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SESA2025 Mechanics of Flight
Phugoid approximations

Lecture 3.7

Phugoid (Long Period) Mode

The pitch angle (constant pitch rate) and airspeed varies but the angle of attack remains constant for moderate oscillations

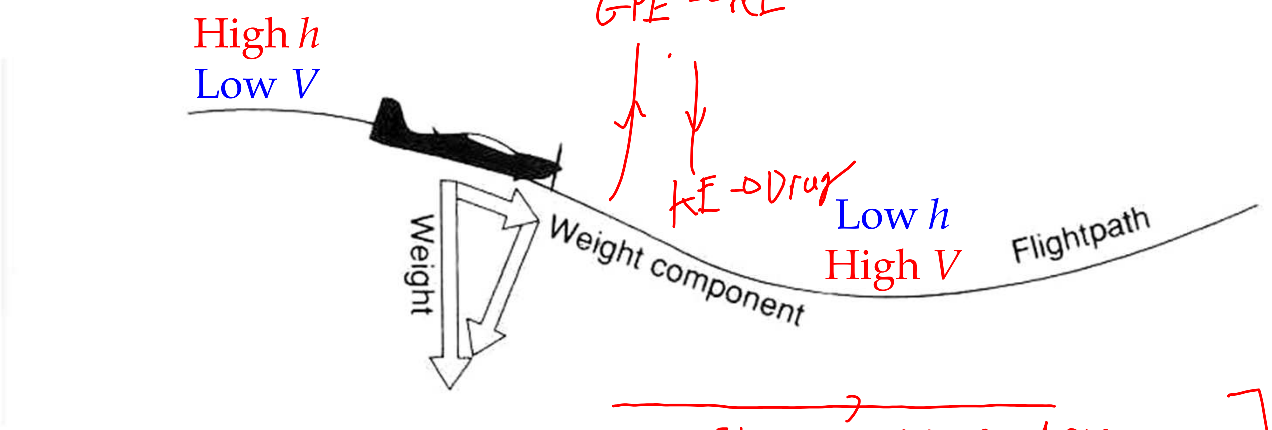
$E_{total} = E_p + E_k = mgh + \frac{1}{2}mV^2 = const.$ (if no dissipation into air)

Exchange between potential energy & kinetic energy

ignoring drag it just continues oscillating as there isn't damping

GPE → KE

KE → Drag



as it progresses, drag dissipates the KE acting as the driver of dampening

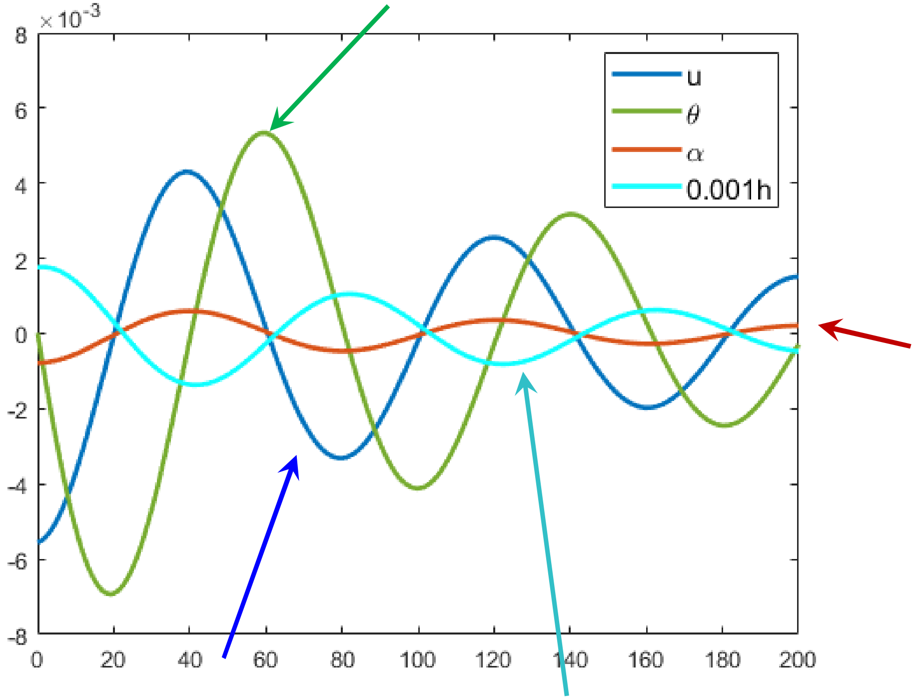
Figure from Barnard and Philpott (2010) *Aircraft Flight* 4th edition Prentice Hall

