SESA2024 Astronautics

UNIVERSITY OF Southampton

Chapter 5: Mission Analysis

Lecture 5 – Orbital motion (part 2)

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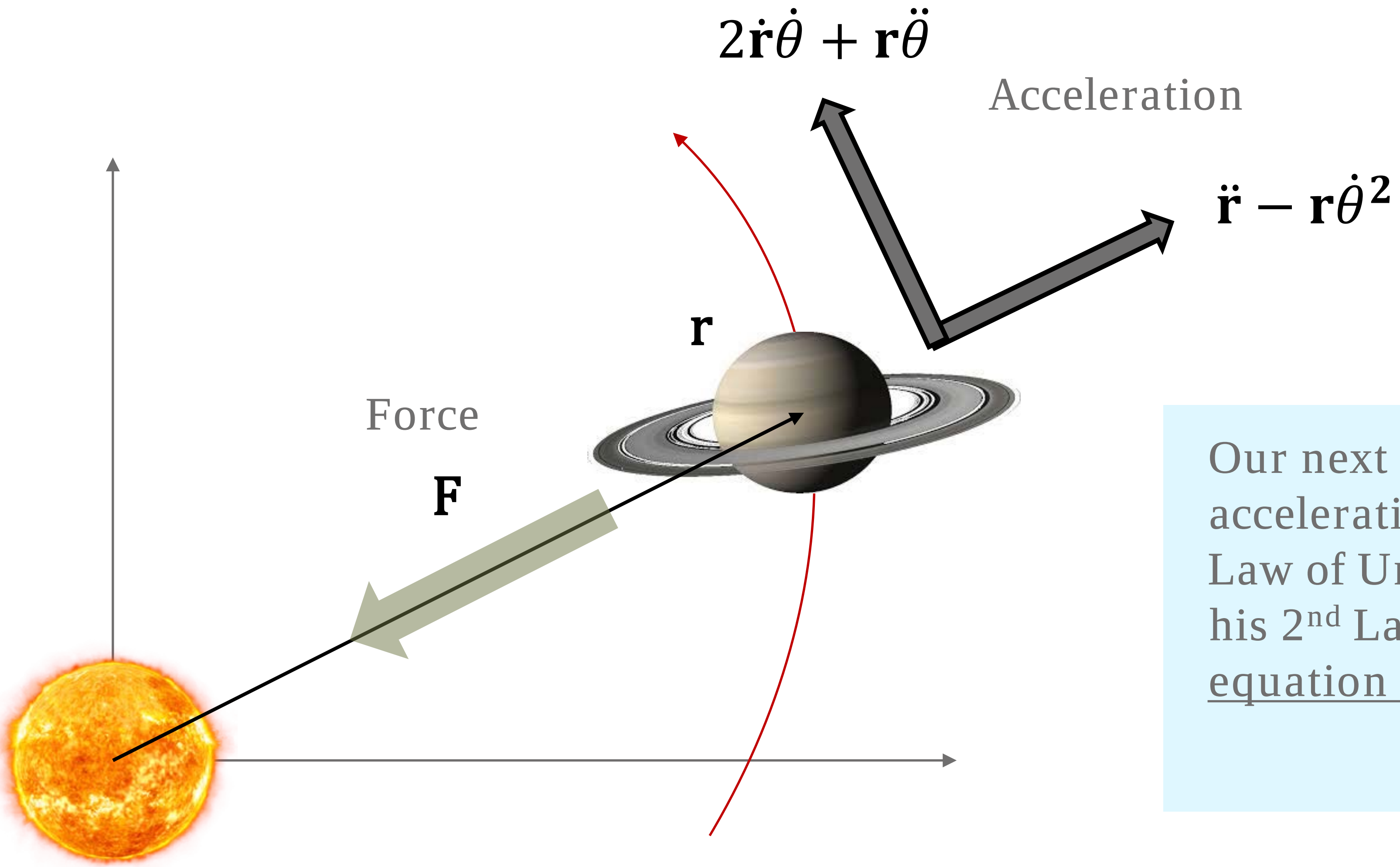
Overview of lecture 5

- This is the second part of a relatively long derivation of an <u>equation of motion</u> that describes mathematically how planets move around the Sun
  - In the previous lecture, we had derived the acceleration of a planet in terms of its position and distance from the Sun.
  - The acceleration was described using a radial component and a tangential component
  - In this lecture we will continue to use methods first developed by Isaac Newton (e.g. calculus, Law of Universal Gravitation)
  - Our ultimate aim is to show mathematically, using Newton's Laws, that orbital trajectories can be described using the ellipse equation and, therefore, that Kepler's 1st Law is correct
    - We will complete the derivation in this lecture

- <u>Understanding the approach at a conceptual level is important, but the full derivation will not be assessed</u>

