FEEG3001 / SESM6047 Notes for Finite Element Analysis in Solid Mechanics

Part 4: Dynamics of Conservative Mechanical Systems in Finite Element Analysis

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1. Introduction to Dynamics

As a reminder, this module presents FEA using an energy approach, as an alternative to using Free Body Diagrams. The aim is to express your 'problem' or model of a structure or system in a general way that is convenient to solve:

- We express the structure or system's energy in a particular form, a quadratic equation of its Degrees of Freedom (DoF)
- from which we can express its characteristic Stiffness Matrix,
- which lets us use the Principle of Minimum Total Potential Energy (PMTPE) to state the Generalised Equation of Equilibrium. Then,
- by applying boundary conditions we can solve it to approximate the structure or system's behaviour.

So far we have only considered statics problems, where we apply forces, moments or torques, and calculate deflections, and then strains and finally stresses. This fourth part of the module will show how we can apply the same principles to analyse dynamics, specifically free vibration. You might use this in design to avoid a structure from failing due to resonance.

2. Analogy between PMTPE in statics and the Lagrangian in dynamics

In statics, we claimed that Newton's laws for equilibrium (the zero summation of forces) is equivalent to PMTPE. However, if we are interested in dynamic movement, this principle is inadequate: now the summation of forces is the product of mass and acceleration. However, we start to apply this principle in a similar way, known as *Hamilton's principle*.

For all conservative systems (where no work is done externally, or is extracted or dissipated), we define the 'Lagrangian' of the system[1], L, where:

$$L = T - U$$

where U represents the system's stored elastic potential strain energy, and T represents the kinetic energy. Then, like we used PMTPE to compare imaginary deformations of a structure, we apply Hamilton's principle to compare imaginary motions (bulk movement or vibration), searching for one true motion result, for which:

$$\delta \int_{t_1}^{t_2} L dt = 0$$

Hamilton's principle applies the same variational principle to state that the integral of the Lagrangian with respect to time will be higher for all the imagined motions than for the true motion. Remember, δ indicates a comparison between variations, not increments in a variable. Hamilton's principle is general, and valid for linear and non-linear problems. Therefore, for all the choices of possible motion, the true motion is the one which gives the lowest value of the action integral $\int_{t_1}^{t_2} L dt$. (Without proof), this means the system must satisfy Lagrange's equations, which we will see next.

[1] Note this is not the same as the total energy we used in statics.

3. Example: a Single DoF Mass-Spring Oscillator

The following example is a cart or truck which can translate in a single DoF / generalised coordinate, and is supported by a grounded spring. No external work is done, so intuition tells us it will either sit at rest or vibrate freely. Its configuration can be described by one generalised coordinate $q(t)$ and its derivative, the generalised velocity $\dot{q}(t)$.

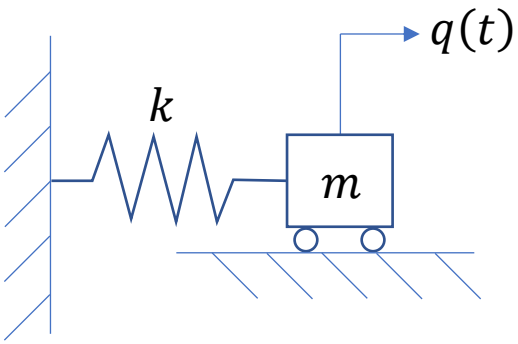

Figure 1: (left) a mass-spring oscillator: a mass which can translate in a single DoF, supported by a grounded spring.

The strain energy is the same as in our spring examples:

$$U = {}^1\!/_2\, kq^2$$

Its kinetic energy is given by:

$$T = {}^1\!/_2\, m\dot{q}^2 \text{ where } \dot{(\cdot)} = \frac{d}{dt}(\cdot)$$

Therefore our Lagrangian is given by:

$$L = {}^1\!/_2\, m\dot{q}^2 - {}^1\!/_2\, kq^2$$

Hamilton's principle defines Lagrange's equation. For a single DoF this is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Applying this to our current problem gives:[2]

$$\frac{d}{dt}(m\dot{q}) + kq = 0$$

$$m\ddot{q} + kq = 0$$

You can see from this simple case how the mass is able to sit in a range of different positions where its energy is shared between kinetic and potential components. For example, it has:

- maximum kinetic energy when the spring is relaxed, and no elastic potential energy is stored; and
- maximum elastic potential energy when the spring is at its maximum extension or compression at its extremes of movement, when the mass is stationary and kinetic energy is zero.

