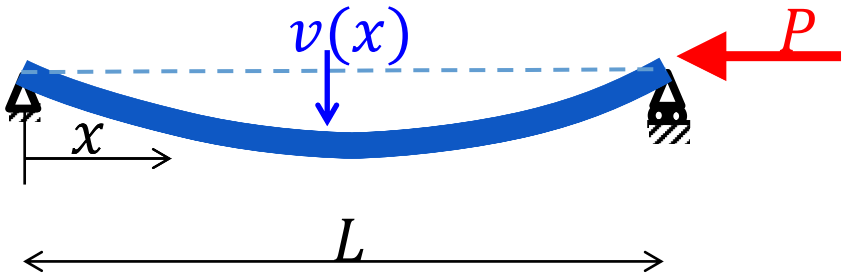
FEEG 2005
Structures: Lecture 5

Buckling 2: Imperfect columns with eccentric loading, initial curvature and lateral loading

Last lecture

- We reviewed Euler buckling theory from Statics: where 'K' depends on the boundary conditions.
  $$\boxed{P_{crit} = \frac{\pi^2 EI}{(KL)^2}}$$
  - Pinned-pinned: $K = 1$
  - Fixed-free: $K = 2$
  - Fixed-fixed: $K = 0.5$
  - Fixed-pinned: $K = 0.7$

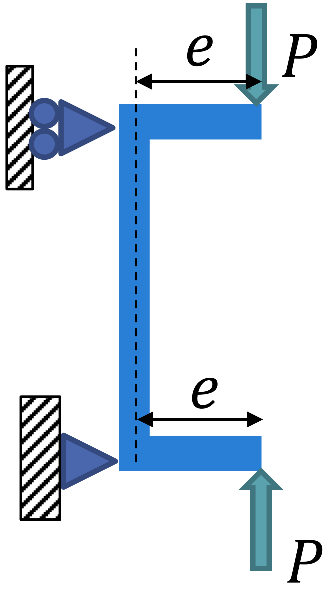


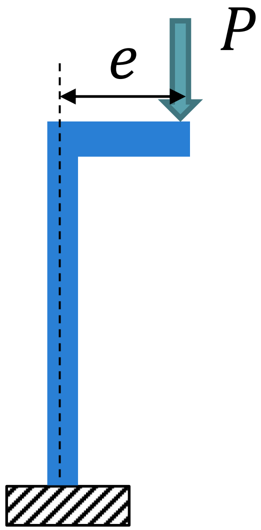
- Assumptions:
  - The column is (initially) perfectly straight with a uniform section
  - Homogenous, isotropic, linear elastic material behaviour
  - The compressive load is applied through the centroid

This lecture

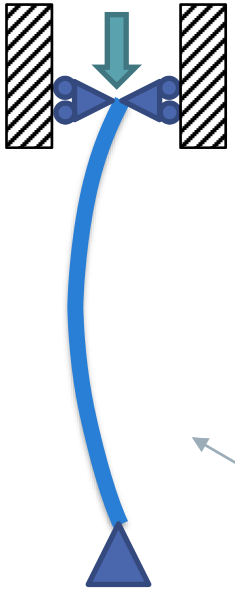
- What happens if…

The column is eccentrically loaded.

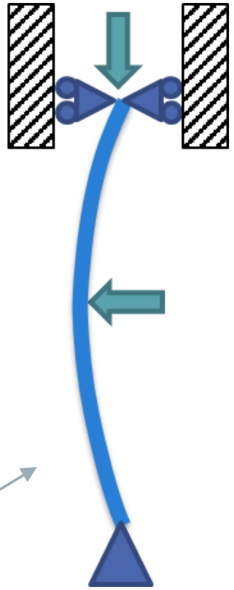




The column is NOT initially straight.



There is a lateral load on the column



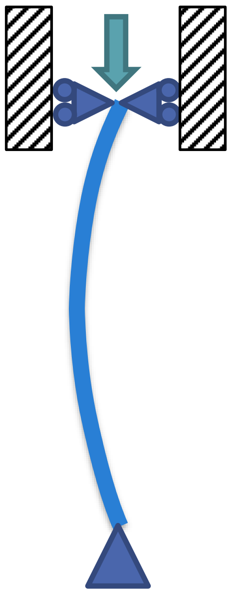
Watch the recorded video and read the slides!

Question

- Which statement(s) is/are correct for a column with initial small curvature under compressive load? (P_c=Euler's critical buckling load for a similar perfect column).

true

1. The maximum load it can take is close to P_c. (usually) true
2. The beam can stay in equilibrium and stable for loads below P_c.
3. For a quasi-static loading, the beam never reaches a bifurcation point so never becomes unstable.



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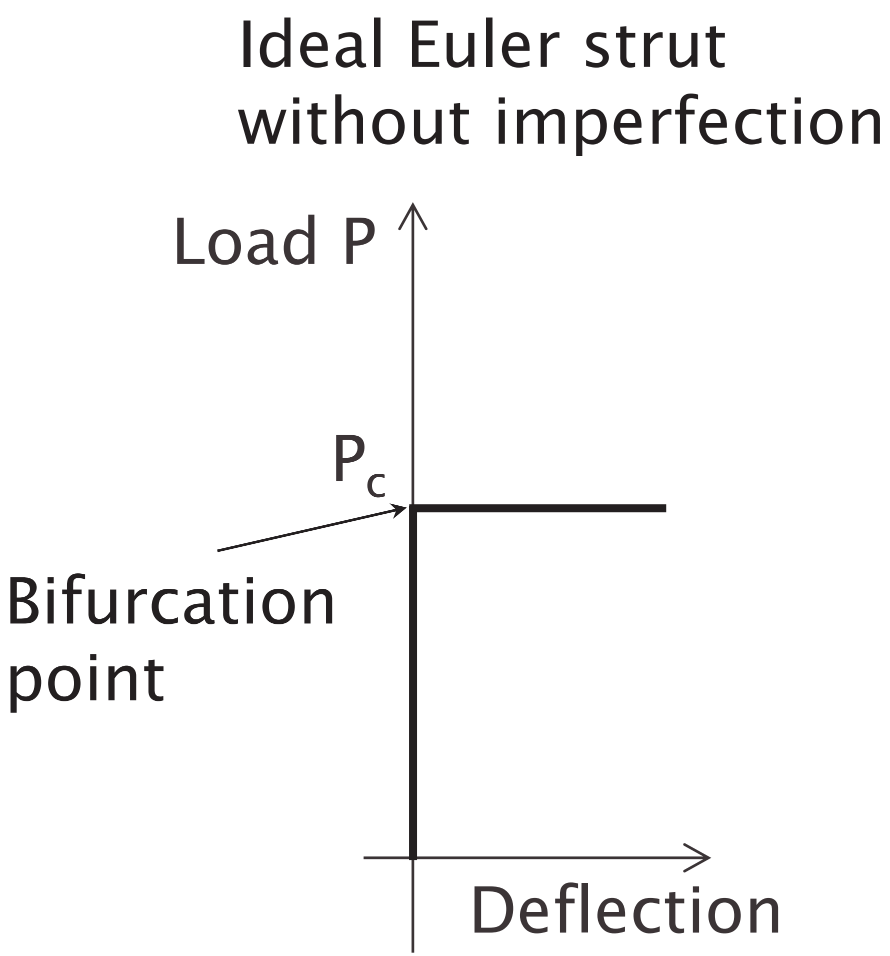
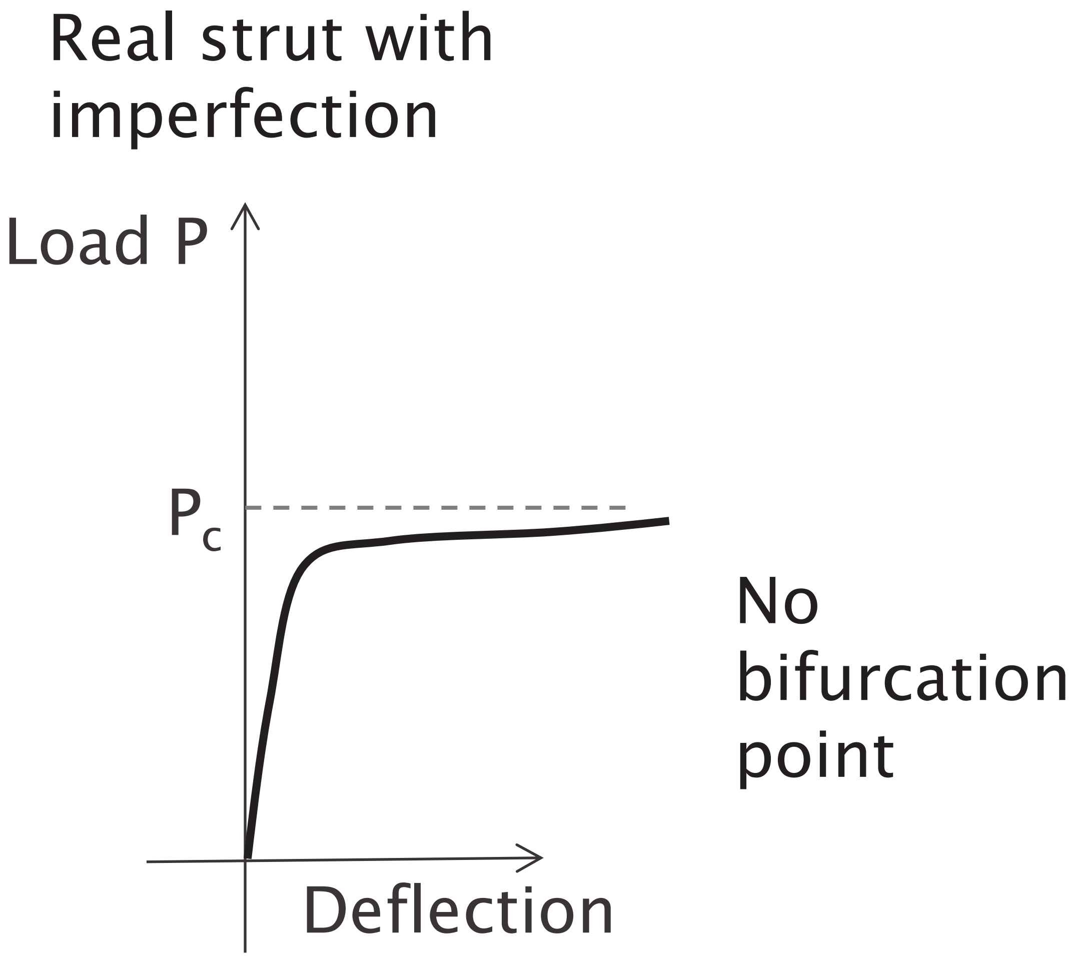
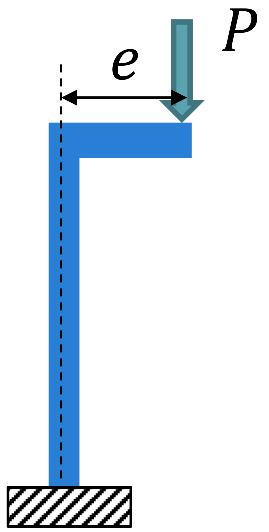
Real column with imperfection

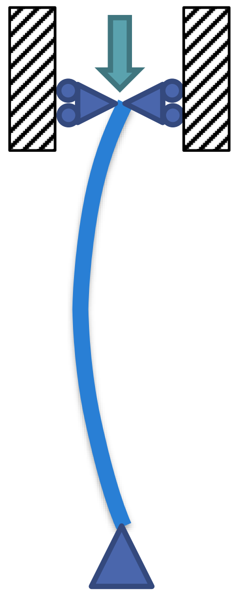




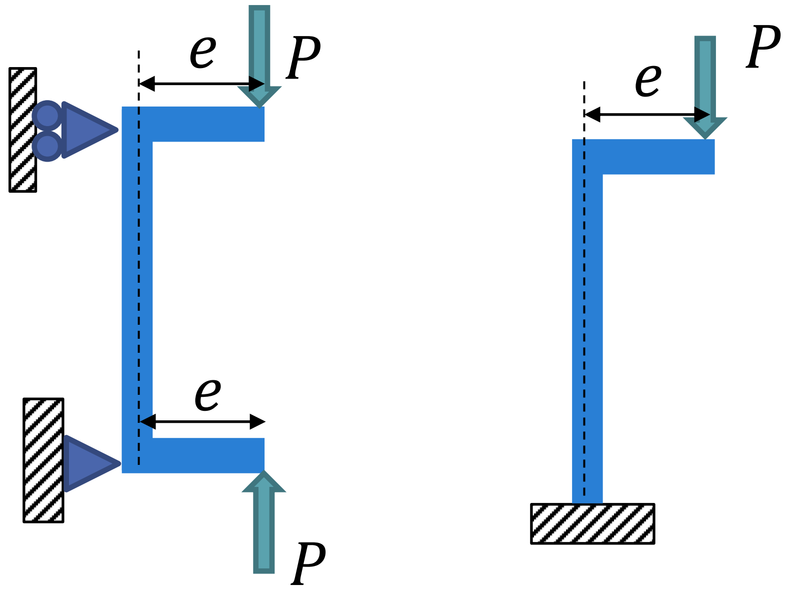
How do we solve buckling problems?



