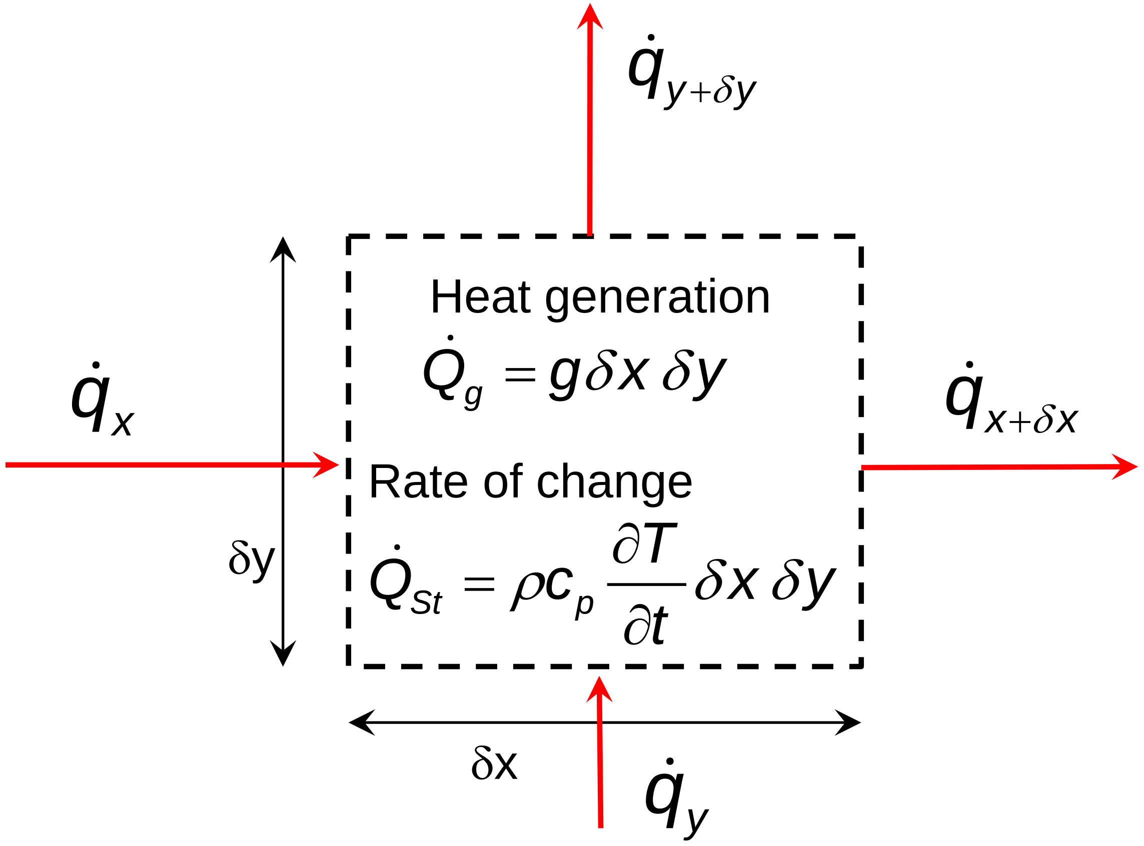
SESA3029 Aerothermodynamics

Lecture 5.5

heat diffusion equation, 1D finite difference methods for conduction

Heat equation



$$\dot{q}_{x+\delta x} = \dot{q}_x + \frac{\partial \dot{q}_x}{\partial x}\delta x$$

$$\dot{q}_{y+\delta y} = \dot{q}_y + \frac{\partial \dot{q}_y}{\partial y}\delta y$$

Energy balance

$$\dot{Q}_{in} - \dot{Q}_{out} + \dot{Q}_g = \dot{Q}_{St}$$

$$\underbrace{(\dot{q}_x - \dot{q}_{x+\delta x})}_{\partial \dot{q}_x}\delta y + \underbrace{(\dot{q}_y - \dot{q}_{y+\delta y})}_{\partial \dot{q}_y}\delta x + g\,\delta x\,\delta y = \rho c_p \frac{\partial T}{\partial t}\delta x\,\delta y$$

$\div \partial x \, \partial y$

$$-\frac{\partial \dot{q}_x}{\partial x} - \frac{\partial \dot{q}_y}{\partial y} + g = \rho c_p \frac{\partial T}{\partial t}$$

Use Fourier's law $\dot{q}_i = -k \dfrac{\partial T}{\partial x_i}$ to obtain

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + g = \rho c_p \frac{\partial T}{\partial t}$$

the heat diffusion equation

$$\boxed{\Delta T + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

with thermal diffusivity $\alpha = \dfrac{k}{\rho c_p}$

Typical boundary conditions

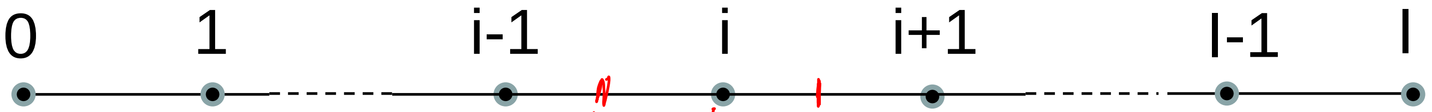
For simplicity: 1D case

- Constant temperature: $$T(0,t) = T_s$$
- Constant surface heat flux: $$-k \left. \frac{\partial T}{\partial x} \right|_{x=0} = \dot{q}_x$$
- Adiabatic or insulated surface: $$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0$$
- Convection surface condition: $$-k \left. \frac{\partial T}{\partial x} \right|_{x=0} = h\left[T_\infty - T(0,t)\right]$$

