

## 2. MODELLING OF DYNAMIC SYSTEMS

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- The idea of using a mathematical model to describe a system we wish to control was introduced in Topic 1.
- This model describes the relationships between the variables of interest and is used to approximate the output of a system given the value(s) of its input(s).
- For a dynamic system, the usual case, differential equations can be used to model its dynamic behaviour.
- The complexity of the model should (normally) be the minimum necessary to represent the system.

## 2.1 Introduction

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Mathematical systems can be **linear** or **nonlinear** and be **time-variant** or **time-invariant**.

Linear systems are those that are described by sets of linear equations. For example,

$$y = 4x + 7$$

is a linear model describing the relationship between the output,  $y$ , and the variable  $x$ .

However, the expression

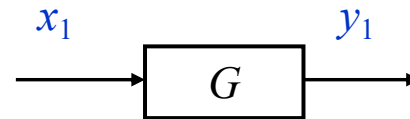
$$y = x^3 + 2x^2 + 12$$

is a nonlinear model describing the relationship between  $y$  and  $x$ .

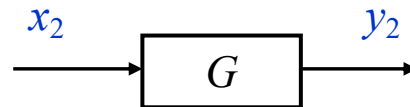
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All linear systems must obey the **principle of superposition**.

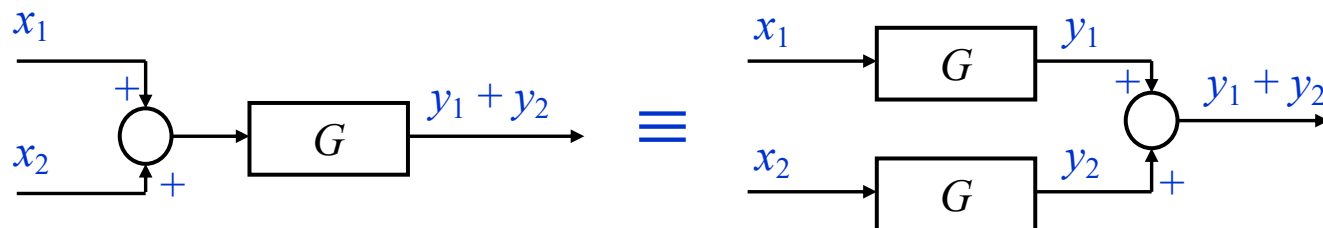
- When a system at rest is subjected to an excitation  $x_1$ , it produces a response  $y_1$ .



- The same system will produce a response  $y_2$  for an excitation  $x_2$ .



- The principle of superposition states that if the system is subjected to the excitation  $x_1 + x_2$  then the response will be equal to  $y_1 + y_2$ .



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A system is said to be time-invariant where the coefficients are constant with respect to time. For example,

$$\alpha_1 \frac{d^2 y}{dx^2} + \alpha_2 \frac{dy}{dx} = c$$

is a time-invariant model as the coefficients of the equation,  $\alpha_1$  and  $\alpha_2$  are not dependent on time,  $t$ .

The system described by the equation

$$t^2 \frac{d^2 y}{dx^2} + y = x$$

where  $t$  is time, is time-variant as one of the coefficients changes with time.

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## Linear differential equations



An  $n^{\text{th}}$  order differential equation takes the form:

$$\begin{aligned} a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \dots + a_1 \frac{dc(t)}{dt} + a_0 c(t) \\ = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \dots + b_1 \frac{dr(t)}{dt} + b_0 r(t) \end{aligned}$$

where  $a_n \neq 0$  or the order would be less than  $n$ .

