Lecture 17 - The Heat Equation

David Gammack and Oscar Dias

Mathematical Sciences,
University of Southampton, UK

MATH2048, Semester 1

Outline

→ Review

University of Southampton
School of Mathematics

- We studied the **wave equation** (hyperbolic PDE)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

- We **solved** it using **separation of variables.**
- The **solutions** of the wave equation are...**travelling waves**:

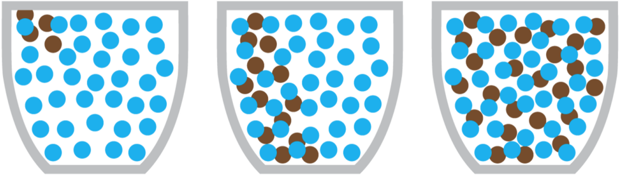
$$y(x,t) \sim \sum_{n=1}^{\infty} C_n \left[\underbrace{\cos\left[n\pi(x - c\,t)\right]}_{\text{Right-mover wave}} + \underbrace{\cos\left[n\pi(x + c\,t)\right]}_{\text{Left-mover wave}} \right].$$

Front of the wave is given by condition that phase vanishes: $(\boldsymbol{x \pm c\,t}) = \mathbf{0}$,

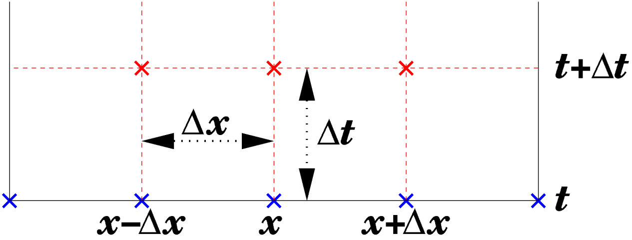
$$\begin{cases} \text{Right-mover wave: } x - c\,t = 0 \Rightarrow x = c\,t \longrightarrow t \nearrow \;\Rightarrow\; x \nearrow \;\Longrightarrow \text{moves to the } \underline{\text{right}} \\ \text{Left-mover wave: } x + c\,t = 0 \Rightarrow x = -c\,t \longrightarrow t \nearrow \;\Rightarrow\; x \searrow \;\Longrightarrow \text{moves to the } \underline{\text{left}} \end{cases}$$

Waves indeed travel with **velocity** c.

→ Today: Diffusion



Outline



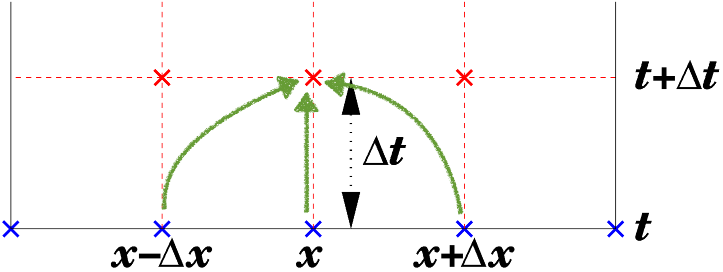
– **At time *t*** a certain number $\boldsymbol{y(x,t)}$ of *Pokémons* (ie funny funky bugs!) are **placed** on a grid **at location *x***.

Similarly, there are $\boldsymbol{y(x \pm \Delta x, t)}$ bugs at $\boldsymbol{x \pm \Delta x}$.

– Next, **two coins are flipped in time Δt:**

1. **Both heads** (1 case): *Pokémons* **from left** (at $x - \Delta x$) **move into *x***.
2. **Both tails** (1 case): *Pokémons* **from right** (at $x + \Delta x$) **move into *x***.
3. **Different** (2 cases): *Pokémons* **stay still**.

Random Walks



By probabilities, the number of *Pokémons* that should be at position $\boldsymbol{x}$ at **later time** $\boldsymbol{t + \Delta t}$ is thus:

$$y(x, t + \Delta t) = \frac{1}{4}\Big[y(x + \Delta x, t) + y(x - \Delta x, t) + 2y(x, t)\Big].$$

The continuum limit

- Consider small $\Delta t \ll 1$ and $\Delta x \ll 1$ and Taylor expand:

$$y(x, t+\Delta t) = y(x,t) + \frac{\partial y}{\partial t}\Delta t + \cdots$$

$$y(x \pm \Delta x, t) = y(x,t) \pm \frac{\partial y}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}(\Delta x)^2 + \cdots$$

- Inserting in the discrete equation we obtain

$$y(x, t+\Delta t) = \frac{1}{4}\Big[y(x+\Delta x, t) + y(x-\Delta x, t) + 2y(x,t)\Big]$$

$$y_{(x,t)} + \frac{\partial y}{\partial t}\Delta t + \cdots = \frac{1}{4}\Big[y_{(x,t)} + \frac{\partial y}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}(\Delta x)^2 + y_{(x,t)} - \frac{\partial y}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}(\Delta x)^2 + 2y_{(x,t)}\Big] + \cdots$$

$$\frac{\partial y}{\partial t} = \left(\frac{(\Delta x)^2}{4\Delta t}\right)\frac{\partial^2 y}{\partial x^2} + \cdots$$

