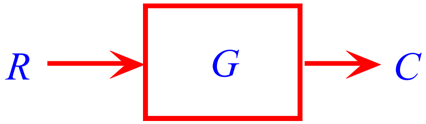
# 4. RESPONSE OF DYNAMIC SYSTEMS

In the input-output relationship $C(s) = G(s)R(s)$, each of $C$, $G$ and $R$ is, in general, a ratio of polynomials in $s$.




$$G(s) = C(s)/R(s)$$

Say we are given TF:

$$G(s) = \frac{2K}{3}\frac{0.5s + 1}{{}^{1}/_{3}\, s^2 + {}^{4}/_{3}\, s + 1}$$

This can be re-arranged:

$$G(s) = \frac{K(s+2)}{(s+1)(s+3)} = K\frac{s+2}{s^2+4s+3}$$

analysing the properties of the numerator and denominator gives you information about system behaviour and stability

where the pre-factors to $s$ are now integers. Some definitions follow:

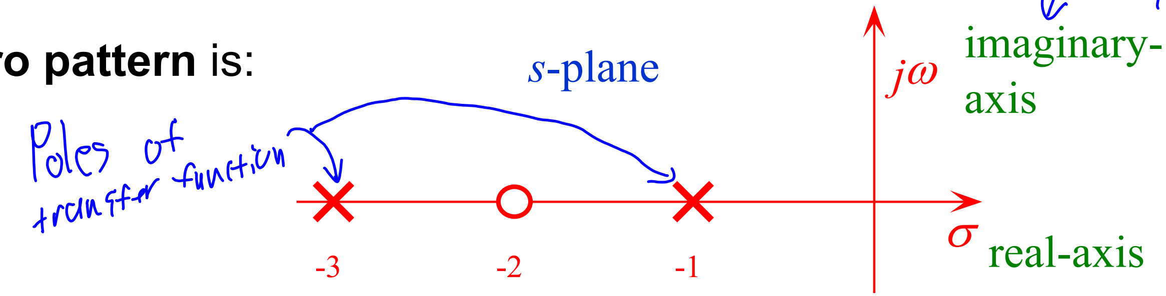
| | |
|---|---|
| Zeros of $C$, $G$, and $R$ | Roots of their numerator polynomials. $(s = -2)$ |
| Poles of $C$, $G$, and $R$ | Roots of their denominator polynomials. $(s = -1, s = -3)$ |
| System zeros and system poles | Those of the system transfer function $G(s)$. |
| System characteristic polynomial | Name used for the denominator of $G(s)$. $(s^2 + 4s + 3)$ |
| System characteristic equation | Result if the system characteristic polynomial is equated to zero. Its roots are obviously the system poles. $(s^2 + 4s + 3 = 0)$ |
| Root locus gain factor | That which results if the coefficients of the highest powers of $s$ in numerator and denominator are made equal to unity. |

- Polynomials have real coefficients then the poles (×) and zeros (⊙) are either real or complex conjugate pairs.
- Poles and zeros are values of $s$ and may be plotted on a complex plane called the $s$-plane.
- Because $s = \sigma + j\omega$, the real axis is the $\sigma$-axis and frequencies, $\omega$, are plotted on the imaginary $j\omega$-axis.
- Axes should be identical when using graphical methods due to the significance of angles in the $s$-plane.

Consider the transfer function: $$G(s) = \frac{2K}{3}\frac{0.5s+1}{{}^1\!/_3\, s^2+{}^4\!/_3\, s+1} = \frac{K(s+2)}{(s+1)(s+3)}$$

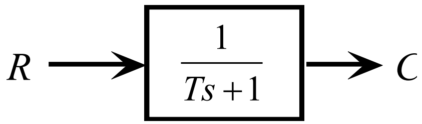
not needed for this particular set of roots

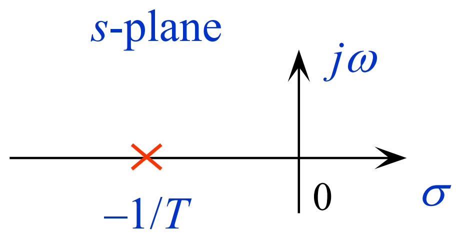
Then the **system pole-zero pattern** is:



# 4.1 First order systems – Impulse response

Consider the simple first order system:




If an impulse change is applied to such a system, how would it respond?

The TF of the system is defined as:

one pole → negative poles show stability

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{Ts + 1}$$

The Laplace transform of an Impulse function ($\delta(t)$-function) is $1$, therefore $R(s) = 1$. Hence:

sudden "kick"

R=1

$$G(s) = C(s) = \frac{1}{Ts + 1}$$

If the system has gain $K$, then:

$$C(s) = \frac{K}{Ts + 1} \tag{4.1}$$

hard to read Laplace transformed functions, need to inverse to get back to time domain

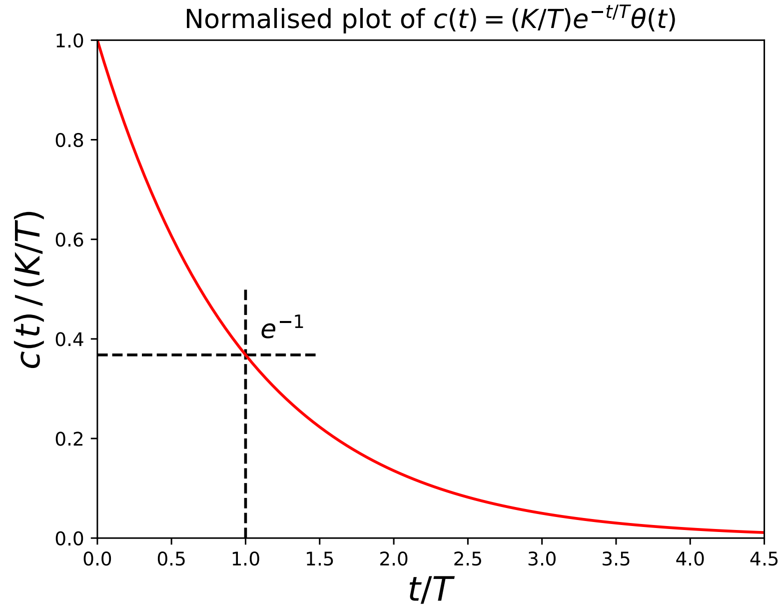
Equation (4.1) tells us very little about the response of the system. To understand the transient response we need to transform the output $C(s)$ into the time domain by using the inverse Laplace transform, i.e. obtain **time dependent response** $\boldsymbol{c(t) = \mathcal{L}^{-1}[C(s)]}$.

Rather than evaluating the inverse Laplace transform integral directly, it may be easier to use:

- Rules of section 3.2 or Table of Laplace transforms
- Software (e.g. Python, Matlab)

$$c(t) = \mathcal{L}^{-1}\left[\frac{K}{Ts+1}\right] = \frac{K}{T}\, e^{-\frac{t}{T}}\,\theta(t)$$

where $\theta(t)$ is the Heaviside (step) function.




Observations:

- At 0 seconds the output of the system $c(t) = K/T$.
- As time increases $t \to \infty$, the value of $c(t) \to 0$.
- The transient is a decaying exponential.
- Long decay time ($T$) implies slow system response.

stable

Commonly used measure of decay speed is **time constant**.

- The **time constant** is the time, in seconds, for the decaying exponential transient to be reduced to $e^{-1} = 0.368$ of its initial value. never reaches 0, so we take some place to characterize it

Since $e^{-t/T} = e^{-1}$ when $t = T$, it can be seen that:

- The time constant for a simple lag $K/(Ts + 1)$ is $T$ seconds.

- The coefficient in front of $s$ then immediately indicates the speed of decay.

- It takes $4T$ seconds for the transient to decay to $1.8\%$ of its initial value ($e^{-4} \approx 0.018$), which defines the **settling time**. AKA "practically stopped"

    - At $t = 2T$, the output is at $13.5\%$ of $K/T$
    - At $t = 3T$ it is $5\%$ of $K/T$.

We can determine the system output at any point in time after the impulse has been applied using the inverse Laplace transform.