The usual convention is to set  $a_n = 1$  and without loss of generality the  $n^{\text{th}}$  order differential equation has the form

$$\begin{aligned} \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \dots + a_1 \frac{dc(t)}{dt} + a_0 c(t) \\ = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \dots + b_1 \frac{dr(t)}{dt} + b_0 r(t) \end{aligned}$$

## 2.2 Electrical circuit example

Consider the RLC circuit

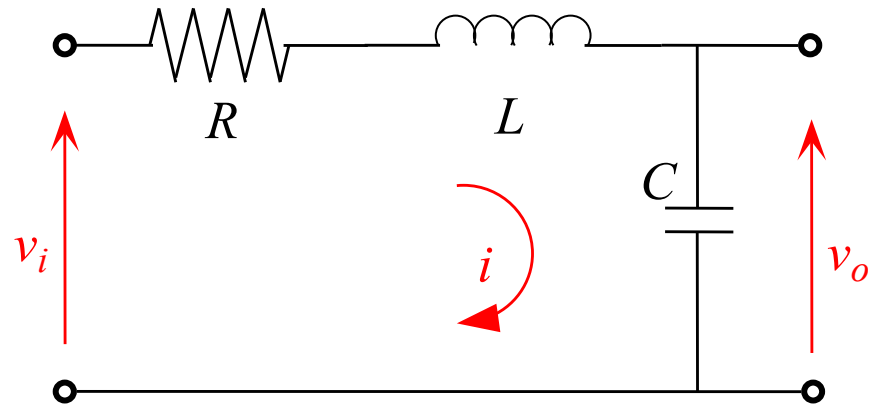
$$v_i = v_o + v_R + v_L$$

The voltage drop across individual components

$$v_R = iR \quad v_L = L \frac{di}{dt} \quad v_o = \frac{1}{C} \int i dt \quad \left[ i = C \frac{dv_o}{dt} \right]$$

$$\text{So } v_i = v_o + CR \frac{dv_o}{dt} + LC \frac{d^2 v_o}{dt^2} \quad \text{or} \quad \frac{d^2 v_o}{dt^2} + \frac{R}{L} \frac{dv_o}{dt} + \frac{1}{LC} v_o(t) = \frac{1}{LC} v_i(t)$$

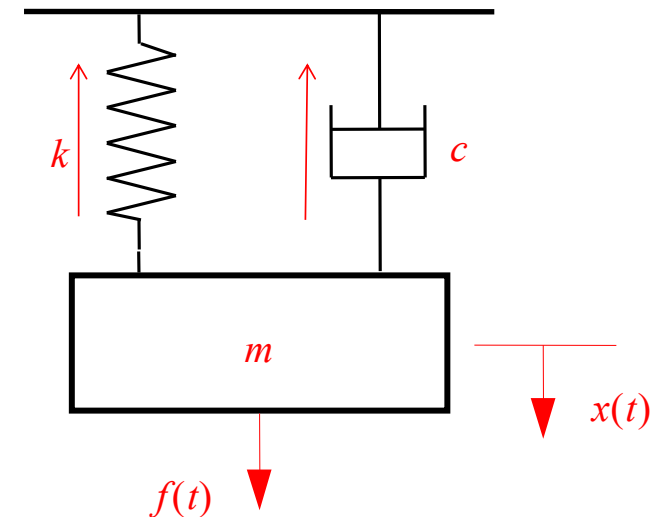
Using the notation from the previous slide and  $n = 2$   $m = 0$



## 2.3 Mechanical system example

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- By Newton's law,  $m\ddot{x}$  equals the resultant of all external forces on  $m$  (in the downward direction).
- As the spring is stretched, spring force  $kx$  is upwards and opposes downward acceleration.
- The mass also moves downwards so the damping force,  $c\dot{x}$ , is upwards and both these terms have a minus sign.
- External force,  $f(t)$ , helps downward acceleration and therefore has a plus sign.



- The net force applied to the mass is  $= f(t) - kx - cv = f(t) - kx - c\frac{dx}{dt}$
- The net force applied to the mass is also equal to  $= ma = m\ddot{x} = m\frac{d^2x}{dt^2}$

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The force equation is therefore:  $m\ddot{x} = -kx - c\dot{x} + f(t)$

Rearranging gives the usual form of the differential equation of motion:

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad \text{or} \quad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

The effects of gravity do not appear and turning the system upside down does not affect the equation.

As  $x = 0$  at the position of static equilibrium weight,  $mg$ , is counterbalanced by the spring force,  $kx$ .

