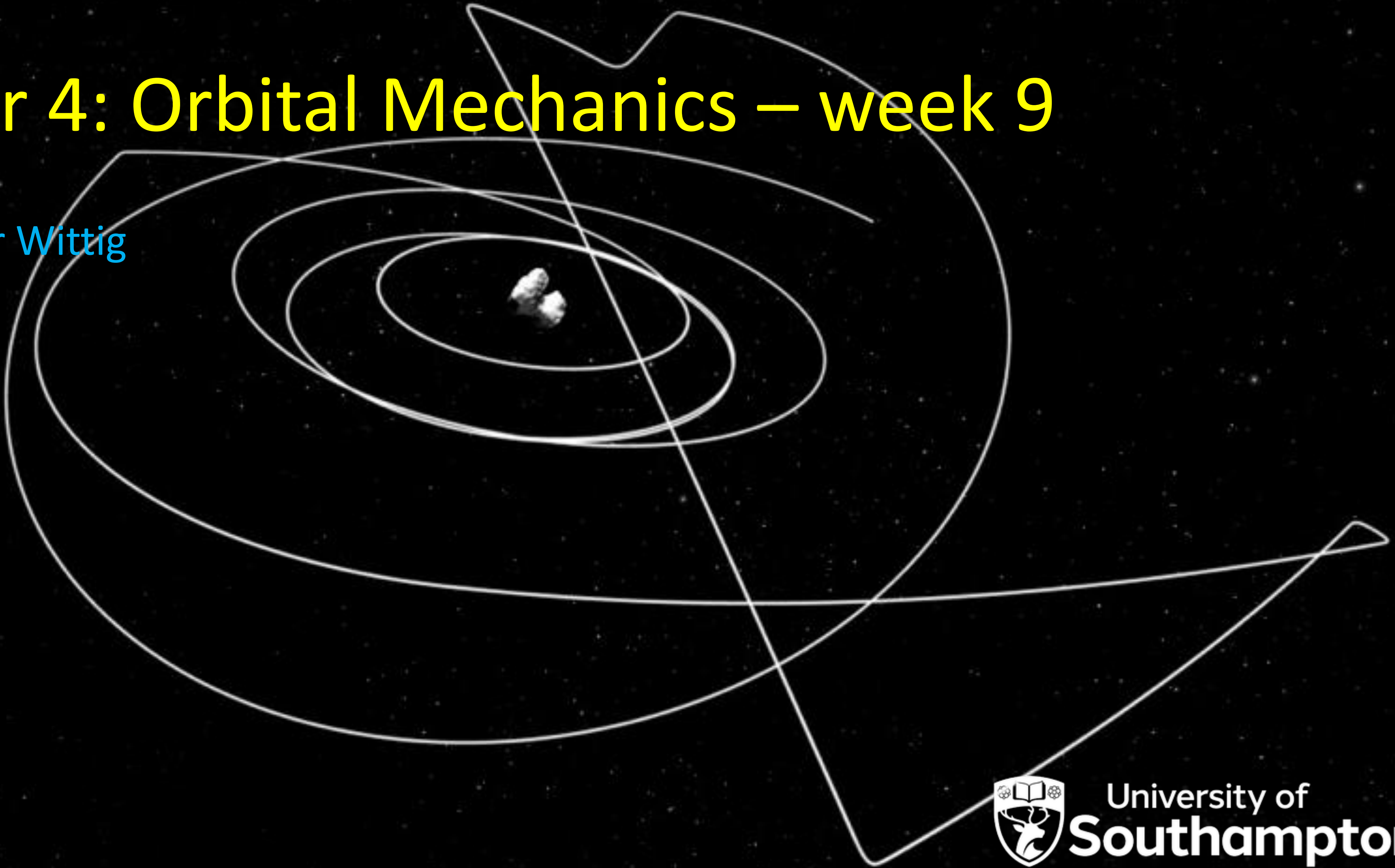
Advanced Astronautics (SESA3039)

Chapter 4: Orbital Mechanics – week 9

Dr. Alexander Wittig



Rosetta lander deployment orbit (ESA)

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Chapter 4 Roadmap

- Week 8: Orbital Motion
  - Math Basics
  - Spherical Trigonometry
  - Keplerian Motion from First Principles
- **Week 9: Orbit Properties**
  - Constants of Motion
  - Eccentricity Vector
  - Conic Sections
- Week 10: Orbit Representation
  - Coordinates
  - Dates & Times
  - Orbital Elements
- Week 11: Time Dependence
  - Eccentric Anomaly
  - Hyperbolic Anomaly
  - Kepler’s Equations

Questions

- Any questions on previous weeks content?

Constants of Motion

Learning outcomes

- Define and recognize dependent and independent constants of motion
- List and prove constants of motion of the 2BP
- Use constants of motion to relate position, velocity, and orbit shape
- Determine orientation of orbit in space

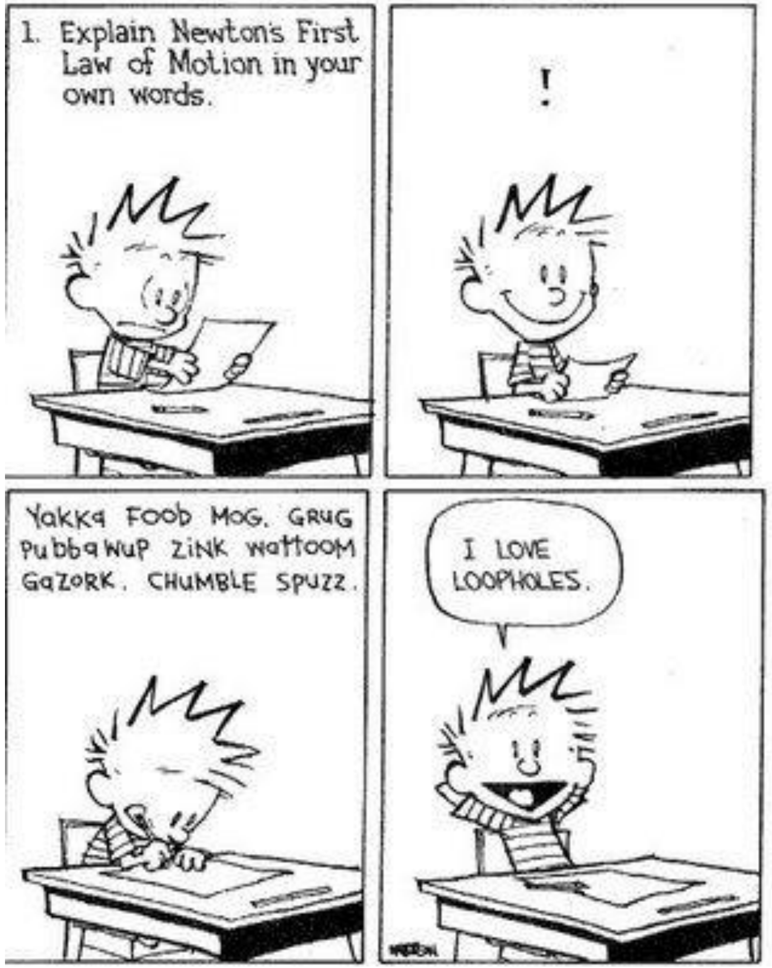
Constants of motion / Integrals of motion

i.e. grand category rather than a single constant

- A *constant of motion* (CoM) in an ODE
  - Can be expressed as functions of position and velocity (*state variables*)
  - Constant along the orbit
  - Can take values in some domain
  - (In)dependence:
    Does one or more constants of motion imply the value of another one?
  - Also called "integral" of motion

hence they can be dependent or independent

describes path not position ∴ constant at all positions. → useful constants of motion can be used to find velocity/position values

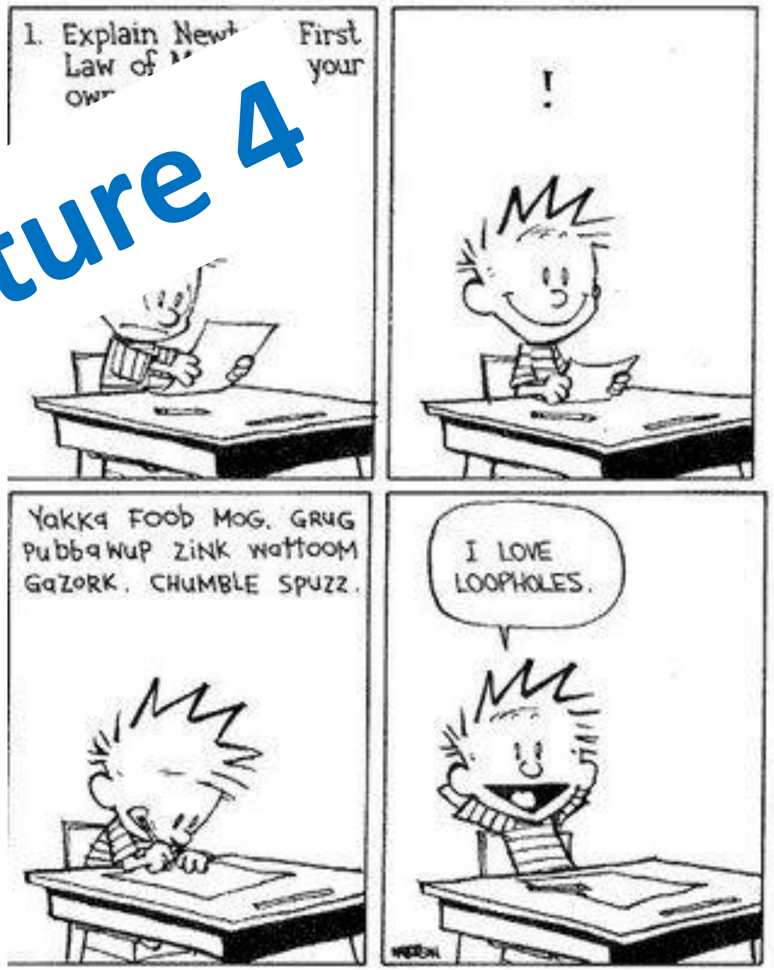


Bill Watterson

A *constant of motion* (CoM) in an ODE

- Can be expressed as functions of position and velocity (*state variables*)
- Constant along the orbit
- Can take values in some domain
- (In)dependence:
  Does one or more con ... on imply the value of another ...
- Also called ... motion

see Chapter 4 Video Lecture 4



Bill Watterson

Dr. Emmy Noether

unknown (Wikipedia)

