SESA2023 Week 3: Gas dynamics

This week we will look at the basics of compressible flow and gas dynamics, where fluid dynamics is coupled to thermodynamics through the (ideal) gas law.

3.1 Learning outcomes

After completing this section you should be able to:

- Explain what the difficulty is to solve compressible flows
- Explain what role the gas law plays in the conservation laws for compressible flow
- Calculate fluid properties along a streamline for compressible flow
- Calculate the stagnation properties for compressible flow
- Calculate the speed of sound of a gas in a given state
- Calculate the Mach number for a given flow condition
- Calculate stagnation and critical properties of a gas at a given Mach number
- Explain what a shock wave is
- Calculate the change in fluid properties across a normal shock
- Explain why flow through a normal shock always goes from supersonic to subsonic
- Explain why a converging nozzle cannot be used to generate supersonic flows
- Explain what choked flow is and why it happens
- Calculate the exit plane properties for a converging nozzle for a given back pressure
- Explain how a converging-diverging nozzle can be used to generate supersonic flows
- Recognize the different regimes of flow through a converging-diverging nozzles and explain how they are formed
- Calculate the mass flow rate through a nozzle given local properties and under choked conditions

3.2 Compressible flow

In this section we will start with identifying the difference between compressible and incompressible flow. We will see that compressibility adds complexity due to the coupling of density to temperature and pressure in the ideal gas law. We will then simplify the compressible equations by looking at steady one-dimensional, inviscid flow without thermal diffusion.

3.2.1 Compressible mass, momentum, and energy conservation

We start by looking at the full set of equations required to describe three-dimensional unsteady flow. We don't have to worry about directly using these equations, but we will use them to arrive at simplified versions that we can more easily use. We have mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j) = 0, \tag{3.1}$$

momentum conservation:

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_j}(\rho U_i U_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{3.2}$$

and energy conservation:

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x_i}(\rho h U_i) = \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + \frac{\mathrm{D}P}{\mathrm{D}t} + \mu\Phi \tag{3.3}$$

where the enthalpy h is a function of only the temperature assuming an ideal gas. These three (sets of) equations are coupled by the ideal gas law

$$P = \rho R T \tag{3.4}$$

For incompressible flow, density, pressure and temperature are not coupled, which means that flows can be solved by considering only the mass and momentum conservation equations. It is the coupling of the equations that makes solving compressible flows more complicated.

3.2.2 Simplified 1D steady compressible flow

We will simplify the compressible equations by considering the following assumptions:

- One-dimensional flow (taking only the x-direction)
- Steady flow (no time dependence, $\frac{\partial}{\partial t} = 0$)
- Inviscid flow ($\frac{\partial \tau_{ij}}{\partial x_j} = 0$)
- No thermal diffusion ($k\frac{\partial T}{\partial x_i} = 0$)
- No viscous heating ($\mu\Phi = 0$)

Resulting in the following equation for mass conservation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho U) = 0 \tag{3.5}$$

The momentum conservation equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho U^2) = -\frac{\mathrm{d}P}{\mathrm{d}x}, \tag{3.6}$$

which can be expanded as

$$\rho U \frac{\mathrm{d}}{\mathrm{d}x}(U) + U\frac{\mathrm{d}}{\mathrm{d}x}(\rho U) = -\frac{\mathrm{d}P}{\mathrm{d}x}. \tag{3.7}$$

From mass conservation, the second term on the left hand side equals zero, so the momentum equation further reduces to

$$\rho U \frac{\mathrm{d}U}{\mathrm{d}x} = -\frac{\mathrm{d}P}{\mathrm{d}x}. \tag{3.8}$$

To simplify the energy equation, we first work out the material derivative of the pressure in 1D

$$\frac{\mathrm{D}P}{\mathrm{D}t} = \frac{\partial P}{\partial t} + U\frac{\partial P}{\partial x} \tag{3.9}$$

Using this, and applying our assumptions to the energy equation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho h U) = U\frac{\mathrm{d}P}{\mathrm{d}x}. \tag{3.10}$$

Also this we can further simplify by expanding the left hand side and using mass conservation

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho h U) = \rho U\frac{\mathrm{d}}{\mathrm{d}x}(h) + h\frac{\mathrm{d}}{\mathrm{d}x}(\rho U) = \rho U\frac{\mathrm{d}}{\mathrm{d}x}(h), \tag{3.11}$$

so we obtain

$$\rho\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}P}{\mathrm{d}x}. \tag{3.12}$$

