# SESA6085 – Advanced Aerospace Engineering Management

Lecture 15

2024-2025

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# Maintenance & Inspection

- An effective maintenance plan can extend the life of a system quite considerably
- But…
  - How often should such maintenance be carried out?
  - How do we keep the system in working order for longer?
  - How do we minimise the cost of such maintenance and repairs?

# Prerequisites

- We can't just tackle the problem of optimal maintenance scheduling head on
- We first need to consider a number of important definitions
  - Mean time to failure
  - Repairable systems
  - Availability
  - Expected number of failures

# Mean Time to Failure

- What is the mathematical definition of the mean time to failure?
- We can also define this as the expected failure time, $E[T]$, in the range $t \in [0, \infty]$
- Mathematically this equates to:

$$\mathrm{MTTF} = E[T] = \int_0^\infty t f(t) dt$$

- Or:

$$\mathrm{MTTF} = E[T] = \int_0^\infty R(t) dt$$

# Repairable Systems

- We've defined a repairable system as one which is repaired upon failure
- This definition includes large and complex systems e.g. aircraft, cars, mainframes, telephone networks etc.
- The two most important performance criteria for such a system are:
  - Availability of the system
  - Mean time before failure (MTBF)
- Of these two availability is probably the most important

# Expected Number of Failures

# Expected Number of Failures, *M(T)*

- The expected number of failures of a system within a time interval $[0, T]$ is an extremely useful quantity
  - Defines an optimal preventive maintenance schedule
  - Criterion for reliability of acceptance
    - A production lot is accepted if the number of expected failures is less than a defined limit
  - Defines warranty policies
    - If we have $M(T)$ expected failures in the time interval T, then the expected warranty cost $C(T)$ is

