SESA2025 Mechanics of Flight
Rigid aircraft model
Six degrees of freedom

Lecture 3.1

Week 7

Lecture 3.1 Equations of motion of a rigid aircraft

Lecture 3.2 Linearisation and decoupling

Lecture 3.3 Longitudinal dynamics: gravity and aerodynamic contributions

Quiz 2 (Part A)

Week 8

Lecture 3.4 Longitudinal model

Lecture 3.5 Phugoid and short period oscillation

Lecture 3.6 SPO approximation

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Week 9

Lecture 3.7 Phugoid approximation

Lecture 3.8 Lateral dynamics

Lecture 3.9 Lateral mode approximations: roll damping, slow spiral and Dutch roll

Quiz 3 (Part B)

Week 10:

Lecture 4.1 Estimation of selected longitudinal derivatives

Lecture 4.2 Estimation of selected lateral derivatives

Lecture 5.1 Control and gust derivatives, steady asymmetric flight paths

Week 11

Lecture 5.2 Control and gust response

Lecture 5.3 Handling qualities, stability augmentation

Lecture 5.4 Extensions to the basic model

Quiz 4 (Part B)

Week 12

Revision week

- can be statically stable but dynamically unstable, like how a driven oscillator can be.
- all dynamically stable systems are statically stable

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Steps to develop a full dynamical model

Six degrees of freedom with six equations (Newton's laws)

$$\mathbf{F}_E = \left(\frac{\mathrm{d}m\mathbf{v}}{\mathrm{d}t}\right)_E$$

$$\mathbf{M}_E = \left(\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\right)_E$$

- three translational
- three rotational

Tait–Bryan angles and rates of change of these angles

- Relate the aircraft (body) co-ordinate system to a fixed (inertial) reference frame, which we will take as the Earth

Using a body-reference frame, work out the changes in the Earth frame of reference

- Hence write down Newton's laws for linear and angular momentum

Define the inertia matrix (tensor) for a 3D rigid rotating body

The aircraft coordinate system is not fixed so is not an inertial reference frame

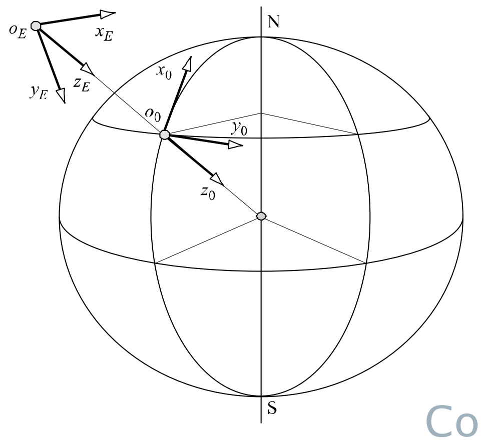
Reference frames

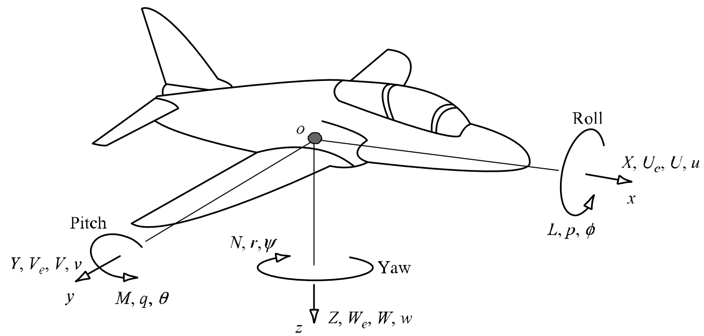
Forces and moments are best found in the body (aircraft) reference frame

We also need accelerations and rotations in this body frame, which may be rotating relative to the Earth

