Outline

Lecture 23 - Gradient and Directional Derivative

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Outline

1. Introductory comments

2. Vector calculus
   - Derivatives of a scalar field (directional derivative & gradient)

3. Summary

Vector Differentiation

We want to be able to take **derivatives of vector fields.** To do this we must consider the **properties of vector fields** with respect to **differential operators**.

Consider, for the moment, a real function of one real variable, $f(x)$, with $a \leqslant x \leqslant b$. The **study** of the properties of this function can be carried out **in two ways:**

- the **integral** of a function $f(x)$ over an interval $a \leqslant x \leqslant b$ requires the knowledge of the function **over the entire interval** and gives **global** properties of the function, for example its average value.
- the **derivative** of a function requires only a **local** knowledge of the function and gives only **local** information: knowing the first derivative of a function at a point x_0 allows us to approximate the function in a **small neighbourhood of** x_0 but does **not** give us any information on the values of the function **away from that point** (no global information).

The **global (integration)** and the **local (differentiation)** approaches are **not unrelated.** The **fundamental theorem of calculus** states that the **derivative of a differentiable functions is (very roughly) the "inverse" of the integral.**

The **same two approaches** can be **used to study vector fields. So far** we have followed the **global approach** and have defined the **line integral of a vector field.**

Today, we start the local approach. In order to study the properties of vector fields we will **define differentiation operations** that either:

- produces a **vector field** by **acting on a scalar function** $f(x, y, z)$ (this operation is called the **gradient**), or
- **acts directly on vector fields.** These two operations are called the **curl** (it produces a **vector field**) and the **divergence** (it generates a **scalar field**) of the vector field.

We will also define an **additional differentiation operator**, called the **Laplacian**, that **acts on scalar or vector fields.**

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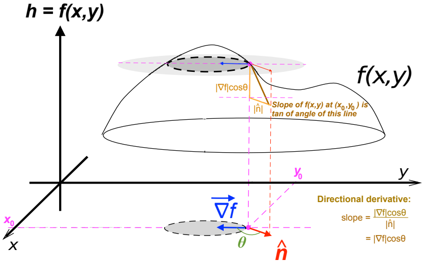
Given a function of two or more variables, $f(x, y)$ for example, we define the **directional derivative of** f at the point $\vec{x}_0 = (x_0, y_0)$ in the direction of the **unit** vector $\hat{\boldsymbol{n}}$ as (it's a scalar field):

$$\left.\frac{\partial f}{\partial \hat{\boldsymbol{n}}}\right|_{\vec{x}_0} = \lim_{t\to 0} \frac{f(\vec{x}_0 + t\hat{\boldsymbol{n}}) - f(\vec{x}_0)}{t}. \qquad \text{For example:} \begin{cases} \text{If } \hat{\boldsymbol{n}} = \hat{\boldsymbol{\imath}}: & \left.\frac{\partial f}{\partial \hat{\boldsymbol{\imath}}}\right|_{\vec{x}_0} = \left.\frac{\partial f}{\partial x}\right|_{\vec{x}_0} \\ \text{If } \hat{\boldsymbol{n}} = \hat{\boldsymbol{\jmath}}: & \left.\frac{\partial f}{\partial \hat{\boldsymbol{\jmath}}}\right|_{\vec{x}_0} = \left.\frac{\partial f}{\partial y}\right|_{\vec{x}_0} \end{cases}$$

Generalization of standard definition of partial derivative derivative along x or y for any direction $\hat{\boldsymbol{n}} = n_x\,\hat{\boldsymbol{\imath}} + n_y\,\hat{\boldsymbol{\jmath}}$

Geometrical interpretation of $\left.\frac{\partial f}{\partial \hat{\boldsymbol{n}}}\right|_{\vec{x}_0}$: it gives the value (*number*) of the **slope of the graph** of $f(x, y)$ **when** standing at point $\vec{x}_0$ and **we move in the direction of vector** $\hat{\boldsymbol{n}}$.



The **gradient** ∇f is a **vector** (*in the $x - y$ plane*) that **points in the direction** (x, y) of **maximum slope** of f. The **curve orthogonal to the gradient** is a **level line** of the graph: a **line along which** $f(x, y)$ **takes the same constant value.**

This interpretation suggests that, to describe the derivative of a function of two or more variables we need:
1) the slope of the graph and
2) the direction along which this slope is measured.

In other words, a complete description of the derivative of a function of two or more variables involves the use of a **vector.** The required vector is called the **gradient of the function**. It is defined as

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\,\hat{k}$$

for a function $f(x, y, z)$. The symbol $\boldsymbol{\nabla}$ is called **grad** or **nabla** and you can think of it as a **vector operator**

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\,\hat{k} \qquad \text{(notation: } \boldsymbol{\nabla} \equiv \vec{\boldsymbol{\nabla}}\text{)}$$

that **acts on** the function $f(x, y, z)$. We drop the arrow, $\boldsymbol{\nabla} \equiv \vec{\boldsymbol{\nabla}}$, because the symbol $\boldsymbol{\nabla}$ ***always*** represents the ***vector*** nabla operator.

