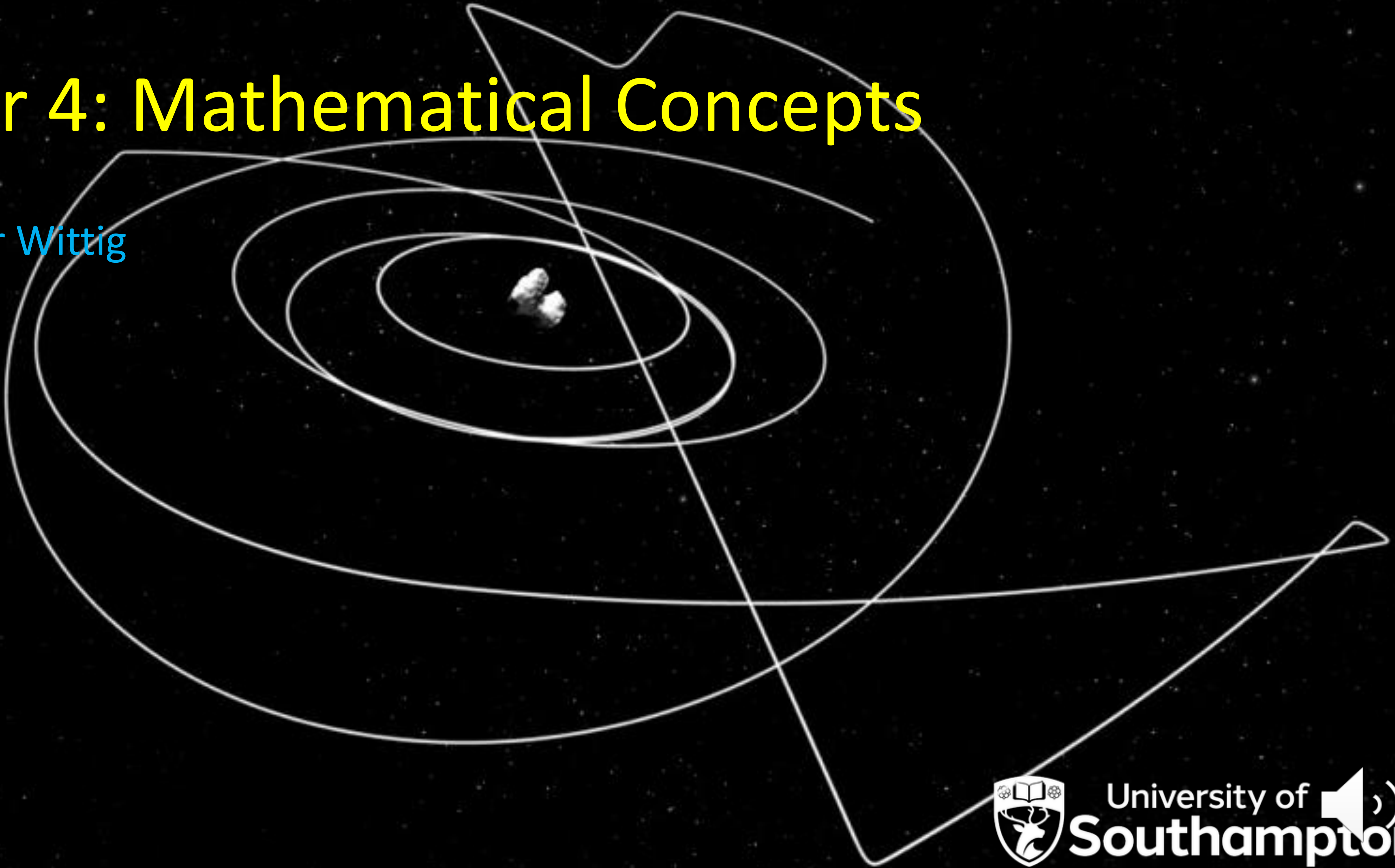
Chapter 4: Mathematical Concepts

Dr. Alexander Wittig



Rosetta lander deployment orbit (ESA)