1. Draw the buckled shape, take a cut and apply equilibrium to get the moment $M(x)$
2. Sub the moment into the bending deflection '$v(x)$' differential eq (DE): $-EI\dfrac{d^2v(x)}{dx^2} = M(x)$
3. Solve the second order DE to obtain the deflection solution '$v(x)$'
4. Use the boundary conditions to get unknown constants!



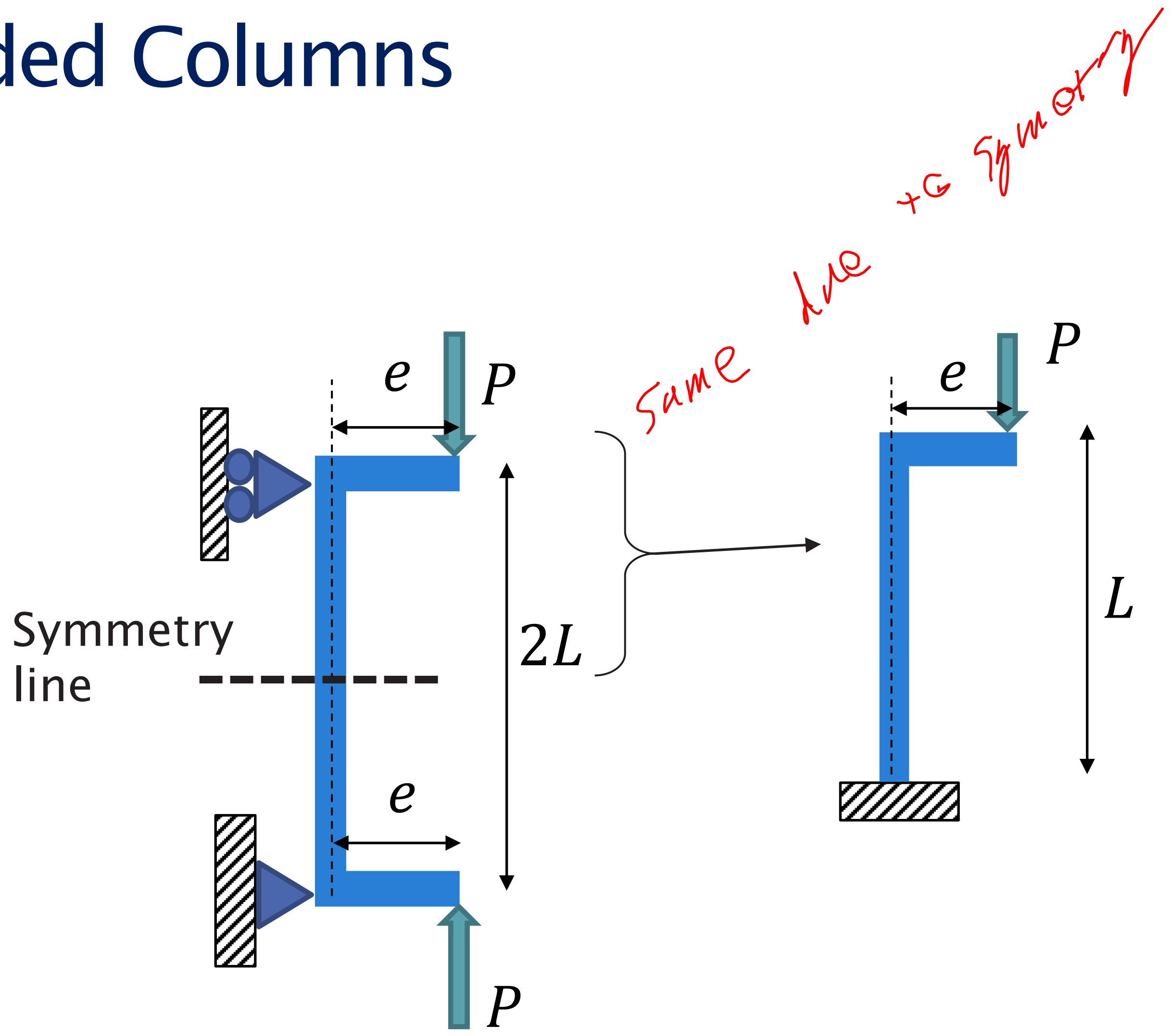
Eccentrically loaded column



Buckling of Eccentrically Loaded Columns

- When the load is not applied through the centroid of the column it is loaded eccentrically.

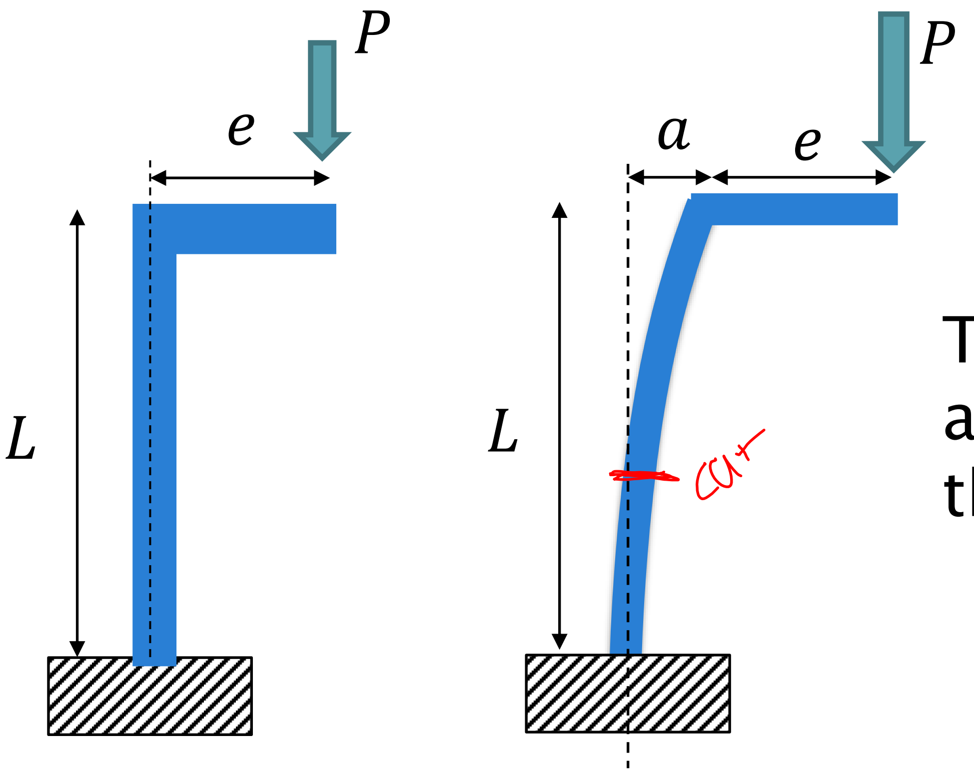
- A pinned-pinned beam with length of 2L is similar to a fixed-pinned end beam with length L.



Bending moment in the beam

V = deformation

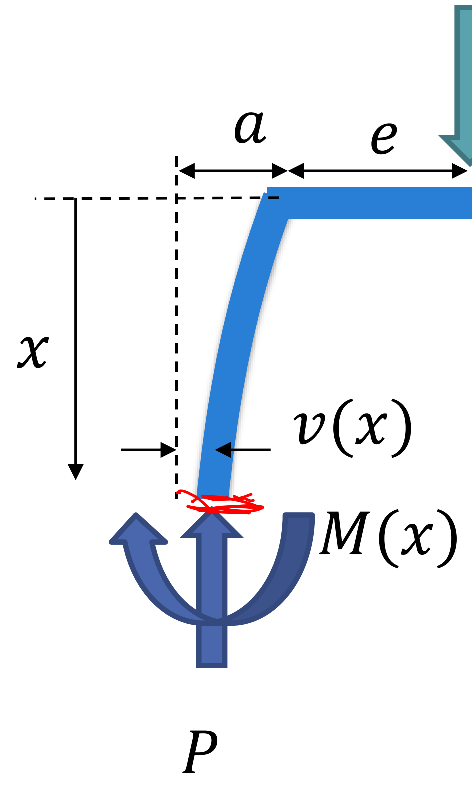
Undeformed Deformed



Take a cut and draw the FBD

Moments at the cut:

$$M(x) = -P(e + a - v(x))$$



Finding the differential equation

2) The deflection equation for the column is:

$$EI\frac{d^2v(x)}{dx^2} = -M(x)$$

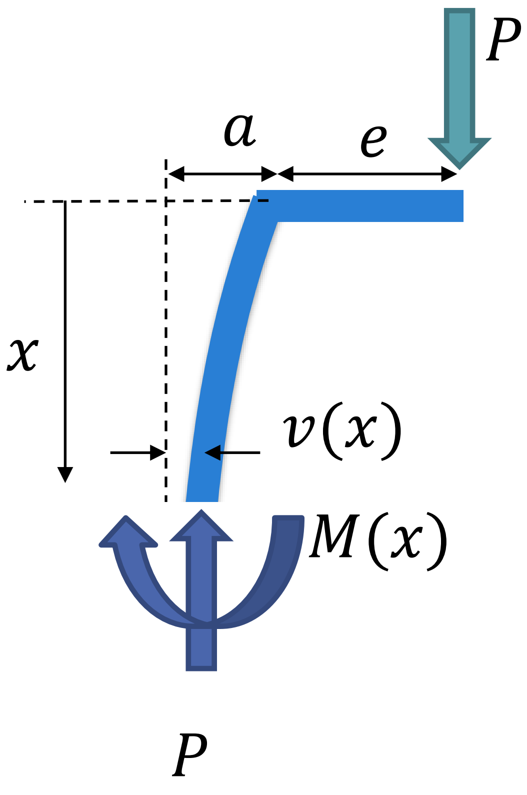
from prev slide

3) Combine these into a single differential eq (DE):

$$EI\frac{d^2v(x)}{dx^2} = P(e + a - v(x))$$

rearrange and create variable μ for simplification

$$\frac{d^2v(x)}{dx^2} + \mu^2 v(x) = \mu^2(e + a) \text{ whereas } \mu^2 = \frac{P}{EI}$$



Solution to the differential equatoin

- $\frac{d^2v(x)}{dx^2} + \mu^2 v(x) = \mu^2(e + a)$
- A complementary solution to the homogeneous DE:

  $v_c(x) = A sin(\mu x) + B cos(\mu x)$

  A DE when the RH side is zero
- The particular integral is: $v_p(x) = e + a$

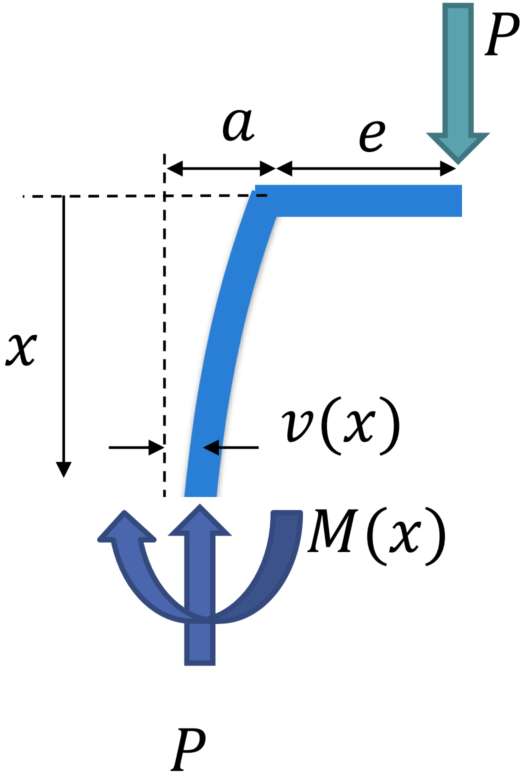
  A solution for the RH side of the DE

Simple calculus stuff

- The solution is a linear combination of these two:

  $v(x) = A sin(\mu x) + B cos(\mu x) + e + a$

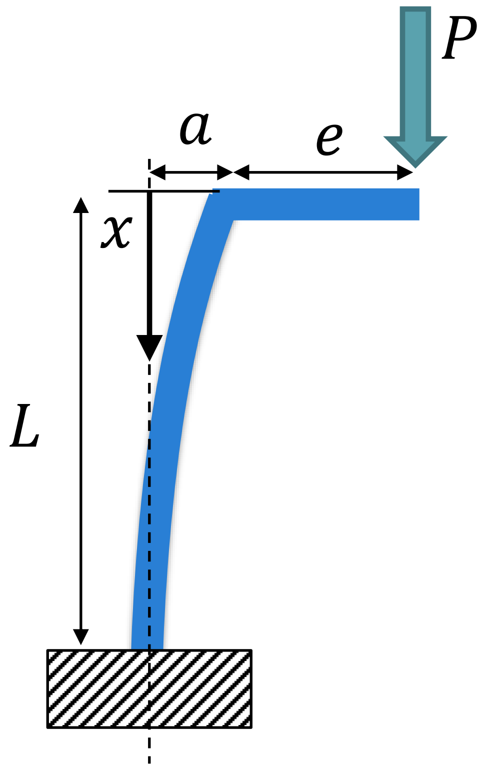
Check this by differentiating and subbing into the DE



Applying the Boundary Conditions

V

- The solution to the buckling DE is:
- $v(x) = A sin(\mu x) + B cos(\mu x) + e + a$
- Obtain A, B and a using the boundary conditions (BCs):
  1. $v(0) = a$ max deformation occurs at top
  2. $v'(L) = 0$
  3. $v(L) = 0$



