SESA3029 Supplementary Notes: Method of Characteristics

These notes are an accompaniment to the lecture material in SESA3029, offering some further discussion of the development of the method of characteristics (MoC) for planar geometries (i.e. geometries defined in the Cartesian x-y plane, infinite in z). We follow a different derivation to that in Anderson (2004) but the MoC material from that textbook (which also appears in a shorter form in his Fundamentals of Aerodynamics book) is still useful if you want to see further discussions of the method. The extension to axisymmetric flows (i.e. nozzles with circular cross sections) is discussed in an appendix to these notes.

To follow the material in these notes, you should be familiar with the Euler equations in *s-n* co-ordinates, the definition of the sound speed, the Mach triangle and the Prandtl-Meyer function, all of which have been covered earlier in the SESA3029 module.

1. Reduction of the governing equations to a system of two equations in two unknowns

We start from the equations for ***homentropic flow*** (entropy constant everywhere in space, not just along streamlines). The stream-direction (*s*) momentum (Euler) equation is

$$\rho u \frac{\partial u}{\partial s} = -\frac{\partial p}{\partial s},$$

where u is the velocity, ρ is the density and p is the pressure. Since we are dealing with flows with uniform entropy we can interchange pressure and density derivatives, using the definition of the sound speed $a^2 = \left(\partial p / \partial \rho\right)$ giving

$$\rho u \frac{\partial u}{\partial s} = -a^2 \frac{\partial \rho}{\partial s} \quad . \tag{1}$$

The equation for mass conservation is given by

$$\frac{\partial \theta}{\partial n} + \frac{1}{u}\frac{\partial u}{\partial s} + \frac{1}{\rho}\frac{\partial \rho}{\partial s} = 0\,, \tag{2}$$

where θ is the flow angle and n is the stream-normal direction. We can rearrange this equation for $\partial \rho / \partial s$ and substitute into the right hand side of (1), giving

$$u \frac{\partial u}{\partial s} = a^2 \left(\frac{\partial \theta}{\partial n} + \frac{1}{u}\frac{\partial u}{\partial s} \right) \; .$$

Dividing through by a^2 and introducing the local Mach number $M = u / a$, this can be written as

$$\left(M^2 - 1\right)\frac{1}{u}\frac{\partial u}{\partial s} = \frac{\partial \theta}{\partial n} \quad . \tag{3}$$

To simplify the final method we introduce a new variable ν, which we have previously seen in connection with Prandtl-Meyer expansion waves (the change in ν is equal to the change in flow angle though an infinitesimal expansion wave). During the Prandtl-Meyer derivation we obtained an equation relating the velocity tangential to the Mach wave to the turning angle, so we can write

$$\frac{1}{u}\frac{\partial u}{\partial s} = \tan\mu \frac{\partial \nu}{\partial s}, \tag{4}$$

where μ is the Mach angle. From the Mach triangle we know $\tan\mu = 1\big/\sqrt{M^2-1}$). Hence equation (3) can be rewritten in a final form

$$\boxed{\frac{\partial \nu}{\partial s} - \tan\mu \frac{\partial \theta}{\partial n} = 0}. \tag{5}$$

The only other equation we need is the equation for irrotational flow (we know the flow is irrotational using Crocco's equation for steady homentropic flow)

$$u\frac{\partial \theta}{\partial s} - \frac{\partial u}{\partial n} = 0.$$

Using the equivalent of (4) for derivatives in the *n*-direction we can write this definition of irrotationality as

$$\boxed{\frac{\partial \theta}{\partial s} - \tan\mu \frac{\partial \nu}{\partial n} = 0}. \tag{6}$$

Equations (5) and (6) can be combined into a matrix system of two equations in two unknowns (the Prandtl Meyer angle ν and the flow angle θ) as

$$\frac{\partial}{\partial s}\begin{pmatrix} \nu \\ \theta \end{pmatrix} + \begin{pmatrix} 0 & -\tan\mu \\ -\tan\mu & 0 \end{pmatrix} \frac{\partial}{\partial n}\begin{pmatrix} \nu \\ \theta \end{pmatrix} = 0. \tag{7}$$

The solution character will be determined by key properties of the 2×2 matrix, which we denote **A**. In particular, the eigenvalues are given by $\det(\mathbf{A}-\mathbf{I}\lambda)=0$, where **I** is the identity matrix. It is straightforward to show that there are two eigenvalues $\lambda = \pm\tan\mu$, which correspond to the slopes of the Mach lines in the flow. In the present context these Mach lines are referred to as characteristic lines or simply the '***characteristics***' of the flow and we denote the positive eigenvalue as the C^+ characteristic and the negative eigenvalue as the C^- characteristic, as shown on Figure 1 for two characteristic lines passing through a point on a streamline *s*. If we imagine we are following the path s, we call the C^+ characteristic the ***left-running characteristic*** and C^- the ***right-running characteristic***.

