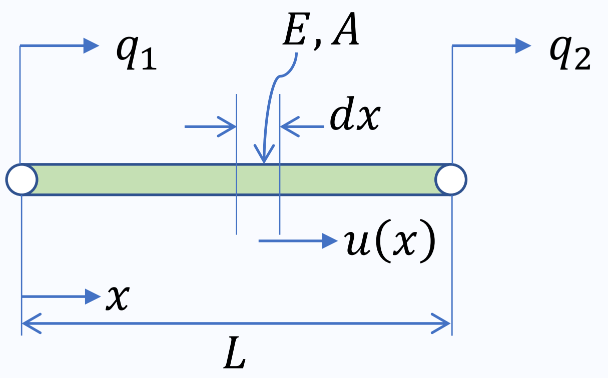
# Part 3: Beams in Bending
# Introduction and Shape Functions

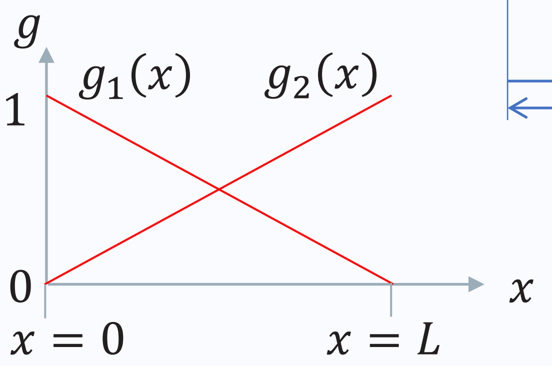
FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

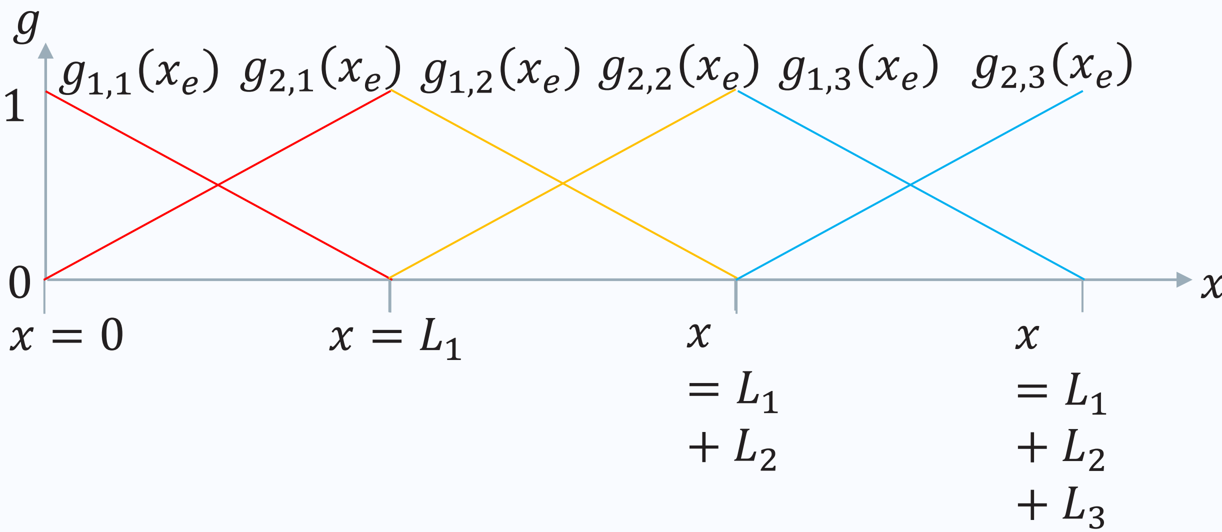
From 22nd October 2024

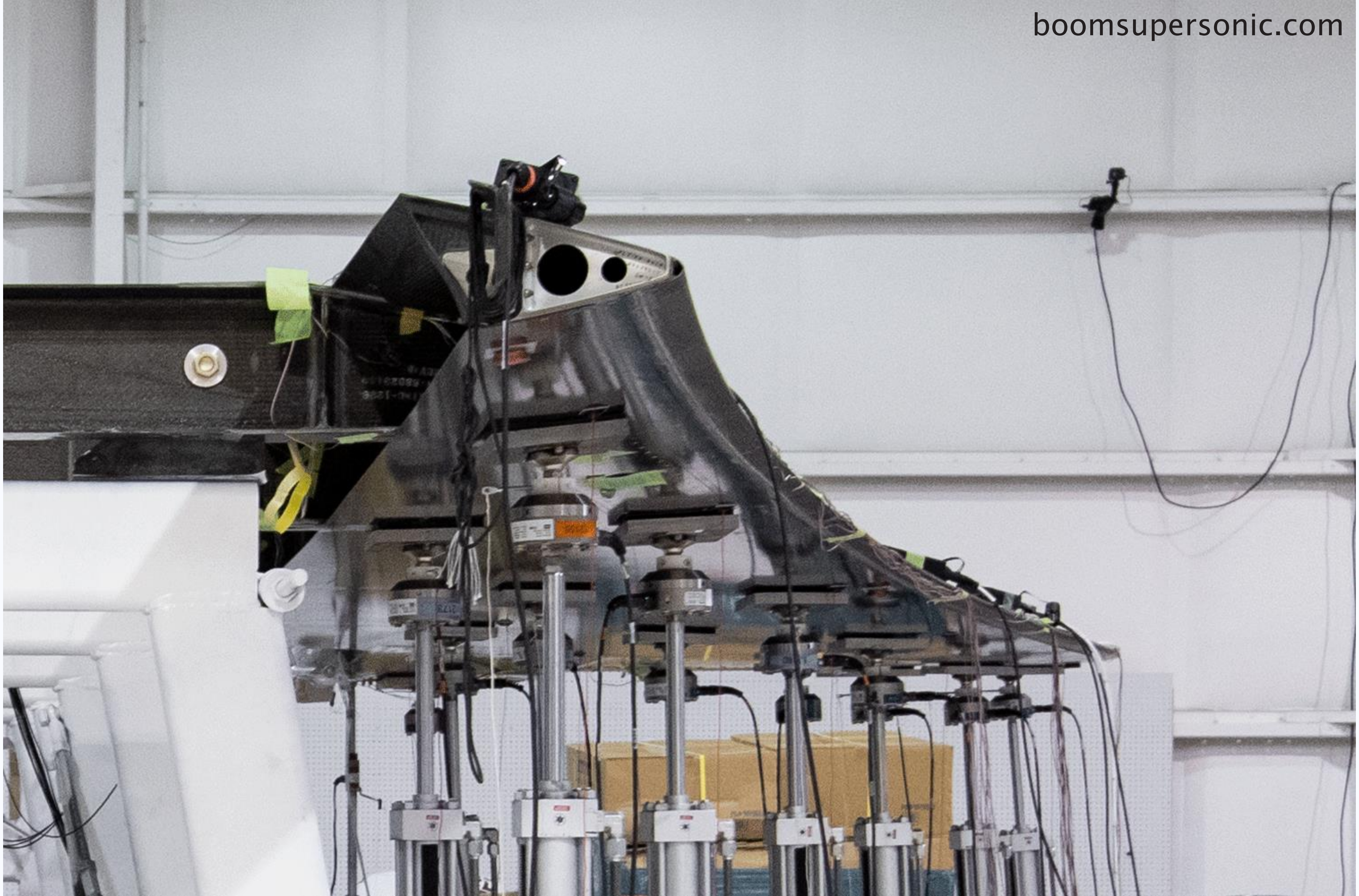
# Reminder of linear interpolation functions for rods in axial tension and compression:




- Why use Shape Functions?
  - A continuum has an infinite number of Degrees of Freedom
  - FEA:
    - describes the mechanics of problems approximately,
    - using an equivalent description that has finite DoF, and
    - describes the displacement field in between using a pre-determined shape function.
- Shape Functions should provide continuity between adjoining elements…