The pressure gradient term we can replace using the momentum equation (3.8), resulting in

$$\frac{\mathrm{d}h}{\mathrm{d}x} + U\frac{\mathrm{d}U}{\mathrm{d}x} = 0 \tag{3.13}$$

Final simplified compressible equations

To summarize, we have the following simplified equations for inviscid compressible steady flow in 1D, without thermal diffusion.
Mass conservation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho U) = 0 \tag{3.14}$$

Momentum conservation:

$$U\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x} = 0 \tag{3.15}$$

Energy conservation:

$$\frac{\mathrm{d}h}{\mathrm{d}x} + U\frac{\mathrm{d}U}{\mathrm{d}x} = 0 \tag{3.16}$$

3.2.3 Streamline analysis

We will now integrate the equations to obtain relations between two points on a streamline. Starting with mass conservation

$$\int_1^2 \mathrm{d}(\rho U) = (\rho U)_2 - (\rho U)_1 = 0, \tag{3.17}$$

or

$$\rho_1 U_1 = \rho_2 U_2. \tag{3.18}$$

Integrating the momentum equation gives

$$\int_1^2 U\mathrm{d}U + \int_1^2 \frac{1}{\rho}\mathrm{d}P = \tfrac{1}{2}U_2^2 - \tfrac{1}{2}U_1^2 + \int_1^2 \frac{1}{\rho}\mathrm{d}P = 0, \tag{3.19}$$

or

$$\tfrac{1}{2}U_2^2 + \int_1^2 \frac{1}{\rho}\mathrm{d}P = \tfrac{1}{2}U_1^2, \tag{3.20}$$

which is the compressible Bernoulli equation. Note that if density was a constant, this equation would turn into the incompressible Bernoulli equation you already know ($U_1^2/2 + P_1/\rho = U_2^2/2 + P_2/\rho$). Finally, integrating the energy equation gives

$$\int_1^2 \mathrm{d}h + \int_1^2 U\mathrm{d}U = h_2 - h_1 + \tfrac{1}{2}U_2^2 - \tfrac{1}{2}U_1^2 = 0, \tag{3.21}$$

or

$$h_1 + \tfrac{1}{2}U_1^2 = h_2 + \tfrac{1}{2}U_2^2. \tag{3.22}$$

3.2.4 Stagnation properties

We are often interested in the properties of a fluid when it has lost all kinetic energy. We will consider a flow from point 1 to point 0 as indicated in figure 3.1. At point 0, the flow stagnates and we can use

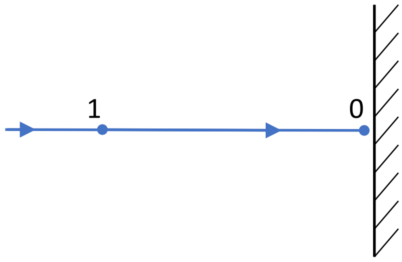

Figure 3.1: Streamline with stagnation point.

the conservation of energy equation directly to define the stagnation enthalpy h_0:

$$\boxed{h_0 = h_1 + \tfrac{1}{2}U_1^2} \tag{3.23}$$

Assuming a perfect gas, we can relate changes in enthalpy to changes in temperature as $h_0 - h_1 = c_P(T_0 - T_1)$, and use this to calculate the stagnation temperature T_0:

$$\boxed{T_0 = T_1 + \frac{U_1^2}{2c_P}} \tag{3.24}$$

The stagnation pressure can be obtained by assuming isentropic flow, which relates temperature to pressure as $T/P^{(\gamma-1)/\gamma} = \text{const.}$:

$$\boxed{P_0 = P_1\left(\frac{T_0}{T_1}\right)^{\frac{\gamma}{\gamma-1}}} \tag{3.25}$$

and equally for the density, using $Pv^\gamma = \text{conts.}$:

$$\boxed{\rho_0 = \rho_1\left(\frac{T_0}{T_1}\right)^{\frac{1}{\gamma-1}}} \tag{3.26}$$

3.3 Speed of sound

In this section we will derive an expression for the speed of sound for a gas in a given state. We will then use this to define the Mach number and critical properties.

3.3.1 Wave speed

We will consider an inviscid fluid in a pipe at rest, with a small disturbance initiated from the left, as shown in figure 3.2. The disturbance is a small increase in velocity δU, which will send a wave through the pipe, travelling at the speed of sound a. To the right of the wave front, the fluid is still at rest and doesn't 'know' about the disturbance yet. To the left of the wave front, the fluid has obtained a velocity δU, as well as changes in the other properties of the fluid (e.g., pressure, density, temperature, enthalpy). We will assume that these changes happen reversibly and adiabatically, and therefore assume no change in entropy.