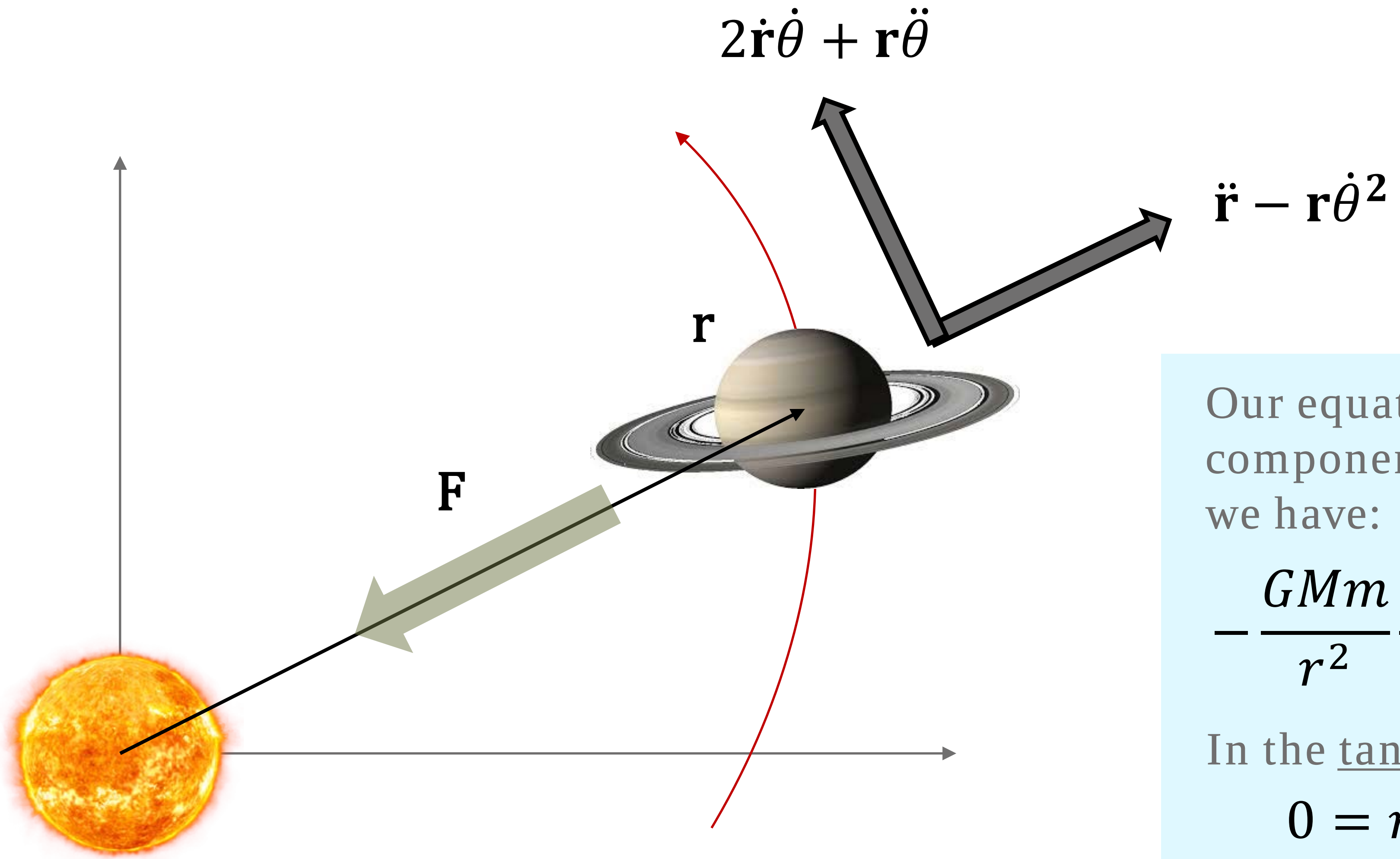
Orbital motion

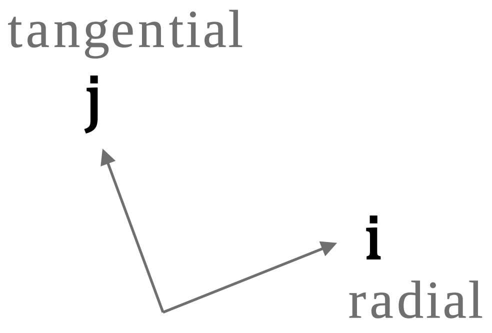


Our next task is to use our acceleration results with Newton's Law of Universal Gravitation and his 2nd Law of Motion to define an <u>equation of motion:</u>

$$\mathbf{F} = m\mathbf{a}$$

Orbital motion





Our equation of motion has two components. In the radial direction we have:

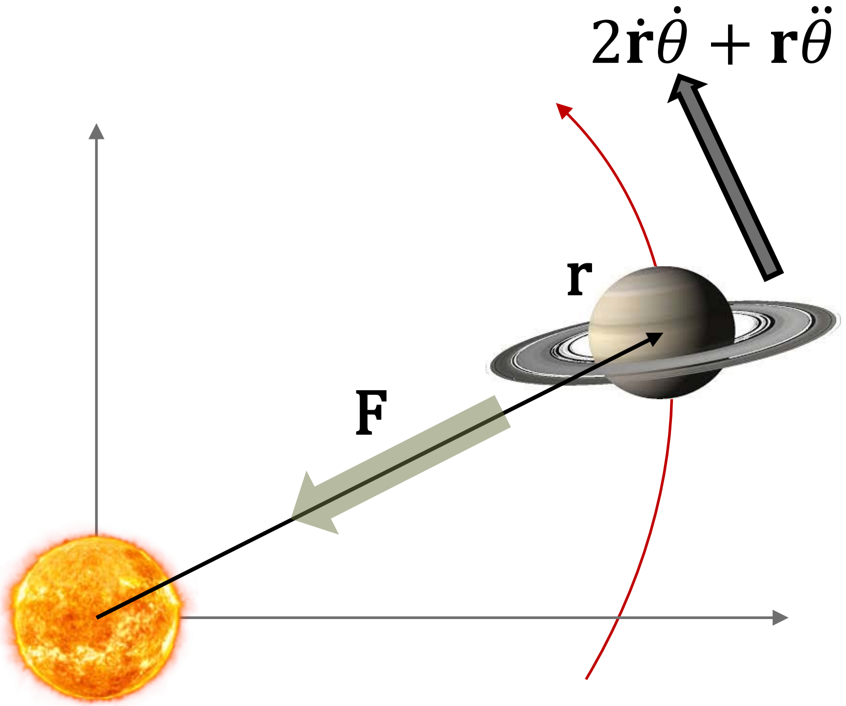
$$-\frac{GMm}{r^2}\frac{\mathbf{r}}{r} = m\left(\ddot{\mathbf{r}} - \mathbf{r}\dot{\theta}^2\right)$$

In the tangential direction we have:

$$0 = m\left(2\dot{\mathbf{r}}\dot{\theta} + \mathbf{r}\ddot{\theta}\right)$$

Orbital motion

- Tangential direction: orbital moment of momentum



- In the tangential direction we have:

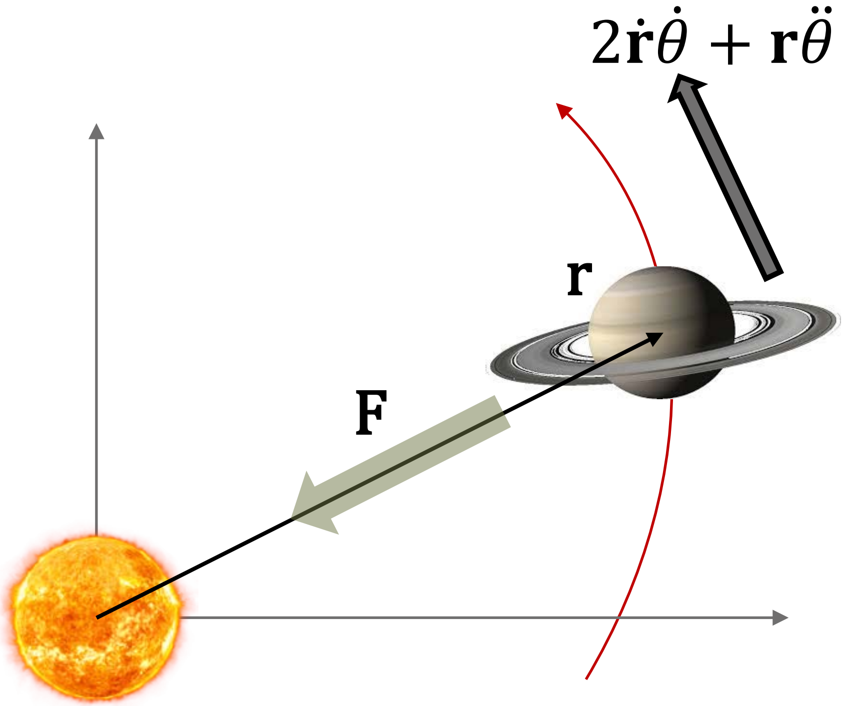
$$0 = m\left(2\dot{\mathbf{r}}\dot{\theta} + \mathbf{r}\ddot{\theta}\right) = 2\dot{\mathbf{r}}\dot{\theta} + \mathbf{r}\ddot{\theta}$$

