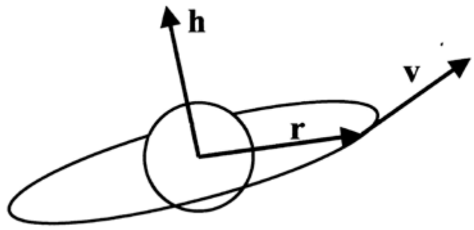
SESA2024 Astronautics

Chapter 5: Mission analysis – Solutions

1. Given $m\mathbf{h} = \mathbf{r} \times m\mathbf{V}$, then by definition **h** is perpendicular to both **r** and **V** (see diagram) in a 'right-handed' sense. Hence **h** is the orbit plane normal and

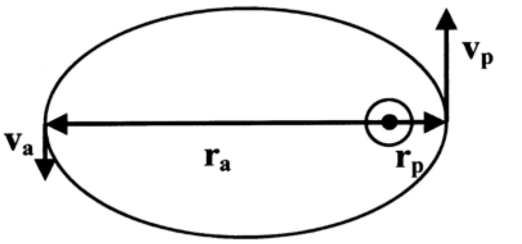


remains constant. Therefore the motion takes place in a plane that is unchanging.

Also, since **h** is conserved, then in an Earth orbit its value at perigee must equal its value at apogee. At each of these points, the position vectors and velocity vectors are orthogonal to each other (see diagram). Hence

$$\mathbf{h_p} = \mathbf{h_a} \;\Rightarrow\; \mathbf{r_p} \times \mathbf{V_p} = \mathbf{r_a} \times \mathbf{V_a} \;\Rightarrow\; r_p V_p \sin 90^{\mathrm{O}} = r_a V_a \sin 90^{\mathrm{O}}$$

$$\therefore\; r_p V_p = r_a V_a$$



2. From Question 1, we have

$$\frac{V_p}{V_a} = \frac{r_a}{r_p}.$$

Also, rearranging the energy equation, we have

$$\frac{(1/2)V_p^2}{(1/2)V_a^2} = \frac{\varepsilon + (\mu / r_p)}{\varepsilon + (\mu / r_a)} \;\Rightarrow\; \frac{r_a^2}{r_p^2} = \frac{\varepsilon + (\mu / r_p)}{\varepsilon + (\mu / r_a)},$$

so that

$$r_a^2 (\varepsilon + \frac{\mu}{r_a}) = r_p^2 (\varepsilon + \frac{\mu}{r_p}) \;\Rightarrow\; \varepsilon(r_a^2 - r_p^2) = \mu(r_p - r_a).$$

$$\therefore\; \varepsilon = \frac{\mu(r_p - r_a)}{(r_a + r_p)(r_a - r_p)} = \frac{-\mu}{(r_a + r_p)} = -\frac{\mu}{2a}.$$

3. From lecture 11, Hohmann transfer delta-V is

$$\Delta V = \sqrt{\frac{\mu}{r_1}}\left\{\sqrt{\frac{2r_2}{r_1+r_2}}-1\right\}+\sqrt{\frac{\mu}{r_2}}\left\{1-\sqrt{\frac{2r_1}{r_1+r_2}}\right\}.$$

Let $V_1 = \sqrt{\mu/r_1}$ and $x = r_1/r_2$. Then

$$\sqrt{\frac{\mu}{r_2}} = \sqrt{\frac{r_1}{r_2}}\sqrt{\frac{\mu}{r_1}} = x^{1/2}V_1, \qquad \sqrt{\frac{2r_2}{r_1+r_2}} = \sqrt{\frac{2}{(r_1/r_2)+1}} = \sqrt{\frac{2}{x+1}}, \quad \text{and}$$

$$\sqrt{\frac{2r_1}{r_1+r_2}} = \sqrt{\frac{2(r_1/r_2)}{(r_1/r_2)+1}} = \sqrt{\frac{2x}{x+1}}.$$

$$\therefore\ \ \Delta V = V_1\left\{\sqrt{\frac{2}{x+1}}-1\right\}+x^{1/2}V_1\left\{1-\sqrt{\frac{2x}{x+1}}\right\}$$

$$= V_1\sqrt{\frac{2}{x+1}}-V_1+V_1x^{1/2}-V_1x\sqrt{\frac{2}{x+1}} \qquad = V_1\sqrt{\frac{2}{x+1}}(1-x)+V_1x^{1/2}-V_1,$$

giving the required result.

4. Clearly, if the unidentified object's apogee is 1000 km, then it cannot be an escaping space probe. To determine the orbit, use the energy equation,

$$\frac{1}{2}V^2-\frac{\mu}{r_a} = -\frac{\mu}{2a}, \text{ where } V = 5.589\ \text{km/s},\ r_a = R_E+1000 = 7378\ \text{km and}$$

$$\mu = 398600\ \text{km}^3/\text{sec}^2 \quad\Rightarrow\quad a = 5189\ \text{km}.$$

Also from the apogee radius equation, $r_a = a(1+e)$, we have $e = (r_a/a)-1 = 0.4218$. Perigee radius is therefore $r_p = a(1-e) = 3000\ \text{km} < R_E$. Hence object is probably a ballistic missile.

5. From lecture 7, we have

$$r = \frac{a(1-e^2)}{1+e\cos\theta}. \quad \text{Now at apogee, } \theta = 180^{\text{O}}. \quad \text{Hence}$$

$$r_{apogee} = \frac{a(1-e^2)}{1+e\cos 180^{\text{O}}} = \frac{a(1+e)(1-e)}{(1-e)} = a(1+e).$$

6. The principal advantage of the Hohmann transfer for interplanetary travel is that it is optimal with respect to minimising delta-V, and therefore fuel usage. The main disadvantage for deep space destinations is the transfer time – for example, to go to Uranus takes 16 years, and to Neptune 30 years. Clearly, over with these transfer periods, the reliable operation of a spacecraft on arrival at the destination planet would be suspect.