$$C(T) = cM(T)$$

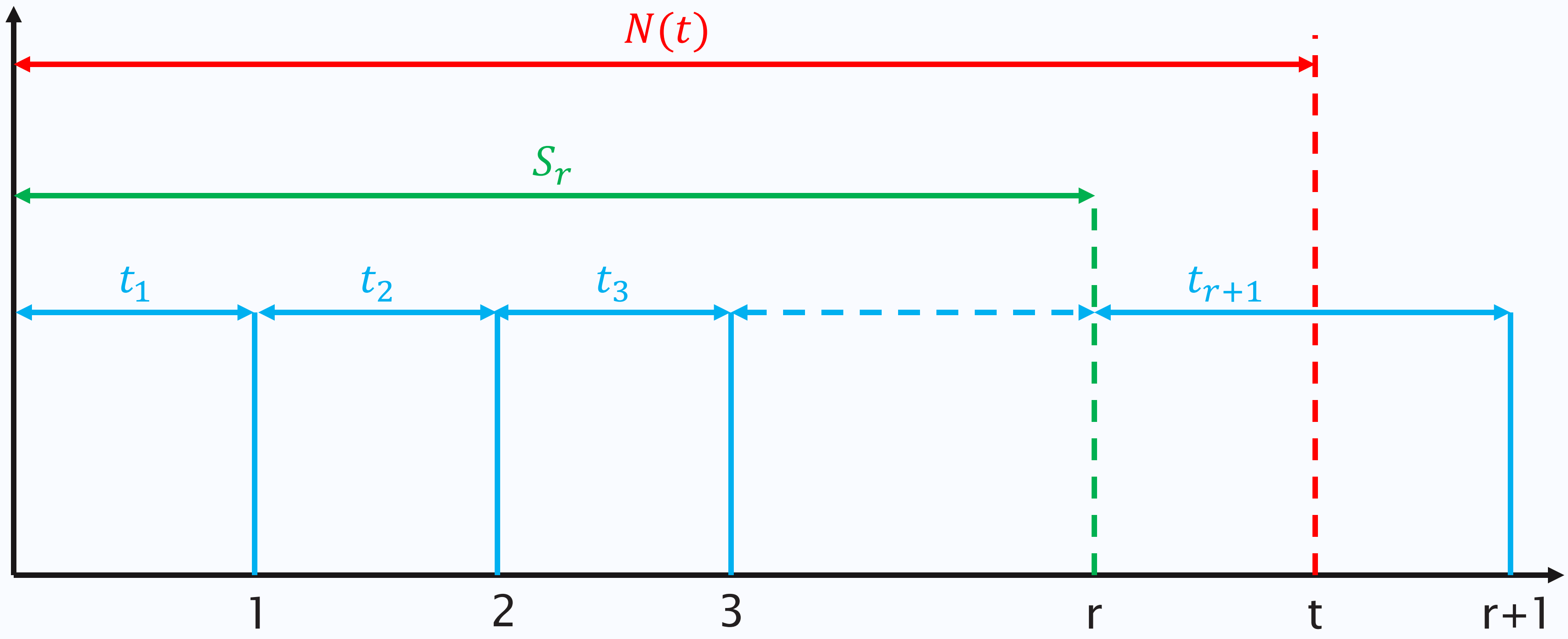
# Expected Number of Failures, *M(T)*

- There are two approaches to calculating $M(T)$
  - *Parametric approach* – the failure-time distribution is known
  - *Nonparametric approach* – the failure-time distribution is unknown and where mean and standard deviation of failure times are known
- In the following we will focus on the parametric approach

# Expected Number of Failures, *M(T)*

- Before beginning lets define some of the notation used
- $N(t)$ – the number of failures in the interval $[0, t]$
- $M(t)$ – the expected number of failures in the interval $[0, t]$, otherwise denoted as $\mathrm{E}[N(t)]$
- $t_i$ – the length of the time interval between failures $i-1$ and $i$
- $S_r$ – the total time up to the *r*th failure - $S_r = \sum_{i=1}^{r} t_i$

# Expected Number of Failures, *M(T)*



# Expected Number of Failures, *M(T)*

- The probability of having a number of failures $N(t) = r$ is the same as the probability that $t$ lies between the $r^{\text{th}}$ and $(r+1)^{\text{th}}$ failure

$$P[N(t) < r] = 1 - F_r(t)$$

- Where $F_r(t)$ is the cumulative distribution function of $S_r$ i.e. $F_r(t) = P[S_r \leq t]$
- Hence:

$$P[N(t) > r] = F_{r+1}(t)$$

# Expected Number of Failures, *M(T)*

- From simple probability theory:

$$P[N(t) < r] + P[N(t) = r] + P[N(t) > r] = 1$$

- Therefore:

$$P[N(t) = r] = F_r(t) - F_{r+1}(t)$$

# Expected Number of Failures, *M(T)*

- Using this we can find an expression for the expected value of $N(t)$

$$M(t) = \mathrm{E}[N(t)] = \sum_{r=0}^{\infty} r\, P[N(t) = r]$$

$$M(t) = \sum_{r=0}^{\infty} r\, [F_r(t) - F_{r+1}(t)]$$

$$M(t) = \sum_{r=0}^{\infty} F_r(t)$$

# Expected Number of Failures, *M(T)*

- This expression for $M(t)$ is termed the renewal function and it can also be written as

$$M(t) = \mathrm{F(t)} + \sum_{r=1}^{\infty} F_{r+1}(t)$$

- $F_{r+1}(t)$ is the convolution of $F_r(t)$ and $F$ therefore

$$F_{r+1}(t) = \int_0^t F_r(t-x) f(x) dx$$

# Expected Number of Failures, *M(T)*

- Substituting this back into our expression for $M(t)$ we get:

$$M(t) = F(\mathrm{t}) + \sum_{r=1}^{\infty} \int_0^t F_r(t-x) f(x) dx$$

- Which equates to:

$$M(t) = F(\mathrm{t}) + \int_0^t M(t-x) f(x) dx$$

- Which is known as the *fundamental renewal equation*

# Renewal Density Equation, *m(t)*

- Although we will not go through its derivation $M(t)$ can be differentiated to give an expression for the renewal density, $m(t)$

$$m(t) = f(\mathrm{t}) + \int_0^t m(t-x)f(x)dx$$

- We can interpret this as the probability that a renewal occurs between $[t, t+\Delta t]$

