SESA2024 Astronautics

Chapter 3: Launch vehicles – Solutions

1. Mechanical environment data is given by the launch agencies to allow spacecraft manufacturers to emulate the launch environment in ground test facilities. The spacecraft is then exposed to these simulated environments to ensure that the spacecraft is sufficiently robust to survive launch.
The launch environment is generally severe, and has a significant input to the requirements for the spacecraft structural design.
For more detail, consult the literature, including the recommended course text.

2. Equations of motion of single-stage-to-orbit launcher.

Statement of Newton's second law for systems with momentum inflow and outflow (jets, rockets, etc.):

$$\frac{d}{dt}(M\mathbf{V}) = \mathbf{F_{ext}} + \{\text{rate of momentum inflow}\}$$
$$- \{\text{rate of momentum outflow}\}.$$

Referring to Figure 1, this becomes

$$\frac{d}{dt}(M\mathbf{V}) = \mathbf{D} + M\mathbf{g} + \mathbf{L} + \{0\} - \{\sigma(\mathbf{V} + \mathbf{V_{ex}})\}.$$

Using the **i**, **j** reference system this can be written

$$\dot{M}\mathbf{V} + M\dot{\mathbf{V}} = -D\mathbf{i} - Mg\cos(90^{\circ} - \gamma)\mathbf{i} - Mg\sin(90^{\circ} - \gamma)\mathbf{j}$$
$$+ L\mathbf{j} - \sigma(V - V_{ex})\mathbf{i},$$

where $\dot{M} = -\sigma$, and $\dot{\mathbf{V}} = \dot{V}\mathbf{i} + V\dot{\gamma}\mathbf{j}$. Hence,

$$-\sigma V\mathbf{i} + M(\dot{V}\mathbf{i} + V\dot{\gamma}\mathbf{j}) = -D\mathbf{i} - Mg\sin\gamma\mathbf{i} - Mg\cos\gamma\mathbf{j} + L\mathbf{j} - \sigma V\mathbf{i} + \sigma V_{ex}\mathbf{i}.$$

Cancelling the $-\sigma V\mathbf{i}$ terms and looking at the **i** component and **j** component gives

Along-track equations of motion: $M\dot{V} = -D - Mg\sin\gamma + \sigma V_{ex}$ (1)

Across-track equations of motion: $MV\dot{\gamma} = -Mg\cos\gamma + L$ (2)
Equation (1) is solved in the notes to give

$$\Delta V = v - u = V_{ex}\log_e\left(\frac{M_0}{M_b}\right) - \int_0^t g\sin\gamma dt - \int_0^t \frac{D}{M}dt$$

$$= \Delta V_{ideal} - \Delta V_g - \Delta V_{D.}$$

When a rocket vehicle is launched, the ascent trajectory needs to be optimised to acquire the maximum speed change ΔV by minimising the losses ΔV_g due to gravity, and ΔV_D due to drag. That is, most of the fuel mass $M_{fuel} = M_0 - M_b$ used to produce ΔV_{ideal} is usefully utilised to produce the maximum physical speed change ΔV .

One way of acquiring orbit would be to ascend vertically to orbital height and then rotate the velocity vector into the horizontal direction to inject into orbit. Although this strategy minimises drag loss (the vehicle climbs through the denser part of the atmosphere quickly), it does however accumulates a large gravity loss. Alternatively, the vehicle could climb to orbit on a 'flat trajectory' with a small climb angle γ . This would minimise the gravity loss term, but the vehicle would spend a long time in the denser part of the atmosphere, so giving a large drag loss. An optimised trajectory, therefore, tries to take a path between these extremes. The shuttle, for example, climbs vertically for a relatively short period to escape the denser part of the atmosphere (minimises drag loss), and then 'rolls over' into a shallow climb (minimises gravity loss) to acquire orbit.

3. Prove that

$$\frac{M_0}{M_b} = \frac{M_p + M_s + M_f}{M_p + M_s} = \frac{1+s}{p+s}.$$

Now

$$\frac{1+s}{p+s} = \frac{1+(M_s/M_f)}{(M_p/M_0)+(M_s/M_f)} = \frac{(M_f+M_s)M_0}{M_pM_f + M_0M_s}$$

$$= \frac{(M_f+M_s)M_0}{(M_0 - M_s - M_f)M_f + M_0M_s} = \frac{(M_f+M_s)M_0}{(M_f+M_s)M_0 - (M_s+M_f)M_f}$$

$$= \frac{M_0}{M_0 - M_f} = \frac{M_p + M_s + M_f}{M_p + M_s} = \frac{M_0}{M_b}.$$

