Part 3: Beams in Bending: Assembling and Solving Problems

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

From 29th October 2024

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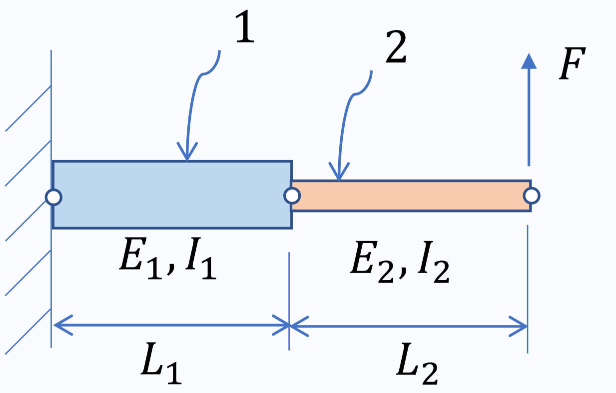
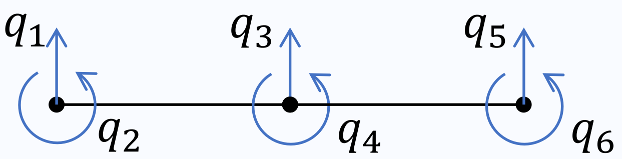
Solving this Problem:

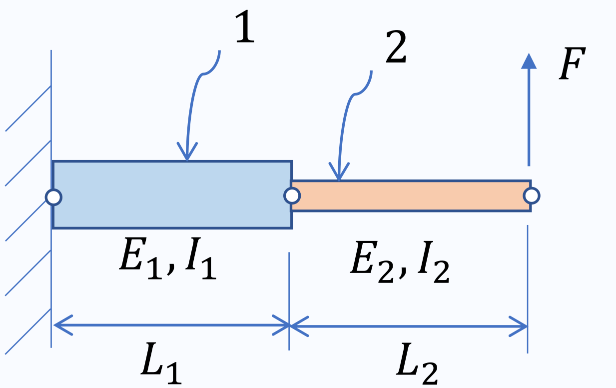
1. State the elemental and global stiffness matrices
2. State the global force vector
3. Apply boundary conditions, and
4. State the reduced governing equation of equilibrium

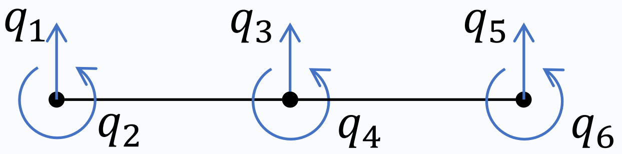




Two-noded beam elements in bending

- Recall our Element Stiffness Matrix (which you do not need to remember):





$$U_1 = {}^1/_2 \begin{Bmatrix} q_1 \\ \vdots \\ q_4 \end{Bmatrix}^T \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_4 \end{Bmatrix}$$

$$U_2 = {}^1/_2 \begin{Bmatrix} q_3 \\ \vdots \\ q_6 \end{Bmatrix}^T \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \begin{Bmatrix} q_3 \\ \vdots \\ q_6 \end{Bmatrix}$$

$$K_1 = \left(\frac{E_1 I_1}{{L_1}^3}\right) \begin{bmatrix} 12 & 6L_1 & -12 & 6L_1 \\ 6L_1 & 4{L_1}^2 & -6L_1 & 2{L_1}^2 \\ -12 & -6L_1 & 12 & -6L_1 \\ 6L_1 & 2{L_1}^2 & -6L_1 & 4{L_1}^2 \end{bmatrix}$$

$$K_2 = \left(\frac{E_2 I_2}{{L_2}^3}\right) \begin{bmatrix} 12 & 6L_2 & -12 & 6L_2 \\ 6L_2 & 4{L_2}^2 & -6L_2 & 2{L_2}^2 \\ -12 & -6L_2 & 12 & -6L_2 \\ 6L_2 & 2{L_2}^2 & -6L_2 & 4{L_2}^2 \end{bmatrix}$$

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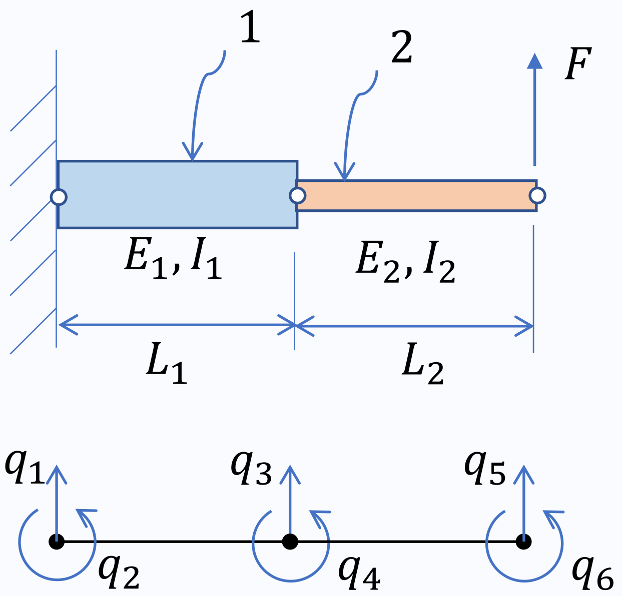
Two-noded beam elements in bending



$$U_1 = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \begin{bmatrix} \quad \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

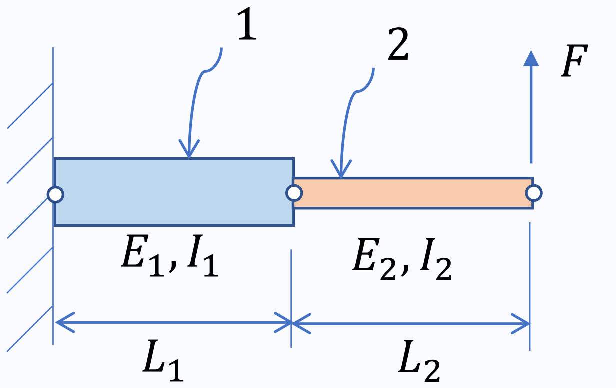
$$U_2 = {}^1\!/_2 \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \begin{bmatrix} \quad \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

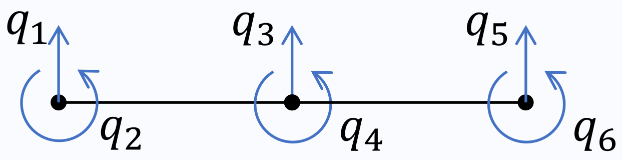
$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \begin{bmatrix} \quad \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

Two-noded beam elements in bending

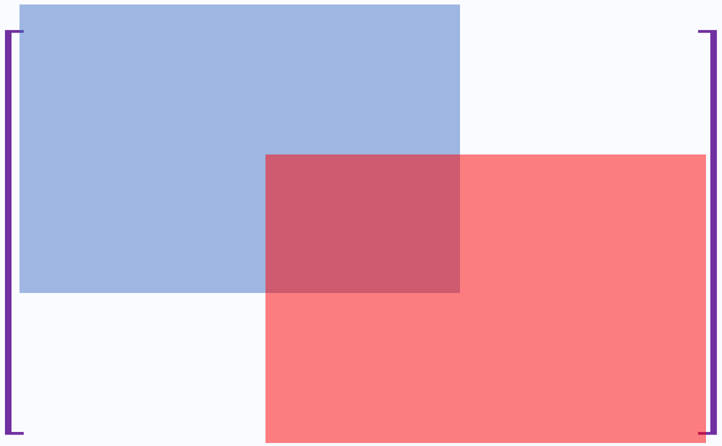
- Assemble our global stiffness matrix and apply PMTPE:

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}^T \begin{bmatrix} & & \\ & 6 \times 6 & \\ & & \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}$$

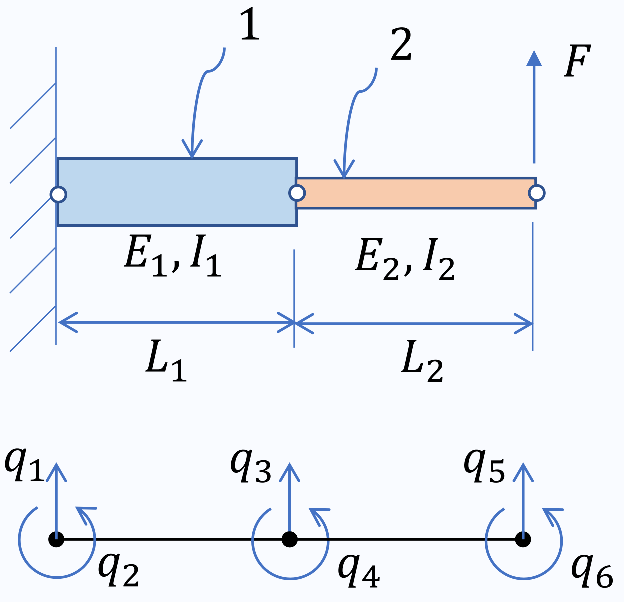




$K =$



Two-noded beam elements in bending



- Assemble our global stiffness matrix and apply PMTPE:

$$U = {}^{1}/_{2} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}^T \left[\quad 6 \times 6 \quad \right] \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}$$

$$\begin{bmatrix} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & \\ & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{Bmatrix}$$

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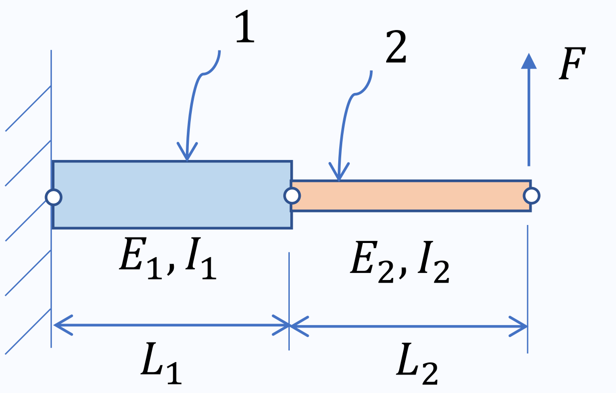
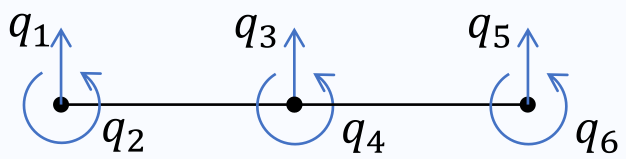
Two-noded beam elements in bending

- Assemble our global stiffness matrix and apply PMTPE:

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}^T \begin{bmatrix} & & \\ & 6 \times 6 & \\ & & \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}$$



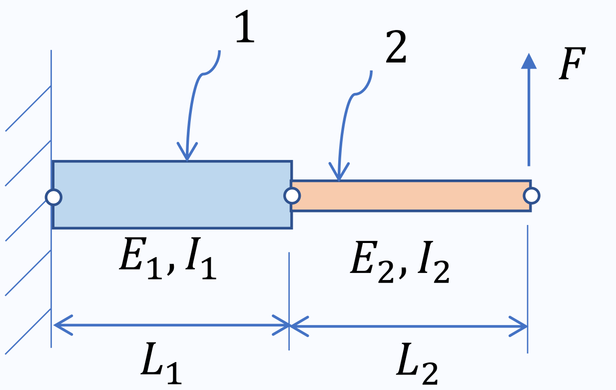


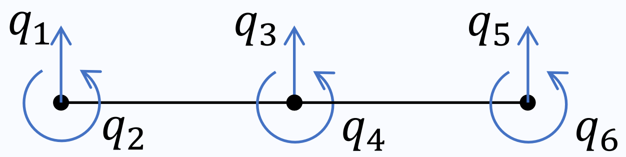
$$\begin{bmatrix} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & \\ & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} R \\ M \\ 0 \\ 0 \\ F \\ 0 \end{Bmatrix}$$

Boundary conditions because:

$q_1 = 0$ and $q_2 = 0$

Two-noded beam elements in bending





- Our *reduced* stiffness matrix is now:

$$\left[\quad\right]\begin{Bmatrix}q_3\\q_4\\q_5\\q_6\end{Bmatrix}=\begin{Bmatrix}0\\0\\F\\0\end{Bmatrix}$$

- and now we can solve for q_{3-6}. You could invert and multiply, but in practice some Gaussian elimination, upper triangulate and back substitute.
- And then return to Global Equation find R and M reactions if you need them.

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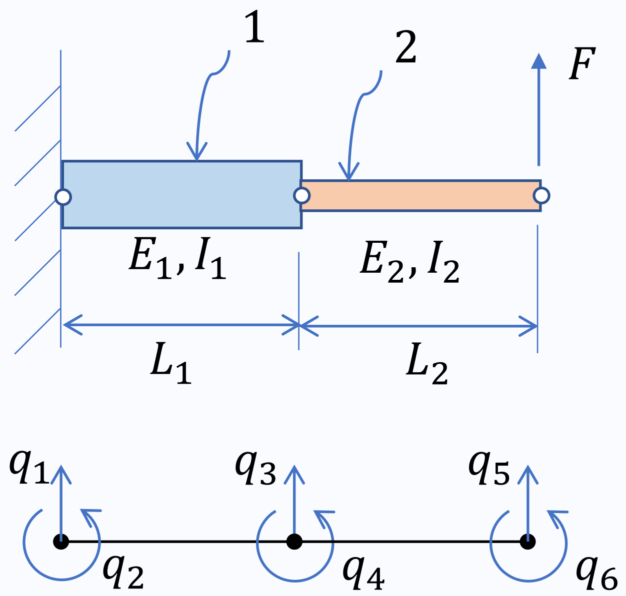
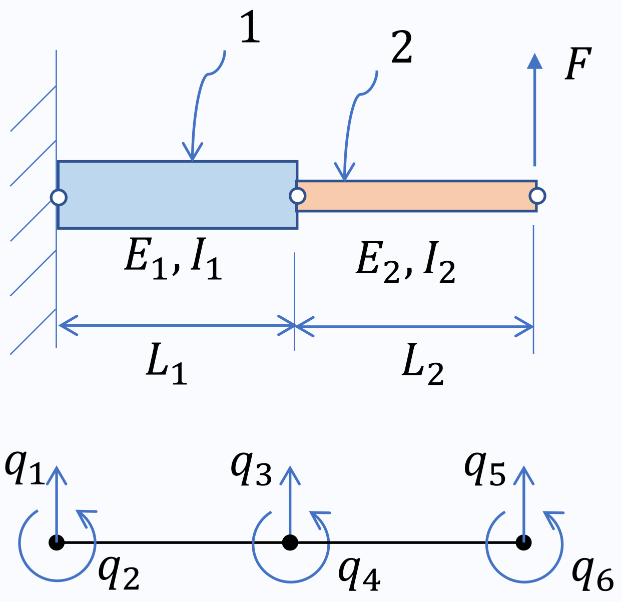
Two-noded beam elements in bending



- Now a challenge: What is the approximate displacement at the middle of the second element?
- … without creating a new node (which costs time)

