SESA2025 Mechanics of Flight
Longitudinal model

Lecture 3.3

Decoupled linearised equations

Longitudinal equations:

so far we're just converted F=ma into 3D
now we are going to define F

$$\Delta X = m\dot{u}$$

$$\Delta Z = m(\dot{w} - qU_\infty)$$

$$\Delta M = I_{yy}\dot{q}$$

Longitudinal equations
three equations
three unknowns: u, w, q

and lateral equations:

$$\Delta Y = m(\dot{v} + rU_\infty)$$

$$\Delta L = I_{xx}\dot{p} - I_{xz}\dot{r}$$

$$\Delta N = -I_{xz}\dot{p} + I_{zz}\dot{r}$$

Lateral equations
three equations
three unknowns: v, p, r

Next steps

Work on the out-of-balance forces and moments

Split the out-of-balance forces into
gravitational and **aerodynamic** contributions

W L, D, T

Introduce **aerodynamic derivatives** (related to aircraft design choices)
to represent the aerodynamic forces

> We will write these in terms of u, w, q (for longitudinal equations)
> and in terms of v, p, r (for lateral equations)
> so that we have a closed set of equations

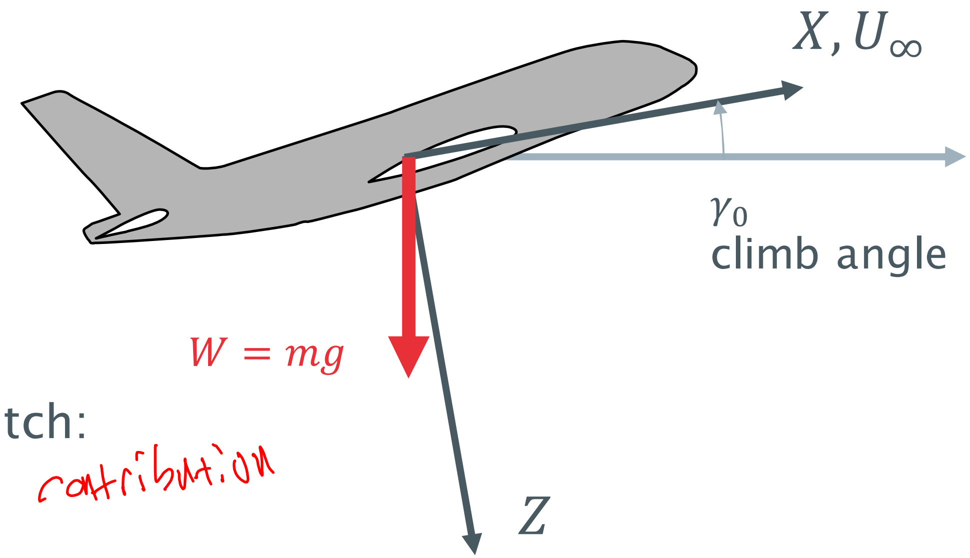
Combine the equations and write them in
matrix (state-space) form.

Gravitational contributions

Gravitational forces in trimmed condition

$$X_{g0} = -mg \sin \gamma_0$$

$$Z_{g0} = mg \cos \gamma_0$$



Gravitational forces for a small perturbation in pitch:

$$X_g = -mg \sin(\gamma_0 + \theta)$$

θ = perturbation contribution

$$Z_g = mg \cos(\gamma_0 + \theta)$$

Gravitational contributions

Gravitational forces for a small perturbation in pitch:

$$X_g = -mg\sin(\gamma_0 + \theta)$$
$$= -mg(\sin\gamma_0 \cos\theta + \cos\gamma_0 \sin\theta)$$
$$= -mg(\sin\gamma_0 + \theta\cos\gamma_0)$$
$$= X_{g0} + \Delta X_g$$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$
$\cos(A + B) = \cos A \cos B - \sin A \sin B$

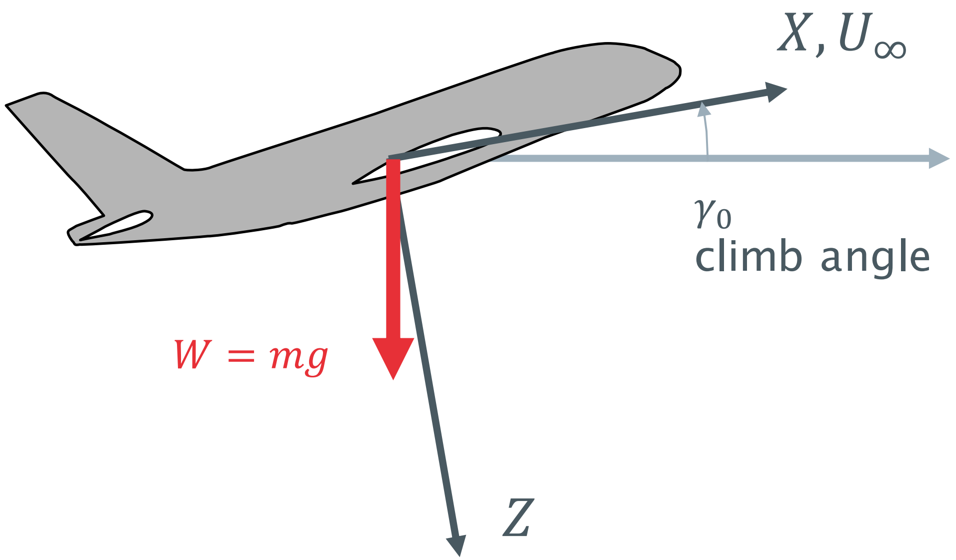
small angles
$\sin\theta =_1 \theta$
$\cos\theta =_1 1$