Newton's laws apply in an inertial frame of reference

We need to transfer between the two





Cook 2007 *Flight Dynamics* 2nd edition, Elsevier

Right handed reference frame

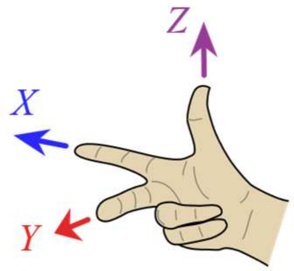
For aircraft dynamics:

Translation, +ve

- x, forward
- y, starboard
- z, down

Rotation, +ve

- ϕ, Y into Z
- θ, Z into X
- ψ, X into Y



Right handed reference frame.
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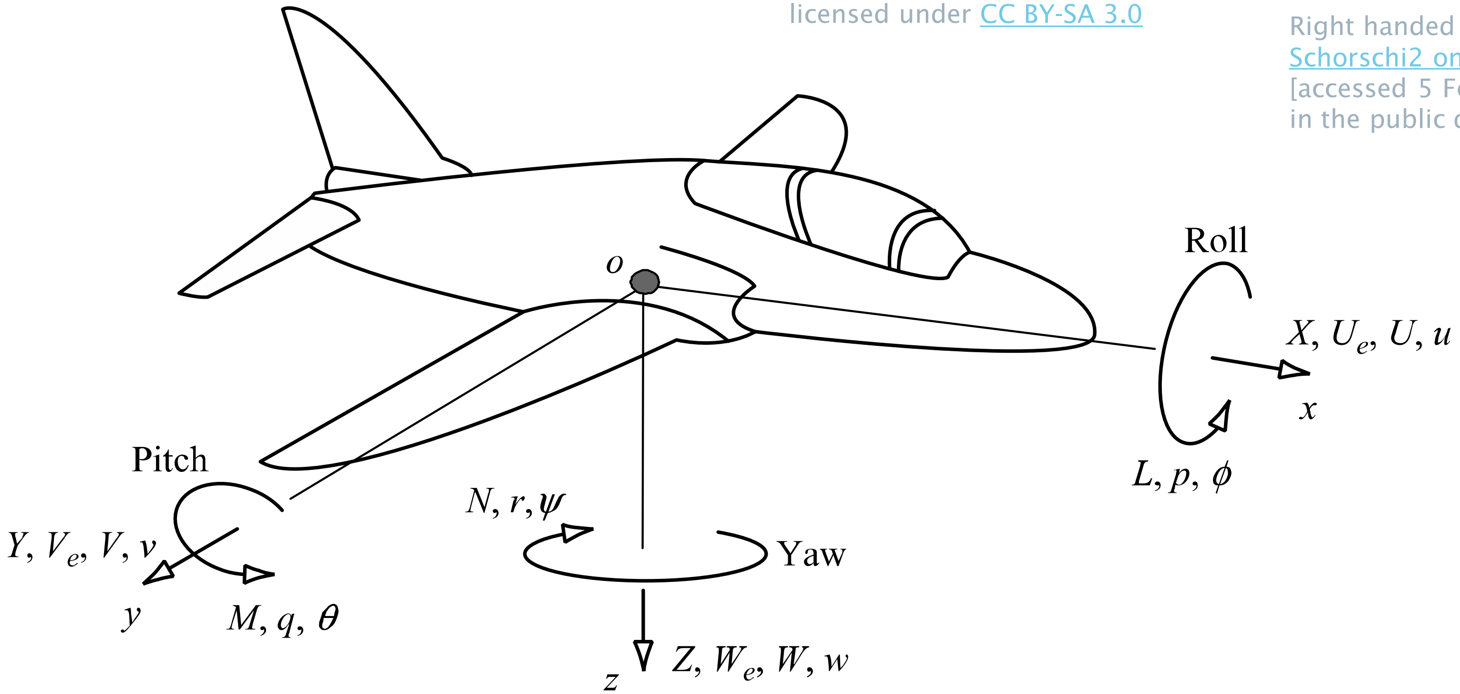

