UNIVERSITY OF Southampton

Chapter 5: Mission Analysis

Lecture 6 – Confirming Kepler's Laws

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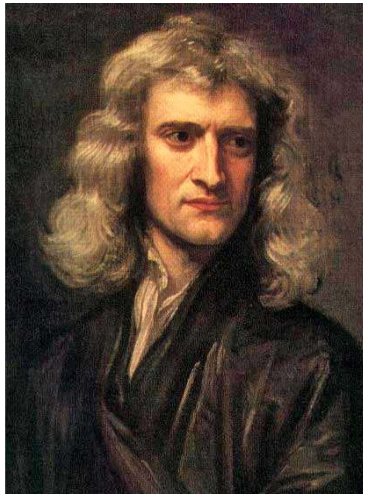
Overview of lecture 6

- This lecture returns to Kepler’s Laws of Planetary Motion:
  - We use the insights gained from the previous two lectures to show that these Laws are true
  - We derive an equation for the orbital period: the time taken for a planet or spacecraft to make one revolution
    - This can be used to determine the time needed to transfer from one orbit to another, which is an important quantity in mission analysis

Orbital motion

- Are orbits ellipses?
  - Our objective was to follow the steps taken by Isaac Newton: use calculus to show that orbits are described by the ellipse equation (in polar form)



- Standard solution to the differential eqn.:

$$r = \frac{h^2}{\mu(1 + e\cos\theta)}$$

- Ellipse equation in polar form:

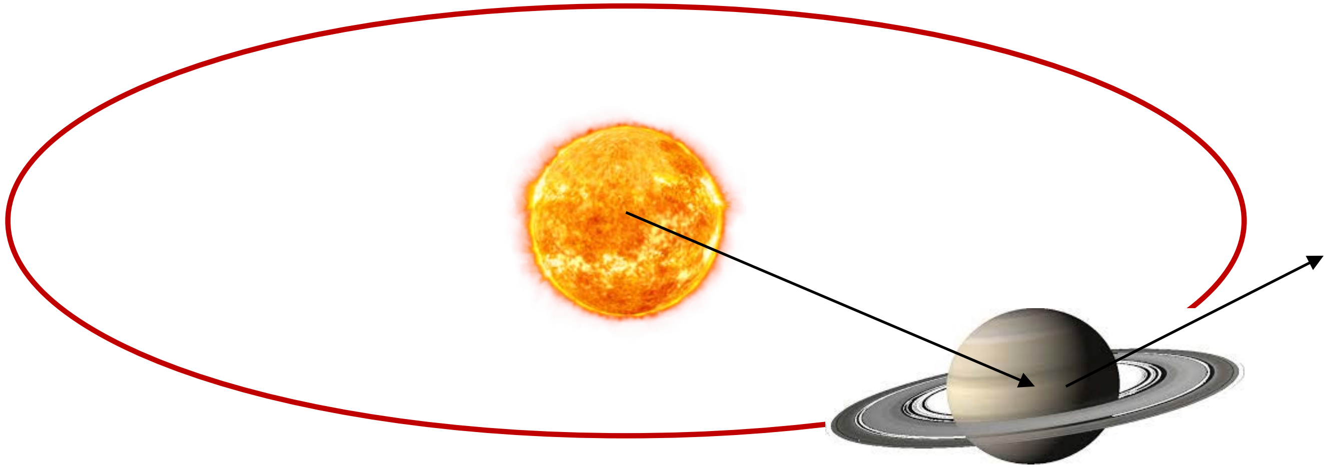
$$r = \frac{a(1-e^2)}{(1 + e\cos\theta)}$$

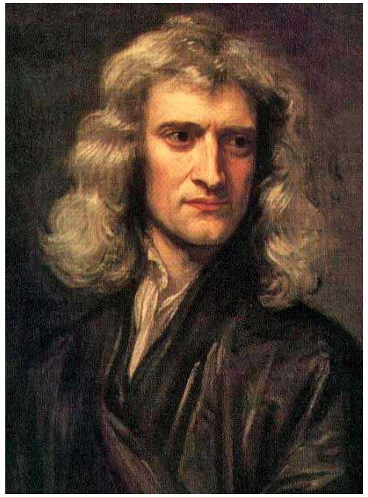
$$a(1-e^2) = \frac{h^2}{\mu}$$

We'll need this later…

Orbital motion

- Are orbits ellipses?
  - <u>YES</u>!