# Solving These Equations

- Both the fundamental renewal equation and the renewal density equation can be solved using Laplace transforms

$$M^*(s) = \frac{f^*(s)}{s[1 - f^*(s)]}$$

$$m^*(s) = \frac{f^*(s)}{1 - f^*(s)}$$

- Where $f^*(s)$ denotes the Laplace transform of the failure-time PDF

# Example

- Define an expression for the number of failures of a system over the interval $[0, t]$ described by

$$f(t) = \lambda e^{-\lambda t}$$

- Taking the Laplace transform of the PDF we obtain

$$f^*(s) = \frac{\lambda}{s + \lambda}$$

- Which upon substitution into the renewal density equation gives:

$$m^*(s) = \frac{\lambda}{s + \lambda - \lambda} = \frac{\lambda}{s}$$

# Example

- The Laplace inverse of this equation gives:

$$m(t) = \lambda$$

- Hence the expression for the number of failures in the interval $[0, t]$ is:

$$M(t) = \int_0^t \lambda \mathrm{dt} = \lambda t$$

- For a component with a constant failure rate of $6 \times 10^{-6}$ per hour we will therefore expect 0.0526 failures over one year

# Calculating M(t)

- If we look at our general expression for $M(t)$,

$$M(t) = F(\mathrm{t}) + \int_0^t M(t-x) f(x) dx$$

- We can see that the function $M(t)$ appears on both sides
- Of course, the equations could be solved computationally
- Researchers have investigated ways of approximating this integral

# Discrete Time Approach

- If there is no closed form solution for $M(t)$ then a discrete time approach may be used where the number of failures at a time period $T$ is:

$$M(t) = \sum_{i=0}^{T-1} [1 + M(T - i - 1)] \int_{i}^{i+1} f(t)dt$$

- Where $M(0) = 0$

# Discrete Calculation Example

- The failure time of a system is normally distributed with mean 4 and standard deviation 1 weeks. Calculate the expected number of failures at the end of week 2

$$M(2) = [1 + M(1)] \int_0^1 f(t)dt + \cdots$$

$$[1 + M(0)] \int_1^2 f(t)dt$$

- With $f(t)$ defined by our known PDF and the integrals calculated using our CDFs

# Discrete Calculation Example

- Calculate the values of $M(t)$ in order up to the point of interest

| *T* | *M(T)* |
|---|---|
| 0 | 0 |
| 1 | 0.001318 |
| 2 | 0.02272 |

- Therefore, we expect 0.0227 failures after 2 weeks

# Optimal Maintenance & Inspection

# Maintenance & Inspection

- Now we've got the basic building blocks in place let's consider our problem of defining maintenance and inspection routines
- In this case we're concerned with optimum preventive maintenance, replacements and inspection (PMRI) schedules
- The primary function of such schedules is to control the condition of the system to ensure availability

# Maintenance & Inspection

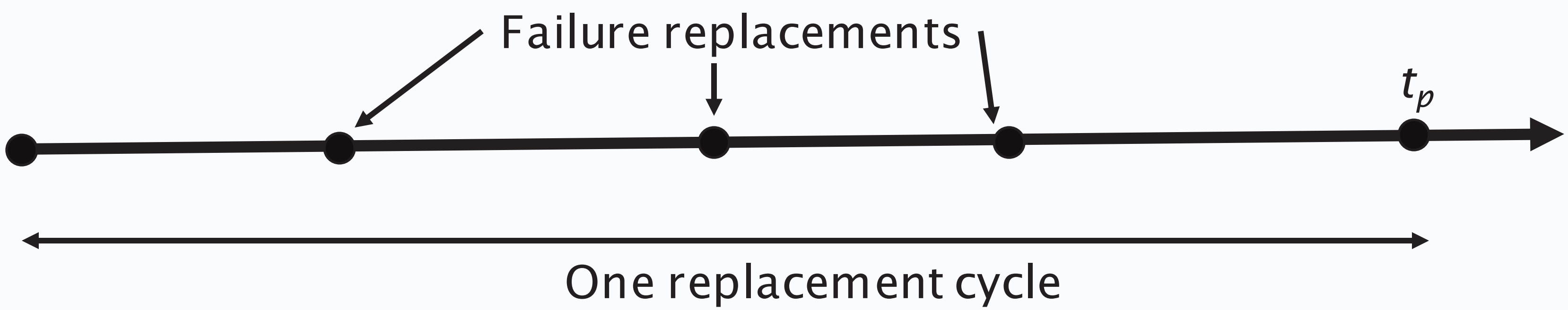
- What do we mean by optimum PRMI?
  - High frequency of PRMI reduces costs due to downtime but increases maintenance costs
  - Low frequency of PRMI reduces maintenance costs but increases costs due to downtime
- Hence there may be a sweet spot for minimum cost
- Other criteria such as availability may drive the optimal schedule