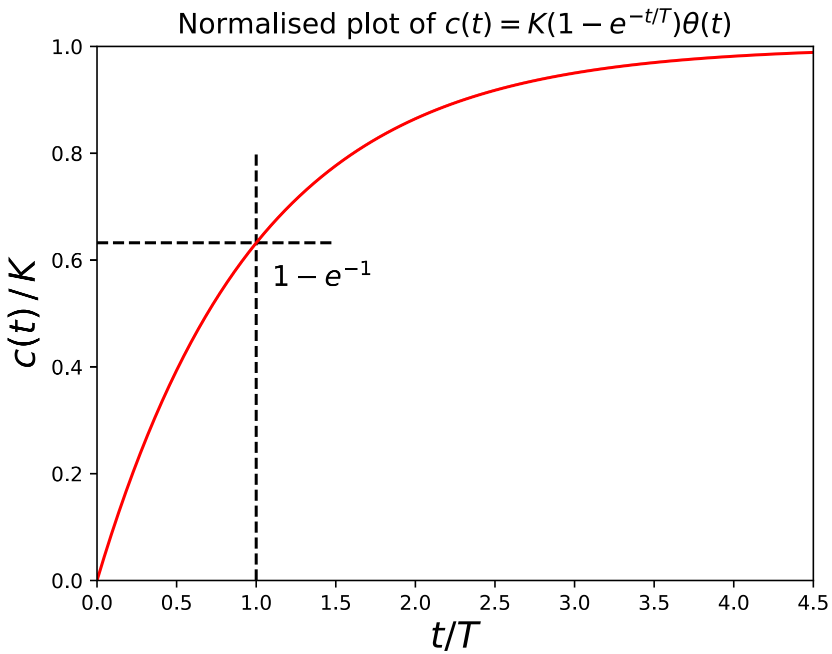
## 4.1 First order systems – Step response

What happens to a simple lag when a step change is applied to the input?

The Laplace transform of a step input is $R(s) = {}^{1}/_{s}$. Hence:

$$C(s) = G(s)R(s) = \frac{K}{Ts+1}R(s)$$

$$= \frac{K}{s(Ts+1)} = \frac{K}{s} - \frac{KT}{Ts+1}$$




Using e.g. the second rule for Laplace transforms from Section 3.2 this can be inverted (in parts) to yield:

$$c(t) = K\left(1 - e^{-t/T}\right)$$

At time $t = 0$, the value of $c(t)$ is zero.

As time $\mathrm{t} \to \infty$ the value of $c(t) \to K$ which is the steady state output.

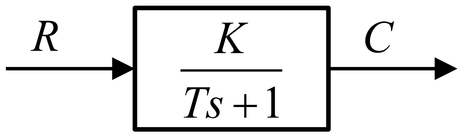
Since $e^{-t/T} = e^{-1} \approx 0.368$ when $t = T$, it can be seen that:

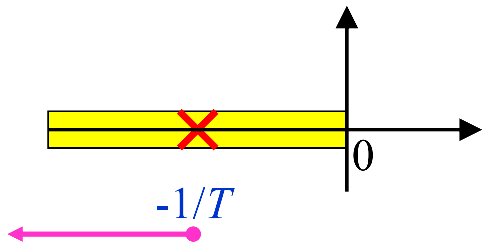
- The time constant for a simple lag $K/(Ts + 1)$ is $T$ seconds.
- The coefficient of $s$ again immediately indicates the speed of decay.
- It takes $4T$ seconds for the transient to decay to $1.8\%$ of its initial value.

At $t = T$, $c(t)$ ≈ 1 – 0.368 = 0.632

- When using the Laplace transform it is often assumed that the initial values of $c$ and $r$ are zero. Otherwise initial conditions need to be included (see previous Topic 3).
- The equation for $c(t)$ is the change in output caused by a change in input.
- Due to superposition, for example, if the system output was $5$ when the step change occurred then it would settle to a value of $5 + K$ as time tended to infinity.

Consider the correlation between this response and the pole position at $s = -1/T$:




For a simple lag, two features are especially important:

- stability:
  - If $-1/T > 0$, the pole lies in the right half of the $s$-plane transient $e^{-t/T}$ then grows steadily as $t \to \infty$.
  - System is therefore unstable and useless.
  - Primary design rule for stability the system:
    - **The pole(s) must lie in the left half of the $s$-plane.**
- speed of response:
  - To speed up the response of the system, i.e. to reduce the time constant $T$:
    - **The pole must be moved to the left !**

## 4.2 Second order systems

since most systems contain some amount of oscillation they are useful for characterising most systems

With no damping, a second order system will continuously oscillate when a change in force is applied.

The frequency of this oscillation is known as the **natural frequency**, $\omega_n$.

The general equation of a second order system is

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = K\omega_n^2 u$$

where $\zeta$ is referred to as the **damping ratio** of the system and is a measure of how oscillatory the system is.

$K$ is the system gain and $u$ is the system input and $y$ the output.

The general form of the Laplace transform of a second order system is

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.2}$$

The second order system is particularly important.

The dynamics of most systems can be approximated by a second order system.

## Step response

For a unit step input, $R(s) = 1/s$, the output transform is

$$C(s) = \frac{K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

This can be inverted by partial fractions (more of these later) by noting

$$C(s) = \frac{1}{s} + \frac{\mathrm{A}_1}{\mathrm{s} - \mathrm{p}_1} + \frac{\mathrm{A}_2}{\mathrm{s} - \mathrm{p}_2}$$

and the inverse Laplace transform is then

$$c(t) = 1 + A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

as $\mathcal{L}[Ae^{-\alpha t}] = A/(s+\alpha)$.

The $p_1$ and $p_2$ are referred to as **poles** and are equal to the roots of the **characteristic equation** of (4.2).

There are three possibilities, depending on the roots of the characteristic equation, i.e.

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{4.3}$$

noting the quadratic form $ax^2 + bx + c = 0$, we expect $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

These system poles depend on $\zeta$:

controls magnitude change

controls oscillatory response

$$\begin{aligned}
\zeta > 1: \text{ over damped: } \quad & p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\
\zeta = 1: \text{ critically damped: } \quad & p_{1,2} = -\omega_n \\
\zeta < 1: \text{ under damped: } \quad & p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \quad \text{(Complex conjugate pair)}
\end{aligned} \tag{4.4}$$

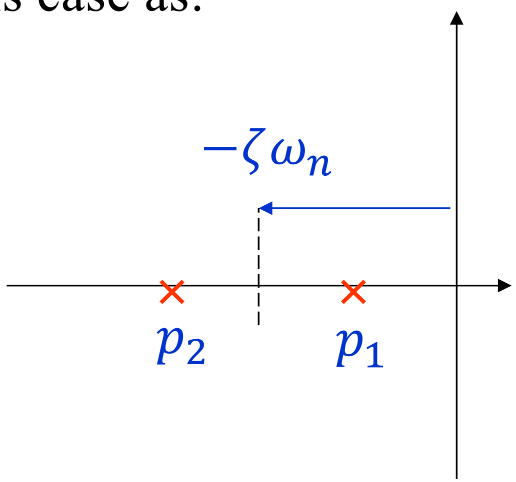
In slide 12, the coefficients $A_1$ and $A_2$ are **uninteresting** constants defined as

$$A_1 = -\frac{K\zeta}{2\sqrt{\zeta^2 - 1}} - \frac{K}{2} \qquad \& \qquad A_2 = \frac{K\zeta}{2\sqrt{\zeta^2 - 1}} - \frac{K}{2}$$

The locations of poles in the $s$-plane for each of these cases:

**$\zeta > 1$ (over damped case)**

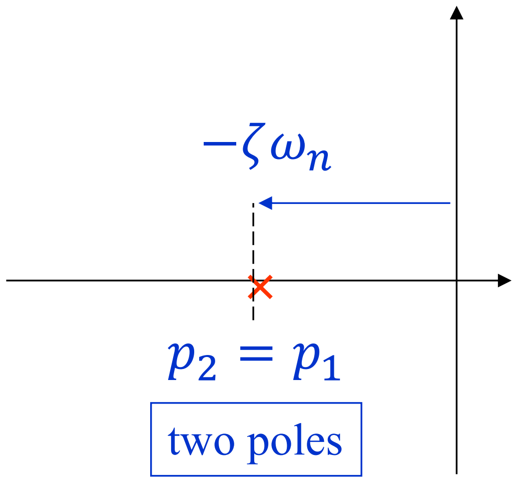
- For $\zeta > 1$ the poles $p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ are always real and negative.
- This is because $\omega_n\sqrt{\zeta^2 - 1} < \omega_n\zeta$ and because $\omega_n > 0$.
- Therefore, we can draw this case as:



The locations of poles in the $s$-plane for each of these cases:

