SESA2023 Propulsion

Lecture 7: Compressible flow and speed of sound

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THIS LECTURE

- Introduction to compressible flow
- Simplifications
- Stagnation properties
- Speed of sound and Mach number
- Critical properties

FROM INCOMPRESSIBLE TO COMPRESSIBLE

too complex for manual calculation, more used in CFD

Mass conservation: $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j) = 0$$

Momentum conservation: $$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_j}(\rho U_i U_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

Energy conservation: $$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x_i}(\rho h U_i) = \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + \frac{\mathrm{D}P}{\mathrm{D}t} + \mu\Phi$$

for incompressible flow these are independent

Equation of state: $$\boxed{P = \rho R T}$$

once its compressible they couple and it becomes hellish (lol) avoiding 3D hell

SIMPLIFICATIONS

avoiding time stuff

- One-dimensional flow
- Steady flow (no time dependence)
- Inviscid flow

no friction (yum)

- No thermal diffusion

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{\partial}{\partial x_j}(\rho U_j) = 0$$

$$\cancel{\frac{\partial}{\partial t}(\rho U_i)} + \frac{\partial}{\partial x_j}(\rho U_i U_j) = -\frac{\partial P}{\partial x_i} + \cancel{\frac{\partial \tau_{ij}}{\partial x_j}}$$

$$\cancel{\frac{\partial}{\partial t}(\rho h)} + \frac{\partial}{\partial x_i}(\rho h U_i) = \frac{\partial}{\partial x_i}\cancel{\left(k\frac{\partial T}{\partial x_i}\right)} + \frac{\mathrm{D}P}{\mathrm{D}t} + \cancel{\mu\Phi}$$

$$P = \rho R T$$

SIMPLIFIED EQUATIONS

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho U) = 0$$

$$U\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} + U\frac{\mathrm{d}U}{\mathrm{d}x} = 0$$

$$P = \rho R T$$

now they are manageable
(still more complex than what we do though)

WHAT WE WILL ACTUALLY BE DEALING WITH…

Compressible

mass conservation

$$\rho_1 U_1 A_1 = \rho_2 U_2 A_2$$

$$h_1 + \tfrac{1}{2}U_1^2 = h_2 + \tfrac{1}{2}U_2^2$$

$$P = \rho R T$$

Incompressible

volume conservation

$$U_1 A_1 = U_2 A_2$$

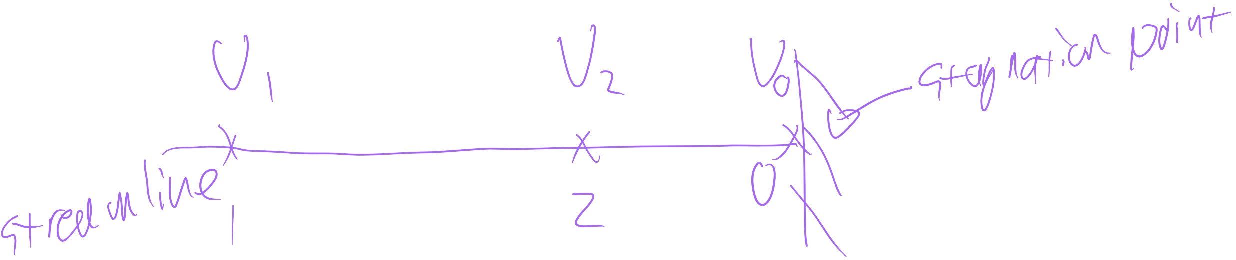
$$\frac{P_1}{\rho} + \frac{1}{2}U_1^2 = \frac{P_2}{\rho} + \frac{1}{2}U_2^2$$

energy conservation

converted to based versions of Bernoulli

STAGNATION PROPERTIES

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2 = h_0 + \frac{1}{2}\cancel{(V_0)}$$ ← $V_0 = 0$



$$h_0 = h + \frac{1}{2}V^2$$

(stagnation enthalpy)

assume perfect gas

$$h_0 - h = C_p(T_0 - T)$$

$$\boxed{T_0 = T + \frac{V^2}{2C_p}}$$

(stagnation temp)

assume isentropic compression: $\boxed{p_0 = p\left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}}$

(stagnation pressure)

— stagnation equations

can now find everything else with $\boxed{\rho = \frac{p}{RT}}$

SPEED OF SOUND

once you get to ~$\frac{1}{3}$ of speed of sound compression starts to be important

$T+\delta T$		T
$P+\delta P$	$\longrightarrow a$	P
$\rho+\delta\rho$		ρ
$U=\delta U+0$		$U=0$

SPEED OF SOUND

$$\xrightarrow{\quad\quad} a$$

$T+\delta T$	T
$P+\delta P$	P
$\rho+\delta\rho$	ρ
$U=\delta U$	$U=0$

SPEED OF SOUND

front moves at speed "a"

