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Lecture 10 - Fourier Transform: Properties and response function

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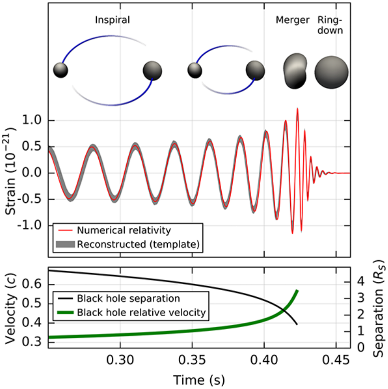
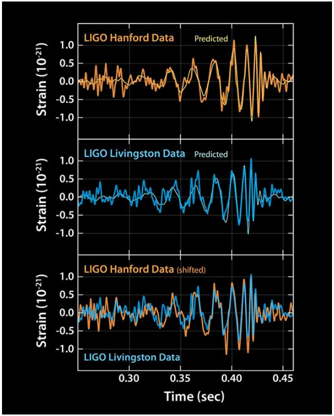
https://www.ligo.caltech.edu/video/ligo20170927v1





Initial black holes: 25 and 31 solar masses
Final black hole: 53 solar masses
$\sim$ 3 solar masses released as gravitational waves

Review

- Inverse **Fourier transforms** extends the concept of Fourier Series to functions that are **not periodic** and that are defined on the **whole real line** (i.e. $-\infty < t < \infty$).
- **Fourier transforms** (FT) were obtained by taking the **infinite period limit** ($T \to \infty$) of **Complex Fourier Series.**
- The **Fourier transform** of $f(t)$ is:

$$F(\omega) \equiv \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{j}\,\omega\, t} \, \mathrm{d}t.$$

Freq

- The **inverse Fourier transform** of $F(\omega)$ is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{+\mathbf{j}\,\omega\, t} \, \mathrm{d}\omega.$$

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- Let $f(t)$ be a function such that $f(t) \to 0$ as $t \to \pm\infty$.

  By definition the Fourier Transform of $\frac{df}{dt}$ is given by

$$\mathcal{F}\left[\frac{df}{dt}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df}{dt} e^{-\mathbf{j}\,\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overbrace{e^{-\mathbf{j}\,\omega t}}^{u} \overbrace{\frac{df}{dt}\, dt}^{dv}$$

integration by parts

$(\searrow \int \frac{df}{dt} dt = f)$ $(\searrow [e^{-\mathbf{j}\,\omega t}]' = -\mathbf{j}\,\omega e^{-\mathbf{j}\,\omega t})$ $(\swarrow \int_A^B u\, dv = [uv]_A^B - \int_A^B v\, du)$

since we assume for a F trans to exist $f(\infty) \to 0$ we can tell this goes to zero

$$= \frac{1}{\sqrt{2\pi}} \left[f(t) e^{-\mathbf{j}\,\omega t}\right]_{-\infty}^{\infty} + \frac{\mathbf{j}\,\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{j}\,\omega t}\, dt$$

$$= 0 + \mathbf{j}\,\omega \mathcal{F}[f(t)]$$

$(\nwarrow \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{j}\,\omega t}\, dt)$

- Hence, if $\lim_{t \to \pm\infty} f(t) \to 0 \Rightarrow \left[f(t) e^{-\mathbf{j}\,\omega t}\right]_{-\infty}^{\infty} \to 0$ we have:

important bit

$$\mathcal{F}\left[\frac{df}{dt}\right] = \mathbf{j}\,\omega \mathcal{F}[f(t)] \qquad \longleftarrow \textbf{FT of the } \underline{\textbf{derivative}} \textbf{ of } f(t)$$

Example: fast way of finding FT

- **What is the Fourier Transform of** the function:

$$g(t) = \begin{cases} \cos(t) & , |t| \leq \pi \\ 0 & , \text{otherwise} \end{cases} \qquad \textbf{?}$$