Provided that the function f is differentiable the **directional derivative** of f in the direction of the unit vector $\hat{\boldsymbol{n}}$ is given by

$$\frac{\partial f}{\partial \hat{\boldsymbol{n}}} = \nabla f \cdot \hat{\boldsymbol{n}}, \qquad |\hat{\boldsymbol{n}}| = 1.$$

The **gradient of a function** thus **points in the direction** of **steepest ascent** (direction in $x - y$ plane of maximum slope). This follows immediately from

$$\frac{\partial f}{\partial \hat{\boldsymbol{n}}} = \nabla f \cdot \hat{\boldsymbol{n}} = |\nabla f|\,|\hat{\boldsymbol{n}}|\cos\theta = |\nabla f|\cos\theta,$$

where θ is the angle between the gradient and $\hat{\boldsymbol{n}}$ (see Fig. of slide 6). The **directional derivative**, i.e. the slope, is **maximal** when $\hat{\boldsymbol{n}}$ is **parallel** to ∇f (ie $\cos\theta = 1$). In general: $\quad -|\nabla f| \leq \frac{\partial f}{\partial \hat{\boldsymbol{n}}} \leq |\nabla f|$.

Note that the surface is level (recall dashed **level line** of Fig. of slide 6) in the **directions orthogonal to the gradient:** if $\hat{\boldsymbol{n}}$ is orthogonal to ∇f then

$$\hat{\boldsymbol{n}}\cdot\nabla f = 0 \implies \frac{\partial f}{\partial \hat{\boldsymbol{n}}} = 0 \qquad \longleftarrow \ \underline{\text{no}}\text{ slope} \ \Rightarrow \ f(x,y) = \text{const along level line}$$

Properties of $\nabla\phi \equiv \text{grad}\phi$

1. Linearity: $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$

Proof: Use Cartesian coordinates

$$\nabla(\phi+\psi) = \frac{\partial(\phi+\psi)}{\partial x}\,\hat{\imath} + \frac{\partial(\phi+\psi)}{\partial y}\,\hat{\jmath} + \frac{\partial(\phi+\psi)}{\partial z}\,\hat{k}$$

and the fact that $\frac{\partial}{\partial x}(\phi+\psi) = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial x}\,, \qquad \frac{\partial}{\partial y}(\phi+\psi) = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y}$

2. Leibnitz rule: $\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$

Proof: Use Cartesian coordinates

$$\nabla(\phi\psi) = \frac{\partial(\phi\psi)}{\partial x}\,\hat{\imath} + \frac{\partial(\phi\psi)}{\partial y}\,\hat{\jmath} + \frac{\partial(\phi\psi)}{\partial z}\,\hat{k}$$

and the fact that $\frac{\partial}{\partial x}(\phi\psi) = \psi\frac{\partial\phi}{\partial x} + \phi\frac{\partial\psi}{\partial x}\,, \qquad \frac{\partial}{\partial y}(\phi\psi) = \psi\frac{\partial\phi}{\partial y} + \phi\frac{\partial\psi}{\partial y}$

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Summary

- **Gradient** (geometric definition) The gradient of the scalar field ϕ is the vector field $\boldsymbol{\nabla}\phi$ given by

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial \hat{\boldsymbol{N}}}\,\hat{\boldsymbol{N}}$$

where $\hat{\boldsymbol{N}}$ is the **unit normal to the surfaces** $\phi = \mathit{const}$ and the scalar $\dfrac{\partial \phi}{\partial \hat{\boldsymbol{N}}}$ is the directional derivative of ϕ in the $\hat{\boldsymbol{N}}$ direction.

- **Gradient** (Cartesian definition) If the scalar field ϕ is given in Cartesian coordinates by $\phi(x, y, z)$ then

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial \phi}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial \phi}{\partial z}\,\hat{k}$$

- **Directional Derivative** The directional derivative

$$\begin{aligned}\frac{\partial \phi}{\partial \hat{\boldsymbol{n}}}(\vec{r}_0) &= \lim_{t \to 0}\left\{\frac{\phi(\vec{r}_0 + t\,\hat{\boldsymbol{n}}) - \phi(\vec{r}_0)}{t}\right\} = \hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\phi \qquad |\hat{\boldsymbol{n}}| = 1\\ &= n_1\frac{\partial \phi}{\partial x} + n_2\frac{\partial \phi}{\partial y} + n_3\frac{\partial \phi}{\partial z} \quad \text{(in Cartesian coordinates)}\end{aligned}$$