Career:

- Doctorate in Mathematics (Universität Erlangen, 1907)
- various Lectureships (Universitäten Erlangen, Göttingen, 1907 - 1933)
- Professor (Princeton University, 1933)

these become constants.

Achievements:

- Noether's theorem: linking symmetries in physical equations to conserved quantities
- "Most important women in history of math" (A. Einstein, J. Dieudonné, H. Weyl, N. Wiener)

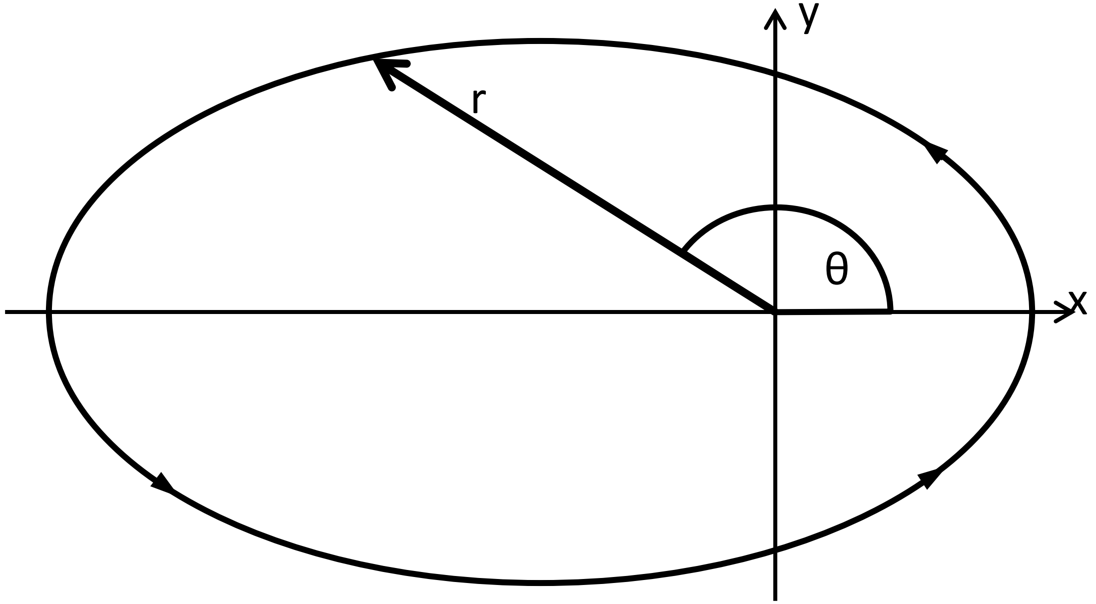
Recap: Central force problem

$$\ddot{\vec{x}} = -\frac{GM}{|\vec{x}|^3}\vec{x} = -\frac{\mu}{|\vec{x}|^3}\vec{x}$$

central force problem

$$r(\theta) = \frac{p}{1 + e\cos(\theta)}$$

Geometrical description of solution



Examples of CoM in 2BP

NASA

$\Rightarrow$ conserved quantities.

- Energy
- Angular momentum
- Areal velocity
- Orbit shape parameters (eccentricity, semi-major axis, orientation)
- …

Specific angular momentum

- Vectorial quantity with 3 independent components

$$\vec{h} = \vec{r} \times \dot{\vec{r}}$$

- Norm is scalar angular momentum

$$|\vec{h}| = h = r^2 \dot{\theta}$$

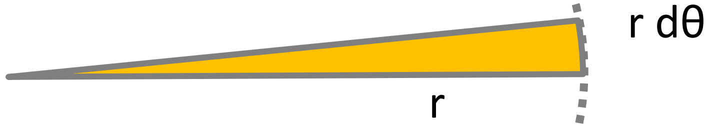
- Orthogonal to position and velocity at all times: planar motion!

$$\vec{h} \perp \vec{r}$$

$$\vec{h} \perp \dot{\vec{r}}$$

Areal velocity

$$dA = \frac{1}{2} r \; r \; d\theta = \frac{1}{2} r^2 \frac{d\theta}{dt} dt = \frac{h}{2} dt$$



areal velocity is not independent from angular momentum, hence it's irrelevant

- Same as h, no new information
- Kepler's second law:
  radial vector sweeps out equal area in equal time

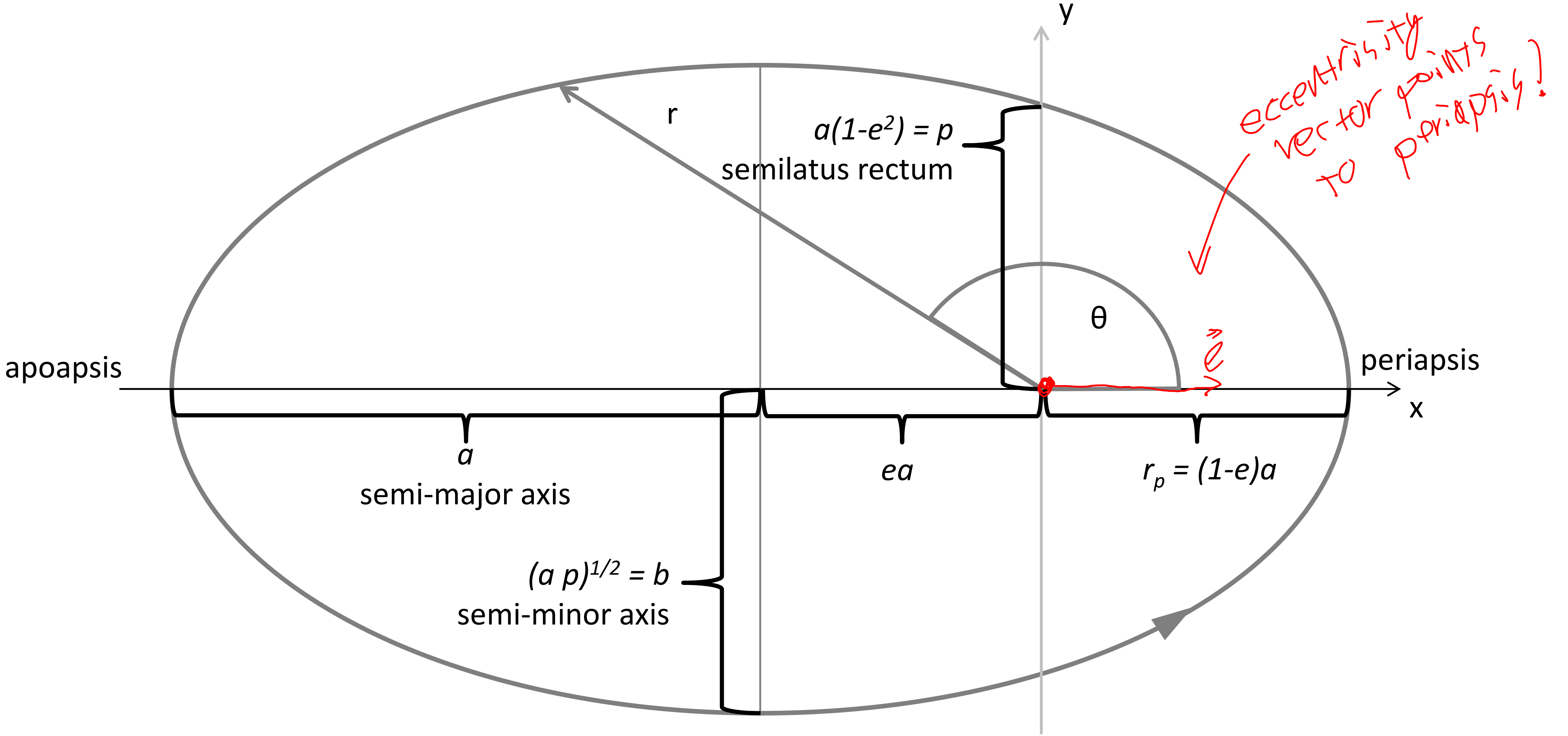
Specific energy

$$\epsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} = -\frac{\mu^2}{2h^2}(1 - e^2)$$

- Vis-viva or energy equation
- Obviously time independent: kinetic + potential energy
- Relates kinematic quantities to orbit shape!