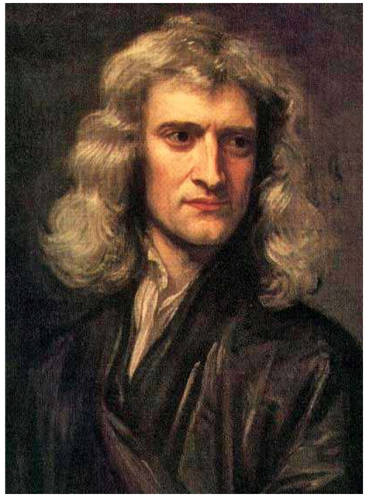




- Kepler's 1st Law:
  - The orbit of each planet is an ellipse with the Sun at one focus

Orbital motion

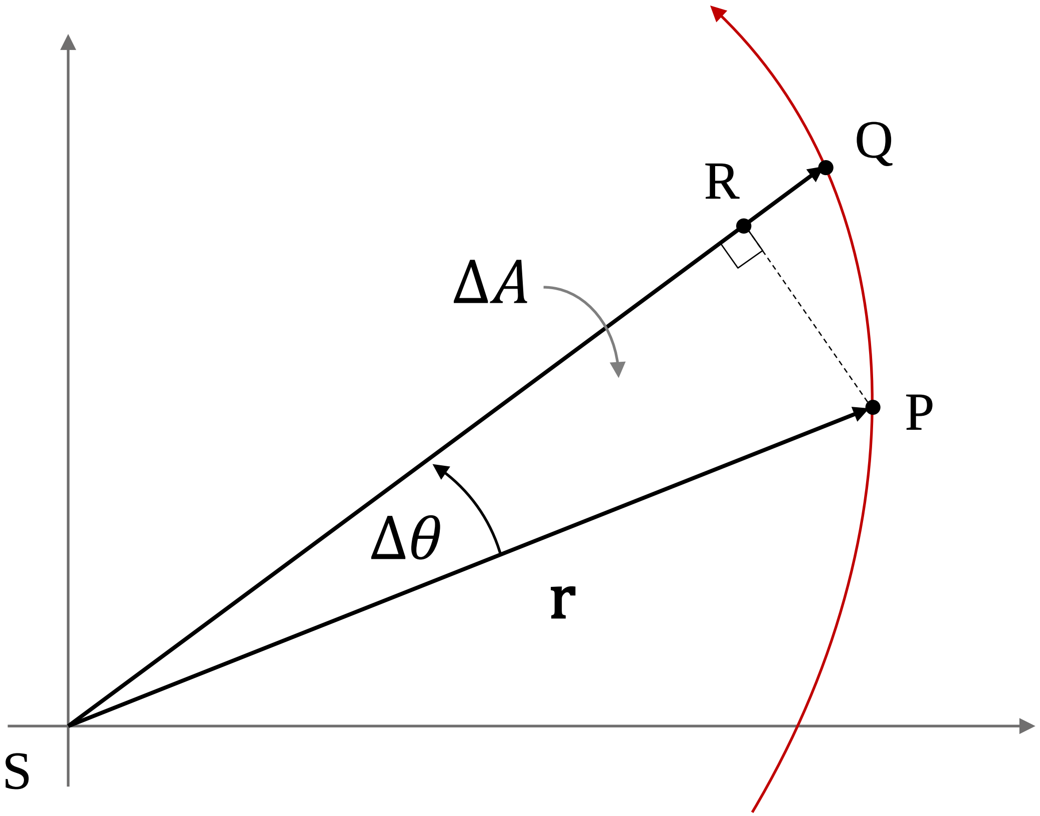
- We also want to show mathematically that Kepler's 2nd Law and 3rd Law also follow from Newton's Laws of Universal Gravitation and Motion.