**$\zeta = 1$ (critically damped case)**

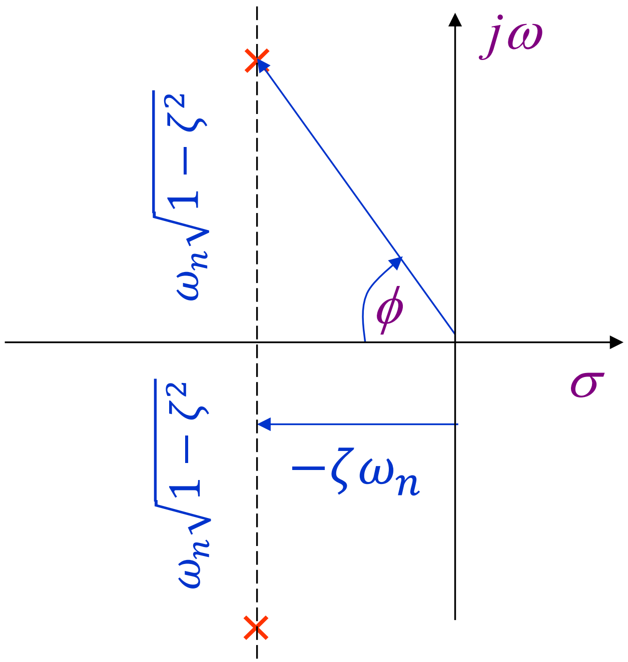
- For $\zeta = 1$ the poles $p_{1,2} = -\omega_n$.
- Therefore, we can draw this case as:



The locations of poles in the $s$-plane for each of these cases:

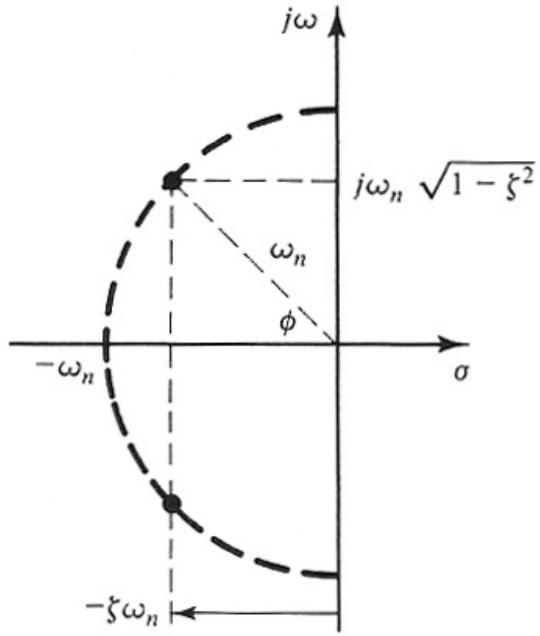
**$\zeta < 1$ (under damped case)**

- For $\zeta < 1$ the poles $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ are complex numbers with negative real part.
- Therefore, we can draw this case as:



From this geometry it can be seen that $cos\phi = (\zeta\omega_n)/\omega_n = \zeta$ and therefore the **damping ratio** is

$$\zeta = \cos\phi$$




where $\phi$ is the angle of the poles to the negative real axis from the origin:

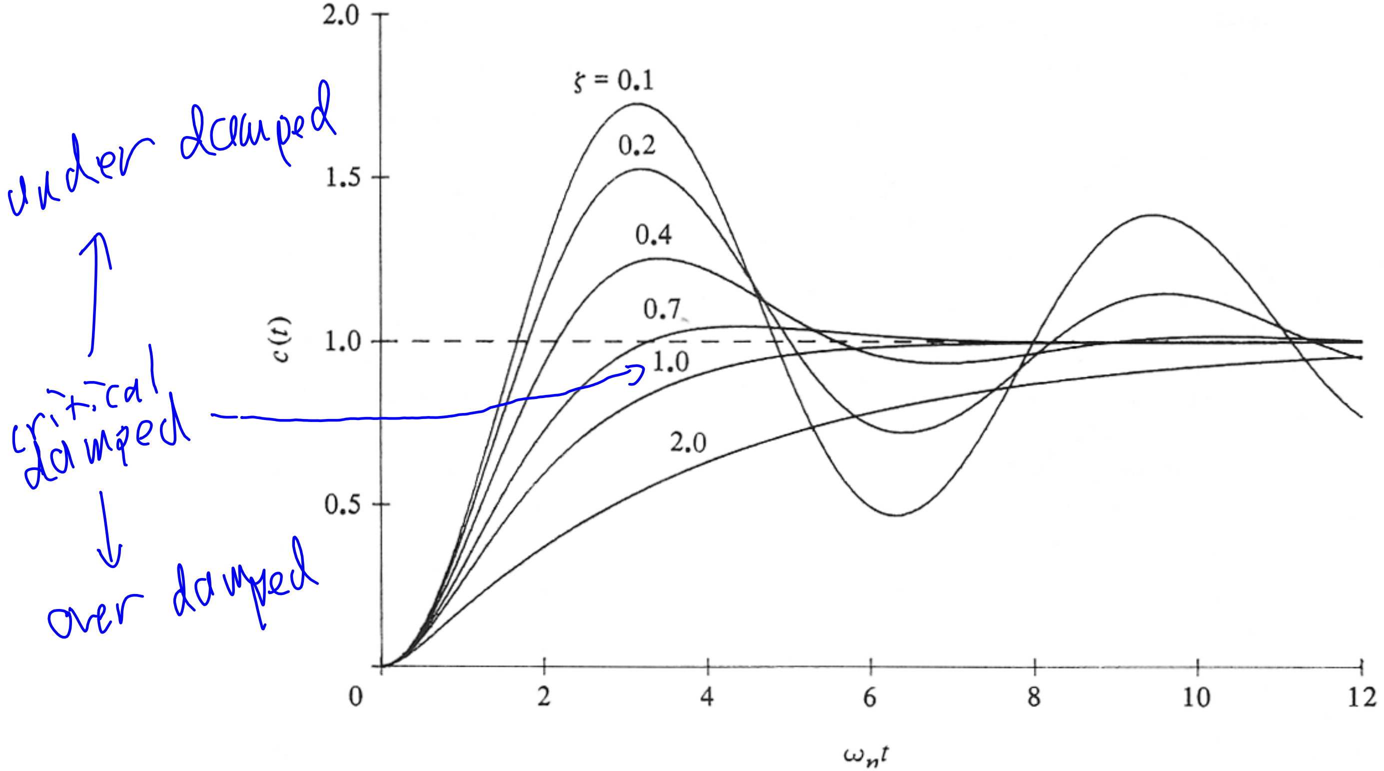
- an angle of $\phi = 45°$ corresponds to $\zeta = 0.707$,
- and $\phi = 60°$ corresponds to $\zeta = 0.5$.

The roots of the characteristic equation form a semi-circle of radius $\omega_n$ and the angle $\phi$ is given by

$$\tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta} \qquad \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

For $\zeta > 1$, when the poles are real and distinct, the transient is a sum of two decaying exponentials, each with its own time constant.

As the damping ratio is increased, the output response becomes slower.



# 4. RESPONSE OF DYNAMIC SYSTEMS (part II) – Examples

***Example 4.1:*** Consider the system described by the transfer function

$$G(s) = \frac{2}{s^2 + 3s + 2} = \frac{2}{(s+1)(s+2)}$$

for a unit step input:

$$C(s) = \frac{2}{s(s+1)(s+2)}$$

Expand into partial fractions:

$$C(s) = \frac{2}{s(s+1)(s+2)} = \underbrace{\frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}}$$

$K_1, K_2, K_3$? these can be found using standard partial fractions solving methods

but it's good to use this new method.

We can find the coefficients $K_1$, $K_2$, $K_3$ by the method of residues, which states:

For a single pole:

$$\boxed{K_i} = \lim_{s \to p_i} (s - p_i) C(s) \tag{4.5}$$

and for higher order poles:

$$K_{i,n-k} = \frac{1}{(n-k)!} \lim_{s \to p_i} \frac{d^{n-k}}{ds^{n-k}} \left( (s - p_i)^n C(s) \right) \tag{4.6}$$

pain in the ass, not on exam!

where $p_i$ is the $i$-th pole, $n$ is the highest order of the pole, and $k < n$ is the actual order of the pole in the given partial fraction. We will demonstrate these formulas in examples below.

Returning to our problem $C(s) = \frac{2}{s(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$ we have only single poles and so:

$$K_1 = \lim_{s \to 0} sC(s) = \lim_{s \to 0} s \frac{2}{s(s+1)(s+2)} = 1$$

$$K_2 = \lim_{s \to -1} (s+1) C(s) = \lim_{s \to -1} (s+1) \frac{2}{s(s+1)(s+2)} = -2$$

$$K_3 = \lim_{s \to -2} (s+2) C(s) = \lim_{s \to -2} (s+2) \frac{2}{s(s+1)(s+2)} = 1$$

not going to prove it, it just works somehow lmao

Therefore, using the fact that Laplace transform $\mathcal{L}[Ae^{-\alpha t}] = \frac{A}{s+\alpha}$, the transient response for:

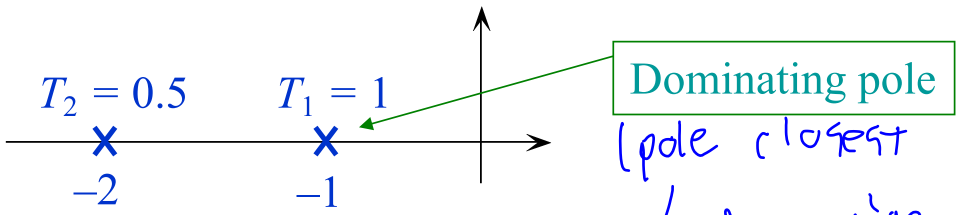
$$C(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

assuming zero initial conditions is:

$$c(t) = 1 - 2e^{-t} + e^{-2t}$$

The transient consists of two exponentials with two time-constants $T_1 = 1$ and $T_2 = 0.5$.