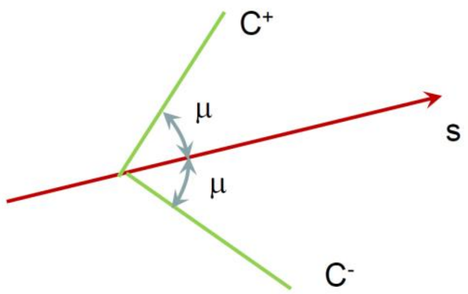

Figure 1. Characteristic lines passing through a point on a streamline.

Since the characteristic lines are at the Mach angle relative to the flow direction, they also represent limits with respect to the propagation of sound waves within the flow. Figure 2 shows how the characteristic lines passing through a point P bound a region downstream in which a sound source at P could be heard. We have previously seen the same 'Mach cone' in connection with the sound produced by a supersonic aircraft. The downstream zone is denoted the '***domain of influence***' for the point P, since changes in the flow at P can propagate throughout this domain by following any of the infinite number of positive and negative characteristics available.

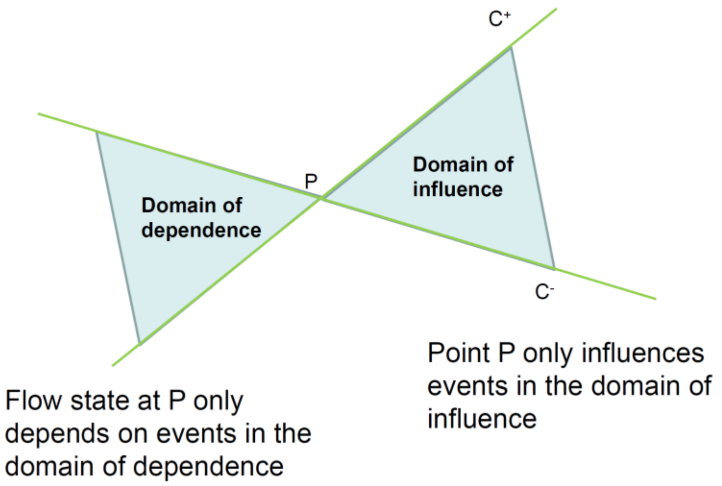

Figure 2. Zones of dependence and influence of a point P.

We can also think about the region upstream of P, also bounded by C^+ and C^-. No sound waves from outside this region can influence points P, since they would be outside the Mach cone. Conversely, any changes at any point within this region can propagate along characteristics to influence the conditions at point P. We call this region the '***domain of dependence***' of point P, since conditions at P will depend only on what is happening within this region. We will see these roles of 'dependence' and 'influence' played out in the method of characteristics.

2. Decoupled system: Compatibility conditions and Riemann invariants.

We have already seen the physical significance of the eigenvalues of the matrix defined in (7). Having formulated the problem in this way, we can now use a standard technique of linear algebra to decouple the equations into two separate equations that will form the basis of the method of characteristics. For each eigenvalue of **A** there is a corresponding eigenvector that satisfies

$$\mathbf{AS} = \mathbf{S\Lambda}\,, \tag{8}$$

where $\mathbf{\Lambda}$ is a diagonal vector of the eigenvalues and **S** is the right eigenvector matrix. There is a corresponding left eigenvector matrix given by the inverse $\mathbf{S}^{-1}$ defined by $\mathbf{S}^{-1}\mathbf{A} = \mathbf{\Lambda S}^{-1}$ (and $\mathbf{SS}^{-1}=\mathbf{I}$, the identity matrix). You should know how to find eigenvalues and eigenvectors from previous modules. There is an arbitrary constant involved in the procedure that may lead to some differences when you compare eigenvectors derived by hand with those that come from a computer package (such as matlab or numpy). This isn't a problem so long as they satisfy (8) and $\mathbf{S}\mathbf{S}^{-1}=\mathbf{I}$. For our current problem a complete set of eigenvalues and eigenvectors is given by

$$\mathbf{S} = \begin{pmatrix} \tfrac{1}{2} & \tfrac{1}{2} \\ -\tfrac{1}{2} & \tfrac{1}{2} \end{pmatrix} \quad \mathbf{S}^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \mathbf{\Lambda} = \begin{pmatrix} \tan\mu & 0 \\ 0 & -\tan\mu \end{pmatrix} \tag{9}$$