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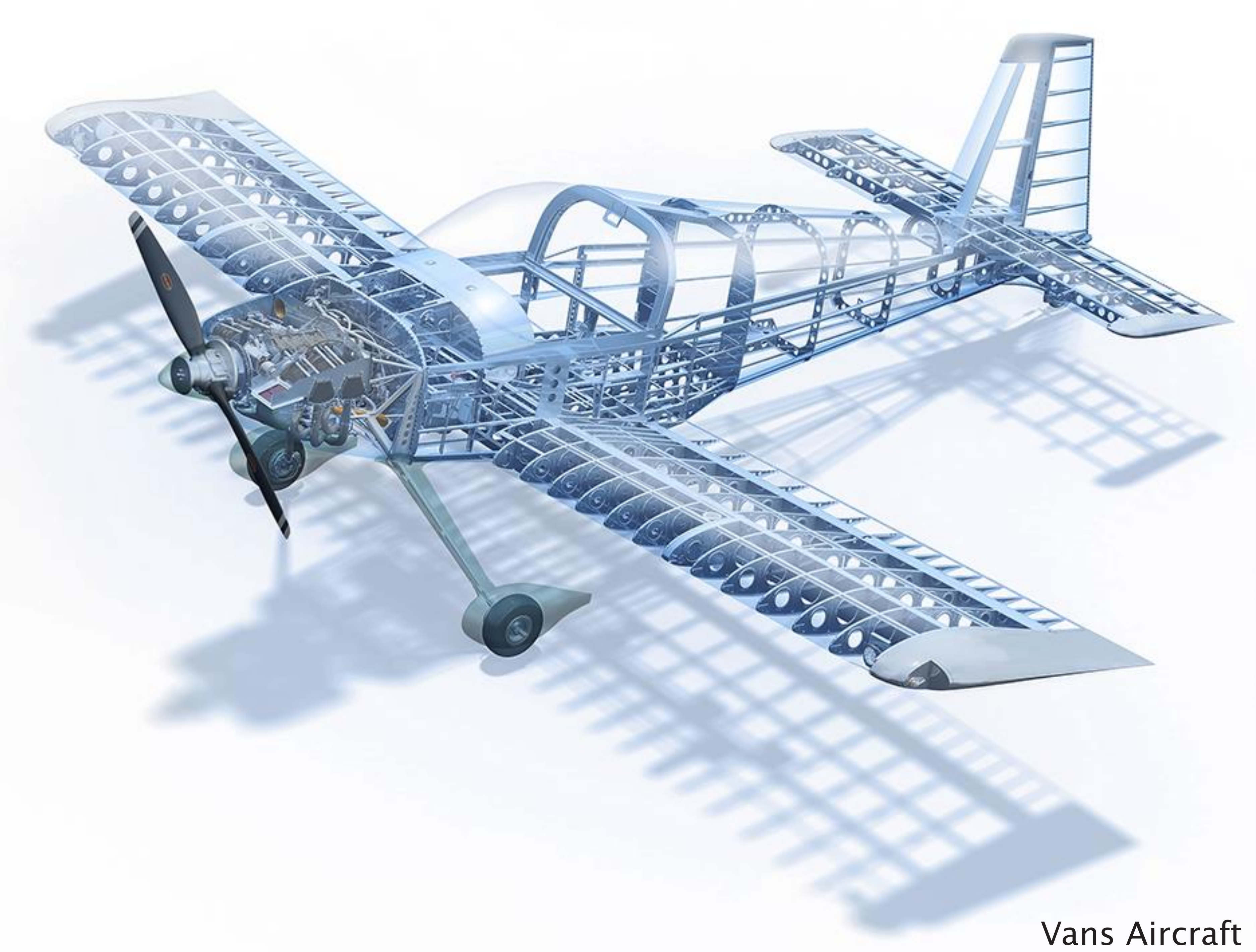
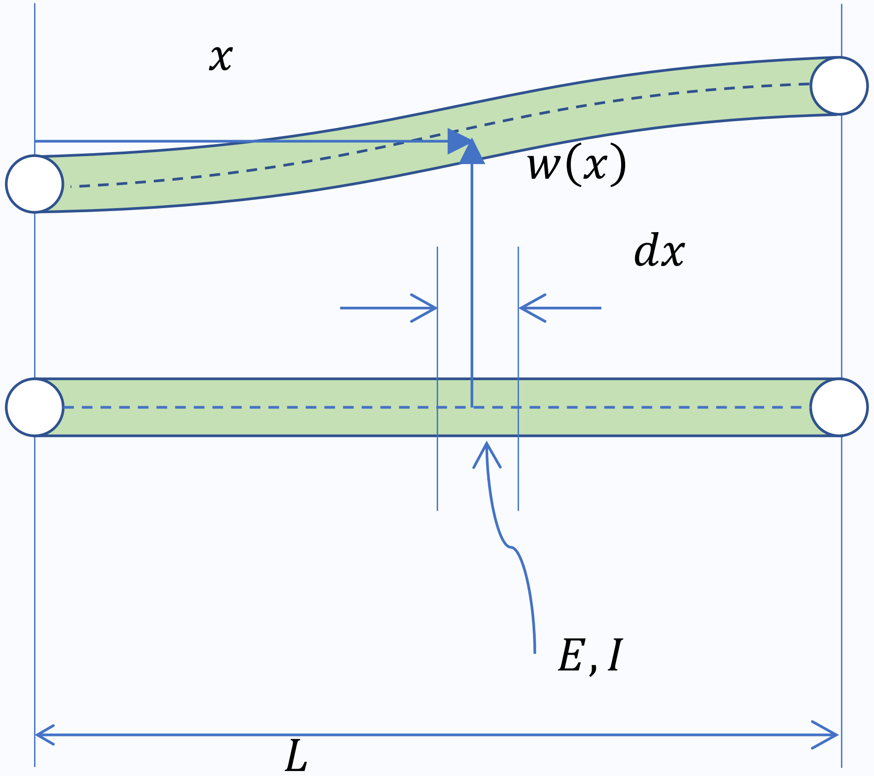
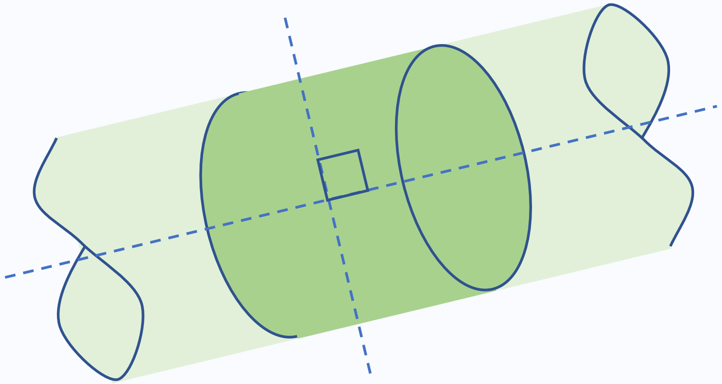


Vans Aircraft

# Two-noded beam elements in bending

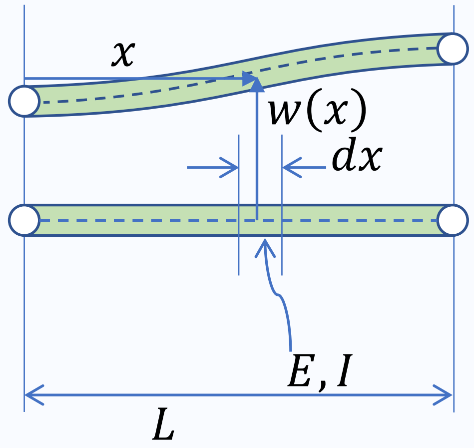
- Similar to what we saw for rods, but displacements $w(x)$ are perpendicular to the element's axis
- What parameters give it its bending properties?
  - $E$, Young's modulus
  - $I$, Second moment of area



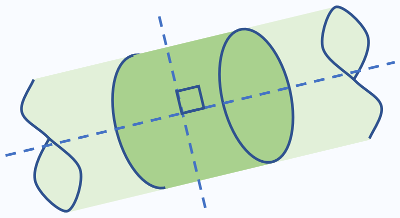
Euler-Bernoulli hypothesis assumptions:
- Cross sections do not change during bending
- Cross section remains perpendicular to the neutral axis during bending

