Part 1: Principle of Minimum Total Potential Energy (PMTPE)

FEEG3001/SESM6047 FEA in Solid Mechanics

Prof A S Dickinson

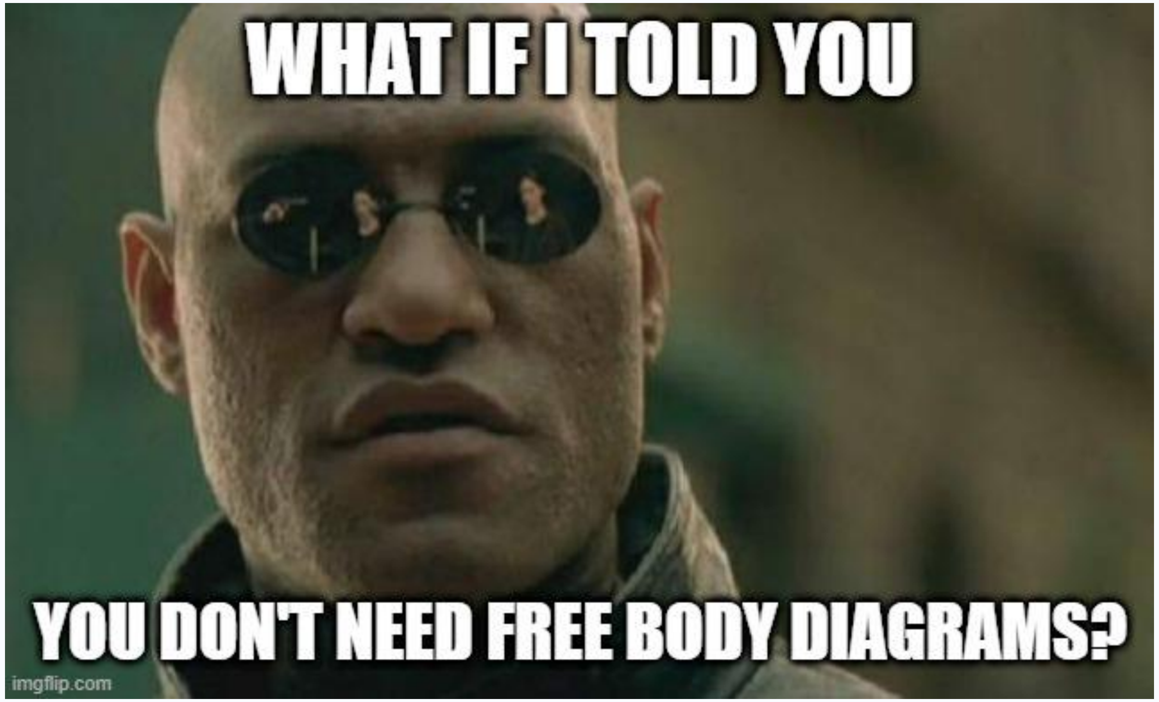
From 3rd October 2024

Principle of Minimum Total Potential Energy

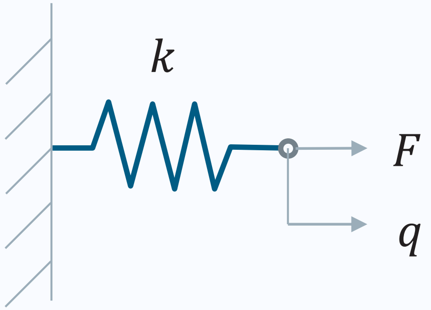
- PMTPE works for Statics:
  - Very simple problems on springs and masses
  - Elastic rods in tension and compression
  - Beam bending formulations (combined)
  - Trusses and Frames
- Hamilton's Principle for structural dynamics
  - Requires Lagrange's equations
  - special cases in Dynamics (free vibration, and rigid body motion)
- Then onto 2D elements (can still be 3D, it just means that the thickness of things like plates is really thing -> sheet type behaviour)

Structural mechanics – how?

- Free body diagrams, reaction forces
- Isolate the object of our attention (free body), and replace rest of universe with two things: Forces, and Moments.
- Classical or Newtonian Mechanics = Newton's Law + Euler's Law
  - Sum of external force = mass x acceleration
  - and Euler equivalent for rotation
- We will not draw FBDs but we will still do Newtonian Mechanics
- FEA goes back to 19th century, pre-computers!



PMTPE for a Spring:



q=displacem
nt

- Elastic structure has deformed under the influence of static forces.

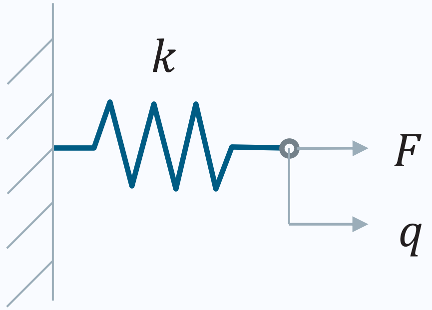
  U represents the stored elastic strain energy in the system

  V represents the potential energy, the negative of work done by external forces

- You can imagine where the material points have moved to (e.g. a wing under pressure, a bridge under gravity, a drive shaft under torque) i.e. deformation.
- We imagine an infinite number of possible responses, and work out the PE for them; the response that corresponds to the minimum of Π is the true answer.

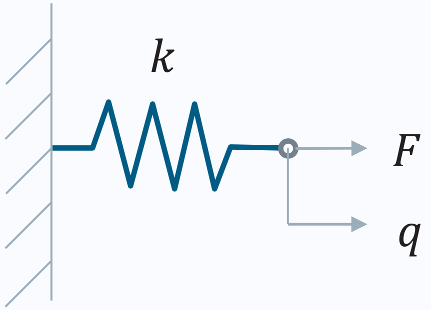
  $\Pi = U + V$

PMTPE for a Spring:



- This means we don't solve the problem, we search for an answer to it.
- The true answer will have lower value of Π than all the values for Π for the wrong answers
- i.e. minimum with respect to *comparisons* with other, wrong answers, not with respect to increments of a function (i.e. d).
- Instead we'll use delta δ. $\boldsymbol{\delta\Pi = 0}$, using principle 'variational calculus'. $\delta\Pi$ is known as the first variation of Π.
- Valid for all mechanical systems under equilibrium (linear, nonlinear, differential, etc).
- We can't prove it, but we can verify it (then inductive inference instead of deduction).

