MATH2048 Mathematics for Engineering and the Environment

FORMULA SHEET FS/MATH2048/2021

FOURIER SERIES

1. If $f(x)$ is a bounded periodic function of period 2ℓ which in any one period has at most a finite number of local maxima and minima and a finite number of points of discontinuity (ie satisfies the Dirichlet conditions) then the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

where

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right) dx \quad \text{and} \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right) dx$$

converges to $f(x)$ at all points where $f(x)$ is continuous and converges to the average of the right- and left- hand limits of $f(x)$ at each point where $f(x)$ is discontinuous.

2. The Fourier series of a periodic function $f(x)$ of period 2ℓ can be differentiated term by term to give the Fourier series of $f'(x)$, provided $f(x)$ is **everywhere** continuous, $f'(x)$ satisfies the Dirichlet conditions (and hence possesses a Fourier series) and $f(-\ell) = f(\ell)$.

3. The Fourier series of any periodic function $f(x)$ may always be integrated term by term to give the Fourier series of the integral of f(x).

FOURIER TRANSFORM

The *Fourier transform* of a function $f(t)$ is given by

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

The *inverse Fourier transform* is then given by

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

PARTIAL DIFFERENTIAL EQUATIONS

Second order linear homogeneous partial differential equations in two independent variables are of the form

$$a\frac{\partial^2 u}{\partial x^2} + 2h\frac{\partial^2 u}{\partial x \partial y} + b\frac{\partial^2 u}{\partial y^2} + 2f\frac{\partial u}{\partial x} + 2g\frac{\partial u}{\partial y} + cu = 0$$

where a, h, b, f, g and c are functions of x and y. The equation is

- i elliptic if $h^2 - ab < 0$
- ii parabolic if $h^2 - ab = 0$
- iii hyperbolic if $h^2 - ab > 0$

In the method of **separation of variables** we seek a solution of the form

$$u(x, y) = X(x)Y(y)$$

and if the equation can be written in the form

$$\frac{1}{X} f\left(x, \frac{dX}{dx}, \frac{d^2X}{dx^2}\right) = \frac{1}{Y} g\left(y, \frac{dY}{dy}, \frac{d^2Y}{dy^2}\right)$$

the equation is said to be **separable**.

LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}[f(t)] \equiv \tilde{f}(s)$
A	$\dfrac{A}{s}\,, \quad \text{Re}(s) > 0$
e^{at}	$\dfrac{1}{s-a}\,, \quad \text{Re}(s) > a$
$t^n, \quad n = 1, 2 \ldots$	$\dfrac{n!}{s^{n+1}}\,, \quad \text{Re}(s) > 0$
$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}\,, \quad \text{Re}(s) > 0$
$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}\,, \quad \text{Re}(s) > 0$
$\sinh \omega t$	$\dfrac{\omega}{s^2 - \omega^2}\,, \quad \text{Re}(s) > \lvert\omega\rvert$
$\cosh \omega t$	$\dfrac{s}{s^2 - \omega^2}\,, \quad \text{Re}(s) > \lvert\omega\rvert$
$t^n f(t)$	$(-1)^n \dfrac{d^n \tilde{f}}{ds^n}$
$\dfrac{df}{dt}$	$s\tilde{f}(s) - f(0)$
$\dfrac{d^2 f}{dt^2}$	$s^2 \tilde{f}(s) - sf(0) - \dfrac{df}{dt}(0)$
$H(t-a)$	$\dfrac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}
$e^{-at} f(t)$	$\tilde{f}(s+a)$
$f(t-a)H(t-a)$	$e^{-as} \tilde{f}(s)$

In the above, H and δ are defined on page 4.

VECTOR DIFFERENTIAL OPERATORS

Cartesian, Cylindrical and Spherical Co-ordinate Systems

Co-ordinates and unit vectors:

Cartesian	Cylindrical	Spherical
x, y, z $\mathbf{i},\ \mathbf{j},\ \mathbf{k}$	R, ϕ, z $\hat{\mathbf{R}},\ \hat{\phi},\ \mathbf{k}$	r, θ, ϕ $\hat{\mathbf{r}},\ \hat{\theta},\ \hat{\phi}$

Relations between them

$$x = R\cos\phi = r\sin\theta\cos\phi$$

$$y = R\sin\phi = r\sin\theta\sin\phi$$

$$z = z = r\cos\theta$$

$$\hat{\mathbf{R}} = \cos\phi\ \mathbf{i} + \sin\phi\ \mathbf{j}$$

$$\hat{\phi} = \cos\phi\ \mathbf{j} - \sin\phi\ \mathbf{i} = \frac{d\hat{\mathbf{R}}}{d\phi}$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\ \mathbf{i} + \sin\theta\sin\phi\ \mathbf{j} + \cos\theta\ \mathbf{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\ \mathbf{i} + \cos\theta\sin\phi\ \mathbf{j} - \sin\theta\ \mathbf{k} = \frac{\partial\hat{\mathbf{r}}}{\partial\theta}$$