# Two-noded beam elements in bending



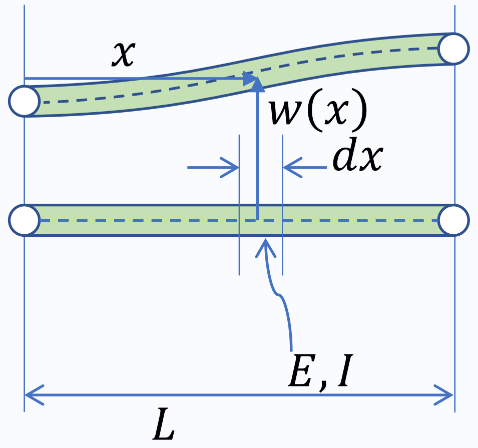

- Without deriving it, we define that the strain energy stored in the bent beam is given by:

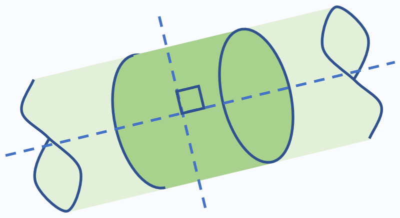
$$U = {}^{1}\!/_{2} \int_0^L EI(w(x)'')^2 dx$$



- where $(\cdot)' = \frac{d}{dx}(\cdot)$

# Two-noded beam elements in bending



- Recalling FEEG1002:

  $EAu'' =$ longitudinal loading (rods, tens/comp)

  $EIw'''' =$ transverse loading (beams)

- the axial rod differential equation has a second derivative of deformation
- the beam bending differential equation has a 4th derivative of deformation
- How do we handle this without solving the differential equation?
- We use a shape or interpolation function again;
- We cannot use linear interpolation – we need ‘cubic’ interpolation (i.e. the order of the D.E. minus 1).

# Two-noded beam elements in bending

- We require continuity of deformation from one element to the next, and continuity of 1st derivative of deformation. Beams: transverse deflection and slope
- If we use cubic interpolation for transverse deflection:

  $M(x) = \mathrm{EI}\dfrac{d^2w(x)}{dx^2}$ can capture linearly varying bending moment within an element

  Since $\sigma = \dfrac{My}{I}$ this means we can capture linearly varying stress within an element

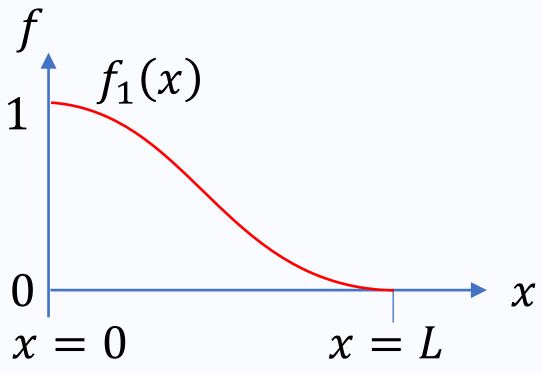
  $\mathrm{V}(x) = \mathrm{EI}\dfrac{d^3w(x)}{dx^3}$ can capture constant shear force within an element

- Though deflection and slope must be continuous from element to element, bending moments and shear force are not. This allows us to apply concentrated moments and forces on nodes.
- FEA is popular because it 'weakens' the restriction on continuity of our interpolation functions (interpolation order is order of the differential equation -1)

# Two-noded beam elements in bending

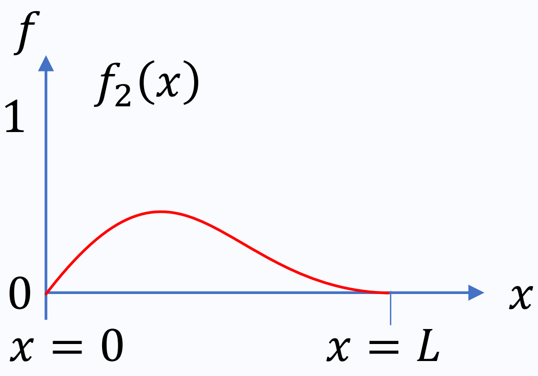
- Now we will define four cubic interpolation functions in the $x = 0$ to $L$ domain, defined by their value and their slope.
- These are called the ‘Hermite cubics’:

| | Left Node | Right Node |
|---|---|---|
| Value | 1 | 0 |
| Slope | 0 | 0 |



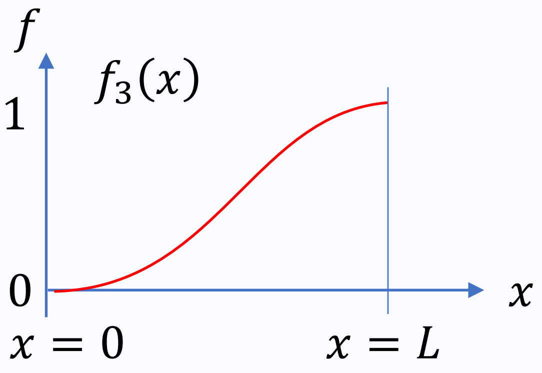

$$f_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$

| | Left Node | Right Node |
|---|---|---|
| Value | 0 | 0 |
| Slope | 1 | 0 |



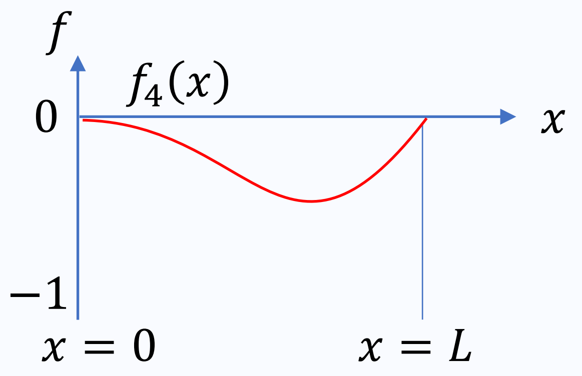

$$f_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}$$

| | Left Node | Right Node |
|---|---|---|
| Value | 0 | 1 |
| Slope | 0 | 0 |



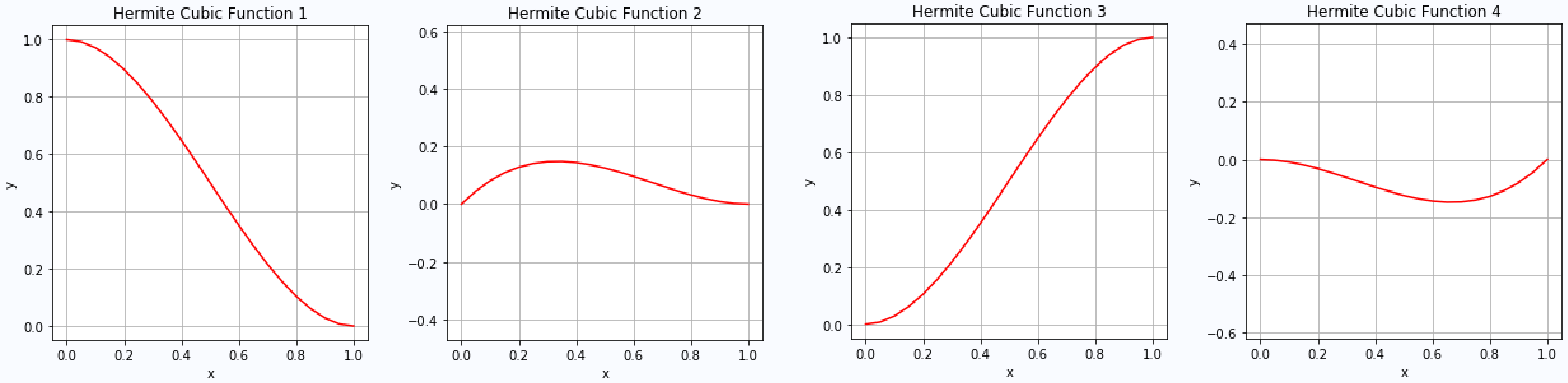

$$f_3(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}$$

| | Left Node | Right Node |
|---|---|---|
| Value | 0 | 0 |
| Slope | 0 | 1 |




$$f_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

# The Hermite Cubics



# Two-noded beam elements in bending

- and we make our approximation by saying the displacement anywhere in the element is approximated as:

$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4$$

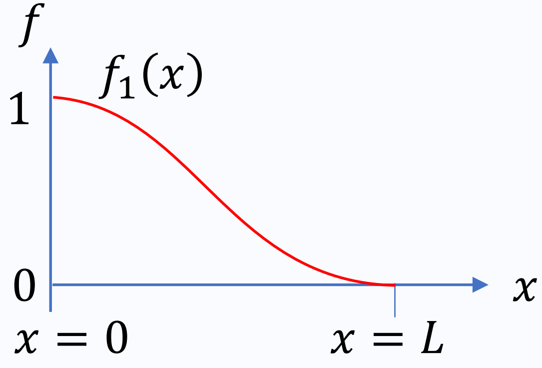
- where $f_i(x)$ are the four shape functions or interpolation functions, each having the form:

$$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3$$

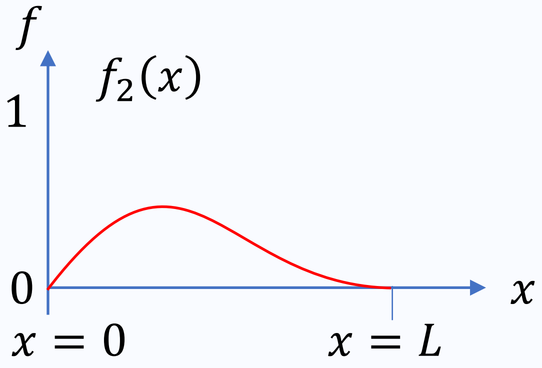
- We also now have four $q_i$ values to find…
- We won't solve it 4 times, and you won't need to remember them, but you should now understand how, based on the axial rod.

# Two-noded beam elements in bending

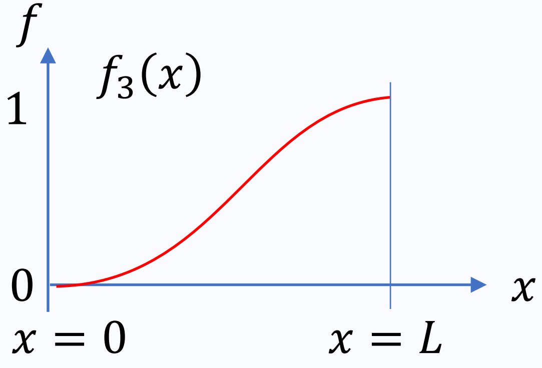
$$f_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$



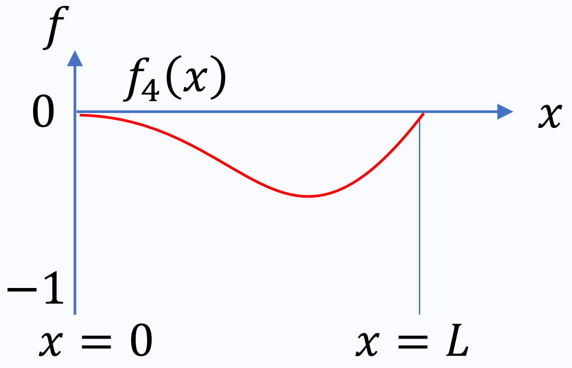

$$f_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}$$




$$f_3(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}$$




$$f_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$




$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4$$

- What can we say about the values of deflection at each end?

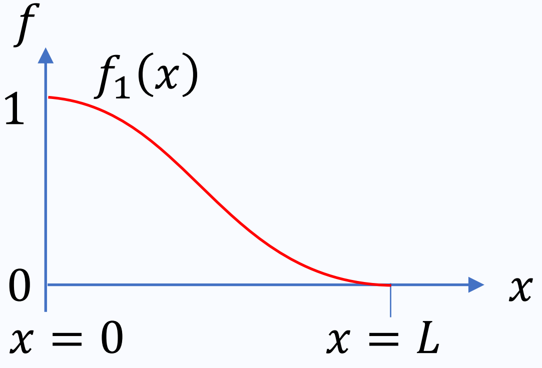
$$w(0) = f_1(0)q_1 + f_2(0)q_2 + f_3(0)q_3 + f_4(0)q_4 = q_1$$

$$w(L) = f_1(L)q_1 + f_2(L)q_2 + f_3(L)q_3 + f_4(L)q_4 = q_3$$

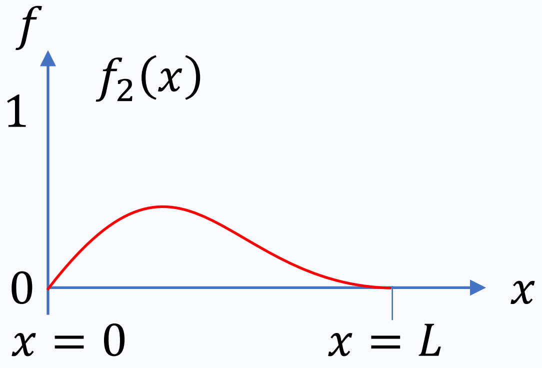

- so these are meaningful results! End deflections: generalised coordinates

# Two-noded beam elements in bending

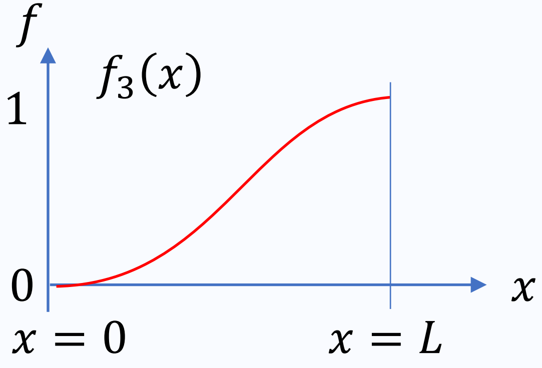
$$f_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$



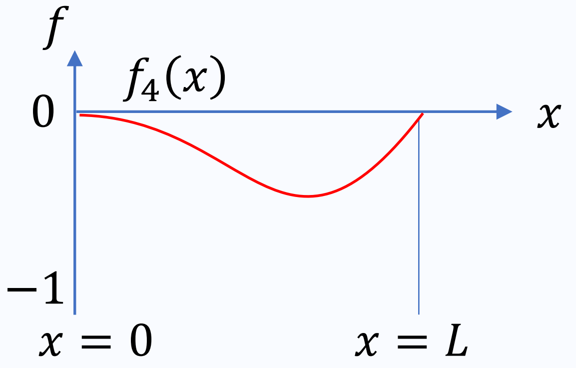

$$f_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}$$




$$f_3(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}$$




$$f_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$




$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4$$

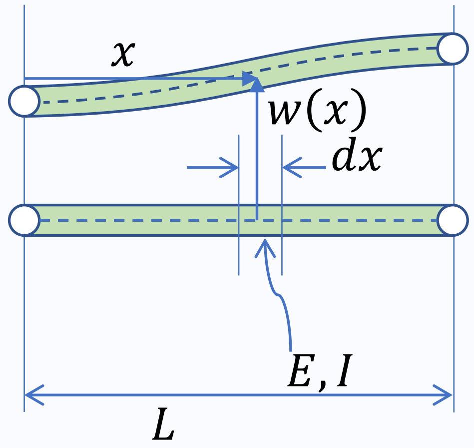
- What can we say about the values of rotation at each end?

$$w'(0) = f_1{}'(0)q_1 + f_2{}'(0)q_2 + f_3{}'(0)q_3 + f_4{}'(0)q_4 = q_2$$

$$w'(L) = f_1{}'(L)q_1 + f_2{}'(L)q_2 + f_3{}'(L)q_3 + f_4{}'(L)q_4 = q_4$$

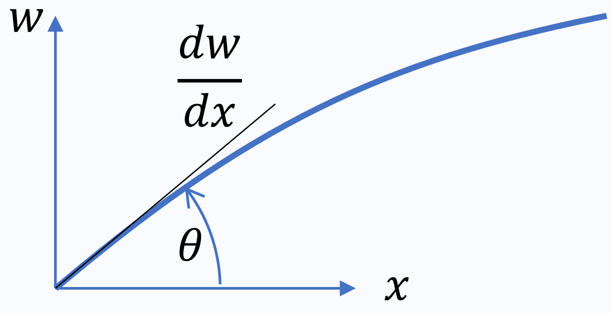

- also meaningful results! End slopes/rotations: generalised coordinates.

