Lecture 13 - Heaviside and Delta functions

David Gammack and Oscar Dias

Mathematical Sciences,
University of Southampton, UK

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Outline

1. Introductory comments
2. Laplace Transforms involving special functions
   - Heaviside function
   - δ-functions
3. Summary

Discontinuous and impulsive sources

- We have seen that Laplace transform methods are useful for solving DEs: we take the Laplace transform (i.e. integral) of the IVP and BCs are straightforwardly implemented.
- Today: want to consider **discontinuous** or "**impulsive**" sources.

  jump

  derivative jump
- **Discontinuous** functions are most useful to model when an **effect suddenly starts or stops**; electrical current controlled by a switch.
- By "**impulsive**" we mean a **"spiky" source** that takes a **non-zero value** for an infinitely small amount of the independent variable (e.g. time), and yet still has a finite overall effect.
- **Impulsive behaviour** is most useful for **"impact" type problems**; think of an elastic collision where two bodies collide, but the **force acts** on each "**instantaneously**".

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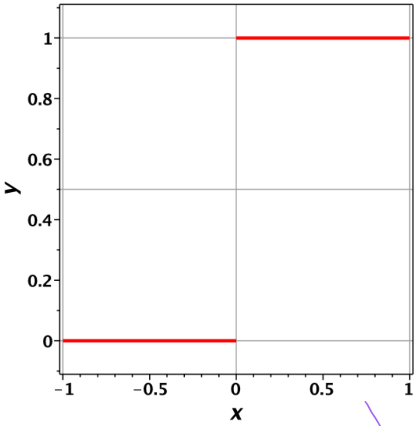
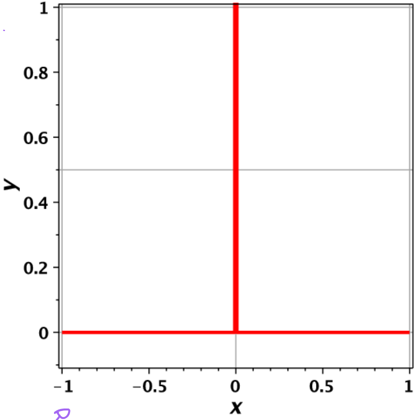
Discontinuous and impulsive sources

step func





derivative
(one interpretation)

The **Heaviside function**

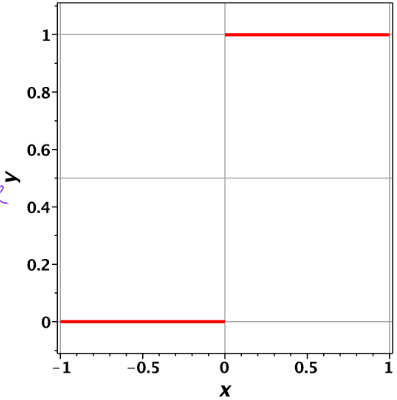
$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

is the simplest discontinuous function.

More generically:

only change is jump loc

$$H(x-a) = \begin{cases} 0, & x < a \\ 1, & x > a \end{cases}$$



Comments about Heaviside function

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- Note that **summing** Heaviside functions can give **switching on and off** behaviour; for example,

combine multiple to get more complex behaviour

$$S(x) = H(x+1) - H(x-1) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

gives the "square wave".

- Equally, if you have an arbitrary function $f(x)$ that you want to "switch on" at $x = a$ (and keep it on for $x \geq a$) you can just multiply f by a Heaviside function:

$$g(x) = f(x)H(x-a)$$

- "Switching off" at $x = a$ (and beyond) uses the same trick:

$$h(x) = f(x)[1 - H(x-a)]$$

Dirac δ-functions

The Dirac δ-function is not a function, but **a functional**. We have:

1. $$\int_{-\infty}^{\infty} \delta(x-c)f(x)\,\mathrm{d}x = f(c),$$ $\therefore \int_{-\infty}^{\infty} \delta(x)f(x)\,dx = f(0)$

2. $$\int_{a}^{b} \delta(\underbrace{x-c}_{c \text{ must fall in the range } a \to b!})f(x)\,\mathrm{d}x = 0 \quad \text{if } c \notin (a,b),$$ (for obvious reasons)

