SESA3029 Aerothermodynamics

Lecture 4.4

Prandtl-Glauert transformation

Recap:

Velocity potential equation

$$\boxed{\left(1-M_\infty^2\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0}$$

- Reduces to Laplace's equation in incompressible flow
- Not valid near $M_\infty=1$
- Not applicable for $M_\infty>5$ (hypersonic flow)
- We will consider solution to this equation in two flow regimes
  - $M_\infty<0.8$ (elliptic equation: Prandtl-Glauert transformation)
  - $1.2<M_\infty<5$ (hyperbolic equation: Ackeret theory)

need to memorize derivation

Prandtl-Glauert transformation

Start from $$\left(1-M_\infty^2\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

scaled velocity potential

Let $\phi_0 = \beta\phi$

$x_0 = x$

$y_0 = \beta y$

with $\beta = \sqrt{1-M_\infty^2}$ ∴ constant

$$\beta^2\frac{\partial^2\left(\phi_0/\beta\right)}{\partial x_0^2}+\frac{\partial^2\left(\phi_0/\beta\right)}{\partial\left(y_0/\beta\right)^2}=0$$

β is a constant so it can be pulled out and cancelled

$$\boxed{\frac{\partial^2\phi_0}{\partial x_0^2}+\frac{\partial^2\phi_0}{\partial y_0^2}=0}$$

i.e. a simple transform takes us back to Laplace's equation

Requires $M_\infty < 1$ so only subsonic

Since $C_p = -2\frac{u'}{U_\infty}$ As in incompressible flow

and $u' = \frac{\partial \phi}{\partial x}$

transform using ϕ_0 and x_0

$$C_p = -\frac{2}{U_\infty}\frac{\partial \phi}{\partial x}$$

$$= -\frac{2}{U_\infty \beta}\frac{\partial \phi_0}{\partial x_0}$$

$$= \frac{C_{p0}}{\sqrt{1-M_\infty^2}}$$

where C_{p0} is the incompressible pressure coefficient

$$\boxed{C_p = \frac{C_{p0}}{\sqrt{1-M_\infty^2}}}$$

Prandtl-Glauert transformation

Prandtl-Glauert transformation

- Allows incompressible results to be used in the compressible flow regime
- Applies also to quantities derived from C_P, for example lift and moment coefficients
- Example: If C_L=0.4 in incompressible flow, find C_L at M_∞=0.6

$$\frac{C_{L0}}{\beta} = C_L = \frac{C_{L0}}{\sqrt{1-M_\infty^2}} = \frac{0.4}{\sqrt{1-0.6^2}} = 0.5$$

Use of P-G to estimate M_{crit}

$$\frac{2(p - p_\infty)}{\rho_\infty M_\infty^2 \underbrace{\gamma R T_\infty}_{a^2}} = C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

and for isentropic flow

$$\frac{p}{p_\infty} = \frac{p_0/p_\infty}{p_0/p} = \left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M^2}\right)^{\frac{\gamma}{\gamma-1}}$$

Critical Mach number is when the local M=1 and $M_\infty = M_{crit}$

Also apply P-G transform

$$C_p = \frac{C_{p0}}{\sqrt{1-M_{crit}^2}} = \frac{2}{\gamma M_{crit}^2}\left(\left(\frac{1+\frac{\gamma-1}{2}M_{crit}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma}{\gamma-1}} - 1\right)$$

also use $C_{p_0} \propto C_p$ relationship

memorise equation or derivation

i.e. Knowing the worst case C_{p0} (the lowest C_p on the airfoil in incompressible flow), we can estimate M_{crit}.

Example

- If $C_{p,min}$=-0.4 in low speed (incompressible) flow of air, estimate M_{crit}

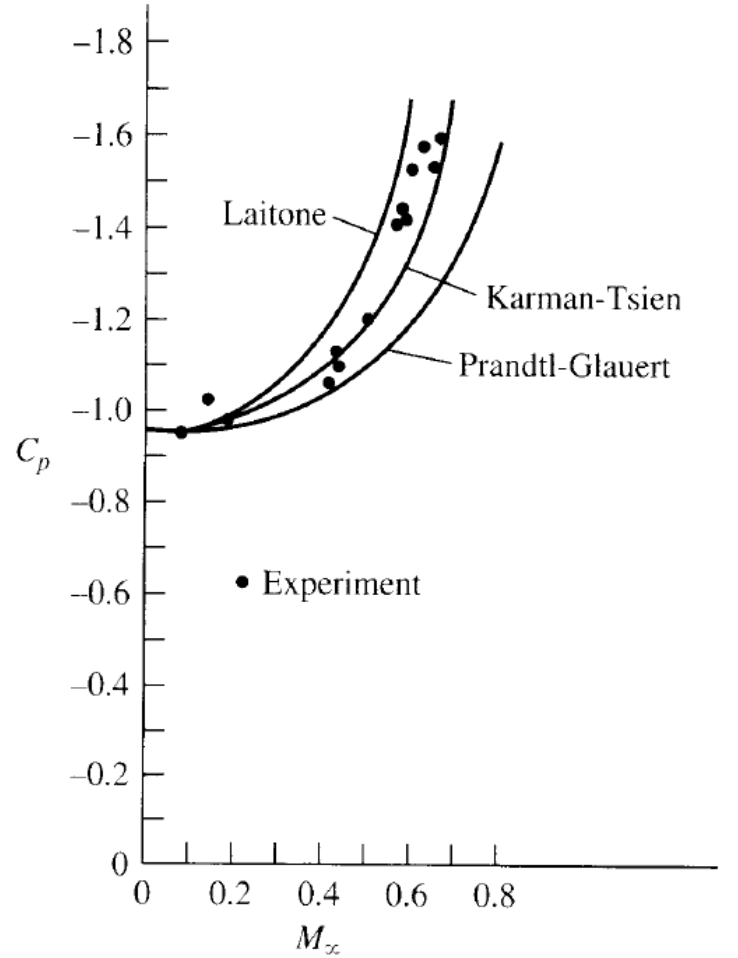
$$\frac{C_{p0}}{\sqrt{1-M_{crit}^2}} = \frac{2}{\gamma M_{crit}^2}\left(\left(\frac{1+\frac{\gamma-1}{2}M_{crit}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma}{\gamma-1}} - 1\right)$$

M_{crit}	LHS	RHS
0.6	-0.50	-1.29
0.7	-0.56	-0.78
0.747	-0.60	-0.60

Critical Mach number is 0.747

Improved correction: Karman-Tsien

$$C_p = \frac{C_{p,0}}{\sqrt{1-M_\infty^2} + \frac{C_{p,0}}{2}\left[\frac{M_\infty^2}{1+\sqrt{1-M_\infty^2}}\right]}$$



Improved fit to experimental data – the formula attempts to include nonlinear effects.

(Anderson, 5th Ed)