We can use this to decouple our system (7), which can be written in matrix notation as

$$\frac{\partial \mathbf{Q}}{\partial s} + \mathbf{A}\frac{\partial \mathbf{Q}}{\partial n} = 0\,,$$

where $\mathbf{Q} = \begin{pmatrix} \nu & \theta \end{pmatrix}^T$ is the solution column vector. Multiplying (8) from the right by $\mathbf{S}^{-1}$ and using $\mathbf{S}\mathbf{S}^{-1}=\mathbf{I}$ we can substitute for $\mathbf{A}$ to give

$$\frac{\partial \mathbf{Q}}{\partial s} + \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\frac{\partial \mathbf{Q}}{\partial n} = 0\,.$$

Now multiplying this equation from the left by $\mathbf{S}^{-1}$ and using $\mathbf{S}^{-1}\mathbf{S}=\mathbf{I}$ we have

$$\mathbf{S}^{-1}\frac{\partial \mathbf{Q}}{\partial s} + \mathbf{\Lambda}\mathbf{S}^{-1}\frac{\partial \mathbf{Q}}{\partial n} = 0\,.$$

At this point we have decoupled the equations and we can see how this has worked by writing out the equations in full using the eigenvector matrix $\mathbf{S}^{-1}$ from (9). This gives

$$\boxed{\begin{aligned} \frac{\partial(\nu-\theta)}{\partial s} + \tan\mu\,\frac{\partial(\nu-\theta)}{\partial n} = 0 \\ \frac{\partial(\nu+\theta)}{\partial s} - \tan\mu\,\frac{\partial(\nu+\theta)}{\partial n} = 0 \end{aligned}} \tag{10}$$

The first of these equations now has only one dependent variable (ν-θ) as a function of *s* and *n*, while the second has only (ν+θ) as the dependent variable. We can see why this is so useful if we look at the rate of change of functions along characteristic lines, as sketched in Figure 3.

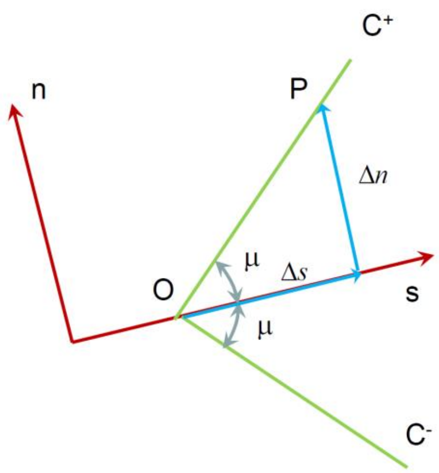

Figure 3. Changes along characteristic lines.

The changes along the C^+ characteristic from point O to point P in figure 3 can be broken down into changes along the *s* and *n* directions, so that a change in a generic function Δf is given by

$$\Delta f = \frac{\partial f}{\partial s}\Delta s + \frac{\partial f}{\partial n}\Delta n .$$

Dividing through by Δs and recognising that $\Delta n / \Delta s = \tan\mu$ we have

$$\frac{\Delta f}{\Delta s} = \frac{\partial f}{\partial s} + \tan\mu \frac{\partial f}{\partial n} .$$

Comparing this with the first equation in (10) we see that we have an exact analogy for $f = \nu - \theta$ and $\Delta f = 0$, i.e. there are no changes in $\nu - \theta$ along C^+. This is known as a ***compatibility condition***. If we repeat the exercise along C^- we find that there are no changes in $\nu + \theta$ along C^-. The quantities that don't change along characteristic lines are known as the ***Riemann invariants*** and we will denote them as

$$\begin{aligned} R^+ &= \nu - \theta \\ R^- &= \nu + \theta \end{aligned} \tag{11}$$

The Riemann invariants are crucial to the method of characteristics because they enable us to work out the flowfield very quickly.

[Aside: if you do look at Anderson's book you will find that he denotes the Riemann invariants as K^+ and K^-. They are connected to our definitions by $K^+ = -R^+$ and $K^- = R^-$. This makes no difference to practical calculations as long as you are consistent. Our signs are consistent with the classic text by Liepmann & Roshko (1957).]