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Vector Algebra

- Basic properties
  - Distributivity of multiplication by a scalar and addition: $$c\left(\vec{a}+\vec{b}\right)=c\vec{a}+c\vec{b}$$
  - Commutativity of addition: $$\vec{a}+\vec{b}=\vec{b}+\vec{a}$$
  - Associativity of addition : $$\vec{a}+\left(\vec{b}+\vec{c}\right)=\left(\vec{a}+\vec{b}\right)+\vec{c}$$
  - Commutativity of scalar (dot) product : $$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$$
  - **Anti**commutativity of vector cross product : $$\boxed{\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}}$$

Just basic recap

add this to notes

Reminder: Cross product

- Orthogonal to both vectors (direction right handed!)
- Magnitude: $\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \left|\vec{b}\right| \sin\left(\measuredangle \vec{a}\vec{b}\right)$
- Calculation: $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

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- As determinant:

$$\det \begin{pmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} & \boldsymbol{x_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

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- As extended vectors:

$$\begin{pmatrix} \cancel{a_1} \\ a_2 \\ a_3 \\ a_1 \\ a_2 \\ \cancel{a_3} \end{pmatrix} \times \begin{pmatrix} \cancel{b_1} \\ b_2 \\ b_3 \\ b_1 \\ b_2 \\ \cancel{b_3} \end{pmatrix}$$

Reminder: Cross product

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- As skew matrix multiplication:

$$\boldsymbol{A}^{\times} \cdot \vec{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \cdot \vec{b}$$

Note

Any skew symmetric 3x3 matrix represents a cross product with some corresponding vector!

Vector Algebra

- Basic properties
  - Distributivity of addition wrt scalar product : $\left(\vec{a}+\vec{b}\right)\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$
  - Distributivity of addition wrt vector cross product : $\left(\vec{a}+\vec{b}\right)\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$
  - Scalar triple product:

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{b}\cdot\left(\vec{c}\times\vec{a}\right)=\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)$$

Cyclic permutation

$$=\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

memorise these (just flashcard em)

note

- Advanced vector identities

  - Vector triple product:

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \cdot \vec{c}\right) \vec{b} - \left(\vec{a} \cdot \vec{b}\right) \vec{c}$$

  - Binet-Cauchy identity:

$$\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = \left(\vec{a} \cdot \vec{c}\right) \left(\vec{b} \cdot \vec{d}\right) - \left(\vec{b} \cdot \vec{c}\right) \left(\vec{a} \cdot \vec{d}\right)$$

  - Lagrange's identity:

$$\left|\vec{a} \times \vec{b}\right|^2 = \left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{a} \times \vec{b}\right) = \left(\vec{a} \cdot \vec{a}\right) \left(\vec{b} \cdot \vec{b}\right) - \left(\vec{a} \cdot \vec{b}\right)^2$$