- We can use a non-intuitive approach to solve this (an approach that makes sense only in hindsight)
- Using the product rule, we can write:

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = \dot{\theta}(2r\dot{r}) + r^2\left(\ddot{\theta}\right)$$
$$= r\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)$$

Orbital motion

- Tangential direction: orbital moment of momentum



- Hence

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = r\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)$$

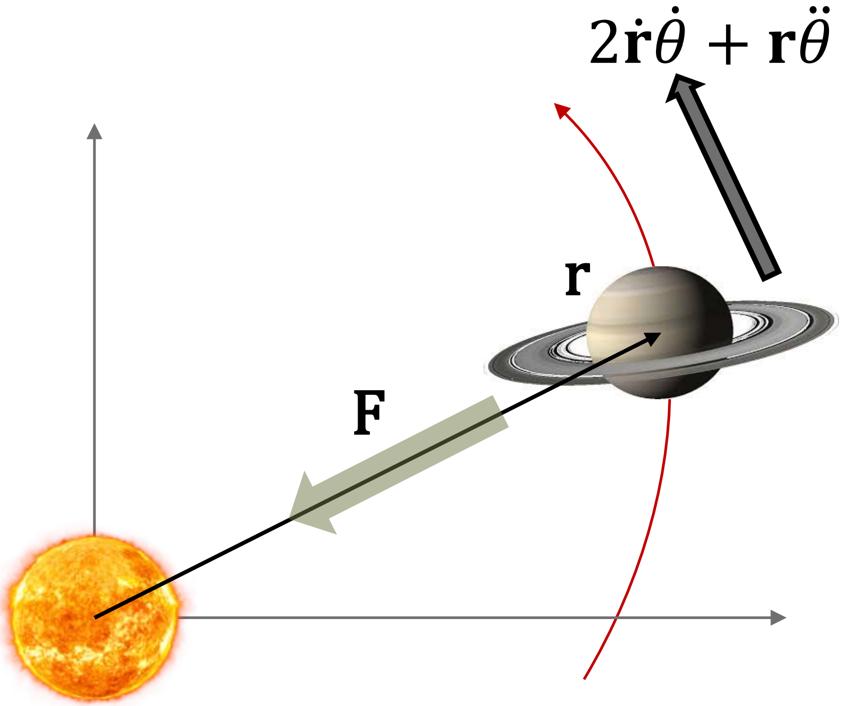
- So, we can write:

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

- The right-hand side of this expression is equal to the acceleration in the tangential direction…

Orbital motion

- Tangential direction: orbital moment of momentum



- The acceleration in the tangential direction is zero:

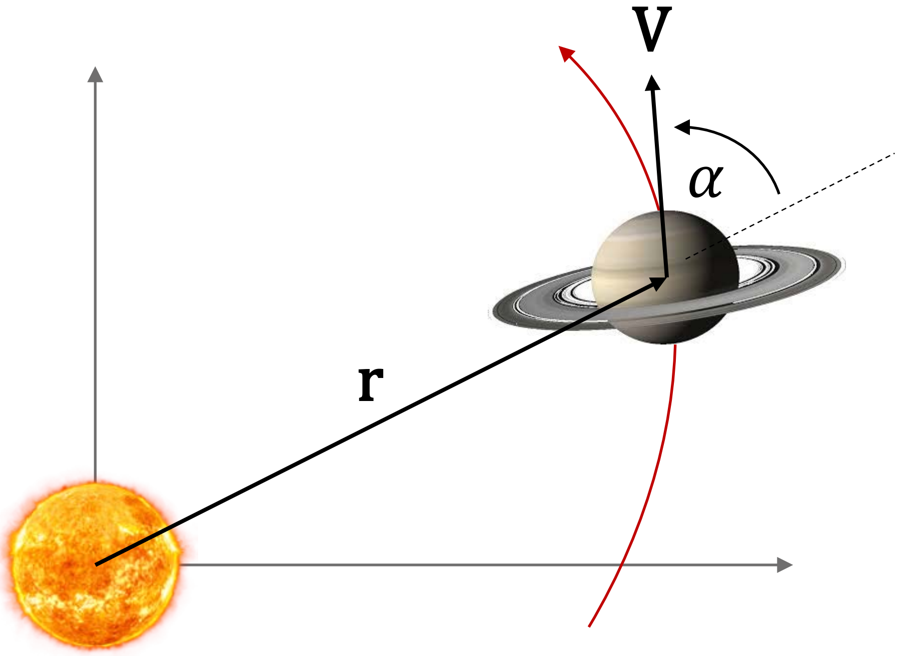
$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

- So, we can write $\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$

- In other words, $r^2\dot{\theta}$ is <u>constant</u>

Orbital motion

- Tangential direction: orbital moment of momentum



Here, the direction of the vector **h** would be out of the screen (using the right-hand rule)

