Lecture 14 - Classification of PDEs

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Outline

We look at second order linear Partial Differential Equations (PDEs):

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + f\,u = 0, \qquad u = u(x, y)$$

where $a, \ldots, f$ are given functions of the independent variables (coordinates) x, y.

Simple PDEs can be classified into three types with distinct behaviour:

1. **Hyperbolic,**
2. **Parabolic,**
3. **Elliptic.**

1. Hyperbolic PDEs

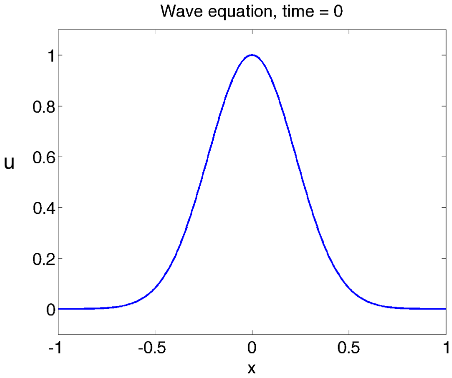
Hyperbolic PDEs are usually associated with **wave propagation**, and are central to hydro- and electro- dynamics.

The prototype is the **wave equation**:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

A **key point** is that information is propagated at finite speed c.

- Other wave propagation problems include acoustics, seismic waves, shallow water theory, general relativity, global atmospheric dynamics, Maxwell's equations, and so on.



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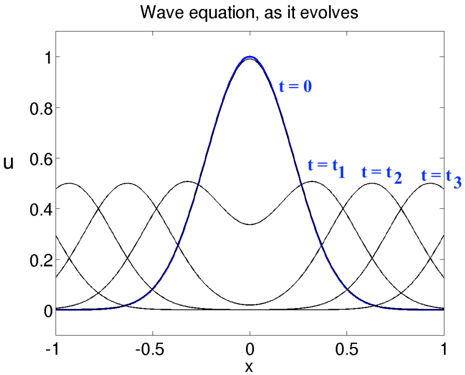

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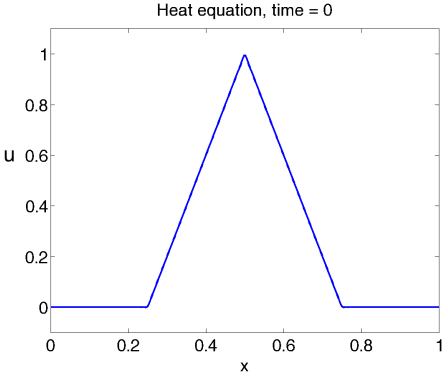
2. Parabolic PDEs

Parabolic PDEs are usually associated with **diffusion problems**.

The prototype is the **heat equation**:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

If you consider information, it is propagated at "*infinite speed*".



- Parabolic PDEs are acausal (not casual) and are used when the small scale (i.e. "microscopic") causal effects happen much faster than the large scale (i.e. "macroscopic") behaviour in which we are interested.
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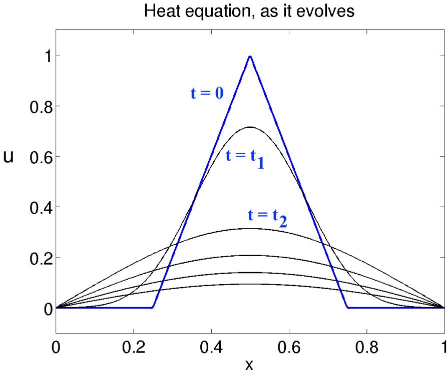

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Read at home: what is diffusion?

• **Diffusion** is the **movement of a substance** from an area of **high concentration** to an area of **low concentration**.

• Diffusion happens in liquids and gases since their **particles move randomly** (while they **collide with each other**) from place to place. Diffusion occurs because moving particles **tend to disperse and spread over the whole container**.