As in the statics implementation of FEA, we solved this problem without a free body diagram, but it could be shown that Lagrange's equations are equivalent to Newton's Law, and Euler's equations of rotational dynamics, combined.

[2] Remember these are partial derivatives, so q is constant for the partial derivative by velocity $\frac{\partial}{\partial \dot{q}}$

4. Example: a Multiple DoF System of Masses supported by Springs

Consider a conservative system with multiple DoFs which stores energy during motion:

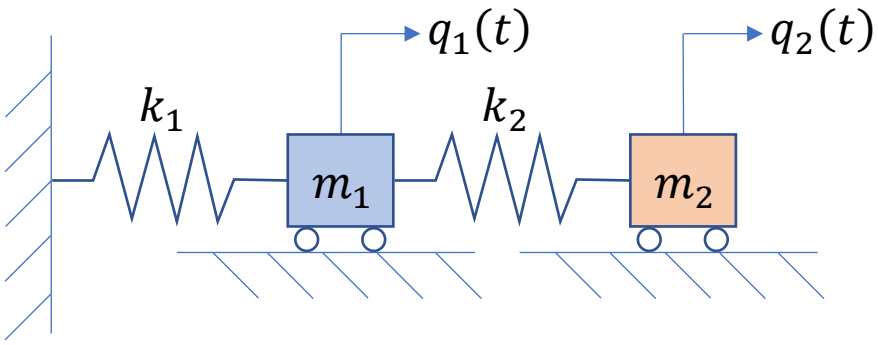

Figure 2: (left) a system of masses which can translate in a single direction, joined by springs and supported by a grounded spring. Note, it has multiple degrees of freedom, even though they are all in only one direction.

the Lagrangian is given as a function of the N generalised coordinates and n generalised velocities:

$$L = L(q_1, q_2, \dots q_n, \dot{q}_1, \dot{q}_2, \dots \dot{q}_n)$$

Hamilton's principle states that

$$\delta \int_{t_1}^{t_2} (T - U) dt = 0$$

so Lagrange's equation in general is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, i = 1,2, \dots N$$

Notice we have N Lagrange's equations for an N degree of freedom system.

The energies take quadratic forms which are recognisable from static FEA:

$$U = {}^1\!/_2\, k_1 {q_1}^2 + {}^1\!/_2\, k_2 (q_2 - q_1)^2 = {}^1\!/_2\, \{q\}^T [K] \{q\}$$

and from a generalisation of kinetic energy, ${}^1\!/_2\, mv^2$:

$$T = {}^1\!/_2\, m_1 {\dot{q}_1}^2 + {}^1\!/_2\, m_2 {\dot{q}_2}^2 = {}^1\!/_2\, \{q\}^T [M] \{q\}$$

where $[K]$ is our system's Stiffness Matrix and $[M]$ is its Mass Matrix.

Our Lagrangian is calculated as:

$$L = \left[{}^1\!/_2\, m_1 {\dot{q}_1}^2 + {}^1\!/_2\, m_2 {\dot{q}_2}^2\right] - \left[{}^1\!/_2\, k_1 {q_1}^2 + {}^1\!/_2\, k_2 (q_2 - q_1)^2\right]$$

Solving Lagrange's equations for the generalised coordinates q_i one by one. We have two generalised coordinates, so we will have two equations.