:: really useful

Geometric quantities



All of these geometric properties are constants of motion, too!

can be used with prev to fully define orbit

$$\vec{e} = \frac{\dot{\vec{r}} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}$$

Exercise

Expand the cross product to write this only in terms of vectors r and v.

- Is constant of motion $\dot{\vec{e}} = 0$
- Lies in plane of motion $\vec{e} \cdot \vec{h} = 0$
- Has norm equal to eccentricity $|\vec{e}| = e$
- Points along line from apoapsis (or focal point) to periapsis

New kid in town: the eccentricity vector

$$\vec{e} = \frac{\dot{\vec{r}} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}$$

Exercise

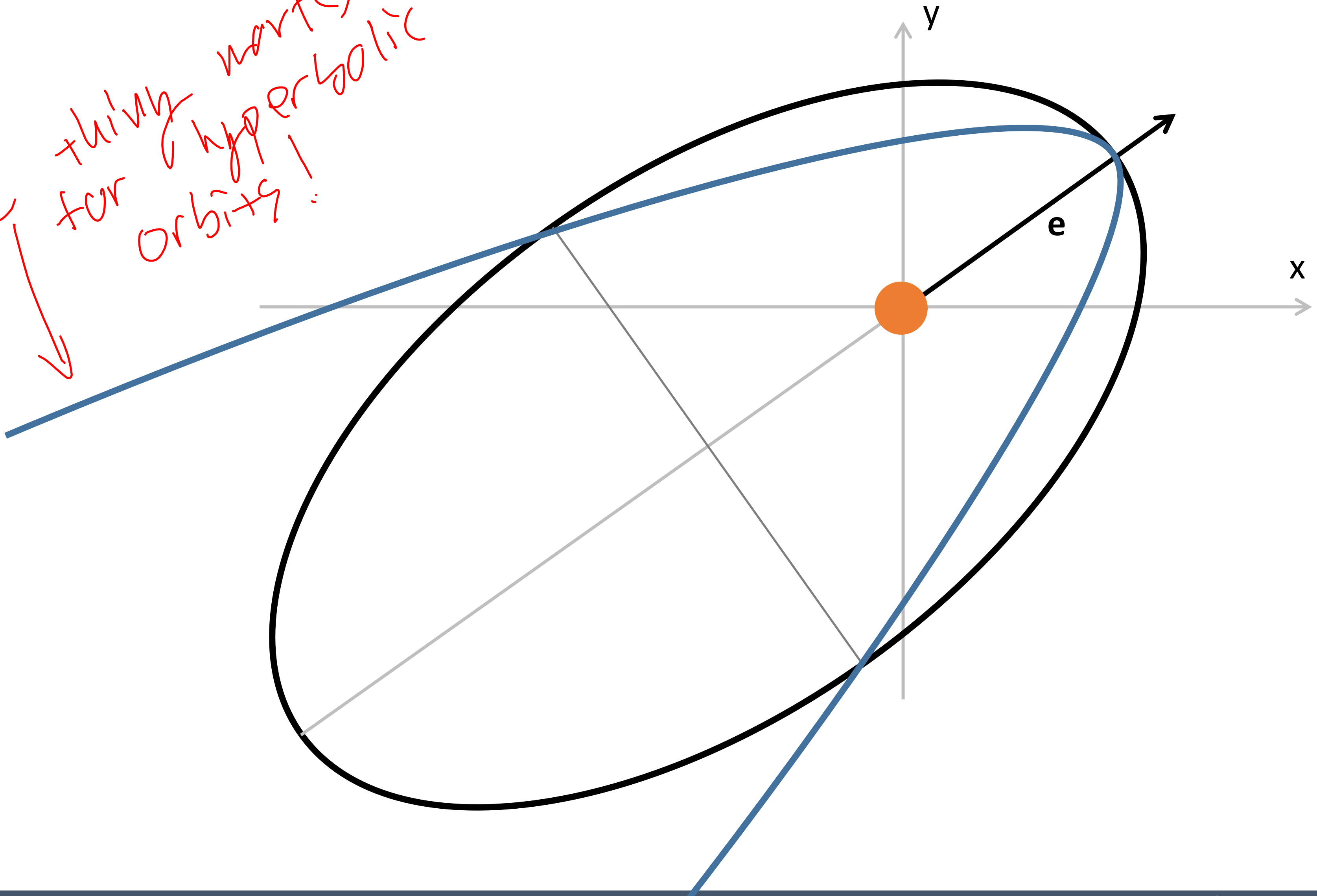
Expa… cross product …nly in terms …s r and v.

see Chapter 4 Video Lecture 5

- Is constant of motion $\dot{\vec{e}} = 0$
- Lies in plane of motio… $\vec{e} \cdot \vec{h} = 0$
- Has norm …ccentricity $|\vec{e}| = e$
- Points along line from apoapsis (or focal point) to periapsis

Eccentricity vector

$\vec{e}$ thing works for hyperbolic orbits!



Fight for independence

Too many constants: 2 vectors with 3 each,
3 geometric ones (a, p, e), 1-2 kinematic

- Can have maximum of 5 independent ones (else everything is constant!)
- Indeed the 2BP has 5 independent constants of motion (hence “*integrable*”)
- All other constants are interdependent

Fight for independence

- 5 independent constants from $\vec{e}, \vec{h}$
  - Interdependent because $\vec{e} \cdot \vec{h} = 0$

are perpendicular!

- Others can be derived from these, e.g.
  - scalar eccentricity $e = |\vec{e}|$
  - scalar angular momentum $h = |\vec{h}|$
  - specific energy $\epsilon = \dfrac{\mu^2(e^2-1)}{2h^2}$
  - semi-latus rectum $p = \dfrac{h^2}{\mu}$

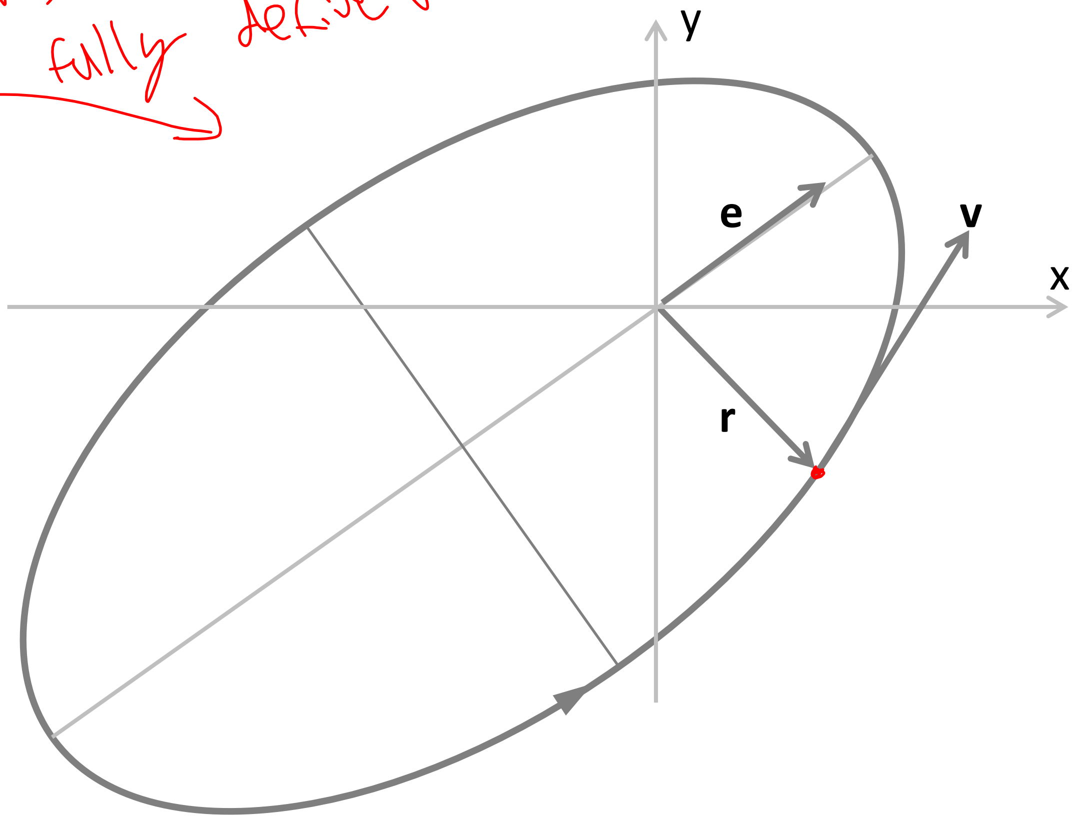
dependent on these

Where is the apogee of a satellite in orbit around Earth (μ=3.986·10^5 km^3/s^2) with state:

all that's needed to fully define orbit.

$$\vec{r} = \begin{pmatrix} 20000 \\ 10000 \\ 0 \end{pmatrix} \text{km}$$

$$\vec{v} = \begin{pmatrix} -3.6 \\ 3 \\ 0 \end{pmatrix} \text{km/s}$$



Sketch! Don't know real orientation and shape yet!

