Part 3a: Introduction to Dynamics

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

From 5th November 2024

University of Southampton

We use your feedback to improve this module and recommend good practice to other modules.
Your feedback is really important to us!

(1) Every comment is read personally by the Programme Leads

(2) For each module the feedback is read and acted on by the relevant module staff

(3) Where we can, we will take actions immediately that will improve the module for you in the current year;

(4) Otherwise, it will be used make a difference for the future

(3) We produce a "You said, we did" summary so you can see how and where we have acted on feedback

(4) A summary of all feedback is presented and discussed at the relevant Education Boards (to which all academic staff are invited) and it feeds into our Staff-Student Liaison committee

Module Feedback FEEG3001

Module Feedback SESM6047

Reminder: what's it all about?

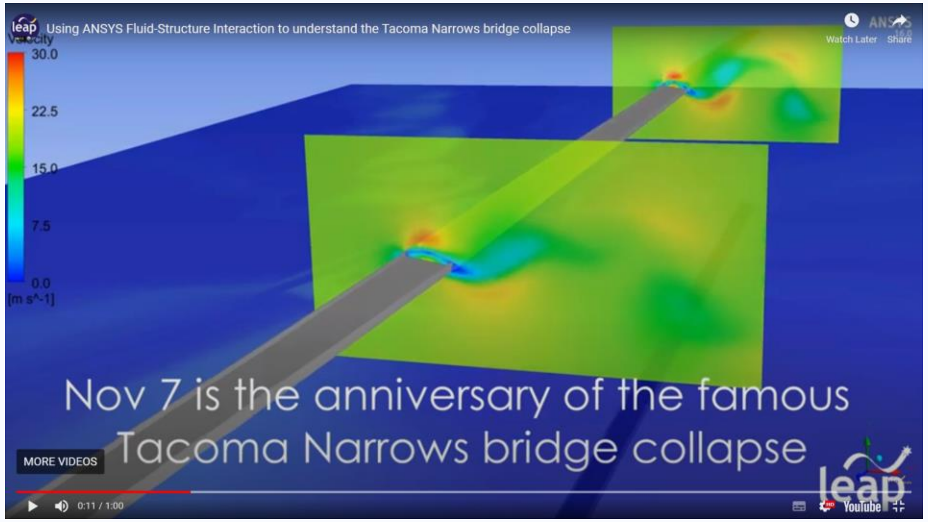
- This module presents FEA using an energy approach, as an alternative to using Free Body Diagrams
- It is about expressing your model in a general way that is convenient to solve. It is about:
  - expressing the energy of a structure or system in a particular form (a quadratic about the DoF),
  - which lets us use the Principle of Minimum Total Potential Energy in order to state the Generalised Equation of Equilibrium using the same characteristic Stiffness Matrix, and then
  - by applying boundary conditions we can solve it and fully describe the structure or system

7/11: anyone know the significance?

1940/11/7?

FEA for Dynamics:

- Finding the frequency of free vibration, without applied load. Why do we care?
- Structures can fail if experiencing conditions below their designed static loading
- Design to avoid resonance
- Even if designing for forced vibration, this is the first step.
- Using ANSYS Fluid-Structure Interaction to understand the Tacoma Narrows bridge collapse - YouTube



What have engineers learned about Fluid-Structure Interaction from the Tacoma Narrows bridge collapse? | Finite Element Analysis (FEA) Blog – LEAP Australia & New Zealand

Why?

- Finding the frequency of free vibration, without applied load. Why do we care?
- Structures can fail if experiencing conditions below their designed static loading
- Design to avoid resonance
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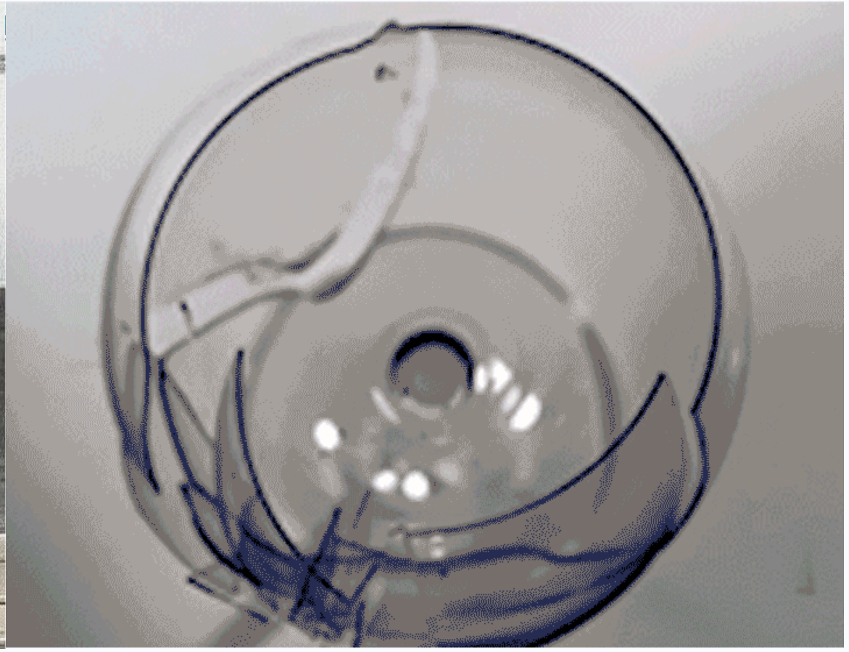


The theory (see the data book):

- Equation of Motion for a Spring-Mass-Damper:

$$m\ddot{q} + \alpha\dot{q} + kq = F\cos\omega t$$

where

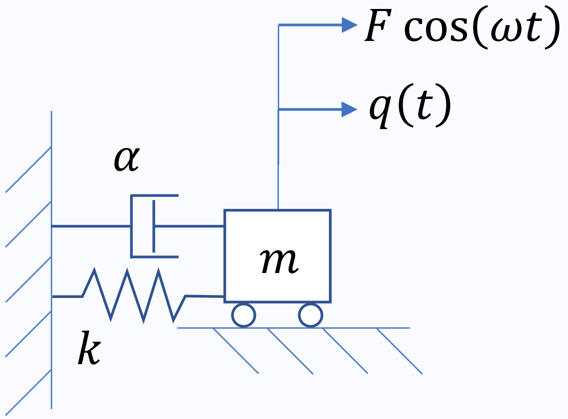
$$\ddot{q} = \frac{d^2q}{dt^2} \text{ and } \dot{q} = \frac{dq}{dt}$$

- Can be expressed as:

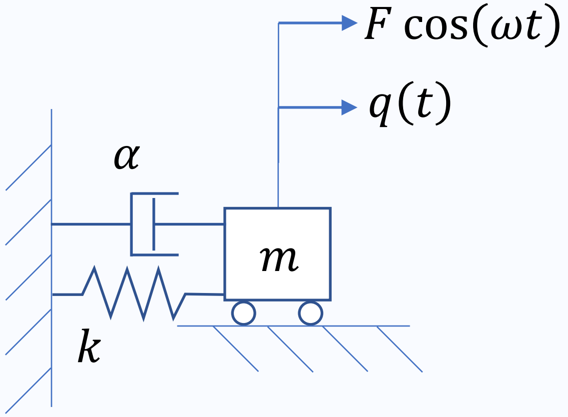
$$\ddot{q} + 2\xi\omega_0\dot{q} + {\omega_0}^2 q = \frac{F}{m}\cos\omega t$$

where

$$\xi = \frac{\alpha}{2\sqrt{mk}} = \frac{\alpha}{2m\omega_0} \text{ and } \omega_0 = \sqrt{\frac{k}{m}}$$