To summarise, what we need to know in connection with figure 1 are the compatibility conditions:

$$\boxed{\begin{array}{l} \text{Along } C^+,\ R^+=\nu-\theta \text{ is constant} \\ \text{Along } C^-,\ R^- =\nu+\theta \text{ is constant} \end{array}}$$

Here, ν is the Prandtl-Meyer function (tabulated in isentropic flow tables as a function of Mach number) and θ is the flow angle. If we know the Riemann invariants we can easily solve (11) for ν and θ.

3. The MoC Unit Process

We can now look at how the method of characteristics works at a detailed level. Referring to figure 4 what we are trying to do is advance a flow solution downstream. In this case we suppose that we know the flow (Mach number and direction) at points A and B and we want to progress the solution downstream to point P. To do this we follow the characteristic lines that pass through points A and B. In particular, given the arrangement of A below B we know that the left-running characteristic from A will cross the right-running characteristic from B at point P as shown in the figure.

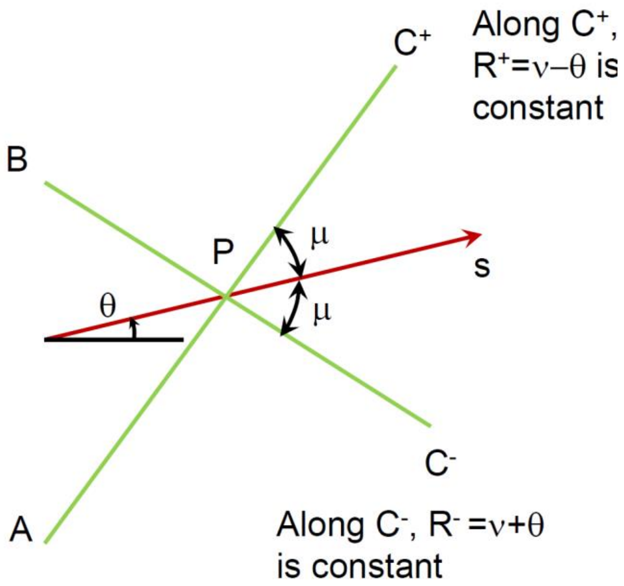

Figure 4. The unit process: how we get from points A and B upstream to point P.

Since the Riemann invariants don't change along the lines we can use this to relate the points according to

$$\begin{aligned} R_A^+ &\equiv \nu_A - \theta_A = \nu_P - \theta_P \\ R_B^- &\equiv \nu_B + \theta_B = \nu_P + \theta_P \end{aligned} \tag{12}$$

We can solve this to find the flow properties at point P from

$$\boxed{\nu_P = \frac{R_A^+ + R_B^-}{2} \quad \theta_P = \frac{R_B^- - R_A^+}{2}}. \tag{13}$$

Once we know ν we can get the Mach number from the Isentropic Flow Tables (IFT) and once we know the Mach number we can find the Mach angle $\mu = \sin^{-1}(1/M)$. Equations (12) and (13) together with the IFT are all we need to implement the method of characteristics to find the flowfield (once we know M other properties follow since the stagnation pressure and density are constant in homentropic flow).

To complete the calculation we need the co-ordinates of the new point P in Figure 4. To avoid having to worry about negative angles in the derivation we make a new sketch with point P above point B in Figure 5. The derived formulae will be applicable for any location of P relative to A and B (P just has to be downstream of A and B and located along intersecting characteristic lines).

In figure 5 the flow angle θ, which is already known from (13), has been added to the sketch at all the points along with the Mach angle μ. In general, characteristic lines are curved and to improve the accuracy of the method we will define the key angles as averages from A to P and from B to P according to

