SESA2025 Mechanics of Flight
Decoupled linear model

Lecture 3.2

Steps to develop a full dynamical model

- Six degrees of freedom with six equations (Newton's laws)
  - three translational
  - three rotational
- Tait–Bryan angles and rates of change of these angles
  - Relate the aircraft (body) co-ordinate system to a fixed (inertial) reference frame, which we will take as the Earth
- Using a body-reference frame, work out the changes in the Earth frame of reference
  - Hence write down Newton's laws for linear and angular momentum
- Define the inertia matrix (tensor) for a 3D rigid rotating body

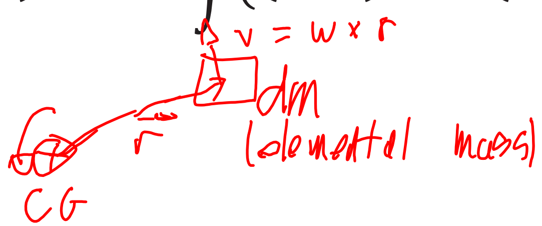
unconstrained system
∴ all rotation about CG

Moment of inertia about the centre of mass of a rigid body

The angular momentum (using a vector identity)

$$\mathbf{h} = \underbrace{\int \mathbf{r} \times \mathbf{v}\, \mathrm{d}m}_{} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, \mathrm{d}m = \int \big((\mathbf{r} \cdot \mathbf{r})\boldsymbol{\omega} - (\mathbf{r} \cdot \boldsymbol{\omega})\mathbf{r}\big)\, \mathrm{d}m$$

where m is mass



can be expanded into:

$$\begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = \int \left[(x^2 + y^2 + z^2) \begin{pmatrix} p \\ q \\ r \end{pmatrix} - (xp + yq + zr) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right] \mathrm{d}m$$

integrated for all elemental masses (way too complex outside of a computer)

we assume a rigid body to simplify the maths. of course wings are not rigid at all but this is suitable for first order modelling.

Moment of inertia about the centre of mass of a rigid body

Moments and products of inertia

$$I_{xx} = \int (y^2 + z^2)\,\mathrm{d}m\,; \quad I_{yy} = \int (x^2 + z^2)\,\mathrm{d}m\,; \quad I_{zz} = \int (x^2 + y^2)\,\mathrm{d}m$$

$$I_{xy} = \int xy\,\mathrm{d}m\,; \quad I_{xz} = \int xz\,\mathrm{d}m\,; \quad I_{yz} = \int yz\,\mathrm{d}m$$

Moment of inertia about the centre of mass of a rigid body

The angular momentum then becomes

$$\begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = \underbrace{\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix}}\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Therefore we can write:

$$\mathbf{h} = \mathbf{I}\boldsymbol{\omega}$$

where $\mathbf{I}$ is the (symmetric) inertia matrix (tensor)

Expanding the equations of motion

Starting from the six-equation system capturing Newton's laws:

$$\mathbf{F}_B = m(\dot{\mathbf{v}}_B + \boldsymbol{\omega}_B \times \mathbf{v}_B)$$

$$\mathbf{M}_B = \dot{\mathbf{h}}_B + \boldsymbol{\omega}_B \times \mathbf{h}_B$$

Where $\mathbf{h}_B = \mathbf{I}_B \boldsymbol{\omega}_B$ and $\mathbf{I}_B$ is the inertia matrix

we use the following:

$$\mathbf{F}_B = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}; \dot{\mathbf{v}}_B = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} U \\ V \\ W \end{pmatrix}$$

$$\mathbf{M}_B = \begin{pmatrix} L \\ M \\ N \end{pmatrix};\ \mathbf{I}_B = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix}; \boldsymbol{\omega}_B = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Expanding the equations of motion

The force equations

$$\mathbf{F}_B = m(\dot{\mathbf{v}}_B + \boldsymbol{\omega}_B \times \mathbf{v}_B)$$

then become

$$X = m\left(\frac{\mathrm{d}U}{\mathrm{d}t} + qW - rV\right)$$

$$Y = m\left(\frac{\mathrm{d}V}{\mathrm{d}t} + rU - pW\right)$$

$$Z = m\left(\frac{\mathrm{d}W}{\mathrm{d}t} + pV - qU\right)$$

Expanding the equations of motion

The Moment equations

$$\mathbf{M}_B = \dot{\mathbf{h}}_B + \boldsymbol{\omega}_B \times \mathbf{h}_B$$