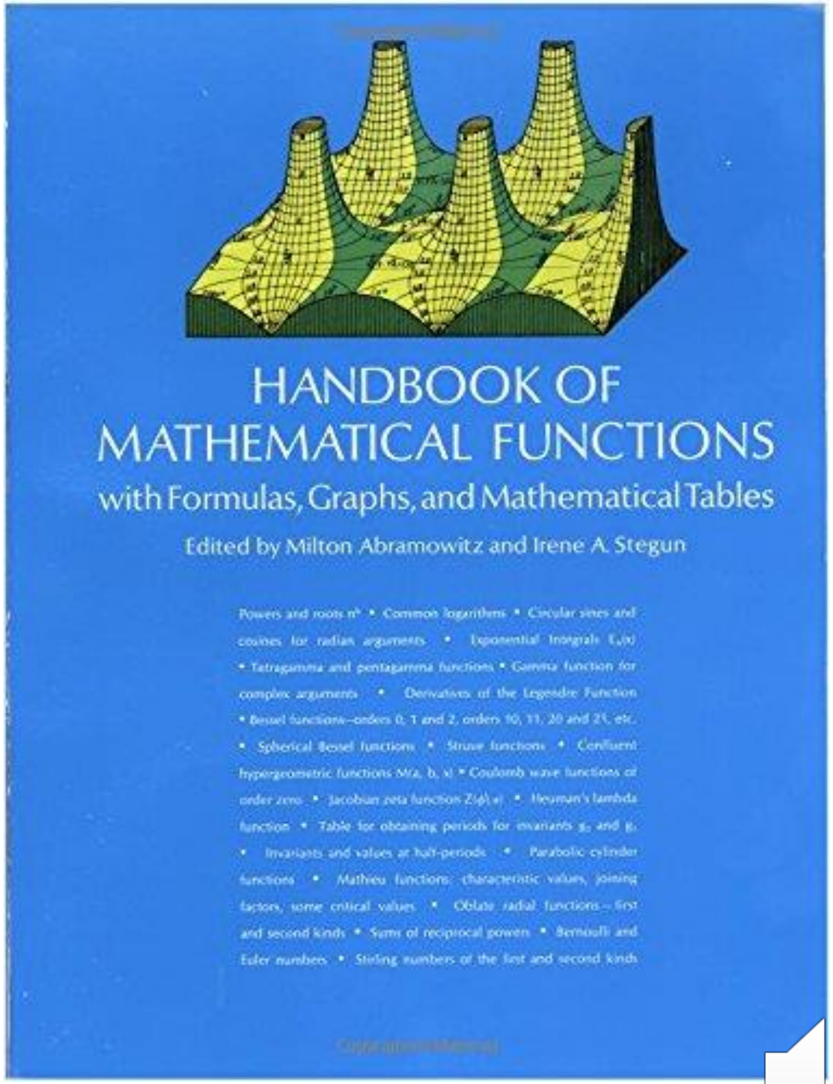
$$= \left|\vec{a}\right|^2 \left|\vec{b}\right|^2 - \left(\vec{a} \cdot \vec{b}\right)^2$$

“Do I have to know these for the exam??????”

YES!

- Identities are not difficult
- Advanced identities are a bit rare, maybe don’t always recall exactly
- Look those up in reference books/notes

- You **must know** that they **exist** and roughly **what they look like** otherwise you can’t recognize them