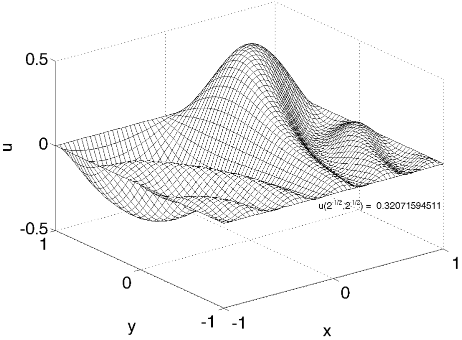
• **Examples of diffusion**: diffusion of coffee in a milk glass, diffusion of pollutant in a lake, diffusion of smoke in the atmosphere, diffusion of water in the soil into the (partially permeable) root air cells of a plant (osmosis), diffusion of CO_2 from a cell into the surrounding blood.

Elliptic PDEs are usually associated with **static or stationary problems**.

The prototype is Laplace's equation:

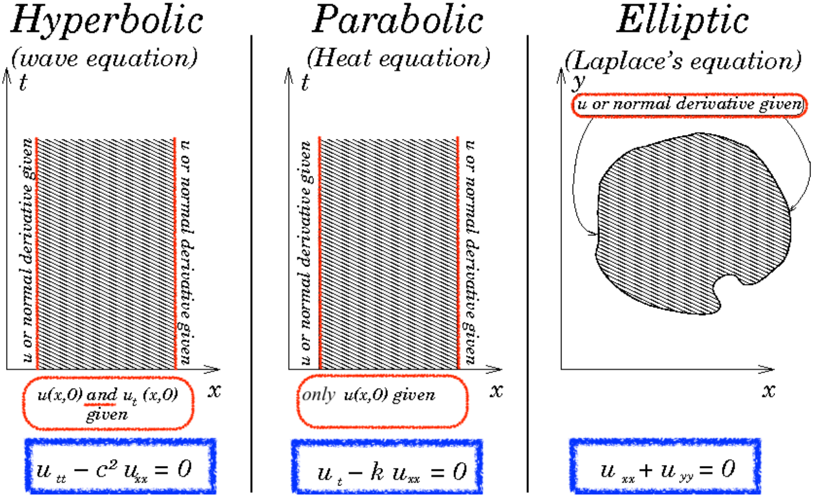
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

These are in a sense generalized boundary value problems.



- Elliptic equations are mathematically exceptionally useful.

• Boundary conditions



Visualize what a boundary condition means (geometrically)!

Dirichlet BCs: $u(0,t) = 0\,, \quad u(L,t) = 0$ (click here)

Neumann BCs: $\partial_x u(0,t) = 0\,, \quad \partial_x u(L,t) = 0$ (click here)

Classification: Criterion to distinguish PDEs

The most general **second order linear PDE** of two independent variables is:

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + f\,u = 0, \qquad u = u(x, y)\,.$$

1. The PDE is said to be **elliptic** if: $b^2 - a\,c < 0$.
2. The PDE is said to be **hyperbolic** if: $b^2 - a\,c > 0$.
3. The PDE is said to be **parabolic** if $b^2 - a\,c = 0$.

However... the character (classification) of a PDE can depend on range of coordinates x or y.

Problems with classification

However... the character (classification) of a PDE can depend on range of coordinates x or y.

Example: The classic *Tricomi equation*:
(used in modelling transonic flows, i.e. flows $\sim 0.7 - 1\ c_{sound}$ $\Rightarrow$ more drag, instabilities)

$$y\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \quad \Rightarrow a(x,y) = y\,,\ \ b(x,y) = 0\,,\ \ c(x,y) = -1\ \left(\text{all other vanish}\right).$$

is

- **hyperbolic** when $y > 0$ since then $b^2 - a\,c = 0 - (-y) = y > 0$;
- **elliptic** when $y < 0$: since then $b^2 - a\,c = 0 - (-y) = y < 0$;
- a real problem (*not* hyperbolic/elliptic/parabolic) when $y = 0$.

Generically the PDE classification above is useful when describing (and solving for) the **local** behaviour (i.e. in a certain local range of x, y) of a PDE.

Outline

→ Wave Equation

Our first PDE to consider will be the 1-D **Wave Equation**:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \Leftrightarrow \quad \frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0\,, \qquad y = y(t, x).$$

The wave function y depends on <u>two</u> independent variables (time t and space coord. x) but we say that it is a 1-D wave equation because it depends on a <u>single spatial</u> coordinate x (and then also on time).

c is the **velocity of propagation of the wave** along x direction

This is a key toy problem for models of **fluids**, **electromagnetism**, **acoustics**, **gravity**, etc.