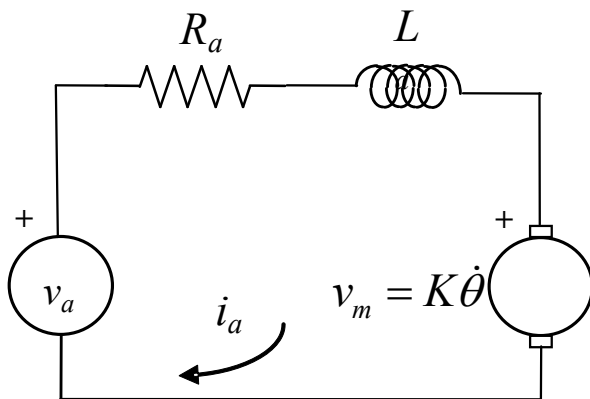


## 2.4 Rotational systems - Armature-controlled dc motor

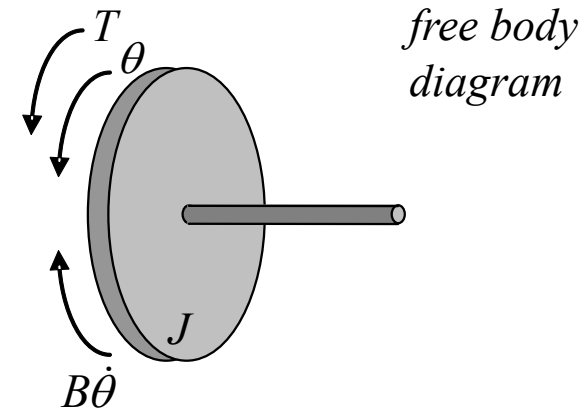
The motor load is assumed an inertia  $J$  and a damper with constant  $B$ .

shaft position,  $\theta$ , and developed motor torque,  $T$ , are related by:

$$T(t) = J\ddot{\theta}(t) + B\dot{\theta}(t)$$



*circuit diagram*



*free body diagram*

The armature loop is described by:

$$v_a = R_a i_a + L \dot{i}_a + v_m$$

counter emf is taken to be proportional to shaft speed:

$$v_m = K_e \dot{\theta}$$

developed torque is proportional to current  $i_a$

$$T = K_t i_a$$

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Using Newton's law combined with Kirchhoff's law:

$$J\ddot{\theta} + B\dot{\theta} = Ki$$

$$T(t) = J\ddot{\theta}(t) + B\dot{\theta}(t)$$

$$T = K_t i_a$$

$$L \frac{di}{dt} + Ri = v_a - K\dot{\theta}$$

$$v_a = R_a i_a + L_a \dot{i}_a + v_m$$

$$v_m = K_e \dot{\theta}$$

where the motor constant is equal to the torque/electromotive force constant of the motor:

$$K = K_t = K_e$$

## 2.5 Non-linear systems

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A nonlinear system is one that is described by a nonlinear equation.

Examples of nonlinear systems are:

$$y = x^2 \qquad \frac{d^2 x}{dt^2} + \left[ \frac{dx}{dt} \right]^2 + Ax = B \sin(\omega t)$$

**In reality ALL systems are nonlinear to some extent.**

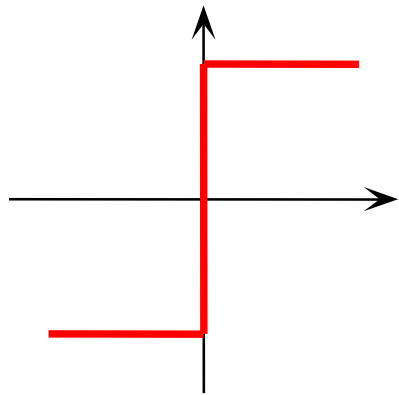
For example, an aircraft's pitch response changes significantly when the stall angle is approached.

Similarly a switch can only take the values on and off and a valve can only vary flow between 0 and 100%.