Possibly one of the most cited books in math

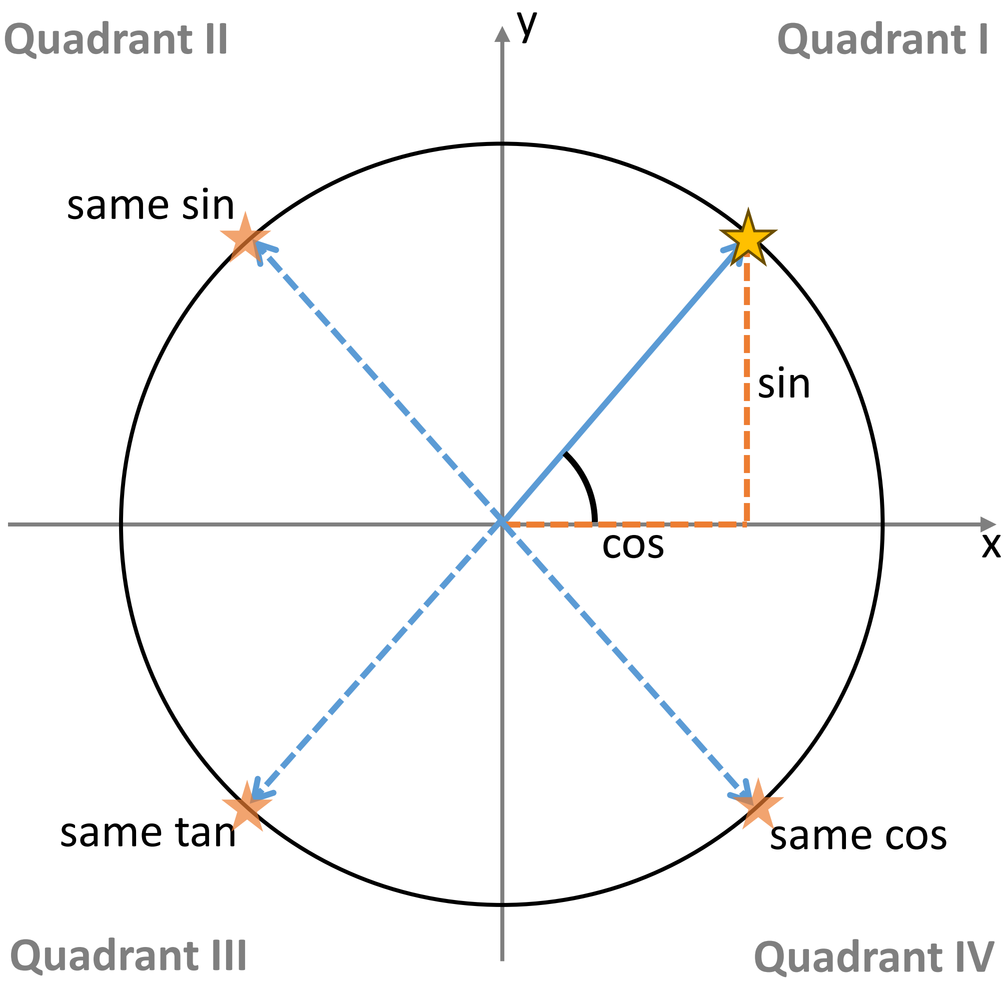
Vector Math on Calculators

Vector math by hand can be tedious and error-prone for longer equations.

- Scientific Computing Environments
  - Python: Numpy, Scipy
  - Matlab, Mathematica, Maple
  - C++: Eigen3
  - C: BLAS/LAPACK, IMSL, GNU Scientific Library
- Scientific calculators
  - Casio fx-991 series
  - Supports vectors/matrices
  - University approved calculator in exams
  - See Video Lecture 13 on Blackboard/Panopto for demonstration

Quadrant Disambiguation

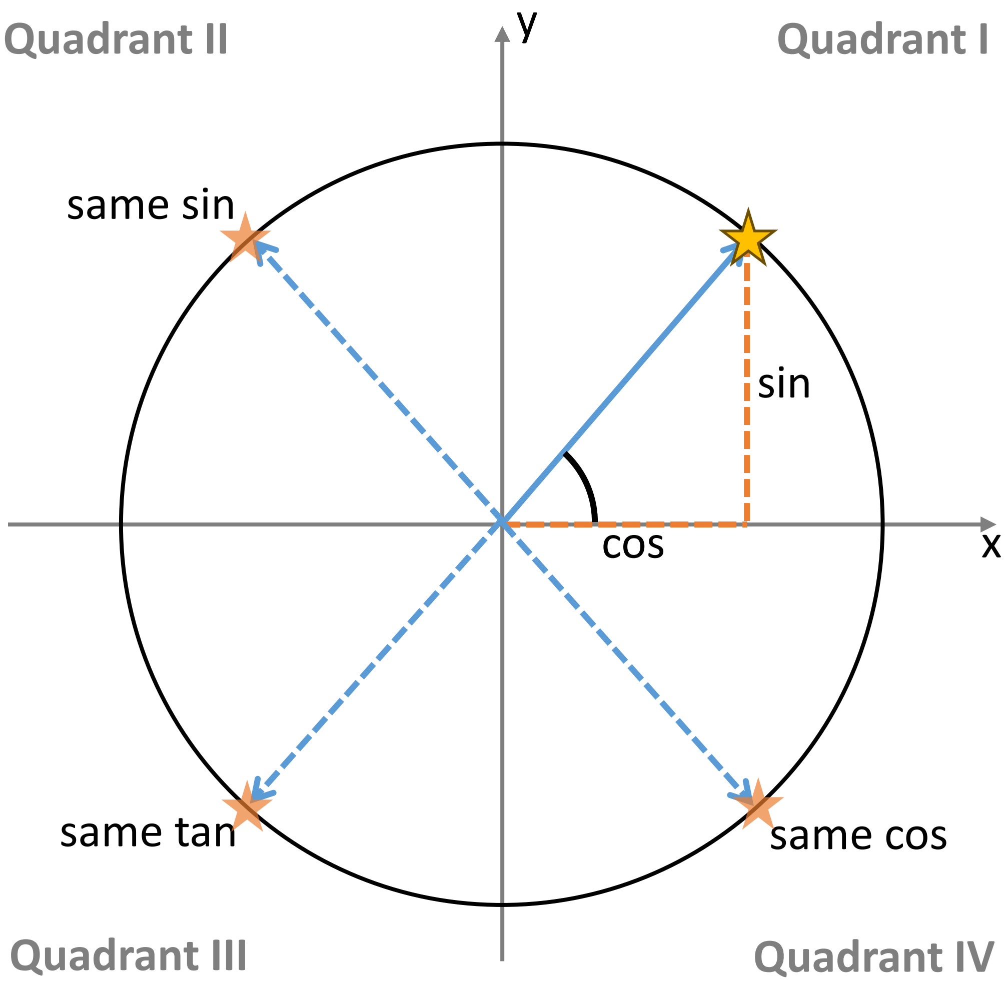
flash card and notes!



