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Mathematics for Engineering and the Environment Part II

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Contents

Chapter 1

Ordinary Differential Equations

1.1 Second order equations

1.1.1 Linear Second Order ODEs

The general linear second order ODE can be written as

$$A(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + B(x)\frac{\mathrm{d}y}{\mathrm{d}x} + C(x)y + D(x) = 0 \tag{1.1}$$

Taking the $D(x)$ term onto the RHS and dividing by $A(x)$ we may write such an equation in the *standard form*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = r(x). \tag{1.2}$$

where $p(x) = B(x)/A(x)$, $q(x) = C(x)/A(x)$ and $r(x) = -D(x)/A(x)$.

Homogeneous Equations

In the case where $r(x) \equiv 0$ the equation becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0. \tag{1.3}$$

and is called *homogeneous*. For such homogeneous equations the solution will be of the form

$$y(x) = c_1y_1(x) + c_2y_2(x) \tag{1.4}$$

where $y_1(x)$ and $y_2(x)$ are *linearly independent* solutions of the homogeneous equation (1.3) and c_1 and c_2 are constants. The functions $y_1(x)$ and $y_2(x)$ are linearly independent if

$$c_1y_1(x) + c_2y_2(x) \equiv 0 \quad \Leftrightarrow \quad c_1 = 0 \text{ and } c_2 = 0. \tag{1.5}$$

The Wronskian determinant

The following section is NOT examinable. All non-examinable material will be in red. Examainable material starts again with black text.

If we have two solutions y_1 and y_2 of the homogeneous linear ODE (1.3), how do we know if they are linearly independent? One way to motivate this is to note that if they were linearly *dependent* then

$$\frac{y_1}{y_2} = \text{const.} \tag{1.6a}$$

$$\Rightarrow \quad \left(\frac{y_1}{y_2}\right)' = 0 \tag{1.6b}$$

$$\Rightarrow \quad y_1 y_2' - y_1' y_2 = 0. \tag{1.6c}$$

We rewrite this using the *Wronskian determinant* $W[y_1, y_2](x)$ which is defined as follows:

$$W(x) = W[y_1, y_2](x) \tag{1.7a}$$

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{1.7b}$$

$$= y_1 y_2' - y_1' y_2. \tag{1.7c}$$

Using the Wronskian one can show that two solutions y_1 and y_2 are linearly independent if, and only if, $W(x) \neq 0$.

In addition, we can straightforwardly check by direct substitution into (1.3) that the Wronskian obeys

$$\frac{\mathrm{d}W}{\mathrm{d}x} + p(x)W = 0. \tag{1.8}$$

This follows from the explicit calculation below:

$$\frac{\mathrm{d}W}{\mathrm{d}x} = (y_1 y_2'' + y_1' y_2') - (y_1'' y_2 + y_1' y_2') \tag{1.9a}$$

$$= y_1 y_2'' - y_1'' y_2 \tag{1.9b}$$

$$= -p(x)\left(y_1 y_2' - y_2 y_1'\right) \tag{1.9c}$$

where we have used that $y_{1,2}$ are solutions of the homogeneous equation (1.3)

$$= -p(x)W. \tag{1.9d}$$

As the simple equation (1.8) for the Wronskian is separable we can write the solution as

$$\int \frac{dW}{W} = -\int p(x)dx \quad \Rightarrow \quad \log W = -\int p(x)dx + C \tag{1.10}$$

So that we may write

$$W = Ke^{-\int p\,\mathrm{d}x} \tag{1.11}$$

where $K = e^C$ is a constant of integration. From this result it immediately follows that the Wronskian is either zero for all x in the range considered (if $K = 0$), or *never* zero for any x in the range considered (if $K \neq 0$).

Homogeneous constant coefficient

After looking at the general theory we now turn to some examples. An important class of second order ODEs are the simple linear, homogeneous, constant coefficient equations

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0, \tag{1.12}$$

where a, b and c are constants. We need to find linearly independent solutions. To do this we assume that they take the form $y \propto e^{\lambda x}$. Substituting this guess into equation (1.12) gives the *auxiliary equation*

$$a\lambda^2 + b\lambda + c = 0. \tag{1.13}$$

Depending on the type of the roots of the auxiliary equation (real, complex, repeated) there are different types of solution:

1. $b^2 - 4ac > 0$: Real roots λ_1 and λ_2 giving independent solutions

   $$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x} \tag{1.14}$$

   and general solution

   $$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \tag{1.15}$$

2. $b^2 - 4ac < 0$: Complex conjugate roots $\lambda = \alpha \pm \mathrm{j}\beta$ giving independent (real) solutions

   $$y_1 = e^{\alpha x}\cos(\beta x), \tag{1.16a}$$
   $$y_2 = e^{\alpha x}\sin(\beta x). \tag{1.16b}$$

   and general solution

   $$y(x) = \left(c_1\sin(\beta x) + c_2\cos(\beta x)\right)e^{\alpha x} \tag{1.17}$$

3. $b^2 - 4ac = 0$: Repeated root λ giving one solution $e^{\lambda x}$. Direct calculation shows that a second solution can be computed to give

   $$y_1 = e^{\lambda x}, \tag{1.18a}$$
   $$y_2 = xe^{\lambda x}. \tag{1.18b}$$

   and general solution

   $$y(x) = (c_1 + c_2 x)e^{\lambda x} \tag{1.19}$$

Euler equations

The *Euler* type of linear, homogeneous equation are also straightforward. These have the form

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + cx\frac{\mathrm{d}y}{\mathrm{d}x} + d\,y = 0 \tag{1.20}$$

where c, d are constants. Again we need to find two independent solutions. We guess that such solutions are $y \propto x^k$ which, on substituting into equation (1.20), gives the condition

$$k^2 + (c-1)k + d = 0. \tag{1.21}$$

If equation (1.21) has distinct real roots k_1 and k_2 then the general solution to the Euler equation (1.20) can be written

$$y = c_1x^{k_1} + c_2x^{k_2}. \tag{1.22}$$

If equation (1.21) has repeated or complex roots it is better to use the change of variables $t = \log(x)$ to transform equation (1.20) to the constant coefficient problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (c-1)\frac{\mathrm{d}y}{\mathrm{d}t} + d\,y = 0 \tag{1.23}$$

which can be solved using the techniques of section 1.1.1.

Reduction of order and the Wronskian

(Not examined)

We have seen that in general the solution of a second order equation requires finding two independent solutions of the homogeneous equation. There are a number of situations where it is possible to find one solution of the equation (often a simple polynomial) but it is not easy to find a second solution. We show below a method of using one given solution to find a second solution. This makes use of the fact that the Wronskian is given by (1.11).

Assume that we have somehow found one solution y_1 of the general linear homogeneous ODE given by equation (1.3). We assume that the second solution y_2 is related to y_1 by

$$y_2(x) = v(x)y_1(x). \tag{1.24}$$

This immediately implies that

$$y_2' = v'y_1 + vy_1' \tag{1.25}$$

and from this we can compute the Wronskian determinant (see section 1.1.1)

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & vy_1 \\ y_1' & v'y_1 + vy_1' \end{vmatrix} \tag{1.26a}$$

$$= v'y_1^2 \tag{1.26b}$$

$$= Ke^{-\int p\,\mathrm{d}x} \tag{1.26c}$$

where the last equation follows from equation (1.11).

This allows us to directly compute v, the function relating the two solutions, as

$$v = K\int \frac{e^{-\int p\,\mathrm{d}x}}{y_1^2}\,\mathrm{d}x + D \tag{1.27}$$

and hence the second solution is given by

$$y_2(x) = v(x)y_1(x) \tag{1.28a}$$

$$= Dy_1(x) + Ky_1(x)\int \frac{e^{-\int p\,\mathrm{d}x}}{y_1^2}\,\mathrm{d}x \tag{1.28b}$$

and, as we do not care about proportionality constants, we can write this as

$$y_2(x) = y_1(x)\int \frac{e^{-\int p\,\mathrm{d}x}}{y_1^2}\,\mathrm{d}x. \tag{1.28c}$$

(Example using Legendre polynomials)

1.1.2 Inhomogeneous Linear Second Order ODEs

If $r(x) \not\equiv 0$ we say that the differential equation is *inhomogeneous*.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = r(x) \tag{1.29}$$

We often call $r(x)$ a *source term* for reasons that will be clear from the examples.

In order to find the *general solution* to an inhomogeneous equation we first need to find a *particular solution* $y_p(x)$ (sometimes called a particular integral in some textbooks). That is some (any) solution of the inhomogeneous equation. If one then looks at the function $\tilde{y}(x) = y(x) - y_p(x)$ then it follows that it satisfies the homogeneous equation

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}\tilde{y}}{\mathrm{d}x} + q(x)\tilde{y} = 0. \tag{1.30}$$

We can therefore write the general solution in the form

$$\tilde{y}(x) = c_1y_1(x) + c_2y_2(x) \tag{1.31}$$

where $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0. \tag{1.32}$$

It follows that we can write the (*general*) solution of the inhomogeneous equation (1.2) in the form

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x). \tag{1.33}$$

So the only question remaining is how to find the particular integral solution of the full inhomogeneous problem in equation (1.29).

Finding a particular solution: undetermined coefficients

All the techniques used above concentrated on homogeneous linear ODEs. We now look at methods for finding particular solutions of the homogeneous equation.

The idea is to use a trial solution (i.e., make an informed guess for the solution) which is based on the form of the inhomogeneous term in equation (1.29), $r(x)$. We leave a number of *undetermined coefficients* in this guess, and substitute the result into the full equation (1.29). If this form is a potential solution there will be a non-trivial solution for the coefficients.

Standard simple trial solutions are:

$r(x)$	Trial function
Any polynomial degree n	Most general polynomial degree n
$e^{\alpha x}$	$Ce^{\alpha x}$
$\sin(\alpha x)$ or $\cos(\alpha x)$	$C\sin(\alpha x) + D\cos(\alpha x)$

Note that if the trial function can be written as a linear combination of the solutions to the homogeneous equation then, of course, the trial function will only be a solution of the *homogeneous* equation. In this case it is usual to multiply the trial solution by the independent variable until the trial solution is independent of the complementary functions.

Examples!

Variation of parameters

(Not examined)

The method of undetermined coefficients is simple and effective when the inhomogeneous term $r(x)$ is simple, but useless in general. A more general method can be found by *assuming* that the particular integral $y_p(x)$ is related to the solutions y_1 and y_2 of the homogeneous problem, by

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x). \tag{1.34}$$

Here v_1 and v_2 are functions to be determined that allow us to compute the unknown particular integral. Note that we have introduced *two* functions v_1 and v_2 in order to find a single function y_p, so that we have an additional degree of freedom. We will use this later on to impose an additional condition on v_1 and v_2 which will reduce things back to one degree of freedom in such a way as to simplify the calculation.

To find v_1 and v_2 We calculate the first and second derivatives of y_p and substitute into (1.2).

$$y_p' = (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2). \tag{1.35}$$

In order to simplify the expression for the second derivative we now use our additional degree of freedom to impose the condition that the term in the second bracket vanishes so that

$$v_1' y_1 + v_2' y_2 = 0. \tag{1.36}$$

With this condition (1.35) becomes

$$y_p' = (v_1 y_1' + v_2 y_2'). \tag{1.37}$$

We can then differentiate again to obtain

$$y_p'' = (v_1 y_1'' + v_2 y_2'') + (v_1' y_1' + v_2' y_2'). \tag{1.38}$$

Substituting into the inhomogeneous ODE (1.29) and remembering that y_1 and y_2 are solutions of the homogeneous ODE (1.32) we can see that

$$r(x) = y_p'' + p y_p' + q y_p \tag{1.39a}$$

$$= v_1 \underbrace{(y_1'' + p y_1' + q y_1)}_{\text{from equation (1.32)}} + v_2 \underbrace{(y_2'' + p y_2' + q y_2)}_{\text{from equation (1.32)}} + (v_1' y_1' + v_2' y_2'). \tag{1.39b}$$

Hence the unknowns v_1 and v_2 are completely determined from our assumption in equation (1.36) and our condition in equation (1.39) which can be written as

$$v_1' y_1 + v_2' y_2 = 0, \tag{1.40a}$$

$$v_1' y_1' + v_2' y_2' = r. \tag{1.40b}$$

The unknown functions $v_{1,2}$ can thus be determined by solving equation (1.40) as a linear system for $v_{1,2}'$ and integrating the results. We find that

$$v_1' = -\frac{r y_2}{y_1 y_2' - y_1' y_2} \tag{1.41a}$$

$$= -\frac{r(x) y_2(x)}{W[y_1, y_2](x)}, \tag{1.41b}$$

and similarly

$$v_2' = \frac{r(x)y_1(x)}{W[y_1, y_2](x)}. \tag{1.42}$$

This thus gives

$$v_1 = \int -\frac{r(x)y_2(x)}{W[y_1, y_2](x)}\,\mathrm{d}x, \tag{1.43a}$$

$$v_2 = \int \frac{r(x)y_1(x)}{W[y_1, y_2](x)}\,\mathrm{d}x \tag{1.43b}$$

and the particular integral of the original inhomogeneous ODE (1.29) follows from equation (1.34).

1.2 Boundary and Initial Value Problems

We have already seen that the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = r(x) \tag{1.44}$$

has general solution $y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x)$ where $y_p(x)$ is a particular solution of the equation and $y_1(x)$ and $y_2(x)$ are independent solutions of the homogeneous equation and c_1 and c_2 are constants.

In order to obtain a unique solution to the differential equation we need some way of fixing the constants c_1 and c_2. One common way of doing this is to fix the value of both $y(x)$ and $y'(x)$ at some point x_0. So that

$$y(x_0) = C, \tag{1.45a}$$

$$y'(x_0) = D. \tag{1.45b}$$

where C and D are given. Equation (1.44) together with the initial conditions (1.45) is called an *initial value problem*.

Rather than give the value of the function y and its derivative y' at a single point x_0 one can give alternatively give the value of y at *two* point x_0 and x_1, so that the solution satisfies the differential equation (1.44) together with the boundary conditions

$$y(x_0) = \alpha, \tag{1.46a}$$

$$y(x_1) = \beta. \tag{1.46b}$$

This is an example of a *boundary value problem*. More generally we can specify the value of some linear combination of the function and its derivatives at two pint x_0 and x_1 and demand that

$$a_0 y(x_0) + b_0 y'(x_0) = \alpha, \tag{1.47a}$$
$$a_1 y(x_1) + b_1 y'(x_1) = \beta. \tag{1.47b}$$

One can then show that the boundary value problem of solving the differential equation (1.44) on the region $x_0 \leq x \leq x_1$ subject to the boundary conditions (1.47) has a unique solution provided the associated homogeneous boundary problem (i.e. the case where $r(x) \equiv 0$, $\alpha = 0$, $\beta = 0$) only has the trivial solution (this result is known as the Fredholm Alternative)

1.2.1 Eigenvalues and functions

We now look at a class of BVPs where the point is to find the conditions under which the problem has a solution. We start by looking at a simple example where we try and find *non-trivial* solutions (i.e. solutions that are not zero everywhere) of the simple BVP

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y = 0, \tag{1.48a}$$
$$y(0) = 0, \tag{1.48b}$$
$$y(\pi) = 0. \tag{1.48c}$$

The *unknown* parameter λ may take any value. However, it may not be true that the BVP given by equation (1.48) has a solution for any value of λ. The task of finding a solution of the BVP involves finding not just solutions y that satisfy the differential equation (1.48a), but also of finding which values of λ allow such a solution to satisfy the boundary conditions given by equations (1.48b-1.48c).

In this simple case we can solve the BVP by inspection. However, to proceed more systematically we split the problem into three separate cases depending on the sign of the parameter λ:

1. $\lambda = -\alpha^2 < 0$. We first find solutions of equation (1.48a). As the differential equation is homogeneous, these combine to form the general solution. In this case the auxiliary equation has real roots $\pm\sqrt{-\lambda} = \pm\alpha$. It follows that the general solution is

   $$y(x) = a_1 e^{\alpha x} + a_2 e^{-\alpha x}. \tag{1.49}$$

   The only solution compatible with the boundary conditions (1.48b-1.48c) is the trivial solution $y \equiv 0$ when $a_1 = 0 = a_2$.

2. $\lambda = 0$. In this case the equation (1.48a) obviously has the general solution

$$y(x) = b_1 x + b_2. \tag{1.50}$$

Again, the only solution compatible with the boundary conditions (1.48b-1.48c) is the trivial solution $y \equiv 0$ when $b_1 = 0 = b_2$.

3. $\lambda = \mu^2 > 0$. In this case the auxiliary equation has imaginary roots $\pm\sqrt{-\lambda} = \pm \mathrm{j}\mu$. It follows that the general solution can be written as

$$y(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x). \tag{1.51}$$

From the left boundary condition, equation (1.48b), we must have that $c_2 = 0$ and so this reduces to

$$y(x) = c_1 \sin(\mu x). \tag{1.52}$$

In order to satisfy the final boundary condition, equation (1.48c), we *either* have $c_1 = 0$ and hence the trivial solution $y \equiv 0$ again, or we have a condition on μ that ensures that

$$\sin(\mu\pi) = 0. \tag{1.53}$$

This condition is obviously that $\mu = n$ for any integer n[1]; given our previous definitions, this gives the condition on λ that

$$\lambda = n^2, \qquad n = 1, 2, \ldots \tag{1.54}$$

Therefore the non-trivial solutions of the BVP (1.48) are

$$y_n(x) = \sin(nx). \tag{1.55}$$

The values of λ for which a solution of the BVP can be obtained are denoted λ_n, in this case are

$$\lambda_n = n^2, \qquad n = 1, 2, \ldots, \tag{1.56}$$

and are called the *eigenvalues* of the problem. The associated solutions $y_n(x)$ are called the *eigenfunctions*. Clearly the eigenvalues are uniquely determined but the eigenfunctions are not; any constant multiple of an eigenfunction y_n is also a solution of the BVP, and is also an eigenfunction.

A number of points should be noted for later consideration.

[1]The condition (1.53) is satisfied with $\mu = \pm n$, where n is a positive integer. Since $\sin(-nx) = -\sin(nx)$, we may restrict to $\mu = n > 0$ without loss of generality.

1. The eigenvalues are all real.

2. The different eigenfunctions are *orthogonal* in this sense:

$$\int_0^\pi \sin(mx)\sin(nx)\,\mathrm{d}x = 0, \qquad \text{if } m \neq n, \tag{1.57a}$$

$$\int_0^\pi \cos(mx)\cos(nx)\,\mathrm{d}x = 0, \qquad \text{if } m \neq n, \tag{1.57b}$$

$$\int_0^\pi \sin(mx)\cos(nx)\,\mathrm{d}x = 0, \qquad \text{for all } m, n, \tag{1.57c}$$

3. The eigenvalues form an increasing sequence of positive numbers that approach ∞.

4. The n^{th} eigenfunction ($\sin(nx)$) has exactly $n-1$ zeros inside the interval $[0, \pi]$ on which the BVP is defined. It also vanishes at the endpoints of the interval.

1.2.2 Sturm-Liouville form

(Not Examined)
We consider the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y}{\mathrm{d}x} + c(x)y = \lambda d(x)y. \tag{1.58}$$

This is a minor modification of previous second order ODEs that we have looked at. We shall later specify boundary conditions in order to turn it in to a BVP.

A standard form is found by multiplying the whole equation by $e^{\int b(x)\,\mathrm{d}x}$ to rewrite equation (1.58) as

$$\left(e^{\int b(x)\,\mathrm{d}x}y'\right)' + c(x)e^{\int b(x)\,\mathrm{d}x}y(x) = \lambda d(x)e^{\int b(x)\,\mathrm{d}x}y(x). \tag{1.59}$$

This is called the *self-adjoint form* of the ODE. The conventional notation is

$$(p(x)y')' - q(x)y + \lambda w(x)y = 0. \tag{1.60}$$

This is often written as

$$\mathcal{L}y = \lambda w(x)y \tag{1.61}$$

where $w(x)$, the *weight function*, is positive ($w \geq 0$) and non-trivial ($w \not\equiv 0$), and the differential operator $\mathcal{L}$ is given by

$$\mathcal{L} : y \to (p(x)y')' - q(x)y. \tag{1.62}$$

A solution y_n of the *regular Sturm-Liouville problem* given by equation (1.60) (with suitable boundary conditions) is called an *eigenfunction* of the problem, and the associated value of λ, λ_n is called the *eigenvalue*. As the notation suggests, we are expecting that more than one such solution exists.

Self-adjoint operators and boundary conditions

A key property is that the differential operator $\mathcal{L}$ is *self-adjoint*. This is true on $x \in [x_0, x_1]$ if, and only if, for all pairs of functions y_A, y_B satisfying the appropriate boundary conditions,

$$\int_{x_0}^{x_1} y_A \mathcal{L} y_B \, \mathrm{d}x = \int_{x_0}^{x_1} y_B \mathcal{L} y_A \, \mathrm{d}x. \tag{1.63}$$

For the Sturm-Liouville equation (1.60) we find that

$$\int_{x_0}^{x_1} (y_A \mathcal{L} y_B - y_B \mathcal{L} y_A) \, \mathrm{d}x = \int_{x_0}^{x_1} \left[-(py_B')' y_A + q y_A y_B + (py_A')' y_B - q y_B y_A \right] \mathrm{d}x \tag{1.64a}$$

$$= \left[-p y_A y_B' + p y_B y_A' \right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \left[p y_B' y_A' - p y_A' y_B' \right] \mathrm{d}x \tag{1.64b}$$

$$= \left[-p y_A y_B' + p y_B y_A' \right]_{x_0}^{x_1} \tag{1.64c}$$

We want this to vanish in order to have a self-adjoint problem. We therefore have a self-adjoint problem if, and only if,

$$\left[-p y_A y_B' + p y_B y_A' \right]_{x_0}^{x_1} = 0. \tag{1.65}$$

This can be guaranteed if at $x = x_0$ and $x = x_1$ we insist on some appropriate combination of the following boundary conditions:

1. $y = 0$: Dirichlet condition.
2. $y' = 0$: Neumann condition.
3. $y + ky' = 0$: Radiation condition.

4. $p = 0$: $x = x_0, x_1$ are singular points of the ODE.

We note that the conditions at the boundaries $x = x_0, x_1$ need not be the same.

There is one final possibility: periodic boundary conditions, where

$$y(x_0) = y(x_1), \quad y'(x_0) = y'(x_1), \quad p(x_0) = p(x_1). \tag{1.66}$$

Any of these are *Sturm-Liouville boundary conditions*, and given these the differential operator $\mathcal{L}$ is *self-adjoint*.

Properties of solutions

Properties 1. *The eigenvalues λ_n are real.*

Proof. Suppose that the eigenvalue λ, and possibly also the eigenfunction y are complex. By definition

$$\mathcal{L}y = \lambda w(x) y. \tag{1.67}$$

If we take the complex conjugate then, as both the operator $\mathcal{L}$ and the weight function $w(x)$ are real, we have

$$\mathcal{L}\bar{y} = \bar{\lambda} w(x) \bar{y}. \tag{1.68}$$

Taking the difference between these two equations and integrating over the domain gives us

$$\int_{x_0}^{x_1} (\bar{y}\mathcal{L}y - y\mathcal{L}\bar{y})\,\mathrm{d}x = (\lambda - \bar{\lambda}) \int_{x_0}^{x_1} w(x) y \bar{y}\,\mathrm{d}x. \tag{1.69}$$

As the operator is self-adjoint, we have from equation (1.63) that the left hand side of equation (1.69) vanishes identically.

As the eigenfunction solution is non-trivial we can see that

$$y\bar{y} = |y|^2 > 0, \tag{1.70}$$

and by assumption we have that $w(x) > 0$. Therefore in order for the right hand side to also vanish, we must have that $\lambda = \bar{\lambda}$, and hence the eigenvalues must be real. Without loss of generality, we can therefore also take the eigenfunctions to be real. □

Properties 2. *The eigenfunctions y_n are real.*

Proof. This is essentially given by the previous argument, as noted above, but an alternative is as follows.

Suppose that $y_n = f_n(x) + \mathrm{j}g_n(x)$ where both f, g are real and the eigenfunction y_n is non-trivial. As the differential equation (and hence operator $\mathcal{L}$) are real it follows that

$$\mathcal{L}(f_n + \mathrm{j}g_n) = \mathcal{L}(f_n) + \mathrm{j}\mathcal{L}(g_n) \tag{1.71a}$$
$$= -\lambda_n w(x)(f_n + \mathrm{j}g_n). \tag{1.71b}$$

By equating real and imaginary parts in (1.71) it follows that

$$\mathcal{L}(f_n) = -\lambda w(x) f_n, \tag{1.72a}$$
$$\mathcal{L}(g_n) = -\lambda w(x) g_n, \tag{1.72b}$$

which, after checking the boundary conditions and using that the original BVP is real, gives that both f_n and g_n must therefore be solutions of the original Sturm-Liouville problem. Therefore we can take the non-zero real function of f_n, g_n to be the real-valued eigenfunction with eigenvalue λ_n (at least one must be non-zero as y_n is non-trivial). □

Properties 3. *Eigenfunctions corresponding to distinct eigenvalues are orthogonal.*

Proof. Suppose that we have distinct eigenfunction solutions y_m, y_n with associated eigenvalues $\lambda_m \neq \lambda_n$. Again we can use the self-adjoint nature of the problem, given by equation (1.63), to show that

$$0 = \int_{x_0}^{x_1} (y_m \mathcal{L} y_n - y_n \mathcal{L} y_m)\,\mathrm{d}x \tag{1.73a}$$
$$= (\lambda_n - \lambda_m) \int_{x_0}^{x_1} w(x) y_m y_n \,\mathrm{d}x. \tag{1.73b}$$

Therefore, as the eigenvalues are distinct, we have that

$$\int_{x_0}^{x_1} w(x) y_m y_n \,\mathrm{d}x = 0. \tag{1.74}$$

We say that the associated eigenfunctions y_m, y_n are *orthogonal with respect to the weight function* $w(x)$. □

Properties 4. *There are infinitely many eigenvalues $\lambda_1 < \lambda_2 < \ldots$ that approach infinity, and the associated eigenfunction $y_n(x)$ has exactly $(n-1)$ zeros in the interval (x_0, x_1).*

Properties 5. *The eigenfunctions are* complete *in the sense that for* any *bounded, piecewise continuous function* $F(x)$ *on* $[x_0, x_1]$ *the* error ϵ_N

$$\epsilon_N = \int_{x_0}^{x_1} w(x) \left[F(x) - \sum_{n=1}^{N} a_n y_n(x) \right] dx \tag{1.75}$$

becomes arbitrarily small as $N \to \infty$.

Chapter 2

Fourier Series

2.1 Fourier series

Motivated by the results for boundary value problems discussed in section 1.2 we now look at how we can represent a particular function as a series of other functions; in particular, as a series of trigonometric functions.

We start by considering periodic functions. Periodic functions occur frequently in engineering problems and their representation in terms of simple periodic functions, which leads to **Fourier series**, is of great practical importance. They are in a sense more general than Taylor series because they permit the representation of **discontinuous** periodic functions which occur in many engineering applications.

A Fourier series in simple terms is an expansion of a function in terms of cos and sin functions. Suppose $f(x)$ is a **periodic function of period** 2π so that

$$f(x + 2\pi) = f(x)$$

Then (provided $f(x)$ is a "fairly nice" function that satisfies certain conditions) it may be shown that it can be expanded in a **Fourier series**

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \tag{2.1}$$

where a_n, b_n are constants called the **Fourier Coefficients** and are such that the series on the right hand side converges for all x.

More generally suppose that $f(x)$ is a periodic function of period $2L$ so that

$$f(x + 2L) = f(x)$$

Then by using the result above and rescaling the x-coordinate we may expand the function in a Fourier series in terms of the $2L$-periodic functions $\cos\left(\frac{n\pi x}{L}\right)$ and

$\sin\left(\frac{n\pi x}{L}\right)$ so that we may write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \tag{2.2}$$

The question then arises: given a $2L$-periodic function $f(x)$, how can we find the Fourier coefficients a_n, b_n in the Fourier Series?

2.2 Orthogonality

Throughout this section (and indeed this chapter) the following identities are repeatedly used; they are sufficiently important that you should know them.

$$\sin(n\pi) = 0, \qquad \sin\left(\left(n+\tfrac{1}{2}\right)\pi\right) = (-1)^n; \tag{2.3a}$$

$$\cos(n\pi) = (-1)^n, \qquad \cos\left(\left(n+\tfrac{1}{2}\right)\pi\right) = 0. \tag{2.3b}$$

Consider the integral

$$I_{mn} = \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{2.4}$$

We can straightforwardly use the formula $\cos(A+B) = \cos A\cos B - \sin A\sin B$ and $\cos(A-B) = \cos A\cos B + \sin A\sin B$ to show that

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

This enables us to write the integral as

$$I_{mn} = \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \tag{2.5a}$$

$$= \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right]\mathrm{d}x \tag{2.5b}$$

$$= \frac{L}{2\pi}\left[\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{m-n} - \frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{m+n}\right]_{-L}^{L} \tag{2.5c}$$

$$= 0 \quad \text{except when } m = n. \tag{2.5d}$$

In the special case when $m = n$ it is straightforward to check (exercise) that

$$I_{nn} = L. \tag{2.6}$$

We can combine these results into one simple equation:

$$I_{mn} = L\delta_{mn} \tag{2.7}$$

where the symbol δ_{mn} represents

$$\delta_{mn} = \begin{cases} 1 & m = n, \\ 0 & m \neq n \end{cases}. \tag{2.8}$$

Similar use of the double angle formula give the results

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = L\delta_{mn}, \tag{2.9}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = 0. \tag{2.10}$$

Note that the $\cos$ relation given by equation (2.9) does not hold for $m = 0$; in that case the result is

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = 2L\delta_{0n} \tag{2.11}$$

Note: If we consider

$$\langle f(x), g(x) \rangle = \int_{-L}^{L} f(x)g(x)\,\mathrm{d}x \tag{2.12}$$

to be an inner product between the functions f and g, then the set of functions

$$\left\{1, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\} \tag{2.13}$$

form an *orthogonal basis* for the space of $2L$-periodic "nice" functions. In fact we know from property 3 of the general theory of Sturm-Liouville boundary value problems (even without doing the calculations above) that these functions are orthogonal. For Fourier series it is just as easy to do the explicit calculations but for other types of expansion (e.g. Fourier-Bessel series which are used to describe vibrating plates) it is much easier to use the general theory.

2.3 Fourier Coefficients

We remember the definition of the Fourier Series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]. \tag{2.2}$$

If we multiply this equation by the appropriate $\sin$ term,

$$\sin\left(\frac{m\pi x}{L}\right) \tag{2.14}$$

and integrate the function over its period $2L$, then the orthogonality relations given by equations (2.7, 2.10) give

$$Lb_m = \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x. \tag{2.15}$$

The orthogonality relations ensure that except for the *single* term b_m, all other terms vanish under integration. Similarly, using the appropriate $\cos$ term and the orthogonality relations given by equations (2.9, 2.10) gives

$$La_m = \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) \mathrm{d}x, \tag{2.16}$$

and this equation holds even for $m = 0$ (due to the factor of $1/2$ in the definition of the Fourier Series, and using equation (2.11)).

We can summarise this to give the *Euler relations*:

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) \mathrm{d}x, \tag{2.17a}$$

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x. \tag{2.17b}$$

Note: Because the function $f(x)$ is periodic, and the range of integration is a single period, then one could equally use any $2L$-periodic range in x; for example, one could equally well use $\int_0^{2L} \mathrm{d}x$.

Note: Because the formulae for the Fourier coefficients a_n, b_n are explicit, the Fourier Series of a function is *unique*.