McDonnell Douglas F-4C Phantom II

Phugoid eigenvector; Mode shape 2nd Mode

```
>> M_inv = np.linalg.inv(M)
>> A = np.dot(M_inv, A_prime)
>> evals,evect = np.linalg.eig(A)
>> evect[:,2] =
   0.9903 + 0.0000i
   0.1384 - 0.0067i
   -0.0006 + 0.0000i
   0.0000 - 0.0079i ]
>> eval[2]
ans =
   −0.0065 + 0.0779i
```

$$\begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$



Altitude out of phase with velocity

McDonnell Douglas F-4C Phantom II

Example

Eigenvalues

$$\lambda_1 = -0.3690 \pm i1.3558$$

$$\lambda_2 = -0.0065 \pm i0.0779$$

First Mode is SPO with

$$\omega_n = \sqrt{0.3960^2 + 1.3558^2} = 1.4052 \text{ rad} \cdot \text{s}^{-1}$$

$$\zeta = \frac{0.3690}{1.4052} = 0.2626$$

Second mode is Phugoid with

$$\omega_n = \sqrt{0.0065 + 0.0779^2} = 0.0781 \text{ rad} \cdot \text{s}^{-1}$$

$$\zeta = \frac{0.0065}{0.0781} = 0.0825$$

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(slightly unstable phugoid, but easily corrected by a pilot)

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Lanchester's phugoid approximation

Suppose total energy is constant = *kinetic* plus *potential*

$$\frac{1}{2}mU_\infty^2 = \frac{1}{2}mU^2 + mgh = const.$$

basic application of conservation of energy

$U_\infty^2 = U^2 + 2gh$

$U_\infty^2 - 2gh = U^2$

Assume angle of attack α and hence C_L are fixed during the motion and $C_L = \frac{mg}{\frac{1}{2}\rho U_\infty^2 S}$

$C_L = f(\alpha)$

$$L = \left(\frac{1}{2}\rho U^2 S\right) C_L = \frac{1}{2}\rho S C_L (U^2)$$

$$= \frac{1}{2}\rho S C_L (U_\infty^2 - 2gh)$$

$$L = mg - \rho S C_L gh$$

here everything is constant except L and h, $\therefore L = f(h)$

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Lanchester's phugoid approximation

we are going to assume lift = weight, ignoring pitching effects.

$$L = mg - \rho S C_L g h$$

$L - mg = -\rho S C_L g h$

Newton's law in vertical direction (altitude, h)

$ma = F$

$$m\ddot{h} = L - mg$$

$$= -\rho S C_L g h$$

Use C_L definition

$$\ddot{h} + \frac{2g^2}{U_\infty^2} h = 0$$

→ eigenvalue problem

$\lambda = \frac{g\sqrt{2}}{V_\infty} \pm 0i$

Which results in

$$\omega_n = \frac{g\sqrt{2}}{U_\infty} \; ; \; \zeta = 0 \qquad \text{(no damping)}$$

so ignoring drag, the phugoid motions frequency is only a function of V_∞

Approximation result for F-4C

$$\omega_n = 0.0779\ \mathrm{rad \cdot s^{-1}}$$

A better phugoid approximation

Not Examinable

Pitch equilibrium:

$\dot{q} = 0$

Zero initial climb angle

$Z_q = Z_{\dot{w}} = M_q = M_{\dot{w}} = 0$

$$\begin{bmatrix} m & 0 & \cancel{0} & 0 \\ 0 & m - \cancel{\mathring{Z}_{\dot{w}}} & \cancel{0} & 0 \\ 0 & -\cancel{\mathring{M}_{\dot{w}}} & \cancel{I_{yy}} & 0 \\ 0 & 0 & \cancel{0} & 1 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \mathring{X}_u & \mathring{X}_w & 0 & -mg\cancelto{1}{\cos\gamma_0} \\ \mathring{Z}_u & \mathring{Z}_w & \cancel{\mathring{Z}_q} + mU_\infty & \cancel{-mg\sin\gamma_0} \\ \mathring{M}_u & \mathring{M}_w & \cancel{\mathring{M}_q} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

A better phugoid approximation

Not Examinable

$$
\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \dot{u} \\ \dot{w} \\ 0 \\ \dot{\theta} \end{bmatrix}
=
\begin{bmatrix} \mathring{X}_u & \mathring{X}_w & 0 & -mg \\ \mathring{Z}_u & \mathring{Z}_w & mU_\infty & 0 \\ \mathring{M}_u & \mathring{M}_w & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}
$$