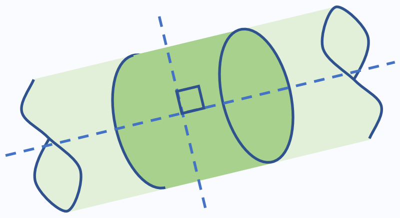
# Two-noded beam elements in bending




- because with small deformations, rotations and slopes are equivalent

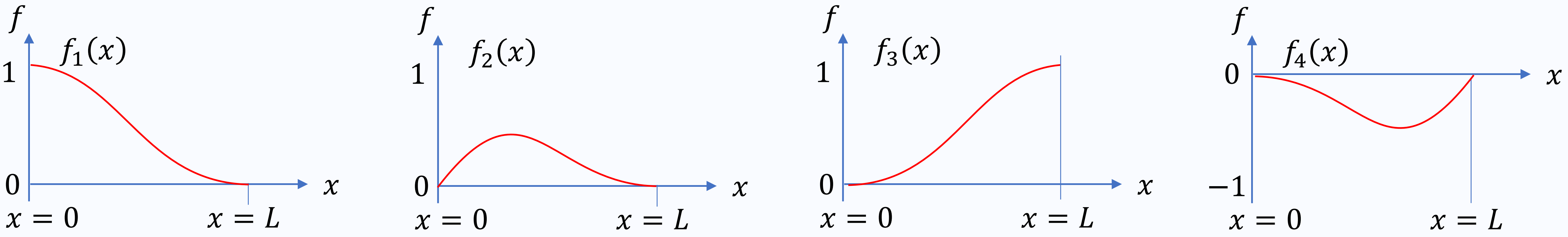
$$w' = \frac{dw}{dx} = \tan\theta \text{ and for small } \theta \approx \sin\theta \approx \tan\theta$$



- recall we don't know what $w(x)$ is, but if we make the elements small enough (refined enough mesh), we can approximate throughout the element using the end deflections and slopes.

# Recap:




- These are the shape functions for the ‘Euler-Bernoulli Beam’
- This neglects transverse shear but often gives adequate predictions of beam deflection and stress with appropriate length : thickness ratios

- Next we will derive and start assembling beam element Stiffness Matrices!

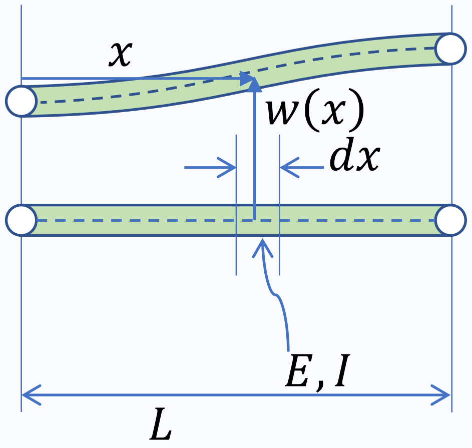
# Part 3: Beams in Bending Stiffness Matrix

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

From 25th October 2024

# Two-noded beam elements in bending




- Going back to our elastic potential energy expression, since:

$$U = {}^1\!/_2 \int_0^L EI \left(\frac{d^2w}{dx^2}\right)^2 dx = {}^1\!/_2 \int_0^L EIw''(x)^2 dx$$

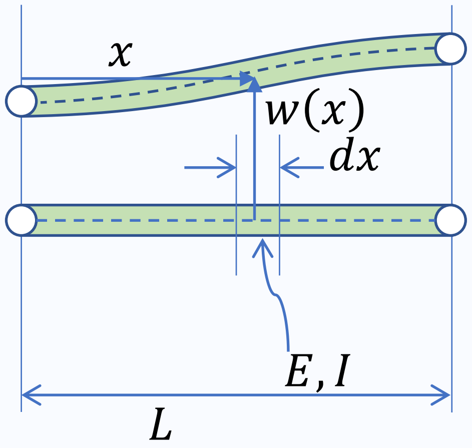
$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4$$

- we need

$$w''(x) = f_1''(x)q_1 + f_2''(x)q_2 + f_3''(x)q_3 + f_4''(x)q_4$$

- Notice dimensional analysis might not seem to work here.
- $q_1$ and $q_3$ are displacements and $q_2$ and $q_4$ are slopes
- but we are working in matrix space, with whatever functions we like; cubics are convenient.

# Two-noded beam elements in bending

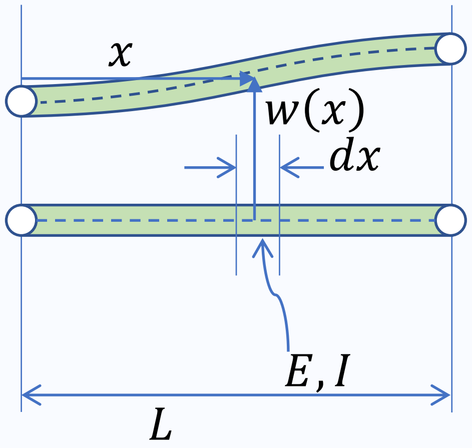



- Since each shape function is a cubic, linear function of x, what can we say about $w(x)$?
- And if we know this about $w(x)$, what about $w''(x)$?
- It means we can substitute $w''(x)$ into our elastic strain energy expression:

$$U = {}^{1}\!/_{2} \int_0^L EI(w(x)'')^2 dx$$

$$U = {}^{1}\!/_{2} \int_0^L EI[f_1''(x)q_1 + f_2''(x)q_2 + f_3''(x)q_3 + f_4''(x)q_4]^2 dx$$

# Two-noded beam elements in bending




$$U = {}^1\!/_2 \int_0^L EI[f_1''(x)q_1 + f_2''(x)q_2 + f_3''(x)q_3 + f_4''(x)q_4]^2 dx$$

- expand out the square for a quadratic with 10 terms in $q_i$…

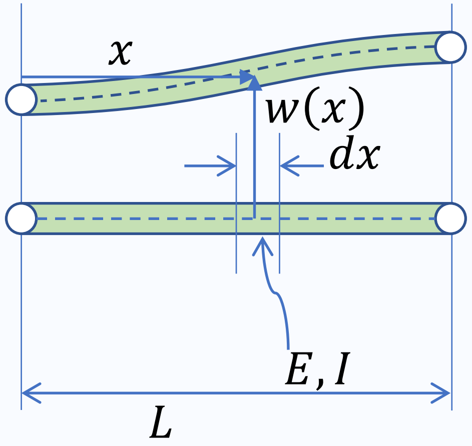
$$U = {}^1\!/_2\, EI \int_0^L [f_1''^2(x){q_1}^2 + f_2''^2(x){q_2}^2 + \cdots + 2f_1''(x)q_1 f_2''(x)q_2 + \cdots]dx$$

- so our integration result will arrive in a form something like:

$$U = {}^1\!/_2\, EI\ [(\cdot){q_1}^2 + (\cdot){q_2}^2 + \cdots + 2(\cdot)q_1 q_2 + \cdots]$$

- There will be 10 integrals. Why?
- We won't go through the full process of these 10 integrals.. but you could! Instead we will just organise it in general:

# Two-noded beam elements in bending




$$U = {}^1\!/_2\, EI \int_0^L \left[ f_1''^2(x) q_1{}^2 + f_2''^2(x) q_2{}^2 + \cdots + 2 f_1''(x) q_1 f_2''(x) q_2 + \cdots \right] dx$$

