FEEG 2005
Structures: Lecture 4

Review, Torsion of circular sections

Arbitrary Closed Thin-Walled Sections

Summary of last lectures

- Completed bending and shear of arbitrary thin-walled sections:
  - Axial bending stress:

$$\boxed{\sigma_{xx} = \frac{\left(M_z I_y - M_y I_{yz}\right)y + \left(M_y I_z - M_z I_{yz}\right)z}{\left(I_y I_z - I_{yz}^2\right)}}$$

$$\boxed{\tan\phi = -\frac{\left(M_y I_z - M_z I_{yz}\right)}{\left(M_z I_y - M_y I_{yz}\right)}}$$

  - Deflection due to bending:

$$\boxed{\frac{d^2 v}{dx^2} = -\frac{M_z I_y - M_y I_{yz}}{E\left(I_y I_z - I_{yz}^2\right)}}$$

$$\boxed{\frac{d^2 w}{dx^2} = -\frac{M_y I_z - M_z I_{yz}}{E\left(I_y I_z - I_{yz}^2\right)}}$$

  - Shear stress due to bending:

Open/Closed

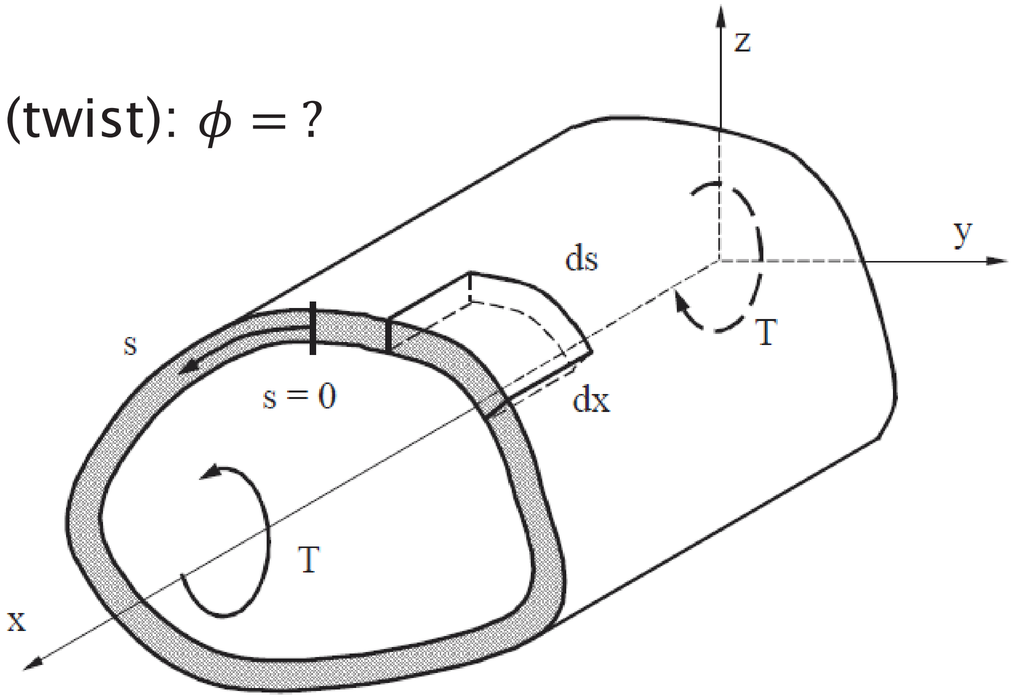
$$\boxed{q_{in} = \frac{\left(Q_y I_y - Q_z I_{yz}\right)\bar{y}A + \left(Q_z I_z - Q_y I_{yz}\right)\bar{z}A}{\left(I_y I_z - I_{yz}^2\right)} + q_{out}}$$

Closed

$$\boxed{\frac{1}{G}\oint \frac{q}{t}\,ds = 0}$$

This lecture

- Develop torsion theory for circular and arbitrary closed thin-walled sections:
  - 1) Stress: $\tau = ?$
  - 2) Deformation (twist): $\phi = ?$



Where does torsion occur?

- Transfer of mechanical power requires drive shafts which are subjected to torsion:

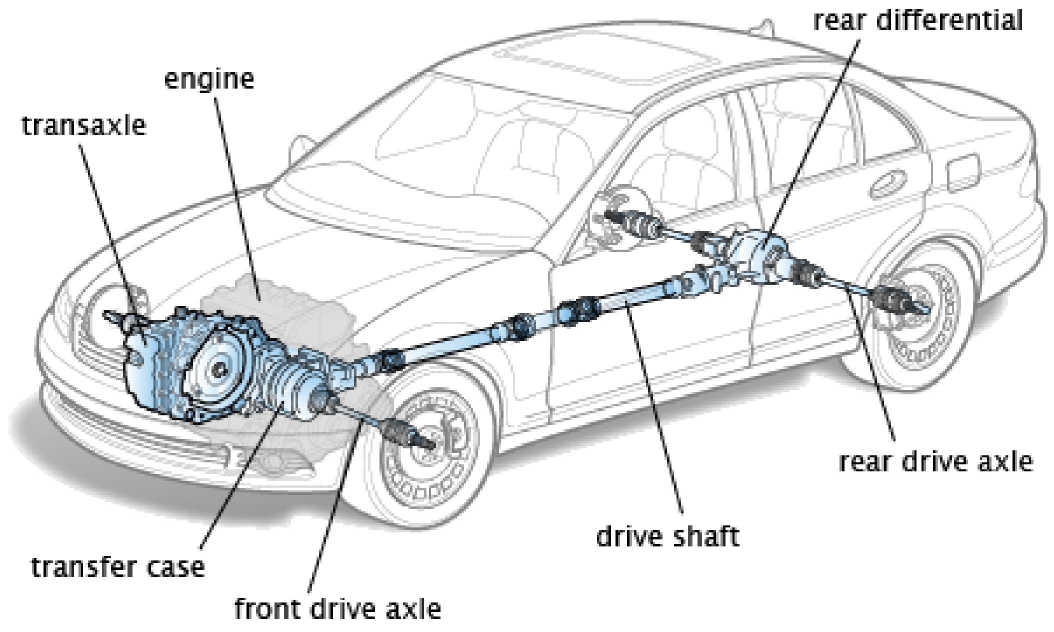
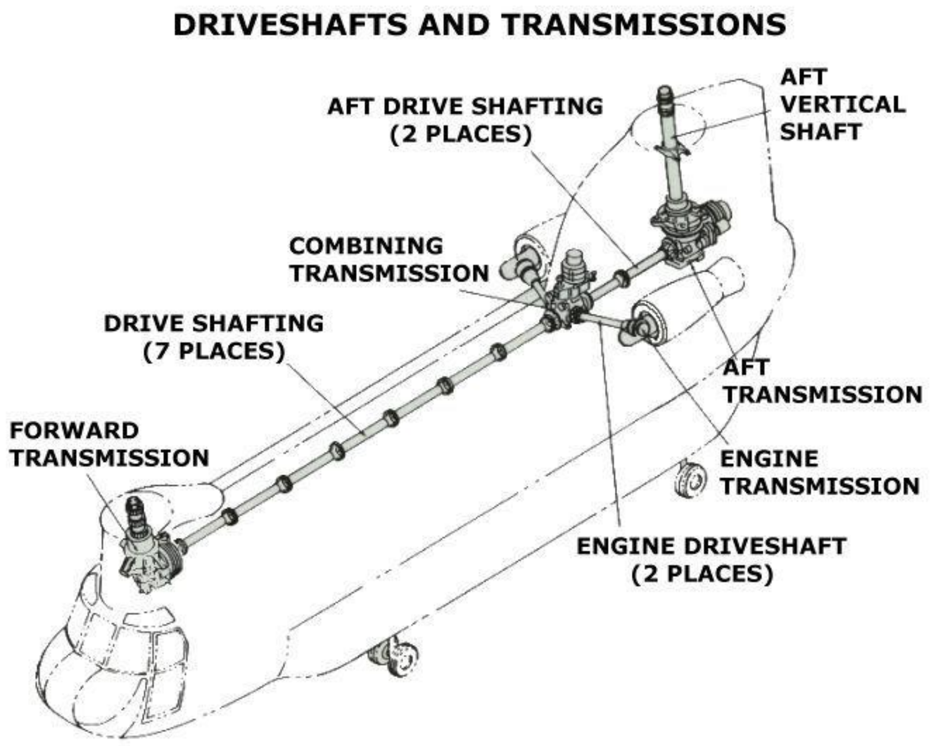


