SESA3029 Aerothermodynamics
Lecture 3.4

Crocco's theorem for supersonic flow

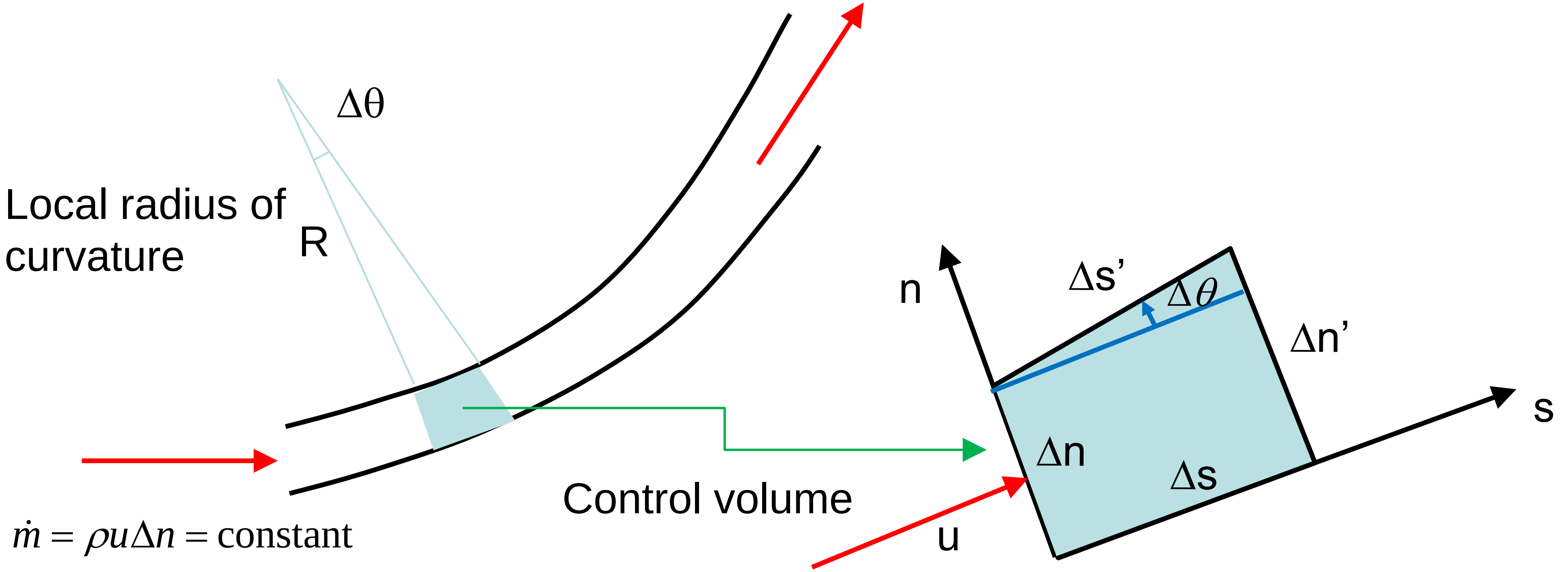
Coursework Q&A

Outline

- Complete set of equations for curved streamtube
  - connecting stagnation pressure entropy and vorticity
- Coursework Q&A

Recap: Curved streamtube

No flow across streamlines, therefore for steady flow
mass flowrate in = mass flowrate out



$$\dot{m} = \rho u \Delta n = \text{constant}$$

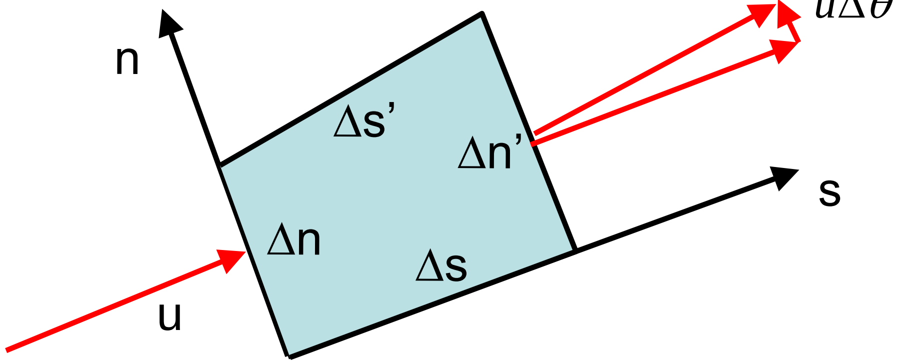
$$\Delta s = R\Delta\theta$$

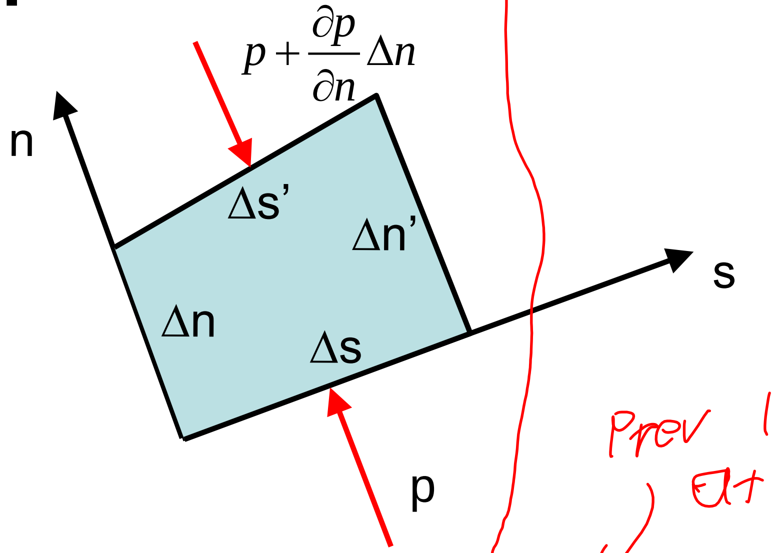
$$\frac{1}{R} = \frac{\partial\theta}{\partial s} \qquad (1)$$