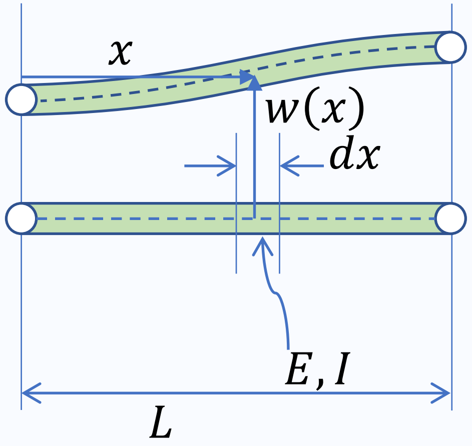
$$U = {}^1\!/_2\, EI \left[ (\cdot) q_1{}^2 + (\cdot) q_2{}^2 + \cdots + 2(\cdot) q_1 q_2 + \cdots \right]$$

- Organising:

$$U = {}^1\!/_2\, EI \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

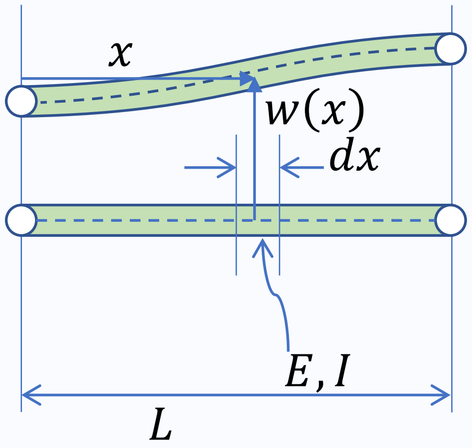
- How to complete it?

# Two-noded beam elements in bending




$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T EI \underbrace{\begin{bmatrix} \int f_1''^2(x)dx & \int f_1''(x)f_2''(x)dx & & \\ \int f_1''(x)f_2''(x)dx & \int f_2''^2(x)dx & & \\ & & \int f_3''^2(x)dx & \\ & & & \int f_4''^2(x)dx \end{bmatrix}}_{\text{Beam Bending Element Stiffness Matrix!}} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

# Two-noded beam elements in bending



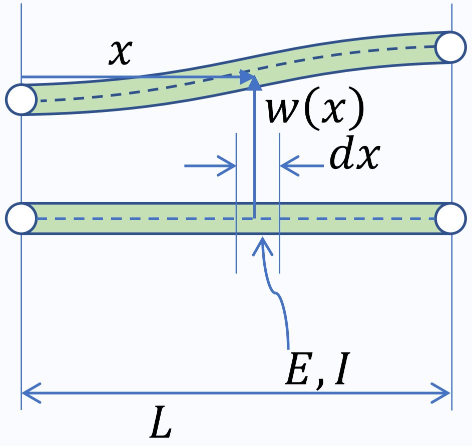

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T EI \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

calculates out to:

$$K = \left(\frac{EI}{L^3}\right) \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Beam Bending Element Stiffness Matrix!

# Two-noded beam elements in bending




$$U = {}^{1}\!/_{2}\,\{q\}_{4\times4}{}^{T}[K]_{4\times4}\{q\}_{4\times4}$$

where

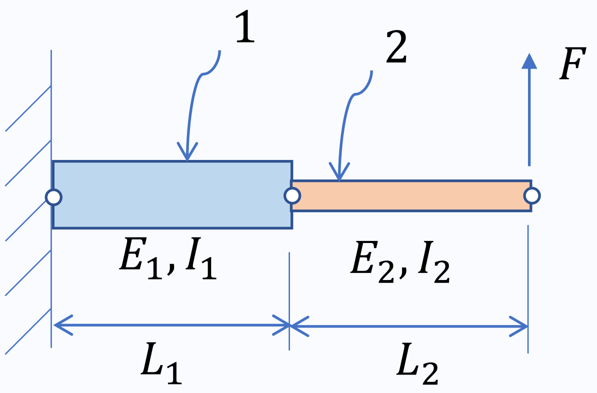
$$[K]_{4\times4} = (EI)\int_0^L f_i''f_j''(x)dx, i = 1..4, j = 1..4$$

and because it is in quadratic form, this is equivalent to saying

$$U = {}^{1}\!/_{2}\sum_{j=1}^{4}\sum_{i=1}^{4} K_{ij}q_iq_j$$

which hopefully you recognise from the $2 \times 2$ system for axial rods in tension and compression!

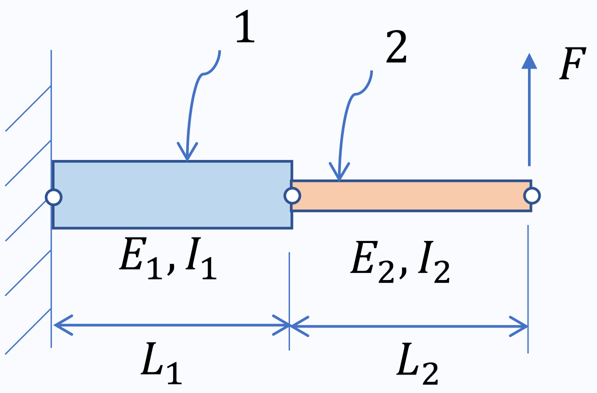
# Two-noded beam elements in bending

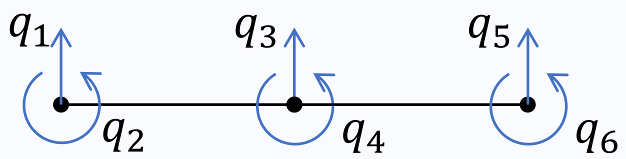



- Now let's try a more complicated example – a stepped *beam*
- You could use differential equations, it would be painful – two different 4th order differential equations. So we use FEM to approximate it
- We calculate the strain energy for each element, *locally*

# Two-noded beam elements in bending

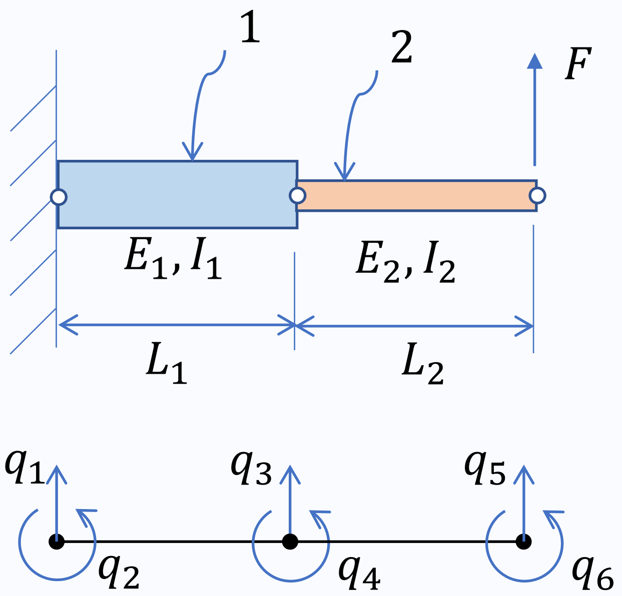
- Now let’s try a more complicated example – a stepped *beam*
- You could use differential equations, it would be painful – two different 4th order differential equations. So we use FEM to approximate it
- We calculate the strain energy for each element, *locally*
- Recall we build our model forgetting the fixed end: 6 d.o.f.
- So how big is our stiffness matrix? And how do we obtain it?