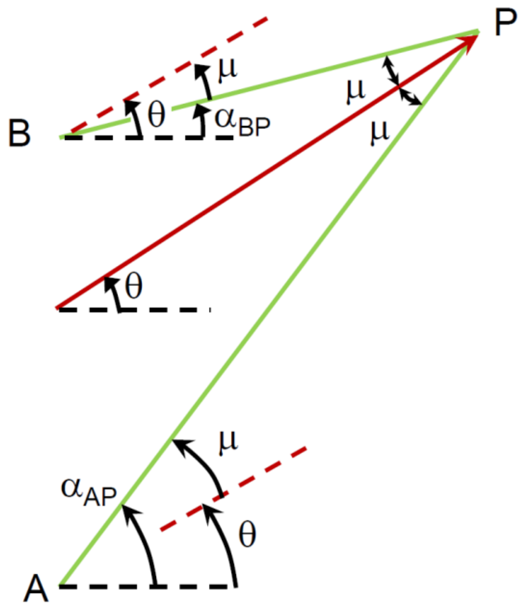

Figure 5. Sketch of the unit process used to derive the co-ordinates of point P

$$\begin{aligned}\alpha_{AP} &= \frac{1}{2}\left[(\theta+\mu)_A + (\theta+\mu)_P\right]\\ \alpha_{BP} &= \frac{1}{2}\left[(\theta-\mu)_B + (\theta-\mu)_P\right]\end{aligned}, \tag{14}$$

From Figure 5, we can relate the angles to the co-ordinates by

$$\tan\alpha_{AP} = \frac{y_P - y_A}{x_P - x_A} \tag{15a}$$

$$\tan\alpha_{BP} = \frac{y_P - y_B}{x_P - x_B} \tag{15b}$$

We can then substitute for y_P from (15a) into (15b) to give

$$(x_P - x_B)\tan\alpha_{BP} = (x_P - x_A)\tan\alpha_{AP} + y_A - y_B$$

and solve for

$$x_p = \frac{x_B\tan\alpha_{BP} - x_A\tan\alpha_{AP} + y_A - y_B}{\tan\alpha_{BP} - \tan\alpha_{AP}}. \tag{16a}$$

Substitution of (16a) back into (15a) gives

$$y_P = y_A + (x_P - x_A)\tan\alpha_{AP}. \tag{16b}$$

With (14) and (16) we now we have a method to calculate the co-ordinates of point *P* and we can repeat this unit process for all the intersections of characteristic lines, allowing us to work our way downstream from some prescribed solution in a flow.

4. Reflecting and non-reflecting walls: simple and uniform flow regions

The unit process described in the previous section can be applied to all characteristic line intersections in a flow. However, we also need to be able to treat solid surfaces. Figure 6 shows what happens when a left-running characteristic C^+ meets a wall point W.

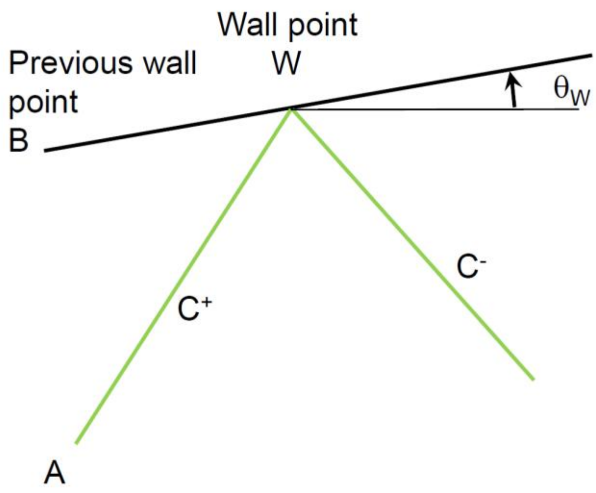

Figure 6. Reflection of a characteristic line from a wall.

To analyse this situation we need to consider the Riemann invariants at the point W. Since point W is connected to point A by C^+ we can write

$$R_A^+ = \nu_A - \theta_A = \nu_W - \theta_W \ ,$$

So we can solve for ν_W and hence the Mach number and Mach angle. For a prescribed wall geometry we would know the flow angle θ_W, since the flow must be tangential to the wall. Hence, we can construct a Riemann invariant for the reflected wave C^-as

$$R_W^- = \nu_W + \theta_W$$

This will then be invariant along C^- until another wall is reached.

A special situation arises if the wall angle is the same as the flow angle from point A, i.e. if $\theta_W = \theta_A$. In this case there is no reflected wave and we say that the characteristic line has been cancelled at the wall. In some circumstances, we want to cancel out characteristics at wall and we can use this condition as a way of determining the wall co-ordinates required for cancellation. We can use the previous geometry method, but redefine

$$\begin{aligned} \alpha_{AW} &= (\theta + \mu)_A \\ \alpha_{BW} &= \frac{1}{2}[\theta_B + \theta_W] \end{aligned} \tag{17}$$

Along the AW line the flow angle and Mach number is constant so we have no need to do the averaging procedure to improve accuracy. For BW we take B as being the previous wall point i.e. this line is tangential to the wall. Using these definitions in equation (16), setting P=W, we can calculate the co-ordinates of the wall point W.

