SESA2024 Astronautics

University of Southampton

# Chapter 5: Mission Analysis

## Lecture 4 – Orbital motion (part 1)

Professor Hugh Lewis

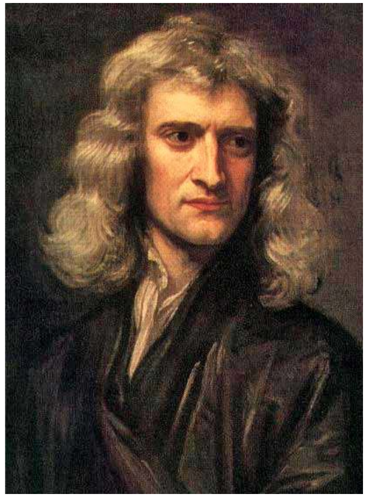
# Overview of lecture 4

- This is a relatively long lecture focused on the derivation of an equation of motion:
  - An equation of motion will ultimately enable us to understand the motion of planets around the Sun, or spacecraft around the Earth, etc.
  - We will adopt methods first developed and used by Isaac Newton (e.g. calculus)
  - Our ultimate aim is to show mathematically, using Newton's Laws, that orbital trajectories can be described using the ellipse equation and, therefore, that Kepler's 1st Law is correct
    - We will complete the derivation over this lecture and the next

- Understanding the approach at a conceptual level is important, but the full derivation will not be assessed

# Orbital motion

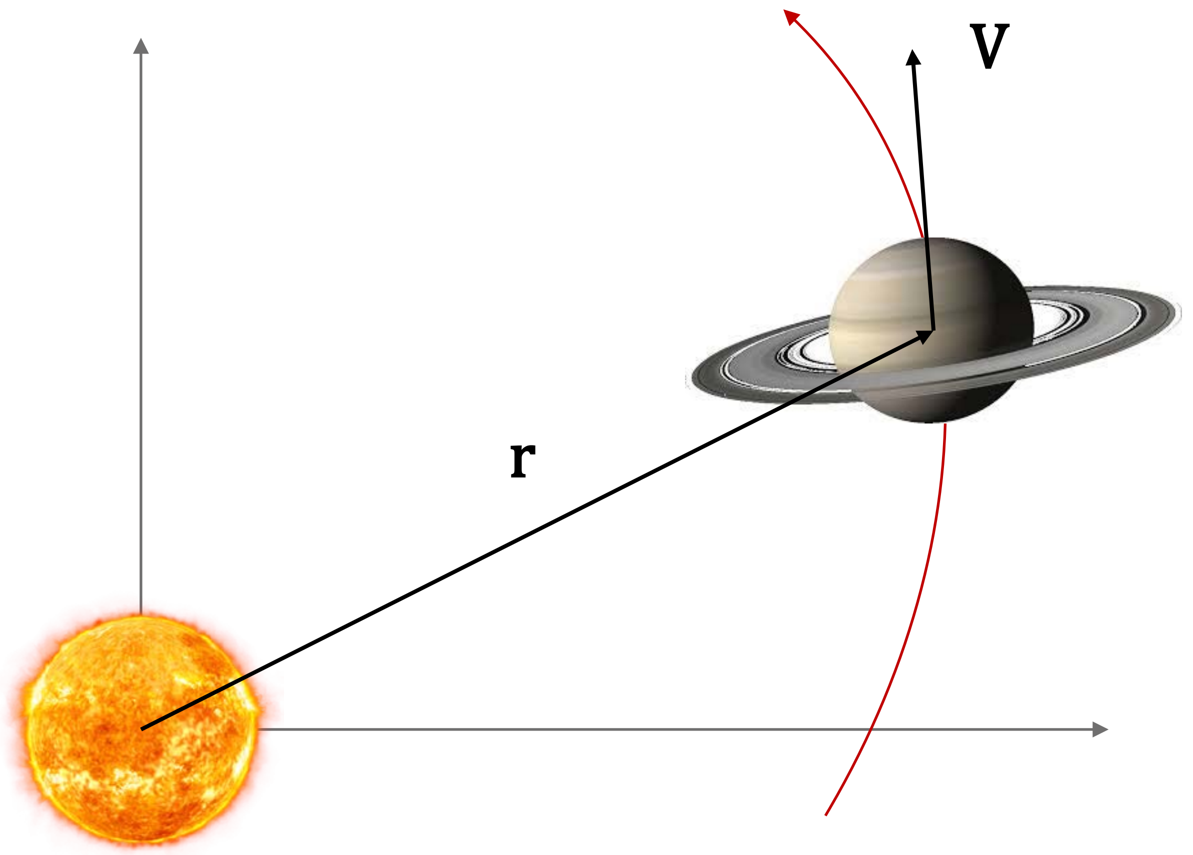
- Kepler's laws originated from <u>observations</u> of the solar system
- In his book 'Philosophiae Naturalis Principia Mathematica' (1687) Isaac Newton established that Kepler's laws follow <u>mathematically</u> from his Law of Universal Gravitation and his Laws of Motion
  - He <u>proved</u> using calculus that orbits are elliptical if the gravitational force is inverse square



Using an approach similar to Newton's, we will use calculus to show that orbits are also described by the ellipse equation – hence proving that Kepler's Laws follow mathematically from Newton's Laws.

# Orbital motion

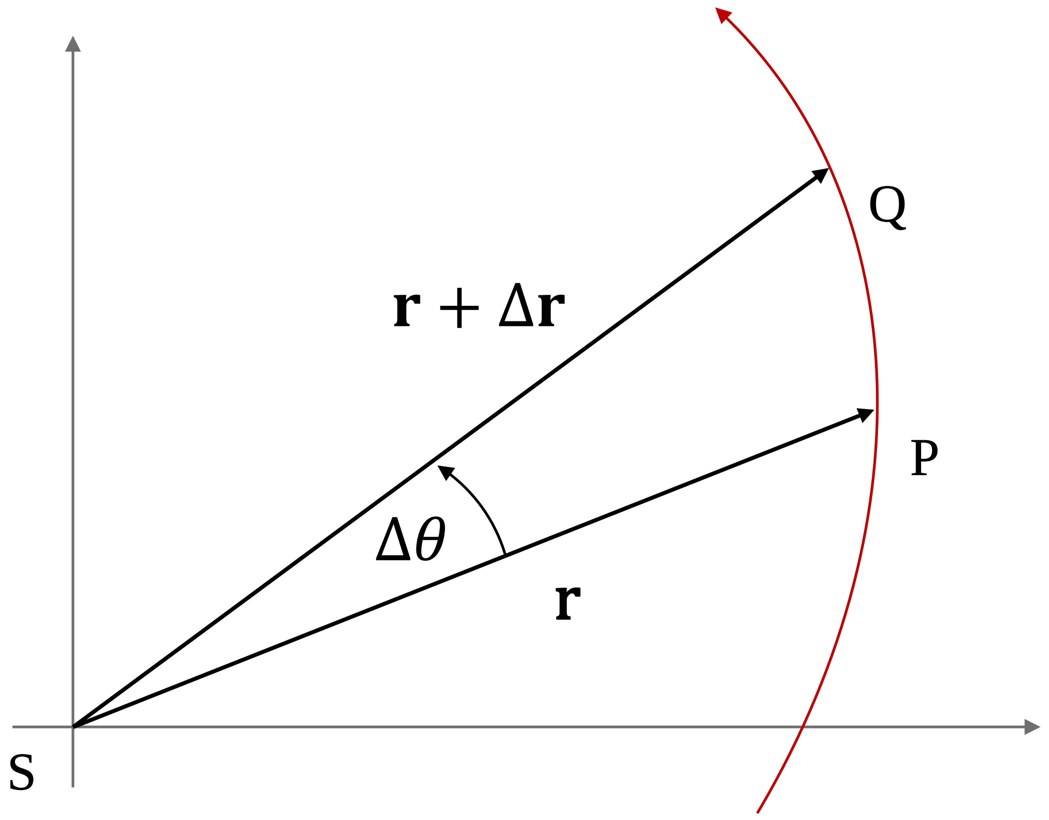
- Planet moving on an orbital trajectory around the Sun:

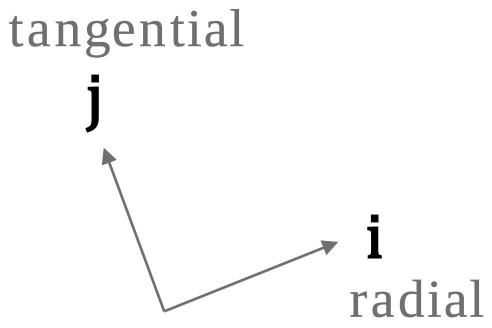



Our first task is to describe the acceleration experienced by the planet. To do this, we will need to find the change in velocity over a very small time interval.

