UNIVERSITY OF Southampton
School of Mathematics

Lecture 16 - Separation of Variables: Examples

David Gammack and Oscar Dias

Mathematical Sciences,
University of Southampton, UK

MATH2048, Semester 1

Outline

Review: Separation of variables in 6 steps

Separation of variables *ansatz* (SoV): $y(x,t) = X(x)T(t)$

1. **Determine ODEs** for X, T.
2. Use **boundary conditions** of y in order to obtain boundary conditions of X.
3. **Solve eigenvalue problem for** X: determine eigenvalues λ_n and eigenfunctions X_n.
4. Insert eigenvalue λ_n in the ODE for T and solve it to obtain T_n.
5. The **normal modes** are $y_n = X_n T_n$ and the **general solution** is obtained by **superposition**

$$y(x,t) = \sum_n X_n(x) T_n(t)$$

6. **Use initial conditions**, $y(x,0), \partial y(x,0)/\partial t$ to **find all undetermined coefficients**. This step **involves Fourier series.**

Outline

→ A simpler example: SoV in a Wave Equation

UNIVERSITY OF Southampton
School of Mathematics

We use **separation of variables** to solve

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad c \text{ constant}$$

with **boundary conditions**:

$$y(0,t) = 0\,, \qquad y(1,t) = 0$$

and **initial data**:

$$y(x,0) = x(1-x), \qquad \frac{\partial y}{\partial t}(x,0) = 0.$$

Step 1: Equations for X, T (*Review*)

We write the solution as

Plug into PDE

$$y(x,t) = X(x)\,T(t),$$

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

giving the **separably constant** equations (ODEs):

$$\begin{cases} \ddot{T} - c^2\lambda\, T = 0\,, \\ X'' - \lambda\, X = 0\,. \end{cases}$$

Step 2: Boundary conditions for $X(x)$ (*Review*)

To solve the ODEs

$$\begin{cases} X'' - \lambda X = 0\,. \\ \ddot{T} - c^2 \lambda T = 0\,, \end{cases}$$

we need **boundary conditions.** We have that (from BCs in previous slide)

$$y(0,t) = 0 \Leftrightarrow X(0)T(t) = 0, \qquad y(1,t) = 0 \Leftrightarrow X(1)T(t) = 0$$
$$\Rightarrow \quad X(0) = 0, \qquad \Rightarrow \quad X(1) = 0.$$

At present we have no sensible boundary conditions for T.

$X'' - \lambda X = 0$ where $X(0)=0$ $X(1)=0$ 3 cases exist $\lambda=0$ $\lambda<0$ $\lambda>0$

$\therefore$ eigenvalue problem.

UNIVERSITY OF Southampton School of Mathematics

We thus have the Eigenvalue Problem

$$X'' - \lambda X = 0; \qquad X(0) = 0, \quad X(1) = 0.$$

λ is the unknown eigenvalue (revisit Lecture 3: there $X'' + \lambda X = 0$!!).
We have to consider the **three cases** (revisit Lecture 3):

1. $\lambda = k^2 > 0$ (distinct real roots $\Rightarrow X = A\,e^{kx} + B\,e^{-kx}$), → trivial solution
2. $\lambda = 0$ ($\Rightarrow X = A + B\,x$), → trivial solution
3. $\lambda = -k^2 < 0$ (complex conjugate roots $\Rightarrow X = A\sin(k\,x) + B\cos(k\,x)$), → useful solution?

we find that only the **third case** gives a **non-trivial** solution. Namely, we get the eigenfunction and eigenvalue:

$$X_n(x) = A_n \sin(n\pi x)\,, \qquad \lambda_n = -(n\pi)^2\,, \quad n = 1, 2, 3, \ldots$$

[Exercise: get this result! (see Lecture Notes ∮ 5.4.2)]

Step 4: Solve the $T(t)$ equation (*Review*)

Southampton UNIVERSITY OF School of Mathematics

- First ODE for $X_n(x)$ ✓ Separation constant λ_n ✓
- But we still have to solve the second **ODE for** $T(t)$:

$$\ddot{T} - c^2\lambda T = 0 \qquad \text{with } \underline{\text{no}} \text{ boundary conditions}$$

but **we now know the value of** $\lambda = -k^2 < 0$. (From last bit)
↳ if not then $X(0)=0$
∴ must be complex

So this is a **constant coefficient ODE** (revisit Lecture 1).