- We also know that:

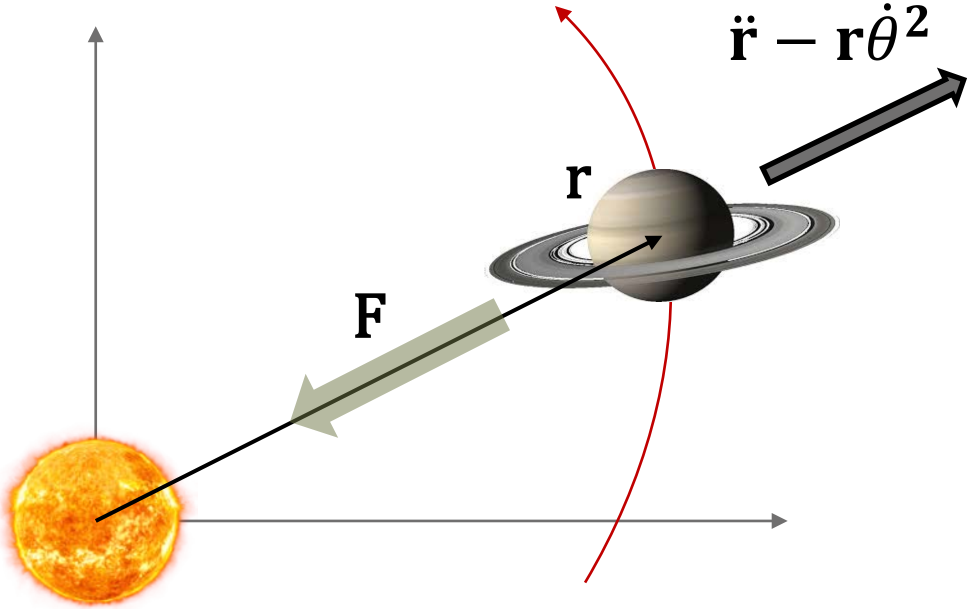
$$r^2\dot{\theta} = r(r\dot{\theta}) = rV\sin\alpha$$

$$= |\mathbf{r} \times \mathbf{V}|$$

- This is the orbital moment of momentum (per unit mass) and it must be constant.
- The orbital moment of momentum vector **h** is at 90° to both the radius vector **r** and the velocity vector **V**.

Orbital motion

- Radial direction: acceleration due to gravity



- In the radial direction we have:

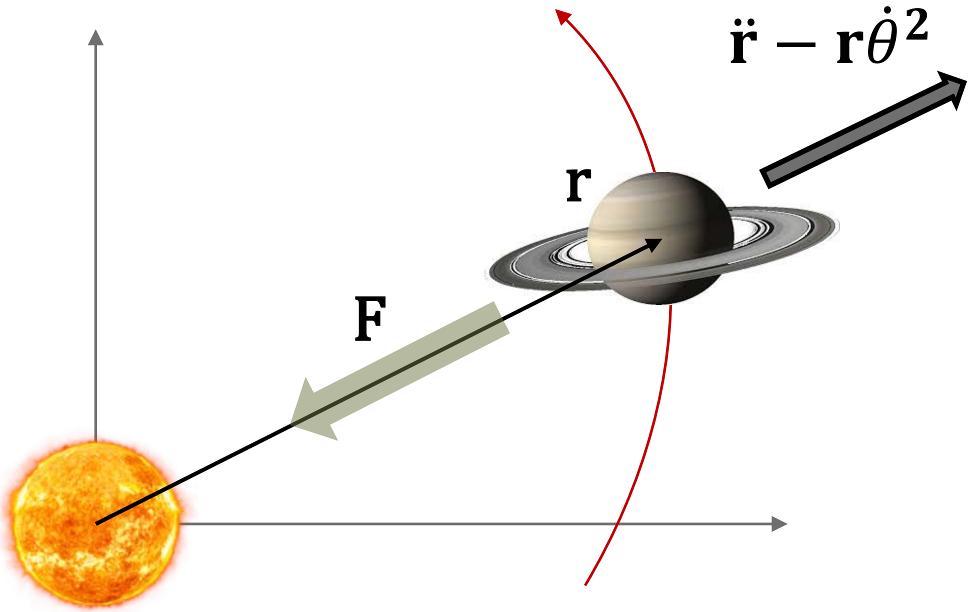
$$-\frac{GMm}{r^2}\frac{\mathbf{r}}{r} = m\left(\ddot{\mathbf{r}} - \mathbf{r}\dot{\theta}^2\right)$$

$$-\frac{GM}{r^2} = -\frac{\mu}{r^2} = \ddot{r} - r\dot{\theta}^2$$

- We need to use another non-intuitive approach to solve this (which again will make sense only in hindsight)

