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SESA6085 – Advanced Aerospace Engineering Management

Lecture 11

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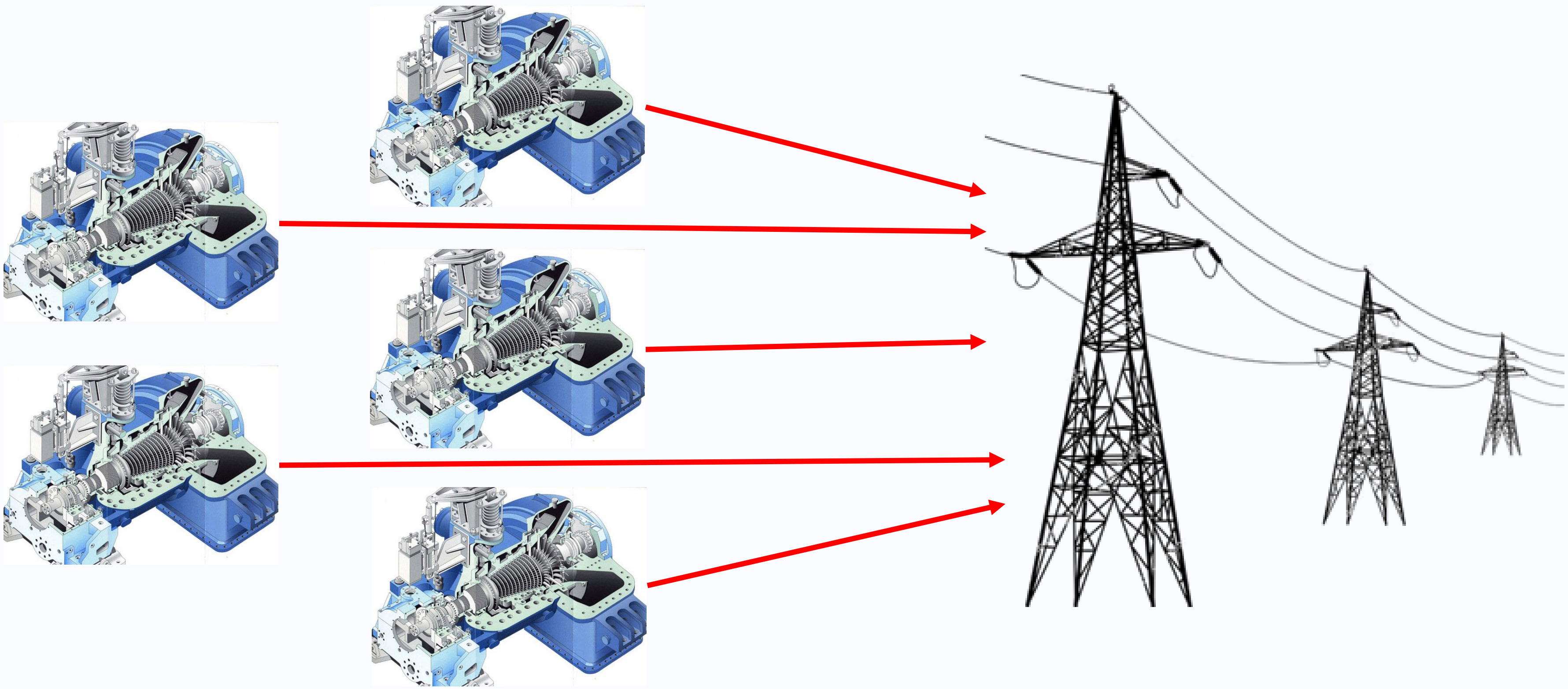
Landing Gear Example

- An aircraft landing gear has 4 tyres
- Tyre bursts probability is a function of the load
- How would you solve this system?
  - 2-out-of-4 redundant?
  - Parallel?
  - Serial?

