SESA3029 Aerothermodynamics
Lecture 3.1

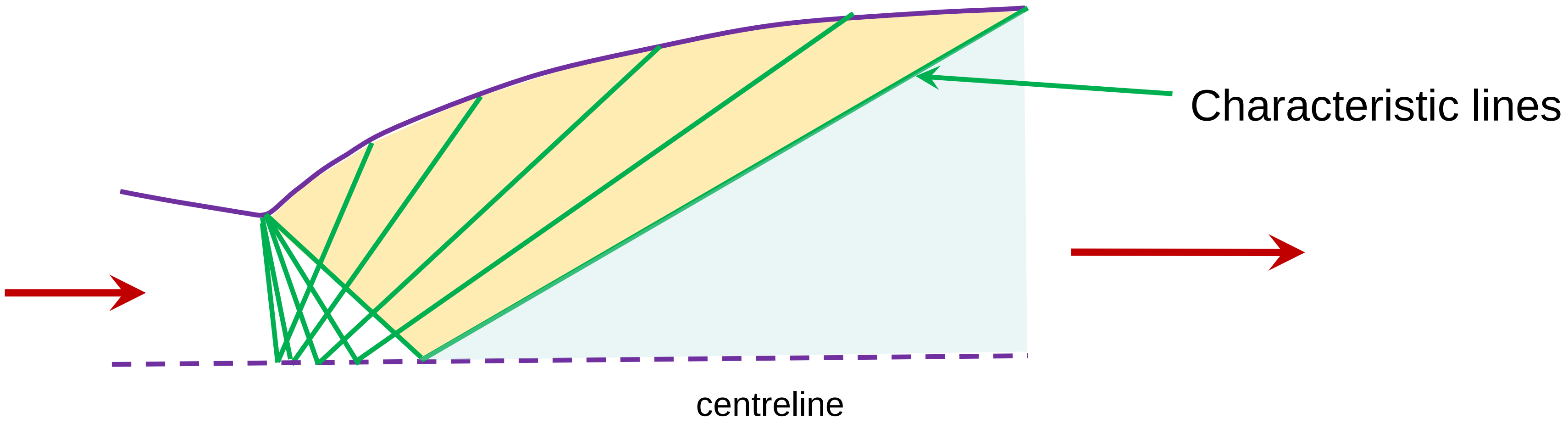
Method of characteristics:

theory (part A)

Objective

everywhere has same entropy (isentropic on everything)

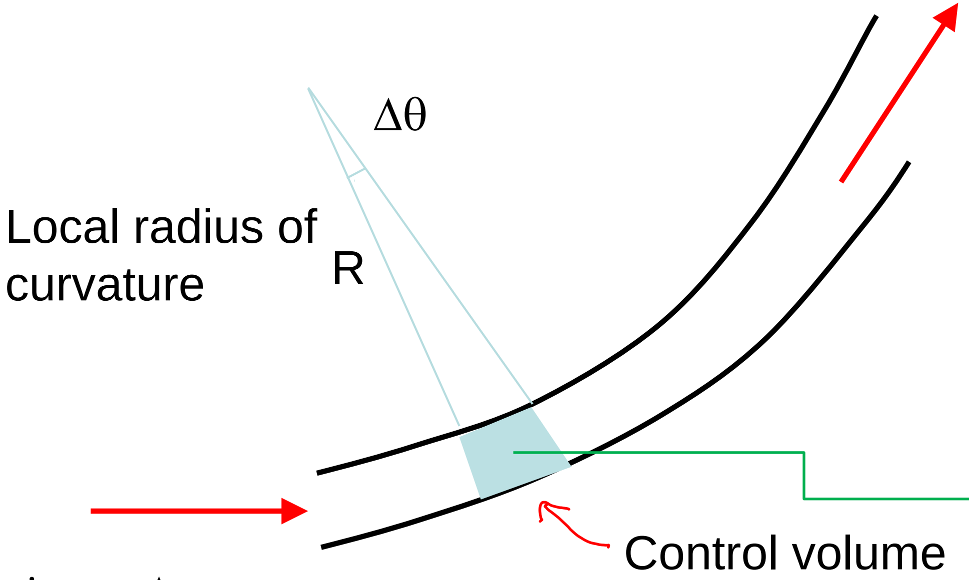
- Introduce the equations for steady flow with entropy constant everywhere (defined as 'homentropic flow')
  - Enables a simpler derivation of the method of characteristics
  - Eventually we will have method for nozzle design