Therefore:

$$\Delta X_g = -mg\theta\cos\gamma_0$$

Repeat for Z_g to obtain:

$$\Delta Z_g = -mg\theta\sin\gamma_0$$



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Aerodynamic contributions

Aerodynamic forces will depend on the details of design and also on the nature of the perturbation

Represented by aerodynamic derivatives

measure of how x force changes with x velocity change (eg vel increase → drag increase ↓ forward force decrease)

$$\Delta X_a = \mathring{X}_u u + \mathring{X}_w w + \cdots$$

Out-of-balance aerodynamic force

Perturbation

Continue for as many terms as needed for an accurate representation

Aerodynamic derivative (circle on top denotes a dimensional quantity)

Taylor series (ignoring HOT)

$$\Delta X_a \approx \frac{dX_a}{du} u + \frac{dX_a}{dw} w$$

← this dot means $\mathring{X}$ dimensional derivatives, we are not working with non dimensionalised forces.

$\mathring{X}_a$ = how x forces change with x vel change

Longitudinal aerodynamic model

Standard representation for longitudinal out-of-balance forces and pitching moment

$$\Delta X_a = \mathring{X}_u u + \mathring{X}_w w$$

note we ignore $\mathring{X}_v v$ since its small

$$\Delta Z_a = \mathring{Z}_u u + \mathring{Z}_w w + \mathring{Z}_q q + \mathring{Z}_{\dot{w}} \dot{w}$$

$$\Delta M_a = \mathring{M}_u u + \mathring{M}_w w + \mathring{M}_q q + \mathring{M}_{\dot{w}} \dot{w}$$

Streamwise velocity perturbation

Normal velocity perturbation – i.e. angle of attack, since $\alpha = w/U_\infty$ for small disturbances

Pitch rate perturbation

Rate of change of angle of attack (turns out to be important)

Some terms are small

$$\mathring{X}_q = \mathring{X}_{\dot{w}} = 0$$

at low speed: $\mathring{M}_u = 0$

something like a drone or UAV

State equation (longitudinal motion)

Combine them all:

$$
\begin{aligned}
\Delta X &= m\dot{u} & & \Delta X_a = \mathring{X}_u u + \mathring{X}_w w & & \Delta X_g = -mg\theta\cos\gamma_0 \\
\Delta Z &= m\left(\dot{w} - qU_\infty\right) & = \quad & \Delta Z_a = \mathring{Z}_u u + \mathring{Z}_w w + \mathring{Z}_q q + \mathring{Z}_{\dot{w}}\dot{w} & + \quad & \Delta Z_g = -mg\theta\sin\gamma_0 \\
\Delta M &= I_{yy}\dot{q} & & \Delta M_a = \mathring{M}_u u + \mathring{M}_w w + \mathring{M}_q q + \mathring{M}_{\dot{w}}\dot{w} & & 0
\end{aligned}
$$

Write them out in full and rearrange:

$q = \frac{d\theta}{dt} = \dot{\theta}$

$$
\begin{aligned}
m\dot{u} &= \overbrace{\mathring{X}_u u + \mathring{X}_w w}^{\text{vehicle response}} - \overbrace{mg\cos\gamma_0\theta}^{\text{weight}} \\
\left(m - \mathring{Z}_{\dot{w}}\right)\dot{w} &= \mathring{Z}_u u + \mathring{Z}_w w + \left(\mathring{Z}_q + mU_\infty\right) q - mg\sin\gamma_0\theta \\
-\mathring{M}_{\dot{w}}\dot{w} + I_{yy}\dot{q} &= \mathring{M}_u u + \mathring{M}_w w + \mathring{M}_q q
\end{aligned}
$$

State equation (longitudinal motion)