2.3.1 Even and odd functions

The function $f(x)$ is an **even** function if

$$\boxed{f(-x) = f(x)}$$

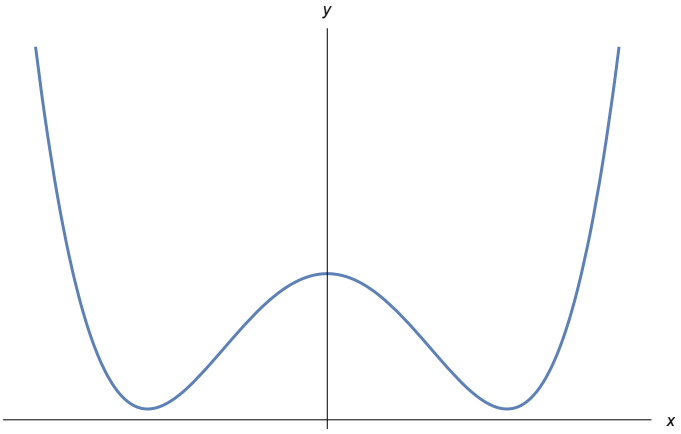


Figure 2.1: Even function. The function is symmetric about the y-axis.

for all x. Graphically this means that the function is symmetric about the y-axis.
The symmetrical integral of an even function can be expressed as follows

$$\int_{-\ell}^{\ell} f(x)dx = 2\int_{0}^{\ell} f(x)dx$$

The function $f(x)$ is said to be an **odd** function if

$$f(-x) = -f(x)$$

for all x. Graphically this means that the function is symmetric under a reflection about the y-axis and the x-axis.
The symmetrical integral of an odd function is zero i.e.

$$\int_{-\ell}^{\ell} f(x)dx = 0$$

Note, in addition, the following important properties

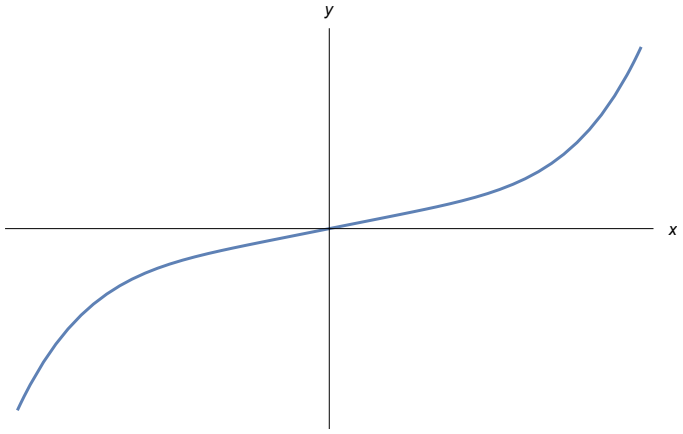


Figure 2.2: Odd function. The function is symmetric under a reflection about the y-axis and the x-axis.

- If $f(x)$, $g(x)$ are even then $f(x)g(x)$ is **even** since
$$f(-x)g(-x) = f(x)g(x)$$
- If $f(x)$, $g(x)$ are odd then $f(x)g(x)$ is **even** since
$$f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x)$$
- If $f(x)$ is even and $g(x)$ is odd then $f(x)g(x)$ is **odd** since
$$f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x)$$

If $f(x)$ is an **even** function then this gives rise to a **Fourier cosine series** only since

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$$

and so

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{with} \quad a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$

If f(x) is an **odd** function then this gives rise to a **Fourier sine series** only since

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

and so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{with} \quad b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

Example.

Find the Fourier series of $f(x)$ which is equal to x in the range $-\pi < x < \pi$ and periodic outside, see Fig. 2.3.
Since $f(x)$ is an odd function, it has a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x d\left(-\frac{\cos nx}{n}\right) \\
&= \frac{2}{\pi}\left[x\frac{-\cos nx}{n}\right] + \frac{2}{n\pi}\int_{0}^{\pi}\cos nx dx \\
&= \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n\right] + \frac{2}{n\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} \\
&= \frac{2}{n}(-1)^{n-1}
\end{aligned}$$

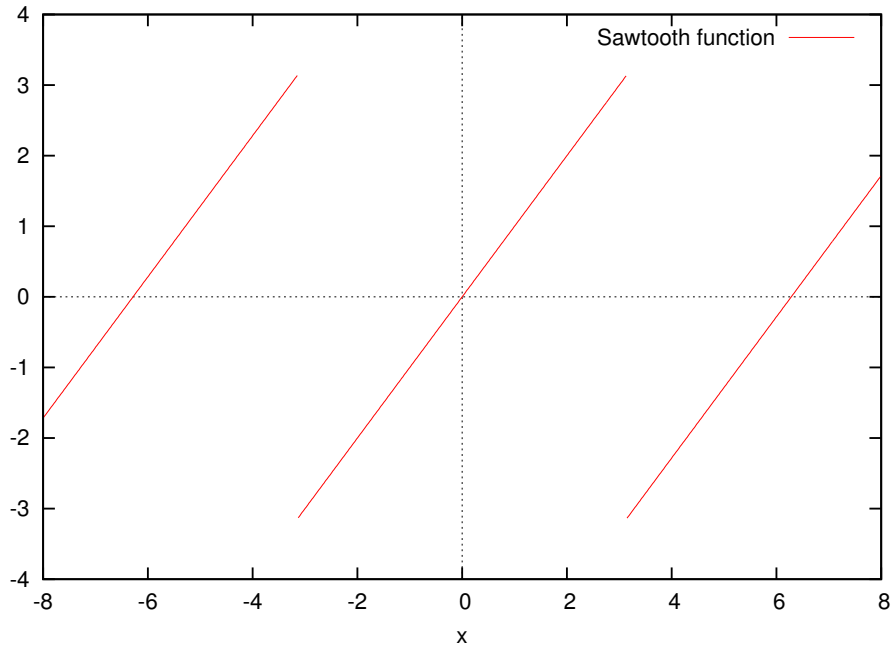


Figure 2.3: The function $f(x) = x$ with range $-\pi < x < \pi$, periodically extended to the real line (Sawtooth function).

Therefore

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n-1} \sin nx \\ &= 2[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots] \end{aligned}$$

If we consider the contribution of the first few terms to the sum we can gain some idea of how the Fourier series approximates to the function in the interval 0 to π.

Then, in particular, at the following points

$$x = 0 \quad : \qquad \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n-1} \sin 0 = f(0) = 0$$

$$x = \pi \quad : \qquad \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n-1} \sin n\pi = \frac{1}{2}[f(\pi - 0) + f(\pi + 0)] = \frac{1}{2}[\pi + (-\pi)] = 0$$

$$x = \frac{\pi}{2} \quad : \qquad \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n-1} \sin \frac{n\pi}{2} = 2[1 - \frac{1}{3} + \frac{1}{5} - \cdots] = f(\frac{\pi}{2}) = \frac{\pi}{2}$$

This last result can be used to show that the sum of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$$

2.3.2 Even and odd extensions: half range series

Suppose that $f(x)$ is given only on $0 \leq x \leq L$; Then we can extend the function f onto $-L < x \leq L$ by making

$$f(-x) = -f(x).$$

We then extend this to all values of x by making the function have period $2L$, see Fig. 2.4.

This is called an **odd periodic extension** of the function. Since the function is **odd** it has a Fourier series which only contains the sine terms. Hence the function is represented by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \tag{2.18}$$

a *Fourier sine series* where

$$a_n = 0, \tag{2.19a}$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x, \tag{2.19b}$$

where we have used the fact that $f(x)$ is odd (and hence $f(x)\sin\left(\frac{m\pi x}{L}\right)$ is even) to write the expression for b_n in terms of an integral over $0 < x < L$ (which is half the period). For this reason such an Fourier sine series is called a *half-range Fourier series*.

Alternatively given $f(x)$ only on $0 \leq x \leq L$ we can extend the function f onto $-L < x \leq L$ by making

$$f(-x) = f(x).$$

and then extend this to all values of x by making the function have period $2L$, see Fig. 2.4.

This is called an **even periodic extension** of the function. Since the function is **even** it has a Fourier series which only contains the cosine terms. Hence the function is represented by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \tag{2.20}$$

a *Fourier cosine series* where

$$b_n = 0, \tag{2.21a}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x \tag{2.21b}$$

and we have again used the fact that the integrand is even to write the expression for a_n as an integral over half the range.

Note: we could also extend the function by making f be periodic with period L, which immediately implies that it is also periodic with period $2L$, see Fig. 2.4. However in applications to differential equations it is the odd and even extensions that are most useful.

Example.

Find the Fourier series of (i) the odd periodic extension and (ii) the even periodic extension of

$$f(x) = \begin{cases} 1, & 0 < x < \frac{\ell}{2} \\ 0, & \frac{\ell}{2} < x < \ell \end{cases}$$

in the range $(0, \ell)$.

1. In $(0, \ell)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

where

$$\begin{aligned} b_n &= \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx \\ &= \frac{2}{\ell}\int_0^{\frac{\ell}{2}} \sin\frac{n\pi x}{\ell}dx \end{aligned}$$

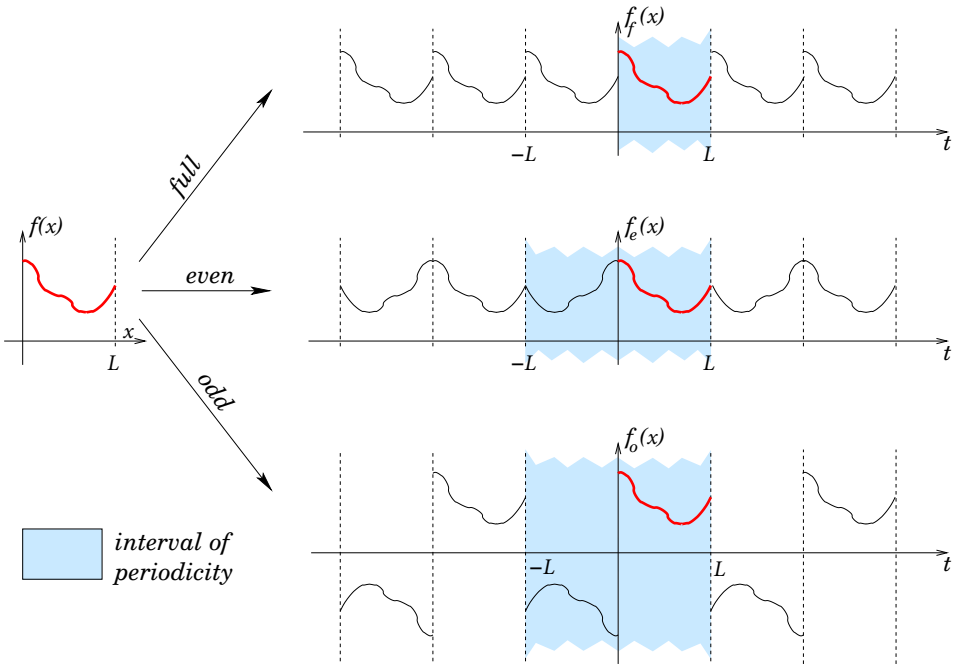


Figure 2.4: Periodic extensions of the function $f(x)$ defined on the half interval, $0 < x < L$ (left panel). Right panel, top: Extension as a periodic function of period L. Right panel, middle: Extension as a even function. Right panel, bottom. Extension as an odd function. The blue shading indicates the interval of periodicity.

$$\begin{aligned} &= -\frac{2}{n\pi}\left[\cos\frac{n\pi x}{\ell}\right]_0^{\frac{\ell}{2}} \\ &= \frac{2}{n\pi}\left[1-\cos\frac{n\pi}{2}\right] \end{aligned}$$

Therefore

$$\begin{aligned} f(x) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-\cos\frac{n\pi}{2}\right]\sin\frac{n\pi x}{\ell} \\ &= \frac{2}{\pi}\left[\sin\frac{\pi x}{\ell}+\sin\frac{2\pi x}{\ell}+\frac{1}{3}\sin\frac{3\pi x}{\ell}+\frac{1}{5}\sin\frac{5\pi x}{\ell}+\cdots\right] \end{aligned}$$

2. In $(0,\ell)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}$$

where

$$\begin{aligned} a_n &= \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx \\ &= \frac{2}{\ell}\int_0^{\frac{\ell}{2}}\cos\frac{n\pi x}{\ell}dx \\ &= \frac{2}{\ell}\frac{\ell}{n\pi}\left[\sin\frac{n\pi x}{\ell}\right]_0^{\frac{\ell}{2}} \\ &= \frac{2}{n\pi}sin\frac{n\pi}{2} \end{aligned}$$

and

$$a_0=\frac{2}{\ell}\int_0^{\frac{\ell}{2}}dx=1$$

Therefore

$$\begin{aligned} f(x) &= \frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{\ell} \\ &= \frac{1}{2}+\frac{2}{\pi}\left[\cos\frac{\pi x}{\ell}-\frac{1}{3}\cos\frac{3\pi x}{\ell}+\frac{1}{5}\cos\frac{5\pi x}{\ell}-\cdots\right] \end{aligned}$$

2.4 Fourier's Theorem

We have said that Fourier Series work for "fairly nice" functions, without specifying what we mean. Here we note that *sufficient* conditions for the existence of a Fourier Series representation, i.e. that the sum given in the definition of the Fourier Series in equation (2.2) converges, is that

1. the function $f(x)$ is bounded, and
2. the function has a finite number of maxima, minima and discontinuities, and
3. the function is $2L$-periodic for some L.

Under these conditions we have the following:

Fourier Convergence Theorem. *Let $f(x)$ be a bounded, $2L$-periodic function with a finite number of maxima, minima and discontinuities. Then the sum*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \tag{2.22}$$

converges for all finite x, and when a_n, b_n are given by the Euler formulas (2.17) *the sum converges to $f(x)$ if, at x, f is continuous. At points x where f is not continuous the sum converges to $\frac{1}{2}\left[f(x_-) + f(x_+)\right]$ where $f(x_-)$ and $f(x_+)$ are the values of the function as we approach from the left and right respectively.*

Note: There may be "hidden" discontinuities at the end of the periodic range. That is, if the function is $2L$-periodic with $f(x) = f(x + 2L)$ over the range $-L \leq x < L$, it may be that $f(-L) \neq f(L)$.

2.5 Calculus and Fourier Series

2.5.1 Differentiation of Fourier Series

It is trivial to take the definition of a Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \tag{2.23}$$

and to differentiate or integrate the right hand side term by term. The question is, under which conditions does the statement

$$f'(x) = \sum_{n=1}^{\infty}\left[-\frac{n\pi}{L}a_n \sin\left(\frac{n\pi x}{L}\right) + \frac{n\pi}{L}b_n \cos\left(\frac{n\pi x}{L}\right)\right] \tag{2.24}$$

hold?

From the conditions given in section 2.4 we would expect that

1. the function $f(x)$ is bounded, and
2. the function has a finite number of maxima, minima and discontinuities, and
3. the function is $2L$-periodic for some L,

to be a good start. Under this set of conditions the Fourier convergence theorem of section 2.4 shows at least that the original definition of equation (2.23) holds everywhere. If in addition we assume that

4. the function $f(x)$ is continuous, and
5. its derivative $f'(x)$ is piecewise smooth for all x

then application of the Fourier convergence theorem to f' gives that

$$f'(x) = \sum_{n=1}^{\infty} \left[\alpha_n \cos\left(\frac{n\pi x}{L}\right) + \beta_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{2.25}$$

for some Fourier coefficients α_n, β_n. It is then straightforward to show that these coefficients are exactly those given by equation (2.24).

Example 1.

In the range $-\pi < x < \pi$

$$f(x) = x \sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \cdots\right]$$

and $f(-\pi) = f(\pi) = 0$.

Since $f(x)$ satisfies the three conditions above we may differentiate both sides to give

$$\sin x + x\cos x = \frac{1}{2}\sin x + 2\left[\frac{2\sin 2x}{1.3} - \frac{3\sin 3x}{2.4} + \cdots\right]$$

from which we get in addition

$$x\cos x = -\frac{1}{2}\sin x + 2\left[\frac{2\sin 2x}{1.3} - \frac{3\sin 3x}{2.4} + \cdots\right]$$

Note: It is important to note that although the conditions above are sufficient and some of them can be relaxed, the example below shows that some of these conditions are crucial.

Example 2.

The periodic extension of $f(x) = x$ in $0 < x < 2\pi$ is

$$x = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

However

$$f(0) = 0 \neq f(2\pi) = 2\pi$$

and therefore the series may **not** be differentiated term by term. If we do differentiate the series term by term we obtain the series

$$g(x) = -2\sum_{n=1}^{\infty} \cos nx$$

which does **not** converge to $f'(x) = 1$ in $0 < x < 2\pi$. Indeed the series is **divergent** since the n^{nth} term does not tend to zero as $n \to \infty$.

2.5.2 Integration of Fourier Series

The Fourier series of $f(x)$ in $-\ell < x < \ell$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right]$$

may **always** be integrated term by term to give

$$\begin{aligned}
\int_{-\ell}^{x} f(x)dx &= \frac{a_0}{2}\int_{-\ell}^{x} dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\ell}^{x} \cos\frac{n\pi x}{\ell}dx + b_n \int_{-\ell}^{x} \sin\frac{n\pi x}{\ell}dx \right] \\
&= \frac{a_0}{2}(x+\ell) + \sum_{n=1}^{\infty} \frac{\ell}{n\pi} a_n \left(\sin\frac{n\pi x}{\ell} - \sin\frac{n\pi(-\ell)}{\ell} \right) \\
&\qquad + \sum_{n=1}^{\infty} \frac{\ell}{n\pi} b_n \left(-\cos\frac{n\pi x}{\ell} + \cos\frac{n\pi(-\ell)}{\ell} \right) \\
&= \frac{a_0}{2}(x+\ell) + \frac{\ell}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[a_n \sin\frac{n\pi x}{\ell} - b_n \cos\frac{n\pi x}{\ell} + (-1)^n b_n \right].
\end{aligned}$$

The proof is straightforward but rather detailed. Of course, the constant term in the new Fourier Series will need to be computed as it appears as the 'constant of integration' when one integrates term by term.

Example.

In the interval $-\pi < x < \pi$, $f(x) = x$ has odd periodic extension

$$x = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n}.$$

Integrating term by term from $-\pi$ to x

$$\left[\frac{x^2}{2}\right]_{-\pi}^{x} = 2\sum_{n=1}^{\infty}(-1)^{n-1}\left[-\frac{\cos nx}{n^2}\right]_{-\pi}^{x}$$

$$\frac{x^2}{2} - \frac{\pi^2}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}[\cos nx - \cos n\pi]$$

$$x^2 - \pi^2 = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}[\cos nx - (-1)^n]$$

$$x^2 = \frac{a_0}{2} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

where

$$\frac{a_0}{2} = \pi^2 - 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

We can determine a_0 either by knowing the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ or directly from the Fourier coefficient formula

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}.\frac{2\pi^3}{3} = \frac{2\pi^2}{3}$$

Hence we get

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

2.6 Complex form of Fourier Series

We can express the Fourier series of a function more succinctly if we employ a complex representation and use the exponential definitions for the trigonometric functions,

$$\cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left(e^{\mathrm{j}n\pi x/L} + e^{-\mathrm{j}n\pi x/L}\right), \tag{2.26a}$$

$$\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2\mathrm{j}}\left(e^{\mathrm{j}n\pi x/L} - e^{-\mathrm{j}n\pi x/L}\right) \tag{2.26b}$$

and substitute these into the definition of the Fourier Series, equation (2.2), then we will end up with

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{j}n\pi x/L}, \tag{2.27}$$

where the complex coefficients c_n are given by

$$c_n = \begin{cases} \frac{1}{2}(a_n - \mathrm{j}b_n) & n > 0, \\ \frac{1}{2}a_0 & n = 0, , \\ \frac{1}{2}(a_{-n} + \mathrm{j}b_{-n}) & n < 0 \end{cases} \tag{2.28}$$

which implies, or can be written as,

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\mathrm{j}n\pi x/L}\,\mathrm{d}x. \tag{2.29}$$

Example.

Find the complex Fourier series for

$$f(x) = e^x, \quad -\pi < x < \pi \quad \text{with } f(x + 2\pi) = f(x)$$

By above

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-\mathrm{j}nx}dx \\ &= \frac{1}{2\pi}\left[\frac{e^{(1-\mathrm{j}n)x}}{1-\mathrm{j}n}\right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}\cdot\frac{1}{1-\mathrm{j}n}\left[e^{(1-\mathrm{j}n)\pi} - e^{-(1-\mathrm{j}n)\pi}\right] \\ &= \frac{(-1)^n}{\pi(1-\mathrm{j}n)}\sinh\pi \end{aligned}$$

Therefore

$$f(x) = \sum_{-\infty}^{\infty}\frac{(-1)^n}{\pi(1-\mathrm{j}n)}e^{\mathrm{j}nx}\sinh\pi$$

2.7 Parseval's Theorem

(not examinable)

Consider the real number

$$\int_{-L}^{L} \{f(x)\}^2 \, \mathrm{d}x. \tag{2.30}$$

If we expand out the Fourier series representation of $f(x)$ given by (2.2) and use the orthogonality relations given by equations (2.7, 2.9, 2.10) then we find

$$\int_{-L}^{L} [f(x)]^2 \, \mathrm{d}x = \int_{-L}^{L} \left\{ \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \right\}^2 \mathrm{d}x \tag{2.31a}$$

$$= \int_{-L}^{L} \left\{ \frac{1}{4}a_0^2 + \sum_{n=1}^{\infty} \left[a_n^2 \cos^2\left(\frac{n\pi x}{L}\right) + b_n^2 \sin^2\left(\frac{n\pi x}{L}\right) \right] \right\} \mathrm{d}x \tag{2.31b}$$

$$= L \left[\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \right]. \tag{2.31c}$$

This can be restated as

$$\text{“}\|f\|^2 = \sum \text{squares of components of } f \text{ on basis vectors”}. \tag{2.32}$$

The norm of f is often called the *power* contained in the function, and *Parseval's Theorem* or *Parseval's Identity*, given by equation (2.31), relates the power to the Fourier coefficients.

2.8 Summary of results

1. **Fourier series of periodic and non-periodic functions**

   (a) **Periodic function** $f(x)$ of period 2π.
   The Fourier series is **valid** for all x.

   (b) **Non-periodic function** $f(x)$ defined over interval $-\ell < x < \ell$, say.
   Periodic extension

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \right)$$

   where

$$a_n = \frac{1}{\ell}\int_{-\ell}^{l} f(x)\cos\frac{n\pi x}{\ell}dx,\; b_n = \frac{1}{\ell}\int_{-\ell}^{l} f(x)\sin\frac{n\pi x}{\ell}dx$$

$$f(x) \quad \text{is even} \longrightarrow b_n = 0$$
$$f(x) \quad \text{is odd} \longrightarrow a_n = 0$$

(c) **Non-periodic function** $f(x)$ defined over interval $0 < x < \ell$, say.

i. **Even periodic extension** (half range expansion)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

where

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx$$

ii. **Odd periodic extension** (half range expansion)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx$$

2. **Differentiation**

The Fourier series for $f(x)$ in $-\ell < x < \ell$ may be differentiated term by term to give a Fourier series for $f'(x)$ if

(a) $f(x)$ is continuous in $-\ell < x < \ell$

(b) $f'(x)$ has a Fourier series

(c) $f(-\ell) = f(\ell)$

3. **Integration**

The Fourier series for $f(x)$ in $-\ell < x < \ell$ may *always* be integrated term by term.

4. **Complex Fourier series**

The complex Fourier series for $f(x)$ in $-\ell < x < \ell$ is

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{jn\pi x}{\ell}}$$

where

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-\frac{\mathrm{j}n\pi x}{\ell}} dx$$

Chapter 3

Fourier transform

The Fourier series is a very powerful tool to analyse periodic functions or functions defined in an interval, using their periodic extensions. However, we now want to extend it to functions defined on the whole real line that are not periodic. We do this by considering the case of a Fourier series of a simple function defined on an interval of length $2L$ and then see what happens in the limit that L tends to infinity.

3.1 An example

Consider the function

$$f_L(t) = \begin{cases} 0 & -L \le t < -1, \\ 1 & -1 \le t < 1, \\ 0 & 1 \le t < L, \end{cases} \quad \text{with} \quad f_L(t+2L) = f_L(t)\, \forall t \in \mathbb{R}. \tag{3.1}$$

As L increases this function tends to the non-periodic function

$$f(t) = \begin{cases} 0 & t < -1, \\ 1 & -1 \le t < 1, \\ 0 & t \ge 1. \end{cases} \tag{3.2}$$

The function $f_L(t)$ has complex Fourier coefficients

$$c_n = \frac{1}{2L} \int_{-L}^{+L} f_L(t) e^{-\mathrm{j}\frac{\pi n}{L}t}\, \mathrm{d}t = \frac{1}{\pi n} \sin\left(\frac{\pi n}{L}\right) \tag{3.3}$$

and complex Fourier series

$$f_L(t) = \sum_{n=-\infty}^{\infty} \frac{1}{n\pi} \sin\left(\frac{n\pi}{L}\right) e^{\mathrm{j}\frac{n\pi}{L}t}. \tag{3.4}$$

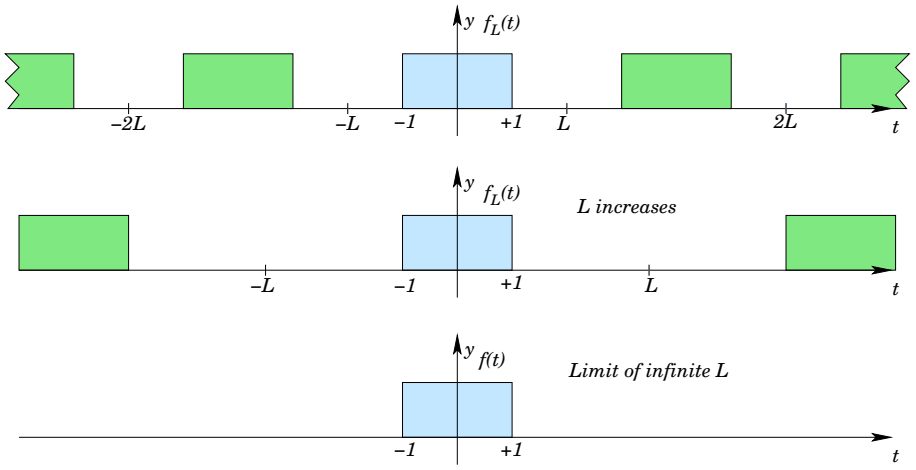


Figure 3.1: *The periodic function $f_L(t)$ defined in equation (3.1), top and middle row, and its limit as $L \to \infty$, $f(t)$ defined by equation (3.2), bottom row.*

We want to see how equation (3.3) and (3.4) change as L tends to infinity. We introduce the variable

$$\omega_n = \frac{n\pi}{L} \tag{3.5}$$

and look at how this changes with n by considering

$$\Delta\omega_n = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}\,. \tag{3.6}$$

Note that as L increases $\Delta\omega_n$ tends to zero and the ω_n variables becomes continuous, rather than discrete. With this notation we also have that

$$\frac{1}{n} = \frac{\Delta\omega_n}{\omega_n}. \tag{3.7}$$

We start with equation (3.4) and rewrite it as

$$f_L(t) = \sum_{\omega_n=-\infty}^{\infty} \frac{1}{\pi}\left(\frac{\Delta\omega_n}{\omega_n}\right)\sin(\omega_n)e^{\mathrm{j}\omega_n t}\,. \tag{3.8}$$

We now take the limit $L \to \infty$ of this expression and obtain

$$f(t) \equiv \lim_{L\to\infty} f_L(t) \tag{3.9a}$$

$$= \lim_{L\to\infty}\left\{\sum_{\omega_n=-\infty}^{\infty} \frac{1}{\pi}\left(\frac{\Delta\omega_n}{\omega_n}\right)\sin(\omega_n)e^{\mathrm{j}\omega_n t}\right\} \tag{3.9b}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega)}{\omega}e^{\mathrm{j}\omega t}\,\mathrm{d}\omega\,. \tag{3.9c}$$

In this way we have expressed $f(t)$ as the integral of the function $\sin(\omega)/\omega$ of the continuous variable ω, multiplied by the complex exponential $\exp(\mathrm{j}\omega t)$. We now need to deal with equation (3.3). We define

$$c(\omega_n) = Lc_n \,. \tag{3.10}$$

Using the same substitutions as for $f_L(t)$ we can write

$$c(\omega_n) = \frac{1}{2} \int_{-L}^{+L} f_L(t) e^{-\mathrm{j}\omega_n t} \,\mathrm{d}t \,, \tag{3.11}$$

which, in the limit $L \to \infty$ gives a function of the continuous variable ω,

$$c(\omega) = \frac{1}{2} \int_{-\infty}^{\infty} f(t) e^{-\mathrm{j}\omega t} \,\mathrm{d}t = \frac{1}{2} \int_{-1}^{+1} e^{-\mathrm{j}\omega t} = \frac{\sin(\omega)}{\omega} \,. \tag{3.12}$$

Note that $c(\omega)$ is the function that appears as integrand in the expression (3.9c) of $f(t)$. Equation (3.12) is the generalisation of (3.3) to a non-periodic function, while equation (3.9c) is the generalisation of (3.4). We will see shortly that the former is called the *Fourier transform* of $f(t)$, while the latter is its *inverse-Fourier transform*.

3.2 Definition

We are now in a position to define the Fourier transform of a function of a real variable, $f(t)$, with $t \in \mathbb{R}$. We assume that the function $f(t)$ satisfies the following conditions:

1. $\int_{-\infty}^{\infty} |f(t)| \,\mathrm{d}t < \infty$, (note that this requires that $|f(t)| \to 0$ as $|t| \to \infty$),
2. $f(t)$ has at most a finite number of maxima, minima and discontinuities in any finite interval.

The function

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \,\mathrm{d}t \,. \tag{3.13}$$

is called the *Fourier transform* of $f(t)$ and we write

$$F(\omega) = \mathcal{F}[f(t)](\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \,\mathrm{d}t. \tag{3.14}$$

Conversely, given $F(\omega)$ we can write $f(t)$ as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, \mathrm{d}\omega \tag{3.15}$$

where this expression converges to

1. $f(t)$ at all points where $f(t)$ is continuous;
2. the average of right- and left-hand limits of $f(t)$ at points where $f(t)$ is discontinuous.

In this respect the function $f(t)$ is called the *inverse Fourier transform* of $F(\omega)$ and we write

$$f(t) = \mathcal{F}^{-1}[F(\omega)](t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, \mathrm{d}\omega \tag{3.16}$$

Remark - The requirement that $\int_{-\infty}^{\infty} |f(t)| \, \mathrm{d}t < \infty$ is necessary to ensure that the integrals (3.13) and (3.15) converge and the Fourier transform and its inverse are defined.

Remark - There are a variety of definitions of Fourier and inverse-Fourier transform: they all differ by the factor in front of the integral. The definition given here is symmetric, the factors in front of the Fourier and inverse-Fourier transform are identical, both equal to $1/\sqrt{2\pi}$. The "definition" given in Section 3.1 has a factor $1/2$ in front of the Fourier transform, eq. (3.12), and a factor $1/\pi$ in front of the inverse-Fourier transform, eq. (3.9c). The important point is that the product of the two factors must be equal to $1/2\pi$.

Example - Find the Fourier transform of the function

$$f(t) = \begin{cases} \sin(t) & -\pi \leq t \leq \pi, \\ 0 & \text{Otherwise}\,. \end{cases} \tag{3.17}$$

$$
\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, \mathrm{d}t = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} \sin(t) e^{-j\omega t}\, \mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \left(-\frac{j}{2}\right) \int_{-\pi}^{+\pi} \left[e^{j(1-\omega)t} - e^{-j(1+\omega)t}\right] \mathrm{d}t \\
&= -\frac{j}{2\sqrt{2\pi}} \left[\frac{e^{j(1-\omega)t}}{j(1-\omega)} - \frac{e^{-j(1+\omega)t}}{-j(1+\omega)}\right]_{-\pi}^{+\pi} \\
&= -\frac{1}{2\sqrt{2\pi}} \left\{\frac{1}{1-\omega}\left[e^{j(1-\omega)\pi} - e^{-j(1-\omega)\pi}\right] + \frac{1}{1+\omega}\left[e^{-j(1+\omega)\pi} - e^{j(1+\omega)\pi}\right]\right\} \\
&= -\frac{1}{2\sqrt{2\pi}} \left[\frac{1}{1-\omega}\left(e^{j\omega\pi} - e^{-j\omega\pi}\right) + \frac{1}{1+\omega}\left(e^{j\omega\pi} - e^{-j\omega\pi}\right)\right] \\
&= -\frac{1}{2\sqrt{2\pi}} \left[\left(\frac{1}{1-\omega} + \frac{1}{1+\omega}\right)\left(e^{j\omega\pi} - e^{-j\omega\pi}\right)\right] \\
&= -\frac{1}{2\sqrt{2\pi}} \frac{2}{1-\omega^2}\, 2j \sin(\omega\pi) = \frac{2j}{(\omega^2-1)\sqrt{2\pi}} \sin(\omega\pi)\,.
\end{aligned}
\tag{3.18}
$$

3.3 Properties

We list here some of the basic properties of Fourier transforms. We indicate with the symbol $\mathcal{F}$ the Fourier transform, in the sense that

$$
\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, \mathrm{d}t = F(\omega)\,. \tag{3.19}
$$

We indicate with the symbol $\mathcal{F}^{-1}$ the inverse Fourier transform.

