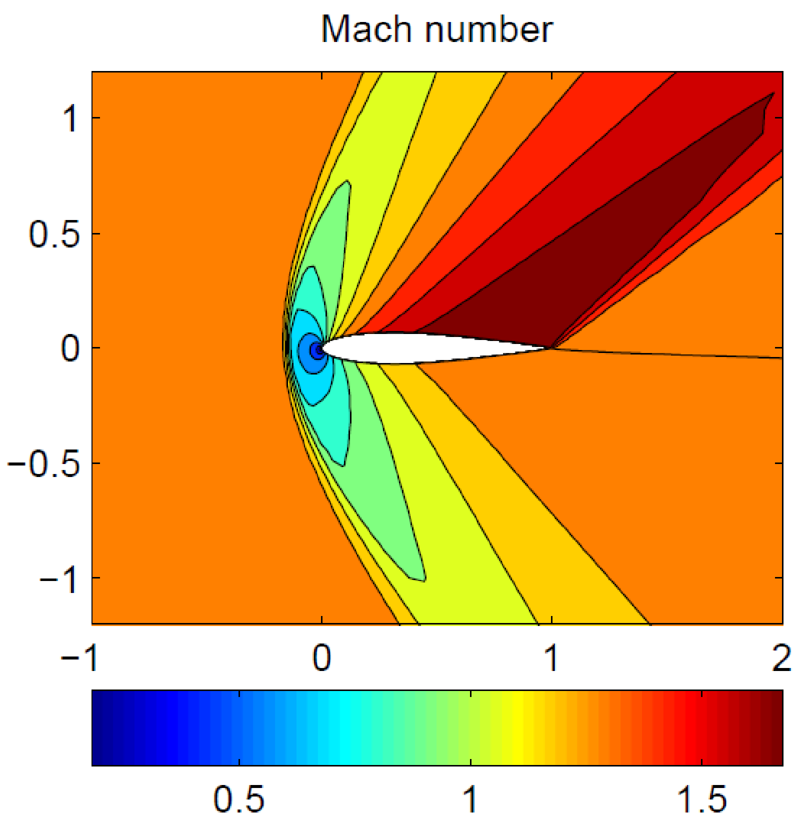
SESA3029 Aerothermodynamics



Lecture 4.3

Potential flow equations for compressible flow

Simplify Euler equation for Potential flow

If a flow is irrotational (curl **u**=0) we can always write $\mathbf{u} = \nabla \phi$

with velocity components $u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$

recall Aerodynamics

where ϕ is a scalar known as the velocity potential
- this works because of the vector identity $\nabla \times (\nabla \phi) = 0$

+ constant stagnation enthalpy implies <u>homentropic</u> flow

We can derive a single equation for compressible potential flow which will be an intermediate step in developing simplified theories.

Start from Euler equations for steady flow:

$$\frac{\partial}{\partial x}\begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vH \end{pmatrix} = 0$$

Rearrange the momentum equations

$$\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial p}{\partial x}=0 \Rightarrow \rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+u\overbrace{\cancel{\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}\right)}}^{\text{mass conservation } \therefore = 0}+\frac{\partial p}{\partial x}=0 \qquad (1)$$

$$\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial p}{\partial y}=0 \Rightarrow \rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+v\underbrace{\cancel{\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}\right)}}_{\text{mass conservation } \therefore = 0}+\frac{\partial p}{\partial y}=0 \qquad (2)$$

Bracketed terms are zero by mass conservation

Take (1) times u + (2) times v

$$\rho u^2\frac{\partial u}{\partial x}+\rho uv\frac{\partial u}{\partial y}+\rho uv\frac{\partial v}{\partial x}+\rho v^2\frac{\partial v}{\partial y}=-\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)$$

$$\rho u^2\frac{\partial u}{\partial x}+\rho uv\frac{\partial u}{\partial y}+\rho uv\frac{\partial v}{\partial x}+\rho v^2\frac{\partial v}{\partial y}=-\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)$$

$$=-a^2\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right)$$

$$=a^2\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

homentropic flow assumption $\left.\frac{dp}{d\rho}\right|_{s=const}=a^2$

mass conservation again

Since for homentropic flow $\frac{\partial p}{\partial x}=a^2\frac{\partial \rho}{\partial x}$ and $\frac{\partial p}{\partial y}=a^2\frac{\partial \rho}{\partial y}$

and for mass conservation $u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$