Questions:



- What is the strain energy in the spring if the end moves by distance q?

  $U = {}^1\!/_2\, kq^2$

- What potential energy is there?

  Negative of the work done by the external force:

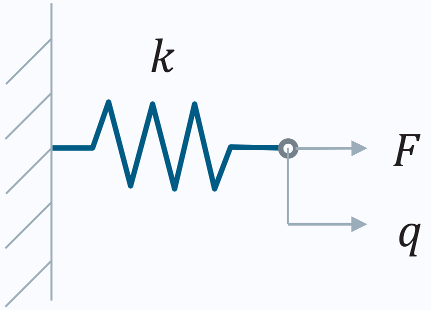
  $V = -W = -Fq$

- Reaction does no work, because displacement is zero. Therefore

  $\Pi = {}^1\!/_2\, kq^2 - Fq$

- q is described as a **'generalised coordinate'** or **'Degree of Freedom' (DoF)** of the problem, because it's not just a coordinate – may not have units of distance or length, etc. But if we know q, we know everything about the mechanical system.

Minimum energy scenario:



- So $\Pi = \Pi(q)$ and we need to find out what q is
- Without proof of variational calculus, by minimisation with respect to comparison with the false answers, the statement:

$$\delta\Pi = 0$$

< This

$\Rightarrow$ (implies)

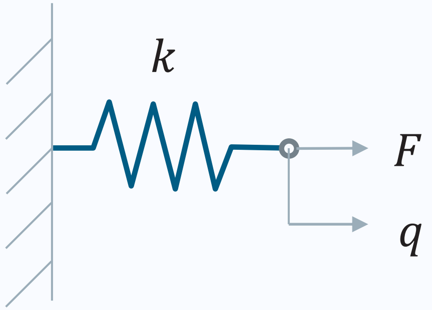
means

$$\frac{\partial \Pi(q)}{\partial q} = 0$$

<This
(because reasons
we skip over)

- (that is, differentiating Π with respect to q and setting it to zero)
- I'm not proving the 'implies' arrow. But if you trust me, the next step is easy:

Minimum energy scenario:



$$\frac{\partial \Pi(q)}{\partial q} = 0$$

$\Pi = {}^1/_2\, kq^2 - Fq$ so

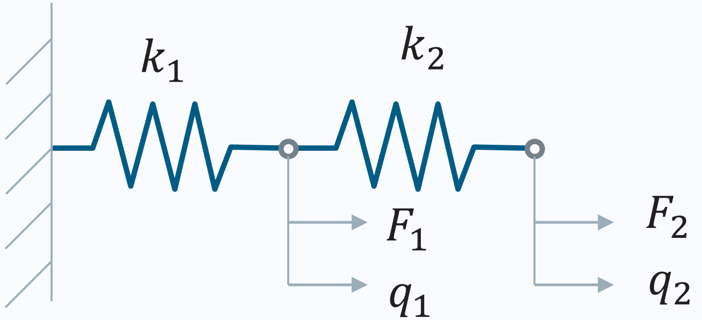
$\frac{\partial \Pi(q)}{\partial q} = {}^1/_2\, k \times 2q - F = 0$ or simply

$$F = kq$$

- Look familiar?
- Imagine the alternative scenarios, if q was larger or smaller

(Both result in non possible cases)

A more interesting problem:



- We can extend this to another problem: how about two springs in series on the same wall:
- Π becomes a function of two generalised coordinates (system has 2 DoF):

$$\Pi = \Pi(q_1, q_2)$$

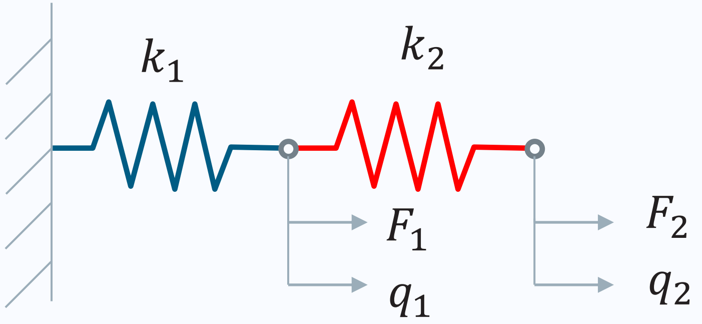
- and the location of BOTH qs must obey the variational, comparison principle of minimum potential energy:

$$\frac{\partial \Pi}{\partial q_1} = 0$$

2 degrees of freedom -> 2 unknowns -> 2 equations

$$\frac{\partial \Pi}{\partial q_2} = 0$$

Again PMTPE:

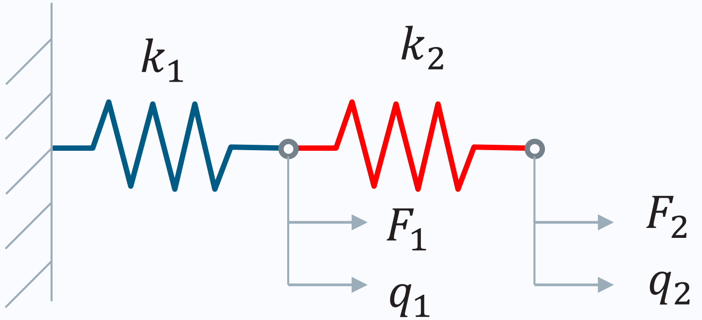


- $\Pi = U + V$ so what is U?

  $U = {}^1\!/_2\, k_1 q_1^2 + {}^1\!/_2\, k_2 (q_2 - q_1)^2$ (strain energy in first spring + second spring)

- Question: is it $q_1 - q_2$ or $q_2 - q_1$?
- The squaring is forgiving, it doesn't matter!