Since $\lambda = -k^2 < 0$ the associated auxiliary equation is a quadratic with two purely imaginary roots

$$\Lambda = \pm\,\mathbf{j}\; n\,\pi\,c\,, \qquad n = 1, 2, 3, \cdots .$$

So its **general solution** $T(t) = T_n(t)$ is:

$$T_n(t) = \tilde{C}_n \cos\left(n\,\pi c\,t\right) + \tilde{D}_n \sin\left(n\,\pi c\,t\right).$$

We have our separation *ansatz* $y(x,t) = X(x)T(t)$
and **normal mode** solutions (each n describes a normal mode):

$$X_n(x) = A_n \sin(n\pi x), \quad T_n(t) = \tilde{C}_n \cos(n\pi c\, t) + \tilde{D}_n \sin(n\pi c\, t).$$

Combining these, $y_n(x,t) = X_n(x)T_n(t)$, and
superposing the **normal modes** $y_n(x,t)$ gives the **general solution**:

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) \quad \Leftrightarrow$$

$$y(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) \quad \Leftrightarrow$$

$$y(x,t) = \sum_{n=1}^{\infty} \Big[C_n \cos(n\pi c\, t) + D_n \sin(n\pi c\, t)\Big] \sin(n\pi x).$$

Just clearing the setup

$(A_n\tilde{C}_n \equiv C_n, \quad A_n\tilde{D}_n \equiv D_n)$

Step 6: Use initial data (*Our job today!*)

UNIVERSITY OF Southampton
School of Mathematics

- **Initial data**: the function y and its time derivative $\dot{y}$ at $t = 0$.
- Our initial data is

Fourier series period 1 (at $t=0$)

from prev slide

$$\underbrace{y(x,0) = x(1-x)}, \qquad \frac{\partial y}{\partial t}(x,0) = 0.$$

Evaluating our general solution at $t = 0$ gives

c vanishes for $t=0$

$$\sum_{n=1}^{\infty} C_n \sin(n\pi x) = x(1-x) \quad (\star), \qquad \sum_{n=1}^{\infty} n\pi c D_n \sin(n\pi x) = 0$$

$$\Rightarrow \quad C_n = ??, \qquad\qquad \Rightarrow \quad D_n = 0$$

How do we find C_n ? ... well equation $(\star)$ is:

$$x(1-x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

Note: if you want to have a deeper understanding of the problem, at home see an **alternative but absolutely equivalent way of computing** C_n that is in the Extra slides 24-25 at the end of this Lecture.

Step 6: Initial data

- C_n ? ... well equation ($\star$) is: $\quad x(1-x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) \quad (\star)$

  which we recognize to be the <u>sine</u> Fourier series (FS) of $f(x) = x(1-x)$ (i.e. the FS of the <u>odd</u> extension of $f(x) = x(1-x)$ with period 2ℓ, $\ell = 1$).

  Why? BCs of original problem are given at $\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{x} = \mathbf{1}$
  $\Rightarrow$ initial data gives us a known function $f(x)$ in the **<u>half-range</u>** $\mathbf{0} \le \boldsymbol{x} \le \mathbf{1}$.

  Moreover, ($\star$) tells that this half-range function has a **<u>sine</u>** FS.
  But **<u>sine</u> FS** $\Rightarrow$ FS of **<u>odd</u> extension** of $f(x)$ with period $\mathbf{2\ell}$. Here, $\ell = 1$.