Image courtesy of ClearMechanic.com



Review, Torsion of circular sections

Question

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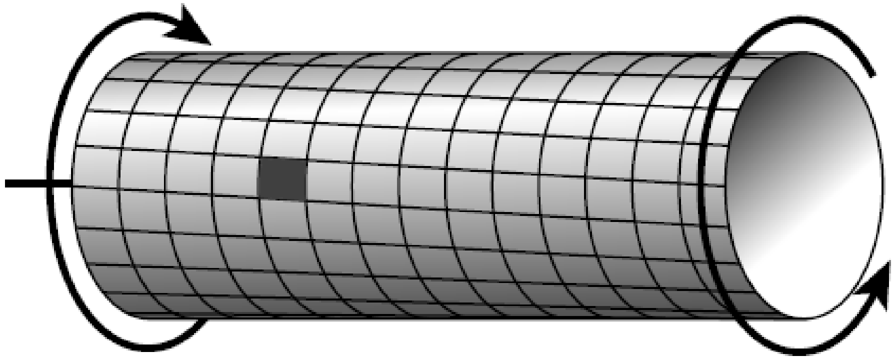
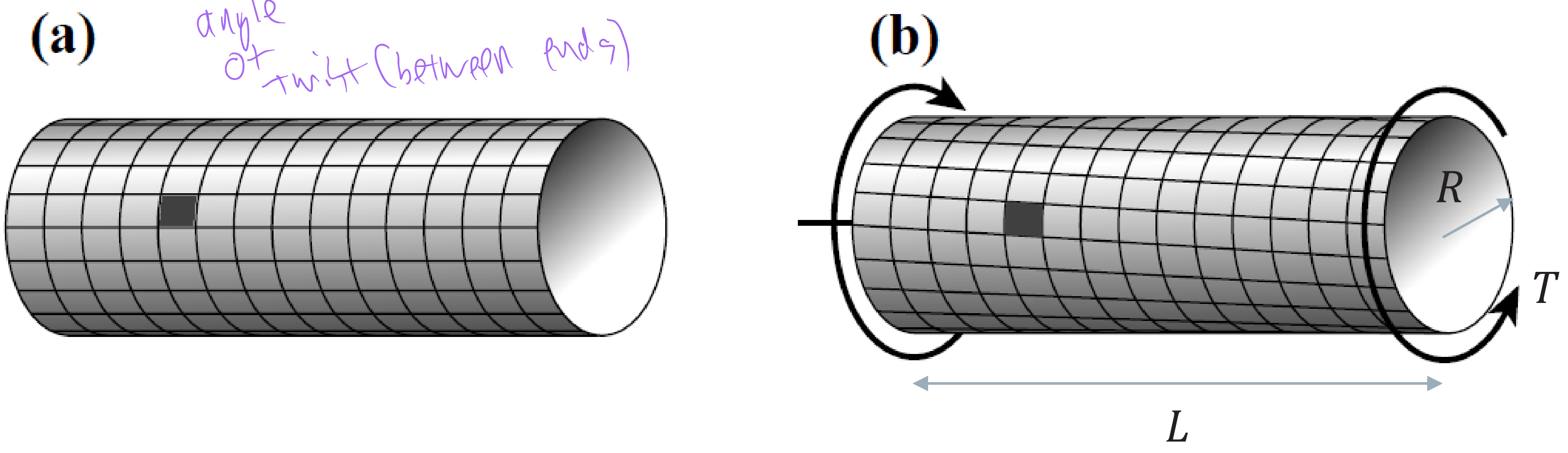
- Which points in a circular rod under twisting torque may fail first?
  - The centreline
  - Outer surface
  - Between the centreline and outer surface



Torsion theory from Statics

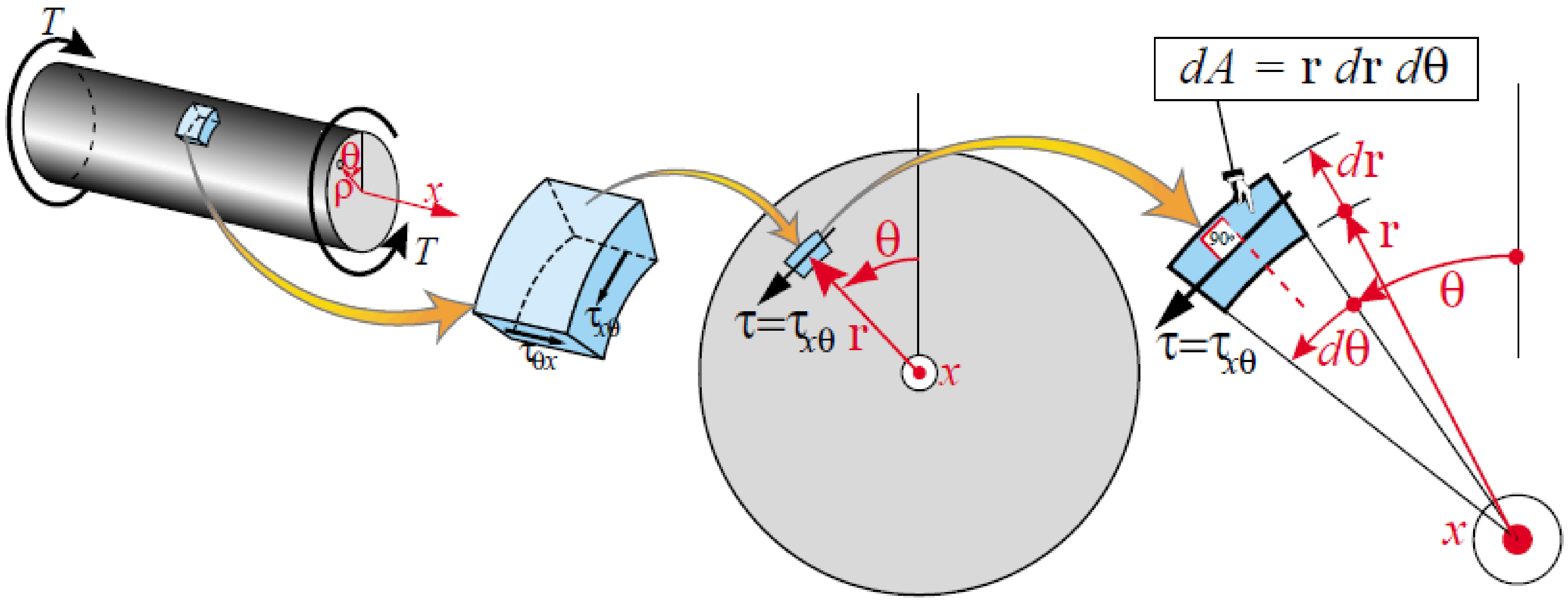
statics (year 1)

