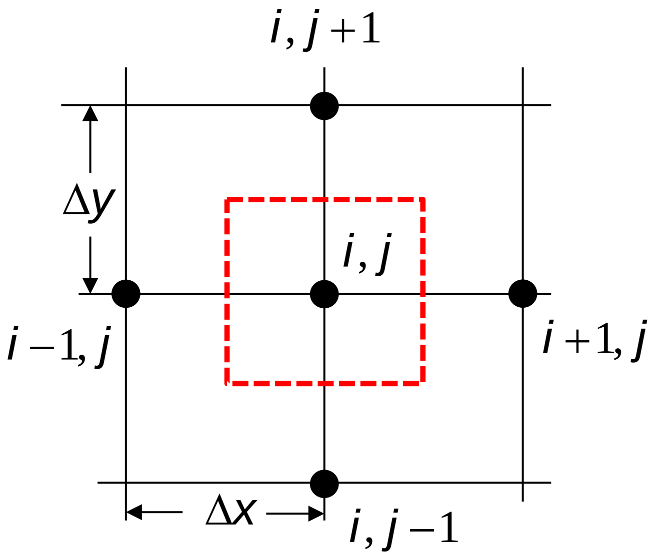
SESA3029 Aerothermodynamics

Lecture 5.7

Finite difference methods for the heat diffusion equation in 2D, example case

2D Discretisation



2D heat diffusion equation, stationary, no heat source

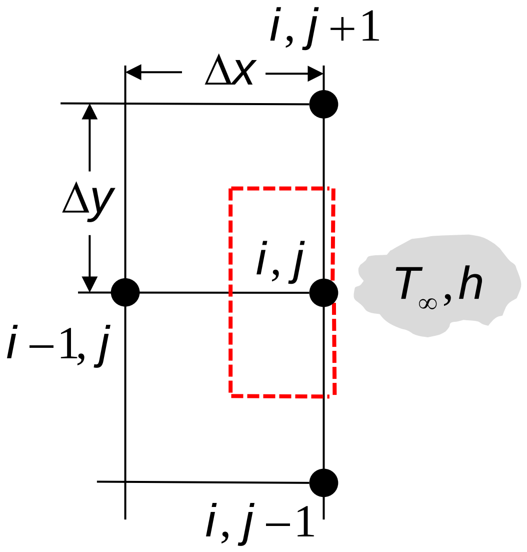
on mesh with uniform **but different** stepsizes $\Delta x, \Delta y$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{\partial T/\partial x\big|_{i+1/2,j} - \partial T/\partial x\big|_{i-1/2,j}}{\Delta x} + \frac{\partial T/\partial y\big|_{i,j+1/2} - \partial T/\partial y\big|_{i,j-1/2}}{\Delta y} = 0$$

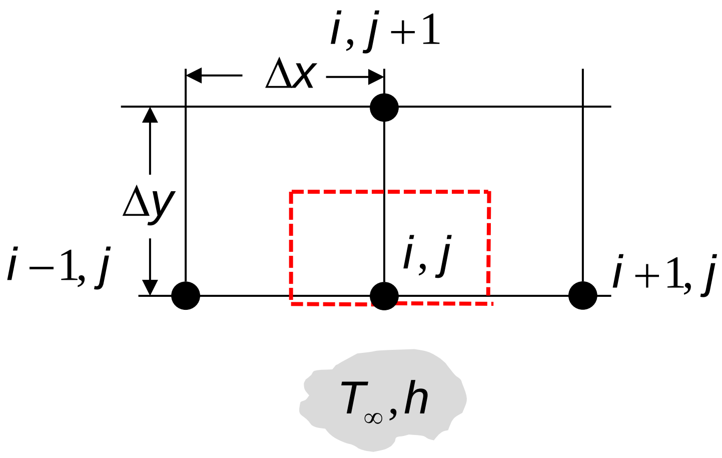
$$\frac{1}{\Delta x^2}\left(T_{i+1,j} - 2T_{i,j} + T_{i-1,j}\right) + \frac{1}{\Delta y^2}\left(T_{i,j+1} - 2T_{i,j} + T_{i,j-1}\right) = 0 \qquad (5)$$

