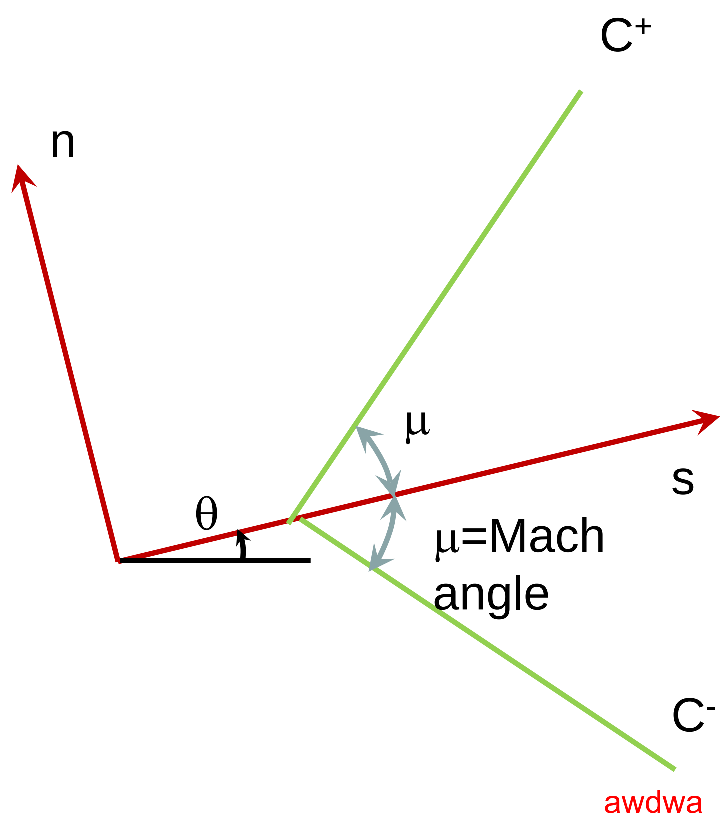
SESA3029 Aerothermodynamics
Lecture 3.2

Method of characteristics:

theory (part B)

Recap: Supersonic flow characteristics



Governing equations for homentropic flow reduced to

$$\frac{\partial \mathbf{Q}}{\partial s} + \mathbf{A}\frac{\partial \mathbf{Q}}{\partial n} = 0$$

$$\mathbf{A} = \begin{pmatrix} 0 & -\tan\mu \\ -\tan\mu & 0 \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} \nu \\ \theta \end{pmatrix}$$

The character of the solution will be determined by the properties of **A**

Eigenvalues of A:

$$\lambda = \pm \tan\mu$$

Diagonalisation of A

Any (non-defective) square matrix can be written as

$$\mathbf{AS} = \mathbf{S\Lambda}$$

S=right eigenvector matrix
S^{-1}=left eigenvector matrix
Λ=diag(λ_1, λ_2)=eigenvalue matrix

$$\mathbf{A} = \begin{pmatrix} 0 & -\tan\mu \\ -\tan\mu & 0 \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} \tfrac{1}{2} & \tfrac{1}{2} \\ -\tfrac{1}{2} & \tfrac{1}{2} \end{pmatrix} \quad \mathbf{S}^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} \tan\mu & 0 \\ 0 & -\tan\mu \end{pmatrix}$$

With $\mathbf{Q} = \begin{pmatrix} \nu \\ \theta \end{pmatrix}$ we had:

$$\frac{\partial \mathbf{Q}}{\partial s} + \mathbf{A}\frac{\partial \mathbf{Q}}{\partial n} = 0$$

$$\frac{\partial \mathbf{Q}}{\partial s} + \mathbf{S\Lambda S}^{-1}\frac{\partial \mathbf{Q}}{\partial n} = 0$$

$$\mathbf{S}^{-1}\frac{\partial \mathbf{Q}}{\partial s} + \mathbf{\Lambda S}^{-1}\frac{\partial \mathbf{Q}}{\partial n} = 0$$

From definitions of **Q** and **S**$^{-1}$ we have decoupled equations:

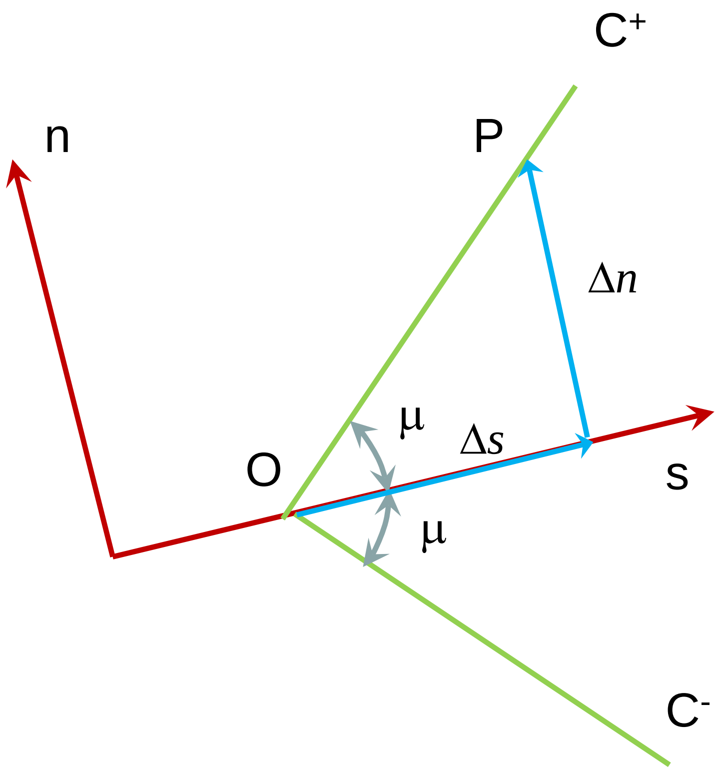
$$\lambda_1: \quad \frac{\partial(\nu-\theta)}{\partial s} + \tan\mu\frac{\partial(\nu-\theta)}{\partial n} = 0$$

$$\lambda_2: \quad \frac{\partial(\nu+\theta)}{\partial s} - \tan\mu\frac{\partial(\nu+\theta)}{\partial n} = 0$$

This is just a more useful form of the version derived pre-matrixification with this form being derived from strange matix manipulation

Riemann invariants

This slide is about what those eqautions mean



Change in some function *f* along OP

First one is simple application of first order taylor series to some arbitrary property f

$$\Delta f = \frac{\partial f}{\partial s}\Delta s + \frac{\partial f}{\partial n}\Delta n$$

$$\frac{\Delta f}{\Delta s} = \frac{\partial f}{\partial s} + \frac{\partial f}{\partial n}\frac{\Delta n}{\Delta s}$$

<----------
Now describing property change gradient along s

$$= \frac{\partial f}{\partial s} + \tan\mu\frac{\partial f}{\partial n}$$

Compare with previous result:

$$\lambda_1 : \quad \frac{\partial(\nu-\theta)}{\partial s} + \tan\mu\frac{\partial(\nu-\theta)}{\partial n} = 0$$

$$\lambda_2 : \quad \frac{\partial(\nu+\theta)}{\partial s} - \tan\mu\frac{\partial(\nu+\theta)}{\partial n} = 0$$

Consider what happens if:
f = \nu-\etheta
If we follow tan then the change in popertys is zero!

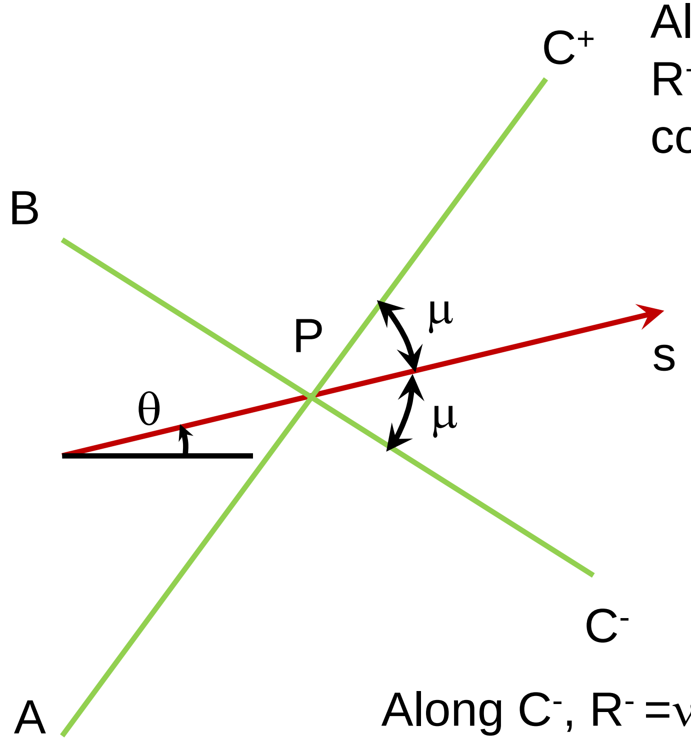
What this translates to is that nu-theta is INVARIENT along a characteristic line

Along C^+, $R^+=\nu-\theta$ is constant
Along C^-, $R^-=\nu+\theta$ is constant

These are called the **compatibility conditions**

R^+ and R^- are called the **Riemann invariants** (N.B. $R^+=-K^+$, $R^-=K^-$ from Anderson)