3. which is sometimes *loosely* written as

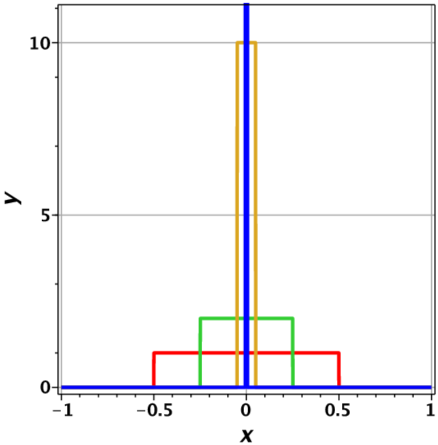
$$\delta(x-c) = \begin{cases} 0 & x \neq c. \\ \infty & x = c. \end{cases}$$

Of course, we can have $c = 0$.

Dirac δ-function as a limit

Loosely the δ function can be thought of as a limit:

$$\text{Area rectangle} = \underline{\text{fixed}}$$
$$= \int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0).$$



We are squeezing the rectangle along x and stretching it along the vertical direction while keeping its area fixed (see Fig.).

However, this procedure is ***not*** useful when manipulating $\delta-$functions in DEs.

Laplace transform of Heaviside function

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The Laplace transform of the Heaviside function $H(x-a)$ is:

$$\mathcal{L}\left[H(x-a)\right] = \frac{e^{-as}}{s} \quad \Leftrightarrow \quad \mathcal{L}^{-1}\left[\frac{e^{-as}}{s}\right] = H(x-a)\,, \qquad \text{if } \mathrm{Re}(s) > 0$$

$\nwarrow$ one more to add to the known list of LT!

Proof:

$$\mathcal{L}\left[H(x-a)\right] = \int_0^\infty H(x-a)e^{-sx}\,\mathrm{d}x$$

expansion of $H(\ldots)$

$$= \cancel{\int_0^a 0 \times e^{-sx}\,\mathrm{d}x} + \int_a^\infty 1 \times e^{-sx}\,\mathrm{d}x$$

$$= \left[-\frac{e^{-sx}}{s}\right]_a^\infty \qquad \swarrow \lim_{x\to\infty} e^{-sx} = 0 \text{ if } \mathrm{Re}(s) > 0 \text{ otherwise LT not defined}$$

$$= \frac{e^{-as}}{s}\,, \qquad \text{if } \mathrm{Re}(s) > 0.$$

Second shift theorem

The **second** shift theorem is:

$$\mathcal{L}\left[f(x-a)H(x-a)\right] = e^{-as}\tilde{f}(s) \quad \Leftrightarrow \quad \mathcal{L}^{-1}\left[e^{-as}\tilde{f}(s)\right] = f(x-a)H(x-a)$$

Alternatively, we can also formulate it as:

$$\mathcal{L}\left[f(x)H(x-a)\right] = e^{-as}\mathcal{L}\left[f(x+a)\right] \quad \Leftrightarrow \quad \mathcal{L}^{-1}\left[e^{-as}\mathcal{L}\left[f(x+a)\right]\right] = f(x)H(x-a)$$

Like the first shift theorem, it is very useful to invert Laplace Transforms.

Proof **:** (of main version; *Exercise*: prove the alternative version similarly)

$$\mathcal{L}\left[H(x-a)f(x-a)\right] = \int_0^\infty H(x-a)f(x-a)e^{-sx}\,\mathrm{d}x = \int_{x=a}^\infty f(x-a)e^{-sx}\,\mathrm{d}x$$

Use the change of variables $\tau = x - a$, to get $d\tau = dx$

$$= \int_{\tau=0}^{\tau=\infty} f(\tau)\underbrace{e^{-s(\tau+a)}}_{= e^{-s\tau} \times e^{-sa}}\,\mathrm{d}\tau$$

$$= e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,\mathrm{d}\tau = e^{-as}\tilde{f}(s).$$