Power Station Example

- A power station has 5 steam turbines providing electricity to the national grid



Power Station Example

- The power station must provide a certain amount of electricity to the grid and the reliability of the turbines is affected by their load
- How would you model the reliability of such a system?
  - Series?
  - Parallel?
  - *m*-out-of-*n* redundant?

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Steel Cable Example

- A steel cable is constructed by winding together 10 solid steel wires
- The failure of each of these wires is described by a known PDF and is a function of the load the wire is subjected to
- How would you model the reliability of the cable?
  - Series?
  - Parallel?
  - *m*-out-of-*n* redundant?

Steel Cable Tension Test

- Carefully watch the following video…

https://www.youtube.com/watch?v=qZ_wdaUn0dg&feature=player_embedded

Issues?

- Do these examples and videos highlight any issues or limitations with our current approaches to reliability modelling?

Load Sharing Systems

Load-Sharing Systems

- The three previous reliability problems are all examples of load-sharing systems
- Such systems include *n* components each sharing a load and if one or more component fails then the load on the remaining components increases accordingly
- Naturally this type of situation is encountered throughout engineering e.g.
  - Mechanical & electronic systems
  - Manufacturing

Load-Sharing Systems

- Modelling the reliability of load sharing systems is therefore extremely important
- But we need a new process in order to achieve this modelling
  - Now the failures of components are no longer independent

Notation & Assumptions

- Before continuing let's first define our notation and assumptions
- n, defines the number of components in our system which are equally sharing the load
- Overall the system fails when every component within it fails but as a component fails its load is now shared amongst the remaining components

Notation & Assumptions

- L, defines the **load per component** ($L \geq 0$)
  - Note we use load per component and not total applied load as it makes the notation much easier
- $F(L)$, defines a failure distribution for individual components
  - $F(L)$ is therefore the probability of a component failing due to load L
- $G_n(L)$, defines the probability of failure of a system of n components due to a load L per component

The Daniels Model

- Developed by H.E. Daniels in 1945[1]
- Originally this model was developed for textiles but can be used for composite materials
- Fibres are not its only applications and the technique could easily be applied to any load sharing system
- Let's now look at formulating $G_n(L)$ for small values of n

[1] – Daniels (1945), "The Statistical Theory of the Strength of Bundles of Threads", Proceedings of the Royal Society, Vol. 183

Simple Cases ($n = 1$)

- The simplest case is when there is only a single component under load
- In this case the:
  - The load per component is L
  - $F(L)$ describes the failure distribution
- What is the probability of failure for the whole system, $G_1(L)$?

$$G_1(L) = F(L)$$

Simple Cases ($n = 2$)

- Now consider a case with 2 components with a load L per component
- What are the situations that the system will fail?
  - If component #1 fails *under* a load L & component #2 fails *under* a load $2L$
  - If component #2 fails *under* a load L & component #1 fails *under* a load $2L$

Simple Cases ($n = 2$)

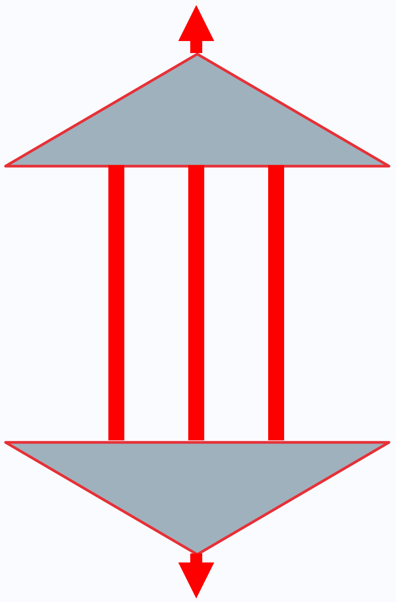
- This leads us to a probability of failure $G_2(L)$ of:

$$G_2(L) = 2F(L)F(2L) - F(L)^2$$

- The probability of component 1 failing under L and component two under $2L$ equals the probability of component 2 failing under L and component one under $2L$
- Remember that these are cumulative probabilities hence the case where both fail under L is counted twice by the first term and so we need to subtract $F(L)^2$