Discretising the heat equation in 1D

1D heat equation, stationary, no heat source $\dfrac{\partial^2 T}{\partial x^2} = 0$

on mesh with uniform stepsize Δx



The gradients can be approximated as

gradient here (assuming liner gradient)

$$\left.\frac{\partial T}{\partial x}\right|_{i-1/2} \approx \frac{T_i - T_{i-1}}{\Delta x}, \qquad \left.\frac{\partial T}{\partial x}\right|_{i+1/2} \approx \frac{T_{i+1} - T_i}{\Delta x}$$

yielding the approximated equation

Second gradient

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_i \approx \frac{\left.\partial T/\partial x\right|_{i+1/2} - \left.\partial T/\partial x\right|_{i-1/2}}{\Delta x} \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} = 0 \qquad (1)$$

derived numerical approximation for second derivative from some arbitrary discretized temperature line

Discrete boundary conditions

start of line

end of line (boundary)

as is typical, There are issues at the boundary

Constant temperature: $T_2 - 2T_1 = -T_{sl}\ ,\quad -2T_{l-1} + T_{l-2} = -T_{sr}$

T_{sl} T_{sr}

Surface convection on the left: $-k\left.\dfrac{\partial T}{\partial x}\right|_0 = h\left[T_\infty - T_0\right]$

surface touching on left

Using $$\frac{\left.\dfrac{\partial T}{\partial x}\right|_{1/2} - \left.\dfrac{\partial T}{\partial x}\right|_0}{\frac{1}{2}\Delta x} = \frac{\dfrac{T_1 - T_0}{\Delta x} - \left(-\dfrac{h}{k}\left[T_\infty - T_0\right]\right)}{\frac{1}{2}\Delta x} = 0$$

gives $$T_1 - \left(1 + \frac{h\Delta x}{k}\right)T_0 = -\frac{h\Delta x}{k}T_\infty \qquad (2)$$

Surface convection on the right (note sign!): $k\left.\dfrac{\partial T}{\partial x}\right|_l = h\left[T_\infty - T_l\right]$

gives $$T_{l-1} - \left(1 + \frac{h\Delta x}{k}\right)T_l = -\frac{h\Delta x}{k}T_\infty \qquad (3)$$

Adiabatic boundary: Simply use $h = 0$ in Eqs. (2), (3)

Constant surface heat flux: Use $h = 0,\quad T_\infty = \dfrac{\dot{q}_x}{h}$ in Eqs. (2), (3)

Solution process

Equation (1) gives tridiagonal linear systems.
Constant temperature boundary conditions:

$$\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} T_1 \\ \\ \vdots \\ \\ T_{l-1} \end{pmatrix} = \begin{pmatrix} -T_{sl} \\ 0 \\ \vdots \\ 0 \\ -T_{sr} \end{pmatrix}$$

Surface convection boundary conditions:

zeros

$$\begin{pmatrix} -\left(1+\frac{h_l \Delta x}{k}\right) & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -\left(1+\frac{h_r \Delta x}{k}\right) \end{pmatrix} \begin{pmatrix} T_0 \\ \\ \vdots \\ \\ T_l \end{pmatrix} = \begin{pmatrix} -\frac{h_l \Delta x}{k} T_{\infty l} \\ 0 \\ \vdots \\ 0 \\ -\frac{h_r \Delta x}{k} T_{\infty r} \end{pmatrix}$$

zeros

Order of accuracy of finite differences

Finite difference representations are based on Taylor series. For instance

$$T_{i+1} = T_i + \left(\frac{\partial T}{\partial x}\right)_i \Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)_i \frac{\Delta x^2}{2} + \left(\frac{\partial^3 T}{\partial x^3}\right)_i \frac{\Delta x^3}{3!} + \ldots$$

can be rearranged to

$$\left(\frac{\partial T}{\partial x}\right)_i = \underbrace{\frac{T_{i+1} - T_i}{\Delta x}}_{\text{Finite difference}} \cancel{\underbrace{- \left(\frac{\partial^2 T}{\partial x^2}\right)_i \frac{\Delta x}{2} - \left(\frac{\partial^3 T}{\partial x^3}\right)_i \frac{\Delta x^2}{3!} + \ldots}_{\text{Truncation error}}} = \frac{T_{i+1} - T_i}{\Delta x} + O(\Delta x)$$

Similarly, we can show that

second order accuracy

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_i = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

Accuracy of basic scheme by Taylor series

Error~Δx^p Grid size Δx, order of accuracy p

1st order: error decreases by a factor of 2 for twice the number of grid points

2nd order: error decreases by a factor of 4 for twice the number of grid points

4th order: error decreases by a factor of 16 for twice the number of grid points

Fewer grid points needed for high order methods for a given accuracy, but does depend on the function being approximated