Outline

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Lecture 12 - Laplace transform properties and solution methods

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Outline

1. Review
2. Laplace Transform Properties
   - Properties
   - Examples and methods
   - Products
3. Summary

Review

- Definition of Laplace Transform (LT):

frequency

$$\mathcal{L}\left[f(x)\right] = \tilde{f}(\boxed{s}) = \int_0^\infty f(x) e^{-sx}\, dx.$$

- First shift theorem:

$$\mathcal{L}\left[e^{-ax} f(x)\right] = \tilde{f}(s+a)$$

- Laplace Transform of Derivatives:

$$\mathcal{L}\left[\frac{df}{dx}\right] = s\tilde{f}(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^2f}{dx^2}\right] = s^2\tilde{f}(s) - s\,f(0) - f'(0)$$

Review

- List of known basic examples (you need to know them by heart!):

not all on formula sheet

$$\mathcal{L}[e^{ax}] = \frac{1}{s-a}$$

$$\mathcal{L}[\sin(a\,x)] = \frac{a}{s^2+a^2},$$

$$\mathcal{L}[\cos(a\,x)] = \frac{s}{s^2+a^2}$$

$$\mathcal{L}[1] = \tfrac{1}{s}$$ [Today's lecture]

$$\mathcal{L}[x] = \frac{1}{s^2}$$ [Today's lecture]

$$\mathcal{L}[x^n] = \frac{n!}{s^{n+1}}$$ [Today's lecture]

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$\rightarrow$ Laplace Transform Properties: Linearity

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The Laplace transform is linear:

if $\mathcal{L}\left[f_{1,2}(x)\right] = \tilde{f}_{1,2}(s)$ and $\mu_{1,2}$ are constants then

$$\mathcal{L}\left[\mu_1 f_1(x) + \mu_2 f_2(x)\right] = \mu_1 \tilde{f}_1(s) + \mu_2 \tilde{f}_2(s).$$

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The Laplace transform of $x\,f(x)$ is related to the derivative of the LT as

$$\mathcal{L}\left[x\,f(x)\right] = -\frac{\mathrm{d}\tilde{f}}{\mathrm{d}s}$$

Proof: differentiating the definition of the Laplace transform:

$\mathcal{L}(f(x)) = \int_0^\infty dx\, e^{-sx} f(x)$

$x\dfrac{d}{ds}$

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty f(x)e^{-s\,x}\,\mathrm{d}x$$

can do this since s and x are independent.

$$= \int_0^\infty f(x)\left(\underbrace{\frac{\mathrm{d}}{\mathrm{d}s}}e^{-s\,x}\right)\mathrm{d}x$$

just differentiation with respect to s

$$= -\int_0^\infty x\,f(x)e^{-s\,x}\,\mathrm{d}x$$

rearrange in terms of Lap trans

$$\frac{d\mathcal{L}[f(x)]}{ds} = \frac{\mathrm{d}\tilde{f}}{\mathrm{d}s} = -\mathcal{L}\left[x\,f(x)\right].$$

Example

As an example we can use the result

assuming $x > 0$

$$\mathcal{L}[1] = \int_0^\infty 1\, e^{-sx}\, dx = \left[-\frac{1}{s}e^{-sx}\right]_0^\infty = \frac{1}{s}$$

This identity

$\therefore = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$

to compute

$$\mathcal{L}[x] = \mathcal{L}[1 \times x] = -\frac{d}{ds}\mathcal{L}[1] = \frac{1}{s^2}$$

and by induction

$\mathcal{L}[x^2] = \frac{2}{s^3} \ldots$

$$\mathcal{L}[x^n] = \frac{n!}{s^{n+1}}.$$

This is proved by repeating above

$\therefore$ using linearity and any polynomial can be found

LT Properties: Functions of parameters

We often have functions depending on parameters or constants, such as $\sin(a\,x)$. A useful trick is to note that

$$\mathcal{L}\left[\frac{\partial f}{\partial a}\right] = \frac{\partial \tilde{f}}{\partial a} \qquad [\leftarrow \mathcal{L}\left[f(x)\right] = \tilde{f}(s)]$$