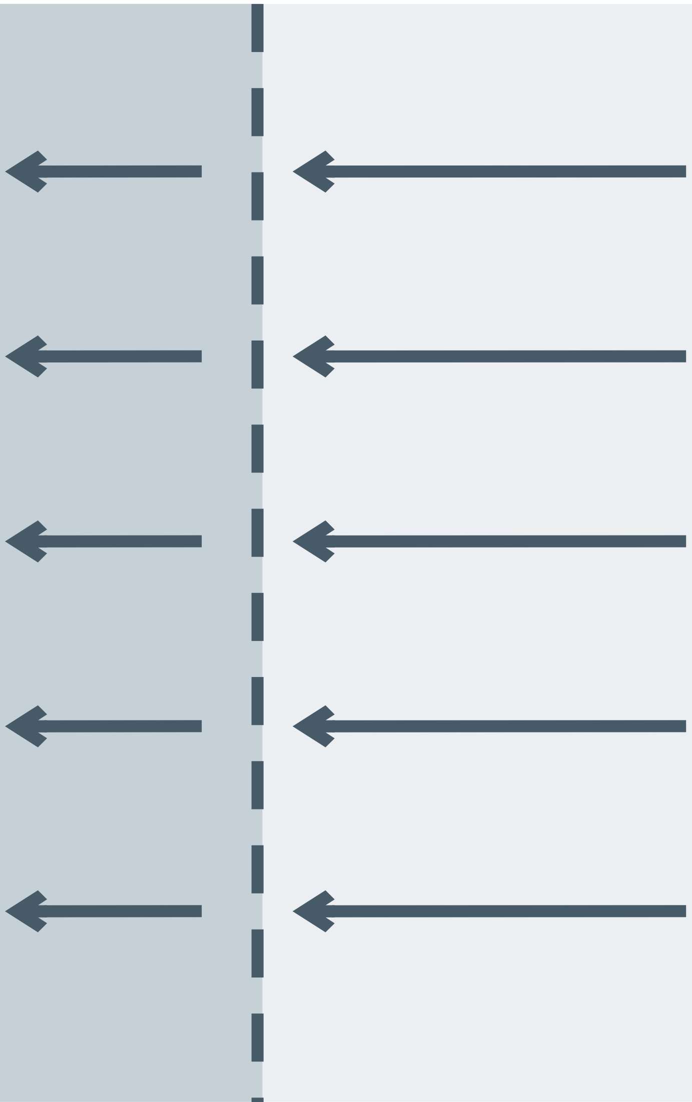
(gas,

$T+\delta T$

$P+\delta P$

$\rho+\delta\rho$

$U=a-\delta U$



T

P

ρ

$U=a$

SPEED OF SOUND

From mass and momentum conservation:

$$\underbrace{a^2}_{\text{speed of sound}} = \frac{\mathrm{d}P}{\mathrm{d}\rho}$$

Assuming isentropic flow:

$$\frac{\mathrm{d}P}{\mathrm{d}\rho} = \gamma R T,$$

to combine

$$\boxed{a = \sqrt{\gamma R T}}$$

MACH NUMBER

$$Ma = \frac{V}{a} = \frac{V}{\sqrt{\gamma R T}} \rightarrow T = \frac{a^2}{\gamma R}$$

by using $\frac{T_0}{T} = 1 + \frac{V^2}{2C_pT}$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} Ma^2$$

EXAMPLE: MACH NUMBER

A supersonic jet is flying at Ma = 1.1, where the local temperature is 220 K.

Keeping the same flight speed, but at a temperature of 300 K, what is the Mach number?

$$Ma = \frac{U}{\sqrt{\gamma R}} \frac{1}{\sqrt{T}}$$

$$Ma\sqrt{T} = \underbrace{\frac{U}{\sqrt{\gamma R}}}_{\text{constant}}$$

$$Ma_1\sqrt{T_1} = Ma_2\sqrt{T_2}$$

$$Ma_1\sqrt{\frac{T_1}{T_2}} = Ma_2 = 0.94198$$

CRITICAL PROPERTIES

Props at $Ma = 1$: T^*, p^*, ρ^*

$$\frac{T_0}{T_*} = 1 + \frac{\gamma - 1}{2} Ma^2 = 1 + \frac{\gamma - 1}{2}$$

$$T^* = T_0 \frac{2}{1+\gamma}$$

$$p^* = p_0 \left(\frac{T^*}{T}\right)^{\frac{\gamma}{\gamma - 1}} \leftarrow \text{(isentropic assumption)}$$

$$p^* = p_0 \left(\frac{2}{1+\gamma}\right)^{\frac{\gamma}{\gamma - 1}}$$

$$\rho^* = \rho_0 \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma - 1}}$$

$\therefore$ can find critical values from stagnation values

SUMMARY

- Difficulty with compressible flows
- Assumptions and simplifications for steady 1D inviscid flow
- Define stagnation properties
  - Enthalpy, temperature, pressure, and density at zero velocity
- Speed of sound
  - Temperature dependence, Mach number
- Critical properties
  - Temperature, pressure, and density at Ma = 1

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Lecture 8: Shocks

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THIS LECTURE

- Introduction to shocks
- Analysis outline
- Shock relations
- Shock properties

SPEED OF SOUND: INFINITESIMAL DISTURBANCE

$$a = \sqrt{\gamma R T}$$

$T + \delta T$	T
$P + \delta P$	P
$\rho + \delta \rho$	ρ
$U = \delta U$	$U = 0$

SHOCK: FINITE DISTURBANCE

$$a = \sqrt{\gamma R (T + \delta T)}$$

$$a = \sqrt{\gamma R T}$$

$T + \delta T$	T
$P + \delta P$	P
$\rho + \delta \rho$	ρ
$U = a - \delta U$	$U = a$

