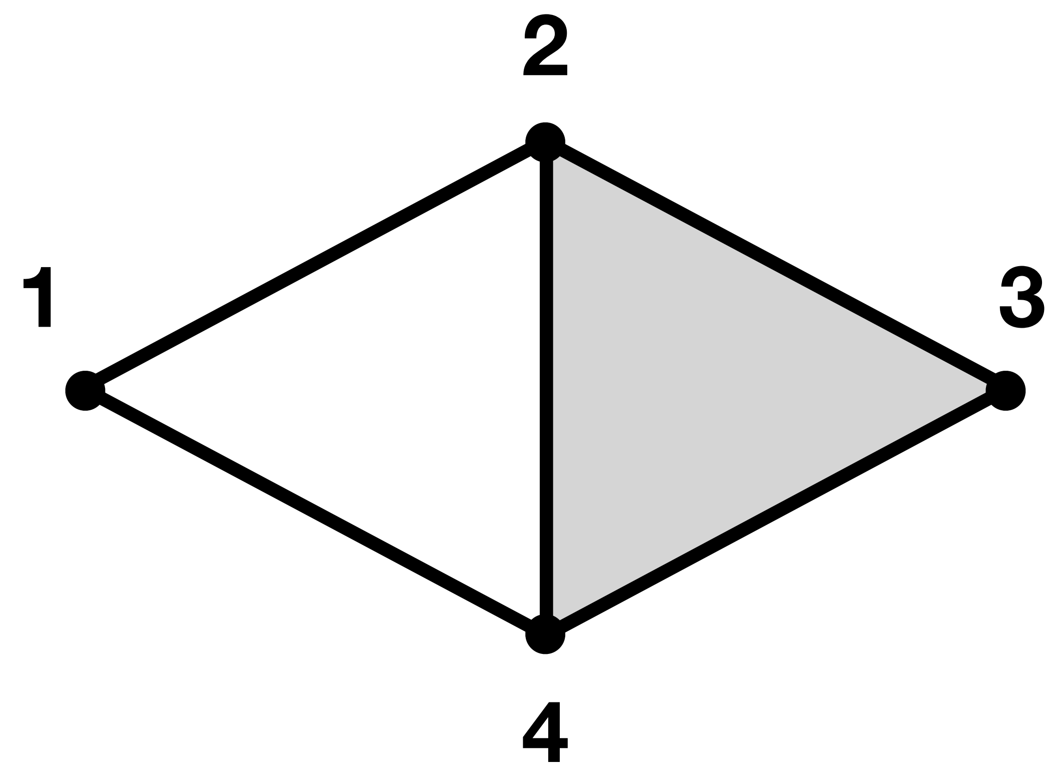
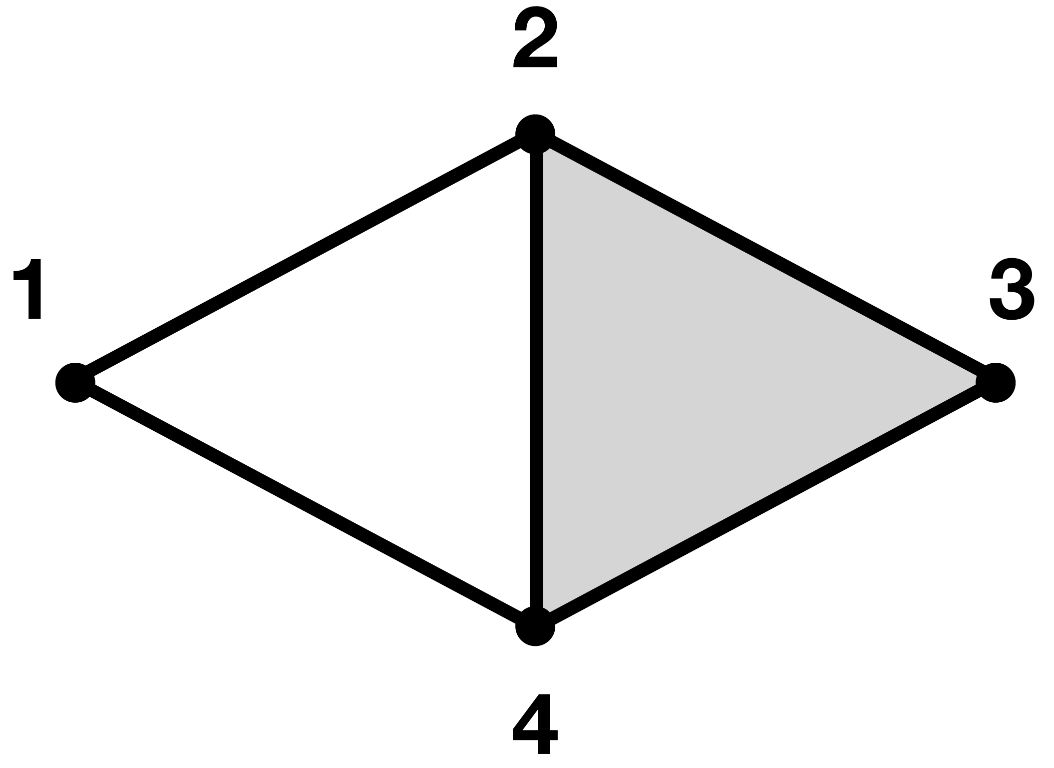
Boundary matrix



