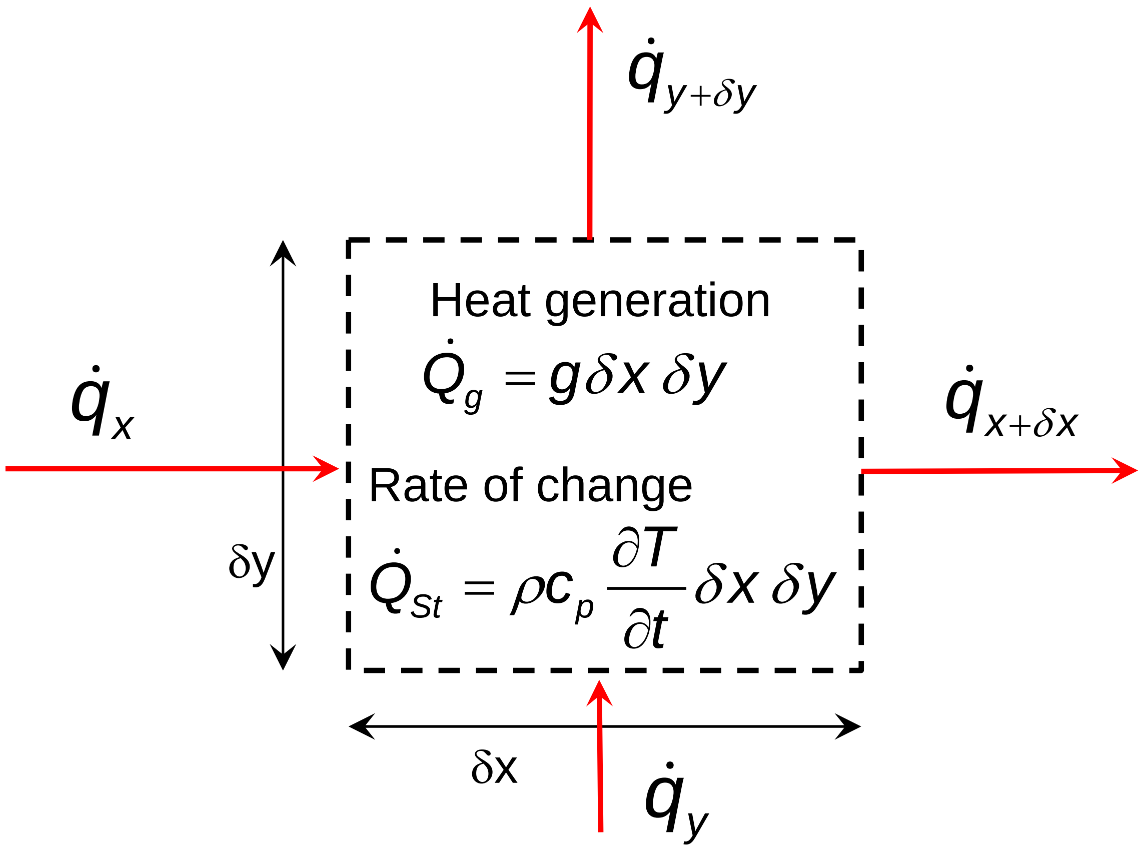
SESA3029 Aerothermodynamics

Lecture 5.5

heat diffusion equation, 1D finite difference methods for conduction

Heat equation



$$\dot{q}_{x+\delta x} = \dot{q}_x + \frac{\partial \dot{q}_x}{\partial x}\delta x$$

$$\dot{q}_{y+\delta y} = \dot{q}_y + \frac{\partial \dot{q}_y}{\partial y}\delta y$$

Energy balance

$$\dot{Q}_{in} - \dot{Q}_{out} + \dot{Q}_g = \dot{Q}_{St}$$

$$(\dot{q}_x - \dot{q}_{x+\delta x})\delta y + (\dot{q}_y - \dot{q}_{y+\delta y})\delta x + g\,\delta x\,\delta y = \rho c_p \frac{\partial T}{\partial t}\delta x\,\delta y$$

$$-\frac{\partial \dot{q}_x}{\partial x}-\frac{\partial \dot{q}_y}{\partial y}+g=\rho c_p\frac{\partial T}{\partial t}$$

Use Fourier's law $\dot{q}_i=-k\frac{\partial T}{\partial x_i}$ to obtain

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+g=\rho c_p\frac{\partial T}{\partial t}$$

the heat diffusion equation

$$\boxed{\Delta T+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

with thermal diffusivity $\alpha=\frac{k}{\rho c_p}$

Typical boundary conditions

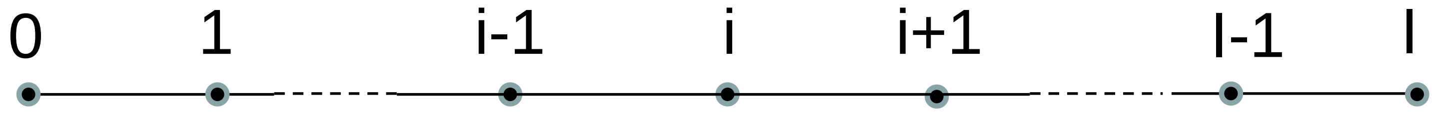
For simplicity: 1D case

- Constant temperature: $$T\left(0,t\right)=T_s$$
- Constant surface heat flux: $$-k\left.\frac{\partial T}{\partial x}\right|_{x=0}=\dot{q}_x$$
- Adiabatic or insulated surface: $$\left.\frac{\partial T}{\partial x}\right|_{x=0}=0$$
- Convection surface condition: $$-k\left.\frac{\partial T}{\partial x}\right|_{x=0}=h\left[T_\infty-T(0,t)\right]$$