SHOCK

$$T_2 = T_1 + \Delta T$$

$$P_2 = P_1 + \Delta P$$

$$\rho_2 = \rho_1 + \Delta \rho$$

$$U_2 = U_1 - \Delta U$$

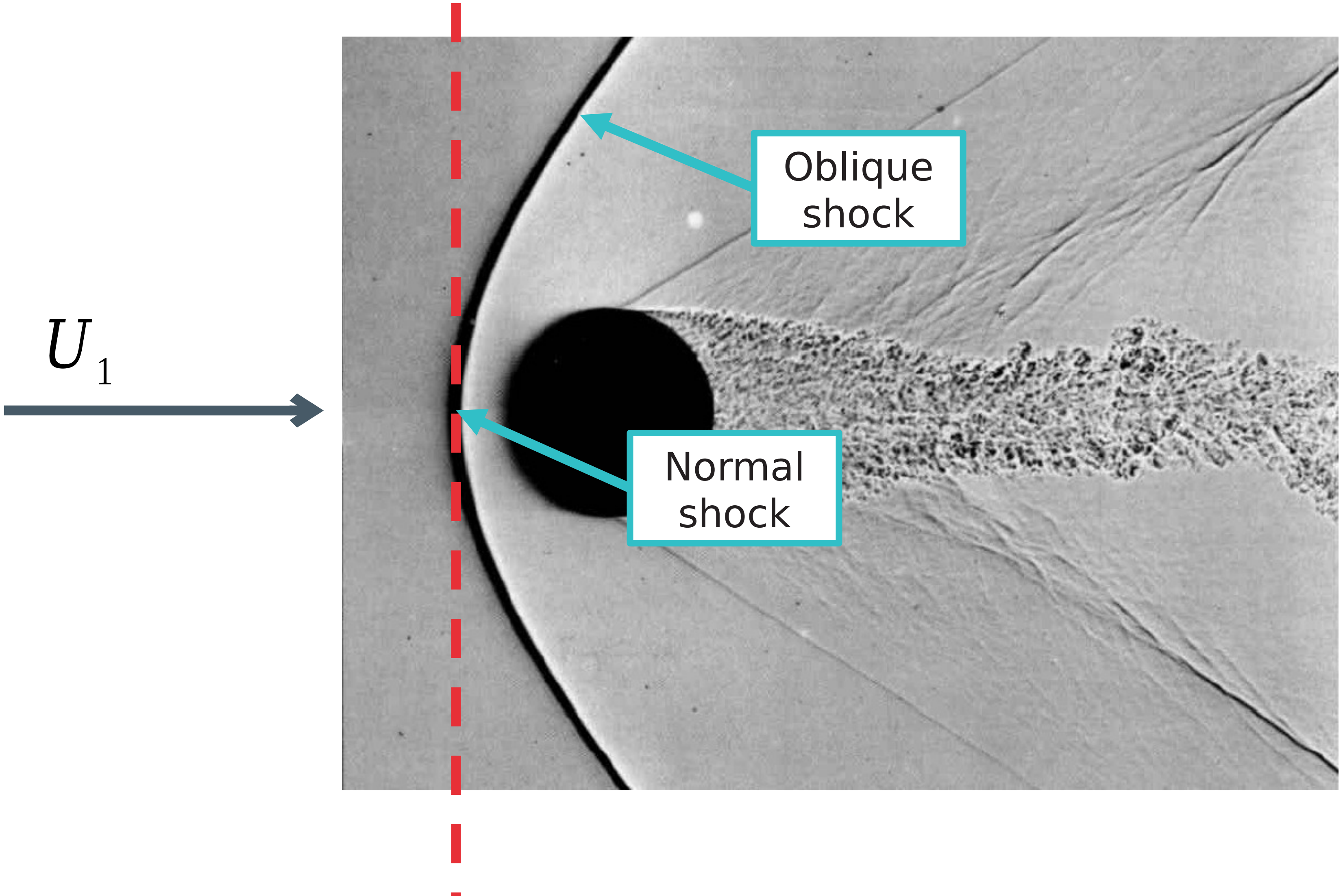
$$T_1$$

$$P_1$$

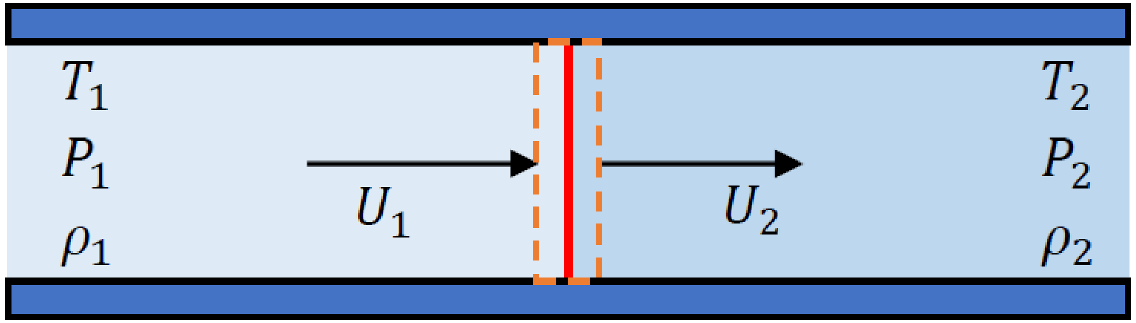
$$\rho_1$$

$$U_1$$

NORMAL SHOCKS AND OBLIQUE SHOCKS



NORMAL SHOCK ANALYSIS



Mass conservation

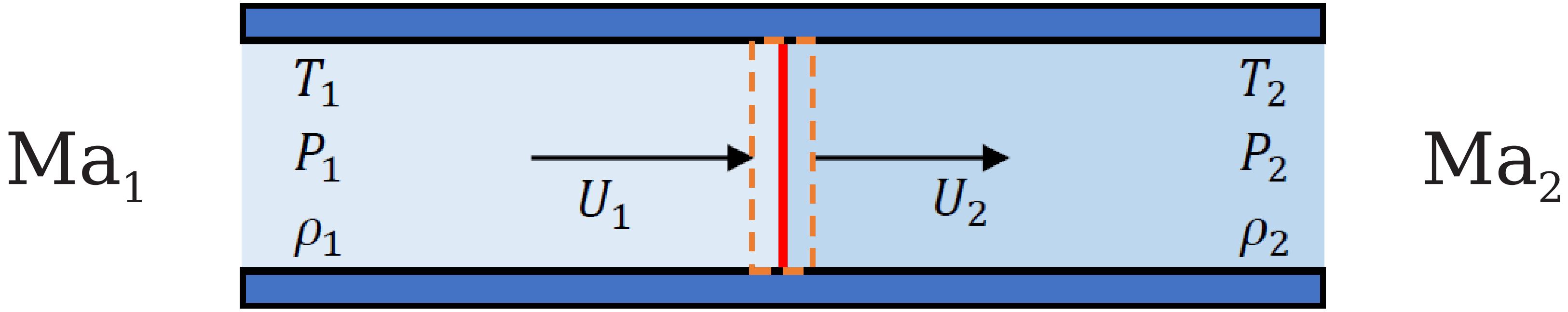
Momentum conservation

Energy conservation

Perfect gas

Not isentropic!

SHOCK RELATIONS



$$\mathrm{Ma}_2^2 = \frac{(\gamma - 1)\mathrm{Ma}_1^2 + 2}{2\gamma \mathrm{Ma}_1^2 + 1 - \gamma}$$

$$\frac{T_2}{T_1} = 1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2}\left(\frac{\gamma \mathrm{Ma}_1^2 + 1}{\mathrm{Ma}_1^2}\right)(\mathrm{Ma}_1^2 - 1)$$