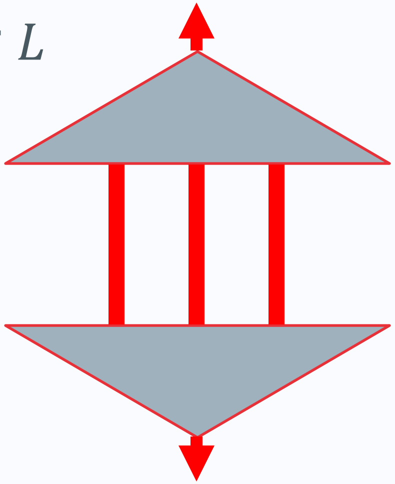
A Harder Case ($n = 3$)

- Now consider the case with 3 components
- Again each component sees a load of L
- What are the conditions for failure?
- How many conditions for failure are there?
  - There are a total of 4 conditions

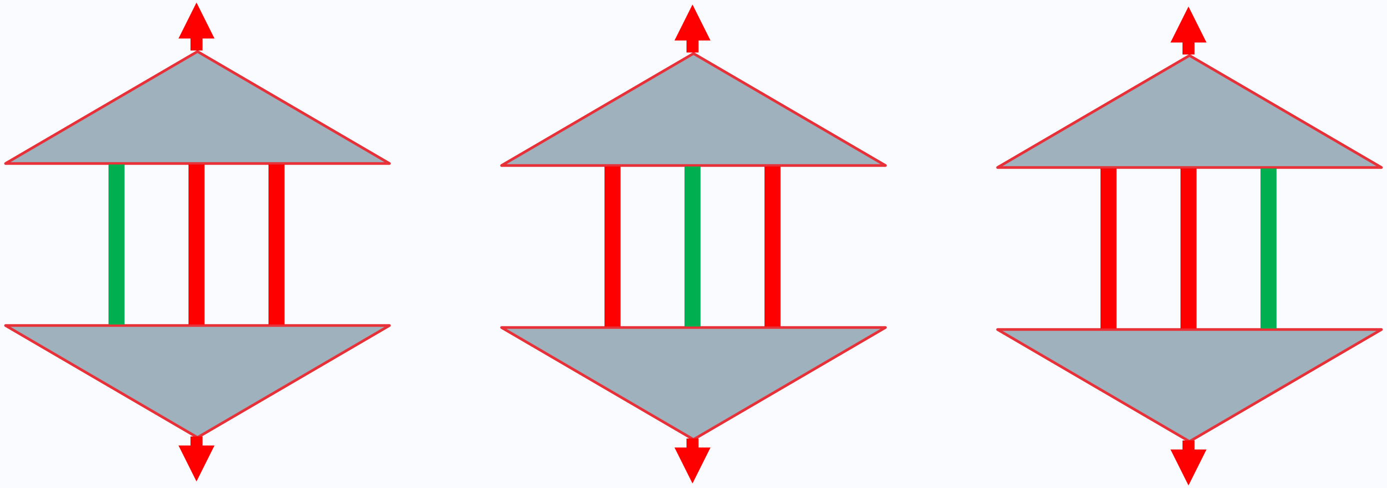


A Harder Case ($n = 3$)