7. In this case, we have
$r_1 = R_E+300 = 6678\ \text{km}, \qquad r_2 = R_{GEO} = 42164\ \text{km}.$
From lecture 11, we have

$$\Delta V_1 = \sqrt{\frac{\mu}{r_1}}\left\{\sqrt{\frac{2r_2}{r_1+r_2}}-1\right\} = 2.426\,\text{km, and}$$

$$\Delta V_2 = \sqrt{\frac{\mu}{r_2}}\left\{1-\sqrt{\frac{2r_1}{r_1+r_2}}\right\} = 1.467\,\text{km.}$$

To estimate final mass deployed in GEO, consider each engine burn in turn.

Mass after first burn: Let initial mass be $M_1 = 2500\,\text{kg}$ and let mass after first engine burn be M_2. From the rocket equation, we have

$$\Delta V_1 = V_{ex}\log_e(M_1/M_2) \;\Rightarrow\; M_2 = M_1\exp(-\Delta V_1/V_{ex}) = 1113.6\,\text{kg.}$$

Discard rocket stage: $M_3 = 1113.6 - 136 = 977.6\,\text{kg}$, where M_3 is the mass prior to the second engine burn.

Final mass after second burn: From the rocket equation, we have

$$\Delta V_2 = V_{ex}\log_e(M_3/M_{final}) \;\Rightarrow\; M_{final} = M_3\exp(-\Delta V_2/V_{ex}) = 599.5\,\text{kg} \approx 600\,\text{kg.}$$

8. ***Comet's speed and distance at perihelion.***

Perihelion distance, $r_{cp} = a(1-e) = (30\times10^8)(1-0.9) = 3\times10^8$ km.

Perihelion speed, V_{cp} - use the energy equation:

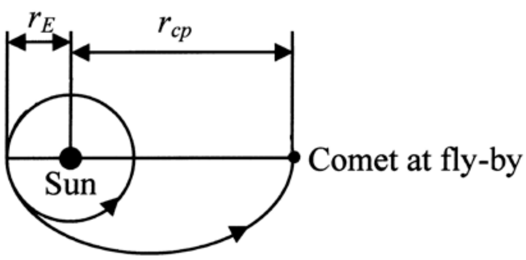
$$\frac{1}{2}V_{cp}^2 - \frac{\mu_{Sun}}{r_{cp}} = -\frac{\mu_{Sun}}{2a} \;\Rightarrow\; V_{cp}^2 = 2\mu_{Sun}\left(\frac{1}{r_{cp}}-\frac{1}{2a}\right) \;\Rightarrow\; V_{cp} = 28.69\,\text{km/sec.}$$

ΔV for Hohmann transfer orbit injection.

Probe speed before burn, $V_{circ} = \sqrt{\mu_{Sun}/r_E} = 29.44\,\text{km/sec.}$

Probe speed after burn = speed at transfer orbit perihelion, V_{Tp}.

Transfer orbit semi-major axis, $a_T = \frac{1}{2}(r_E + r_{cp}) = 2.25\times10^8$ km (see diagram)



Use the energy equation to obtain V_{Tp}:

$$\frac{1}{2}V_{Tp}^2 - \frac{\mu_{Sun}}{r_E} = -\frac{\mu_{Sun}}{2a_T} \;\Rightarrow\; V_{Tp}^2 = 2\mu_{Sun}\left(\frac{1}{r_E}-\frac{1}{2a_T}\right) \;\Rightarrow\; V_{Tp} = 33.99\,\text{km/sec.}$$

Hence, ΔV for transfer orbit injection is

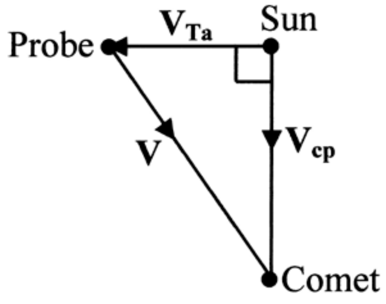
$$\Delta V = V_{Tp} - V_{circ} = 33.99 - 29.44 = 4.55\,\text{km/sec.}$$

Time of engine firing.

Probe's engine must be fired at a time equal to the Hohmann transfer time before the comet's perihelion. The probe will transfer from Earth orbit to comet encounter in half the transfer orbit's period. Hence time of engine firing prior to comet encounter is

$$T_{firing} = \tfrac{1}{2}\tau_T = \tfrac{1}{2}(2\pi\sqrt{a_T^3/\mu_{Sun}}) = 2.9407\times10^7\ \text{sec} \approx 340\,\text{days.}$$

Speed of comet relative to probe at fly-by.
See relative velocity diagram, where



$\mathbf{V_{Ta}}$ = Velocity of probe relative to the Sun at transfer orbit aphelion,
$\mathbf{V_{cp}}$ = Velocity of comet relative to the Sun at its perihelion, and
$\mathbf{V}$ = Velocity of comet relative to probe.

Use the energy equation to find speed V_{Ta},

$$\frac{1}{2}V_{Ta}^2 - \frac{\mu_{Sun}}{r_{cp}} = -\frac{\mu_{Sun}}{2a_T} \;\Rightarrow\; V_{Ta}^2 = 2\mu_{Sun}\left(\frac{1}{r_{cp}} - \frac{1}{2a_T}\right) \;\Rightarrow\; V_{Ta} = 17.00 \text{ km/sec.}$$

From relative velocity diagram, the speed of the comet relative to the probe is

$$V = \sqrt{V_{Ta}^2 + V_{cp}^2} = 33.35 \text{ km/sec.}$$