Two-noded beam elements in bending



- Now a challenge: What is the approximate displacement at the middle of the second element?
- … without creating a new node (which costs time)
- We have all the qs, nodal deformations
- We use the combined interpolation function:

$$w^{E2}(x) = f_1(x)q_3 + f_2(x)q_4 + f_3(x)q_5 + f_4(x)q_6$$

- So the approximate answer is:

$$w^{E2}\left(x = \frac{L_2}{2}\right) = f_1\left(\frac{L_2}{2}\right)q_3 + f_2\left(\frac{L_2}{2}\right)q_4 + f_3\left(\frac{L_2}{2}\right)q_5 + f_4\left(\frac{L_2}{2}\right)q_6$$

- We could solve this, because we defined f_i and we found q_i

Part 3: Beams in Bending
Strain and Stress Calculations

FEEG3001/SESM6047 FEA in Solid Mechanics

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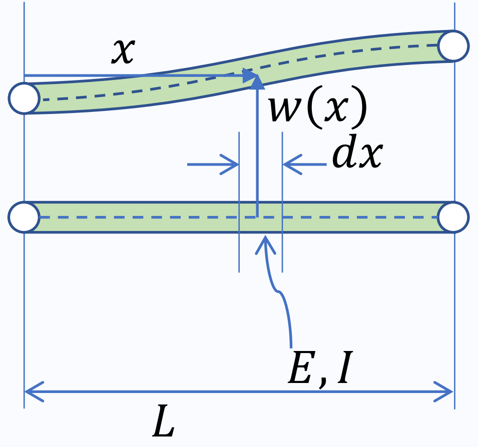
Elemental Strain and Stress:



- From statics: with small deformations, rotations and slopes are equivalent

  $\frac{dw}{dx} = \tan\phi$ and for small $\phi \approx \sin\phi \approx \tan\phi$

- So for small rotations we can state the axial displacements u on the planar surface in terms of ϕ:

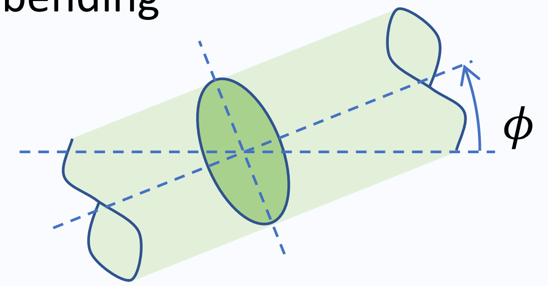
$$\varepsilon_x = \frac{du}{dx}$$

$$u = -y\phi = -y\frac{dw}{dx}$$

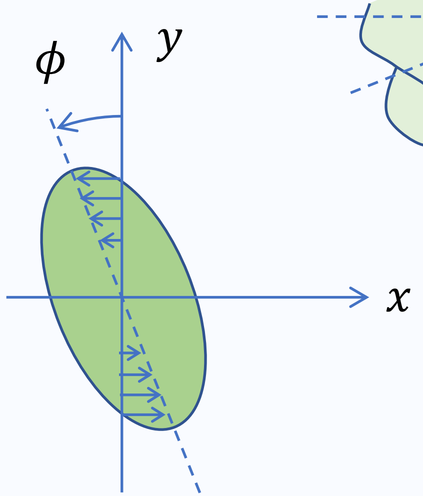
$$\varepsilon_x = -y\frac{d^2w}{dx^2}$$

Euler-Bernoulli hypothesis assumptions:

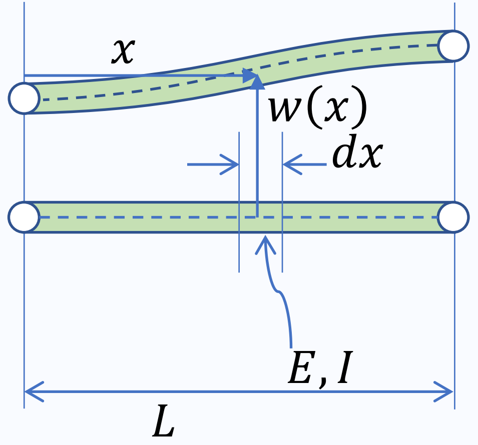
- Cross sections do not change during bending
- Cross section remains perpendicular to the neutral axis during bending



$$u = -y\phi$$



Elemental Strain and Stress:



- This allows us to state strain using a ‘matrix of operations’:

$\varepsilon_x = -y\left[\frac{d^2}{dx^2}\right]\{w\}$ or

$$\varepsilon_x = -y\left[\frac{d^2}{dx^2}\right]\begin{bmatrix} f_1(x) & f_2(x) & f_3(x) & f_4(x) \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$\varepsilon_x = -y\begin{bmatrix} f_1''(x) & f_2''(x) & f_3''(x) & f_4''(x) \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Euler-Bernoulli hypothesis assumptions:

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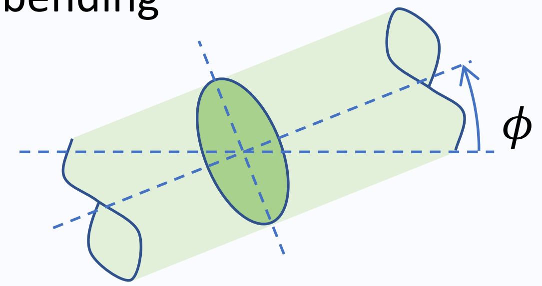

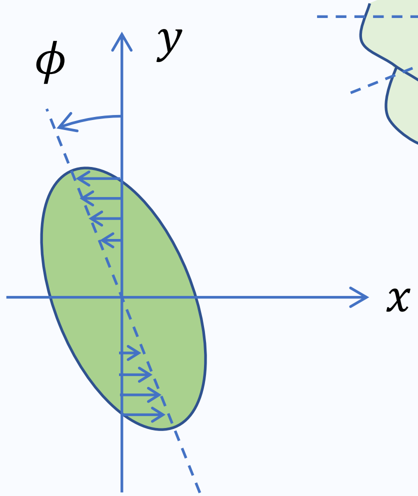

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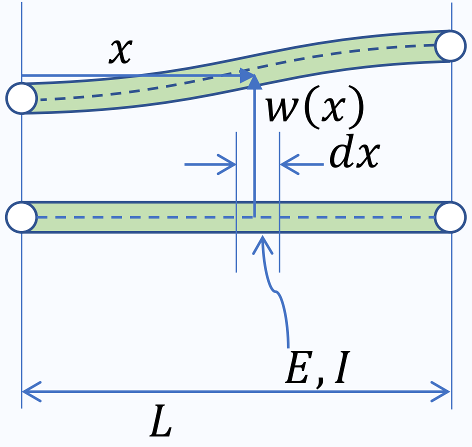
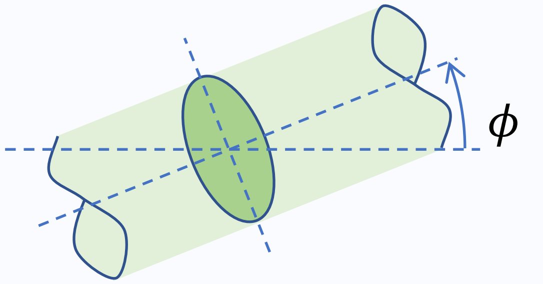
Elemental Strain and Stress:

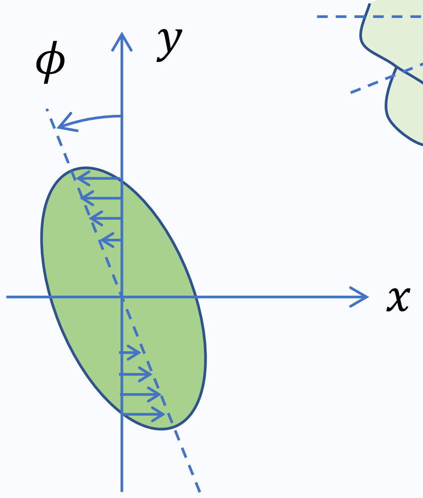


- We can calculate those second derivatives of our shape functions to create an approximation of strain called the 'B matrix' $[B]$:

$$\varepsilon_x = -y\left[\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right) \quad \left(-\frac{4}{L}+\frac{6x}{L^2}\right) \quad \left(\frac{6}{L^2}-\frac{12x}{L^3}\right) \quad \left(-\frac{2}{L}+\frac{6x}{L^2}\right)\right]\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$



$$\varepsilon_x = [B]\{q\}$$



- Finally the stress is estimated using the 'D matrix' $[D]$, in this simple, 1D case equal to the Young's modulus E:

$$\sigma_x = [D][B]\{q\} = E[B]\{q\}$$

Update: Content and Assessment Plan

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

From 1st November 2024

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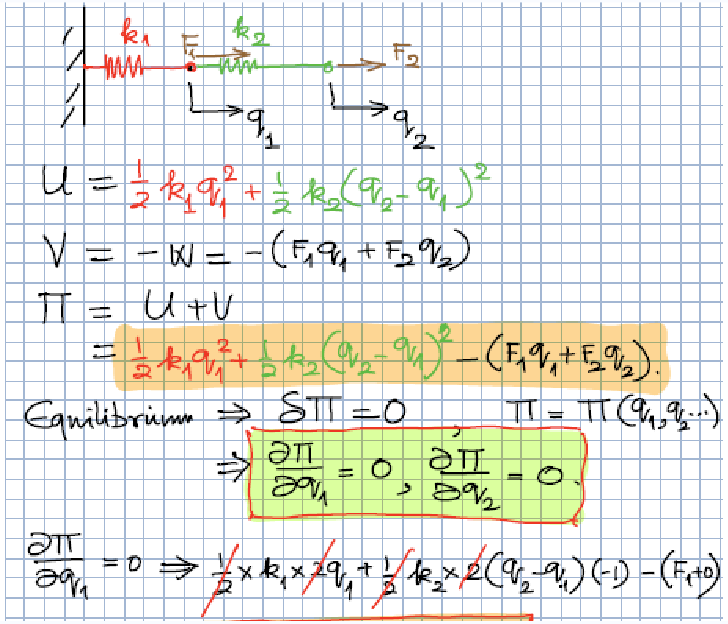
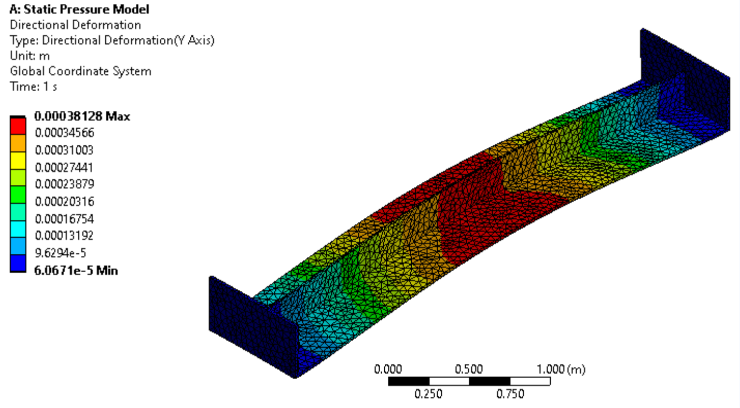
Course Overview

- Final assessment worth 50%
  - Online, 4hr, open book
  - Mixture of Short (MCQ, T/F) and Long Answer
  - More example Short Questions coming soon…

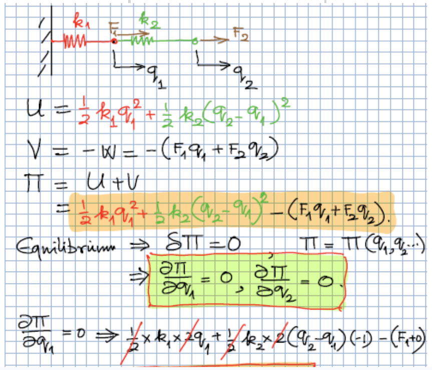
- Coursework worth 50%
  - Assignment based on labs, 40%, set 8th November, due 29th November
  - Quiz on Blackboard, 10%, Tues 10th December, 5pm (during lecture slot)

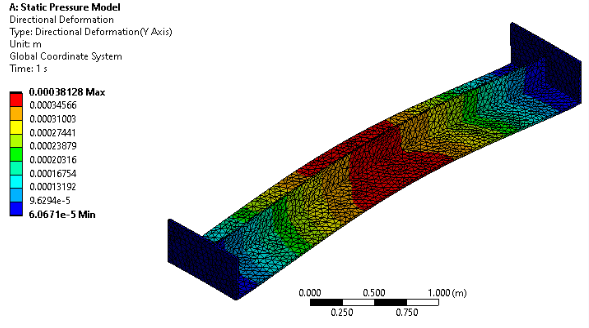




Course Overview: UPDATED!

Commencing	30/09/2024	07/10/2024	14/10/2024	21/10/2024	28/10/2024	04/11/2024	11/11/2024	18/11/2024	25/11/2024	02/12/2024	09/12/2024	06/01/2025
Week	1	2	3	4	5	6	7	8	9	10	11	12
Lectures	1 to 3	4 to 6	7 to 9	10 to 12	13 to 15	16 to 18	19 to 21	22 to 24	25 to 27	28 to 30	31 to 33	34 to 36
Topic	1: PMTPE	1: PMTPE	2: Rods	Beams	3: Beams	4: Dynamics	4: Dynamics	4: Dynamics	5: Trusses	5: Trusses	6: Frames	Dimensions
Session 1	Intro	Assembling, Solution	Combining Rods & Springs	Beam SFs: HCs	Beam Assembly, Solution	Lagrange, Hamilton, 1, 2 DOF Systems	Dynamic Beams, Solving Dynamics	Trusses, CS	Frames, CS	Const. Strain Tris	MCQs, Exam Conditions	Revision
Session 2	PMTPE, [K], Springs	Elastic Rods [K]	Distributed Loads	Beam [K]	Distributed Loads	Dynamic Rods, Reporting FEA	Mode Shapes, Rods, Shafts, Strings	TMs, Assembly	Frames and Stress Calculations	Const. Strain Tris	4-Node Rectangles	Revision
Session 3	Lab 1 intro	Shape Functions	Lab 2 intro	Rod Questions	Lab 3 intro, Modal Demo	Beam Questions, Form. MCQs	Formative MCQs	Dynamics Ex.	Python Trusses Risk-Based FEA	Truss Ex.Stress Calc, Python	Revision	Q&A, Examples
Labs		Solid Static Models		Solid Modal and Buckling Models		Shell Models		3D Models? Help/Extra		Spare. Python/ANSYS comparison?		
Coursework						Assigned 08/11/24, Submitted 29/11/24						





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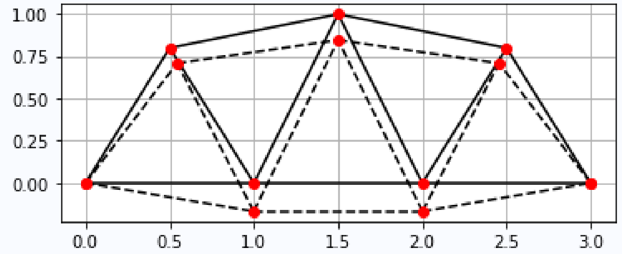