1. All 3 components fail under L

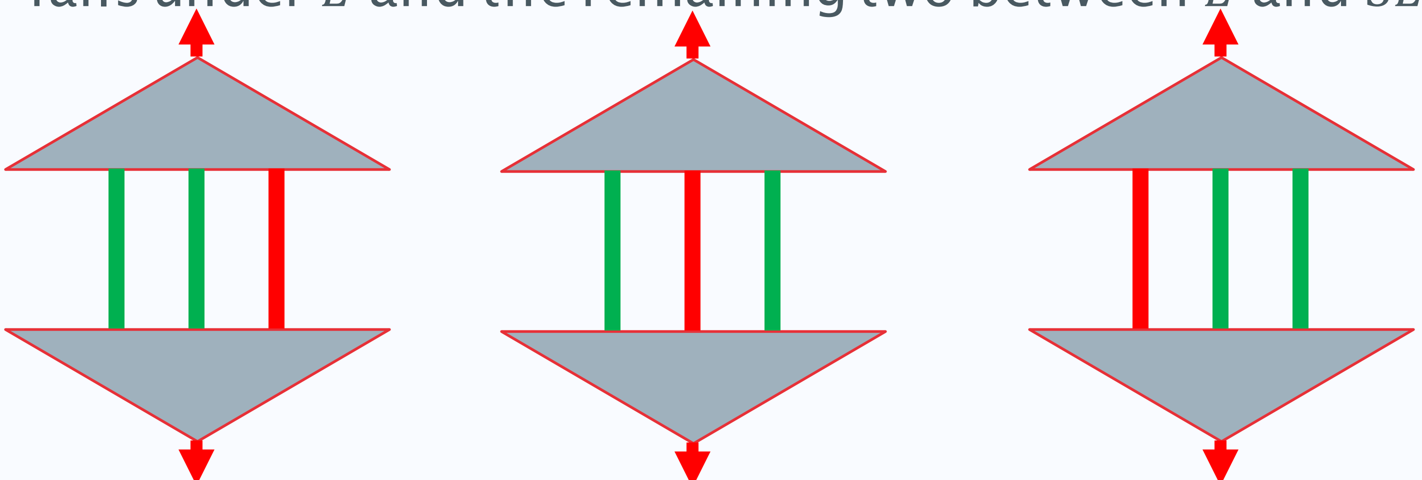


2. 2 components fail under L and the third between L and $3L$

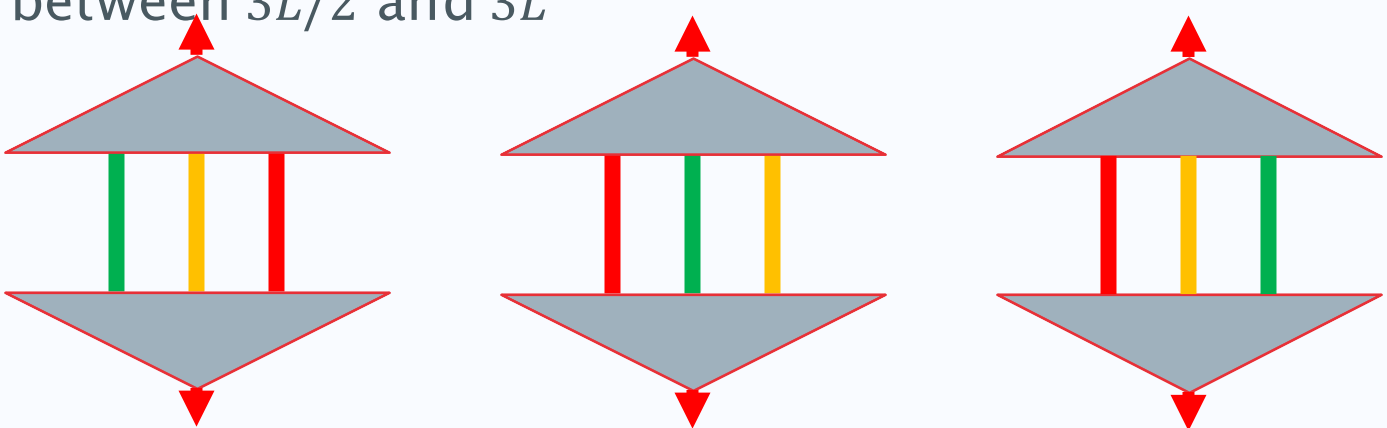


A Harder Case ($n = 3$)