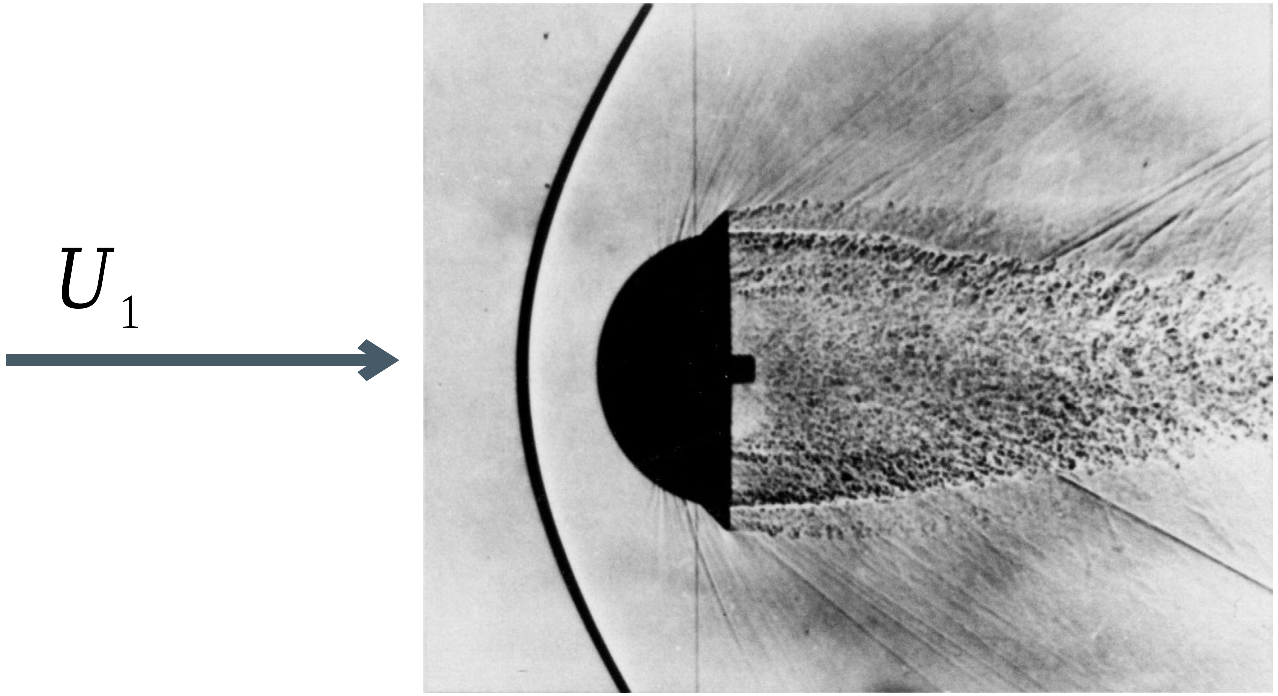
$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)\mathrm{Ma}_1^2}{(\gamma - 1)\mathrm{Ma}_1^2 + 2}$$

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma + 1}(\mathrm{Ma}_1^2 - 1)$$

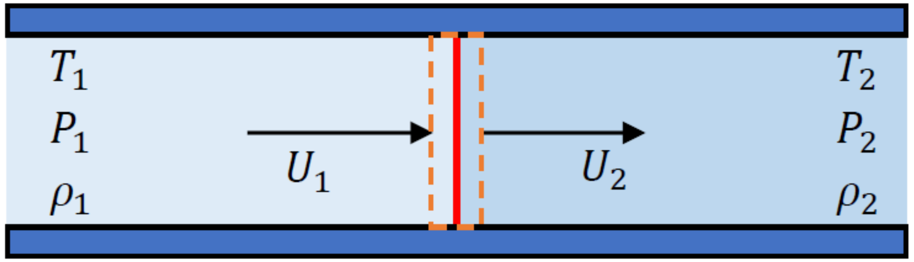
EXAMPLE

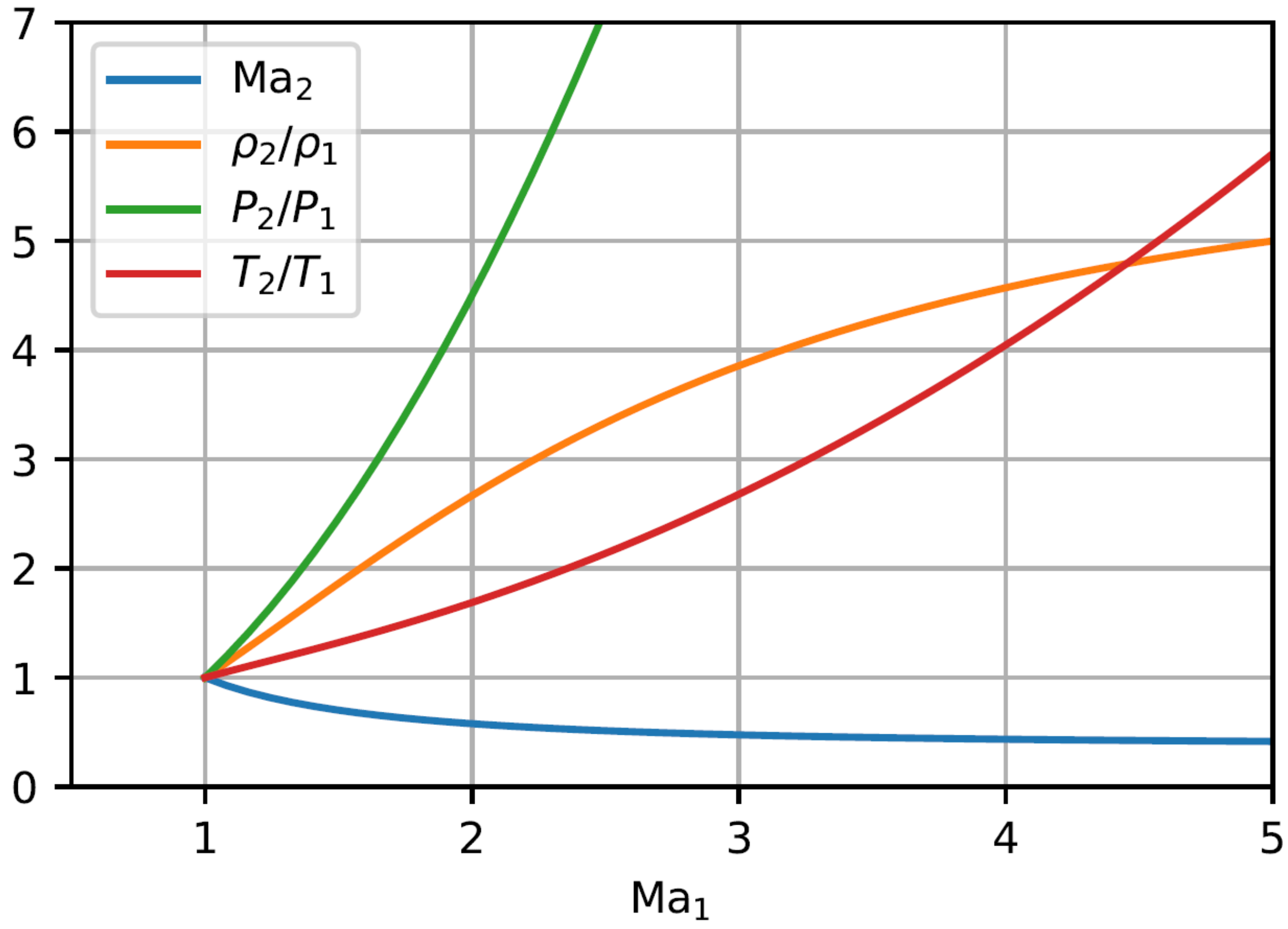
An airflow T = 250 K, P = 0.5 bar, U = 600 m/s, encounters a shock.