- 1) Stress: $\tau = \frac{Tr}{J}$ where $J = \frac{\pi R^4}{2}$ in a circular section and r is the distance of a point from the centre
- 2) Deformation: $\underbrace{\phi}_{\text{angle of twist (between ends)}} = \frac{\overbrace{T}^{\text{torque}}L}{GJ}$ where G = Shear modulus



Torsion theory: Polar co-ordinates

- Torsion is often applied to circular drive shafts, so it is easier to work in polar coordinate



Equilibrium of a circular rod loaded in torsion

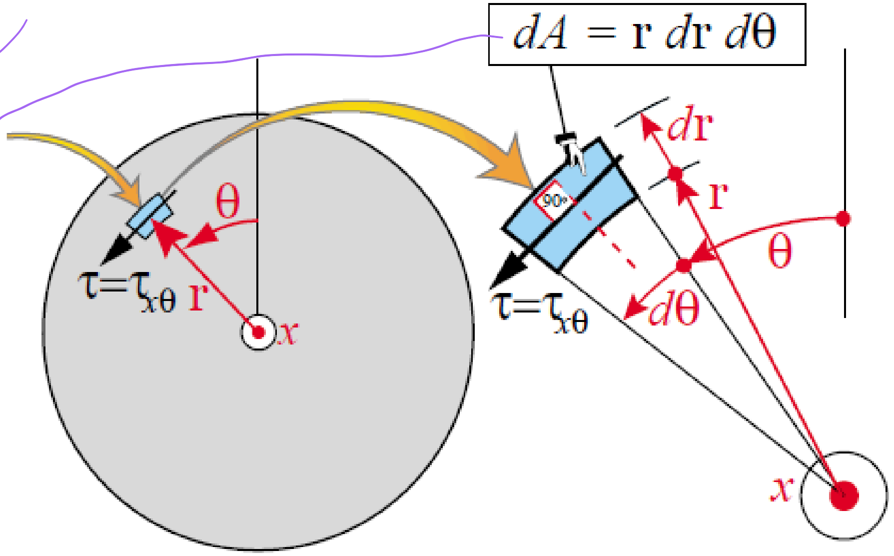
we assume a linear relationship between τ and τ_{max} such that $\tau(r) = \frac{r}{R}\tau_{max}$

$$T = \int_A (\text{shear stress} \times \text{element area}) \times \text{lever arm}$$

$$T = \int_A (\tau\, dA) r = \int_A \frac{r}{R}\tau_{max} r\, dA = \frac{\tau_{max}}{R}\int_A r^2 dA$$

$$T = \frac{\tau_{max}}{R}\int_A r^2 (r dr\, d\theta) = \frac{\tau_{max}}{R}\int_A r^3 dr\, d\theta = \frac{\tau_{max}\, J}{R}$$

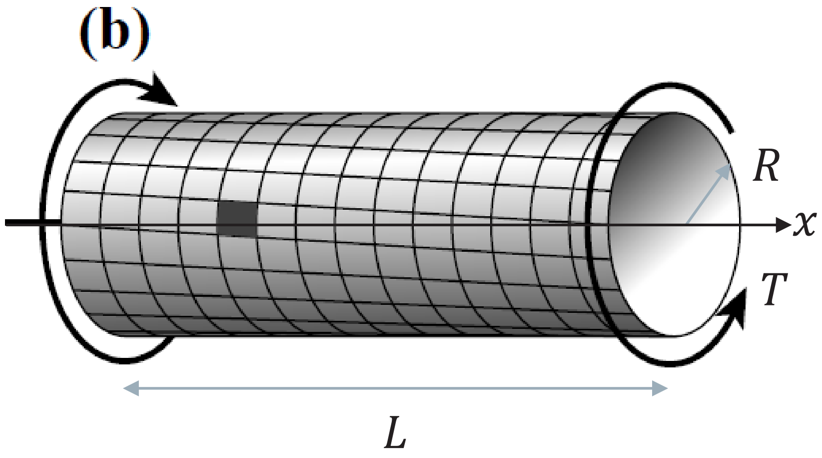
Where J is the **polar second moment of area** about the 'x' axis: $J = \int_A r^2 dA = \int_A r^3 dr d\theta$



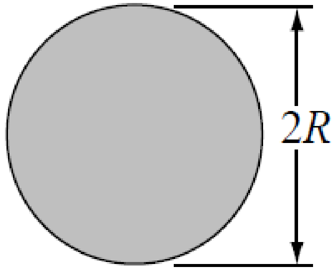
Shear stress in a circular rod loaded in pure torsion

- The shear stress at radial location 'r' is given by: $\boxed{\tau = \frac{Tr}{J}}$

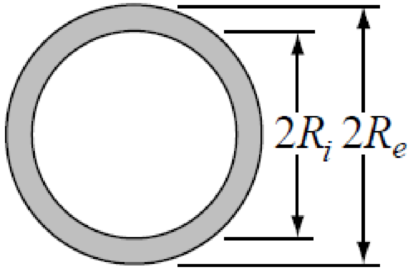
- The max shear stress occurs at the outer radius 'R': $\boxed{\tau_{max} = \frac{TR}{J}}$



- Where:



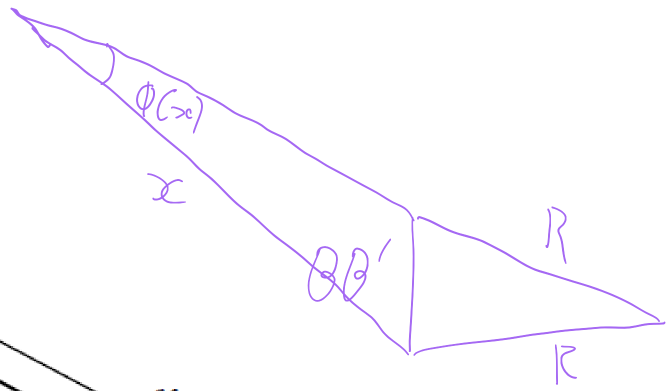
$$J = \frac{\pi R^4}{2}$$

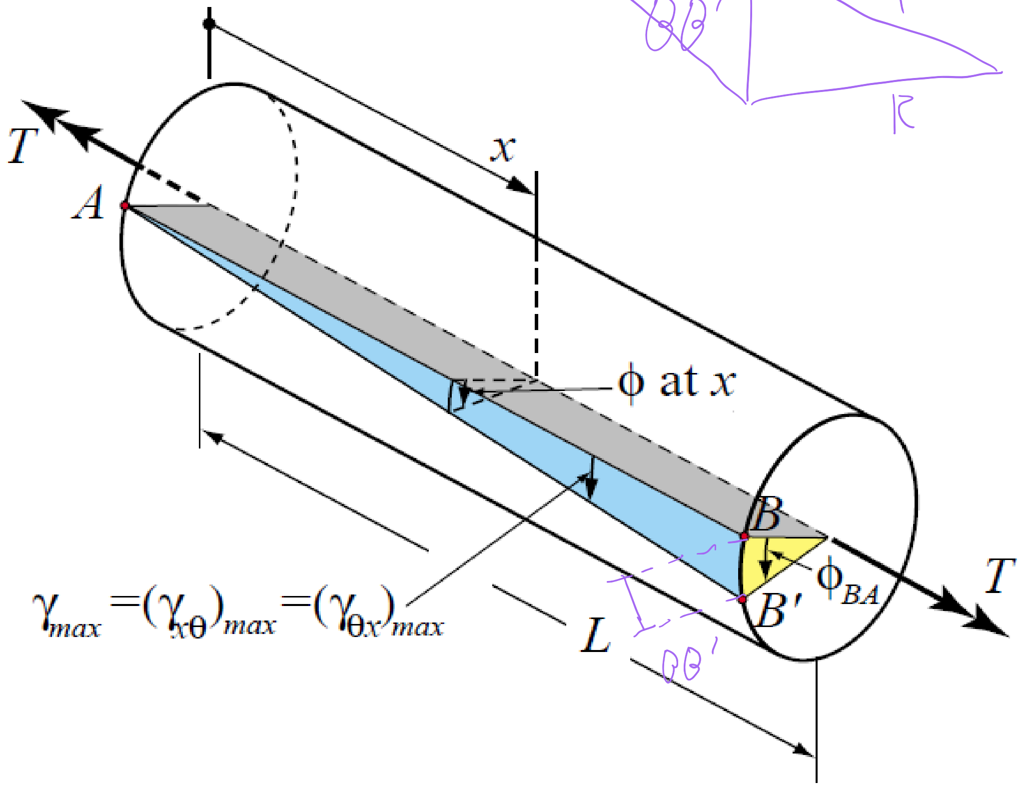


$$J = \frac{\pi(R_e^4 - R_i^4)}{2}$$

Rate of twist of a circular rod

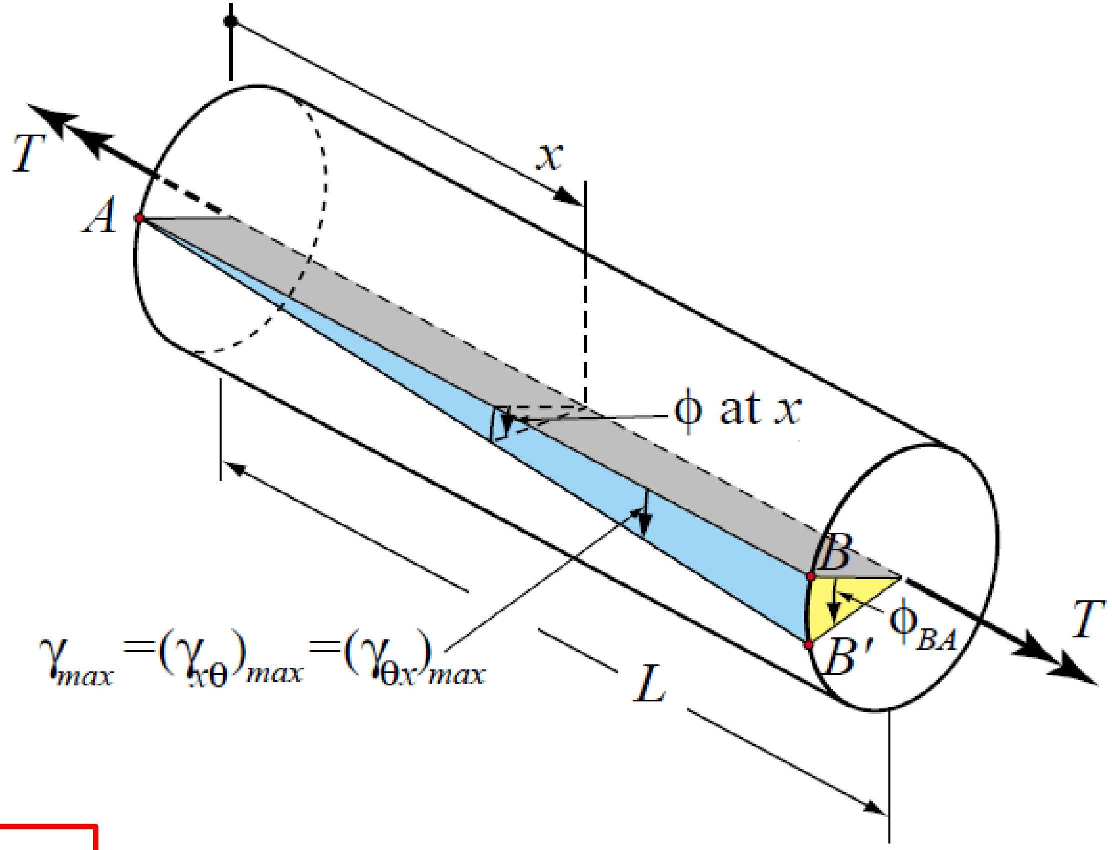
- The relative rotation of one end of a torqued rod with respect to the other is given by the twist angle: ϕ
- The rate of twist is given by the amount of twist per unit length of the rod: $\frac{d\phi}{dx}$
- For small angles: $BB' \approx R\phi_{BA} \approx L\gamma_{max}$
- At an arbitrary distance 'x' from the free end: $r\phi \approx x\gamma$, $\frac{\phi}{x} \approx \frac{\gamma}{r}$
- So, the rate of twist is: $\frac{d\phi}{dx} = \frac{\gamma}{r}$





Rate of twist of a circular rod

- Combine these three equations:
  - Hooke's law in shear: $\tau = G\gamma$
  - Shear stress due to torsion: $\tau = \frac{Tr}{J}$
  - Rate of twist: $\frac{d\phi}{dx} = \frac{\gamma}{r}$
- Therefore, the rate of twist is:



$$\boxed{\frac{d\phi}{dx} = \frac{T}{GJ}}$$

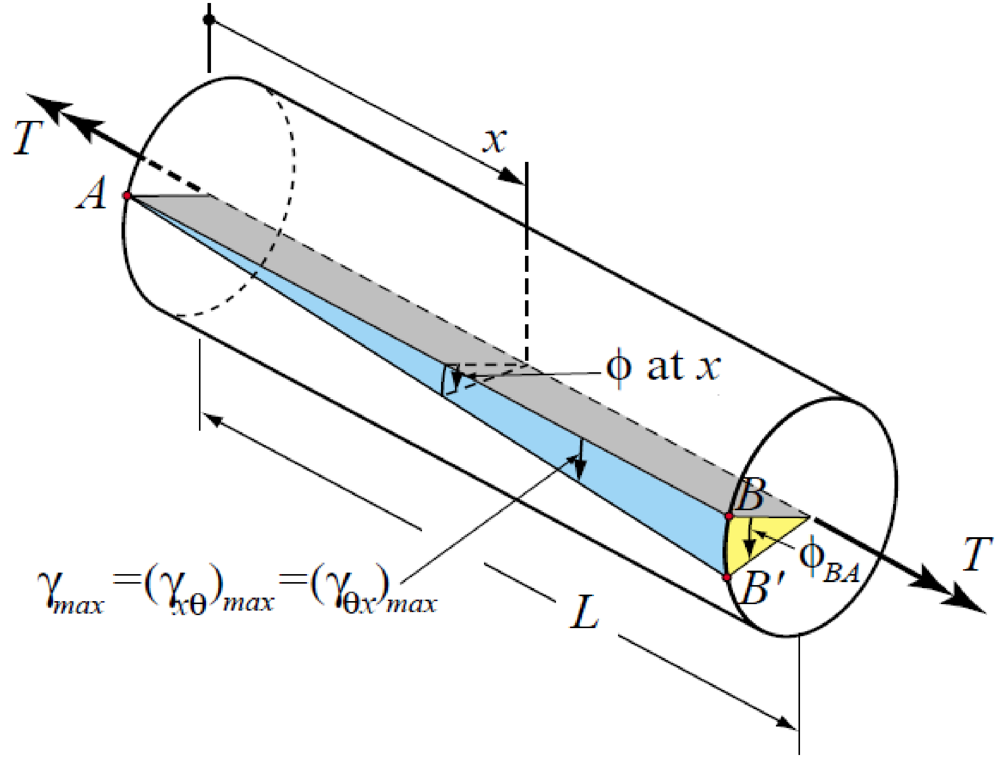
Angle of twist of a circular rod

- To get the angle of twist for the whole rod we integrate along the rod:

$$\phi = \int_0^L \frac{d\phi}{dx} dx = \int_0^L \frac{T}{GJ} dx$$

- If T, G and J are constant along the length then:

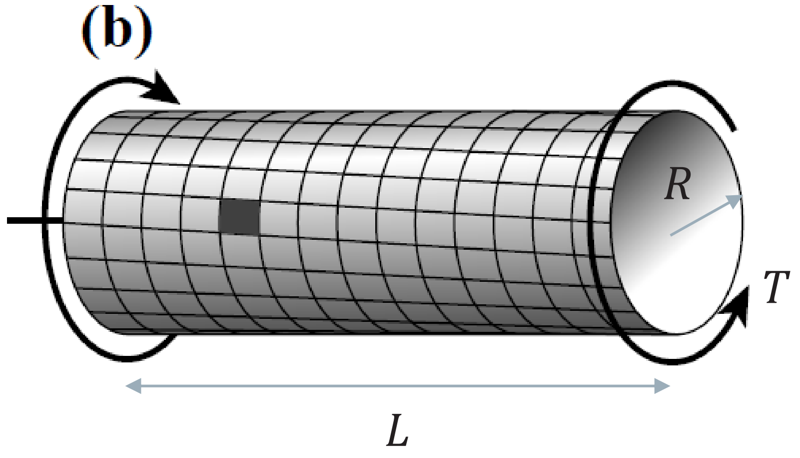
$$\phi = \frac{T}{GJ} \int_0^L dx \Rightarrow \boxed{\phi = \frac{TL}{GJ}}$$

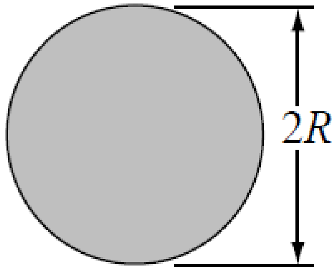


Summary: Torsion of a circular rod

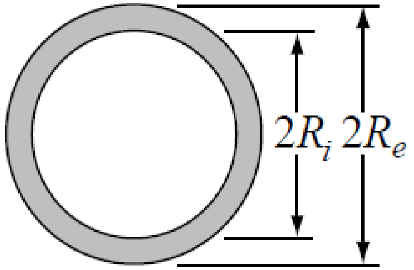
- 1) Stress: $\boxed{\tau = \frac{Tr}{J}}$ $\boxed{\tau_{max} = \frac{TR}{J}}$

- 2) Deformation: Rate of twist: $\boxed{\frac{d\phi}{dx} = \frac{T}{GJ}}$ Angle of twist: $\boxed{\phi = \frac{TL}{GJ}}$





$$J = \frac{\pi R^4}{2}$$

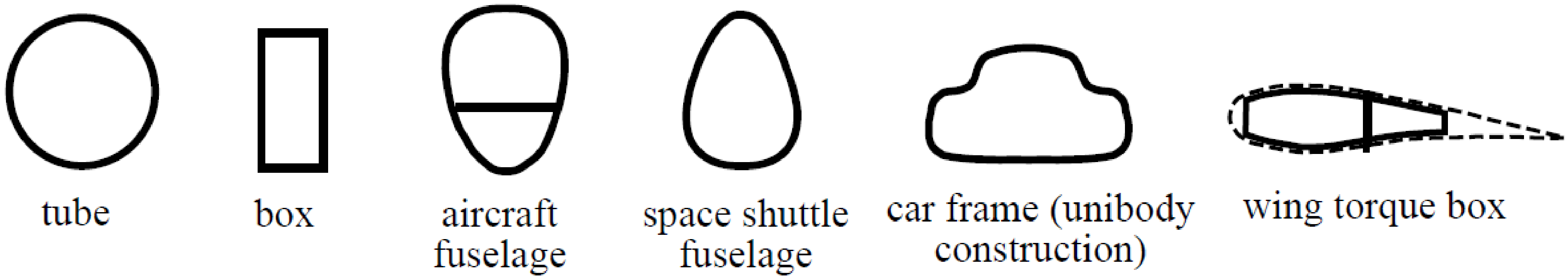


$$J = \frac{\pi(R_e^4 - R_i^4)}{2}$$

Arbitrary Closed Thin-Walled Sections

Arbitrary closed sections under torsion

- A closed thin-walled section is one in which uninterrupted circuits of shear flow ‘q’ can occur



- The shear flow $q = \tau t$ resists the applied torque T

Question

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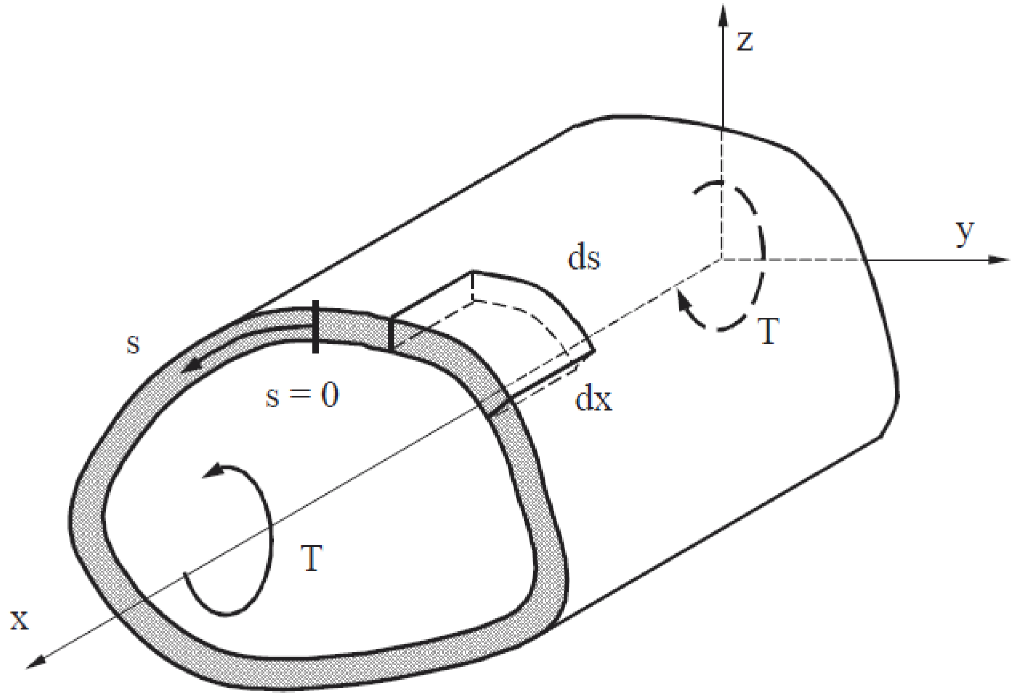
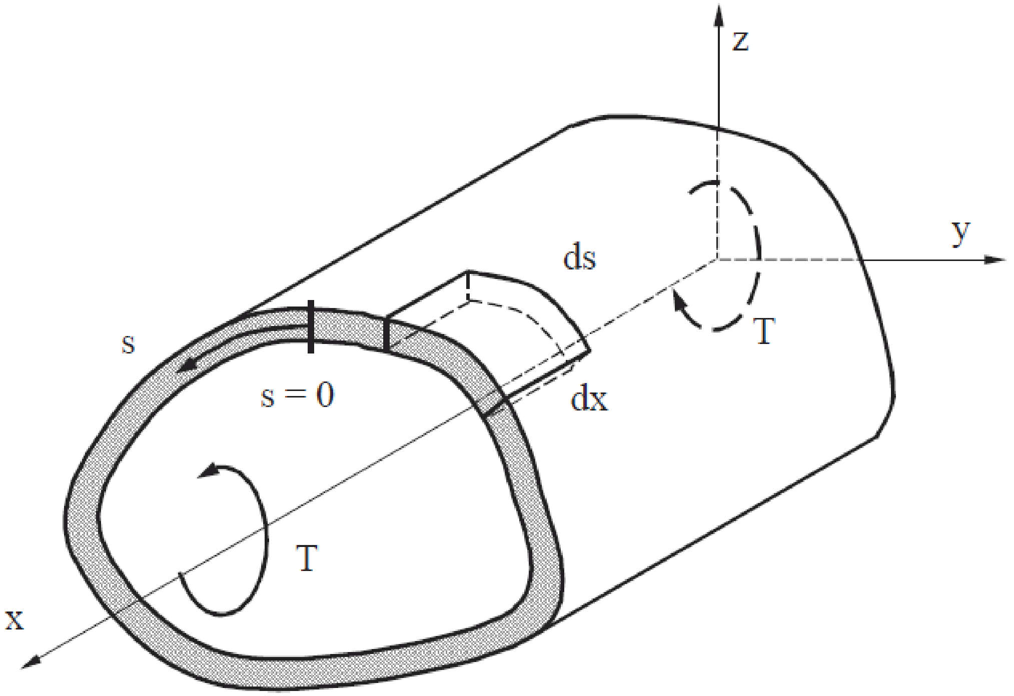
- Which point(s) in an arbitrary thin-walled structure under a twisting torque experience the highest shear stress?
  - The points furthest away from the centreline
  - The inner surface
  - For a section with a constant low thickness wall, all points on a cross section will reach max stress more or less similarly.