To simplify our analysis we will use a reference frame travelling with the wave front at speed a. Figure 3.3 shows the wave front with a small control volume drawn around it. Fluid enters the control volume from the right at speed a, and leaves the control volume on the left at a speed $a - \delta U$. The

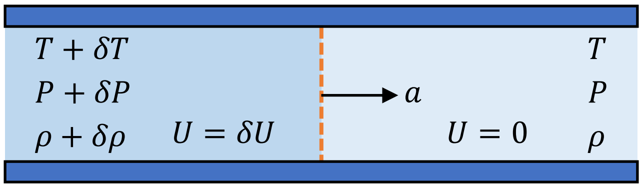

Figure 3.2: Wave travelling at the speed of sound.

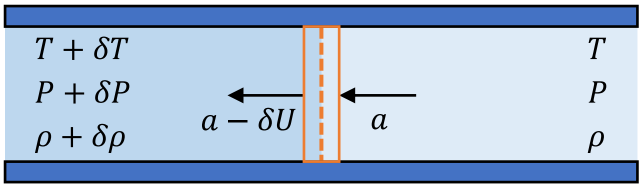

Figure 3.3: Control volume analysis in a frame of reference moving with the wave.

fluid at rest has a density ρ and a pressure P, while the fluid to the left of the wave front has a density $\rho+\delta\rho$ and a pressure $P+\delta P$. We will use mass and momentum conservation to determine the speed a of the wave.

We start with mass conservation, stating that the mass entering the control volume must equal that leaving the control volume:

$$\begin{aligned}
\dot{m}_{in} &= \dot{m}_{out} &(3.27)\\
\rho_{in}U_{in}A &= \rho_{out}U_{out}A &(3.28)\\
\rho a &= (\rho+\delta\rho)(a-\delta U) &(3.29)\\
\rho a &= \rho a-\rho\delta U+a\delta\rho-\delta\rho\delta U &(3.30)\\
a\delta\rho &= \rho\delta U+\delta\rho\delta U &(3.31)\\
a\delta\rho &= \rho\delta U &(3.32)
\end{aligned}$$

note that the cross-sectional area A has dropped out, and that we've neglected the higher order term $\delta\rho\delta U$ because we are looking at very small disturbances. We will do the same for the momentum conservation (not showing the higher order terms)

$$\begin{aligned}
\dot{M}_{out}-\dot{M}_{in} &= \sum F &(3.33)\\
\dot{m}_{out}U_{out}-\dot{m}_{in}U_{in} &= P_{\text{left}}A-P_{\text{right}}A &(3.34)\\
-(\rho+\delta\rho)(a-\delta U)^2A+\rho a^2A &= (P+\delta P)A-PA &(3.35)\\
-\rho a^2-a^2\delta\rho+2\rho a\delta U+\rho a^2 &= P+\delta P-P &(3.36)\\
2\rho a\delta U-a^2\delta\rho &= \delta P &(3.37)
\end{aligned}$$

Combining the mass and momentum equations, we can remove the δU and obtain

$$\frac{a\delta\rho}{\rho}=\frac{\delta P+a^2\delta\rho}{2\rho a}, \tag{3.38}$$

which we can reduce to

$$\frac{\delta P}{\delta\rho}=a^2. \tag{3.39}$$

In the limit of infinitesimal changes, we have the following expression for the speed of sound:

$$a^2=\frac{\mathrm{d}P}{\mathrm{d}\rho} \tag{3.40}$$

We can find an expression for $\frac{dP}{d\rho}$ using our isentropic assumption, which relates pressure to density as

$$Pv^\gamma = \frac{P}{\rho^\gamma} = \text{const.}, \tag{3.41}$$

so we can write

$$\frac{dP}{d\rho} = (\text{const.})\gamma\rho^{\gamma-1} = \frac{P}{\rho^\gamma}\gamma\rho^{\gamma-1} = \gamma\frac{P}{\rho} \tag{3.42}$$

