}$

The force equation is therefore: $m\ddot{x} = -kx - c\dot{x} + f(t)$

Rearranging gives the usual form of the differential equation of motion:

$$m\ddot{x} + c\dot{x} + kx = f(t) \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

The effects of gravity do not appear and turning the system upside down does not affect the equation.

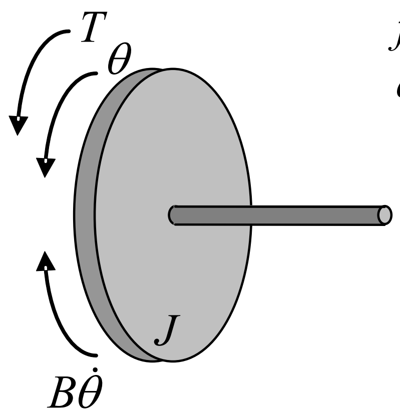
As $x = 0$ at the position of static equilibrium weight, mg, is counterbalanced by the spring force, kx.

2.4 Rotational systems - *Armature-controlled dc motor*

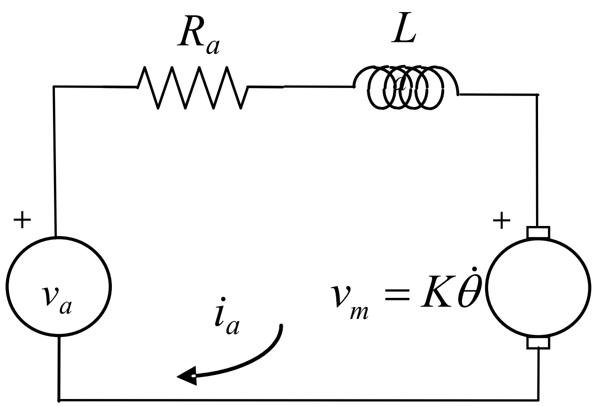
The motor load is assumed an inertia J and a damper with constant B.

shaft position, θ, and developed motor torque, T, are related by:

$$T(t) = J\ddot{\theta}(t) + B\dot{\theta}(t)$$



free body diagram



circuit diagram

The armature loop is described by:

$$v_a = R_a i_a + L_a \dot{i}_a + v_m$$

counter emf is taken to be proportional to shaft speed:

$$v_m = K_e \dot{\theta}$$

developed torque is proportional to current i_a

$$T = K_t i_a$$

Using Newton's law combined with Kirchhoff's law:

$$J\ddot{\theta} + B\dot{\theta} = Ki$$

$$T(t) = J\ddot{\theta}(t) + B\dot{\theta}(t)$$

$$T = K_t i_a$$

$$L\frac{di}{dt} + Ri = v_a - K\dot{\theta}$$

$$v_a = R_a i_a + L_a \dot{i}_a + v_m$$

$$v_m = K_e \dot{\theta}$$

where the motor constant is equal to the torque/electromotive force constant of the motor:

$$K = K_t = K_e$$

2.5 Non-linear systems

A nonlinear system is one that is described by a nonlinear equation.

Examples of nonlinear systems are:

$$y = x^2 \qquad\qquad \frac{d^2x}{dt^2} + \left[\frac{dx}{dt}\right]^2 + Ax = B\sin(\omega t)$$

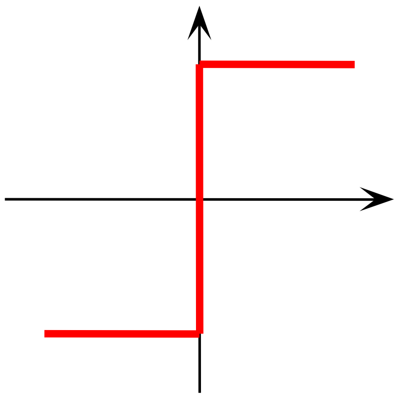
In reality ALL systems are nonlinear to some extent.

For example, an aircraft's pitch response changes significantly when the stall angle is approached.

Similarly a switch can only take the values on and off and a valve can only vary flow between 0 and 100%.