Linearity

The Fourier transform is a *linear operator*:

$$
\mathcal{F}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{F}[f(t)] + \beta \mathcal{F}[g(t)]\,. \tag{3.20}
$$

The same apply to the inverse Fourier transform.

Differentiation

If $f(t)$ is continuous everywhere and its derivative has at most isolated jump discontinuities then

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = j\omega\mathcal{F}[f(t)]\,. \tag{3.21}$$

The generalisation of this formula is

$$\mathcal{F}\left[\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right] = (j\omega)^n\mathcal{F}[f(t)]\,. \tag{3.22}$$

To prove (3.21) we compute the Fourier transform of the derivative of $f(t)$:

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\mathrm{d}f}{\mathrm{d}t}e^{-j\omega t}\,\mathrm{d}t\,. \tag{3.23}$$

The next step is to integrate by parts and use the fact that $|f(t)| \to 0$ as $|t| \to \infty$ (as required to ensure convergence of the Fourier transform). For simplicity we consider here the case that the function is differentiable everywhere. In this case, integration by parts gives

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = \frac{1}{\sqrt{2\pi}}\left[f(t)e^{-j\omega t}\right]_{t\to-\infty}^{t\to+\infty} + j\omega\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,\mathrm{d}t \;= j\omega\,F(\omega) \tag{3.24}$$

Equation (3.22) can be obtained by applying (3.21) n times.

Parseval's theorem

(not examinable)

Starting from the definition of Fourier transform and using the δ-function it is possible to prove the following equality, that goes under the name of *Parseval's theorem*[1]

$$\int_{-\infty}^{\infty}|f(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty}|F(\omega)|^2\,\mathrm{d}\omega \tag{3.25}$$

This relation has a clear physical meaning. We have already mentioned when studying Parseval's theorem for Fourier series that the moduli of the coefficients of the series, $|a_n|$, $|b_n|$ are a measure of the strength of the frequency component ω_n. A similar interpretation can be given for the Fourier transform: if $f(t)$ is some physical signal, e.g. the amplitude of a sound or electromagnetic wave measured

[1] Marc-Antoine Parseval des Chênes, French mathematician (1755-1836)

as a function of time, then the modulus square of its Fourier transform, $|F(\omega)|^2$, is the energy density in frequency space, i.e. $|F(\omega)|^2\delta\omega$ is the energy of the signal in a frequency band of width $\delta\omega$ centred around the frequency ω (in appropriate units). The total energy of the wave is the integral of this energy density, i.e.

$$\text{Total energy of wave} = \int_{-\infty}^{\infty} |F(\omega)|^2 \, \mathrm{d}\omega \, . \tag{3.26}$$

On the other hand, the total energy of a wave of amplitude $f(t)$ is given by integrating $|f(t)|^2$ over time (in appropriate units), i.e.

$$\text{Total energy of wave} = \int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t \, . \tag{3.27}$$

These two expressions must be equal. In this interpretation, equation (3.25), is just a statement that the total energy measured must be the same whether we integrate over time, equation (3.27), or over frequency, equation (3.26).

Shift properties

The Fourier transform of a shifted signal is equal to the Fourier transform of the original signal, multiplied by a phase factor proportional to the shift:

$$\mathcal{F}[f(t-t_0)] = e^{-j\omega t_0} F(\omega). \tag{3.28}$$

The converse of (3.28) is

$$\mathcal{F}\left[e^{j\omega_0 t} f(t)\right] = F(\omega - \omega_0) \, . \tag{3.29}$$

The signal processing interpretation of this result is that the multiplication of $f(t)$ by a constant frequency term, $\exp(j\omega_0 t)$, is equivalent to shifting the entire frequency spectrum of the signal by an amount ω_0. This is the mathematical basis of the process of *modulation*, where a high frequency carrier at frequency ω_0 is modulated by a low frequency signal $f(t)$.

3.4 Fourier transform and the response function

The Fourier transform can be used to study the response function of a linear system to generic modulations. It is particular useful to analyse its frequency response. We show this by considering the following example [based on Glyn James, *Modern Engineering Mathematics*].

A dynamical system $y(t)$ is modelled by the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 7y(t) = 3\frac{\mathrm{d}u}{\mathrm{d}t} + 2u(t), \tag{3.30}$$

where $u(t)$ is a given signal. We can use the Fourier transform to study the response of this system to the modulation due to the function $u(t)$. We let $Y(\omega) = \mathcal{F}[y(t)]$ and $U(\omega) = \mathcal{F}[u(t)]$ denote the Foureir transform of $y(t)$ and $u(t)$ respectively. Taking the Fourier transform of both sides of (3.30) we obtain

$$(j\omega)^2 Y(\omega) + 3j\omega Y(\omega) + 7Y(\omega) = 3j\omega U(\omega) + 2U(\omega) \tag{3.31a}$$

$$\implies \qquad Y(\omega) = \frac{2 + 3j\omega}{7 - \omega^2 + 3j\omega} U(\omega) \tag{3.31b}$$

$$= G(\omega)U(\omega), \tag{3.31c}$$

where

$$G(\omega) = \frac{2 + 3j\omega}{7 - \omega^2 + 3j\omega}\,. \tag{3.32}$$

The function $G(\omega)$ is the *transfer function* of the system: it determines how the system responds to the various frequency components of the forcing term $u(t)$. In particular, we have

$$|Y(\omega)| = |G(\omega)|\,|U(\omega)|\,. \tag{3.33}$$

In other words, the strength of the frequency component ω of the output is equal to that of the input multiplied by the modulus of the transfer function. In the case of the function $G(\omega)$ defined by (3.32) we have

$$|G(\omega)| = \sqrt{\frac{9\,\omega^2 + 4}{49 - 5\,\omega^2 + \omega^4}} \tag{3.34}$$

This quantity is plotted in Figure 3.2: from the graph we can deduce that the system acts like a band pass filter with maximal response at

$$\omega = \frac{1}{3}\sqrt{-4 + 7\sqrt{85}} \simeq 2.59\,. \tag{3.35}$$

An example of the effect of the response function on an input signal is shown in Figure 3.3. In this case the input signal consists in a Gaussian pulse in time (solid line, Figure 3.3 right). Its Fourier power spectrum is also Gaussian (solid line, Figure 3.3 left). The power spectrum of the output (dashed line, Figure 3.3 left) is the product $Y(\omega) = G(\omega)U(\omega)$ where $G(\omega)$ is given by equation (3.32). The time dependent output $y(t)$ (dashed line, Figure 3.3 right) is the inverse Fourier transform of $Y(\omega)$. A more detailed analysis of the response of a linear system can

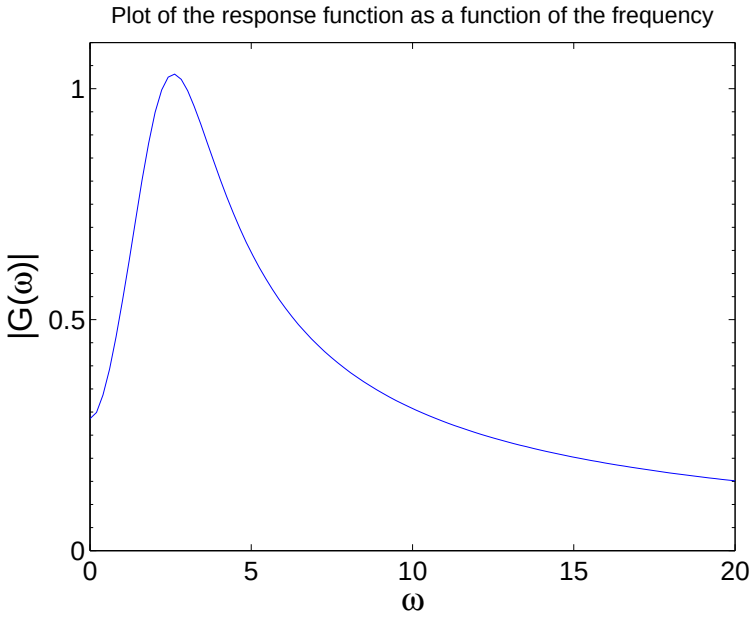


Figure 3.2: *Plot of the modulus of the response function, equation (3.34), as a function of the frequency.*

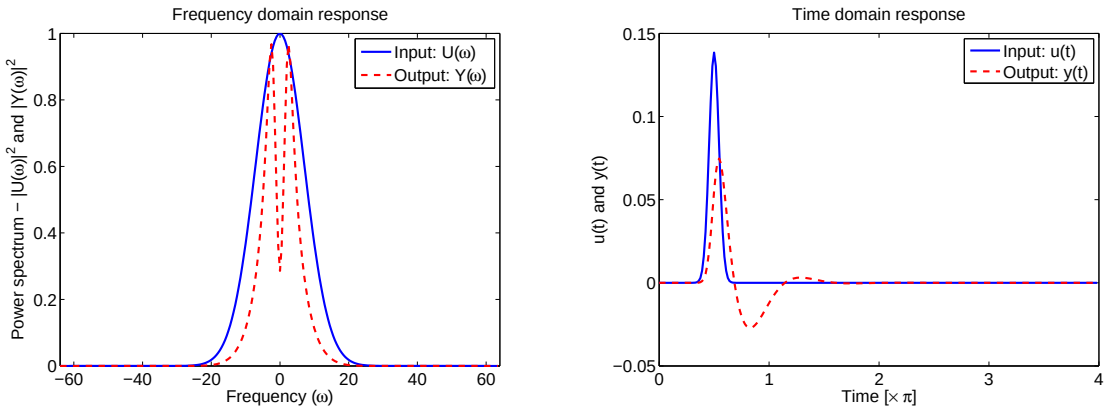


Figure 3.3: *Input and output signal in the frequency (left) and time (right) domain for equation (3.30). The relation between the Fourier series of the input $U(\omega)$ and output $Y(\omega)$ signal is $Y(\omega) = G(\omega)U(\omega)$, with the response function $G(\omega)$ given by equation (3.32).*

be obtained by studying the singularities of the transfer function in the complex plane.

In the next chapter we will look at a related integral transform called the *Laplace Transform*. This is defined by

$$\mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}dt$$

This is up to a factor just the Fourier transform where $\omega = js$. It is used more widely then the Fourier transform for solving differential equations as it is simpler to use when one can claculate the inverse transform. However unlike the Fourier transform there is no simple formula for calculating the inverse of a Lapalce transform - one either has to look it up in a table (which does not always work) or else regard it as a "complex Fourier transform" in which case one can calculate the inverse using complex integration.

3.5 Summary

The Fourier transform is important because it we can use it to transform an ordinary differential equation into an algebraic equation (or a PDE into an ODE). We can then solve the algebraic equation and find the solution to the original ODE by applying the inversse Fourier transform.

- **Definition**
  The **Fourier transform** of a function $f(t)$ is defined by

$$\mathcal{F}[f(t)](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t = F(\omega)\,.$$

- **Inverse Fourier transform**
  The inverse Fourier transform of a function $F(\omega)$ is defined by

$$\mathcal{F}^{-1}[F(\omega)](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,\mathrm{d}\omega = f(t)$$

The key properties of the Fourier transform are the following:

- **Linearity**

$$\mathcal{F}[\alpha f(t) + \beta g(t)] = \alpha\mathcal{F}[f(t)] + \beta\mathcal{F}[g(t)]\,.$$

- **Derivatives** The Fourier transform of the derivative of a function $f'(t)$ is given by
$$\mathcal{F}[f'(t)] = j\omega\mathcal{F}[f(t)]\,.$$
More generally the Fourier transform of the n-th derivative of a function $f^{(n)}(t)$ is given by
$$\mathcal{F}[f^{(n)}(t)] = (j\omega)^n\mathcal{F}[f(t)]\,.$$

- **First Shift Theorem**
$$\mathcal{F}[f(t-t_0)] = e^{-j\omega t_0}F(\omega).$$

- **Second Shift Theorem**
$$\mathcal{F}\left[e^{j\omega_0 t}f(t)\right] = F(\omega-\omega_0)\,.$$

Chapter 4

Laplace transforms

4.1 Introduction

A standard method for solving differential equations is to transform the problem into an equivalent, simpler problem. For a broad range of differential equations (especially linear differential equations) the *integral transforms*

$$\tilde{y}(s) = \int_{\alpha}^{\beta} K(s,x)y(x)\,\mathrm{d}x \tag{4.1}$$

convert the problem from a complex differential equation for $y(x)$ to a (hopefully) simpler equation for $\tilde{y}(s)$. Here the *kernel* $K(s,x)$ is a given function which together with the real integration limits α, β defines the transform.

The broad outline of the method is the same for all types of integral transform:

1. apply the transform to the original equation(s) to get a simpler problem for $\tilde{y}$;
2. solve the simpler problem for $\tilde{y}$;
3. invert the transform to get y.

In certain cases it may only be formally possible to invert the transform; that is, we may not be able to compute the solution in closed form, but only as an integral.

Here we will only study one type of integral transform, the Laplace transform.

4.2 Definition and existence

For a suitably well-behaved function $f(x)$ we define the Laplace transform of $f(x)$ to be

$$\tilde{f}(s) = \int_0^\infty f(x)e^{-sx}\,\mathrm{d}x \tag{4.2a}$$

$$= \mathcal{L}\left[f(x)\right]. \tag{4.2b}$$

We note that a wide variety of notation is used; the Laplace transform is also often denoted $\bar{f}(s)$ or $F(s)$.

4.2.1 Existence

Strictly the definition is not complete. For unrestricted functions f it is not at all obvious that the integral in the definition (4.2) converges for any given value of s. In fact, it is simple to check (using the strict definition of the unbounded integral) that the integral need not converge for some values of s for certain simple functions.

We look at e^{cx} where c is some real nonzero constant. We therefore have

$$\int_0^\infty e^{cx}\,\mathrm{d}x = \lim_{A\to\infty}\int_0^A e^{cx}\,\mathrm{d}x \tag{4.3a}$$

$$= \lim_{A\to\infty}\left[\frac{e^{cx}}{c}\right]_0^A \tag{4.3b}$$

$$= \lim_{A\to\infty}\frac{1}{c}\left(e^{cA}-1\right). \tag{4.3c}$$

We immediately see that the limit is finite, and hence the integral converges, if $c < 0$, and that the integral diverges if $c > 0$. The case $c = 0$ is not covered by this, but it is trivial to check that the integral diverges in this case as well.

This immediately motivates:

Theorem 4.2.1. *If $f(x)$ is a piecewise continuous function on $0 \le x \le A$ for all $A > 0$, and $|f(x)| \le Ke^{ax}$ for all $x \ge M$, where K, a, M are some real constants and K, M are both positive, then the Laplace transform $\mathcal{L}\left[f(x)\right] = \tilde{f}(s)$ as defined by* (4.2) *exists for $s > a$.*

The proof is given, for example, in Boyce & DiPrima. We note that if the conditions of the theorem hold then

$$\lim_{x\to\infty} f(x)e^{-sx} = 0. \tag{4.4}$$

4.3 General properties

There are a number of key basic properties that can easily be checked, and which are essential in calculations involving Laplace transforms.

1. If $\mathcal{L}[f(x)] = \tilde{f}(s)$ and C is a constant, then
$$\mathcal{L}[Cf(x)] = C\tilde{f}(s). \tag{4.5}$$

2. If $\mathcal{L}[f_1(x)] = \tilde{f}_1(s)$ and $\mathcal{L}[f_2(x)] = \tilde{f}_2(s)$ then
$$\mathcal{L}[f_1(x) + f_2(x)] = \tilde{f}_1(s) + \tilde{f}_2(s). \tag{4.6}$$
Both of these properties follow immediately from the linearity of integration.

3. If $\mathcal{L}[f(x)] = \tilde{f}(s)$ then
$$\mathcal{L}\left[e^{-ax}f(x)\right] = \tilde{f}(s+a). \tag{4.7}$$
This is the **first shift theorem** and is a simple application of the definition (4.2). It is extremely useful when inverting transforms.

4. If $\mathcal{L}[f(x)] = \tilde{f}(s)$ then
$$\mathcal{L}[xf(x)] = -\frac{\mathrm{d}\tilde{f}}{\mathrm{d}s}(s). \tag{4.8}$$
This follows using the definition (4.2) and integration by parts. Less useful in practice, it illustrates the difficulties that arise when transforming nonlinear equations.

5. If $\mathcal{L}[f(x,a)] = \tilde{f}(s,a)$ then
$$\mathcal{L}\left[\frac{\partial f}{\partial a}\right] = \frac{\partial \tilde{f}}{\partial a}. \tag{4.9}$$
This is an obvious consequence of the definition (4.2), and integration by parts gives the additional result
$$\mathcal{L}\left[\int_0^x f(u)\,\mathrm{d}u\right] = \frac{1}{s}\mathcal{L}[f(x)]. \tag{4.10}$$

4.3.1 Transforming derivatives

As our key application of Laplace transforms is going to be to differential equations, we are most interested in how a Laplace transform affects a derivative.

We directly apply the definition to find

$$\mathcal{L}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}x} e^{-sx}\,\mathrm{d}x \tag{4.11a}$$

$$= \left[f(x)e^{-sx}\right]_0^\infty + s\int_0^\infty f(x)e^{-sx}\,\mathrm{d}x. \tag{4.11b}$$

We then recall from section 4.2.1 that in order for the Laplace transform to exist we have

$$\lim_{x\to\infty} f(x)e^{-sx} = 0. \tag{4.4}$$

We therefore get

$$\mathcal{L}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = s\tilde{f}(s) - f(0). \tag{4.12}$$

This allows us to transform first derivatives of f into algebraic expressions involving $\tilde{f}$ and the initial data $f(0)$.

We can repeat this exercise for the second derivative, finding that

$$\mathcal{L}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right] = \mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\right] \tag{4.13a}$$

$$= s\mathcal{L}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] - f'(0) \tag{4.13b}$$

$$= s\left[s\tilde{f}(s) - f(0)\right] - f'(0) \tag{4.13c}$$

$$= s^2\tilde{f}(s) - sf(0) - f'(0). \tag{4.13d}$$

We note that in taking these steps we have assumed that the Laplace transform for f' exists, and hence assumed that

$$\lim_{x\to\infty} f'(x)e^{-sx} = 0. \tag{4.14}$$

Clearly we can extend this result to higher order by repeating these methods, finding

$$\mathcal{L}\left[\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right] = s^n\tilde{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0), \tag{4.15}$$

where we require

$$\lim_{x\to\infty} f^{(k)}(x)e^{-sx} = 0, \qquad k = 0,\ldots,n. \tag{4.16}$$

4.4 Inverse Transform

We first ask the key question of whether two different functions can have the same Laplace transform. If the two functions $f_1(x)$, $f_2(x)$ both transform to $\tilde{f}(s)$ then we have that

$$\int_0^\infty e^{-sx} f_1(x)\,\mathrm{d}x = \tilde{f}(s) = \int_0^\infty e^{-sx} f_2(x)\,\mathrm{d}x, \tag{4.17}$$

and hence

$$0 = \int_0^\infty e^{-sx}\left(f_1(x) - f_2(x)\right)\mathrm{d}x \tag{4.18a}$$

$$\Rightarrow \qquad 0 = f_1(x) - f_2(x) \tag{4.18b}$$

$$\Rightarrow \qquad f_1(x) = f_2(x). \tag{4.18c}$$

A truly rigorous proof of this requires various restrictions on the continuity of f_1, f_2 and on their behaviour as $x \to \infty$, but continuity plus the restrictions given in section 4.2.1 are sufficient.

Once we have that there is a one-to-one correspondence between a function and its Laplace transform, we see that the simplest method to invert a Laplace transform $\tilde{f}$ is simply to find or spot the function f that transforms to it. This is usually done using tables given in books or in the Course Summary Notes. With the addition of certain results, such as those given in section 4.3 and those to come later, most simple transforms can rapidly be inverted.

There does exist a general formula giving the inverse transform:

$$f(x) = \frac{1}{2\pi\mathrm{j}} \underbrace{\int_{\gamma-\mathrm{j}\infty}^{\gamma+\mathrm{j}\infty} \tilde{f}(s) e^{sx}\,\mathrm{d}x}_{\text{Contour integral}}. \tag{4.19}$$

This *Bromwich inversion formula* is beyond the scope of this course.

4.5 Special functions

4.5.1 Heaviside function

The *Heaviside step function*

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \tag{4.20}$$

is discontinuous at $x = 0$ and is useful in modelling problems where an effect suddenly "switches on". It also has a number of useful mathematical properties when combined with Laplace transform.

We can use the definition to show

$$\mathcal{L}\left[H(x-a)\right] = \int_0^\infty H(x-a)e^{-sx}\,\mathrm{d}x \tag{4.21a}$$

$$= \int_0^a 0 \times e^{-sx}\,\mathrm{d}x + \int_a^\infty 1 \times e^{-sx}\,\mathrm{d}x \tag{4.21b}$$

$$= \left[-\frac{e^{-sx}}{s}\right]_a^\infty \tag{4.21c}$$

$$= \frac{e^{-sa}}{s}. \tag{4.21d}$$

But more usefully we can multiply the Heaviside function by *any* function $f(x)$ that has a Laplace transform $\tilde{f}(s)$, to find

$$\mathcal{L}\left[H(x-a)f(x-a)\right] = \int_0^\infty H(x-a)f(x-a)e^{-sx}\,\mathrm{d}x \tag{4.22a}$$

$$= \int_{x=a}^\infty f(x-a)e^{-sx}\,\mathrm{d}x \tag{4.22b}$$

$$= \int_{\tau=0}^{\tau=\infty} f(\tau)e^{-s(\tau+a)}\,\mathrm{d}\tau \tag{4.22c}$$

using the change of variables $\tau = x - a$, to get

$$= e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}\,\mathrm{d}\tau \tag{4.22d}$$

$$= e^{-sa}\tilde{f}(s). \tag{4.22e}$$

This is the **second shift theorem**, which should be compared to the first shift theorem given in section 4.3.

These theorems are most useful to invert Laplace transforms, as they show that

$$\mathcal{L}^{-1}\left[\tilde{f}(s+a)\right] = e^{-sa}f(x), \tag{4.23a}$$

$$\mathcal{L}^{-1}\left[e^{-sa}\tilde{f}(s)\right] = H(x-a)f(x-a). \tag{4.23b}$$

4.5.2 The Dirac δ-function

The Dirac δ-"function" is not a genuine function at all. It is used in modelling instantaneous impulses, and as with the Heaviside function has a number of useful mathematical properties as well. There are a number of ways of defining the δ-function, but its essential properties are that

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x = f(0), \tag{4.24a}$$

$$\int_{a}^{b} \delta(x) f(x)\,\mathrm{d}x = 0 \qquad 0 \notin [a,b]. \tag{4.24b}$$

In particular, we note that to be strict the δ-function only makes sense when integrated.

One way of thinking about the δ-function is that it takes the values

$$\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0. \end{cases} \tag{4.25}$$

This is obviously an unsatisfactory definition at $x = 0$. It is more usual to rely on definitions based on limits, such as

$$\delta(x) = \lim_{\epsilon \to 0} D(x, \epsilon) \tag{4.26a}$$

where

$$D(x,\epsilon) = \begin{cases} 1/\epsilon & -\frac{\epsilon}{2} \leq x \leq \frac{\epsilon}{2} \\ 0 & \text{otherwise,} \end{cases}, \tag{4.26b}$$

which gives

$$\int_{-\infty}^{\infty} D(x,\epsilon)\,\mathrm{d}x = \int_{\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{1}{\epsilon}\,\mathrm{d}x \tag{4.26c}$$

$$= 1 \qquad \forall \epsilon. \tag{4.26d}$$

Note that $\lim_{\epsilon \to 0} D(0,\epsilon) = \infty$.

We see that the δ-function, when multiplied by f and integrated, "picks out" the value of f at the value of x where the argument of the δ-function vanishes. Thus the Laplace transform has the form

$$\mathcal{L}\left[\delta(x-a)\right] = \int_{0}^{\infty} \delta(x-a) e^{-sx}\,\mathrm{d}x \tag{4.27a}$$

$$= e^{-as}, \tag{4.27b}$$

and in particular $\mathcal{L}[\delta(x)] = 1$.

There is a sense in which the δ function is the derivative of the Heaviside function. This is certainly intuitively appealing and for certain arguments can be made rigorous.

4.6 Convolution

(Not examined)

One point that we have not emphasized but which should be clear is that

$$\mathcal{L}[f(x) \times g(x)] \neq \mathcal{L}[f(x)] \times \mathcal{L}[g(x)]. \tag{4.28}$$

It should also be clear that

$$\mathcal{L}^{-1}\left[\tilde{f}(s) \times \tilde{g}(s)\right] \neq \mathcal{L}^{-1}\left[\tilde{f}(s)\right] \times \mathcal{L}^{-1}[\tilde{g}(s)]. \tag{4.29}$$

As in practical examples we will frequently find the expression $\tilde{h}$ that we wish to invert is a product $\tilde{f}\tilde{g}$, it is useful to find an expression for this case.

The method relies on the *convolution theorem*, which can be expressed

$$\mathcal{L}\left[\int_0^x f(u)g(x-u)\,\mathrm{d}u\right] = \tilde{f}(s)\tilde{g}(s). \tag{4.30}$$

In particular this implies

$$\mathcal{L}^{-1}\left[\tilde{f}(s)\tilde{g}(s)\right] = \int_0^x f(u)g(x-u)\,\mathrm{d}u. \tag{4.31}$$

We define the *convolution* operation $f * g$ as

$$(f * g)(x) = \int_0^x f(u)g(x-u)\,\mathrm{d}u, \tag{4.32}$$

and say that the Laplace transform of the convolution is the product of the Laplace transforms.

Convolution Theorem.

$$\mathcal{L}[(f * g)(x)] = \tilde{f}(s)\tilde{g}(s). \tag{4.30}$$

Proof. We directly apply the definition of the Laplace transform to find

$$\mathcal{L}\left[(f*g)(x)\right] = \mathcal{L}\left[\int_0^x f(u)g(x-u)\,\mathrm{d}u\right] \tag{4.33a}$$

$$= \int_0^\infty e^{-sx}\left(\int_0^x f(u)g(x-u)\,\mathrm{d}u\right)\mathrm{d}x \tag{4.33b}$$

and, by interchanging the integration order, find

$$= \int_0^\infty f(u)\left(\int_u^\infty e^{-sx}g(x-u)\,\mathrm{d}x\right)\mathrm{d}u \tag{4.33c}$$

$$= \int_0^\infty f(u)\left(\int_u^\infty e^{-s(x-u)}e^{-su}g(x-u)\,\mathrm{d}x\right)\mathrm{d}u \tag{4.33d}$$

and, by introducing $z = x - u$

$$= \int_0^\infty f(u)e^{-su}\left(\int_{z=0}^\infty e^{-sz}g(z)\,\mathrm{d}z\right)\mathrm{d}u \tag{4.33e}$$

$$= \left(\int_0^\infty f(u)e^{-su}\,\mathrm{d}u\right)\left(\int_0^\infty e^{-sz}g(z)\,\mathrm{d}z\right) \tag{4.33f}$$

$$= \tilde{f}(s)\tilde{g}(s) \tag{4.33g}$$

as required. □

Convolution is often used to write the final solution $y(x)$ in terms of a formal integral solution.

4.7 Applications

4.7.1 Solving ODEs

Laplace transforms may be used to solve ODEs. To do this we follow the following steps.

1. We apply the Laplace transform to the equation we want to solve.

   Using linearity this amounts to rewriting the equation in terms of the Laplace transform of the function we want to obtain.

For example, given the 2^{nd} order differential equation

$$ay''(x) + by'(x) + cy(x) = f(x) \tag{4.34}$$

where a, b and c are constants, taking the Laplace transform gives:

$$\mathcal{L}\left[ay''(x) + by'(x) + cy(x)\right] = a\mathcal{L}\left[y''\right] + b\mathcal{L}\left[y'\right] + c\mathcal{L}\left[y\right] = \mathcal{L}\left[f\right] \tag{4.35}$$

2. We use the formula for the Laplace transform of derivatives and insert the initial condition.

   In the above example, we use $\mathcal{L}\left[y'\right] = s\tilde{y} - y(0)$ and $\mathcal{L}\left[y''\right] = s^2\tilde{y} - sy(0) - y'(0)$ in (4.35) to arrive at

$$a\left(s^2\tilde{y} - sy(0) - y'(0)\right) + b\left(s\tilde{y} - y(0)\right) + c\tilde{y} = \tilde{f} \tag{4.36}$$

   Collecting terms we have

$$\left(as^2 + bs + c\right)\tilde{y} = \tilde{f} + ay(0)s + (ay'(0) + by(0)) \tag{4.37}$$

3. This leads to an algebraic equation for the Laplace transform of the solution we want to find, which we solve.

   In our example we have:

$$\tilde{y} = \frac{\tilde{f} + ay(0)s + (ay'(0) + by(0))}{as^2 + bs + c} \tag{4.38}$$

4. We determine the inverse Laplace transform to find the solution.

$$y(t) = \mathcal{L}^{-1}\left[\frac{\tilde{f} + ay(0)s + (ay'(0) + by(0))}{as^2 + bs + c}\right] \tag{4.39}$$

Example – Let us solve the following differential equation using Laplace transform,

$$y'' + y = 0, \qquad y(0) = 0, \quad y'(0) = 1. \tag{4.40}$$

The first step is to apply the Laplace transform to the equation:

$$0 = \mathcal{L}\left[y'' + y\right] = \mathcal{L}\left[y''\right] + \mathcal{L}\left[y\right] \tag{4.41}$$

In step 2 we use (4.13),

$$\mathcal{L}\left[y''\right] = s^2\tilde{y}(s) - sy(0) - y'(0) = s^2\tilde{y}(s) - 1 \tag{4.42}$$

where in the last equality we used the boundary conditions. Inserting (4.42) in (4.41) (step 3) yields an algebraic equation for $\tilde{y}(s)$, which we solve:

$$0 = s^2\tilde{y}(s) - 1 + \tilde{y}(s) \qquad \Rightarrow \qquad \tilde{y}(s) = \frac{1}{1+s^2} \tag{4.43}$$

Finally (step 4), we need to determine the inverse Laplace transform of $\tilde{y}(s)$. Looking at the Table of Laplace transforms in the Formula Sheet we find

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{1+s^2}\right](t) = \sin t \tag{4.44}$$

4.7.2 Systems of ODEs

(Not Examinable)
We know that we can transform a single linear ODE such as

$$\frac{\mathrm{d}y}{\mathrm{d}x} - ay = f(x) \tag{4.45}$$

to the algebraic equation in terms of its Laplace transform $\tilde{y}(s)$,

$$(s-a)\,\tilde{y}(s) = \tilde{f}(s) + y(0). \tag{4.46}$$

We could similarly transform the system of equations

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}x} - A\boldsymbol{y} = \boldsymbol{f}(x), \tag{4.47}$$

where A is a matrix compatible with $\boldsymbol{y}$, $\boldsymbol{f}$, to the system of algebraic equations

$$(sI - A)\,\tilde{\boldsymbol{y}}(s) = \tilde{\boldsymbol{f}}(s) + \boldsymbol{y}(0). \tag{4.48}$$

Given the appropriate initial conditions for $\boldsymbol{y}$ we could then solve this vector equation for fixed s, as it is a standard linear algebra problem. In particular we note that the solution

$$\tilde{\boldsymbol{y}}(s) = (sI - A)^{-1}\left(\tilde{\boldsymbol{f}}(s) + \boldsymbol{y}(0)\right) \tag{4.49}$$

only exists when $sI - A$ is non-singular, which is when s is not an eigenvalue of A.

4.7.3 PDEs

As Laplace transforms are integral transforms they affect only one variable at a time. We can therefore use a Laplace transform to convert a PDE to an ODE. For example, if we consider the wave equation in spherical symmetry

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}(r,t) = \frac{1}{r}\frac{\partial^2 (ry)}{\partial r^2}(r,t), \tag{4.50}$$

we can perform a Laplace transform with respect to t to find

$$\frac{1}{c^2}\left(s^2\tilde{y}(s,r) - sy(r,0) - \frac{\partial y}{\partial t}(r,0)\right) = \frac{1}{r}\frac{\partial^2 (r\tilde{y})}{\partial r^2}(s,r). \tag{4.51}$$

Once we have initial conditions for y and its time derivative then we will have an ODE for $\tilde{y}$. In particular if we make the simplest choice that y and its time derivative vanish at $t = 0$ then we are left with the ODE

$$\frac{\mathrm{d}^2 (r\tilde{y})}{\mathrm{d}r^2} - \left(\frac{s}{c}\right)^2 (r\tilde{y}) = 0, \tag{4.52}$$

which obviously has the solution

$$r\tilde{y} = De^{\frac{sr}{c}} + Ee^{-\frac{sr}{c}}. \tag{4.53}$$

As we are assuming that $c > 0$, and $r, s > 0$ by definition, in order for the solution to be regular as $r \to \infty$ we must have $D \equiv 0$.