Method of characteristics



Along C^+, $R^+=\nu-\theta$ is constant

Along C^-, $R^-=\nu+\theta$ is constant

Suppose we know the flow state at points A and B

$$R_A^+ = \nu_A\left(M_A\right) - \theta_A$$

$$R_B^- = \nu_B\left(M_B\right) + \theta_B$$

At point P we must have

$$\nu_P - \theta_P = R_A^+$$

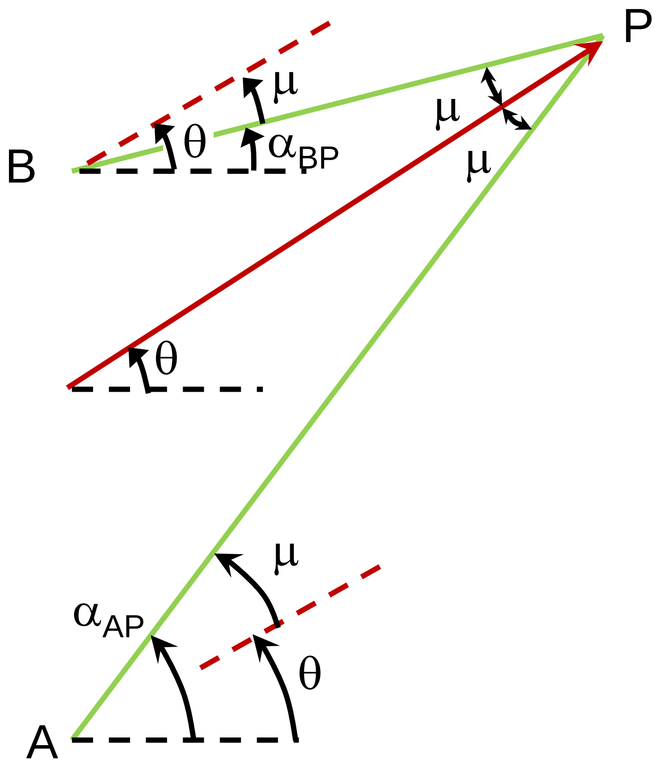
$$\nu_P + \theta_P = R_B^-$$

Hence, at point P

$$\nu_P = \frac{R_A^+ + R_B^-}{2} \qquad \theta_P = \frac{R_B^- - R_A^+}{2}$$

From IFT we can find M_P, hence we have marched the solution downstream if we want to know the value of alpha along these lines we need to keep in mind that the green lines curve on large scales, so it's actually more accurate to calculate the value at B and P then use the average instead of just one or the other,

Geometry



Average angles

$$\alpha_{AP} = \frac{1}{2}\left[(\theta+\mu)_A + (\theta+\mu)_P\right]$$

$$\alpha_{BP} = \frac{1}{2}\left[(\theta-\mu)_B + (\theta-\mu)_P\right]$$

Geometry

$$\tan\alpha_{AP} = \frac{y_P - y_A}{x_P - x_A}$$

$$\tan\alpha_{BP} = \frac{y_P - y_B}{x_P - x_B}$$

$$(x_P - x_B)\tan\alpha_{BP} = (x_P - x_A)\tan\alpha_{AP} + y_A - y_B$$

$$x_p = \frac{x_B\tan\alpha_{BP} - x_A\tan\alpha_{AP} + y_A - y_B}{\tan\alpha_{BP} - \tan\alpha_{AP}}$$ and $$y_P = y_A + (x_P - x_A)\tan\alpha_{AP}$$

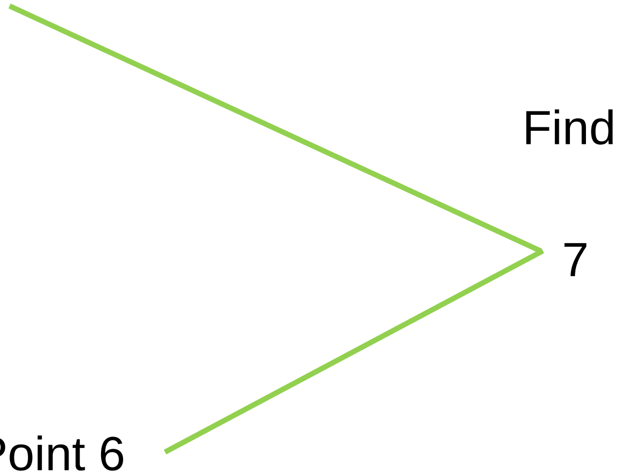
This process of using 2 points to get the properties of a downstream point is the "unit process", it evaluates the values downstream using upstream conditions. That being said fundamentally it relys on linear approximations of curved lines, which means smaller units yield a more accurate approximation