Proof: As the integrals in the definition do not depend on the parameter a at all, the partial derivative commutes with the integrals:

$$\mathcal{L}\left[\frac{\partial f}{\partial a}\right] = \int_0^\infty \frac{\partial f}{\partial a} e^{-s\,x}\,\mathrm{d}x = \frac{\partial}{\partial a}\int_0^\infty f(x) e^{-s\,x}\,\mathrm{d}x = \frac{\partial}{\partial a}\tilde{f}.$$

a is independent

LT Properties: An odd corollary

For integral equations, or strange inversion problems, the following is useful:

$$\mathcal{L}\left[\int_0^x f(z)\,\mathrm{d}z\right] = \frac{1}{s}\mathcal{L}\left[f(x)\right].$$

Proof: This requires an unusual integration by parts step. Start from

$$\mathcal{L}\left[f(x)\right] = \int_0^\infty \overbrace{e^{-s\,x}}^{u}\,\overbrace{f(x)\mathrm{d}x}^{dv} \qquad \longleftarrow \quad dv = f(x)\mathrm{d}x \;\Rightarrow\; v = \int_0^x f(z)\,\mathrm{d}z$$

Now perform the unusual integration by parts: ($\swarrow$ $\int_A^B u\,dv = [uv]_A^B - \int_A^B v\,du$)

$$= \left[\overbrace{e^{-s\,x}}^{u}\overbrace{\int_0^x f(z)\,\mathrm{d}z}^{v}\right]_0^\infty - \int_0^\infty \left[\overbrace{\left(-s\,e^{-s\,x}\right)}^{du}\overbrace{\int_0^x f(z)\,\mathrm{d}z}^{v}\right]\mathrm{d}x$$

$$= 0 + s\,\mathcal{L}\left[\int_0^x f(z)\,\mathrm{d}z\right].$$

At the lower bound ($x = 0$) the integral vanishes, and at the upper bound we assume that the integral of f is bounded, and so the decaying exponential kills this term too.

LT Properties: Examples and methods

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With LT properties we bagged we can solve ODEs indirectly. For example:

$$\textbf{BVP:}\quad y''+2y'+5y=2+5x; \qquad y(0)=0, \quad y'(0)=3.$$

since are known

Taking Laplace transform of the ODE & (considerable) rearrangement:

$$\mathcal{L}[y'']+2\mathcal{L}[y']+5\mathcal{L}[y]=2\mathcal{L}[1]+5\mathcal{L}[x] \qquad \begin{cases}\mathcal{L}[y''(x)]=s^2\tilde{y}(s)-s\,y(0)-y'(0)\\ \mathcal{L}[y'(x)]=s\,\tilde{y}(s)-y(0)\end{cases}$$

intermittant step

$$\Leftrightarrow\ \tilde{y}\left(s^2+2s+5\right)=3+\frac{2}{s}+\frac{5}{s^2} \quad\Rightarrow\quad \tilde{y}(s)=\frac{1}{s^2}+\frac{2}{(s+1)^2+4}.$$

sub and simplification

$\tilde{y}=\frac{3s^2+2s+5}{s^2(s^2+2s+5)}=\frac{As+B}{s^2}+\frac{Cs+D}{s^2+2s+5}$

rearrange + partial fractions to simplify using $\tilde{y}=\frac{As+B}{s^2}+\frac{Cs+D}{s^2+2s+5}$