In order to compute the final solution we *cannot* directly invert the result that follows from equation (4.53),

$$\tilde{y}(s) = E\frac{e^{-\frac{sr}{c}}}{r}, \tag{4.54}$$

as the "constant" E that follows from solving equation (4.52) may be $E \equiv E(s)$. Instead we have to use an appropriate boundary condition in order to fix E. If we have a boundary condition such as

$$\left.\frac{\partial y}{\partial r}\right|_{r=a} = g(t) \tag{4.55}$$

then we can Laplace transform this, finding

$$\left.\frac{\mathrm{d}\tilde{y}}{\mathrm{d}r}(r,s)\right|_{r=a} = \tilde{g}(s) \tag{4.56a}$$

$$\Rightarrow \quad -E\frac{e^{-\frac{sa}{c}}}{a^2}\left(\frac{a^2 s}{c} + 1\right) = \tilde{g}(s) \tag{4.56b}$$

$$\Rightarrow \quad E = -\frac{a^2 c\tilde{g}(s)}{a^2 s + c}e^{\frac{sa}{c}}. \tag{4.56c}$$

We can then form our Laplace transformed solution,

$$\tilde{y}(s) = -\frac{a^2 c \tilde{g}(s)}{r\left(a^2 s + c\right)} e^{\frac{s(a-r)}{c}}, \tag{4.57}$$

which can (in principle) be inverted to find the solution.

Chapter 5

Wave equation

5.1 Classification of PDEs

We consider second order, linear PDEs of the form

$$\begin{aligned} a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} \\ + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = 0. \end{aligned} \tag{5.1}$$

We assume that $c \neq 0$ and perform the general change of variables

$$\xi = x + \beta y, \quad \eta = x + \delta y, \quad \delta \neq \beta. \tag{5.2}$$

This transforms the general form of equation (5.1) to

$$\begin{aligned} \left(a + 2b\beta + c\beta^2\right)\frac{\partial^2 u}{\partial \xi^2} + 2\left(a + b(\delta+\beta) + c\beta\delta\right)\frac{\partial^2 u}{\partial \xi \partial \eta} + \\ \left(a + 2b\delta + c\delta^2\right)\frac{\partial^2 u}{\partial \eta^2} = G\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta, F\right). \end{aligned} \tag{5.3}$$

We now use our freedom to choose β, δ to be roots of the quadratic equation

$$c\lambda^2 + 2b\lambda + a = 0. \tag{5.4}$$

This implies that the coefficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ in equation (5.3) vanish, so that it simplifies to the *canonical form*

$$\frac{4}{c}\left(ac - b^2\right)\frac{\partial^2 u}{\partial \xi \partial \eta} = G. \tag{5.5}$$

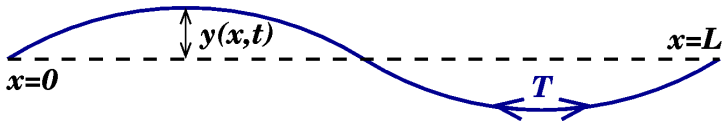


Figure 5.1: An elastic string is stretched between two fixed points at $x = 0, L$. The tension is constant (T) and the density (ρ) is constant. The wave equation describes the motion of the string through the behaviour of the ("small") displacement $y(x, t)$.

As we will see with the wave equation, when a PDE can be reduced to this form it has simple solutions (at least locally). Clearly the type of the solution will depend strongly on the qualitative form of the values β, δ used to change to the canonical form. In particular if

- $b^2 > ac$: the roots of equation (5.4), β and δ are real and distinct. The equation is called *hyperbolic*.
- $b^2 = ac$: the roots of equation (5.4), β and δ are real and repeated. The equation is called *parabolic*.
- $b^2 < ac$: the roots of equation (5.4), β and δ are complex conjugates. The equation is called *elliptic*.

5.2 Small amplitude vibrations of a stretched string

Consider a purely elastic string having a uniform density ρ per unit length, stretched at tension T, constant, between the fixed points $x = 0$ and $x = L$, subject to a small deflection $y(x, t)$.

Because of the fixed points, the boundary conditions in space are

$$y(0, t) = 0 = y(L, t). \tag{5.6}$$

Given an initial shape and velocity, i.e. prescribed initial conditions (or boundary conditions in time)

$$y(x, 0) = p(x), \quad \frac{\partial y}{\partial t}(x, 0) = q(x), \tag{5.7}$$

what is the shape at subsequent times?

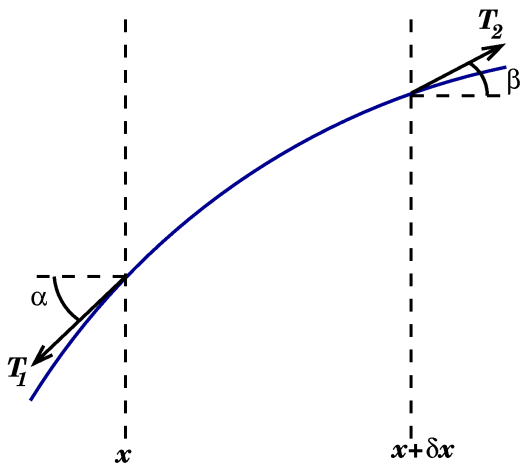


Figure 5.2: A small section of the string and the forces acting on it. By examining the forces over this small section and insisting that the string not break we find the wave equation holds for small displacements.

5.2.1 Equations of motion

We consider motions with small slopes

$$\left|\frac{\partial y}{\partial x}\right| \ll 1 \qquad \forall x, t, \tag{5.8}$$

as for large slopes the methods used below will not work and the assumptions made above do not hold. In the case of small slopes then we can use the small angle expansion of the trigonometric functions to see that

$$\alpha \approx \left.\frac{\partial y}{\partial x}\right|_x . \tag{5.9}$$

We resolve the forces on the string in the x-direction to find

$$T_1 \cos\alpha \approx T_2 \cos\beta \qquad \text{as} \quad \delta x \to 0. \tag{5.10}$$

But we can again use the small angle expansion to see that this implies that $T_1 \approx T_2$, i.e. the tension T is constant along the length of the string (as assumed above).

Resolving the forces in the y-direction gives

$$\rho\,\delta x\,\frac{\partial^2 y}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha \tag{5.11a}$$

$$\approx T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) \tag{5.11b}$$

$$\approx T\frac{\partial^2 y}{\partial x^2}\,\delta x. \tag{5.11c}$$

This immediately gives

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{5.12}$$

the 1-dimensional wave equation, where the *wave speed* c is given by

$$c^2 = \frac{T}{\rho}. \tag{5.13}$$

5.2.2 Energy of vibrating string

The total *kinetic* energy, that is the energy due to the motion of the string, is given by

$$KE = \int_0^L \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x. \tag{5.14}$$

The elastic *potential* energy, that is the energy due to the tension caused by the change in the length of the string, is given by

$$PE = T\int_0^L \left[\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1\right]\mathrm{d}x \tag{5.15a}$$

$$\approx T\int_0^L \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x \tag{5.15b}$$

as the slopes, and hence the derivatives, are small (equation (5.8)).

Hence the total energy E is

$$E = \int_0^L \frac{1}{2}\rho\left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x. \tag{5.16}$$

As there is no friction in the system, the total energy must be constant in time. It is a straightforward exercise (check!) to prove from the wave equation and the boundary conditions that the energy is constant.

5.3 D'Alembert's solution

We now look at the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{5.17}$$

in the absence of boundaries.

We introduce a new coordinate system, sometimes called *characteristic coordinates*:

$$\xi = x + ct, \tag{5.18a}$$
$$\eta = x - ct. \tag{5.18b}$$

The motivation for these coordinates should be clear from section 5.1, but for the moment we will just rewrite the wave equation (5.17) in terms of $y = y(\xi, \eta)$ and see where it takes us.

Firstly we note that the chain rule gives us

$$\frac{\partial}{\partial x} = 1 \cdot \frac{\partial}{\partial \xi} + 1 \cdot \frac{\partial}{\partial \eta}, \tag{5.19a}$$
$$\frac{\partial}{\partial t} = c \cdot \frac{\partial}{\partial \xi} - c \cdot \frac{\partial}{\partial \eta}. \tag{5.19b}$$

Thus the wave equation (5.17) becomes

$$\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 y = \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)^2 y \tag{5.20}$$

which gives

$$\frac{\partial^2 y}{\partial \xi \partial \eta} = 0. \tag{5.21}$$

This can immediately be solved one coordinate at a time to give, e.g.,

$$\frac{\partial y}{\partial \xi} = h(\xi) \tag{5.22}$$

and hence

$$y = f(\xi) + g(\eta) \tag{5.23}$$

where the functions f, g are arbitrary. Therefore the general solution of the wave equation (5.17), ignoring initial and boundary conditions, can be written as

$$y(x,t) = f(x + ct) + g(x - ct). \tag{5.24}$$

5.3.1 Travelling waves

The solution given by $y = g(x - ct)$ is a *travelling* or *progressive* wave. It is given this name as on the lines in the (x, t) plane where $x - ct$ is constant, called *characteristics*, the solution is constant. Therefore the wave retains its shape and moves with constant speed c. Characteristics can be similarly defined for solutions $y = f(x + ct)$ which travel in the opposite direction. This property is useful when the string is considered to be infinite so that the effects of the boundary conditions, which are neglected here, are unimportant. However, it is generally true that for the wave equation *information propagates at speed* c.

There is an even more special case solution, called a *harmonic* travelling wave

$$y = Ae^{\mathrm{j}\omega(t-x/c)}. \tag{5.25}$$

Here A is a *complex* constant, and it is understood that we always take the real part of the right hand side to get the solution. If we write the complex constant A as $A = |A|e^{\mathrm{j}\phi}$ then we see that

$$y = \underbrace{|A|}_{\text{amplitude}} \cos\Big(\underbrace{\omega}_{\text{frequency}}(t - x/c) + \underbrace{\phi}_{\text{phase}}\Big). \tag{5.26}$$

The key quantity is typically the *frequency* of the wave, ω, which implicitly defines its *wavelength* $\frac{2\pi c}{\omega}$.

5.3.2 Wave reflection and transmission

Wave reflection and transmission arises where the "natural" properties, such as the density of the string ρ, change discontinuously. This will lead to a discontinuous change in the wave speed $c = \sqrt{T/\rho}$. Typically a travelling wave is *incident* on (i.e., hits) the discontinuity and is partially reflected and transmitted. We want to predict the amount of reflection and transmission.

For simplicity we assume that the discontinuity arises in the density ρ only, and we choose our coordinates such that it occurs at $x = 0$. Thus to the left, $x < 0$, we have $\rho = \rho_-$ leading to the wave speed c_-, whilst to the right, $x > 0$, we have $\rho = \rho_+$ leading to the wave speed c_+. We assume that the incident wave is harmonic, i.e.,

$$y = Ae^{\mathrm{j}\omega(t-x/c_-)}, \tag{5.27}$$

and that the transmitted and reflected waves are given by travelling waves f, g respectively, i.e.

$$y \to f\left(t - \frac{x}{c_+}\right) + g\left(t + \frac{x}{c_-}\right). \tag{5.28}$$

From our assumption that the string does not break we have that y is continuous at $x = 0$. This tells us that the sum of the incident wave and the reflected wave is the transmitted wave at $x = 0$ for all times t.

We can also use the force balance at $x = 0$ and that the tension in the string is constant to give

$$T \left.\frac{\partial y}{\partial x}\right|_{x=0_-} = T \left.\frac{\partial y}{\partial x}\right|_{x=0_+}, \tag{5.29}$$

which implies that $\frac{\partial y}{\partial x}$ is continuous at $x = 0$ as well.

In order to get a solution we have to assume a specific form for the transmitted and reflected waves. We look for harmonic travelling wave solutions for these as well, where

$$f = De^{\mathrm{j}\omega(t-x/c_+)}, \tag{5.30a}$$

$$g = Be^{\mathrm{j}\omega(t+x/c_-)}. \tag{5.30b}$$

The continuity conditions at $x = 0$ give

$$y \text{ continuous:} \qquad A + B = D \tag{5.31a}$$

$$\frac{\partial y}{\partial x} \text{ continuous:} \qquad -\frac{\mathrm{j}\omega A}{c_-} + \frac{\mathrm{j}\omega B}{c_-} = -\frac{\mathrm{j}\omega D}{c_+}. \tag{5.31b}$$

We can solve these equations to get the unknown coefficients B, D in terms of the incident wave defined by A to get

$$B = \frac{c_+ - c_-}{c_+ + c_-} A, \tag{5.32a}$$

$$D = \frac{2c_+}{c_+ + c_-} A. \tag{5.32b}$$

Given that the problem is linear we could have taken A to be 1 from the outset, which would give the amplitudes and phases of the reflected (B) and transmitted (D) waves relative to the incident wave.

Note three special cases:

1. $c_+ = c_-$: In this case $B = 0$ and $D = A$, which is exactly what we would expect. The material properties do not change and information is perfectly transmitted with no reflection.

2. $\rho_+/\rho_- \to \infty$, or equivalently $c_+/c_- \to 0$: In this case $D = 0$ and $B = -A$. This corresponds to no transmission or perfect reflection; the reflected wave has the same amplitude but is out of phase by π. This is "reflection from a solid wall".

3. $\rho_+/\rho_- \to 0$: In this case $D = 2A$ and $B = A$. This corresponds to the string becoming "massless".

5.4 Separation of variables

5.4.1 Ansatz

We wish to solve the 1-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \quad t > 0, \tag{5.33}$$

with the boundary conditions

$$y(0,t) = 0 = y(L,t), \qquad t > 0, \tag{5.34}$$

and initial conditions

$$y(x,0) = p(x), \quad \frac{\partial y}{\partial t}(x,0) = q(x) \tag{5.35}$$

for given p, q.

We will firstly seek a solution in the very special form

$$y(x,t) = X(x)T(t), \tag{5.36}$$

i.e. with variables x and t separated. We substitute this ansatz into the wave equation (5.33) finding

$$X\ddot{T} = c^2 X'' T \tag{5.37a}$$

or, on dividing by $y = XT$,

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X}. \tag{5.37b}$$

Now we note that the left hand side of this equation depends solely on T and its derivatives with respect to time; that is, the left hand side is solely a function of t. However, the right hand side depends solely on X and its derivatives with respect to space; that is, the right hand side is solely a function of x.

How is it possible for a function solely of t to equal a function solely of x? If both functions are trivial, i.e. equal to zero, then it is obviously true. If both functions are constant then this gives us a constraint; the two functions must be equal to the *same* constant. However, if either function varies then this equation cannot hold, as (say) the left hand side will vary as t varies whilst the right hand side stays constant. Therefore this variables separable form immediately implies that both sides of the equation must be *separately constant*. In general, if the equation can be written as the sum of functions of different variables then each function must be separately constant.

In this case we have, from equation (5.37), that both sides must be equal to the same constant. Let us call it λ. Then we can re-arrange our variables separable form of equation (5.37) to give

$$X'' - \lambda X = 0, \tag{5.38a}$$

$$\ddot{T} - \lambda T = 0. \tag{5.38b}$$

So in this very special case we have reduced the *partial* DE defining the wave equation, equation (5.33), into a pair of *ordinary* DEs which are coupled through the *separation constant* λ.

5.4.2 Solving the separated form: spatial dependence

In order to solve the wave equation we have to solve the ODEs given by equation (5.38). Let us consider equation (5.38a) for the spatial behaviour. At present it is not completely specified, because we have not given any boundary conditions. We do not have any boundary conditions for X as it has no physical meaning. Instead we have to find boundary conditions for X that are compatible with the boundary conditions for y given by equation (5.34). As these boundary conditions force y to vanish at $x = 0, L$ we could impose that $X = 0$ at $x = 0, L$. If we consider what would happen if we assumed that $X \neq 0$ at one of these boundaries we see that the only way for $y = XT$ to vanish would be for T to vanish *for all time* t, which would give the trivial solution $y \equiv 0$. Hence we need to solve

$$X'' - \lambda X = 0, \qquad X(0) = 0, \quad X(L) = 0. \tag{5.39}$$

This is a second order ODE with one additional unknown, the value of the separation constant λ. So we do not expect to be able to completely solve the problem with the information provided, as we only have two boundary conditions. We do need to ensure that we find the value of the separation constant λ, however, as it is this that couples the two ODEs that we are trying to solve in equation (5.38).

We find that it is best to consider the solution of the ODE given by equation (5.39) in terms of three separate cases.

1. $\lambda > 0$. In this case the general solution to the (constant coefficient, linear, second order, homogeneous) ODE is

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}. \tag{5.40}$$

   We need this to be compatible with the boundary conditions. By setting

$x = 0, L$ we immediately get

$$x = 0 : \qquad A + B = 0, \tag{5.41a}$$
$$x = L : \qquad Ae^{2\sqrt{\lambda}L} + B = 0, \tag{5.41b}$$
$$\Rightarrow \quad A = B = 0. \tag{5.41c}$$

Hence we only have the solution $X \equiv 0$ which implies that $y \equiv 0$, the trivial solution. This is of no use.

2. $\lambda = 0$. In this case the general solution to the (constant coefficient, linear, second order, homogeneous) ODE is

$$X(x) = A + Bx. \tag{5.42}$$

We need this to be compatible with the boundary conditions. By setting $x = 0, L$ we immediately get

$$x = 0 : \qquad A = 0, \tag{5.43a}$$
$$x = L : \qquad A + BL = 0, \tag{5.43b}$$
$$\Rightarrow \quad A = B = 0. \tag{5.43c}$$

Hence we only have the solution $X \equiv 0$ which implies that $y \equiv 0$, the trivial solution. This is of no use.

3. $\lambda < 0$. In this case the general solution to the (constant coefficient, linear, second order, homogeneous) ODE is

$$X(x) = A\cos\left(\sqrt{-\lambda}x\right) + B\sin\left(\sqrt{-\lambda}x\right). \tag{5.44}$$

We need this to be compatible with the boundary conditions. If we impose the boundary condition at $x = 0$ then we immediately see that $A = 0$. If we impose the boundary condition at $x = L$ then we see that

$$0 = B\sin\left(\sqrt{-\lambda}L\right). \tag{5.45}$$

If this were to imply that $B = 0$ then once again we would only have the trivial solution $y = 0$. However, there is another possibility. It could be that the $\sin$ term vanishes instead. We know that $\sin$ vanishes whenever its argument is an integer multiple of π. In other words, there is a *non-trivial* solution to this equation if

$$\sqrt{-\lambda}L = n\pi, \qquad n \in \mathbb{Z}. \tag{5.46}$$

Hence, by looking at all the possible cases, we see that the only non-trivial solutions to the ODE for the spatial dependence, equation (5.39), are

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = -\left(\frac{n\pi}{L}\right)^2, \qquad n \in \mathbb{Z}. \tag{5.47}$$

Here A_n is an undetermined constant.

5.4.3 Solving the separated form: time dependence

We still have not found solutions to the full wave equation, as we have only solved for the spatial dependence in our separated problem given by equation (5.38). However, we have found the separation constant λ; it can take infinitely many different values, all discrete, depending on n as given by equation (5.47). So we can use this value for the separation constant when solving for the time dependence of the solution.

Substituting the value of λ into the ODE governing the time dependence, equation (5.38b), we find

$$\ddot{T} + \left(\frac{n\pi}{L}\right)^2 T = 0. \tag{5.48}$$

We can immediately write down the solution of this (linear, second order, constant coefficient, homogeneous) ODE as the coefficients have definite sign, to get

$$T_n(t) = C_n \cos\left(\frac{n\pi ct}{L}\right) + D_n \sin\left(\frac{n\pi ct}{L}\right). \tag{5.49}$$

We have not yet used any conditions that could determine the value of the constant coefficients C_n, D_n.

5.4.4 Normal modes

We now know, up to various free constants, the behaviour of the space (equation (5.47)) and time (equation (5.49)) dependence of the separated solution. Therefore we can multiply the results together to get *a* solution (or set of solutions) to the wave equation,

$$y_n(x,t) = \left[C_n \cos\left(\frac{n\pi ct}{L}\right) + D_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right). \tag{5.50}$$

Here we have slightly abused notation by absorbing the free A_n coefficient in the spatial dependence in equation (5.47) into the C_n, D_n coefficients in the time dependence, but without changing the names of the coefficients.

The set of solutions given by equation (5.50) satisfy the wave equation itself, and the boundary conditions, but *not* the initial conditions. They are a special set of solutions that have the property of retaining a constant *shape* of the form of a $\sin$ function with only the amplitude varying in time. These are called *normal modes*.

For reasons that are related to Sturm-Liouville theory, the spatial shape $\sin\left(\frac{n\pi x}{L}\right)$ are also called the *eigenfunctions*, whilst the separation constants λ_n are called the *eigenvalues*.

5.4.5 Superposition

We still do not have the solution to the problem that we really want, which is a complete solution of the wave equation (5.33) satisfying the boundary conditions of equation (5.34) and the initial conditions of equation (5.35). The normal mode solutions given by equation (5.50) solve the wave equation and the boundary conditions, but say nothing about the initial conditions.

However, we can exploit the fact that the wave equation and the boundary conditions are *linear*. Therefore if we have two solutions y_1, y_2 that satisfy the wave equation and the boundary conditions then any linear combination

$$y = A_1 y_1 + A_2 y_2 \tag{5.51}$$

will also satisfy both the wave equation and the boundary conditions. Therefore we can take a *general* solution of the wave equation to be a linear combination of the normal mode solutions given by equation (5.50):

$$y(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos\left(\frac{n\pi ct}{L}\right) + D_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right). \tag{5.52}$$

This is a *superposition* of the normal mode solutions. This general solution satisfies the wave equation and the boundary conditions, but has sufficient freedom in the choice of the C_n, D_n coefficients to also satisfy the *initial* condition given by equation (5.35).

To actually use this in practise we simply set $t = 0$ in our general solution which, when matched against the initial data of equation (5.35), gives

$$p(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right). \tag{5.53}$$

This is just a Fourier sine series for the C_n coefficients. Similarly, by taking the *time* derivative of our general solution given by equation (5.52) and then setting

$t = 0$, we find

$$q(x) = \sum_{n=1}^{\infty} D_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right). \tag{5.54}$$

This is again just a Fourier sine series for the D_n coefficients.

Using the standard Euler formulas for the coefficients of Fourier series we can explicitly write

$$C_n = \frac{2}{L} \int_0^L p(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \tag{5.55a}$$

$$D_n = \frac{2}{n\pi c} \int_0^L q(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x. \tag{5.55b}$$

5.4.6 Additional notes

Travelling waves

If we consider the case where the initial "velocity" of the wave as given by $\frac{\partial y}{\partial t}$ vanishes, i.e. $q \equiv 0 \equiv D_n$, then our solution reduces to

$$y(x,t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \tag{5.56a}$$

and using the compound angle formulas for $\sin$ we write this as

$$= \frac{1}{2} \sum_{n=1}^{\infty} C_n \left[\sin\left(\frac{n\pi(x+ct)}{L}\right) + \sin\left(\frac{n\pi(x-ct)}{L}\right)\right] \tag{5.56b}$$

$$= \frac{1}{2}\left[p(x+ct) + p(x-ct)\right], \tag{5.56c}$$

where in the last step we have noted that

$$p(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right). \tag{5.57}$$

We can interpret this by noting that $p(x)$ was the initial "shape" of the wave, i.e. $y(x,0) = p(x)$. Therefore $p(x+ct)$ is just the original shape of the wave with the origin of the x coordinate moved back from $x = 0$ to $x = -ct$. Therefore the solution given by equation (5.56) represents the initial data splitting into two travelling waves, one travelling to the left and one to the right, both being the same "shape" as the initial data but with half the amplitude. Each wave moves with speed c, giving the meaning of the *wave speed*.

It should be noted that this solution is not genuinely in the separation of variables form, however, this form, suitably re-interpreted, is d'Alembert's solution as seen in section 5.3.

Energy

We showed earlier in section 5.2.2 that the energy of the wave is

$$E = \int_0^L \frac{1}{2}\rho \left[\left(\frac{\partial y}{\partial t} \right)^2 + c^2 \left(\frac{\partial y}{\partial x} \right)^2 \right] \mathrm{d}x. \tag{5.58}$$

If we substitute in our general solution given by equation (5.52) then we find, on integration over the domain, that all the cross terms between $\sin$ and $\cos$ or between $\sin$ and $\sin$ with different arguments, for example, will vanish. Hence we find that

$$\text{Total energy at time } t = \sum_{\substack{\text{normal} \\ \text{modes}}} \text{Energy in normal mode } n \text{ at time } t. \tag{5.59}$$

It is a straightforward exercise to check, by using the equation for a single normal mode, equation (5.50), in the equation for the energy, equation (5.58), that

$$\text{Energy in normal mode } n = \frac{1}{4}\rho \left(\frac{n\pi c}{L} \right)^2 \left(C_n^2 + D_n^2 \right). \tag{5.60}$$

This is independent of time. In particular, this shows that there is *no* exchange of energy between different normal modes.

Chapter 6

Heat equation

6.1 Heat conduction in a rod

The model problem for all *diffusive* equations is that of heat transport in a solid rod. The heat can only move along the axis, as we assume that it is perfectly insulated in the other directions. The key assumption, which is borne out by experiment, is that if we consider two parallel cross sections of area A separated by a small distance d, an amount of heat (per unit time) will pass from the warmer section to the cooler. In particular we will assume that the amount of heat transferred is proportional to the area A and the temperature difference $|T_2 - T_1|$ and inversely proportional to the separation d, giving

$$\text{Amount of heat transferred per unit time} = \kappa \frac{A|T_2 - T_1|}{d}. \tag{6.1}$$

The proportionality constant κ, called the *thermal conductivity*, is (to first approximation) dependent only on the material of the rod. This is an empirical relation which has its problems from the mathematical physics point of view, but leads to an equation that produces accurate results in many cases.

We will consider the temperature as a function of x and t only, assuming that it is constant in any cross section at constant x. We consider the section of the bar such that $x \in [x_0, x_0 + \delta x]$, where x_0 is arbitrary and δx small. From the empirical law in equation (6.1) we have that the instantaneous rate of heat transfer $H(x_0, t)$ from left to right into the section under consideration through the surface at $x = x_0$ is given by

$$H(x_0, t) = -\lim_{d \to 0} \kappa A \frac{T\left(x_0 + \frac{d}{2}, t\right) - T\left(x_0 - \frac{d}{2}, t\right)}{d} \tag{6.2a}$$

$$= -\kappa A \frac{\partial T}{\partial x}(x_0, t). \tag{6.2b}$$

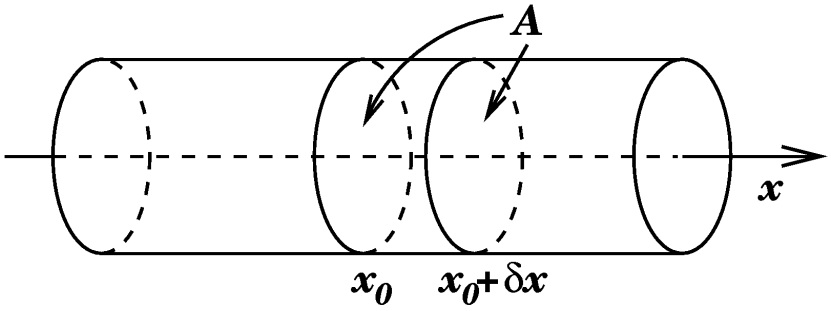


Figure 6.1: Schematic picture of heat flow in a section of a rod

This indicates that there will be positive heat flow into the section from the left if the temperature is greater to the left of $x = x_0$, as we would expect. Repeating the calculation for heat flow out of the section to the right through the surface at $x = x_0 + \delta x$ gives

$$H(x_0 + \delta x, t) = -\kappa A \frac{\partial T}{\partial x}(x_0 + \delta x, t). \tag{6.3}$$

Therefore the net rate at which heat flows into (or out of) the section of the bar between x_0 and $x_0 + \delta x$ is given by

$$Q = H(x_0, t) - H(x_0 + \delta x, t) \tag{6.4a}$$

$$= \kappa A \left[\frac{\partial T}{\partial x}(x_0, t) - \frac{\partial T}{\partial x}(x_0 + \delta x, t) \right], \tag{6.4b}$$

and hence the amount of heat entering the section in time δt is

$$Q\,\delta t = \kappa A \left[\frac{\partial T}{\partial x}(x_0, t) - \frac{\partial T}{\partial x}(x_0 + \delta x, t) \right] \delta t. \tag{6.5}$$

This has to be balanced by the amount of heat actually absorbed by this section of the bar. The average change in temperature δT in time δt is proportional to the heat that flows into the section, $Q\,\delta t$ and inversely proportional to the mass of the section. As the mass is proportional to the volume of the section $A\,\delta x$, we write this as

$$\delta T = \frac{Q\,\delta t}{sA\,\delta x}, \tag{6.6}$$

where s is related to (but is not exactly) the *specific heat* of the material of the bar. Now, for a sufficiently small section of bar the average temperature change in the section will be

$$\delta T \approx \frac{\partial T}{\partial t}\,\delta t \tag{6.7a}$$

$$= \frac{Q\,\delta t}{sA\,\delta x} \tag{6.7b}$$

by using equation (6.6). Matching this against the expression for the heat flowing in to the bar, equation (6.5), gives

$$\frac{\partial T}{\partial t}\,\delta t \approx \frac{\kappa}{s}\frac{\left[\frac{\partial T}{\partial x}(x_0,t) - \frac{\partial T}{\partial x}(x_0+\delta x,t)\right]\,\delta t}{\delta x\,\delta t}. \tag{6.8}$$

Taking the limits $\delta x \to 0$, $\delta t \to 0$ gives

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2}. \tag{6.9}$$

We note that the standard notation that we will use from now on is

$$\frac{\partial y}{\partial t}(x,t) = \kappa^2 \frac{\partial^2 y}{\partial x^2}(x,t). \tag{6.10}$$

Here κ is some constant related to the thermal conductivity of the material, or in general the diffusivity of the problem that we are interested in.

6.2 Separation of Variables

We start with the simplest problem of the heat equation

$$\frac{\partial y}{\partial t}(x,t) = \kappa^2 \frac{\partial^2 y}{\partial x^2}(x,t) \tag{6.10}$$

with Dirichlet boundary conditions

$$y(0,t) = 0 = y(L,t), \qquad t > 0, \tag{6.11}$$

and initial conditions

$$y(x,0) = p(x) \tag{6.12}$$

for given p.

The standard ansatz

$$y(x,t) = X(x)T(t) \tag{6.13}$$

when substituted into the heat equation (6.10) gives

$$X\dot{T} = \kappa^2 X'' T \tag{6.14a}$$

or, on dividing by $y = XT$,

$$\frac{1}{\kappa^2}\frac{\dot{T}}{T} = \frac{X''}{X}. \tag{6.14b}$$

Again the two sides are separately constant, and by the same argument made as for the wave equation in section 5.4.1 we can write the separated equations as

$$X'' - \lambda X = 0, \tag{6.15a}$$

$$\dot{T} - \kappa^2 \lambda T = 0. \tag{6.15b}$$

Again, λ is some constant whose value is to be determined.

At this point we need boundary conditions in order to solve the ODEs give by equation (6.15). Exactly as in the wave equation case the boundary conditions in equation (6.11) imply Dirichlet boundary conditions for X at $x = 0, L$, giving the fully-posed ODE problem

$$X'' - \lambda X = 0, \qquad X(0) = 0, \quad X(L) = 0. \tag{6.16}$$

This is identical to the problem in the wave equation case (equation (5.39) in section 5.4.2). Hence we can write down the solution as

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = -\left(\frac{n\pi}{L}\right)^2, \qquad n \in \mathbb{Z}. \tag{6.17}$$

Here A_n is an undetermined constant.

We now need to solve for the time dependence, which is given by equation (6.15b). Once the eigenvalue λ is known, this is a simple first order ODE. Given that the results for the spatial dependence, summarized in equation (6.17), tell us that λ is negative, we can write down the solution to the time dependence as

$$T_n(t) = C_n \exp\left(\lambda_n \kappa^2 t\right) \tag{6.18a}$$

$$= C_n \exp\left(-\left(\frac{n\pi}{L}\right)^2 \kappa^2 t\right). \tag{6.18b}$$

Again, C_n is an undetermined constant.

