

Lecture 16 - Separation of Variables: Examples

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1 Review

2 Examples

- Basic Example
- More complex example

3 Summary

Review: Separation of variables in 6 steps

Separation of variables *ansatz* (SoV): $y(\mathbf{x}, t) = X(\mathbf{x})T(t)$

- 1 **Determine ODEs** for X, T .
- 2 Use **boundary conditions** of y in order to obtain boundary conditions of X .
- 3 **Solve eigenvalue problem for X** : determine eigenvalues λ_n and eigenfunctions X_n .
- 4 Insert eigenvalue λ_n in the ODE for T and solve it to obtain T_n .
- 5 The **normal modes** are $y_n = X_n T_n$ and the **general solution** is obtained by **superposition**

$$y(\mathbf{x}, t) = \sum_n X_n(\mathbf{x}) T_n(t)$$

- 6 **Use initial conditions**, $y(\mathbf{x}, 0), \partial y(\mathbf{x}, 0)/\partial t$ to **find all undetermined coefficients**. This step **involves Fourier series**.

1 Review

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→ *A simpler example: SoV in a Wave Equation*

We use **separation of variables** to solve

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad c \text{ constant}$$

with **boundary conditions**:

$$y(0, t) = 0, \quad y(1, t) = 0$$

and **initial data**:

$$y(x, 0) = x(1 - x), \quad \frac{\partial y}{\partial t}(x, 0) = 0.$$

Step 1: Equations for X , T (Review)

We write the solution as

$$y(x, t) = X(x)T(t),$$

← Plug into PDE
 $\frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$

giving the **separably constant** equations (ODEs):

$$\begin{cases} \ddot{T} - c^2 \lambda T = 0, \\ X'' - \lambda X = 0. \end{cases}$$

Step 2: Boundary conditions for $X(x)$

(Review)

To solve the ODEs

$$\begin{cases} X'' - \lambda X = 0, \\ \ddot{T} - c^2 \lambda T = 0, \end{cases}$$

we need **boundary conditions**. We have that (from BCs in previous slide)

$$\begin{aligned} y(0, t) = 0 &\Leftrightarrow X(0)T(t) = 0, & y(1, t) = 0 &\Leftrightarrow X(1)T(t) = 0 \\ \Rightarrow X(0) &= 0, & \Rightarrow X(1) &= 0. \end{aligned}$$

At present we have no sensible boundary conditions for T .

$X'' - \lambda X = 0$ where $X(0)=0$ $X(1)=0$ 3 cases exist $\lambda > 0$ $\lambda < 0$ $\lambda = 0$
∴ eigenvalue problem.



Step 3: Eigenvalue Problem for $X(x)$ (Review)

We thus have the **Eigenvalue Problem**

$$X'' - \lambda X = 0; \quad X(0) = 0, \quad X(1) = 0.$$

λ is the **unknown eigenvalue** (revisit Lecture 3: there $X'' + \lambda X = 0$!!).

We have to consider the **three cases** (revisit Lecture 3):

- ① $\lambda = k^2 > 0$ (distinct real roots $\Rightarrow X = A e^{kx} + B e^{-kx}$), *trivial solution*
- ② $\lambda = 0$ ($\Rightarrow X = A + Bx$), *trivial solution*
- ③ $\lambda = -k^2 < 0$ (complex conjugate roots $\Rightarrow X = A \sin(kx) + B \cos(kx)$), *useful solution*

we find that only the **third case** gives a **non-trivial** solution. Namely, we get the eigenfunction and eigenvalue:

$$X_n(x) = A_n \sin(n\pi x), \quad \lambda_n = -(n\pi)^2, \quad n = 1, 2, 3, \dots$$

[Exercise: get this result! (see Lecture Notes § 5.4.2)]

Step 4: Solve the $T(t)$ equation (Review)

- First ODE for $X_n(x)$ ✓ Separation constant λ_n ✓
- But we still have to solve the second **ODE for $T(t)$** :

$$\ddot{T} - c^2 \lambda T = 0 \quad \text{with no boundary conditions}$$

but **we now know the value of** $\lambda = -k^2 < 0$. (From last bit)
↳ if not then $x(0)=0$
 \therefore must be complex

So this is a **constant coefficient ODE** (revisit Lecture 1).