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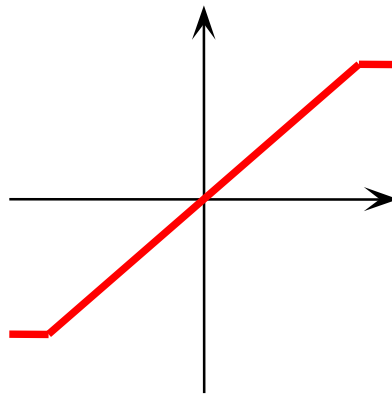
## Examples of system nonlinearities

(a) Relay



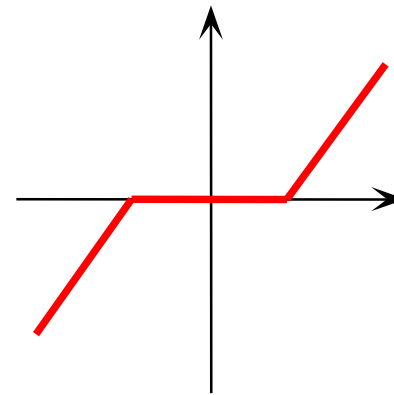
switch  
solenoid

(b) Saturation



limit max/min  
flow, speed,  
acceleration

(c) Dead-zone



gear backlash  
neutral zones  
tolerances

nonlinearities frequently occur in actuating components of systems

## 2.6 Linearization of nonlinear systems

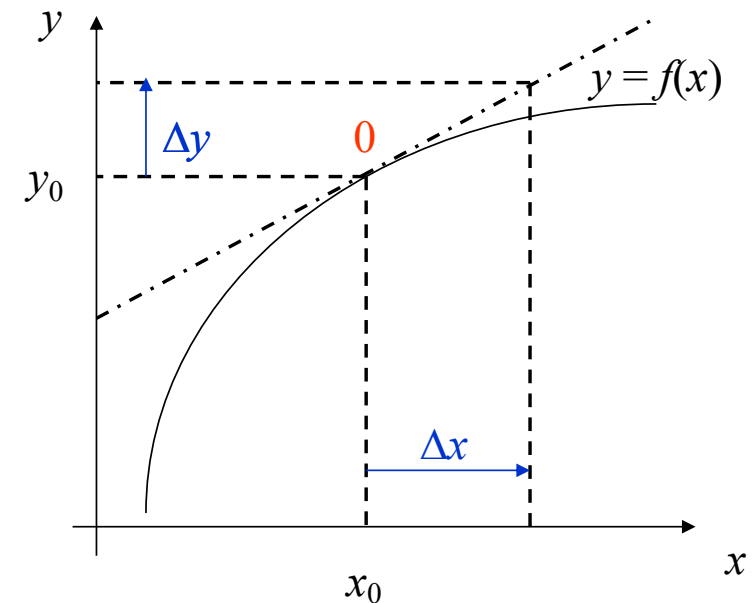
It is normally possible to represent a nonlinear system using a linear (or small-signal) approximation.

If the variation of  $x$  about  $x_0$  is small enough, the nonlinear curve can be approximated by its tangent at point 0.

Defining  $\Delta x = x - x_0$  and  $\Delta y = y - y_0$ , the linearized model of  $f(x)$  about  $x_0$  is

$$\Delta y = \left( \frac{df}{dx} \right)_{x_0} \Delta x \quad (2.1)$$

where  $\left( \frac{df}{dx} \right)_{x_0}$  is the slope of the tangent at  $x_0$ .



The model is then said to be linearized about a steady state operating point, 0 ( $x_0$ ).