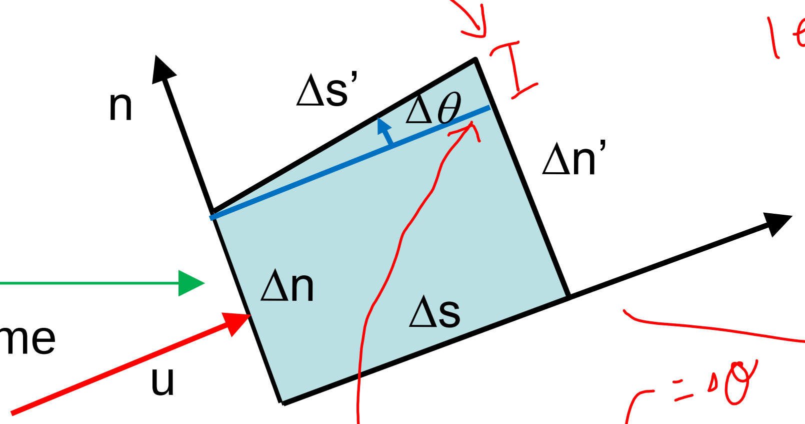
Curved streamtube

similar to last year aerodynamics

No flow across streamlines, therefore for steady flow
mass flowrate in = mass flowrate out



$$\dot{m} = \rho u \Delta n = \text{constant}$$



$= \Delta s \, \Delta\theta$

shrinking Δθ to zero will lead into integration

not entropy!

sn coordinate system

$$\Delta s = R\Delta\theta$$

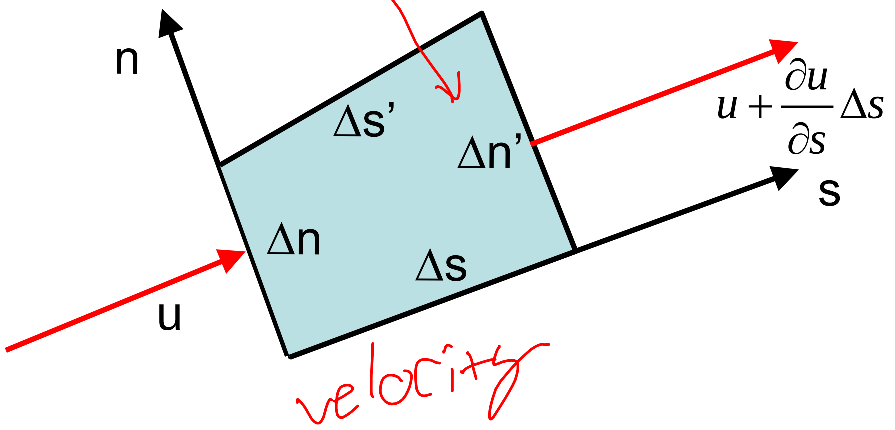
$$\frac{1}{R} = \frac{\partial\theta}{\partial s} \qquad (1)$$