Again PMTPE:



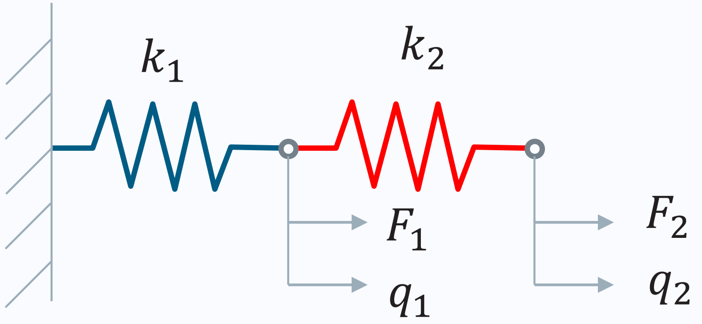
$V = -W$ negative of work done by external forces

- Fixed end doesn't move, so doesn't do work.
- The other points move, so do work

$W = F_1 q_1 + F_2 q_2$

- Question: why $F_2 q_2$, and not $F_2(q_2 - q_1)$?
- q_1 and q_2 are displacements in your reference frame; absolute displacement. Work done is just force x how far it moves.

Thank You



- $U = {}^{1}/_{2}\, k_1 q_1^2 + {}^{1}/_{2}\, k_2 (q_2 - q_1)^2$
- $V = -(F_1 q_1 + F_2 q_2)$

- Next session: we will combine, differentiate, and present in a matrix form…

Part 1b: PMTPE: a ‘Stiffness Matrix’ for more complex problems

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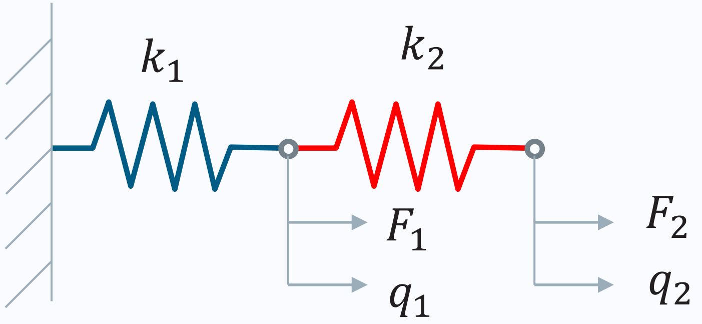
From 3rd October 2024

Reminder:



- Springs as Finite Elements
- What are the values of q_1 and q_2 when equilibrium is achieved?

$\Pi = U + V$ (Total potential energy: strain energy plus external loading energy)

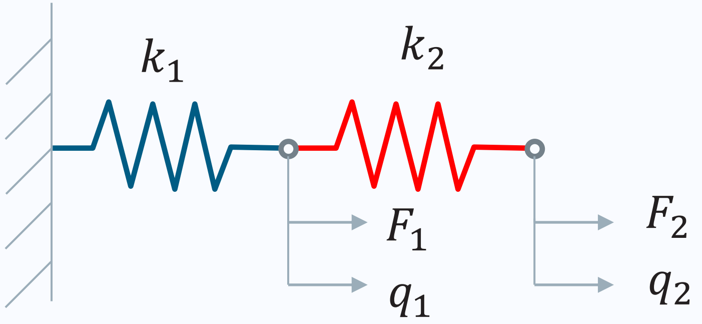
$U = {}^1/_2\, k_1 q_1^2 + {}^1/_2\, k_2 (q_2 - q_1)^2$ (strain energy in first spring + second spring)

$V = -W$ (negative of work done by external forces)

$V = -(F_1 q_1 + F_2 q_2)$

- Combine, differentiate, and present in a matrix form…

Apply PMTPE:



$\Pi = {}^1\!/_2\, k_1 q_1^2 + {}^1\!/_2\, k_2 (q_2 - q_1)^2 - (F_1 q_1 + F_2 q_2)$ so now we have

$\Pi = \Pi(q_1, q_2)$ Two qs, so two degrees of freedom (DOF).

- Equilibrium says

$\delta\Pi(q_1, q_2) = 0 \Rightarrow \dfrac{\partial \Pi}{\partial q_1} = 0$ and $\dfrac{\partial \Pi}{\partial q_2} = 0$

- (partial derivatives – not proven, but trusting variational calculus)

$\dfrac{\partial \Pi}{\partial q_1} = k_1 q_1 + k_2(q_1 - q_2) - F_1 = 0$ (expanding brackets above: chain rule), or

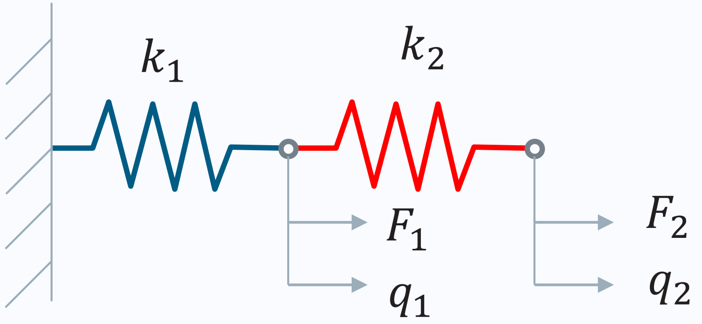
$F_1 = (k_1 + k_2) q_1 - k_2 q_2$

and

$\dfrac{\partial \Pi}{\partial q_2} = 0 + k_2(q_2 - q_1) - F_2 = 0$ or

$F_2 = -k_2 q_1 + k_2 q_2$

Apply PMTPE:



$F_1 = (k_1 + k_2)q_1 - k_2 q_2$ and

$F_2 = -k_2 q_1 + k_2 q_2$

- What kind of equations are these? qs are unknowns, Fs and ks are knowns…
- Linear, simultaneous, algebraic equations. They can be expressed in matrix form where:

$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$ Experience tells us the size: where each row comes from one equation:

$\begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$ or in general terms,

$[K]\{q\} = \{F\}$

$\{F\}$ is the ***force vector***

$\{q\}$ is the ***displacement vector***, and

$[K]$ is the ***stiffness matrix***

- We use a computers to solve matrix equations fast when there are many more DOFs.