Assumptions: Torsion of arbitrary closed sections

- The cross section does not vary along the length (i.e. along the x axis)
- The cross section is closed
- The wall thickness ‘t’ can vary about the section but must be small compared to other dimensions (i.e. a factor of 10 smaller)
  - NOTE: this implies the shear stress can be assumed to be constant through the wall thickness
- The member is subjected to end torques or distributed torques only
- The ends are free to warp



Equilibrium of an Arbitrary Closed Section under Torsion

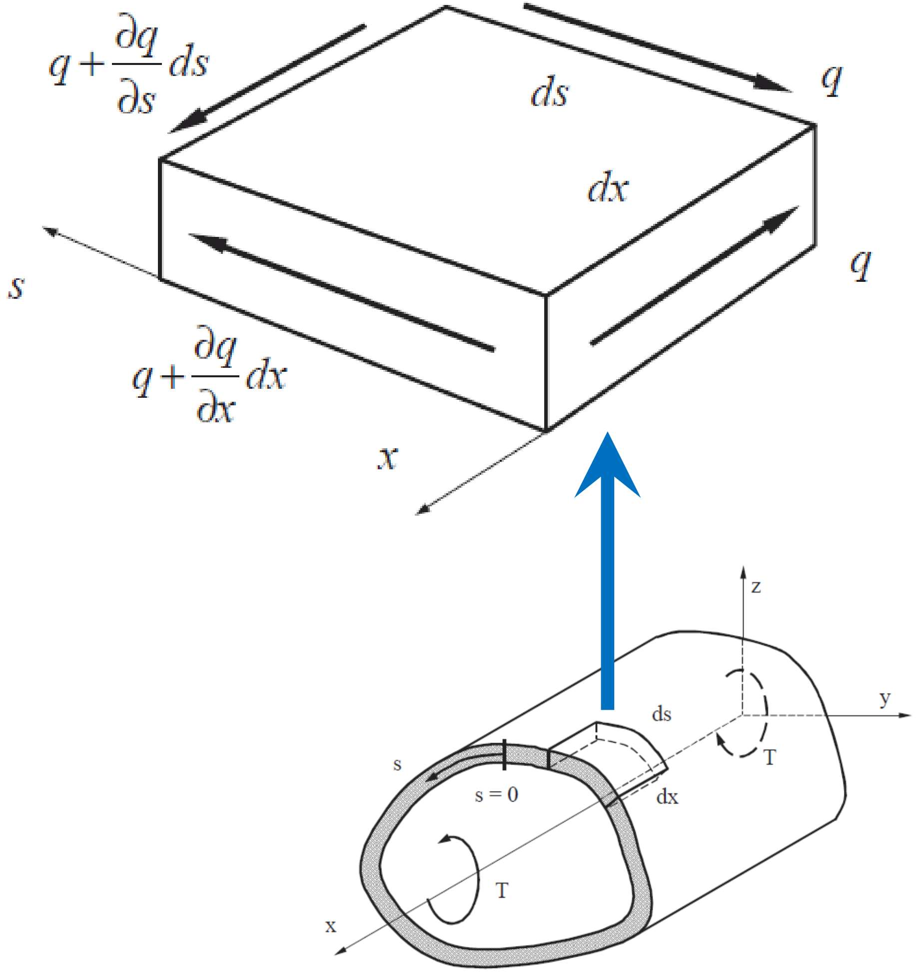
- Consider a stress element for the case of pure shear/torsion and apply equilibrium:

$$\sum F_x = \left(q + \frac{dq}{ds}ds\right)dx - qdx = 0 \Rightarrow \frac{dq}{ds} = 0$$

$$\sum F_s = \left(q + \frac{dq}{dx}dx\right)ds - qds = 0 \Rightarrow \frac{dq}{dx} = 0$$

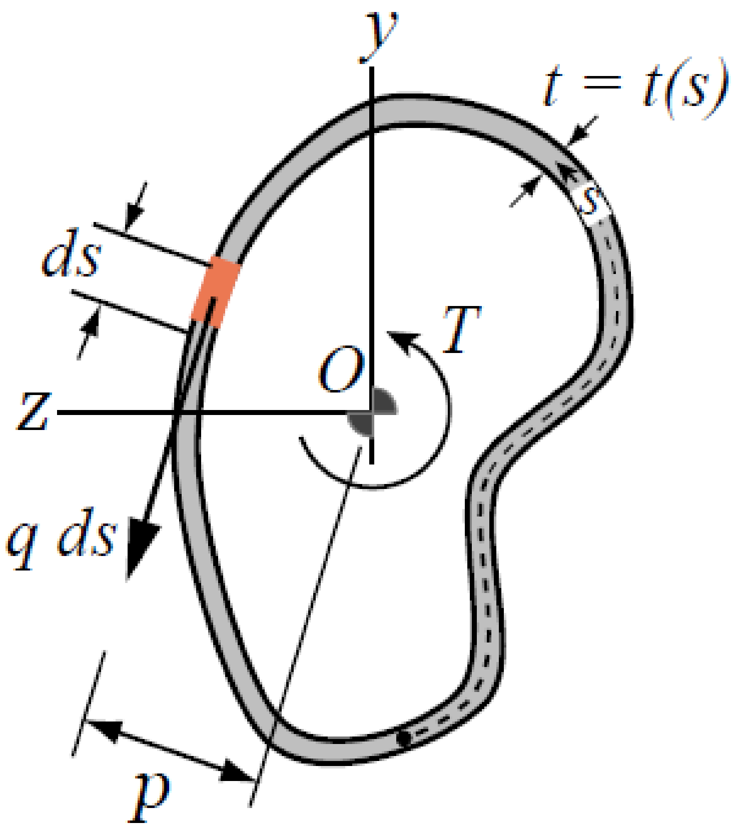
- The rate of change of the shear flow about the section is zero so the shear flow due to an applied torque must be constant:

$$\boxed{q = \tau t = constant}$$

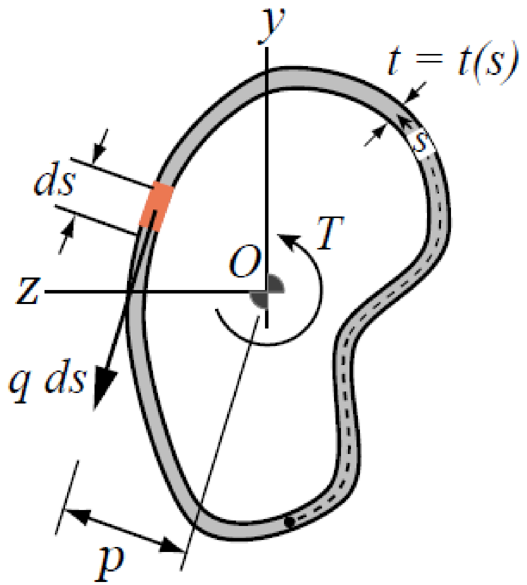


Equilibrium of an arbitrary closed section under torsion

- This constant shear flow ‘q’ must be in equilibrium with the applied torque ‘T’.
- Consider a small slice about the perimeter of the section ‘ds’, the differential torque ‘dT’ is then given by: $dT = pq\ ds$
- where ‘p’ is the perpendicular moment arm to the median line of the slice ‘ds’

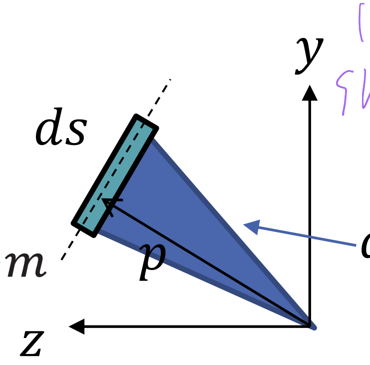


Equilibrium of an Arbitrary Closed Section under Torsion



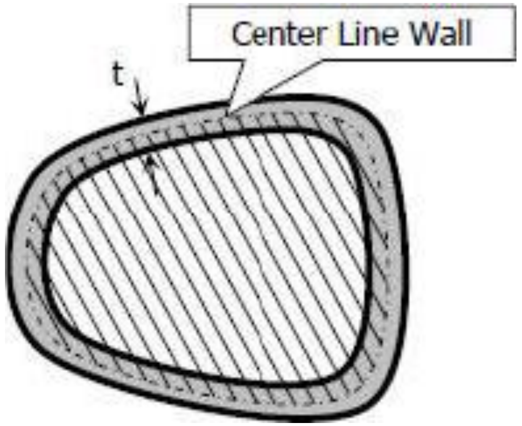
- The total torque is obtained if we integrate around the whole section (equilibrium): $T = \oint \underbrace{q}_{\text{shear}}\underbrace{p}_{\text{perpendicular distance}} \overbrace{ds}^{\text{small length}} = q \oint p\, ds$ - $\underbrace{\text{‘}q\text{’}}_{\text{as shown prev}}$ is constant

- $p ds = 2\, dA_m \Rightarrow T = 2q \oint dA_m$



$dA_m = \dfrac{p\, ds}{2}$

- $\oint \underbrace{dA_m}_{\text{area of small section}} = A_m \Rightarrow \boxed{T = 2qA_m = 2\tau t A_m}$



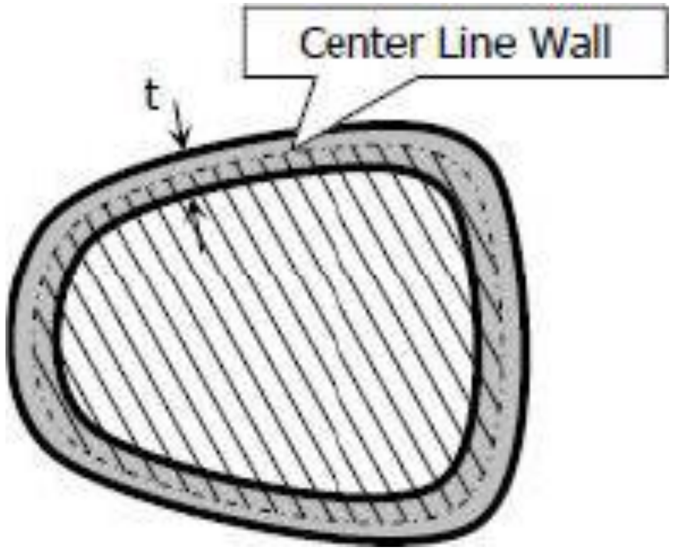
- **NOTE:** A_m is the area enclosed by the median line!

Shear stress in closed sections

- The shear stress in a thin walled closed section subjected to pure torsion is:

$$\boxed{\tau = \frac{T}{2A_m t}}$$

- NOTE: A_m is the area enclosed by the median line!



- The max shear stress occurs where the torque is highest and the walls are thin:

$$\boxed{\tau_{max} = \frac{T_{max}}{2A_m t_{min}}}$$

Twist of thin-walled closed sections

- Closed thin-walled sections will generally 'twist' and 'warp' so the total shear strain is (from geometry): $\gamma = \gamma_t + \gamma_w$

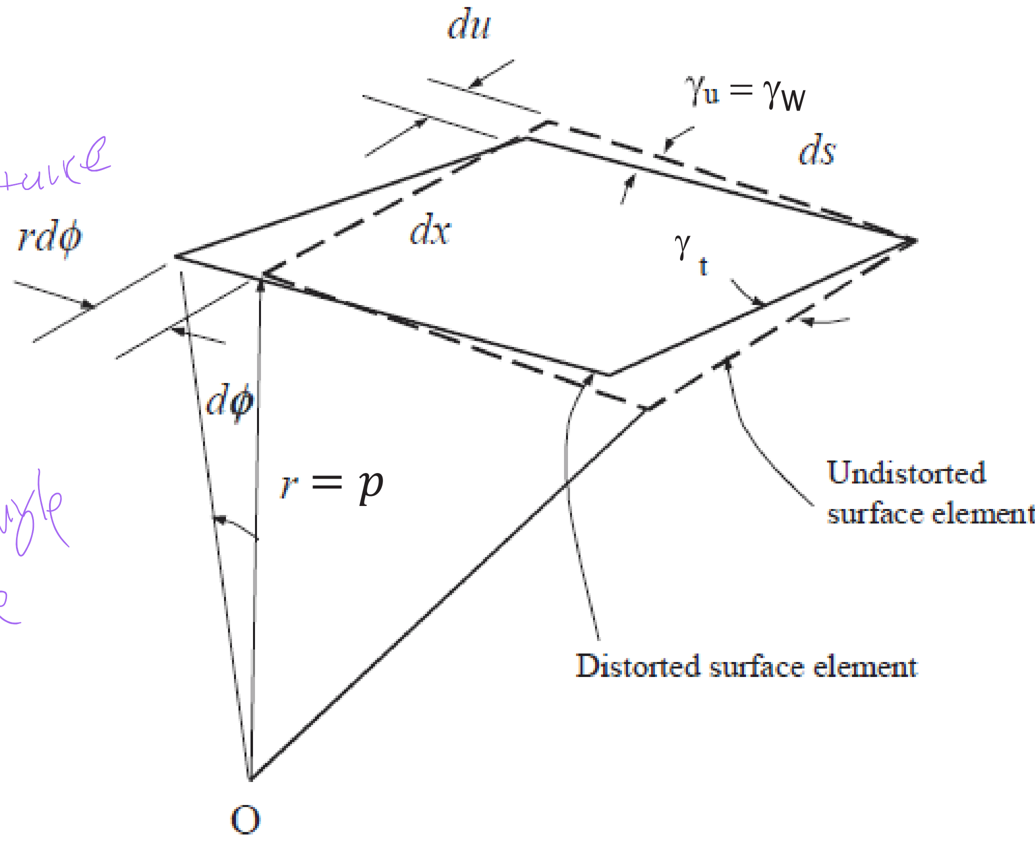
- Shear strain due to twist is: $\gamma_t \approx r\frac{d\phi}{dx}$ or