Since $\lambda = -k^2 < 0$ the associated **auxiliary equation** is a quadratic with two purely imaginary roots

$$\Lambda = \pm j n \pi c, \quad n = 1, 2, 3, \dots$$

So its **general solution** $T(t) = T_n(t)$ is:

$$T_n(t) = \tilde{C}_n \cos(n \pi c t) + \tilde{D}_n \sin(n \pi c t).$$

Step 5: The general solution $y(x, t) = X(x)T(t)$ (Review)

We have our separation *ansatz* $y(x, t) = X(x)T(t)$
and **normal mode** solutions (each n describes a normal mode):

$$X_n(x) = A_n \sin(n\pi x), \quad T_n(t) = \tilde{C}_n \cos(n\pi c t) + \tilde{D}_n \sin(n\pi c t).$$

Combining these, $y_n(x, t) = X_n(x)T_n(t)$, and
superposing the **normal modes** $y_n(x, t)$ gives the **general solution**:

$$y(x, t) = \sum_{n=1}^{\infty} y_n(x, t) \quad \Leftrightarrow$$

$$y(x, t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) \quad \Leftrightarrow$$

$$y(x, t) = \sum_{n=1}^{\infty} \left[C_n \cos(n\pi c t) + D_n \sin(n\pi c t) \right] \sin(n\pi x).$$

Just cleaning
the setup

$(A_n \tilde{C}_n \equiv C_n, \quad A_n \tilde{D}_n \equiv D_n)$

Step 6: Use initial data (Our job today!)

- **Initial data:** the function y and its time derivative \dot{y} at $t = 0$.

- Our initial data is

Fourier series period 1 (at $t=0$)

$$y(x, 0) = x(1 - x),$$

from prev slide

$$\frac{\partial y}{\partial t}(x, 0) = 0.$$

Evaluating our general solution at $t = 0$ gives

c vanishes for $t=0$

$$\sum_{n=1}^{\infty} C_n \sin(n\pi x) = x(1 - x) \quad (\star),$$

$$\sum_{n=1}^{\infty} n\pi c D_n \sin(n\pi x) = 0$$

$$\Rightarrow C_n = ??,$$

$$\Rightarrow D_n = 0$$

How do we find C_n ? ... well equation (\star) is:

$$x(1 - x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

Note: if you want to have a deeper understanding of the problem, at home see an **alternative but absolutely equivalent way of computing C_n** that is in the Extra slides 24-25 at the end of this Lecture.

Step 6: Initial data

- C_n ? ... well equation (\star) is: $x(1-x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) \quad (\star)$

which we recognize to be the **sine Fourier series** (FS) of $f(x) = x(1-x)$ (i.e. the FS of the odd extension of $f(x) = x(1-x)$ with period 2ℓ , $\ell = 1$).

Why? BCs of original problem are given at $x = 0$ and $x = 1$

\Rightarrow initial data gives us a known function $f(x)$ in the **half-range** $0 \leq x \leq 1$.

Moreover, (\star) tells that this half-range function has a **sine** FS.

But **sine FS** \Rightarrow FS of **odd extension** of $f(x)$ with period 2ℓ . Here, $\ell = 1$.

Since (\star) is a sine FS of the odd extension of $f(x) = x(1-x)$ with period 2ℓ , $\ell = 1$) we can apply the associated **Euler formulae** to find C_n :

$$C_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx = 2 \int_0^1 x(1-x) \sin(n\pi x) dx = \frac{4}{(n\pi)^3} [1 - (-1)^n]$$

... (1077 or 1045)

[\nearrow slide 11 of Lecture 6 with $b_n \equiv C_n$]

[Exercise: check it at home! \nwarrow]

- So, **solution** to our **original problem**, that obeys the BCs & initial data, is:

$$y(x, t) = 4 \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{(n\pi)^3} \cos(n\pi c t) \sin(n\pi x).$$

Exercise to do at home:

Show that C_n is indeed given by $C_n = \frac{4}{(n\pi)^3} [1 - (-1)^n]$:

(✓ you did similar exercises in FS lectures)

$$\begin{aligned} C_n &= 2 \int_0^1 \overbrace{x(1-x)}^u \overbrace{\sin(n\pi x)}^{dv} dx \\ &= 2 \left\{ \left[-x(1-x) \frac{\cos(n\pi x)}{(n\pi)} \right]_0^1 + \int_0^1 (1-2x) \frac{\cos(n\pi x)}{(n\pi)} dx \right\} \\ &= 2 \left\{ \left[-x(1-x) \frac{\cos(n\pi x)}{(n\pi)} + (1-2x) \frac{\sin(n\pi x)}{(n\pi)^2} \right]_0^1 + 2 \int_0^1 \frac{\sin(n\pi x)}{(n\pi)^2} dx \right\} \\ &= 2 \left[-x(1-x) \frac{\cos(n\pi x)}{(n\pi)} + (1-2x) \frac{\sin(n\pi x)}{(n\pi)^2} - 2 \frac{\cos(n\pi x)}{(n\pi)^3} \right]_0^1 \\ &= \frac{4}{(n\pi)^3} [1 - (-1)^n]. \end{aligned}$$

→ A more complex example: SoV in Wave Equation

We use **separation of variables** to solve (same as in previous example)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad c \text{ constant}$$

with **boundary conditions** (✓ different from previous example)

$$\frac{\partial y}{\partial x}(0, t) = 0, \quad \frac{\partial y}{\partial x}(1, t) = 0$$

and **initial data** (same as in previous example)

$$y(x, 0) = x(1 - x), \quad \frac{\partial y}{\partial t}(x, 0) = 0.$$

- Once more we assume $y = XT$ which gives

$$\begin{aligned}X'' - \lambda X &= 0, \\ \ddot{T} - c^2 \lambda T &= 0\end{aligned}$$

- Boundary conditions (BCs) (✓ different from previous example)

$$\begin{aligned}\frac{\partial y}{\partial x}(0, t) = 0 &\Leftrightarrow X'(0)T(t) = 0, \quad \forall t &\Rightarrow X'(0) = 0, \\ \frac{\partial y}{\partial x}(1, t) = 0 &\Leftrightarrow X'(1)T(t) = 0, \quad \forall t &\Rightarrow X'(1) = 0.\end{aligned}$$

Step 3: Eigenvalue problem for $X(x)$

2 non-trivial

$$X'' - \lambda X = 0; \quad X'(0) = 0, \quad X'(1) = 0.$$

- The solution for $\lambda = 0$ is

$$X(x) = Ex + F \Rightarrow X' = E.$$

BCs $\Rightarrow E = 0$. Thus, we have a **non-trivial solution**: $X(x) = F$.

- The solution for $\lambda = -k^2 < 0$ is

$$X(x) = A \sin(kx) + B \cos(kx) \Rightarrow X'(x) = k [A \cos(kx) - B \sin(kx)]$$

BC at $x = 0 \Rightarrow A = 0$.

BC at $x = 1 \Rightarrow$ either **trivial solution** $A = B = 0$ or $k = n\pi$.

- The solution for $\lambda = k^2 > 0$ is

$$X = A e^{kx} + B e^{-kx} \Rightarrow X' = k (A e^{kx} - B e^{-kx})$$

BC at $x = 0 \Rightarrow A = B$. BC at $x = 1 \Rightarrow A = 0$.

Thus, we **only** have the **trivial solution** $A = B = 0$ in this case.

Step 4: Solve for $T(t)$

Constant coefficient ODE for $T(t)$ with known λ (revisit Lecture 1)

$$\ddot{T} - c^2 \lambda T = 0$$

We now have to **solve two cases**:

- For $\lambda = 0$ we have

$$T = \tilde{G}t + \tilde{H}.$$

- For $\lambda = -k^2 < 0$ we have $k \equiv k_n = n\pi$ and:

$$T_n(t) = \tilde{C}_n \cos(n\pi c t) + \tilde{D}_n \sin(n\pi c t).$$

- NO need to solve $\lambda > 0$ since $X(x) = 0$.

Step 5: The general solution

So we have:

$$\left\{ \begin{array}{ll} \text{If } \lambda = 0: & X(x) = F, \quad T(t) = \tilde{G}t + \tilde{H} \\ \text{If } \lambda < 0: & X(x) = B_n \cos(n\pi x), \quad T(t) = \sum_{n=1}^{\infty} [\tilde{C}_n \cos(n\pi ct) + \tilde{D}_n \sin(n\pi ct)] \\ \text{If } \lambda > 0: & \text{Only trivial solution} \end{array} \right.$$