Using the ideal gas law $P = \rho RT$, we can then also write this as

$$\frac{dP}{d\rho} = \gamma RT, \tag{3.43}$$

and therefore the speed of sound is

$$\boxed{a = \sqrt{\gamma RT}} \tag{3.44}$$

Note that for an *ideal* gas, the speed of sound is a function of only the temperature. If we now define the Mach number as the ratio of a velocity U to the speed of sound a

$$\boxed{\text{Ma} = \frac{U}{a}} \tag{3.45}$$

we see that the Mach number is a function of the speed of the fluid (or of an object in a fluid) and the temperature of the fluid. If we look back at the stagnation temperature equation 3.24

$$\frac{T_0}{T_1} = 1 + \frac{U_1^2}{2c_P T_1}, \tag{3.46}$$

then we can now express this as a function of the Mach number, by using $T_1 = a_1^2/(\gamma R)$ from the speed of sound:

$$\frac{T_0}{T_1} = 1 + \frac{\gamma R U_1^2}{2c_P a_1^2} = 1 + \frac{\gamma(c_P - c_V)}{2c_P}\text{Ma}_1^2, \tag{3.47}$$

which we can further simplify to

$$\boxed{\frac{T_0}{T_1} = 1 + \frac{\gamma-1}{2}\text{Ma}_1^2} \tag{3.48}$$

3.3.2 Critical properties

Critical properties are the properties (temperature, pressure, density) that the fluid has at a Mach number of 1. We will derive expressions of these properties with respect to the stagnation properties. Using equation 3.48, we define the critical temperature at $\text{Ma}_1 = 1$, where the temperature T_1 becomes

$$(T_1)_{\text{Ma}_1=1} = T_0\frac{1}{1+\frac{\gamma-1}{2}} = \frac{2T_0}{\gamma+1} \tag{3.49}$$

We denote this critical temperature as T^*:

$$\boxed{T^* = T_0\frac{2}{\gamma+1}} \tag{3.50}$$

The same can be done for the pressure:

$$\boxed{P^* = P_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}} \tag{3.51}$$

and the density:

$$\boxed{\rho^* = \rho_0\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}} \tag{3.52}$$

3.4 Shocks

In section 3.3 we analysed the propagation of an infinitesimal disturbance to find the speed of sound. We will now look at shocks, which are *finite* disturbances. The analysis will initially be similar, but because of the finite disturbance we need to consider a change in entropy.

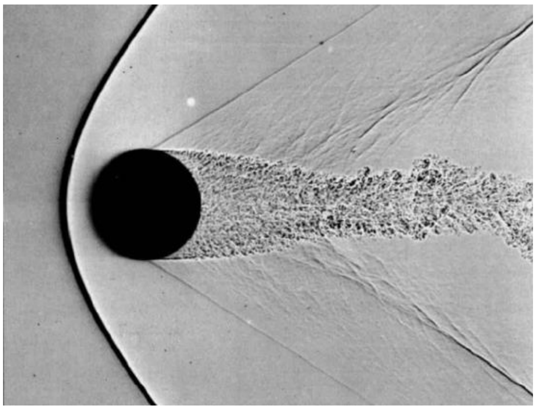

Figure 3.4: Sphere with shock wave (Van Dyke, An Album of Fluid Motion).

Figure 3.4 shows an example of a shock wave formed in front of a sphere moving at Ma = 1.53. We will use this example as a basis for our analysis. Moving with the sphere, we encounter a flow from left to right that passes through a shock wave that is stationary in our frame of reference. We will restrict ourselves to normal shocks, where the shock wave is perpendicular to the direction of flow as shown in figure 3.5. This corresponds to the center part of the shock in figure 3.4.

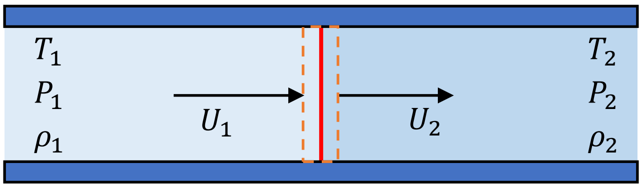

Figure 3.5: Normal shock analysis.

We will analyse the shock using conservation of mass, momentum, and energy. Because shocks are very thin (typically of the order of a micrometer), we don't have to consider any change in area on either side of the shock. We therefore can write for the mass conservation,

$$\rho_1 U_1 = \rho_2 U_2, \tag{3.53}$$

momentum conservation

$$\rho_1 U_1^2 + P_1 = \rho_2 U_2^2 + P_2, \tag{3.54}$$

and energy conservation

$$\tfrac{1}{2} U_1^2 + h_1 = \tfrac{1}{2} U_2^2 + h_2. \tag{3.55}$$

If we consider the inlet conditions (P_1, ρ_1, U_1, h_1) known, then we have four unknown variables with three equations. We can close this problem by assuming a perfect gas

$$h_2 - h_1 = c_P(T_2 - T_1) \tag{3.56}$$

$$h_2 - h_1 = \frac{\gamma R}{\gamma - 1}\left(\frac{P_2}{\rho_2 R} - \frac{P_1}{\rho_1 R}\right) \tag{3.57}$$

$$h_2 - h_1 = \frac{\gamma}{\gamma - 1}\left(\frac{P_2}{\rho_2} - \frac{P_1}{\rho_1}\right), \tag{3.58}$$

which eliminates the change in enthalpy from the energy equation

$$\frac{\gamma}{\gamma-1}\left(\frac{P_2}{\rho_2}-\frac{P_1}{\rho_1}\right)=\frac{1}{2}\left(U_1^2-U_2^2\right). \tag{3.59}$$