- Last Lecture 9: we showed that the Fourier transform of

$$f(t) = \begin{cases} \sin t & , |t| \leq \pi \\ 0 & , \text{otherwise} \end{cases}$$

known

is

$$F(\omega) = \mathbf{j}\sqrt{\frac{2}{\pi}}\frac{\sin(\omega\pi)}{\omega^2 - 1}$$

- Well,... note that $g(t) = f'(t)$! So the FT of $g(t)$ can be quickly obtained using the FT derivative property $\mathcal{F}\left[f'(t)\right] = \mathbf{j}\,\omega\mathcal{F}\left[f(t)\right]$:

need to be careful that deriv not taken so much such that it no longer tends to zero as $t \to \infty$

$$\mathcal{F}\left[g(t)\right] = \mathcal{F}\left[f'(t)\right] = \mathbf{j}\,\omega\mathcal{F}\left[f(t)\right] = -\frac{\omega}{\omega^2 - 1}\sqrt{\frac{2}{\pi}}\sin(\omega\pi).$$

- Differentiation:

obvious identity

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = \mathbf{j}\,\omega\,\mathcal{F}\left[f(t)\right] \implies \mathcal{F}\left[\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right] = (\mathbf{j}\,\omega)^n\,\mathcal{F}\left[f(t)\right].$$

($\nearrow$ Just apply n times the property $\mathcal{F}\left[f'(t)\right] = \mathbf{j}\,\omega\mathcal{F}\left[f(t)\right]$)

- Linearity:

α, β either constants or functions independent of t. eg $\alpha(\omega)$

$$\mathcal{F}\left[\alpha f(t) + \beta g(t)\right] = \alpha\mathcal{F}\left[f(t)\right] + \beta\mathcal{F}\left[g(t)\right]$$

→ Application of FT: Response function & Solving ODEs

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- **Given an ODE** connecting the **input** $u(t)$ to the **output** $y(t)$:

$$\mathcal{L}_y y(t) = \mathcal{L}_u u(t),$$

Its **Fourier transform** gives the **transfer** or **response function** $\boldsymbol{G(\omega)}$:

$$\mathcal{F}[\mathcal{L}_y y(t)] = \mathcal{F}[\mathcal{L}_u u(t)] \quad \Rightarrow \quad Y(\omega) = G(\omega)U(\omega)$$

(↖ Output = Response to Input)

where $\mathcal{F}[\boldsymbol{y(t)}] \equiv \boldsymbol{Y}(\omega)$ and $\mathcal{F}[\boldsymbol{u(t)}] \equiv \boldsymbol{U}(\omega)$

- Solution of the ODE is given by the inverse Fourier transform of $Y(\omega)$:

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Y(\omega) e^{\mathbf{j}\,\omega t}\, \mathrm{d}\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega)U(\omega) e^{\mathbf{j}\,\omega t}\, \mathrm{d}\omega$$

$\mathcal{L}_y: y \to \ddot{y} + \gamma\dot{y} + \omega_0^2 y$
$\mathcal{L}_u: u \to u$

- *Example*: **resonance of the damped oscillator**

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}y(t)}{\mathrm{d}t} + \omega_0^2\, y(t) = u(t) \quad \longleftarrow \quad \begin{cases} \mathcal{L}_y = \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} + \dfrac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2 \\ \mathcal{L}_u = 1 \end{cases}$$

Resonance of the damped oscillator

y is unknown
ω_0, γ is constant

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- To **compute the response** we take the Fourier transform **of the** **ODE** and use **linearity**:

use of linearity from earlier

$$\mathcal{F}\left[\frac{d^2y}{dt^2}+\gamma\frac{dy}{dt}+\omega_0^2 y(t)\right]=\mathcal{F}\left[u(t)\right]$$

unknown

$\mathcal{F}[y(t)] \equiv Y(\omega)$

$[\ \swarrow \mathcal{F}[u(t)] \equiv U(\omega)\ ]$

known

$\therefore$ also known

notation simplification

(split it up)

$$\mathcal{F}\left[\frac{d^2y}{dt^2}\right]+\gamma\mathcal{F}\left[\frac{dy}{dt}\right]+\omega_0^2\mathcal{F}\left[y(t)\right]=U(\omega)$$