- Sine, Cosine, Tangent not uniquely invertible on whole unit circle
- Additional information needed to pick correct branch
  - Any two trig functions
  - Physical/engineering constraints
  - Prior knowledge

memorise this

Quadrant Disambiguation



Given ⭐ = (x, y) find angle from x-axis

- Scientific programming: **atan2(y, x)**
  - Result in rad, range (-π, π]
  - Excel is stupid (order reversed)
- Casio scientific calculators: **Pol(x, y)**
  1. Pol(**SHIFT +**
  2. Enter x value
  3. , **SHIFT)**
  4. Enter y value
  5. =
  6. Read off theta (0° – 360°, in deg or rad per settings, scroll as needed)

Mechanics

Newton’s equations of motion

- Equations of motion for point mass in **inertial** reference frame & **Cartesian**

  - Position $\vec{x}(t)$
  - Velocity $\vec{v}(t) = \frac{d}{dt}\vec{x}(t) = \dot{\vec{x}}(t)$
  - Acceleration $\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \dot{\vec{v}}(t) = \frac{d^2}{dt^2}\vec{x}(t) = \ddot{\vec{x}}(t)$

  - Newton’s second law**:

$$\vec{F}(t) = \frac{d}{dt}\vec{p}(t) = \frac{d}{dt}m(t)\vec{v}(t) =^{*} m\vec{a}(t)$$

Note

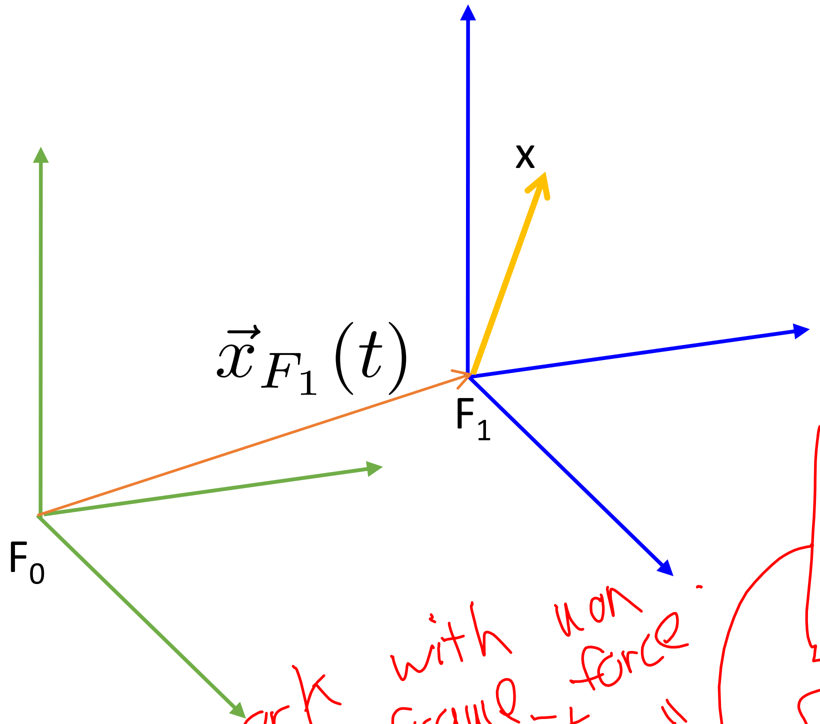
What’s an inertial reference frame?