# Cost Minimisation

Constant Interval Replacement

# Cost Minimisation - CIRP

- The simplest preventive replacement policy is where a component is replaced after a constant interval
  - Constant Interval Replacement Policy (CIRP)
- This approach has two actions:
  1. A component is replaced on failure
  2. A component is replaced at a predetermined interval regardless of age



# Cost Minimisation - CIRP

- In order to optimise this function we need some sort of objective
- As we're interested in minimising cost lets use the total replacement cost per unit time

$$c(t_p) = \frac{\text{Total expected cost } [0, t_p]}{\text{Length of the interval}}$$

- The total cost consists of:
  - The cost of preventative replacement
  - The expected cost of failure replacements

# Cost Minimisation - CIRP

- During the interval $[0, t_p]$ we therefore have
  - A single preventative replacement of cost, $c_p$
  - $M(t_p)$ failure replacements of cost, $c_f$ each
  - We already know our expected interval length is $t_p$
- Hence:

$$c(t_p) = \frac{c_p + c_f M(t_p)}{t_p}$$

# CIRP Example

- A bearing in a rotating shaft wears out according to a normal distribution with a mean of $1 \times 10^6$ cycles and standard deviation of $1 \times 10^5$ cycles
- The cost of preventative replacement is $\$50$
- The cost of failure replacement $\$100$
- Assuming discrete time intervals of $1 \times 10^5$ cycles determine the optimum replacement interval
- How do we proceed?

# CIRP Example

- First define a list of discrete intervals
- Next use the PDF to calculate $M(t_p)$ at each interval
- Use the costs along with $M(t_p)$ and $t_p$ to calculate the cost per interval