- In the continuum limit, $\Delta x \to 0$, $\Delta t \to 0$ with $\lim\limits_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} \frac{(\Delta x)^2}{4\Delta t} \to \kappa^2$, we obtain

$$\frac{\partial y}{\partial t} = \kappa^2 \frac{\partial^2 y}{\partial x^2} \qquad \longleftarrow \qquad \text{This is the diffusion or heat equation}$$

Outline

Main take away is movement of quantity over time is done statistically. (eg heat diffusion on macro level)

UNIVERSITY OF Southampton
School of Mathematics

- We use **separation of variables** to solve

$$\frac{\partial y}{\partial t} = \kappa^2 \frac{\partial^2 y}{\partial x^2}, \qquad \kappa \text{ constant}$$

with **boundary conditions**

$$\frac{\partial y}{\partial x}(0,t) = 0, \qquad \frac{\partial y}{\partial x}(1,t) = 0$$

and **initial data**

$$y(x,0) = x(1-x).$$

- *Note*: only a **single** initial condition – for $y(x,0)$ – because a **parabolic PDE** only has a **first derivative in time** $\frac{\partial y}{\partial t}$ (but no $\frac{\partial^2 y}{\partial t^2}$)

Steps 1 & 2: Equations & boundary conditions

UNIVERSITY OF Southampton
School of Mathematics

- Heat or Diffusion equation: $\frac{\partial y}{\partial t} = \kappa^2 \frac{\partial^2 y}{\partial x^2}$, κ constant
- Again, assume the *separation ansatz* $y(x,t) = X(x)T(t)$ yielding

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial (X\,T)}{\partial t} = X\frac{dT}{dt} = X\dot{T} \\ \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 (X\,T)}{\partial x^2} = T\frac{d^2X}{dx^2} = TX'' \end{cases} \Rightarrow X\dot{T} = \kappa^2 TX'' \quad \Leftrightarrow \quad \frac{1}{\kappa^2}\frac{\dot{T}}{T} = \frac{X''}{X}.$$

- Thus, both sides must be separately constant which gives two ODEs:

$$\begin{cases} \frac{X''}{X} = \lambda\,, \\ \frac{1}{\kappa^2}\,\frac{\dot{T}}{T} = \lambda\,. \end{cases} \Leftrightarrow \quad \begin{cases} X'' - \lambda X = 0\,, \\ \dot{T} - \kappa^2\lambda\,T = 0\,. \end{cases}$$

- Boundary conditions

$$\frac{\partial y}{\partial x}(0,t) = 0 \quad \Leftrightarrow \quad X'(0)T(t) = 0 \qquad \Rightarrow X'(0) = 0,$$
$$\frac{\partial y}{\partial x}(1,t) = 0 \quad \Leftrightarrow \quad X'(1)T(t) = 0 \qquad \Rightarrow X'(1) = 0.$$

Steps 3: Eigenvalue problem

$$X'' - \lambda X = 0; \qquad X'(0) = 0, \quad X'(1) = 0.$$

- The solution for $\boldsymbol{\lambda = 0}$ is

$$X(x) = Ex + F \quad \Rightarrow \quad X' = E.$$

BCs $\Rightarrow \boldsymbol{E = 0}$. Thus, we have a **non-trivial solution**: $X(x) = F$.

- The solution for $\boldsymbol{\lambda = -k^2 < 0}$ is

$$X(x) = A\sin(k\,x) + B\cos(k\,x) \quad \Rightarrow X'(x) = k\Big[A\cos(k\,x) - B\sin(k\,x)\Big]$$

BC at $x = 0 \Rightarrow \boldsymbol{A = 0}$.
BC at $x = 1 \Rightarrow$ either **trivial solution** $\boldsymbol{A = B = 0}$ or $\boldsymbol{k = n\,\pi}$.

- The solution for $\boldsymbol{\lambda = k^2 > 0}$ is

$$X = A\,e^{k\,x} + B\,e^{-k\,x} \quad \Rightarrow \quad X' = k\left(A\,e^{k\,x} - B\,e^{-k\,x}\right)$$

BC at $x = 0 \;\Rightarrow\; \boldsymbol{A = B}$. BC at $x = 1 \;\Rightarrow\; \boldsymbol{A = 0}$.
Thus, we **only** have the **trivial solution** $\boldsymbol{A = B = 0}$ in this case.