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Part 3: Beams in Bending
Distributed Loads

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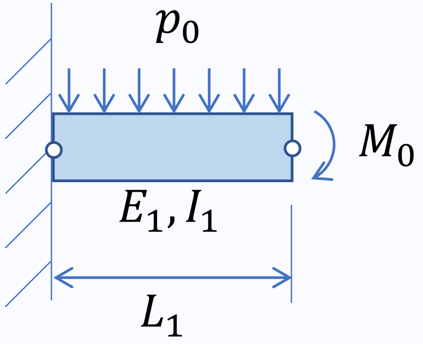
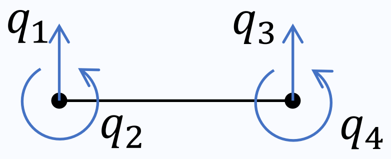
Two-noded beam elements in bending





- One more example: take a single element, with a moment M_0 applied at its free end and a uniformly distributed load p_0. What is its tip displacement?
- Due to PMTPE, the equilibrium equation will look like:

$$[K]\{q\} = \{F\} \text{ or } \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \\ \\ \\ \\ \end{Bmatrix}$$

- Only the forces have changed, so we only need to modify the right hand side. First, forget about the distributed force and consider the moment. Which concentrated forces and moments do work?

$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \\ \\ \\ \\ \end{Bmatrix} + \begin{Bmatrix} R \\ M \\ 0 \\ -M_0 \end{Bmatrix}$$

- and the distributed force? What work does it do? Take a slice dx

Two-noded beam elements in bending

- One more example: take a single element, with a moment M_0 applied at its free end and a uniformly distributed load p_0. What is its tip displacement?
- The force within a slice dx is $p_0 dx$, and the work done by the distributed force on a slice dx is $p_0 dx \times w(x)$
- Overall the work done is:

$$V = -\int_0^{L_E} p_0 w(x)\, dx$$

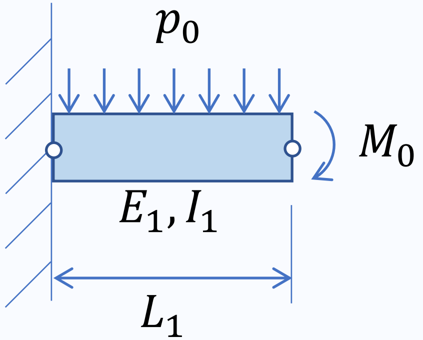
- and

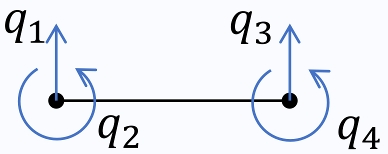
$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4$$

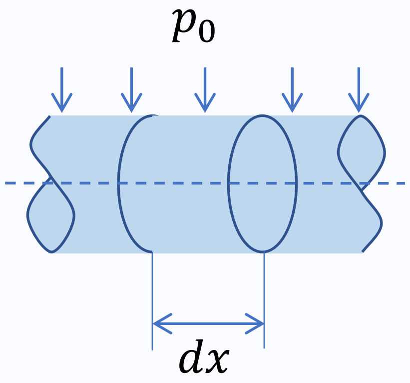
$$V = -\int_0^{L_E} p_0[f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4\,]\, dx$$

$$V = -\left(\int_0^{L_E} p_0\, f_1(x)dx\right)q_1 - \left(\int_0^{L_E} p_0\, f_2(x)dx\right)q_2 - \cdots$$

- so…







Two-noded beam elements in bending

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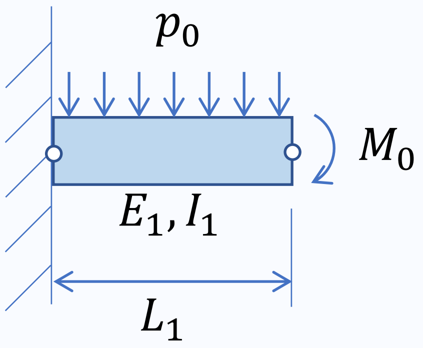
- The force within a slice dx is $p_0 dx$, and the work done by the distributed force on a slice dx is $p_0 dx \times w(x)$
- Overall the work done is:

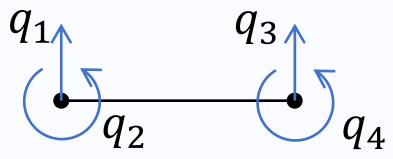
$$V = -\left[\left(p_0 \int_0^{L_E} f_1(x)dx\right) \quad \left(p_0 \int_0^{L_E} f_2(x)dx\right) \quad \dots \quad \dots\right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

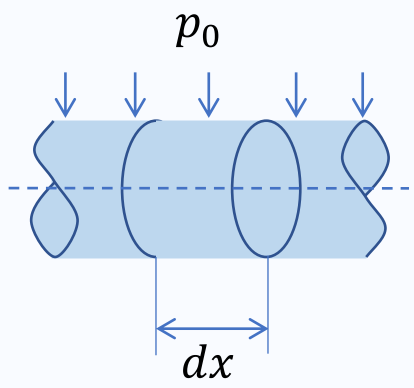
- which has the original form we recognise

$$V = -\{F\}^T\{q\}$$

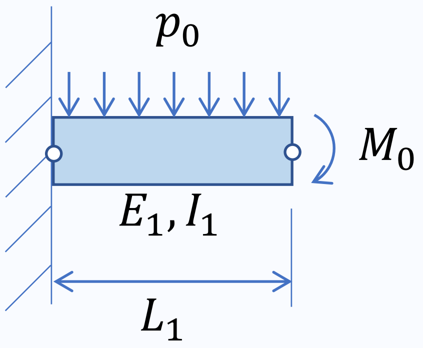
- So the force contributions are integrals of our Hermite cubics f_i weighting each of our generalised coordinates q_i, giving:



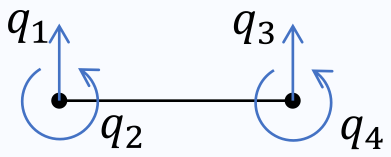




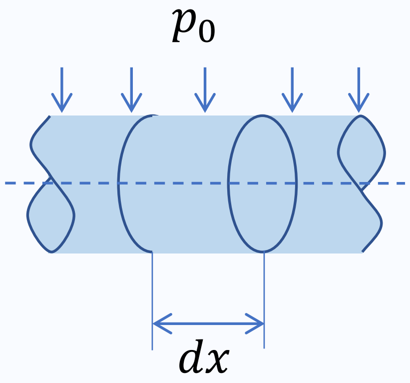
Two-noded beam elements in bending



- One more example: take a single element, with a moment M_0 applied at its free end and a uniformly distributed load p_0. What is its tip displacement?
- So the force contributions are integrals of our Hermite cubics f_i weighting each of our generalised coordinates q_i, giving:



$$\left[\qquad\qquad\qquad\right]\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} p_0 \int_0^{L_E} f_1(x)dx \\ p_0 \int_0^{L_E} f_2(x)dx \\ p_0 \int_0^{L_E} f_3(x)dx \\ p_0 \int_0^{L_E} f_4(x)dx \end{Bmatrix} + \begin{Bmatrix} R \\ M \\ 0 \\ -M_0 \end{Bmatrix}$$



