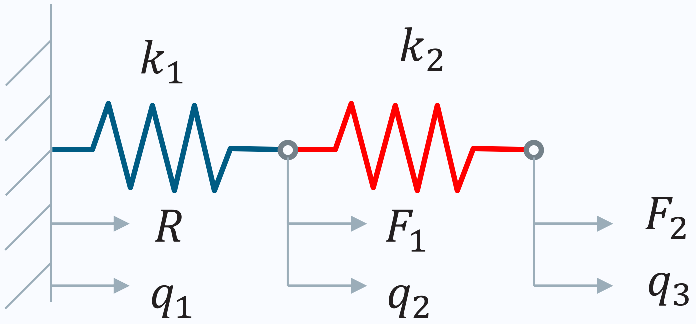
Part 2a: Elastic Rods in Tension and Compression

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

From 11th October 2024

Reminder from last time:

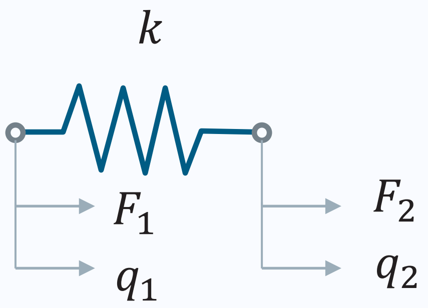


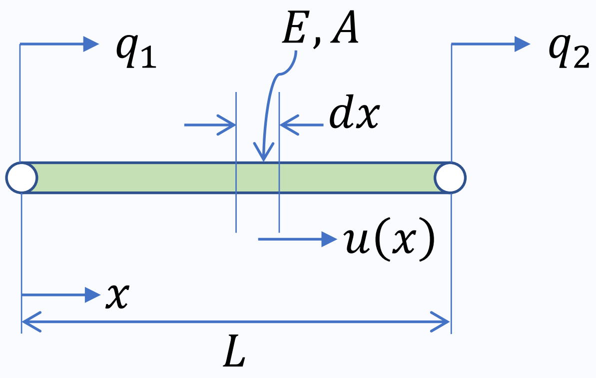
- We search for solutions for q_i in problems like this:
  - We express the system's Elastic Strain Energy U and its Potential Energy V
  - by finding U_i for each element and V based on all applied loads.
  - We could apply PMTPE and calculate $\frac{\partial \Pi(q_i)}{\partial q_i} = 0$ to obtain $\{F\} = [K]\{q\}$
  - but because we can express U_i in quadratic form, we can take a shortcut where $U_i = 1/2\{q\}^T[K_i]\{q\}$
  - and assemble our elemental $[K_i]$ into a global $[K]$
  - and by expressing V in matrix form too, we find $\{F\}$ and can therefore solve $\{F\} = [K]\{q\}$
  - by inverting the stiffness matrix, allowing $\{q\} = [K]^{-1}\{F\}$

This week:

- Similarities:
  - PMTPE gives us our governing equation of equilibrium (a set of linear quadratic equations)
  - How we fetch and assemble our element stiffness matrices into an assembled stiffness matrix
- Differences:
  - Unlike for springs, we have some approximation, for ‘elastic rods’

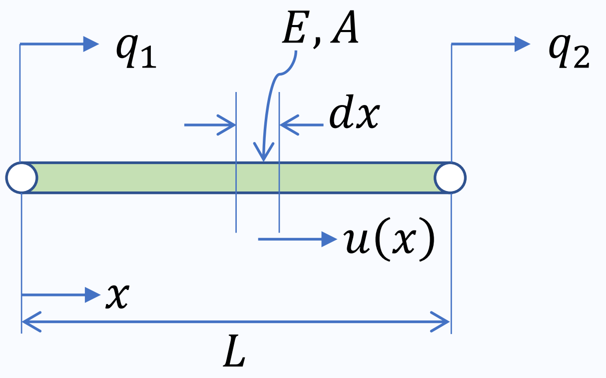
Elastic Rods





- Can deform only in tension or compression
- Unlike our spring, properties are described by E, A (cross section) and L (length), and the axial displacement of the endpoints q_1 and q_2 in coordinate system x
- We look at a small portion $\mathrm{d}x$, and its deformation $u(x)$ depends on where we take the slice
- Every point on the cross-section slice displaces the same amount
- So x is the ‘label’ (which cross section) and $u(x)$ is the displacement.

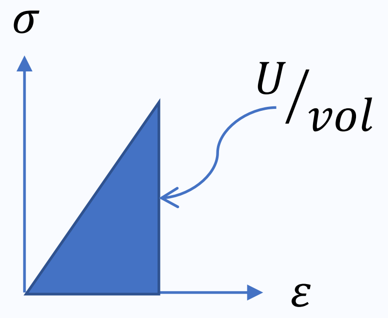
Elastic Rods



- PMTPE requires an expression for the strain energy U in terms of the displacement field in the rod u.

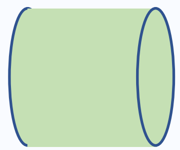
$$U = {}^{1}\!/_{2} \int_0^L EA \left(\frac{du}{dx}\right)^2 dx$$

- Where does this come from? You should remember…
- Elastic strain energy per unit volume is the area under the stress-strain curve, for simplified 1D (uniaxial) elasticity:



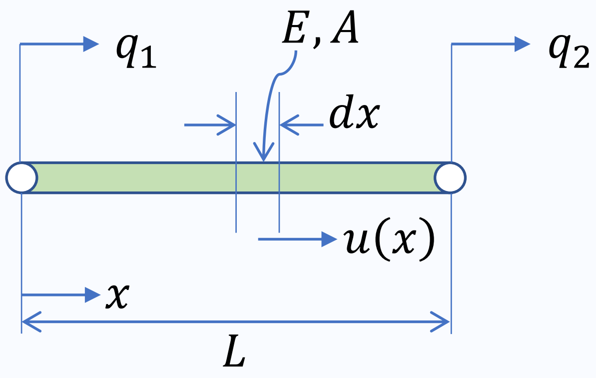
$$U = {}^{1}\!/_{2} \int \sigma_x \varepsilon_x dVol$$

$$U = {}^{1}\!/_{2} \int \sigma_x \varepsilon_x A dx$$



$dVol = Adx$

Elastic Rods

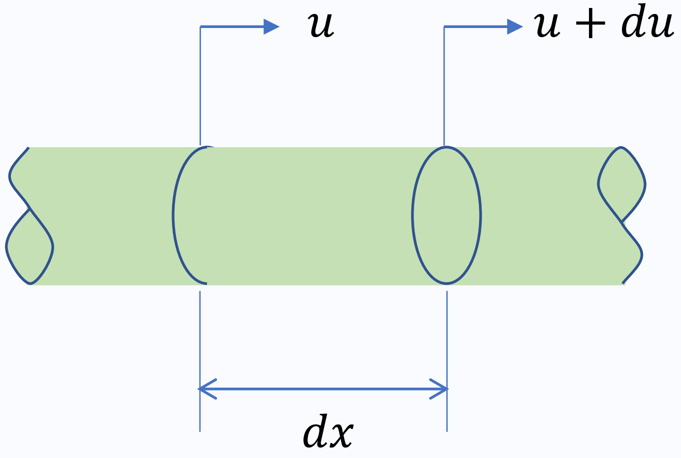


- and because in 1D elasticity (highly simplified case) we can say that $\sigma_x = E\varepsilon_x$,

$$U = {}^1\!/_2 \int E\varepsilon_x \times \varepsilon_x \times A dx$$

$$U = {}^1\!/_2 \int EA{\varepsilon_x}^2 dx$$

- What is ε_x given the displacement field?

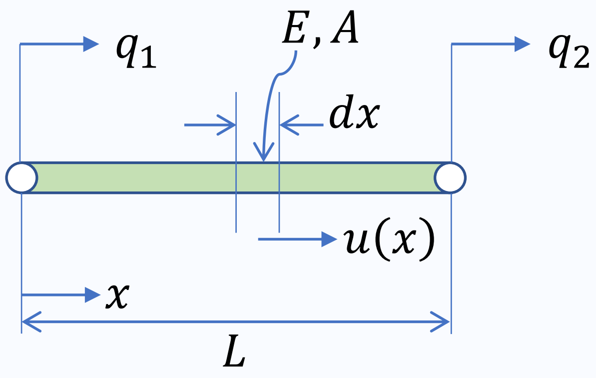


$$\varepsilon_x = \frac{(u + du) - u}{dx} = \frac{du}{dx}$$

- Hence

$$U = {}^1\!/_2 \int EAu'^2 dx$$

Elastic Rods



- **Aim:** we want to express the displacement throughout the element $u(x)$ from the nodal displacements, q_1 and q_2, (currently unknown).
- Now our first approximation: let's say $u(x)$ is a linear function of x. i.e.:

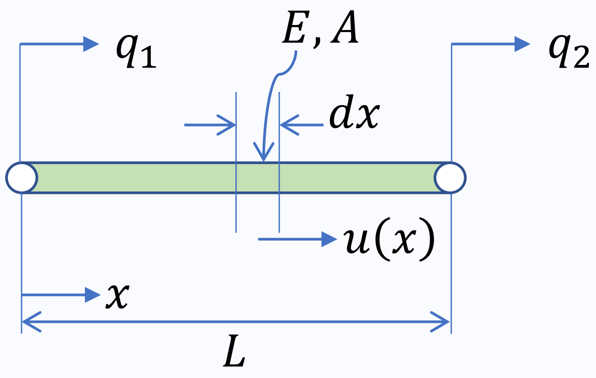
$$u(x) = a + bx$$

- (We can make this linear function assumption for 1D elasticity, but we couldn't make that assumption for more complex cases).
- This is inconvenient though, as a and b have no meaning. A more convenient version, more general:

$$u(x) = N_1(x)q_1 + N_2(x)q_2$$

- q_1 and q_2 are unknowns, N_1 and N_2 are prescribed ***shape functions*** or ***interpolation functions*** for approximation: a common approach taken in all finite elements.

Elastic Rods



$$u(x) = N_1(x)q_1 + N_2(x)q_2$$

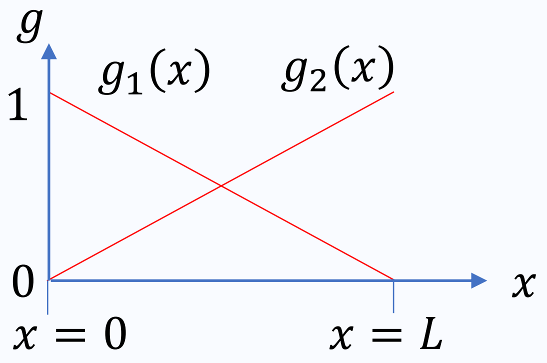
- The trick is we choose for these shape functions N_1 and N_2 some linear functions, g_1 and g_2. (We could choose other functions…)

$$u(x) = g_1(x)q_1 + g_2(x)q_2$$

- We can define these by drawing our domain 0 to L:

$$g_1(x) = 1 - \frac{x}{L}$$

$$g_2(x) = \frac{x}{L}$$



$$u(x) = \left(1 - \frac{x}{L}\right) q_1 + \left(\frac{x}{L}\right) q_2$$

Elastic Rods

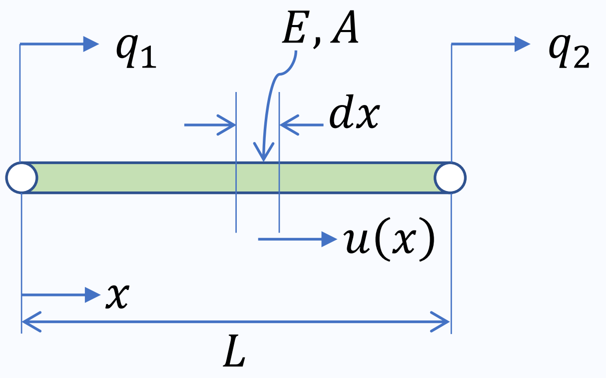
$$u(x) = \left(1 - \frac{x}{L}\right) q_1 + \left(\frac{x}{L}\right) q_2$$

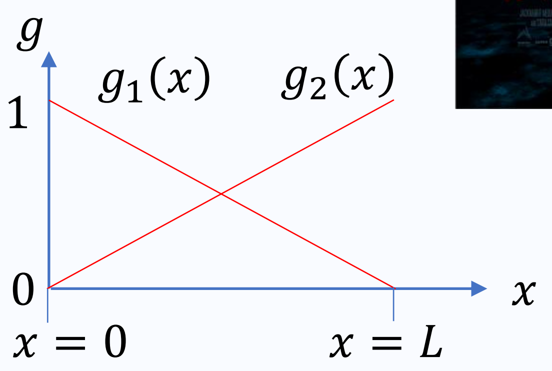
- But what are these qs?
- Why didn't we just use $u(x) = a + bx$?
- Work out $u(x)$ at each end:

$$u(0) = \left(1 - \frac{0}{L}\right) q_1 + \left(\frac{0}{L}\right) q_2 = q_1$$

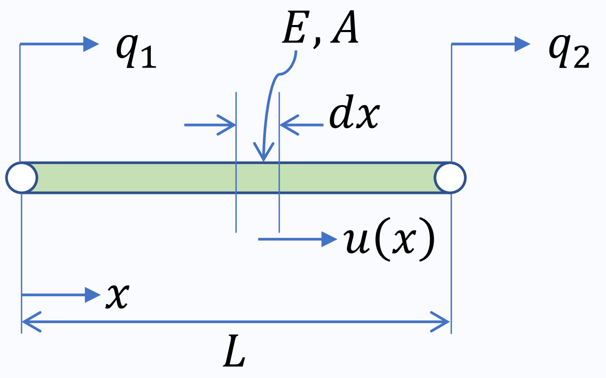
$$u(L) = \left(1 - \frac{L}{L}\right) q_1 + \left(\frac{L}{L}\right) q_2 = q_2$$

- The qs, which aren't functions of x, describe the end displacements! This is why we choose shape functions in that form.





