Outline

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Lecture 19 - Inhomogeneous boundary conditions

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Outline

1. Review

2. Inhomogeneous boundary conditions
   - Simplest case: inhomogeneous BCs are constants
   - General strategy: inhomogeneous BCs are functions
   - Example

3. Summary

$\rightarrow$ Review

- Previously, we solved **homogeneous** and **<u>in</u>homogeneous parabolic PDEs**
- **... with <u>simple</u> (i.e. homogeneous) boundary conditions**
- but there are parabolic PDE problems with **<u>more elaborated</u> boundary conditions (i.e. <u>in</u>homogeneous).**

  How do we solve them ? ($\longleftarrow$ Today's Lecture)
- Don't get confused: the PDE and/or the BCs can be ***<u>independently</u>*** homogeneous or inhomogeneous.

Outline

→ Inhomogeneous boundary conditions

The (homogeneous) **heat equation**

$$\frac{\partial y}{\partial t} = \kappa^2 \frac{\partial^2 y}{\partial x^2}$$

is straightforward to solve when **simple boundary conditions (BCs)** are imposed. **But** the problem with **inhomogeneous BCs**:

$$\frac{\partial y}{\partial t} = \kappa^2 \frac{\partial^2 y}{\partial x^2}\,; \qquad y(0,t) = f_0(t), \quad y(1,t) = f_1(t)$$

Inhomogeneous BCs means RHS of BC is not 0

for known $f_0(t), f_1(t)$ is **much harder to solve.**

Simplest case: inhomogeneous BCs are constants

- Simplest problem with **inhomogeneous boundary conditions (BCs)** is:

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}; \qquad y(0,t) = \boldsymbol{T_0}, \quad y(1,t) = \boldsymbol{T_1}.$$

Inhomogeneous BCs means RHS of BC is not 0 ↗ ↗

where T_0 and T_1 are **constants** (this is why it is the simplest case).

- Our **standard separation of variables *ansatz*** is often of not much use in these cases...

- Instead, consider the **steady state** (i.e. $\boldsymbol{\partial_t y = 0}$) $\Rightarrow$ PDE simplifies:

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial^2 y(x,t)}{\partial x^2} \Leftrightarrow 0 = \frac{d^2 y(x)}{dx^2} \quad \Longrightarrow \quad y(x) = (T_1 - T_0)x + T_0 \equiv y_P(x)$$

This is a ***particular solution*** $y_P(x)$ of the problem.

Using our particular solution

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \qquad y_P = (T_1 - T_0)x + T_0.$$

If we **define:** [$\searrow$ we can always “replace" an unknown function $y(x,t)$ by another unknown function $v(x,t)$!!]

$$y(x,t) = v(x,t) + y_P(x) \tag{1}$$

$$\Rightarrow \begin{cases} \frac{\partial y}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial y_P}{\partial t} = \frac{\partial v}{\partial t} \\ \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 y_P}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} \end{cases} \qquad \begin{cases} \boldsymbol{y(0,t) = T_0} \Leftrightarrow v(0,t) + y_P(0) = T_0 \Leftrightarrow v(0,t) + \cancel{T_0} = \cancel{T_0} \Leftrightarrow \boldsymbol{v(0,t) = 0}, \\ \boldsymbol{y(1,t) = T_1} \Leftrightarrow v(1,t) + y_P(1) = T_1 \Leftrightarrow v(1,t) + \cancel{T_1} = \cancel{T_1} \Leftrightarrow \boldsymbol{v(1,t) = 0} \end{cases}$$

then the **new auxiliary variable** $v(x,t)$ obeys the (homogeneous) heat equation and **homogeneous Dirichlet** boundary conditions,

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}\,; \qquad v(0,t) = 0, \quad v(1,t) = 0.$$

So we can solve for $v(x,t)$ as we have been doing!

... and then insert it into (1) to find $y(x,t)$.