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### Example: Linearization of a nonlinear spring

The differential equation that describes the position  $x(t)$  of the mass  $m$  in response to an external force  $f(t)$  is

$$m\ddot{x} + c\dot{x} + kx = f$$

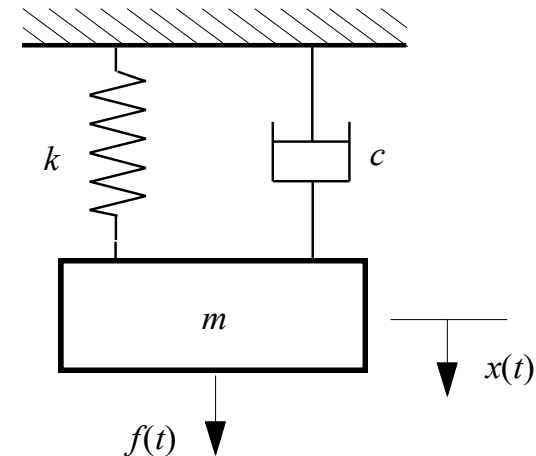
Spring force may not change linearly with  $x$  and a nonlinear equation is possible, e.g.

$$m\ddot{x} + c\dot{x} + \textcolor{red}{kx^3} = f + mg$$

where  $\textcolor{red}{h(x) = kx^3}$  is a nonlinear spring force. Weight,  $\textcolor{red}{mg}$ , must also be added to the downward force  $f(t)$  and  $x = 0$  at the point of zero spring force.

If at a steady-state operating point where  $f = f_0$  is a constant, then the solution  $x = x_0$  is also a constant, i.e.

$$h_0 = h(x_0) = kx_0^3 = f_0 + mg \quad \text{or} \quad x_0 = \left( \frac{f_0 + mg}{k} \right)^{1/3} \quad \textcolor{red}{(\text{Force due to spring} = \text{constant force} + \text{weight})}$$



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To perform linearisation of the spring equation for small variations  $\Delta x = x - x_0$  and  $\Delta h = h - h_0$  about an operational point:

$$h(x) = kx^3$$

we use Eq. (2.1), which in terms of the function  $h$  is  $\Delta h = \left(\frac{dh}{dx}\right)_{x_0} \Delta x$ , and obtain:

$$h(x) = h(x_0) + \Delta h = h(x_0) + \left(\frac{dh}{dx}\right)_{x_0} \Delta x = kx_0^3 + 3kx_0^2\Delta x$$

Then as  $(\Delta \dot{x}) = \dot{x}$  and  $(\Delta \ddot{x}) = \ddot{x}$ , the linear model of  $m\ddot{x} + c\dot{x} + h(x) = f + mg$  is:

$$m(\Delta \ddot{x}) + c(\Delta \dot{x}) + kx_0^3 + 3kx_0^2\Delta x = f_0 + \Delta f + mg$$

But since according to previous slide  $kx_0^3 = f_0 + mg$ , we obtain:

$$m(\Delta \ddot{x}) + c(\Delta \dot{x}) + 3kx_0^2\Delta x = \Delta f$$

which is a linear equation expressed in terms of variations  $\Delta x$  and  $\Delta f$  about  $x_0$  and  $f_0$ . If the actual positions are needed,  $x_0$  must be added to the solution of this differential equation.

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Taylor series expansion, ignoring all terms of order 2 or higher, can also be used for nonlinear approximation:

$$f(x) = f(x_0) + \left(\frac{df}{dx}\right)_{x_0} \frac{x - x_0}{1!} + \left(\frac{d^2f}{dx^2}\right)_{x_0} \frac{(x - x_0)^2}{2!} + \dots + \left(\frac{d^n f}{dx^n}\right)_{x_0} \frac{(x - x_0)^n}{n!}$$

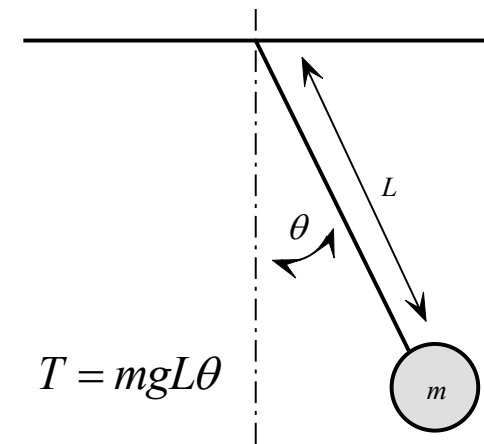
Consider the pendulum on the left. The torque is  $T = mgL \sin \theta$ . Nonlinear part is  $\sin \theta$ , which approximated around the point  $\theta = \theta_0$  is:

$$\sin \theta = \sin \theta_0 + \cos \theta_0 (\theta - \theta_0) + \dots$$

Taking  $\theta_0 = 0$  we obtain  $\sin \theta \approx \theta$ .