For q_1:

$$m_1 \ddot{q}_1 + (k_1 + k_2) q_1 - k_2 q_2 = 0$$

For q_2:

$$m_2 \ddot{q}_2 - k_2 q_1 + k_2 q_2 = 0$$

These are linear, ordinary, homogeneous differential equations with constant coefficients. They are also *coupled*. We can collect these and organise them into matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Simply, the implication of Hamilton's Principle is that if our Lagrangian is the difference of two quadratics, which correspond with our Kinetic and Elastic Potential energies, without any integration we can state our Governing Equation of Motion using the Stiffness and Mass Matrices:

$$[M]_{n\times n}\{\ddot{q}\}_{n\times 1} + [K]_{n\times n}\{q\}_{n\times 1} = \{0\}_{n\times 1}$$

5. Solving the Dynamics of Linear Oscillatory Systems in Absence of External Force

For our 1-DoF system, our generalised equation of motion was:

$$m\ddot{q} - kq = 0$$

Any homogeneous ordinary differential equation with constant coefficients has a solution of the form:

$$q(t) = e^{st}$$

and substitution gives:

$$ms^2 e^{st} + ke^{st} = 0$$

$$(ms^2 + k)e^{st} = 0$$

Our trivial solution is $e^{st} = 0$, where there is no motion, so instead our characteristic equation is:

$$ms^2 + k = 0$$

$$s = \pm i\sqrt{\frac{k}{m}}$$

giving our solutions for the generalised coordinates:

$$q(t) = e^{i\sqrt{\frac{k}{m}}t}$$

$$q(t) = e^{-i\sqrt{\frac{k}{m}}t}$$

The general solution is a linear combination of these two independent solutions:

$$q(t) = Ae^{i\sqrt{\frac{k}{m}}t} + Be^{-i\sqrt{\frac{k}{m}}t}$$

We call $\sqrt{\frac{k}{m}} = \omega$, the natural frequency of the system, and we would find A and B from the initial conditions. Recalling Euler's identity, which states $e^{i\theta} = \cos\theta + i\sin\theta$ we can assert that $q(t)$ must be real, and

$$q(t) = C\cos(\omega t) + D\sin(\omega t)$$

For multi-DoF systems, we saw that the equations of motions are coupled and take the form:

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$$

Systems like this display *synchronous motion*, where all the generalised coordinates move such that their extremities and equilibria are reached at the same time. You could show that synchronous motion is simple harmonic, i.e.[3]

$$\{q(t)\} = \{A\}\sin(\omega t)$$

where $\{A\}$ is a list (vector) of amplitudes, conventionally referred to as $\{u\}$. Therefore,

$$\{q(t)\} = \{u\}\sin(\omega t)$$

$$\{\dot{q}(t)\} = \omega\{u\}\cos(\omega t)$$

$$\{\ddot{q}(t)\} = -\omega^2\{u\}\sin(\omega t)$$

so

$$[-\omega^2[M]\{u\} + [K]\{u\}]\sin(\omega t) = \{0\}$$

When $\sin(\omega t) = \{0\}$ we have our trivial solution with no motion, so we discard this. Instead the more interesting solutions are obtained when the remaining terms are expressed as a Generalised Eigenvalue Problem:

$$[-\omega^2[M]\{u\} + [K]\{u\}] = \{0\}$$

$$[K]\{u\} = \lambda[M]\{u\} \text{ where } \omega^2 = \lambda$$

Where the mass matrix can be inverted, this equation takes the form of a Standard Eigenvalue Problem:

$$[M]^{-1}[K]\{u\} = \lambda\{u\}$$

for which you are likely to have encountered the solution in your Maths studies.

This means that the n DoF system has n natural frequencies ω_i which are the roots of the solutions λ from our Eigenvalue Problem. $\omega_i \geq 0$, and when $\omega_i = 0$ we have a 'rigid body mode' where the whole structure is moving without oscillation. A structure which is not anchored (imagine the system in Figure 2 without the grounded spring) must have at least one rigid body mode.