This could have been predicted from the system pole-zero plot.




(pole closest to zero)
↳ determines decay time?

- The system is a series connection of two simple lags.
- The exponential corresponding to the pole closest to the origin has the largest time constant and therefore takes the longest to decay.
- This is called the *dominating pole* and to improve the speed of the system response it would have to be moved towards the left.

In terms of general expression $C(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega s+\omega_n^2)}$, the above result applies to case $\zeta > 1$.

For $\zeta = 1$, a repeated root occurs at $-\omega_n$ and the result can be calculated as in Example 4.3.

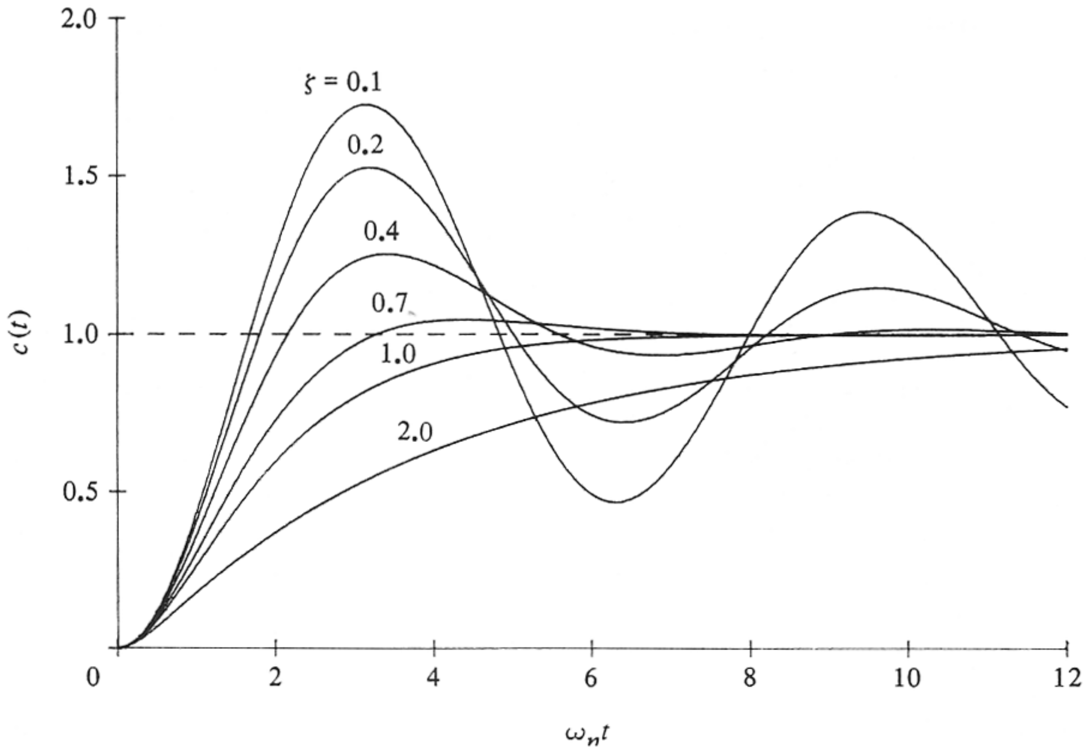
For $\zeta < 1$ the result in the Laplace transform Table 3.1 provided earlier implies:

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \qquad \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \phi \tag{4.7}$$

can rep as $c = 1 - Ae^{Bt}\sin(Dt+E)$

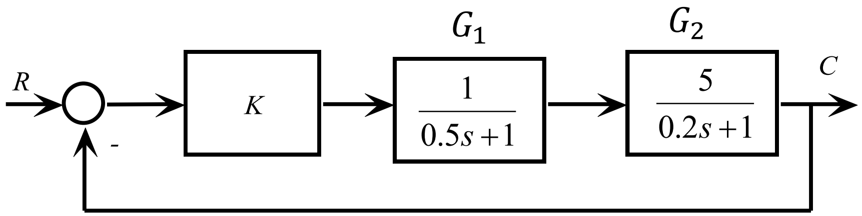
The transient term is an oscillation of damped natural frequency $\omega_n\sqrt{1-\zeta^2}$ of which the amplitude decays according to $e^{-\zeta\omega_n t}$ i.e. with time constant $T = 1/(\zeta\omega_n)$.

Thus for an underdamped ($\zeta < 1$) second-order system the response is




In a similar manner to the simple lag, the amplitude decays to 2% of its initial value in $4T$ seconds.

***Example 4.2:*** Find the unit step response of the system for a loop gain of $K = 0.025$.



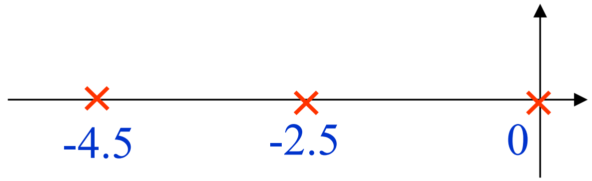

First find the closed loop transfer function:

$$\frac{C}{R} = \frac{KG_1G_2}{1 + KG_1G_2} = \frac{5K}{5K + (0.5s + 1)(0.2s + 1)} = \frac{1.25}{(s + 2.5)(s + 4.5)}$$

apply the unit step input $R(s) = 1/s$:

$$C(s) = \frac{1.25}{s(s + 2.5)(s + 4.5)} = \frac{K_1}{s} + \frac{K_2}{s + 2.5} + \frac{K_3}{s + 4.5}$$

The pole-zero map of $C(s) = \frac{1.25}{s(s+2.5)(s+4.5)}$ now yields:




The coefficients of partial fractions in $C(s) = \frac{K_1}{s} + \frac{K_2}{s+2.5} + \frac{K_3}{s+4.5}$ can be found by using the **residue rule**:

$$K_1 = \lim_{s\to 0} sC(s) = \left.\frac{1.25}{(s+2.5)(s+4.5)}\right|_{s=0} = \frac{1}{9}$$

$$K_2 = \lim_{s\to -2.5} (s+2.5)C(s) = \left.\frac{1.25}{s(s+4.5)}\right|_{s=-2.5} = -\frac{1}{4}$$

$$K_3 = \lim_{s\to -4.5} (s+4.5)C(s) = \left.\frac{1.25}{s(s+2.5)}\right|_{s=-4.5} = \frac{1.25}{9}$$

And thus the system response to a step input is:

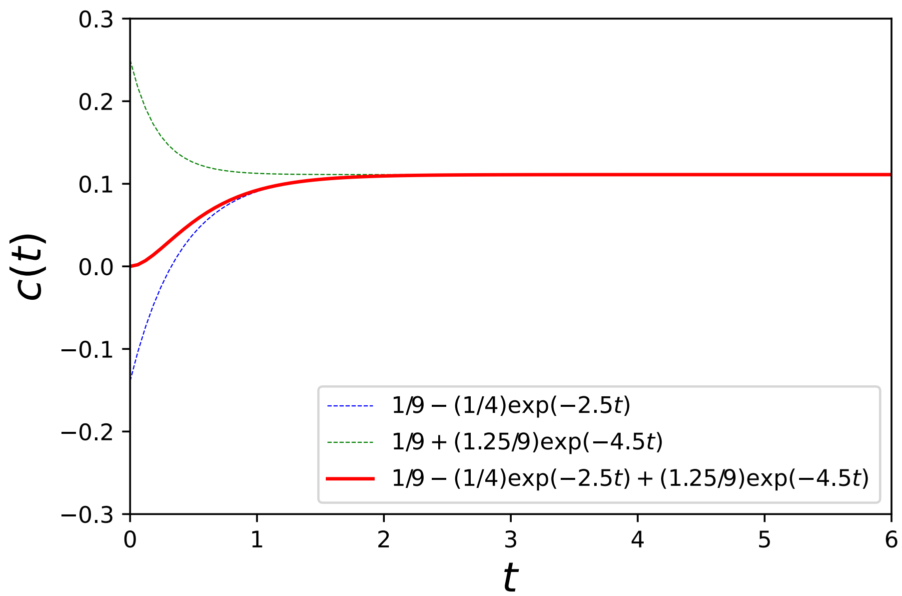
$$C(s) = \frac{1/9}{s} - \frac{1/4}{s+2.5} + \frac{1.25/9}{s+4.5}$$

Calculating the inverse Laplace transform gives the following transient response:

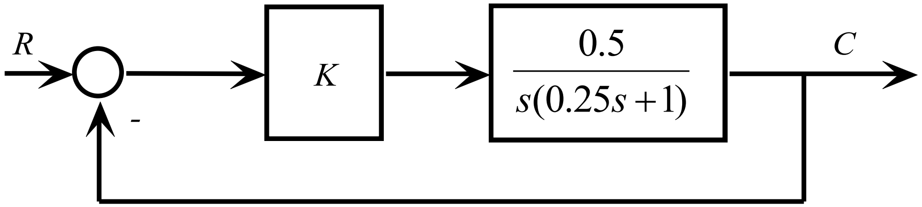
$$c(t) = \frac{1}{9} - \frac{1}{4}e^{-2.5t} + \frac{1.25}{9}e^{-4.5t}$$

under the assumption of zero initial conditions.