# Orbital motion

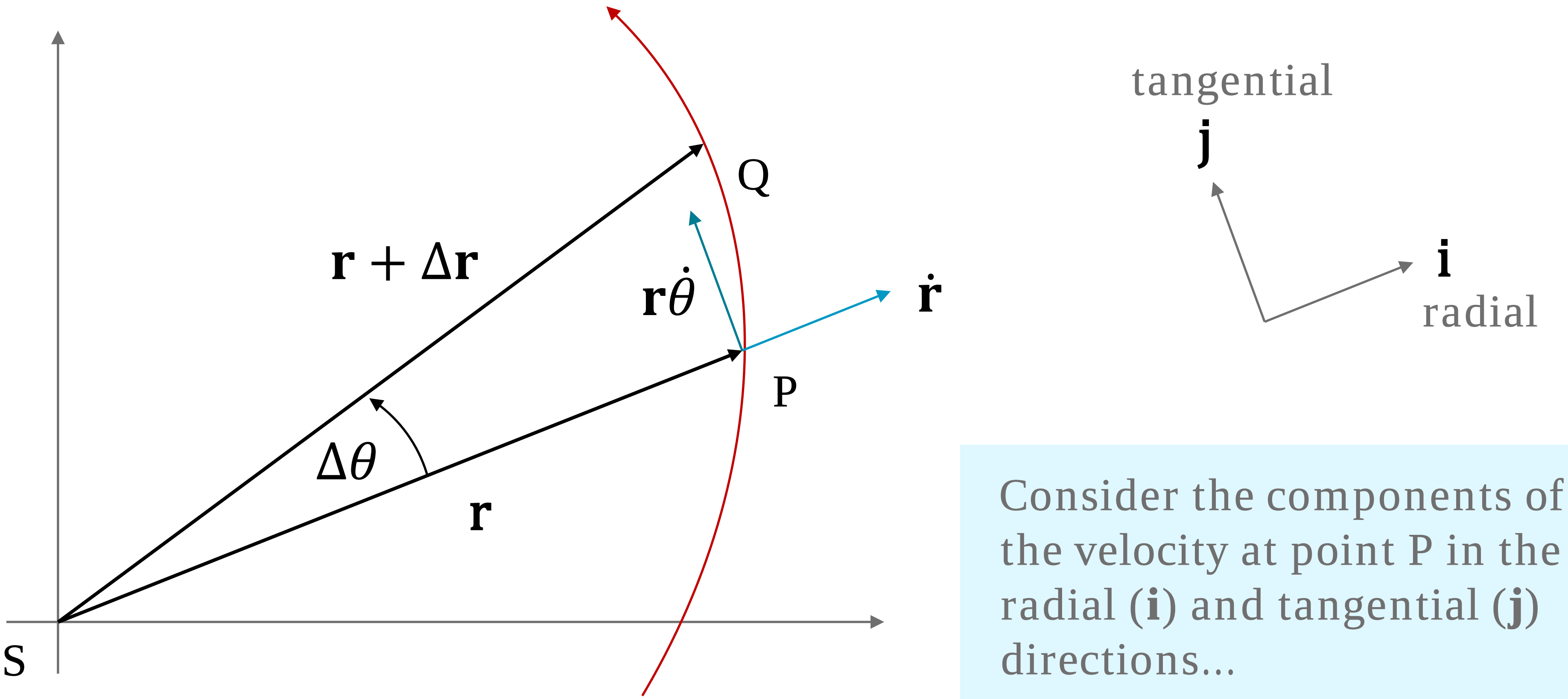
- Look at the motion from point P to point Q (assuming P and Q are very close):



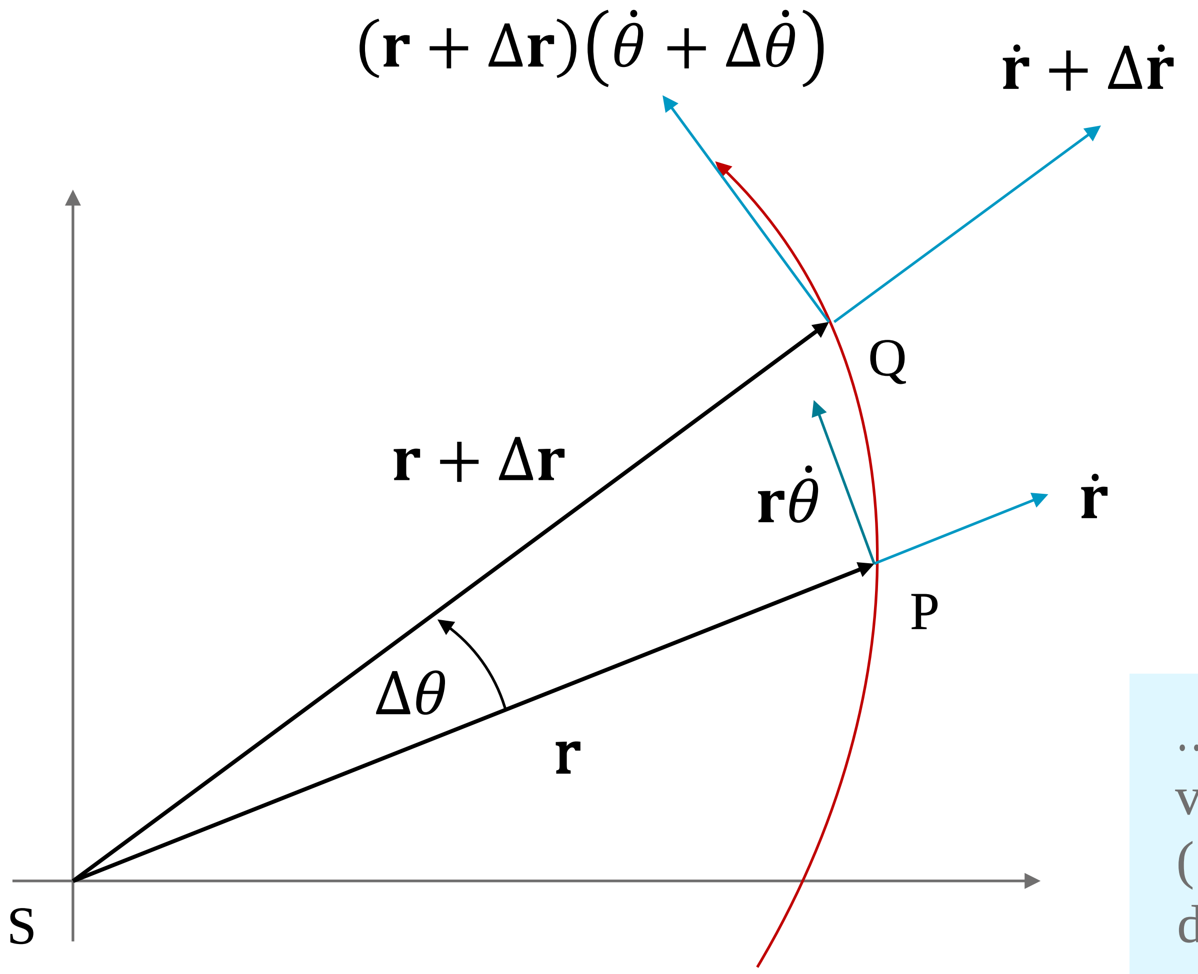

We will focus on the change in velocity in a rotating coordinate system as the planet moves from point P to point Q.

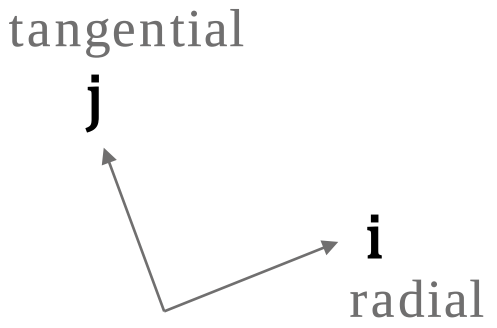
# Orbital motion




Consider the components of the velocity at point P in the radial (**i**) and tangential (**j**) directions...