Combining these, $y_n = X_n T_n$, and **superposing** these **normal modes** y_n gives the **general solution**:

$$y(x, t) = (\tilde{G}t + \tilde{H})F + \sum_{n=1}^{\infty} [\tilde{C}_n \cos(n\pi ct) + \tilde{D}_n \sin(n\pi ct)] B_n \cos(n\pi x)$$

$$y(x, t) = Gt + H + \sum_{n=1}^{\infty} [C_n \cos(n\pi ct) + D_n \sin(n\pi ct)] \cos(n\pi x).$$

$$(\nwarrow G \equiv \tilde{G}F, H \equiv \tilde{H}F, C_n \equiv \tilde{C}_n B_n, D_n \equiv \tilde{D}_n B_n)$$

Step 6: Initial data

$$y(x, t) = Gt + H + \sum_{n=1}^{\infty} \left[C_n \cos(n\pi c t) + D_n \sin(n\pi c t) \right] \cos(n\pi x).$$

$$\longrightarrow G, H, C_n, D_n = ??$$

- Our **initial data** is

$$y(x, 0) = x(1 - x), \quad \frac{\partial y}{\partial t}(x, 0) = 0.$$

Evaluating our general solution at **$t = 0$** gives

$$y(x, 0) = x(1 - x) \quad \Leftrightarrow \quad H + \sum_{n=1}^{\infty} C_n \cos(n\pi x) = x(1 - x), \quad (1)$$

$$\frac{\partial y}{\partial t}(x, 0) = 0 \quad \Leftrightarrow \quad G + \sum_{n=1}^{\infty} n\pi c D_n \cos(n\pi x) = 0. \quad (2)$$

Step 6: Initial data

- Equation (2) \Rightarrow each of the $\cos(n\pi x)$ terms vanish separately:

$$G = D_n = 0. \quad (\searrow \text{From BCs, } x \in [0, 1])$$

- Equation (1) is $x(1-x) = H + \sum_{n=1}^{\infty} C_n \cos(n\pi x)$ with $x \in [0, 1]$

which we recognize to be the cosine Fourier series of $f(x) = x(1-x)$ (ie the FS of the even extension of $f(x) = x(1-x)$ w/ period 2ℓ , $\ell = 1 \Leftarrow x \in [0, 1]$).

Thus, $f(x) = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{\ell}\right)$ with $\ell = 1$ & we apply **Euler formulae**:

[✓ see slide 10 of Lecture 6 with $a_0 \equiv C_0$ and $a_n \equiv C_n$]

$$H \equiv \frac{1}{2}C_0 = \frac{1}{2} \left(\frac{2}{\ell} \int_0^{\ell} f(x) dx \right) = \frac{1}{2} \left(2 \int_0^1 x(1-x) dx \right) = \frac{1}{6}, \quad [\downarrow \text{Exercise: check it! similar to slide 13}]$$

$$C_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx = 2 \int_0^1 x(1-x) \cos(n\pi x) dx = -\frac{2}{(n\pi)^2} [1 + (-1)^n]$$

- Solution** to our **original problem**, that obeys BCs and initial data, is:

$$y(x, t) = \frac{1}{6} - 2 \sum_{n=1}^{\infty} \frac{[1 + (-1)^n]}{(n\pi)^2} \cos(n\pi c t) \cos(n\pi x).$$

Wave equation describes...travelling waves

Solution of wave equation (& its BCs + initial data) is

$$y(x, t) = \frac{1}{6} - 2 \sum_{n=1}^{\infty} \frac{[1 + (-1)^n]}{(n\pi)^2} \cos(n\pi x) \cos(n\pi c t).$$

↘ Trigonometric identity: $2 \cos A \cos B = \cos(A - B) + \cos(A + B)$:

$$= \frac{1}{6} - \sum_{n=1}^{\infty} \frac{[1 + (-1)^n]}{(n\pi)^2} \left[\underbrace{\cos[n\pi(x - ct)]}_{\text{Right-mover wave}} + \underbrace{\cos[n\pi(x + ct)]}_{\text{Left-mover wave}} \right].$$

- **Front of the wave** is given by condition that phase vanishes: $(x \pm ct) = 0$,

$\left\{ \begin{array}{l} \text{Right-mover wave: } x - ct = 0 \Rightarrow x = ct \rightarrow t \nearrow \Rightarrow x \nearrow \Rightarrow \text{moves to the right} \\ \text{Left-mover wave: } x + ct = 0 \Rightarrow x = -ct \rightarrow t \nearrow \Rightarrow x \searrow \Rightarrow \text{moves to the left} \end{array} \right.$

- Waves indeed travel with **velocity** c .
- **Periodic wave propagation**: $\cos[(x \pm ct) + 2\pi] = \cos[x \pm ct]$
- Go back to **slide 4 of Lecture 14** and see its **figure**: now we understand it!