- which we can apply BCs to and solve for q_3

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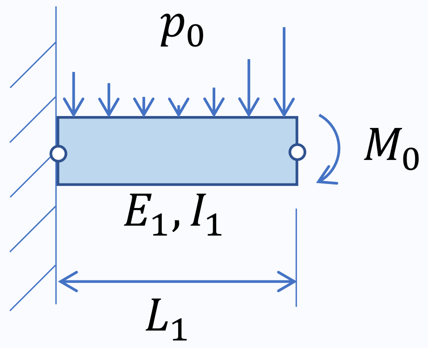
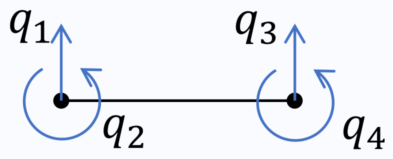
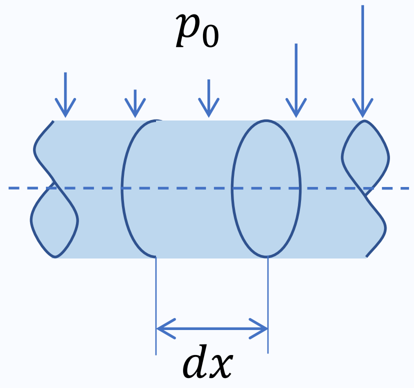
Two-noded beam elements in bending



- How could we include a distributed load p_0 which is **not** uniform, but a function of x?
- Simply need to consider $p_0(x)$ within the integration:



$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & \dots & \dots \\ 6L & 4L^2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \int_0^{L_E} p_0(x) f_1(x)\, dx \\ \int_0^{L_E} p_0(x) f_2(x)\, dx \\ \int_0^{L_E} p_0(x) f_3(x)\, dx \\ \int_0^{L_E} p_0(x) f_4(x)\, dx \end{Bmatrix} + \begin{Bmatrix} R \\ M \\ 0 \\ -M_0 \end{Bmatrix}$$



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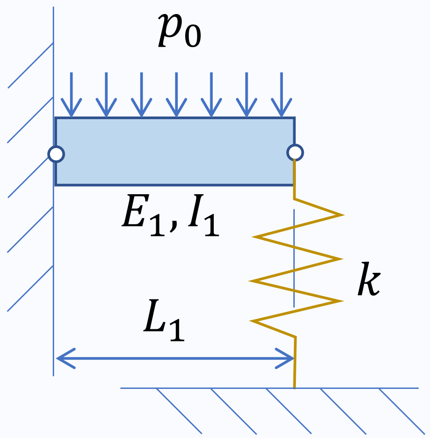
Two-noded beam elements in bending

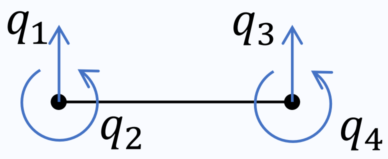


- What if we add a grounded spring to the free end? As before,

$$U = U_1 + U_{spring}$$

$$U_{spring} = {}^{1}\!/_{2}\, k q_3{}^2$$

- How is this contribution to stiffness accommodated?



$$U = {}^{1}\!/_{2} \frac{EI}{L^3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \begin{bmatrix} 12 & 6L & \dots & \dots \\ 6L & 4L^2 & \dots & \dots \\ \dots & \dots & \dots + \frac{L^3}{EI} k & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Can simplify with $k^* = \frac{L^3}{EI} k$

- Note this does NOT affect the force vector, as the spring support is considered 'yielding' and therefore provides no reaction. The force in the spring would be found using the spring constitutive equation, $R_{spring} = kq_3$

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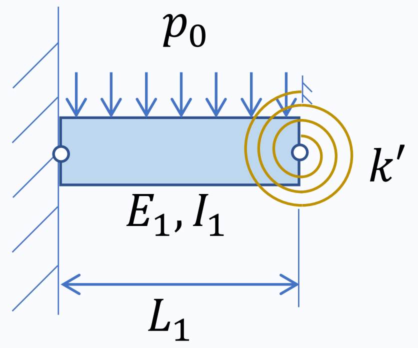
Two-noded beam elements in bending

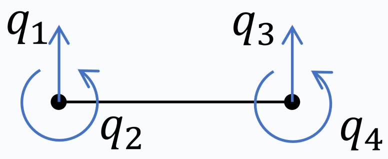


- or a grounded rotational spring?

$$U = U_1 + U_{spring}$$

$$U_{spring} = {}^1\!/_2\, k' {q_4}^2$$

- How is this contribution to stiffness accommodated?



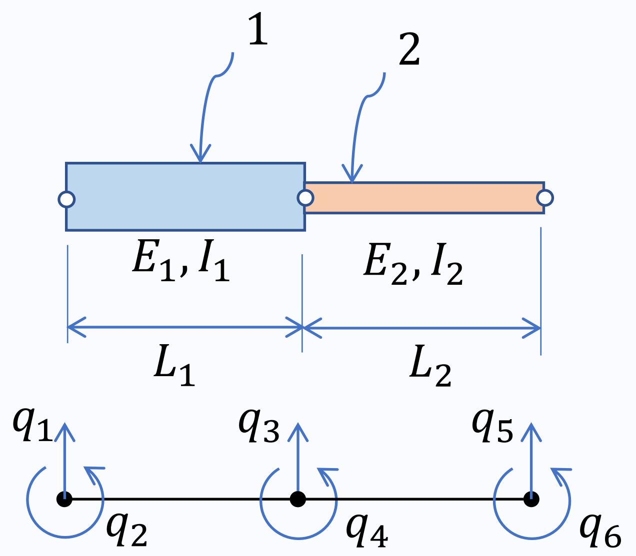
$$U = {}^1\!/_2 \frac{EI}{L^3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \begin{bmatrix} 12 & 6L & \dots & \dots \\ 6L & 4L^2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots + \frac{L^3}{EI} k' \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

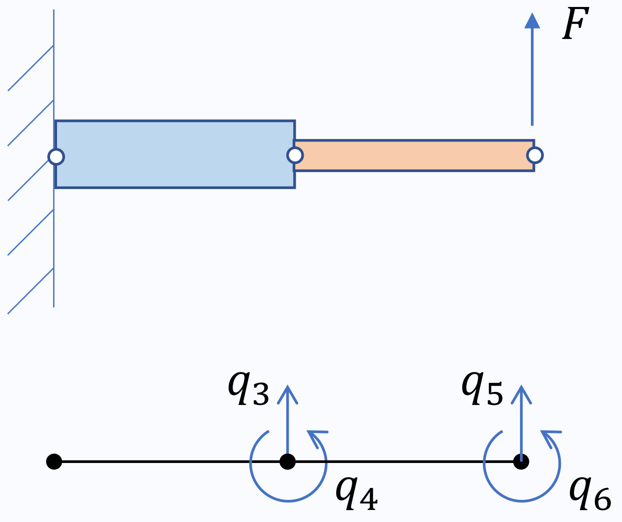
Can simplify with $k^* = \frac{L^3}{EI} k'$

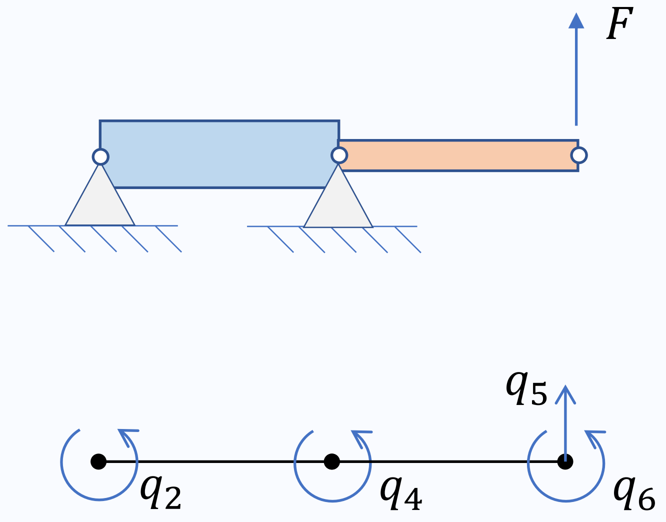
- So hopefully you are now more confident the two tricks in FEA are:
  - using the quadratic form for a given element and approximation for displacement within the element; and
  - book-keeping (formally, ‘Assembly’; computation)