These equations can now be solved for any of the properties after the shock. They are commonly expressed as a function of the Mach number of the flow approaching the shock Ma_1, called the Rankine-Hugoniot equations:

$$\boxed{\mathrm{Ma}_2^2=\frac{(\gamma-1)\mathrm{Ma}_1^2+2}{2\gamma\mathrm{Ma}_1^2+1-\gamma}} \tag{3.60}$$

$$\boxed{\frac{\rho_2}{\rho_1}=\frac{(\gamma+1)\mathrm{Ma}_1^2}{(\gamma-1)\mathrm{Ma}_1^2+2}} \tag{3.61}$$

$$\boxed{\frac{P_2}{P_1}=1+\frac{2\gamma}{\gamma+1}\left(\mathrm{Ma}_1^2-1\right)} \tag{3.62}$$

$$\boxed{\frac{T_2}{T_1}=1+\frac{2(\gamma-1)}{(\gamma+1)^2}\left(\frac{\gamma\mathrm{Ma}_1^2+1}{\mathrm{Ma}_1^2}\right)\left(\mathrm{Ma}_1^2-1\right)} \tag{3.63}$$

Figure 3.6 shows how the property ratios depend on the upstream Mach number Ma_1, as given by the Rankine-Hugoniot equations for air ($\gamma = 1.4$). Note that the downstream Mach number Ma_2 is always lower than 1 and appears to reach a plateau value at very large upstream Mach numbers. In other words, a shock always converts a supersonic flow to a subsonic flow. Extrapolating the downstream Mach number for $\mathrm{Ma}_1 < 1$ would appear to show a transition from subsonic to supersonic, but we will show below that this regime is not possible due to the second law of thermodynamics. Looking at the other properties, we see that the density ratio also reaches a plateau value, while the pressure and temperature ratios keep increasing.

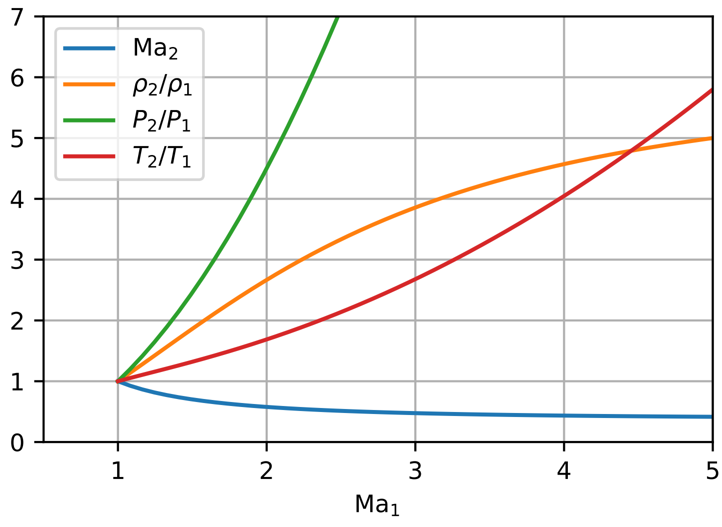

Figure 3.6: Mach number and property ratios across a normal shock.

As mentioned at the start of this section, shocks are not isentropic. We can investigate the change in entropy using the property ratios from the Rankine-Hugoniot equations and one of the relations we have for entropy differences, for example

$$\Delta s = s_2 - s_1 = c_P \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right), \tag{3.64}$$

which we can write in dimensionless form as

$$\frac{\Delta s}{R} = \frac{\gamma}{\gamma - 1} \ln\left(\frac{T_2}{T_1}\right) - \ln\left(\frac{P_2}{P_1}\right). \tag{3.65}$$

Figure 3.7 shows how the entropy change depends on the Mach number. Note that at $\mathrm{Ma}_1 = 1$ the entropy change is zero, which is where we recover the sound wave analysed in section 3.3 for infinitesimal disturbances. Note also that shock waves at $\mathrm{Ma} < 1$ cannot exist, because a decrease in entropy would violate the second law of thermodynamics. This is why we're only showing results for $\mathrm{Ma}_1 > 1$ in figure 3.6.