Example: LT with Heaviside function

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Solve

$$\textbf{IVP:}\quad y'' + y = H(x-1);\qquad y(0)=0,\quad y'(0)=1.$$

Taking the Laplace Transform and rearranging terms gives

$$\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[H(x-1)] \qquad \mathcal{L}[y''(x)] = s^2\tilde{y}(s) - s\,y(0) - y'(0)$$

$$\Leftrightarrow (s^2+1)\tilde{y} - 1 = \frac{e^{-s}}{s} \quad \Leftrightarrow \quad \tilde{y}(s) = \frac{1}{s^2+1} + \frac{e^{-s}}{s(s^2+1)}.$$

rearrange

Now we invert this Laplace transform to get the solution $y(x) = \mathcal{L}^{-1}[\tilde{y}(s)]$:

$$y(x) = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] + \mathcal{L}^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right]$$

$$\begin{cases} \mathcal{L}[\sin(a\,x)] = \frac{a}{s^2+a^2} \Leftrightarrow \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin(a\,x) \\ \frac{e^{-s}}{s(s^2+1)} = \frac{e^{-s}}{s} - \frac{e^{-s}s}{s^2+1} \end{cases}$$

(Partial Fractions: practice!!!)

$$= \sin(x) + \mathcal{L}^{-1}\left[\frac{e^{-s}}{s}\right] - \mathcal{L}^{-1}\left[\frac{e^{-s}s}{s^2+1}\right]$$

$$\begin{cases} \mathcal{L}[H(x-a)] = \frac{e^{-a\,s}}{s} \\ \quad \Leftrightarrow \mathcal{L}^{-1}\left[\frac{e^{-a\,s}}{s}\right] = H(x-a) \end{cases}$$

soln n simplify

$$= \sin(x) + H(x-1)\left[1 - \cos(x-1)\right]$$

$$\begin{cases} \mathcal{L}[\cos(a\,x)] = \frac{s}{s^2+a^2}, \\ \mathcal{L}^{-1}\left[e^{-a\,s}\tilde{f}(s)\right] = f(x-a)H(x-a), \\ \Rightarrow \mathcal{L}^{-1}\left[\frac{e^{-a\,s}s}{s^2+a^2}\right] = \cos(x-a)H(x-a) \end{cases}$$

Laplace Transform of $\delta-$function

By the definition of the δ function and the Laplace transform we have

$$\mathcal{L}\left[\delta(x-a)\right] = e^{-as} \qquad \longleftarrow \qquad \text{one more to add to the known list of LT!}$$

This only works for $a > 0$.

Proof: Just apply the definition of LT to $f(x) = \delta(x-a)$ to find:

$$\mathcal{L}\left[f(x)\right] = \int_0^\infty e^{-sx} f(x)\,\mathrm{d}x \;\longrightarrow\; \mathcal{L}\left[\delta(x-a)\right] = \int_0^\infty e^{-sx}\delta(x-a)\,\mathrm{d}x$$

$$= e^{-as},$$

where the final step uses the definition of the δ-function but only holds if $a > 0$. Why?

$$\int_{-\infty}^{+\infty} \delta(x-a) f(x) dx = f(a) \quad \Rightarrow \text{If } a > 0 : \begin{cases} \int_0^{+\infty} \delta(x-a) f(x) dx = f(a) \\ \\ \int_{-\infty}^{0} \delta(x-a) f(x) dx = 0 \end{cases}$$

Example

periodic instantaneous force on spring

A mass on a spring is struck by a hammer at times $t = n\pi$. Its motion follows:

$$\textbf{IVP:} \qquad \ddot{y} + y = \sum_n \delta(t - n\pi); \qquad \underbrace{y(0) = 0, \quad \dot{y}(0) = 0.}_{\text{starting from rest}}$$

Taking the Laplace Transform and rearranging terms gives

$$\mathcal{L}[\ddot{y}] + \mathcal{L}[y] = \mathcal{L}\left[\sum_n \delta(t - n\pi)\right]$$

$$\left\{\begin{array}{l} \mathcal{L}[\ddot{y}(t)] = s^2\tilde{y}(s) - s\,y(0) - \dot{y}(0) \\ \mathcal{L}[y(t)] = \tilde{y}(s) \\ \text{LT is linear: } \mathcal{L}\left[\sum_n f_n(t)\right] = \sum_n \mathcal{L}[f_n(t)] \end{array}\right.$$

$$s^2\tilde{y}(s) + \tilde{y}(s) = \sum_n \mathcal{L}[\delta(t - n\pi)] \qquad \left(\mathcal{L}[\delta(t-a)] = e^{-as},\ a \equiv n\pi\right)$$