Where $\mathbf{h}_B = \mathbf{I}_B \boldsymbol{\omega}_B$ and $\mathbf{I}_B$ is the inertia matrix

then become

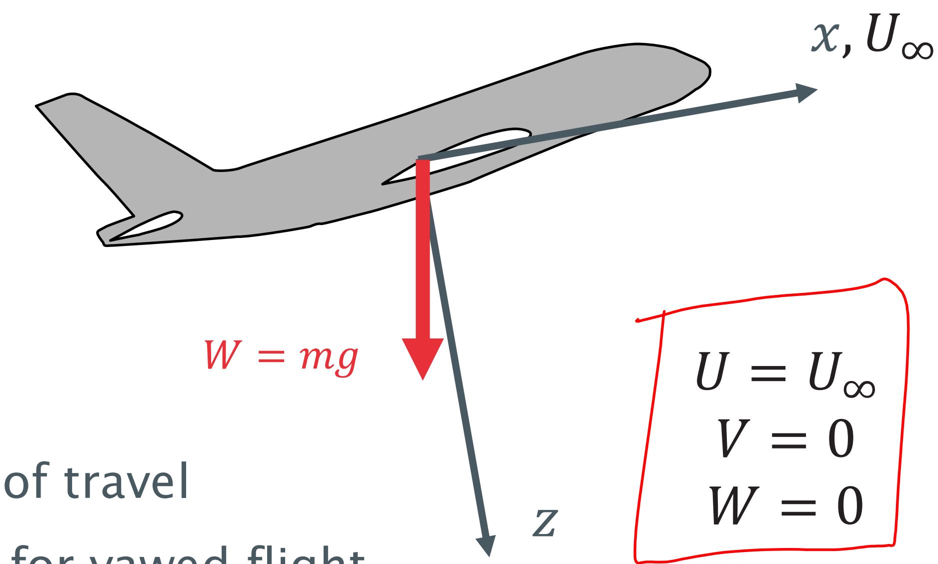
$$L = I_{xx}\frac{\mathrm{d}p}{\mathrm{d}t} - I_{xy}\frac{\mathrm{d}q}{\mathrm{d}t} - I_{xz}\frac{\mathrm{d}r}{\mathrm{d}t} + q\left(-I_{xz}p - I_{yz}q + I_{zz}r\right) - r\left(-I_{xy}p + I_{yy}q - I_{yz}r\right)$$

$$M = -I_{xy}\frac{\mathrm{d}p}{\mathrm{d}t} + I_{yy}\frac{\mathrm{d}q}{\mathrm{d}t} - I_{yz}\frac{\mathrm{d}r}{\mathrm{d}t} + r\left(I_{xx}p - I_{xy}q - I_{xz}r\right) - p\left(-I_{xz}p - I_{yz}q + I_{zz}r\right)$$

$$N = -I_{xz}\frac{\mathrm{d}p}{\mathrm{d}t} - I_{yz}\frac{\mathrm{d}q}{\mathrm{d}t} + I_{zz}\frac{\mathrm{d}r}{\mathrm{d}t} + p\left(-I_{xy}p + I_{yy}q - I_{yz}r\right) - q\left(I_{xx}p - I_{xy}q - I_{xz}r\right)$$

Stability axes

A convenient co-ordinate system is one where the axes are aligned with the U_∞ direction, known as stability axes



Still a body-fixed co-ordinate system that moves with the aircraft

During the perturbed motion the axes will not in general be aligned with the direction of travel