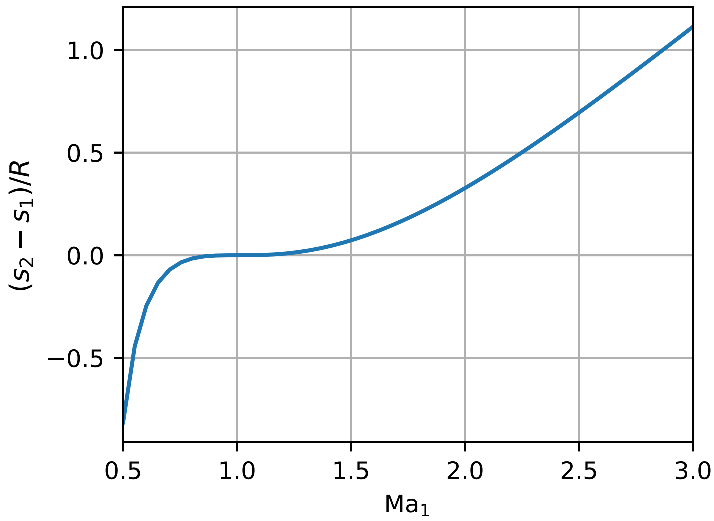

Figure 3.7: Entropy change across a normal shock.

3.5 Duct flow (nozzles)

In the last section of gas dynamics we will look at compressible flow through ducts, which has its main application in nozzles and diffusers. The main function of nozzles is to accelerate flows, in order to increase the thrust. We will see that the nozzle design completely changes when we aim to accelerate fluids to supersonic velocities.

3.5.1 Mass and momentum conservation analysis

We will start our analysis by assuming a quasi-1D flow through a radially symmetric duct. In this case this means that we assume the flow profile to be uniform with no velocity components in the radial direction, but the area of the duct can change in the axial direction. So both the velocity and the area depend only on the axial coordinate x. We further assume

- Steady flow
- Gradual changes in the area
- No friction
- No heat transfer (adiabatic)
- Isentropic flow (except when shocks appear)

Starting with continuity (mass conservation), we can state that the mass flow rate is constant with respect to x:

$$\rho U A = \dot{m} = \text{const.} \tag{3.66}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho U A) = 0, \tag{3.67}$$

which we can expand to

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}x} + \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = 0. \tag{3.68}$$

Starting now with the momentum conservation equation

$$U\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x} = 0, \tag{3.69}$$

we can rewrite this as

$$U\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}x} = 0. \tag{3.70}$$

Recognizing that $\mathrm{d}P/\mathrm{d}\rho = a^2$, we can write this now as

$$U\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{a^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}x} = 0. \tag{3.71}$$

Dividing both sides by U^2

$$\frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{1}{\mathrm{Ma}^2}\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}x} = 0. \tag{3.72}$$

We can use this now to replace the $\mathrm{d}\rho/\mathrm{d}x$ in the continuity equation 3.68, so we obtain

$$-\mathrm{Ma}^2\frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = 0, \tag{3.73}$$

which we can rearrange as

$$\boxed{\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}x}\left(\mathrm{Ma}^2 - 1\right)} \tag{3.74}$$

This equation shows how a change in area results in a change in speed. Note that the term on the right ($\mathrm{Ma}^2 - 1$) changes sign when going from subsonic ($\mathrm{Ma} < 1$) to supersonic ($\mathrm{Ma} > 1$). What this means is, that as long as the flow is subsonic, a *decrease* in area results in an *increase* in velocity, which is what we are used to in incompressible flows. However, for supersonic flows, the exact opposite is true: to further increase the velocity of a supersonic flow, the area must *increase*.

3.5.2 Nozzle design

Converging nozzles

From the previous analysis we conclude that the highest Mach number that can be achieved with a converging nozzle is $\mathrm{Ma} = 1$. Remembering that $\mathrm{Ma} = 1$ corresponds to the critical conditions, we now know that there is also a minimum temperature, pressure, and density that can be achieved using a converging nozzle, given by equations (3.50-3.52) for the critical values T^*, P^* and ρ^*. Figure 3.8 illustrates this principle for a nozzle connected to a large reservoir containing a gas at a pressure P_0. We look at the pressure at the exit plane of the nozzle P_e as we vary the back pressure P_b, which is the pressure of the undisturbed gas outside of the nozzle as indicated in the figure. For back pressures $P_b \geq P^*$, the exit pressure P_e will be equal to the back pressure. However, because the pressure in a converging nozzle cannot be smaller than P^*, when the back pressure drops below the critical pressure ($P_b < P^*$), the pressure at the exit plane will remain at P^*. This means that for $P_b < P^*$, the exit pressure, and therefore also the flow in the nozzle, are *independent* of the back pressure. We call this last condition *choked flow*, because the mass flow rate cannot be further increased by decreasing the back pressure.