The theory (see the data book):



- Equation of Motion for a Spring-Mass-Damper:

  $$m\ddot{q} + \alpha\dot{q} + kq = F\cos\omega t$$

  where

  $$\ddot{q} = \frac{d^2q}{dt^2} \text{ and } \dot{q} = \frac{d^2q}{dt^2}$$

- If there is no damping, the system is described by:

  $$\ddot{q} + {\omega_0}^2 q = \frac{F}{m}\cos\omega t$$

  where

  $$\omega_0 = \sqrt{\frac{k}{m}}$$

The theory (see the data book):

- A particular integral solution to the damped system is:

$$q(t) = \frac{F}{m\sqrt{({\omega_0}^2 - \omega^2)^2 + (2\omega\omega_0\xi)^2}} \cos(\omega t + \varphi)$$

or, without damping

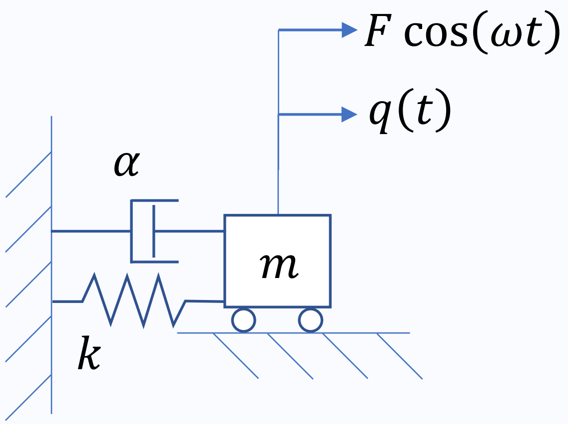
$$q(t) = \frac{F}{m\sqrt{({\omega_0}^2 - \omega^2)^2}} \cos(\omega t + \varphi)$$

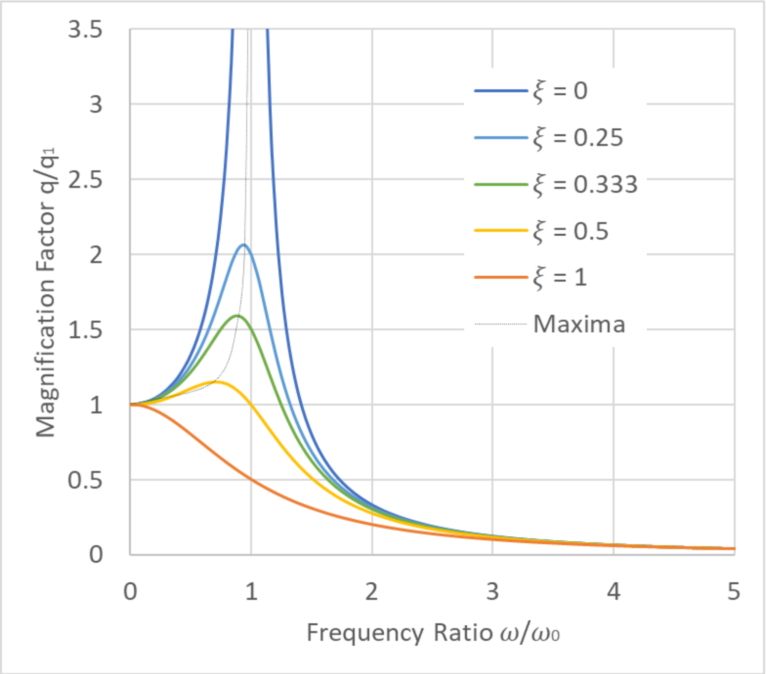
- Reference displacement could be:

$q_1 = \frac{F}{k}$ allowing a magnification factor $\mathrm{A} = \frac{q}{q_1}$

- The system becomes unstable when:

${\omega_0}^2 \to \omega^2$ and $q(t) \to \infty$





What about Dynamics in FEA? Energy Laws

- In Statics, we started by claiming Newton's laws for equilibrium: $(\sum Forces = 0)$ is equivalent to PMTPE.
- So what if we say that the summation of forces = mass x acceleration?
- But what if we still want the bodies to be able to deform (i.e. store elastic energy)?
- We are now concerned with motion, so PMTPE is now inadequate.

What about Dynamics in FEA? Energy Laws

- We have a similar starting point for Dynamics though, valid for all conservative systems (where no work is done externally or extracted, dissipated; that is, it stores energy during motion):

  U represents the potential energy, the stored elastic strain energy in the system

  T represents the kinetic energy

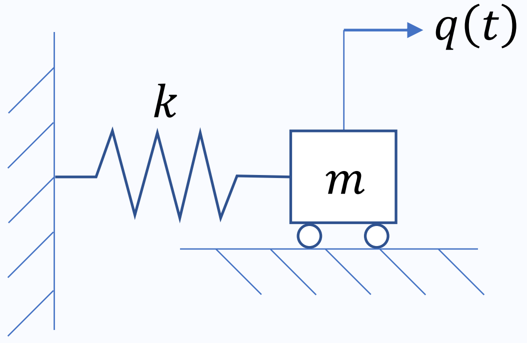
- A new term is the Lagrangian of the system, L, where:

  $L = T - U$

- Note the Lagrangian is *not* the same as the total energy we used in statics.
- (also, here 'L' does not denote length…)

Energy Laws

- Conservative, dynamic systems (which store energy during motion):
- You can think of this as occupying different possible states where the system's bodies have some balance of kinetic and elastic potential energy, which can change over time.
- For example, at extremes, in this system the mass has:
  - maximum kinetic energy when the spring is relaxed, with no elastic potential energy, and
  - no kinetic energy when the spring's elastic potential energy is at its maximum, when maximally extended or compressed
- Like PMTPE, Hamilton's principle *compares* imaginary motions
  - valid for all conservative systems, linear and nonlinear



Energy Laws

- Like in PMTPE, instead of solving the differential equations, we search for a feasible solution of all the potential solutions. How?

  $\delta \int_{t_1}^{t_2} L dt = 0$, known as Hamilton's Principle

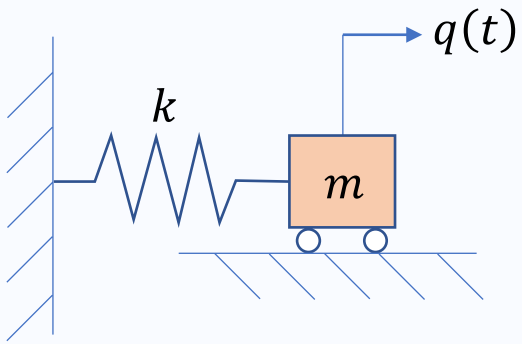
- Previously we imagined deformations – now we imagine motions
- This might be bulk motion or vibratory motion
- There is a true, observed motion, and many imagined alternatives
- And by our variational principle (δ, for comparisons) the integral of the Lagrangian will be higher for imagined cases than the true case
- Again, without the proof (requiring 3 classes of variational calculus), *Hamilton's principle* gives rise to *Lagrange's Equations*…

An example:

- A single DoF system with a mass, which can translate, supported by a spring
- No external work done; free vibration
- What are the energies, T and U?
- Again, the system's configuration can be described by one generalised coordinate, $q(t)$ and its derivative $q(\dot{t})$

$L = T - U$ (the Lagrangian)

$$L = {}^{1}\!/_{2}\, m\dot{q}^2 - {}^{1}\!/_{2}\, kq^2$$



$T = {}^{1}\!/_{2}\, m\dot{q}^2$, where $(\dot{\cdot}) = \frac{d}{dt}(\cdot)$

$U = {}^{1}\!/_{2}\, kq^2$

Lagrange's Equation (1DoF):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

An example:

$$L = L(q, \dot{q}) = {}^1\!/_2\, m\dot{q}^2 - {}^1\!/_2\, kq^2$$

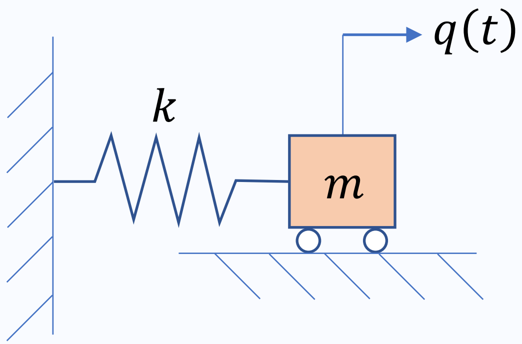
- When we find the partial derivative of Lagrange's equation w.r.t. $\dot{q}$, the q terms behave as constants (and vice versa):

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

$$\frac{\partial L}{\partial q} = -kq$$

$$m\ddot{q} + kq = 0$$

- Again, we did this without a free body diagram as Newton's Law would require



Lagrange's Equation for 1 DoF:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

or for multiple DoFs:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, i = 1,2,..N$$

…as in data books

Part 3b: Introduction to Dynamics: 1 and 2 DoF Systems

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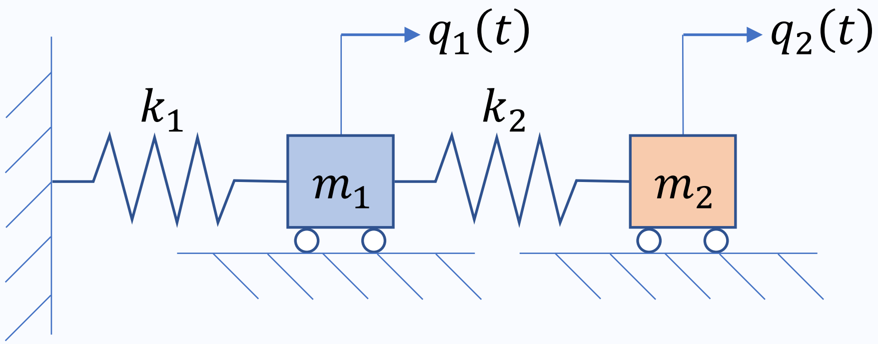
From 8th November 2024

Another example:



- A *multiple DoF* conservative system (which stores energy during motion): by comparisons, the First Variational

$$\delta \int_{t_1}^{t_2} (T - U)dt = 0$$

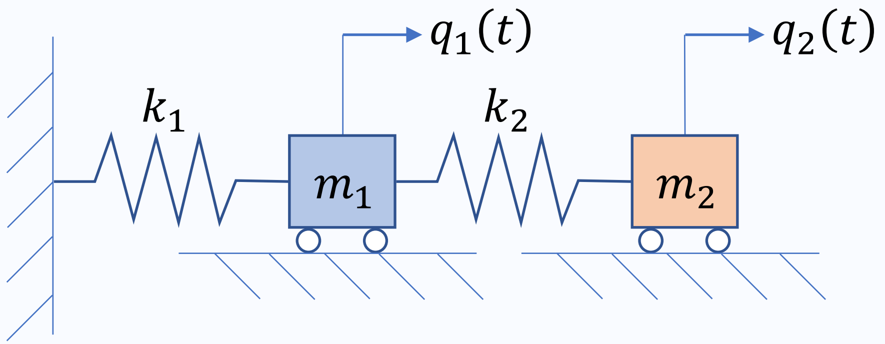
- leads to Lagrange's equation, in general:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, i = 1,2,..N$$

- where the system's Lagrangian, for a system with N DoF:

$$L = L(q_1, q_2 .. q_N, \dot{q}_1, \dot{q}_2, .. \dot{q}_N)$$

Another example:



- Our energies are calculated as:

$U = ?$

$U = {}^1\!/_2\, k_1 q_1{}^2 + {}^1\!/_2\, k_2 (q_2 - q_1)^2$

$U = {}^1\!/_2\, \{q\}^T [K] \{q\}$

- Familiar?

$T = ?$

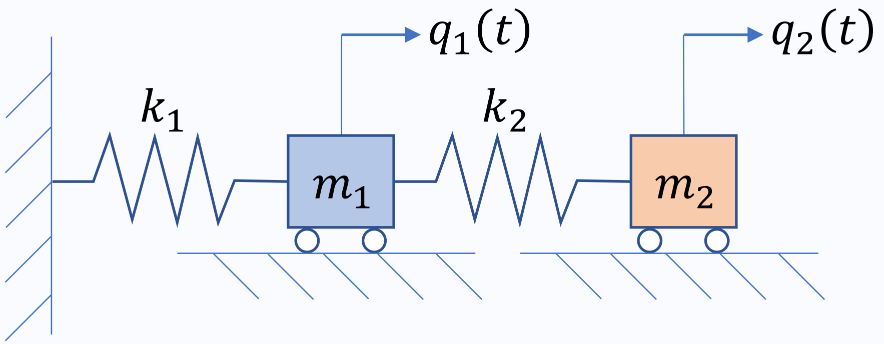
$T = {}^1\!/_2\, m_1 \dot{q}_1{}^2 + {}^1\!/_2\, m_2 \dot{q}_2{}^2$

$T = {}^1\!/_2\, \{\dot{q}\}^T [M] \{\dot{q}\}$

- So our Lagrangian is calculated as:

$L = [{}^1\!/_2\, m_1 \dot{q}_1{}^2 + {}^1\!/_2\, m_2 \dot{q}_2{}^2\ ] - [{}^1\!/_2\, k_1 q_1{}^2 + {}^1\!/_2\, k_2 (q_2 - q_1)^2]$

Another example:



$$L = [{}^1\!/_2\, m_1\dot{q}_1{}^2 + {}^1\!/_2\, m_2\dot{q}_2{}^2\,] - [{}^1\!/_2\, k_1 q_1{}^2 + {}^1\!/_2\, k_2(q_2 - q_1)^2]$$

- So solving Lagrange’s equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, i = 1,2$$

- first for $i = 1$:

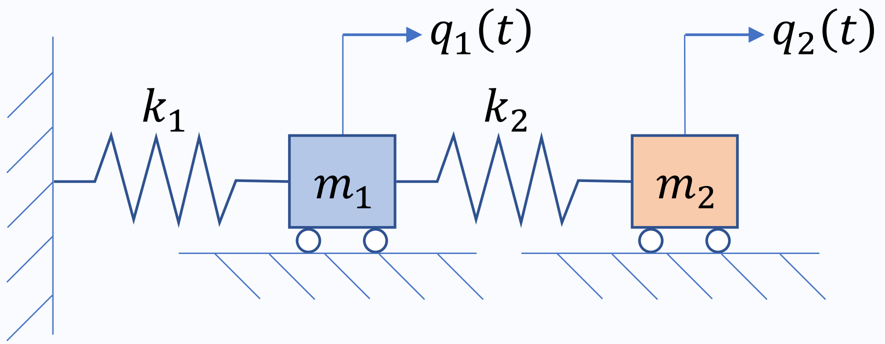
- Differentiate the first term w.r.t. time, and collect our q_i terms:

$$\frac{\partial L}{\partial \dot{q}_1} = {}^1\!/_2\, m_1 \times 2\dot{q}_1 = m_1\dot{q}_1$$

$$\frac{\partial L}{\partial q_1} = -[{}^1\!/_2\, k_1 \times 2q_1 + {}^1\!/_2\, k_2(2q_1 - 2q_2)] = -k_1q_1 - k_2(q_1 - q_2)$$

$$m_1\ddot{q}_1 + (k_1 + k_2)q_1 - k_2q_2 = 0$$

Another example:



$$L = [{}^1/_2\, m_1 \dot{q}_1{}^2 + {}^1/_2\, m_2 \dot{q}_2{}^2\,] - [{}^1/_2\, k_1 q_1{}^2 + {}^1/_2\, k_2 (q_2 - q_1)^2]$$

- So solving Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, i = 1,2$$

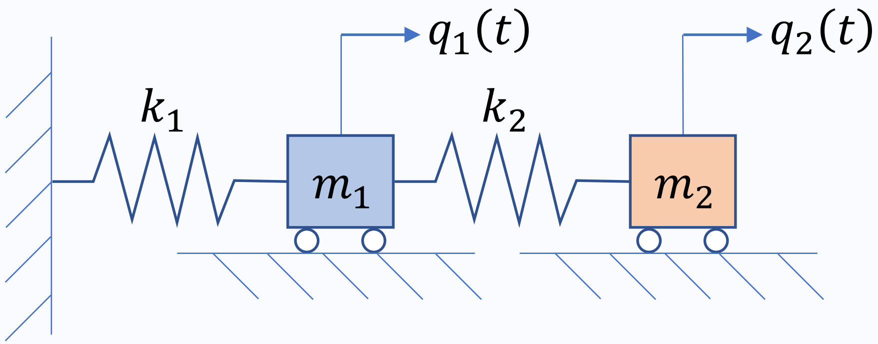
- first for $i = 2$:
- Differentiate the first term w.r.t. time, and collect our q_i terms:

$$\frac{\partial L}{\partial \dot{q}_2} = {}^1/_2\, m_2 \times 2\dot{q}_2 = \ m_2 \dot{q}_2$$

$$\frac{\partial L}{\partial q_2} = -[{}^1/_2\, k_2 (2q_2 - 2q_1)] = -k_2(q_2 - q_1)$$

$$m_2 \ddot{q}_2 - k_2 q_1 + k_2 q_2 = 0$$

Another example:



- Collecting:

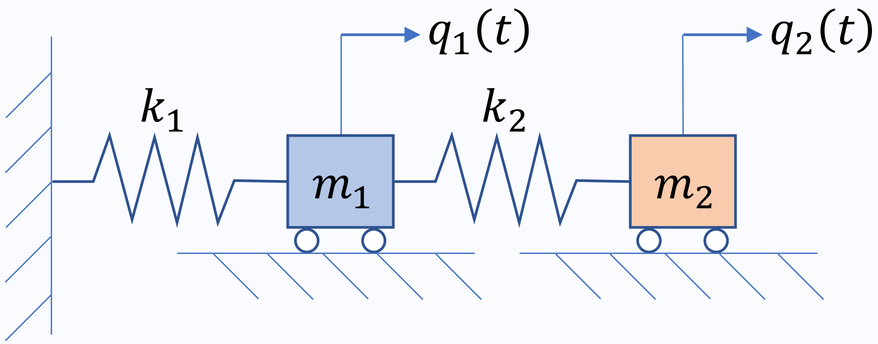
$$m_1\ddot{q}_1 + (k_1 + k_2)q_1 - k_2 q_2 = 0$$

$$m_2\ddot{q}_2 - k_2 q_1 + k_2 q_2 = 0$$

- So, we can organise into matrix form:

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} ? \\ ? \end{Bmatrix}$$

Another example:



- Collecting:

$$m_1\ddot{q}_1 + (k_1 + k_2)q_1 - k_2 q_2 = 0$$

$$m_2\ddot{q}_2 - k_2 q_1 + k_2 q_2 = 0$$

- So, we can organise into matrix form:

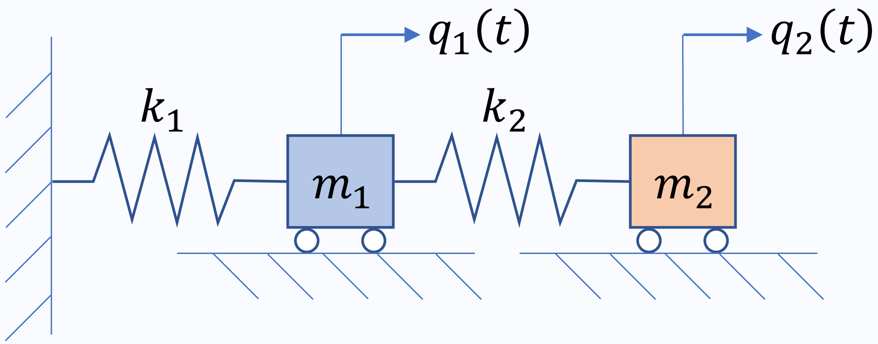
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & +k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- or

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$$

- Which includes $[M]$ our Mass Matrix and $[K]$ our Stiffness Matrix
- The terms look like they correspond with T and U…!

Why?



- Since we have our energy terms in quadratic form:

$$U = {}^1/_2\{q\}^T[K]\{q\}$$

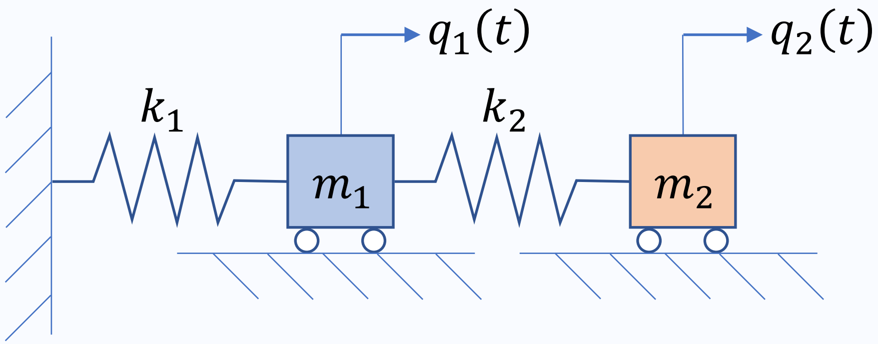
$$T = {}^1/_2\{\dot{q}\}^T[M]\{\dot{q}\}$$

- and since our Lagrangian is the difference of two quadratics:

$$L = {}^1/_2\{\dot{q}\}^T[M]\{\dot{q}\} - {}^1/_2\{q\}^T[K]\{q\}$$

- this look like our simple models of kinetic and elastic potential energy

Why?