Elastic Rods



- and assuming we have solved to find the q unknowns, we can use

$$u(x) = \left(1 - \frac{x}{L}\right) q_1 + \left(\frac{x}{L}\right) q_2$$

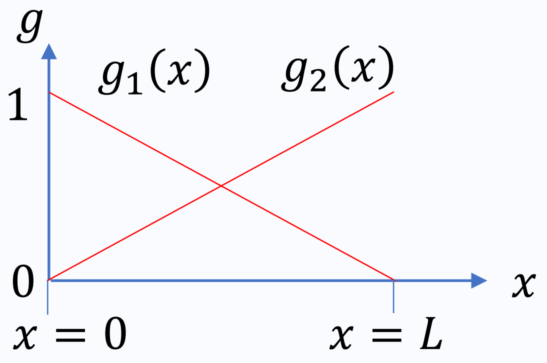
- to give us values for strain:

$$\varepsilon(x) = \frac{du}{dx} = \left(-\frac{1}{L}\right) q_1 + \left(\frac{1}{L}\right) q_2$$

- and in this simple 1D case, for stress:

$$\sigma(x) = E\varepsilon(x) = \left(-\frac{E}{L}\right) q_1 + \left(\frac{E}{L}\right) q_2$$

- (check dimensional analysis if you like!)



Elastic Rods

- or in matrix form (as more complex element types will require):

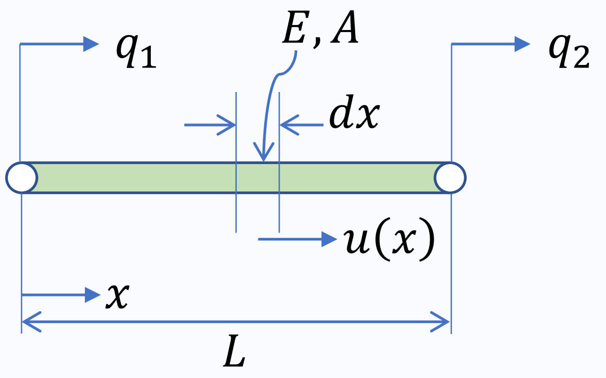
$$\varepsilon(x) = \frac{du}{dx} = \left(-\frac{1}{L}\right)q_1 + \left(\frac{1}{L}\right)q_2$$

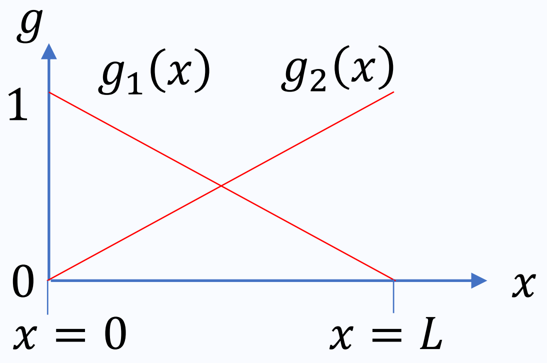
$$\varepsilon(x) = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

- and

$$\sigma(x) = E\varepsilon(x) = \left(-\frac{E}{L}\right)q_1 + \left(\frac{E}{L}\right)q_2$$

$$\sigma(x) = [E]\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

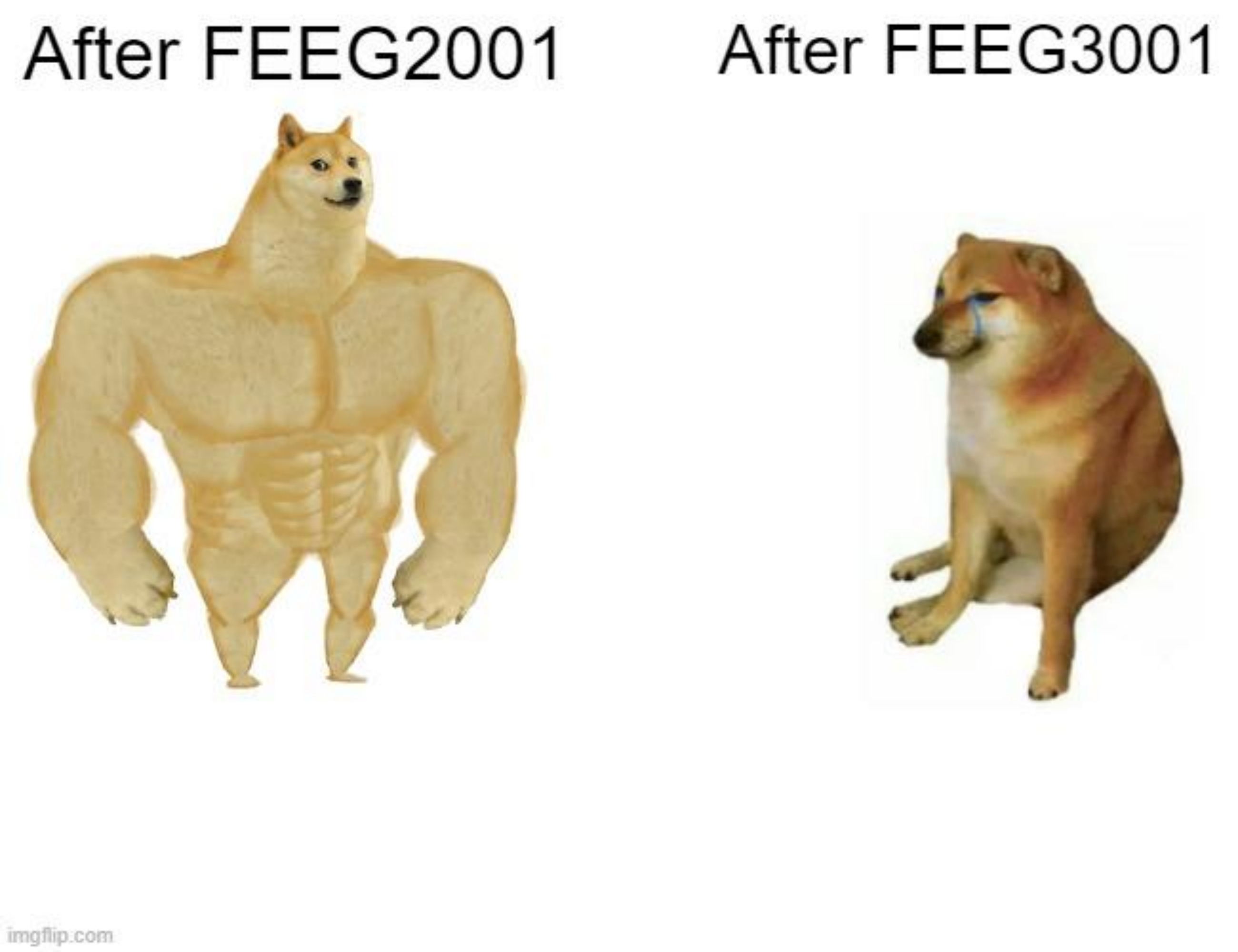




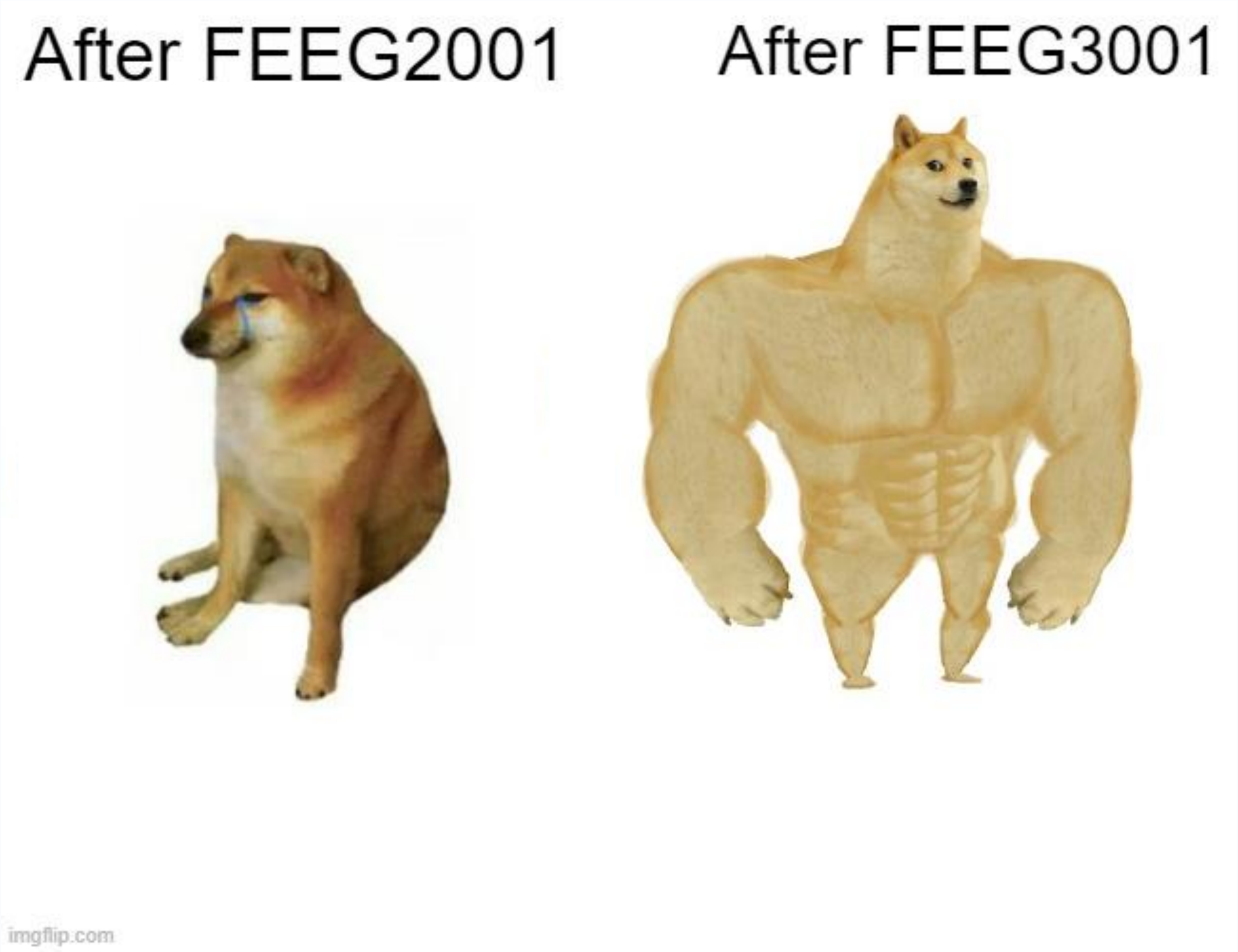
Summary:

- We have written the internal field of displacement $u(x)$ in terms of two chosen functions that happen here to be linear (Shape Functions) $g_1(x)$ and $g_2(x)$.
- $g_1(x)$ has a property where its value is 1 at left and 0 at right, and
- $g_2(x)$ has a value 0 at left and 1 at right.
- Because they are linear functions, what about their combination? It must also be a linear function, if q values do not depend on x (they are single, nodal values).
- Next class will continue towards how we find values for q!

How you might feel today:



Where we are going:




MUSIC BAND
HOW DO YOU DO, FELLOW KIDS?

Part 2b: Elastic Rods in Tension and Compression

FEEG3001/SESM6047 FEA in Solid Mechanics

Dr A S Dickinson

From 15th October 2024

Elastic Rods

- Remember we had a rod of length L: with a cross section at x where the displacement is $u(x)$ and the properties are given by E and A.
- And we defined ***shape functions*** which vary linearly along the length and have the properties:

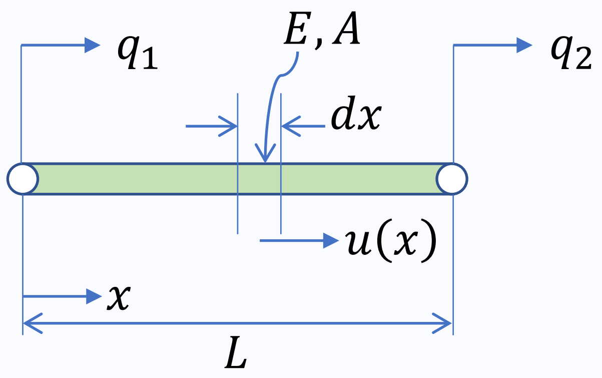
$$g_1(x) = 1 - \frac{x}{L}$$

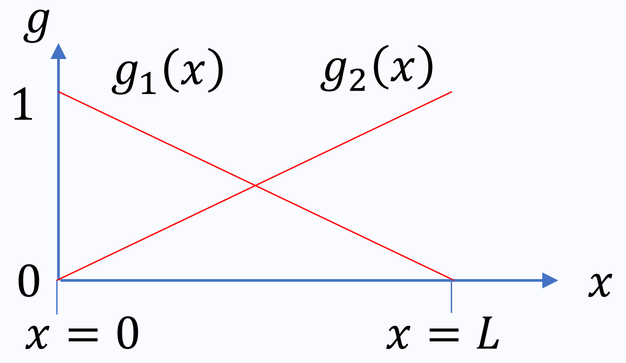
$$g_2(x) = \frac{x}{L}$$

- And because we can combine them linearly:

$$u(x) = \left(1 - \frac{x}{L}\right) q_1 + \left(\frac{x}{L}\right) q_2$$

- so to estimate $u(x)$ anywhere we are allowed to use a linear interpolation or '***approximation***'.