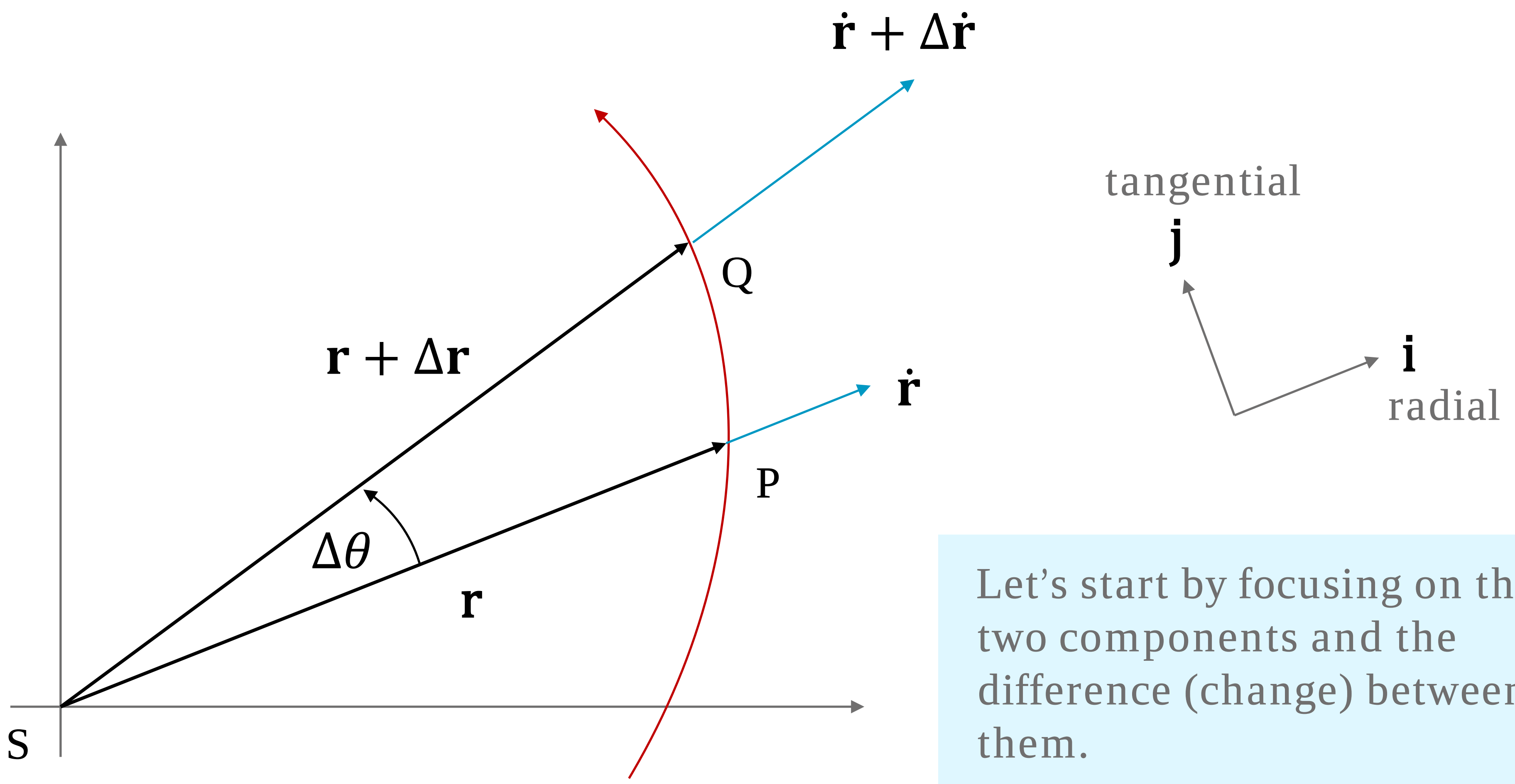
# Orbital motion



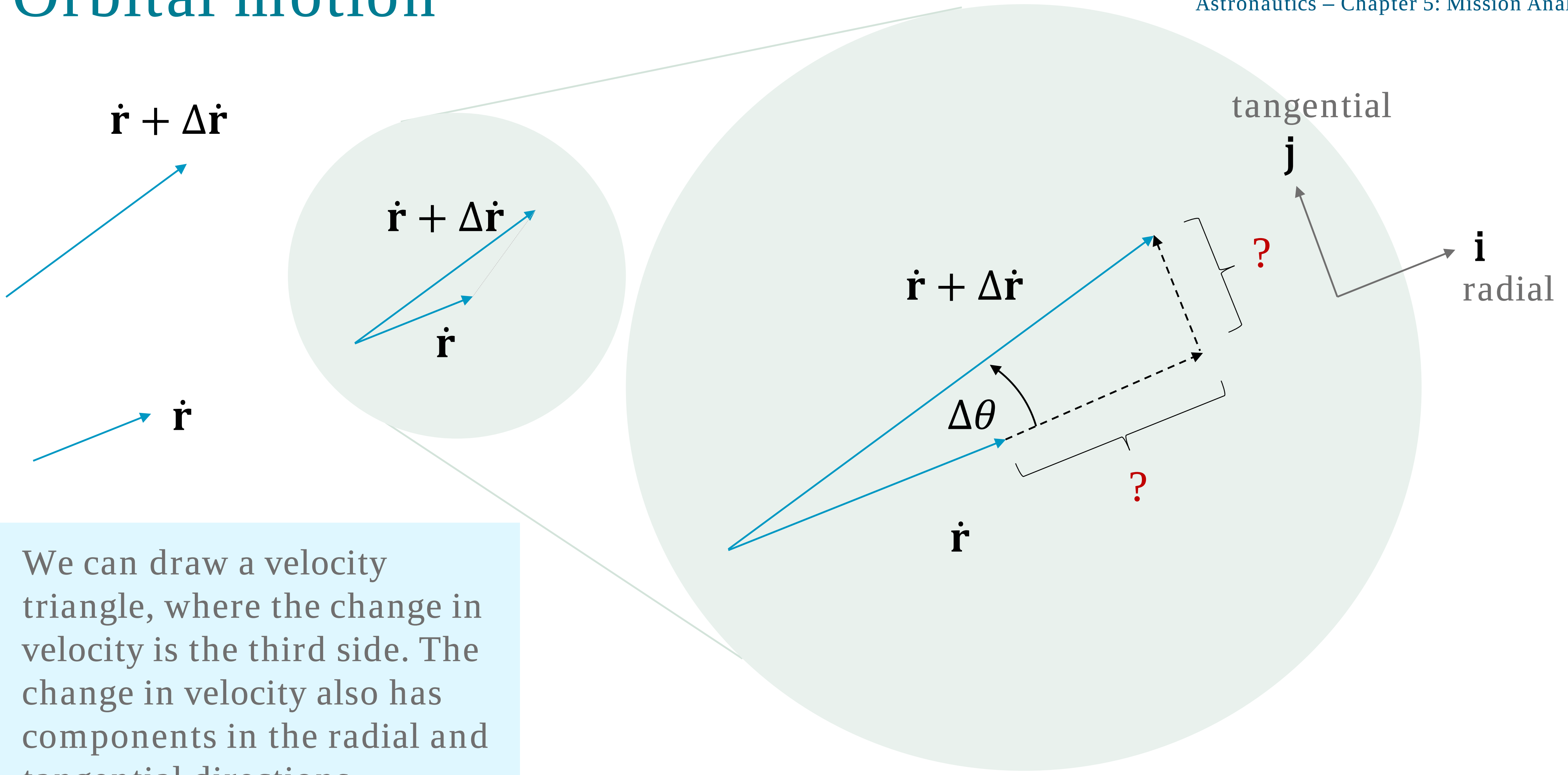

...and the components of the velocity at point Q in the radial ($\mathbf{i}$) and tangential ($\mathbf{j}$) directions.