$B =$

	1	2	3	4	12	14	23	24	34	234
0		B_0								
1										
2							B_1			
3										
4										
12										
14										
23										B_2
24										
34										

Boundary matrix

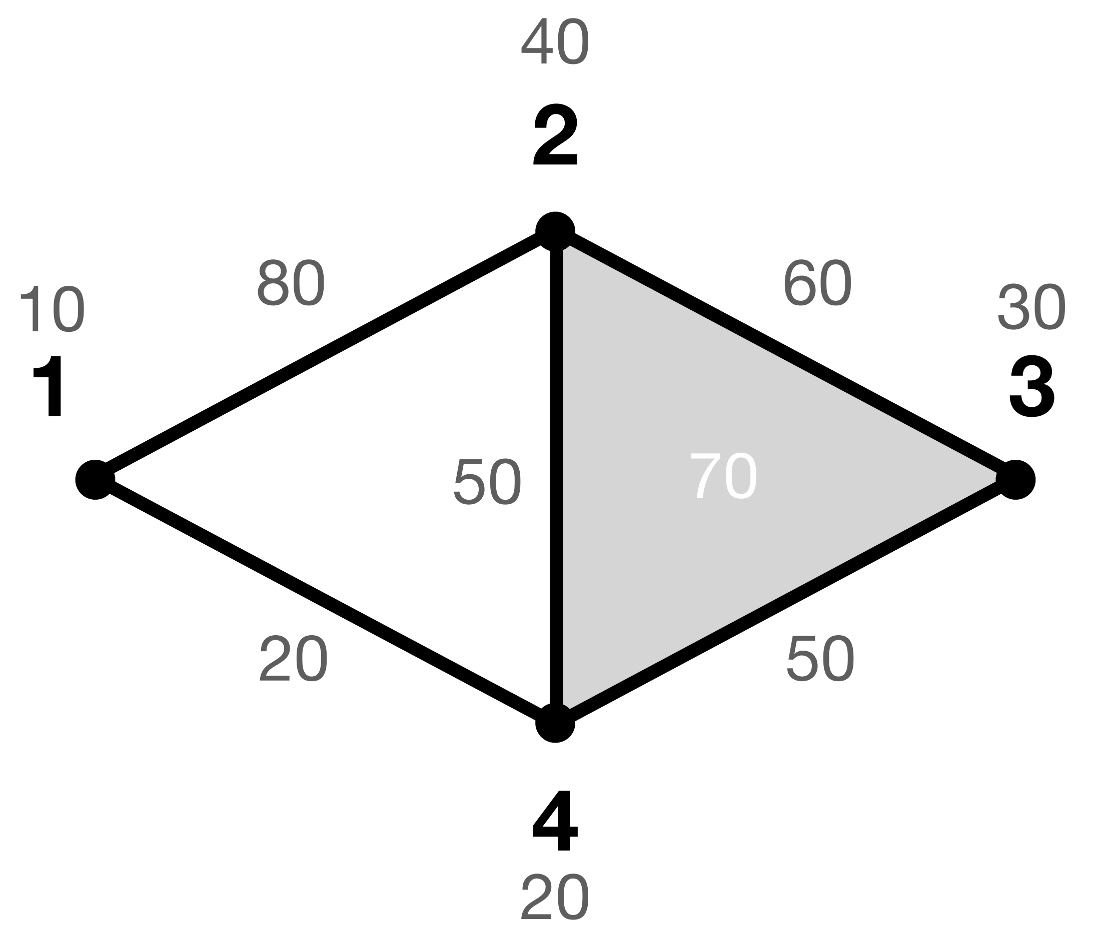


$\mathbf{B} =$

	1	2	3	4	12	14	23	24	34	234
0	1	1	1	1						
1					1	1				
2					1		1	1		
3							1		1	
4						1		1	1	
12										
14										
23										1
24										1
34										1

One may skip adding $\mathbf{B}_0$ to the full matrix or zero out it

Filtration function