Note there are two time-constants $\tau = \frac{1}{4.5}$ and $\tau = \frac{1}{2.5}$ (compare the green dotted line and blue dotted line in the figure)



**Example 4.3:** The block diagram for a dc servo system is




The closed-loop transfer function is:

$$\frac{C(s)}{R(s)} = \frac{2K}{s^2 + 4s + 2K}$$

The system poles are: $s_{1,2} = -2 \pm \sqrt{4 - 2K}$

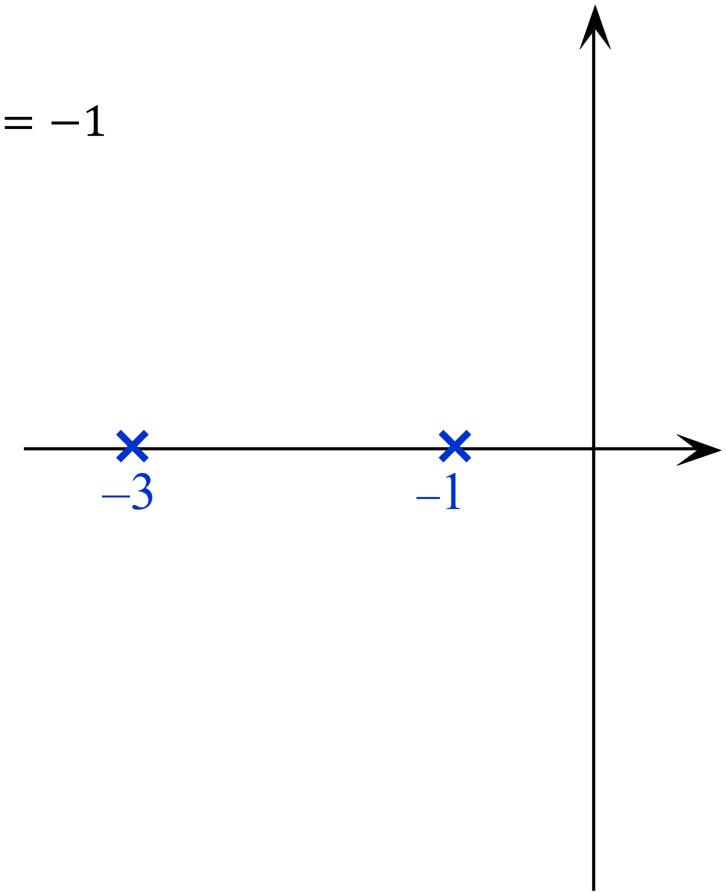
For $K < 2$, poles lie along the negative real axis in the $s$-plane.
This corresponds to $\zeta > 1$ and the response will therefore be overdamped and not oscillate.

Real part of poles is $-2$ for all $K \geq 2$.

Consider $K = 1.5$:

Characteristic equation reads: $s^2 + 4s + 3 = 0$ and roots are: $s_1 = -1$ and $s_2 = -3$.

The unit step response is:




$$C(s) = \frac{3}{s(s+1)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+3}$$

$$A_1 = \lim_{s\to 0} sC(s) = \lim_{s\to 0} \frac{s\cdot 3}{s(s+1)(s+3)} = 1$$

$$A_2 = \lim_{s\to -1} (s+1)C(s) = \lim_{s\to -1} \frac{(s+1)\cdot 3}{s(s+1)(s+3)} = -\frac{3}{2}$$

$$A_3 = \lim_{s\to -3} (s+3)C(s) = \lim_{s\to -3} \frac{(s+3)\cdot 3}{s(s+1)(s+3)} = \frac{1}{2}$$

And so the time dependent response is: $c(t) = 1 - \frac{3}{2} e^{-t} + \frac{1}{2} e^{-3t}$

Consider $K = 2$:

Characteristic equation reads: $s^2 + 4s + 2 = 0$ and roots are: $s_1 = -2$ and $s_2 = -2$.

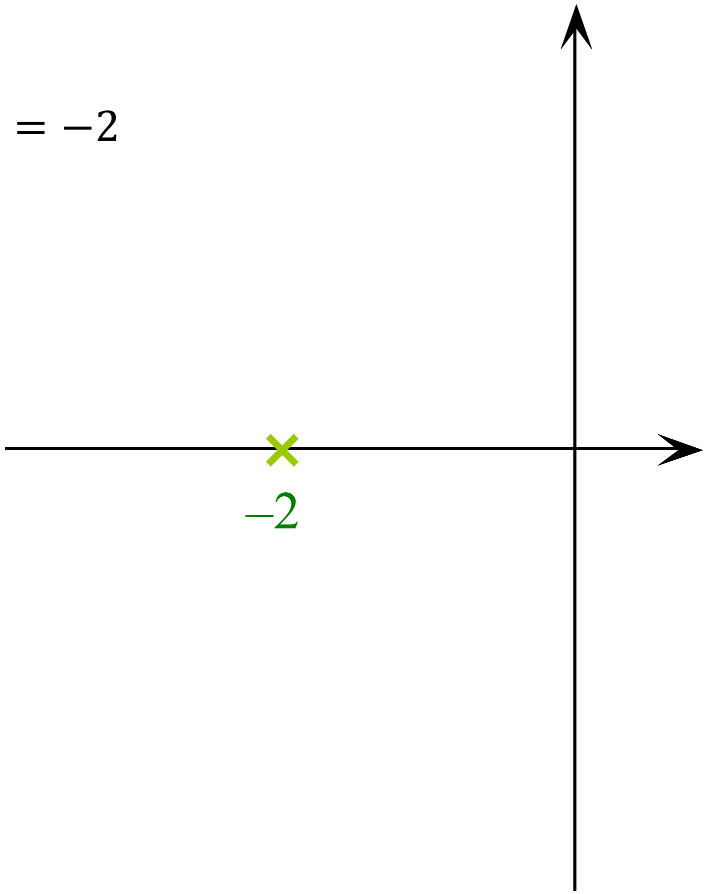
The unit step response is:

$$C(s) = \frac{4}{s(s+2)^2} = \frac{A_1}{s} + \frac{A_2}{(s+2)^2} + \frac{A_3}{s+2}$$

$$A_1 = \lim_{s\to 0} sC(s) = \lim_{s\to 0} \frac{s \cdot 4}{s(s+2)^2} = 1$$

$$A_2 = \lim_{s\to -2} (s+2)^2 C(s) = \lim_{s\to -2} \frac{(s+2)^2 \cdot 4}{s(s+2)^2} = -2$$

$$A_3 = \lim_{s\to -2} \frac{d}{ds}\left((s+2)^2 C(s)\right) = \lim_{s\to -2} \frac{d}{ds} \frac{(s+2)^2 \cdot 4}{s(s+2)^2} = \lim_{s\to -2} -\frac{4}{s^2} = -1$$




And so the time dependent response is: $c(t) = 1 - 2te^{-2t} - e^{-2t}$

Consider $K = 8$:

Characteristic equation reads: $s^2 + 4s + 16 = 0$ and (complex conjugate) roots are: $s_1 = -2 + 2\sqrt{3}i$ and $s_2 = -2 - 2\sqrt{3}i$.