2D Boundary conditions



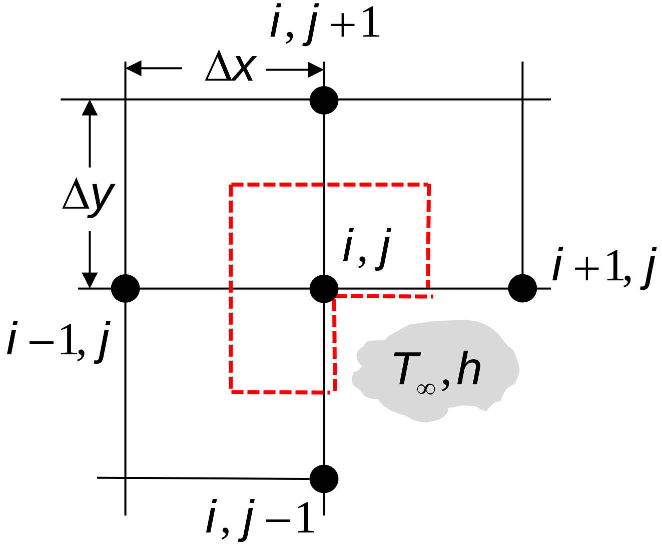
Node at a plane surface with convection

$$\frac{2}{\Delta x^2}T_{i-1,j}+\frac{1}{\Delta y^2}\left(T_{i,j+1}+T_{i,j-1}\right)$$
$$-2T_{i,j}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{h}{k\Delta x}\right)=-\frac{2h}{k\Delta x}T_\infty \qquad (6)$$



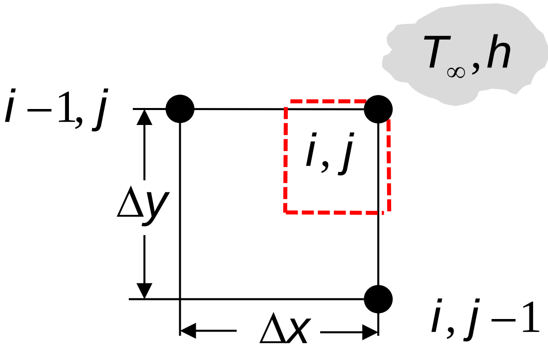
$$\frac{1}{\Delta x^2}\left(T_{i+1,j}+T_{i-1,j}\right)+\frac{2}{\Delta y^2}T_{i,j+1}$$
$$-2T_{i,j}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{h}{k\Delta y}\right)=-\frac{2h}{k\Delta y}T_\infty \qquad (7)$$

Node at an internal corner



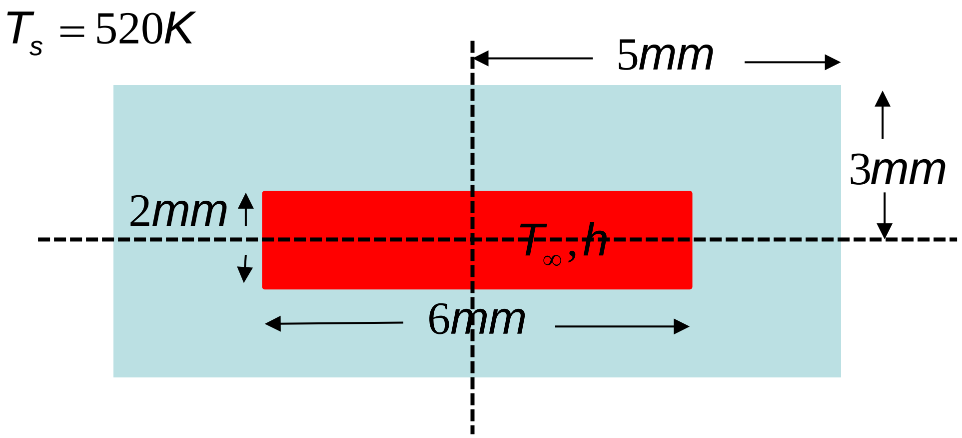
$$\frac{1}{\Delta x^2}\left(T_{i+1,j}+2T_{i-1,j}\right)+\frac{1}{\Delta y^2}\left(2T_{i,j+1}+T_{i,j-1}\right)$$
$$-T_{i,j}\left(3\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)+\frac{h}{k}\left(\frac{1}{\Delta x}+\frac{1}{\Delta y}\right)\right)$$
$$=-\frac{h}{k}\left(\frac{1}{\Delta x}+\frac{1}{\Delta y}\right)T_\infty \qquad (8)$$