Elastic Rods

$u(x) = \left(1 - \frac{x}{L}\right) q_1 + \left(\frac{x}{L}\right) q_2$, or more generally:

$u(x) = g_1(x) q_1 + g_2(x) q_2$

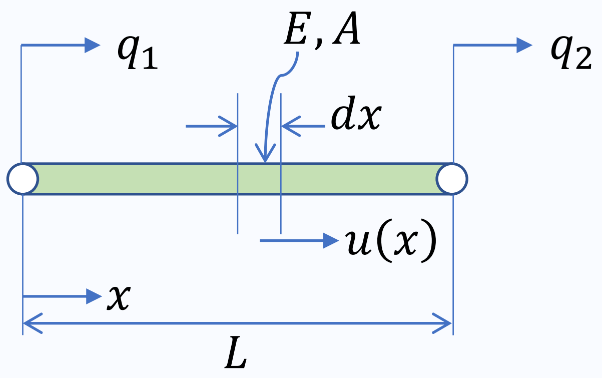
- Sometimes this will be exact, but often it is an approximation. We will see many functions of this form.
- We want to find q_1 and q_2 which are unknowns but which do not depend on x. Remember we had:

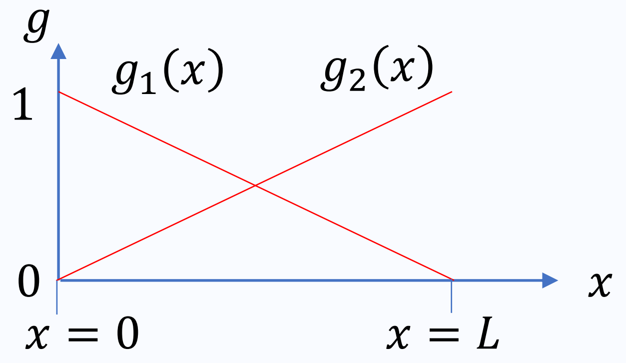
$U = {}^1\!/_2 \int EAu'^2 dx$

$u'(x) = g_1{}'(x) q_1 + g_2{}'(x) q_2$ so

$u'(x) = \left(-\frac{1}{L}\right) q_1 + \left(\frac{1}{L}\right) q_2$

- Now we substitute into U:





Elastic Rods

$$U = {}^1\!/_2 \int EA\left[\left(-\frac{1}{L}\right)q_1 + \left(\frac{1}{L}\right)q_2\right]^2 dx$$

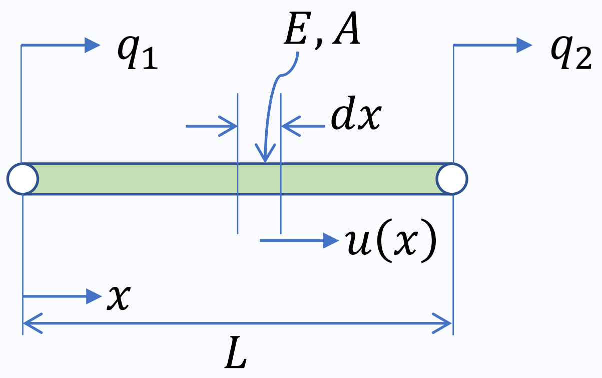
- and in this case we are integrating a constant in x so

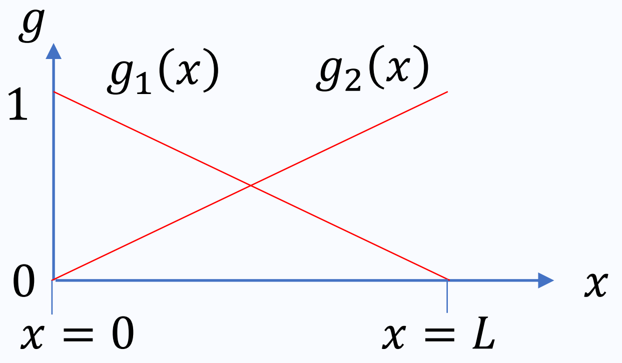
$$U = {}^1\!/_2 \left(\frac{EA}{L^2}\right)(-q_1 + q_2)^2 \int_0^L 1dx$$

$$U = {}^1\!/_2 \left(\frac{EA}{L}\right)(-q_1 + q_2)^2 \text{ so expanding,}$$

$$U = {}^1\!/_2 \left(\frac{EA}{L}\right)({q_1}^2 - 2q_1\, q_2 + {q_2}^2)$$

- Can we reorganise this as we did for springs, using quadratic forms?





Elastic Rods

$$U = {}^{1}\!/_{2}\left(\frac{EA}{L}\right)(q_1{}^2 - 2q_1\, q_2 + q_2{}^2)$$

- In quadratic form:

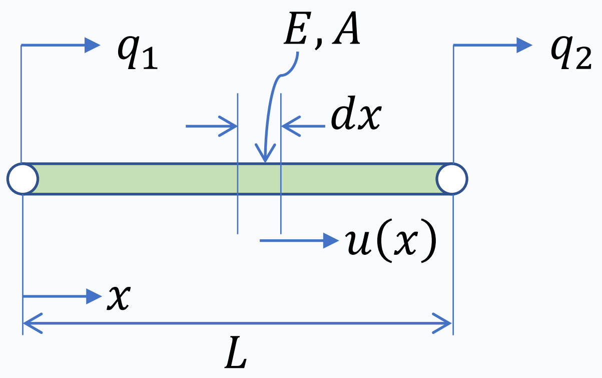
$$U = {}^{1}\!/_{2}\left(\frac{EA}{L}\right)\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$U = {}^{1}\!/_{2}\left(\frac{EA}{L}\right)\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

- or alternatively:

$$U = {}^{1}\!/_{2}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

- Can you spot our new ***element stiffness matrix***?



Elastic Rods

- Finally, and as before, we need to find the values of our generalised coordinates q_1 and q_2, for a given set of loads and BCs, to solve the problem:

$$U = {}^{1}/_{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$[K] = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}$$

Part 2c: Solving Elastic Rods in Generalised Coordinates

FEEG3001/SESM6047 FEA in Solid Mechanics

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From 15th October 2024

Elastic Rods

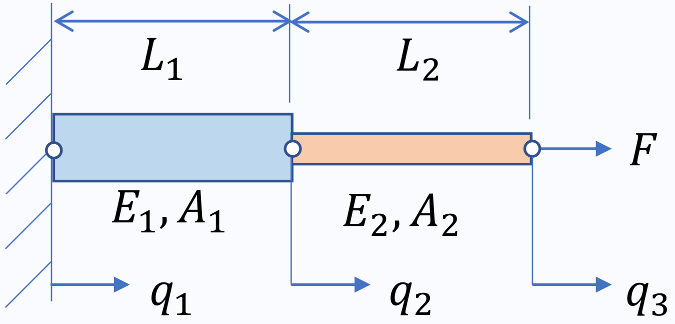
- Now again, like with the springs, we analysed **a typical rod**, which we can use over again, without needing to re-calculate the behaviour each time. So for any, general rod we have:

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

- containing our **element stiffness matrix**, a 2x2 matrix.
- Like last time, we can assemble this into the ***assembled stiffness matrix*** for a ***structure***.

Assembling Elemental Stiffness Matrices:

- Take a simple example of a rod with changing properties along its length
- Considering the elastic strain energy:



$$U = U_1 + U_2$$

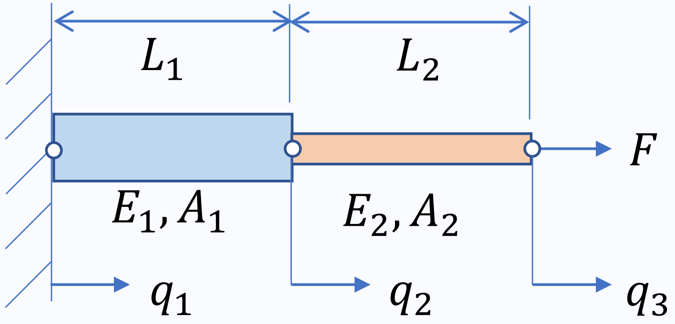
$$U_1 = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$U_2 = {}^1\!/_2 \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^T \begin{bmatrix} \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T_{1\times 3} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}_{3\times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1}$$

Assembling Elemental Stiffness Matrices:

- Take a simple example of a rod with changing properties along its length
- Considering the elastic strain energy:



$$U = U_1 + U_2$$

$$U_1 = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$U_2 = {}^1\!/_2 \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^T \begin{bmatrix} \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T_{1\times 3} \begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} + \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix}_{3\times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1}$$

Assembling Elemental Stiffness Matrices:

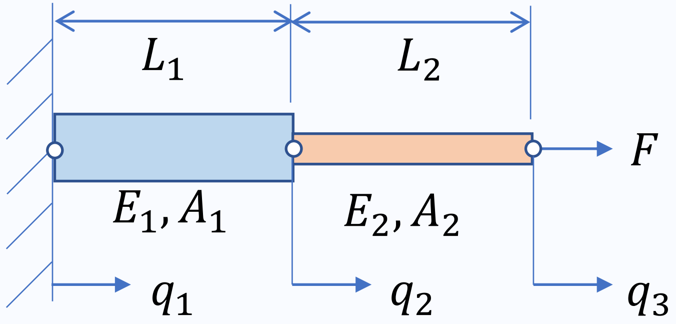
- Considering the work done by the external forces:
- F is the only force which is doing work, so

$$V = -Fq_3$$

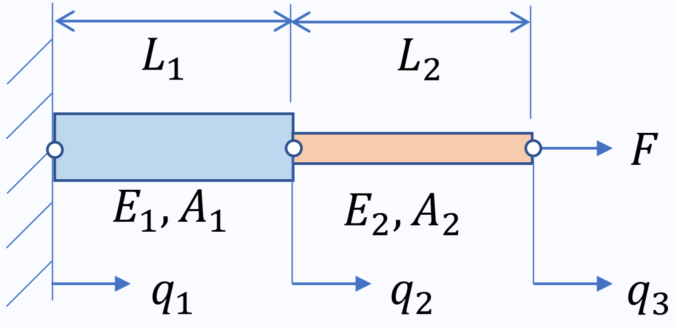
- can be written in the form:

$$V = -\{\qquad\qquad\}_{1\times 3}\{q\}_{3\times 1}$$

$$V = -\{R \quad 0 \quad F\}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$



Assembling Elemental Stiffness Matrices:



- So we have been able to represent the whole system in the matrix form:

$$U = {}^1/_2\, \{q\}^T [K]\{q\}$$

- and

$$V = -\{F\}^T \{q\}$$

- and since:

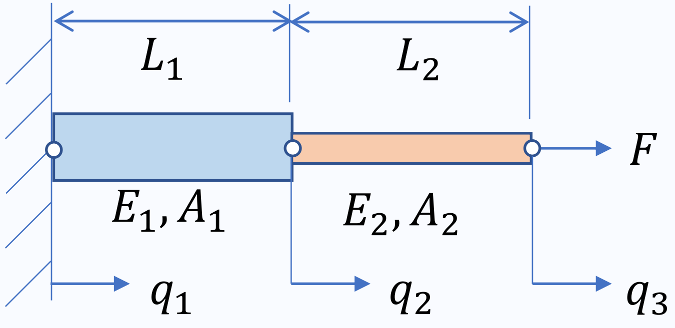
$\Pi = U + V$ so

$$\Pi = {}^1/_2\, \{q\}^T [K]\{q\} - \{F\}^T \{q\}$$

- We want to apply PMTPE, where in i notation Equilibrium says

$$\delta\Pi(q_i) = 0 \Rightarrow \frac{\partial \Pi}{\partial q_i} = 0, i = 1, 2, \dots$$

Assembling Elemental Stiffness Matrices:



- And we know that where we have this quadratic form,

  $\Pi = {}^1/_2 \{q\}^T [K]\{q\} - \{F\}^T\{q\}$

- that the partial derivative of the total potential energy leads to

  $[K]\{q\} = \{F\}$

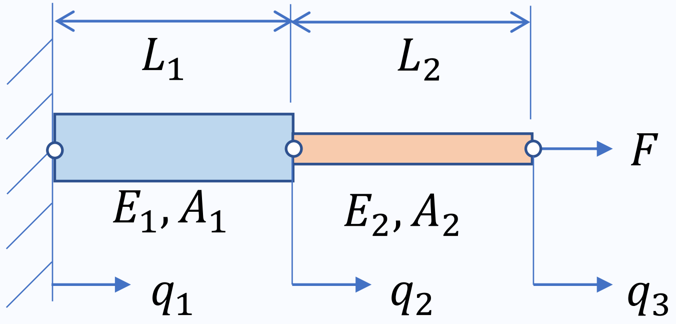
- (i.e. we don't need to derive it all over again, we can just write out our equation of equilibrium):

$$\begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} + \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ F \end{Bmatrix}$$

- And recall we cannot solve this – our stiffness matrix $[K]$ is *singular* and cannot be inverted (its determinant is zero), so alone it cannot be solved. So what do we do?