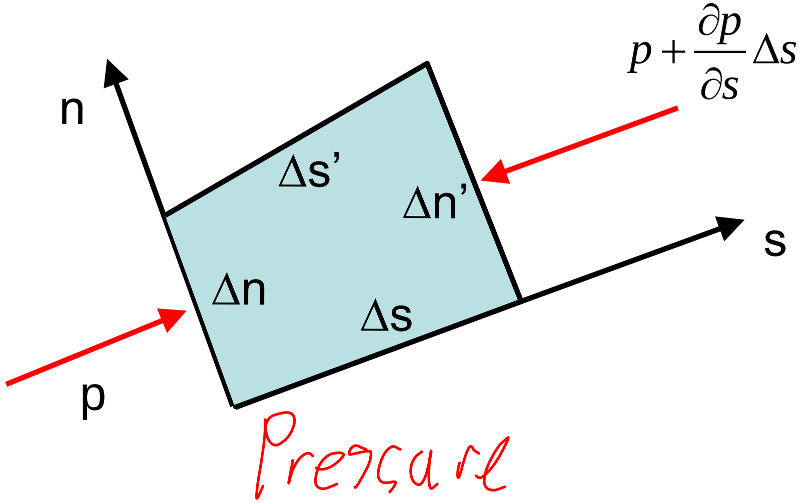
$$\Delta n' - \Delta n = \Delta\theta\Delta s = \Big(\underbrace{\frac{\partial\theta}{\partial n}\Delta n}_{=\Delta\theta}\Big)\Delta s$$

$$\frac{1}{\Delta n}\frac{\partial\Delta n}{\partial s} = \frac{\partial\theta}{\partial n} \qquad (2)$$

can be diverging or converging (compressible)

Euler s-equation





s-momentum out – s-momentum in = force applied in s direction

$$\dot{m}\left[u+\frac{\partial u}{\partial s}\Delta s\right]-\dot{m}u=-\left(\frac{\partial p}{\partial s}\Delta s\right)\Delta n$$

(highest order term only on RHS)

$$\dot{m}=\rho u \Delta n$$

sub in

becomes

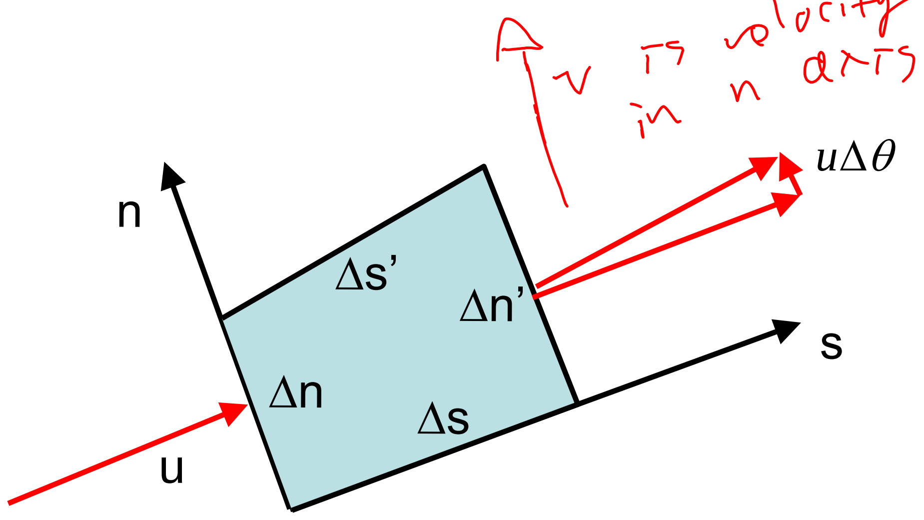
hence

$$\boxed{\rho u\frac{\partial u}{\partial s}=-\frac{\partial p}{\partial s}}$$

euler equation

which integrates for Bernoulli's equation for constant density

Vorticity ω



Cartesian co-ordinate form

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

conversion

Here, if x,y are originally aligned with s,n we have dv=udθ

$$\omega = u\frac{\partial \theta}{\partial s} - \frac{\partial u}{\partial n}$$

Mass conservation

$$\dot{m} = \rho u \Delta n = \left(\rho + \frac{\partial \rho}{\partial s}\Delta s\right)\left(u + \frac{\partial u}{\partial s}\Delta s\right)\left(\Delta n + \frac{\partial \Delta n}{\partial s}\Delta s\right)$$

$$= \rho u \Delta n + \rho u \frac{\partial \Delta n}{\partial s}\Delta s + \rho \Delta n \frac{\partial u}{\partial s}\Delta s + u \Delta n \frac{\partial \rho}{\partial s}\Delta s$$