What is the stagnation temperature and pressure before and after the shock?



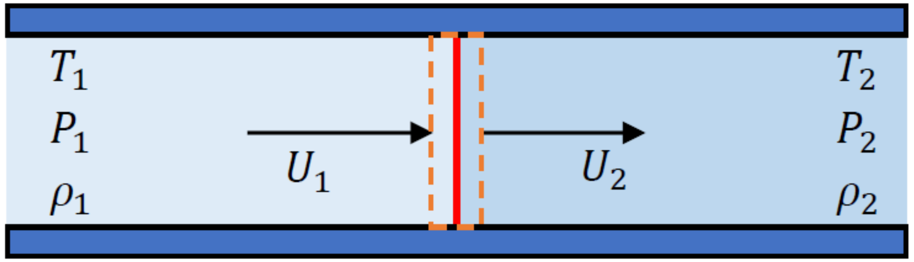
SHOCK PROPERTIES

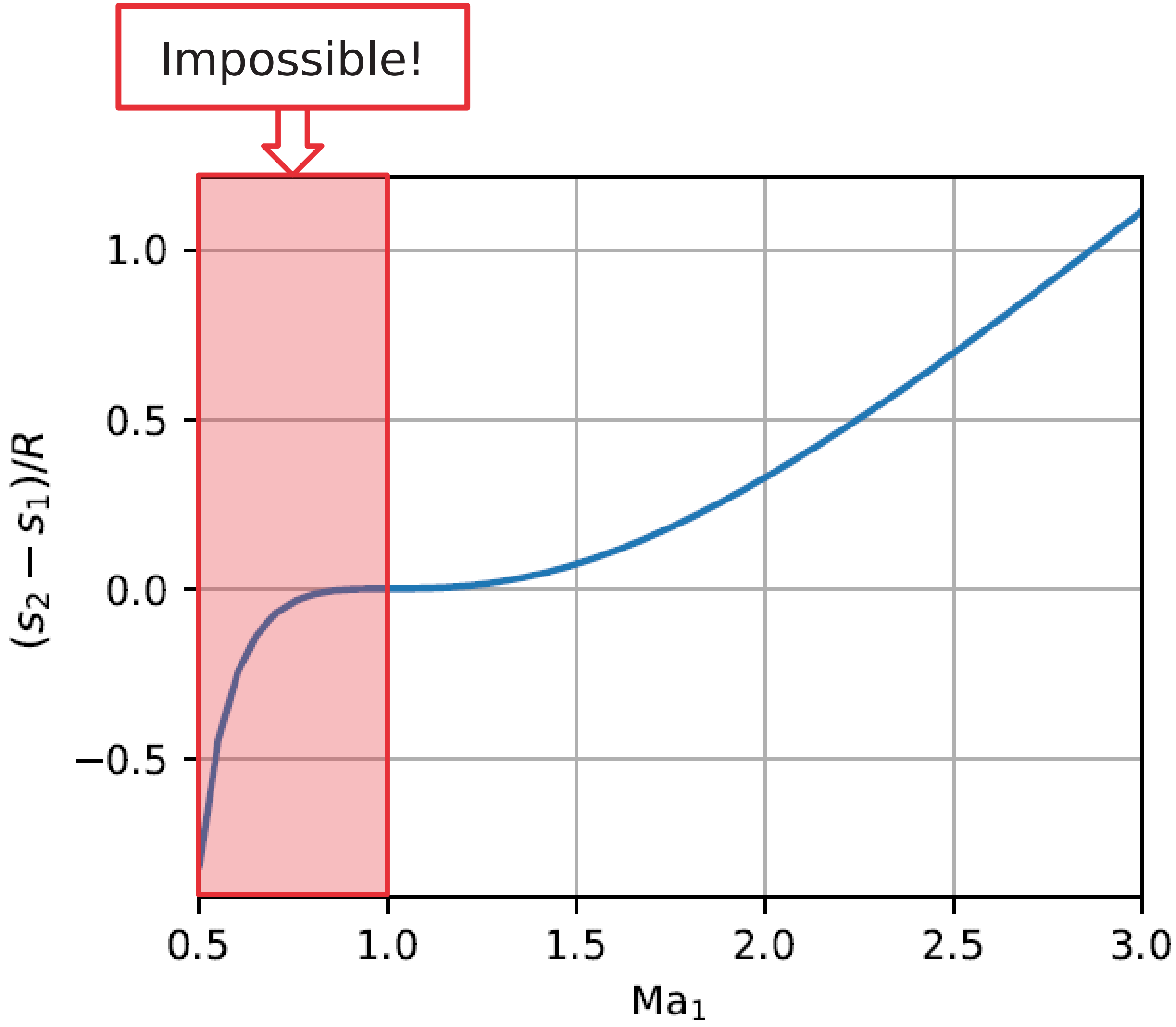




ENTROPY CHANGE

$$\Delta s = s_2 - s_1 = c_P \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right)$$





SHOCK RELATIONS

Normal shock relations

1. Mach number after the shock

In [2]:
```
def shock_Ma2(Ma1, gamma=1.4):
    A = (gamma-1)*Ma1**2+2
    B = 2*gamma*Ma1**2+1-gamma
    return (A/B)**0.5
```