Assembling Elemental Stiffness Matrices:

- We can apply our boundary condition because $q_1 = 0$:



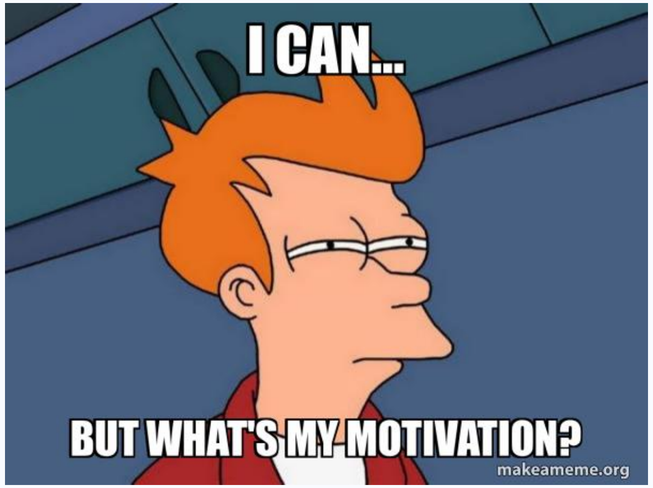
$$\begin{bmatrix} \frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ F \end{Bmatrix}$$ leaving

$$\begin{bmatrix} \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

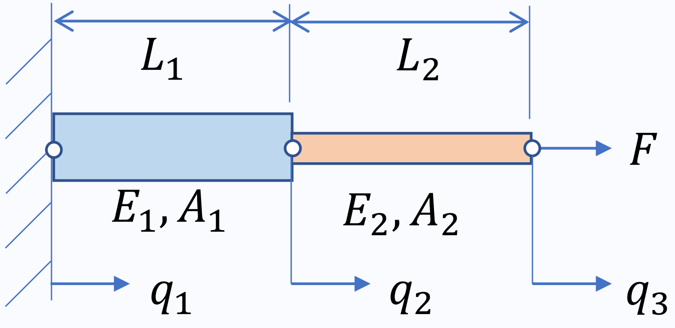
- leaving a 2x2 system (two equations, two unknowns) which we can solve

Why do you solve matrix equations like that?

- Question:
  - Why do we solve our equations in a strange way, by crossing out rows and columns? Why is it OK to ignore those rows and columns?
- Answer:
  - We are not actually ignoring them. We are taking advantage of a simplification possible due to a 4th equation (in a 3x3 system) which comes from our boundary condition (BC), i.e. $q_1 = 0$
  - If you set non-zero displacement BCs, you would need another solution method…!



Assembling Elemental Stiffness Matrices:



- To calculate the reaction force R, we could use intuition – and here it is easy enough we could do it by inspection, but with a complex system we don't have that luxury.
- Instead, we use the rows we crossed out from the general equation of equilibrium:

$$\begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ F \end{Bmatrix}$$

$$R = \frac{E_1A_1}{L_1} q_1 - \frac{E_1A_1}{L_1} q_2 + 0q_3 = -\frac{E_1A_1}{L_1} q_2$$

- That is, you can calculate your reaction solution by using the boundary condition rows.

Finally, a new problem:

- To show you how quickly this can be done, for a less intuitive problem, add a grounded spring (i.e. there is no q_4):

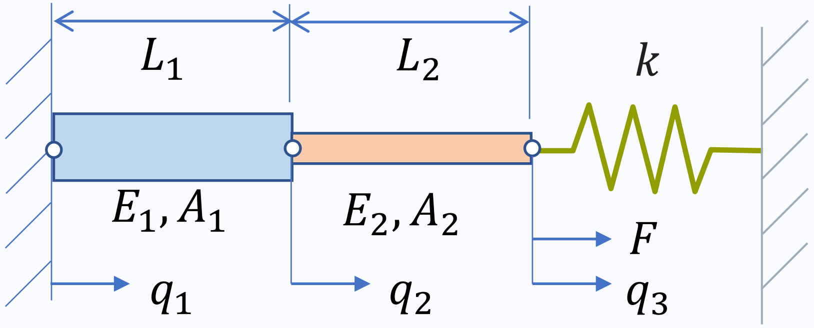
$$U = U_1 + U_2 + U_{spring}$$

$$U_{spring} = {}^1\!/_2\, k {q_3}^2$$

- U_1 and U_2 are the same as before! No work to do. Now assemble the matrix. Where does the k contribute?

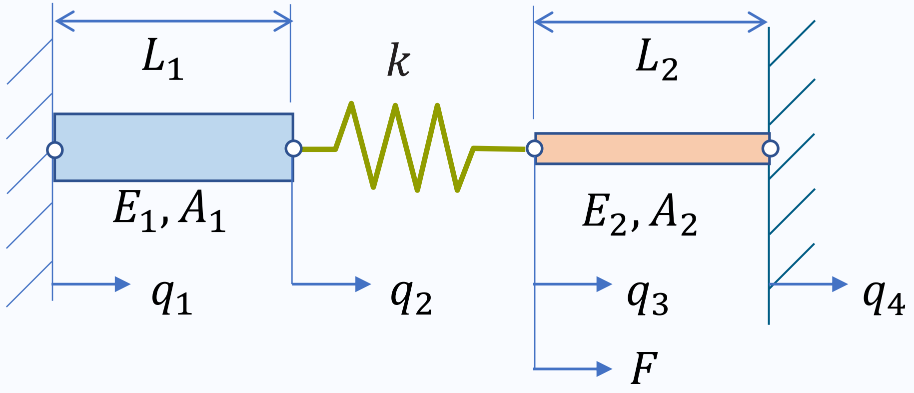
$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T \begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} + \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} + k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

- This spring's stiffness contribution is 'lumped', or lies on the diagonal.
- Note this would NOT affect the force vector, as the spring support is considered 'yielding' and therefore provides no reaction. The ground reaction at the spring would be found using the spring constitutive equation, $R_{spring} = kq_3$



To try at home:

- and what if the spring was not grounded?
- Write down the equation of equilibrium in matrix form, with:
  - the degrees of freedom as shown
  - just one element for each of the two elastic rods in tension/compression, and the spring



Part 2d: Elastic Rods with Distributed Loading

FEEG3001/SESM6047 FEA in Solid Mechanics

Dr A S Dickinson

From 18th October 2024

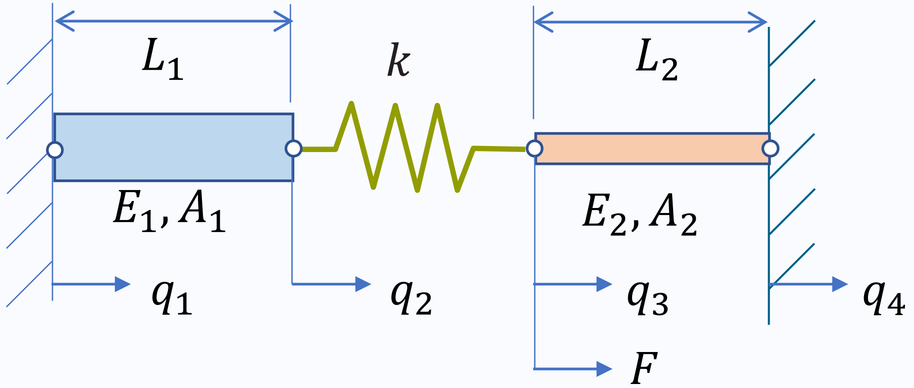
Recap

- Last time we looked at how to derive and then assemble stiffness matrices for elastic rods in tension and compression
- This was quite simple because we assumed a linear interpolation function (or ‘shape function’) for how the elements deform
- This is the simplest interpolation we can get away with
- You could try quadratic or cubic… here linear is adequate, and the stiffness matrix becomes messy if we make it more complicated.
- In the (near) future, we will meet situations where linear interpolation is not sufficient…

But why…?

Solution:

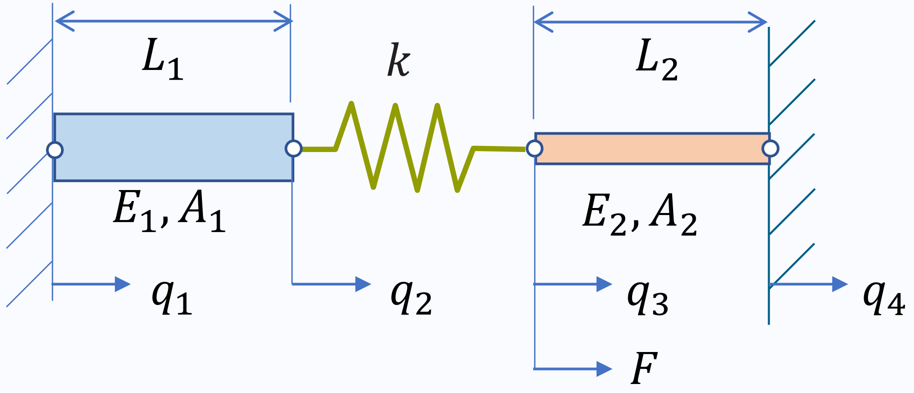
- and what if the spring was not grounded?
- Short-Hand:
- Where does the k contribute to the stiffness matrix?



$$U = {}^1/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \begin{bmatrix} \frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} & 0 & 0 \\ -\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1} + k & -k & 0 \\ 0 & -k & \frac{E_2 A_2}{L_2} + k & -\frac{E_2 A_2}{L_2} \\ 0 & 0 & -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

- Why?

Solution:



- Long-hand:

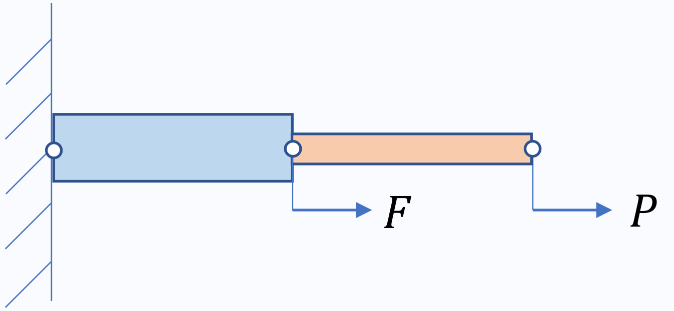
$$U = U_1 + U_2 + U_{spring}$$

$$U = U_1 + U_2 + {}^1\!/_2\, k(q_3 - q_2)^2$$

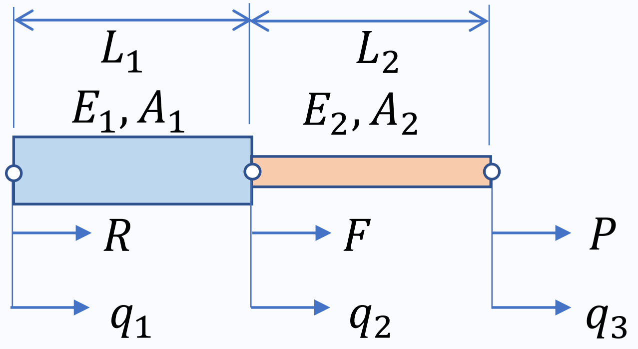
$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} + k & -k & 0 \\ 0 & -k & \frac{E_2A_2}{L_2} + k & -\frac{E_2A_2}{L_2} \\ 0 & 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

- This spring's stiffness contribution is not 'lumped'.

Recap of our solution sequence in FEM:



- If our problem is as sketched on the right, we can formulate it without the fixed wall on the left, with a variable for that node's displacement, and an unknown reaction force.