Orbital motion

- Radial direction: acceleration due to gravity



- Let $u = \frac{1}{r}$ (i.e. $r = \frac{1}{u}$)
- Using the chain rule, we have:

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{dt} = \frac{d}{du}\left(\frac{1}{u}\right)\frac{du}{dt}$$

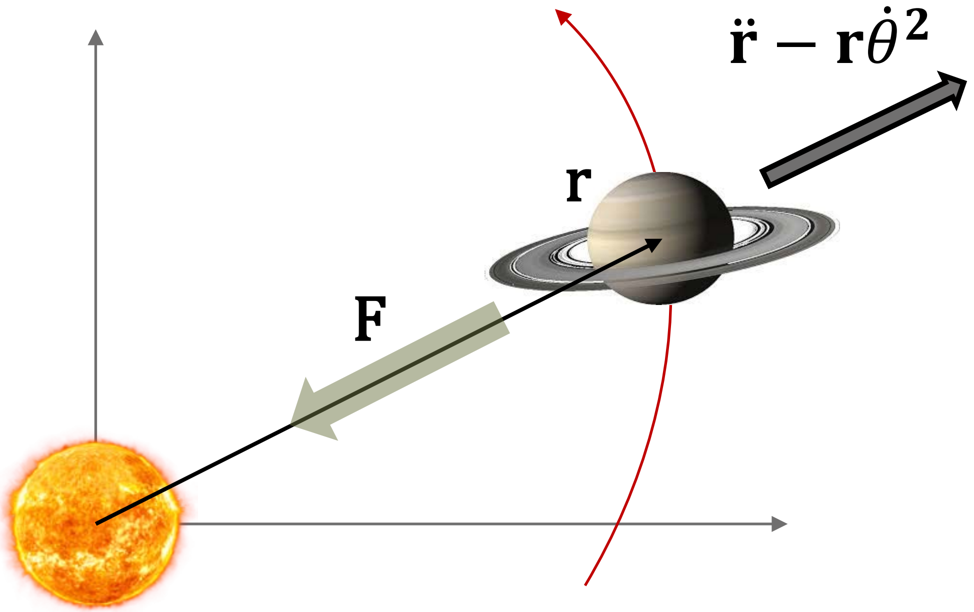
- Hence

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -r^2\dot{\theta}\frac{du}{d\theta} = -h\frac{du}{d\theta}$$

Orbital motion

- Radial direction: acceleration due to gravity



- Hence, we can write:

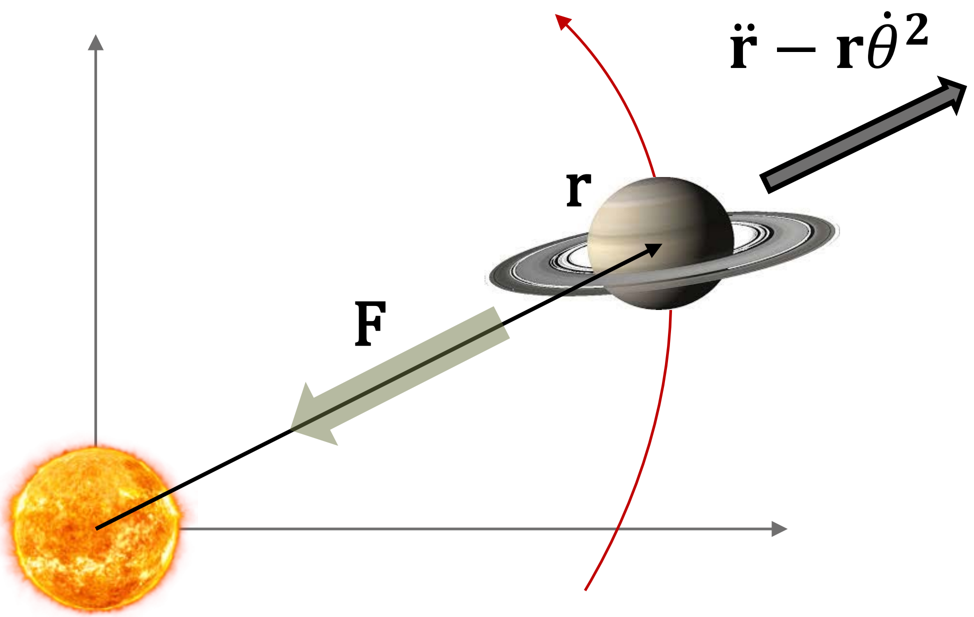
$$\ddot{r} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$$

- Let $x = -h\dfrac{du}{d\theta}$ & using the chain rule:

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

Orbital motion

- Radial direction: acceleration due to gravity



- Recall that h is constant, and we can use the product rule (which has the effect of moving it outside the brackets)

$$\ddot{r} = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}\dot{\theta}$$

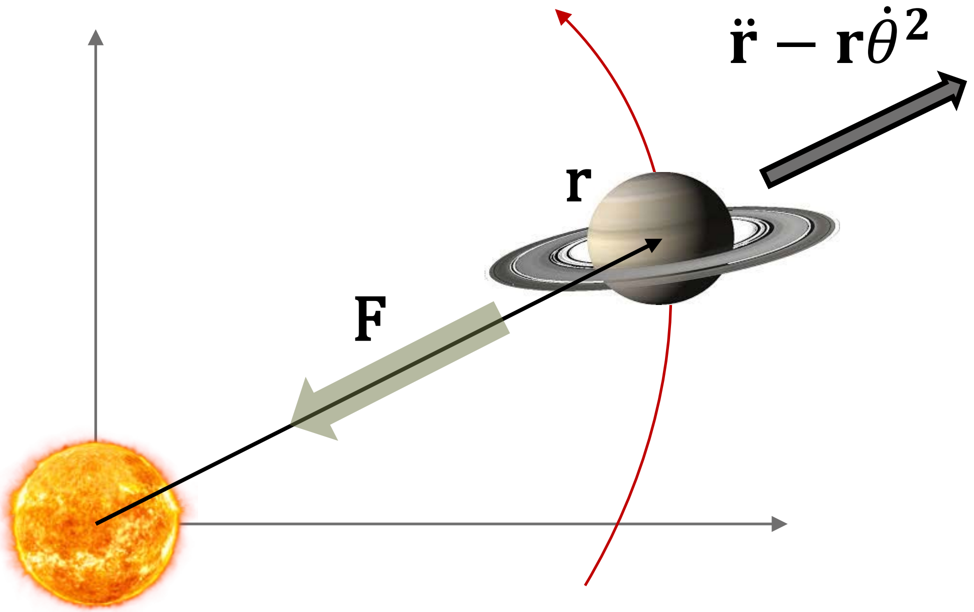
- We know $h = r^2\dot{\theta}$ (i.e. $\dot{\theta} = \frac{h}{r^2}$), so:

$$\ddot{r} = -\frac{h^2}{r^2}\frac{d^2u}{d\theta^2}$$

- We can now go back to the radial component of the acceleration…

Orbital motion

- Radial direction: acceleration due to gravity



- We can now go back to the radial component of the acceleration:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

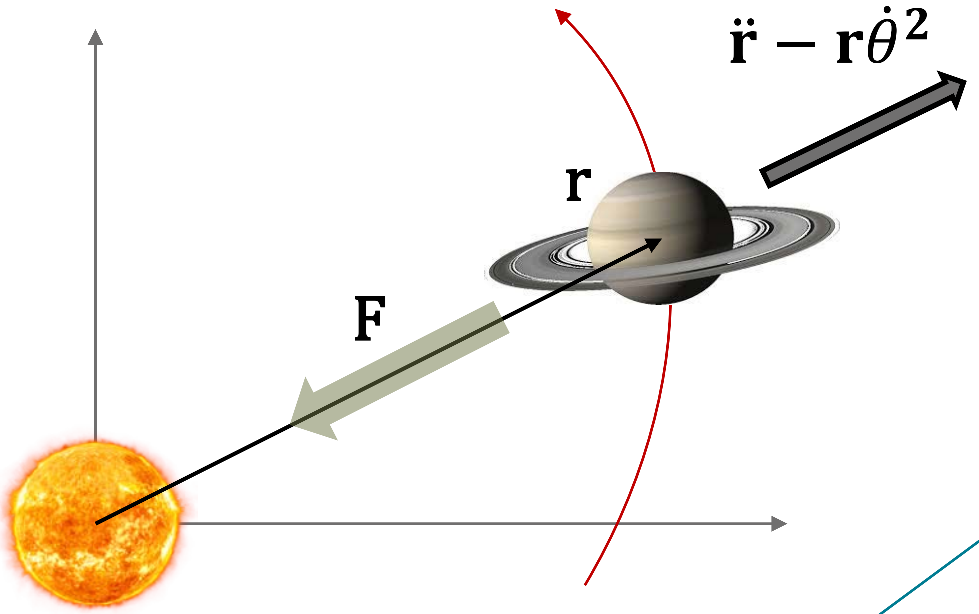
- Substituting $\ddot{r} = -\frac{h^2}{r^2}\frac{d^2u}{d\theta^2}$ gives:

$$-\frac{h^2}{r^2}\frac{d^2u}{d\theta^2} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

- And $\dot{\theta}^2 = \frac{h^2}{r^4}$ so we can write…

Orbital motion

- Radial direction: acceleration due to gravity



Type this into wolframalpha.com:
(d^2y)/(dx^2)+y=(a/h^2)