2. Temperature after the shock

Returns the temperature ratio T2/T1

In [3]:
```
def shock_T(Ma1, gamma=1.4):
    A = 2*(gamma-1)/(gamma+1)**2
    B = (gamma*Ma1**2+1)/Ma1**2
    C = Ma1**2-1
    return 1+A*B*C
```

SHOCK TABLES

M	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{0_2}}{p_{0_1}}$	$\frac{p_{0_2}}{p_1}$	M_2
0.2000 + 01	0.4500 + 01	0.2667 + 01	0.1687 + 01	0.7209 + 00	0.5640 + 01	0.5774 + 00
0.2050 + 01	0.4736 + 01	0.2740 + 01	0.1729 + 01	0.6975 + 00	0.5900 + 01	0.5691 + 00
0.2100 + 01	0.4978 + 01	0.2812 + 01	0.1770 + 01	0.6742 + 00	0.6165 + 01	0.5613 + 00
0.2150 + 01	0.5226 + 01	0.2882 + 01	0.1813 + 01	0.6511 + 00	0.6438 + 01	0.5540 + 00
0.2200 + 01	0.5480 + 01	0.2951 + 01	0.1857 + 01	0.6281 + 00	0.6716 + 01	0.5471 + 00

M	$\frac{T}{T_0}$	$\frac{p}{p_0}$	$\frac{\rho}{\rho_0}$	$\frac{V}{\sqrt{c_p T_0}}$	$\frac{\dot{m}\sqrt{c_p T_0}}{A p_0}$	$\frac{\dot{m}\sqrt{c_p T_0}}{A p}$	$\frac{F}{\dot{m}\sqrt{c_p T_0}}$	$\frac{4 c_f L_{max}}{D}$	$\frac{\frac{1}{2}\rho V^2}{p_0}$	M_s	$\frac{p_{0s}}{p_0}$	$\frac{p_s}{p}$	$\frac{p_{0s}}{p}$	$\frac{T_s}{T}$	ν_{PM} (deg.)	M
1.960	0.5655	0.1360	0.2405	0.9322	0.7846	5.7695	1.1055	0.2929	0.3657	0.5844	0.7395	4.3152	5.4378	1.6553	25.27	**1.9600**
1.970	0.5630	0.1339	0.2378	0.9349	0.7782	5.8118	1.1069	0.2960	0.3638	0.5826	0.7349	4.3611	5.4881	1.6633	25.55	**1.9700**
1.980	0.5605	0.1318	0.2352	0.9375	0.7718	5.8542	1.1084	0.2990	0.3618	0.5808	0.7302	4.4071	5.5386	1.6713	25.83	**1.9800**
1.990	0.5580	0.1298	0.2326	0.9402	0.7655	5.8969	1.1098	0.3020	0.3598	0.5791	0.7255	4.4535	5.5894	1.6794	26.10	**1.9900**
2.000	0.5556	0.1278	0.2300	0.9428	0.7591	5.9397	1.1112	0.3050	0.3579	0.5774	0.7209	4.5000	5.6404	1.6875	26.38	**2.0000**

SUMMARY

- What is a shock, how is it formed
- A shock converts a supersonic flow to a subsonic flow
- Entropy increases across a shock
- The Mach number determines change in speed, pressure, temperature and density
- Shock relations
  - Equations
  - Tables

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Lecture 9: Duct flow

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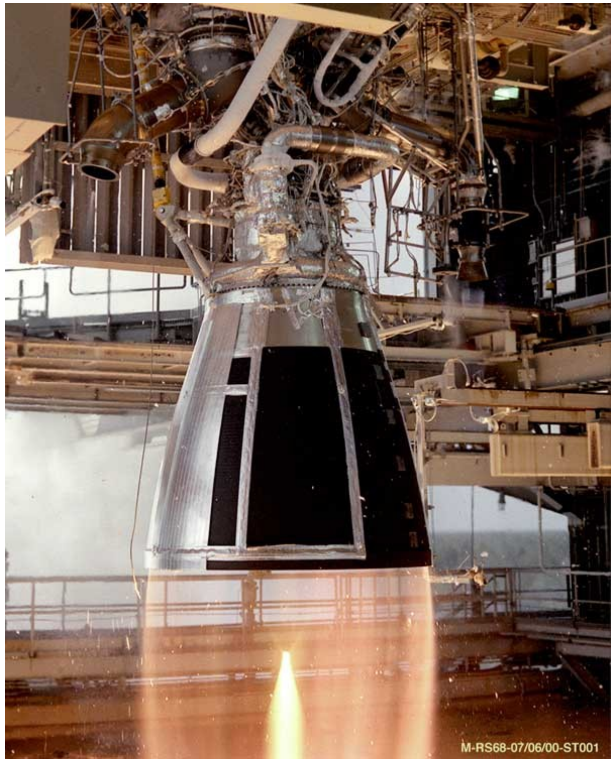
THIS LECTURE

- Nozzle applications
- Nozzle analysis assumptions
- Limitations of converging nozzles
- Converging-diverging nozzles
- Mass flow rate

NOZZLE APPLICATIONS

Goal: accelerate flows to increase thrust. Ideally to supersonic speeds.