4. Referring to Figure 2, and noting that the payload of the i th stage is the total mass above the i th stage, we have

$$p_3 = M_{p3} / M_{03}, \qquad p_2 = M_{p2} / M_{02} = M_{03} / M_{02},$$
$$p_1 = M_{p1} / M_{01} = M_{02} / M_{01}.$$

Therefore,

$$p_1 p_2 p_3 = \frac{M_{p3}}{M_{03}} \frac{M_{03}}{M_{02}} \frac{M_{02}}{M_{01}} = \frac{M_{p3}}{M_{01}} = P.$$

5. For a three-stage vehicle:
The ideal delta-V for stage 1 is

$$\Delta V_{ideal,1} = V_{ex,1} \log_e \left(\frac{1 + s_1}{p_1 + s_1} \right) \text{ where } s_1 = M_{s1} / M_{f1} \text{ and } p_1 = M_{p1} / M_{01}.$$

The ideal delta-V is given by similar expressions for stages 2 and 3, so that the sum is

$$\Delta V_{ideal} = \sum_{i=1}^{3} V_{ex,i} \log_e \left(\frac{1 + s_i}{p_i + s_i} \right).$$

Insertion of numerical values gives

$$\Delta V_{ideal} = 3(3 \text{ km/sec}) \log_e \left(\frac{1.1}{0.315} \right) = 11.254 \text{ km/sec}.$$

Overall payload fraction $P = p_1 p_2 p_3 = (0.215)^3 \approx 0.01.$

6. Referring to the definitions of M_{pi}, M_{si}, M_{fi} and M_{0i} in Figure 2, and using the data in Table 1, we have the following:

First stage: $\Delta V_{ideal,1} = V_{ex,1} \log_e \left(\frac{1 + s_1}{p_1 + s_1} \right)$, where

$s_1 = M_{s1} / M_{f1} = 18.0/251.0 = 0.0717, \quad p_1 = M_{p1} / M_{01}.$

Here

$M_{p1} = (3.3 + 35.0) + (1.3 + 10.8) + (2.1) = 52.5$ tonnes,

$M_{01} = (52.5) + (18.0 + 251.0) = 321.5$ tonnes.

Therefore $p_1 = 52.5/321.5 = 0.1633,$ and

$$\Delta V_{ideal,1} = (2.454\ \text{km/sec})\log_e\left(\frac{1.0717}{0.1633+0.0717}\right) = 3.724\ \text{km/sec.}$$

Second stage: $\Delta V_{ideal,2} = V_{ex,2}\log_e\left(\dfrac{1+s_2}{p_2+s_2}\right)$, where

$s_2 = M_{s2}/M_{f2} = 3.3/35.0 = 0.0943,\quad p_2 = M_{p2}/M_{02}$.

Here

$M_{p2} = (1.3+10.8)+(2.1) = 14.2$ tonnes,

$M_{02} = (14.2)+(3.3+35.0) = 52.5$ tonnes.

Therefore $p_2 = 14.2/52.5 = 0.2705,$ and

$$\Delta V_{ideal,2} = (2.884\ \text{km/sec})\log_e\left(\frac{1.0943}{0.2705+0.0943}\right) = 3.168\ \text{km/sec.}$$

Third stage: $\Delta V_{ideal,3} = V_{ex,3}\log_e\left(\dfrac{1+s_3}{p_3+s_3}\right)$, where

$$s_3 = M_{s3}/M_{f3} = 1.3/10.8 = 0.1204,\quad p_3 = M_{p3}/M_{03} = \frac{2.1}{[(1.3+10.8)+2.1]} = 0.1479.$$

Therefore,

$$\Delta V_{ideal,3} = (4.356\ \text{km/sec})\log_e\left(\frac{1.1204}{0.1479+0.1204}\right) = 6.226\ \text{km/sec.}$$

Total ideal delta-V is:

$\Delta V_{ideal} = 3.724+3.168+6.226 = 13.118$ km/sec.

Total speed change is:

$\Delta V = \Delta V_{ideal} - \Delta V_g - \Delta V_D = 13.118 - 2.6 = 10.518$ km/sec.

Estimate semi-major axis a and eccentricity e of resulting orbit (payload is injected at perigee of orbit, since velocity vector is horizontal at burnout):

Use energy equation to calculate *a* –

$\dfrac{1}{2}V^2 - \dfrac{\mu}{r_p} = \dfrac{-\mu}{2a}$, where $r_p = R_E + h = 6378+300\ \text{km} = 6678\ \text{km}$.

Therefore,

$$a = \frac{-\mu}{2}\left(\frac{1}{2}V^2 - \frac{\mu}{r_p}\right)^{-1} = 45561\ \text{km, using } \mu = 398600\ \text{km}^3/\text{sec}^2.$$

Use perigee equation to calculate *e* –

$$r_p = a(1-e) \;\Rightarrow\; e = 1 - \left(r_p / a\right) = 0.853.$$