General strategy: inhomogeneous BCs are functions

$$\frac{\partial y}{\partial t} = \kappa^2 \frac{\partial^2 y}{\partial x^2}\,; \qquad y(0,t) = f_0(t), \quad y(1,t) = f_1(t)$$

In general we cannot find a particular solution. However, we can consider the **(time-*dependent*, spatially linear) "solution"**

$$y_P(x,t) = \Big[f_1(t) - f_0(t)\Big]x + f_0(t). \qquad \longleftarrow \text{does not need to be real solution!}$$

If we again set $\boldsymbol{y(x,t) = v(x,t) + y_P(x,t)}$ then $v(x,t)$ satisfies **homogeneous Dirichlet** boundary conditions. But this time (unlike previous simple case), it satisfies the ***inhomogeneous* heat equation**

$$\frac{\partial v}{\partial t} = \kappa^2 \frac{\partial^2 v}{\partial x^2} \underbrace{- \Big[\dot{f}_1(t) - \dot{f}_0(t)\Big]x - \dot{f}_0(t)}_{\text{Forcing term } F(x,t) \text{ of Lecture 18}}\,.$$

which is not so easy to solve...but we can still solve it using the techniques of the last Lecture 18 to solve for $v(x,t)$ and then $y(x,t)$.

Example

Consider the **(homogeneous) diffusion equation**
with **inhomogeneous BCs** and initial data given by:

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}; \quad y(0,t) = \overbrace{\tfrac{1}{2}\left[1-\cos(t)\right]}^{f_0(t)}, \quad y(1,t) = \overbrace{\ 0\ }^{f_1(t)}; \qquad y(x,0) = 0.$$

We have: $\swarrow$ Previous slide: $y_P(x,t) = \Big[f_1(t) - f_0(t)\Big]x + f_0(t)$

$$y_P = \frac{1}{2}(1-x)\left[1-\cos(t)\right] \quad \Rightarrow \quad \dot{y}_P = \frac{1}{2}(1-x)\sin(t),$$

Change of variable, $\boldsymbol{y(x,t) = v(x,t) + y_P(x,t)}$ $\Rightarrow$ v satisfies:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \underbrace{-\frac{1}{2}(1-x)\sin(t)}_{\text{Forcing term } F(x,t)}; \quad \underbrace{v(0,t) = 0, \quad v(1,t) = 0}_{\text{Homogeneous (Dirichlet) BCs}}; \quad v(x,0) = 0.$$

Solving the problem

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \underbrace{-\frac{1}{2}(1-x)\sin(t)}_{\text{Forcing term } F(x,t)}; \quad \underbrace{v(0,t)=0, \quad v(1,t)=0}_{\text{Homogeneous (Dirichlet) BCs}}; \quad v(x,0)=0.$$

The *educated guess* is: [↙ From slide 8 of Lecture 18. Review it!]

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x); \qquad \overbrace{-\frac{1}{2}(1-x)\sin(t)}^{F(x,t)} = \sum_{n=1}^{\infty} F_n(t)\sin(n\pi x).$$

FS in x, ie $\sin(t)$ is "to be interpreted as a constant" here

This is the sine Fourier series (FS) in x of $F(x,t) = -\frac{1}{2}(1-x)\sin(t)$ (i.e. the FS of the odd extension of $F(x,t)$ with period 2ℓ, $\ell = 1$). So, apply the associated **Euler formulae** to find $F_n(t)$ [revisit slide 11 of Lecture 6 with $b_n \equiv F_n(t)$]:

$$F_n(t) = \frac{2}{\ell}\int_0^{\ell} F(x,t)\sin(n\pi x)\,dx = 2\int_0^1 \left[\frac{-1}{2}(1-x)\sin(t)\right]\sin(n\pi x)\,dx = \cdots = -\frac{\sin(t)}{n\pi}$$

Equation for $T_n(t)$ is (**1**) of slide 11 Lecture 18 (Review it!): $\boldsymbol{\dot{T}_n + (n\pi)^2 T_n = -\frac{\sin(t)}{n\pi}}$.