ANALYSIS AND ASSUMPTIONS

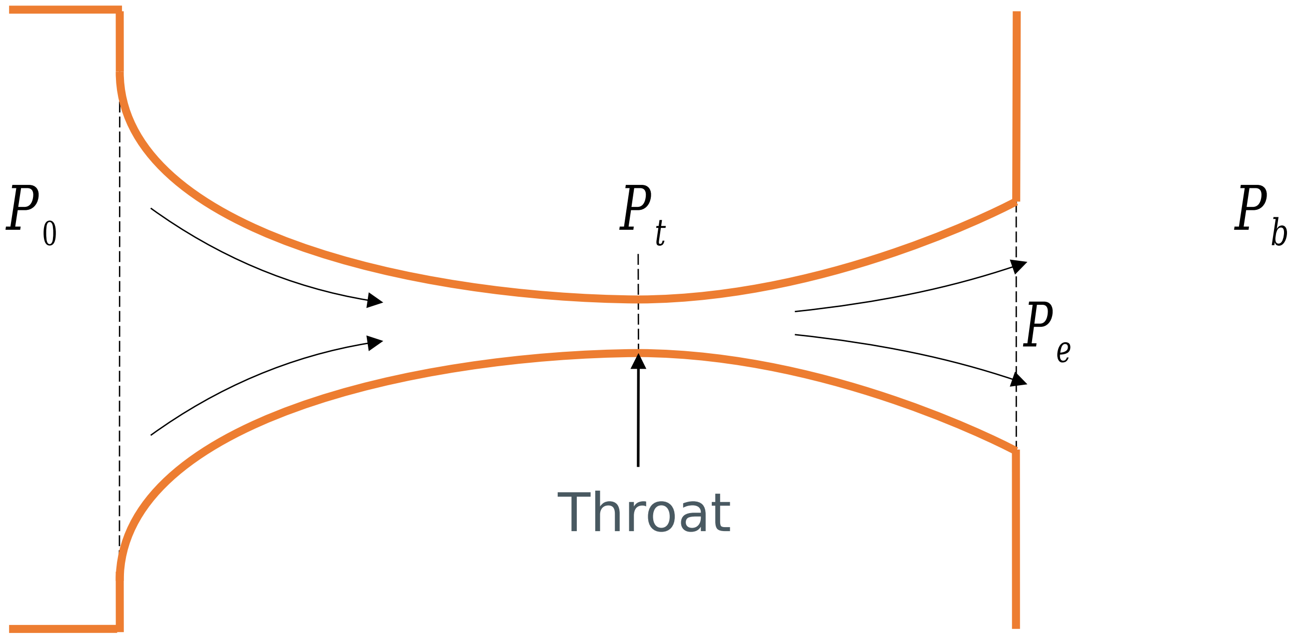
- Steady Flow
- Quasi 1D: Gradual changes in area
- No friction
- Adiabatic
- Isentropic (except for shocks!)

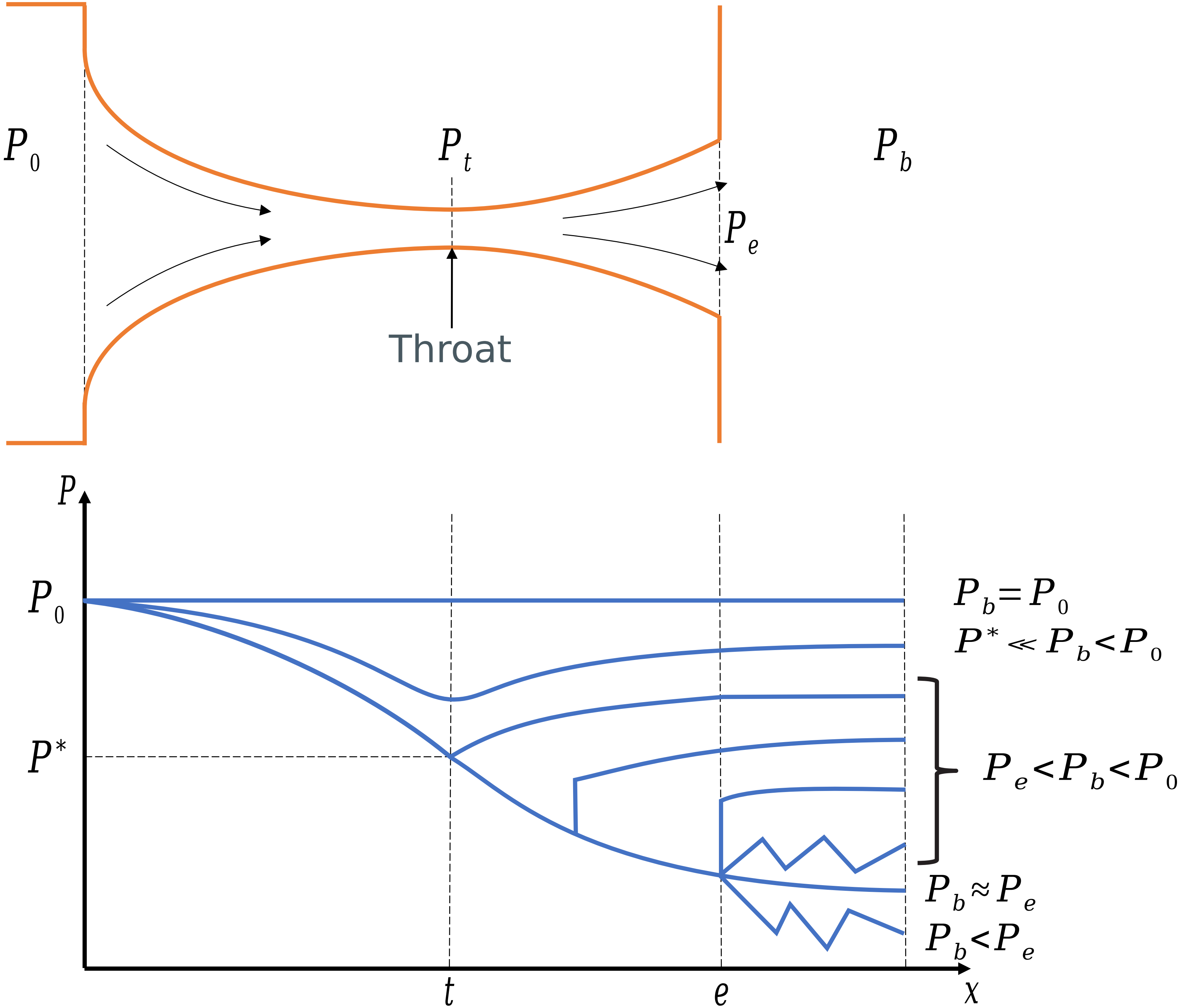
Mass and momentum conservation:

$$\boxed{\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}x}\left(\mathrm{Ma}^2 - 1\right)}$$