Last Thought: Boundary Conditions

- Remember, a model needs to be constrained in all Degrees of Freedom
- Note some constraints might be indirect, e.g. fixing rotation by fixing two translations







Part 3: Beams in Bending
Beam-Rod or Beam-Column Elements

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

From 1st November 2024

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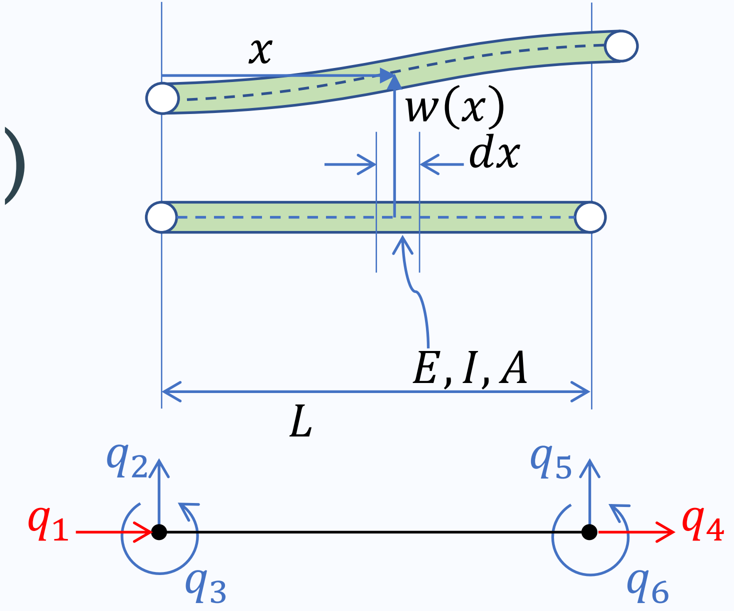
University of Southampton

Beam-Rod elements (or Beam-Column)



- Combine bending with stretching in tension and compression
- Note how we have renumbered our generalised coordinates (DOF) systematically, for convenience

$$U = U_{stretch} + U_{bending}$$

$$U_{stretch} = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_4 \end{Bmatrix}^T \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_4 \end{Bmatrix}$$ (which uses linear interpolation)

$$U_{bending} = {}^1\!/_2 \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}^T \begin{bmatrix} \quad & \quad \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}$$ (which uses cubic interpolation)

- so

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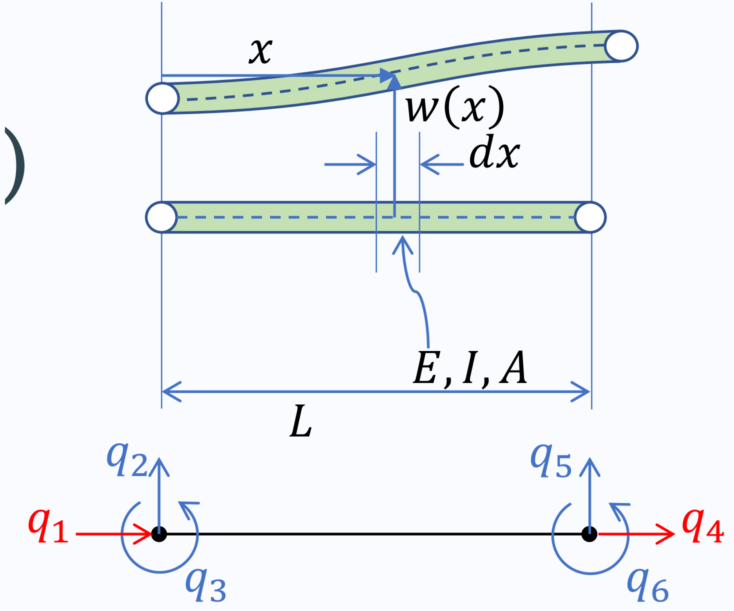
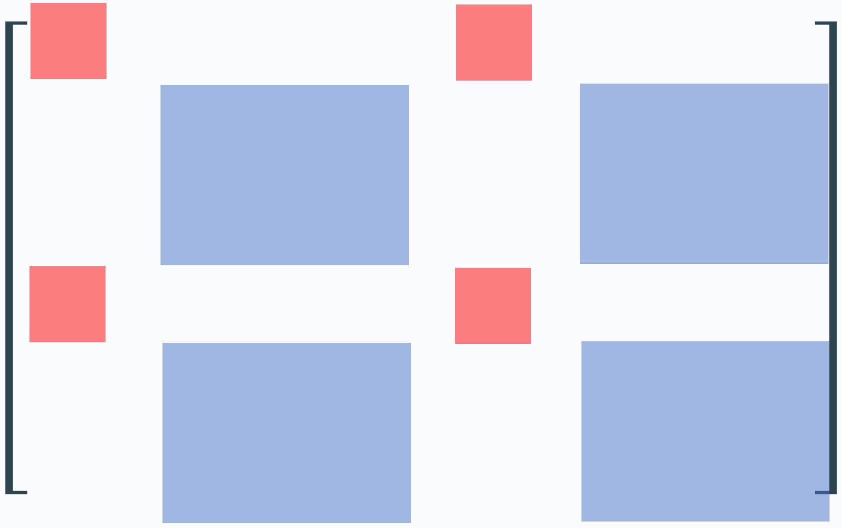
Beam-Rod elements (or Beam-Column)

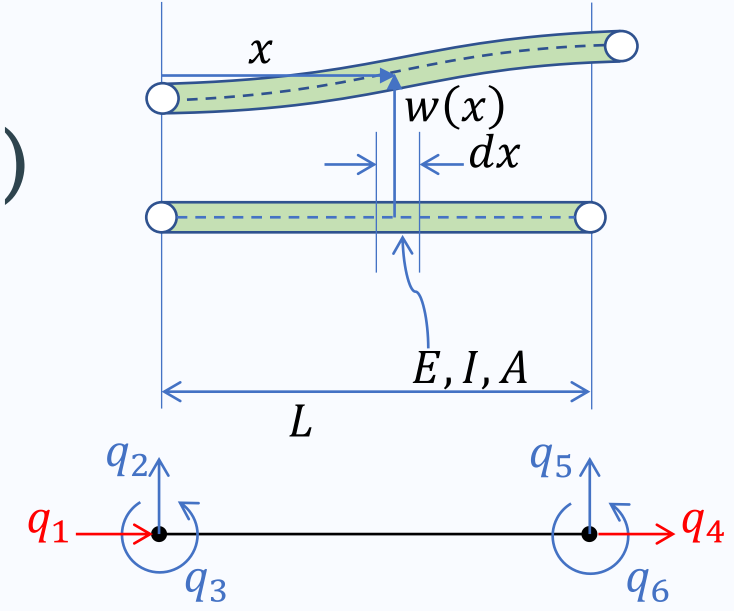


- What form does the stiffness matrix take?
- Where do these contributions fit into the stiffness matrix?

$$U = {}^{1}/_{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \left[\quad \right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$



Beam-Rod elements (or Beam-Column)



- Note you may be able to solve bending and stretch parts of the problem separately, though! We combine for computation, and stress calculation.