Node at an external corner with convection



$$\frac{1}{\Delta x^2}T_{i-1,j}+\frac{1}{\Delta y^2}T_{i,j-1}$$
$$-T_{i,j}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{h}{k}\left(\frac{1}{\Delta x}+\frac{1}{\Delta y}\right)\right)$$
$$=-\frac{h}{k}\left(\frac{1}{\Delta x}+\frac{1}{\Delta y}\right)T_\infty \qquad (9)$$

2D heat diffusion example



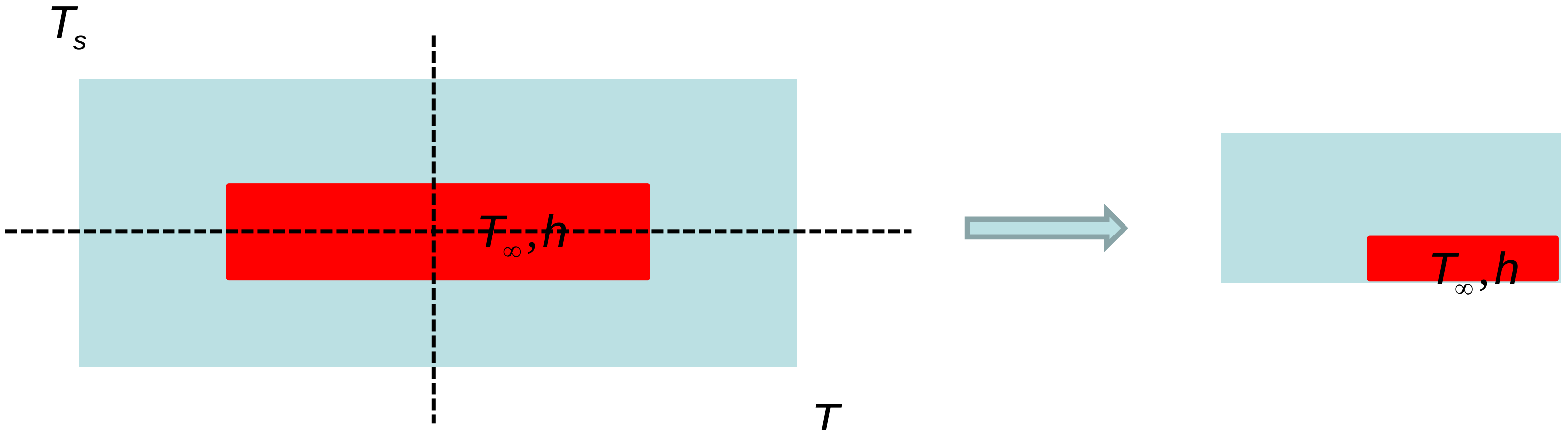
A rectangular glas channel ($k = 2\ \mathrm{W/(mK)}$) with 2mm wall thickness is exposed to an outside constant temperature $T_s = 520K$.