The unit step response is:

$$C(s) = \frac{16}{s(s-s_1)(s-s_2)} = \frac{A_1}{s} + \frac{A_2}{s-s_1} + \frac{A_3}{s-s_2}$$

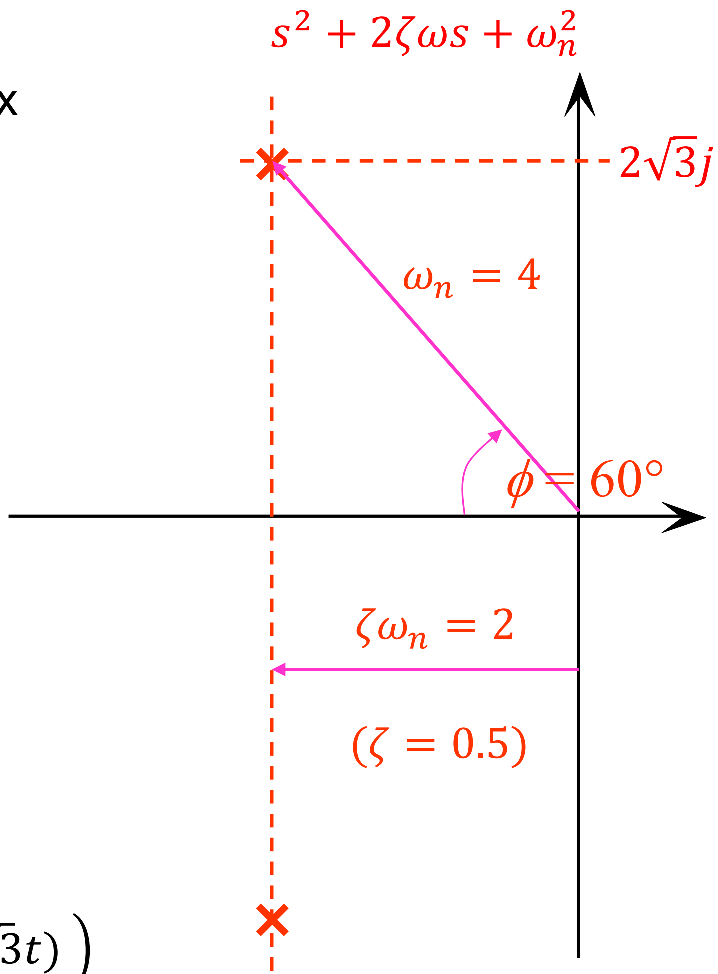
$$A_1 = \lim_{s\to 0} sC(s) = \lim_{s\to 0}\frac{s\cdot 16}{s(s-s_1)(s-s_2)} = 1$$

$$A_2 = \lim_{s\to s_1}(s-s_1)C(s) = \lim_{s\to s_1}\frac{(s-s_1)\cdot 16}{s(s-s_1)(s-s_2)} = -\frac{1}{2} + \frac{\sqrt{3}}{6}j$$

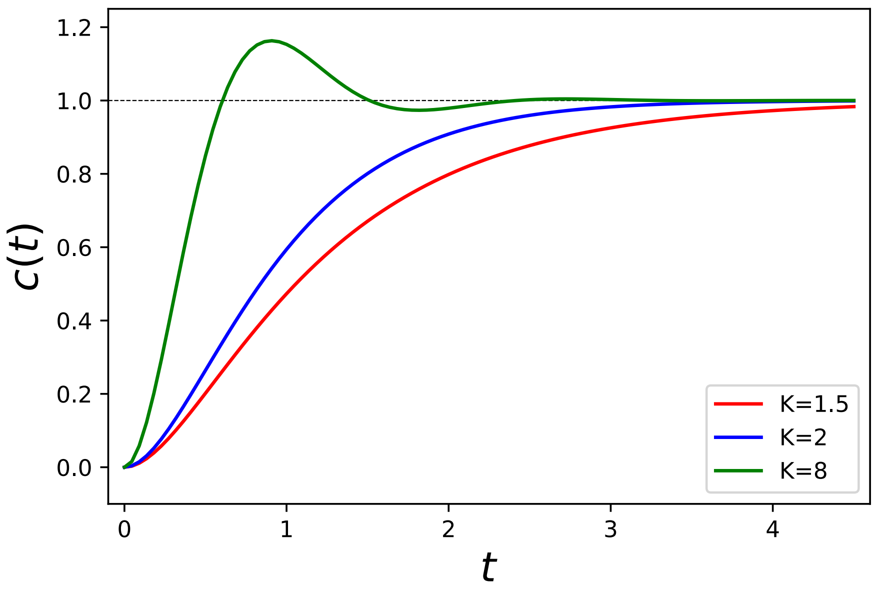
$$A_3 = \lim_{s\to s_2}(s-s_2)C(s) = \lim_{s\to s_2}\frac{(s-s_2)\cdot 16}{s(s-s_1)(s-s_2)} = -\frac{1}{2} - \frac{\sqrt{3}}{6}j$$

And so the time dependent response is:

$$c(t) = 1 - A_2e^{s_1t} - A_3e^{s_2t} = 1 - e^{-2t}\left(\cos(2\sqrt{3}t) + \frac{\sqrt{3}}{3}\sin(2\sqrt{3}t)\right)$$



- Time constant and settling time for are the same for $K = 2$ and $K = 8$.
- Response for $K = 8$ has a higher response speed ($\omega_n$).
- Difference is due to smaller peak time and rise time associated with larger values of $K$ and $\omega_n$.




```
import numpy as np
import pylab as py

t = np.linspace(0, 4.5, 100)

y = 1-3/2*np.exp(-t)+1/2*np.exp(-3*t)

y1 = 1-2*t*np.exp(-2*t)-np.exp(-2*t)

y2 = 1-np.exp(-2*t)*(np.cos(2*np.sqrt(3)*t) +
                     np.sqrt(3)/3*np.sin(2*np.sqrt(3)*t))

py.plot(t, y, 'r')
py.plot(t, y1, 'b')
py.plot(t, y2, 'g')
py.hlines(1, -0.1, 4.5, linestyles='dashed', linewidth=0.5)

py.axis([-0.1, 4.6, -0.1, 1.25])
py.xlabel('$t$', fontsize=18)
py.ylabel('$c(t)$', fontsize=18)

py.legend(('K=1.5', 'K=2', 'K=8'))

py.savefig("fig_slide8.pdf", bbox="tight")
```

# 4.4 Dynamic performance criteria

how long does it take to cross the equilibrium first?

General solution for:

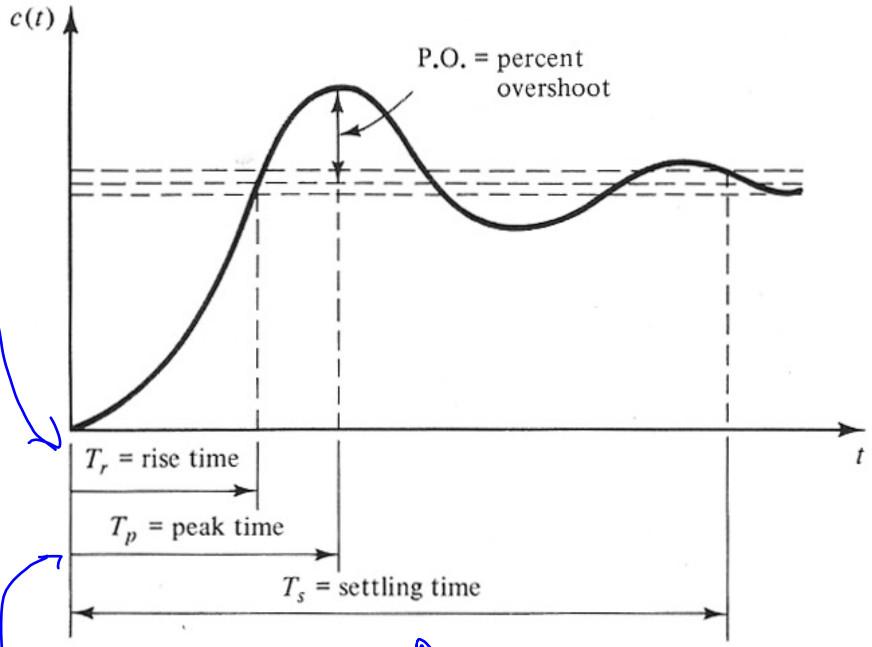
$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega s + \omega_n^2)}$$

is:

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

(compare this expression with the previous result for $\omega_n = 4$ and $\zeta = 0.5$)

There are a number of characteristics that help define the response of a system:




how long to peak?

how long till "settled"

**Rise time**, $T_r$:

the time for the response to reach (a certain percentage of) the steady state value.

**Peak time**, $T_p$:

the time for the response to rise to the first peak.

**Maximum overshoot**, $PO$:

the maximum percentage of the response above the steady state value.

**Settling time**, $T_s$:

the time for the response to reach and stay within a range of the final value (e.g. $2\%$, $5\%$).

The maximum $PO$ occurs at the same time as the peak time, i.e. when $\frac{dc}{dt} = 0$.