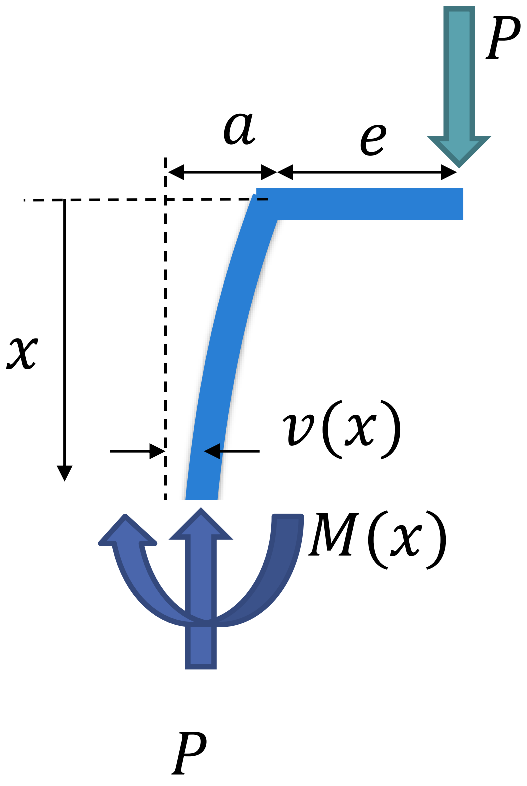
1. $v(0) = a$ gives $B = -e$
2. $v'(L) = 0$ and $v'(x) = \mu A cos(\mu x) - \mu B sin(\mu x)$ therefore $v'(L) = \mu A\, cos(\mu L) + \mu e\, sin(\mu L) = 0$ and finally: $A = -e\, tan(\mu L)$

Applying the third boundary condition

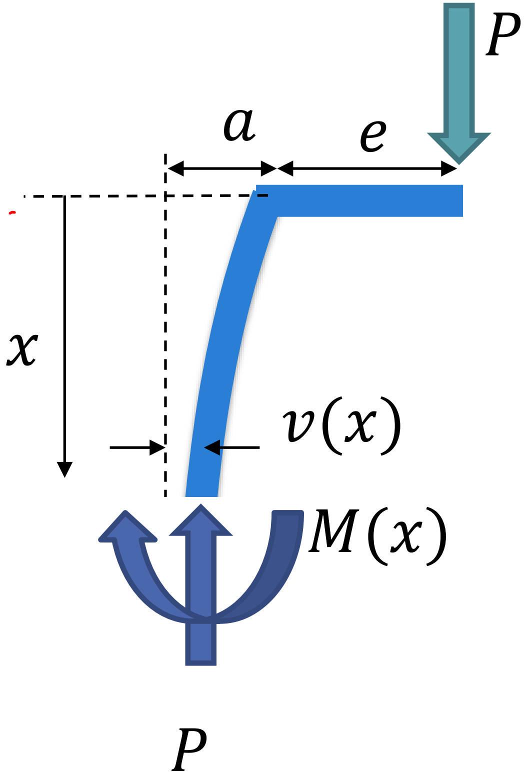
3. Obtain the final constant with the third BC: $v(L) = 0$

- We already have $A = -e\,tan(\mu L)$ and $B = -e$:
- $v(x) = -e\,tan(\mu L)\,sin(\mu x) - e\,cos(\mu x) + e + a$
- $v(L) = -e\,tan(\mu L)\,sin(\mu L) - e\,cos(\mu L) + e + a = 0$
- Therefore: $a = e\,[tan(\mu L)\,sin(\mu L) + cos(\mu L) - 1]$

- Simplify with trig identities: $a = e\,[\sec(\mu L) - 1]$



Final solution for the deformation $v(x)$



- Combining the result for all constants:
- $v(x) = -e\,tan(\mu L)\,sin(\mu x) - e\,cos(\mu x) + \cancel{e} + e\ \sec(\mu L) - \cancel{e}$
- $v(x) = e[-\,tan(\mu L)\,sin(\mu x) -\ cos(\mu x) + \sec(\mu L)]$
- $v(x) = e[-\,\frac{\sin(\mu L)}{\cos(\mu L)}\ sin(\mu x) -\ cos(\mu x)\frac{\cos(\mu L)}{\cos(\mu L)} + \frac{1}{\cos(\mu L)}]$
- $v(x) = \frac{e}{\cos(\mu L)}[-\sin(\mu L)\sin(\mu x)\ - \cos(\mu L)\,cos(\mu x) + 1]$
- $v(x) = e\sec(\mu L)\ [-(\sin(\mu L)\sin(\mu x) + \cos(\mu L)\,cos(\mu x)) + 1]$

Angle sum identity for cos

- Finally: $\boxed{v(x) = e\sec(\mu L)\ [1 - \cos(\mu L - \mu x)]}$

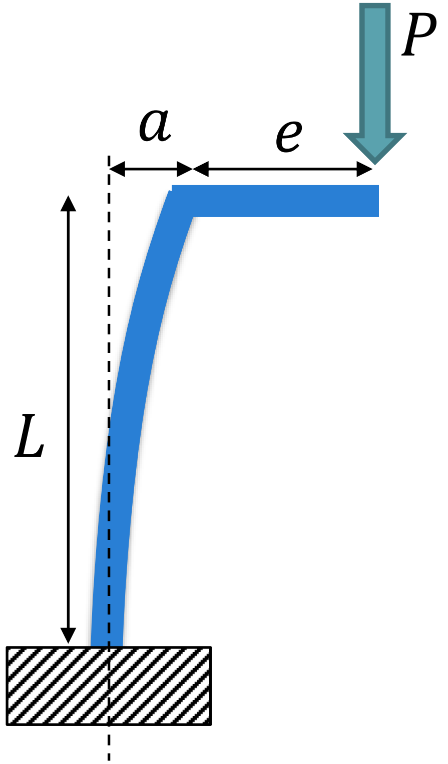
$\mu = \sqrt{P/EI}$

Maximum deflection of eccentrically loaded columns

- The max deflection 'v_{max}' occurs at the top of the column '$x = 0$' so it is equal to 'a':
- $v_{max} = a = e\,[\sec(\mu L) - 1]$

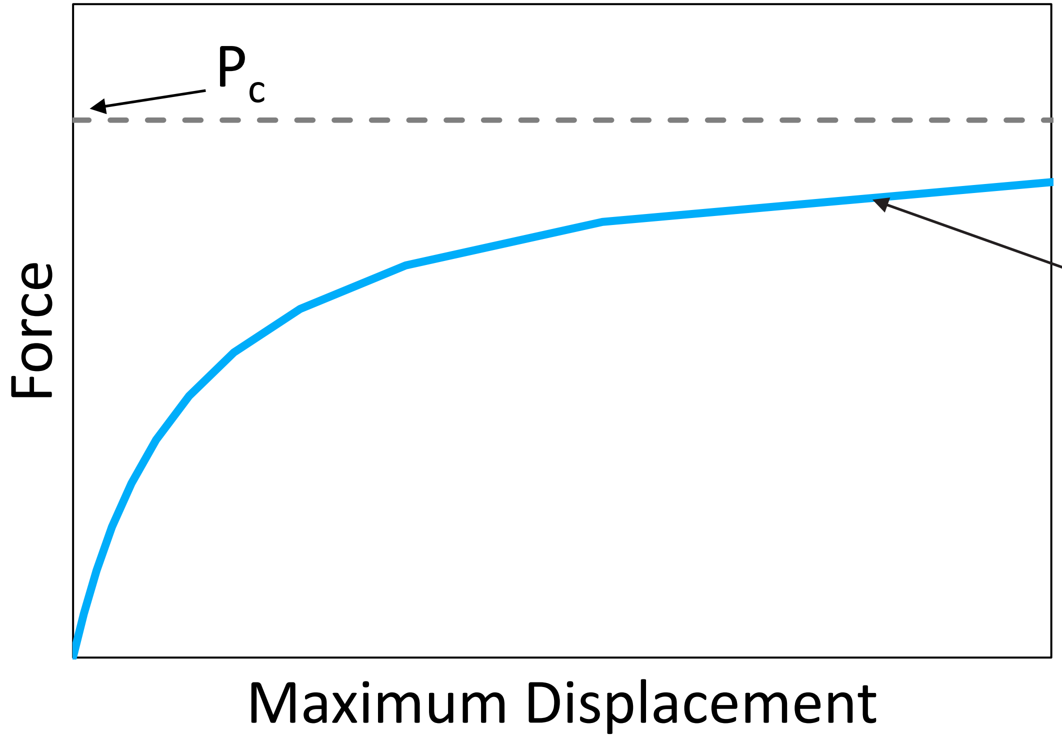
$$\boxed{v_{max} = e\,[\sec(\mu L) - 1]}$$

- NOTE: only applies if $e > 0$. If $e = 0$ use Euler's formula!



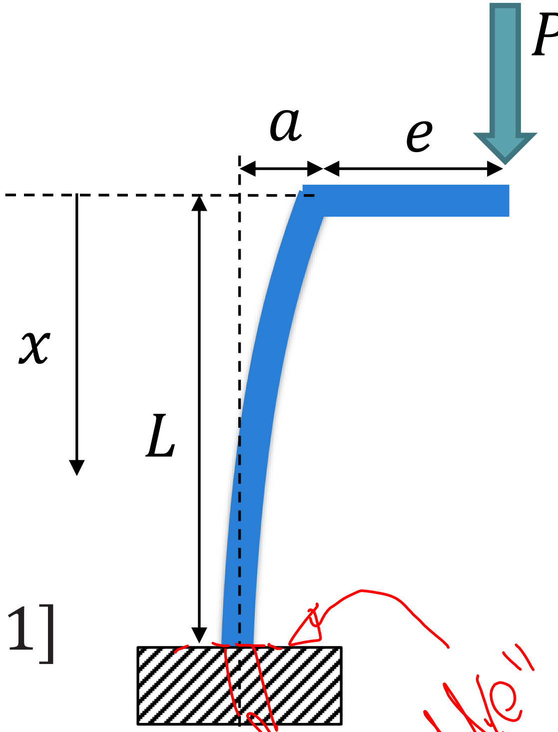
No bifurcation, no instability

- Unlike the perfect column, eccentrically loaded columns load-deflection doesn't have a bifurcation point



$$\boxed{v_{max} = e\,[\sec(\mu L) - 1]}$$

Stress in Eccentrically Loaded Columns



- Stress in the column comes from:

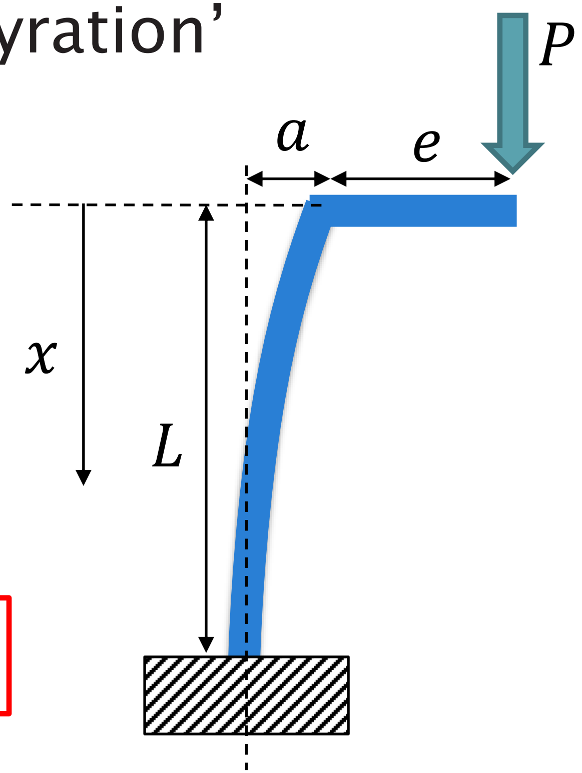
  Total stress = axial stress + bending stress

- Max comp. stress in the column at the fixed end, $x = L$ where bending moment is maximum.
- $M(x) = -P(e + a - v(x))$ with $v(L) = 0$ and $a = e\,[\sec(\mu L) - 1]$
- $M(L) = -P(e + e\ \sec(\mu L) - e) \rightarrow M(L) = -Pe\ \sec(\mu L)$
- Sub into the bending stress eq: $\sigma_b = \pm \frac{Pey}{I} \sec(\mu L)$
- So, the total comp. stress is: $\sigma_{tot\_comp} = -\frac{P}{A} - \frac{Pey}{I} \sec(\mu L)$

max load occurs at "middle" of beam

Max stress and deflection

- Max comp. stress in the column at the fixed end $x = L$:
- $\sigma_{max} = -\frac{P}{A} - \frac{Pey}{I}\sec(\mu L)$. Let define $\kappa^2 = \frac{I}{A}$ as 'radius of gyration'
- $\sigma_{max} = -\frac{P}{A}\left[1 + \frac{ey}{\kappa^2}\sec(\mu L)\right]$ with: $\mu^2 = \frac{P}{EI}$



- Summary:

1) Stress (compressive)

$$\boxed{\sigma_{max} = -\frac{P}{A}\left[1 + \frac{ey}{\kappa^2}\sec(\mu L)\right]}$$