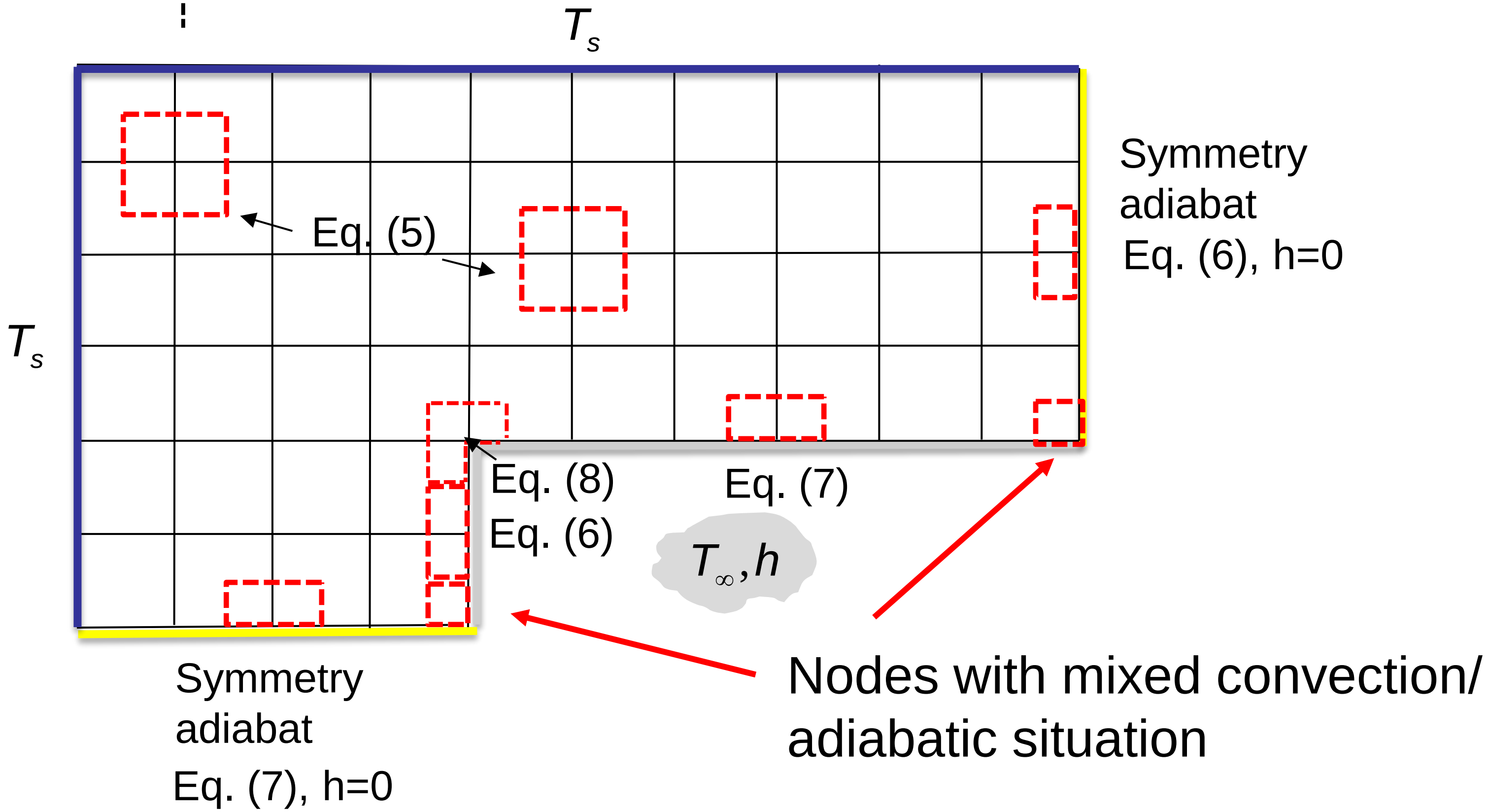
The channel is water cooled at constant temperature $T_\infty = 300K$
with a constant convective heat transfer coefficient $h = 150\ \mathrm{W/(m^2K)}$.

Approximate the temperature field in the walls and study the convergence of the predicted values at selected channel locations.