3. 1 fails under L and the remaining two between L and $3L/2$



4. 1 fails under L, another between L and $3L/2$ and the third between $3L/2$ and $3L$



…

A Harder Case ($n = 3$)

- All 3 components fail under L

$$F(L)^3$$

- 2 components fail under L and the third between L and $3L$

$$3 \times F(L)^2\big(F(3L) - F(L)\big)$$

- 1 fails under L and the remaining two between L and $3L/2$

$$3 \times F(L)(F(3L/2) - F(L))^2$$

- 1 fails under L, another between L and $3L/2$ and the third between $3L/2$ and $3L$

$$6 \times \mathrm{F(L)}\big(\mathrm{F}(3L/2) - \mathrm{F(L)}\big)\big(\mathrm{F(3L)} - \mathrm{F}(3L/2)\big)$$

A Harder Case ($n = 3$)

- The probability of system failure therefore becomes:

$$G_3(L) = 6\mathrm{F(L)}\big(\mathrm{F}(3L/2) - \mathrm{F(L)}\big)\big(\mathrm{F(3L)} - \mathrm{F}(3L/2)\big) +$$

$$3F(L)(F(3L/2) - F(L))^2 +$$

$$3F(L)^2\big(F(3L) - F(L)\big) +$$

$$F(L)^3$$

Cases for $n > 3$

- Clearly as the number of components, n, increases the complexity of this calculation increases quite considerably
- We therefore require a method of solving such systems more efficiently
- This is achieved via a recursive formula

Load Sharing with *n* Components

- The generalised probability of system failure for a system with *n* components is given by the following recursive formula:

$$G_n(L) = \sum_{r=1}^{n} \binom{n}{r} (-1)^{r+1} F^r(L) G_{n-r}\left(\frac{nL}{n-r}\right)$$

- If *r* components fail then this leaves *n-r* components each with a load of $\frac{nL}{n-r}$
- The binomial coefficient here denotes the number of ways that a particular combination of failures occurs

Load Sharing with 2 Components

- Lets use this recursive formula to define the equation for 2 components
- First let us recall that: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Filling in n=2 into our recursive formula gives us:

$$G_2(L) = \sum_{r=1}^{2} \binom{2}{r} (-1)^{r+1} F^r(L) G_{2-r}\left(\frac{2L}{2-r}\right)$$

Load Sharing with 2 Components

- Expanding this out we obtain:

$$G_2(L) = \binom{2}{1}(-1)^2 F^1(L) G_1(2L) + \cdots$$

$$\binom{2}{2}(-1)^3 F^2(L) G_0$$

- We can observe that we need to use our general formula again to calculate $G_1(2L)$
- What is G_0 equal to?
- What should $G_1(2L)$ equal?

Load Sharing with 2 Components

- $G_0 = 1$
- $G_1(2L)$ should equal $F(2L)$ but lets confirm it

$$G_1(2L) = \binom{1}{1}(-1)^2 F^1(2L)G_0$$

$$G_1(2L) = \mathrm{F}(2\mathrm{L})$$

Load Sharing with 2 Components

- Substituting this back into our expression for $G_2(L)$ we obtain:

$$G_2(L) = 2F(L)F(2L) - F^2(L)$$

- Which matches the formula we obtained previously for $G_2(L)$ just by considering the different potential failure modes
- Now lets try the case where *n=3*

Load Sharing with 3 Components

- This time our expression is given by:

$$G_3(L) = \sum_{r=1}^{3} \binom{3}{r} (-1)^{r+1} F^r(L) G_{3-r}\left(\frac{3L}{3-r}\right)$$

- Which after expanding out equals:

$$G_3(L) = 3F(L)G_2\left(\frac{3L}{2}\right) - 3F^2(L)G_1(3L) + F^3(L)$$

- Once again we need to recursively calculate both $G_2\left(\frac{3L}{2}\right)$ and $G_1(3L)$