Same as wind axes here, but would be different for yawed flight (wind axes are aligned with the wind tunnel flow)

$$\boxed{\begin{aligned} U &= U_\infty \\ V &= 0 \\ W &= 0 \end{aligned}}$$

in equilibrium

Simplifications due to symmetry

For symmetric aircraft we can simplify the inertia matrix:

$$I_{xy} = \int xy \, \mathrm{d}m = 0$$

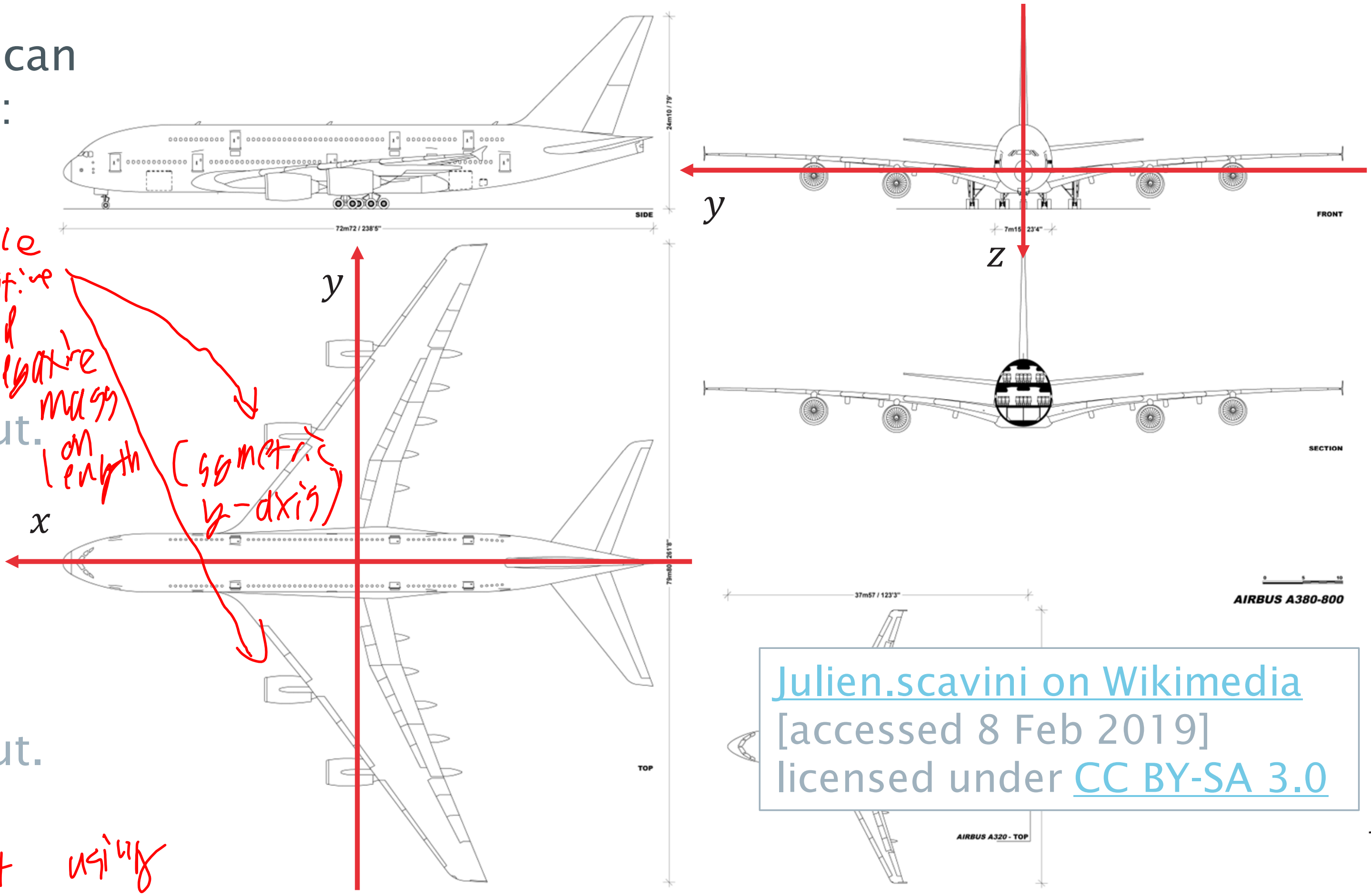
since positive and negative mass on length (symmetric y-axis)

For each section along the x−axis contributions along the y−axis cancel out.

$$I_{yz} = \int yz \, \mathrm{d}m = 0$$

For each section along the z−axis contributions along the y−axis cancel out.