$$\Delta n' - \Delta n = \Delta\theta\Delta s = \left(\frac{\partial\theta}{\partial n}\Delta n\right)\Delta s$$

$$\frac{1}{\Delta n}\frac{\partial \Delta n}{\partial s} = \frac{\partial\theta}{\partial n} \qquad (2)$$

Euler n-equation





Prev looked at s direction

n-momentum out – n-momentum in = force applied in n direction

$$\rho u \Delta n \quad \dot m\left[u\frac{\partial\theta}{\partial s}\cancel{\Delta s}\right]-\cancel{0}=-\left(\frac{\partial p}{\partial n}\cancel{\Delta n}\right)\cancel{\Delta s}$$

$$\dot m=\rho u\Delta n$$

$$\rho u^2\frac{\partial\theta}{\partial s}=-\frac{\partial p}{\partial n}$$

hence

$$\boxed{\rho u^2\frac{\partial\theta}{\partial s}=-\frac{\partial p}{\partial n} \quad \text{or} \quad \frac{\rho u^2}{R}=-\frac{\partial p}{\partial n}}$$

using (1) (as expected)

← direction change is direct func of normal pressure

Full set of governing equations

Euler-s equation
$$\rho u \frac{\partial u}{\partial s} = -\frac{\partial p}{\partial s} \qquad (1)$$

compressible flow relation

Euler-n equation
$$\rho u^2 \frac{\partial \theta}{\partial s} = -\frac{\partial p}{\partial n} \qquad (2)$$

Mass conservation
$$\frac{\partial \theta}{\partial n} + \frac{1}{u}\frac{\partial u}{\partial s} + \frac{1}{\rho}\frac{\partial \rho}{\partial s} = 0 \qquad (3)$$

derived recently

Energy conservation
$$h + \frac{u^2}{2} = h_0 \qquad (4)$$

obvious (stagnation enthalpy)

Entropy

Recall Gibbs' equation
(with capital S for entropy)

specific

$$TdS = dh - \frac{dp}{\rho}$$

Substitute for enthalpy from energy equation

$$TdS = dh_0 - udu - \frac{dp}{\rho}$$

Along our streamtube (using s-momentum)

$$T\frac{dS}{ds} = \underbrace{\frac{dh_0}{ds}}_{=0} - \underbrace{\left(u\frac{\partial u}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} \right)}_{=0} = 0$$

$\times = 0$

$\therefore \frac{dS}{ds} = 0$

i.e. entropy is constant along an (adiabatic) streamtube

Entropy normal to streamline

Along our streamtube
$$T\frac{\mathrm{d}S}{dn}=\frac{\mathrm{d}h_0}{\mathrm{d}n}-\left(u\frac{\partial u}{\partial n}+\underbrace{\frac{1}{\rho}\frac{\partial p}{\partial n}}\right)$$

from derived equation at start

Using Euler n-equation
$$=\frac{\mathrm{d}h_0}{\mathrm{d}n}-\underbrace{\left(u\frac{\partial u}{\partial n}-u^2\frac{\partial\theta}{\partial s}\right)}$$

$$\boxed{T\frac{\mathrm{d}S}{\mathrm{d}n}=\underbrace{\frac{\mathrm{d}h_0}{\mathrm{d}n}}+u\omega}$$

with the vorticity given by
$$\omega=u\frac{\partial\theta}{\partial s}-\frac{\partial u}{\partial n}$$

can be non zero

(eg exhaust)

Entropy, vorticity and stagnation pressure

vortizity can be created from a curved shockwave!

$$T\frac{\mathrm{d}S}{\mathrm{d}n} = \frac{\mathrm{d}h_0}{\mathrm{d}n} + u\omega$$

For constant h_0, the vorticity is directly related to the entropy gradient

Also (from lecture 1.1)

$$S_2 - S_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

For constant h_0, let states 1 and 2 be brought to rest isentropically

$$S_2 - S_1 = -R\ln\left(\frac{p_{0,2}}{p_{0,1}}\right)$$

i.e. entropy gradient is also linked to changes in *stagnation* pressure

Crocco's theorem

Special form
(derived here)

$$T\frac{\mathrm{d}S}{\mathrm{d}n} = \frac{\mathrm{d}h_0}{\mathrm{d}n} + u\omega$$

General form
(unsteady)

$$T\nabla S + \mathbf{u}\times\boldsymbol{\omega} = \nabla h_0 + \frac{\partial \mathbf{u}}{\partial t}$$

lecture doesn't prove

If h_0 is constant everywhere and the flow is irrotational and steady, then the entropy is constant everywhere.

This is called homentropic flow

In a homentropic flow, the isentropic flow relation holds everywhere (not just along streamlines), i.e.

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

You should now have an appreciation of:

- A theoretical framework which links the effects of entropy, vorticity and stagnation pressure in compressible flows
  - For example the link between entropy and stagnation pressure is the basis for discussion of ‘loss’ in turbomachinery
- A deeper understanding of what is meant by the homentropic flow assumption that we use to develop MoC