- Kepler's 2nd Law:
  - The line joining the planet to the Sun sweeps out equal areas in equal times
- Kepler's 3rd Law:
  - The square of the period of the planet is proportional to the cube of its mean distance from the Sun

Orbital motion

- Kepler's 2nd Law:



- The area swept out by radius vector r during an interval of time is the area of the sector of the ellipse SPQ
- For a small interval of time the area of the sector can be approximated by the area of the triangle SPR:
  - Half × height × base

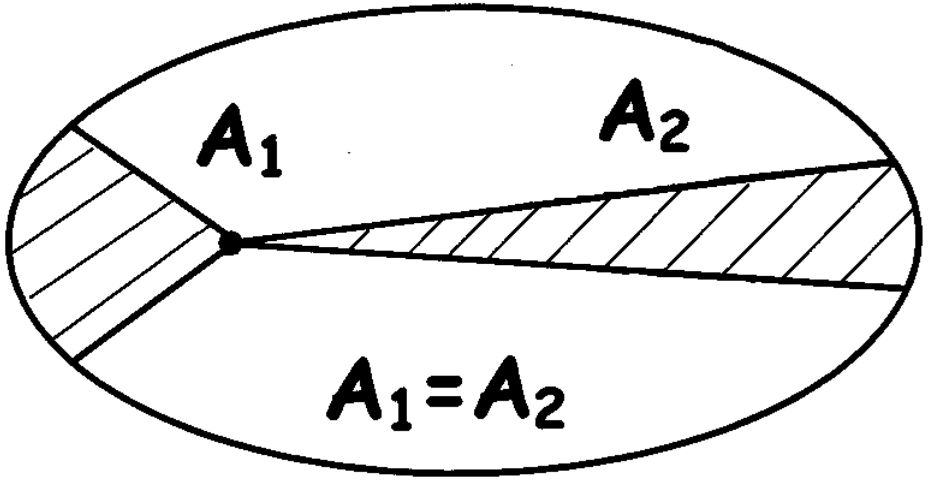
$$\Delta A = \frac{1}{2} r (r \Delta\theta)$$

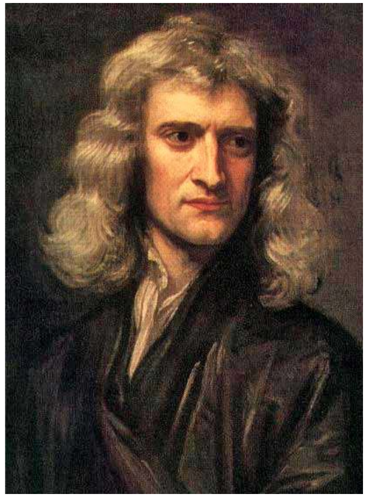
- In the limit, as $\Delta t \rightarrow 0$

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h = \text{constant}$$

Orbital motion

- Kepler's 2nd Law:
  - The line joining the planet to the Sun sweeps out equal areas in equal times

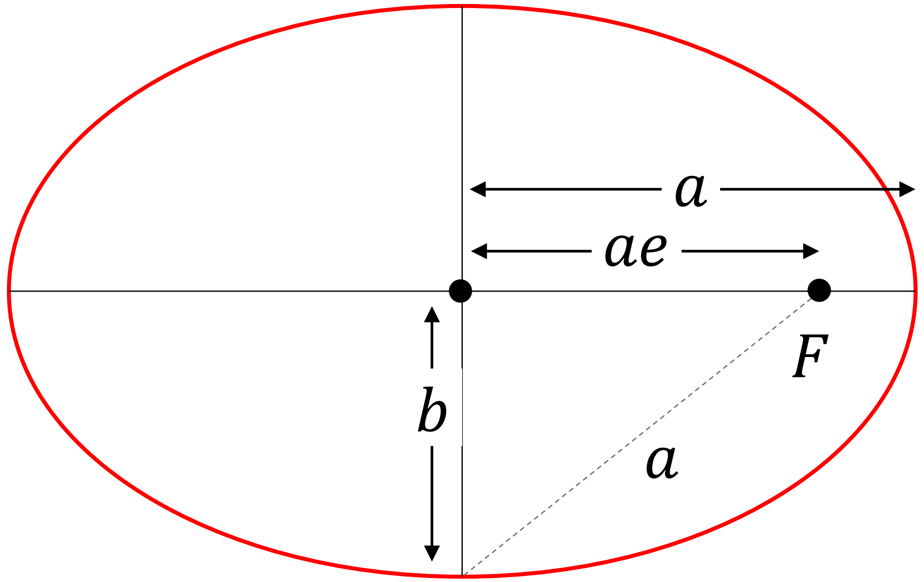




  - This must also mean that a planet will travel faster when closest to the Sun and slower when farthest from the Sun