Summary

- Two vectorial constants $\vec{e}, \vec{h}$
- Provide full information about solution
- Other physical, kinematic and geometric constants can be derived from there
- Relations between constants allow conversion from any set of known constants to another

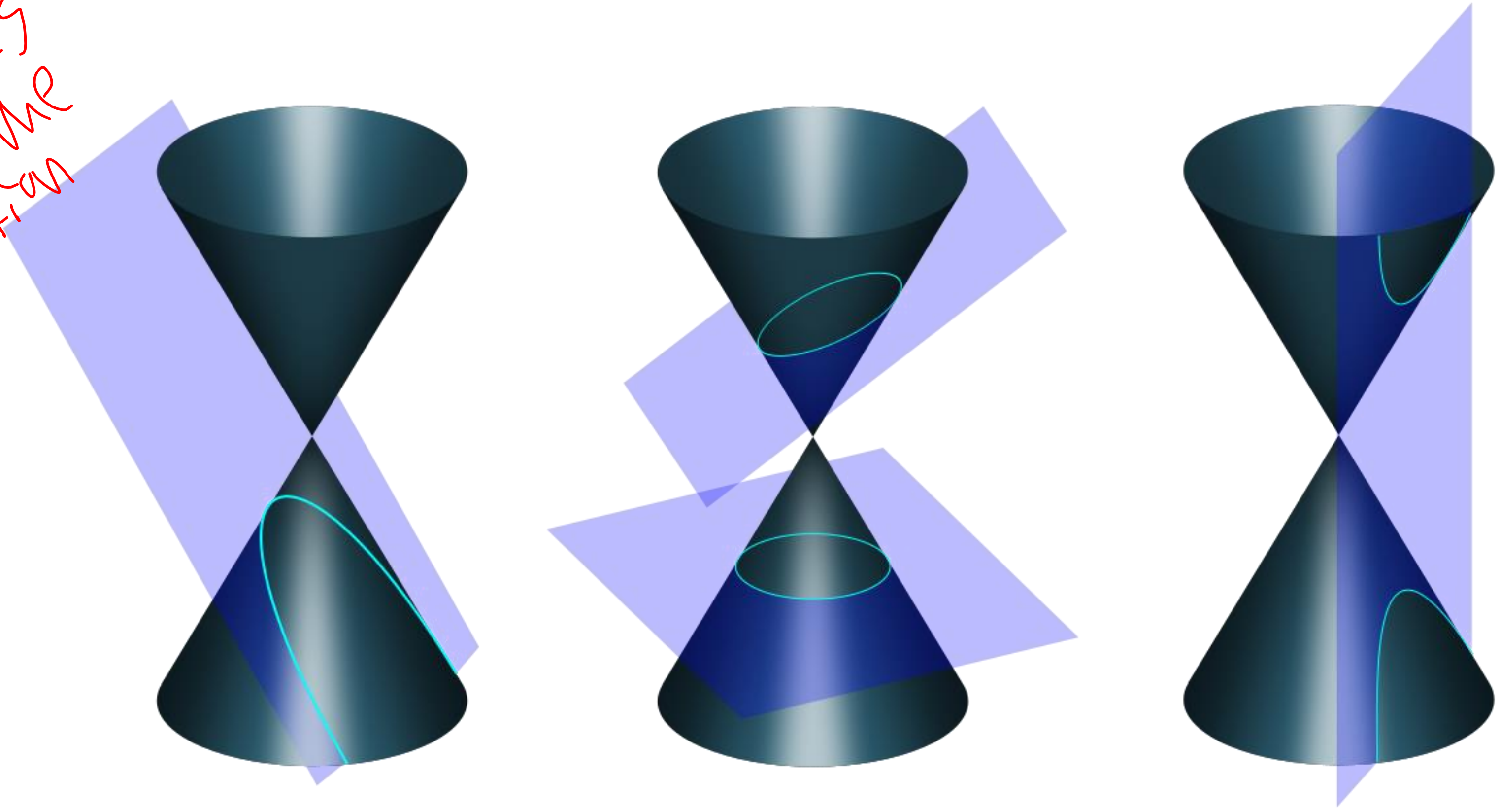
Conic Sections

Geometry of the solution

- Solution of the two body problem: Conic Section

don't need notes in volume just explanation

$$r(\theta) = \frac{p}{1 + e\cos(\theta)}$$



Wikipedia/Pbroks13

https://upload.wikimedia.org/wikipedia/commons/9/9a/Conic_section_interactive_visualisation.svg

Types of Conics

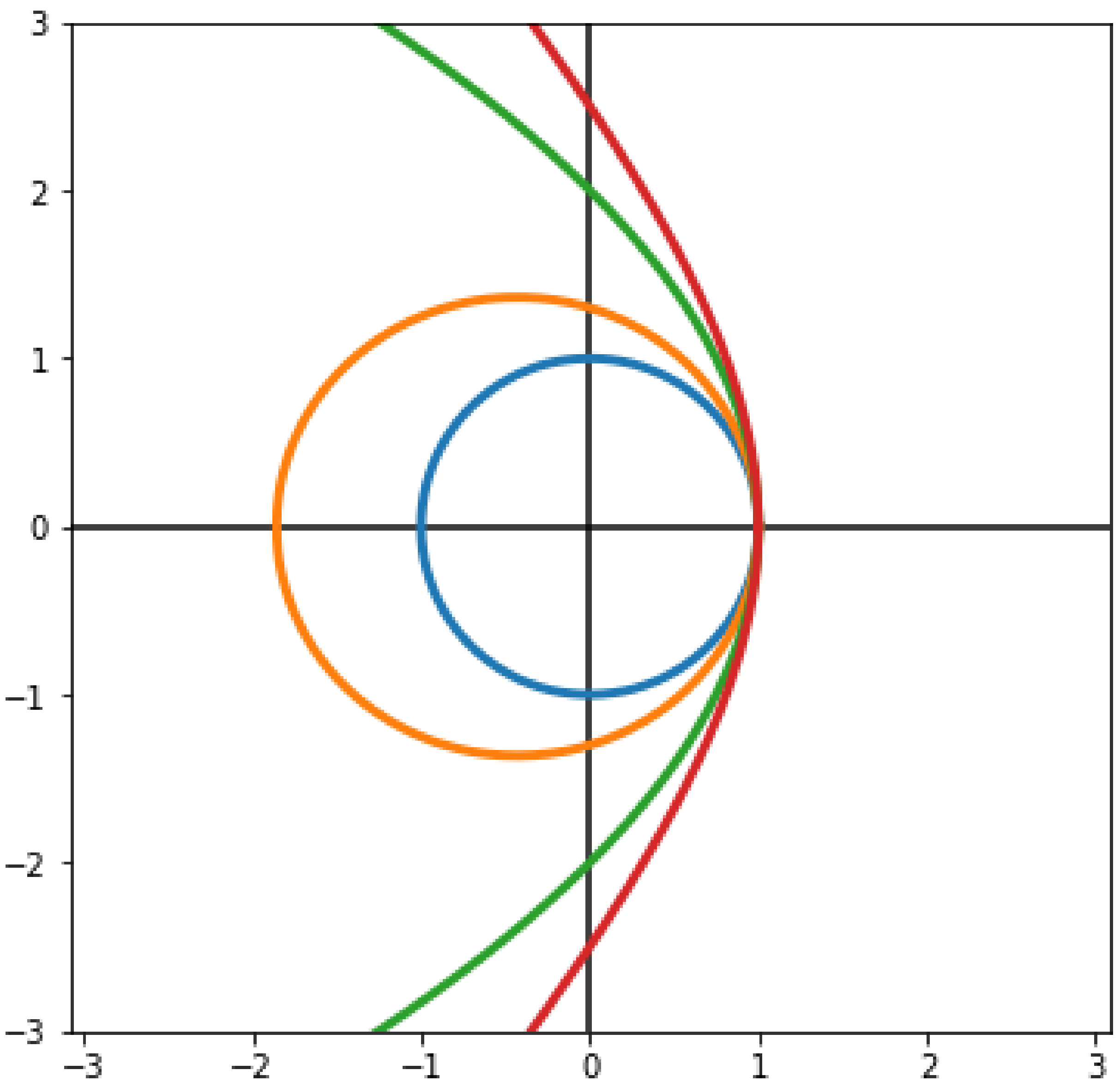
- $e=0$ circle
- $0<e<1$ ellipse
- $e=1$ parabola
- $e>1$ hyperbola