Solving for $v(x,t)$

The ODE for $T_n(t)$ is solved reviewing slide 11 of Lecture 18:

$$\dot{T}_n + (n\pi)^2 T_n = -\frac{\sin(t)}{n\pi} \qquad \swarrow \; (\star)\; T_n(t) = e^{-(\kappa\, n\,\pi)^2 t}\left[C_n + \int e^{+(\kappa\, n\,\pi)^2 t} F_n(t)\,\mathrm{d}t\right] \quad \swarrow \; \kappa = 1$$

whose solution is $(\star)$ of slide 11 Lecture 18 (Review it!)

$$T_n = C_n\, e^{-(n\pi)^2 t} + \frac{\cos(t) - (n\pi)^2\sin(t)}{(n\pi)\left[1 + (n\pi)^4\right]} \longleftarrow \begin{cases} e^{-(n\pi)^2 t}\int e^{+(n\pi)^2 t} F_n(t)\,\mathrm{d}t \\ = e^{-(n\pi)^2 t}\int \underbrace{e^{+(n\pi)^2 t}}_{u}\underbrace{\left(-\frac{\sin(t)}{n\pi}\right)\mathrm{d}t}_{dv} \\ = \frac{\cos(t)-(n\pi)^2\sin(t)}{(n\pi)\left[1+(n\pi)^4\right]} \end{cases}$$

[$\nwarrow$ Exercise: check it!]

giving the general solution for $v(x,t)$:

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$$

$$v(x,t) = \sum_{n=1}^{\infty}\left[C_n\, e^{-(n\pi)^2 t} + \frac{\cos(t) - (n\pi)^2\sin(t)}{(n\pi)\left(1 + (n\pi)^4\right)}\right]\sin(n\pi x)\,. \qquad (2)$$

$C_n = ??$ $\longrightarrow$ The **initial data** gives (after evaluating (2) at $t = 0$):

$$v(x,0) = 0 \;\Leftrightarrow\; \sum_{n=1}^{\infty}\left[C_n + \frac{1-0}{(n\pi)\left(1+(n\pi)^4\right)}\right]\sin(n\pi x) = 0 \;\Leftrightarrow\; C_n = -\frac{1}{(n\pi)\left[1+(n\pi)^4\right]}.$$

General solution for $y(x, t)$

Putting together the solution for $v(x, t)$ (including the values for C_n) and $y_P(x, t)$ one gets, from slide 9, the solution for original function $y(x, t)$:

$$
\begin{aligned}
y(x,t) &= v(x,t) + y_P(x,t) \\
y(x,t) &= v(x,t) + \frac{1}{2}(1-x)\left[1-\cos(t)\right] \\
y(x,t) &= \sum_{n=1}^{\infty}\left[\cos(t) - (n\pi)^2\sin(t) - e^{-(n\pi)^2 t}\right]\frac{\sin(n\pi x)}{(n\pi)\left[1+(n\pi)^4\right]} \\
&\quad + \frac{1}{2}(1-x)\left[1-\cos(t)\right].
\end{aligned}
$$

In this case the **inhomogeneous boundary condition dominates the behaviour** (sine & cosines of t contributions);
the transient behaviour (ie the exp decay contribution $e^{-(n\pi)^2 t}$ from homogeneous PDE with homogeneous BCs) is comparatively small.

Outline

Summary

- Separation of variables does not work if **inhomogeneous boundary conditions** are imposed on the PDE.
- We can however use a **particular "solution"** y_P that is compatible with the boundary conditions. It does ***not*** need to be a genuine solution of the equation!
- By **changing variable** $y \to v$ we recover a problem with **homogeneous boundary conditions**, but it **may** add an **inhomogeneous source term.**
- This problem for v can be solved using the techniques of the previous lecture 18; from that and y_P we find the original solution $y(x, t)$.

Exercise: Homework

1) Find/check, using Euler formula, the Fourier coefficients of slide 10:

$$
\begin{aligned}
F_n(t) &= 2\int_0^1 \left[-\frac{1}{2}(1-x)\sin(t)\right]\sin(n\pi x)\,\mathrm{d}x \\
&= -\sin(t)\int_0^1 \overbrace{(1-x)}^{u}\,\overbrace{\sin(n\pi x)\,\mathrm{d}x}^{dv} \\
&= -\sin(t)\left\{\left[-(1-x)\frac{\cos(n\pi x)}{n\pi}\right]_0^1 - \int_0^1 \frac{\cos(n\pi x)}{n\pi}\,\mathrm{d}x\right\} \\
&= -\sin(t)\left[-(1-x)\frac{\cos(n\pi x)}{n\pi} - \frac{\sin(n\pi x)}{(n\pi)^2}\right]_0^1 \\
&= -\frac{\sin(t)}{n\pi}.
\end{aligned}
$$

2) Use integration by parts to obtain the integral of slide 11