Write them in matrix form:

$$\begin{bmatrix} m\dot{u} & 0 & 0 & 0 \\ 0 & \left(m - \mathring{Z}_{\dot{w}}\right)\dot{w} & 0 & 0 \\ 0 & -\mathring{M}_{\dot{w}}\dot{w} & I_{yy}\dot{q} & 0 \\ 0 & 0 & 0 & \dot{\theta} \end{bmatrix} = \begin{bmatrix} \mathring{X}_u u & \mathring{X}_w w & 0 & -mg\cos\gamma_0\theta \\ \mathring{Z}_u u & \mathring{Z}_w w & \left(\mathring{Z}_q + mU_\infty\right)q & -mg\sin\gamma_0\theta \\ \mathring{M}_u u & \mathring{M}_w w & \mathring{M}_q q & 0 \\ 0 & 0 & q & 0 \end{bmatrix}$$

and rewrite them in **state space** matrix form:

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m - \mathring{Z}_{\dot{w}} & 0 & 0 \\ 0 & -\mathring{M}_{\dot{w}} & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \mathring{X}_u & \mathring{X}_w & 0 & -mg\cos\gamma_0 \\ \mathring{Z}_u & \mathring{Z}_w & \mathring{Z}_q + mU_\infty & -mg\sin\gamma_0 \\ \mathring{M}_u & \mathring{M}_w & \mathring{M}_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

State equation (longitudinal motion)

State space matrix form for decoupled linearised longitudinal motion:

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m - \mathring{Z}_{\dot{w}} & 0 & 0 \\ 0 & -\mathring{M}_{\dot{w}} & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \mathring{X}_u & \mathring{X}_w & 0 & -mg\cos\gamma_0 \\ \mathring{Z}_u & \mathring{Z}_w & \mathring{Z}_q + mU_\infty & -mg\sin\gamma_0 \\ \mathring{M}_u & \mathring{M}_w & \mathring{M}_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

Which can also be written as:

$\mathbf{M}\dot{\mathbf{x}} = \mathbf{A}'\mathbf{x}$

or

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{A} = \mathbf{M}^{-1}\mathbf{A}'$

Dimensional vs dimensionless aerodynamic derivatives

Important for later

We need to be able to convert between dimensional derivatives (with circles), needed for stability calculations, and dimensionless derivatives based on dimensionless aircraft properties (C_L, C_D etc)

$$\Delta Z_a = \mathring{Z}_u u + \mathring{Z}_w w + \mathring{Z}_q q + \mathring{Z}_{\dot{w}} \dot{w}$$

Equivalent dimensionless form:

$$\frac{\Delta Z_a}{\frac{1}{2}\rho U_\infty^2 S} = Z_u\left(\frac{u}{U_\infty}\right) + Z_w\left(\frac{w}{U_\infty}\right) + Z_q\left(\frac{qc}{U_\infty}\right) + Z_{\dot{w}}\left(\frac{\dot{w}c}{U_\infty^2}\right)$$

By multiplication we can equate the different forms eg:

N.B. Different conversion factors for different derivatives (and moment needs an extra chord factor)

$$\begin{aligned}\mathring{Z}_q &= Z_q \cdot \frac{1}{2}\rho U_\infty^2 S\left(\frac{c}{U_\infty}\right) \\ &= Z_q \cdot \frac{1}{2}\rho U_\infty S c\end{aligned}$$

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Dimensional vs dimensionless aerodynamic derivatives

Important for later

$$\mathring{Z}_u = Z_u \cdot \frac{1}{2}\rho U_\infty^2 S \left(\frac{1}{U_\infty}\right) \qquad \mathring{Z}_q = Z_q \cdot \frac{1}{2}\rho U_\infty^2 S \left(\frac{c}{U_\infty}\right) \qquad \mathring{Z}_{\dot{w}} = Z_{\dot{w}} \cdot \frac{1}{2}\rho U_\infty^2 S \left(\frac{c}{U_\infty^2}\right)$$

$$= Z_u \cdot \frac{1}{2}\rho U_\infty S \qquad = Z_q \cdot \frac{1}{2}\rho U_\infty S c \qquad = Z_{\dot{w}} \cdot \frac{1}{2}\rho S c$$

$$\mathring{M}_u = M_u \cdot \frac{1}{2}\rho U_\infty^2 S c \left(\frac{1}{U_\infty}\right) \qquad \mathring{M}_q = M_q \cdot \frac{1}{2}\rho U_\infty^2 S c \left(\frac{c}{U_\infty}\right) \qquad \mathring{M}_{\dot{w}} = M_{\dot{w}} \cdot \frac{1}{2}\rho U_\infty^2 S c \left(\frac{c}{U_\infty^2}\right)$$

$$= M_u \cdot \frac{1}{2}\rho U_\infty S c \qquad = M_q \cdot \frac{1}{2}\rho U_\infty S c^2 \qquad = M_{\dot{w}} \cdot \frac{1}{2}\rho S c^2$$