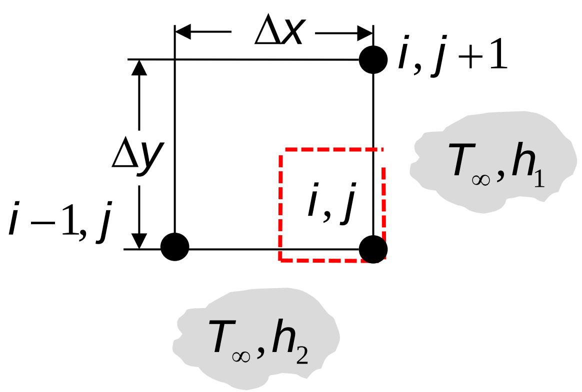
See also Bergman et al., page 251ff / Incropera et al. page 231ff (different values)

Domain and boundary conditions





Node at an external corner with mixed convection



$$\frac{1}{\Delta x^2}T_{i-1,j}+\frac{1}{\Delta y^2}T_{i,j+1}$$

$$-T_{i,j}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{k}\left(\frac{h_1}{\Delta x}+\frac{h_2}{\Delta y}\right)\right)$$

$$=-\frac{1}{k}\left(\frac{h_1}{\Delta x}+\frac{h_2}{\Delta y}\right)T_\infty$$

Lower left: $h_1=h,\quad h_2=0$

$$\frac{1}{\Delta x^2}T_{i-1,j}+\frac{1}{\Delta y^2}T_{i,j+1}-T_{i,j}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{h}{k\Delta x}\right)=-\frac{h}{k\Delta x}T_\infty$$

Upper right : $h_1=0,\quad h_2=h$

$$\frac{1}{\Delta x^2}T_{i-1,j}+\frac{1}{\Delta y^2}T_{i,j+1}-T_{i,j}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{h}{k\Delta y}\right)=-\frac{h}{k\Delta y}T_\infty$$

Matrix organization

$$
\begin{array}{cc}
\begin{array}{c} \\ \\ j \cdot I + i \\ \\ \\ \end{array} &
\left(\begin{array}{ccccccccccc}
1 & 0 & & \cdots & & \cdots & & \cdots & & & 0 \\
 & & & & \ddots & & & & & & \\
\cdots & 0 & \frac{1}{\Delta y^2} & 0\cdots 0 & \frac{1}{\Delta x^2} & -2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) & \frac{1}{\Delta x^2} & 0\cdots 0 & \frac{1}{\Delta y^2} & 0 & \cdots \\
 & & & & & & \ddots & & & & \\
0 & & & \cdots & & \cdots & & \cdots & & 0 & 1
\end{array}\right)
\\
 & \begin{array}{ccccc} (j-1)\cdot I + i & j \cdot I + i - 1 & j \cdot I + i & j \cdot I + i + 1 & (j+1)\cdot I + i \end{array}
\end{array}
\begin{pmatrix} T_{0,0} \\ T_{1,0} \\ \vdots \\ T_{i,j-1} \\ \vdots \\ T_{i-1,j} \\ T_{i,j} \\ T_{i+1,j} \\ \vdots \\ T_{i,j+1} \\ \vdots \\ T_{I,J} \end{pmatrix}
=
\begin{pmatrix} T_s \\ \vdots \\ 0 \\ \vdots \\ T_s \end{pmatrix}
$$

Define an index function to transform the 2D solution into a 1D vector:

```
def index(i,j,Nx):
  return j*Nx+i
```

Matrix assembly:

```
A[index(i,j,Nx),index(i,j,Nx)]=-2.0*(dx2i+dy2i)
A[index(i,j,Nx),index(i+1,j,Nx)]=dx2i
A[index(i,j,Nx),index(i-1,j,Nx)]=dx2i
A[index(i,j,Nx),index(i,j+1,Nx)]=dy2i
A[index(i,j,Nx),index(i,j-1,Nx)]=dy2i
C[index(i,j,Nx)]=0.0
```

```
dx2i = 1.0/(dx*dx)
dy2i = 1.0/(dy*dy)
```