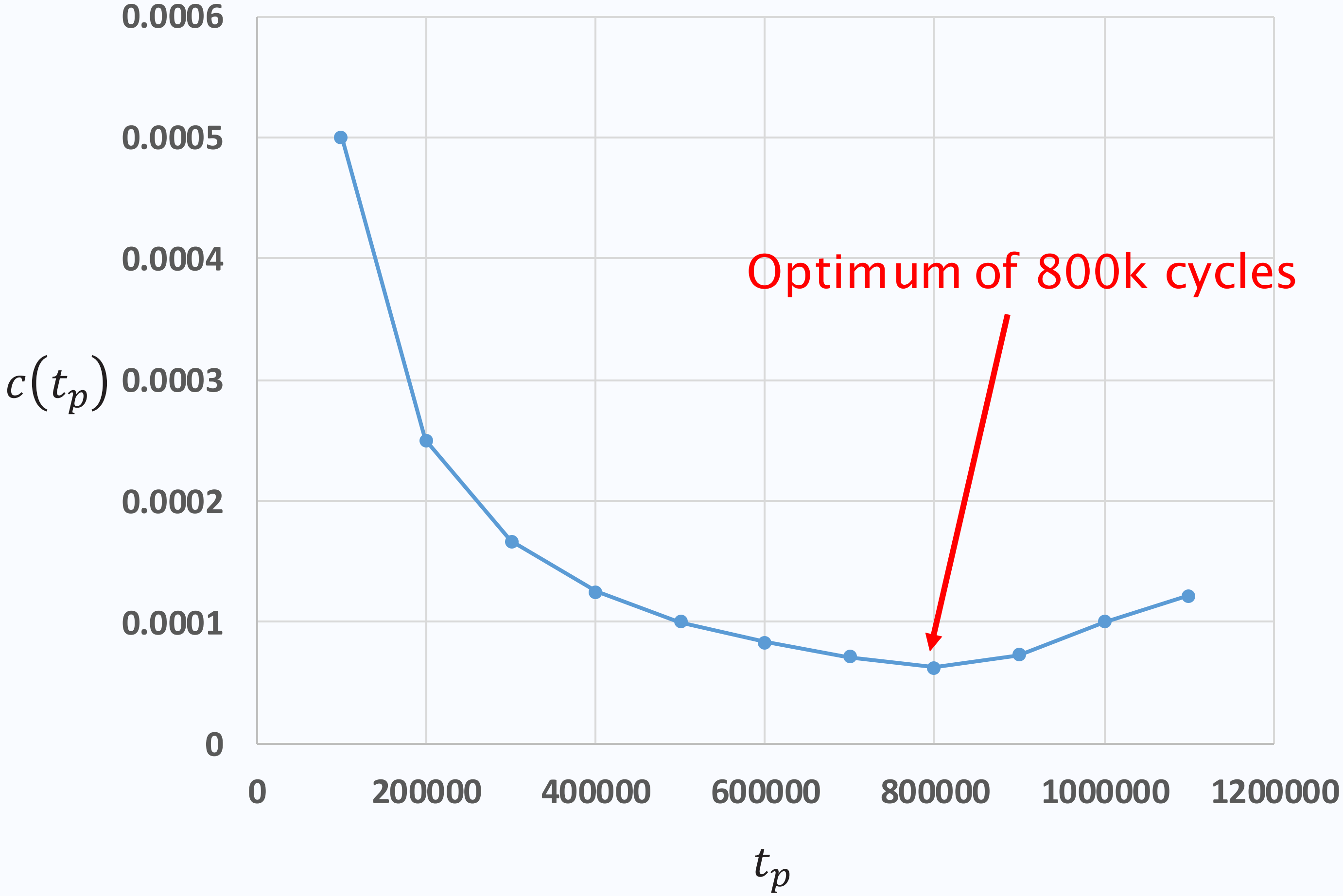
# CIRP Example

- Doing so for the first 11 intervals gives

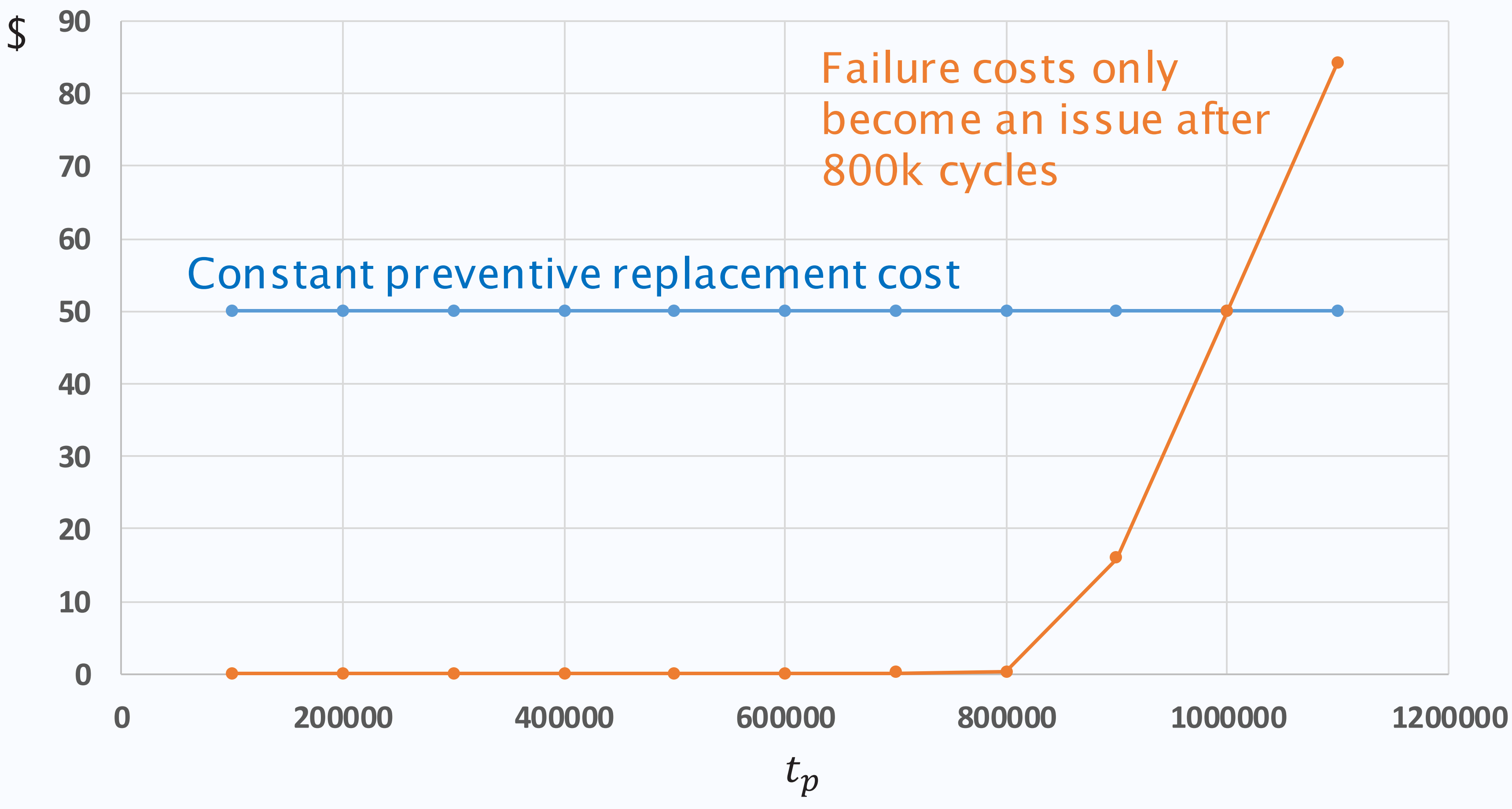
| $t_p$ | $M(t_p)$ | $c(t_p)$ |
|---|---|---|
| 100,000 | 0 | 0.000500 |
| 200,000 | 0 | 0.000250 |
| 300,000 | 0 | 0.000166 |
| 400,000 | 0 | 0.000125 |
| 500,000 | 0 | 0.000100 |
| 600,000 | 0 | 0.000083 |
| 700,000 | 0.00140 | 0.000072 |
| 800,000 | 0.00275 | 0.000065 |
| 900,000 | 0.15875 | 0.000073 |
| 1,000,000 | 0.50005 | 0.0001 |
| 1,100,000 | 0.84135 | 0.000121 |