1. We can assemble a stiffness matrix to solve it:

$$\begin{bmatrix} a & b & 0 \\ c & d & e \\ 0 & f & g \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R \\ F \\ P \end{Bmatrix}$$

2. giving us simultaneous equations:

$$aq_1 + bq_2 + 0q_3 = R$$

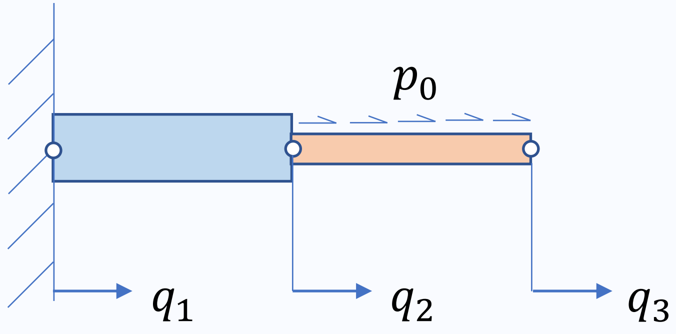
$$cq_1 + dq_2 + eq_3 = F$$
$$0q_1 + fq_2 + gq_3 = P$$

5. then finally solve this row for R

3. Apply BCs: i.e. say $q_1 = 0$

4. Solve these for q_2 and q_3

Today a new problem:

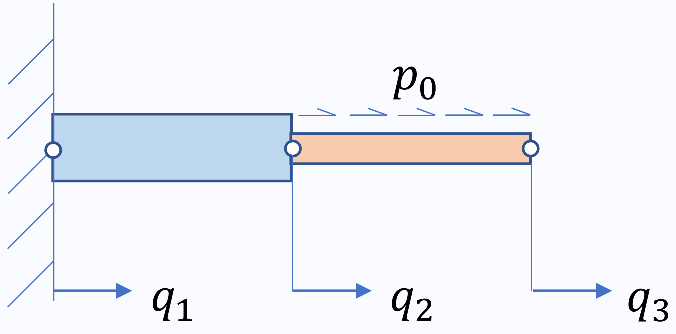


- Three generalised coordinates, variables needed to describe the configuration of the mechanical system
- The properties of the rods don't matter yet.
- How will the Stiffness Matrix be affected?
- As before:

$$U = {}^{1}/_{2}\{q\}^T[K]\{q\}$$

- but how to deal with a distributed force/unit length, p_0?
- Where do the forces come from? Unlike the previous version, the forces do not act at a node here…

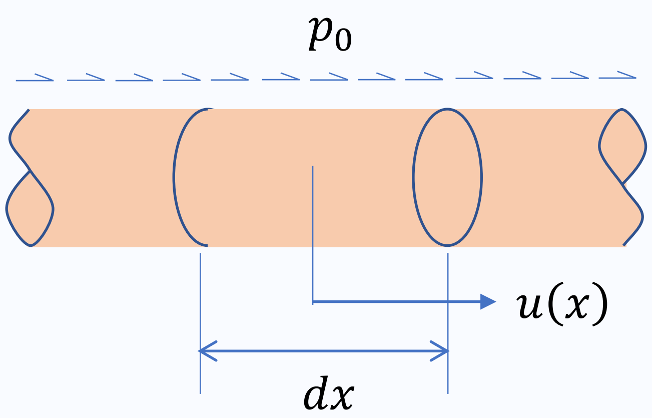
Forces which are not applied at nodes:



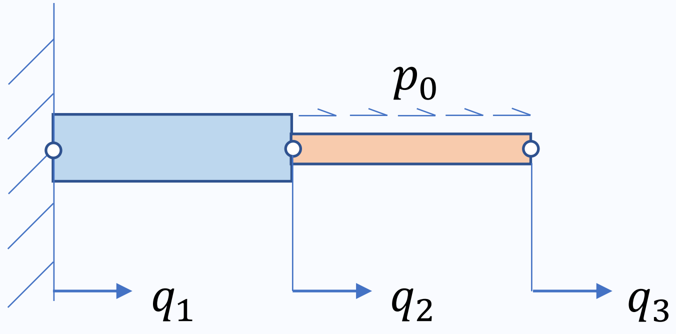
- Where do the forces come from? Unlike the previous version, the forces do not act at a node here…
- What is the work done by distributed force p_0?
- What is the work done on a small slice dx?

$$V = -W = -\int_{x=0}^{L_E} \underbrace{p_0 dx}_{\text{the force}} \times \underbrace{u(x)}_{\text{the unknown}}$$

in terms of the *element's x*



Forces which are not applied at nodes:

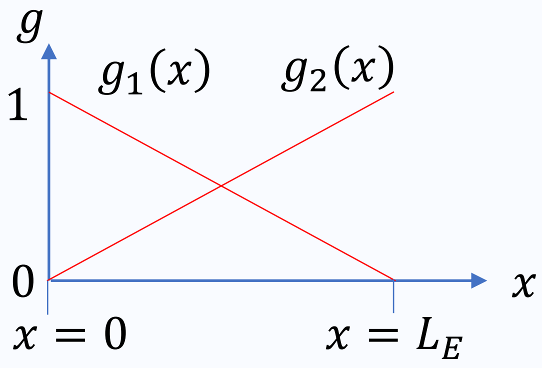


- What is the work done on a small slice $u(x)$?

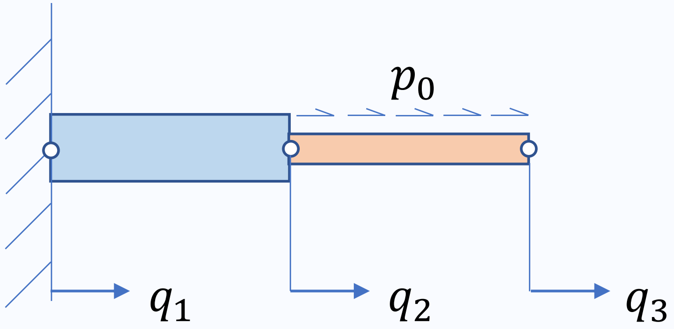
$$V = -W = -\int_{x=0}^{L_E} p_0 dx \times u(x)$$

- Recall $u(x)$ is unknown, but we agreed before to approximate it as a linear function, locally within the element. So we can interpolate as:

$$u(x) = \left(1 - \frac{x}{L_E}\right) q_2 + \left(\frac{x}{L_E}\right) q_3$$



Forces which are not applied at nodes:



- So substituting:

$$V = -W = -\int_{x=0}^{L_E} p_0 \left[\left(1 - \frac{x}{L_E}\right) q_2 + \left(\frac{x}{L_E}\right) q_3 \right] dx$$

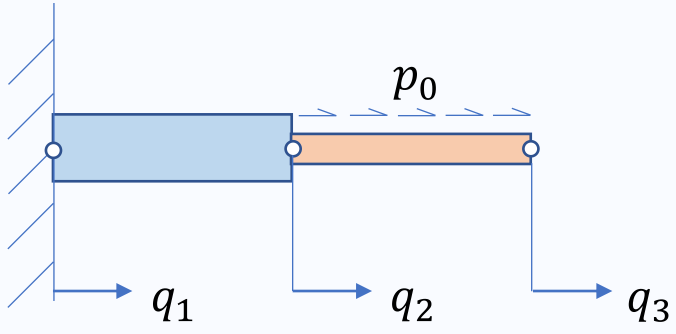
$$V = -p_0 \left[\left(x - \frac{x^2}{2L_E}\right) q_2 + \left(\frac{x^2}{2L_E}\right) q_3 \right]_0^{L_E}$$

$$V = -p_0 \left[\left(L_E - \frac{L_E}{2}\right) q_2 + \left(\frac{L_E}{2}\right) q_3 \right]$$

$$V = -p_0 \left[\left(\frac{L_E}{2}\right) q_2 + \left(\frac{L_E}{2}\right) q_3 \right]$$

- so we see an intuitive (but not general!) phenomenon: the total force can be split half and half between the degrees of freedom.

Forces which are not applied at nodes:



$$V = -\left(p_0 \frac{L_E}{2}\right) q_2 - \left(p_0 \frac{L_E}{2}\right) q_3$$

- and recall that we expected V to have a form something like:

$$V = -\{F\}^T\{q\}$$

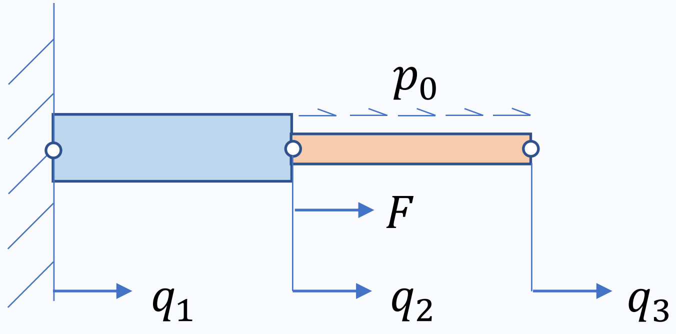
- and in that case we known PMTPE will give us a solution of the form:

$$[K]\{q\} = \{F\}$$

- So without needing to solve every step, we can say that our generalised equation of equilibrium for the system is:

$$\begin{bmatrix} & & 0 \\ & & \\ 0 & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R \\ p_0 \dfrac{L_E}{2} \\ p_0 \dfrac{L_E}{2} \end{Bmatrix}$$

Another example (to show it's easy!):



- Suppose we add a further nodal force F:

$$V = -\left(p_0 \frac{L_E}{2} + F\right) q_2 - \left(p_0 \frac{L_E}{2}\right) q_3$$

- so

$$\begin{bmatrix} & & 0 \\ & & \\ 0 & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R \\ p_0 \dfrac{L_E}{2} + F \\ p_0 \dfrac{L_E}{2} \end{Bmatrix}$$

Practice Questions

- A typical sequence:

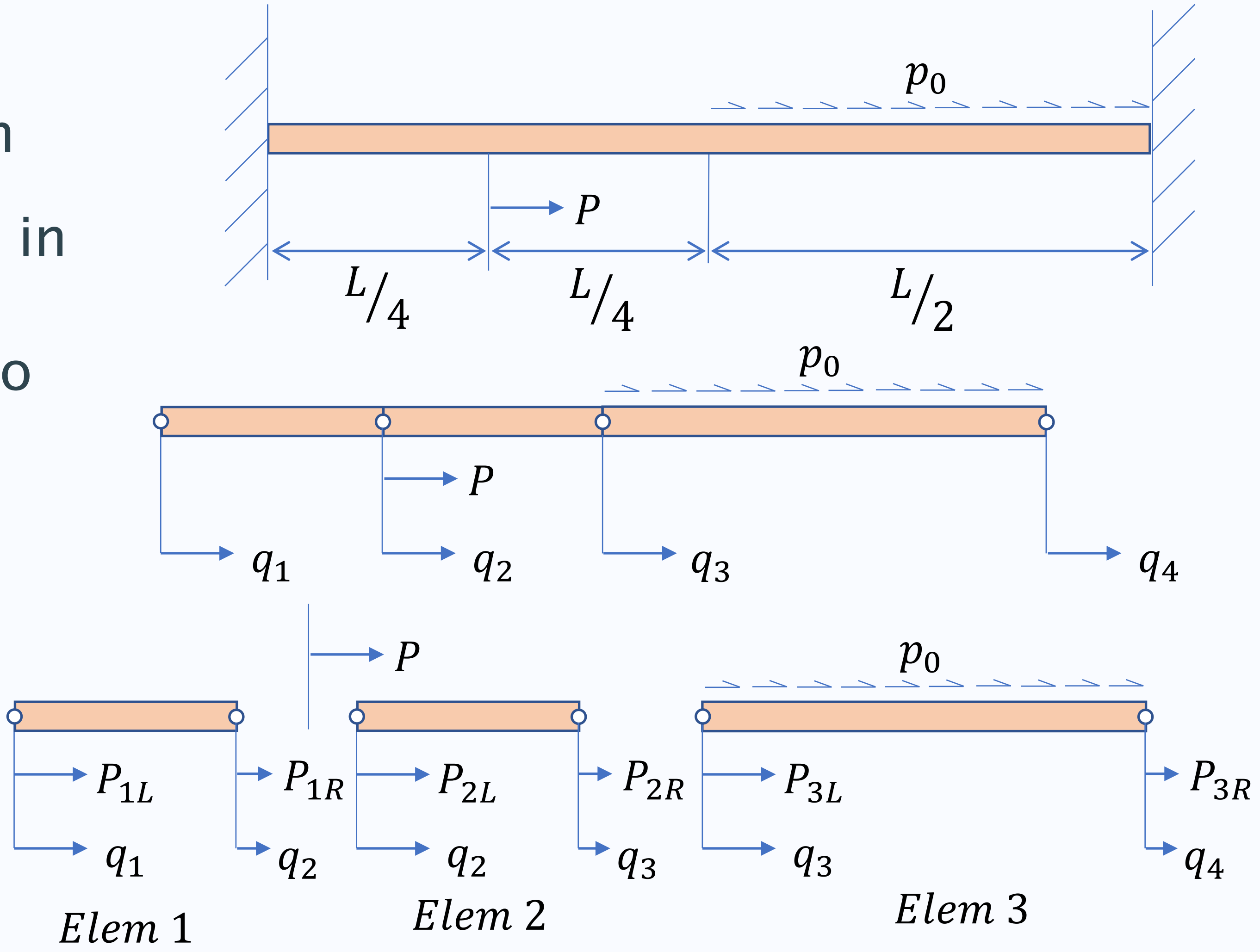
1. Draw a Diagram
2. Idealise the Elements, calculating equivalent nodal forces
3. Assemble the Global Stiffness Matrix and Equilibrium Equation
4. Invert the Stiffness Matrix and solve for Nodal Displacements
5. Solve for Displacement across elements
6. Solve for Strain across elements
7. Solve for Stress across elements

An example question:

- The choice of elements might be defined by the loads we wish to apply, and our required resolution of deformation, strain and stress results, instead of by the changes in rod cross section.
- How many elements do we need to model this scenario?
- At least 3 elements
- (not all working is shown!)
- Nodal forces:

$$P_{1L} = R_L \qquad P_{1R} + P_{2L} = P$$

$$P_{2R} + P_{3L} = \frac{p_0 L}{4} \qquad P_{3R} = R_R + \frac{p_0 L}{4}$$



An example question:

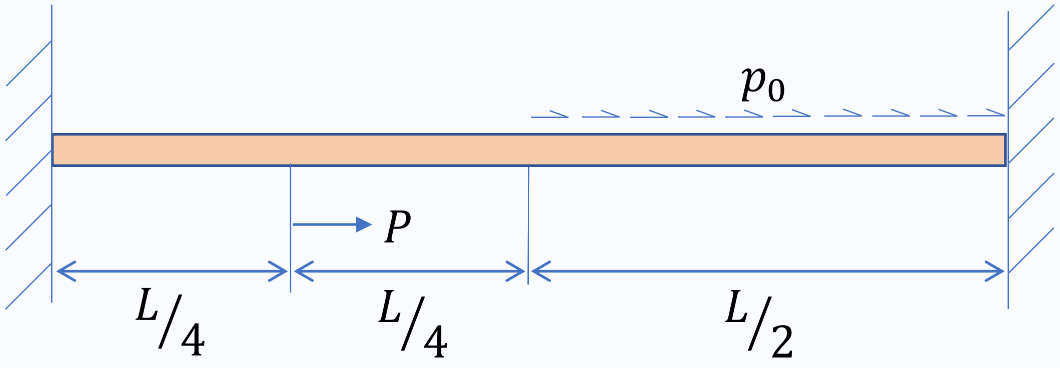
- Find the generalised equation of equilibrium

$$\frac{EA}{L}\begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} ? \\ ? \\ ? \\ ? \end{Bmatrix}$$

- Apply boundary conditions to obtain the reduced equation of equilibrium

$$\frac{EA}{L}\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} ? \\ ? \end{Bmatrix}$$

which can be solved, as $\left|K^{reduced}\right| \neq 0$.