normal orbits

escape trajectory

$V_\infty = 0$

$V_\infty \neq 0$

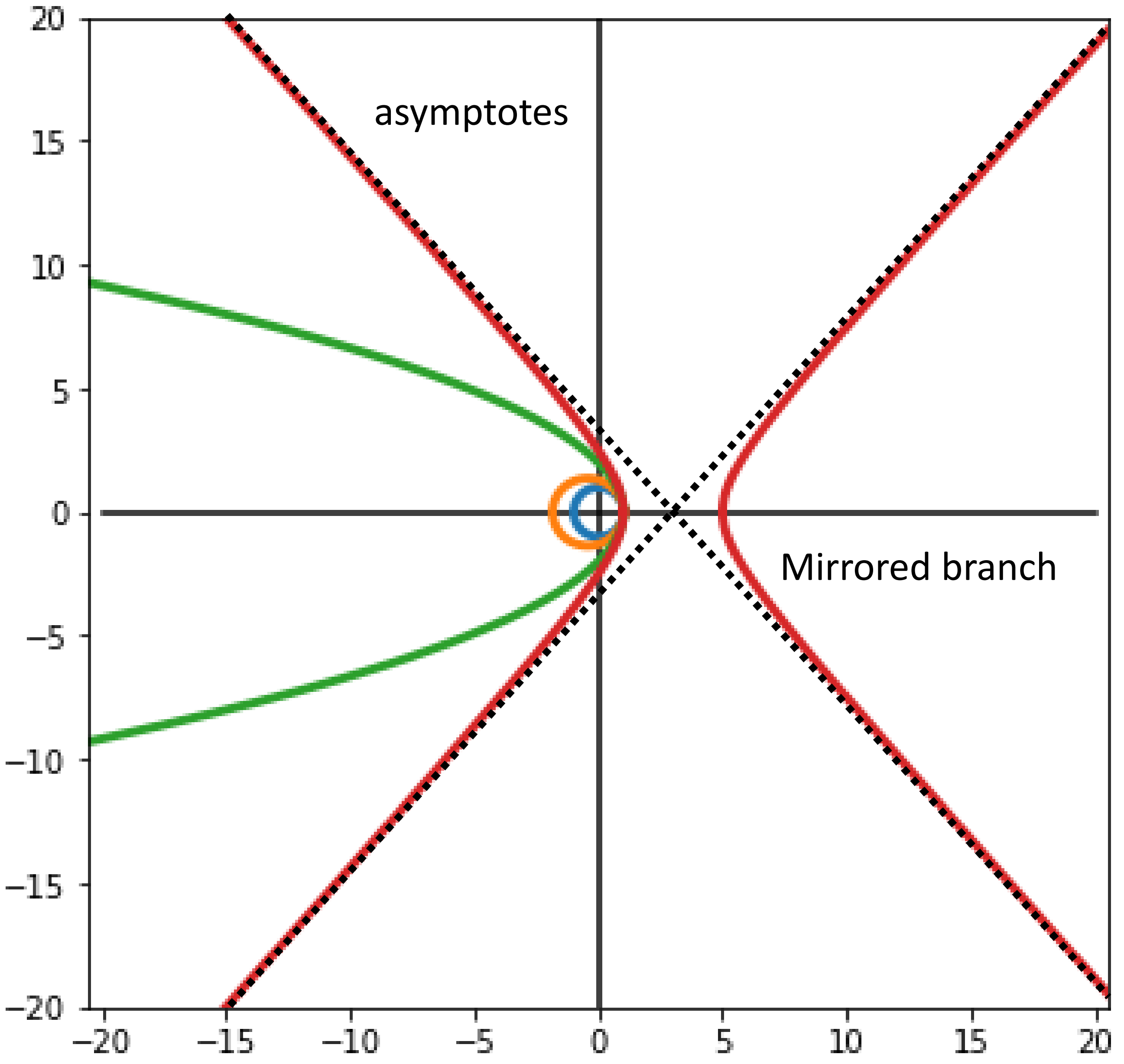


Types of Conics

- $e=0$ circle
- $0<e<1$ ellipse
- $e=1$ parabola
- $e>1$ hyperbola

not realistic, not relevant

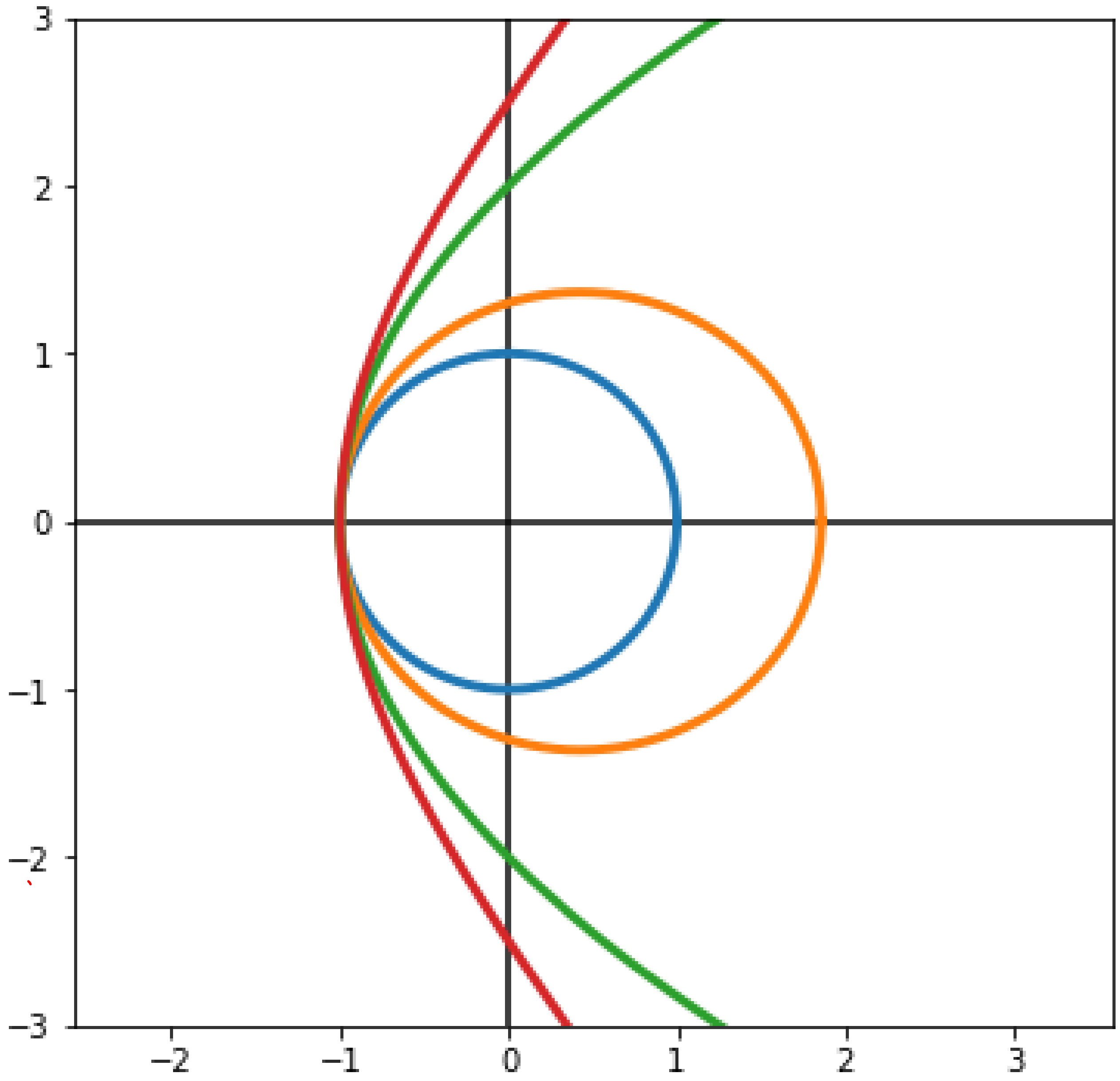
What happens for $e<0$??



- *e=0* circle
- *-1<e<0* ellipse
- *e=-1* parabola
- *e<-1* hyperbola

$$r(\theta) = \frac{p}{1 + e\cos(\theta)} = \frac{p}{1 \boxed{- e\cos(\theta + 180^\circ)}}$$

negative eccentricity flips the periapsis and apoapsis!