$$
\mathrm{U} = {}^{1}/_{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T
\begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & \frac{EI}{L^3}(12) & \frac{EI}{L^3}(6L) & 0 & \frac{EI}{L^3}(-12) & \frac{EI}{L^3}(6L) \\
0 & \frac{EI}{L^3}(6L) & \frac{EI}{L^3}(4L^2) & 0 & \frac{EI}{L^3}(-6L) & \frac{EI}{L^3}(2L^2) \\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & \frac{EI}{L^3}(-12) & \frac{EI}{L^3}(-6L) & 0 & \frac{EI}{L^3}(12) & \frac{EI}{L^3}(-6L) \\
0 & \frac{EI}{L^3}(6L) & \frac{EI}{L^3}(2L^2) & 0 & \frac{EI}{L^3}(-6L) & \frac{EI}{L^3}(4L^2)
\end{bmatrix}
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}
$$

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Example Questions for Beams in Statics:

2016/17 Question 4

5 FEEG3001 W1

Q4. A beam of length $3l$ and bending stiffnes EI, fixed at the two ends, supported at a distance l from the left end is subjected to a concentrated moment, as shown in Figure Q4. Write the global stiffness matrix and the force vector using TWO finite elements.The element stiffness matrix for a beam bending element of length l and bending stiffness EI, is given by

$$K_e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Calculate the slope of the beam at the point of application of the concentrated moment. The order of degrees-of-freedom numbering for a typical element is shown in the accompanying figure below.

[18]

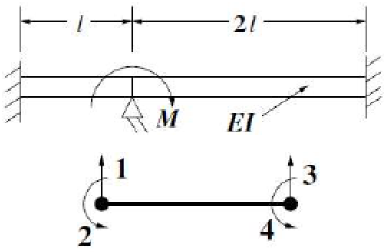

FIGURE Q4

2017/18 Question 4

5 FEEG3001W1

Q4. For the stepped beam shown in Figure Q4, write the global stiffness matrix and the force vector using TWO finite elements.The element stiffness matrix for a beam bending element of length l and bending stiffness EI, is given by

$$K_e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

The node numbering scheme in a typical element is shown in the sub-fugure below.

[16]

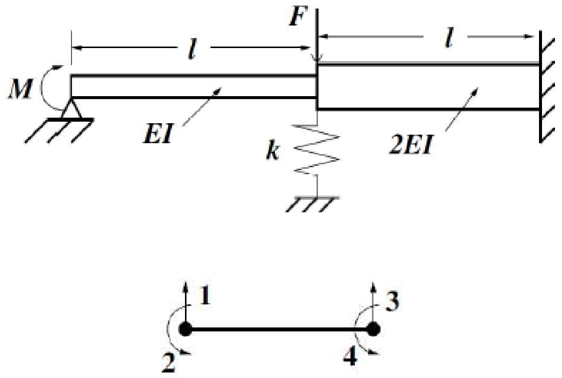

FIGURE Q4

For Fun: Stiffness Matrix Elements from Integrating Hermite Cubics

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

From 2nd November 2023

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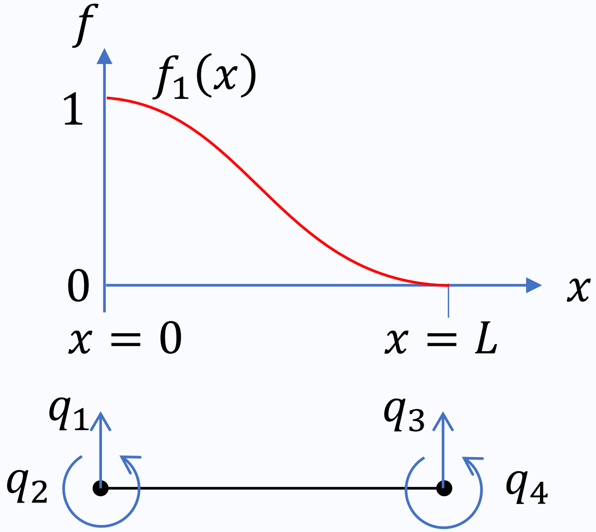
Ex. Stiffness Matrix Derivation



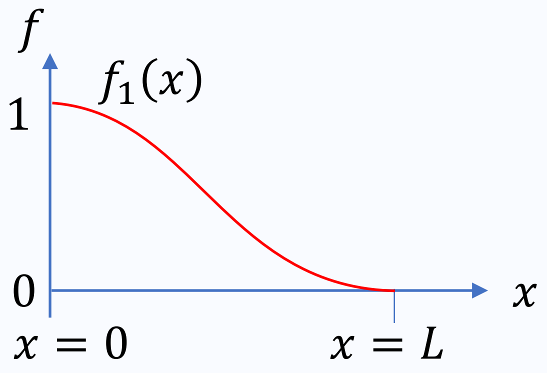
$$f_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$

- Can you derive an example element of the Beam stiffness matrix, e.g. $K_{1,1}$?
- Assume E, I are constants

$$K_{1,1} = \frac{12EI}{L^3}$$

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \underbrace{EI \begin{bmatrix} \int f_1''^2(x)dx & \int f_1''(x)f_2''(x)dx & \dots & \dots \\ \int f_1''(x)f_2''(x)dx & \int f_2''^2(x)dx & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}}_{\text{Beam Bending Element Stiffness Matrix!}} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Ex. Stiffness Matrix Derivation



$$K_{1,1} = EI \int_0^L f_1''^2(x)dx$$

$$f_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$

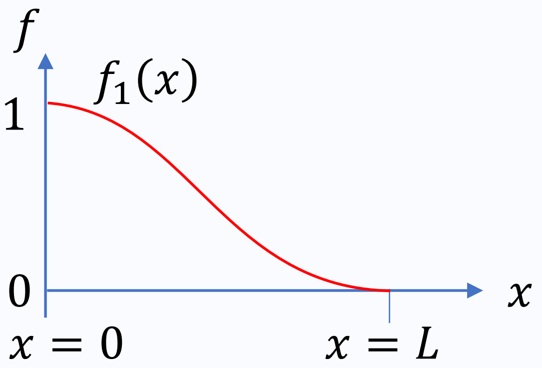
$$f_1'(x) = 0 - 6\frac{x}{L^2} + 6\frac{x^2}{L^3}$$

$$f_1''(x) = 0 - \frac{6}{L^2} + 12\frac{x}{L^3}$$

So:

$$K_{1,1} = EI \int_0^L \left(-\frac{6}{L^2} + \frac{12x}{L^3}\right)^2 dx$$

Ex. Stiffness Matrix Derivation



$$f_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$

$$K_{1,1} = EI\int_0^L \left(-\frac{6}{L^2} + \frac{12x}{L^3}\right)^2 dx$$

$$K_{1,1} = EI\int_0^L \left(\frac{36}{L^4} - \frac{144x}{L^5} + \frac{144x^2}{L^6}\right) dx$$

$$K_{1,1} = EI\left[\frac{36x}{L^4} - \frac{72x^2}{L^5} + \frac{48x^3}{L^6}\right]_0^L$$

$$K_{1,1} = EI\left[\left(\frac{36L}{L^4} - \frac{72L^2}{L^5} + \frac{48L^3}{L^6}\right) - \left(\frac{36 \times 0}{L^4} - \frac{72 \times 0^2}{L^5} + \frac{48 \times 0^3}{L^6}\right)\right]$$

$$K_{1,1} = EI\left[\left(\frac{36}{L^3} - \frac{72}{L^3} + \frac{48}{L^3}\right) - (0)\right]$$

$$K_{1,1} = \frac{12EI}{L^3}$$

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