Orbital motion

- Kepler's 3rd Law:



a= semi-major axis b= semi-minor axis

e= eccentricity

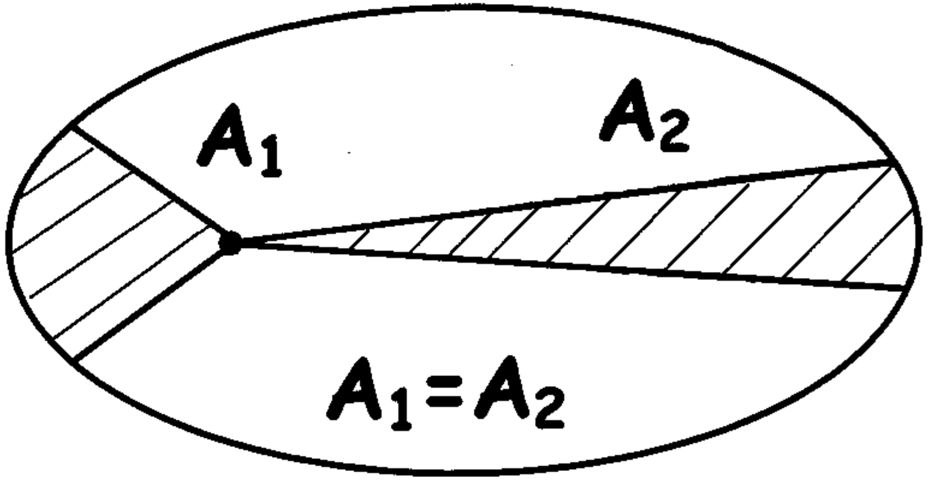
- Two ellipse properties we need:
  - Semi-minor axis: $b^2 = a^2(1 - e^2)$
  - Area: $A = \pi ab = \pi a^2\sqrt{1 - e^2}$
- The orbit period is the time taken for the radius vector to sweep out the full area of the ellipse:

$$\tau = \frac{A}{\dot{A}}$$

- From our proof of Kepler's 2nd Law we know that the rate at which the area is swept is constant...

Orbital motion

- Kepler's 3rd Law:



- From our proof of Kepler's 2nd Law, we know the rate at which the area is swept out:

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h = \text{constant}$$

- Hence the orbital period is:

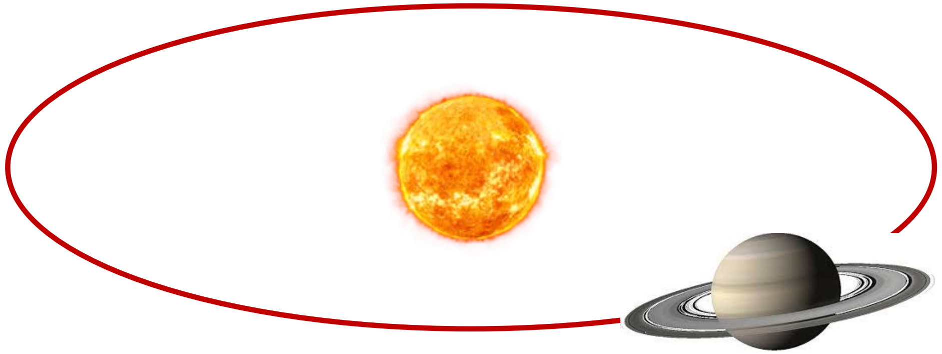
$$\tau = \frac{\pi a^2 \sqrt{1 - e^2}}{\frac{1}{2} h}$$