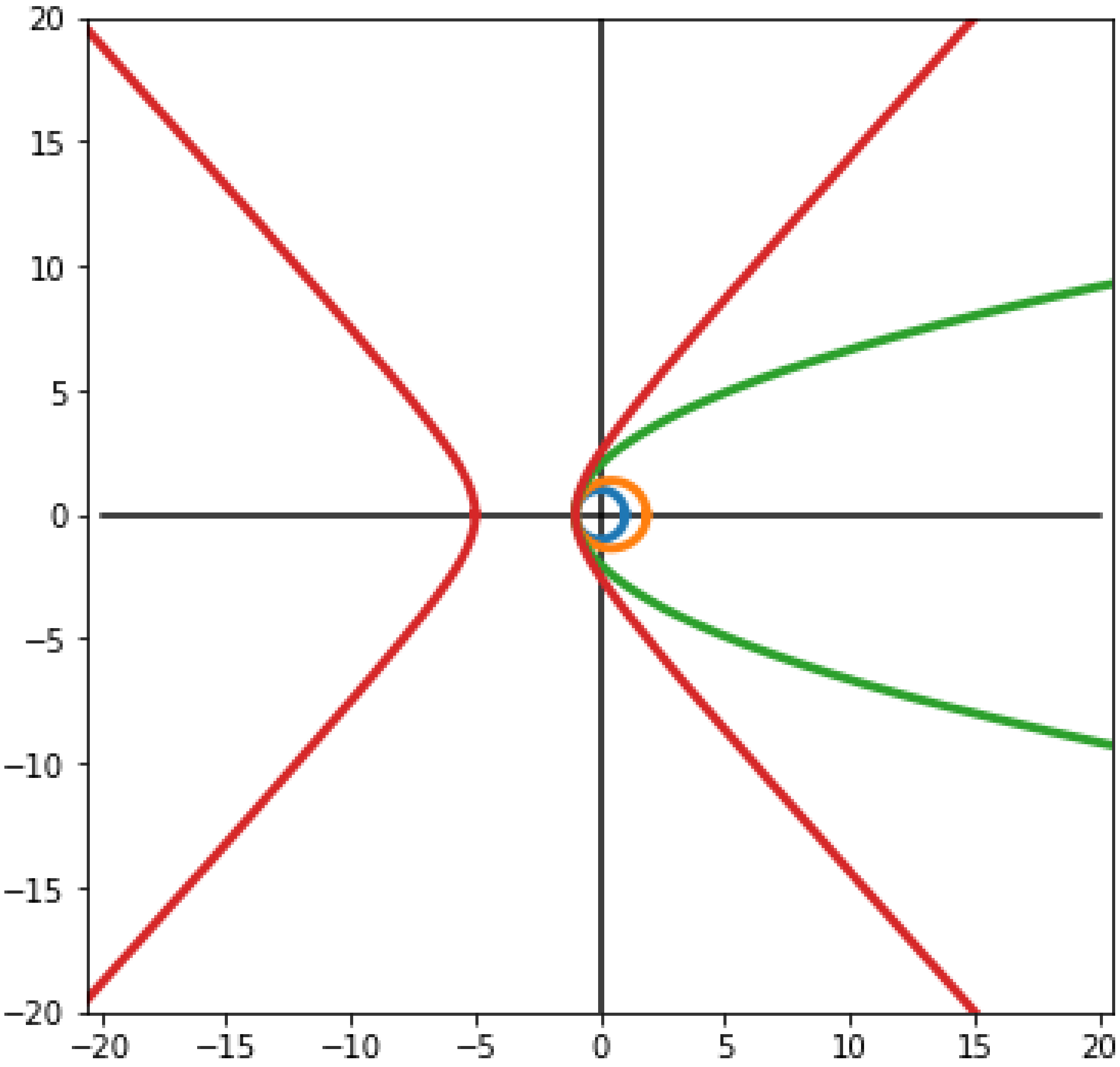


Types of Conics

- *e=0* circle
- *-1<e<0* ellipse
- *e=-1* parabola
- *e<-1* hyperbola

- No new information:
  same shapes, just mirrored
- Can restrict to positive *e*

for convenience, though negative works



Circular orbits

- Eccentricity *e=0*
- Constant radius *p*

$$r(\theta) = \frac{p}{1 + e\cos(\theta)} = p$$

$$a = p$$

$$r_p = p$$

$$r_a = p$$

- From energy equation:

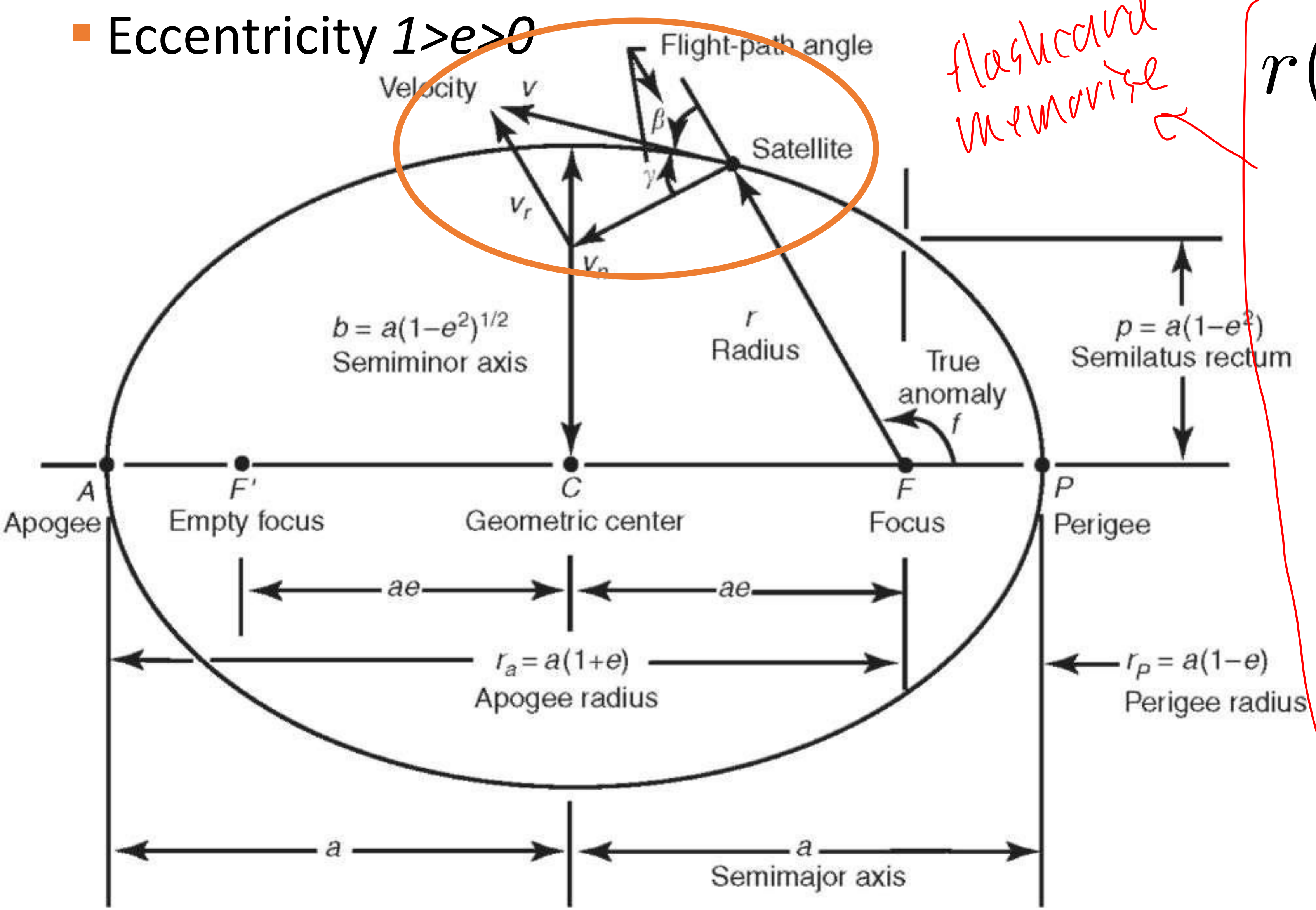
$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{r}\right) = \frac{\mu}{r}$$

- From energy equation:

$$\epsilon = -\frac{\mu}{2a} < 0$$

Elliptic orbits

- Eccentricity *1>e>0*



flashcard memorise

$$r(\theta) = \frac{p}{1 + e\cos(\theta)}$$

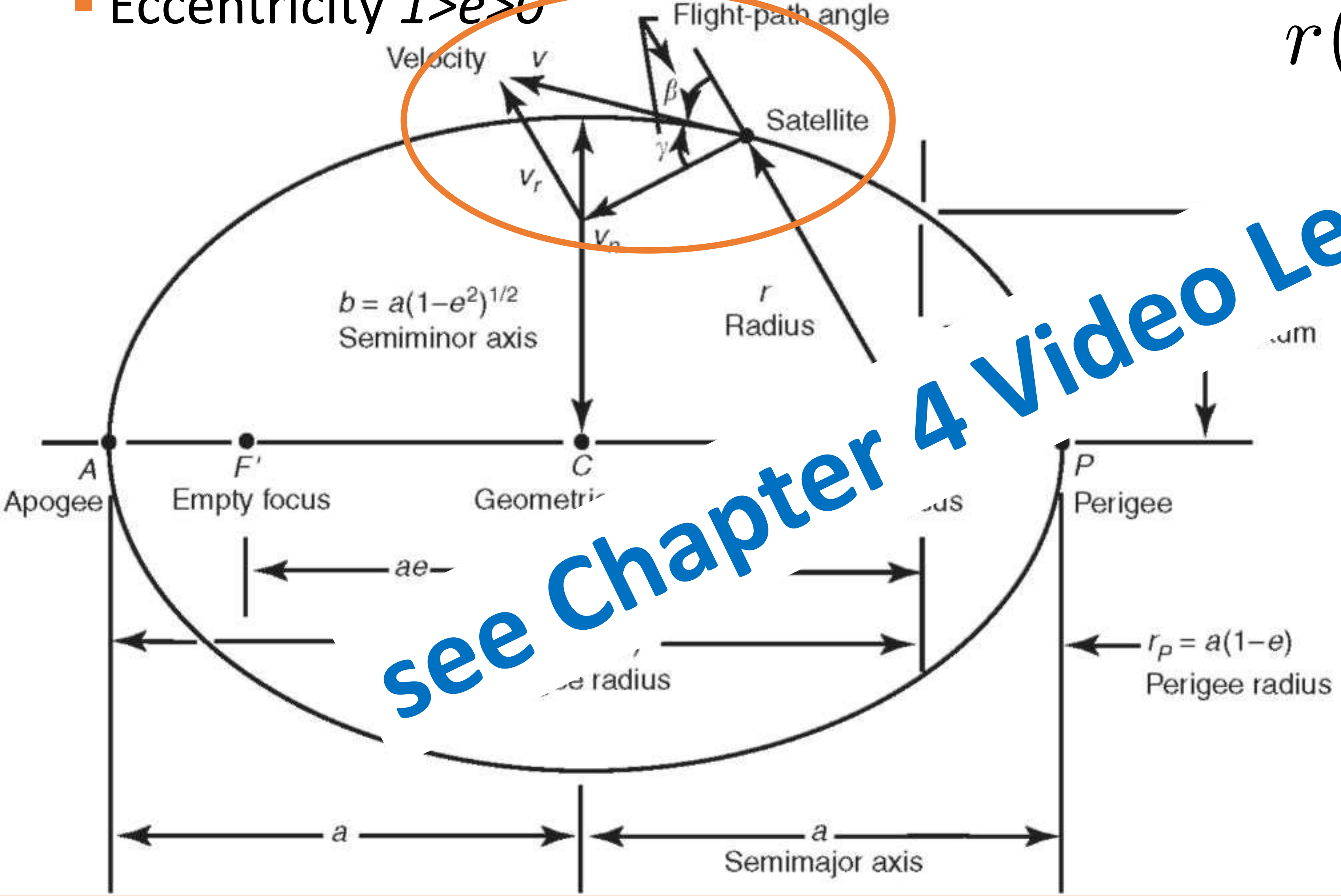
$$a = \frac{p}{1 - e^2}$$

$$r_p = a(1 - e)$$

$$r_a = a(1 + e)$$

$$\epsilon = -\frac{\mu}{2a} < 0$$

- Eccentricity *1>e>0*



see Chapter 4 Video Lecture 6

$$r(\theta) = \frac{p}{1 + e\cos(\theta)}$$

$$a = \frac{p}{1-e^2}$$

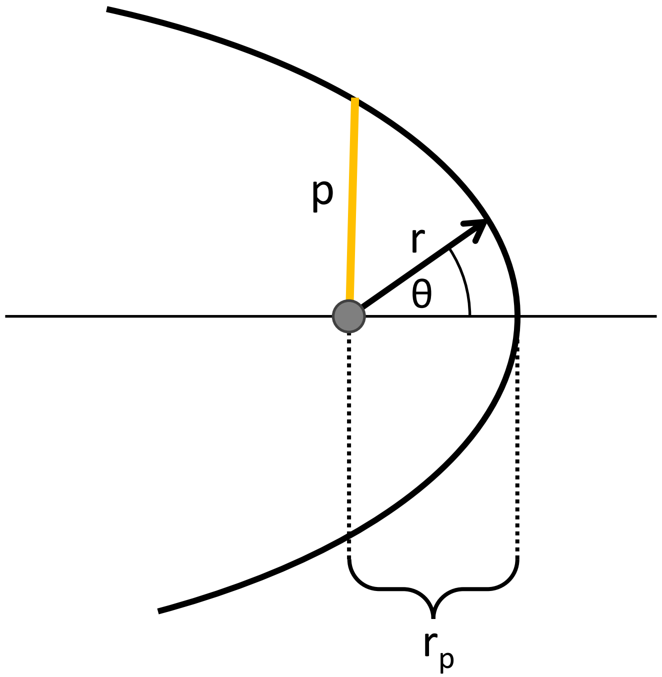
$$r_p = a(1-e)$$

$$r_a = a(1+e)$$

$$\epsilon = -\frac{\mu}{2a} < 0$$

Parabolic orbits

- Eccentricity *e=1*



$$r(\theta) = \frac{p}{1+\cos(\theta)}$$

$$a = \frac{p}{1-e^2} \rightarrow \pm\infty$$

$$r_p = p/2$$

$$\epsilon = 0$$

$$r(\theta \rightarrow 180^\circ) \rightarrow \infty$$

- Minimum escape velocity

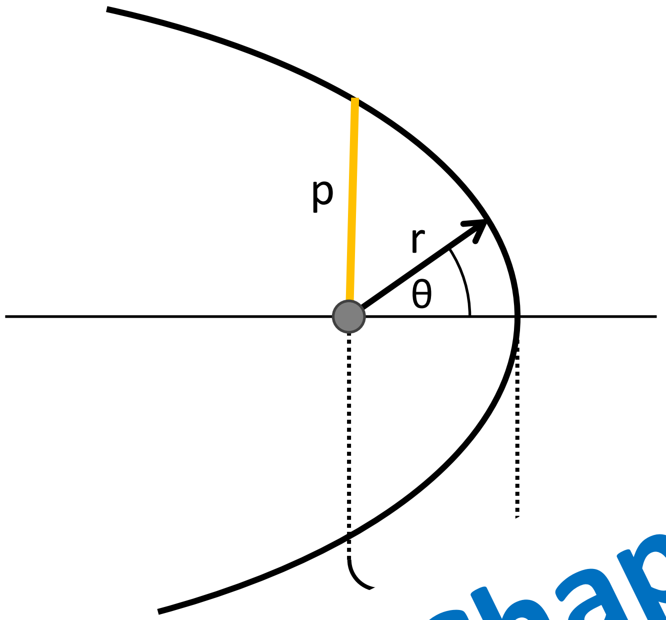
$$v_{esc,min} = \sqrt{\frac{2\mu}{r}}$$

equivilent to a parabolic orbit!