The pendulum dynamics are then modelled as  $T = mgL\theta$ .

- As the pendulum moves further from  $\theta = 0$ , the greater the error in the approximation.
- This is one reason why linearization is often associated with small signal responses.





## 2.7 Summary

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- Dynamic systems can be modelled with differential equations.
- Linear differential equations can be used to describe the input-output properties of many simple systems
- The use of linear and time-invariant differential equations implies that the system is linear and, this is unlikely to be the case in practise.
- System linearization can be used to approximate the system dynamics about a specific operating point.

## 2.8 Problems

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1. Classify the following differential equations as to whether they are time-varying or time-invariant:

a)  $\frac{d^2 y}{dt^2} + 7y = 0$

b)  $\frac{d}{dt}(t^2 y) + y = 0$

c)  $\left(\frac{1}{t+1}\right)\frac{d^2 y}{dt^2} + \left(\frac{1}{t+y}\right)\frac{dy}{dt} + \left(\frac{1}{t+1}\right)y = 0$

d)  $\frac{d^2 y}{dt^2} + (\cos t)y = 0$

2. Classify the following differential equations as to whether they are linear or non-linear.

a)  $\frac{dy}{dt} + y = 0$  linear

b)  $\frac{d^2 y}{dt^2} + y = 0$  linear

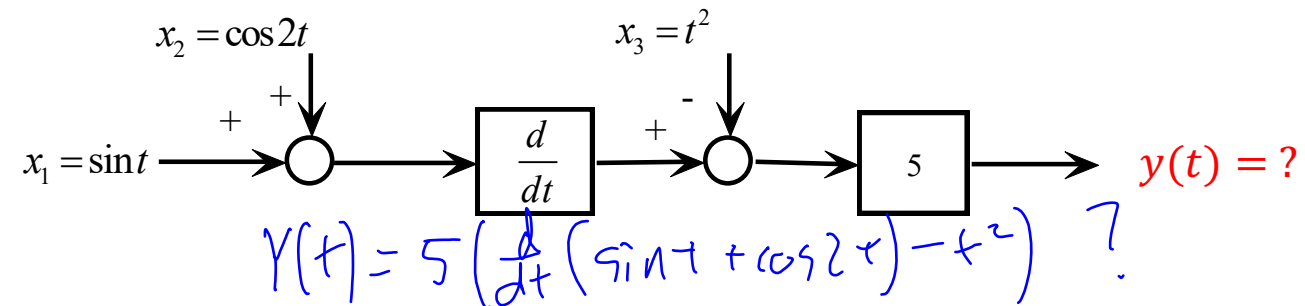
c)  $\frac{dy}{dt} + y^2 = 0$  non

d)  $\cos y \frac{d^2 y}{dt^2} + \sin 2y = 0$  non

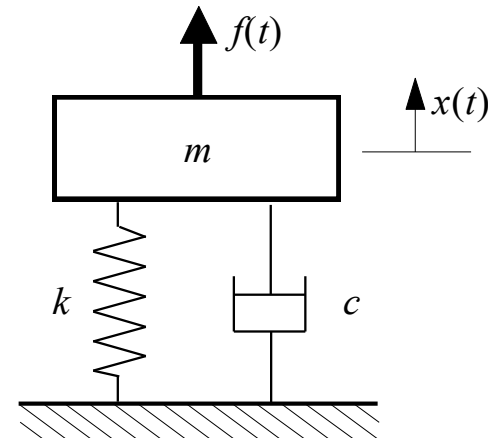
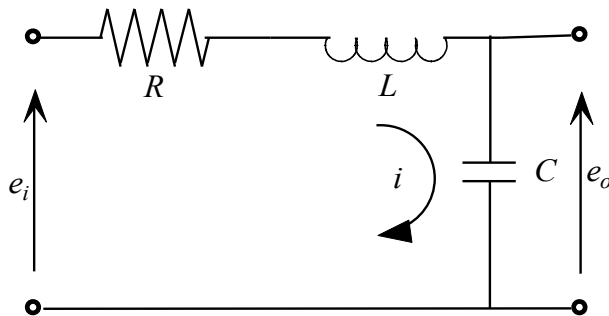
e)  $\cos x(t) \frac{d^2 y}{dt^2} + \sin x(t) = 0$  depends on  $x$