- The big implication is that if our Lagrangian is the difference of two quadratics:

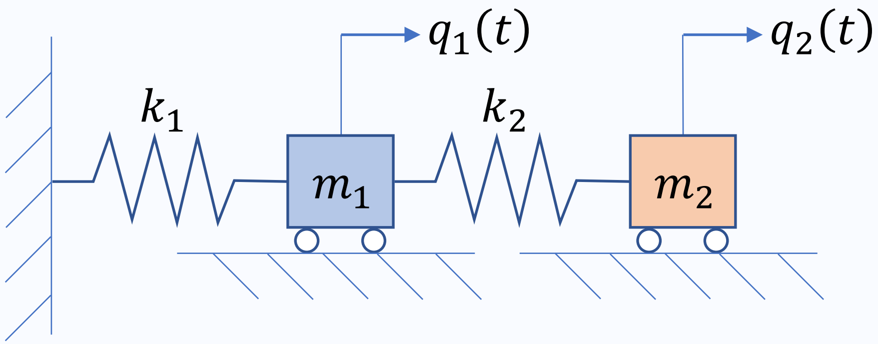
$$L = {}^1\!/_2\,\{\dot{q}\}^T[M]\{\dot{q}\} - {}^1\!/_2\,\{q\}^T[K]\{q\}$$

- this means we can immediately state, without integration, Lagrange's Equations leads to our Governing Equation of Motion:

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$$

- Much like we could use PMTPE in statics as our FEM shortcut

And to help with computation:



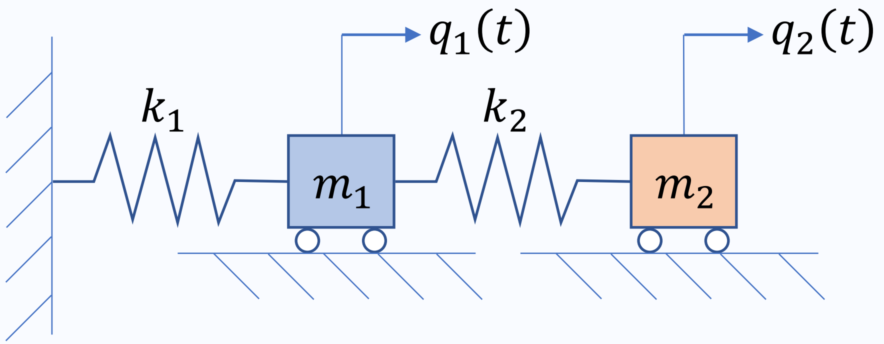
- And like in our Statics scenarios, you can express U in matrix index notation terms:

$$U = {}^1/_2 \{q\}^T[K]\{q\} = {}^1/_2 \sum_{j=1}^{N}\sum_{i=1}^{N} K_{ij} q_i q_j$$

- Plus a new, equivalent statement for T:

$$T = {}^1/_2 \{\dot{q}\}^T[M]\{\dot{q}\} = {}^1/_2 \sum_{j=1}^{N}\sum_{i=1}^{N} M_{ij}\, \dot{q}_i \dot{q}_j$$

Important Note!



- Terminology vs. your other modules might be confusing.
- In a dynamics course you might consider the Degrees of Freedom as the axes in which a rigid body can move (here, 1: x translation).
- Here though, we consider elastic, flexible bodies (the springs connecting the carts) so the *structure* has multiple DoF even if they only translate in x.

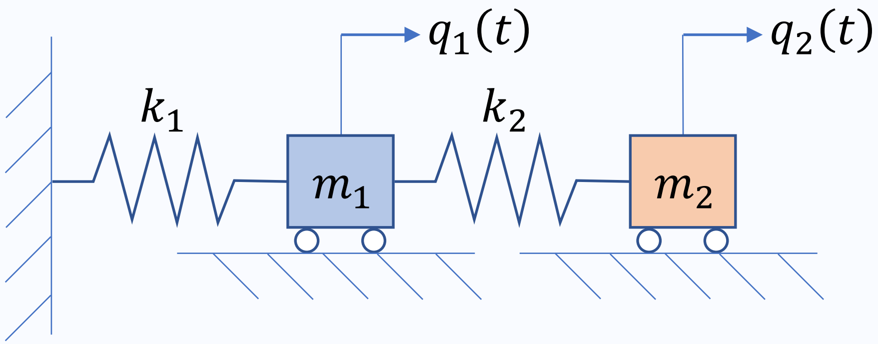
Part 3c: Dynamics of Rods

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

From 8th November 2024

Reminder:



- Our system's kinetic and strain energy terms have quadratic form:

$$U = {}^1/_2\,\{q\}^T[K]\{q\}$$

$$T = {}^1/_2\,\{\dot{q}\}^T[M]\{\dot{q}\}$$

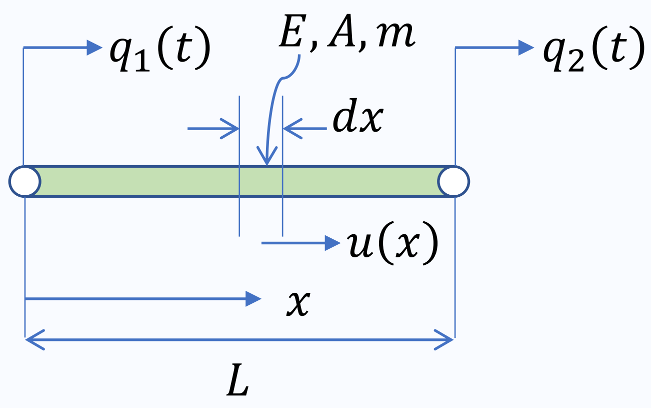
- and since our Lagrangian is the difference of two quadratics:

$$L = {}^1/_2\,\{\dot{q}\}^T[M]\{\dot{q}\} - {}^1/_2\,\{q\}^T[K]\{q\}$$

- similar to the PMTPE principle, this means Lagrange's Equations lead to our Governing Equation of Motion:

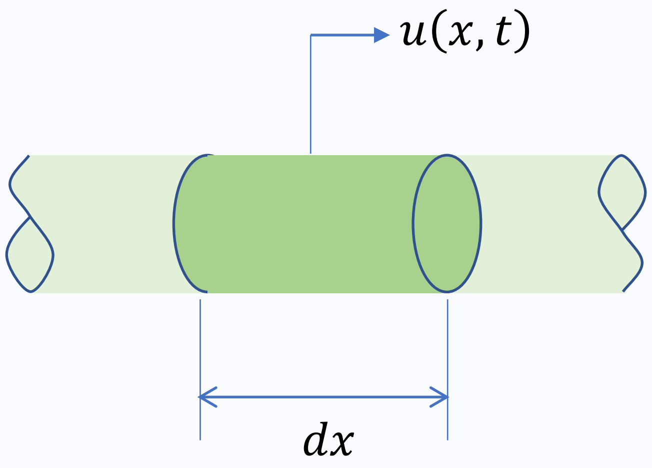
$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$$

Elastic Rods – in Dynamics



- A rod vibrating along its length; same assumptions
- E is Young's modulus, A is cross-sectional area, and m is the mass *per unit length*
- What is the governing equation of motion?
- We need the kinetic and potential energies
- For a slice dx:

$T = {}^1\!/_2\, m dx (\dot{u})^2$ (mdx is the mass *of the slice*)