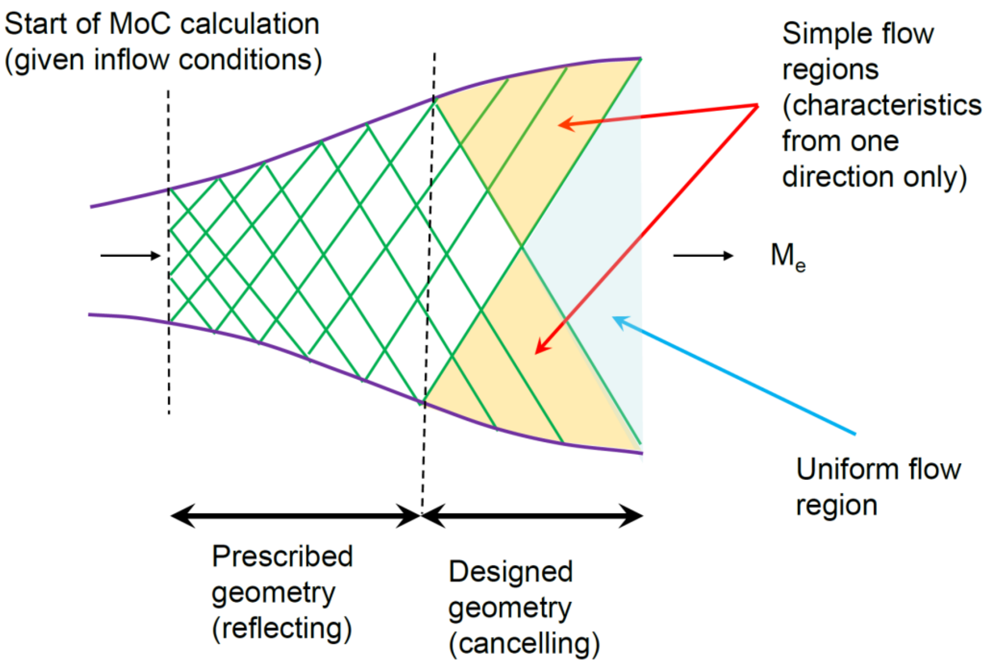

Figure 7. Addition of a designed section to a nozzle to achieve a uniform exit Mach number.

We can now put together everything we have learned and consider a practical nozzle flow. Figure 7 shows part of the diverging section of a nozzle that accelerates flow up to a supersonic exit Mach number M_e. We can start our MoC calculation from a plane where we have known conditions. This could be just downstream of the throat (at the throat itself the Mach number is one and the characteristic lines are at 90° to the flow direction, so that wouldn't make a good starting point in this example). During the first section of the nozzle (denoted as 'prescribed geometry' in the figure), we can work out the Riemann invariants along all the characteristic lines, we can deal with reflections at the walls and we can calculate the co-ordinates of all the intersection and wall points. This is essentially using MoC as a cheap flow prediction method. The clever bit comes when we start designing the geometry using the cancelling approach. In Figure 7 this is shown as the second section of the nozzle. In this section, when characteristic lines reach the all they are not reflected, resulting in the amber shaded regions in which the characteristic lines are all in one direction, with no crossings. These regions are called '***simple'*** regions. Finally, we reach a region where there are no characteristics at all, where the flow is ***uniform***. At this point, what we have succeeded in doing is design a nozzle wall to get the whole flow to the uniform exit Mach number, which is advantageous in terms of nozzle flow efficiency (no losses due to shock waves or turbulent mixing in non-uniform flows). An extreme example of this design process is discussed in the next section.

Before moving on, we can note that the flow in figure 7 has a plane of symmetry that we can exploit to halve the effort. On the symmetry plane we must have $\theta = 0$ so $R^+=R^-$. Hence, at these points we can just set a reflected characteristic with Riemann invariant equal to that of the incoming characterstic. Figure 8 illustrates what is happening, where point A is the reflection of point B in the symmetry plane $y=0$.