- By comparing the ellipse equation with our equation of motion we have

$$a(1 - e^2) = \frac{h^2}{\mu}$$

Orbital motion

- Kepler's 3rd Law:



- Finally, we can write the orbital period:

$$\tau = \frac{\pi a^2\sqrt{1-e^2}}{\frac{1}{2}\sqrt{\mu a(1-e^2)}} = 2\pi\sqrt{\frac{a^3}{\mu}}$$

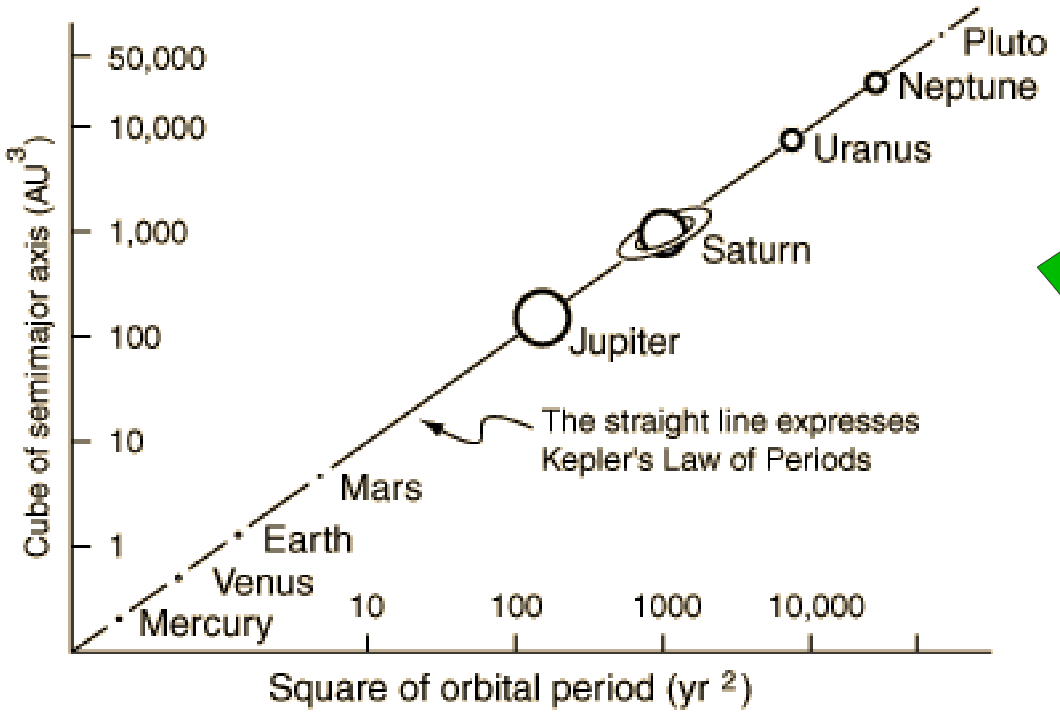
- Or:

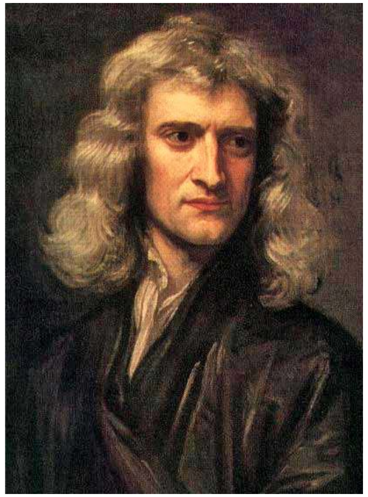
$$\tau^2 = \frac{4\pi^2}{\mu}a^3$$

- Which means "the square of the period of the planet is proportional to the cube of its mean distance from the Sun"