f)  $\cos x(y) \frac{d^2 y}{dt^2} + \sin 2x(y)t^2 = 0$

3. Use the principle of superposition to determine the output of the following system:



4. Obtain mathematical models for the systems below and show that they are analogous (Hint: let  $i = dq/dt$ ).

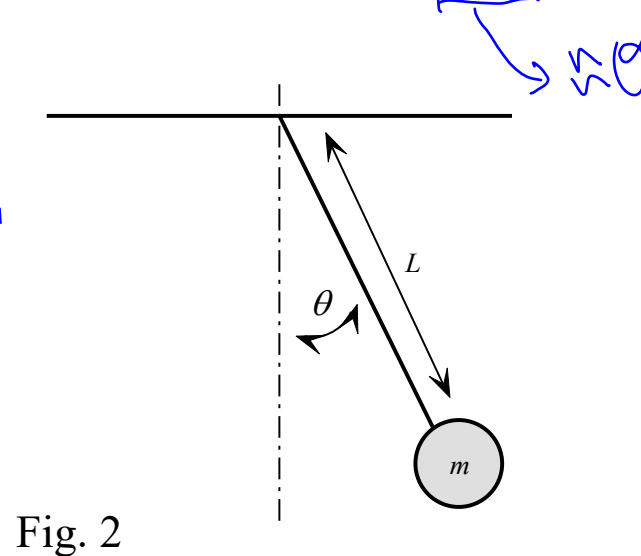
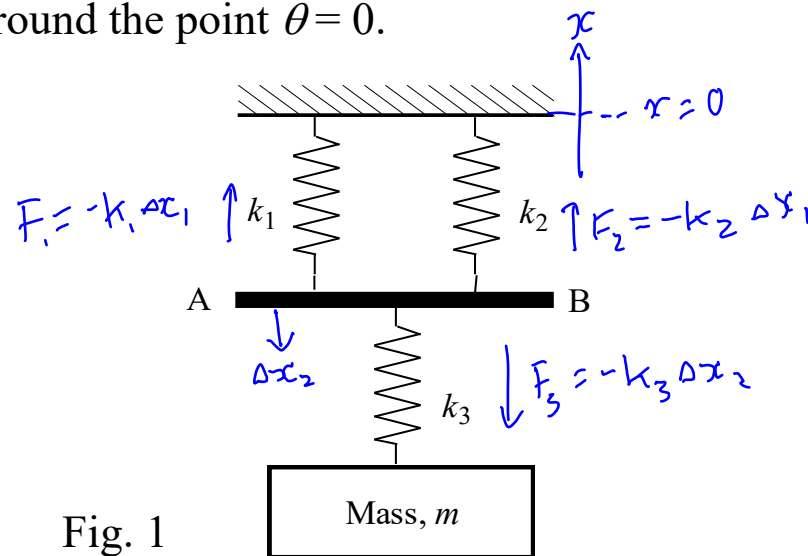


5. Write down the equations of motion for the system shown in Fig. 1 assuming that the bar AB is rigid, massless and constrained to remain horizontal.

6. Linearize the following nonlinear differential equation using Taylor series expansion

$$\frac{d^2 y}{dx^2} + y \cos y = x \quad \text{with } y = 0 \text{ when } x = 0$$

7. The nonlinear equation describing the pendulum in Fig. 2 below is  $\frac{d^2 \theta}{dt^2} + \frac{g}{L} \sin \theta = 0$ . Linearize the system around the point  $\theta = 0$ .



$ma = \sum F$   
 (better detail in  
 solutions make notes it's  
 on the exam)