Parametric equation of a curve

$$\mathbf{r} = \mathbf{r}(t)$$

Line element

$$\mathbf{dr} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

$$\mathbf{dr} = \frac{d\mathbf{r}}{dt}dt$$

Gradient, Divergence and Curl

'Del' operator $\nabla = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}$

$$\nabla\phi = \text{grad}\ \phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\cdot\mathbf{a} = \text{div}\ \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\nabla\times\mathbf{a} = \text{curl}\ \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

If $\nabla\cdot\mathbf{a} = 0$, $\mathbf{a}$ is 'solenoidal'

If $\nabla\times\mathbf{a} = 0$, $\mathbf{a}$ is 'irrotational'

$\nabla-$ identities:

$$
\begin{aligned}
\nabla \cdot \nabla \times \mathbf{a} &= \mathrm{div}(\mathrm{curl}\ \mathbf{a}) = 0 \\
\nabla \times \nabla \phi &= \mathrm{curl}(\mathrm{grad}\ \phi) = \mathbf{0} \\
\nabla \cdot (\phi \mathbf{a}) &= \mathrm{div}(\phi \mathbf{a}) = \phi \nabla \cdot \mathbf{a} + (\nabla \phi) \cdot \mathbf{a} = \phi(\mathrm{div}\ \mathbf{a}) + (\mathrm{grad}\ \phi) \cdot \mathbf{a} \\
\nabla \times (\phi \mathbf{a}) &= \mathrm{curl}(\phi \mathbf{a}) = \phi \nabla \times \mathbf{a} + (\nabla \phi) \times \mathbf{a} \\
&= \phi\ \mathrm{curl}\ \mathbf{a} + (\mathrm{grad}\ \phi) \times \mathbf{a} \\
\nabla \times (\nabla \times \mathbf{a}) &= \mathrm{curl}(\mathrm{curl}\ \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} = \mathrm{grad}(\mathrm{div}\ \mathbf{a}) - \nabla^2 \mathbf{a} \\
\nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \mathrm{div}(\mathbf{a} \times \mathbf{b}) = (\nabla \times \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \\
&= (\mathrm{curl}\ \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\mathrm{curl}\ \mathbf{b}) \\
\nabla \times (\mathbf{a} \times \mathbf{b}) &= \mathrm{curl}(\mathbf{a} \times \mathbf{b}) \\
&= \mathbf{a}(\nabla \cdot \mathbf{b}) - (\nabla \cdot \mathbf{a})\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} \\
&= \mathbf{a}(\mathrm{div}\ \mathbf{b}) - (\mathrm{div}\ \mathbf{a})\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} \\
\nabla(\mathbf{a} \cdot \mathbf{b}) &= \mathrm{grad}(\mathbf{a} \cdot \mathbf{b}) \\
&= (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \\
&= (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times \mathrm{curl}\ \mathbf{b} + \mathbf{b} \times \mathrm{curl}\ \mathbf{a}
\end{aligned}
$$

Conservative Force Theorem

The following statements are equivalent:

(i) $\mathbf{a}$ is a conservative

(ii) the line integral of $\mathbf{a}$ between any two points is independent of the path taken

(iii) $\oint_C \mathbf{a} \cdot \mathbf{dr} = 0$ for any closed curve C

(iv) $\mathbf{a} = \mathrm{grad}\ \phi$ for some ϕ defined up to an additive constant

(v) $\mathrm{curl}\ \mathbf{a} = \mathbf{0}$

Divergence theorem

$$\int_V \nabla \cdot \mathbf{F} dV = \int_S \mathbf{n} \cdot \mathbf{F} dS$$

where $\mathbf{n}$ is the outward vector normal to the surface S which encloses the volume V.

The Heaviside and Delta functions:

The Heaviside function is defined as:

$$H(x-a) = \begin{cases} 0, & x < a \\ 1, & x > a \end{cases}$$

The $\delta-$function has the following properties

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a); \\
&\int_p^q \delta(x-a) f(x) dx = f(a) \quad \text{if } 0 \in (p,q); \qquad \int_p^q \delta(x-a) f(x) dx = 0 \quad \text{if } 0 \notin (p,q).
\end{aligned}
$$

and is sometimes loosely written as

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0. \end{cases}$$

STATISTICS

1. Statistical Tables

For a standard normal ($\mu = 0, \sigma = 1$) random variable Z, the distribution function $F(z) = P(Z \leq z)$ is given below for various values of $z \geq 0$.