$I_{zz} \neq 0$, not using y axis



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Linearisation

new assumption, hence model
bad for large perturbations.

We suppose that the perturbations in velocity and rotation rates are small, such that products of small quantities can be neglected

$$U = U_\infty + u$$

$$V = 0 + v$$

$$W = 0 + w$$

where $u, v, w \ll U_\infty$

Out of balance forces and moments are also small and denoted by:

$\Delta X \qquad \Delta L$

$\Delta Y \qquad \Delta M$

$\Delta Z \qquad \Delta N$

Linearisation for a symmetric aircraft

Then

$$X = m\left(\frac{\mathrm{d}U}{\mathrm{d}t} + qW - rV\right)$$

becomes

neglected because small ($qw << \dot{u}$)

$$\Delta X = m(\dot{u} + \underbrace{qw - rv}_{\text{neglected}}) =_1 m\dot{u}$$

hence

$=_1$ means we take the first-order approximation, ie neglect higher order terms (also called linearisation)

For a symmetric aircraft: $I_{xy} = I_{yz} = 0$

and

$$L = I_{xx}\frac{\mathrm{d}p}{\mathrm{d}t} - I_{xy}\frac{\mathrm{d}q}{\mathrm{d}t} - I_{xz}\frac{\mathrm{d}r}{\mathrm{d}t} + q\left(-I_{xz}p - I_{yz}q + I_{zz}r\right) - r\left(-I_{xy}p + I_{yy}q - I_{yz}r\right)$$

small small

becomes

$$\Delta L =_1 I_{xx}\dot{p} - I_{xz}\dot{r}$$

Linearisation for a symmetric aircraft

So we get

$$\Delta X =_1 m\dot{u}$$

$$\Delta Y =_1 m(\dot{v} + rU_\infty)$$

$$\Delta Z =_1 m(\dot{w} - qU_\infty)$$

and

$$\Delta L =_1 I_{xx}\dot{p} - I_{xz}\dot{r}$$

$$\Delta M =_1 I_{yy}\dot{q}$$

$$\Delta N =_1 -I_{xz}\dot{p} + I_{zz}\dot{r}$$

somewhat practical to solve but has lots of assumptions.

$=_1$ means we take the first-order approximation, ie neglect higher order terms (also called linearisation).

From now we will use the linearised equations and omit the subscript for convenience.

Decoupling of the linearised equations

Now we split/regroup the equations by **longitudinal and lateral equations**

$\Delta X = m\dot{u}$ Longitudinal equation

$\Delta Y = m(\dot{v} + rU_\infty)$ Lateral equation

$\Delta Z = m(\dot{w} - qU_\infty)$ Longitudinal equation

and:

$\Delta L = I_{xx}\dot{p} - I_{xz}\dot{r}$ Lateral equation

$\Delta M = I_{yy}\dot{q}$ Longitudinal equation

$\Delta N = -I_{xz}\dot{p} + I_{zz}\dot{r}$ Lateral equation

Decoupling of the linearised equations

Longitudinal equations:

$$\Delta X = m\dot{u}$$

$$\Delta Z = m(\dot{w} - qU_\infty)$$

$$\Delta M = I_{yy}\dot{q}$$

and lateral equations:

$$\Delta Y = m(\dot{v} + rU_\infty)$$

$$\Delta L = I_{xx}\dot{p} - I_{xz}\dot{r}$$

$$\Delta N = -I_{xz}\dot{p} + I_{zz}\dot{r}$$

Longitudinal equations
three equations
three unknowns: u, w, q

Lateral equations
three equations
three unknowns: v, p, r

Next steps

Work on the out-of-balance forces and moments

Split the out-of-balance forces into
gravitational
and **aerodynamic** contributions

Introduce **aerodynamic derivatives** (related to aircraft design choices) to represent the aerodynamic forces

> We will write these in terms of u, w, q (for longitudinal equations)
> and in terms of v, p, r (for lateral equations)
> so that we have a closed set of equations

Combine the equations and write them in
matrix (state-space) form.