  Since ($\star$) is a sine FS of the <u>odd</u> extension of $f(x) = x(1-x)$ with period 2ℓ, $\ell = 1$) we can apply the associated **Euler formulae** to find C_n:

  ...(lots of maths)

$$C_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n x \pi}{\ell}\right) dx = 2\int_0^1 x(1-x)\sin(n\pi x)\,\mathrm{d}x = \frac{4}{(n\pi)^3}\left[1-(-1)^n\right]$$

  [$\nearrow$ slide 11 of Lecture 6 with $b_n \equiv C_n$] [<u>Exercise</u>: check it at home! $\nwarrow$]

- So, **solution** to our **original problem**, that obeys the BCs & initial data, is:

$$y(x,t) = 4\sum_{n=1}^{\infty} \frac{\left[1-(-1)^n\right]}{(n\pi)^3}\cos(n\pi c\,t)\sin(n\pi x)\,.$$

Exercise to do at home:

Southampton UNIVERSITY OF School of Mathematics

Show that C_n is indeed given by $C_n = \frac{4}{(n\pi)^3}\left[1-(-1)^n\right]$:

($\swarrow$ you did similar exercises in FS lectures)

$$C_n = 2\int_0^1 \overbrace{x(1-x)}^{u}\,\overbrace{\sin(n\pi x)\,\mathrm{d}x}^{dv}$$

$$= 2\left\{\left[-x(1-x)\frac{\cos(n\pi x)}{(n\pi)}\right]_0^1 + \int_0^1 (1-2x)\frac{\cos(n\pi x)}{(n\pi)}\,\mathrm{d}x\right\}$$

$$= 2\left\{\left[-x(1-x)\frac{\cos(n\pi x)}{(n\pi)} + (1-2x)\frac{\sin(n\pi x)}{(n\pi)^2}\right]_0^1 + 2\int_0^1 \frac{\sin(n\pi x)}{(n\pi)^2}\,\mathrm{d}x\right\}$$

$$= 2\left[-x(1-x)\frac{\cos(n\pi x)}{(n\pi)} + (1-2x)\frac{\sin(n\pi x)}{(n\pi)^2} - 2\frac{\cos(n\pi x)}{(n\pi)^3}\right]_0^1$$

$$= \frac{4}{(n\pi)^3}\left[1-(-1)^n\right].$$

$\rightarrow$ *A more complex example*: SoV in Wave Equation

We use **separation of variables** to solve (same as in previous example)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad c \text{ constant}$$

with **boundary conditions** ($\swarrow$ different from previous example)

$$\frac{\partial y}{\partial x}(0,t) = 0, \qquad \frac{\partial y}{\partial x}(1,t) = 0$$

and **initial data** (same as in previous example)

$$y(x,0) = x(1-x), \qquad \frac{\partial y}{\partial t}(x,0) = 0.$$

Step 1 & 2: ODEs and boundary conditions

UNIVERSITY OF Southampton
School of Mathematics

- Once more we assume $y = XT$ which gives

$$X'' - \lambda X = 0,$$
$$\ddot{T} - c^2 \lambda T = 0$$

- Boundary conditions (BCs) ($\swarrow$ different from previous example)

$$\frac{\partial y}{\partial x}(0,t) = 0 \quad \Leftrightarrow \quad X'(0)T(t) = 0, \;\; \forall t \qquad \Rightarrow X'(0) = 0,$$
$$\frac{\partial y}{\partial x}(1,t) = 0 \quad \Leftrightarrow \quad X'(1)T(t) = 0, \;\; \forall t \qquad \Rightarrow X'(1) = 0.$$

Step 3: Eigenvalue problem for $X(x)$

UNIVERSITY OF Southampton
School of Mathematics

2 non trivials

$$X'' - \lambda X = 0; \qquad X'(0) = 0, \quad X'(1) = 0.$$

- The solution for $\lambda = 0$ is

$$X(x) = Ex + F \quad \Rightarrow \quad X' = E.$$

BCs $\Rightarrow$ $\boldsymbol{E = 0}$. Thus, we have a **non-trivial solution**: $X(x) = F$.