2) Deflection

$$\boxed{v_{max} = e\,[\sec(\mu L) - 1]}$$

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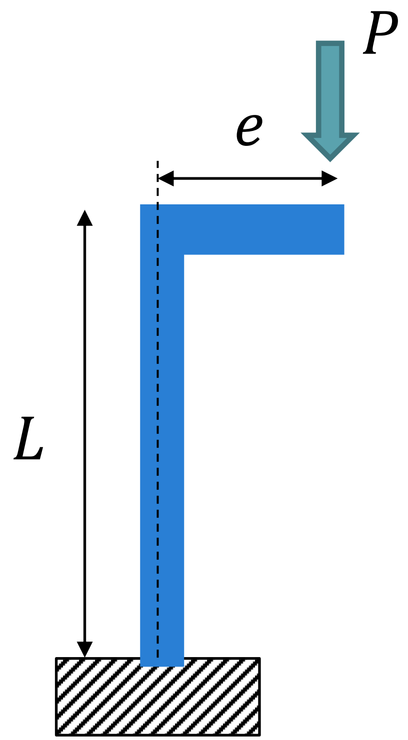
The Secant Formula vs Euler's critical load

- Max stress/deflection in the column :
- $\sigma_{max} = -\frac{P}{A}\left[1 + \frac{ey}{\kappa^2}\sec(\mu L)\right]$ with $\mu^2 = \frac{P}{EI}$ and $\kappa^2 = \frac{I}{A}$
- $v_{max} = e[sec(\mu L) - 1]$
- $\sec(\mu L) = \frac{1}{\cos\left(L\sqrt{\frac{P}{EI}}\right)}$. So, as $L\sqrt{\frac{P}{EI}} \to \frac{\pi}{2}$ then $\sec(\mu L) \to \infty$
- If $P \to P_c = \frac{\pi^2 EI}{4L^2}$ then $v_{max} \to \infty$ and $\sigma_{max} \to \infty$

suggests failure, consistent with analysis done for P_c stuff

Question

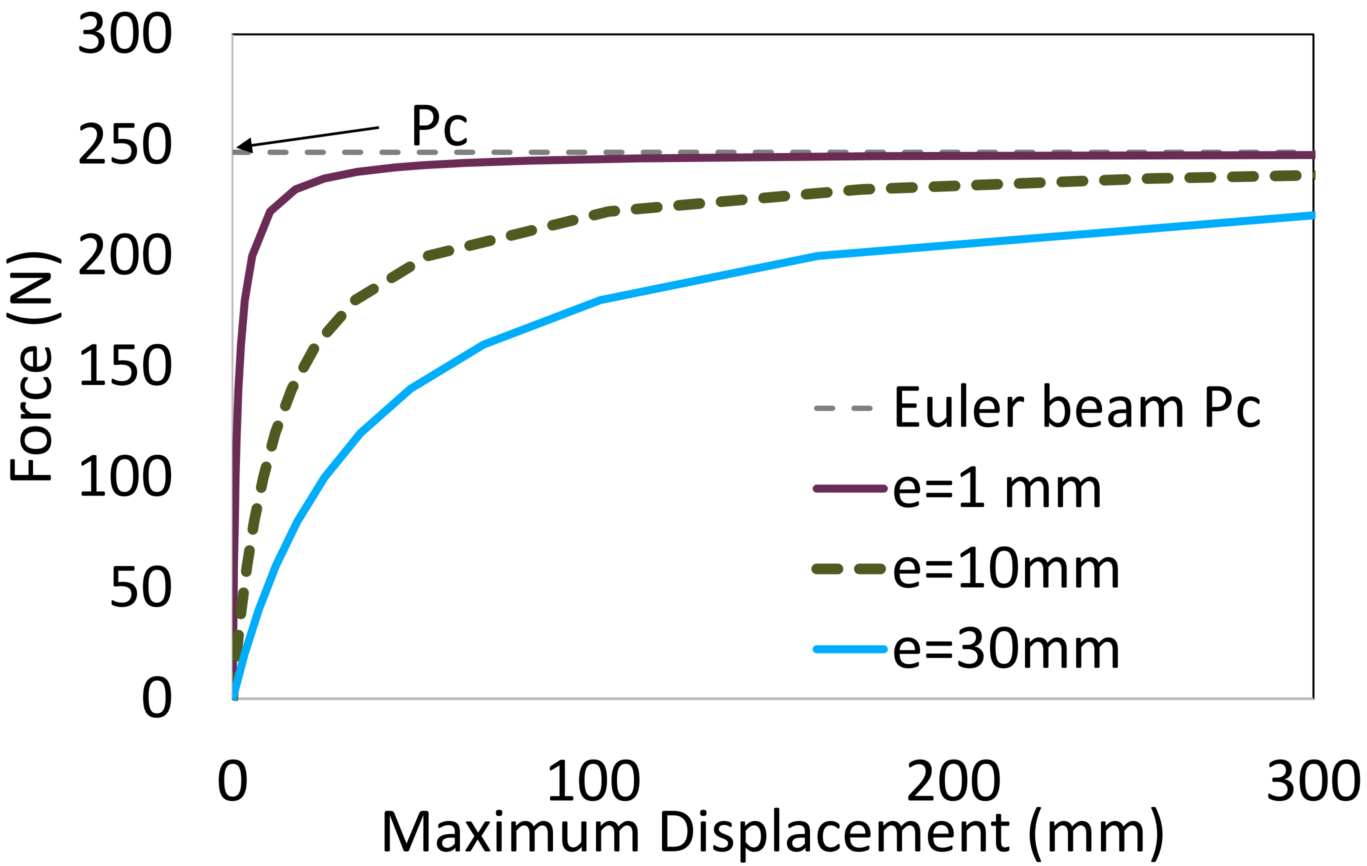
- What happens if the value of e, eccentricity, is very small?

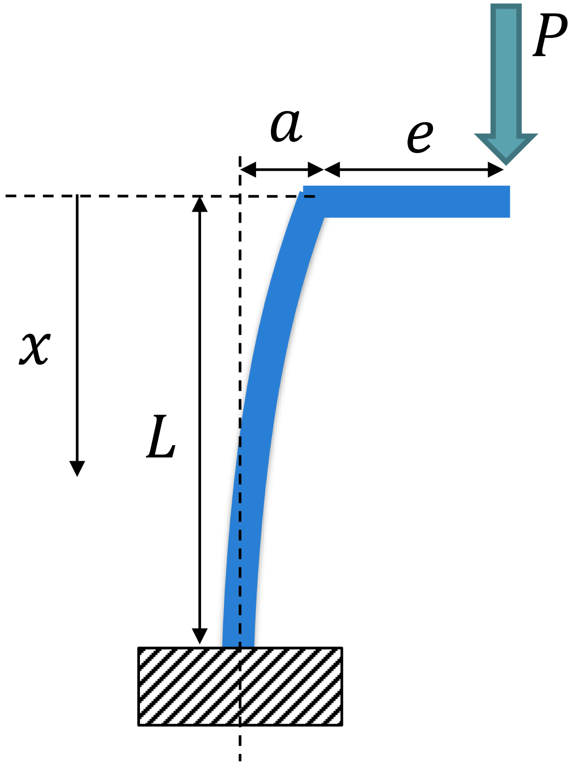


Please send your answers using the link below?

https://vevox.app/#/m/184613719

Max deflection versus applied load P

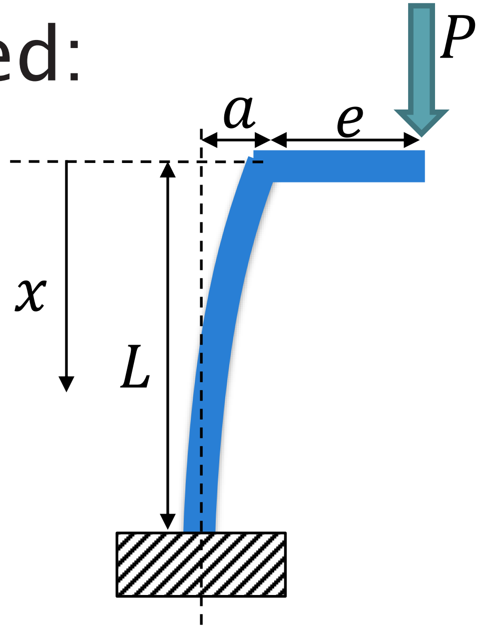




- Decreasing eccentricity (e) pushes the deflection graph towards perfect beam response.

Eccentric Loading: Fixed vs Pinned

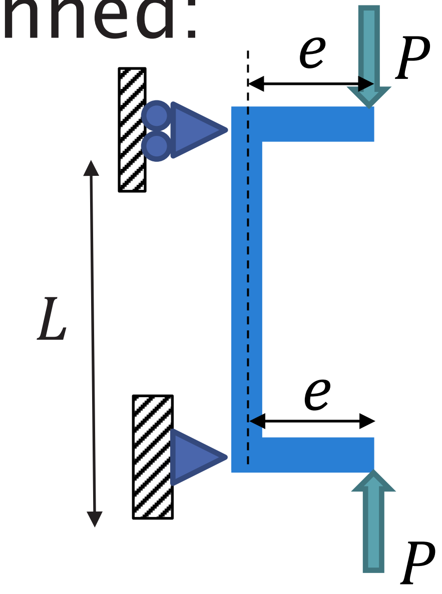
1) Fixed:



$$v_{max} = e\left[\sec(\mu L) - 1\right]$$

$$\sigma_{max} = -\frac{P}{A}\left[1 + \frac{ey}{\kappa^2}\sec(\mu L)\right]$$

2) Pinned-pinned:



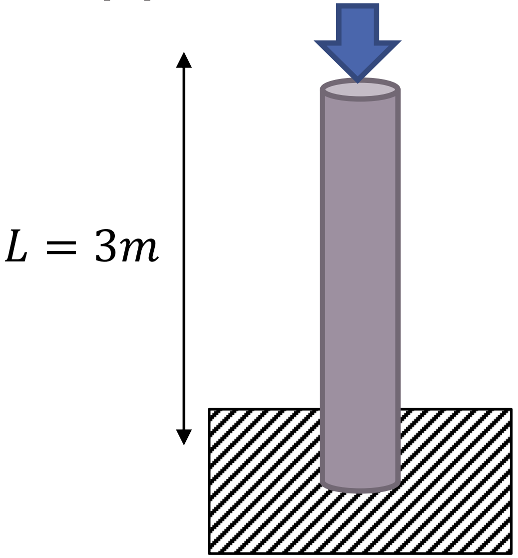
$$v_{max} = e\left[\sec(\mu L/2) - 1\right]$$

$$\sigma_{max} = -\frac{P}{A}\left[1 + \frac{ey}{\kappa^2}\sec(\mu L/2)\right]$$

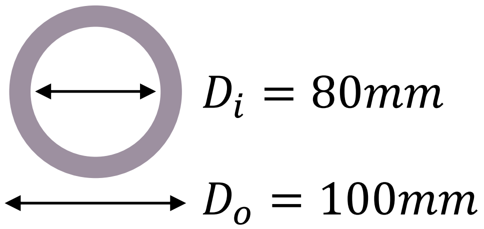
Replaced L by L/2 in the equation!

Next: Example – Have a go first!

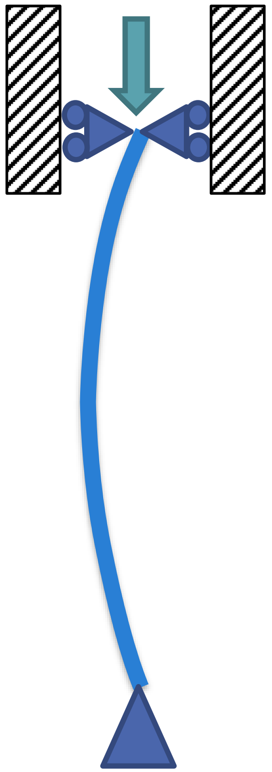
- Consider the hollow circular column shown in the figure. The column is made of steel ($E = 200\ GPa\ , \sigma_{ys} = 250\ MPa$). Find the compressive load that will cause failure if:
- The load is applied through the centroid
- The load is applied 10 mm away from the centroid



Colum cross section:

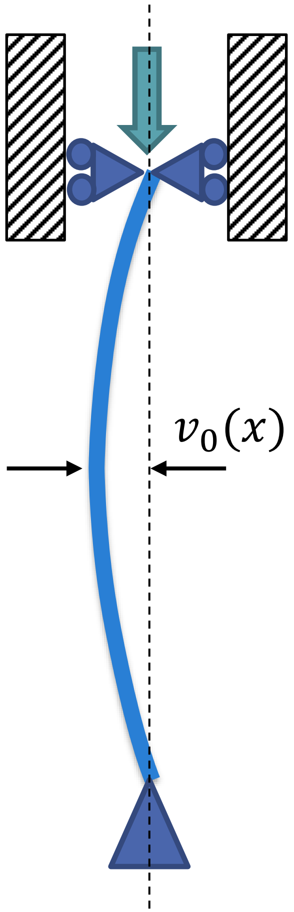


Initially curved columns



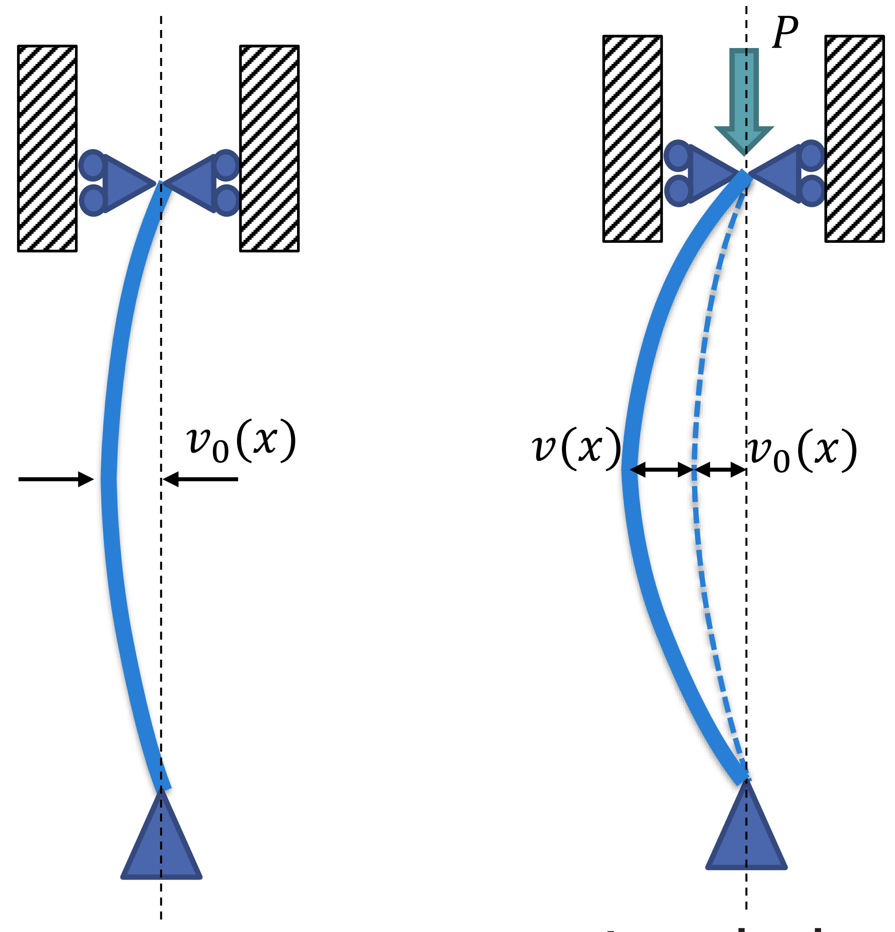
How to find the load-deflection equation

1. Draw the buckled shape, take a cut and apply equilibrium to get the moment $M(x)$
2. Sub the moment into the bending deflection '$v(x)$' DE:
   $$-EI\frac{d^2v(x)}{dx^2} = M(x)$$
3. Solve the second order DE to obtain the deflection solution '$v(x)$'
4. Using the boundary conditions to get unknown constants!



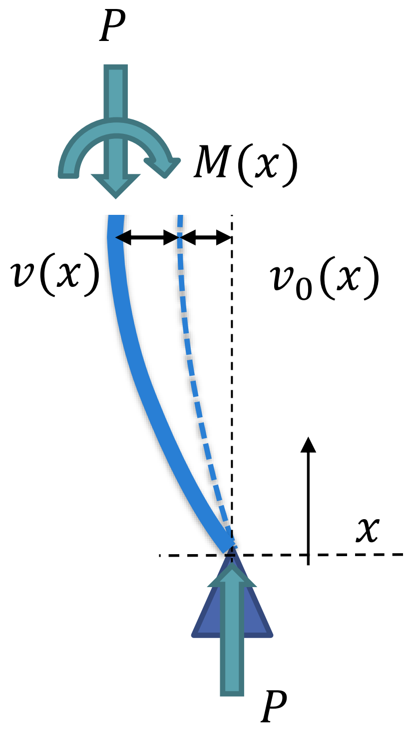
Free body diagram of a cut section

Draw the buckled shape



Take a cut and apply equilibrium:

$$M(x) = P(v(x) + v_0(x))$$



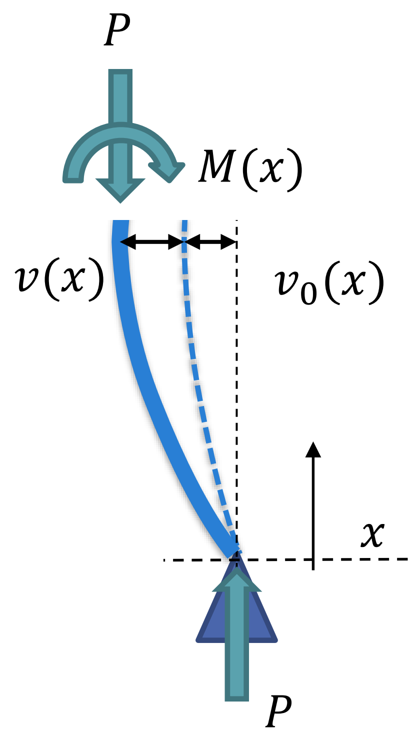
Finding the differential equation

Sub the moment eq into the deflection DE:

$$-EI\frac{d^2v(x)}{dx^2} = M(x) \qquad \text{with: } M(x) = P(v(x) + v_0(x))$$

$$\frac{d^2v(x)}{dx^2} + \frac{P}{EI}v(x) = -\frac{P}{EI}v_0(x) \qquad \text{let: } \mu^2 = \frac{P}{EI}$$

$$\frac{d^2v(x)}{dx^2} + \mu^2 v(x) = -\mu^2 v_0(x)$$



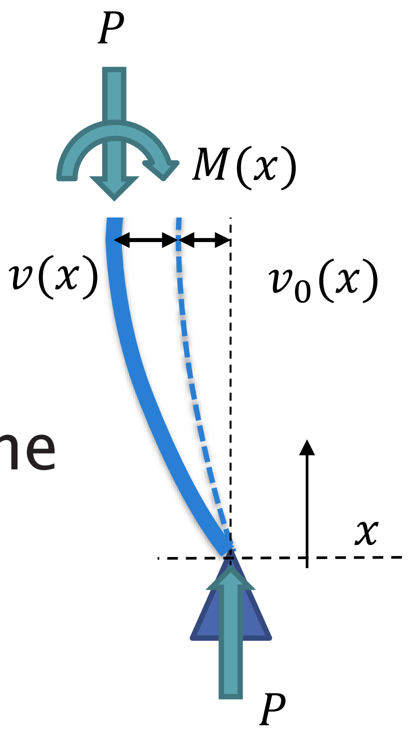
Solutions to the DE

- $\frac{d^2v(x)}{dx^2} + \mu^2 v(x) = -\mu^2 v_0(x)$ with $\mu^2 = \frac{P}{EI}$
- A complementary solution to the homogeneous DE:
- $v_c(x) = Asin(\mu x) + Bcos(\mu x)$

A solution when the RH side is zero

- Assume the initial deflection is: $v_0(x) = a\ sin\left(\frac{\pi x}{L}\right)$ Where a is the 'amplitude' of the initial deflection (constant)
- So, the particular integral is: $Csin\left(\frac{\pi x}{L}\right)$

A solution to the RH side of the DE



Solving the DE – finding C

- The solution is a linear combination of these:

$$v(x) = A sin(\mu x) + B cos(\mu x) + C sin\left(\frac{\pi x}{L}\right)$$

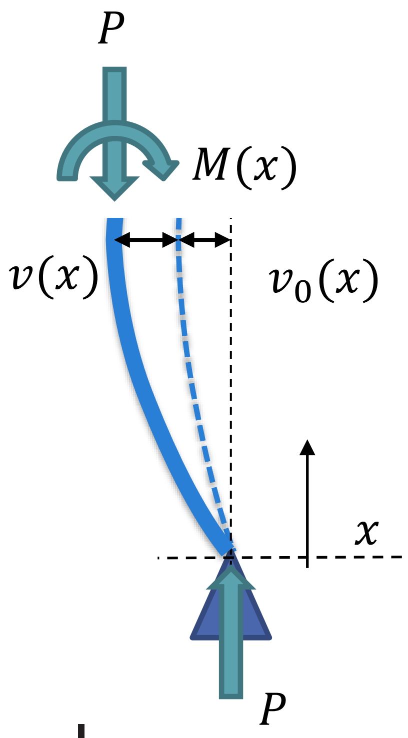
- Differentiate this twice:

$$v'(x) = \mu A\, cos(\mu x) - \mu B\, sin(\mu x) + C\frac{\pi}{L} cos\left(\frac{\pi x}{L}\right)$$

$$v''(x) = -\mu^2 A\, sin(\mu x) - \mu^2 B\, cos(\mu x) - C\frac{\pi^2}{L^2} sin\left(\frac{\pi x}{L}\right)$$

- Substitute this into the original DE, the A and B terms cancel so:

$$-C\frac{\pi^2}{L^2} sin\left(\frac{\pi x}{L}\right) + \mu^2\left(C sin\left(\frac{\pi x}{L}\right)\right) = -\mu^2 v_0(x) \text{ where: } v_0(x) = a\, sin\left(\frac{\pi x}{L}\right)$$

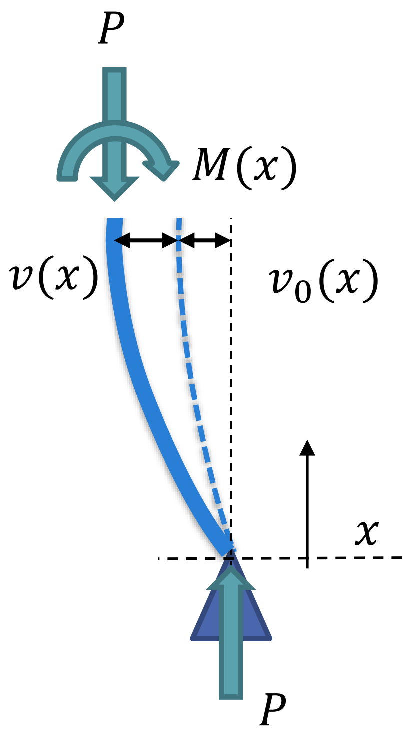


Solving the DE - finding C

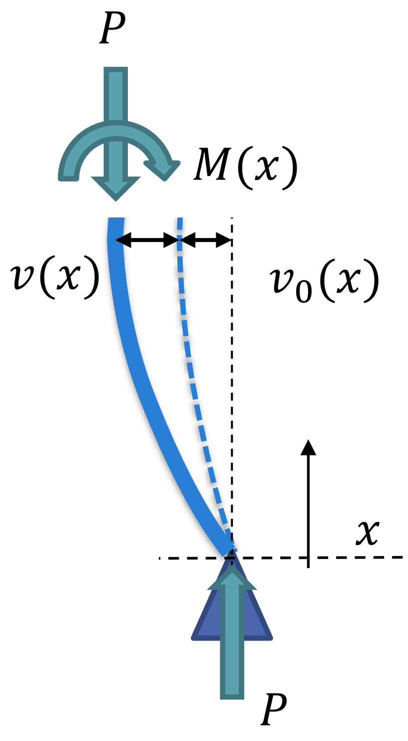
- $-C\frac{\pi^2}{L^2} sin\left(\frac{\pi x}{L}\right) + \mu^2 C sin\left(\frac{\pi x}{L}\right) = -\mu^2 a\ sin\left(\frac{\pi x}{L}\right)$
- Cancel $\sin\frac{\pi x}{L}$ and multiple both side by -1 :

$$C\frac{\pi^2}{L^2} - \mu^2 C = \mu^2 a$$

- Finally: $C = \frac{\mu^2 a}{\left(\frac{\pi^2}{L^2} - \mu^2\right)}$ Where a is the ‘amplitude’ of the initial deflection (constant) and $\mu^2 = \frac{P}{EI}$



Applying BCs - Finding A and B



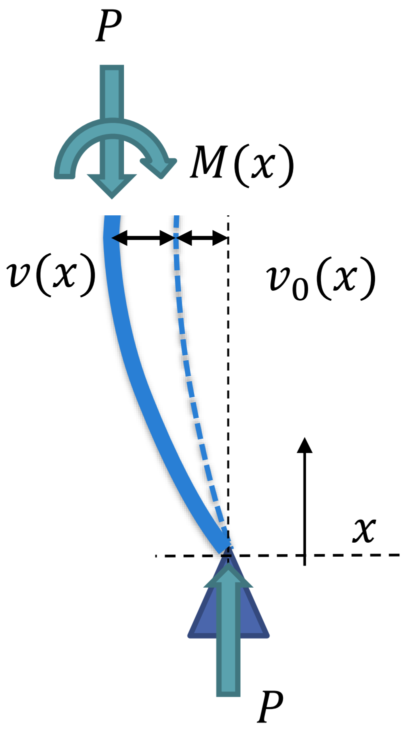
- Now we can get A and B from the boundary conditions.
- The column is pinned at both ends: $v(0) = 0$ and $v(L) = 0$
- Recall: $v(x) = Asin(\mu x) + Bcos(\mu x) + Csin\left(\frac{\pi x}{L}\right)$ and $C = \frac{\mu^2 a}{\left(\frac{\pi^2}{L^2} - \mu^2\right)}$
- $v(0) = 0$ gives $B = 0$
- $v(L) = 0$ gives $Asin(\mu L) = 0$ so $A = 0$
- $v(x) = \frac{\mu^2}{\left(\frac{\pi^2}{L^2} - \mu^2\right)} \underbrace{a\, sin\left(\frac{\pi x}{L}\right)}_{v_0(x)}$

$$\frac{\mu^2}{\left(\frac{\pi^2}{L^2} - \mu^2\right)} \times \frac{\frac{1}{\mu^2}}{\frac{1}{\mu^2}} = \frac{1}{\left(\frac{\pi^2}{L^2\mu^2} - 1\right)}$$