$\downarrow \div \rho u \Delta n$ and rearrange

$$0 = \frac{1}{\Delta n}\frac{\partial \Delta n}{\partial s} + \frac{1}{u}\frac{\partial u}{\partial s} + \frac{1}{\rho}\frac{\partial \rho}{\partial s}$$

Finally, using (2)

prev equation

$$\boxed{\frac{\partial \theta}{\partial n} + \frac{1}{u}\frac{\partial u}{\partial s} + \frac{1}{\rho}\frac{\partial \rho}{\partial s} = 0}$$

Combine mass conservation and Euler-s equation

Euler s-equation

$$\rho u \frac{\partial u}{\partial s} = -\frac{\partial p}{\partial s}$$

Homentropic flow

$$\rho u \frac{\partial u}{\partial s} = -\underbrace{\left(\frac{\partial p}{\partial \rho}\right)_s}_{} \frac{\partial \rho}{\partial s}$$

constant entropy pressure-density gradient

$$\rho u \frac{\partial u}{\partial s} = -a^2 \frac{\partial \rho}{\partial s}$$

Continuity

$$\frac{\partial \theta}{\partial n} + \frac{1}{u}\frac{\partial u}{\partial s} + \frac{1}{\rho}\frac{\partial \rho}{\partial s} = 0$$

$$\frac{\partial \rho}{\partial s} = -\rho\left(\frac{\partial \theta}{\partial n} + \frac{1}{u}\frac{\partial u}{\partial s}\right)$$

substitute

$$u\frac{\partial u}{\partial s} = a^2\left(\frac{\partial \theta}{\partial n} + \frac{1}{u}\frac{\partial u}{\partial s}\right)$$

just re-arrange

$$\left(\frac{u^2}{a^2} - 1\right)\frac{1}{u}\frac{\partial u}{\partial s} = \frac{\partial \theta}{\partial n}$$

mach number

2 equations in 2 unknowns

combine

Recalling that $\tan\mu = \dfrac{1}{\sqrt{M^2-1}}$ and $\tan\mu\,\mathrm{d}\nu = \dfrac{\mathrm{d}u}{u}$

expansion fan proof

From Mach triangle

mach angle
doppler type stuff → μ is mach cone thingy

from Prandtl-Meyer
(lecture 2.4, slide 4): $\dfrac{\mathrm{d}V}{V} = \dfrac{\mathrm{d}\theta_{PM}}{\sqrt{M^2-1}} = \tan\mu\,\mathrm{d}\nu$

$$\left(M^2-1\right)\frac{1}{u}\frac{\partial u}{\partial s} = \frac{\partial\theta}{\partial n}$$ becomes $$\frac{1}{\tan\mu}\frac{\partial\nu}{\partial s} = \frac{\partial\theta}{\partial n}$$

mach triangle

Notes!

We already have another equation relating u and θ from the definition of vorticity

$$\omega = u\frac{\partial\theta}{\partial s} - \frac{\partial u}{\partial n} = 0$$ (irrotational flow)

hence $\dfrac{\partial\nu}{\partial s} - \tan\mu\dfrac{\partial\theta}{\partial n} = 0$ and $\dfrac{\partial\theta}{\partial s} - \tan\mu\dfrac{\partial\nu}{\partial n} = 0$ Set of hyperbolic equations for u,ν

Homentropic flow in (θ,ν) variables

diagonalising equations?