  centre distance

  rotation angle rate

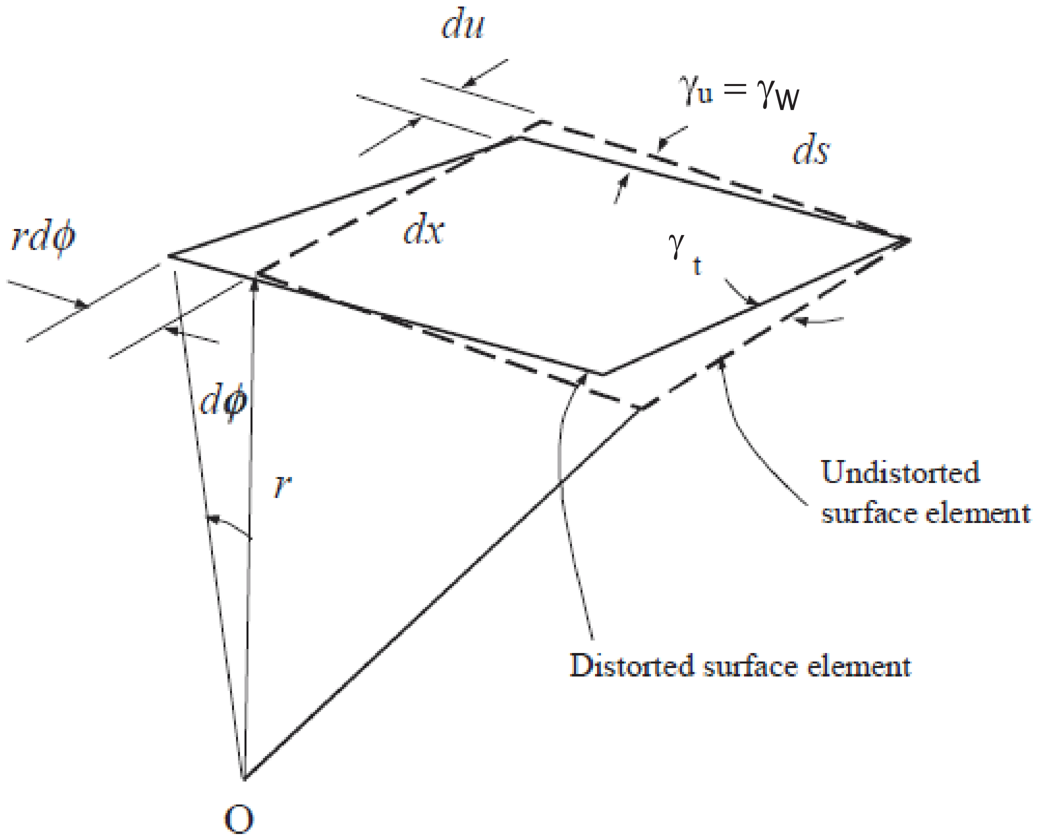
  $\gamma_t \approx p\frac{d\phi}{dx}$

- Shear strain due to warping is: $\gamma_w \approx \frac{du}{ds}$



Twist of thin-walled closed sections

- Combining the equations for shear strain due to 'twist' and 'warp': $\gamma = p\dfrac{d\phi}{dx} + \dfrac{du}{ds}$
- Now combine with:
  1. Hooke's Law in shear: $\tau = G\gamma$
  2. Shear stress: $\tau = \dfrac{T}{2A_m t}$
- $\dfrac{T}{2A_m G t} = p\dfrac{d\phi}{dx} + \dfrac{du}{ds}$



Twist of thin-walled closed sections

G=shear stiffness

- Integrate about the whole section:

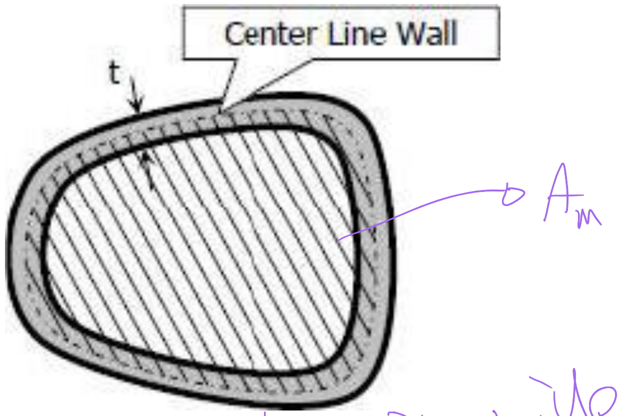
$$\frac{1}{2A_m}\oint\frac{T}{Gt}ds=\frac{d\phi}{dx}\overset{2A_m}{\boxed{\oint p\,ds}}+\overset{0\text{ (continuity)}}{\cancel{\oint\frac{du}{ds}ds}}\Rightarrow\frac{1}{2A_m}\oint\frac{T}{Gt}ds=2A_m\frac{d\phi}{dx}$$

take away; ignore warping since its impact is negligible

- For the general case: $\boxed{\dfrac{d\phi}{dx}=\dfrac{1}{4A_m^2}\oint\dfrac{T}{Gt}ds}$

- If G is constant: $\boxed{\dfrac{d\phi}{dx}=\dfrac{T}{GJ}}$ where $\boxed{J=\dfrac{4A_m^2}{\oint\dfrac{ds}{t}}}$

Summary: Torsion of Thin-Walled Closed Sections



- 1) Stress: $\boxed{\tau = \dfrac{T}{2A_m t}}$ $\boxed{\tau_{max} = \dfrac{T_{max}}{2A_m t_{min}}}$
- NOTE: A_m is the area enclosed by the median line! → So make it wide
- 2) Deformation (Twist):

$$\boxed{\frac{d\phi}{dx} = \frac{T}{4A_m^2}\oint \frac{1}{Gt}\,ds}$$

If G is constant:

$$\boxed{\frac{d\phi}{dx} = \frac{T}{GJ}} \;+\; \boxed{J = \frac{4A_m^2}{\oint \frac{ds}{t}}}$$

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Example – Have a go first!

- The light-alloy stabilizing strut of a high-wing monoplane is 2 m long (L=2 m) and has the cross-section shown in the figure.
- Determine the torque that can be sustained if the maximum shear stress is limited to 28 MPa
- Determine the corresponding angle of twist if G = 27 GPa