# Orbital motion



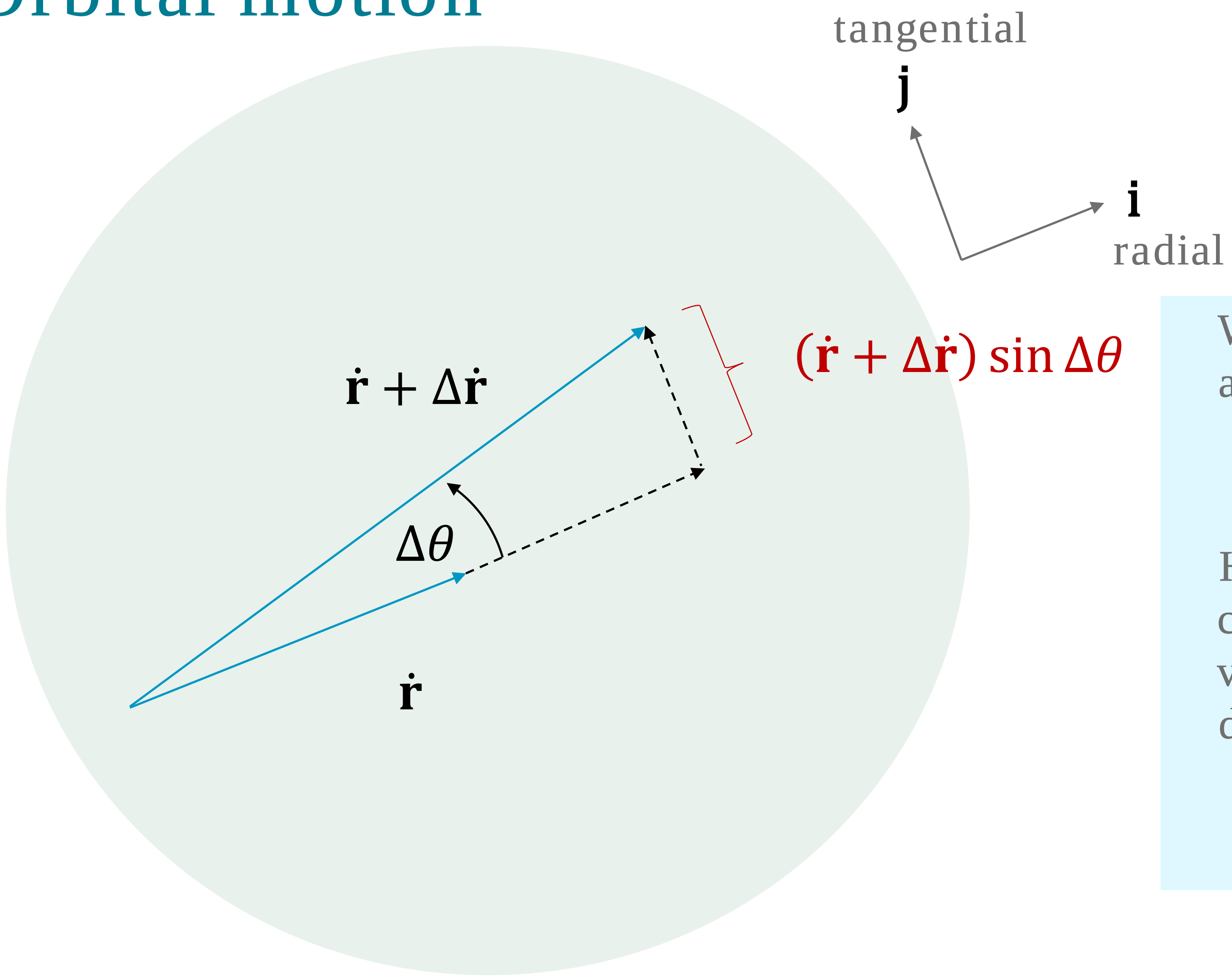

Let's start by focusing on these two components and the difference (change) between them.

# Orbital motion




We can draw a velocity triangle, where the change in velocity is the third side. The change in velocity also has components in the radial and tangential directions.

# Orbital motion



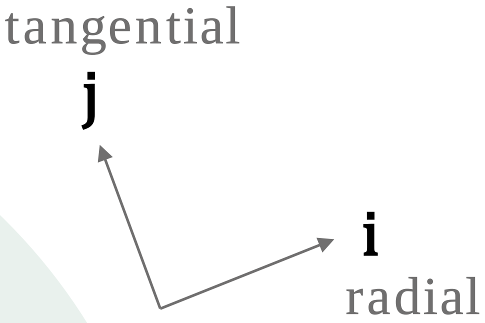

We can use the small angle approximation:

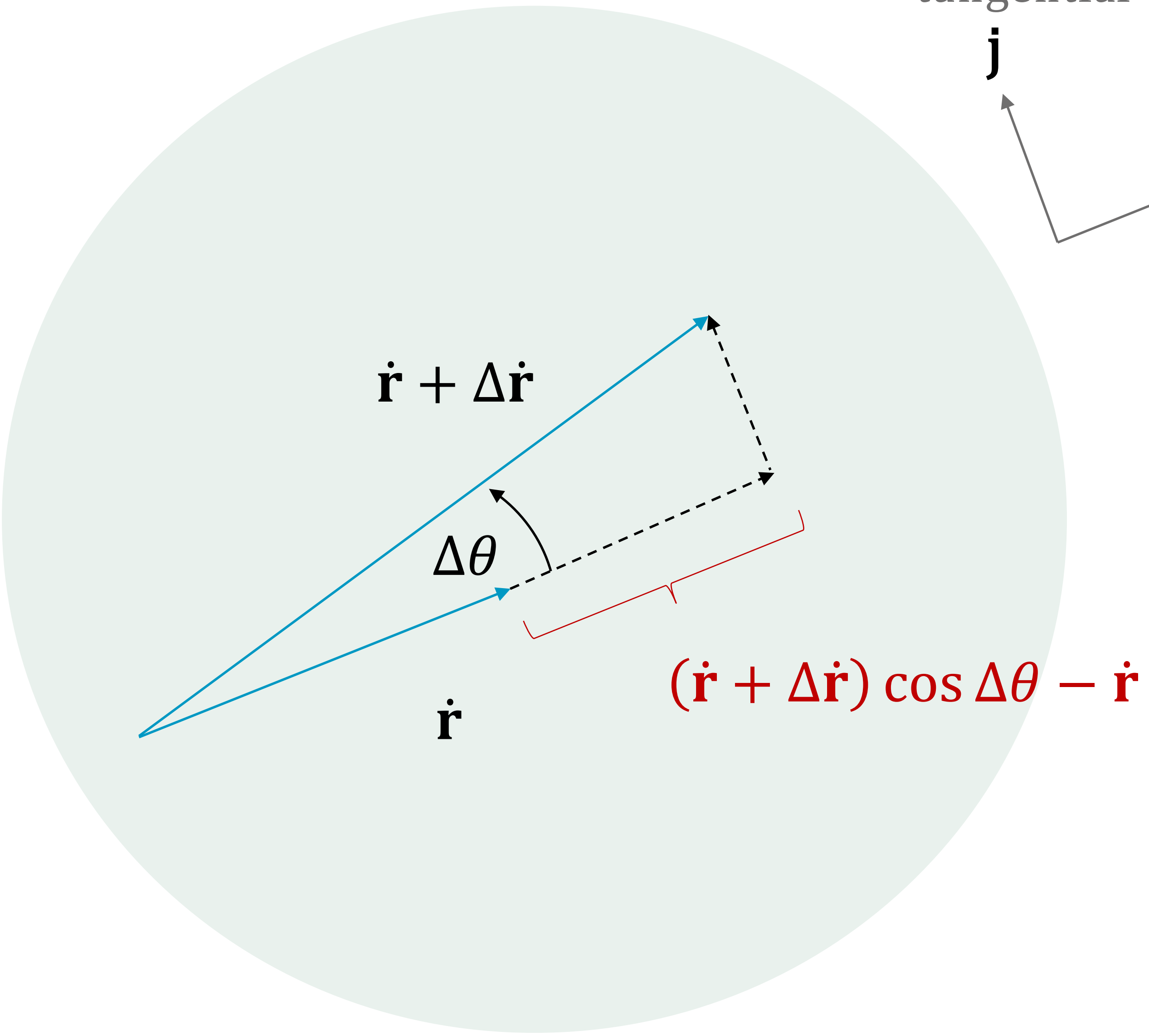
$$\sin \Delta\theta \approx \Delta\theta$$

Hence the magnitude of this component of the change in velocity in the <u>tangential</u> direction is:

$$(\dot{\mathbf{r}} + \Delta\dot{\mathbf{r}})\Delta\theta \approx \dot{\mathbf{r}}\Delta\theta$$

