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School of Mathematics

Lecture 24 - Divergence and Curl

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Outline

1. Review
2. Vector calculus
   - Divergence
   - Curl
3. Summary

Review

- We studied the vector derivative of a scalar field: the **gradient.**
  - The gradient of the scalar field ϕ is the vector field $\boldsymbol{\nabla}\phi$ given by

    $$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial\hat{\boldsymbol{N}}}\,\hat{\boldsymbol{N}} \qquad \text{(recall: } \boldsymbol{\nabla} \equiv \vec{\nabla}\text{)}$$

    where $\hat{\boldsymbol{N}}$ is the **unit normal to the surfaces** $\phi =$ *const* and the scalar $\dfrac{\partial\phi}{\partial\hat{\boldsymbol{N}}}$ is the directional derivative of ϕ in the $\hat{\boldsymbol{N}}$ direction.
  - For a scalar field ϕ given in Cartesian coordinates by $\phi(x, y, z)$:

    $$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial\phi}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial\phi}{\partial z}\,\hat{k}$$

- The **directional derivative** along $\hat{\boldsymbol{n}}$ at point $\vec{r}_0$ is the scalar field:

$$\frac{\partial\phi}{\partial\hat{\boldsymbol{n}}}(\vec{r}_0) = \lim_{t\to 0}\left\{\frac{\phi(\vec{r}_0 + t\,\hat{\boldsymbol{n}}) - \phi(\vec{r}_0)}{t}\right\} = \hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\phi \qquad |\hat{\boldsymbol{n}}| = 1$$

$$= n_1\frac{\partial\phi}{\partial x} + n_2\frac{\partial\phi}{\partial y} + n_3\frac{\partial\phi}{\partial z} \quad \text{(in Cartesian coordinates)}$$

Outline

$\rightarrow$ Divergence

We have already seen how to use $\boldsymbol{\nabla}$ to compute the **gradient** ∇. The gradient differentiates a scalar function, like $f(x, y, z)$, and produces a vector field ∇f

The same approach can be used to **introduce new "derivatives"**, that **differentiate vector fields**.

One such derivative is the **divergence** $\boldsymbol{\nabla}\cdot\vec{F}$ of a vector field $\vec{F}$ which **gives a scalar derivative of a vector field**.

We **define** the **divergence** to satisfy the following **two conditions:**

1. If $\vec{c}$ is a **constant** vector, and ϕ is a **scalar** field then

$$\boldsymbol{\nabla}\cdot(\phi\,\vec{c}) = (\boldsymbol{\nabla}\phi)\cdot\vec{c} \quad \Leftrightarrow \quad \mathrm{div}\,(\phi\,\vec{c}) = (\mathrm{grad}\,\phi)\cdot\vec{c} \tag{1}$$

2. If $\vec{F}$ and $\vec{G}$ are vector fields then the divergence is *linear*. Hence:

$$\boldsymbol{\nabla}\cdot(\vec{F}+\vec{G}) = \boldsymbol{\nabla}\cdot\vec{F} + \boldsymbol{\nabla}\cdot\vec{G} \tag{2}$$

Divergence in Cartesian Coordinates: definition

We can use properties (1) and (2) to **define the divergence in Cartesian coordinates.**

Let $\quad \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\imath} + F_2(x,y,z)\,\hat{\jmath} + F_3(x,y,z)\,\hat{k}$

Then divergence of $\vec{F}$, $\operatorname{div}\vec{F} \equiv \nabla\cdot\vec{F}$, is

$\nabla\cdot\vec{F} = \nabla\cdot(F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k})$ $\quad\swarrow$ by condition (2)

$= \nabla\cdot(F_1\,\hat{\imath}) + \nabla\cdot(F_2\,\hat{\jmath}) + \nabla\cdot(F_3\,\hat{k})$ $\quad\swarrow$ by condition (1)

$= (\nabla F_1)\cdot\hat{\imath} + (\nabla F_2)\cdot\hat{\jmath} + (\nabla F_3)\cdot\hat{k}$ $\quad\swarrow$ $\nabla F_n \equiv \mathrm{grad}\,F_n$ is a vector

$$\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \text{since} \quad \begin{cases} (\nabla F_n)\cdot\hat{\imath} = \partial_x F_n \\ (\nabla F_n)\cdot\hat{\jmath} = \partial_y F_n \\ (\nabla F_n)\cdot\hat{k} = \partial_z F_n \end{cases} \qquad \left(\partial_x \equiv \frac{\partial}{\partial x}\right)$$