How to solve PMTPE…?

$[K]\{q\} = \{F\}$ (with unknowns). This might be expressed as

$$\{q\} = [K]^{-1}\{F\}$$

- Matrix inversion is very inefficient computationally, so often algorithms will use methods related to Gaussian elimination: upper triangulate or lower triangulate the matrix then back substitute.
- Mostly in this module we will be breaking down mechanical problems into this form. This one is really simple, but we do this for bones, whole ships, etc.

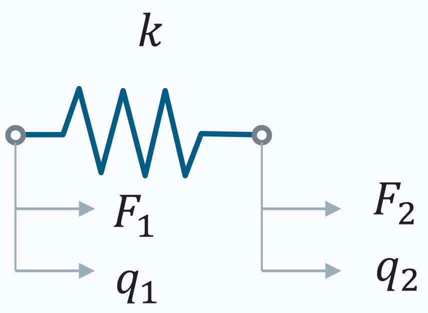
How to formulate mechanical problems using PMTPE

- (remember, no free body diagrams – but you would get to the same result)
- Can we do this without needing to repeat the integration? Can we make a generic Finite Element, a tidy, generic package we can use more quickly?
- The spring element: simplest mechanical element that stores strain energy
- We formulate the element, then we collect information about it. A spring floating alone would have:

$$U = {}^{1}\!/_{2}\, k(q_2 - q_1)^2$$

- And we can use a special form:

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T_{1\times 2} \begin{bmatrix} \quad & \quad \end{bmatrix}_{2\times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2\times 1}$$

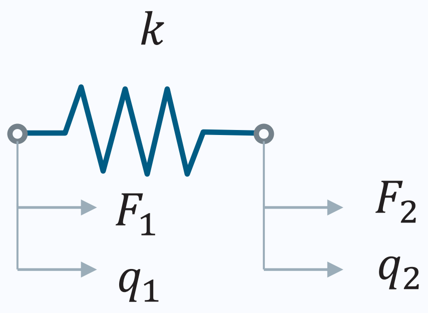


How to formulate mechanical problems using PMTPE

Quadratic form is:

$$U = {}^{1}\!/_{2}\,(Aq_1{}^2 + Bq_1q_2 + Cq_2q_1 + Dq_2{}^2)$$

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$



The square terms come from the diagonal, and the mixed terms from the off-diagonal

$$U = {}^{1}\!/_{2}\,k_1(q_2 - q_1)^2$$

$$U = {}^{1}\!/_{2}\,(k_1q_1{}^2 - 2k_1q_1q_2 + k_1q_2{}^2)$$

$$U = {}^{1}\!/_{2}\,(k_1q_1{}^2 - k_1q_1q_2 - k_1q_1q_2 + k_1q_2{}^2)$$

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

How to formulate mechanical problems using PMTPE

Quadratic form is:

$$U = {}^1/_2\,(Aq_1{}^2 + Bq_1q_2 + Cq_2q_1 + Dq_2{}^2)$$

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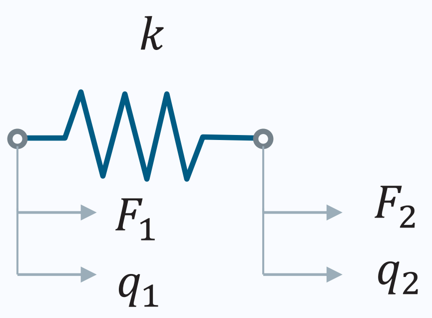


The square terms come from the diagonal, and the mixed terms from the off-diagonal

$$U = {}^1/_2\,k_1(q_2 - q_1)^2$$

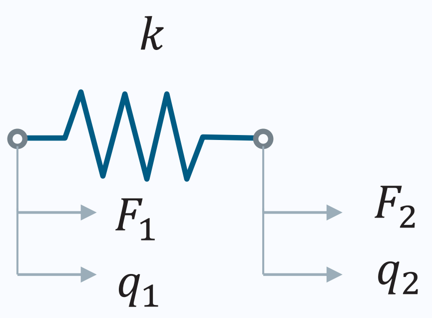
$$U = {}^1/_2\,(k_1q_1{}^2 - 2k_1q_1q_2 + k_1q_2{}^2)$$

$$U = {}^1/_2\,(k_1q_1{}^2 - k_1q_1q_2 - k_1q_1q_2 + k_1q_2{}^2)$$

$$U = {}^1/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

How to formulate mechanical problems using PMTPE

$$U = {}^{1}\!/_{2}\, k(q_2 - q_1)^2$$



can be written as:

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Much easier to prove by back multiplication than forwards!

Recognise the structure from maths? *Quadratic form* in matrix algebra (i.e. will *only* give terms in ${q_1}^2$, ${q_2}^2$ and $(q_1 q_2)^2$; no terms in just q_1 or q_2).

$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$ is known as the ***element stiffness matrix***, $[K]$

In short-hand we can write this as something which will reappear frequently:

$$U = {}^{1}\!/_{2} \{q\}^T [K] \{q\}$$

What was the point of all this?