# Orbital motion



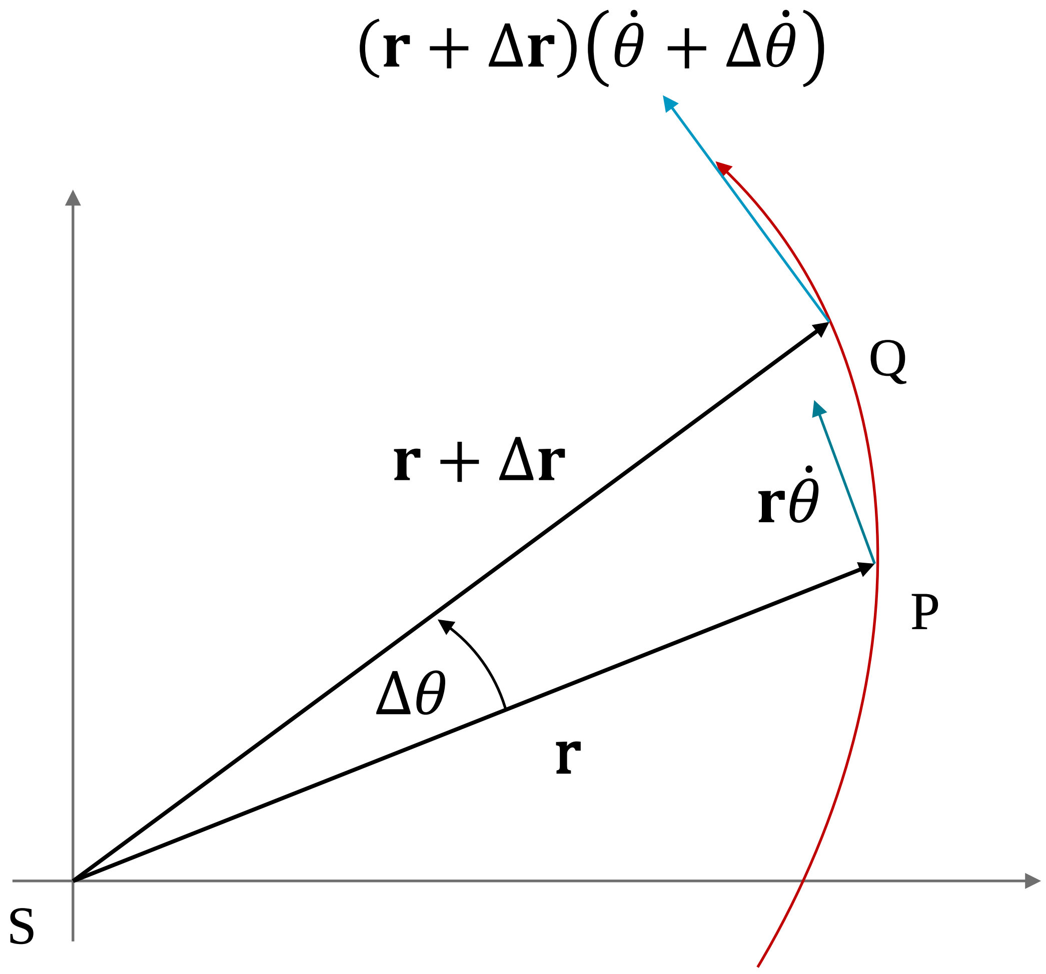

Again, we can use the small angle approximation:

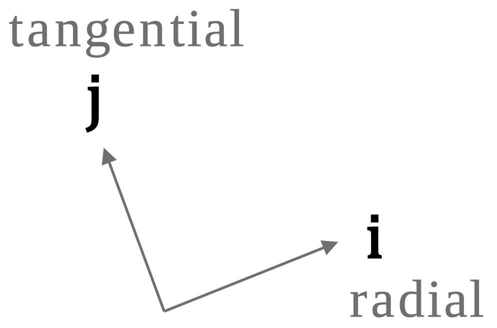
$$\cos \Delta\theta \approx 1$$

Hence the magnitude of this component of the change in velocity in the radial direction is:

$$(\dot{r} + \Delta\dot{r}) - \dot{r} = \Delta\dot{r}$$

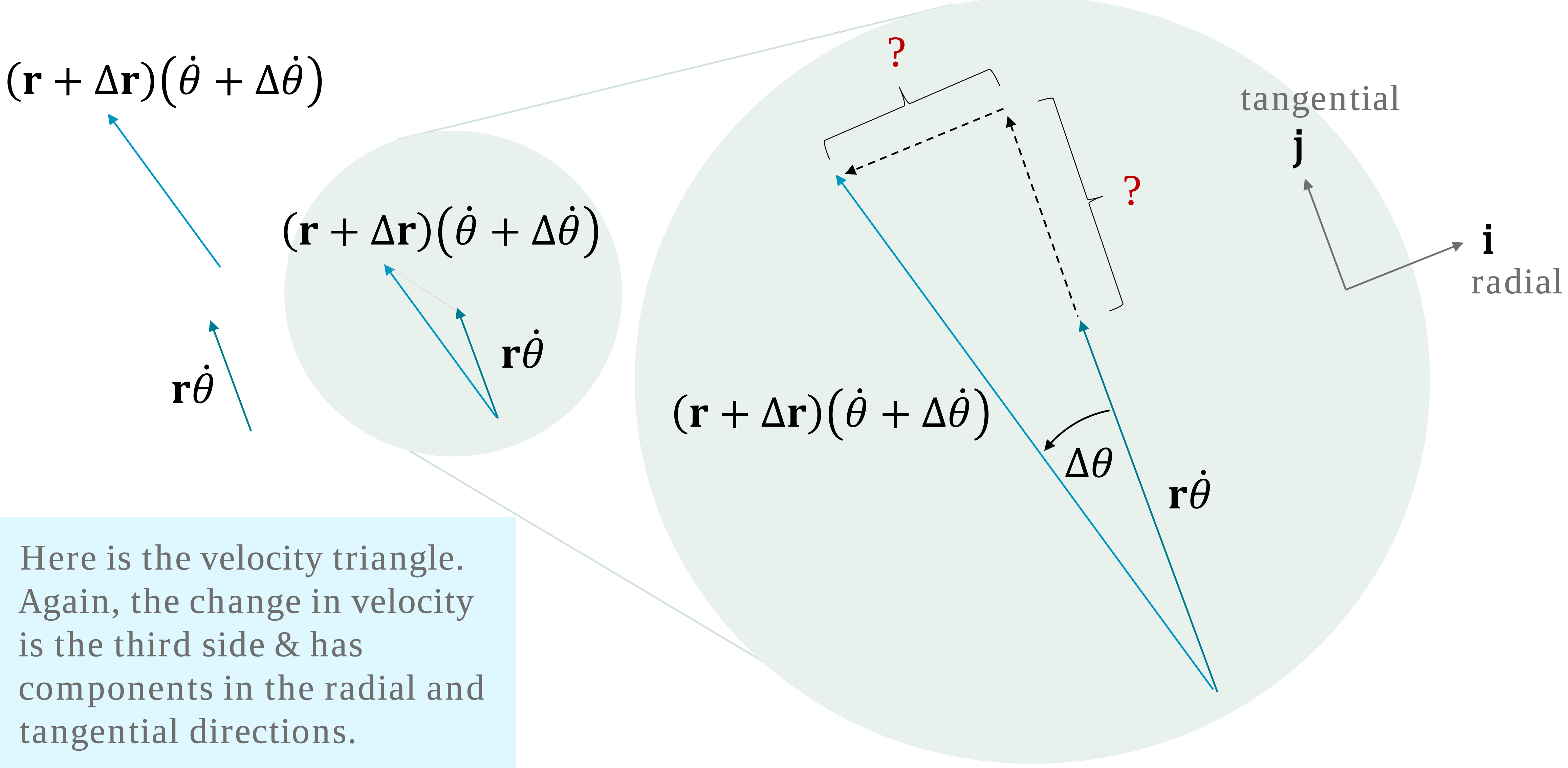
# Orbital motion



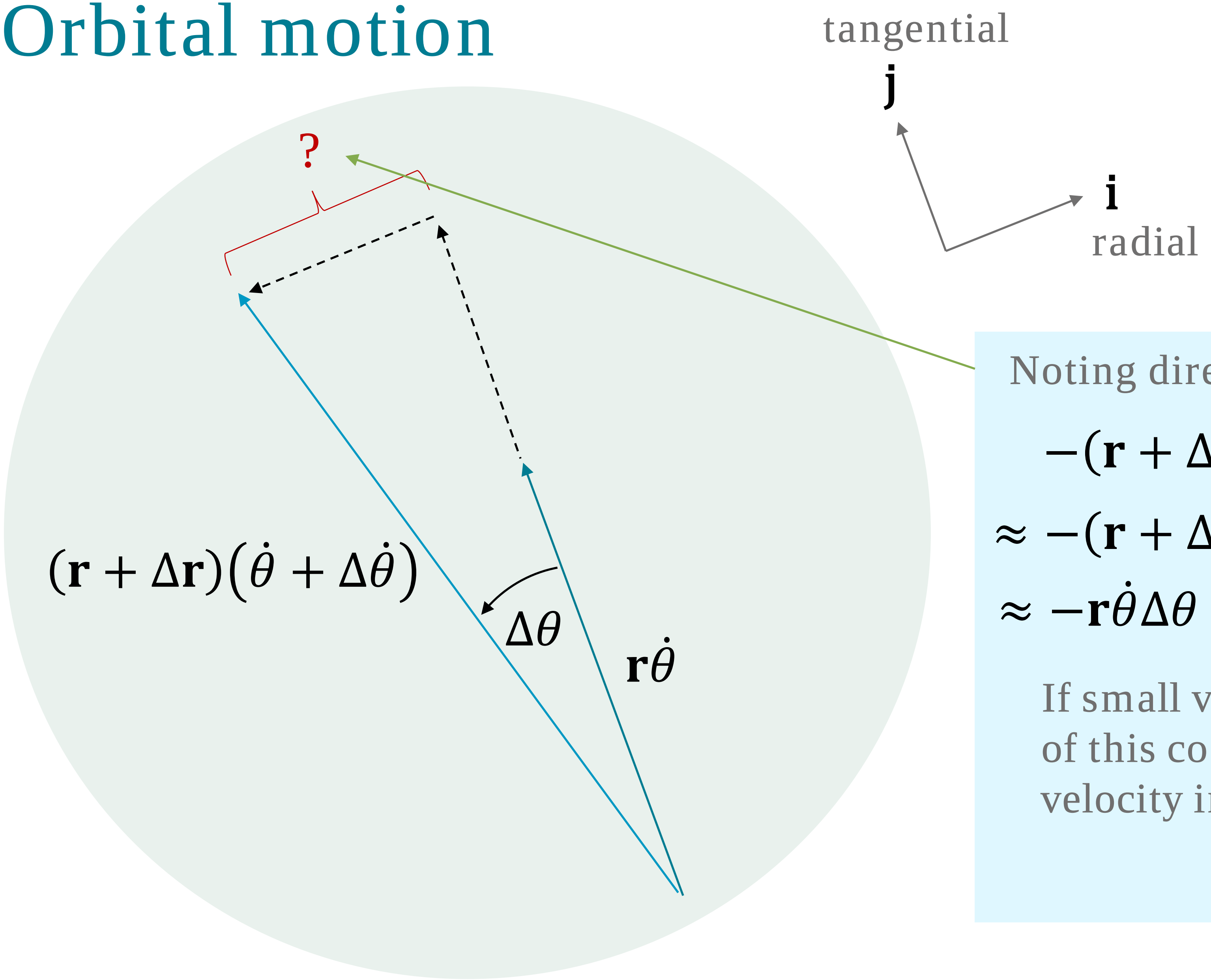

Now we focus on these remaining two components and the difference (change) between them.

# Orbital motion




Here is the velocity triangle. Again, the change in velocity is the third side & has components in the radial and tangential directions.

# Orbital motion




Noting direction of this vector:

$$-(\mathbf{r}+\Delta\mathbf{r})(\dot{\theta}+\Delta\dot{\theta})\sin\Delta\theta$$

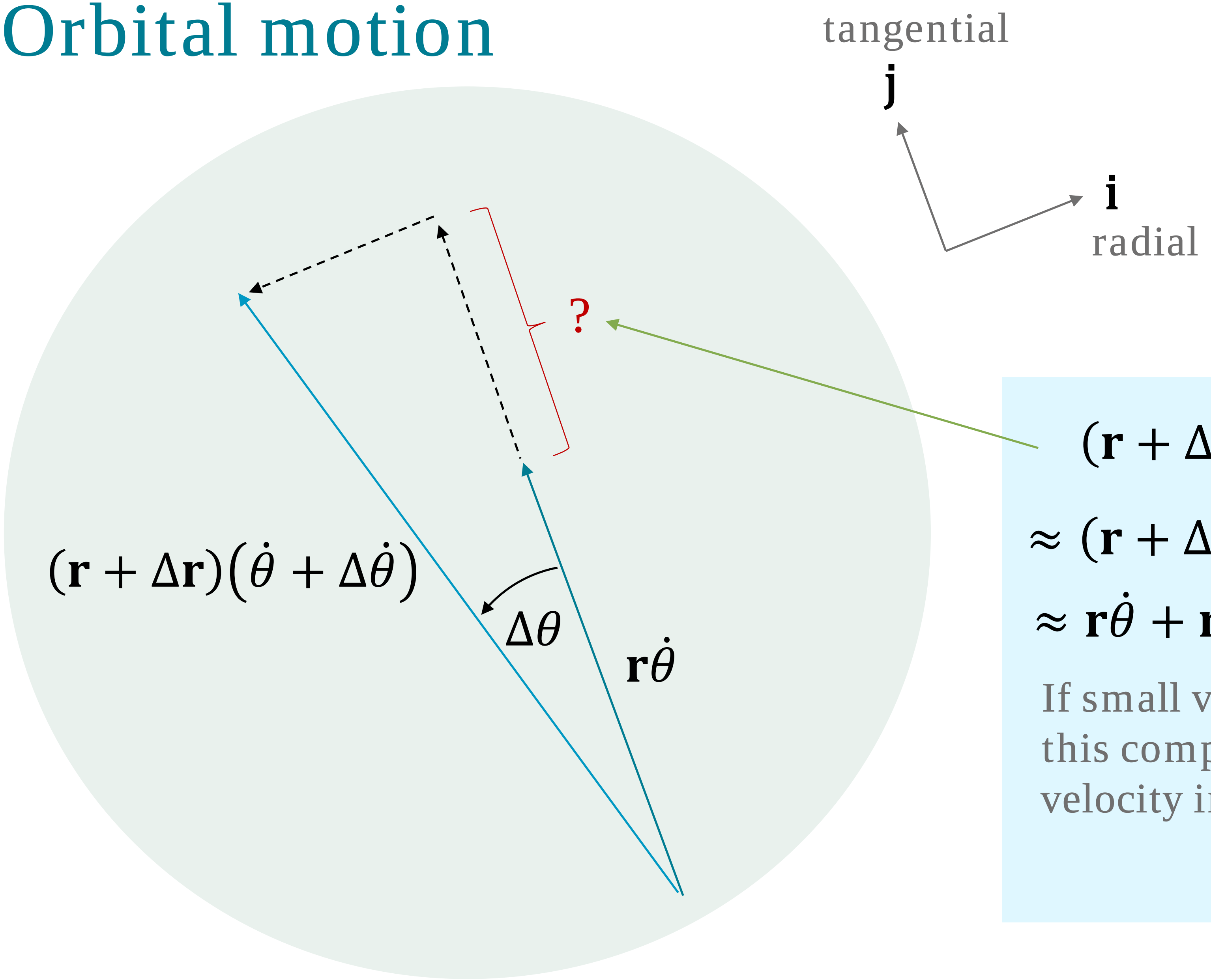
$$\approx -(\mathbf{r}+\Delta\mathbf{r})(\dot{\theta}+\Delta\dot{\theta})\Delta\theta$$

$$\approx -\mathbf{r}\dot{\theta}\Delta\theta - \mathbf{r}\Delta\dot{\theta}\Delta\theta - \Delta\mathbf{r}\dot{\theta}\Delta\theta - \Delta\mathbf{r}\Delta\dot{\theta}\Delta\theta$$

If small values $\approx 0$ the magnitude of this component of the change in velocity in the radial direction is:

$$-r\dot{\theta}\Delta\theta$$

# Orbital motion




$$(\mathbf{r}+\Delta\mathbf{r})(\dot\theta+\Delta\dot\theta)\cos\Delta\theta-\mathbf{r}\dot\theta$$

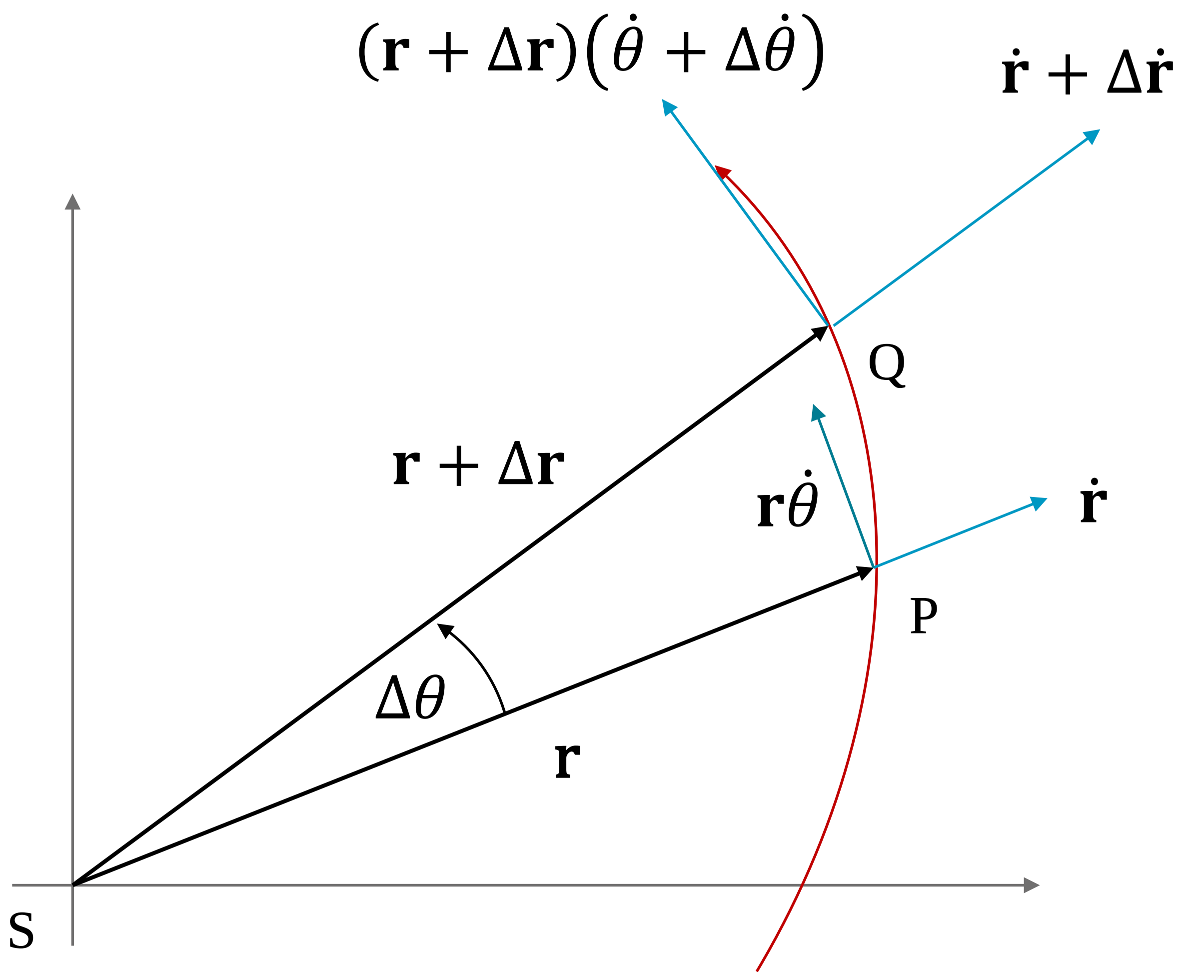
$$\approx(\mathbf{r}+\Delta\mathbf{r})(\dot\theta+\Delta\dot\theta)-\mathbf{r}\dot\theta$$