$\nwarrow$ Divergence of a vector field is thus a scalar field

Examples of calculating the Divergence

$$\text{div}\,\vec{F} \equiv \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Example: Let $\vec{F} = y^2 z\,\hat{\imath} + xz\,\hat{\jmath} - y^2\,\hat{k}$. Then

$$\begin{aligned}\nabla\cdot\vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{\partial(y^2 z)}{\partial x} + \frac{\partial(xz)}{\partial y} + \frac{\partial(-y^2)}{\partial z} \\ &= 0 + 0 + 0 = 0\end{aligned}$$

So, for $\vec{F} = y^2 z\,\hat{\imath} + xz\,\hat{\jmath} - y^2\,\hat{k}$ one has $\nabla\cdot\vec{F} = 0$.

This example shows that:

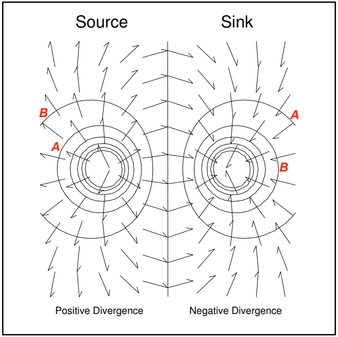
div $\vec{F} = 0$ does not imply that $\vec{F}$ is constant.

A **field with zero divergence** is called **solenoidal**.

Properties of the divergence: Geometric interpretation

The divergence of a field is associated with the:

$\begin{cases} \text{sources of vector field: expansion (divergence) of the vector flow} \\ \text{sinks of vector field: contraction (convergence) of the vector flow} \end{cases}$



A **source** is a region in space from which *field lines* tangent to $\vec{F}$ flow outward (e.g., the neighbourhood of a positive charge or a "source" of water). In such a region the **divergence is *positive*** (field lines diverge/expand).

A **sink** is a region of space where the *field lines* converge (e.g., the neighbourhood of a negative charge or a "hole" into which water disappears). In such a region the **divergence is *negative***. (field lines converge/contract)

If the divergence vanishes there is **no overall change** in the volume of the surface orthogonal to the field lines (see $A \to B$ in Fig.) as we move along the flow (or, in 2D, area of the curves orthogonal to the field lines).

$\rightarrow$ Curl

A further derivative is the **curl of a vector field $\vec{F}$**: $\quad \text{curl}\,\vec{F} \equiv \nabla\times\vec{F}\,,$
which gives a **vector derivative of a vector field.**

We **define** the **curl** to satisfy the following **two conditions:**

1. If $\vec{c}$ is a **constant** vector, and ϕ is a scalar field then

$$\nabla\times(\phi\,\vec{c}) = (\nabla\phi)\times\vec{c} \quad \Leftrightarrow \quad \text{curl}\,(\phi\,\vec{c}) = (\text{grad}\,\phi)\times\vec{c} \qquad (1)$$

2. If $\vec{F}$ and $\vec{G}$ are vector fields then the divergence is *linear*:

$$\nabla\times(\vec{F}+\vec{G}) = \nabla\times\vec{F} + \nabla\times\vec{G} \qquad (2)$$

Curl in Cartesian Coordinates: definition

We can use properties (1) and (2) to **define the curl in Cartesian coordinates.**

Let $\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\imath} + F_2(x,y,z)\,\hat{\jmath} + F_3(x,y,z)\,\hat{k}$

Then curl of $\vec{F}$, curl $\vec{F} \equiv \nabla \times \vec{F}$, is:

$$\begin{aligned}
\nabla\times\vec{F} &= \nabla\times(F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}) && \swarrow \text{ by condition (2)} \\
&= \nabla\times(F_1\,\hat{\imath}) + \nabla\times(F_2\,\hat{\jmath}) + \nabla\times(F_3\,\hat{k}) && \swarrow \text{ by condition (1)} \\
&= (\nabla F_1)\times\hat{\imath} + (\nabla F_2)\times\hat{\jmath} + (\nabla F_3)\times\hat{k} && \swarrow \text{ def. cross prod: Lecture 21} \\
&= \left(-\frac{\partial F_1}{\partial y}\,\hat{k} + \frac{\partial F_1}{\partial z}\,\hat{\jmath}\right) + \left(\frac{\partial F_2}{\partial x}\,\hat{k} - \frac{\partial F_2}{\partial z}\,\hat{\imath}\right) + \left(-\frac{\partial F_3}{\partial x}\,\hat{\jmath} + \frac{\partial F_3}{\partial y}\,\hat{\imath}\right)
\end{aligned}$$

$$\nabla\times\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$\nwarrow$ Curl of a vector field is thus a vector field