Combining our solutions and superposing gives the general solution

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 \kappa^2 t\right). \tag{6.19}$$

Once again we have abused notation to absorb the free constants into the single set of constants C_n.

Again we determine the free constants by noting that our initial conditions given by equation (6.12) must match our general solution of equation (6.19) when evaluated at $t = 0$, to give

$$p(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right). \tag{6.20}$$

This is exactly the same Fourier Series problem as in the wave equation case.

6.3 More complex material properties

In the initial derivation of the heat equation given in section 6.1 we assumed that all of the material properties of the rod, such as the conductivity κ, the cross-sectional area A and the specific heat s were constant within the rod. It is a key part of the derivation that they are constant within any cross-section. However, a first generalization would allow them to depend on the x location within the rod.

Allowing the parameters to vary in this fashion obviously modifies the heat flux used in equation (6.2) and later to give, for example,

$$H(x_0, t) = -\kappa(x_0)A(x_0)\frac{\partial T}{\partial x}(x_0, t). \tag{6.21}$$

When these expressions are used to compute the net heat transfer Q and balanced against the heat absorption we obtain a more general heat conduction equation

$$r(x)\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial y}{\partial x}\right). \tag{6.22}$$

The spatial dependence of the various parameters has been collected into the functions r and p which are positive on empirical grounds.

Additional complexity can be added by relaxing the assumption that the rod is perfectly insulated and adding a *source* or *sink* term that adds or removes heat. To make the problem tractable it is typically assumed that this source term $G(x, t, T)$ depends linearly on the temperature T and that the linear term is time independent, i.e.

$$G(x, t, T) = F(x, t) - q(x)T(x, t). \tag{6.23}$$

When adding this source term to our increasingly complex heat conduction model of equation (6.22) we obtain the *generalized heat conduction* equation

$$r(x)\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial y}{\partial x}\right) - q(x)y + F(x, t). \tag{6.24}$$

6.3.1 Separation of variables

We take our generalized heat conduction equation

$$r(x)\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial y}{\partial x}\right) - q(x)y + F(x, t) \tag{6.24}$$

and for the moment consider the homogeneous problem where $F \equiv 0$, giving

$$r(x)\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial y}{\partial x}\right) - q(x)y. \tag{6.25}$$

We use our standard separation of variables ansatz, $y = X(x)T(t)$ to obtain

$$r(x)X(x)\dot{T}(t) = (p(x)X'(x))'T(t) - q(x)X(x)T(t). \tag{6.26}$$

Dividing both sides by $r(x)y(x,t) = r(x)X(x)T(t)$ gives

$$\frac{\dot{T}}{T}(t) = \frac{(p(x)X'(x))'}{r(x)X(x)} - \frac{q(x)}{r(x)}. \tag{6.27}$$

Once again we can note that both sides must be separately constant, leading to the ODEs

$$\dot{T} + \lambda T = 0, \tag{6.28a}$$

$$(p(x)X'(x))' - q(x)X(x) + \lambda r(x)X(x) = 0. \tag{6.28b}$$

Once again we have an eigenvalue problem for the unknown separation constant λ. However, the spatial dependence governed by equation (6.28b) is clearly far more complex than before. However, we can note that equation (6.28b) is exactly the form of a Sturm-Liouville problem as given in section 1.2.2; in particular, equation (6.28b) should be compared to equation (1.60).

Particular results from Sturm-Liouville theory can immediately be used in the solution of more complex boundary value problems such as this. For example, the theory immediately tells us which boundary conditions guarantee a solution (see section 1.2.2), that eigenvalues and eigenfunctions with qualitatively similar properties to those seen earlier exist (see section 1.2.2), and that there are orthogonality conditions that can be used in place of the Fourier Series techniques to determine the resulting free coefficients. Therefore once the solution to the spatial dependence given by equation (6.28b) is found, all of the standard separation of variables steps can be applied.

6.4 More complex boundary conditions

The results from Sturm-Liouville theory in section 1.2.2 suggest that we should be able to solve the standard heat equation

$$\frac{\partial y}{\partial t}(x,t) = \kappa^2 \frac{\partial^2 y}{\partial x^2}(x,t) \tag{6.10}$$

with a broader range of boundary conditions than the simple Dirichlet conditions

$$y(0,t) = 0 = y(L,t), \qquad t > 0. \tag{6.11}$$

For the generalized (homogeneous) heat equation

$$r(x)\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial y}{\partial x}\right) - q(x)y \tag{6.25}$$

we can compare the results of the separation of variables ansatz in section 6.3.1 to those of Sturm-Liouville theory in section 1.2.2 to see that any appropriate combination of

1. $y = 0$: Dirichlet condition;
2. $y' = 0$: Neumann condition;
3. $y + ky' = 0$: Radiation condition;
4. $p = 0$: $x = x_0, x_1$ are singular points of the ODE;
5. y periodic;

fits the conditions necessary for separation of variables to be a practical technique.

For the standard heat equation (6.10) we have $p = 1$ so either the solution is periodic or some appropriate combination of the Dirichlet, Neumann and Radiation conditions must hold. We consider applying the general combined boundary condition

$$\alpha_1 y(0) + \beta_1 y'(0) = 0, \qquad \alpha_2 y(L) + \beta_2 y'(L) = 0. \tag{6.29}$$

Applied to the standard heat equation and using separation of variables as normal gives the generalization of the ODE in equation (6.16)

$$X'' - \lambda X = 0, \qquad \alpha_1 X(0) + \beta_1 X'(0) = 0, \quad \alpha_2 X(L) + \beta_2 X'(L) = 0. \tag{6.30}$$

We look at the case $\lambda = \mu^2 > 0$ where the auxiliary equation has distinct real roots $\pm\mu$. Rather than writing the general solution to the ODE in terms of exponentials we instead write it as

$$X(x) = A\sinh(\mu x) + B\cosh(\mu x). \tag{6.31}$$

It is straightforward to check that the boundary conditions imply that

$$0 = X(0) = \tilde{\alpha}_1 \sinh(\mu 0) + \tilde{\beta}_1 \cosh(\mu 0), \tag{6.32a}$$

$$0 = X(L) = \tilde{\alpha}_2 \sinh(\mu L) + \tilde{\beta}_2 \cosh(\mu L). \tag{6.32b}$$

It is straightforward to check that the only solution compatible with these boundary conditions is the trivial solution. The same is true if we attempt to impose that

the solution y is periodic; this implies that X is periodic, which is not possible for the hyperbolic trigonometric functions that make up this solution.

In contrast, the case $\lambda = 0$ may have a solution. The general solution of equation (6.30) when $\lambda = 0$ is obviously

$$X(x) = Ax + B. \tag{6.33}$$

In most cases we can still check that $A = 0$ (but this need not be true, for example, for the radiation condition!), but the solution where $B \neq 0$ is compatible with periodic boundary conditions, and with a range of Dirichlet and radiation conditions. This leads to additional solutions for the time dependence that are similarly of the form

$$T(t) = Ct + D. \tag{6.34}$$

Typically the solution being bounded either in the past or the future implies that $C = 0$. However, the constant solution is again frequently valid. It should be noted that in more complex domains these additional solutions often lead to considerably more complex behaviour that must be considered.

6.5 Inhomogeneous problems

6.5.1 Inhomogeneous equations

In this section we consider a small extension to the standard heat equation

$$\frac{\partial y}{\partial t}(x,t) = \kappa^2 \frac{\partial^2 y}{\partial x^2}(x,t) \tag{6.10}$$

suggested by the generalized heat equation (6.24); we add the source term F to get

$$\frac{\partial y}{\partial t}(x,t) = \kappa^2 \frac{\partial^2 y}{\partial x^2}(x,t) + F(x,t). \tag{6.35}$$

For simplicity and for later use we will assume trivial Dirichlet boundary conditions

$$y(0,t) = 0 = y(1,t) \tag{6.36}$$

on the simple domain $x \in [0,1]$.

Clearly for general source term F we cannot use separation of variables, as our ansatz $y = XT$ does not lead to an equation containing only separably constant terms. Instead we make the *assumption* that the spatial dependence of the solution can still be described by a Fourier Series, and because of the simple boundary

conditions of equation (6.36), this implies a Fourier Sine Series. This gives us the more general ansatz

$$y(x,t) = \sum_{n=1}^{\infty} y_n(t) \sin(n\pi x). \tag{6.37}$$

We still cannot start solving our problem without making additional assumptions. However, we can substitute our ansatz into the inhomogeneous heat equation (6.35) and see what results. We find

$$\sum_{n=1}^{\infty} \left[\dot{y}_n(t) - (n\pi)^2 y_n(t)\right] \sin(n\pi x) = F(x,t). \tag{6.38}$$

At any instant in time the left hand side is a Fourier Sine Series. Therefore in order to be consistent the source term F must also be a Fourier Sine Series. Making that assumption,

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin(n\pi x), \tag{6.39}$$

we can substitute this in to the inhomogeneous heat equation as well to find

$$\sum_{n=1}^{\infty} \left[\dot{y}_n(t) - (n\pi)^2 y_n(t) - F_n(t)\right] \sin(n\pi x) = 0. \tag{6.40}$$

By using the standard orthogonality relations we immediately see that

$$\dot{y}_n(t) - (n\pi)^2 y_n(t) - F_n(t) = 0. \tag{6.41}$$

This is a set of ODEs to solve for the unknown $y_n(t)$ in terms of the F_n, which can be computed from the known source term F. In particular, this linear ODE has the obvious solution

$$y_n(t) = C_n \exp\left[-(n\pi)^2 t\right] + \int_0^t F_n(s) \exp\left[-(n\pi)^2 s\right] \mathrm{d}s. \tag{6.42}$$

Once F_n has been computed we can compute y_n and our original ansatz in equation (6.37) gives the solution.

6.5.2 Inhomogeneous boundary conditions

In this section we consider the standard heat equation

$$\frac{\partial y}{\partial t}(x,t) = \kappa^2 \frac{\partial^2 y}{\partial x^2}(x,t) \tag{6.10}$$

with the inhomogeneous boundary conditions

$$y(0,t) = f(t), \tag{6.43a}$$
$$y(1,t) = g(t). \tag{6.43b}$$

We have assumed that the domain has length 1 for simplicity and that f, g are given functions.

It should immediately be obvious that separation of variables fails. This is because we can no longer solve the separated problem for the spatial dependence, as the boundary conditions are functions of time. However, we can use a simple trick inspired by techniques for solving inhomogeneous ODE problems to reduce this inhomogeneous PDE to a form that can be solved by separation of variables.

First we note that the heat equation (6.10) is linear so we can superpose solutions, even when we have complex boundary conditions such as equation (6.43). Therefore if we had a *particular* solution $y_P(x,t)$ that satisfies the heat equation and the boundary conditions (6.43), we could superpose it with another solution $y_G(x,t)$ of the heat equation, and the combined solution

$$y(x,t) = y_P(x,t) + y_G(x,t) \tag{6.44}$$

will satisfy the heat equation. More importantly, we note that if the solution $y_G(x,t)$ *vanishes* at the boundaries $x = 0, 1$ then the combined solution $y(x,t)$ also satisfies the boundary conditions (6.43).

Unfortunately it is just as difficult to find a single solution to the full inhomogeneous problem as it is to find the general solution. Instead we suppose we have a particular solution y_P that satisfies the boundary conditions but *not* the heat equation. When we form the combined solution using equation (6.44) we find that the combined solution satisfies the boundary conditions but not the heat equation (6.10). It will however satisfy the *inhomogeneous* heat equation (6.35), where the source term $F(x,t)$ can be computed directly from y_P.

Thus we have reduced the problem of finding the general solution of the heat equation (6.10) with complex boundary conditions (6.43) to the problem of finding a simple function y_P that satisfies the complex boundary conditions (6.43), plus the problem of solving the inhomogeneous heat equation (6.35) with trivial Dirichlet boundaries as studied in section 6.5.1.

We still have the problem of finding a sufficiently simple particular solution y_P. However, this can itself be simplified by assuming that it separates, $y_P(x,t) = X_P(x)T_P(t)$. In particular, if we choose X_P to be a *linear* polynomial, then it trivially vanishes under the second derivative in the heat equation. We then choose T to satisfy the boundary conditions. In particular the choice

$$y_P(x,t) = xg(t) + (1-x)f(t) \tag{6.45}$$

has the correct behaviour at the boundaries and vanishes under the second spatial derivative. Therefore we can see that this solution satisfies the equation

$$\frac{\partial y_P}{\partial t}(x,t) = \frac{\partial^2 y_P}{\partial x^2}(x,t) + x\dot{g}(t) + (1-x)\dot{f}(t). \tag{6.46}$$

This gives the simple source term

$$F(x,t) = x\dot{g}(t) + (1-x)\dot{f}(t), \tag{6.47}$$

meaning that the general solution of the heat equation with inhomogeneous boundary conditions (6.43) reduces to

$$y(x,t) = y_P(x,t) + y_G(x,t) \tag{6.48}$$

where y_G is the general solution of the inhomogeneous heat equation (6.35) with source term (6.47) and trivial Dirichlet boundary conditions.

Chapter 7

Laplace's equation

7.1 Introduction

Laplace's equation is simply

$$\nabla^2 \phi = 0. \tag{7.1}$$

In $\mathbb{R}^1$ and Cartesian coordinates Laplace's equation reduces to

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}x^2} = 0, \tag{7.2}$$

where $\phi(x)$ represents, e.g., the static shape of a stretched string, or the steady temperature in a conducting bar, or something similar.

In $\mathbb{R}^2$ and Cartesian coordinates Laplace's equation reduces to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0, \tag{7.3}$$

where $\phi(x, y)$ represents, e.g., the static shape of a stretched membrane such as a soap film, or the steady temperature in a conducing sheet, or something similar.

In $\mathbb{R}^3$ and Cartesian coordinates Laplace's equation reduces to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 0, \tag{7.4}$$

where $\phi(x, y, z)$ represents, e.g., the steady temperature in a conducing solid, or the velocity potential in an inviscid fluid, or the electrostatic potential in a charge-free region, or the gravitational potential in a mass-free region, or something similar.

In general we usually call ϕ the *potential*, and $\boldsymbol{F} = (-)\boldsymbol{\nabla}\phi$ is the corresponding *field*.

7.1.1 Boundary conditions

In order to pick out a unique solution it is necessary to specify boundary conditions at the edges of the domain. The *natural* boundary conditions are to specify either

1. ϕ: Dirichlet boundary conditions, *or*
2. $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}\phi$: Neumann boundary conditions,

at each point of the bounding surface. We note that this is two points in $\mathbb{R}^1$, a bounding curve in $\mathbb{R}^2$, and a bounding surface in $\mathbb{R}^3$. The (unit) vector $\hat{\boldsymbol{n}}$ is the normal to the bounding surface.

7.2 Separable solutions of Laplace's equation

We consider the two dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0 \tag{7.3}$$

on the unit square with trivial Dirichlet boundary conditions,

$$\begin{gathered}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0, \quad 0 \leq x, y \leq 1, \\ \phi(0,y) = \phi(1,y) = \phi(x,0) = \phi(x,1) = 0.\end{gathered} \tag{7.5}$$

We make the standard separation of variables assumption, $\phi(x,y) = X(x)Y(y)$. As usual this gives two ODEs

$$X'' - \lambda X = 0, \tag{7.6a}$$

$$Y'' + \lambda Y = 0. \tag{7.6b}$$

Both are BVPs with trivial boundary conditions. This is different to the situation for the heat and wave equations where only one of the ODEs was a BVP.

We solve the eigenvalue problem for the x dependence as normal, finding that the only non-trivial solution is

$$\lambda_n = -(n\pi)^2, \qquad X_n(x) = \sin(n\pi x). \tag{7.7}$$

We are left with the solution for the y dependence, and using this value of the separation constant λ_n in the ODE of equation (7.6b) gives

$$Y_n = A_n \sinh(n\pi y) + B_n \cosh(n\pi y). \tag{7.8}$$

The trivial Dirichlet boundary conditions immediately imply that the only solution compatible with the boundary conditions is the trivial solution $Y_n \equiv 0$ and hence $\phi(x, y) \equiv 0$.

This illustrates two points. One is the *maximum principle*: solutions of Laplace's equation always take their extreme values (both maxima and minima) at the boundaries of the domain. This holds in all dimensions and (reasonable) domains. Although not proved or used in this course, it is easily motivated by considering the stationary limit of the heat equation.

The second point is that for the vast majority of interesting problems using Laplace's equation we must expect them to be inhomogeneous, either through their boundary conditions or a source term (Poisson's equation). There is a straightforward method of dealing with these problems, exploiting the linearity of the equation.

7.2.1 Cartesians

For simplicity we explain the technique using the two-dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0 \tag{7.3}$$

on the unit square (hence using Cartesian coordinates) with some set of boundary conditions

$$\phi(x, 1) = \Phi_1(x), \tag{7.9a}$$
$$\phi(1, y) = \Phi_2(y), \tag{7.9b}$$
$$\phi(x, 0) = \Phi_3(x), \tag{7.9c}$$
$$\phi(0, y) = \Phi_4(y), \tag{7.9d}$$

which implicitly number the boundaries of the domain in a clockwise sense.

The technique has two steps.

1. Change variable from $\phi(x, y)$ to a function $u(x, y) = \phi(x, y) - v(x, y)$ which satisfies the Laplace equation in the interior and also satisfies boundary conditions U_i on boundary i that vanish in the corners of the domain.

2. Solve for $u(x, y)$ by writing it as

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y), \tag{7.10}$$

where each u_i satisfies Laplace's equation in the interior, satisfies the boundary condition U_i on boundary i and vanishes (i.e., satisfies a trivial Dirichlet boundary condition) on all other boundaries.

Once we have found each of the u_i, we can reconstruct the full solution to the original problem using

$$\phi(x,y) = v(x,y) + u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y). \tag{7.11}$$

Finding the difference function

The first required step is to find the difference function $v(x,y)$. This is constructed so that u has boundary conditions taking trivial values in the corners of the domain. In general it should be clear that as ϕ satisfies Laplace's equation, and we want u also to do so, v must satisfy Laplace's equation. It should also be clear that v must take the same value as the boundary conditions Φ_i in the corners of the domain.

This is a considerably easier problem, as we can take any solution of Laplace's equation that is constrained at only four points. In Cartesians the simplest solutions of Laplace's equation are the low order polynomials $1, x, y, xy$. Therefore we can define our difference function to be

$$v(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy. \tag{7.12}$$

The values of the α coefficients is set by the restrictions given at the corners of the domain, so that

$$\begin{aligned}
v(0,0) &= \alpha_1 &&= \Phi_3(0) &&= \Phi_4(0) && \text{(7.13a)}\\
v(1,0) &= \alpha_1 + \alpha_2 &&= \Phi_2(0) &&= \Phi_3(1) && \text{(7.13b)}\\
v(0,1) &= \alpha_1 + \alpha_3 &&= \Phi_1(0) &&= \Phi_4(1) && \text{(7.13c)}\\
v(1,1) &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &&= \Phi_1(1) &&= \Phi_2(1). && \text{(7.13d)}
\end{aligned}$$

This simple linear system can be solved by forward substitution, provided the boundary conditions are consistent in the corners.

Solving the sub-problems

We now have a Laplace equation problem for u subject to boundary conditions that vanish in the corners. We have to solve the four sub-problems for u_i; for example, the problem for $u_1(x,y)$ is

$$\begin{gathered}
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_1 = 0,\\
u_1(x,1) = U_1(x), \quad u_1(1,y) = u_1(x,0) = u_1(0,y) = 0.
\end{gathered} \tag{7.14}$$

Using standard separation of variables techniques as in section 7.2 we can write $u_1 = X(x)Y(y)$, and again compute the separation constant and x dependent behaviour to find

$$\lambda_n = -(n\pi)^2, \qquad X_n(x) = \sin(n\pi x) \tag{7.15}$$

as in equation (7.7). We then solve for the y dependence as in equation (7.8), and the trivial boundary condition at $y = 0$ restricts the solution to

$$Y_n = A_n \sinh(n\pi y). \tag{7.16}$$

We can once again superpose solutions, finding the result

$$u_1(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh(n\pi y), \tag{7.17}$$

where the value of the coefficients A_n is set by the boundary condition at $y = 1$. This gives

$$U_1(x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) \tag{7.18}$$

where the new constant C_n is

$$C_n = A_n \sinh(n\pi). \tag{7.19}$$

Using standard Fourier Series techniques we can find C_n, from which we can find A_n, and hence find u_1. Repeating this for all boundaries gives the solution to each sub-problem u_i, hence the solution for u, from which we can reconstruct the solution to the original problem.

7.2.2 Other domains

The essential features of the problem remain as in Cartesian coordinates even when the domain is not a simple Cartesian rectangle. However, it may be simpler to solve. For example, if the problem is Laplace's equation in two dimensions on a circular disk, or on an annulus, then it is simpler to express Laplace's equation in polar coordinates as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0, \tag{7.20}$$

where $\phi \equiv \phi(r, \theta)$. The domains for the disk D and annulus A are then simply expressed as

$$D = \{(r, \theta) \,|\, 0 \leq r < R,\ 0 \leq \theta < 2\pi\}, \tag{7.21a}$$

$$A = \{(r, \theta) \,|\, R_1 \leq r < R_2,\ 0 \leq \theta < 2\pi\}. \tag{7.21b}$$

The constant R defines the radius of the disk and the constants $R_{1,2}$ the inner and outer radii of the annulus.

There are two key points to note. The first is that the boundaries are smooth, so there are no corners to worry about as in the Cartesian case. This means that the step of finding the difference function as in section 7.2.1 is considerably simpler – only one boundary needs considering in the case of the annulus, and the step is completely unnecessary for the disk.

The second point is that there is no explicit boundary condition in the angular direction, as there is no explicit boundary. However, in order to have a single-valued solution we need ϕ to be 2π-periodic. In addition, in the case of the disk we require the solution to be finite at $r = 0$. These combine with the standard separation of variables techniques to give solutions of the form

$$\phi(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) \tag{7.22}$$

for the disk problem, and imposing boundary conditions at $r = R$ and using orthogonality will give the free constants a_n, b_n.

Chapter 8

Vector Calculus

8.1 Vector and scalar valued functions and their derivatives

8.1.1 Curves and vector valued functions of 1-variable

A vector valued function is a vector $\boldsymbol{r}$ which depends on a real variable t say. We write such a vector valued function as $\boldsymbol{r}(t)$. Sometimes t will take on all real values and sometimes t will be in some interval, I. If we take $a < t < b$ then I is an open interval and we write $I = (a, b)$ however if we include the end points so that $a \leq t \leq b$ then I is a closed interval and we write $I = [a, b]$.

More formally we think of a vector valued function as a map

$$I \to \mathbb{R}^3$$
$$t \mapsto \boldsymbol{r}(t)$$

For most of theses notes you may assume, unless specified otherwise, that the map is smooth. This means that one can differentiate the map as often as one likes. However in some of the theorems we need to be careful about the degree of differentiability of functions. A continuous function is said to be C^0, a differentiable function with continuous first derivative is said to be C^1, whilst a twice differentiable function with continuous second derivative is sad to be C^2, and so on.

As t varies between a and b the vector $\boldsymbol{r}(t)$ traces out a curve.

We call this set of points in $\mathbb{R}^3$ a **curve**. However we refer to the map

$$I \to \mathbb{R}^3$$
$$t \mapsto \boldsymbol{r}(t)$$

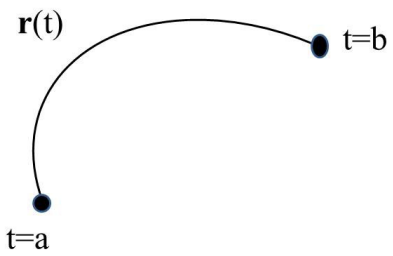


Figure 8.1: As t varies between a and b the vector $\boldsymbol{r}(t)$ traces out a curve.

as a **parameterised curve**. This is because it is possible to obtain the same curve in $\mathbb{R}^3$ with a different parameterisation. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonic increasing (smooth) function. Let $f(c) = a$ and $f(d) = b$ and define $\tilde{I} = [c, d]$. Then we may define a new vector valued function $\tilde{\boldsymbol{r}}(s) = \boldsymbol{r}(f(s))$ for $c \leq s \leq d$. So that

$$\begin{aligned} \tilde{I} &\to \mathbb{R}^3 \\ s &\mapsto \tilde{\boldsymbol{r}}(s) \end{aligned}$$

is also a parameterised curve. However as s varies between c and d, $\tilde{\boldsymbol{r}}$ traces out the **same** set of points as $\boldsymbol{r}$. Thus $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ are simply different parameterisations of the same curve.

The tangent to a parametrised curve $\boldsymbol{r}(t)$ is given by

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = \lim_{h \to 0} \left\{ \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h} \right\}$$

If we give the components of $\boldsymbol{r}(t)$ as

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

The $\dot{\boldsymbol{r}}(t)$ has components

$$\dot{\boldsymbol{r}}(t) = \dot{x}(t)\boldsymbol{i} + \dot{y}(y)\boldsymbol{j} + \dot{z}(t)\boldsymbol{k}$$

If we think of the parameter t representing time, then $\dot{\boldsymbol{r}}(t)$ is the velocity and $v(t) = |\dot{\boldsymbol{r}}(t)|$ is the speed. This gives us the following formula for the length of the curve between the point A with position vector $\boldsymbol{r}(t_0)$ and the point B with position vector $\boldsymbol{r}(t_1)$:

$$L = \int_{t_0}^{t_1} v(t)dt = \int_{t_0}^{t_1} |\dot{\boldsymbol{r}}(t)|dt \int_{t_0}^{t_1} \left\{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2\right\}^{1/2} dt$$

What happens if we use some other parameterisation of the same curve? Let $s = f(t)$ and $\tilde{\boldsymbol{r}}(s) = \boldsymbol{r}(f(s))$ as above. Then by the chain rule

$$\frac{d\tilde{\boldsymbol{r}}(s)}{ds} = \frac{d\boldsymbol{r}(t)}{dt}\frac{dt}{ds}$$

Let $f(t_0) = s_0$ and $f(t_1) = s_1$. Then the length of the curve from A to B using the s parameterisation is given by

$$\begin{aligned}
\tilde{L} &= \int_{s_0}^{s_1} \left|\frac{d\tilde{\boldsymbol{r}}(s)}{ds}\right| ds \\
&= \int_{s_0}^{s_1} \left|\frac{d\boldsymbol{r}(t)}{dt}\frac{dt}{ds}\right| ds \\
&= \int_{s_0}^{s_1} \left|\frac{d\boldsymbol{r}(t)}{dt}\right| \frac{dt}{ds} ds \\
&= \int_{t_0}^{t_1} \left|\frac{d\boldsymbol{r}(t)}{dt}\right| dt \\
&= L
\end{aligned}$$

Hence the *length of the curve is independent of the parameterisation*. This feature of the integral is called re-parameterisation invariance.

8.1.2 Scalar Fields

A scalar field is simply a scalar valued function that depend upon position. For example the temperature in a room. In 2-dimensional space a scalar field is a map

$$\begin{aligned}
&\phi : \mathbb{R}^2 \to \mathbb{R} \\
&(x, y) \mapsto \phi(x, y)
\end{aligned}$$

An example of a scalar field in 2-dimensions is therefore given by

$$\phi(x, y) = \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}$$

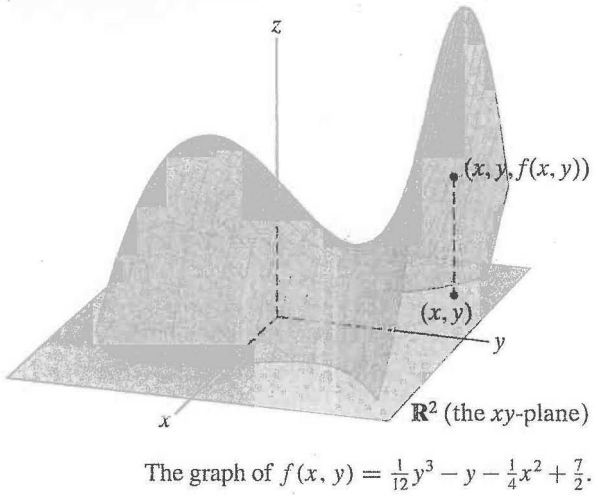


Figure 8.2: Three dimensional plot of the function $\phi(x, y) = \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}$.

To visualise this function we can plot the graph of the surface given by

$$z = \phi(x, y)$$

using Maple, for example, see Fig. 8.2.

An alternative is to plot the *contour lines* or *level surfaces* given by $z = c$ =constant. Thus we plot the curves

$$\frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2} = c$$

for a number of equally spaced contours. For the above function this gives the picture in Fig. 8.3.

Example 2: For the scalar field given by $f(x, y) = 4 - x^2 - y^2$ plot the graph $z = f(x, y)$ and the contour lines $f(x, y) = c$.

Often it is easier to draw the contour lines so we start with these. For this example they are given by $x^2 + y^2 = 4 - c$. These are a series of circles centre the origin radius $\sqrt{4 - c}$. Note that for real solutions we must have $c \leq 4$, we also note that the curves get closer together as the radius of the circles increases. This indicates that the surface gets steeper as we move away from the origin.

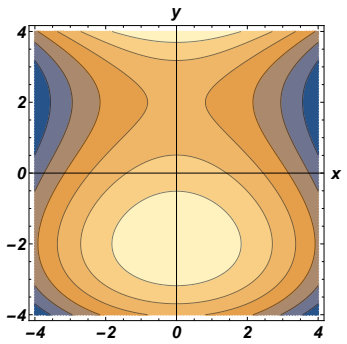


Figure 8.3: Contour plot of the function $\phi(x, y) = \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}$.

We now consider scalar functions in 3-dimensions. These are maps

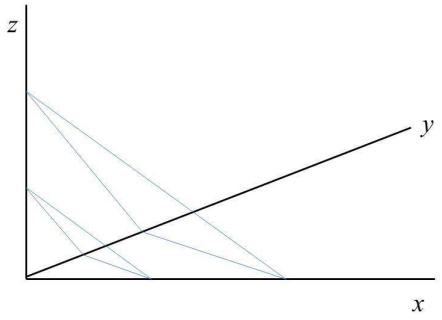
$$\phi : \mathbb{R}^3 \to \mathbb{R}$$
$$(x, y, z) \mapsto \phi(x, y, z)$$

The graphs of this function $w = \phi(x, y, z)$ are 3-dimensional hypersurfaces in 4-dimensional space - which are not so easy to draw! Instead we draw the *level surfaces* given by

$$\phi(x, y, z) = c$$

As with contour lines we can visualise the function by drawing a series of **equally spaced** level surfaces. These are 2-dimensional surfaces in $\mathbb{R}^3$ so drawing them is feasible.

Example: Draw the level surfaces of the function $\phi(x, y, z) = x + y + z$. These are simply given by $x + y + z = c$, which we know from chapter 1 consist of planes with normal $\boldsymbol{n} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.



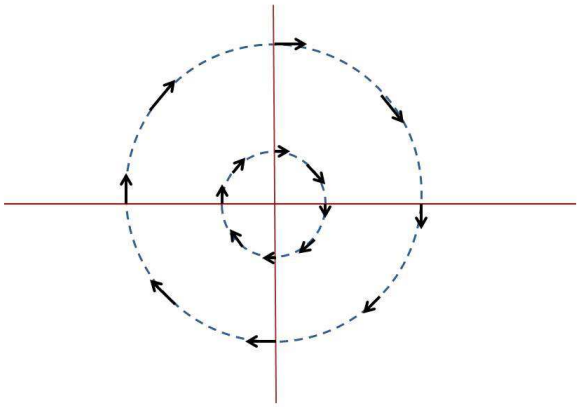
8.1.3 Vector fields

A vector field is a vector valued function that depends upon position. In 2-dimensions it is a map

$$\begin{aligned}&\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2\\ &(x, y) \mapsto \mathbf{F}(x, y) = F_1(x, y)\boldsymbol{i} + F_2(x, y)\boldsymbol{j}\end{aligned}$$

For example $\mathbf{F}(x, y) = y\boldsymbol{i} - x\boldsymbol{j}$. How do we visualise a vector field?