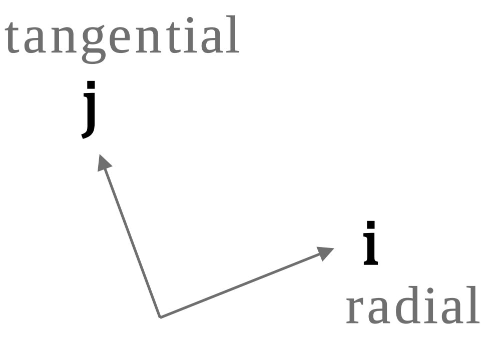
$$\approx\mathbf{r}\dot\theta+\mathbf{r}\Delta\dot\theta+\Delta\mathbf{r}\dot\theta+\Delta\mathbf{r}\Delta\dot\theta-\mathbf{r}\dot\theta$$

If small values $\approx 0$ the magnitude of this component of the change in velocity in the tangential direction is:

$$r\Delta\dot\theta+\Delta r\dot\theta$$

# Orbital motion



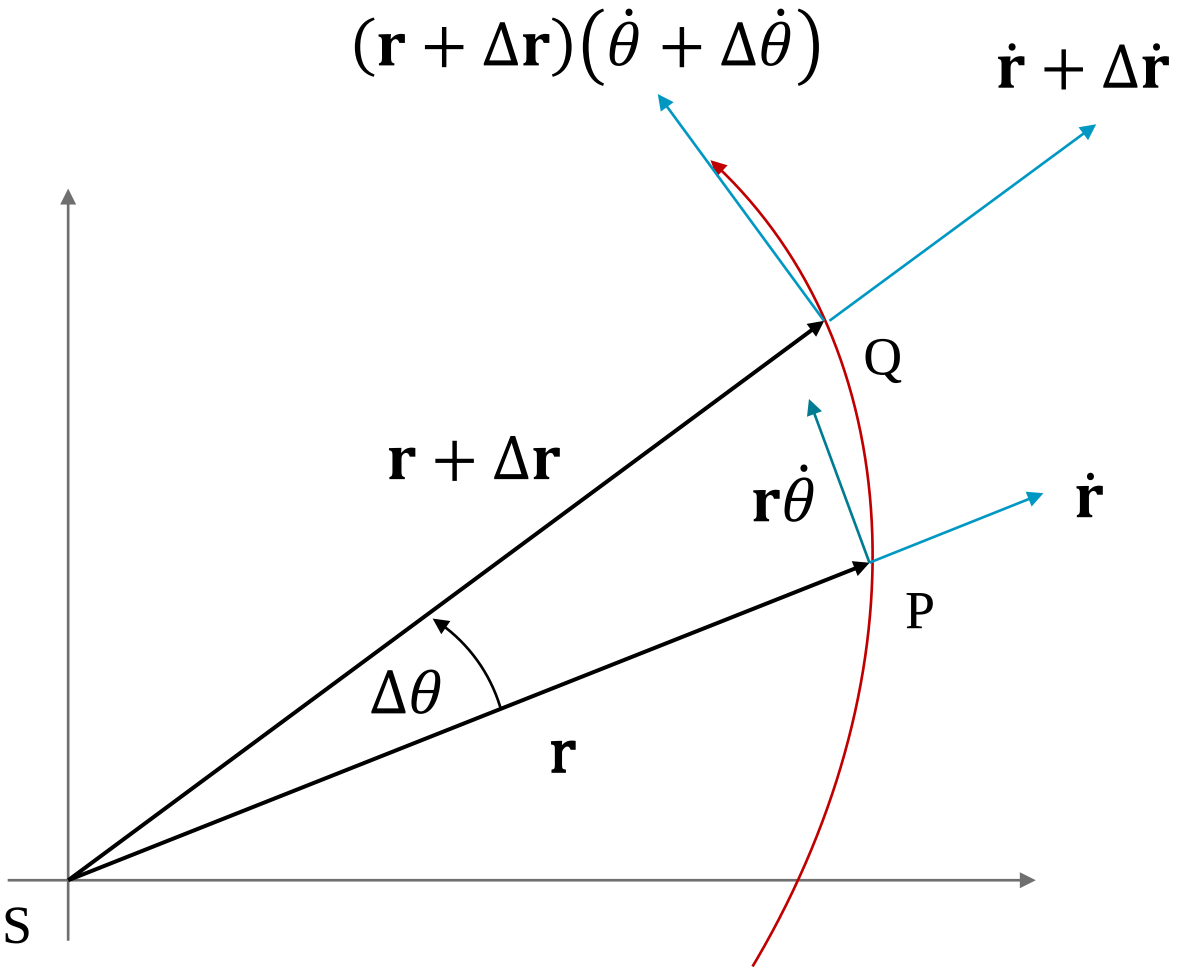



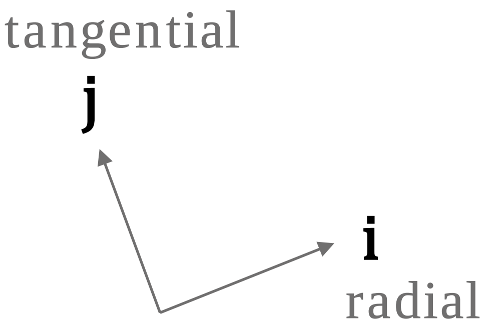

The change in velocity over a small interval is:

Radial: $\Delta\dot{\mathbf{r}}-\mathbf{r}\dot{\theta}\Delta\theta$

Tangential: $\dot{\mathbf{r}}\Delta\theta+\mathbf{r}\Delta\dot{\theta}+\Delta\mathbf{r}\dot{\theta}$