Examples of calculating the Curl

Let $\vec{F}$ be a vector field with components

$$\vec{F} = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}$$

Then

$$\operatorname{curl}\vec{F} \equiv \nabla\times\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

This may be **formally written using a determinant** as:

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{1}$$

<u>*Example*</u>: Let $\vec{F} = x\,y^2\,\hat{\imath} + e^z\,\hat{\jmath} + y\,e^z \sin x\,\hat{k}$. Using (1) yields

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & e^z & y\sin x\, e^z \end{vmatrix} = e^z(\sin x - 1)\,\hat{\imath} - (y\cos x\, e^z)\,\hat{\jmath} - 2xy\,\hat{k}$$

Properties of the curl & Geometric interpretation

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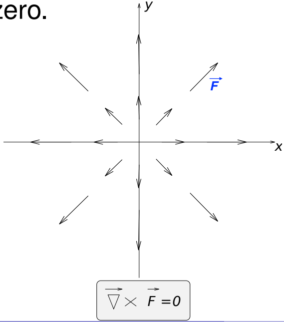
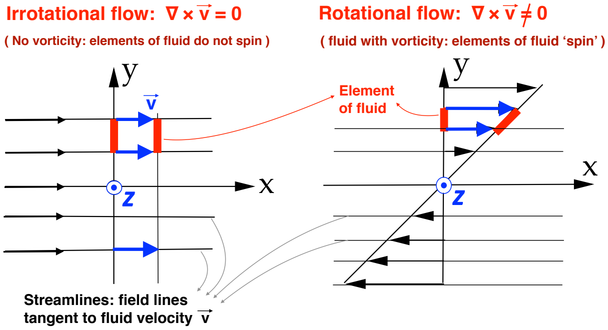
- $\nabla \times (g\vec{F}) = g(\nabla \times \vec{F}) + \nabla g \times \vec{F}$, where g is a scalar function and $\vec{F}$ a vector field. Note that (1) in slide 9 is a special case of this identity when $\vec{F} \equiv \vec{c} =$ constant vector $\Rightarrow \nabla \times \vec{F} = 0$
- $\nabla \times (\nabla f) = 0 \quad \Leftrightarrow \quad$ curl of grad of a scalar **vanishes**
- The curl of a **centrally symmetric** field $\vec{F}$ is **zero**:

$$\vec{F} = f(r)\vec{r} \quad \Longrightarrow \quad \nabla \times \vec{F} = 0.$$

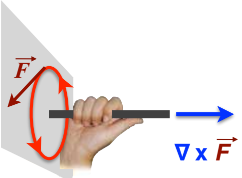
This result is in line with the **interpretation of the curl as a measure of the rotation of a vector field**. Since a centrally symmetric field does not “rotate” its curl must be zero.



- If $\vec{F} \equiv \vec{v}$ is **fluid velocity** then $\vec{\omega} = \nabla \times \vec{F}$ is **vorticity**



- **Direction of** $\nabla \times \vec{F}$ given by **right hand rule (version 2):**



Outline

Summary

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- Divergence

$$\text{div}\,\vec{F} \equiv \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- Curl

$$\text{curl}\,\vec{F} \equiv \nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- Differential operators: ($\swarrow$ recall: $\nabla \equiv \vec{\nabla}$)

grad $\equiv \nabla$ $\longleftarrow$ acts on a <u>scalar</u> field & produces a <u>vector</u> field

div $\equiv \nabla\cdot$ $\longleftarrow$ acts on a <u>vector</u> field & produces a <u>scalar</u> field

curl $\equiv \nabla\times$ $\longleftarrow$ acts on a <u>vector</u> field & produces a <u>vector</u> field