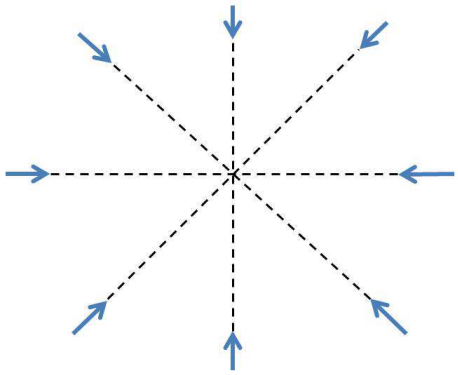
At the point (x, y) we draw the vector $\mathbf{F}(x, y)$. For the above example this gives the picture:



Example 2: Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ denote a typical point in $\mathbb{R}^2$. Let the vector field $\mathbf{F}(\boldsymbol{r})$ be defined by

$$\mathbf{F}(\boldsymbol{r}) = -k\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} = -\frac{k}{r^2}\hat{\boldsymbol{r}}$$

which represents an inverse square force law (such as gravitation or electromagnetism). The picture we get of this vector field is as follows:



Note: The vector field is not defined at the origin.

8.1.4 Directional derivatives

Let $\phi(\boldsymbol{r})$ be a scalar field. What is the derivative of ϕ? It depends on what direction you move in!

Let $\boldsymbol{v}$ be some fixed vector, then the directional derivative of ϕ in the $\boldsymbol{v}$ direction, at the point $\boldsymbol{r}_0$ is given by

$$\nabla_{\boldsymbol{v}}\phi(\boldsymbol{r}_0) = \lim_{t\to 0}\left\{\frac{\phi(\boldsymbol{r}_0 + t\boldsymbol{v}) - \phi(\boldsymbol{r}_0)}{t}\right\}$$

This measures the change in ϕ in the $\boldsymbol{v}$ direction.

This formal definition is not very useful when it comes to doing calculations so we will convert it into something that we can use.

To do this we think of $\boldsymbol{r}_0$ and $\boldsymbol{v}$ as fixed and define the real valued function $f(t) = \phi(\boldsymbol{r}_0 + t\boldsymbol{v})$. Then

$$\begin{aligned}\nabla_{\boldsymbol{v}}\phi(\boldsymbol{r}_0) &= \lim_{t\to 0}\left\{\frac{\phi(\boldsymbol{r}_0 + t\boldsymbol{v}) - \phi(\boldsymbol{r}_0)}{t}\right\}\\ &= \lim_{t\to 0}\left\{\frac{f(t) - f(0)}{t}\right\}\\ &= f'(0)\end{aligned}$$

We can write $f(t)$ as:

$$f(t) = \phi(x, y, z) \tag{8.1}$$

where

$$\begin{aligned} x &= r_1 + tv_1 \\ y &= r_2 + tv_2 \\ x &= r_3 + tv_3 \end{aligned}$$

and $\boldsymbol{r}_0 = r_1\boldsymbol{i} + r_2\boldsymbol{j} + r_3\boldsymbol{k}$ and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$.

Differentiating (8.1) with respect to t using the chain rule gives

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} \\ &= v_1\frac{\partial\phi}{\partial x} + v_2\frac{\partial\phi}{\partial x} + v_3\frac{\partial\phi}{\partial z} \end{aligned}$$

If we introduce the vector field $\mathrm{grad}\phi$ (denoted $\boldsymbol{\nabla}\phi$) defined by

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

Then we see that

$$\nabla_{\boldsymbol{v}}\phi = \boldsymbol{v}\cdot\boldsymbol{\nabla}\phi$$

Example: Let $\phi(x, y, z) = xy + z^2$. What is the rate of change of ϕ in the direction of $\hat{\boldsymbol{v}}$, where $\boldsymbol{v} = 3\boldsymbol{i} + 4\boldsymbol{k}$, at the point $x = 1, y = 2, z = 1$?

We first calculate $\boldsymbol{\nabla}\phi$;

$$\begin{aligned} \boldsymbol{\nabla}\phi &= \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k} \\ &= y\boldsymbol{i} + x\boldsymbol{j} + 2z\boldsymbol{k} \\ &= 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k} \quad \text{at the point } (1, 2, 1) \end{aligned}$$

Now $|\boldsymbol{v}| = 5$ so that

$$\hat{\boldsymbol{v}} = \frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{k}$$

and hence

$$\nabla_{\boldsymbol{v}}\phi(1, 2, 1) = \left(\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{k}\right)\cdot(2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) = \frac{14}{5}$$

8.1.5 Gradient and steepest ascent

When calculating $\boldsymbol{\nabla}\phi$ using the formula above we need to know ϕ as a function of the Cartesian coordinates x, y and z. However we sometimes use other coordinate systems. In this section we give a geometrical definition of $\mathrm{grad}\phi$ which enables us to calculate it in any coordinate system.

Given a scalar field $\phi(\boldsymbol{r})$ in what direction is it increasing most? Let $\hat{\boldsymbol{v}}$ be a unit vector then

$$\begin{aligned}\nabla_{\boldsymbol{v}}\phi &= \hat{\boldsymbol{v}}\cdot\boldsymbol{\nabla}\phi \\ &= |\hat{\boldsymbol{v}}||\boldsymbol{\nabla}\phi|\cos\theta \\ &= |\boldsymbol{\nabla}\phi|\cos\theta\end{aligned}$$

This is a maximum when $\theta = 0$ and a minimum when $\theta = \pi$, so that

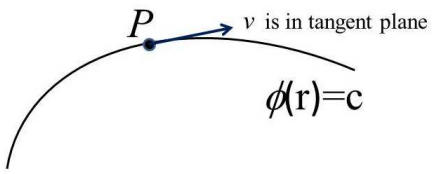
$$-|\boldsymbol{\nabla}\phi| \leq \nabla_{\boldsymbol{v}}\phi \leq |\boldsymbol{\nabla}\phi|$$

We have therefore established the following theorem:

Theorem 1: The directional derivative $\nabla_{\boldsymbol{v}}\phi$ is maximised when $\hat{\boldsymbol{v}}$ points in the same direction $\boldsymbol{\nabla}\phi$ and minimised when it points in the opposite direction. Furthermore the max and min values are given by $\pm|\boldsymbol{\nabla}\phi|$.

8.1.6 Gradients and level surfaces

We want to look at $\boldsymbol{\nabla}\phi$ at the point P with coordinates (x_0, y_0, z_0). Let $\phi(x_0, y_0, z_0) = c_0$. Now consider the level surface through the point P which is given by the set of points (x, y, z) such that $\phi(x, y, z) = c_0$.



Now let $\boldsymbol{v}$ be a tangent vector at P (ie $\boldsymbol{v}$ is an element of the tangent plane at P). Then since $\boldsymbol{v}$ is tangent to the level surface $\phi(x, y, z) = c_0$, the derivative of ϕ in

the $\boldsymbol{v}$ direction is zero. Thus

$$\begin{aligned}
&\nabla_{\boldsymbol{v}}\phi = 0 \\
\Rightarrow &\boldsymbol{v}\cdot\boldsymbol{\nabla}\phi = 0 \\
\Rightarrow &\text{the tangent vector } \boldsymbol{v} \text{ is perp. to } \boldsymbol{\nabla}\phi \\
\Rightarrow &\boldsymbol{\nabla}\phi \text{is perpendicular to the tangent space at } P \\
\Rightarrow &\boldsymbol{\nabla}\phi \text{is normal to the level surface } \phi = c_0
\end{aligned}$$

We have therefore established the following:

Theorem 2: The gradient of ϕ, $\boldsymbol{\nabla}\phi$, is normal to the surfaces ϕ =const.

We now use the above two theorems to obtain a new formula for the gradient.

By Theorem 2, since $\boldsymbol{\nabla}\phi$ is normal to the surface ϕ=const. we know that

$$\boldsymbol{\nabla}\phi = k\hat{\boldsymbol{n}} \tag{8.2}$$

where $k = |\boldsymbol{\nabla}\phi|$.

On the other hand we know from Theorem 1 that the maximum value of $\boldsymbol{\nabla}_{\boldsymbol{v}}\phi$ is $|\boldsymbol{\nabla}\phi|$ and that it obtains this value when $\boldsymbol{v}$ points in the same direction as $\boldsymbol{\nabla}\phi$, i.e. when $\hat{\boldsymbol{v}} = \hat{\boldsymbol{n}}$. Thus

$$|\boldsymbol{\nabla}\phi| = \nabla_{\hat{\boldsymbol{n}}}\phi \tag{8.3}$$

Where we have taken $\hat{\boldsymbol{n}}$ to be the choice of normal that points in the direction of **increasing** ϕ.

Combining formula (8.2) with formula (8.3) we obtain

$$\boldsymbol{\nabla}\phi = (\nabla_{\hat{\boldsymbol{n}}}\phi)\,\hat{\boldsymbol{n}}$$

To simplify notation we write $\dfrac{\partial\phi}{\partial n}$ to denote the directional derivative in the normal direction $\nabla_{\hat{\boldsymbol{n}}}\phi$. Hence we can write

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial n}\hat{\boldsymbol{n}}$$

To summarise:

Gradient (geometric definition)

The gradient of the scalar field ϕ is the vector field $\boldsymbol{\nabla}\phi$ given by

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial n}\hat{\boldsymbol{n}}$$

where $\hat{\boldsymbol{n}}$ is the unit normal to the surfaces $\phi = const$ and $\dfrac{\partial\phi}{\partial n}$ is the directional derivative of ϕ in the $\hat{\boldsymbol{n}}$ direction.

Gradient (Cartesian definition)

If the scalar field ϕ is given in Cartesian coordinates by $\phi(x, y, z)$ then

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

Directional Derivative

The directional derivative

$$\begin{aligned}\nabla_{\boldsymbol{v}}\phi(\boldsymbol{r}_0) &= \lim_{t\to 0}\left\{\frac{\phi(\boldsymbol{r}_0 + t\boldsymbol{v}) - \phi(\boldsymbol{r}_0)}{t}\right\} \\ &= \boldsymbol{v}\cdot\boldsymbol{\nabla}\phi \\ &= v_1\frac{\partial\phi}{\partial x} + v_2\frac{\partial\phi}{\partial y} + v_3\frac{\partial\phi}{\partial z} \quad \text{(in Cartesian coordinates)}\end{aligned}$$

Example 1: Let a general point be denoted $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, and let $r = |\boldsymbol{r}|$. Consider the scalar field $\phi(\boldsymbol{r}) = |\boldsymbol{r}| = r$. What is $\boldsymbol{\nabla}\phi$?

Cartesian method:

$\phi(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$. So that

$$\frac{\partial\phi}{\partial x} = 2x\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} = \frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r}.$$

Similarly $\dfrac{\partial\phi}{\partial y} = \dfrac{y}{r}$, and $\dfrac{\partial\phi}{\partial z} = \dfrac{z}{r}$. Hence

$$\begin{aligned}\boldsymbol{\nabla}\phi &= \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k} \\ &= \frac{x}{r}\boldsymbol{i} + \frac{y}{r}\boldsymbol{j} + \frac{z}{r}\boldsymbol{k} \\ &= \frac{1}{r}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = \frac{\boldsymbol{r}}{r} = \hat{\boldsymbol{r}}\end{aligned}$$

Geometric method:

The level surfaces of ϕ =const. are the surfaces r =const. which are simply spheres centre the origin. Hence the normal to the surface points in the same direction as the position vector. Thus $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$. Hence

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial n}\hat{\boldsymbol{n}} = \frac{\partial\phi}{\partial r}\hat{\boldsymbol{r}} = 1.\hat{\boldsymbol{r}}$$

More generally if the scalar field ϕ only depends upon the distance r from the origin and not the direction then we can write $\phi(\boldsymbol{r}) = f(r)$. The level surfaces are still spheres centre the origin and hence we still have $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$. Thus

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial n}\hat{\boldsymbol{n}} = \frac{\partial\phi}{\partial r}\hat{\boldsymbol{r}} = \frac{df}{dr}\hat{\boldsymbol{r}} = f'(r)\hat{\boldsymbol{r}}$$

Example 2: Calculate the equation of the tangent plane to the surface S given by $x^3y - yz^2 + z^5 = 9$ at the point P given by $(3, -1, 2)$.

Let $\phi = x^3y - yz^2 + z^5$ then at P, the value of $\phi = 9$ so that S is the level surface given by $\phi(x, y, z) = 9$. Then

$$\boldsymbol{\nabla}\phi = 3x^2y\boldsymbol{i} + (x^3 - z^2)\boldsymbol{j} + (5z^4 - 2yz)\boldsymbol{k}$$

So that at P, a normal to the surface is given by $\boldsymbol{n} = \boldsymbol{\nabla}\phi = -27\boldsymbol{i} + 23\boldsymbol{j} + 84\boldsymbol{k}$. The equation of the tangent plane at $\boldsymbol{r}_0$ is $\boldsymbol{n}\cdot(\boldsymbol{r} - \boldsymbol{r}_0) = 0$ which is

$$(-27\boldsymbol{i} + 23\boldsymbol{j} + 84\boldsymbol{k})\cdot((x-3)\boldsymbol{i} + (y+1)\boldsymbol{j} + (z-2)\boldsymbol{k}) = 0$$

or $-27x + 23y + 84z = 64$.

8.1.7 Properties of gradϕ

(1.) Linearity: $\boldsymbol{\nabla}(\phi + \psi) = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\psi$

Proof: Use Cartesian coordinates and the fact that $\dfrac{\partial}{\partial x}(\phi + \psi) = \dfrac{\partial\phi}{\partial x} + \dfrac{\partial\psi}{\partial x}$

(2.) Leibnitz rule: $\boldsymbol{\nabla}(\phi\psi) = \psi\boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla}\psi$

Proof: Use Cartesian coordinates and the fact that $\dfrac{\partial}{\partial x}(\phi\psi) = \psi\dfrac{\partial\phi}{\partial x} + \phi\dfrac{\partial\psi}{\partial x}$

8.1.8 The divergence of a vector field

In the previous section we saw how to calculate a vector field $\mathrm{grad}\phi$ by taking the derivative of a scalar field. In this section we show how to calculate a scalar field by taking the derivative of a vector field $\mathbf{F}$. We call this derived scalar field the **divergence** of $\mathbf{F}$ and denote it $\mathrm{div}\mathbf{F}$ or $\boldsymbol{\nabla}\cdot\mathbf{F}$.

For the **special** case where $\mathbf{F}$ is given by

$$\mathbf{F} = \phi\boldsymbol{c}$$

where ϕ is a scalar field and $\boldsymbol{c}$ is a **constant** vector, we define $\mathrm{div}\mathbf{F}$ by the formula

$$\mathrm{div}\mathbf{F} = \boldsymbol{\nabla}\phi\cdot\boldsymbol{c}$$

We also require that the divergence is a linear operator, so that if $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are two vector fields then

$$\mathrm{div}(\mathbf{F} + \tilde{\mathbf{F}}) = \mathrm{div}\mathbf{F} + \mathrm{div}\tilde{\mathbf{F}}$$

Using the fact that we may write a **general** vector field in a Cartesian basis as

$$\mathbf{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$$

Then

$$\begin{aligned}\mathrm{div}\mathbf{F} &= \mathrm{div}(F_1\boldsymbol{i}) + \mathrm{div}(F_2\boldsymbol{j}) + \mathrm{div}(F_3\boldsymbol{k}) \\ &= \boldsymbol{\nabla}(F_1)\cdot\boldsymbol{i} + \boldsymbol{\nabla}(F_2)\cdot\boldsymbol{j} + \boldsymbol{\nabla}(F_3)\cdot\boldsymbol{k} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\end{aligned}$$

Hence we have obtained the following result

Divergence in Cartesian coordinates Let $\mathbf{F}$ be a vector field with components

$$\mathbf{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$$

Then the **divergence** of $\mathbf{F}$ is given by

$$\mathrm{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note if we introduce the differential operator

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

Then we may write the above as

$$\mathrm{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Example 1: Let $\mathbf{F} = y^2z\boldsymbol{i} + xz\boldsymbol{j} - y^2\boldsymbol{k}$ Then

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{\partial y^2z}{\partial x} + \frac{\partial xz}{\partial y} + \frac{\partial - y^2}{\partial z} \\ &= 0 + 0 + 0 = 0\end{aligned}$$

So that for $\mathbf{F} = y^2z\boldsymbol{i} + xz\boldsymbol{j} - y^2\boldsymbol{k}$ then $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$. This example shows that $\mathrm{div}\mathbf{F} = 0$ does not imply that $\mathbf{F}$ is constant.

In fact the divergence measure the effect of the vector field on the volume of a small sphere. If the flow lines are moving apart they diverge and the volume of a small sphere moving with the flow increases, and the divergence is positive. However if the flow lines are moving together they converge and the volume of a small sphere moving with the flow decreases and the divergence is negative. However it is possible for the flow lines to converge in some directions and diverge in others so that there is no overall change in volume of a small sphere. In this case the divergence vanishes (even if the sphere is distorted by the flow).

Proposition:

Let ϕ be a scalar field and $\mathbf{F}$ a vector field then

$$\boldsymbol{\nabla}\cdot(\phi\mathbf{F}) = (\boldsymbol{\nabla}\phi)\cdot\mathbf{F} + \phi\boldsymbol{\nabla}\cdot\mathbf{F}$$

Proof: Use cartesian coordinates to calculate both sides.

8.1.9 The Laplacian

If we start with a scalar field ϕ we may first take the gradient to obtain $\boldsymbol{\nabla}\phi$. This is a vector field so we may take its divergence to obtain a scalar field, $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi)$. We call this scalar field the "Laplacian" of ϕ.

Definition:

The Laplacian of a scalar field ϕ is denoted $\nabla^2\phi$ and defined by

$$\nabla^2\phi = \mathrm{div}(\mathrm{grad}\phi) = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi)$$

If we do the above calculation in Cartesian coordinates we have

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

Taking the divergence of this we get

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi) &= \frac{\partial\phi}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial\phi}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial\phi}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)\\ &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\end{aligned}$$

Hence in Cartesian coordinates

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Example:

Let $\phi(x, y, z) = e^x y^3 \sin z$, calculate $\nabla^2\phi$.

$$\begin{aligned}\boldsymbol{\nabla}\phi &= \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k} \\ &= e^x y^3 \sin z\boldsymbol{i} + 3e^x y^2 \sin z\boldsymbol{j} + e^x y^3 \cos z\boldsymbol{k}\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{\partial(e^x y^3 \sin z)}{\partial x} + \frac{\partial(3e^x y^2 \sin z)}{\partial y} + \frac{\partial(e^x y^3 \cos z)}{\partial z} \\ &= e^x y^3 \sin z + 6e^x y \sin z - e^x y^3 \sin z\end{aligned}$$

Alternatively

$$\begin{aligned}\nabla^2\phi &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\ &= \frac{\partial^2(e^x y^3 \sin z)}{\partial x^2} + \frac{\partial^2(e^x y^3 \sin z)}{\partial y^2} + \frac{\partial^2(e^x y^3 \sin z)}{\partial z^2} \\ &= e^x y^3 \sin z + 6e^x y \sin z - e^x y^3 \sin z\end{aligned}$$

8.1.10 The curl of a vector field

We have already considered a differential operator called the divergence that may be symbolically written as $\boldsymbol{\nabla}\cdot$ in this section we consider a differential operator called the "curl" or "rotation" which may be symbolically written as $\boldsymbol{\nabla}\times$. As its name suggests the "curl" measures the extent to which a small sphere rotates as it moves along the flow lines of the vector field $\mathbf{F}$.

For the **special** case where $\mathbf{F}$ is given by

$$\mathbf{F} = \phi\boldsymbol{c}$$

where ϕ is a scalar field and $\boldsymbol{c}$ is a **constant** vector, we define $\mathrm{curl}\mathbf{F}$ by the formula

$$\mathrm{curl}\mathbf{F} = \boldsymbol{\nabla}\phi\times\boldsymbol{c}$$

We also require that the curl is a linear operator, so that if $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are two vector fields then

$$\mathrm{curl}(\mathbf{F}+\tilde{\mathbf{F}}) = \mathrm{curl}\mathbf{F} + \mathrm{curl}\tilde{\mathbf{F}}$$

Using the fact that we may write a **general** vector field in a Cartesian basis as

$$\mathbf{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$$

Then

$$\begin{aligned}
\mathrm{curl}\mathbf{F} &= \mathrm{curl}(F_1\boldsymbol{i}) + \mathrm{curl}(F_2\boldsymbol{j}) + \mathrm{curl}(F_3\boldsymbol{k}) \\
&= \boldsymbol{\nabla}(F_1)\times\boldsymbol{i} + \boldsymbol{\nabla}(F_2)\times\boldsymbol{j} + \boldsymbol{\nabla}(F_3)\times\boldsymbol{k} \\
&= \left(-\frac{\partial F_1}{\partial y}\boldsymbol{k} + \frac{\partial F_1}{\partial z}\boldsymbol{j}\right) + \left(\frac{\partial F_2}{\partial x}\boldsymbol{k} - \frac{\partial F_2}{\partial z}\boldsymbol{i}\right) + \left(\frac{\partial F_3}{\partial x}\boldsymbol{j} - \frac{\partial F_3}{\partial y}\boldsymbol{i}\right) \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}
\end{aligned}$$

Hence we have obtained the following result

Curl in Cartesian coordinates Let $\mathbf{F}$ be a vector field with components

$$\mathbf{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$$

Then

$$\mathrm{curl}\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}$$

This may be formally written using a determinant as

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Example 1: Let $\mathbf{F} = xy^2\boldsymbol{i} + e^z\boldsymbol{j} + y\sin xe^z\boldsymbol{k}$. Calculate $\mathrm{curl}\mathbf{F}$. Using the above formula

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & e^z & y\sin xe^z \end{vmatrix} = (\sin xe^z - e^z)\boldsymbol{i} + (y\cos xe^z)\boldsymbol{j} + 2xy\boldsymbol{k}$$

Example 2: Let $\phi = x^2yz^3$ and let $\mathbf{F} = \boldsymbol{\nabla}\phi$. Calculate $\boldsymbol{\nabla}\times\mathbf{F}$.

$$\mathbf{F} = \boldsymbol{\nabla}\phi = 2xyz^3\boldsymbol{i} + x^2z^3\boldsymbol{j} + 3x^2yz^2\boldsymbol{k}$$

Then using the above formula

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix} \\ &= (3x^2z^2 - 3x^2z^2)\boldsymbol{i} + (-6xyz^2 + 6xyz^2)\boldsymbol{j} + (2xz^3 - 2xz^3)\boldsymbol{k} \quad = \mathbf{0}\end{aligned}$$

So in this example we find that $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = \mathbf{0}$. In fact this is not just chance but is a general result.

Theorem: Let $\mathbf{F}$ be a C^2 (ie twice differentiable with continuous second derivative) vector field then:

$$\text{curl}(\text{grad}\phi) = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = \mathbf{0}$$

Proof:

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

So that

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)\boldsymbol{i} + \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right)\boldsymbol{j} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)\boldsymbol{k} \\ &= \mathbf{0}\end{aligned}$$

where the C^2 condition is required to show that, for example, $\dfrac{\partial^2\phi}{\partial y\partial z} = \dfrac{\partial^2\phi}{\partial z\partial y}$

Example: Let $\mathbf{F} = x^2e^y\boldsymbol{i} + x\ln z\boldsymbol{j} + (x+z)\boldsymbol{k}$. Calculate $\boldsymbol{G} = \boldsymbol{\nabla}\times\mathbf{F}$ and also $\boldsymbol{\nabla}\cdot\boldsymbol{G}$.

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2e^y & x\ln z & (x+z) \end{vmatrix} = (0 - x/z)\boldsymbol{i} + (0-1)\boldsymbol{j} + (\ln z - x^2e^y)\boldsymbol{k}$$

Hence $\boldsymbol{G} = -\frac{x}{z}\boldsymbol{i} - \boldsymbol{j} + (\ln z - x^2 e^y)\boldsymbol{k}$. Calculating the divergence of this we obtain

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{G} &= \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \\ &= -\frac{1}{z} + 0 + \frac{1}{z} = 0\end{aligned}$$

Hence in this example $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0$. Again this is not just chance but is a general result:

Theorem: Let $\mathbf{F}$ be a C^2 vector field, then

$$\mathrm{div}(\mathrm{curl}\mathbf{F}) = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0$$

Proof:

Let $\mathbf{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$, then

$$\begin{aligned}\boldsymbol{G} = \boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}\end{aligned}$$

Now $\mathrm{div}\boldsymbol{G}$ is given by

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{G} &= \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \\ &= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y} \\ &= 0\end{aligned}$$

Where we have again used the C^2 condition to ensure that the mixed second derivatives commute.

8.1.11 Vector Identities

In this section we collect together some useful vector identities:

(1.) $\boldsymbol{\nabla}(\phi\psi) = \psi\boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla}\psi$

(2.) $\boldsymbol{\nabla}\cdot(\phi\mathbf{F}) = \boldsymbol{\nabla}\phi\cdot\mathbf{F} + \phi\boldsymbol{\nabla}\cdot\mathbf{F}$

(3.) $\boldsymbol{\nabla}\times(\phi\mathbf{F}) = \boldsymbol{\nabla}\phi\times\mathbf{F} + \phi\boldsymbol{\nabla}\times\mathbf{F}$

(4.) $\boldsymbol{\nabla}\cdot(\mathbf{F}\times\boldsymbol{G}) = \boldsymbol{G}\cdot\boldsymbol{\nabla}\times\mathbf{F} - \mathbf{F}\cdot\boldsymbol{\nabla}\times\boldsymbol{G}$

(5.) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = \mathbf{0}$

(6.) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0$

(7.) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi) = \nabla^2\phi$

(8.) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) - \nabla^2\mathbf{F}$

where $\nabla^2\mathbf{F} = \nabla^2 F_1\boldsymbol{i} + \nabla^2 F_2\boldsymbol{j} + \nabla^2 F_3\boldsymbol{k}$.

8.2 Integrals of vector and scalar fields

8.2.1 Line integrals

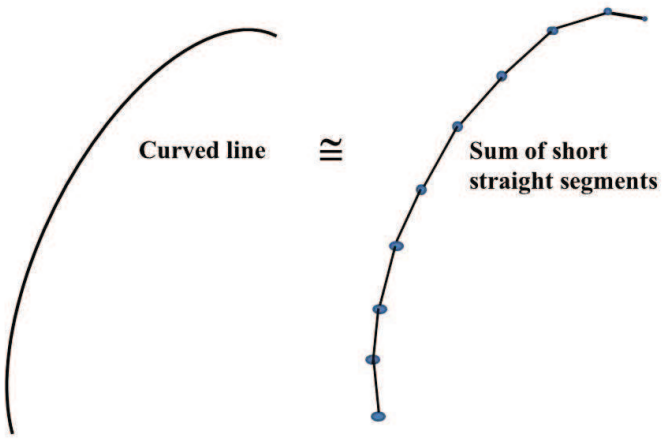
Let $\mathbf{F}(\boldsymbol{r})$ be a (smooth) vector field and let $\gamma(t)$ for $t_0 \leq t \leq t_1$ be a curve parameterised by t. Let $\boldsymbol{r}(t)$ denote the position vector of the point on the curve given by $\gamma(t)$.

In this section we give a definition of

$$\int_\gamma \mathbf{F}\cdot d\boldsymbol{r}$$

the line integral of $\mathbf{F}$ along the curve γ. We will see below that this integral depends upon the curve γ, but not upon the parameterisation of the curve - only upon the set of points on the curve.

In the usual way when considering integrals we can break the curve γ up into a series of m small straight line segments.



Let $\delta\boldsymbol{r}_k$ denote the line segment joining $\boldsymbol{r}_{k-1}$ to $\boldsymbol{r}_k$, and let $\mathbf{F}_k = \mathbf{F}(\boldsymbol{r}_k)$. Then we may define the line integral by

$$\int_\gamma \mathbf{F}\cdot d\boldsymbol{r} = \lim_{m\to\infty} \sum_{j=1}^{m} \mathbf{F}_j \cdot \delta\boldsymbol{r}_j$$

However this definition is not very useful when it comes to calculations. In oder to calculate a line integral we convert it to an ordinary 1-dimensional integral.

We start be looking at the parameterised curve

$$\gamma : t \mapsto \boldsymbol{r}(t), \quad a \leq t \leq b$$

Then

$$d\boldsymbol{r} = \frac{d\boldsymbol{r}}{dt}dt$$

We also look at the value of the vector field along the curve

$$\mathbf{F} = \mathbf{F}(\boldsymbol{r}(t))$$

Combining these the integrand becomes

$$\mathbf{F}\cdot d\boldsymbol{r} = \mathbf{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}dt$$

Finally the integral may be written:

$$\int_\gamma \mathbf{F}\cdot d\boldsymbol{r} = \int_{t=a)}^{b} \left(\mathbf{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}\right) dt$$

Notice that $\left(\mathbf{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}\right)$ is simply some function of t, so that the final integral has the form $\int_{t=a}^{b} f(t)dt$ of an ordinary integral.

In order to show that the integral only depends upon the curve γ and not upon the parameterisation, we introduce a new parameterisation as in section 2.3

Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonic increasing (smooth) map. Let $f(c) = a$ and $f(d) = b$. Then we may define a new vector valued function $\tilde{\boldsymbol{r}}(s) = \boldsymbol{r}(f(s))$ for $c \leq s \leq d$. As s varies between c and d then $\tilde{\boldsymbol{r}}(s)$ traces out the same set of points as $\boldsymbol{r}(t)$, so it is also a parameterisation of the same curve γ. In terms of this parameterisation the line integral is given by

$$\int_\gamma \mathbf{F}\cdot d\boldsymbol{r} = \int_{s=c)}^{d} \left(\mathbf{F}(\tilde{\boldsymbol{r}}(s))\cdot\frac{d\tilde{\boldsymbol{r}}}{ds}\right) ds$$

However by the chain rule

$$\frac{d\tilde{\boldsymbol{r}}}{ds} = \frac{d\boldsymbol{r}}{dt}\frac{dt}{ds}$$

So changing variable in the integral we get

$$\begin{aligned}\int_\gamma \mathbf{F}\cdot d\boldsymbol{r} &= \int_{s=c)}^{d} \left(\mathbf{F}(\tilde{\boldsymbol{r}}(s))\cdot\frac{d\tilde{\boldsymbol{r}}}{ds}\right) ds \\ &= \int_{s=c)}^{d} \left(\mathbf{F}(\tilde{\boldsymbol{r}}(s))\cdot\frac{d\boldsymbol{r}}{dt}\right) \frac{dt}{ds} ds \\ &= \int_{s=a)}^{b} \left(\mathbf{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}\right) dt\end{aligned}$$

which agrees with the previous formula. Hence the line integral is independent of the parameterisation of the curve as claimed.

Example 1a: Let $\mathbf{F} = 2xy\boldsymbol{i} + (x^2 - z^2)\boldsymbol{j} - 3xz^2\boldsymbol{k}$. Evaluate the line integral $\int_\gamma \mathbf{F}\cdot d\boldsymbol{r}$, where γ is the curve given by

$$\boldsymbol{r}(t) = 2t\boldsymbol{i} + t^3\boldsymbol{j} + 3t^2\boldsymbol{k}, \quad 0 \le t \le 1$$

going from the point $(0, 0, 0)$ to the point $(2, 1, 3)$.

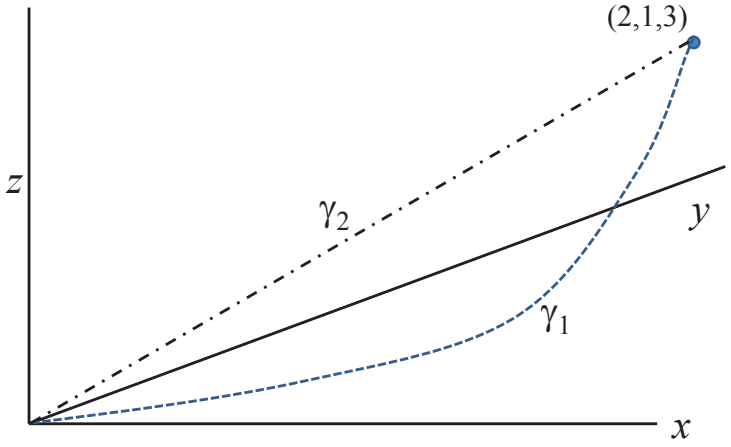


Figure 8.4: Geometry for Examples 1a (γ_1) and 1b (γ_2).