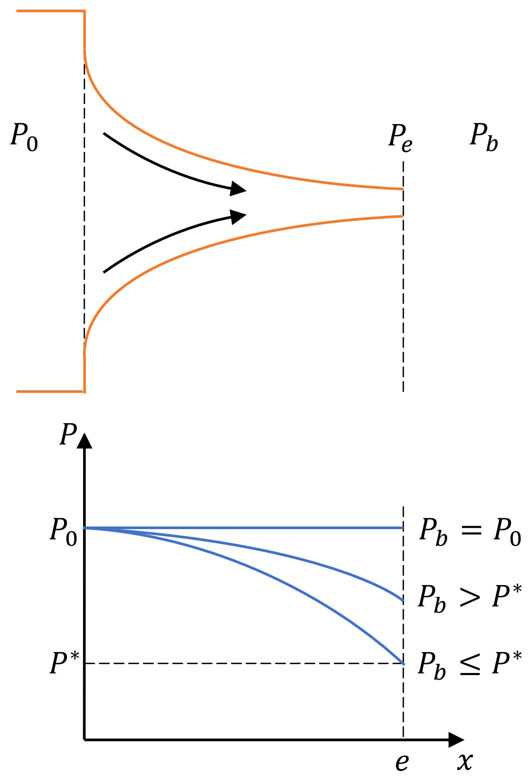

Figure 3.8: Pressure profiles in a converging nozzle.

Converging-diverging nozzles

If we want to generate a supersonic flow, then we need to increase the area after reaching Ma = 1. Figure 3.9 shows an example of such a nozzle, which is called a converging-diverging, or *de Laval* nozzle. The nozzle first converges, until it reaches a minimum area, called the 'throat' of the nozzle. After the throat, the nozzle expands until the exit plane. The design condition is such that sonic (or critical) conditions are reached at the throat, followed by supersonic acceleration in the diverging part of the nozzle, with a pressure at the exit plane close to the back pressure. Figure 3.9 shows the different possible pressure profiles that can form, from high to low back pressures:

1. $P_b = P_0$ no pressure gradient, no flow.
2. Back pressure is not low enough to reach sonic conditions at the throat. Flow is subsonic everywhere.
3. Back pressure is *just* low enough to reach sonic conditions at the throat, but not low enough to further accelerate the fluid in the diverging section. Flow is subsonic everywhere except at the throat, where it is exactly sonic. The flow is choked, any further lowering of the back pressure will not change anything of the flow in the converging section, but will change what happens in the diverging section.
4. Back pressure is low enough to initiate supersonic acceleration after the throat, but too high to sustain the acceleration because the pressure difference between the flow in the nozzle and the back pressure is too big. A normal shock will form in the nozzle that brings the pressure back up to P_b. The position of the shock will depend on the back pressure. Flow is supersonic between the throat and the shock, but subsonic after the shock.
5. Further lowering of the pressure will move the normal shock towards the exit plane of the nozzle, until the normal shock is exactly at the nozzle exit. Flow is supersonic between the throat and the exit, but leaves the nozzle subsonic due to the shock.