A function $f : K \to \mathbb{R}$ is called a filtration function iff, either

$$f(\tau) \leq f(\sigma) \iff \tau \subseteq \sigma$$ (sublevel filtration)

$$f(\tau) \geq f(\sigma) \iff \tau \supseteq \sigma$$ (superlevel filtration)

$$K_t = \{\sigma \in K \mid f(\sigma) \leq t\}$$

sublevel set $t \in (-\infty, +\infty)$

$$K^t = \{\sigma \in K \mid f(\sigma) \geq t\}$$

superlevel set $t \in (+\infty, -\infty)$

A filtration is a sequence of sublevel (superlevel) sets s.t.

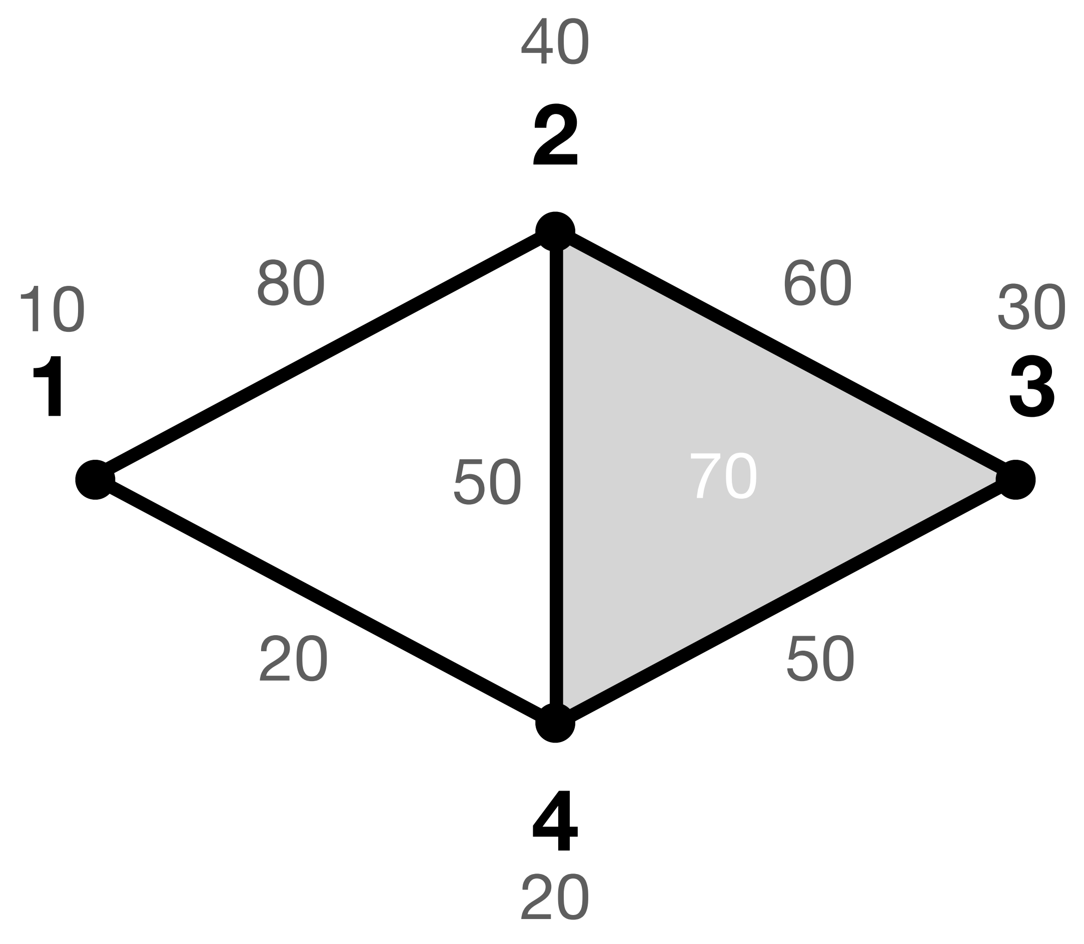
$$\emptyset \subset K_1 \subset K_2 \subset \ldots \subset K$$