The last equation is just $\dot{\theta} = q$, so we can remove it and we also don't need q on the right-hand-side (ie replace with $\dot{\theta}$)

$$
\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \dot{u} \\ \dot{w} \\ 0 \end{bmatrix}
=
\begin{bmatrix} \mathring{X}_u & \mathring{X}_w & 0 & -mg \\ \mathring{Z}_u & \mathring{Z}_w & mU_\infty & 0 \\ \mathring{M}_u & \mathring{M}_w & 0 & 0 \end{bmatrix}
\begin{bmatrix} u \\ w \\ \dot{\theta} \\ \theta \end{bmatrix}
$$

A better phugoid approximation

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Now rewrite back into state space form

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathring{X}_u & \mathring{X}_w & 0 & -mg \\ \mathring{Z}_u & \mathring{Z}_w & mU_\infty & 0 \\ \mathring{M}_u & \mathring{M}_w & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ \dot{\theta} \\ \theta \end{bmatrix}$$

to obtain

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & -mU_\infty \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \mathring{X}_u & \mathring{X}_w & -mg \\ \mathring{Z}_u & \mathring{Z}_w & 0 \\ \mathring{M}_u & \mathring{M}_w & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix}$$

A better phugoid approximation

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Then using

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & -mU_\infty \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \mathring{X}_u & \mathring{X}_w & -mg \\ \mathring{Z}_u & \mathring{Z}_w & 0 \\ \mathring{M}_u & \mathring{M}_w & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix}$$

and let $w = w_0 e^{\lambda t}$ etc.

to obtain:

$$\begin{bmatrix} m\lambda - \mathring{X}_u & -\mathring{X}_w & mg \\ -\mathring{Z}_u & m\lambda - \mathring{Z}_w & -mU_\infty \lambda \\ -\mathring{M}_u & -\mathring{M}_w & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ w_0 \\ \theta_0 \end{bmatrix} = 0$$

Finally, nontrivial solutions exist when the matrix determinant is zero.

A better phugoid approximation

Not Examinable

Determine characteristic equation, expand out and compare with standard form.

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

how pitch changes with vel change (machtuch)

↳ only large if $Ma \to 0.6$ so $\mathring{M}_u = 0$ for us

to obtain:

$$\omega_n^2 = \frac{g}{mU_\infty}\left(\frac{\cancel{\mathring{M}_u}}{\mathring{M}_w}\mathring{Z}_w - \mathring{Z}_u\right)$$

how drag increases as function of velocity, will be negative

Phugoid approximation including damping

and:

$$\zeta = \frac{1}{2\omega_n}\frac{1}{m}\left(\frac{\cancel{\mathring{M}_u}}{\mathring{M}_w}\left(\cancel{\mathring{X}_w} - \frac{mg}{U_\infty}\right) - \mathring{X}_u\right)$$

so if $\mathring{M}_u$ is near zero it'll basically always lead to a stable phugoid

but if $Ma \to 0.6$ then $\mathring{M}_u \neq 0$ ∴ phugoid may be unstable for fast aircraft.

A better phugoid approximation

Undamped natural frequency:

$$\omega_n^2 = \frac{g}{mU_\infty}\left(\frac{\mathring{M}_u}{\mathring{M}_w}\mathring{Z}_w - \mathring{Z}_u\right)$$

Damping ratio:

$$\zeta = \frac{1}{2\omega_n}\frac{1}{m}\left(\frac{\mathring{M}_u}{\mathring{M}_w}\left(\mathring{X}_w - \frac{mg}{U_\infty}\right) - \mathring{X}_u\right)$$

Aircraft details	
Flight Speed (m/s)	178
Aircraft mass (kg)	17642
g (m/s^2)	9.81

$$\mathring{X}_u = -126.86 \text{ Ns/m}$$

$$\mathring{X}_w = 80.62 \text{ Ns/m}$$

$$\mathring{Z}_u = -1214.01 \text{ Ns/m}$$

$$\mathring{Z}_w = -5215.44 \text{ Ns/m}$$

$$\mathring{M}_u = 277.47 \text{ Ns}$$

$$\mathring{M}_w = -1770.07 \text{ Ns}$$

Approximation results for F-4C

$$\omega_n = 0.0797 \text{ rad} \cdot \text{s}^{-1}$$

$$\zeta = 0.0949$$

McDonnell Douglas F-4C Phantom II

Phugoid approximation results

Lanchester's approximation result

$$\omega_n = 0.0779 \text{ rad} \cdot \text{s}^{-1}$$

$$(\zeta = 0)$$

Approximation results:

$$\omega_n = 0.0797 \text{ rad} \cdot \text{s}^{-1}$$

$$\zeta = 0.0949$$

approximation is not that bad

Full matrix results results:

$$\omega_n = 0.0781 \text{ rad} \cdot \text{s}^{-1}$$

$$\zeta = 0.0852$$

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Extra questions: What is the half-life? and what is the period?