Divide by ρ

$$\boxed{u^2\frac{\partial u}{\partial x}+uv\frac{\partial u}{\partial y}+uv\frac{\partial v}{\partial x}+v^2\frac{\partial v}{\partial y}=a^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}$$

with $u=\frac{\partial \phi}{\partial x}$ $v=\frac{\partial \phi}{\partial y}$ this is a velocity potential equation

Linearisation of the energy equation

Let $u = U_\infty + u'$ with $u', v' \ll U_\infty$

$\cancel{v = v'}$

$v = 0 + v'$

$T_0 = T\left(1 + \frac{\gamma - 1}{2} M^2\right) \xrightarrow[T \propto a^2]{} a_0^2 = a^2\left(1 + \frac{\gamma-1}{2} M^2\right)$

$a_0^2 = a^2 + \frac{\gamma - 1}{2}\left(u^2 + v^2\right)$

$a_0^2 = a^2 + \frac{\gamma - 1}{2} U_\infty^2$

e.g. energy equation $$a_\infty^2 + \frac{\gamma - 1}{2} U_\infty^2 = a^2 + \frac{\gamma - 1}{2}\left(u^2 + v^2\right)$$

$$= a^2 + \frac{\gamma - 1}{2}\left[\left(U_\infty + u'\right)^2 + \left(v'\right)^2\right]$$

$$= a^2 + \frac{\gamma - 1}{2}\left[U_\infty^2 + 2u'U_\infty + \left(u'\right)^2 + \left(v'\right)^2\right]$$

Remove small terms

$$\boxed{a_\infty^2 = a^2 + (\gamma - 1) u' U_\infty}$$

Linearised energy equation

Linearisation of combined momentum equations

$$u^2\frac{\partial u}{\partial x}+uv\frac{\partial u}{\partial y}+uv\frac{\partial v}{\partial x}+v^2\frac{\partial v}{\partial y}=a^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

with $a^2=a_\infty^2-(\gamma-1)u'U_\infty$

Decompose (noting that U_∞ is independent of *x* and *y*):

small, neglected

$$(U_\infty+u')^2\frac{\partial u'}{\partial x}+(U_\infty+u')\underbrace{v'\frac{\partial u'}{\partial y}}_{\text{neglected}}+(U_\infty+u')\overbrace{v'\frac{\partial v'}{\partial x}}+\overbrace{v'^2\frac{\partial v'}{\partial y}}=a^2\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)$$

Substitute for a^2 and remove products of small terms

$$U_\infty^2\frac{\partial u'}{\partial x}=a_\infty^2\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)$$

$\div a_\infty^2$

$$u'=\frac{\partial \phi}{\partial x}\quad v'=\frac{\partial \phi}{\partial y}\quad \Rightarrow \quad M_\infty^2\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\quad \Rightarrow \quad \boxed{\left(1-M_\infty^2\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0}$$

Velocity potential equation

Velocity potential equation $\boxed{\left(1-M_\infty^2\right)\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0}$

- Reduces to Laplace's equation in incompressible flow
- Not valid near $M_\infty=1$
  - we would have needed to carry around the next order of terms to get a valid equation there
- Not applicable for $M_\infty>5$ (hypersonic flow)
- We will consider solution to this equation in two flow regimes
  - $M_\infty<0.8$ (elliptic equation: Prandtl-Glauert transformation)
  - $1.2<M_\infty<5$ (hyperbolic equation: Ackeret theory)

Linearised pressure coefficient

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

From linearised energy equation

$$a_\infty^2 = a^2 + (\gamma - 1)u' U_\infty \qquad \text{with} \qquad u' \ll U_\infty$$

hence

$$\frac{p}{p_\infty} = \left(\frac{a^2}{a_\infty^2}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 - (\gamma - 1)M_\infty^2 \frac{u'}{U_\infty}\right)^{\frac{\gamma}{\gamma-1}}$$

$$= 1 - \gamma M_\infty^2 \frac{u'}{U_\infty} + \cdots$$

so

$$\boxed{C_p = -2\frac{u'}{U_\infty}}$$