- The solution for $\lambda = -k^2 < 0$ is

$$X(x) = A\sin(k\,x) + B\cos(k\,x) \quad \Rightarrow X'(x) = k\Big[A\cos(k\,x) - B\sin(k\,x)\Big]$$

BC at $x = 0 \Rightarrow \boldsymbol{A = 0}$.
BC at $x = 1 \Rightarrow$ either **trivial solution** $\boldsymbol{A = B = 0}$ or $\boldsymbol{k = n\pi}$.

- The solution for $\lambda = k^2 > 0$ is

$$X = A\,e^{k\,x} + B\,e^{-k\,x} \quad \Rightarrow \quad X' = k\left(A\,e^{k\,x} - B\,e^{-k\,x}\right)$$

BC at $x = 0 \Rightarrow \boldsymbol{A = B}$. BC at $x = 1 \Rightarrow \boldsymbol{A = 0}$.
Thus, we **only** have the **trivial solution** $\boldsymbol{A = B = 0}$ in this case.

Step 4: Solve for $T(t)$

Constant coefficient ODE for $T(t)$ with known λ (revisit Lecture 1)

$$\ddot{T} - c^2 \lambda T = 0$$

We now have to **solve two cases:**

- For $\boldsymbol{\lambda = 0}$ we have
$$T = \tilde{G}\, t + \tilde{H}.$$
- For $\boldsymbol{\lambda = -k^2 < 0}$ we have $k \equiv k_n = n\pi$ and:
$$T_n(t) = \tilde{C}_n \cos\left(n\pi c\, t\right) + \tilde{D}_n \sin\left(n\pi c\, t\right).$$
- NO need to solve $\boldsymbol{\lambda > 0}$ since $X(x) = 0$.

Step 5: The general solution

So we have:

$$\begin{cases} \text{If } \lambda = 0: & X(x) = F\,, & T(t) = \tilde{G}t + \tilde{H} \\ \text{If } \lambda < 0: & X(x) = B_n \cos(n\pi x)\,, & T(t) = \sum_{n=1}^{\infty} \left[\tilde{C}_n \cos(n\pi ct) + \tilde{D}_n \sin(n\pi c\,t)\right] \\ \text{If } \lambda > 0: & \text{Only trivial solution} \end{cases}$$

Combining these, $y_n = X_n T_n$, and
superposing these **normal modes** y_n gives the **general solution**:

$$y(x,t) = \left(\tilde{G}\,t + \tilde{H}\right)F + \sum_{n=1}^{\infty} \left[\tilde{C}_n \cos(n\pi ct) + \tilde{D}_n \sin(n\pi c\,t)\right] B_n \cos(n\pi x)$$

$$y(x,t) = G\,t + H + \sum_{n=1}^{\infty} \left[C_n \cos(n\pi c\,t) + D_n \sin(n\pi c\,t)\right] \cos(n\pi x)\,.$$

$$(\nwarrow\ G \equiv \tilde{G}F,\ H \equiv \tilde{H}F,\ C_n \equiv \tilde{C}_n B_n,\ D_n \equiv \tilde{D}_n B_n\ )$$

Step 6: Initial data

$$y(x,t) = G\,t + H + \sum_{n=1}^{\infty} \Big[C_n \cos(n\pi c\,t) + D_n \sin(n\pi c\,t) \Big] \cos(n\pi x)\,.$$

$$\longrightarrow \quad \boldsymbol{G, H, C_n, D_n = ??}$$

- Our **initial data** is

$$y(x,0) = x(1-x), \qquad \frac{\partial y}{\partial t}(x,0) = 0.$$

Evaluating our general solution at $\boldsymbol{t = 0}$ gives

$$y(x,0) = x(1-x) \quad \Leftrightarrow \quad H + \sum_{n=1}^{\infty} C_n \cos(n\pi x) = x(1-x), \qquad (1)$$

$$\frac{\partial y}{\partial t}(x,0) = 0 \quad \Leftrightarrow \quad G + \sum_{n=1}^{\infty} n\pi c\, D_n \cos(n\pi x) = 0. \qquad (2)$$