Load Sharing with 3 Components

- Fortunately we've just calculated an expression for G_2 which we can reuse replacing L with $\frac{3L}{2}$
- We also know the expression for G_1
- Hence G_3 is calculated to be:

$$G_3 = 6F(L)F\left(\frac{3L}{2}\right)F(3L) - 3F(L)F^2\left(\frac{3L}{2}\right) - \cdots$$

$$3F^2(L)F(3L) + F^3(L)$$

Load Sharing with 3 Components

- Previously we saw that the probability of failure for 3 load sharing components was:

$$G_3(L) = 6\mathrm{F}(\mathrm{L})\big(\mathrm{F}(3L/2) - \mathrm{F}(\mathrm{L})\big)\big(\mathrm{F}(3\mathrm{L}) - \mathrm{F}(3L/2)\big) +$$

$$3F(L)(F(3L/2) - F(L))^2 +$$

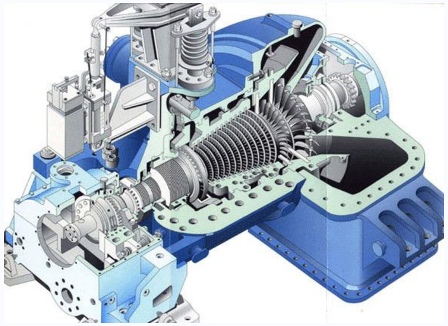
$$3F(L)^2\big(F(3L) - F(L)\big) +$$

$$F(L)^3$$

- While these look very different they are actually equivalent results. I would encourage you to confirm this for yourself

Example 1

- An engineer is tasked with designing a powerplant capable of producing 1.5 MW of electricity with a reliability of >95%



- The engineer has a model of gas turbine available to him whose failures are normally distributed with μ = 1.0MW and σ = 0.25MW
- How many of these gas turbines are required for the plant to produce 1.5MW with >95% reliability?

Example 1

- How would you solve this problem?
- What is the load on the system?
  - 1.5MW
- How many gas turbines do we have?
  - Unknown so lets try a number of different options
- What is the load per turbine?
  - Depends on the number of turbines
- How do we define the reliability of the system?

$$R(L) = 1 - G_n(L)$$

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Example 1

- Let's consider n=1, 2 & 3
- We already know the expressions for these cases
- It's therefore only a matter of calculating:
  - $F(L) = ?$
  - $F(2L) = ?$
  - $F(3L) = ?$
  - $F(3L/2) = ?$
- Where L=1.5/n in our case

How do we calculate these values?

Example 1

- The values for these expressions come from the CDF for our gas turbine
- Hence, for example,