- Crucially:
  - The Stiffness Matrix in the generalised equilibrium equation $\{q\} = [K]^{-1}\{F\}$
  - Is exactly the same Stiffness Matrix in the energy equation $U = {}^{1}/_{2}\{q\}^T[K]\{q\}$
  - and using this variational calculus observation, of energy expressions in the quadratic form, means we can skip the derivation!
- Next week we will join these stiffness matrices together…

Part 1c: Assembling Stiffness Matrices for multiple element problems

FEEG3001/SESM6047 FEA in Solid Mechanics

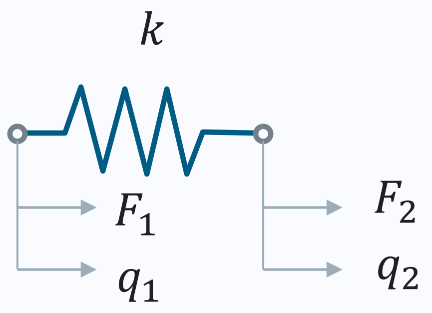
Prof A S Dickinson

From 8th October 2024

How to formulate mechanical problems using PMTPE

For one spring element we have:

$$U = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

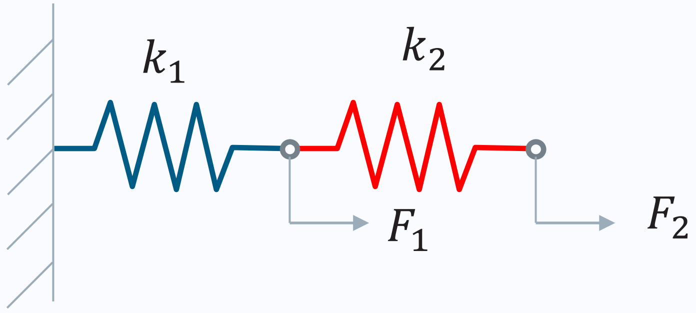


which is the same as:

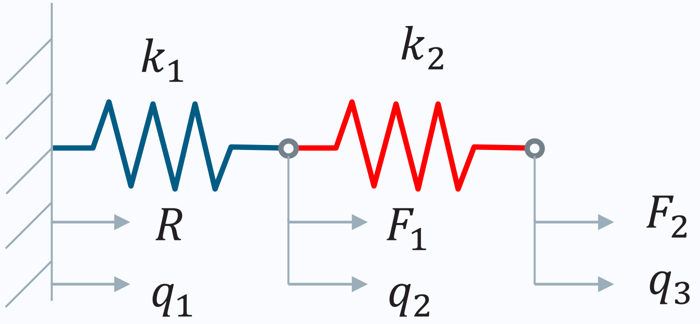
$$U = {}^1\!/_2 \{q\}^T [K] \{q\}$$

But what about when we have many? Let’s chain two springs as before:

and standardise them (like software would)



How to formulate mechanical problems using PMTPE



Redefine our q generalised coordinates:

(Ignoring for now that we know $q_1 = 0\ldots$)

$U_1 = ?$

$U_1 = {}^1\!/_2\, k_1(q_2 - q_1)^2$

$U_2 = ?$

$U_2 = {}^1\!/_2\, k_2(q_3 - q_2)^2$

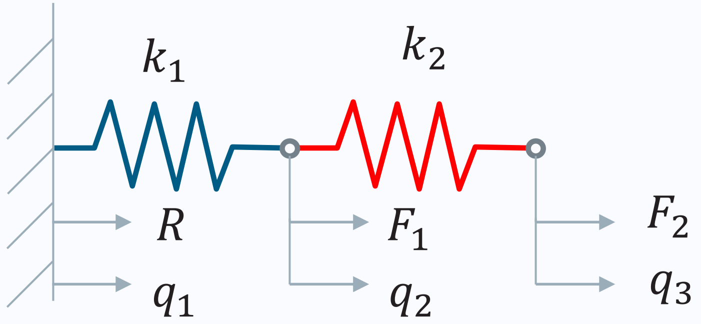
so that we can apply our general, off-the-shelf spring elements.

If we put these into quadratic form:

$U = {}^1\!/_2\, \{q\}^T[K]\{q\}$

we should get:

How to formulate mechanical problems using PMTPE



$$U_1 = {}^1\!/_2\, k_1 (q_2 - q_1)^2$$

can be expressed as:

$$U_1 = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$ (very similar, but now in k_1 instead of k)

and

$$U_2 = {}^1\!/_2\, k_2 (q_3 - q_2)^2$$

can be expressed as:

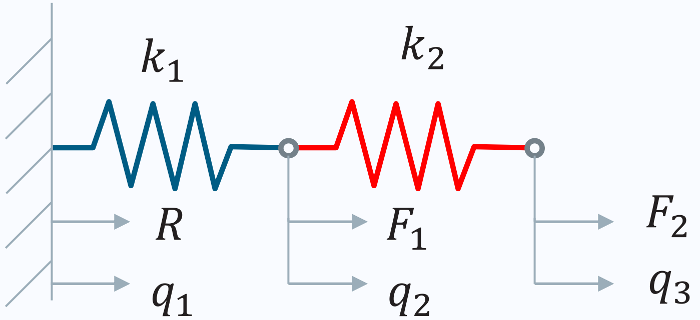
$$U_2 = {}^1\!/_2 \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^T \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

How to formulate mechanical problems using PMTPE

A fundamental principle of finite element analysis is that the strain energy of the system will be the sum of its parts, because it is a simple, additive quantity (unlike e.g. temperature):

$$U = U_1 + U_2$$

So, for the whole system this time:



$$U = {}^1\!/_2 \left\{ \quad \right\}^T_{1\times 3} \left[\qquad \right]_{3\times 3} \left\{ \quad \right\}_{3\times 1}$$

Because we have 3 degrees of freedom, it becomes a 3 sized. The energy is scalar and there are 3 of em

What is the product of a 1x3, a 3x3 and a 3x1 matrix?

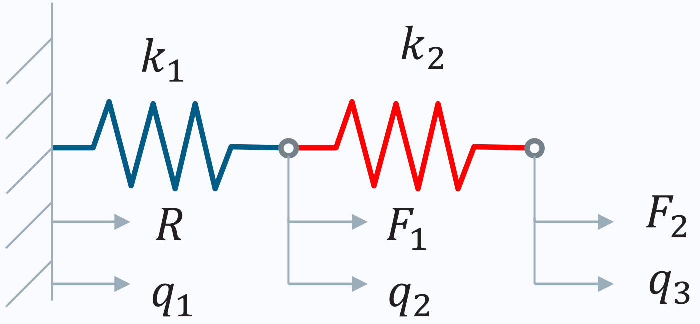
A 1x1. Makes sense!