Step 4: Solve for T

Constant coefficient ODE for $T(t)$ with known λ (revisit Lecture 1)

$$\dot{T} - \kappa^2 \lambda T = 0$$

We now have to **solve two cases**:

- For $\boldsymbol{\lambda = 0}$ we have

$$\dot{T} = 0 \quad \Rightarrow \quad T(t) = \tilde{H} \qquad \longleftarrow \qquad \tilde{H} \text{ is a constant}$$

- For $\boldsymbol{\lambda = -k^2 < 0}$ we have $k \equiv k_n = n\pi$ and:

$$\dot{T}_n + (n\pi\kappa)^2\, T_n = 0 \quad \longleftarrow \quad \text{first order ODE}$$

didn't use IF here

↳ can solve using integrating factors

with solution

$$\int \frac{\dot{T}_n}{T_n} = -(n\pi\kappa)^2 \Leftrightarrow \ln T_n = K_n - (n\pi\kappa)^2\, t \Leftrightarrow T_n = e^{K_n - (n\pi\kappa)^2 t} = e^{K_n} e^{-(n\pi\kappa)^2 t}$$

antiderivative formula used

$$T_n(t) = \tilde{C}_n\, e^{-(n\pi\kappa)^2 t}, \quad \text{with} \quad e^{K_n} \equiv \tilde{C}_n.$$

- NO need to solve $\boldsymbol{\lambda > 0}$ since $X(x) = 0$.

Step 5: The general solution

So we have:

$$\begin{cases} \text{For } \lambda = 0: \quad X(x) = F\,, & T(t) = \tilde{H} \\ \text{For } \lambda_n = -(n\pi)^2 < 0: \quad X_n(x) = B_n \cos(n\pi x)\,, & T_n(t) = \tilde{C}_n e^{-(n\pi\kappa)^2 t} \\ \text{For } \lambda > 0: \quad \text{Only trivial solution} & \end{cases}$$

Combining these, $y = XT$, and **superposing** gives the **general solution:**

$$y(x,t) = \tilde{H}\,F + \sum_{n=1}^{\infty} \tilde{C}_n e^{-(n\pi\kappa)^2 t} B_n \cos(n\pi x)$$

$$y(x,t) = H + \sum_{n=1}^{\infty} C_n e^{-(n\pi\kappa)^2 t} \cos(n\pi x)\,.$$

$(H \equiv \tilde{H}F,\ C_n \equiv \tilde{C}_n B_n \nearrow)$

$\longrightarrow$ $\boldsymbol{H, C_n =}$ **??** $\longrightarrow$ fixed by initial data

Step 6: Initial data. Parabolic PDE: only $y(x,0)$ is given!

- **Initial data:** $y(x,0) = x(1-x)$. **Evaluating** $y(x,t)$ at $\boldsymbol{t=0}$ gives

$$x(1-x) = H + \sum_{n=1}^{\infty} C_n \cos(n\pi x) \qquad \text{with } x \in [0,1] \quad \leftarrow \text{from BCs}$$

- We recognize this! It's the cosine Fourier series of $f(x) = x(1-x)$ (ie the FS of the even extension of $f(x) = x(1-x)$ w/ period 2ℓ, $\ell = 1 \Leftarrow x \in [0,1]$).
- Thus, $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\,x\,\pi}{\ell}\right)$ with $\ell = 1$ & we apply **Euler formulae**:

[↙ see slide 10 of Lecture 6]

$$H \equiv \frac{1}{2}a_0 = \frac{1}{2}\left(\frac{2}{\ell}\int_0^{\ell} f(x)dx\right) = \frac{1}{2}\left[2\int_0^1 x(1-x)dx\right] = \frac{1}{6},$$

[↓ Exercise: check it!] (like previous lecture)

$$C_n \equiv a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\,x\,\pi}{\ell}\right)dx = 2\int_0^1 x(1-x)\cos(n\pi x)\,dx = -\frac{2}{(n\pi)^2}\left[1+(-1)^n\right]$$

- The solution to our original problem (PDE+BCs+Initial Data) is thus

$$y(x,t) = \frac{1}{6} - 2\sum_{n=1}^{\infty} \frac{\left[1+(-1)^n\right]}{(n\pi)^2} e^{-(n\pi\kappa)^2 t} \cos(n\pi x) \leftarrow \begin{cases} \textbf{exponential decay in time}; \\ \textbf{characteristic of heat equation} \end{cases}$$

Outline

Summary

- **Diffusive behaviour** giving the change in the distribution of a quantity, such as heat flow, stock volatility or species or disease spread, is usually modelled by **parabolic equations**.
- The heat equation
$$\frac{\partial y}{\partial t} = \kappa^2 \frac{\partial^2 y}{\partial x^2}.$$
is the model parabolic equation.
- Separation of variables works for simple boundary conditions in the same way as for the wave equation:
  - The **spatial behaviour is identical**;
  - **Initial data:** single condition since PDE only has first derivative in t

    [the wave equation has second derivative in $t \Rightarrow$ initial data must give conditions for $y(x, 0)$ and $\partial_t y(x, 0)$]
  - The **time behaviour leads to exponential decay**, the **key qualitative difference** between heat and wave equations.