Deflection function $v(x)$

$$v(x) = \frac{1}{\left(\frac{\pi^2}{L^2\mu^2} - 1\right)} a \sin\left(\frac{\pi x}{L}\right) \quad \text{or} \quad v(x) = \frac{v_0(x)}{\left(\frac{\pi^2}{L^2\mu^2} - 1\right)}$$

with: $\mu^2 = \dfrac{P}{EI}$ so: $$v(x) = \frac{v_0(x)}{\left(\underbrace{\frac{\pi^2 EI}{L^2}}_{P_{crit}} \frac{1}{P} - 1\right)}$$



The critical buckling load 'P_{crit}' is:

$$\boxed{P_{crit} = \frac{\pi^2 EI}{L^2}}$$

Finally: $$\boxed{v(x) = \frac{v_0(x)}{\left(\frac{P_{crit}}{P} - 1\right)}}$$

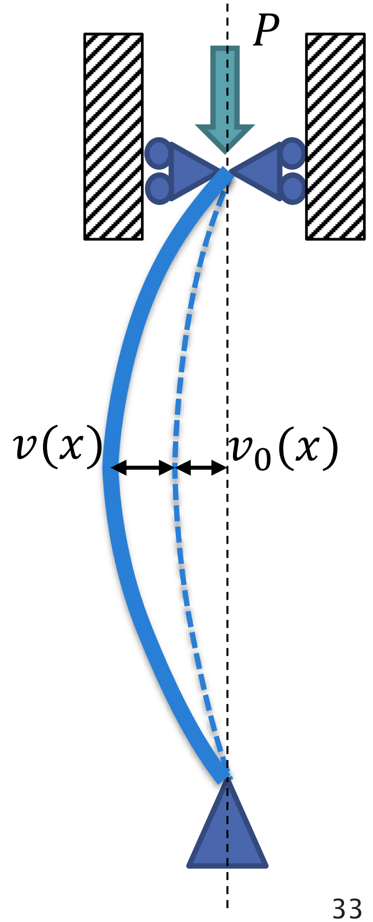
Maximum deflection due to load $v(L/2)$

The max deflection 'v_{max}' occurs at the mid span '$x = L/2$':

$$v(x) = \frac{v_0(x)}{\left(\frac{P_{crit}}{P} - 1\right)} \quad \xrightarrow{v_0(x) = a\, sin\left(\frac{\pi x}{L}\right)} \quad v(x) = \frac{a\, sin\left(\frac{\pi x}{L}\right)}{\left(\frac{P_{crit}}{P} - 1\right)}$$

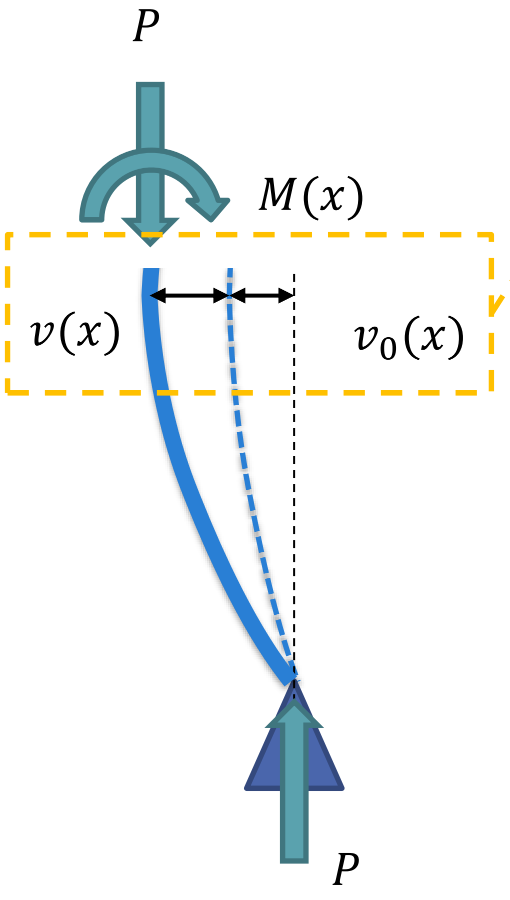
$$v(L/2) = \frac{a\, \cancelto{1}{sin\left(\frac{\pi}{2}\right)}}{\left(\frac{P_{crit}}{P} - 1\right)} \quad \Longrightarrow \quad v(L/2) = \frac{a}{\left(\frac{P_{crit}}{P} - 1\right)}$$

$$v(L/2) = \frac{a}{\left(\frac{P_{crit}}{P} - 1\right)} \times \frac{P}{P} \quad \Longrightarrow \quad v(L/2) = \frac{a\,P}{(P_{crit} - P\,)}$$



Max total deflection $v_{total}(L/2)$

- The max total deflection 'v_{max}' occurs at the mid span '$x = L/2$':



$$v_{total}(x) = v(x) + v_0(x)$$

$$v_0(x) = a\,sin\left(\frac{\pi x}{L}\right)$$

$$v_0(L/2) = a$$

$$v_{total}(L/2) = v_{max} = v(L/2) + v_0(L/2)$$

$$v_{max} = \frac{a\,P}{P_{crit} - P} + a \quad \Rightarrow \quad v_{max} = a\left(\frac{P}{P_{crit} - P} + 1\right)$$

$$v_{max} = a\left(\frac{P}{P_{crit} - P} + \frac{P_{crit} - P}{P_{crit} - P}\right)$$

$$\boxed{v_{max} = a\frac{P_{crit}}{P_{crit} - P}} \quad \text{with:} \quad \boxed{P_{crit} = \frac{\pi^2 EI}{L^2}}$$

Max compressive stress

- Stress in the column comes from:

$$\sigma_{a(xial)} = \frac{F}{A} + \sigma_{b(ending)} = \frac{My}{I} = \sigma_{t(otal)} = \frac{F}{A} + \frac{My}{I}$$

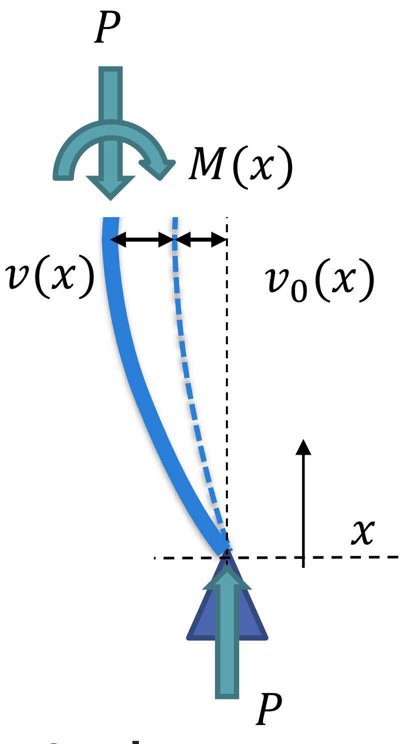
Moment is dependent on deflection which is max at the midspan:

$$M(L/2) = P\big(v(L/2) + v_0(L/2)\big) \dashrightarrow v_{max} = a\frac{P_{crit}}{P_{crit} - P}$$

$$\sigma_b = a\frac{P_{crit}}{P_{crit} - P}\frac{Py}{I}$$

'a' is the initial amplitude of the column deflection

$$\sigma_{max} = \frac{P}{A} + a\frac{P_{crit}}{P_{crit} - P}\frac{Py}{I} \Rightarrow \boxed{\sigma_{max} = \frac{P}{A}\left(1 + \frac{ay}{\kappa^2}\frac{P_{crit}}{P_{crit} - P}\right)}$$



'radius of gyration'

$$\kappa^2 = \frac{I}{A}$$

Buckling of an Initially Curved Column: Summary

- The initial deflection of the column was assumed to be:
- $v_0(x) = a\,sin\left(\frac{\pi x}{L}\right)$ Where a is the 'amplitude' of the initial deflection (constant)

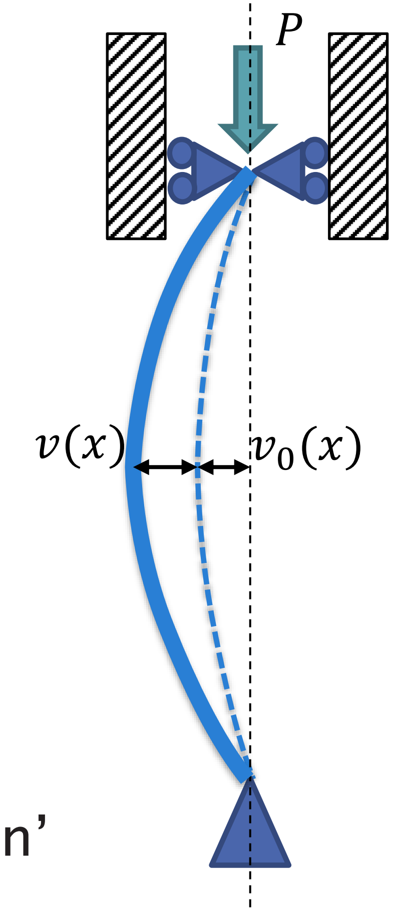
1) Max total deflection $\boxed{v_{max} = a\frac{P_{crit}}{P_{crit} - P}}$

2) Max stress $\boxed{\sigma_{max} = \frac{P}{A}\left(1 + \frac{ay}{\kappa^2}\frac{P_{crit}}{P_{crit} - P}\right)}$

$$\boxed{P_{crit} = \frac{\pi^2 EI}{L^2}}$$

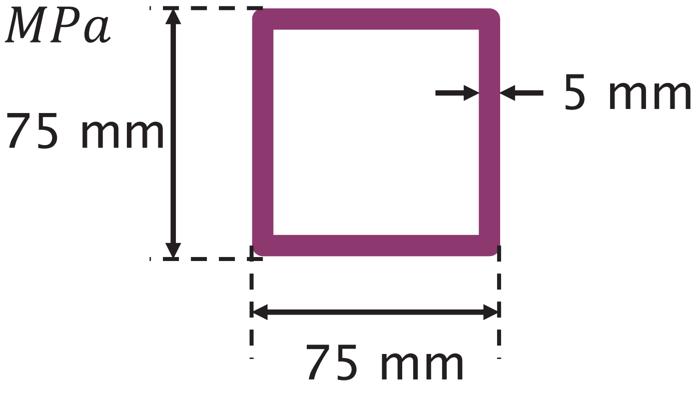
$$\kappa^2 = \frac{I}{A}$$

'radius of gyration'

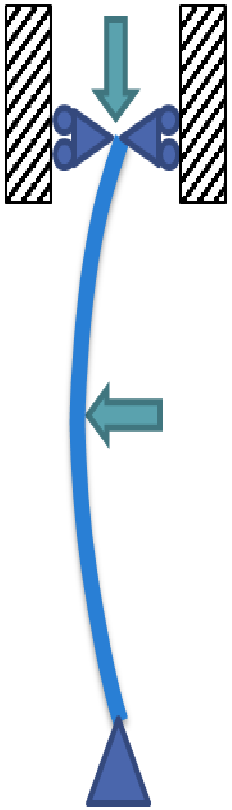


Next: Example – Have a go first!

- The hollow square section shown below is used for a column that needs to support an axial load of 120kN in compression. The column is 3 m long and pinned on its ends. Find:
  - The axial stress in the column and the safety factor against buckling if the column is perfectly straight.
  - The percentage increase in the axial stress if the column is initially curved with an amplitude of 0.5% of its length and the corresponding safety factor.
  - The column is made of steel with $E = 200\ GPa\ , \sigma_{ys} = 250\ MPa$

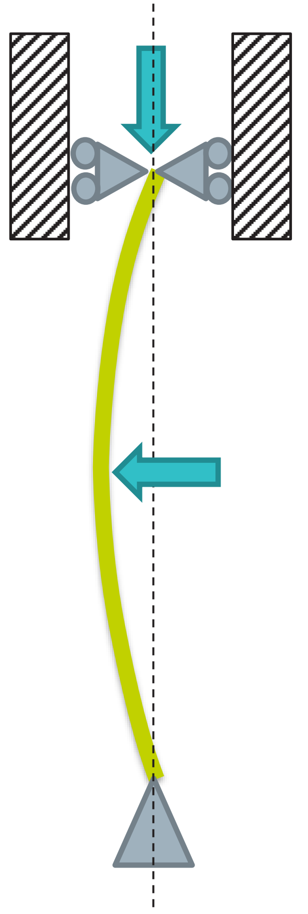


laterally loaded columns



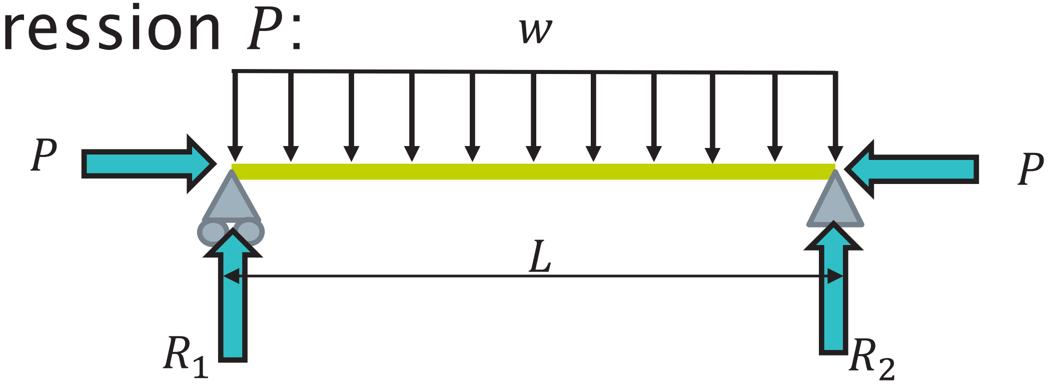
How to find the load-deflection equation

1. Draw the buckled shape, take a cut and apply equilibrium to get the moment $M(x)$
2. Sub the moment into the bending deflection '$v(x)$' DE:
   $$-EI\frac{d^2v(x)}{dx^2} = M(x)$$
3. Solve the second order DE to obtain the deflection solution '$v(x)$'
4. Using the boundary conditions to get unknown constants!