$$\frac{\partial \nu}{\partial s} - \tan\mu \frac{\partial \theta}{\partial n} = 0$$

$$\frac{\partial \theta}{\partial s} - \tan\mu \frac{\partial \nu}{\partial n} = 0$$

In matrix form: $$\frac{\partial}{\partial s}\begin{pmatrix} \nu \\ \theta \end{pmatrix} + \begin{pmatrix} 0 & -\tan\mu \\ -\tan\mu & 0 \end{pmatrix} \frac{\partial}{\partial n}\begin{pmatrix} \nu \\ \theta \end{pmatrix} = 0$$

Q is a single vector

$$\frac{\partial \mathbf{Q}}{\partial s} + \mathbf{A}\frac{\partial \mathbf{Q}}{\partial n} = 0$$

The character of the solution will be determined by the properties of A

Eigenvalues of **A**

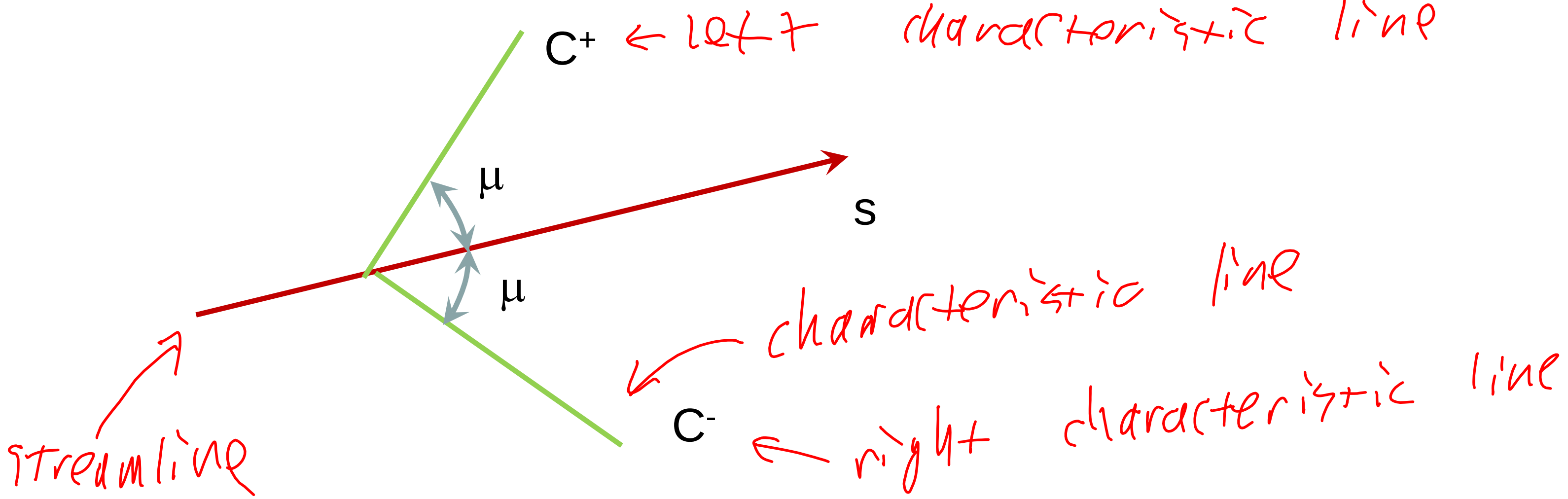
$$\det(\mathbf{A} - \mathbf{I}\lambda) = 0$$

solve determinant (easy)