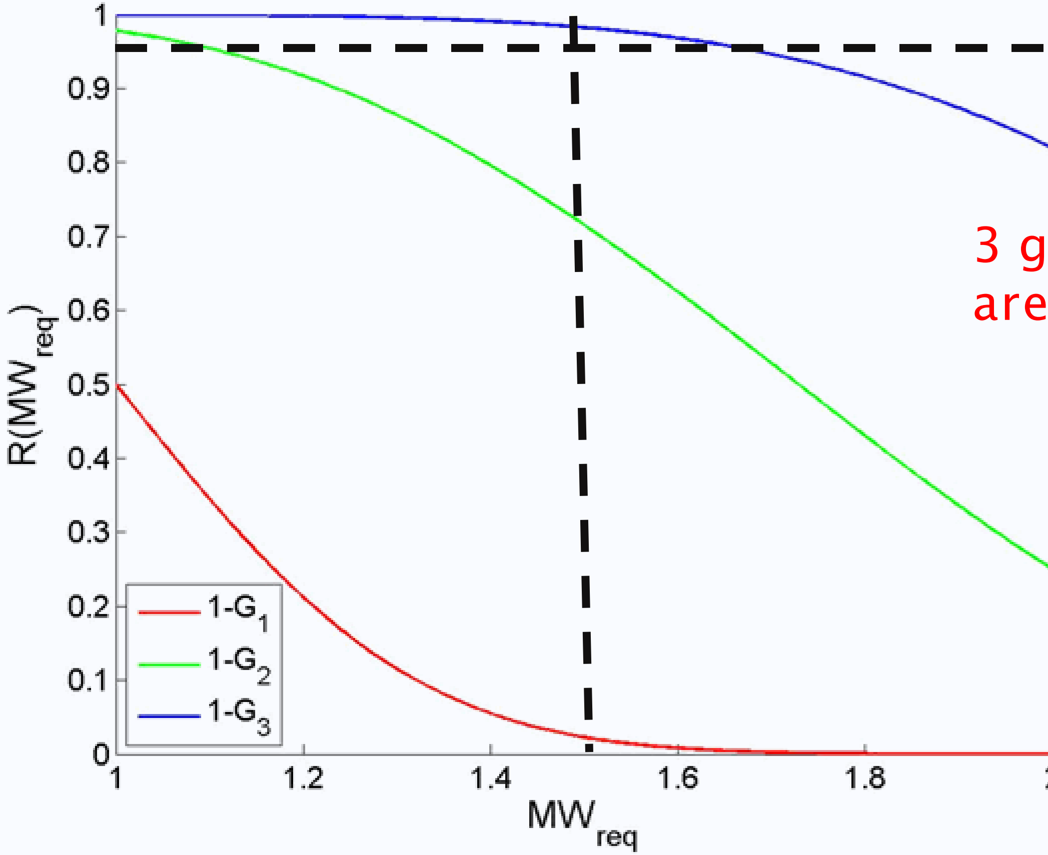
$$F(L) = \int_0^L f(l)dl$$

$$F(2L) = \int_0^{2L} f(l)dl$$

- And so on…

Example 1

- The reliabilities for n=1, 2 & 3 systems can be calculated and used to identify those which give the desired reliability



3 gas turbines are required

Recall each turbine has reliability described by μ = 1.0MW & σ = 0.25MW

Example 1

- Remember that for each case the initial load per gas turbine will change
- The required load is 1.5MW
  - If n=1, the load per turbine is 1.5MW
  - If n=2, the load per turbine is 0.75MW
  - If n=3, the load per turbine is 0.5MW
- Hence there are large changes in reliability as the number of turbines increases

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Example 2

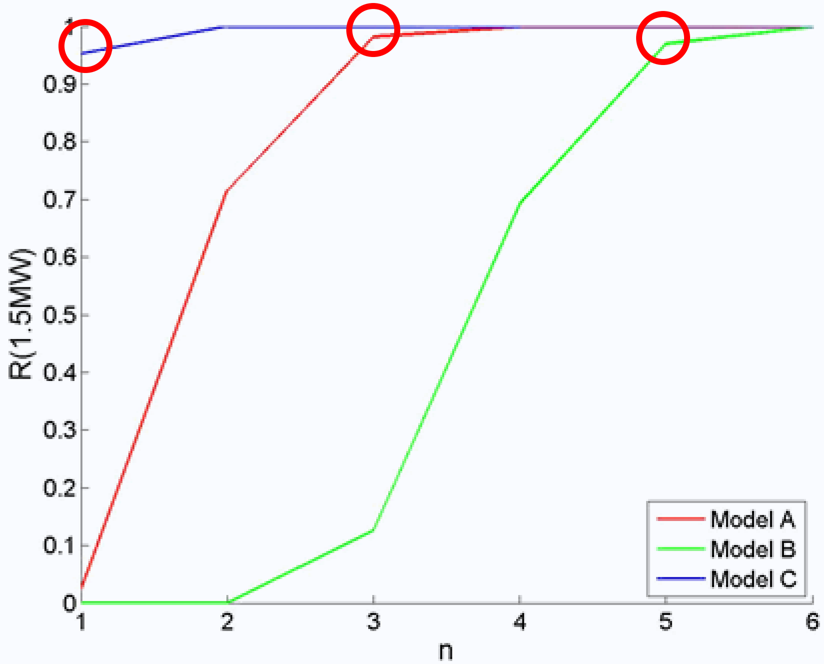
- Lets make this example a little harder…
- The engineer has two additional models of gas turbine to choose from with failures defined by normal distributions