Free body diagram

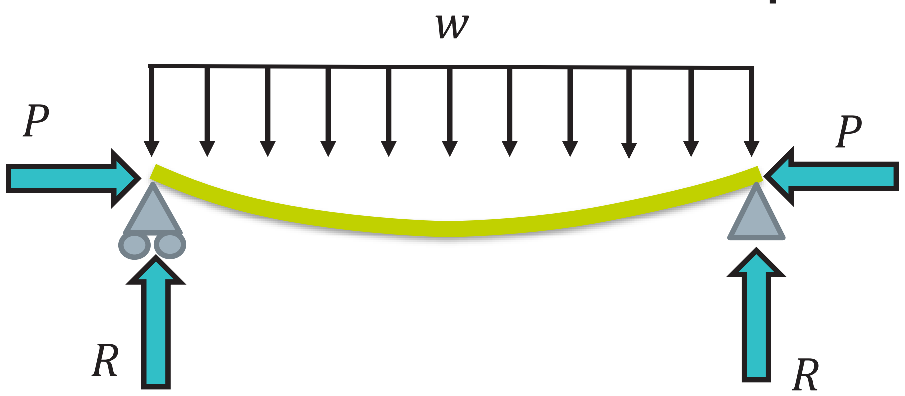
- Consider a beam loaded by ‘self weight uniform ’ w + axial compression P:

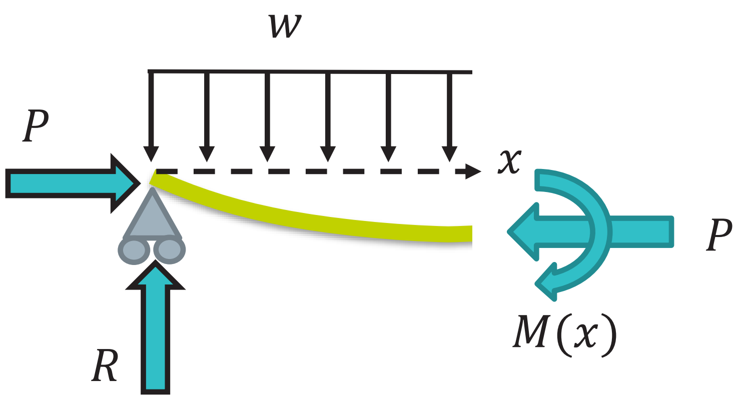


Apply **equilibrium** to get the vertical reactions at the pins:

$$R_1 = R_2 = R = \frac{wL}{2}$$

- Consider the buckled shape and take a ‘cut’ at ‘x’:





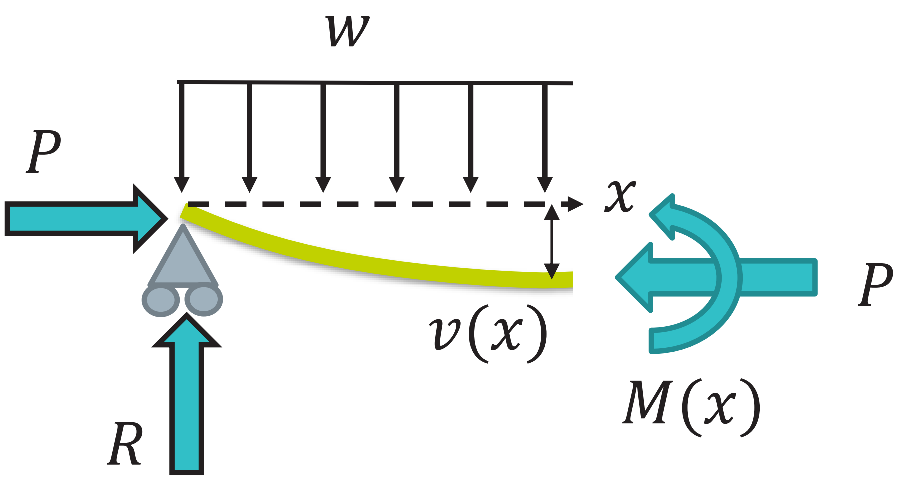
Finding the internal bending moment $M(x)$

- Take moments about the cut to obtain $M(x)$:

$$M(x) - R\,x - P\,v(x) + wx\left(\frac{x}{2}\right) = 0 \qquad R = \frac{wL}{2}$$

$$M(x) = \frac{wLx}{2} - \frac{wx^2}{2} + P\,v(x)$$

$$M(x) = \frac{w}{2}(Lx - x^2) + P\,v(x)$$



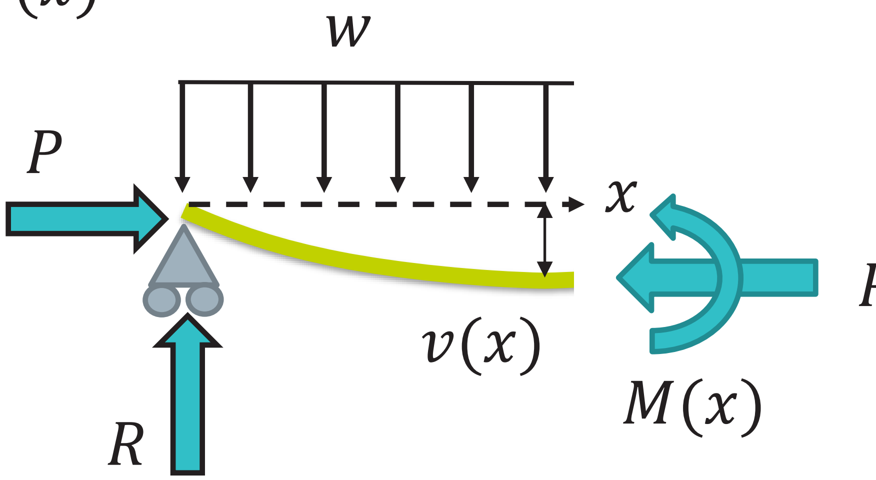
The differential equation

- Substitute the equation for the moment into the DE for the beam deflection:

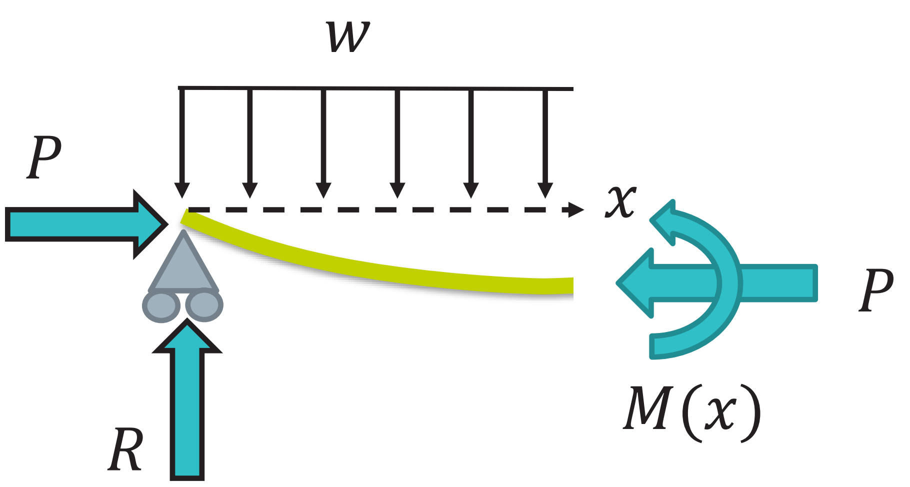
$$-EI\frac{d^2v(x)}{dx^2} = M(x) \quad \text{with} \quad M(x) = \frac{w}{2}(Lx - x^2) + P\,v(x)$$

$$EI\frac{d^2v(x)}{dx^2} + P\,v(x) = -\frac{w}{2}(Lx - x^2)$$

$$\frac{d^2v(x)}{dx^2} + \mu^2\,v(x) = -\frac{w}{2EI}(Lx - x^2) \quad \text{with: } \mu^2 = \frac{P}{EI}$$



Finding the solution to the DE



- 3. Solve the second order DE:

$$\frac{d^2v(x)}{dx^2} + \mu^2\, v(x) = -\frac{w}{2EI}(Lx - x^2) \text{ with } \mu^2 = \frac{P}{EI}$$

- A complementary solution to the homogeneous DE:

$$v_c(x) = A\sin(\mu x) + B\cos(\mu x)$$

A solution when the RH side is zero

- The particular integral is a second order polynomial:

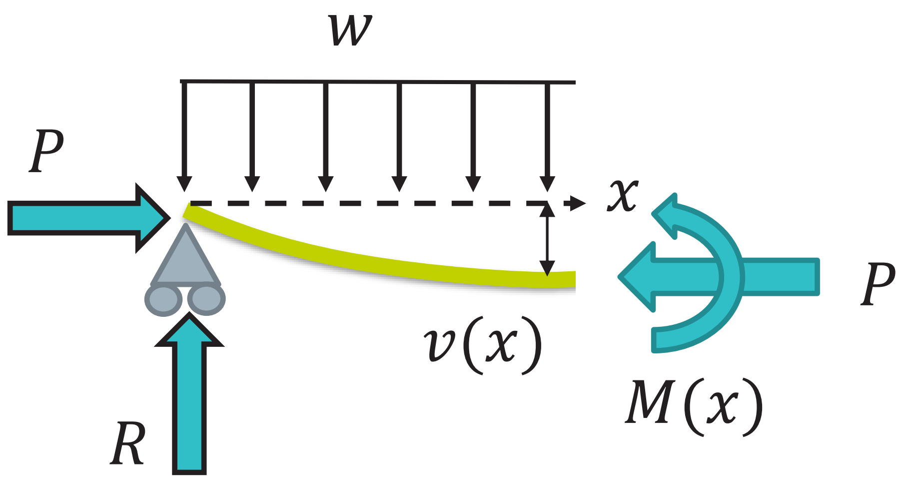
$$v_p(x) = Cx^2 + Dx + F$$

A solution to the RH side of the DE

- Differentiating twice: $v_p'(x) = 2Cx + D$ and $v_p''(x) = 2C$

Finding the particular solution

- Substitute the particular integral into the DE:
- $\dfrac{d^2 v(x)}{dx^2} + \mu^2\, v(x) = -\dfrac{w}{2EI}(Lx - x^2)$
- $(2C) + \mu^2\,(Cx^2 + Dx + F) = -\dfrac{w}{2EI}(Lx - x^2)$
- $\boxed{\dfrac{P}{EI}}Cx^2 + \dfrac{P}{EI}Dx + \dfrac{P}{EI}F + 2C = \boxed{\dfrac{w}{2EI}}x^2 - \dfrac{wL}{2EI}x$
- Equate the coefficients of the two quadratics:
  - x^2coefficients : $\dfrac{P}{EI}C = \dfrac{w}{2EI}$ gives $C = \dfrac{w}{2P}$
  - Similarly, $D = \dfrac{-wL}{2P}$ and $F = -\dfrac{wEI}{P^2}\ \ or\ -\dfrac{w}{P\mu^2}$



Applying BCs and finding the complete solution

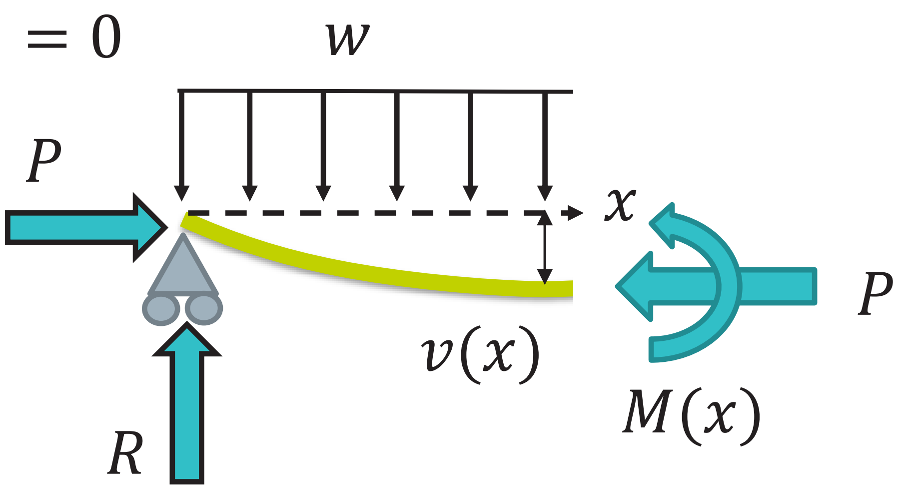
- The general solution to the DE is a linear combination of these:

$$v(x) = v_c(x) + v_p(x) = Asin(\mu x) + Bcos(\mu x) + \frac{w}{2P}\left(x^2 - Lx - \frac{2}{\mu^2}\right)$$