# CIRP Example

- Plotting this result gives the following…



# CIRP Example



# Cost Minimisation

Replacement at Predetermined Age

# Replacement at Predetermined Age

- CIRP may result in preventative replacements being performed shortly after failure replacements
- Replacement at predetermined age is a slightly different preventative policy
- Under this policy:
  - Components are replaced on failure
  - Components are replaced after a set age, $t_{pa}$
  - Or whichever comes first

# Replacement at Predetermined Age

- As before we need to select the length of time, $t_{pa}$, in order to minimise the cost of the preventative maintenance
- As before we therefore need some form of objective function to minimise
- Again, we'll use a cost per unit time

$$c(t_{pa}) = \frac{\text{Total expected replacement cost per cycle}}{\text{Expected length of cycle}}$$

# Replacement at Predetermined Age

- As noted previously we have two modes of operation
  1. The component reaches $t_{pa}$ and is replaced
  2. The component fails before $t_{pa}$ and is replaced
- The numerator in our above equation includes:
  - The cost of preventative replacement × the probability of survival to $t_{pa}$
  - The cost of failure replacement × the probability of failure before $t_{pa}$
  - Note that the expected number of failures is not present because we restart our cycle from the time of a failure

# Replacement at Predetermined Age

- The numerator is therefore:

$$c_p R(t_{pa}) + c_f[1 - \mathrm{R}(t_{pa})]$$

- The denominator is defined from:
  - Length of a preventive cycle times the probability of that cycle
  - Expected length of a failure cycle times the probability of a failure cycle

# Replacement at Predetermined Age

- This gives us the following denominator

$$t_{pa}R(t_{pa}) + [1 - \mathrm{R}(t_{pa})] \int_{-\infty}^{t_{pa}} tf(t)dt$$

- And finally, an expression for the cost per unit time

$$c(t_{pa}) = \frac{c_p R(t_{pa}) + c_f [1 - \mathrm{R}(t_{pa})]}{t_{pa}R(t_{pa}) + [1 - \mathrm{R}(t_{pa})] \int_{-\infty}^{t_{pa}} tf(t)dt}$$

- Our optimum is therefore the value of predetermined age which gives us the lowest cost per cycle

# Predetermined Age Example

- Previously in the CIRP example we considered a shaft with failure-time described by a normal distribution with a mean of $1 \times 10^6$ cycles and standard deviation of $1 \times 10^5$ cycles
- The cost of preventative replacement is $\$50$
- The cost of failure replacement $\$100$
- For CIRP we saw an optimum $t_p$ of 800,000 cycles
- What is the predetermined age, $t_{pa}$, for this case?
- How to proceed?

