Outline

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Lecture 4 - Fourier Series

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Outline

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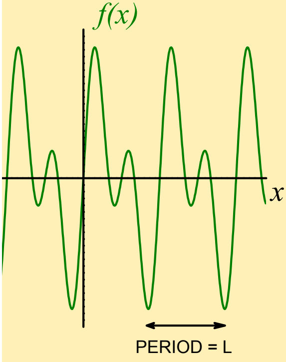
either work everywhere or no where (in terms of x for a function)

- Today we start a new topic:
  Fourier Series (FS)
- Main idea: write a **periodic** function in terms of **sums** of **sin** and **cos**
- First: **definition** of a Fourier series
- Second: look at some **properties** of **trig functions**
- Fundamental property of FS: "**orthogonality**"
  only works on periodic functions
- Finally: **Euler formulae**
  (this is **how we calculate FS**)



→ Fourier Series: definition

- Let $f(x)$ be a **2π periodic function** so that

$$f(x+2\pi) = f(x). \tag{1}$$

Note that equation (1) implies that

$$f(x+2k\pi) = f(x) \quad \text{where } k \text{ is an integer.}$$

∴ any valid period

- Our **aim**: write such a 2π periodic function $f(x)$ in terms of a **<u>sum</u>** of simple **periodic functions involving** sin and cos.

- More precisely we want to write (**Definition** of Fourier Series):

any periodic function

arbitrary

cos(nx) has period $\frac{2\pi}{n}$ ← basic stuff

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\,x) + b_n\sin(n\,x)\right],$$

defined by a potentially inf numb of constants

with a_n, b_n constants to be fixed appropriately later.

- The Fourier Series consists of **simple functions** ⇒ **easy to manipulate**.

Fourier Series: restrictions - periodicity

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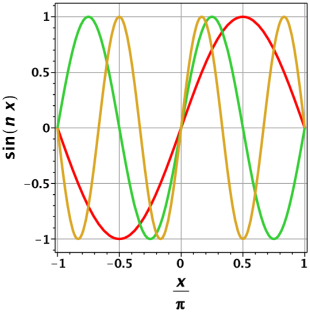

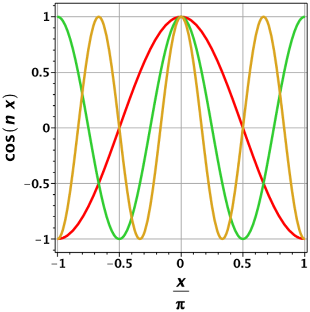

Red: $n = 1$; Green: $n = 2$; Yellow: $n = 3$.

The Fourier Series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big[a_n\cos(n\,x) + b_n\sin(n\,x)\Big]$$

contains terms with period 2π, so ***f* must have period 2π**:

$$\begin{cases}\cos\left[n(x+2\pi)\right] = \cos(n\,x)\\ \sin\left[n(x+2\pi)\right] = \sin(n\,x)\end{cases} \Rightarrow \quad \text{FS applies to } f(x+2\pi) = f(x)$$

Basic identities: Cosine

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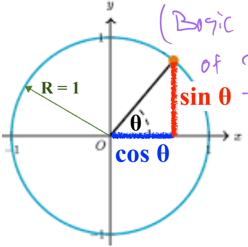
As we are using trigonometric functions, the following are *essential* knowledge: (n is integer)

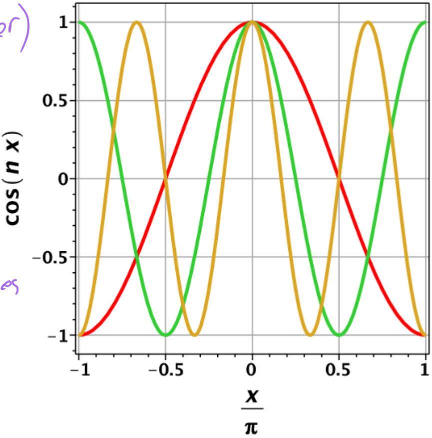
$$\cos(n\pi) = (-1)^n,$$

$$\underbrace{\cos\left[\left(n+\tfrac{1}{2}\right)\pi\right] = 0.}_{\text{cos vanishes at every } \frac{\pi}{2}}$$

(Basic stuff)

of course same applies to sin





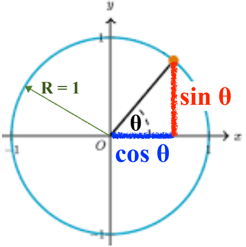
Red: $n = 1$; Green: $n = 2$; Yellow: $n = 3$.