# For next time:

1. State the elemental and global stiffness matrices
2. State the global force vector
3. Apply boundary conditions, and
4. State the reduced governing equation of equilibrium



# Part 3: Beams in Bending: Assembling and Solving Problems

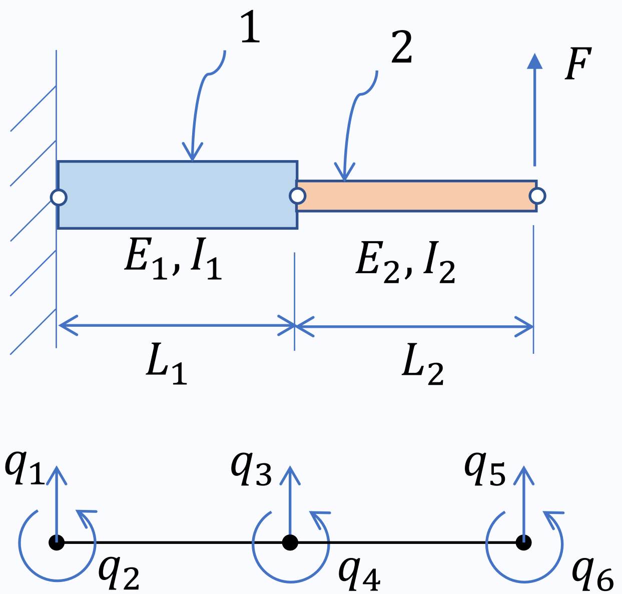
FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

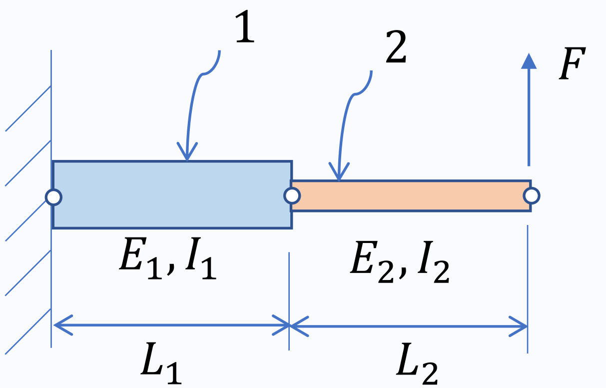
From 29th October 2024

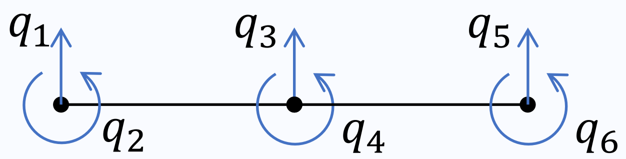
# Solving this Problem:

1. State the elemental and global stiffness matrices
2. State the global force vector
3. Apply boundary conditions, and
4. State the reduced governing equation of equilibrium



# Two-noded beam elements in bending




- Recall our Element Stiffness Matrix (which you do not need to remember):

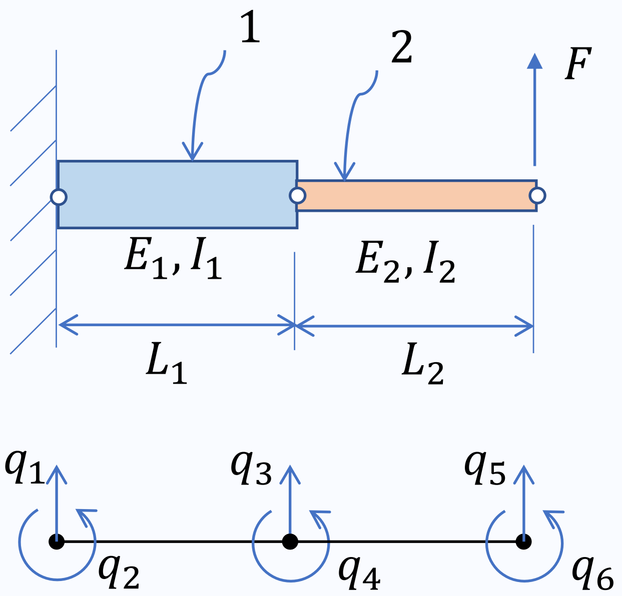
$$U_1 = {}^1\!/_2 \begin{Bmatrix} q_1 \\ \vdots \\ q_4 \end{Bmatrix}^T \left[ \qquad\qquad \right] \begin{Bmatrix} q_1 \\ \vdots \\ q_4 \end{Bmatrix}$$

$$U_2 = {}^1\!/_2 \begin{Bmatrix} q_3 \\ \vdots \\ q_6 \end{Bmatrix}^T \left[ \qquad\qquad \right] \begin{Bmatrix} q_3 \\ \vdots \\ q_6 \end{Bmatrix}$$

$$K_1 = \left(\frac{E_1 I_1}{{L_1}^3}\right) \begin{bmatrix} 12 & 6L_1 & -12 & 6L_1 \\ 6L_1 & 4{L_1}^2 & -6L_1 & 2{L_1}^2 \\ -12 & -6L_1 & 12 & -6L_1 \\ 6L_1 & 2{L_1}^2 & -6L_1 & 4{L_1}^2 \end{bmatrix}$$

$$K_2 = \left(\frac{E_2 I_2}{{L_2}^3}\right) \begin{bmatrix} 12 & 6L_2 & -12 & 6L_2 \\ 6L_2 & 4{L_2}^2 & -6L_2 & 2{L_2}^2 \\ -12 & -6L_2 & 12 & -6L_2 \\ 6L_2 & 2{L_2}^2 & -6L_2 & 4{L_2}^2 \end{bmatrix}$$

# Two-noded beam elements in bending




$$U_1 = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \left[ \quad \right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

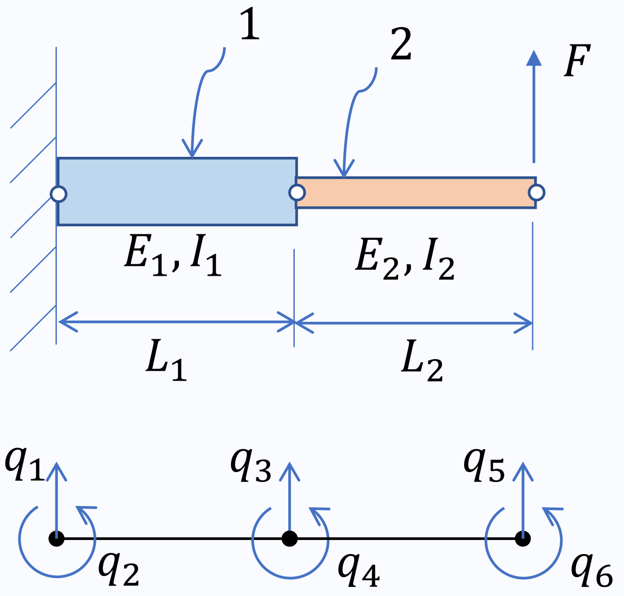
$$U_2 = {}^1\!/_2 \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \left[ \quad \right] \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \left[ \quad \right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

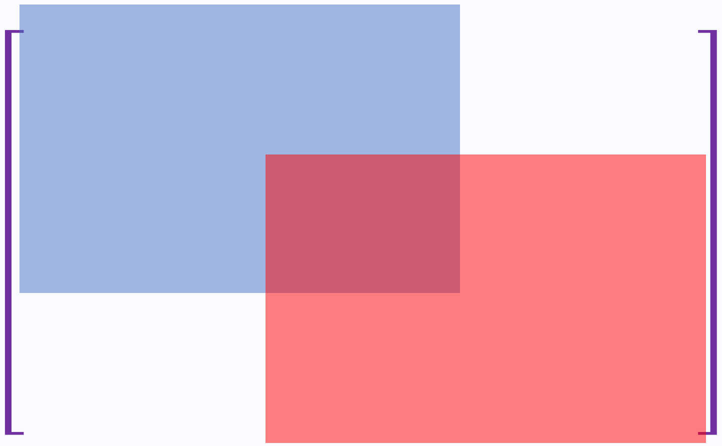
# Two-noded beam elements in bending

- Assemble our global stiffness matrix and apply PMTPE:

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}^T \begin{bmatrix} & & \\ & 6 \times 6 & \\ & & \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}$$