- Velocity at infinity

$$v_\infty = 0$$

- Eccentricity *e=1*



$$r(\theta) = \frac{p}{1 + \cos(\theta)}$$

$$a = \frac{p}{1 - e^2} \to \pm\infty$$

$$r_p = p/2$$

$$\epsilon = 0$$

$$r(\theta \to 180^\circ) \to \infty$$

see Chapter 4 Video Lecture 6

- Minimum escape velocity

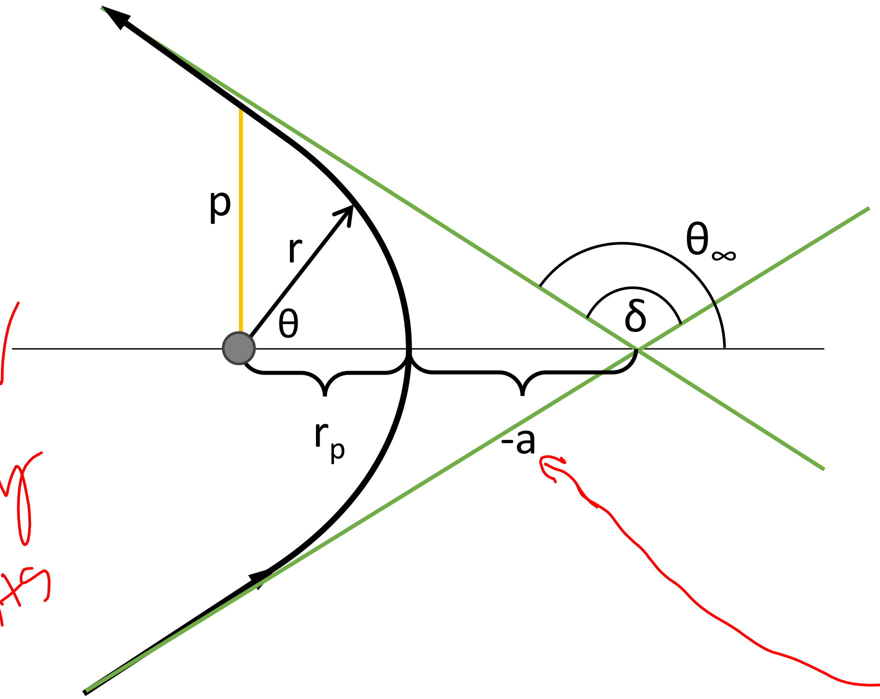
$$v_{esc,min} = \sqrt{\frac{2\mu}{r}}$$

- Velocity at infinity

$$v_\infty = 0$$

Hyperbolic orbits

- Eccentricity *e>1*



used for gravity assists

physical interpretation

$$r(\theta) = \frac{p}{1 + e\cos(\theta)}$$

$$a = \frac{p}{1-e^2} < 0$$

$$r_p = p/(1+e)$$

$$\epsilon = -\frac{\mu}{2a} > 0$$

- hyperbolic excess velocity

$$v_\infty = \sqrt{-\mu/a}$$

- escape angle

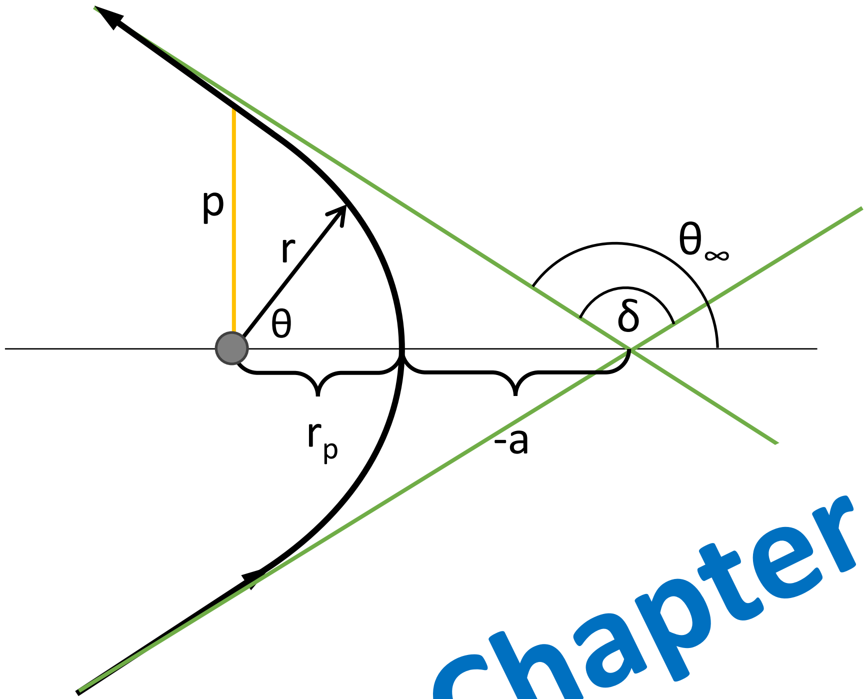
$$\theta_\infty = \arccos(-1/e)$$

- deflection angle

rotation change from incoming to outgoing trajectory

$$\delta = 2\arcsin(1/e)$$

- Eccentricity *e>1*



- hyperbolic excess velocity
- escape angle
- deflection angle

$$r(\theta) = \frac{p}{1+e\cos(\theta)}$$

$$a = \frac{p}{1-e^2} < 0$$

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$$v_\infty = \sqrt{-\mu/a}$$

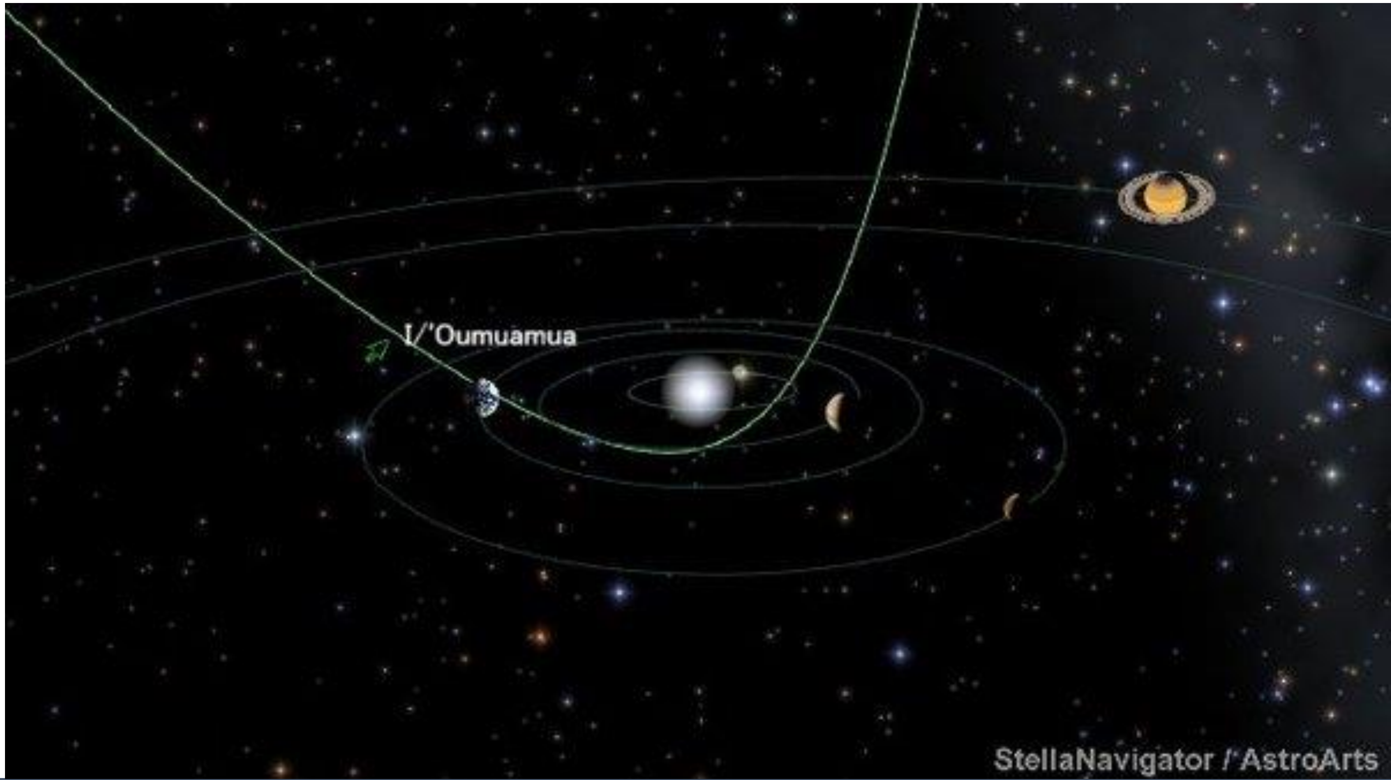
$$\theta_\infty = \arccos(-1/e)$$

$$\delta = 2\arcsin(1/e)$$

see Chapter 4 Video Lecture 6

Summary

- Orbit shape is defined by eccentricity
- Expressions with semi-latus rectum p are always valid
- Expressions with a need to be treated with care
- Each shape has different geometric properties
- Negative eccentricity just mirrors orbits
- Know relations between shape and kinetic quantities!



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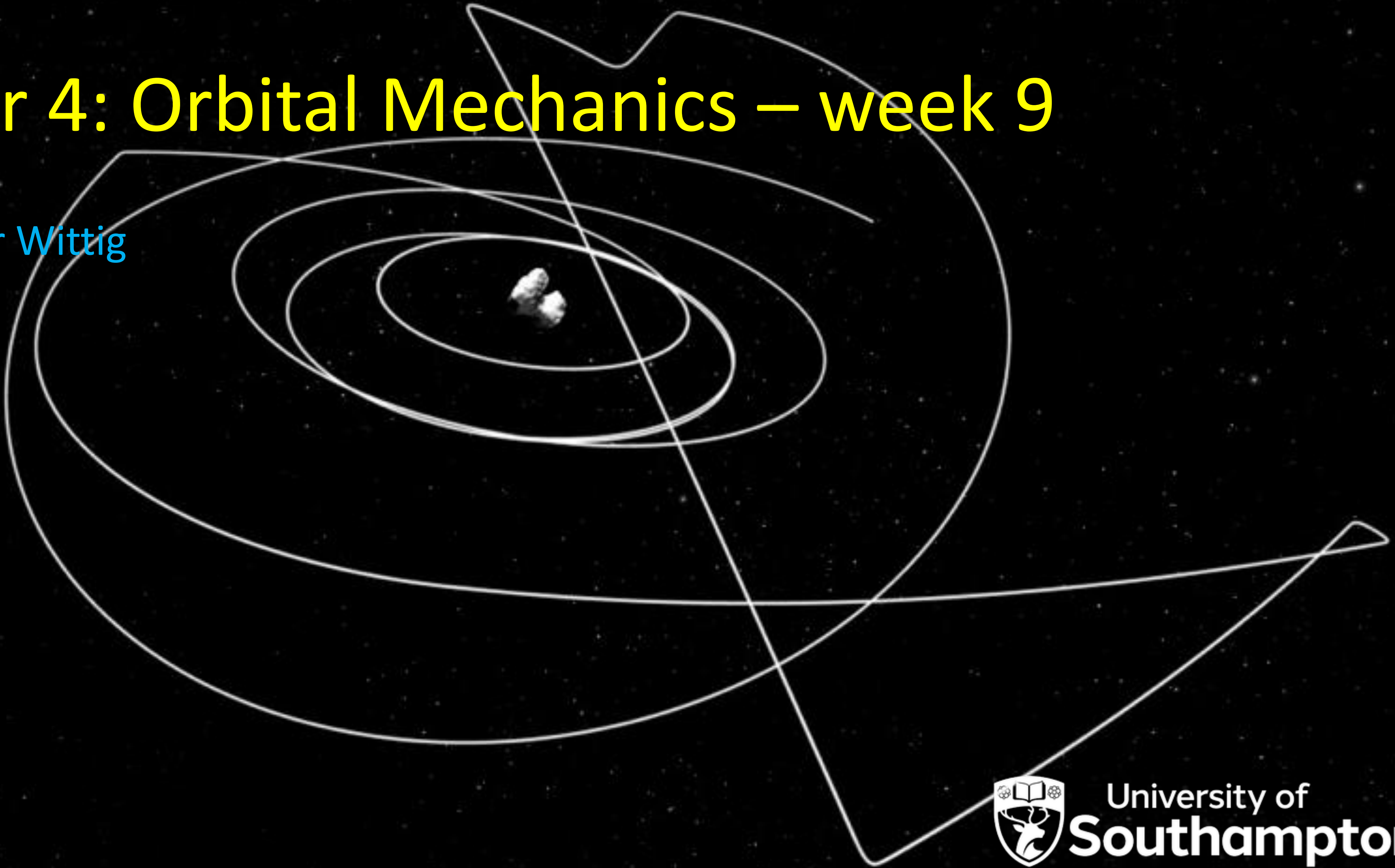

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