[3] We could just as well use $\{q(t)\} = \{A\}\cos(\omega t)$

6. Dynamics of Rods

The same methods can be used for FEA of structures assembled from rods, beams and other element types. For example, consider a rod vibrating along its length subject to the same assumptions:

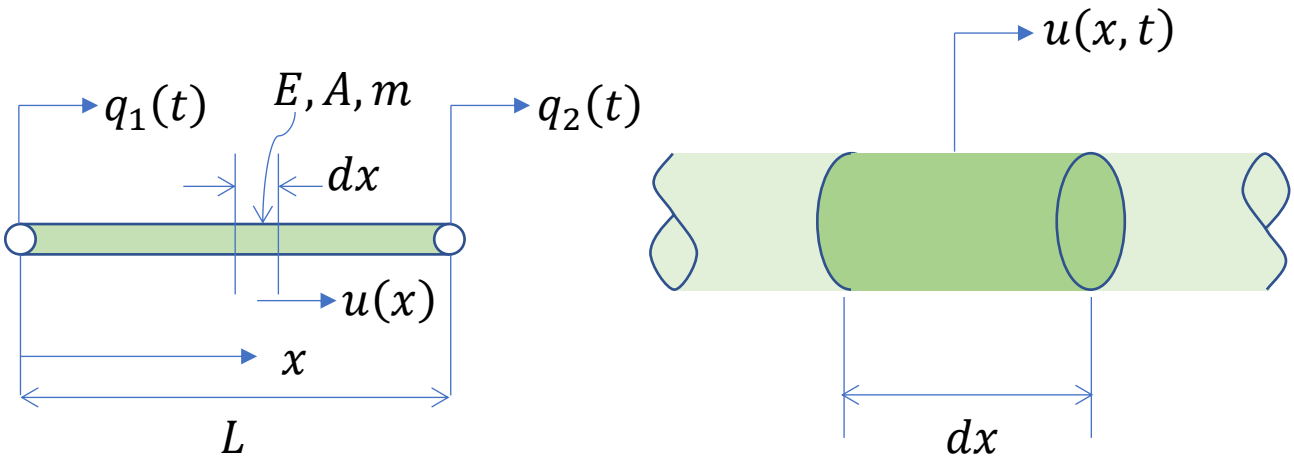

Figure 3: (left) a rod vibrating along its length.

To find its governing equation of motion, we need to find its kinetic and elastic potential energies. For a slice dx the kinetic energy is given by:

$$T = {}^1\!/_2\, m dx \left(\dot{u}(x,t)\right)^2$$

and overall we integrate to obtain:

$$T = {}^1\!/_2 \int_0^L m\left(\dot{u}(x,t)\right)^2 dx$$

and as in static FEA we use approximation to find $u(x,t)$, from our shape functions. The shape functions remain only a spatial approximation (functions of x), so the time variation is introduced to the generalised coordinates:

$$u(x,t) = g_1(x)q_1(t) + g_2(x)q_2(t)$$

with the same, linear shape functions we used in statics:

$$g_1(x) = 1 - \frac{x}{L},\ g_2(x) = \frac{x}{L}$$

Therefore,

$$\dot{u}(x,t) = g_1(x)\dot{q_1}(t) + g_2(x)\dot{q_2}(t)$$

so the kinetic energy expression yields:

$$T = {}^1\!/_2 \int_0^L m[g_1(x)\dot{q_1}(t) + g_2(x)\dot{q_2}(t)]^2 dx$$

$$T = {}^1\!/_2 \int_0^L m[{g_1}^2\dot{q_1}^2 + {g_2}^2\dot{q_2}^2 + 2g_1g_2\dot{q_1}\dot{q_2}]dx$$

which is integrated term-by-term, to result in:

$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T \begin{bmatrix} \frac{mL}{3} & \frac{mL}{6} \\ \frac{mL}{6} & \frac{mL}{3} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

This is our elemental mass matrix $[M]$.

Our elastic potential energy is identical to that obtained in static FEA, except that our generalised coordinates $q_i(t)$ are now functions of time:

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

and so our elemental stiffness matrix $[K]$ is also the same.

Once again, we have a Lagrangian of the form:

$$L = {}^1\!/_2 \{\dot{q}\}^T [M] \{\dot{q}\} - {}^1\!/_2 \{q\}^T [K] \{q\}$$

so, due to Hamilton's Principle, can bypass the working and state that the governing equation of motion is:

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$$

The process for multiple rod element structures is the same as in statics, where the elements are assembled, globalising the elemental stiffness and mass matrices and adding them. Similarly, 'lumped' masses can be added to the generalised coordinates and corresponding addresses in the mass matrix as springs can be added to the generalised coordinates and corresponding addresses in the stiffness matrix.