Discretising the heat equation in 1D

1D heat equation, stationary, no heat source $\qquad \dfrac{\partial^2 T}{\partial x^2} = 0$

on mesh with uniform stepsize $\quad \Delta x$



The gradients can be approximated as

$$\left.\frac{\partial T}{\partial x}\right|_{i-1/2} \approx \frac{T_i - T_{i-1}}{\Delta x}, \qquad \left.\frac{\partial T}{\partial x}\right|_{i+1/2} \approx \frac{T_{i+1} - T_i}{\Delta x}$$

yielding the approximated equation

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i} \approx \frac{\left.\partial T/\partial x\right|_{i+1/2} - \left.\partial T/\partial x\right|_{i-1/2}}{\Delta x} \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} = 0 \qquad (1)$$

Discrete boundary conditions

Constant temperature: $T_2 - 2T_1 = -T_{sl}$, $\quad -2T_{l-1} + T_{l-2} = -T_{sr}$

Surface convection on the left: $$-k\left.\frac{\partial T}{\partial x}\right|_0 = h\left[T_\infty - T_0\right]$$

Using
$$\frac{\left.\frac{\partial T}{\partial x}\right|_{1/2} - \left.\frac{\partial T}{\partial x}\right|_0}{\frac{1}{2}\Delta x} = \frac{\frac{T_1 - T_0}{\Delta x} - \left(-\frac{h}{k}\left[T_\infty - T_0\right]\right)}{\frac{1}{2}\Delta x} = 0$$

gives
$$T_1 - \left(1 + \frac{h\Delta x}{k}\right)T_0 = -\frac{h\Delta x}{k}T_\infty \qquad (2)$$

Surface convection on the right (note sign!): $$k\left.\frac{\partial T}{\partial x}\right|_l = h\left[T_\infty - T_l\right]$$

gives
$$T_{l-1} - \left(1 + \frac{h\Delta x}{k}\right)T_l = -\frac{h\Delta x}{k}T_\infty \qquad (3)$$

Adiabatic boundary: Simply use $h = 0$ in Eqs. (2), (3)

Constant surface heat flux: Use $h = 0, \quad T_\infty = \frac{\dot{q}_x}{h}$ in Eqs. (2), (3)

Solution process

Equation (1) gives tridiagonal linear systems.
Constant temperature boundary conditions:

$$\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} T_1 \\ \\ \vdots \\ \\ T_{l-1} \end{pmatrix} = \begin{pmatrix} -T_{sl} \\ 0 \\ \vdots \\ 0 \\ -T_{sr} \end{pmatrix}$$

Surface convection boundary conditions:

$$\begin{pmatrix} -\left(1+\dfrac{h_l \Delta x}{k}\right) & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -\left(1+\dfrac{h_r \Delta x}{k}\right) \end{pmatrix} \begin{pmatrix} T_0 \\ \\ \vdots \\ \\ T_l \end{pmatrix} = \begin{pmatrix} -\dfrac{h_l \Delta x}{k} T_{\infty l} \\ 0 \\ \vdots \\ 0 \\ -\dfrac{h_r \Delta x}{k} T_{\infty r} \end{pmatrix}$$

Order of accuracy of finite differences

Finite difference representations are based on Taylor series. For instance

$$T_{i+1} = T_i + \left(\frac{\partial T}{\partial x}\right)_i \Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)_i \frac{\Delta x^2}{2} + \left(\frac{\partial^3 T}{\partial x^3}\right)_i \frac{\Delta x^3}{3!} + \ldots$$

can be rearranged to

$$\left(\frac{\partial T}{\partial x}\right)_i = \underbrace{\frac{T_{i+1} - T_i}{\Delta x}}_{\text{Finite difference}} \underbrace{- \left(\frac{\partial^2 T}{\partial x^2}\right)_i \frac{\Delta x}{2} - \left(\frac{\partial^3 T}{\partial x^3}\right)_i \frac{\Delta x^2}{3!} + \ldots}_{\text{Truncation error}} = \frac{T_{i+1} - T_i}{\Delta x} + O(\Delta x)$$

Similarly, we can show that

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_i = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

Accuracy of basic scheme by Taylor series

Error~Δx^p Grid size Δx, order of accuracy p

1st order: error decreases by a factor of 2 for twice the number of grid points

2nd order: error decreases by a factor of 4 for twice the number of grid points

4th order: error decreases by a factor of 16 for twice the number of grid points

Fewer grid points needed for high order methods for a given accuracy, but does depend on the function being approximated