Step 6: Initial data

- Equation (2) $\Rightarrow$ each of the $\cos(n\pi x)$ terms vanish separately:

  $G = D_n = 0.$ ($\searrow$ From BCs, $x \in [0,1]$)

- Equation (1) is $\quad x(1-x) = H + \sum_{n=1}^{\infty} C_n \cos(n\pi x) \quad$ with $x \in [0,1]$

  which we recognize to be the cosine Fourier series of $f(x) = x(1-x)$ (ie the FS of the even extension of $f(x) = x(1-x)$ w/ period 2ℓ, $\ell = 1 \Leftarrow x \in [0,1]$).

  Thus, $f(x) = \frac{1}{2} C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n x \pi}{\ell}\right)$ with $\ell = 1$ & we apply **Euler formulae**:

  [$\swarrow$ see slide 10 of Lecture 6 with $a_0 \equiv C_0$ and $a_n \equiv C_n$]

$$H \equiv \frac{1}{2} C_0 = \frac{1}{2}\left(\frac{2}{\ell}\int_0^{\ell} f(x)dx\right) = \frac{1}{2}\left(2\int_0^1 x(1-x)dx\right) = \frac{1}{6},$$

[$\downarrow$ Exercise: check it!] similar to slide 13

$$C_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n x \pi}{\ell}\right) dx = 2\int_0^1 x(1-x)\cos(n\pi x)\,\mathrm{d}x = -\frac{2}{(n\pi)^2}\left[1+(-1)^n\right]$$

- **Solution** to our **original problem**, that obeys BCs and initial data, is:

$$y(x,t) = \frac{1}{6} - 2\sum_{n=1}^{\infty} \frac{[1+(-1)^n]}{(n\pi)^2} \cos(n\pi c\, t)\cos(n\pi x)\,.$$

Wave equation describes...travelling waves

Solution of wave equation (& its BCs + initial data) is

$$y(x,t) = \frac{1}{6} - 2\sum_{n=1}^{\infty} \frac{[1+(-1)^n]}{(n\pi)^2} \cos(n\pi x)\cos(n\pi c\,t)\,.$$

$\searrow$ Trigonometric identity: $2\cos A\cos B = \cos(A-B) + \cos(A+B)$:

$$= \frac{1}{6} - \sum_{n=1}^{\infty} \frac{[1+(-1)^n]}{(n\pi)^2} \left[\underbrace{\cos\left[n\pi(x-c\,t)\right]}_{\text{Right-mover wave}} + \underbrace{\cos\left[n\pi(x+c\,t)\right]}_{\text{Left-mover wave}} \right].$$

- **Front of the wave** is given by condition that phase vanishes: $\boldsymbol{(x \pm c\,t) = 0}$,

$$\begin{cases} \text{Right-mover wave: } x - c\,t = 0 \Rightarrow x = c\,t \longrightarrow t \nearrow \Rightarrow x \nearrow \implies \text{moves to the right} \\ \text{Left-mover wave: } x + c\,t = 0 \Rightarrow x = -c\,t \longrightarrow t \nearrow \Rightarrow x \searrow \implies \text{moves to the left} \end{cases}$$

- Waves indeed travel with **velocity** c.
- **Periodic wave propagation:** $\cos\left[(x \pm c\,t) + 2\pi\right] = \cos\left[x \pm c\,t\right]$
- Go back to **slide 4 of Lecture 14** and see its **figure**: now we understand it!