From the formula for $\boldsymbol{r}(t)$ above we see that

$$x = 2t, \quad y = t^3, \text{and } z = 3t^2$$

substituting into the formula for $\mathbf{F}(\boldsymbol{r})$ this gives

$$\mathbf{F}(\boldsymbol{r}(t)) = 4t^4\boldsymbol{i} + (4t^2 - 9t^4)\boldsymbol{j} - 54t^5\boldsymbol{k}$$

Also differentiating the expression for $\boldsymbol{r}(t)$ we get

$$\frac{d\boldsymbol{r}}{dt} = 2\boldsymbol{i} + 3t^2\boldsymbol{j} + 6t\boldsymbol{k}$$

Hence

$$\begin{aligned}
\int_\gamma \mathbf{F}\cdot d\boldsymbol{r} &= \int_{t=0}^{1} \left(\mathbf{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}\right) dt \\
&= \int_{t=0}^{1} (4t^4\boldsymbol{i} + (4t^2 - 9t^4)\boldsymbol{j} - 54t^5\boldsymbol{k})\cdot(2\boldsymbol{i} + 3t^2\boldsymbol{j} + 6t\boldsymbol{k})dt \\
&= \int_{t=0}^{1} (8t^4 + 12t^4 - 27t^6 - 324t^6)dt \\
&= \left[4t^5 - \frac{351}{7}\right]_0^1 \\
&= -\frac{323}{7}
\end{aligned}$$

Example 1b: Calculate the line integral for the bf same vector field $\mathbf{F} = 2xy\boldsymbol{i} + (x^2 - z^2)\boldsymbol{j} - 3xz^2\boldsymbol{k}$, but this time along the straight line connecting $(0, 0, 0)$ to $(2, 1, 3)$.

This time the curve is given by

$$\boldsymbol{r}(t) = 2t\boldsymbol{i} + t\boldsymbol{j} + 3t\boldsymbol{k}, \quad 0 \leq t \leq 1$$

Hence

$$x = 2t, \quad y = t, \text{and } z = 3t$$

substituting into the formula for $\mathbf{F}(\boldsymbol{r})$ this gives

$$\mathbf{F}(\boldsymbol{r}(t)) = 4t^2\boldsymbol{i} - 5t^2\boldsymbol{j} - 54t^3\boldsymbol{k}$$

Also differentiating the expression for $\boldsymbol{r}(t)$ we get

$$\frac{d\boldsymbol{r}}{dt} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$$

Hence

$$\begin{aligned}\int_\gamma \mathbf{F}\cdot d\boldsymbol{r} &= \int_{t=0}^{1}\left(\mathbf{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}\right)dt\\ &= \int_{t=0}^{1}(4t^2\boldsymbol{i}-5t^2\boldsymbol{j}-54t^3\boldsymbol{k})\cdot(2\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k})dt\\ &= \int_{t=0}^{1}(6t^8+36t^8+12t^8)dt\\ &= \left[\frac{54}{9}t^9\right]_0^1\\ &= 6\end{aligned}$$

This example shows the important fact that in general *the value of the line integral depends upon the path taken, not just the end points*.

Example 2: Let $\mathbf{F}=\boldsymbol{\nabla}\phi$ where $\phi = xy^2z$. Calculate the line integral of $\mathbf{F}$ for both of the curves used in example 1.

$$\mathbf{F}=\boldsymbol{\nabla}\phi = y^2z\boldsymbol{i}+2xyz\boldsymbol{j}+xy^2\boldsymbol{k}$$

For part (a)

$$\boldsymbol{r}(t)=2t\boldsymbol{i}+t^3\boldsymbol{j}+3t^2\boldsymbol{k},\quad 0\le t\le 1$$

So

$$\mathbf{F}(\boldsymbol{r}(t)=3t^8\boldsymbol{i}+12t^6\boldsymbol{j}+2t^7\boldsymbol{k}$$

and

$$\frac{d\boldsymbol{r}}{dt}=2\boldsymbol{i}+3t^2\boldsymbol{j}+6t\boldsymbol{k}$$

Hence

$$\begin{aligned}\int_\gamma \mathbf{F}\cdot d\boldsymbol{r} &= \int_{t=0}^{1}\left(\mathbf{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}\right)dt\\ &= \int_{t=0}^{1}(3t^8\boldsymbol{i}+12t^6\boldsymbol{j}+2t^7\boldsymbol{k})\cdot(2\boldsymbol{i}+3t^2\boldsymbol{j}+6t\boldsymbol{k})dt\\ &= \int_{t=0}^{1}(6t^8+36t^8+12t^8)dt\\ &= \left[\frac{54}{9}t^9\right]_0^1\\ &= 6\end{aligned}$$

For part (b)

$$\boldsymbol{r}(t) = 2t\boldsymbol{i} + t\boldsymbol{j} + 3t\boldsymbol{k}, \quad 0 \leq t \leq 1$$

Hence

$$\mathbf{F}(\boldsymbol{r}(t)) = 3t^3\boldsymbol{i} + 12t^3\boldsymbol{j} + 2t^3\boldsymbol{k}$$

and

$$\frac{d\boldsymbol{r}}{dt} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$$

Hence

$$\begin{aligned}
\int_\gamma \mathbf{F}\cdot d\boldsymbol{r} &= \int_{t=0}^{1} \left(\mathbf{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}\right) dt \\
&= \int_{t=0}^{1} (3t^3\boldsymbol{i} + 12t^3\boldsymbol{j} + 2t^3\boldsymbol{k})\cdot(2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})dt \\
&= \int_{t=0}^{1} (6t^3 + 12t^3 + 6t^3)dt \\
&= \left[\frac{24}{4}t^4\right]_0^1 \\
&= 6
\end{aligned}$$

So that in this example the two line integrals give the same answer. In fact this is true for the line integral of any vector field that is the form $\mathrm{grad}\phi$.

Proposition:

If $\mathbf{F}$ is a **gradient** vector field so that $\mathbf{F} = \boldsymbol{\nabla}\phi$, for some C^1 scalar field ϕ, then the line integral $\displaystyle\int_\gamma \mathbf{F}\cdot d\boldsymbol{r}$ only depends upon the end points A and B of γ, not on the path taken. In fact

$$\int_\gamma (\boldsymbol{\nabla}\phi)\cdot d\boldsymbol{r} = \phi(\boldsymbol{r}_B) - \phi(\boldsymbol{r}_A)$$

where $\phi(\boldsymbol{r}_A)$ and $\phi(\boldsymbol{r}_B)$ are the values of ϕ at the start and end points of the curve γ.

Proof: Let γ be parameterised by $\boldsymbol{r}(t)$, $t_0 \leq t \leq t_1$ then

$$\int_\gamma (\boldsymbol{\nabla}\phi)\cdot d\boldsymbol{r} = \int_{t=t_0}^{t_1} \left(\boldsymbol{\nabla}\phi(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}\right) dt$$

Now

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

And if we write

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}, \quad t_0 \leq t \leq t_1$$

then

$$\frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$$

Hence

$$\begin{aligned}\boldsymbol{\nabla}\phi\boldsymbol{\cdot}\frac{d\boldsymbol{r}}{dt} &= \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} \\ &= \frac{d\phi(\boldsymbol{r}(t))}{dt}\end{aligned}$$

Thus

$$\begin{aligned}\int_\gamma (\boldsymbol{\nabla}\phi)\boldsymbol{\cdot}d\boldsymbol{r} &= \int_{t=t_0}^{t_1} \frac{d\phi(\boldsymbol{r}(t))}{dt}dt \\ &= [\phi(\boldsymbol{r}(t))]_{t=t_0}^{t_1} \\ &= \phi(\boldsymbol{r}(t_1)) - \phi(\boldsymbol{r}(t_0)) \\ &= \phi(\boldsymbol{r}_B) - \phi(\boldsymbol{r}_A)\end{aligned}$$

8.2.2 Conservative vector fields

Why is the result of the previous section significant? One important application of line integrals is to find out the work done by a force. For a constant force $\mathbf{F}$ moving an from A to B, the work done is "the force times the distance moved in the direction of the force". The mathematical formula for this is

$$W = \mathbf{F}\boldsymbol{\cdot}\boldsymbol{r}, \quad \text{where } \boldsymbol{r} = \boldsymbol{AB}$$

However if the force and path are changing, we have to divide the path into small sections, and then take the limit

$$W = \lim_{m\to\infty} \sum_{j=1}^{m} \mathbf{F}_j \delta\boldsymbol{r}_j = \int_\gamma \mathbf{F}(\boldsymbol{r})\boldsymbol{\cdot}d\boldsymbol{r}$$

in order to obtain the formula for the work done by the vector field $\mathbf{F}(\boldsymbol{r})$ in moving from A to B along the path γ.

If we can find a scalar field such that

$$\mathbf{F} = -\boldsymbol{\nabla}\phi \quad \textbf{NB note the minus sign}$$

then we say that ϕ is a **potential** for the force $\mathbf{F}$ and the change in ϕ simply represents the change in potential energy. In such a situation any loss of potential energy results in a gain in kinetic energy (and vice-versa) so that the total energy is conserved. For this reason a gradient vector field is often called a **conservative** vector field, and we will use this name from now onwards in this course. In section 4.2 of these notes we will look at conditions which guarantee that a vector field is conservative.

8.2.3 Surfaces

We start by giving a number of different definitions of a 2-dimensional surface in three dimensional space $\mathbb{R}^3$. There are essentially three different ways of specifying a surface

(1.) As the graph of a function of 2 variables, $z = f(x, y)$

(2.) As a level surface of a function of 3 variables $F(x, y, z) = c$

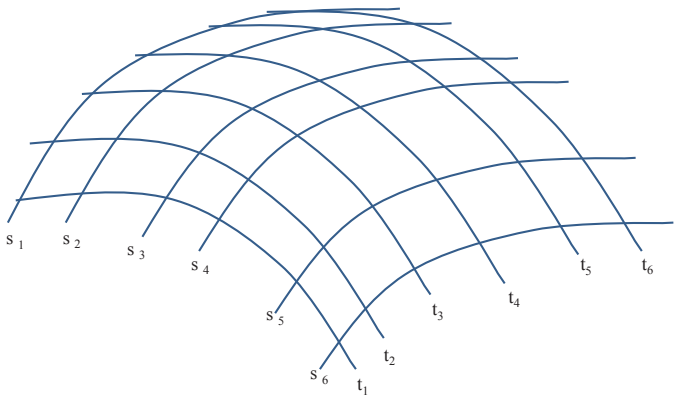
(3). As a parameterised surface $(s, t) \mapsto (x(s, t), y(s, t), z(s, t))$

For the purpose of calculating surface integrals we will mostly use definition (3) and think of surfaces as (smooth) maps

$$S \to \mathbb{R}^3$$
$$(s, t) \mapsto \boldsymbol{r}(s, t)$$

where $S \subset \mathbb{R}^2$ is some 2-dimensional set which specifies the region in which the parameters s and t lie in. In many of the applications this will simply involve $s_0 \leq s \leq s_1$ and $t_0 \leq t \leq t_1$ so that S will be given by the rectangle $[s_0, s_1] \times [t_0, t_1]$.

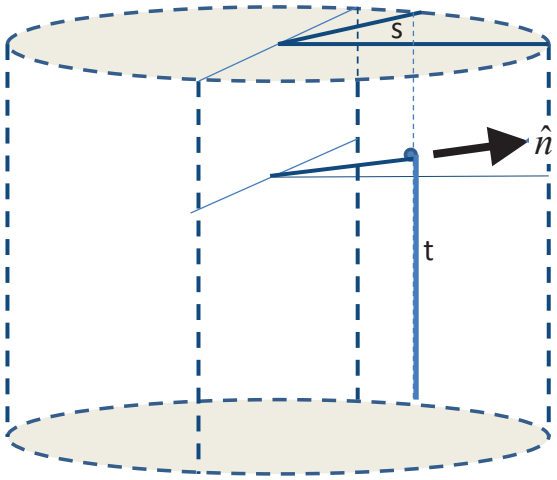
If we draw a picture of this map



we see that the lines s =const. and t =const. are simply the coordinate lines on the surface.

Example 1: A cylinder radius a and height h

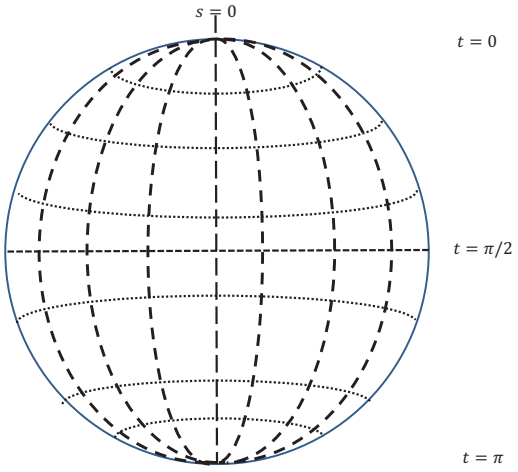
$$\boldsymbol{r}(s,t) = a\sin s\boldsymbol{i} + a\cos s\boldsymbol{j} + t\boldsymbol{k}, \quad -\pi \leq s < \pi, 0 \leq t \leq h$$



The lines t =const. are circles and the lines s =const. are straight vertical lines on the surface of the cylinder.

Example 2: A sphere radius a centre the origin

$$\boldsymbol{r}(s,t) = a\cos s\sin t\boldsymbol{i} + a\sin s\sin t\boldsymbol{j} + a\cos t\boldsymbol{k}, \quad -\pi \leq s < \pi, 0 \leq t \leq \pi$$



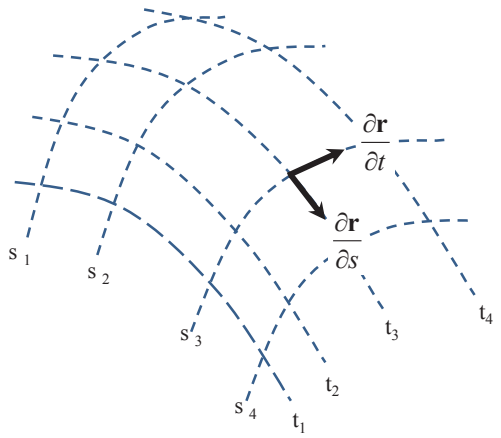
The lines s =const. are the lines of longitude and the lines t =const. are the lines of latitude on the surface of the sphere. Thus the s-coordinate is just the ϕ-coordinate of spherical polar coordinates, while the t-coordinate is just the θ-coordinate of spherical polar coordinates.

8.2.4 The normal to a surface

We now calculate an expression for the normal to a general surface.

The vector $\dfrac{\partial \boldsymbol{r}}{\partial s}$ is tangent to the curve t =const.

While the vector $\dfrac{\partial \boldsymbol{r}}{\partial t}$ is tangent to the curve s =const.



A normal to the surface is therefore given by

$$\boldsymbol{n} = \frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t}$$

We calculate the normal to the surface in the two examples above:

Example 1: A cylinder radius a and height h

$$\begin{aligned} \boldsymbol{r}(s,t) &= a\cos s\boldsymbol{i} + a\sin s\boldsymbol{j} + t\boldsymbol{k} \\ \frac{\partial \boldsymbol{r}}{\partial s} &= -a\sin s\boldsymbol{i} + a\cos s\boldsymbol{j} + 0\boldsymbol{k} \\ \frac{\partial \boldsymbol{r}}{\partial t} &= 0\boldsymbol{i} + 0\boldsymbol{j} + \boldsymbol{k} \end{aligned}$$

Hence

$$\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin s & a\cos s & 0 \\ 0 & 0 & 1 \end{vmatrix} = a\cos s\boldsymbol{i} + a\sin s\boldsymbol{j}$$

Note this is simply the $(\boldsymbol{i}, \boldsymbol{j})$-part of $\boldsymbol{r}(s,t)$ as one would expect from the picture above.

Example 2: A sphere radius a centre the origin

$$\begin{aligned} \boldsymbol{r}(s,t) &= a\cos s\sin t\boldsymbol{i} + a\sin s\sin t\boldsymbol{j} + a\cos t\boldsymbol{k} \\ \frac{\partial \boldsymbol{r}}{\partial s} &= -a\sin s\sin t\boldsymbol{i} + a\cos s\sin t\boldsymbol{j} + a\cot t\boldsymbol{k} \\ \frac{\partial \boldsymbol{r}}{\partial t} &= a\cos s\cos t\boldsymbol{i} + a\sin s\cos t\boldsymbol{j} - a\sin t\boldsymbol{k} \end{aligned}$$

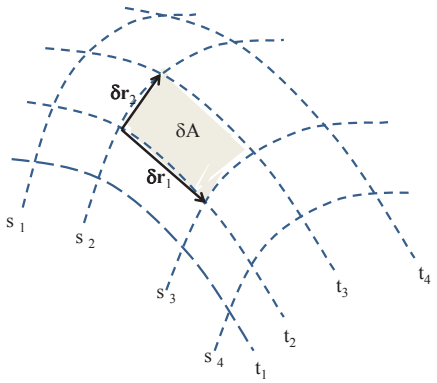
Hence

$$\begin{aligned} \boldsymbol{n} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin s\sin t & a\cos s\sin t & a\cot t \\ a\cos s\cos t & a\sin s\cos t & -a\sin t \end{vmatrix} \\ &= -a^2\sin t(\cos s\sin t\boldsymbol{i} + \sin s\sin t\boldsymbol{j} + \cos t\boldsymbol{k}) \\ &= -a^2\sin t\boldsymbol{r} \end{aligned}$$

Thus $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$ (as one would expect!).

8.2.5 Surface integrals

In order to calculate "surface integrals" we first need to find an expression for the element of surface area dA. To do this we look again at the picture of the coordinate lines on a surface.



Suppose we start at the point $\boldsymbol{r}(s,t)$ and move to the nearby point $\boldsymbol{r}(s+\delta s,t)$. Then the vector connecting these two points is

$$\delta\boldsymbol{r}_1 = \boldsymbol{r}(s+\delta s,t) - \boldsymbol{r}(s,t)$$

Now by Taylor's theorem

$$\boldsymbol{r}(s+\delta s,t) = \boldsymbol{r}(s,t) + \delta s\frac{\partial \boldsymbol{r}}{\partial s}(s,t) + O(\delta s^2)$$

Hence

$$\delta\boldsymbol{r}_1 = \delta s\frac{\partial \boldsymbol{r}}{\partial s}(s,t) + O(\delta s^2)$$

Similarly if we move in the t direction to the nearby point $\boldsymbol{r}(s,t+\delta t)$ then the vector measuring the displacement from $\boldsymbol{r}(s,t)$ is

$$\delta\boldsymbol{r}_2 = \boldsymbol{r}(s,t+\delta t) - \boldsymbol{r}(s,t)$$

Again using Taylor's theorem we have

$$\boldsymbol{r}(s,t+\delta t) = \boldsymbol{r}(s,t) + \delta t\frac{\partial \boldsymbol{r}}{\partial t}(s,t) + O(\delta t^2)$$

and hence

$$\delta\boldsymbol{r}_2 = \delta t\frac{\partial \boldsymbol{r}}{\partial t}(s,t) + O(\delta t^2)$$

Thus the area of the parallelogram with sides $\delta\boldsymbol{r}_1$ and $\delta\boldsymbol{r}_2$ is

$$\begin{aligned}\delta A &= |\delta\boldsymbol{r}_1 \times \delta\boldsymbol{r}_2| \\ &\approx \left|\frac{\partial \boldsymbol{r}}{\partial s}\times\frac{\partial \boldsymbol{r}}{\partial s}\right| \delta s\delta t\end{aligned}$$

So in the limit we get

$$dA = \left|\frac{\partial \boldsymbol{r}}{\partial s}\times\frac{\partial \boldsymbol{r}}{\partial s}\right| dsdt$$

Example 1: we now use this result to calculate the area of a cylinder. The formula for a parameterised cylinder of radius a and height h is given by

$$\boldsymbol{r}(s,t) = a\cos s\boldsymbol{i} + a\sin s\boldsymbol{j} + t\boldsymbol{k}, \quad -\pi \le s < \pi, 0 \le t \le h$$

So that

$$\begin{aligned}\frac{\partial \boldsymbol{r}}{\partial s}\times\frac{\partial \boldsymbol{r}}{\partial s} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin s & a\cos s & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= a\cos s\boldsymbol{i} + a\sin s\boldsymbol{j}\end{aligned}$$

and hence

$$\begin{aligned}\left|\frac{\partial \boldsymbol{r}}{\partial s}\times\frac{\partial \boldsymbol{r}}{\partial s}\right| &= a(\cos^2 s+\sin^2 s)^{1/2}\\ &= a\end{aligned}$$

Therefore by the above formula for the surface of a cylinder

$$dA = adsdt$$

Hence the area is given by

$$A=\int_{s=-\pi}^{\pi}\int_{t=0}^{h} adsdt = 2\pi ah$$

Hence we have shown that the surface area of a cylinder of height h and radius a is $2\pi ah$ (in agreement with the standard formula).

Example 2: We now use the formula for dA to calculate the surface area of a sphere. Recall that a parameterised sphere is given by

$$\boldsymbol{r}(s,t) = a\cos s\sin t\boldsymbol{i} + a\sin s\sin t\boldsymbol{j} + a\cos t\boldsymbol{k}, \quad -\pi\le s<\pi, 0\le t\le\pi$$

So that

$$\begin{aligned}\frac{\partial \boldsymbol{r}}{\partial s}\times\frac{\partial \boldsymbol{r}}{\partial s} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -a\sin s\sin t & a\cos s\sin t & a\cot t\\ a\cos s\cos t & a\sin s\cos t & -a\sin t\end{vmatrix}\\ &= -a^2\sin t(\cos s\sin t\boldsymbol{i}+\sin s\sin t\boldsymbol{j}+\cos t\boldsymbol{k})\end{aligned}$$

and hence

$$\begin{aligned}\left|\frac{\partial \boldsymbol{r}}{\partial s}\times\frac{\partial \boldsymbol{r}}{\partial s}\right| &= a^2\sin t\left(\cos^2 s\sin^2 t+\sin^2 s\sin^2 t+\cos^2 t\right)^{1/2}\\ &= a^2\sin t\left((\cos^2 s+\sin^2 s)\sin^2 t+\cos^2 t\right)^{1/2}\\ &= a^2\sin t\end{aligned}$$

Therefore we have shown that for a sphere

$$dA = a^2\sin t dsdt$$

Note that a is the radius of the sphere and (s,t) correspond to the usual spherical polar coordinates (ϕ,θ), so that the above formula agrees with the standard result that

$$dA = r^2 \sin\theta d\theta d\phi$$

We now use this to calculate the area of a sphere

$$\begin{aligned} A &= \int_{s=-\pi}^{\pi} \int_{t=0}^{\pi} a^2 \sin t ds dt \\ &= 2\pi a^2 \int_{t=0}^{\pi} \sin t dt \\ &= 2\pi a^2 \left[-\cos t\right]_{t=0}^{\pi} \\ &= 4\pi a^2 \end{aligned}$$

Hence we have shown that the area of a sphere radius a is $4\pi a^2$ (in agreement with the standard formula).

8.2.6 Flux integrals

A flux integral measures the "flow" of a vector field $\mathbf{F}$ through a surface S

$$\iint_S \mathbf{F}\cdot\hat{\boldsymbol{n}} dA$$

where $\hat{\boldsymbol{n}}$ is the unit normal to the surface.

If we think of $\mathbf{F}$ as giving the velocity of a fluid, then the flux integral measures the net rate of flow of the fuid through the surface S.

As with line integrals the method of calculating flux integrals is to use the paramterisation of the surface to convert them into ordinary integrals.

If we introduce the **vector element of area**

$$d\boldsymbol{S} = \hat{\boldsymbol{n}} dA$$

then we may write the flux integral as

$$\iint_S \mathbf{F}\cdot d\boldsymbol{S}$$

Then $d\boldsymbol{S}$ points in the same direction as $\boldsymbol{n}$ and has modulus $|d\boldsymbol{S}| = dA$. From this we see that

$$d\boldsymbol{S} = \left(\frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t}\right) dsdt$$

This is because the above expression points in the same direction as $\boldsymbol{n}$, since

$$\boldsymbol{n} = \frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t}$$

and has modulus dA since

$$dA = \left|\frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t}\right|$$

Using this formula for $d\boldsymbol{S}$ we may write the flux integral as

$$\iint_S \mathbf{F} \cdot d\boldsymbol{S} = \iint_S \left\{\mathbf{F}(\boldsymbol{r}(s,t)) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t}\right)\right\} dsdt$$

where the integrand

$$\mathbf{F}(\boldsymbol{r}(s,t)) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t}\right)$$

is simply a function of s and t, so the flux integral has been converted into an ordinary 2-dimensional integral.

Note: There is an important point to note about this result: the formula we use to calculate the flux integral depends upon the paramterisation. However one can show, as with the case of the line integral, that the final answer is independent of the parameterisation chosen. We do not give the details of the calculation here, but not that the essential point is that if we calculate $d\boldsymbol{S}$ using a different paramterisation, $(\tilde{s}, \tilde{t})$, it differs from the original one by being multiplied by the the Jacobian determinant of the transformation from (s,t) to $(\tilde{s}, \tilde{t})$. This is exactly the factor one needs when going from $dsdt$ to $d\tilde{s}d\tilde{t}$.

Example: Let S be the parameterised surface given by

$$\boldsymbol{r}(s,t) = s\cos t\boldsymbol{i} + s\sin t\boldsymbol{j} + t\boldsymbol{k}, \quad 0 \leq s \leq 1, 0 \leq t \leq 2\pi$$

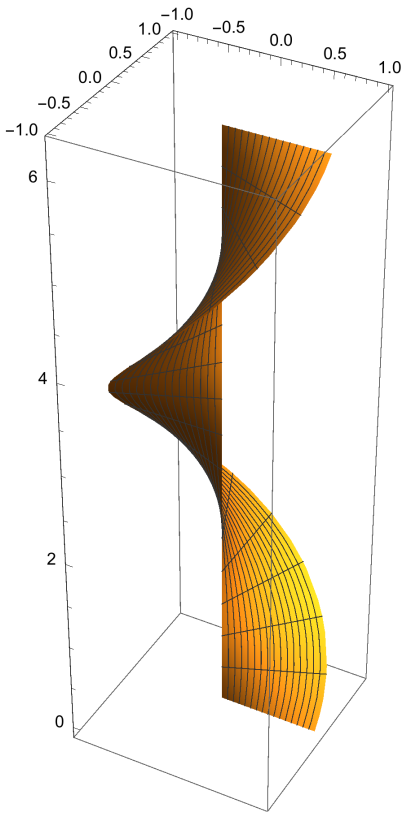


Figure 8.5: Helicoid.

(a twisted ribbon or "helicoid"). Let $\mathbf{F}(\boldsymbol{r})$ be the vector field given by $\mathbf{F} = x\boldsymbol{i} + y\boldsymbol{j} + (z - 2y)\boldsymbol{k}$. Evaluate the flux integral $\iint_S \mathbf{F}\cdot d\boldsymbol{S}$. We first evaluate the vector field on the surface

$$\boldsymbol{r}(s,t) = s\cos t\boldsymbol{i} + s\sin t\boldsymbol{j} + t\boldsymbol{k}$$

Now on the surface

$$x = s\cos t, \quad y = s\sin t, \text{and } z = t$$

So that substituting into the formula $\mathbf{F} = x\boldsymbol{i} + y\boldsymbol{j} + (z - 2y)\boldsymbol{k}$ we get

$$\mathbf{F}(\boldsymbol{r}(s,t)) = s\cos t\boldsymbol{i} + s\sin t\boldsymbol{j} + (t - 2s\sin t)\boldsymbol{k}$$

We now calculate $d\boldsymbol{S}$. Differentiating the expression for $\boldsymbol{r}(s,t)$ gives

$$\frac{\partial \boldsymbol{r}}{\partial s} = \cos t\boldsymbol{i} + \sin t\boldsymbol{j} + 0\boldsymbol{k}$$
$$\frac{\partial \boldsymbol{r}}{\partial t} = -s\sin t\boldsymbol{i} + s\cos t\boldsymbol{j} + \boldsymbol{k}$$

Now

$$\begin{aligned} d\boldsymbol{S} &= \left(\frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t}\right) dsdt \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{vmatrix} dsdt \\ &= \left\{\sin t\boldsymbol{i} + \cos t\boldsymbol{j} + (s\cos^2 t + s\sin^2 t)\boldsymbol{k}\right\} dsdt \\ &= (\sin t\boldsymbol{i} - \cos t\boldsymbol{j} + s\boldsymbol{k})dsdt \end{aligned}$$

Hence

$$\begin{aligned} \iint_S \mathbf{F}\cdot d\boldsymbol{S} &= \int_{s=0}^{1}\int_{t=0}^{2\pi} (s\cos t\boldsymbol{i} + s\sin t\boldsymbol{j} + (t - 2s\sin t)\boldsymbol{k})\cdot(\sin t\boldsymbol{i} - \cos t\boldsymbol{j} + s\boldsymbol{k})dsdt \\ &= \int_{s=0}^{1}\int_{t=0}^{2\pi} (st - 2s^2\sin t)dsdt \\ &= \int_{s=0}^{1}\left[\frac{1}{2}st^2 + 2s^2\cos t\right]_{t=0}^{2\pi} ds \\ &= \int_{s=0}^{1} 2s\pi^2 ds \\ &= \left[s^2\pi^2\right]_{s=0}^{1} \\ &= \pi^2 \end{aligned}$$

Hence

$$\iint_S \mathbf{F}\cdot d\boldsymbol{S} = \pi^2$$

We end this section by giving the expression for a flux integral in the *special case* that the surface is the graph of a function $z = f(x, y)$. To calculate the flux integral in this case we take x and y themselves to be the paramters used to describe the surface. Hence the parameterised surface is given by

$$\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j} + f(x, y)\boldsymbol{k}, \quad (x, y) \in S$$

Then

$$\frac{\partial \boldsymbol{r}}{\partial x} = \boldsymbol{i} + 0\boldsymbol{j} + \frac{\partial f}{\partial x}$$
$$\frac{\partial \boldsymbol{r}}{\partial y} = 0\boldsymbol{i} + \boldsymbol{j} + \frac{\partial f}{\partial y}$$

Hence

$$\begin{aligned} d\boldsymbol{S} &= \left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right) dxdy \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} dxdy \\ &= \left(-\frac{\partial f}{\partial x}\boldsymbol{i} - \frac{\partial f}{\partial y}\boldsymbol{j} + \boldsymbol{k}\right) dxdy \end{aligned}$$

Thus for the case of a suface which is the graph of a function we have the formula

$$\iint\limits_S \mathbf{F}\cdot d\boldsymbol{S} = \iint \mathbf{F}(x, y, f(x, y)\cdot \left(-\frac{\partial f}{\partial x}\boldsymbol{i} - \frac{\partial f}{\partial y}\boldsymbol{j} + \boldsymbol{k}\right) dxdy$$

As a final remark in this section we point out that it often saves time to think a bit about the geometry of a flux integral before going ahead and mechanically doing the calculation.

Example: Let $\mathbf{F} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, and S the surface of the sphere $x^2 + y^2 + z^2 = a^2$. Calculate the flux integral $\displaystyle\iint\limits_S \mathbf{F}\cdot d\boldsymbol{S}$.

In this example a little thought shows that $\mathbf{F} = \boldsymbol{r}$ and that $d\boldsymbol{S} = \hat{\boldsymbol{n}}dA = \hat{\boldsymbol{r}}dA$. Hence

$$\mathbf{F}\cdot d\boldsymbol{S} = \boldsymbol{r}\cdot\hat{\boldsymbol{r}}dA = r(\hat{\boldsymbol{r}})\cdot(\hat{\boldsymbol{r}})dA = rdA$$

Hence

$$\iint\limits_S \mathbf{F}\cdot d\boldsymbol{S} = \iint\limits_S rdA = a\iint\limits_S dA = a.4\pi a^2 = 4\pi a^3$$

(since $r = a$ on S).

8.3 Integral Theorems

In this section we look at the relationship between line integrals, flux integrals and volume integrals using grad, div and curl.

8.3.1 Stokes's Theorem

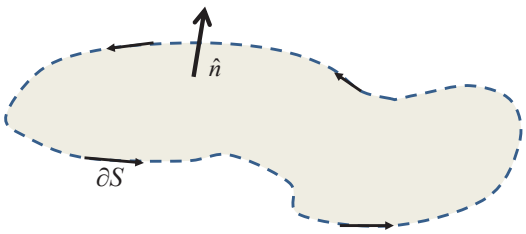
We start by looking at the relationship between the line integral round a closed loop and the flux flowing through the loop.