A reference frame where these equations of motion hold!

* only for constant mass!
** Not on cosmic scale (general relativity) or nano scale (quantum mechanics). Terms and conditions apply.

Newton's equations of motion

- Equations of motion for point mass in **accelerating** reference frame



- Motion expressed in F_0 frame:

$$\vec{x}(t)_0 = \vec{x}_{F_1}(t) + \vec{x}(t)_1$$

moving frame

$$\vec{v}(t)_0 = \dot{\vec{x}}_{F_1}(t) + \dot{\vec{x}}(t)_1$$

$$\vec{a}(t)_0 = \ddot{\vec{x}}_{F_1}(t) + \ddot{\vec{x}}(t)_1$$

- F_0 is an inertial frame, so:

$$m\vec{a}(t)_0 = \vec{F} = m\ddot{\vec{x}}_{F_1}(t) + m\ddot{\vec{x}}(t)_1$$

- In F_1 therefore:

"fictitious force"

$$m\ddot{\vec{x}}(t)_1 = m\vec{a}(t)_1 = \vec{F}(t) \boxed{- m\ddot{\vec{x}}_{F_1}(t)}$$

can work with non inertial frame equations if "fake force" included

Ordinary differential equation:

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), t)$$

Order reduction

- Second order ODE:

$$\frac{d^2}{dt^2}\vec{x}(t) = \vec{f}\left(\vec{x}(t), \frac{d}{dt}\vec{x}(t), t\right)$$

$$\vec{X}(t) = \begin{pmatrix} \vec{x}(t) \\ \frac{d}{dt}\vec{x}(t) \end{pmatrix}$$

$$\frac{d}{dt}\vec{X}(t) = \begin{pmatrix} \frac{d}{dt}\vec{x}(t) \\ \frac{d^2}{dt^2}\vec{x}(t) \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}\vec{x}(t) \\ \vec{f}(\vec{X}(t), t) \end{pmatrix} = \vec{F}(\vec{X}(t), t)$$

first order ODE of twice the dimension.

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$$\vec{X}(t) = \begin{pmatrix} \vec{x}(t) \\ \frac{d}{dt}\vec{x}(t) \end{pmatrix}$$

← state vector in the phase space of the ODE

$$\frac{d}{dt}\vec{X}(t) = \begin{pmatrix} \frac{d}{dt}\vec{x}(t) \\ \frac{d^2}{dt^2}\vec{x}(t) \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}\vec{x}(t) \\ \vec{f}(\vec{X}(t), t) \end{pmatrix} = \vec{F}(\vec{X}(t), t)$$

first order ODE of twice the dimension.