$$\tilde{y}(s) = \sum_n \frac{e^{-n\pi s}}{s^2 + 1}$$

To invert, we use the second shift theorem:

$$y(t) = \mathcal{L}^{-1}[\tilde{y}(s)] = \mathcal{L}^{-1}\left[\sum_n \frac{e^{-n\pi s}}{s^2+1}\right] = \sum_n \mathcal{L}^{-1}\left[\boxed{\frac{e^{-n\pi s}}{s^2+1}}\right]$$

$$y(t) = \sum_n H(t - n\pi)\sin(t - n\pi) \qquad \leftarrow \left\{\begin{array}{l} \mathcal{L}^{-1}\left[e^{-as}\tilde{f}(s)\right] = f(t-a)H(t-a) \\ \mathcal{L}[\sin(ax)] = \frac{a}{s^2+a^2} \Rightarrow \sin(ax) = \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] \end{array}\right.$$

Example (cont.)

A mass on a spring is struck by a hammer at times $t = n\pi$. Its motion follows:

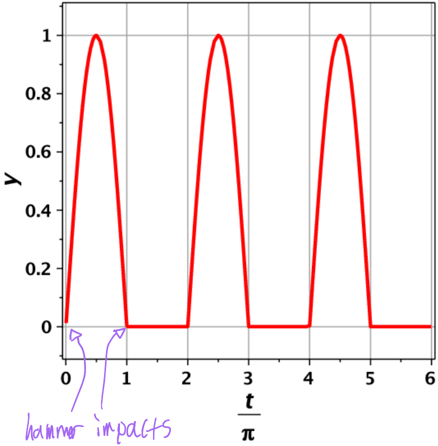
$$\textbf{IVP:} \qquad \ddot{y} + y = \sum_n \delta(t - n\pi), \qquad y(0) = 0, \quad \dot{y}(0) = 0.$$

Taking the Laplace Transform and rearranging terms gives in the end of the day:

$$\tilde{y}(s) = \sum_n \frac{e^{-n\pi s}}{s^2 + 1}$$

and the second shift theorem gives:

$$y(t) = \sum_n H(t - n\pi)\sin(t - n\pi).$$



periodic hammer impacts

Useful trigonometric identities in LT context

Review slide: check it as homework.

$$\cos(-x) = \cos(x), \qquad \cos(x \pm 2\pi) = \cos(x),$$

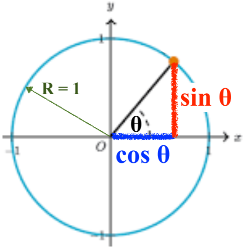
$$\sin(-x) = -\sin(x), \qquad \sin(x \pm 2\pi) = \sin(x),$$

$$\cos(x \pm \pi) = -\cos(x), \qquad \cos\left(x \pm \frac{\pi}{2}\right) = \mp \sin(x),$$

$$\sin(x \pm \pi) = -\sin(x), \qquad \sin\left(x \pm \frac{\pi}{2}\right) = \pm \cos(x).$$

$$\ldots$$

But you do not need to memorise them! ... instead keep in your mind that they might be useful/required and always use the trigonometric unit circle for the specific problem at hand.



Outline

Summary

- Complex source terms, especially discontinuous (or worse) ones, in DEs are most easily dealt with using Laplace transforms.
- Standard example for discontinuous functions: the Heaviside function.
- The Heaviside function appears in the second shift theorem, essential in inverting transforms containing exponentials.
- The δ-function is not a function and must be treated with care.
- The δ-function is useful in modelling impulsive behaviour.
- Add-ons to the List of Results:

$$\mathcal{L}\left[H(x-a)\right] = \frac{e^{-as}}{s};$$

$$\mathcal{L}\left[\delta(x-a)\right] = e^{-as};$$

$$\mathcal{L}\left[f(x-a)H(x-a)\right] = e^{-as}\tilde{f}(s) \quad \underline{\text{or}} \quad \mathcal{L}\left[f(x)H(x-a)\right] = e^{-as}\mathcal{L}\left[f(x+a)\right]$$

$\nwarrow$ 2^{nd} shift theorem.

Be familiar with List of known LT Results in the **Formula Sheet**: your best friend on exam day but only if you practiced using it solving many examples!