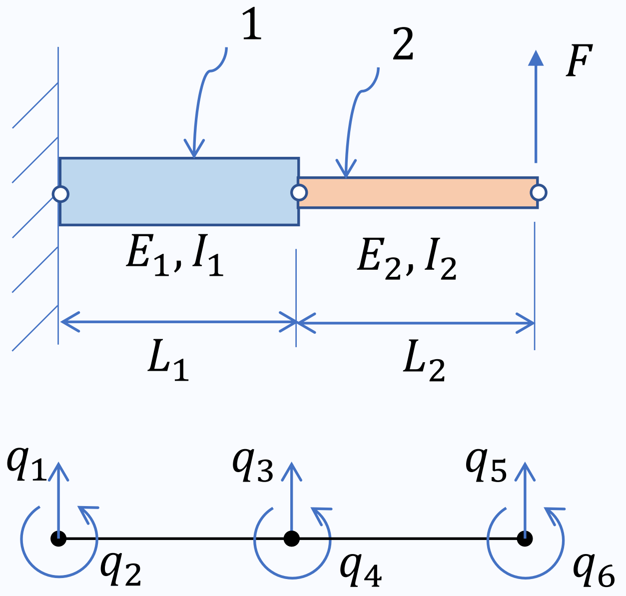

$K =$



# Two-noded beam elements in bending

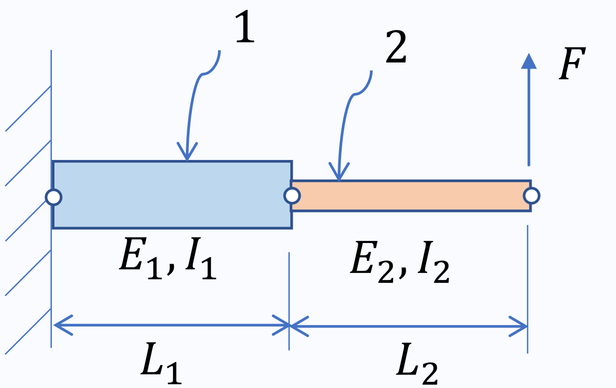
- Assemble our global stiffness matrix and apply PMTPE:

$$U = {}^{1}\!/_{2}\begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}^T \begin{bmatrix} & & \\ & 6\times 6 & \\ & & \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}$$



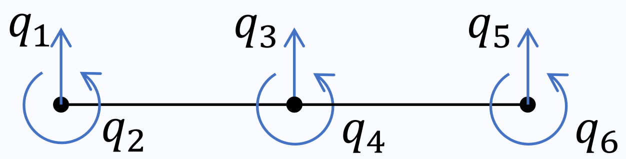

$$\begin{bmatrix} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & \\ & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{Bmatrix}$$

# Two-noded beam elements in bending




- Assemble our global stiffness matrix and apply PMTPE:

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}^T \begin{bmatrix} & & \\ & 6 \times 6 & \\ & & \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}$$

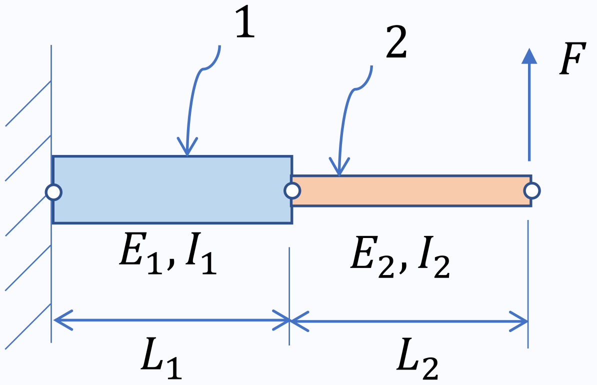


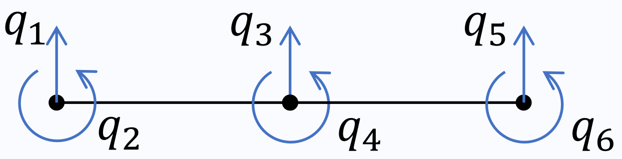

$$\begin{bmatrix} & & & & \cancel{0} & \cancel{0} \\ & & & & \cancel{0} & \cancel{0} \\ & & & & & \\ & & & & & \\ \cancel{0} & \cancel{0} & & & & \\ \cancel{0} & \cancel{0} & & & & \end{bmatrix} \begin{Bmatrix} \cancel{q_1} \\ \cancel{q_2} \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} \cancel{R} \\ \cancel{M} \\ 0 \\ 0 \\ F \\ 0 \end{Bmatrix}$$

Boundary conditions because:

$q_1 = 0$ and $q_2 = 0$

# Two-noded beam elements in bending



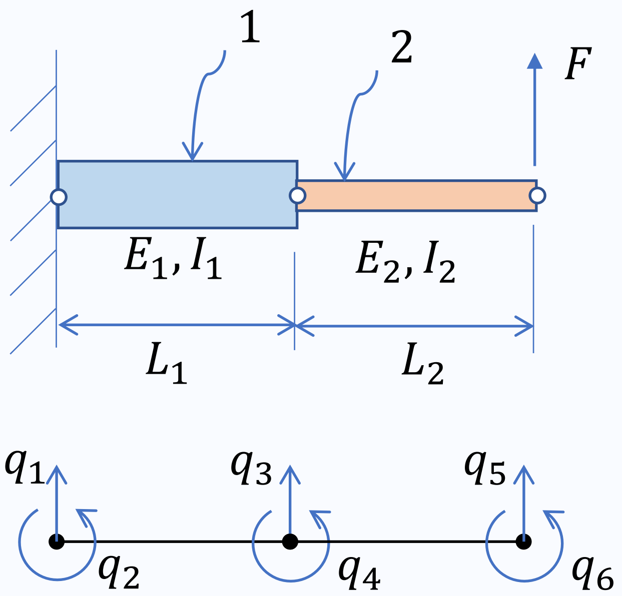




- Our *reduced* stiffness matrix is now:

$$\begin{bmatrix} \quad \end{bmatrix}\begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \\ 0 \end{Bmatrix}$$

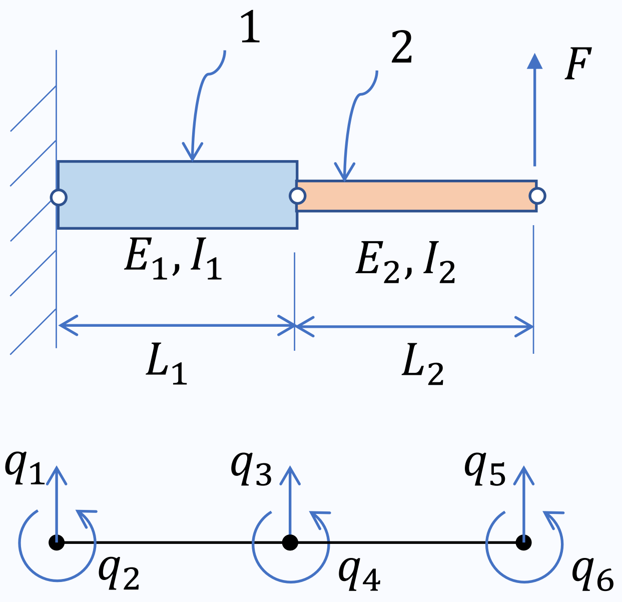
- and now we can solve for $q_{3-6}$. You could invert and multiply, but in practice some Gaussian elimination, upper triangulate and back substitute.
- And then return to Global Equation find $R$ and $M$ reactions if you need them.

# Two-noded beam elements in bending




- Now a challenge: What is the approximate displacement at the middle of the second element?
- … without creating a new node (which costs time)

# Two-noded beam elements in bending




- Now a challenge: What is the approximate displacement at the middle of the second element?
- … without creating a new node (which costs time)
- We have all the $q$s, nodal deformations
- We use the combined interpolation function:

$$w^{E2}(x) = f_1(x)q_3 + f_2(x)q_4 + f_3(x)q_5 + f_4(x)q_6$$

- So the approximate answer is:

$$w^{E2}\left(x = \frac{L_2}{2}\right) = f_1\left(\frac{L_2}{2}\right)q_3 + f_2\left(\frac{L_2}{2}\right)q_4 + f_3\left(\frac{L_2}{2}\right)q_5 + f_4\left(\frac{L_2}{2}\right)q_6$$

- We could solve this, because we defined $f_i$ and we found $q_i$