# Predetermined Age Example

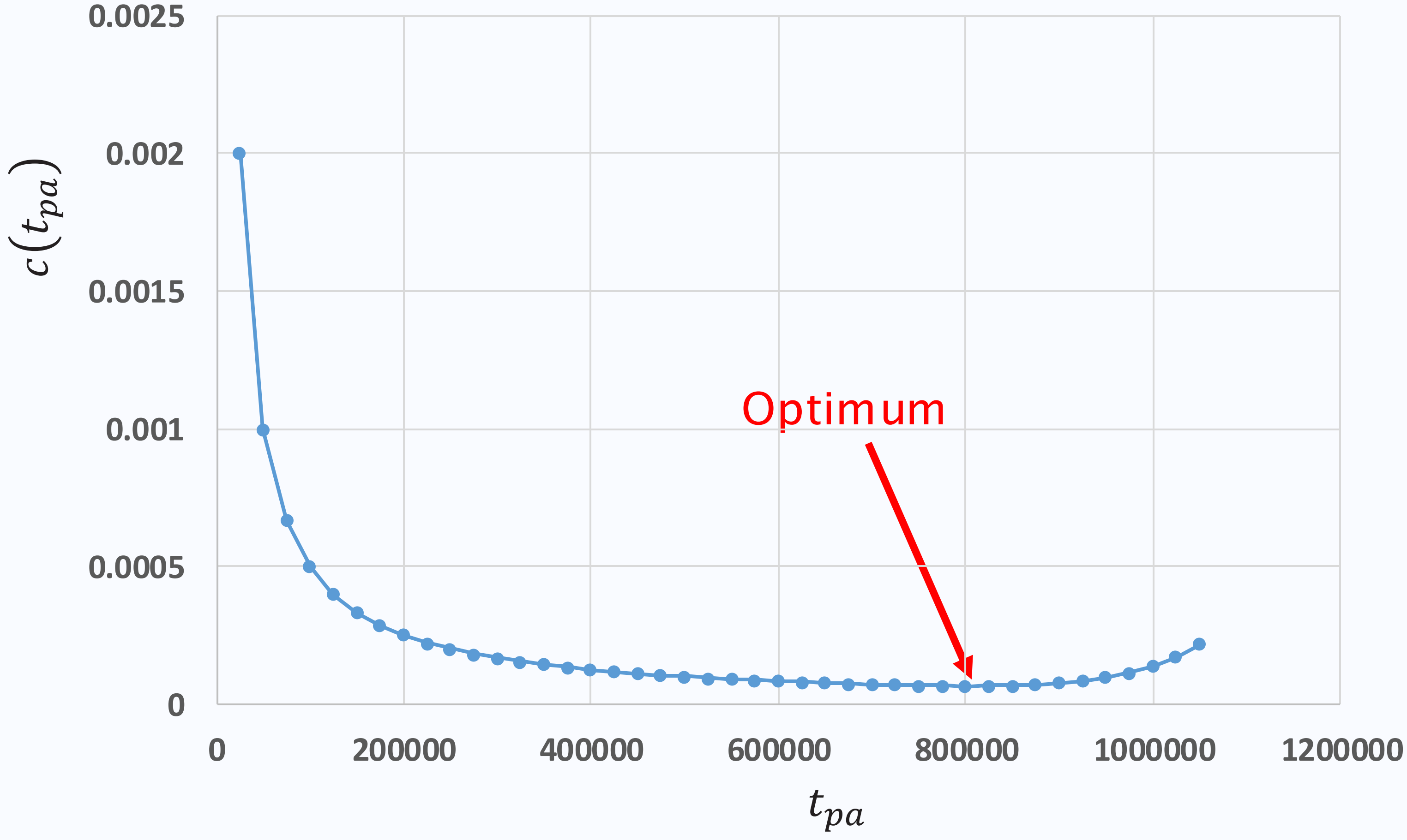
- Define a range of $t_{pa}$ to consider
- Calculate $R(t_{pa})$, $F(t_{pa})$, $f(t_{pa})$ and $\int_{-\infty}^{t_{pa}} tf(t)dt$
- Calculate the numerator and denominator
- Finally calculate $c(t_{pa})$
- Select the value of $t_{pa}$ which gives the minimum
- How can we calculate $\int_{-\infty}^{t_{pa}} tf(t)dt$?
  - Matlab or Python can be used but a very simple trapezoidal approach will suffice