From (4.7) and equating the derivative of $c(t)$ to zero we get:

$$\tan\left(\omega_n\sqrt{1-\zeta^2}t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \frac{\sqrt{1-\zeta^2}}{\zeta} \tag{4.8}$$

which implies that at the peaks

$$\omega_n\sqrt{1-\zeta^2}t = i\pi \qquad i = 1, 3, \ldots$$

(hint: use identity $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ to show that (4.8) indeed holds for this condition).

The time at the maximum peak ($i = 1$), the peak time $T_p$, is:

key output

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}} \tag{4.9}$$

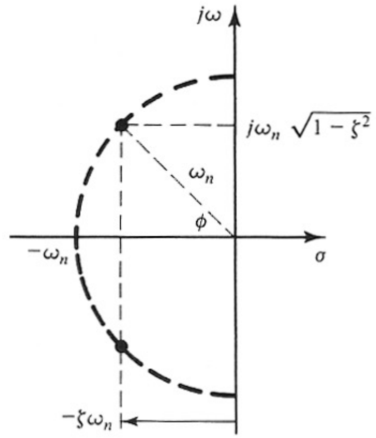
which is a function of natural frequency and damping factor.

The $PO$ is given by:

rise

$$PO = \frac{c_{max} - c_{min}}{c_{final}} \times 100 \qquad (4.10)$$

equilibrium




If the $\tan\phi = \sqrt{1-\zeta^2}/\zeta$ in (4.7), then $\sin\phi = \sqrt{1-\zeta^2}$ :

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

substituting $T_p$ for $t$ into $c(t)$, using $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and the identity $\sin\phi = \sqrt{1-\zeta^2}$ gives the percentage overshoot

$$\boxed{PO = 100\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \qquad (4.11)$$

useful result

which is a function of damping factor only.

To find the settling time, we must determine the time for which $c(t)$ reaches and stays within some percentage bound, e.g. $\pm 2\%$.

A suitable approximation will be given by

$$T_s = 3T \quad (5\%)$$

$$T_s = 4T \quad (2\%) \qquad (4.12)$$

where $T = 1/\zeta\omega_n$, which is a function of natural frequency and damping factor

Finding an analytical expression relating rise time to $\omega_n$ and $\zeta$ is quite complex in the general case.

All the responses tend to rise at roughly the same rate and $T_r$ is always before $T_p$ (assuming the system is not over damped).

Assuming an average damping factor of $\zeta = 0.5$, the rise time from 10% to 90% of $c(t)$) is approximately $\omega_n T_r = 1.8$ then

$$T_r = \frac{1.8}{\omega_n} \tag{4.13}$$

which is a conservative estimate.

**Is this useful?** Determine the approximate transfer function for the unit step response shown.

Exhibits some oscillation and noting that $PO = 100\exp\left(\frac{-\pi\xi}{\sqrt{1-\zeta^2}}\right) \approx 35\%$ $\Rightarrow 100e^{\left(\frac{5.4}{4}\right)}$

Damping factor can be obtained from

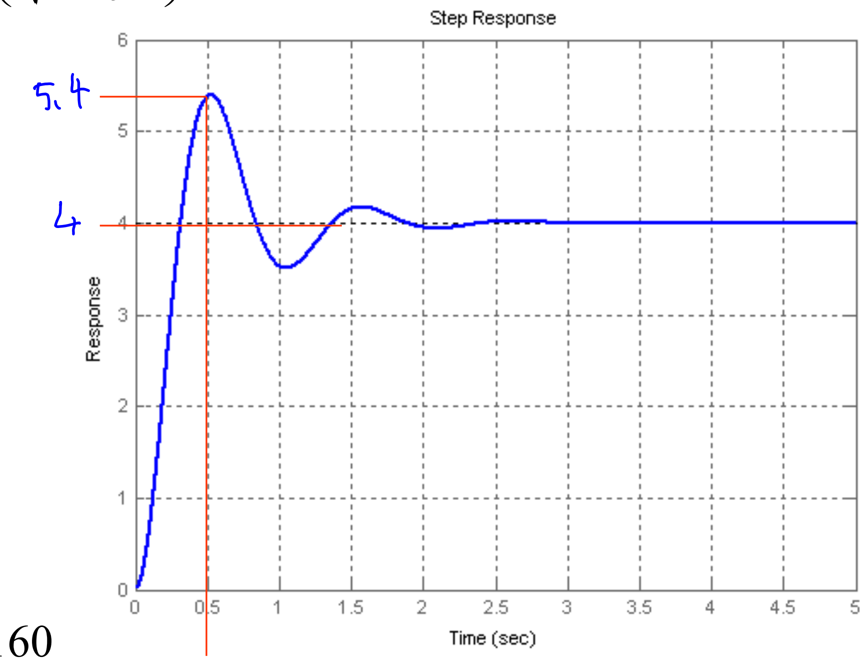
$$\zeta = \frac{-\ln(PO/100)}{\sqrt{\pi^2 + \ln^2(PO/100)}} = 0.32$$

The steady-state output, at $t = 5$, is 4.

So the loop gain $K = 4$

Peak time, $T_p \approx 0.5$ sec, $\omega_n$ can be found as

$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = 6.3$$

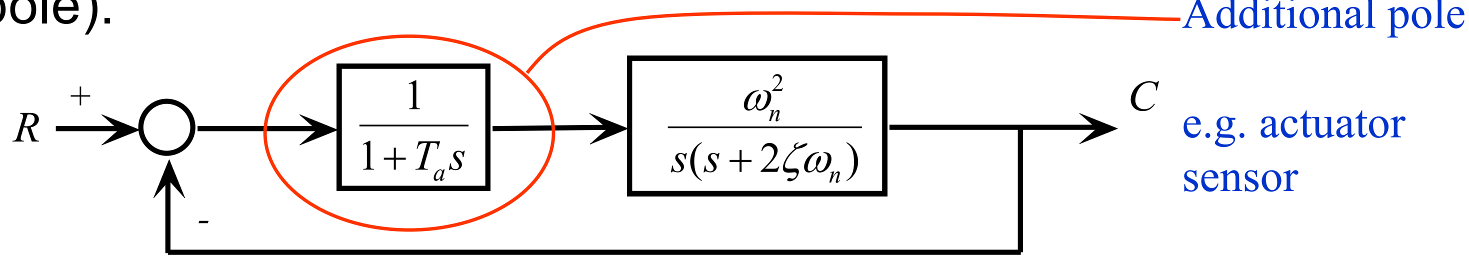



approximate transfer function $\frac{K\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = \frac{160}{s^2 + 4s + 40}$

# 4.5 Additional system poles

We have considered how varying the system gain, $K$, influences the dynamic response of a feedback system.

Consider a second order system placed in series with a first order one (an additional pole).




Without the additional dynamics term, the closed loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

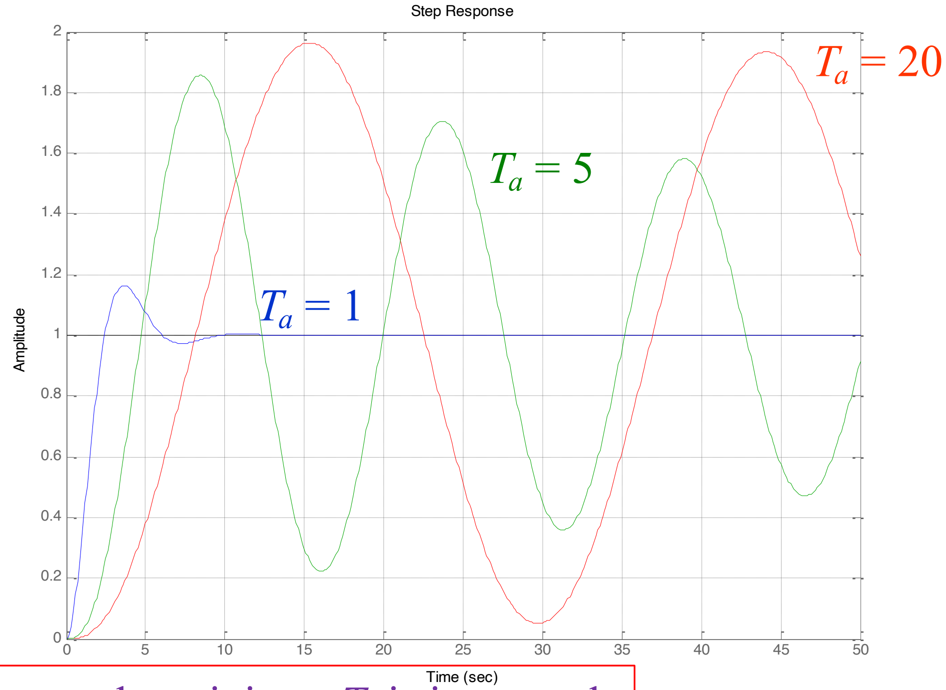
and when the additional dynamics are included, it becomes

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(1+T_a s)(s^2+2\zeta\omega_n s)+\omega_n^2} = \frac{\omega_n^2}{T_a s^3 + (1+2\zeta\omega_n T_a)s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\zeta = 0.5$ and $\omega_n = 1$

effect of the extra pole is to:

- slow the system down
- increase the rise time
- increase the overshoot




Dominant pole gets closer to the origin as $T_a$ is increased

## 4.6 Relationship between system poles and system response

Relating dynamic performance to the pole locations:

1. *Absolute stability*:

   poles must lie in the left half of the $s$-plane.