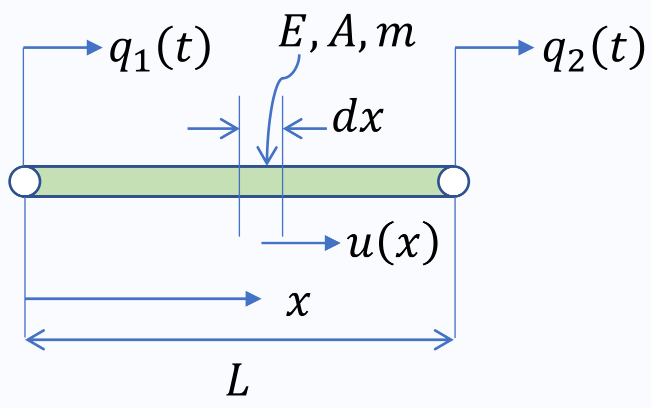


- and overall:

$$T = {}^1\!/_2 \int_0^L m\big(\dot{u}(x,t)\big)^2 dx$$

- So as before we use approximation to find $u(x,t)$

Elastic Rods – in Dynamics



- So as before we use approximation to find $u(x,t)$, with the same shape functions:

  $u(x) = g_1(x)q_1 + g_2(x)q_2$ becomes

  $$u(x,t) = g_1(x)q_1(t) + g_2(x)q_2(t)$$

- and because we need the *velocity*:

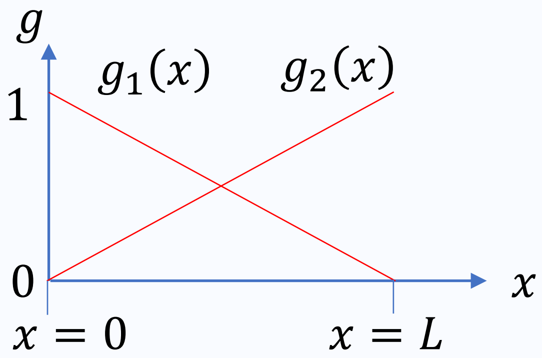
  $$\dot{u}(x,t) = g_1(x)\dot{q}_1(t) + g_2(x)\dot{q}_2(t)$$

- so

  $$T = {}^1\!/_2 \int_0^L m[g_1\dot{q}_1 + g_2\dot{q}_2]^2 dx$$

  $$T = {}^1\!/_2 \int_0^L m[{g_1}^2{\dot{q}_1}^2 + {g_2}^2{\dot{q}_2}^2 + 2g_1\dot{q}_1g_2\dot{q}_2]dx$$

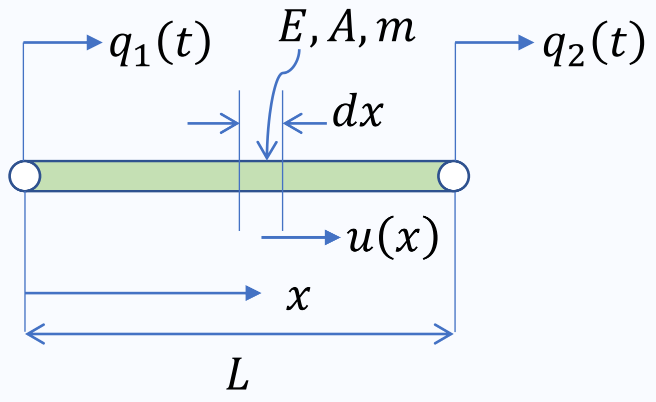
- We will need to integrate this term by term



$$g_1(x) = 1 - \frac{x}{L}$$

$$g_2(x) = \frac{x}{L}$$

Elastic Rods – in Dynamics



- Term 1:

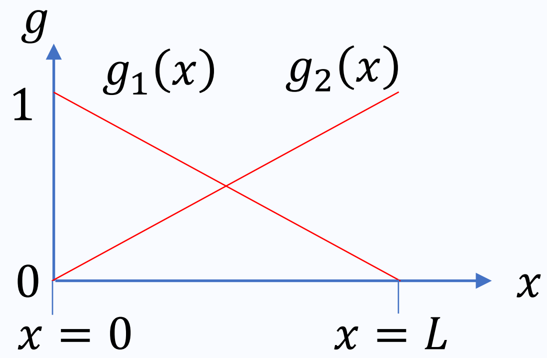
$$T = {}^1\!/_2 \int_0^L m[{g_1}^2{\dot{q}_1}^2]dx$$

$$T = {}^1\!/_2 \int_0^L m\left(1-\frac{x}{L}\right)^2 {\dot{q}_1}^2 dx$$

$$T = {}^1\!/_2\, m{\dot{q}_1}^2 \int_0^L \left(1-\frac{2x}{L}+\frac{x^2}{L^2}\right)dx$$

$$T = {}^1\!/_2\, m{\dot{q}_1}^2 \left[\left(x-\frac{x^2}{L}+\frac{x^3}{3L^2}\right)\right]_0^L$$

$$T = {}^1\!/_2 \left(\frac{mL}{3}\right){\dot{q}_1}^2$$



$$g_1(x) = 1 - \frac{x}{L}$$

$$g_2(x) = \frac{x}{L}$$

Elastic Rods – in Dynamics

- Term 2:

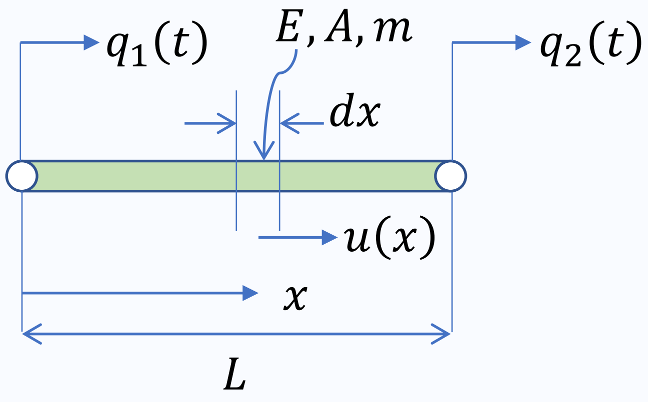
$$T = {}^1\!/_2 \int_0^L m[g_2{}^2 \dot{q}_2{}^2]dx$$

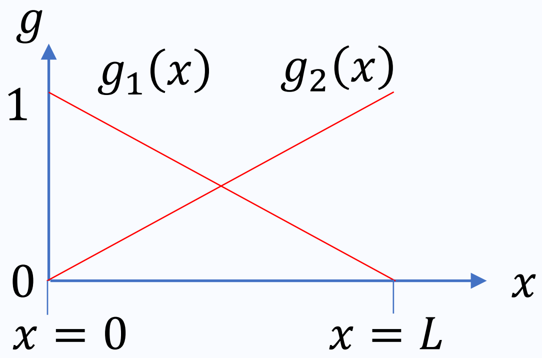
$$T = {}^1\!/_2 \int_0^L m\left(\frac{x}{L}\right)^2 \dot{q}_2{}^2 dx$$

$$T = {}^1\!/_2\, m\dot{q}_2{}^2 \int_0^L \left(\frac{x^2}{L^2}\right) dx$$

$$T = {}^1\!/_2\, m\dot{q}_2{}^2 \left[\frac{x^3}{3L^2}\right]_0^L$$

$$T = {}^1\!/_2 \left(\frac{mL}{3}\right) \dot{q}_2{}^2$$

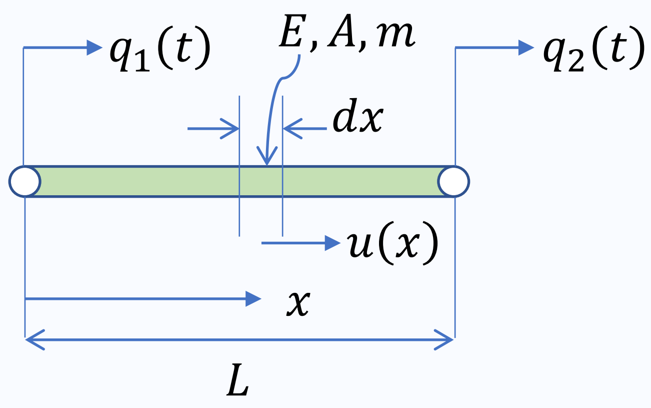




$$g_1(x) = 1 - \frac{x}{L}$$

$$g_2(x) = \frac{x}{L}$$

Elastic Rods – in Dynamics



- Term 3:

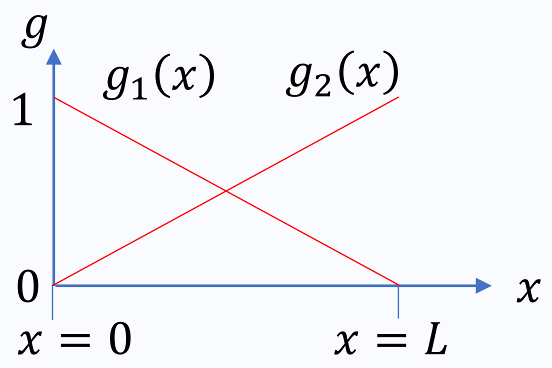
$$T = {}^1\!/_2 \int_0^L m[2g_1\dot{q}_1 g_2\dot{q}_2]dx$$

$$T = {}^1\!/_2\, m \int_0^L 2\left(1 - \frac{x}{L}\right)\left(\frac{x}{L}\right)\dot{q}_1\dot{q}_2 dx$$

$$T = {}^1\!/_2\, m\dot{q}_1\dot{q}_2 \int_0^L 2\left(\frac{x}{L} - \frac{x^2}{L^2}\right)dx$$

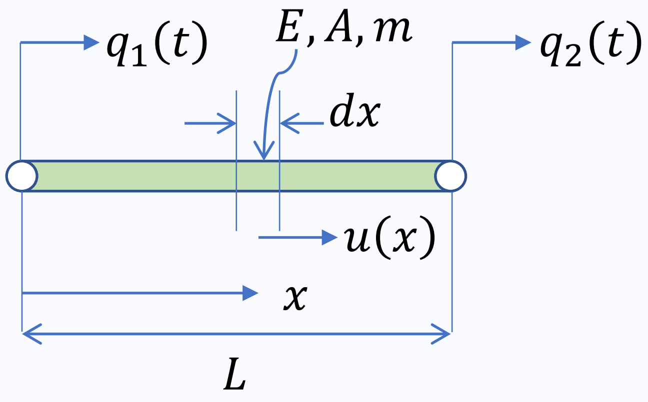
$$T = {}^1\!/_2\, m\dot{q}_1\dot{q}_2 \left[2\left(\frac{x^2}{2L} - \frac{x^3}{3L^2}\right)\right]_0^L$$

$T = {}^1\!/_2 \times 2 \times \left(\frac{mL}{6}\right)\dot{q}_1\dot{q}_2$ (keeping the ${}^1\!/_2$ out…)



$$g_1(x) = 1 - \frac{x}{L}$$
$$g_2(x) = \frac{x}{L}$$

Elastic Rods – in Dynamics



- Overall:

$$T = {}^1\!/_2 \left[\left(\frac{mL}{3}\right) \dot{q}_1{}^2 + \left(\frac{mL}{3}\right) \dot{q}_2{}^2 + 2 \left(\frac{mL}{6}\right) \dot{q}_1 \dot{q}_2 \right]$$

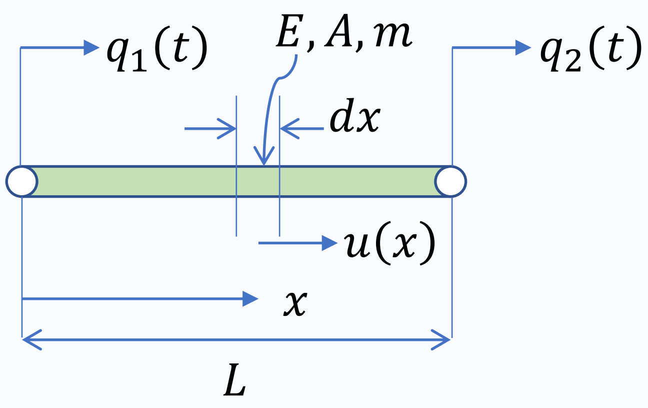
- Hopefully by now you recognise quadratic form:

$$T = {}^1\!/_2 \sum_{j=1}^{2} \sum_{i=1}^{2} M_{ij}\, \dot{q}_i \dot{q}_j = {}^1\!/_2 \{\dot{q}\}^T [M] \{\dot{q}\}$$

$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T \begin{bmatrix} \dfrac{mL}{3} & \dfrac{mL}{6} \\ \dfrac{mL}{6} & \dfrac{mL}{3} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

- Our elemental *mass matrix* $[M]$! From the same shape functions as we used for statics.

Elastic Rods – in Dynamics



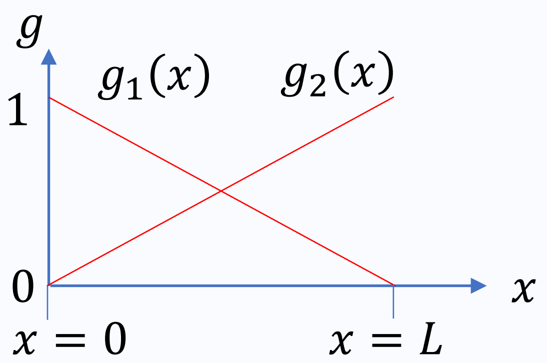
- And for elastic potential energy:

$$u(x,t) = g_1(x)q_1(t) + g_2(x)q_2(t)$$

$$U = {}^1/_2 \int_0^L EAu'^2 dx \text{ where} (\cdot)' = \frac{\partial}{\partial x}(\cdot)$$

$$U = {}^1/_2 \int_0^L EA[g_1'(x)q_1(t) + g_2'(x)q_2(t)]^2 dx$$

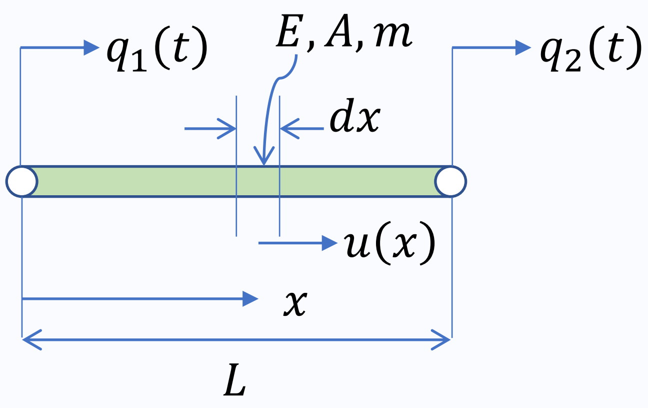
- (No need to show the working on this one, because this is identical to the Statics case except that $q_i(t)$ are now functions of time, so…)



$$g_1(x) = 1 - \frac{x}{L}$$

$$g_2(x) = \frac{x}{L}$$

Elastic Rods – in Dynamics



- Overall:

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

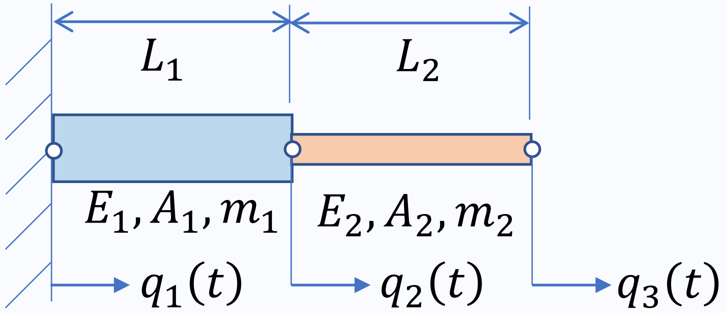
- Our elemental *stiffness matrix* $[K]$!
- Alongside our elemental *mass matrix* $[M]$

$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T \begin{bmatrix} \frac{mL}{3} & \frac{mL}{6} \\ \frac{mL}{6} & \frac{mL}{3} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

- We can now express the governing equation of motion:

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$$

Now, a Two-Rod System:



- Using the same approximation for a single element, what is the governing equation of motion?
- Start with the kinetic energy:

$$T = T_1 + T_2$$

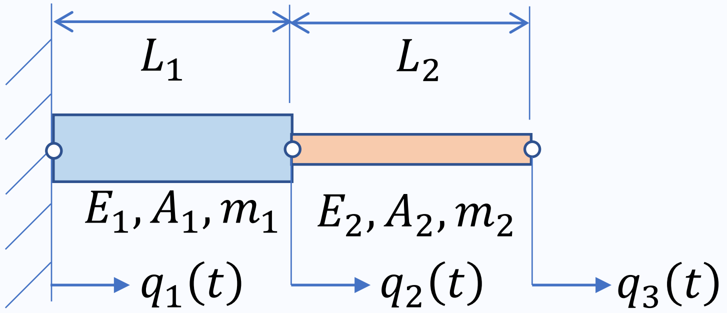
$$T_1 = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T [M_1] \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

$$T_2 = {}^1\!/_2 \begin{Bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}^T [M_2] \begin{Bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}$$

- Just as before we use assembly to create our system mass matrix $[M]$:

$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}^T_{1\times 3} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}_{3\times 3} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}_{3\times 1}$$

Now, a Two-Rod System:



- Using the same approximation for a single element, what is the governing equation of motion?
- Start with the kinetic energy:

$$T = T_1 + T_2$$

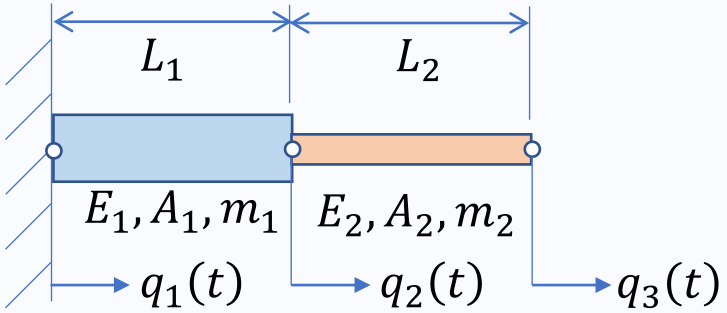
$$T_1 = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T [M_1] \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

$$T_2 = {}^1\!/_2 \begin{Bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}^T [M_2] \begin{Bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}$$

- Just as before we use assembly to create our system mass matrix $[M]$:

$$T = {}^1\!/_2 \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}^T_{1\times3} \begin{bmatrix} \frac{m_1L_1}{3} & \frac{m_1L_1}{6} & 0 \\ \frac{m_1L_1}{6} & \frac{m_1L_1}{3} + \frac{m_2L_2}{3} & \frac{m_2L_2}{6} \\ 0 & \frac{m_2L_2}{6} & \frac{m_2L_2}{3} \end{bmatrix}_{3\times3} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}_{3\times1}$$

Now, a Two-Rod System:



- Using the same approximation for a single element, what is the governing equation of motion?

$$U = U_1 + U_2$$

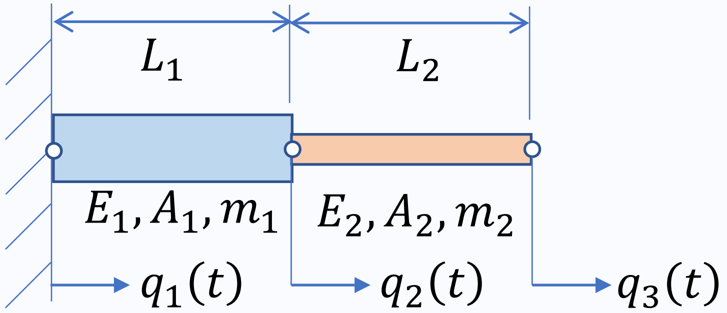
$$U_1 = {}^1/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T [K_1] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$U_2 = {}^1/_2 \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^T [K_1] \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

- We can simply reuse our system stiffness matrix $[K]$:

$$U = {}^1/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T_{1\times3} \begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} + \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix}_{3\times3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times1}$$

Now, a Two-Rod System:



- Using the same approximation for a single element, what is the governing equation of motion?
- Due to Hamilton's Principle, since we have a Lagrangian of the form:

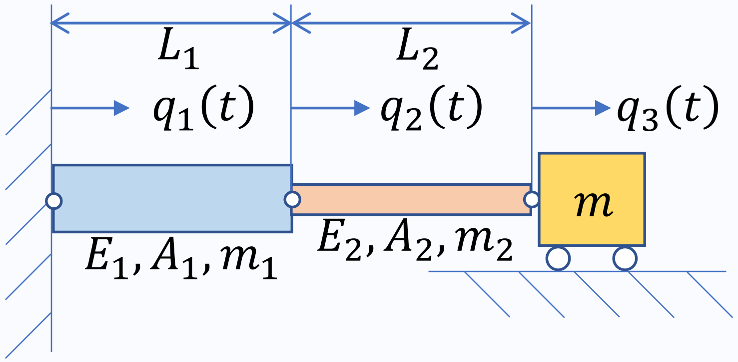
$$L = {}^1\!/_2\{\dot{q}\}^T[M]\{\dot{q}\} - {}^1\!/_2\{q\}^T[K]\{q\}$$

- We can say the governing equation of motion is

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$$

$$\begin{bmatrix} \frac{m_1L_1}{3} & \frac{m_1L_1}{6} & 0 \\ \frac{m_1L_1}{6} & \frac{m_1L_1}{3} + \frac{m_2L_2}{3} & \frac{m_2L_2}{6} \\ 0 & \frac{m_2L_2}{6} & \frac{m_2L_2}{3} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} + \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For next time: what if we add a mass?



- Ask yourself these questions, the same as usual:
- How many DoF do we have now?
- What external forces do we have?
- How is the stiffness matrix assembled?
- How is the mass matrix assembled?