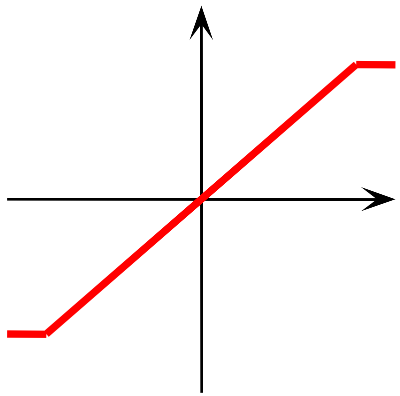
Examples of system nonlinearities

(a) Relay



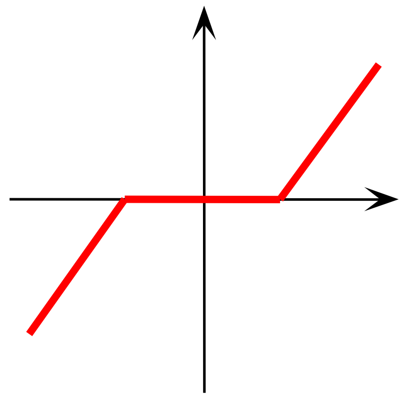
switch
solenoid

(b) Saturation



limit max/min
flow, speed,
acceleration

(c) Dead-zone



gear backlash
neutral zones
tolerances

nonlinearities frequently occur in actuating components of systems

2.6 Linearization of nonlinear systems

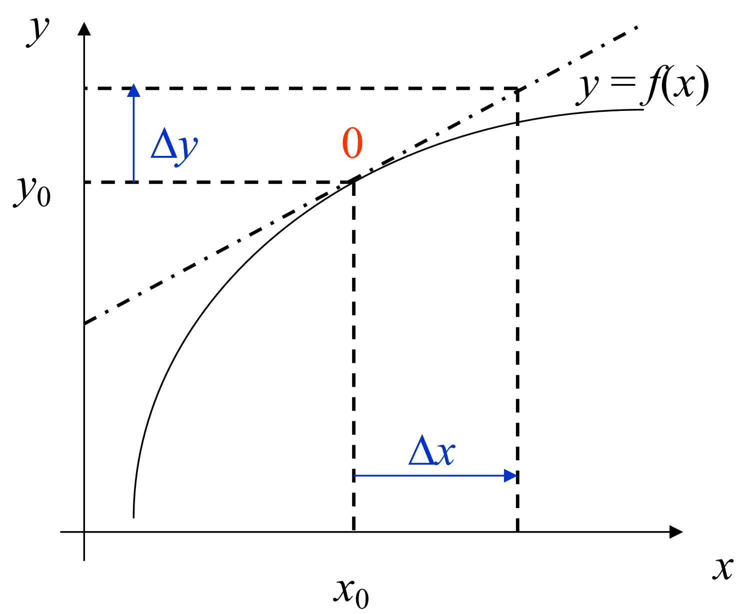
It is normally possible to represent a nonlinear system using a linear (or small-signal) approximation.

If the variation of x about x_0 is small enough, the nonlinear curve can be approximated by its tangent at point 0.

Defining $\Delta x = x - x_0$ and $\Delta y = y - y_0$, the linearized model of $f(x)$ about x_0 is

$$\Delta y = \left(\frac{df}{dx}\right)_{x_0} \Delta x \qquad (2.1)$$

where $\left(\frac{df}{dx}\right)_{x_0}$ is the slope of the tangent at x_0.



The model is then said to be linearized about a steady state operating point, 0 (x_0).

Example: Linearization of a nonlinear spring

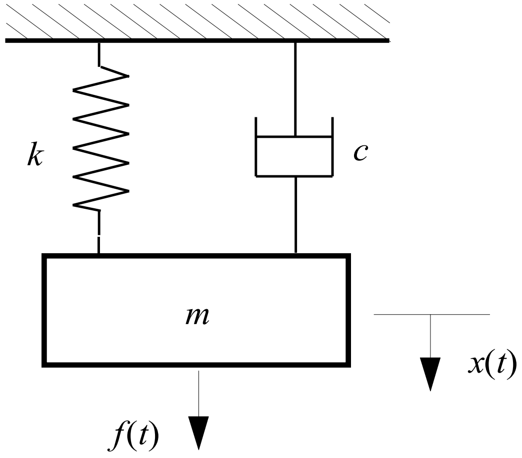
The differential equation that describes the position $x(t)$ of the mass m in response to an external force $f(t)$ is

$$m\ddot{x} + c\dot{x} + kx = f$$

Spring force may not change linearly with x and a nonlinear equation is possible, e.g.

$$m\ddot{x} + c\dot{x} + kx^3 = f + mg$$

where $h(x) = kx^3$ is a nonlinear spring force. Weight, mg, must also be added to the downward force $f(t)$ and $x = 0$ at the point of zero spring force.