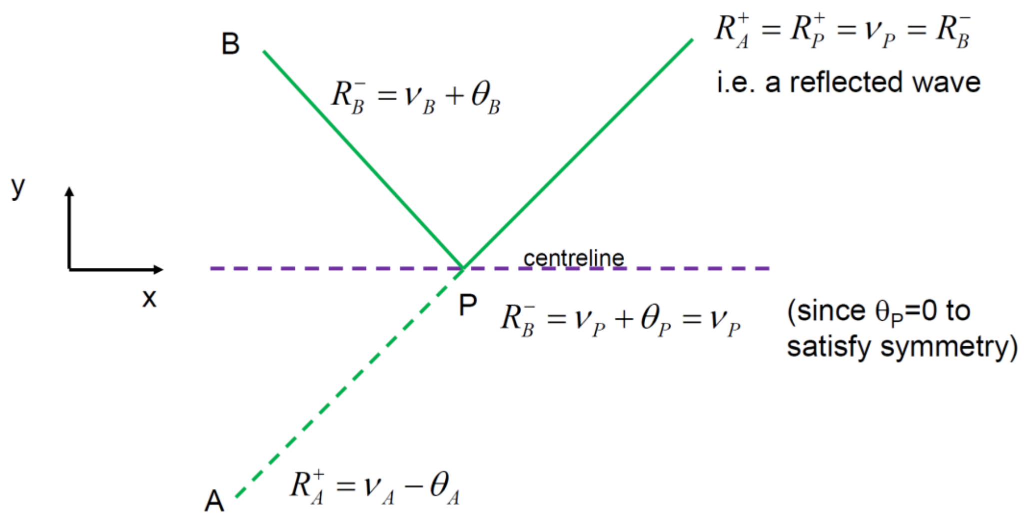

Figure 8. Treatment of a symmetry plane at y=0

The geometry calculation for a point on a symmetry plane simplifies compared to the general unit process. In particular, since $\theta_P = 0$, $y_P = 0$ and we already know the Mach angle from $\mu_P = \sin^{-1}\left(1/M_P\right)$ we only need

$$\alpha_{BP} = \frac{1}{2}\left[(\theta - \mu)_B - \mu_P\right] \tag{18a}$$

and

$$x_P = x_B - \frac{y_B}{\tan \alpha_{BP}}. \tag{18b}$$

5. Case study: Minimum length nozzle

Looking back at Figure 7, we can consider what happens when we move the point where we start the design process all the way back to the nozzle throat. What we end up with is the situation sketched in Figure 9, where we still have the same general arrangement as in Figure 7 of complex zones (with crossing characteristics), simple regions (with only one characteristic) and uniform flow at the design Mach exit number M_e.

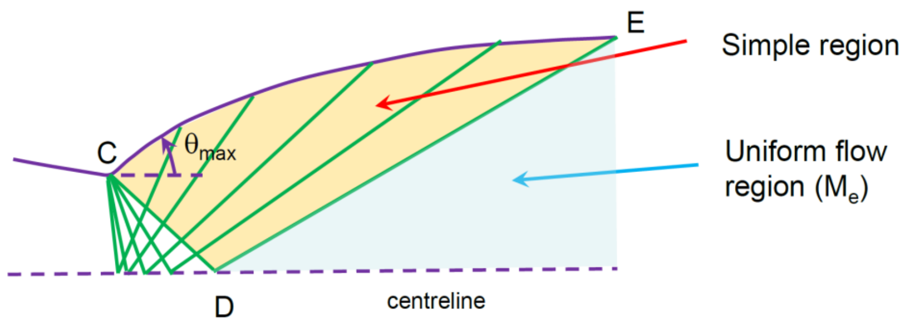

Figure 9. Characteristic lines for a minimum length nozzle.

Consider first just the characteristic line C-D-E, which has the same Riemann invariant all along it, including where it reflects from the symmetry plane at point D, where the flow angle is zero. We can easily look up the Prandtl-Meyer angle ν_e corresponding to the exit Mach number M_e, hence for the characteristic line C-D we have

$$R_C^- = \nu_e = \theta_{\max} + \nu_C \ , \tag{19}$$

where we have denoted the flow angle after point C as $\theta_{\max}$, since the only way to get from M=1 in the throat to the required conditions for the C-D characteristic is by an expansion fan with an overall turning angle of $\theta_{\max}$. Additional characteristic lines within the expansion fan are sketched in Figure 9. We can write down the Prandtl-Meyer relation for this turning of the flow as

$$\theta_{\max} = \nu_C - \nu\left(M = 1\right),$$

which reduces to $\theta_{\max} = \nu_C$ since we know from the IFT that $\nu\left(M = 1\right) = 0$. Substituting for ν_C back into (19) gives

$$\boxed{\theta_{\max} = \frac{\nu_e}{2}}. \tag{20}$$

We now have everything we need to complete the design of a minimum length nozzle:

1. Calculate $\theta_{\max}$ from (20) for the design Mach number.
2. Decide on a number of characteristic lines and a set of initial flow angles within the Prandtl-Meyer expansion, from $\theta = \delta$ up to $\theta = \theta_{\max}$. Note that δ should be a small number (so that the first geometry point is close to the throat), but not zero since $\theta = 0$ would correspond to a characteristic that just bounces backwards and forwards across the throat, without serving any useful design purpose.
3. For an expansion fan starting from M=1 we know that $\nu = \theta$ so we can compute the Riemann invariants of all the C^- lines emanating from the expansion fan.
4. Apply the MoC Unit Process (section 3 above) to find the flowfield (θ, ν and hence M) at all the downstream points.
5. Working from left to right, compute the coordinates of all the characteristic intersections. There are three types of point to consider: (a) interior points following equation (16), (b) symmetry plane points following equation (18) and (c) designed wall points following equation (17).

An example is shown on figure 10 for a calculation with 20 characteristic lines (equally spaced in θ). Such minimum length nozzles are practically important in aerospace application, since from materials considerations the shortest length nozzle is in practice likely to be the minimum weight nozzle.

The accuracy of the method improves with an increasing number of characteristics. While small number of characteristics can be computed with a hand calculator, any number above about 6 is more suited to a spreadsheet approach or to a program in a high-level programming language. Writing such a program is a good exercise in programming as well as in aerothermodynamics.

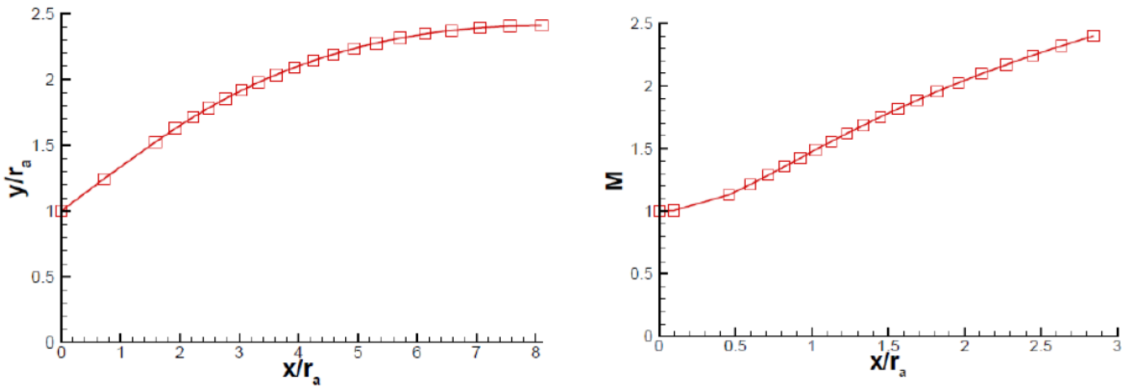

Figure 10. Computed wall geometry (left) and centreline Mach number for a 2D nozzle to reach a design Mach number of 2.4.

References

Anderson, J.D. 'Modern Compressible Flow with Historical Perspective', 3rd Edition, 2004, McGraw-Hill.

Liepmann, H.W. and Roshko, A. 'Elements of Gasdynamics' 1957, John Wiley & Sons.

Appendix: Axisymmetric MoC

For axisymmetric flow (i.e. in an *x*-*r* co-ordinate system), the method is slightly more complicated because we need to include an additional term that involves the radius *r*. This originates in the mass conservation, where we now have $\dot{m} = \rho u 2\pi r \Delta n$. The radius varies with *s* and we have an extra term in the mass conservation equation, given by $(1/r)(\partial r/\partial s)$ which is equal to $(1/r)\sin\theta$. For large *r*, this term is small and we revert to the planar case. The additional term propagates through the MoC derivation and the practical consequence is that we need to update our R^+ and R^- quantities along the characteristic lines, according to

$$\frac{\partial R^{\pm}}{\partial \eta} = \frac{\sin\mu \sin\theta}{r} ,$$

where η is a co-ordinate along the relevant characteristic line. Since the geometry is now involved, we have to update both our co-ordinates and flow quantities simultaneously during the unit process (i.e. we can't do the flowfield first and then come back to get the geometry).

To solve this numerically the simplest approach would be the explicit Euler space method. For more accuracy a predictor-corrector approach could be adopted, using the Euler method as a predictor and then improving the accuracy of the geometry using the averaging approach (14) as part of the corrector step.

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