Outline

Summary: Separation of variables

$$y(x,t) = X(x)T(t)$$

1. Determine equations for X, T.
2. Use boundary conditions of y in order to obtain boundary conditions of X.
3. Solve eigenvalue problem for X: determine eigenvalues λ_n and eigenfunctions X_n.
4. Insert eigenvalue λ_n in the T equation and solve it to obtain T_n.
5. The normal modes are $y_n = X_n T_n$ and the general solution is obtained by superposition

$$y(x,t) = \sum_n X_n(x)T_n(t)$$

6. Use initial conditions, $y(x,0), \partial y(x,0)/\partial t$ to determine all undetermined coefficients. This step involves Fourier series.

EXTRA: an alternative way to solve problem of slides 11 & 12

• **Imagine** that we have **not** learned Fourier Series. Could we still solve the problem of slide 11 and get the final solution given in slide 12? The answer is yes and we give it here. *Of course, the way we solve it in slides 11 & 12 is perfectly good and enough! This extra appendix is just for you to reflect about it at home and understand more deeply how things work and their origin.*

• Our initial data is: $y(x,0) = x(1-x), \qquad \frac{\partial y}{\partial t}(x,0) = 0.$

Evaluating our general solution at $\boldsymbol{t=0}$ gives

$$\sum_{n=1}^{\infty} C_n \sin(n\pi x) = x(1-x) \quad (\star), \qquad \sum_{n=1}^{\infty} n\pi c\, D_n \sin(n\pi x) = 0$$

$$\Rightarrow \quad C_n = ??, \qquad \Rightarrow \quad D_n = 0$$

To find C_n recall the inner product for functions to get projections (slide 9 of Lecture 4)!

Projection (orthogonality) of $f(x) = x(1-x)$ **over** $g(x) = \sin(k\pi x)$: multiply $(\star)$ by $\sin(k\pi x)$ & **integrate between** 0 and 1 (since $x \in [0,1]$ by BCs) we get

$$\int_0^1 \left[\sum_{n=1}^{\infty} C_n \sin(n\pi x) \right] \sin(k\pi x)\, dx = \int_0^1 x(1-x) \sin(k\pi x)\, dx$$

↗ Inner product for functions defined in $x \in [0,\ell]$: $\langle f(x), g(x) \rangle = \frac{1}{\ell}\int_0^\ell f(x)\, g(x)\, dx$ (slide 9, Lecture 4)

(*think about it:... this is exactly how we derived Euler formulae in lecture 4!!*)

UNIVERSITY OF Southampton
School of Mathematics

$$\int_0^1 \left[\sum_{n=1}^{\infty} C_n \sin(n\pi x)\right] \sin(k\pi x)\,dx = \int_0^1 x(1-x)\sin(k\pi x)\,dx$$

$$\sum_{n=1}^{\infty} \int_0^1 C_n \sin(n\pi x)\sin(k\pi x)\,dx = \int_0^1 x(1-x)\sin(k\pi x)\,dx$$ [↙ Done in PS 1]

Do the integral: $\int_0^1 \sin(n\pi x)\sin(k\pi x)\,dx = \left[\frac{1}{2} - \frac{\sin(2\pi n)}{4\pi n}\right]\delta_{kn} = \frac{1}{2}\delta_{kn}$ for $n = 1, 2, 3, \cdots$

$$\sum_{n=1}^{\infty} C_n \frac{1}{2}\delta_{kn} = \int_0^1 x(1-x)\sin(k\pi x)\,dx$$ (↙ $\delta_{kn} = 1|_{\text{if } n=k}$; $\delta_{kn} = 0|_{\text{if } n \neq k}$)

$$\frac{1}{2}C_k = \int_0^1 x(1-x)\sin(k\pi x)\,dx$$ (↙ k is dummy variable: can do $k \to n$)

$$C_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx = \frac{4}{(n\pi)^3}\left[1-(-1)^n\right]$$ [as shown in slide 13]

The **solution** to our **original problem** is thus ↖ ↙ exactly what we got in slide 12!

$$y(x,t) = 4\sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{(n\pi)^3}\cos(n\pi c\,t)\sin(n\pi x)\,.$$