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Lecture 9 - Fourier Series and Transforms

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Outline

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→ Outline/Review

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- In previous lectures we studied the Fourier series of **periodic** functions, or functions defined in an interval, using their periodic extensions.
- Here we will extend these results to functions that are defined on the whole real line and are **not** periodic.
- Our **starting point** will be the **Complex Fourier series:**

harmonic

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{\,\mathbf{j}\,\frac{2\pi\, n\, t}{T}}$$

period

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-\mathbf{j}\,\frac{2\pi\, n\, t}{T}}\, dt$$

(Relative to our earlier notation, $x \to t$, $T = 2\ell$)

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Outline

→ Taking limits of FS to get Fourier transform

- **Idea: 1)** start with complex Fourier series; **2) take** the **limit** $\boldsymbol{T \to \infty}$ (T: period), **3)** we end up with the **Fourier Transform**.

unlike series which is a constant

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{\mathbf{j}\,\frac{2\pi n}{T} t}, \qquad c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-\mathbf{j}\,\frac{2\pi n}{T} t} \, \mathrm{d}t$$

- Define:

equiv   equiv

$$k \equiv \frac{2\pi n}{T} = n\,\frac{2\pi}{T} \Rightarrow \Delta k \equiv k|_{n+1} - k|_n = \frac{2\pi(n+1)}{T} - \frac{2\pi n}{T} = \frac{2\pi}{T} \Rightarrow k = n\,\Delta k$$

- This gives (note that $\Delta k = \frac{2\pi}{T} \Rightarrow \frac{1}{2\pi} T \Delta k = 1$):

changed in terms of k

changed interms of k not n and multiplying by "1" →

$$f(t) = \sum_{k=-\infty}^{\infty} \tilde{c}_k \, e^{\mathbf{j}\,k\,t} \overbrace{\frac{1}{2\pi} T \Delta k}^{=1}, \; =1 \qquad \tilde{c}_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-\mathbf{j}\,k\,t} \, \mathrm{d}t$$

rearrange

$$\Rightarrow \quad f(t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left(\frac{T\,\tilde{c}_k}{\sqrt{2\pi}} \right) e^{\mathbf{j}\,k\,t} \Delta k$$

Taking limits

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \underbrace{\left(\frac{T\,\tilde{c}_k}{\sqrt{2\pi}}\right)}_{\text{short notation}} e^{\mathbf{j}\,k\,t} \Delta k, \qquad \tilde{c}_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-\mathbf{j}\,k\,t} \mathrm{d}t$$

- Define:

short notation

rearrange

$$\boxed{F(k)} \equiv \left(\frac{T\,\tilde{c}_k}{\sqrt{2\pi}}\right) \quad \Leftrightarrow \quad \tilde{c}_k = \frac{\sqrt{2\pi}}{T} F(k)$$

becomes an integral because it is

Then

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} F(k) e^{\mathbf{j}\,k\,t} \Delta k, \qquad F(k) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} f(t) e^{-\mathbf{j}\,k\,t} \mathrm{d}t$$

- **Taking the limit** $T \to \infty$, i.e. $\Delta k = \frac{2\pi}{T} \to 0$, gives:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{\mathbf{j}\,k\,t} \mathrm{d}k, \qquad F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{j}\,k\,t} \mathrm{d}t.$$

Taking the small rectangles ∴ is integration

$$\left[\ \boxed{\mathrm{d}k = \lim_{\Delta k \to 0} \Delta k}, \qquad \boxed{\lim_{\Delta k \to 0} \left[\sum_{k=-\infty}^{\infty} (\cdots) \Delta k\right]} = \int_{-\infty}^{\infty} (\cdots) \mathrm{d}k\ \right]$$

Fourier Transform (FT): the definition

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just dummy var change

The **Fourier Transform** of $f(t)$ is: (✓ replacing k by ω)

notation of fourier transform

$$F(\omega) \equiv \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{j}\,\omega\, t} \mathrm{d}t.$$

The **inverse Fourier transform** is:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{+\mathbf{j}\,\omega\, t} \mathrm{d}\omega.$$

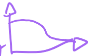
Existence of Fourier Transforms

The Fourier Transform

$$F(\omega) = \mathcal{F}\left[f(t)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{j}\,\omega\, t}\, \mathrm{d}t$$

exists if all of the following conditions are satisfied:

has min and max

1. $f(t)$ is bounded;
2. $\int_{-\infty}^{\infty} |f(t)|\, \mathrm{d}t < \infty$; area of sig, requires decaying func eg 

   max's/min's (turning points)
3. $f(t)$ has a finite number of extrema and finite number of discontinuities in any finite interval.