If at a steady-state operating point where $f = f_0$ is a constant, then the solution $x = x_0$ is also a constant, i.e.

$$h_0 = h(x_0) = kx_0^3 = f_0 + mg \quad \text{or} \quad x_0 = \left(\frac{f_0 + mg}{k}\right)^{1/3}$$

(Force due to spring = constant force + weight)

To perform linearisation of the spring equation for small variations $\Delta x = x - x_0$ and $\Delta h = h - h_0$ about an operational point:

$$h(x) = kx^3$$

we use Eq. (2.1), which in terms of the function h is $\Delta h = \left(\frac{dh}{dx}\right)_{x_0} \Delta x,$ and obtain:

$$h(x) = h(x_0) + \Delta h = h(x_0) + \left(\frac{dh}{dx}\right)_{x_0} \Delta x = kx_0^3 + 3kx_0^2 \Delta x$$

Then as $(\dot{\Delta x}) = \dot{x}$ and $(\ddot{\Delta x}) = \ddot{x}$, the linear model of $m\ddot{x} + c\dot{x} + h(x) = f + mg$ is:

$$m(\ddot{\Delta x}) + c(\dot{\Delta x}) + kx_0^3 + 3kx_0^2 \Delta x = f_0 + \Delta f + mg$$

But since according to previous slide $kx_0^3 = f_0 + mg$, we obtain:

$$m(\ddot{\Delta x}) + c(\dot{\Delta x}) + 3kx_0^2 \Delta x = \Delta f$$

which is a linear equation expressed in terms of variations Δx and Δf about x_0 and f_0. If the actual positions are needed, x_0 must be added to the solution of this differential equation.

Taylor series expansion, ignoring all terms of order 2 or higher, can also be used for nonlinear approximation:

$$f(x) = f(x_0) + \left(\frac{df}{dx}\right)_{x_0} \frac{x - x_0}{1!} + \left(\frac{d^2 f}{dx^2}\right)_{x_0} \frac{(x - x_0)^2}{2!} + \ldots + \left(\frac{d^n f}{dx^n}\right)_{x_0} \frac{(x - x_0)^n}{n!}$$

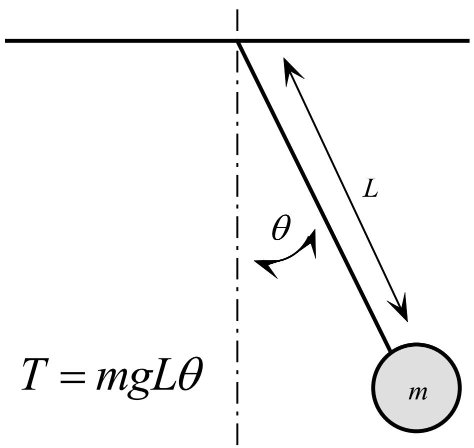
Consider the pendulum on the left. The torque is $T = mgL\sin\theta$. Nonlinear part is $\sin\theta$, which approximated around the point $\theta = \theta_0$ is:

$$\sin\theta = \sin\theta_0 + \cos\theta_0\,(\theta - \theta_0) + \cdots$$

Taking $\theta_0 = 0$ we obtain $\sin\theta \approx \theta$.

The pendulum dynamics are then modelled as $T = mgL\theta$.

- As the pendulum moves further from $\theta = 0$, the greater the error in the approximation.
- This is one reason why linearization is often associated with small signal responses.



2.7 Summary

- Dynamic systems can be modelled with differential equations.
- Linear differential equations can be used to describe the input-output properties of many simple systems
- The use of linear and time-invariant differential equations implies that the system is linear and, this is unlikely to be the case in practise.
- System linearization can be used to approximate the system dynamics about a specific operating point.

2.8 Problems

1. Classify the following differential equations as to whether they are time-varying or time-invariant:

a) $\dfrac{d^2y}{dt^2}+7y=0$

b) $\dfrac{d}{dt}(t^2y)+y=0$

c) $\left(\dfrac{1}{t+1}\right)\dfrac{d^2y}{dt^2}+\left(\dfrac{1}{t+y}\right)\dfrac{dy}{dt}+\left(\dfrac{1}{t+1}\right)y=0$