- We can write:

$$-\frac{h^2}{r^2}\frac{d^2u}{d\theta^2} - \frac{rh^2}{r^4} = -\frac{\mu}{r^2}$$

- Re-arranging and simplifying:

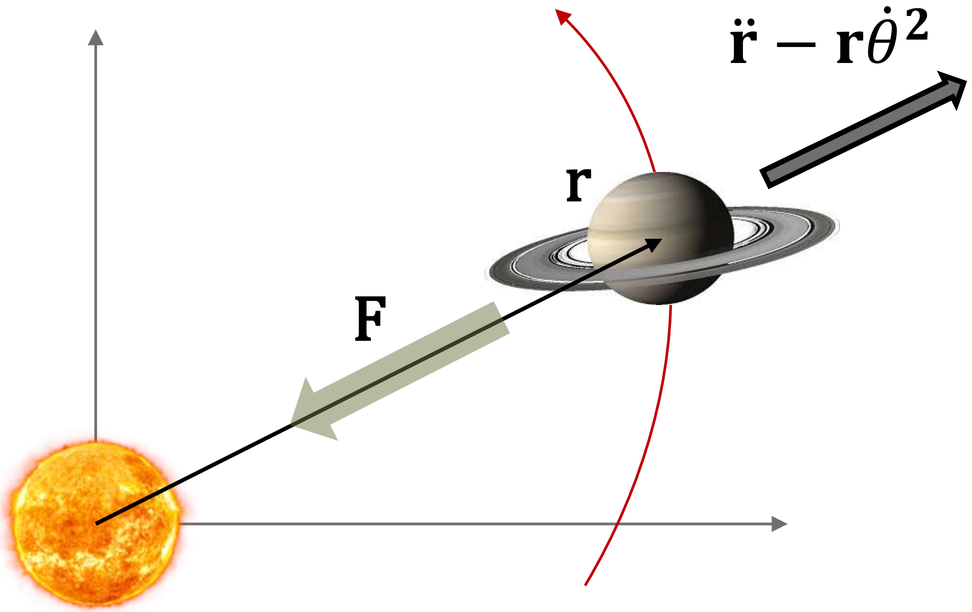
$$\frac{d^2u}{d\theta^2} + \frac{1}{r} = \frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

- The standard solution to a DE like this is:

$$u = A\cos\theta + B\sin\theta + \frac{\mu}{h^2}$$

Orbital motion

- Radial direction: acceleration due to gravity



- Standard solution to the differential eqn.:

$$u = A\cos\theta + B\sin\theta + \frac{\mu}{h^2}$$

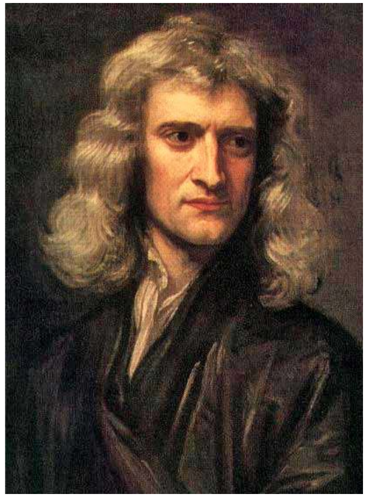
- Apply boundary conditions to determine the constants A and B (this working is not shown here)
- The solution then takes the form:

$$u = \frac{1}{r} = \frac{\mu}{h^2}(1 + e\cos\theta)$$

- Finally, we compare this to the ellipse equation in polar form (remember that?)

Orbital motion

- Are orbits ellipses?
  - Our objective was to follow the steps taken by Isaac Newton: use calculus to show that orbits are described by the ellipse equation (in polar form)



- Standard solution to the differential eqn.:

$$r = \frac{h^2}{\mu(1 + e\cos\theta)}$$

- Ellipse equation in polar form:

$$r = \frac{a(1 - e^2)}{(1 + e\cos\theta)}$$

$$a(1 - e^2) = \frac{h^2}{\mu}$$

We'll need this later…

Recap of lecture 5

- With our equation of motion, which used Newton's Law of Universal Gravitation, we were able to show that orbital motion can be described using:

$$r = \frac{h^2}{\mu(1 + e\cos\theta)}$$

- This has the same form as the ellipse equation in polar form: $r = \dfrac{a(1-e^2)}{(1 + e\cos\theta)}$

- We were also able to show that the orbital moment of momentum (angular momentum) is constant:

$$\frac{d}{dt}(h) = \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

- In the next lecture, we will return to Kepler's Laws, show that they are correct, and describe them using mathematics

Activity

- Visit wolframalpha:
  - https://www.wolframalpha.com/
- Confirm the standard solution to the following differential equation:

$$\frac{d^2u}{d\theta^2} - u = \frac{\mu}{h^2}$$