Orbital motion

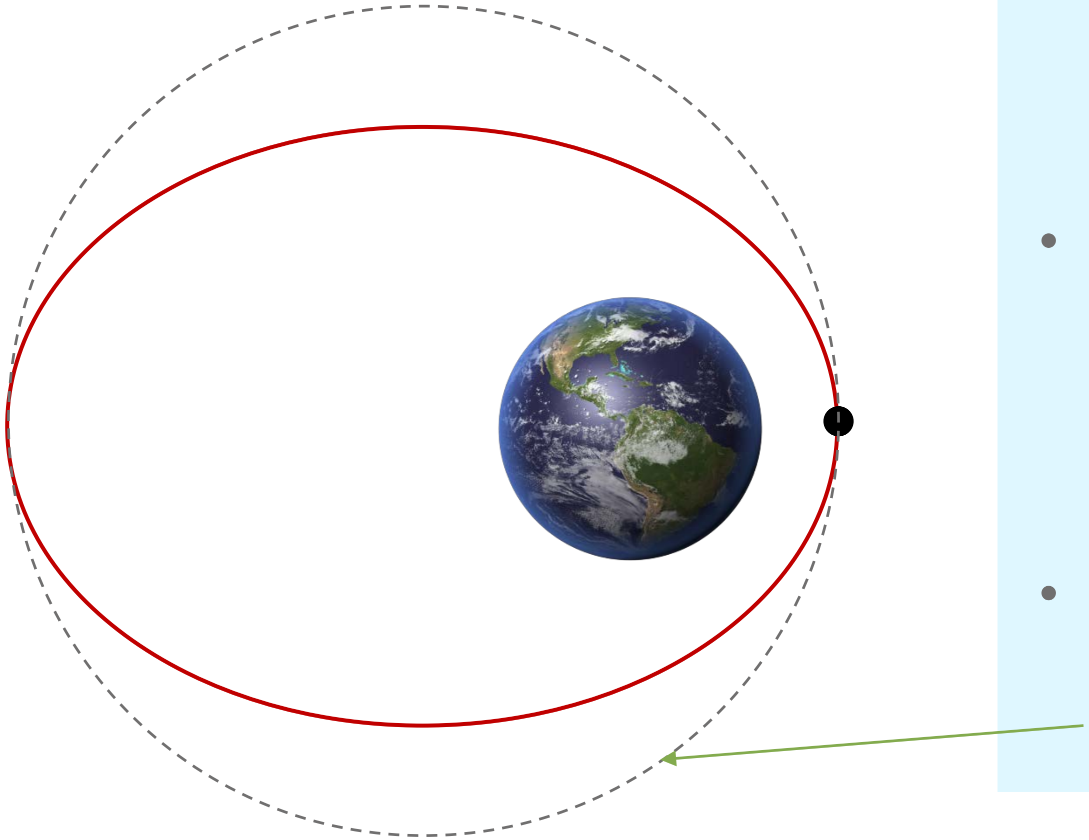
- Kepler's 3rd Law:
  - The square of the period of the planet is proportional to the cube of its mean distance from the Sun





Orbital motion

- Mean motion



- We can also define the orbital mean motion:

$$n = \frac{2\pi}{\tau}$$

- The mean motion is constant for a given value of semi-major axis (as τ is only a function of a, the semi-major axis) with units of °/sec, radians/sec or revolutions/day (for example)
- The mean motion describes the rate at which a satellite or planet moves around the auxiliary circle

Activity 1

- Activity based on Kepler's 3rd Law:
  - Find the appropriate semi-major axis values and gravitational constants (e.g. using Google, Wikipedia, NASA website, etc.) and create a spreadsheet in Microsoft Excel to calculate or confirm:
    - The length of one year for Mars
    - The length of one year for Pluto
    - The semi-major axis of the Earth's orbit around the Sun, if the length of one year is 365.25 days
    - The number of orbits made by the International Space Station in one day
    - The semi-major axis of a Geostationary orbit if a satellite takes 23 hours and 56 minutes to make one orbit of the Earth

$$\tau = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Note: be careful with the units associated with the values above and for the values you find online. A good tip is to convert everything to SI units (because some orbits can be so large, using kilometres can also work)

Activity 2

- The mission analysis topic is covered in the early parts of chapter 5 of Fortescue, Stark & Swinerd:
  - Read this chapter (up to and including the "Mission Analysis" section) in preparation for the next few lectures & to support your learning of this topic
  - Access to the e-book is available via the Library website: https://onlinelibrary.wiley.com/doi/book/10.1002/9781119971009