LIMITATIONS OF CONVERGING NOZZLES

CONVERGING-DIVERGING NOZZLES





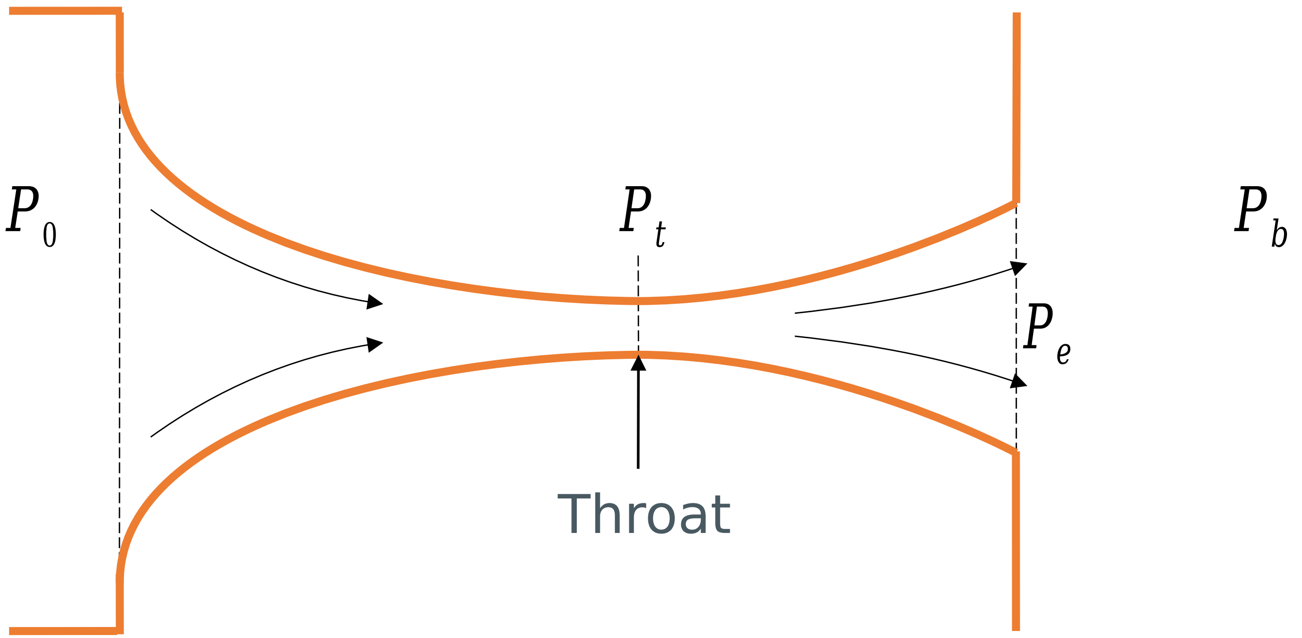
P_0
P_t
P_b
P_e
Throat
P
P_0
P^*
$P_b = P_0$
$P^* \ll P_b < P_0$
$P_e < P_b < P_0$
$P_b \approx P_e$
$P_b < P_e$
t
e
x

XCOR
NASA
ATK

EXAMPLE: ISENTROPIC NOZZLE

Reservoir conditions: air at p_0 = 10 bar, T_0 = 400 K.

What is the exit pressure for Ma_e= 2.0?



MASS FLOW RATE

From mass flow rate and **isentropic** relations:

$$\boxed{\frac{\dot{m}}{\rho_0\left(2c_P T_0\right)^{1/2}} = A\left(\frac{P}{P_0}\right)^{\frac{1}{\gamma}}\left[1-\left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}}$$

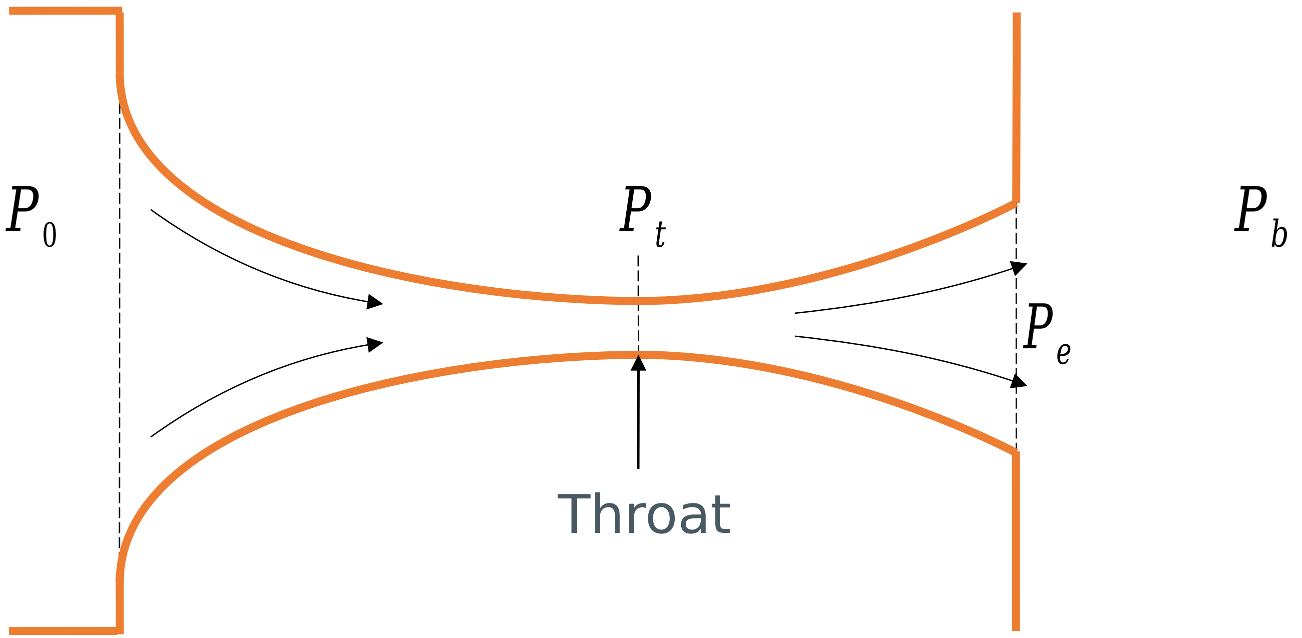
For choked flow we know that at the throat: $\dfrac{P^*}{P_0} = \left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$

So mass flow rate is limited by:

$$\boxed{\frac{\dot{m}}{\rho_0\left(2c_P T_0\right)^{1/2}} = A_t\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}}$$

EXAMPLE: THRUST

For the nozzle from the previous example, what is the thrust generated if the throat area is 0.1 m²?



SUMMARY

- Nozzle applications
- Converging nozzles
  - Acceleration for $Ma < 1$
  - Deceleration for $Ma > 1$
- Choked flow
  - Critical conditions ($Ma = 1$) at throat
- Converging-diverging nozzles for supersonic exit velocity
  - Performance with changing back pressure
- Mass flow rate, area-pressure relation