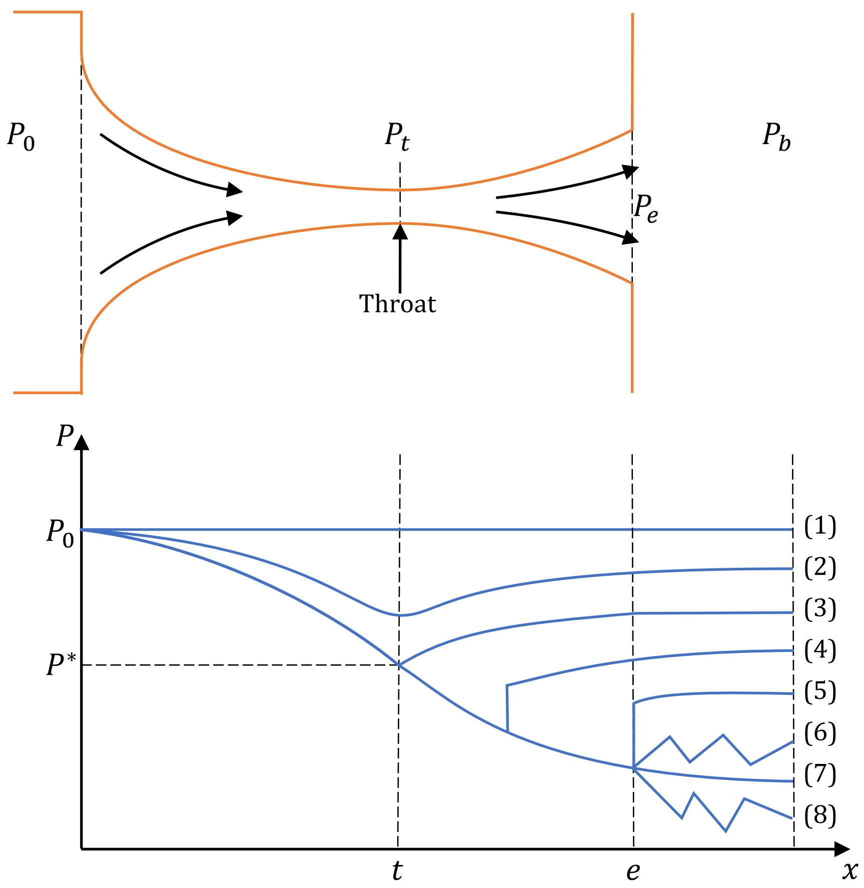

Figure 3.9: Pressure profiles in a converging-diverging nozzle.

6. Back pressure is low enough to have no normal shock in the nozzle, but is still higher than the design condition. Flow leaves the nozzle supersonic. Oblique shocks will form outside of the nozzle due to the pressure difference outside of the nozzle, often forming a diamond pattern ('shock diamonds') that can sometimes be seen at the exhaust of jet engines.

7. Back pressure is at the design condition. The back pressure closely matches the exit pressure, and no shocks are formed. Flow is supersonic at all positions beyond the throat.

8. Back pressure is lower than the nozzle exit pressure. Flow leaves the nozzle supersonic. Expansion waves will form due to the pressure difference outside of the nozzle.

3.5.3 Mass flow rate

Mass flow rate through a nozzle is of specific interest because, together with the nozzle exit velocity determines the thrust. For a nozzle operating with a supersonic exit velocity, we know that the flow is choked at the throat, so we are specifically interested in the mass flow rate in that condition. We will first derive a general expression for the mass flow rate, and will find that is only depends on the area and a single local property of the fluid. After that we will use it to find the mass flow rate for choked conditions.

Starting with the general expression for mass flow rate

$$\dot{m} = \rho A U, \tag{3.75}$$

which, using equation 3.24 for stagnation temperature we can also write as

$$\dot{m} = \rho A \sqrt{2 c_P (T_0 - T)}. \tag{3.76}$$

We can replace the local density and temperature by pressure using our isentropic relations

$$\frac{T}{T_0} = \left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}} \tag{3.77}$$

and

$$\frac{\rho}{\rho_0} = \left(\frac{P}{P_0}\right)^{\frac{1}{\gamma}} \tag{3.78}$$

so we get

$$\dot{m} = A\rho_0 \left(\frac{P}{P_0}\right)^{\frac{1}{\gamma}} \left[2 c_P T_0 \left(1 - \left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right)\right]^{1/2}. \tag{3.79}$$

or:

$$\boxed{\frac{\dot{m}}{\rho_0 \left(2 c_P T_0\right)^{1/2}} = A \left(\frac{P}{P_0}\right)^{\frac{1}{\gamma}} \left[1 - \left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}} \tag{3.80}$$

Note that there is only a single local fluid property in this equation, the local pressure. The equation can also be expressed in temperature or density using equations (3.25) and (3.26). The other parameters that determine the mass flow rate are the local area, the stagnation conditions and gas constants.

If we now look at a nozzle in a choked condition, we know that at the throat we will have the critical pressure P^*. We can therefore replace the pressure ratio in equation (3.80) with

$$\frac{P^*}{P_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \tag{3.81}$$

so that for choked conditions we have the mass flow rate

$$\boxed{\frac{\dot{m}}{\rho_0 \left(2 c_P T_0\right)^{1/2}} = A_t \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}} \tag{3.82}$$

with A_t the area of the throat. This equation shows that for a given set of stagnation properties (e.g. the temperature and pressure in a reservoir), the only parameter that can change the mass flow rate through a nozzle in choked conditions is the area at the throat. As we have seen, whenever the flow in the diverging section of a converging-diverging nozzle is supersonic, we must have critical conditions at the throat, and therefore can use equation (3.82) to calculate the mass flow rate.