How to formulate mechanical problems using PMTPE

$$U_1 = {}^1\!/_2\, k_1 (q_2 - q_1)^2$$

$$U_1 = {}^1\!/_2\, k_1 ({q_1}^2 - 2 q_1 q_2 + {q_2}^2)$$

$$U_1 = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$



$$U_2 = {}^1\!/_2\, k_2 (q_3 - q_2)^2$$

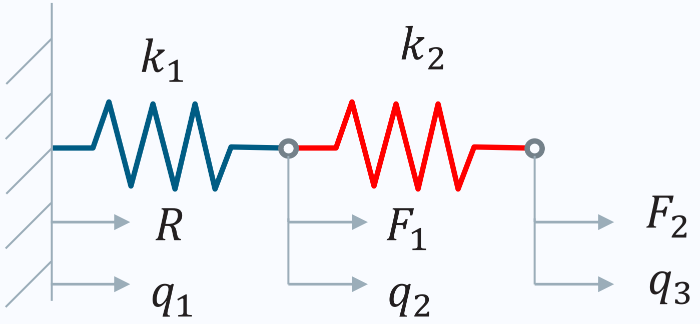
$$U_2 = {}^1\!/_2\, k_2 ({q_2}^2 - 2 q_2 q_3 + {q_3}^2)$$

$$U_2 = {}^1\!/_2 \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^T \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

$$U = {}^1\!/_2 \left\{ \quad \right\}^T_{1\times3} \left[\qquad \qquad \right]_{3\times3} \left\{ \quad \right\}_{3\times1}$$

Expressing this requires a bit of imagination and matrix algebra…

How to formulate mechanical problems using PMTPE



Remember: the form is:

$$U_1 = {}^1/_2\,(Aq_1{}^2 + Bq_1q_2 + Cq_2q_1 + Dq_2{}^2)$$

$$U_1 = {}^1/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

The square terms come from the diagonal, and the mixed terms from the off-diagonal

$$U_1 = {}^1/_2\, k_1(q_2 - q_1)^2$$

$$U_1 = {}^1/_2\,(k_1q_1{}^2 - k_1q_1q_2 - k_1q_1q_2 + k_1q_2{}^2)$$

$$U_1 = {}^1/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Assembly of Elements

$$U_1 = {}^{1}/_{2}\, k_1 (q_2 - q_1)^2$$

$$U_1 = {}^{1}/_{2}\, k_1 (q_1{}^2 - 2q_1 q_2 + q_2{}^2)$$

$$U_1 = {}^{1}/_{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

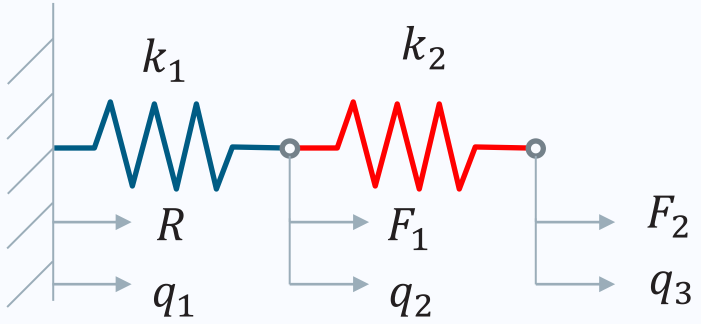
$$U_2 = {}^{1}/_{2}\, k_2 (q_3 - q_2)^2$$

$$U_2 = {}^{1}/_{2}\, k_2 (q_2{}^2 - 2q_2 q_3 + q_3{}^2)$$

$$U_2 = {}^{1}/_{2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^T \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

$$U = U_1 + U_2$$

$$U = {}^{1}/_{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T_{1\times 3} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}_{3\times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1}$$



Basically we need to put the constituent parts of the 2x2 unit spring into the correct parts of the 3x3

Assembly of Elements

$$U_1 = {}^1/_2\, k_1(q_2 - q_1)^2$$

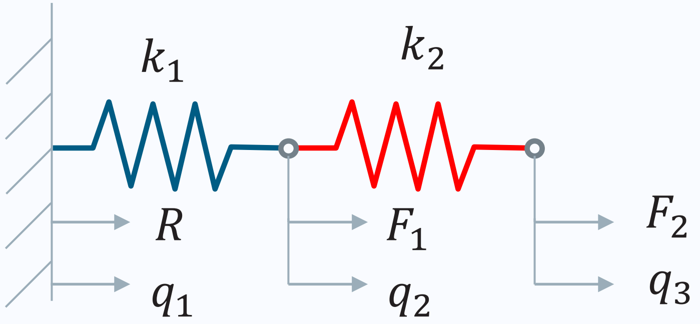
$$U_1 = {}^1/_2\, k_1({q_1}^2 - 2q_1q_2 + {q_2}^2)$$

$$U_1 = {}^1/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$U_2 = {}^1/_2\, k_2(q_3 - q_2)^2$$

$$U_2 = {}^1/_2\, k_2({q_2}^2 - 2q_2q_3 + {q_3}^2)$$

$$U_2 = {}^1/_2 \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^T \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$



In the 3x3 matrix, we kind of have a "q1" and "q2" row and also column

$$U = U_1 + U_2$$

$$U = {}^1/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T_{1\times 3} \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}_{3\times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1}$$

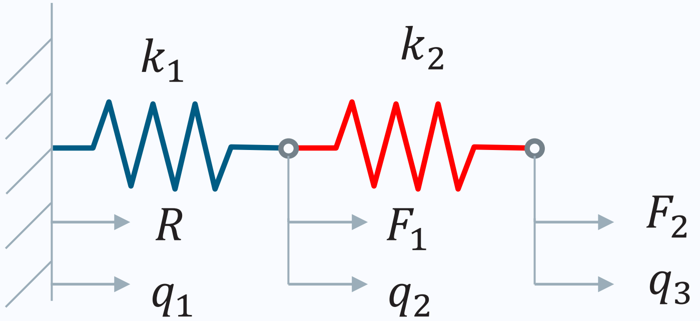
which could be checked by multiplying out.

Why are there zeroes on the two corners?