- Next we use the formula for the **Fourier transform of derivatives**,

use of derivative identity from earlier

$$(\mathbf{j}\,\omega)^2 Y(\omega)+\gamma(\mathbf{j}\,\omega)Y(\omega)+\omega_0^2 Y(\omega)=U(\omega) \qquad \leftarrow \mathcal{F}\left[y(t)\right]\equiv Y(\omega)$$

$$\Leftrightarrow \left[-\omega^2+\gamma\,\mathbf{j}\,\omega+\omega_0^2\right] Y(\omega)=U(\omega) \qquad \nwarrow \mathcal{F}\left[\frac{d^ny}{dt^n}\right]=(\mathbf{j}\,\omega)^n\mathcal{F}\left[y(t)\right]$$

$$\Leftrightarrow Y(\omega)=G(\omega)U(\omega).$$

with the response function $G(\omega)$ being

colour meaning:
Green: known
Red: response function
Blue: unknown

transfer function

given the input how much of the output comes through

$$G(\omega)=\frac{1}{\omega_0^2-\omega^2+\gamma\,\mathbf{j}\,\omega}.$$

Resonance of the damped oscillator

- When is the magnitude of the response function maximized?

we're working with complex numbers hence the need to do this

$|G(\omega)| = (G(\omega)) \times (\bar{G}(\omega))$ complex conjugate

$$\text{Max of } |G(\omega)|^2 \text{ ?} \quad \longrightarrow \quad |G(\omega)|^2 = \frac{1}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

minimal clearly at $\pm\infty$

$$\frac{d|G(\omega)|^2}{d(\omega^2)} = -\frac{-2(\omega_0^2-\omega^2)+\gamma^2}{\left[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right]^2} \quad \Rightarrow \frac{d|G(\omega)|^2}{d(\omega^2)} \propto -2(\omega_0^2-\omega^2)+\gamma^2 = 0$$

since defined in terms of w^2 it's faster to derive w^2 at w then w

$$\text{Max of } |G(\omega)|^2 \Leftrightarrow \frac{d|G(\omega)|^2}{d(\omega^2)} = 0 \quad \Rightarrow \boldsymbol{\omega^2_{\text{max}} = \omega_0^2 - \tfrac{1}{2}\gamma^2}.$$

For small damping $\gamma \ll \omega_0$, response is maximized when $\boldsymbol{\omega \approx \omega_0}$.

shows conditions for resonance for any function

- Solution of the original ODE is given by the inverse Fourier transform:

$$y(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overbrace{G(\omega)\,U(\omega)}^{Y(\omega)}\, e^{\mathbf{j}\,\omega t}\, d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{U(\omega)}{\omega_0^2-\omega^2+\gamma\,\mathbf{j}\,\omega}\, e^{\mathbf{j}\,\omega t}\, d\omega.$$

$\Rightarrow$ when the **input (source) frequency** ω is near ω_{max} (i.e. near the **natural frequency** ω_0 for small γ) we have a **resonance** since $\boldsymbol{y(t)}$ **grows very large !!**

Summary

- Properties of Fourier transforms:
  - Differentiation:
  $$\mathcal{F}\left[\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right] = (\,\mathbf{j}\,\omega)^n\,\mathcal{F}\left[f(t)\right].$$
  - Linearity:
  $$\mathcal{F}\left[\alpha f(t) + \beta g(t)\right] = \alpha\mathcal{F}\left[f(t)\right] + \beta\mathcal{F}\left[g(t)\right]$$
- Fourier transforms may be used to solve ODEs through the response function $G(\omega)$.