# Predetermined Age Example

| $t_{pa}$ | $f(t_{pa})$ | $F(t_{pa})$ | $R(t_{pa})$ | $c(t_{pa})$ |
|---|---|---|---|---|
| 0 | 7.69e-28 | 7.62e-24 | 1.0 | |
| 100,000 | 1.03e-23 | 1.12e-19 | 1.0 | 0.0005 |
| 200,000 | 5.05e-20 | 6.22e-16 | 1.0 | 0.00025 |
| 300,000 | 9.13e-17 | 1.28e-12 | 1.0 | 0.000167 |
| 400,000 | 6.08e-14 | 9.87e-10 | 1.0 | 0.000125 |
| 500,000 | 1.49e-11 | 2.87e-7 | 1.0 | 0.0001 |
| 600,000 | 1.34e-9 | 3.17e-5 | 0.99997 | 8.33E-05 |
| 700,000 | 4.43e-8 | 0.00135 | 0.99865 | 7.16E-05 |
| 800,000 | 5.40e-7 | 0.02275 | 0.97725 | 6.54E-05 |
| 900,000 | 2.42e-6 | 0.15866 | 0.84135 | 7.45E-05 |
| 1,000,000 | 3.99e-6 | 0.5 | 0.5 | 0.000115 |
| 1,100,000 | 2.42e-6 | 0.84134 | 0.15866 | 0.000203 |

# Predetermined Age Example

- Graphically…



# Downtime Minimisation

# Downtime Minimisation

- There are many situations where availability is much more important than the cost of repair or maintenance
  - In such cases the consequences of any downtime may exceed any measurable cost
- In these cases, it’s more appropriate to minimise the total downtime experienced by the system
- In such cases the equations to be minimised take a very similar form to those for cost minimisation
  - We consider time to replace instead of cost to replace

# CIRP Downtime Minimisation

- The total downtime is due to…
  - Downtime due to failure
  - Downtime due to preventive replacement

$$D_{CIRP}(t_p) = \frac{M(t_p)T_f + T_p}{T_p + t_p}$$

$T_f$ - time to perform a replacement after failure

$T_p$ - time to perform a preventive replacement

$M(t_p)$ - Expected number of failures in the interval, $[0, t_p]$

- The goal is to find an interval, $t_p$, which minimises this function

# Predetermined Age Downtime Minimisation

- As before this policy involves:
  - Replacement when a component fails
  - Replacement when a component reaches its planned replacement age

$$D_{ARP}(t_{pa}) = \frac{T_p R(t_{pa}) + T_f[1 - R(t_{pa})]}{(t_{pa} + T_p)R(t_{pa}) + \left[\int_{-\infty}^{t_{pa}} t f(t)dt + T_f\right][1 - R(t_{pa})]}$$

# Other Approaches

# Other Approaches & Models

- There are of course numerous other models for preventative repair and maintenance
- Minimal number of spares
- Minimal repair models:
  - Assume that the system is no longer as good as new after a repair
- Systems subject to shocks:
  - Apply to systems which are subject to shocks causing cumulative damage

# Other Approaches & Models

- Group maintenance
  - Considers the maintenance of a group of similar machines at once
  - This may be more economical than considering each machine separately
- Periodic inspection
  - Reliability can be improved further by coupling preventative maintenance with periodic inspection
- Condition monitoring or condition-based maintenance
  - Replacement when an indicator says failure is imminent

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