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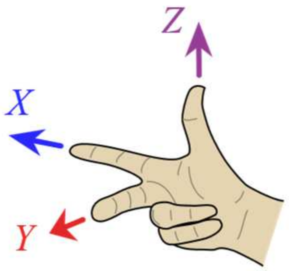

Right handed reference frame.
Modified from Acdx on Wikimedia
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Right handed rotation.
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Unit Vectors (Body Axes)	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$
Force Components (N)	X	Y	Z
Moment Components (N m)	L	M	N
Angles of Rotation (rad)	ϕ (roll)	θ (pitch)	ψ (yaw)
Velocity of CG (m/s)	U	V	W
Angular Velocity (rad/s)	p (roll, $= \dot{\phi}$)	q (pitch, $= \dot{\theta}$)	r (yaw, $= \dot{\psi}$)
Moments of Inertia (kg m^2)	I_{xx}	I_{yy}	I_{zz}

Table 1: Aircraft coordinates

From Part B Derivations Document

Tait–Bryan ~~Euler~~ angles

Sequence of rotations to get from the
Earth reference frame (*E*) to the aircraft (body *B*) frame of reference

Convention to get from *E* to *B*:
yaw first (angle ψ), then pitch (angle θ), then roll (angle ϕ), e.g. for velocity

$$\mathbf{v}_B = \mathbf{R}_{BE}\mathbf{v}_E \text{ with } \mathbf{R}_{BE} = \mathbf{R}_x(\phi)\mathbf{R}_y(\theta)\mathbf{R}_z(\psi)$$

earth to body by multiplying by R_{BE}

$$\mathbf{v}_E = \mathbf{R}_{EB}\mathbf{v}_B \text{ with } \mathbf{R}_{EB} = \mathbf{R}_z(-\psi)\mathbf{R}_y(-\theta)\mathbf{R}_x(-\phi)$$

body to earth

Rotation matrices are given in the handout

(recall that combinations of matrix multiplications are implemented from right to left)

Rotation rates

For completeness, the rotation rates of Tait-Bryan ~~Euler~~ angles to the rotation rate

$$\boldsymbol{\omega} = (p\ q\ r)^T = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

of the aircraft are related by

$$\boldsymbol{\omega} = \dot{\phi}\mathbf{i} + \mathbf{R}_x\dot{\theta}\mathbf{j} + \mathbf{R}_x\mathbf{R}_y\dot{\psi}\mathbf{k}$$

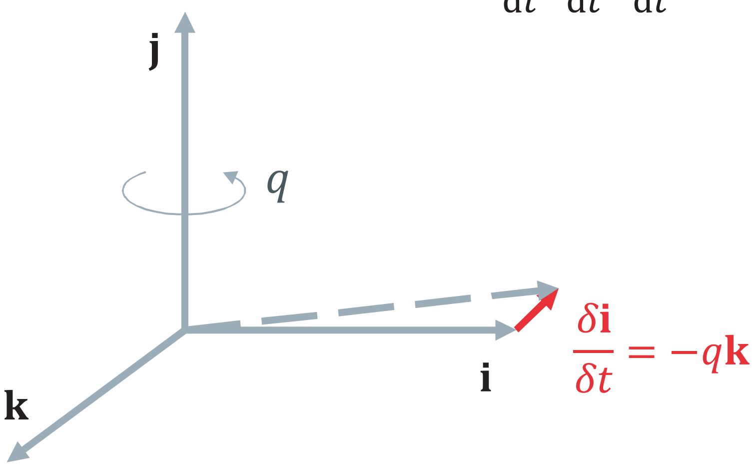
To allow using newtons second law we will use the transformation to convert from local coordinates to inertial coordinates.