Assembly of Elements

$$U = {}^{1}\!/_{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T_{1\times 3} \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}_{3\times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1}$$

which contains $[K]$, the *assembled stiffness matrix*



- In performing assembly of the elements, we added the energies, but we did not simply add the matrices. Why?
- The dimensions and/or $\{q\}$ vectors don't match.
- Practically, how?

How to formulate mechanical problems using PMTPE

$$U_1 = {}^1\!/_2\, k_1 (q_2 - q_1)^2$$

$$U_1 = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$U_2 = {}^1\!/_2\, k_2 (q_3 - q_2)^2$$

$$U_2 = {}^1\!/_2 \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^T \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

$$U = U_1 + U_2$$

$$U = {}^1\!/_2 \begin{Bmatrix} \\ \\ \end{Bmatrix}^T_{1\times 3} \begin{bmatrix} & & \\ & & \end{bmatrix}_{3\times 3} \begin{Bmatrix} \\ \\ \end{Bmatrix}_{3\times 1}$$

A practical, computational step, ‘padding’.

$$U_1 = {}^1\!/_2\, k_1 (q_2 - q_1)^2$$

$$U_1 = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$U_2 = {}^1\!/_2\, k_2 (q_3 - q_2)^2$$

$$U_2 = {}^1\!/_2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$U = U_1 + U_2$$

$$U = 1/2 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T_{1\times 3} \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}_{3\times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1}$$

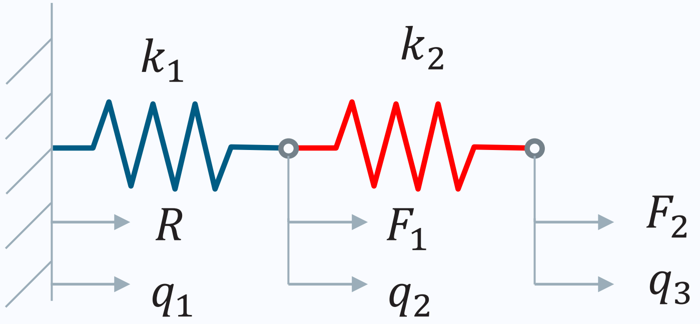
Now, we will introduce the forces

What about the external loading terms?

Work done by external forces:

$$V = -W = ?$$

$$V = (Rq_1 + F_1 q_2 + F_2 q_3)$$ (missing minus sign)



Again we aim for a more general matrix form:

$$V = -\{? \quad ? \quad ?\}_{1\times3}\{q\}_{3\times1}$$

$$V = -\{R \quad F_1 \quad F_2\}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

So together with:

$$U = 1/2\{q\}^T_{1\times3}[K]_{3\times3}\{q\}_{3\times1}$$

$$V = -\{F\}^T_{1\times3}\{q\}_{3\times1}$$

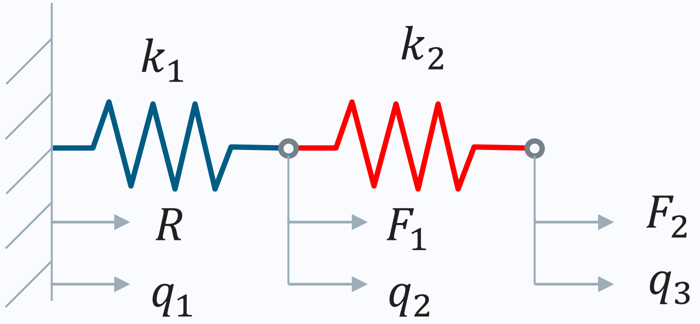
again both will multiply out to scalar energy values, which makes sense!

How to formulate mechanical problems using PMTPE

So applying PMTPE, in i notation:

$$\Pi = U + V$$

Equilibrium says $\delta\Pi(q_i) = 0 \Rightarrow \frac{\partial \Pi}{\partial q_i} = 0, i = 1, 2, \ldots$



So using these quite general energy terms:

$$U = {}^1\!/_2 \{q\}^T_{1\times3}[K]_{3\times3}\{q\}_{3\times1}$$

$$V = -\{F\}^T_{1\times3}\{q\}_{3\times1}$$

with potential energy terms taking quadratic form, our system is linear.

How to formulate mechanical problems using PMTPE

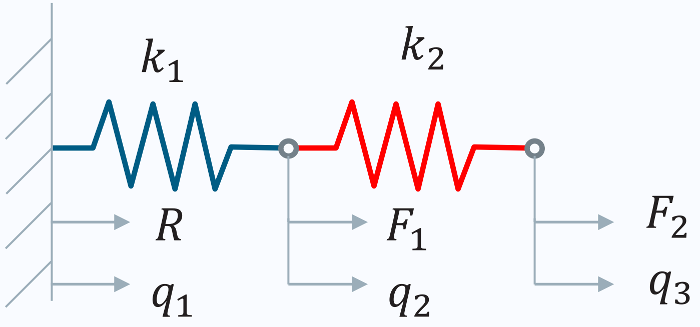
Do the partial differentiation if you want to… the ***governing equation of equilibrium*** takes the form:

$$[K]\{q\} = \{F\}$$

where $[K]$ is our assembled stiffness matrix, and we can recycle the elemental stiffness matrices for each spring in our model, to *assemble* them (instead of simply adding them)

Note that now we have a reaction force!

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R \\ F_1 \\ F_2 \end{Bmatrix}$$

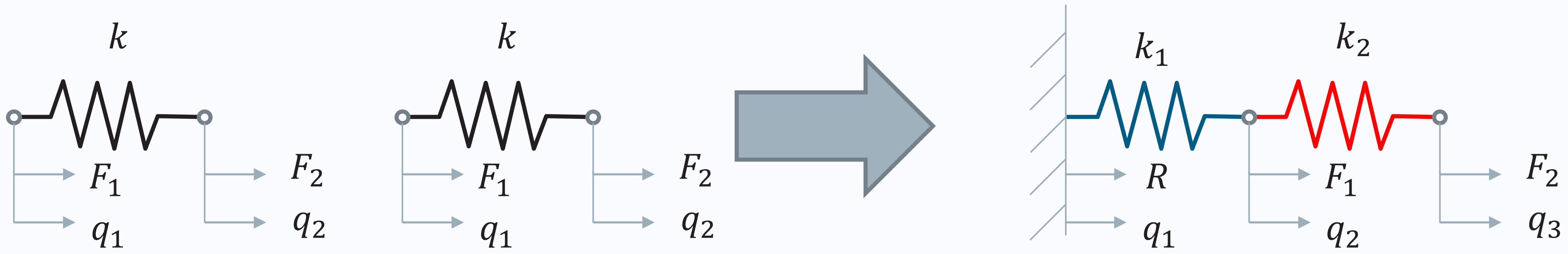


What is our motivation?