# Orbital motion



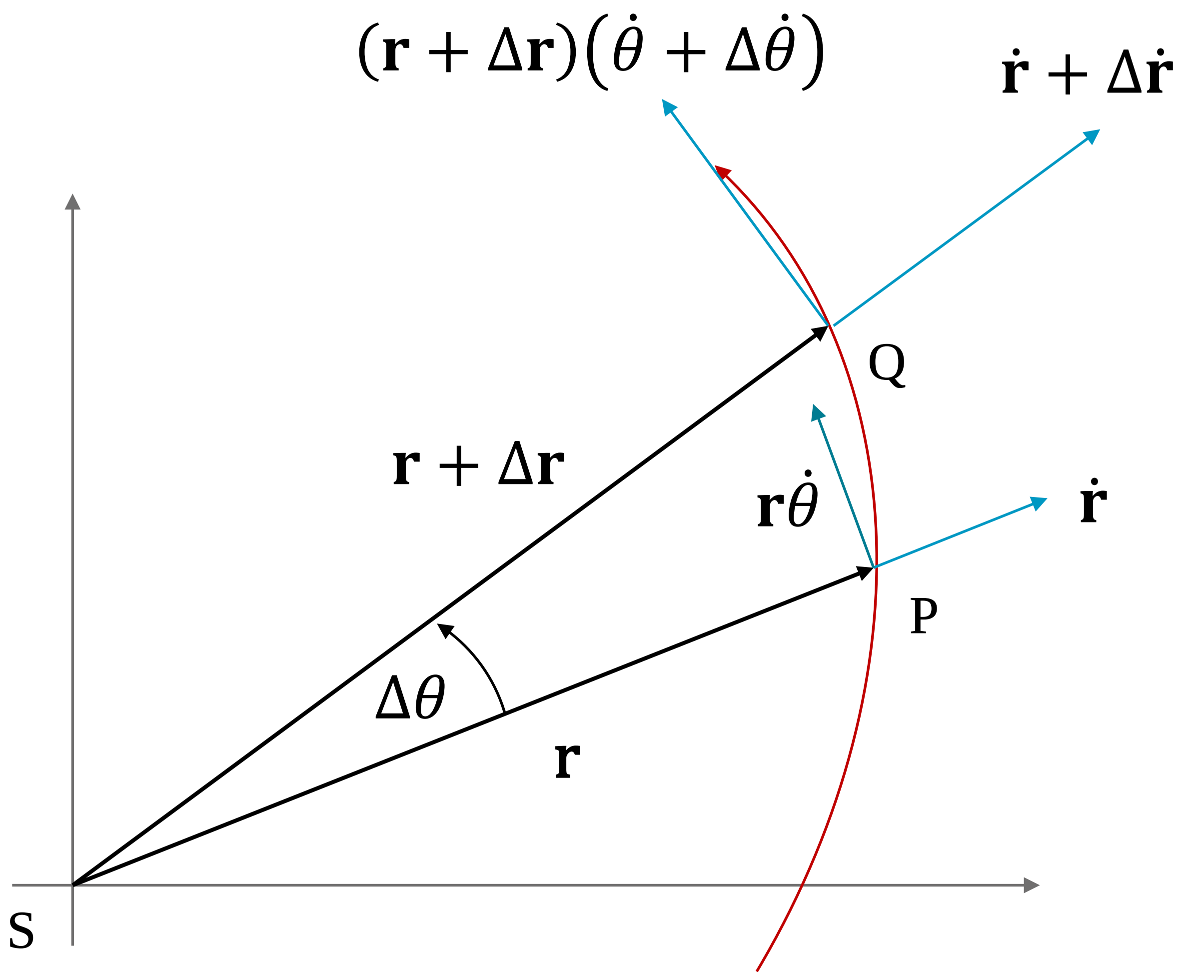



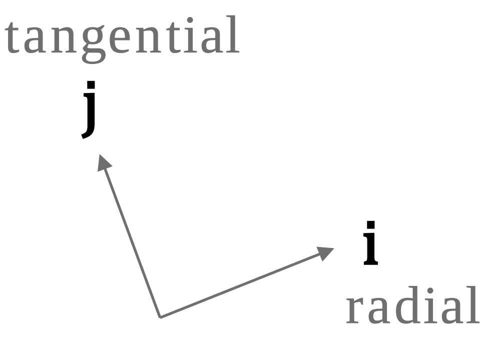

Consider the radial direction: in the limit $\Delta\theta \to 0$ and differentiating with respect to time, the acceleration is

$$\frac{d\dot{\mathbf{r}}}{dt} - \mathbf{r}\dot{\theta}\frac{d\theta}{dt} = \ddot{\mathbf{r}} - \mathbf{r}\dot{\theta}\dot{\theta}$$

$$= \ddot{\mathbf{r}} - \mathbf{r}\dot{\theta}^2$$

# Orbital motion



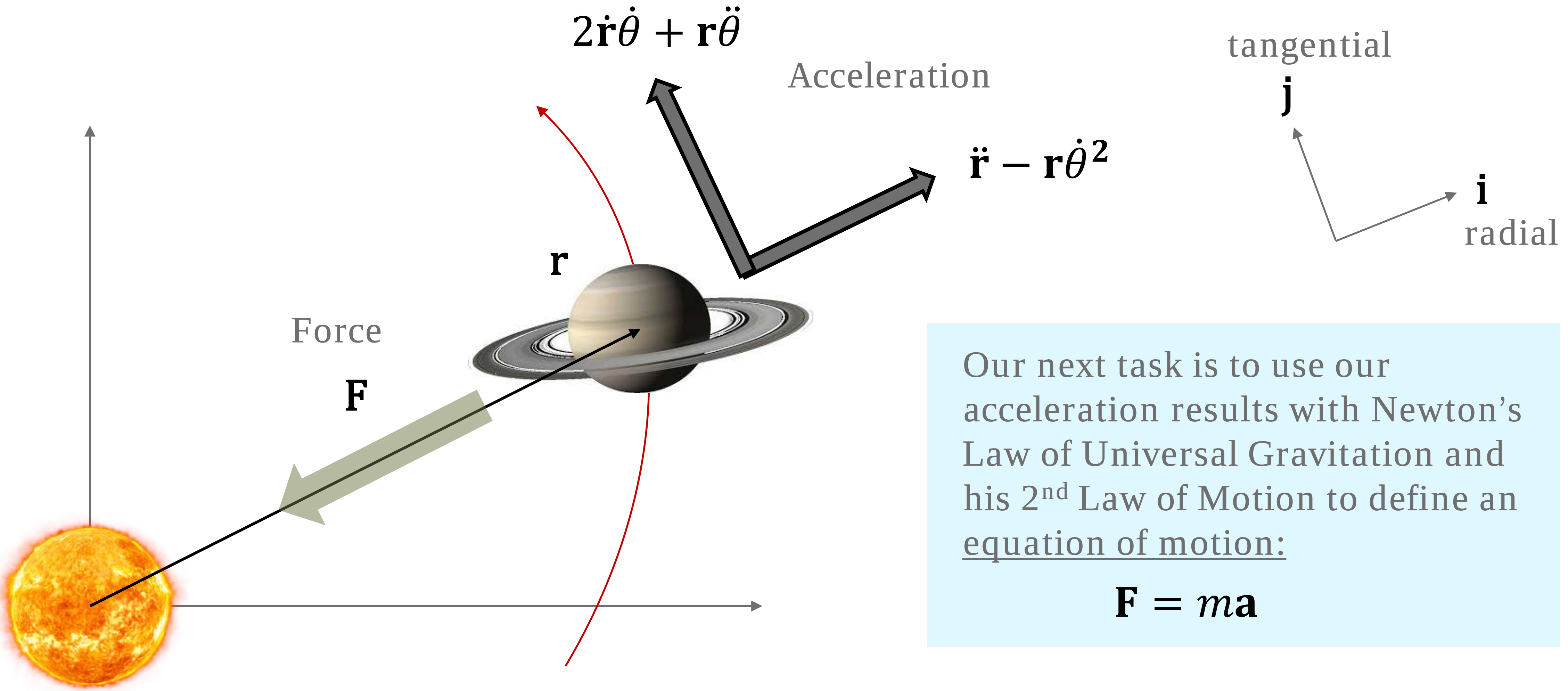




Consider the tangential direction: in the limit $\Delta\theta \rightarrow 0$ and differentiating with respect to time, the acceleration is

$$\dot{\mathbf{r}}\frac{d\theta}{dt} + \mathbf{r}\frac{d\dot{\theta}}{dt} + \frac{d\mathbf{r}}{dt}\dot{\theta}$$

$$= \dot{\mathbf{r}}\dot{\theta} + \mathbf{r}\ddot{\theta} + \dot{\mathbf{r}}\dot{\theta} = 2\dot{\mathbf{r}}\dot{\theta} + \mathbf{r}\ddot{\theta}$$

# Orbital motion




Our next task is to use our acceleration results with Newton's Law of Universal Gravitation and his 2nd Law of Motion to define an <u>equation of motion:</u>

$$\mathbf{F} = m\mathbf{a}$$

# Recap of lecture 4

- This lecture focused on determining the acceleration of a planet moving along an orbital trajectory around the Sun:

  - In the radial direction: $\ddot{\mathbf{r}} - \mathbf{r}\dot{\theta}^{\mathbf{2}}$
  - In the tangential direction: $2\dot{\mathbf{r}}\dot{\theta} + \mathbf{r}\ddot{\theta}$

- To do this we identified the change in velocity as the planet moves over a small angle $\Delta\theta$, then we took the limit $\Delta\theta \rightarrow 0$ & differentiated w.r.t time to determine the acceleration

  - This has enabled us to define the equation of motion using Newton's 2nd Law: $\mathbf{F} = m\mathbf{a}$

- Remember: our overall aim here is to show mathematically, using fundamental physical principles, that orbital trajectories can be described using the ellipse equation and to prove that Kepler's 1st Law is correct
  - We will continue this process in lecture 5

# Activity

- Consolidate your understanding by producing your own sketches, showing the different velocity components for an object moving on a circular orbit:
  - The derivation in this lecture has looked at the general case, for an elliptical orbit; the sketches for a circular orbit will be a little simpler

- Use your sketches to derive the acceleration in the radial and tangential directions