1 Review

2 Examples

- Basic Example
- More complex example

3 Summary

$$y(x, t) = X(x)T(t)$$

- 1 Determine equations for X , T .
- 2 Use boundary conditions of y in order to obtain boundary conditions of X .
- 3 Solve eigenvalue problem for X : determine eigenvalues λ_n and eigenfunctions X_n .
- 4 Insert eigenvalue λ_n in the T equation and solve it to obtain T_n .
- 5 The normal modes are $y_n = X_n T_n$ and the general solution is obtained by superposition

$$y(x, t) = \sum_n X_n(x) T_n(t)$$

- 6 Use initial conditions, $y(x, 0)$, $\partial y(x, 0)/\partial t$ to determine all undetermined coefficients. This step involves Fourier series.

EXTRA: an alternative way to solve problem of slides 11 & 12

• **Imagine** that we have **not** learned Fourier Series. Could we still solve the problem of slide 11 and get the final solution given in slide 12? The answer is yes and we give it here. *Of course, the way we solve it in slides 11 & 12 is perfectly good and enough! This extra appendix is just for you to reflect about it at home and understand more deeply how things work and their origin.*

• Our initial data is: $y(x, 0) = x(1 - x), \quad \frac{\partial y}{\partial t}(x, 0) = 0.$

Evaluating our general solution at $t = 0$ gives

$$\sum_{n=1}^{\infty} C_n \sin(n\pi x) = x(1 - x) \quad (\star),$$

$$\Rightarrow C_n = ??,$$

$$\sum_{n=1}^{\infty} n\pi c D_n \sin(n\pi x) = 0$$

$$\Rightarrow D_n = 0$$

To find C_n recall the inner product for functions to get projections (slide 9 of Lecture 4)!

Projection (orthogonality) of $f(x) = x(1 - x)$ **over** $g(x) = \sin(k\pi x)$: multiply (\star) by $\sin(k\pi x)$ & **integrate between** 0 and 1 (since $x \in [0, 1]$ by BCs) we get

$$\int_0^1 \left[\sum_{n=1}^{\infty} C_n \sin(n\pi x) \right] \sin(k\pi x) dx = \int_0^1 x(1 - x) \sin(k\pi x) dx$$

↗ Inner product for functions defined in $x \in [0, \ell]$: $\langle f(x), g(x) \rangle = \frac{1}{\ell} \int_0^{\ell} f(x) g(x) dx$ (slide 9, Lecture 4)

(think about it:... this is exactly how we derived Euler formulae in lecture 4!!)

$$\int_0^1 \left[\sum_{n=1}^{\infty} C_n \sin(n\pi x) \right] \sin(k\pi x) dx = \int_0^1 x(1-x) \sin(k\pi x) dx$$

$$\sum_{n=1}^{\infty} \int_0^1 C_n \sin(n\pi x) \sin(k\pi x) dx = \int_0^1 x(1-x) \sin(k\pi x) dx \quad [\checkmark \text{ Done in PS 1 }]$$

Do the integral: $\int_0^1 \sin(n\pi x) \sin(k\pi x) dx = \left[\frac{1}{2} - \frac{\sin(2\pi n)}{4\pi n} \right] \delta_{kn} = \frac{1}{2} \delta_{kn}$ for $n = 1, 2, 3, \dots$

$$\sum_{n=1}^{\infty} C_n \frac{1}{2} \delta_{kn} = \int_0^1 x(1-x) \sin(k\pi x) dx \quad (\checkmark \delta_{kn} = 1 \text{ if } n = k ; \delta_{kn} = 0 \text{ if } n \neq k)$$

$$\frac{1}{2} C_k = \int_0^1 x(1-x) \sin(k\pi x) dx \quad (\checkmark k \text{ is dummy variable: can do } k \rightarrow n)$$

$$C_n = 2 \int_0^1 x(1-x) \sin(n\pi x) dx = \frac{4}{(n\pi)^3} [1 - (-1)^n] \quad [\text{as shown in slide 13}]$$

The **solution** to our **original problem** is thus

$\nwarrow \checkmark$ exactly what we got in slide 12!

$$y(x, t) = 4 \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{(n\pi)^3} \cos(n\pi c t) \sin(n\pi x).$$