- Now get A and B from the boundary conditions:
- The beam is pinned so: $v(0) = 0$ and $v(L) = 0$

$$v(0) = 0 \Rightarrow B = \frac{w}{P\,\mu^2}$$

$$v(L) = 0 \Rightarrow A = \frac{w}{P\,\mu^2}\left(\frac{1-\cos(\mu L)}{\sin(\mu L)}\right)$$



Solution for the deflection

- The general solution to the DE is:

$$v(x) = \frac{w}{P\,\mu^2}\left(\frac{1-\cos(\mu L)}{\sin(\mu L)}\right)\sin(\mu x) + \frac{w}{P\,\mu^2}\cos(\mu x) + \frac{w}{2P}\left(x^2 - Lx\; - \frac{2}{\mu^2}\right)$$

- The max deflection 'v_{max}' occurs at the midspan '$x = L/2$':

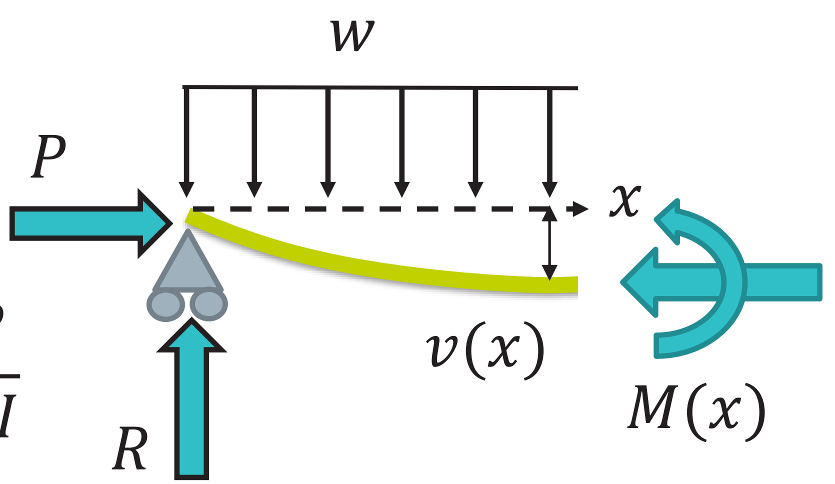
$$v(L/2) = \frac{w}{P\,\mu^2}\left[\left(\frac{1-\cos(\mu L)}{\sin(\mu L)}\right)\sin\left(\frac{\mu L}{2}\right) + \cos\left(\frac{\mu L}{2}\right)\right] + \frac{w}{2P}\left[\left(\frac{L}{2}\right)^2 - L\left(\frac{L}{2}\right) - \frac{2}{\mu^2}\right]$$

$$= \frac{w}{P\,\mu^2}\left[\frac{\sin^2\left(\frac{\mu L}{2}\right) + \cos^2\left(\frac{\mu L}{2}\right)}{\cos\left(\frac{\mu L}{2}\right)} - \frac{\mu^2 L^2}{8} - 1\right]$$

Maximum deflection

- Finally, the max deflection is:

$$\boxed{v_{max} = \frac{w}{P\,\mu^2}\left[\sec\left(\frac{\mu L}{2}\right) - \frac{\mu^2 L^2}{8} - 1\right]}$$ with $\mu^2 = \frac{P}{EI}$



- For small values of P, the deflection approaches the solution for the lateral load by itself

- For $P \rightarrow P_{crit}$ the term $\sec\left(\frac{\mu L}{2}\right) \rightarrow \infty$, where $P_{crit} = \frac{\pi^2 EI}{L^2}$

Max Stress

- The max stress comes from the compressive axial load + bending
- The max bending stress occurs at the location with max moment (the midspan of the beam, $x = L/2$):
- $M(x) = \frac{w}{2}(Lx - x^2) + P\,v(x)$ so $M(L/2) = M_{max} = \frac{wL^2}{8} + P\,v_{max}$
- $M_{max} = \frac{wL^2}{8} + P\,\boxed{v_{max}}$ we derived this on the previous slide
- $M_{max} = \frac{wL^2}{8} + \frac{w}{\mu^2}\left[\sec\left(\frac{\mu L}{2}\right) - \frac{\mu^2 L^2}{8} - 1\right]$
- $M_{max} = \frac{w}{\mu^2}\left[\sec\left(\frac{\mu L}{2}\right) - 1\right]$ or $M_{max} = \boxed{\frac{wL^2}{8}}\boxed{\left[\frac{8(\sec(\mu L/2) - 1)}{\mu^2 L^2}\right]}$

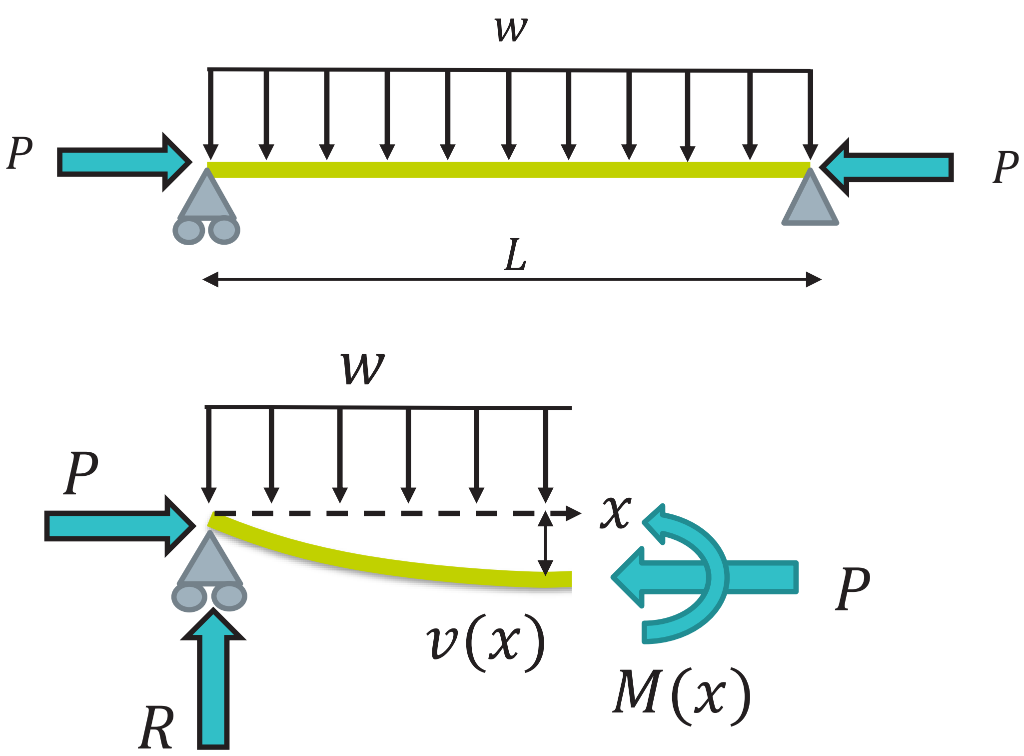
Moment for just the distrib. load 'w'

Amplification due to comp. load 'P'

Max Stress

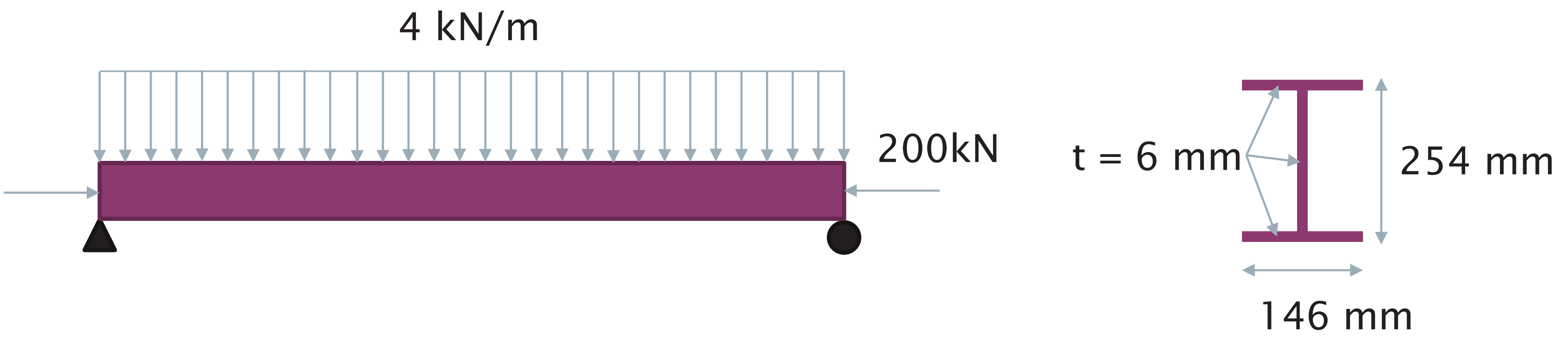
- The max stress comes from the compressive axial load + bending:
- $\sigma_{max} = -\frac{P}{A} - \frac{M_{max}y}{I}$ with $M_{max} = \frac{w}{\mu^2}\left[\sec\left(\frac{\mu L}{2}\right) - 1\right]$ and $\mu^2 = \frac{P}{EI}$
- Finally, the max stress is:

$$\boxed{\sigma_{max} = -\frac{P}{A} - \frac{wy}{I\mu^2}\left[\sec\left(\frac{\mu L}{2}\right) - 1\right]}$$



Next: Example – Have a go first!

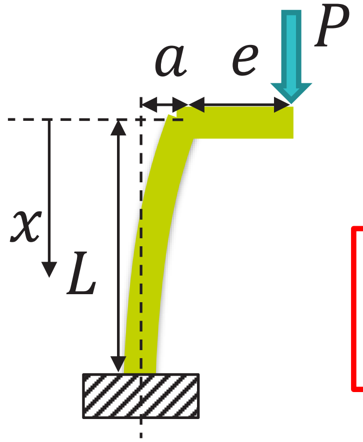
The simply support I-beam shown in the figure is designed to carry a distributed load of 4 kN/m in addition to its own weight. The beam is made of steel with: $\rho = 7850\ kg.m^{-3}$, $E = 200\ GPa$, $\sigma_{ys} = 250\ MPa$. It is 4m long and has the cross-sectional geometry shown. Determine the safety factor against failure if the beam is also required to support a compressive axial load of 200kN. HINT: check buckling along the y and z axis of the beam.



Summary: Buckling of Imperfect Columns

1) Max Comp. Stress: | **2) Max Deflection:**

Eccentric:



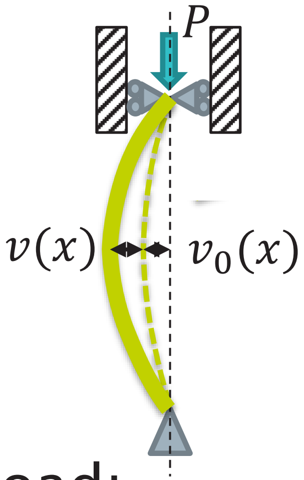
$$\sigma_{max} = -\frac{P}{A}\left[1 + \frac{ey}{\kappa^2}\sec(\mu L)\right]$$

$$v_{max} = e\,[\sec(\mu L) - 1]$$

$$\mu^2 = \frac{P}{EI}$$

$$\kappa^2 = \frac{I}{A}$$

Initially Curved:

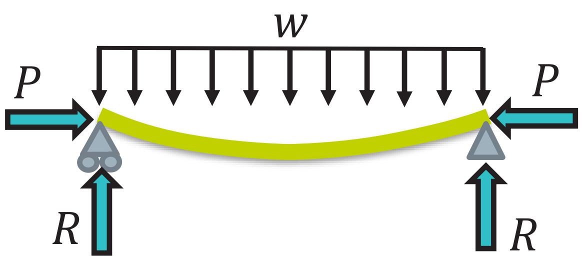


$$\sigma_{max} = -\frac{P}{A}\left(1 + \frac{ay}{\kappa^2}\frac{P_{crit}}{P_{crit} - P}\right)$$

$$v_{max} = a\frac{P_{crit}}{P_{crit} - P}$$

$$P_{crit} = \frac{\pi^2 EI}{L^2}$$

Lateral Load:



$$\sigma_{max} = -\frac{P}{A} - \frac{wy}{I\mu^2}\left[\sec\left(\frac{\mu L}{2}\right) - 1\right]$$

$$v_{max} = \frac{w}{P\,\mu^2}\left[\sec\left(\frac{\mu L}{2}\right) - \frac{\mu^2 L^2}{8} - 1\right]$$