Theorem (Stokes's Theorem)

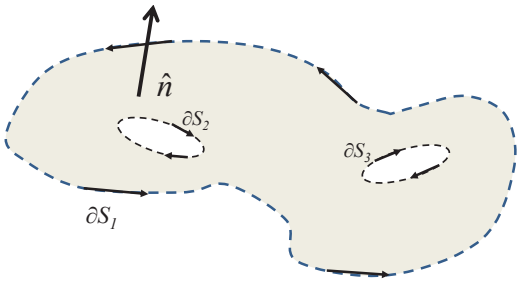
Let S be a bounded, piecewise smooth, orientable, surface in $\mathbb{R}^3$, with boundary ∂S, which consists of a finite number of piecewise C^1 simple closed curves. Let $\mathbf{F}$ be a C^1 vector field whose domain includes S. Then

$$\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} = \oint_{\partial S} \mathbf{F}\cdot d\boldsymbol{r}$$

Note 1: The $\oint$ symbol indicates that the line integral is round a *closed* curve. This line integral round ∂S is computed in a clockwise direction relative to the normal $\boldsymbol{n}$ to S used to define $d\boldsymbol{S}$. (If the opposite normal is chosen both integrals therefore change sign).



Note 2: In the picture above the boundary to S consists of just one component. However the theorem allows for the boundary to consist of several components (e.g. three as shown in the picture below).



In such a case the line integral is the sum of the line integral for the various components, so in the example above

$$\oint_{\partial S} \mathbf{F}\cdot d\boldsymbol{r} = \oint_{\partial S_1} \mathbf{F}\cdot d\boldsymbol{r} + \oint_{\partial S_2} \mathbf{F}\cdot d\boldsymbol{r} + \oint_{\partial S_3} \mathbf{F}\cdot d\boldsymbol{r}$$

where the direction of integration round all three loops ∂S_i is clockwise relative to the normal $\boldsymbol{n}$ to S.

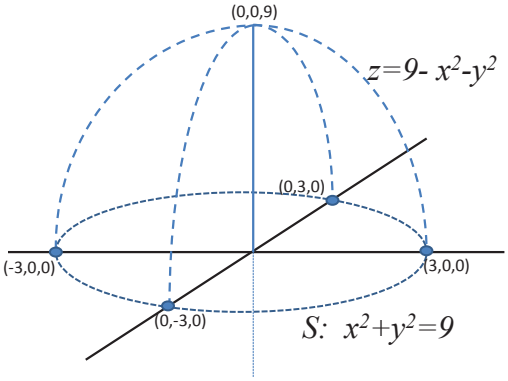
Note 3: Stokes's Theorem contains some *fine print* concerning the nature of the surface "piecewise smooth, orientable", the nature of the boundary "finite number of piecewise C^1 simple closed curves" and concerning the nature of the vector field $\mathbf{F}$ "C^1 on domain that includes S". All these restrictions are important and the theorem is not valid without them (or equivalent) conditions. For further explanation on the precise meaning of these terms see the Appendix.

Example: Let S be the paraboloid

$$z = 9 - x^2 - y^2, \quad z \geq 0$$

Verify Stokes's theorem for the vector field

$$\mathbf{F} = (2z - y)\boldsymbol{i} + (x + z)\boldsymbol{j} + (3x - 2y)\boldsymbol{k}$$



We first evaluate the flux integral of $\boldsymbol{\nabla}\times\mathbf{F}$.

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z-y & x+z & 3x-2y \end{vmatrix} \\ &= (-2-1)\boldsymbol{i} + (2-3)\boldsymbol{j} + (1- \ -1)\boldsymbol{k} \\ &= -3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}\end{aligned}$$

S is the graph of $z = f(x,y) = 9 - x^2 - y^2$. Also from the diagram the condition $z \geq 0$ is equivalent to $x^2 + y^2 \leq 9$ so we parameterise S as:

$$\boldsymbol{r}(x,y) = x\boldsymbol{i} + y\boldsymbol{j} + (9 - x^2 - y^2)\boldsymbol{k}, \quad x^2 + y^2 \leq 9$$

Then

$$\begin{aligned}d\boldsymbol{S} &= \left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right) dxdy \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} dxdy \\ &= (2x\boldsymbol{i} + 2y\boldsymbol{j} + \boldsymbol{k})\, dxdy\end{aligned}$$

Hence

$$\begin{aligned}\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} &= \int\int_{x^2+y^2\leq 9} (-3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})\cdot(2x\boldsymbol{i} + 2y\boldsymbol{j} + \boldsymbol{k})dxdy \\ &= \int\int_{x^2+y^2\leq 9} (-6x - 2y + 2)dxdy \\ &= \int\int_{x^2+y^2\leq 9} 2dxdy, \quad \text{(since the other terms cancel by symmetry)} \\ &= 2 \times \text{area} = 2.9\pi = 18\pi\end{aligned}$$

To verify Stokes's theorem we now calculate the line integral. From the diagram we see that $C = \partial S$ is a circle radius 3 centre the origin and lying in the xy-plane.. Since the normal $\boldsymbol{n}$ to S point upwards we parameterise C as:

$$\boldsymbol{r}(t) = 3\cos t\boldsymbol{i} + 3\sin t\boldsymbol{j} + 0\boldsymbol{k}, quad 0 \leq t < 2\pi$$

Then

$$\mathbf{F}(\boldsymbol{r}(t)) = -3\sin t\boldsymbol{i} + 3\cos t\boldsymbol{j} + (9\cos t - 6\sin t)\boldsymbol{k}$$

and

$$\frac{d\boldsymbol{r}}{dt} = -3\sin t\boldsymbol{i} + 3\cos t\boldsymbol{j} + 0\boldsymbol{k}$$

Thus

$$\begin{aligned}
\oint_{\partial S} \mathbf{F}\cdot d\boldsymbol{r} &= \int_{t=0}^{2\pi} \left(\mathbf{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}(t)}{dt}\right) dt \\
&= \int_{t=0}^{2\pi} ((-3\sin t\boldsymbol{i} + 3\cos t\boldsymbol{j} + (9\cos t - 6\sin t)\boldsymbol{k})\cdot(-3\sin t\boldsymbol{i} + 3\cos t\boldsymbol{j} + 0\boldsymbol{k}))\, dt \\
&= \int_{t=0}^{2\pi} (9\sin^2 t + 9\cos^2 t)dt \\
&= \int_{t=0}^{2\pi} 9dt \\
&= [9t]_{t=0}^{2\pi} = 18\pi
\end{aligned}$$

Hence in this example we have shown

$$\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} = 18\pi = \oint_{\partial S} \mathbf{F}\cdot d\boldsymbol{r}$$

Example 2: Let $\tilde{S}$ be the disc $x^2 + y^2 \leq 9$, $z = 0$. Calculate $\displaystyle\iint_{\tilde{S}} (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S}$ where $\mathbf{F}$ is as before and $\boldsymbol{n}$ is the upward normal.

To do this we again uses Stokes's theorem

$$\iint_{\tilde{S}} (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} = \oint_{\partial\tilde{S}} \mathbf{F}\cdot d\boldsymbol{r}$$

But S and $\tilde{S}$ both have the same boundary given by C, a circle radius 3 in the xy-plane. So that $\partial\tilde{S} = \partial S$. Hence

$$\iint_{\tilde{S}} (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} = \oint_{\partial\tilde{S}} \mathbf{F}\cdot d\boldsymbol{r} = \oint_{\partial S} \mathbf{F}\cdot d\boldsymbol{r} = 18\pi$$

The method used in this example is often useful so we state it in the general case as a consequence of Stokes's theorem.

Corollary to Stokes's Theorem

Let S and $\tilde{S}$ be two bounded, piecewise smooth, orientable, surfaces in $\mathbb{R}^3$, with the same boundary (consisting of a finite number of piecewise C^1 simple closed curves). Let $\mathbf{F}$ be a C^1 vector field whose domain includes both S and $\tilde{S}$. Then

$$\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} = \iint_{\tilde{S}} (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S}$$

8.3.2 Conservative Vector Fields

We start by defining a vector field with path-independent integral.

Definition: A continuous vector field $\mathbf{F}$ has path-independent line integral in the region U if

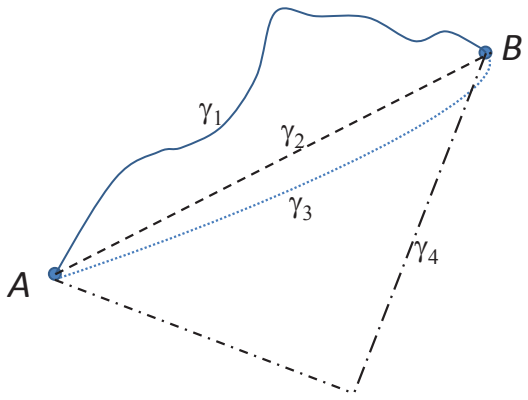
$$\int_{\gamma_1} \mathbf{F}\cdot d\boldsymbol{r} = \int_{\gamma_2} \mathbf{F}\cdot d\boldsymbol{r}$$

for any 2 piecewise C^1, oriented curves γ_1 and γ_2 which lie in the set U and both start at the same point A, and end at both end at the same point B.

Proposition: Let $\mathbf{F}$ be a continuous vector field. Then $\mathbf{F}$ has path-independent integrals in U if and only if

$$\oint_\gamma \mathbf{F}\cdot d\boldsymbol{r} = 0$$

for all simple closed piecewise C^1 curves γ in U.



Proof: Suppose that $\mathbf{F}$ is a path-independent vector field and γ is a closed curve. Then we may split it up into two curves γ_1 and γ_2 that start at A and end at B. Then

γ is just γ_1 followed by $-\gamma_2$ (ie γ_2 traversed in the opposite direction). Hence

$$\oint_\gamma \mathbf{F}\cdot d\boldsymbol{r} = \int_{\gamma_1} \mathbf{F}\cdot d\boldsymbol{r} - \int_{\gamma_2} \mathbf{F}\cdot d\boldsymbol{r} = 0$$

Conversely suppose the line integral round any closed loop vanishes and γ_1 and γ_2 are two curves that start and finish at the same point. Then γ_1 followed by $-\gamma_2$ (ie γ_2 traversed in the opposite direction) is just γ so that

$$\int_{\gamma_1} \mathbf{F}\cdot d\boldsymbol{r} - \int_{\gamma_2} \mathbf{F}\cdot d\boldsymbol{r} = \oint_\gamma \mathbf{F}\cdot d\boldsymbol{r} = 0$$

We now give a theorem that relates path-independent vector fields to conservative (or gradient) vector fields.

Theorem:

Let $\mathbf{F}$ be a continuous vector field in the *connected* open region U of $\mathbb{R}^3$. Then there exists a C^1 scalar field ϕ such that $\mathbf{F} = \boldsymbol{\nabla}\phi$ if and only if $\mathbf{F}$ has path independent integral in U.

Proof: We have already shown that if $\mathbf{F} = \boldsymbol{\nabla}\phi$ then

$$\int_{\gamma_{AB}} \mathbf{F}\cdot d\boldsymbol{r} = \int_{\gamma_{AB}} (\boldsymbol{\nabla}\phi)\cdot d\boldsymbol{r} = \phi(\boldsymbol{r}_B) - \phi(\boldsymbol{r}_A)$$

and hence the integral is independent of the path.

If $\mathbf{F}$ has path independent integral then we may define a scalar field ϕ by

$$\phi(\boldsymbol{r}) = \int_\gamma \mathbf{F}\cdot d\boldsymbol{r}$$

where γ is some path from some (arbitrary) fixed point A to the point P with position vector $\boldsymbol{r}$. Note that because the vector field $\mathbf{F}$ has path independent integral this is well defined without having to specify the path γ.

We now show by direct computation that

$$\frac{\partial\phi}{\partial x} = F_1, \quad \frac{\partial\phi}{\partial x} = F_1, \quad \frac{\partial\phi}{\partial x} = F_1, \quad \text{ie } \boldsymbol{\nabla}\phi = \mathbf{F}.$$

Let

$$\phi(x,y,z) = \int_\gamma \mathbf{F}\cdot d\boldsymbol{r}$$

Then

$$\phi(x+h,y,z) = \int_\gamma \mathbf{F}\cdot d\boldsymbol{r} + \int_{\gamma_1} \mathbf{F}\cdot d\boldsymbol{r}$$

where γ_1 is a straight line parallel to the x-axis going from x to $x+h$. Hence

$$\begin{aligned}\phi(x+h,y,z)-\phi(x,y,z) &= \int_x^{x+h} \mathbf{F}\cdot d\boldsymbol{r} \\ &= \int_x^{x+h} F_1 dx \text{(since } d\boldsymbol{r} = \boldsymbol{i}dx \text{ on } \gamma_1) \\ &= hF_1(x,y,z)+O(h^2)\end{aligned}$$

Hence

$$\frac{\partial\phi}{\partial x} = \lim_{h\to 0}\left(\frac{\phi(x+h,y,z)-\phi(x,y,z)}{h}\right) = \lim_{h\to 0}\left(F_1(x,y,z)+O(h)\right) = F_1(x,y,z)$$

Similarly

$$\frac{\partial\phi}{\partial y} = F_2, \text{and } \frac{\partial\phi}{\partial z} = F_3$$

Note that different choices of the arbitrary initial point A simply change ϕ by a constant, but this has no effect on $\boldsymbol{\nabla}\phi$ which is unchanged.

The above theorem tells us that a conservative vector field is equivalent to one with path-independent integral. Unfortunately this condition is not very easy to check as we need to know it is true for any possible path! The following theorem gives a more useful criterion, which is easy to check.

Theorem: Let $\mathbf{F}$ be a C^1 vector field defined on the *simply connected* region U in $\mathbb{R}^3$. Then

There exists some C^2 scalar field ϕ such that $\mathbf{F}=\boldsymbol{\nabla}$ if and only if $\boldsymbol{\nabla}\times\mathbf{F}=\mathbf{0}$

Note: The *fine print* in this theorem is again important to note. In particular the simply connected condition (which means we can continuously shrink any closed path down to a point) is required - see the exercise sheets for a counterexample to this theorem when U is not simply connected.

Proof:

We know from chapter 2 that $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi)=\mathbf{0}$, so if $\mathbf{F}=\boldsymbol{\nabla}\phi$ then we know that $\boldsymbol{\nabla}\times\mathbf{F}=\mathbf{0}$.

On the other hand if $\boldsymbol{\nabla}\times\mathbf{F}=\mathbf{0}$ then we know from Stokes's theorem that

$$\oint_C \mathbf{F}\cdot d\boldsymbol{r} = \iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} = 0$$

where S is some spanning surface with boundary C. (Note that such a spanning surface S exists precisely because of the simply-connected condition.)

Hence by the first proposition in this section $\mathbf{F}$ has path-independent integral, and thus by the previous theorem $\mathbf{F} = \boldsymbol{\nabla}\phi$, as required.

8.3.3 Finding potentials for conservative vector fields

In the previous section we showed how to test for a conservative vector field in this section we show how one can actually find a potential for the vector field.

Example: $\mathbf{F} = (2xy+\cos 2y)\boldsymbol{i}+(x^2+2y-2x\sin 2y)\boldsymbol{j}$, show that $\mathbf{F}$ is conservative and find a scalar field ϕ such that $\boldsymbol{\nabla}\phi = \mathbf{F}$.

To show that $\mathbf{F}$ is conservative we compute $\boldsymbol{\nabla}\times\mathbf{F}$.

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+\cos 2y & x^2+2y-2x\sin 2y & 0 \end{vmatrix} \\ &= 0\boldsymbol{i} + 0\boldsymbol{j} + (2x - 2\sin 2y - 2y + 2\sin 2y)\boldsymbol{k} \\ &= \mathbf{0}\end{aligned}$$

Hence by the theorem of the previous section $\mathbf{F}$ is conservative. We can therefore try and find a scalar field such that $\boldsymbol{\nabla}\phi = \mathbf{F}$. Since $\mathbf{F}$ only depends upon the x and y coordinates and has no $\boldsymbol{k}$ terms, in this example we may assume that $\phi = \phi(x, y)$. Hence we want to find a scalar field ϕ such that

$$\frac{\partial\phi}{\partial x} = F_1 = 2xy + \cos 2y \tag{8.1}$$

$$\frac{\partial\phi}{\partial y} = F_2 = x^2 + 2y - 2x\sin 2y \tag{8.2}$$

Integrating (1) we get

$$\phi = x^2y + x\cos 2y + f(y) \quad \text{(where } f(y) \text{ is a "constant of integration")} \tag{8.3}$$

Differentiating (3) with respect to y we get

$$\frac{\partial\phi}{\partial y} = x^2 - 2x\sin 2y = f'(y) \tag{8.4}$$

Comparing (2) and (4) we see that

$$f'(y) = 2y \quad \Rightarrow \quad f(y) = y^2 + c$$

Substituting for $f(y)$ in (3) gives

$$\phi(x, y) = x^2 y + x \cos 2y + y^2 + c$$

We end this section by summarising the four equivalent ways of characterising a C^1 conservative vector field (in a simply connected region U).

(1.) Path-independent integral

$$\int_{\gamma_1} \mathbf{F} \cdot d\boldsymbol{r} = \int_{\gamma_2} \mathbf{F} \cdot d\boldsymbol{r}$$

For all paths γ_i starting at A and ending at B

(2.) Vanishing integral round closed curve

$$\oint_{\gamma} \mathbf{F} \cdot d\boldsymbol{r} = 0$$

for all line integral round a simple closed curve γ

(3.) Gradient vector field

$$\mathbf{F} = \boldsymbol{\nabla} \phi$$

(4.) Curl free vector field

$$\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$$

8.3.4 Stokes's theorem in the plane

Stokes's theorem states that

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\boldsymbol{S} = \oint_{\partial S} \mathbf{F} \cdot d\boldsymbol{r}$$

In this section we look at this in the case that the vector field $\mathbf{F}$ depends only upon x and y and both it and the surface S lie in the xy-plane. Since $\mathbf{F}$ lies in the xy-plane and only depends upon x and y, then

$$\mathbf{F} = F_1(x, y)\boldsymbol{i} + F_2(x, y)\boldsymbol{j}$$

Then

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x,y) & F_2(x,y) & 0 \end{vmatrix} \\ &= 0\boldsymbol{i} + 0\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} \\ &= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}\end{aligned}$$

Also for a surface lying in the xy-plane

$$d\boldsymbol{S} = \hat{\boldsymbol{n}}dA = \boldsymbol{k}dxdy$$

Hence

$$\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

On the other hand if we only consider points in the xy-plane then

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} \quad \Rightarrow \quad d\boldsymbol{r} = dx\boldsymbol{i} + dy\boldsymbol{j}$$

So that

$$\oint_{\partial S} \mathbf{F}\cdot d\boldsymbol{r} = \oint_{\partial S} (F_1 dx + F_2 dy)$$

Hence in the xy-plane Stokes's theorem becomes

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_{\partial S} (F_1 dx + F_2 dy)$$

this result is known as Green's theorem:

Green's Theorem Let S be a bounded, piecewise smooth, orientable, surface in $\mathbb{R}^2$, with boundary ∂S, which consists of a finite number of piecewise C^1 simple closed curves. Let $\mathbf{F}$ be a C^1 vector field in $\mathbb{R}^2$ whose domain includes S. Then

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_{\partial S} (F_1 dx + F_2 dy)$$

We have seen that one can obtain Green's theorem from Stokes's theorem by restricting things to be in a 2-dimensional space. However it is possible to derive

Stokes' theorem from Green's theorem. This follows from the fact that if one parameterises the surface S in Stokes's theorem in terms of s and t by

$$\boldsymbol{r}(s,t), \quad \text{with } (s,t) \in \tilde{S}$$

then one finds

$$\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} = \iint_{\tilde{S}} \left(\frac{\partial F_t}{\partial s} - \frac{\partial F_s}{\partial t}\right) dudv$$

where $F_s = \mathbf{F}\cdot\dfrac{\partial \boldsymbol{r}}{\partial s}$ and $F_t = \mathbf{F}\cdot\dfrac{\partial \boldsymbol{r}}{\partial t}$. It then follows from Green's theorem in the st-plane that

$$\iint_{\tilde{S}} \left(\frac{\partial F_t}{\partial s} - \frac{\partial F_s}{\partial t}\right) dudv = \oint_{\partial\tilde{S}} (F_s ds + F_t dt)$$

On the other hand

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial r} dr + \frac{\partial \boldsymbol{r}}{\partial s} ds$$

So that $\mathbf{F}\cdot d\boldsymbol{r} = F_r dr + F_t dt$ and hence

$$\oint_{\partial\tilde{S}} (F_s ds + F_t dt) = \oint_{\partial S} \mathbf{F}\cdot d\boldsymbol{r}$$

Putting all this together we get

$$\begin{aligned}
\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} &= \iint_{\tilde{S}} \left(\frac{\partial F_t}{\partial s} - \frac{\partial F_s}{\partial t}\right) dudv \\
&= \oint_{\partial\tilde{S}} (F_s ds + F_t dt) \quad \text{(by Green's theorem)} \\
&= \oint_{\partial S} \mathbf{F}\cdot d\boldsymbol{r}
\end{aligned}$$

Hence we have used Green's theorem in the st-plane to prove Stokes's theorem:

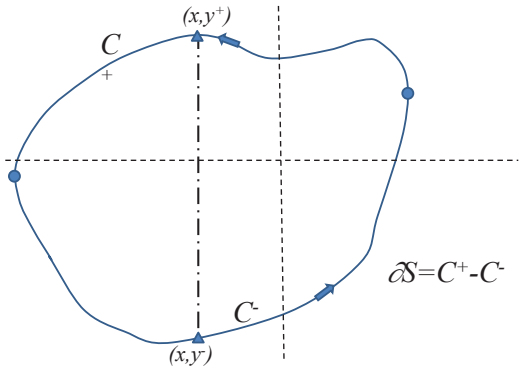
$$\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\boldsymbol{S} = \oint_{\partial S} \mathbf{F}\cdot d\boldsymbol{r}$$

8.3.5 The proof of Green's theorem

In this section we prove Green's theorem (you do not require to know this proof for the exam). As we have seen this also allows us to prove Stokes's theorem.

Outline proof of Greens' Theorem

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_{\partial S} (F_1 dx + F_2 dy)$$



Let C^+ be the upper curve and C^- be the lower curve in the above diagram. Let $(x, y^+(x))$ denote points on the graph of the upper curve and $(x, y^-(x))$ denote points on the graph of the lower curve. Then if we traverse the closed curve in a clockwise direction we move from left to right on the bottom curve and from right to left on the upper curve. Hence

$$\begin{aligned} \int_{C^-} F_1 dx &= \int_{x=a}^{b} F_1(x, y^-(x)) dx \\ \int_{C^+} F_1 dx &= \int_{x=b}^{a} F_1(x, y^+(x)) dx \\ &= -\int_{x=a}^{b} F_1(x, y^+(x)) dx \end{aligned}$$

Thus

$$\begin{aligned}\oint_C F_1 dx &= \int_{C^-} F_1 dx + \int_{C^+} F_1 dx \\ &= -\int_{x=a}^{b} \left(F_1(x, y^+(x)) - F_1(x, y^-(x))\right) dx \\ &= -\int_{x=a}^{b} \left[F_1(x, y)\right]_{y=y^-(x)}^{y^+(x)} dx \\ &= -\int \left\{ \int_{y=y^-(x)}^{y^+(x)} \frac{\partial F_1}{\partial y} dy \right\} dx\end{aligned}$$

Hence

$$\oint_C F_1 dx = -\iint_S \frac{\partial F_1}{\partial y} dx dy$$

Similarly

$$\oint_C F_2 dy = \iint_S \frac{\partial F_2}{\partial x} dx dy$$

Hence

$$\oint_C F_1 dx + \oint_C F_2 dy = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dx dy \right) dx dy$$

as required.

8.3.6 The divergence theorem

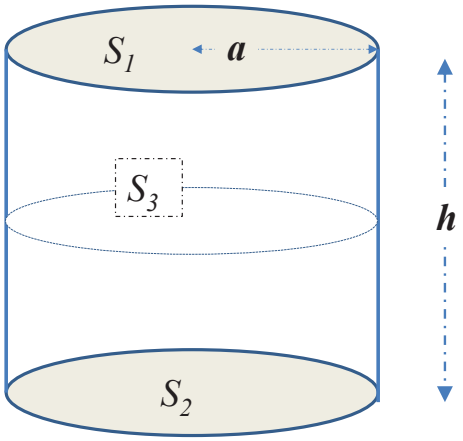
Stokes's theorem relates a flux integral of the curl of a vector field over a surface S to a line integral round the boundary ∂S. In this section we consider an integral theorem which relates a volume integral of the divergence of a vector field over a region V to a flux integral over the boundary surface ∂V. This theorem is called Gauss's theorem or the divergence theorem.

Theorem (Gauss's theorem or the divergence theorem)

Let V be a bounded solid region in $\mathbb{R}^3$ with boundary surface ∂V which consists of a finite number of piecewise smooth, closed orientable surfaces (with the orientation chosen so that the normal point out of the surface). Let $\mathbf{F}$ be a C^1 vector field then

$$\iiint_V (\boldsymbol{\nabla} \cdot \mathbf{F}) dv = \oiint_{\partial V} \mathbf{F} \cdot d\boldsymbol{S}$$

where the $\oiint$ symbol is used to indicate that the integral is over a closed surface.



Example 1: Let $\mathbf{F}$ be the vector field $\mathbf{F} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Let S be the closed surface of the cylinder radius a and height h (with the centre of the base at the origin). Calculate the flux integral

$$F = \oiint_S \mathbf{F}\cdot d\boldsymbol{S}$$

The *hard* way is to calculate the flux integral directly. To do this we observe that S consists of three surfaces: the top S_1, the bottom S_2 and the curved surface S_3 so that

$$\oiint_S \mathbf{F}\cdot d\boldsymbol{S} = \iint_{S_1} \mathbf{F}\cdot d\boldsymbol{S} + \iint_{S_2} \mathbf{F}\cdot d\boldsymbol{S} + \iint_{S_3} \mathbf{F}\cdot d\boldsymbol{S}$$

On S_1 we have $d\boldsymbol{S} = \boldsymbol{k}dxdy$ so that $\mathbf{F}\cdot d\boldsymbol{S} = zdxdy$

$$\iint_{S_1} \mathbf{F}\cdot d\boldsymbol{S} = \iint_{S_1} hdxdy = \pi a^2 h$$

On S_2 we have $d\boldsymbol{S} = -\boldsymbol{k}dxdy$ so that $\mathbf{F}\cdot d\boldsymbol{S} = -zdxdy$

$$\iint_{S_2} \mathbf{F}\cdot d\boldsymbol{S} = \iint_{S_2} 0dxdy = 0$$

On S_3 we have $d\boldsymbol{S} = (x\boldsymbol{i} + y\boldsymbol{j})(x^2 + y^2)^{-1/2}$

$$\iint_{S_3} \mathbf{F}\cdot d\boldsymbol{S} = \iint_{S_3} (x^2 + y^2)^{1/2} dA == \iint_{S_3} adA = a.\text{Area of } S_3 = 2\pi a^2 h$$

Hence

$$F = \iint_S \mathbf{F}\cdot d\boldsymbol{S} = \pi a^2 h + 0 + 2\pi a^2 h = 3\pi a^2 h$$

The *easy* way is to use the divergence theorem

$$F = \iint_S \mathbf{F}\cdot d\boldsymbol{S} = \iiint_V (\boldsymbol{\nabla}\cdot\mathbf{F})dv$$

Now

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= 1 + 1 + 1 = 3\end{aligned}$$

Hence

$$F = \iint_S \mathbf{F}\cdot d\boldsymbol{S} = \iiint_V (\boldsymbol{\nabla}\cdot\mathbf{F})dv = \iiint_V 3dv = 3.\text{volume of } V = 3\pi a^2 h$$

In fact this is a general result

Proposition

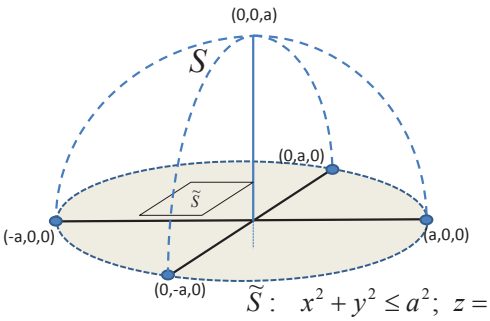
Let S be a closed surface then

$$\iint_S \boldsymbol{r}\cdot d\boldsymbol{S} = 3 \times \text{volume of region enclosed by } S$$

Proof: $\boldsymbol{\nabla}\cdot\boldsymbol{r} = 3$ so bt the divergence theorem

$$\iint_S \boldsymbol{r}\cdot d\boldsymbol{S} = \iiint_V 3dv = 3 \times \text{volume of region enclosed by } S$$

Example 2: Use the divergence theorem to calculate $\iint_S \mathbf{F}\cdot d\boldsymbol{S}$ where $\mathbf{F} = 2xy^2\boldsymbol{i} + z^3\boldsymbol{j} - x^2y\boldsymbol{k}$ and S is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.



$\tilde{S}$: $x^2 + y^2 \leq a^2;\ z = 0$

S: $x^2 + y^2 + z^2 = a^2$

Since S is not a closed surface we cannot immediately apply the divergence theorem. However if we add on the disc $\tilde{S}$ given by $x^2+y^2 \leq a^2$, $z=0$ (see diagram), then $S \cup \tilde{S}$ **is** a closed surface. Hence

$$\iint_S \mathbf{F}\cdot d\boldsymbol{S} + \iint_{\tilde{S}} \mathbf{F}\cdot d\boldsymbol{S} = \iiint_V (\boldsymbol{\nabla}\cdot\mathbf{F})dv \qquad (*)$$

where V is the region enclosed by S and $\tilde{S}$.

We start by calculating the integral on the RHS.

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2y^2 + 0 + 0 = 2y^2$$

Hence

$$\iiint_V (\boldsymbol{\nabla}\cdot\mathbf{F})dv = \iiint_V 2y^2 dydxdz$$

If we do the integral with respect to x and z first we integrate over a semicircular slice of radius $(a^2-y^2)^{1/2}$, which has area $\frac{1}{2}\pi(a^2-y^2)$

$$\begin{aligned}\iiint_V 2y^2 dydxdz &= \int_{-a}^{a} 2y^2 \frac{1}{2}\pi(a^2-y^2)dy \\ &= \pi\int_{-a}^{a}(a^2y^2 - y^4)dy \\ &= \frac{4\pi}{15}a^5\end{aligned}$$

Now

$$\iint_{\tilde{S}} \mathbf{F}\cdot d\boldsymbol{S} = \iint_{\tilde{S}} \mathbf{F}\cdot\boldsymbol{k}dA = \iint_{\tilde{S}} x^2 y dxdy = 0 \quad \text{(by symmetry)}$$

Hence substituting into (*) gives

$$\iint_S \mathbf{F}\cdot d\boldsymbol{S} + \iint_{\tilde{S}} \mathbf{F}\cdot d\boldsymbol{S} = \iiint_V (\boldsymbol{\nabla}\cdot\mathbf{F})dv$$

$$\iint_S \mathbf{F}\cdot d\boldsymbol{S} + 0 = \frac{4\pi}{15}a^5$$

Hence

$$\iint_S \mathbf{F}\cdot d\boldsymbol{S} = \frac{4\pi}{15}a^5$$

8.3.7 Vector potential

For a C^2 scalar field we know that $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0$ so that if $\mathbf{F} = \boldsymbol{\nabla}\phi$ then $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}$. However we have also seen that in a simply connected region the converse is true and if $\boldsymbol{\nabla}\times\mathbf{F} = 0$ there exists a scalar field ϕ such that $\mathbf{F} = \boldsymbol{\nabla}\phi$.

In this section we examine the consequences of the identity

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0$$

It follows from this that

$$\boldsymbol{G} = \boldsymbol{\nabla}\times\mathbf{F} \quad \Rightarrow \quad \boldsymbol{\nabla}\cdot\boldsymbol{G} = 0$$

However in a simply connected region one can also show (although we do not give the proof in this course) that

$$\boldsymbol{\nabla}\cdot\boldsymbol{G} = 0 \quad \Rightarrow \text{there exists a vector field } \mathbf{F} \text{ such that } \boldsymbol{G} = \boldsymbol{\nabla}\times\mathbf{F}$$

We call such a vector field $\mathbf{F}$ a **vector potential** for $\boldsymbol{G}$.

Note that if $\boldsymbol{G}$ is divergence free then it follows from the divergence theorem that

$$\oiint_S \boldsymbol{G}\cdot d\boldsymbol{S} = \iiint_V (\boldsymbol{\nabla}\cdot\boldsymbol{G})dv = \iiint_V 0dv = 0$$

Finally the above result shows that if $\boldsymbol{G}$ is divergence free then

$$\iint_{S_1} \boldsymbol{G}\cdot d\boldsymbol{S} = \iint_{S_2} \boldsymbol{G}\cdot d\boldsymbol{S}$$

where S_1 and S_2 are two open surfaces with the same boundary $C = \partial S_i$. Note, that we have already come across this result when we considered Stokes's theorem in section 4.1.