To invert $\tilde{y}(s)$ and get $y(x)$, we use known results & the first shift theorem:

identity used + first shift theorem

$$\tilde{y}(s)=\frac{1}{s^2}+\frac{2}{(s+1)^2+2^2}=\mathcal{L}[x]+\mathcal{L}\left[e^{-x}\sin(2x)\right] \qquad \begin{cases}\mathcal{L}[x]=\frac{1}{s^2}\\ \mathcal{L}[\sin(ax)]=\frac{a}{s^2+a^2}\\ \mathcal{L}\left[e^{-\beta x}f(x)\right]=\tilde{f}(s+\beta)\end{cases}$$

identity used

reverse linearity

$$\Rightarrow\quad y(x)=x+e^{-x}\sin(2x).$$

We used: $\mathcal{L}[\sin(ax)]=\frac{a}{s^2+a^2}=\tilde{f}(s)$ & $\frac{a}{(s+\beta)^2+a^2}=\overbrace{\tilde{f}(s+\beta)=\mathcal{L}\left[e^{-\beta x}\sin(ax)\right]}^{\text{1st shift theorem }[\,a=2,\ \beta=1\,]}$

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There is a systematic approach to solve ODEs using LT in such a way that we can invert the LT of the ODE $\tilde{y}(s)$ to get the solution $y(t)$ of the original BVP.

- Step 1: *If possible*, rearrange $\tilde{y}(s)$ in the partial fractions form:

$$\tilde{y}(s) = \frac{\text{Numerator}_{\text{(a Rational Polynomial)}}}{(s+a_1)(s+a_2)^2\cdots(s+a_n)^n\,[(s+c)^2+b^2]\,[(s+\alpha)^2+\beta^2]}$$
$$= \frac{C_1}{s+a_1}+\frac{C_2}{(s+a_2)^2}+\cdots+\frac{C_n\,n!}{(s+a_n)^n}+\frac{K_1\,b}{(s+c)^2+b^2}+\frac{K_2\,s}{(s+\alpha)^2+\beta^2}$$

- Step 2: use known results & 1st shift theorem to invert $\tilde{y}(s)$ & get $y(x)$:

$$\tilde{y}(s) \sim \underbrace{\frac{1}{s+a}}_{\to\mathcal{L}[e^{-ax}]} + \underbrace{\frac{1}{(s+a)^2}}_{\to\mathcal{L}[x\,e^{-ax}]} + \underbrace{\frac{n!}{(s+a)^n}}_{\to\mathcal{L}[x^{n-1}e^{-ax}]} + \underbrace{\frac{b}{(s+c)^2+b^2}}_{\to\mathcal{L}[e^{-cx}\sin(bx)]} + \underbrace{\frac{s}{(s+\alpha)^2+\beta^2}}_{\to\mathcal{L}[e^{-\alpha x}\cos(\beta x)]}.$$

We can always write the denominator of a rational polynomial as a product of (powers of) linear and (irreducible) quadratic terms by finding its (real) roots. For example,

$$s^2+2\alpha s+C^2 = s^2+2\alpha s+\alpha^2+(C^2-\alpha^2) = (s+\alpha)^2+\underbrace{(C^2-\alpha^2)} \equiv (s+\alpha)^2+\beta^2$$

LT Properties: Partial fractions examples

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$$\textbf{BVP:} \quad y'' + 3y' + 2y = x + e^{-x}; \qquad y(0) = 0, \quad y'(0) = 0.$$

Taking the Laplace Transform of the ODE gives

$$\mathcal{L}[y'' + 3y' + 2y] = \mathcal{L}\left[x + e^{-x}\right] \qquad \swarrow \text{ Linearity: } \mathcal{L}[\alpha f(x) + \beta g(x)] = \alpha \mathcal{L}[f(x)] + \beta \mathcal{L}[g(x)]$$

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[x] + \mathcal{L}\left[e^{-x}\right] \qquad \swarrow \begin{cases} \mathcal{L}[y''(x)] = s^2 \tilde{y}(s) - s\,y(0) - y'(0) \\ \mathcal{L}[y'(x)] = s\,\tilde{y}(s) - y(0) \end{cases}$$

$$\Leftrightarrow \quad \tilde{y}(s^2 + 3s + 2) = \frac{1}{s^2} + \frac{1}{s+1} \quad \Leftrightarrow \quad \tilde{y}(s+2)(s+1) = \frac{1}{s^2} + \frac{1}{s+1}.$$

$$\nearrow \text{ Quadratic equation: } \quad a\,s^2 + b\,s + c = 0 \Rightarrow s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The required partial fraction is (*Exercise*: check at home using next slide!)