An example question:

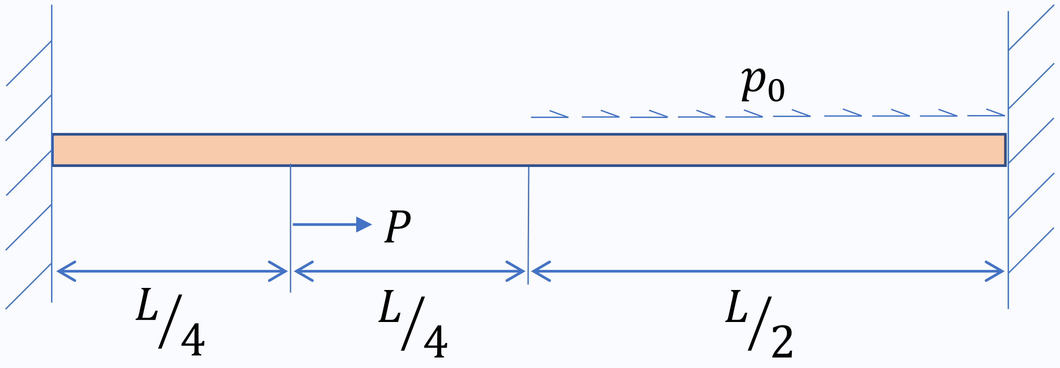
- Find the generalised equation of equilibrium

$$\frac{EA}{L}\begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 4+4 & -4 & 0 \\ 0 & -4 & 4+2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} R_L \\ P \\ \frac{p_0 L}{4} \\ \frac{p_0 L}{4} + R_R \end{Bmatrix}$$

- Apply boundary conditions to obtain the reduced equation of equilibrium

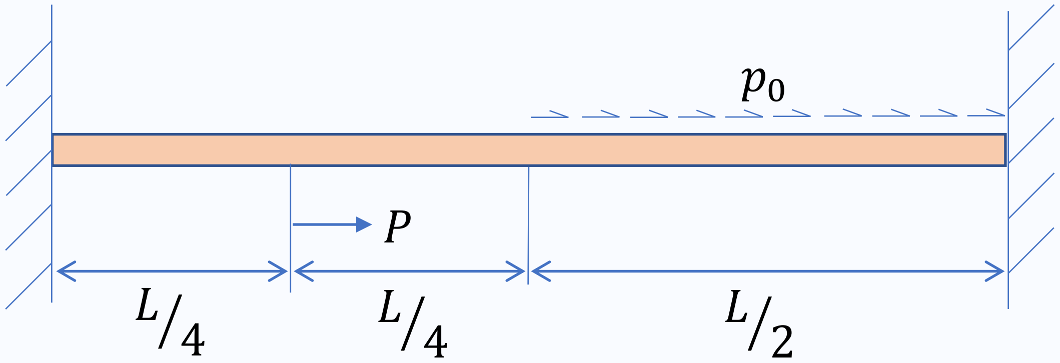
$$\frac{EA}{L}\begin{bmatrix} 8 & -4 \\ -4 & 6 \end{bmatrix}\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} P \\ \frac{p_0 L}{4} \end{Bmatrix}$$

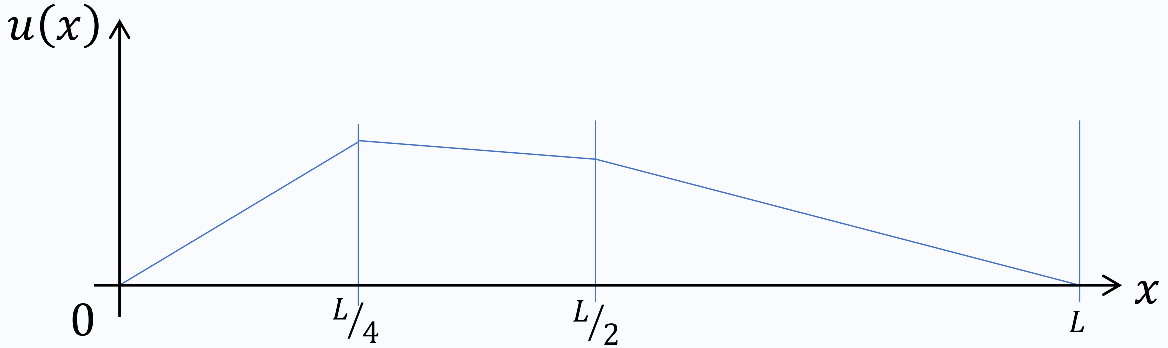
which can be solved, as $\left|K^{reduced}\right| \neq 0$.

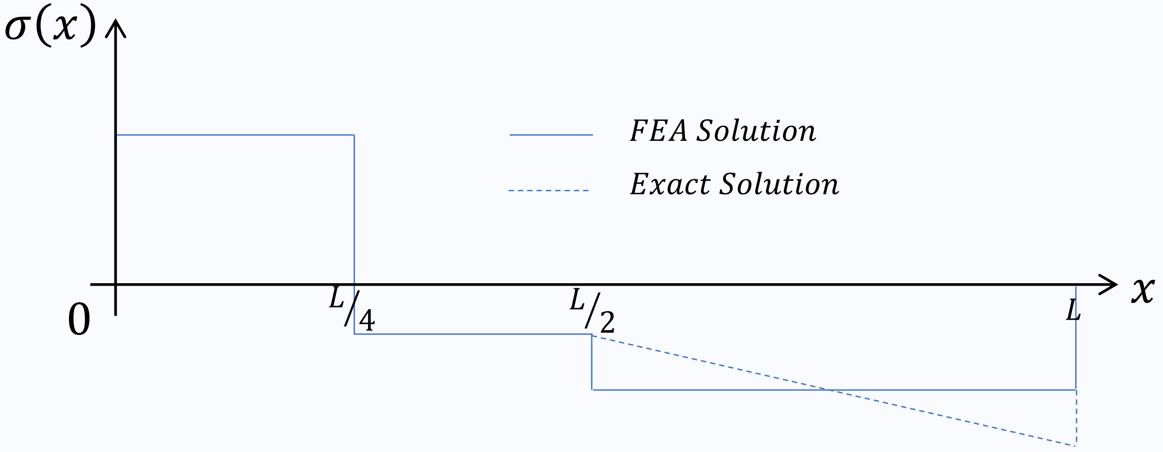


An example question:

- Finally you could calculate and plot the displacement field and the stress field in the structure.



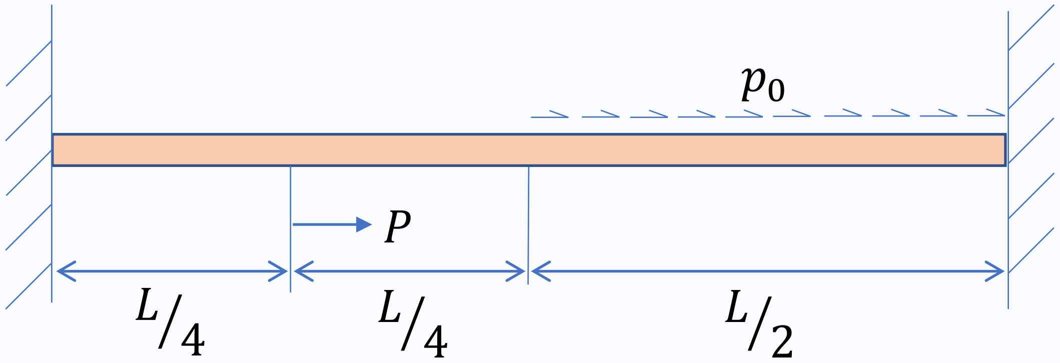


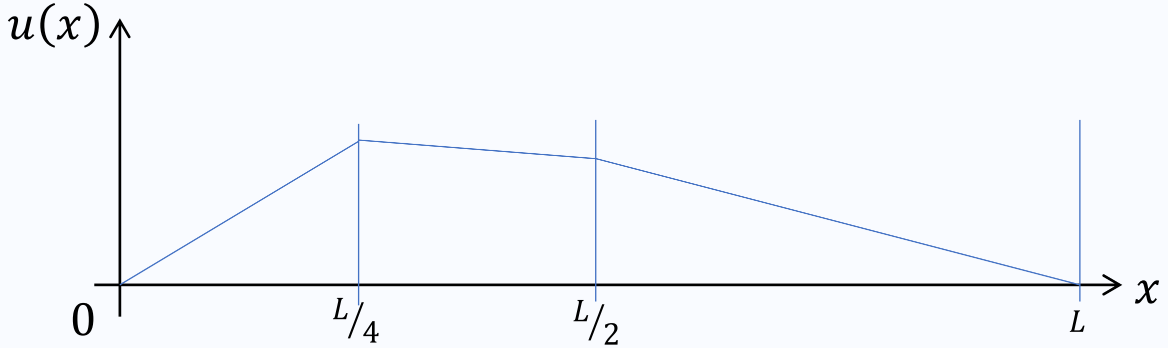


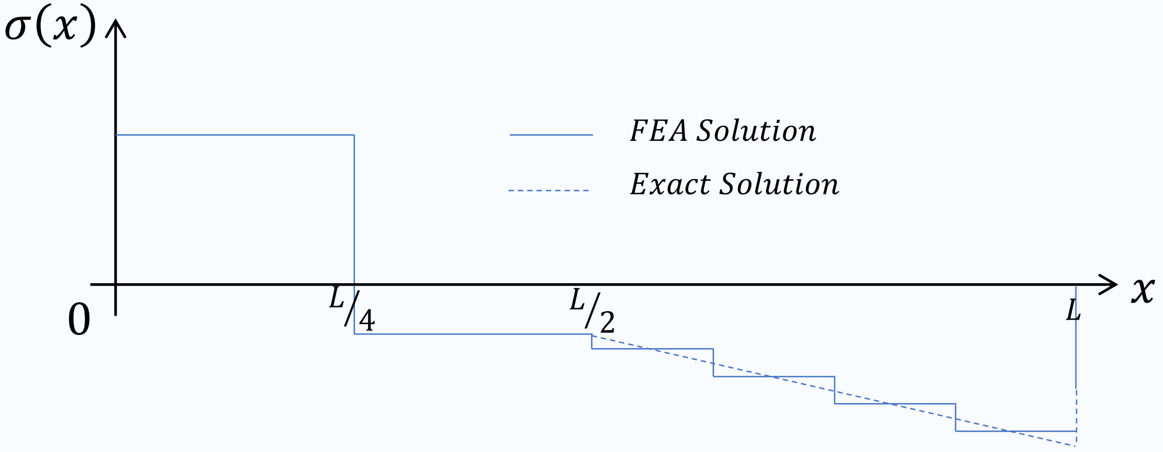
- Plot $\sigma(x)$. How does the exact solution compare?
- How could correspondence be improved?

An example question:

- Finally you could calculate and plot the displacement field and the stress field in the structure.







- Plot $\sigma(x)$. How does the exact solution compare?
- More subdivisions to improve agreement?