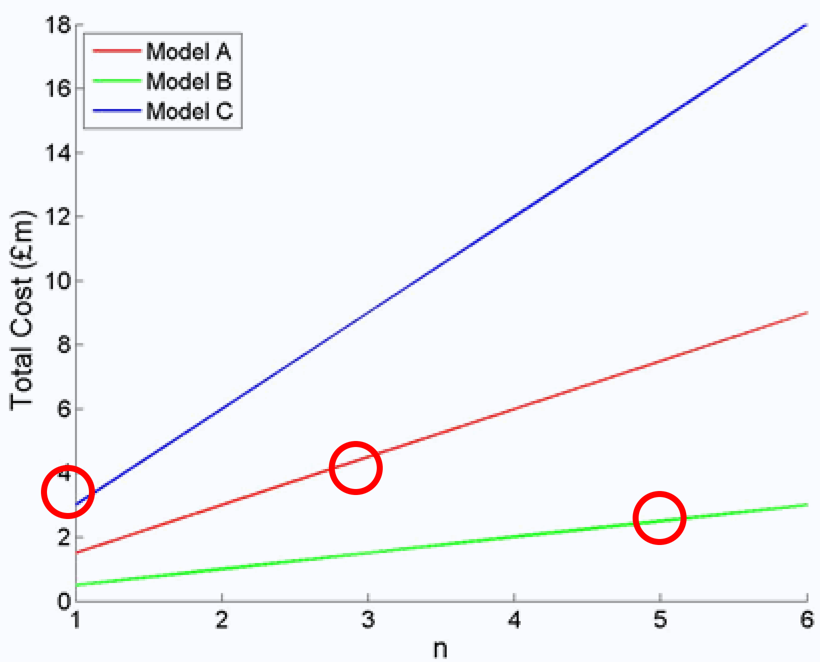
Model	μ	σ	Cost (£m)
A	1.0	0.25	1.5
B	0.5	0.1	0.5
C	2.0	0.3	3.0

- The power plant must achieve >95% reliability at 1.5MW
- But the cost of power plant should be minimal
- How can we solve this case?
- What do you think the answer might be and why?

Example 2

- Plotting out the reliability for power plants using the three models with different numbers of turbines and the costs:





Example 2

- We therefore have the following results:

Model	Reliability	n	Cost (£m)
A	0.982	3	4.5
B	0.969	5	2.5
C	0.955	1	3

- With a power plant constructed from 5 gas turbines of model B being the cheapest and meeting the reliability requirement
- Of course this is a simple case and we've not considered…
  - Other costs e.g. maintenance, infrastructure etc.

Load Sharing with *n* Components

- The recursive formula is relatively easy to program and fast to compute

$$G_n(L) = \sum_{r=1}^{n} \binom{n}{r} (-1)^{r+1} F^r(L) G_{n-r}\left(\frac{nL}{n-r}\right)$$

- However the above formula can become unstable as *n* increases
  - Why might this be the case?
- For cases with more than 40 components the formula produces long series of terms with alternating signs

Load Sharing with *n* Components

- For cases with large *n* values this numerical instability is a serious problem
- For these cases there are a number of numerical approximations which can be used to calculate failure probabilities
- These approximations, of which the Daniels model is one, vary in complexity and accuracy

Extensions of Load Sharing Systems

- Beyond the basic load sharing system described there are a number of further extensions in the literature
  - The inclusion of random slack into fibre bundles
  - The inclusion of random changes in material properties e.g. change from elastic to plastic
  - See Crowder et al. for more information on this
- Of course, we've assumed that the components in the system are identical in terms of their failure distribution which may not always be the case
  - Monte Carlo analysis can help in these situations

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