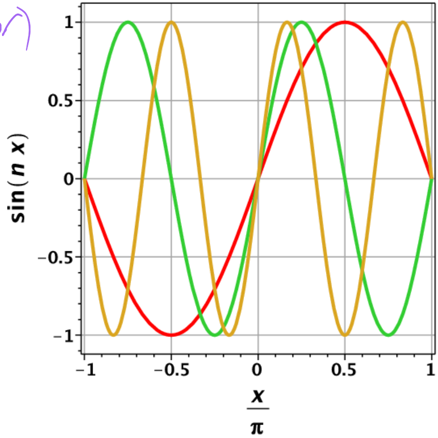
Basic identities: Sine

As we are using **trigonometric functions**, the following are *essential* knowledge: (n is integer)

$$\sin(n\pi) = 0,$$

$$\sin\left[\left(n + \tfrac{1}{2}\right)\pi\right] = (-1)^n.$$





Red: $n = 1$; Green: $n = 2$; Yellow: $n = 3$.

Computing a Fourier Series: I

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

assume:
$f(x)=f(x+2\pi)$
then it
has anti derivative
$F(x)$ so that
$F(x) \to f(x)=F'(x)$
where
$F(x)=F(x+2\pi)$

To **find the Fourier series** we **need to compute** a_m, b_m.
We do this using the following **key integral identities**:

$$\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,\mathrm{d}x = \pi\delta_{mn},$$

$$\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{mn}, \quad m = 0 \text{ exception: } \int_{-\pi}^{\pi}\cos(nx)\,\mathrm{d}x = \mathbf{2}\pi\delta_{\mathbf{0}n}$$

hence: $\int_A^{A+2\pi} f(x) = [F(x)]_A^{A+2\pi} = F(A+2\pi) - F(A) = 0$

$$\int_{-\pi}^{\pi}\sin(mx)\cos(nx)\,\mathrm{d}x = 0,$$

$\therefore$ by integration you can get rid of periodic terms

where δ_{mn} is the *Kronecker symbol*: $\quad \delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}.$

can show is periodic by integrating to 0

Computing a Fourier Series: II

$\rightarrow$ These identities demonstrate **orthogonality** of $\sin(n\,x)$ and $\cos(n\,x)$. They **follow from the trig formulae:**

- $\boldsymbol{m = n}$ **:**

$\cos^2 x + \sin^2 x = 1$

$$\sin^2(m\,x) = \tfrac{1}{2}\Big[1 - \cos(2m\,x)\Big], \qquad \cos^2(m\,x) = \tfrac{1}{2}\Big[1 + \cos(2m\,x)\Big].$$

Now, $\begin{cases} \int_{-\pi}^{\pi} \frac{1}{2}\,dx = \frac{1}{2}\left[x\right]_{-\pi}^{\pi} = \pi, \\ \int_{-\pi}^{\pi} \cos(2m\,x)\,dx = 0 \quad \text{(area vanishes!)}. \end{cases} \Rightarrow \int_{-\pi}^{\pi} \sin^2(m\,x)\,dx = \pi + 0 = \pi$

and similarly for the cos case with $m = 0$ exception: $\int_{-\pi}^{\pi} \cos(0)\,dx = \mathbf{2}\pi \quad \underbrace{\Rightarrow \tfrac{1}{2} a_0}_{(\textit{later})}$

- $\boldsymbol{m \neq n}$ **:**

$$2\sin(m\,x)\sin(n\,x) = \cos\left[(m-n)x\right] - \cos\left[(m+n)x\right],$$
$$2\cos(m\,x)\cos(n\,x) = \cos\left[(m-n)x\right] + \cos\left[(m+n)x\right],$$
$$2\sin(m\,x)\cos(n\,x) = \sin\left[(m+n)x\right] + \sin\left[(m-n)x\right].$$

Integrating these over $-\pi$ **to** π gives the above identities.