Unit process example

Point 3

M_3=1.53
θ_3=12°
x_3=0.69
y_3=0.62

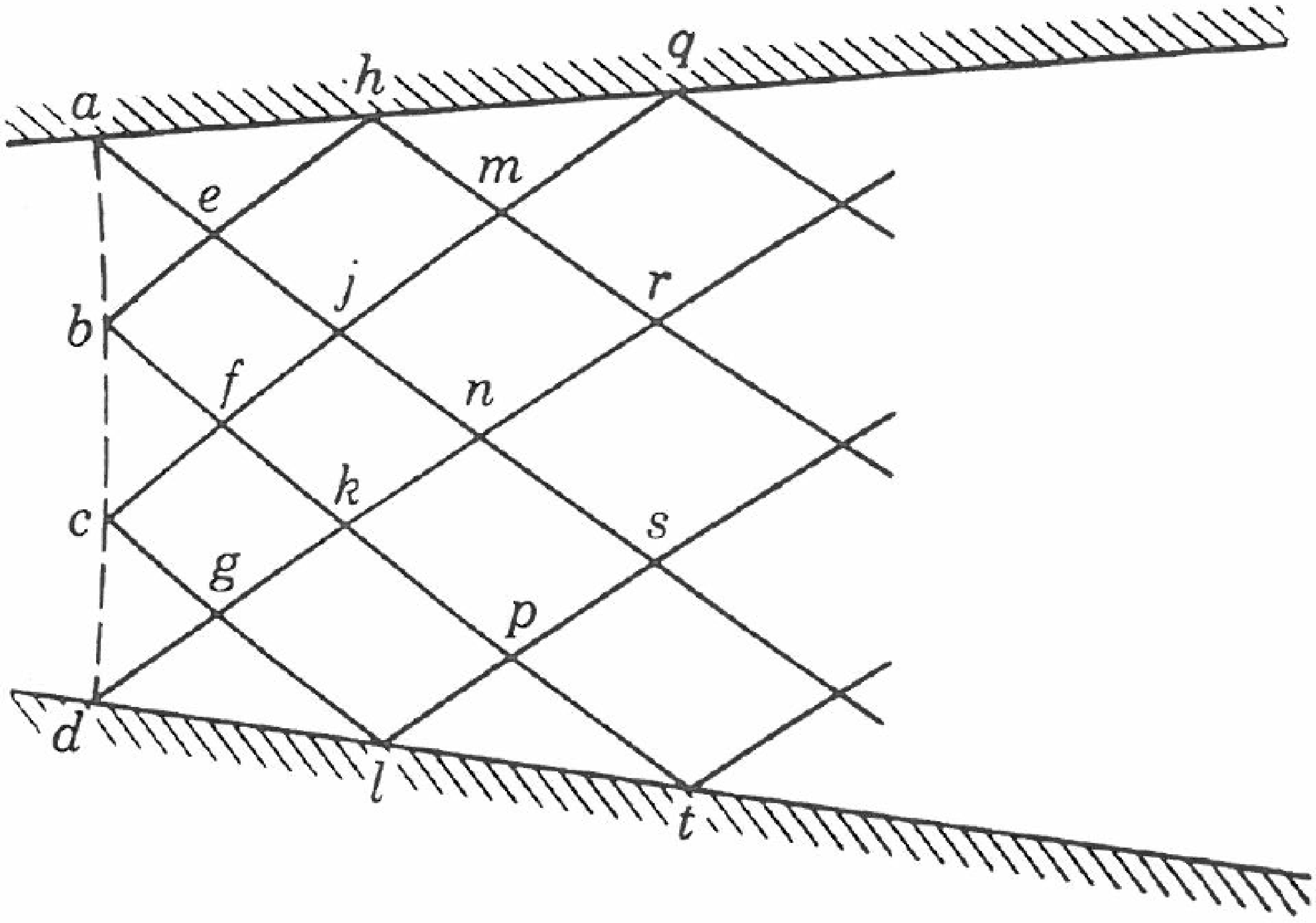
Find M_7, θ_7, x_7, y_7



Point 6

M_6=1.53
θ_6=0°
x_6=1.08
y_6=0.0

Repeat unit process to construct complete flowfield



Calculation method

1. Calculate $\nu(M)$,θ and μ at a number of points on a starting line from known conditions (u,v,T) and hence find R^+ and R^- for all points
2. Calculate the angles $\theta+\mu$ and $\theta-\mu$ at each point.
3. March downstream and identify crossing points. Calculate $\nu(M)$ and θ at these points.
4. Calculate M (from ν) and hence $\mu=\sin^{-1}(1/M)$
5. Find angles α and hence the co-ordinates of the new points
6. Repeat the process moving downstream