Inertial rates of change, seen from body perspective

Total time derivative based on velocity and rotation rates in body frame

$$\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\right) = \frac{\mathrm{d}v_x}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\mathbf{k} + v_x\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} + v_y\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} + v_z\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t}$$

So besides the change of velocity, we need to take the change of the direction of the reference frame into account, ie: $\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t}$; $\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t}$; $\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t}$.



$$\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} = r\mathbf{j} - q\mathbf{k}$$

etc

Since the planes mass changes slowly relative to dynamic effects we can approximate it as a constant.

time derivative of each vel component

Inertial rates of change, seen from body perspective

cross product

Total time derivative based on velocity and rotation rates in body frame

$$\underbrace{\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)}_{\text{velocity}} = \dot{\mathbf{v}} + (qv_z - rv_y)\mathbf{i} + (rv_x - pv_z)\mathbf{j} + (pv_y - qv_x)\mathbf{k} = \dot{\mathbf{v}} + \underbrace{\boldsymbol{\omega}}_{\text{angular vel vector}} \times \underbrace{\mathbf{v}}_{\text{vel vec}}$$

For more details of proofs, see handout

angular vel vector (cause axis change)

Introducing a more precise notation, noting that the left hand side is the inertial acceleration, rotated into the body frame of reference

$$\mathbf{R}_{BE}\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)_E = \dot{\mathbf{v}}_B + \boldsymbol{\omega}_B \times \mathbf{v}_B \longrightarrow \dot{\vec{V}}_E = R_{EB}(\dot{\vec{V}}_B + \vec{\omega}_B \times \vec{V}_B)$$

Translational Equations of Motion

Newton's 2nd law: Applied in Earth reference frame then transformed to body frame.
Force is equal to the time rate of change of linear momentum:

$$\mathbf{F}_E = \left(\frac{\mathrm{d}m\mathbf{v}}{\mathrm{d}t}\right)_E = m\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)_E \qquad (\mathbf{F} = m\mathbf{a})$$

$$\mathbf{R}_{BE}\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)_E = \dot{\mathbf{v}}_B + \boldsymbol{\omega}_B \times \mathbf{v}_B$$

$$\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)_E = \mathbf{R}_{EB}(\dot{\mathbf{v}}_B + \boldsymbol{\omega}_B \times \mathbf{v}_B)$$

$$\mathbf{R}_{EB}\mathbf{F}_B = m\mathbf{R}_{EB}(\dot{\mathbf{v}}_B + \boldsymbol{\omega}_B \times \mathbf{v}_B)$$

$$\Big\downarrow \; \times(R_{EB})^{-1}$$

$$\mathbf{F}_B = m(\dot{\mathbf{v}}_B + \boldsymbol{\omega}_B \times \mathbf{v}_B)$$

Rotational Equations of Motion

Newton's 2nd law: Applied moment is equal to the rate of change of angular momentum **h** (in an inertial reference frame) :

$$\mathbf{M}_E = \left(\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\right)_E \qquad \left(\mathbf{M}_E = \mathbf{I}_E \frac{\mathrm{d}\boldsymbol{\omega}_E}{\mathrm{d}t}\right)$$

Following the same approach:

$$\mathbf{R}_{EB}\mathbf{M}_B = \mathbf{R}_{EB}\left(\dot{\mathbf{h}}_B + \boldsymbol{\omega}_B \times \mathbf{h}_B\right)$$

$$\mathbf{M}_B = \dot{\mathbf{h}}_B + \boldsymbol{\omega}_B \times \mathbf{h}_B$$

Summary

Six-equation system capturing Newton's laws:

$$\mathbf{F}_B = m(\dot{\mathbf{v}}_B + \boldsymbol{\omega}_B \times \mathbf{v}_B)$$

$$\mathbf{M}_B = \dot{\mathbf{h}}_B + \boldsymbol{\omega}_B \times \mathbf{h}_B$$

Where $\mathbf{h}_B = \mathbf{I}_B \boldsymbol{\omega}_B$ and $\mathbf{I}_B$ is the inertia matrix

Reminder of vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$