$\rightarrow$ $\{\sin(n\,x), \cos(n\,x)\}$ form a **basis** for the **space** of **periodic functions** $\boldsymbol{f(x)}$

Euler formulae: I

(proved that for the correct period you can get things to cancel)

- Take our Fourier Series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big[a_n \cos(n\,x) + b_n \sin(n\,x)\Big] \qquad (2)$$

and the two identities involving sines (from previous slide),

$$\int_{-\pi}^{\pi} \sin(m\,x)\sin(n\,x)\,\mathrm{d}x = \pi\delta_{mn}, \qquad \int_{-\pi}^{\pi} \sin(m\,x)\cos(n\,x)\,\mathrm{d}x = 0.$$

- **Projection** of $f(x)$ over an element of the basis: If we multiply equation (2) by $\sin(m\,x)$ and integrate between $-\pi$ and π we get

$$\pi b_m = \int_{-\pi}^{\pi} f(x)\sin(m\,x)\,\mathrm{d}x$$

for **one, single** term m.

$\nwarrow$ $\Big\langle f(x), g(x) \Big\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x$

$\nwarrow$ inner product for functions

Derivation of this result

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The key is to note that we **can take the integral inside the sum**. So:

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \int_{-\pi}^{\pi}\left(\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]\right)\sin(mx)\,dx$$

$$= \tfrac{1}{2}a_0\int_{-\pi}^{\pi}\sin(mx)\,dx$$

$\swarrow$ $\sum$ and $\int$ commute

$$+\sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos(nx)\sin(mx)\,dx$$

$$+\sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx$$

$$= \mathbf{0} + \sum_{n=1}^{\infty} a_n\times\mathbf{0} + \underbrace{\sum_{n=1}^{\infty} b_n\,\pi\delta_{mn}}_{=b_m \text{ (single term!)}}$$

$$= b_m\pi.$$

$\nwarrow$ only term that does not vanish is $n=m$

Similar steps hold for the a_m terms.

Euler formulae: II

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big[a_n\cos(n\,x) + b_n\sin(n\,x)\Big]$$

Similar results for the **cosine (a_m) terms** give the full **Euler formulae:**

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(m\,x)\,\mathrm{d}x,$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(m\,x)\,\mathrm{d}x.$$

These **hold for all terms** in the Fourier Series, including the $\boldsymbol{a_0}$ term (hence the factor of $\mathbf{1/2}$ in the definition!).

Why $\frac{1}{2}a_0$? Well, note that $f(x) = 1$ is a periodic function!

Let $f(x) = 1$ then $a_m = 0 = b_m$ if $m \neq 0$, but ... $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\,\mathrm{d}x = \frac{1}{\pi}\mathbf{2}\pi = \mathbf{2}$
So $f(x) = \frac{1}{2}a_0 + 0 + 0 \;\Leftrightarrow\; 1 = \frac{1}{2}\mathbf{2} \;\Leftrightarrow\; 1 = 1$ ✓

Summary

- Fourier Series are just another way of representing a function,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big[a_n\cos(n\,x) + b_n\sin(n\,x)\Big]$$

- The representation is in terms of periodic functions; the function $f(x)$ you are representing must be periodic.
- The Euler formulae:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(m\,x)\,\mathrm{d}x,$$

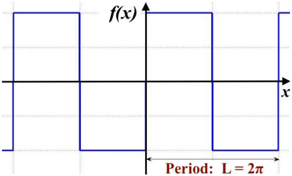
$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(m\,x)\,\mathrm{d}x.$$

- These formulae are valid when the **period** of the function is $\mathbf{2\pi}$ ($\ell = \pi$), see the Lecture Notes for the **general case** of period 2ℓ.

Summary

- Fourier Series of periodic $f(x)$:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$



- The Euler formulae:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x,$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x.$$