We use this like putting these ingredients together in a recipe,

'assembling' them,

so our generalised coordinates of the generic element q_1 and q_2 are 'mapped' onto the generalised coordinates of the new problem.

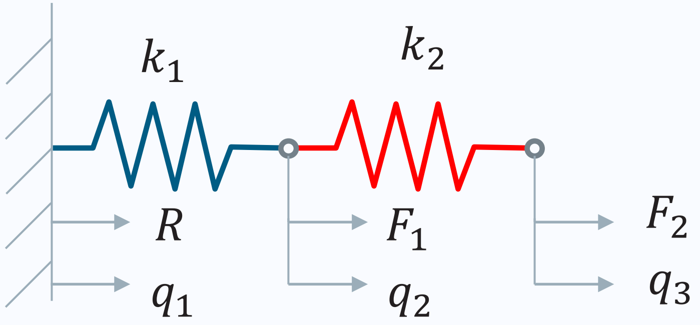


How to formulate mechanical problems using PMTPE

- Can we solve it though? Can we invert the stiffness matrix?
- Remember the properties for a matrix to be invertible?
- The stiffness matrix is *singular* (its determinant is zero), so it cannot be inverted.
- The same is true for the elemental stiffness matrices.

(we can't solve the following, it's singular! So we need to add another constraint)

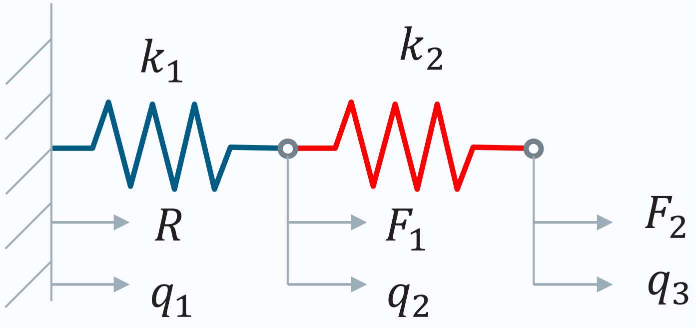
$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R \\ F_1 \\ F_2 \end{Bmatrix}$$



Is $[K]$ invertible? What to do…?

$$[A] = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$



$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

$$|K| = k_1\begin{vmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{vmatrix} - -k_1\begin{vmatrix} -k_1 & -k_2 \\ 0 & k_2 \end{vmatrix}$$

$$|K| = k_1\left((k_1 + k_2)k_2 - {k_2}^2\right) + k_1(-k_1k_2 - 0)$$

$$|K| = k_1(k_1k_2) + k_1(-k_1k_2)$$

$$|K| = 0$$

How to formulate mechanical problems using PMTPE

We need to assert a ***boundary condition***:

$$q_1 = 0$$

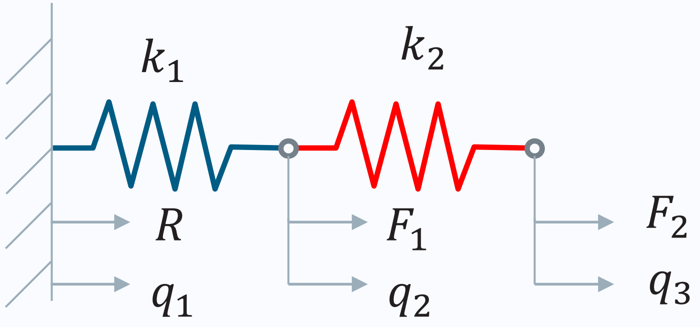
This zero-BC case allows us to use an (unproven) trick where we strike out corresponding rows and columns of our governing equation:

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R \\ F_1 \\ F_2 \end{Bmatrix}$$

and rewrite what is left:

We can still calculate R! Using q_1(=0) and q_2, hence just q_2

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

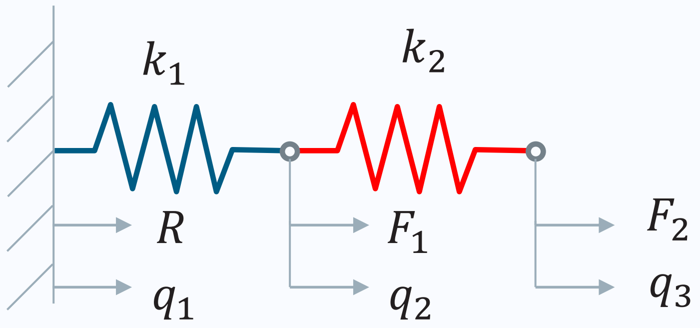


Looks familiar?

Is $[K]$ invertible? What to do…?

Can we invert this reduced stiffness matrix?

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$



$$|K| = \left((k_1 + k_2)k_2 - {k_2}^2\right)$$

$$|K| = \left(k_1 k_2 + {k_2}^2 - {k_2}^2\right)$$

$$|K| = k_1 k_2 \neq 0$$

So we CAN solve the reduced stiffness matrix!

We still haven't actually hit the FEA yet, since the problems simple enough that we can still solve it analytically and get the exact solution!

Conclusion:

- Now you have a first element type on the shelf to pick up and assemble into a model. We don’t have to make a new element for each new spring.
- But is this really Finite Element Analysis? Not really…
  - The FEM solves boundary value differential equations approximately
  - This is an ideal spring element, so there is no approximation: it is exact
- Next we will formulate a new type of element, an elastic rod in tension or compression