Part 2e: Elastic Rod Example Questions

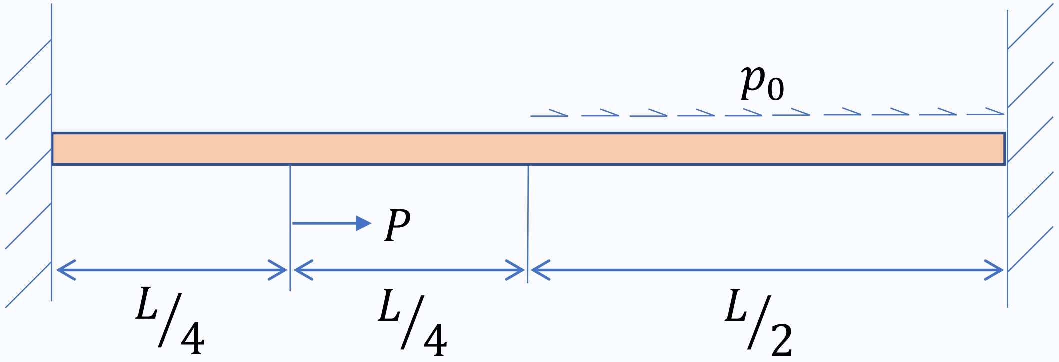
FEEG3001/SESM6047 FEA in Solid Mechanics

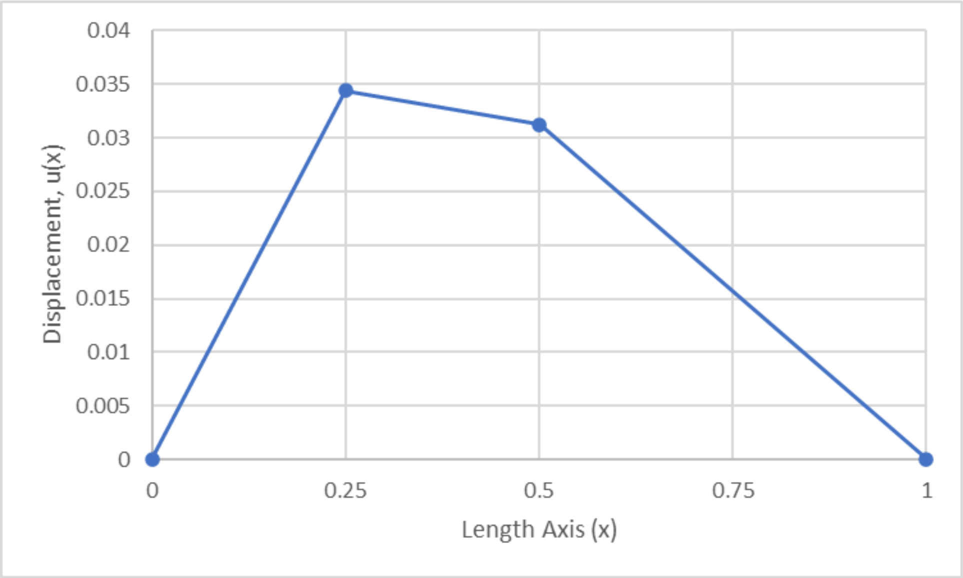
Dr A S Dickinson

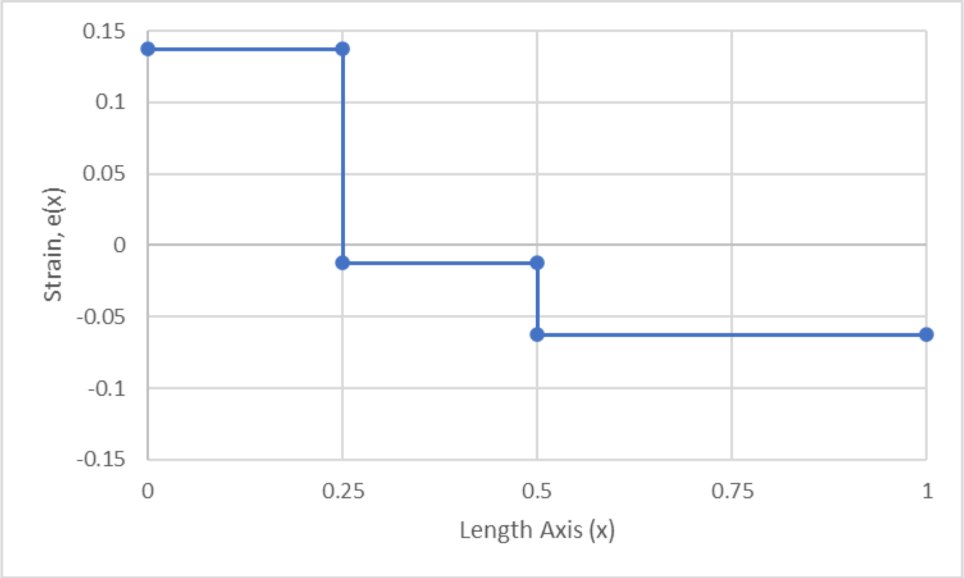
From 25th October 2024

Queries from last week:

- Where do the lines on this graph come from?
- What do $u(x)$, q_1, q_2 etc. mean?
- x refers to the undeformed length

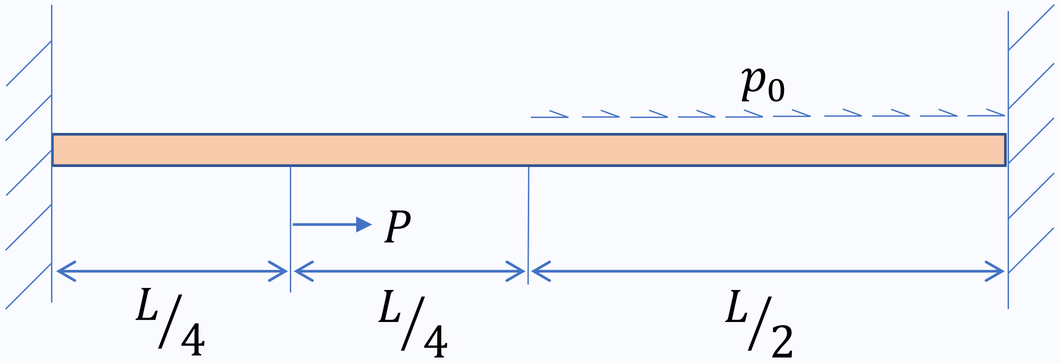


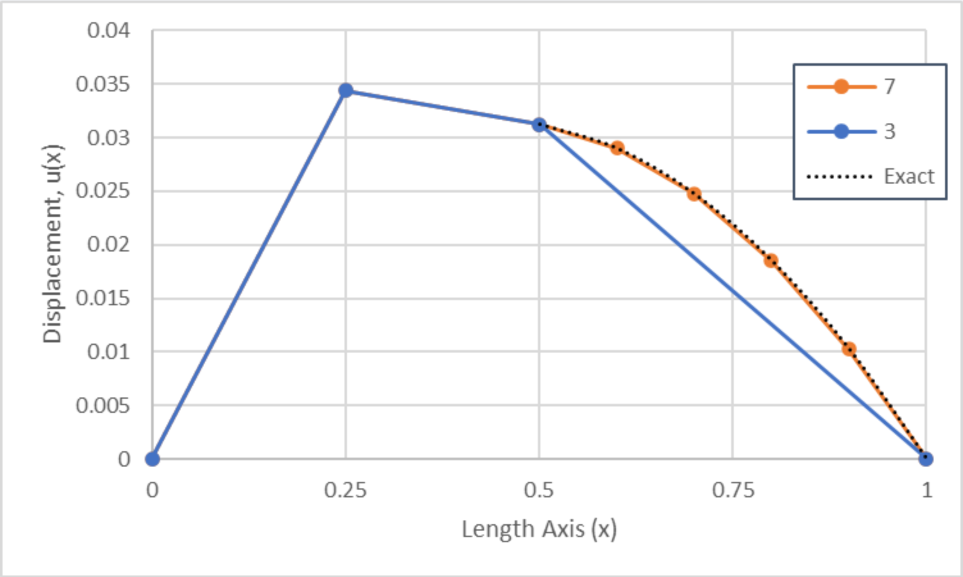


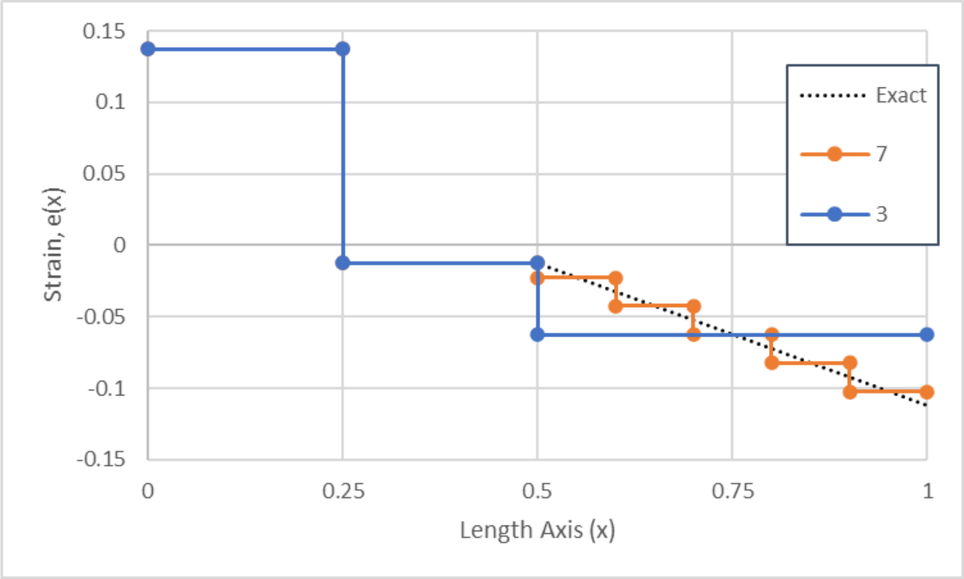


Queries from last week:

- Where do the lines on this graph come from?
- What do $u(x)$, q_1, q_2 etc. mean?
- x refers to the undeformed length



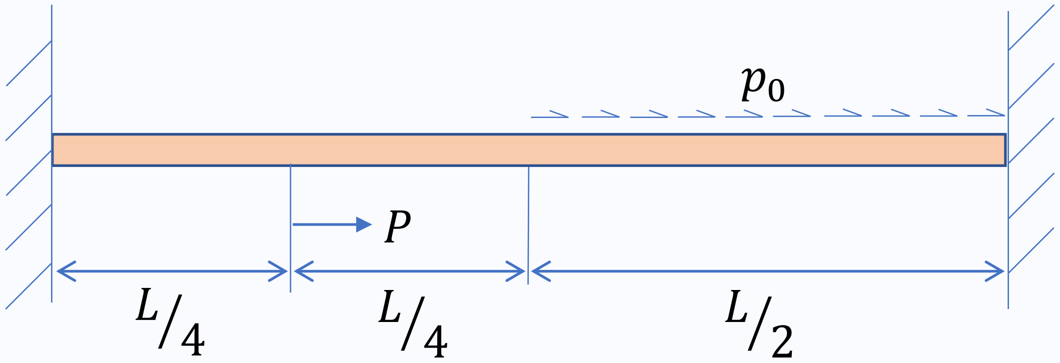




Queries from last week:

- How did we get from here:

$$\frac{EA}{L}\begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} ? \\ ? \\ ? \\ ? \end{Bmatrix}$$



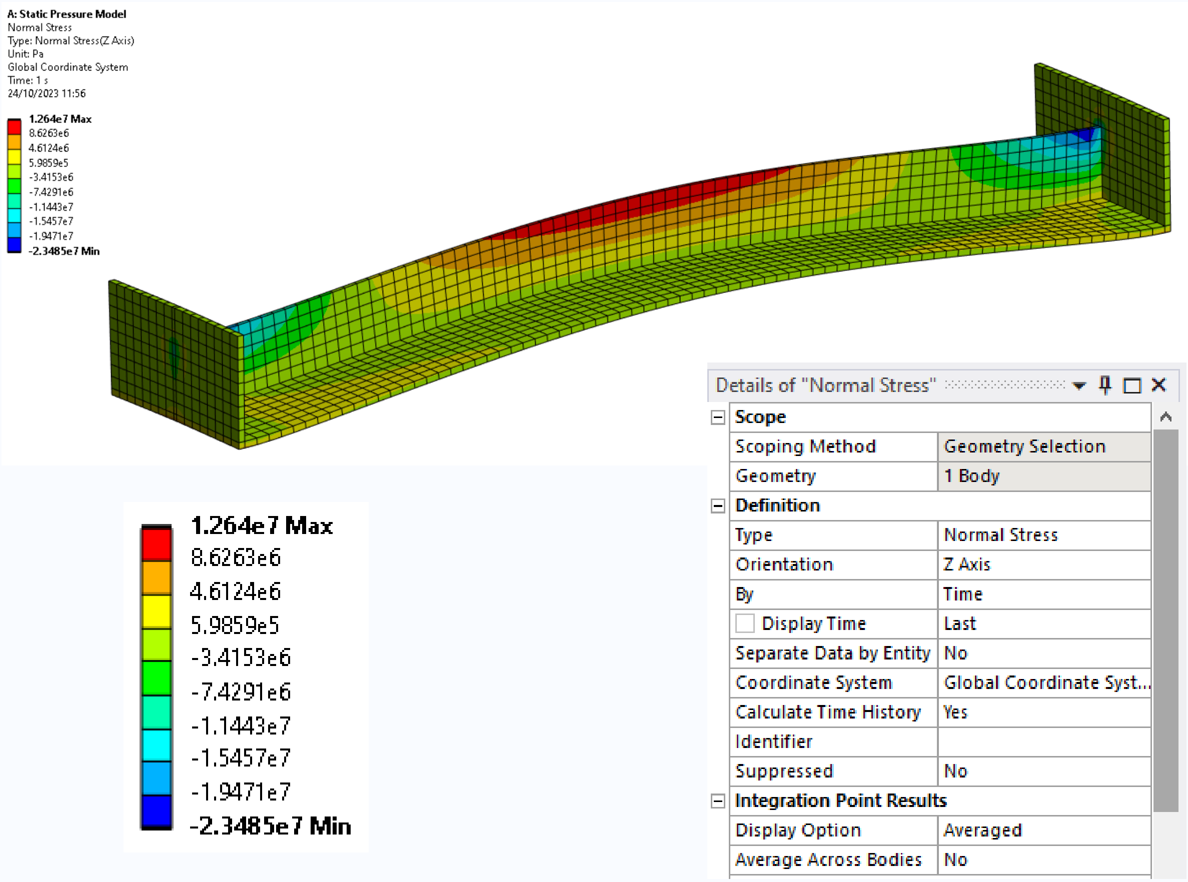
- to here:

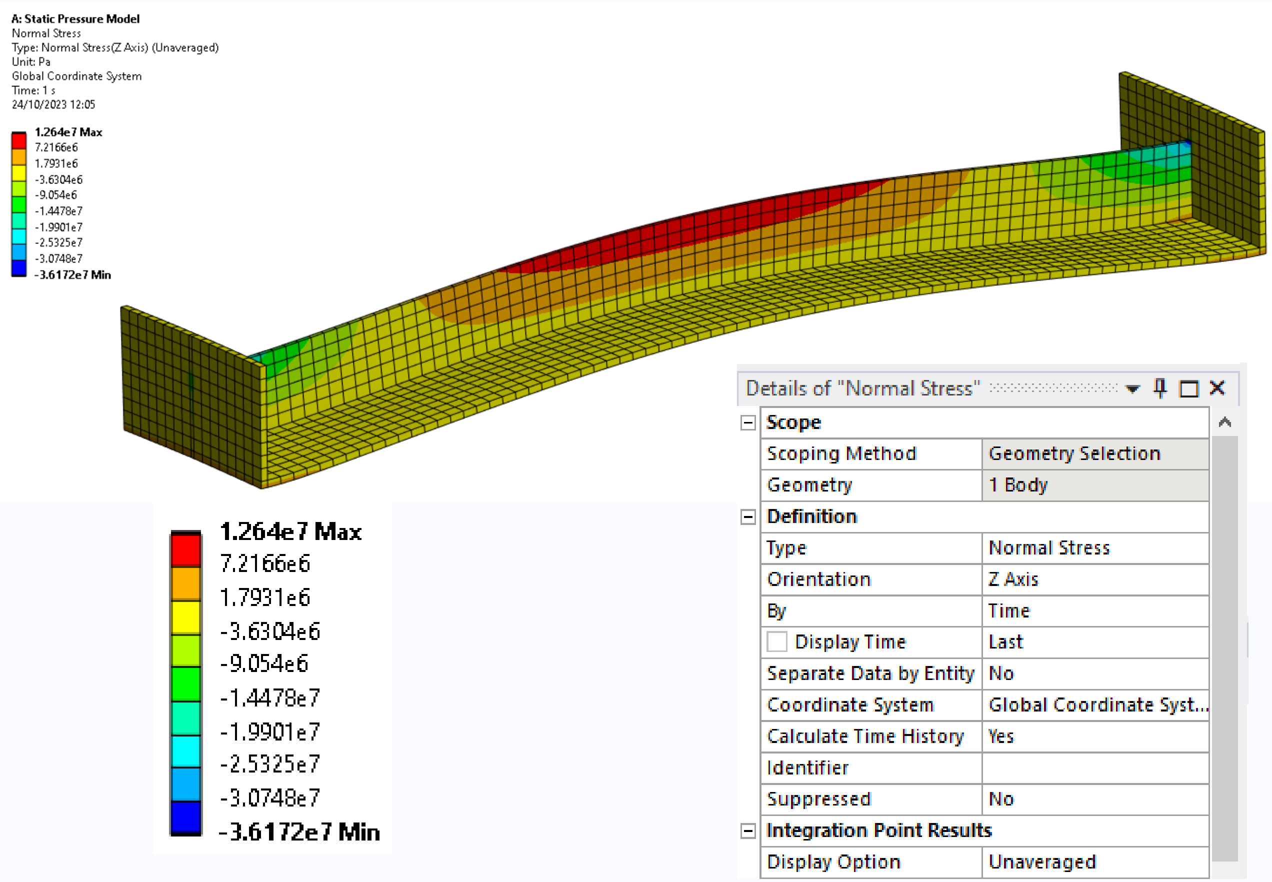
$$\frac{EA}{L}\begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 4+4 & -4 & 0 \\ 0 & -4 & 4+2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} R_L \\ P \\ \dfrac{p_0 L}{4} \\ \dfrac{p_0 L}{4} + R_R \end{Bmatrix}$$

- ?

Queries from last week:

- What did he mean by averaged and unaveraged, or nodal and elemental results?





Queries from last week:

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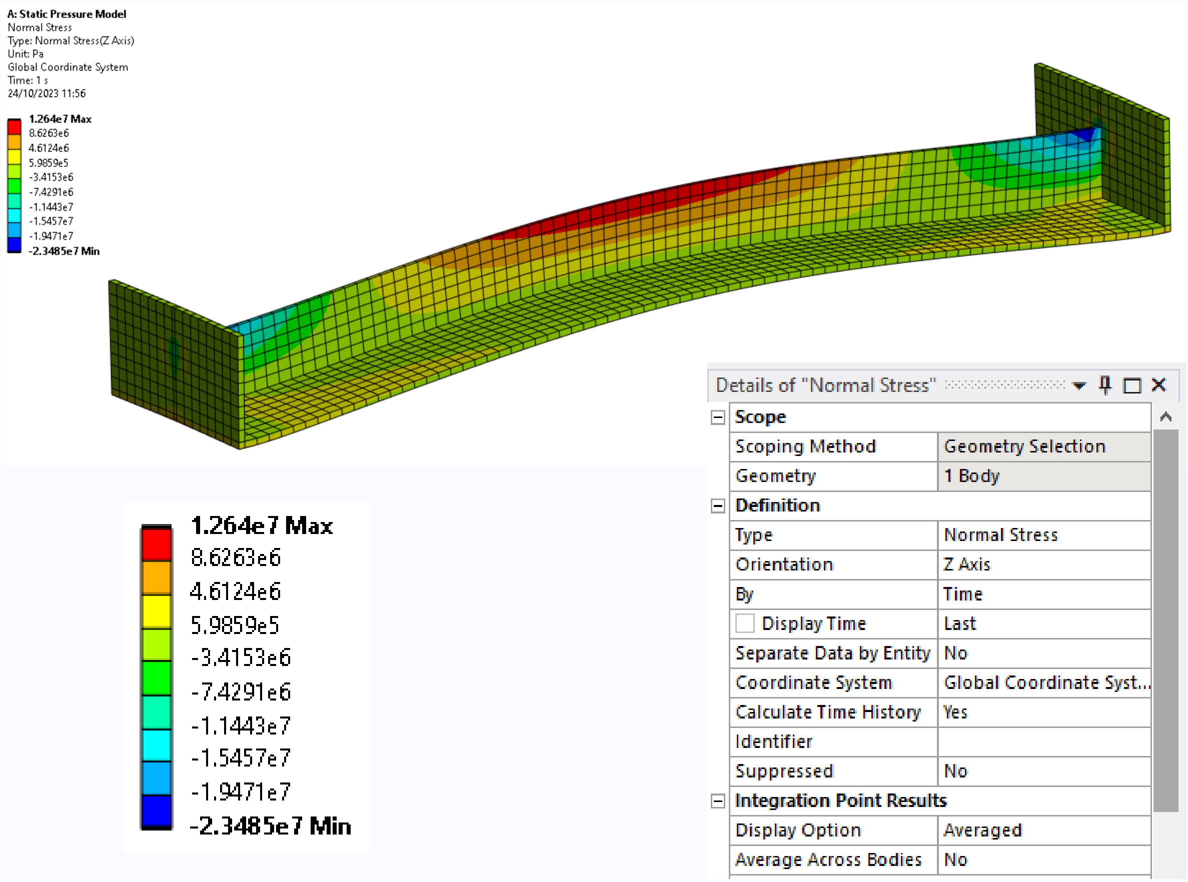
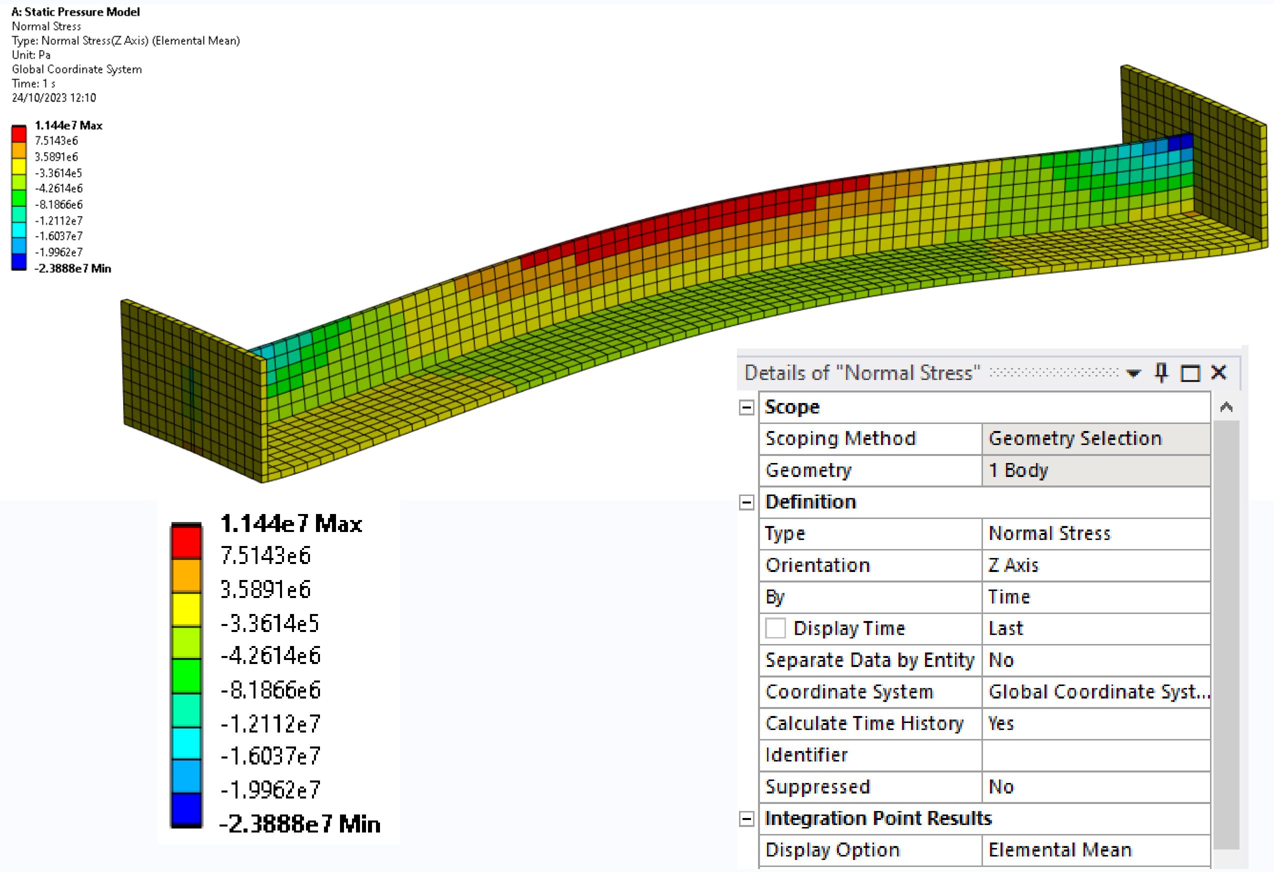


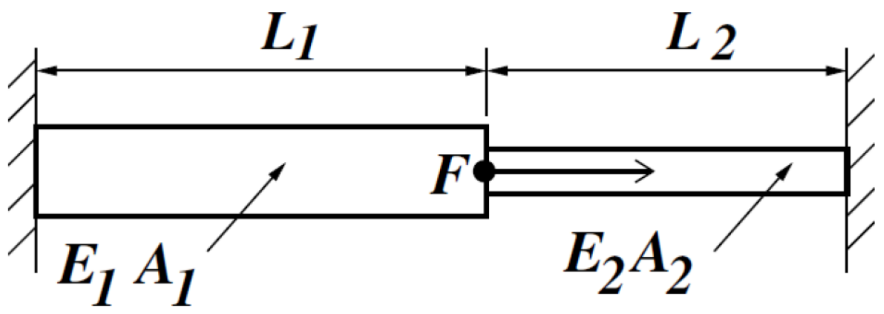


Past Exam Questions:

- 2017/18 Question 3
- 2018/19 Question 5

Q3. Use two finite elements to model the stepped structure in tension and compression as shown in Figure Q3. Calculate reactions at the two fixed ends as well as displacement at the point of application of the external load F, which is the junction of step change in the cross-section.

[12]



Q5. (a) A rod, of total length $2L$, in tension and compression, is fixed at the left end and sprung at the right end, as shown in Figure Q5. Determine the displacement at the centre of the rod and at the point of connectivity of the rod with the spring. A concentrated force F acts at the centre of the rod, whereas a distributed force p_0 per unit length acts on the right half of the rod, in the directions shown in Figure Q5. The axial stiffness of each half is EA, as shown.

[17]

(b) Determine the reaction at the fixed end of the rod.

[3]

(c) Determine the value of the force F for which the spring does not store any strain energy. Comment on this value, concerning the relationship between F and p_0, for the condition of zero energy stored in the spring.

[3]