2nd decimal place of z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

The following Table gives values of the percentiles $t_{k,p}$ where k represents the degrees of freedom, and $p = P(t_k \geq t_{k,p})$.

df (k)	$\mathbf{t_{k,0.1}}$	$\mathbf{t_{k,0.05}}$	$\mathbf{t_{k,0.025}}$	$\mathbf{t_{k,0.01}}$	$\mathbf{t_{k,0.005}}$
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
35	1.306	1.690	2.030	2.438	2.724
40	1.303	1.684	2.021	2.423	2.704
50	1.299	1.676	2.009	2.403	2.678

2. Probability densities and cumulative distribution functions

The Normal distribution with mean μ and standard deviation σ has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

for $-\infty < x < \infty$.

The Weibull distribution has probability density function

$$f(x) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} \exp(-(x/\beta)^\alpha)$$

and cumulative distribution function

$$F(x) = 1 - \exp(-(x/\beta)^\alpha)$$

for $x > 0$.

The Exponential distribution is a Weibull distribution where $\alpha = 1$.

The Largest Extreme Value distribution has probability density function

$$f(x) = \frac{1}{\beta} \exp\left(-\frac{x-\alpha}{\beta} - \exp\left(-\frac{x-\alpha}{\beta}\right)\right)$$

and cumulative distribution function

$$F(x) = \exp\left(-\exp\left(-\frac{x-\alpha}{\beta}\right)\right)$$

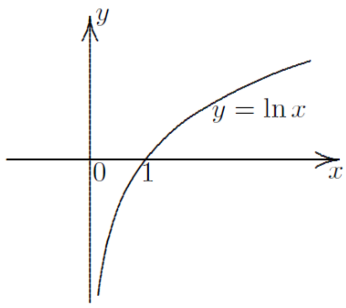
for $-\infty < x < \infty$.

OTHER USEFUL FORMULAE

1. Logarithm and exponential function

$$a^x = e^{x \ln a}$$

The graph of $y = \ln x$ is



2. Trigonometry and hyperbolic functions

Trigonometric identities

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \ \pm \ \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \ \mp \ \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \ \sin\left(\frac{A+B}{2}\right) \ \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \ \cos\left(\frac{A+B}{2}\right) \ \sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2 \ \cos\left(\frac{A+B}{2}\right) \ \cos\left(\frac{A-B}{2}\right)$$

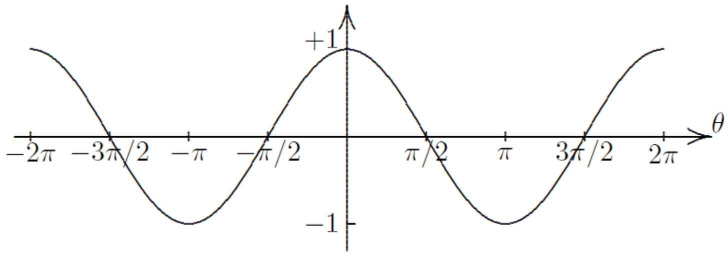
$$\cos A - \cos B = -2 \ \sin\left(\frac{A+B}{2}\right) \ \sin\left(\frac{A-B}{2}\right)$$

$$2 \sin A \cos B = \sin(A-B) + \sin(A+B)$$

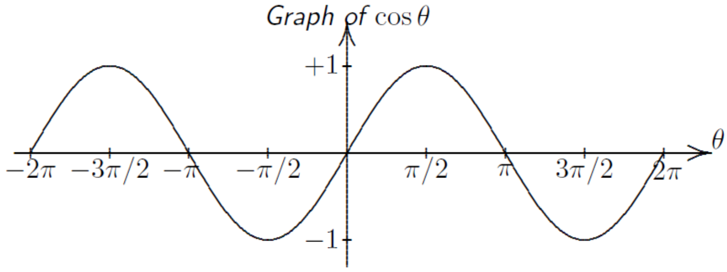
$$2 \cos A \cos B = \cos(A-B) + \cos(A+B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

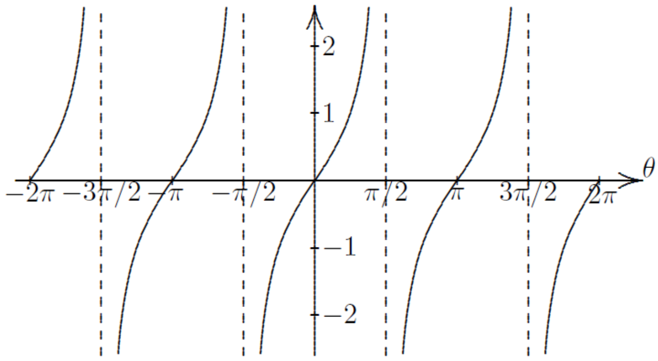
The graphs of the three elementary trigonometric functions between -2π and 2π are:



Graph of $\cos\theta$

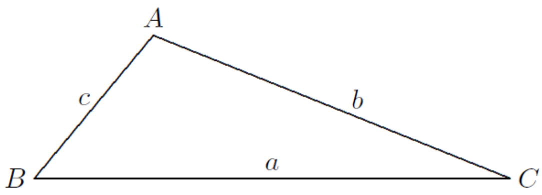


Graph of $\sin\theta$



Graph of $\tan\theta$

Relationships for Plane Triangle



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

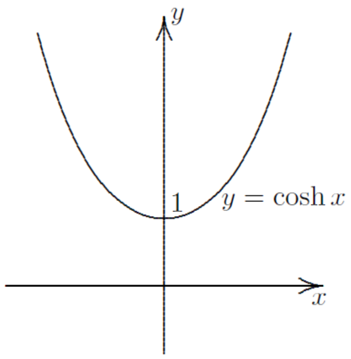
$$a^2 = b^2 + c^2 - 2bc\cos A, \qquad b^2 = c^2 + a^2 - 2ca\cos B, \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

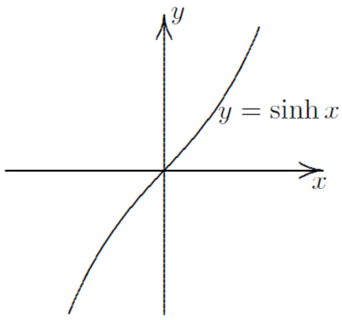
Hyperbolic functions

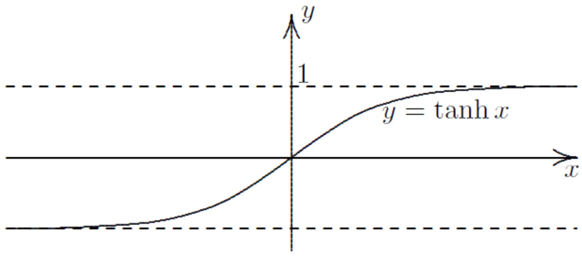
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right), \qquad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right).$$

$$\cosh^2 x - \sinh^2 x = 1$$

The graphs of $\sinh x$, $\cosh x$ and $\tanh x$ are







Formulae involving the hyperbolic functions may be obtained from the corresponding trigonometric ones on the previous page by replacing every product of sines and implied sines by minus the corresponding product of hyperbolic sines. Thus, for instance, $\tan A \tan B$ is an implied product of sines since it can be written $(\sin A \sin B)/(\cos A \cos B)$ and so, to obtain the corresponding formula, terms in $\tan A \tan B$ would be replaced by $-\tanh A \tanh B$.

3. Differentiation

function	*derivative*
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$

4. Complex numbers

Euler's formula

$$e^{\pm j\theta} = \cos\theta \ \pm \ j \ \sin\theta.$$

5. Vectors

Triple vector product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

6. Matrices

The inverse of a matrix $\mathbf{A}$ can be expressed

$$\mathbf{A^{-1}} = \frac{\mathrm{adj}\mathbf{A}}{|\mathbf{A}|}, \quad \text{where adj}\mathbf{A} \text{ is the transpose of the matrix of cofactors of } \mathbf{A}.$$

7. Newton Raphson iteration formula

If x_n is an approximation to a root of $f(x) = 0$ then a better approximation is usually

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

8. Approximate integration

In the following approximations $f_r = f(x_r)$, where $x_r = x_0 + rh$.

1. Trapezium rule

$$\int_{x_0}^{x_n} f(x)\,dx \approx h\left(\frac{1}{2}f_0 + f_1 + f_2 + \;\ldots\; + f_{n-1} + \frac{1}{2}f_n\right)$$

2. Simpson's rule (in which n must be even)

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{1}{3}h\Big(f_0 + 4\,(f_1 + f_3 + \;\ldots\; + f_{n-1}) + 2\,(f_2 + f_4 + \;\ldots\; + f_{n-2}) + f_n\Big)$$

9. Power series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \binom{n}{r}x^r + \ldots,$$

where

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\,.$$

If n is a positive integer the above series terminates and is convergent for all x.

If n is not a positive integer the series is infinite and converges for $|x| < 1$.

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad \text{for } -1 < x \le 1$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \qquad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{for all } x$$

Taylor's theorem

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x)\,,$$

where

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(c), \qquad a < c < x$$

End of Formula Sheet