$$\tilde{y}(s) = \frac{1 + s + s^2}{s^2(s+1)^2(s+2)} = -\frac{3}{4}\underbrace{\frac{1}{s}}_{\mathcal{L}[1]} + \frac{3}{4}\underbrace{\frac{1}{s+2}}_{\mathcal{L}[e^{-2x}]} + \frac{1}{2}\underbrace{\frac{1}{s^2}}_{\mathcal{L}[x]} + \underbrace{\frac{1}{(s+1)^2}}_{\mathcal{L}[x\,e^{-x}]}.$$

Inverting this LT $\tilde{y}(s)$ we get the solution $y(x)$ of the **BVP**:

$$y(x) = -\frac{3}{4} + \frac{3}{4}e^{-2x} + \frac{1}{2}x + x\,e^{-x}.$$

LT Properties: be best friends with Partial Fractions !

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To actually do the partial fractions we write out the most general possibility:

$$\frac{1+s+s^2}{s^2(s+1)^2(s+2)} = \frac{a\,s+b}{s^2} + \frac{c\,s+d}{(s+1)^2} + \frac{f}{(s+2)}.$$

Multiply both sides by the denominator:

$$1+s+s^2 = (a\,s+b)(s+1)^2(s+2) + (c\,s+d)s^2(s+2) + f\,s^2(s+1)^2.$$

You then have two routes. One is to expand both sides and match terms in powers of s. The other is to note that this must hold for all s and pick particular values of s to find constraints. For large problems the second approach is faster.
There are three obvious choices: the roots $s = -2, 0, 1$:

$$\begin{aligned} s=-2: &\quad 3 = 4f &&\Rightarrow \quad f = 3/4, \\ s=0: &\quad 1 = 2b &&\Rightarrow \quad b = 1/2, \\ s=-1: &\quad 1 = d-c &&\Rightarrow \quad d = c+1. \end{aligned}$$

At this point we have no more “easy” values to choose, so try say $s = 1$ and $s = 2$ to get

$$\begin{aligned} 3 &= 12(a+b) + 3(c+d) + 4f &&= 12a + 6c + 12 \\ 7 &= 36(2a+b) + 16(2c+d) + 36f &&= 72a + 48c + 61, \end{aligned}$$

which imply that $a = -3/4$ and $c = 0$, giving $d = 1$.

LT Properties: Products

A final property - a **negative** result - is that

$$\mathcal{L}\left[f(x) \times g(x)\right] \neq \mathcal{L}\left[f(x)\right] \times \mathcal{L}\left[g(x)\right].$$

This should be clear from earlier results; e.g.

$$\mathcal{L}\left[x\right] = \frac{1}{s^2}, \qquad \mathcal{L}\left[x^2\right] = \frac{2}{s^3} \neq \mathcal{L}\left[x\right] \times \mathcal{L}\left[x\right].$$

Outline

Summary

- We derived many key properties of Laplace Transforms:

$$\mathcal{L}\left[\mu_1 f_1(x) + \mu_2 f_2(x)\right] = \mu_1 \tilde{f}_1(s) + \mu_2 \tilde{f}_2(s).$$

$$\mathcal{L}\left[x\, f(x)\right] = -\frac{\mathrm{d}\tilde{f}}{\mathrm{d}s}$$

$$\mathcal{L}\left[\frac{\partial f}{\partial a}\right] = \frac{\partial \tilde{f}}{\partial a}$$

$$\mathcal{L}\left[\int_0^x f(u)\,\mathrm{d}u\right] = \frac{1}{s}\mathcal{L}\left[f(x)\right].$$

- The use of partial fractions is often essential in inverting Laplace Transforms $\tilde{f}(s) = \mathcal{L}\left[f(x)\right]$ to get $f(x)$.
- Practice is really useful
  ... actually it is fundamental no matter how good you are!