Outline

Example of FT: I

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- Consider the function

$$f(t) = \begin{cases} 1 & , |t| < 1 \\ 0 & , \text{otherwise} \end{cases}$$

graph

- Its Fourier transform, $F(\omega) = \mathcal{F}[f(t)]$, is computed by

is zero outside -1 to 1

so can simplify

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{j}\,\omega\,t} \mathrm{d}t = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\mathbf{j}\,\omega\,t} \mathrm{d}t$$

$$= -\frac{1}{\mathbf{j}\,\omega\sqrt{2\pi}} \left[e^{-\mathbf{j}\,\omega\,t} \right]_{-1}^{1} = \sqrt{\frac{2}{\pi}} \frac{\sin(\omega)}{\omega}$$

$[\ \int e^{-\mathbf{j}\,\omega t} dt = \mathbf{j}\ \frac{1}{\omega} e^{-\mathbf{j}\,\omega t} \nearrow$ $\qquad \nwarrow \sin \omega = \mathbf{j}\ \frac{1}{2} \left(e^{-i\omega} - e^{i\omega} \right)\ ]$

identity

Thus the FT of $f(t)$ is:

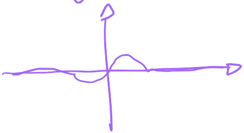
$$F(\omega) = \mathcal{F}[f(t)] = \sqrt{\frac{2}{\pi}} \frac{\sin(\omega)}{\omega}.$$

Example of FT: II

- Consider the function

$$f(t) = \begin{cases} \sin t & , |t| < \pi \\ 0 & , \text{otherwise} \end{cases}$$

graph



- Its Fourier Transform, $F(\omega) = \mathcal{F}\left[f(t)\right]$, is:

same simplification

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{j}\,\omega\, t} \mathrm{d}t = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin(t) e^{-\mathbf{j}\,\omega\, t} \mathrm{d}t$$

$$\swarrow \ \sin(t) = \mathbf{j}\ \tfrac{1}{2}\left(e^{-\mathbf{j}\,t} - e^{\mathbf{j}\,t}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2\,\mathbf{j}}\right) \int_{-\pi}^{\pi} \left(e^{-\mathbf{j}\,(\omega - 1)t} - e^{-\mathbf{j}\,(1+\omega)t}\right) \mathrm{d}t$$

$$= \mathbf{j} \sqrt{\frac{2}{\pi}} \frac{\sin(\omega\pi)}{\omega^2 - 1} \qquad \nwarrow \int e^{-\mathbf{j}\,\alpha t} dt = \mathbf{j}\ \tfrac{1}{\alpha} e^{-\mathbf{j}\,\alpha t}$$

$$\left[\ \nearrow e^{a+b} = e^a e^b, \quad e^{\pm \mathbf{j}\,\pi} = -1, \qquad \nwarrow \sin\beta = \mathbf{j}\ \tfrac{1}{2}\left(e^{-\mathbf{j}\,\beta} - e^{\mathbf{j}\,\beta}\right)\ \right]$$

Outline

Summary

- We discussed how to obtain **Fourier transforms** (FT) by taking the infinite period limit of **Fourier Series.**
- The Fourier transform of $f(t)$ is:

$$F(\omega) = \mathcal{F}\left[f(t)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathbf{j}\,\omega\, t} \, \mathrm{d}t.$$

- The inverse Fourier transform is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{\mathbf{j}\,\omega\, t} \, \mathrm{d}\omega.$$