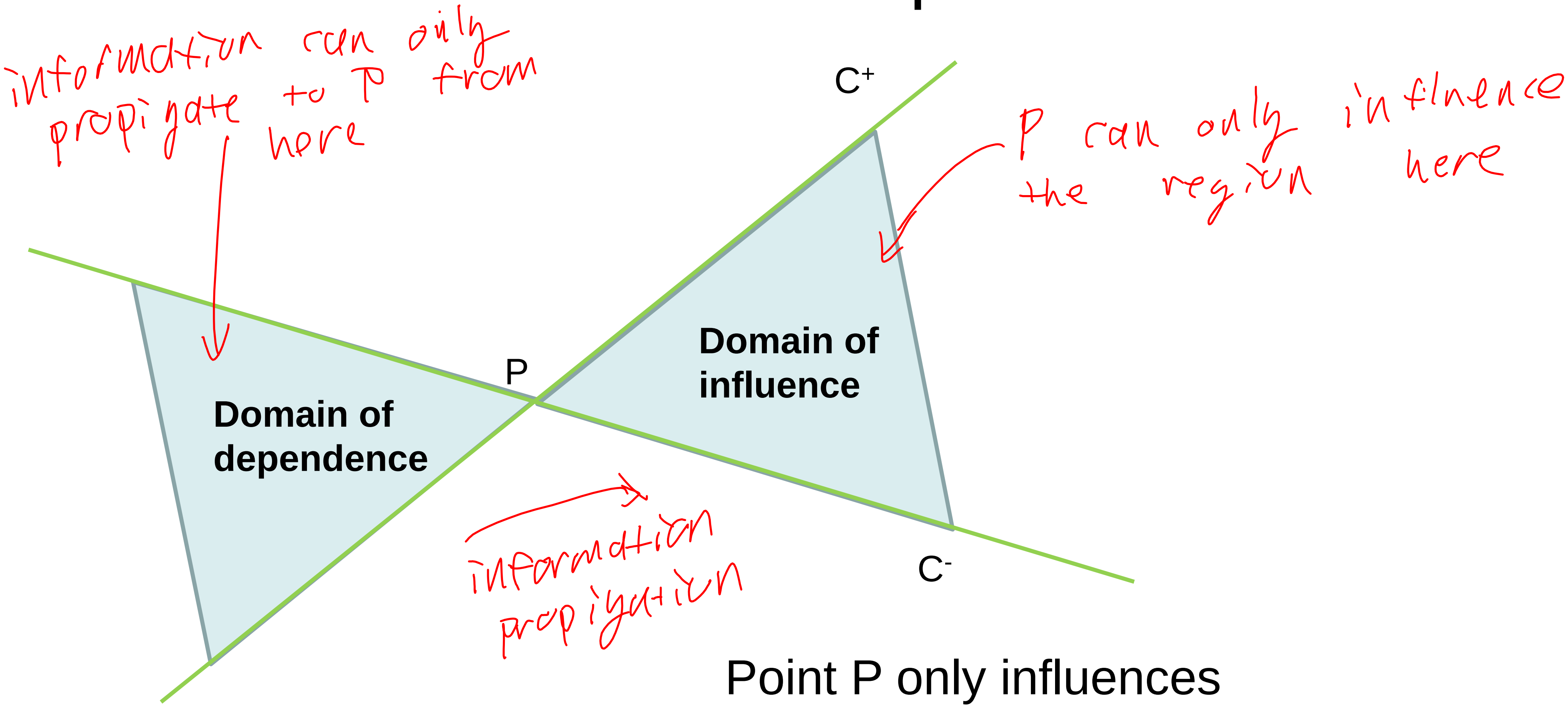
$$\begin{vmatrix} -\lambda & -\tan\mu \\ -\tan\mu & -\lambda \end{vmatrix} = 0$$

$$\lambda = \pm\tan\mu$$

i.e. the slopes of the Mach lines are the eigenvalues



Domains of influence and dependence



Flow state at P only depends on events in the domain of dependence

Point P only influences events in the domain of influence

Summary

- Homentropic irrotational steady supersonic flow has an underlying structure governed by its eigenvalues
- Characteristics are oriented at the Mach angle relative to the flow direction
- Next time: exploit this to form the Method of Characteristics (MoC)