Initial Value Problem

- Initial Value Problem
  - $\vec{x}_0$ is the **initial condition** at t_0
  - has unique solution for smooth RHS
  - Forward and backward in time
  - full state in phase space at one time determines state at all times

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), t)$$

$$\vec{x}(t_0) = \vec{x}_0$$

$$\vec{x}(t) = ???$$

- Can be solved analytically or approximated numerically (ODE integration)
  - Euler step

$$\cancel{\vec{x}(t + \Delta t) \simeq \vec{x}(t) + \vec{f}(\vec{x}(t), t) \cdot \Delta t}$$

  - Runge-Kutta methods

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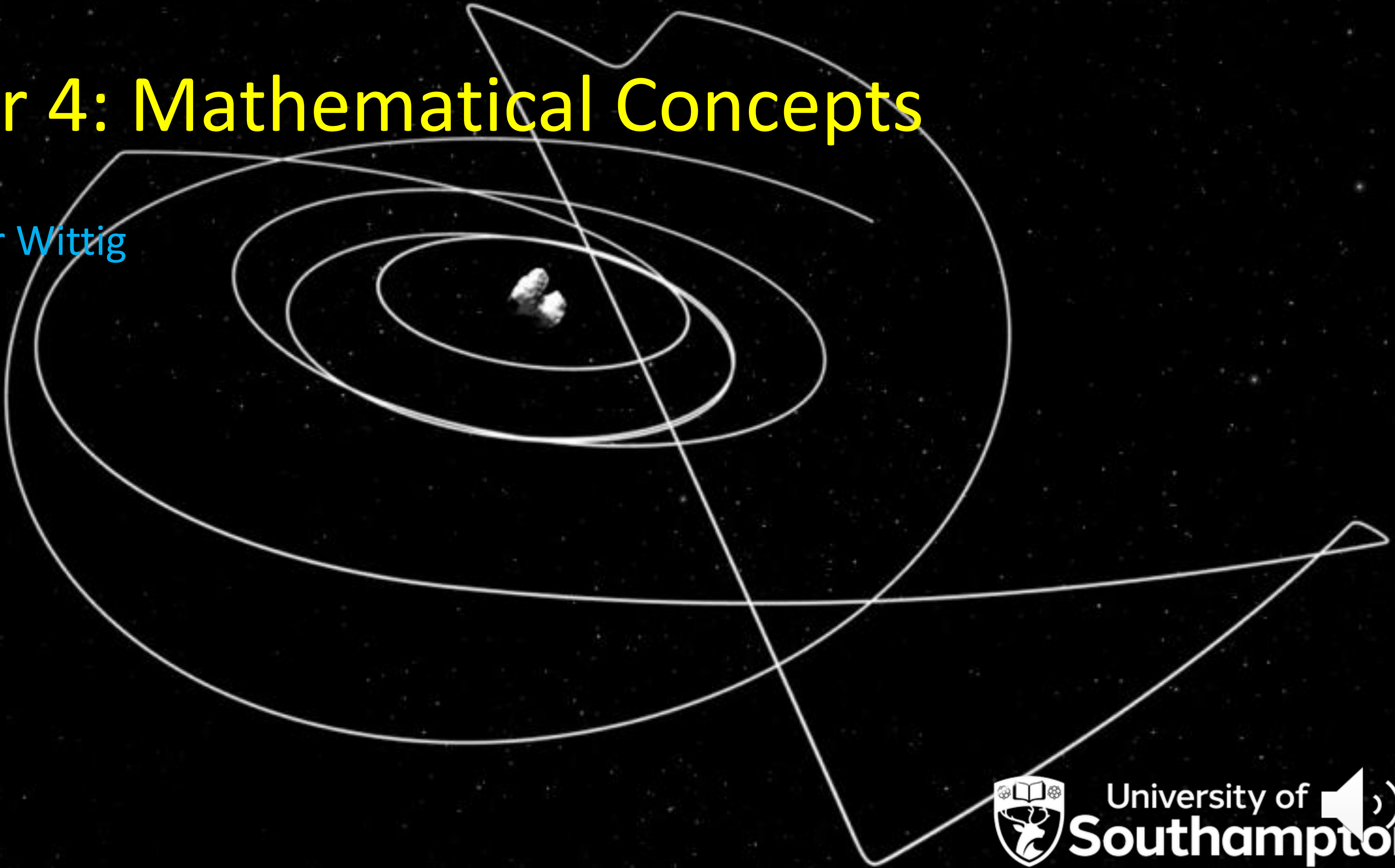

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