2. *Relative stability*:

   to avoid excessive overshoot and unduly oscillatory behaviour, $\zeta$ must be adequate and $\phi$ should not be close to 90°.

3. *Settling time*:

   time constant and settling time can be reduced by increasing the poles real part

4. *Frequency of transient oscillations* :

resonant or damped natural frequency, equals the imaginary part of the pole positions.

5. *Undamped natural frequency* :

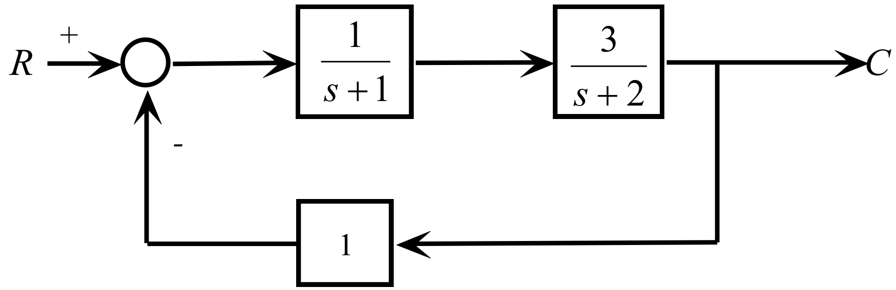
equal to the distance of the poles from the origin. Moving poles further from the origin with constant $\phi$ or $\zeta$ reduces settling time, peak time and rise time while PO remains constant.

6. *Speed of response*:

for a constant real part, improved by increasing the imaginary part until $\zeta$ is reduced to a permissible level, reducing peak and rise times.

## 5.7 Problems

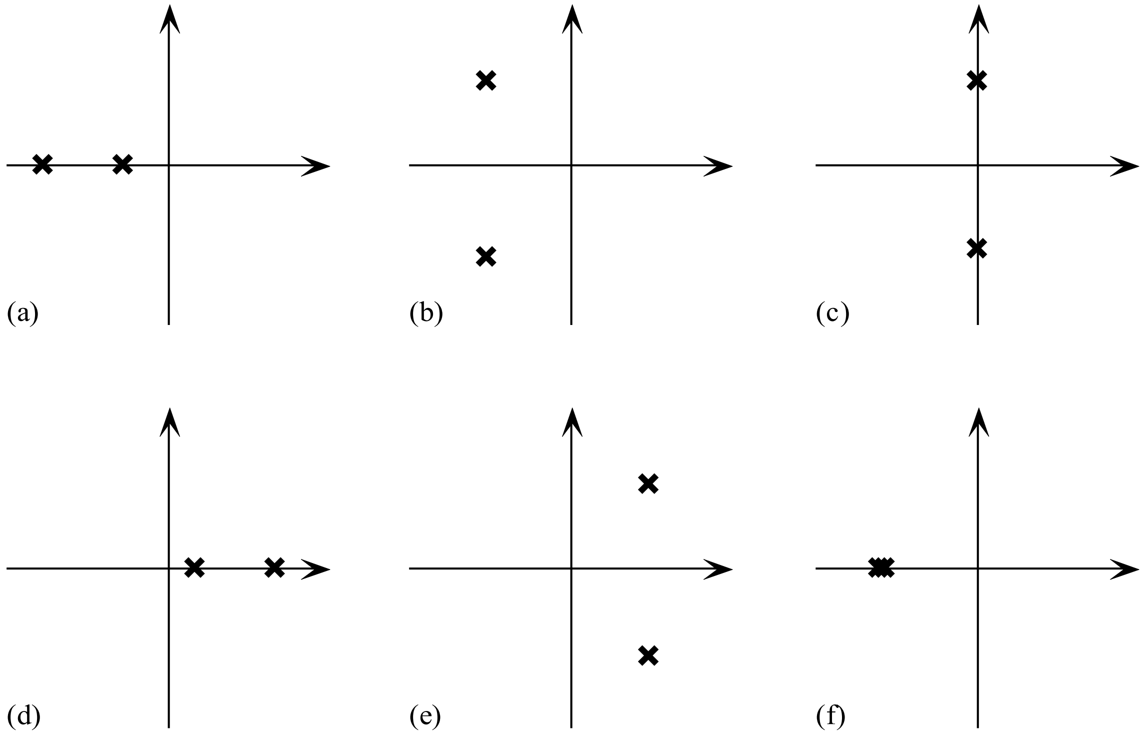
1. Sketch the response of the system below to a unit step input, $R$




2. Sketch responses for the following functions to impulse and unit step inputs

a) $G(s)=\dfrac{2}{10s+1}$ b) $G(s)=\dfrac{5}{s^2+2s+1}$ c) $G(s)=\dfrac{1}{s^2+2s+8}$

3. Approximate the value of $\zeta$ and sketch the response to a unit step input of systems with the following pole locations:



4. By considering the pole locations, sketch the step response of the idle speed controller in an automotive engine management system represented by

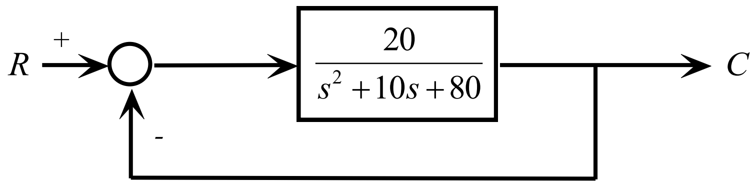
$$G(s) = \frac{50(s+1)(s+2)}{(s^2+5s+40)(s^2+0.03s+0.06)}$$

5. The dynamics of a gas turbine engine are approximated by

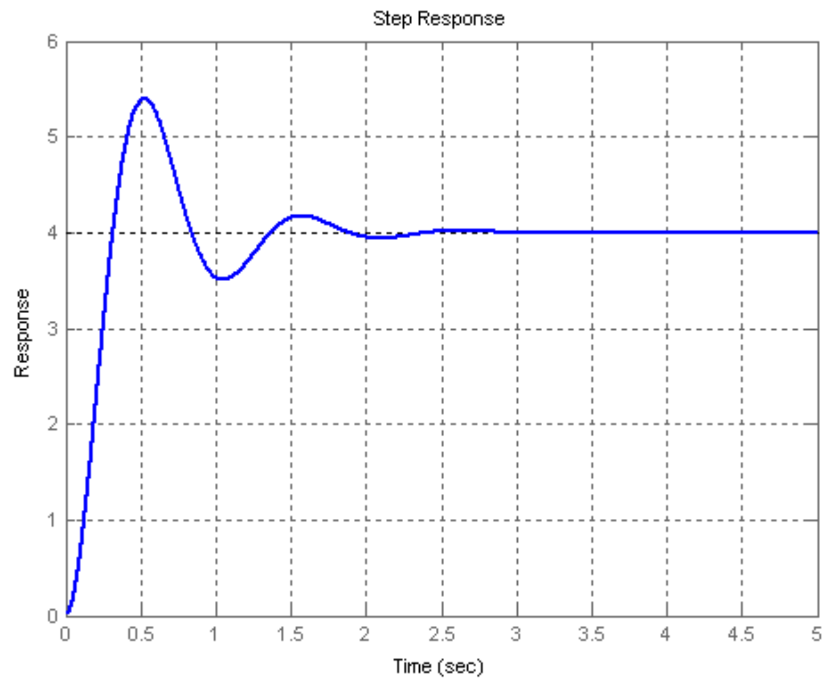
$$T\frac{dy(t)}{dt} + y(t) = Ku(t)$$

$T$ and $K$ are constants, $u(t)$ is the input (fuel) and $y(t)$ the output (jet pipe temperature) both of which are initially zero. A unit step change in fuel is applied at time $t = 0$. Find the solution of $y(t)$.

6. Find the rise time, settling time, overshoot and peak time when a step input is applied to the system below. Draw the time response as accurately as possible.



7. Determine the approximate transfer function for the unit step response shown below. (Q5, 2005 **[9 marks])**




8. Use Matlab/Python to compare the step response for the system in Section 4.5 without the additional (actuator) dynamics (i.e. $T_a = 0$) and when the actuator has $T_a = 1, 5$, and $10$. What happens to the step response if the second order part of the system has the damping factor increased from $\zeta = 0.5$ to $\zeta = 0.707$.