Derivation of Wave Equation on Strings

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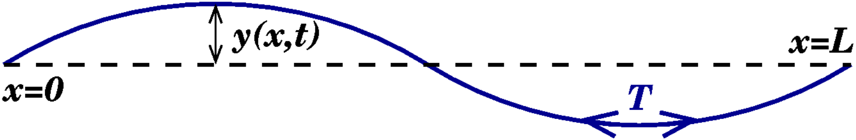
Consider a moving string with mass density ρ under tension $\boldsymbol{T}$ (e.g. in a guitar):



The string extends along the x direction from $\boldsymbol{x} = \boldsymbol{0}$ to $\boldsymbol{x} = \boldsymbol{L}$. At rest (no t dependence), the string thus has **length $\boldsymbol{L}$** and its vertical position is $y(x) = 0$. But strings are elastic and can be stretched: if we apply a tension we produce a displacement of the string along the vertical direction y and it will oscillate in time. At a given time t we can take a photo and the vertical displacement at a given point x (and t of photo) is $\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{t})$.

How do we find the law (PDE) whose solution describes the displacement $y(x, t)$?

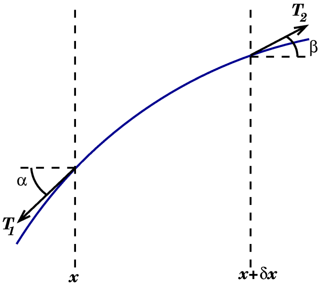
Well, from Newton's second law (Force=mass $\times$ acceleration), one knows that the tension in the string (tension is a force) produces an acceleration (acceleration $=$ second derivative in time of vertical displacement: $a = \frac{\partial^2 y}{\partial t^2}$).

Derivation of Wave Equation on Strings using Newton's laws

Consider an infinitesimal section of the string $(x, x+\delta x)$, $\delta x \ll x$. Newton's second law:

$$\underbrace{\rho\,\delta x}_{\text{Mass}}\ \underbrace{\frac{\partial^2 y}{\partial t^2}}_{\text{Acceleration}} = \underbrace{T_2\sin(\beta) - T_1\sin(\alpha)}_{\text{Force}}$$

Only vertical motion!



For infinitesimal displacements, $\delta y \ll 1$, we can:

1. set $T_1 \approx T_2 \equiv T$;
2. $\sin\beta = \frac{\delta y}{\delta x}\Big|_{x+\delta x} \approx \frac{\partial y}{\partial x}\Big|_{x+\delta x}$ (similarly: $\sin\alpha \approx \frac{\partial y}{\partial x}\big|_x$)

Inserting these approximations:

$$\rho\,\delta x\,\frac{\partial^2 y}{\partial t^2} \approx T\left(\frac{\partial y}{\partial x}\Big|_{x+\delta x} - \frac{\partial y}{\partial x}\Big|_{x}\right) \approx T\,\delta x\,\frac{\partial^2 y}{\partial x^2}. \quad \longleftarrow \begin{cases} \text{Def. derivative: } f'(x) = \lim\limits_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} \\ \text{Let } f = y' \Rightarrow y'' = \lim\limits_{\delta x\to 0}\frac{y'(x+\delta x)-y'(x)}{\delta x} \end{cases}$$

Cancelling the factor of δx and dividing by ρ we finally get:

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} \quad \longrightarrow \text{Got our **Wave Equation** with } \boldsymbol{c = \sqrt{T/\rho}}$$

Outline

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Summary

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- For linear PDEs there are three simple types (hyperbolic, parabolic, elliptic) based on a transformation to a standard form.
- The types have qualitatively different solution behaviour.
- The three types require qualitatively different boundary conditions to have a well posed problem. For example, hyperbolic PDEs appear typically in initial value problems while elliptic PDEs appear typically in boundary value problems.
- The **wave equation (hyperbolic PDE)** is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

c is the (finite) velocity of the wave (information propagates at finite speed)