d) $\dfrac{d^2y}{dt^2}+(\cos t)y=0$

2. Classify the following differential equations as to whether they are linear or non-linear.

a) $\dfrac{dy}{dt}+y=0$

b) $\dfrac{d^2y}{dt^2}+y=0$

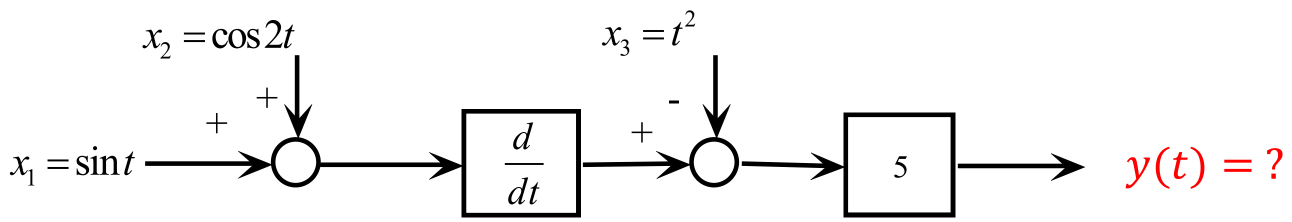
c) $\dfrac{dy}{dt}+y^2=0$

d) $\cos y\dfrac{d^2y}{dt^2}+\sin 2y=0$

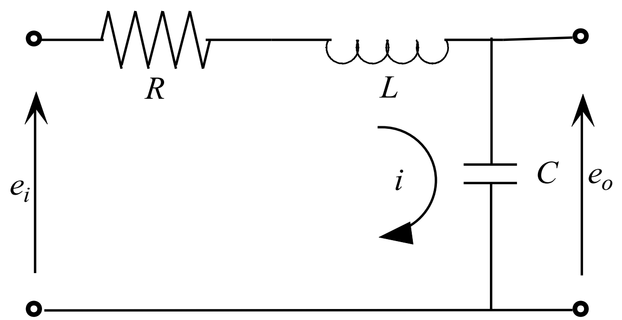
e) $\cos x(t)\dfrac{d^2y}{dt^2}+\sin x(t)=0$

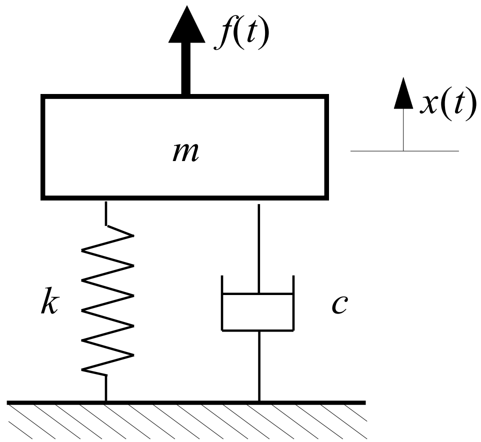
f) $\cos x(y)\dfrac{d^2y}{dt^2}+\sin 2x(y)t^2=0$

3. Use the principle of superposition to determine the output of the following system:



4. Obtain mathematical models for the systems below and show that they are analogous (Hint: let $i = dq/dt$).





5. Write down the equations of motion for the system shown in Fig. 1 assuming that the bar AB is rigid, massless and constrained to remain horizontal.

6. Linearize the following nonlinear differential equation using Taylor series expansion

$$\frac{d^2y}{dx^2} + y\cos y = x \qquad \text{with } y = 0 \text{ when } x = 0$$

7. The nonlinear equation describing the pendulum in Fig. 2 below is $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$. Linearize the system around the point $\theta = 0$.

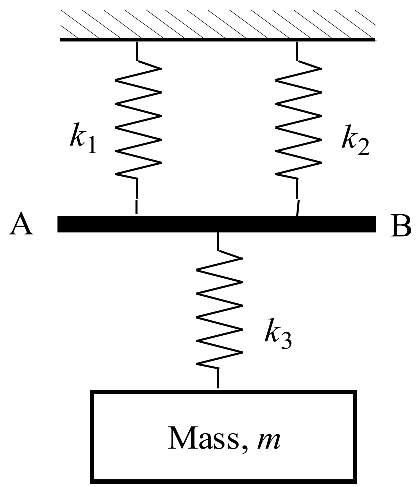


Fig. 1

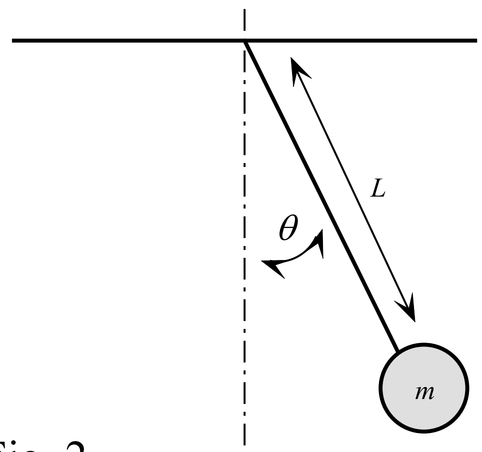


Fig. 2