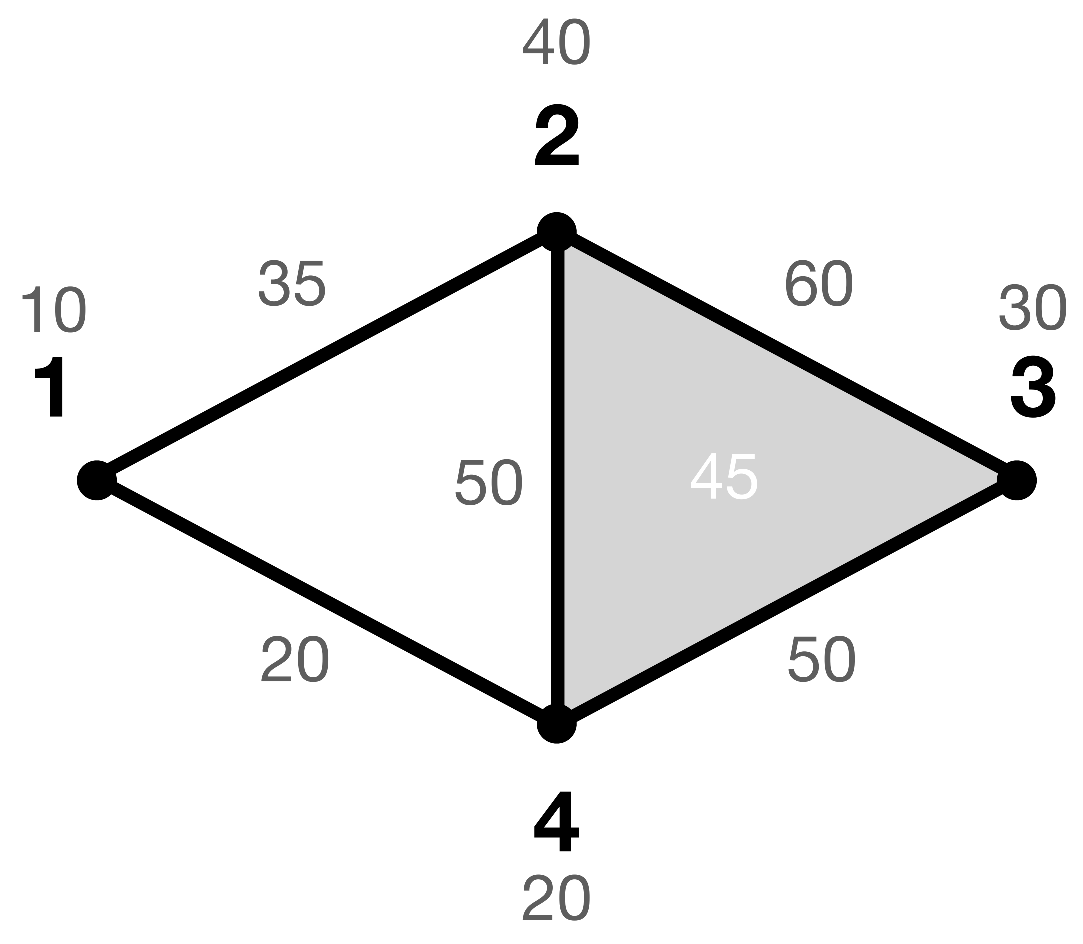
sublevel filtration

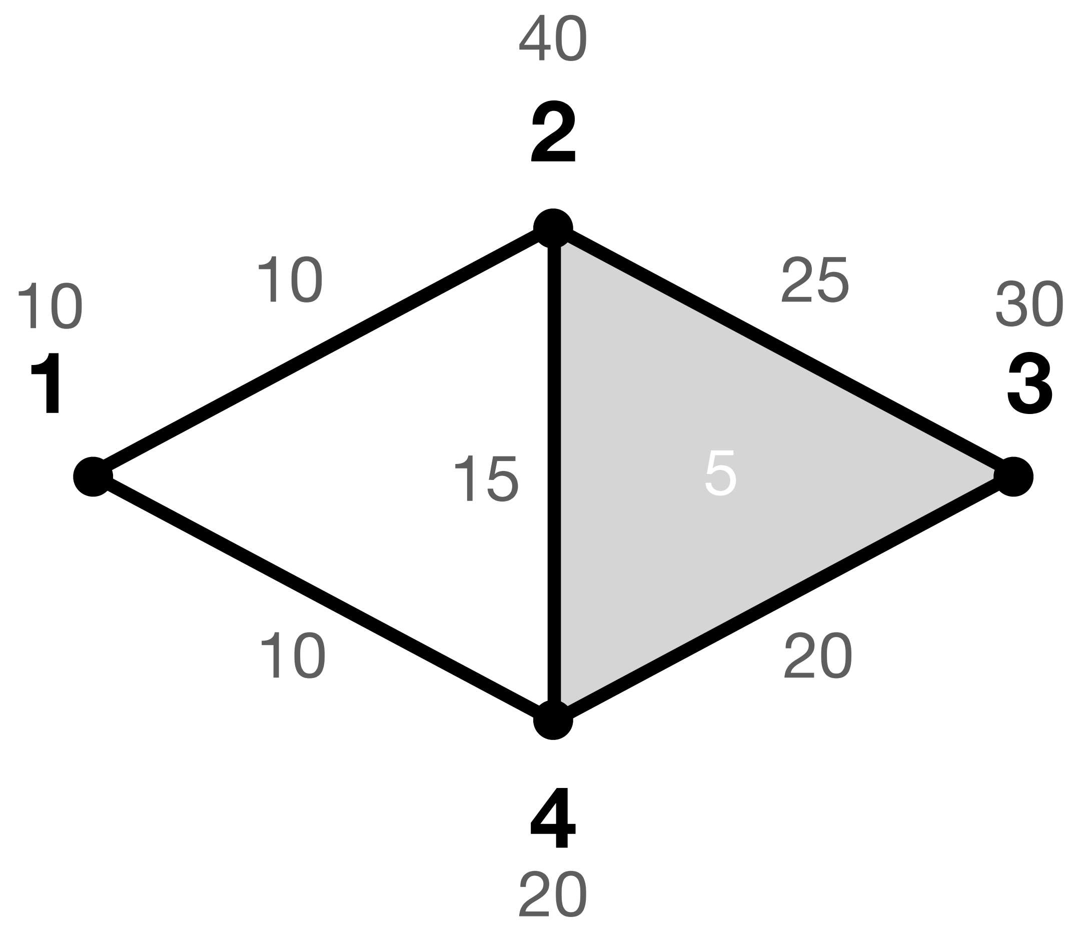
$$\emptyset \subset K^1 \subset K^2 \subset \ldots \subset K$$

superlevel filtration

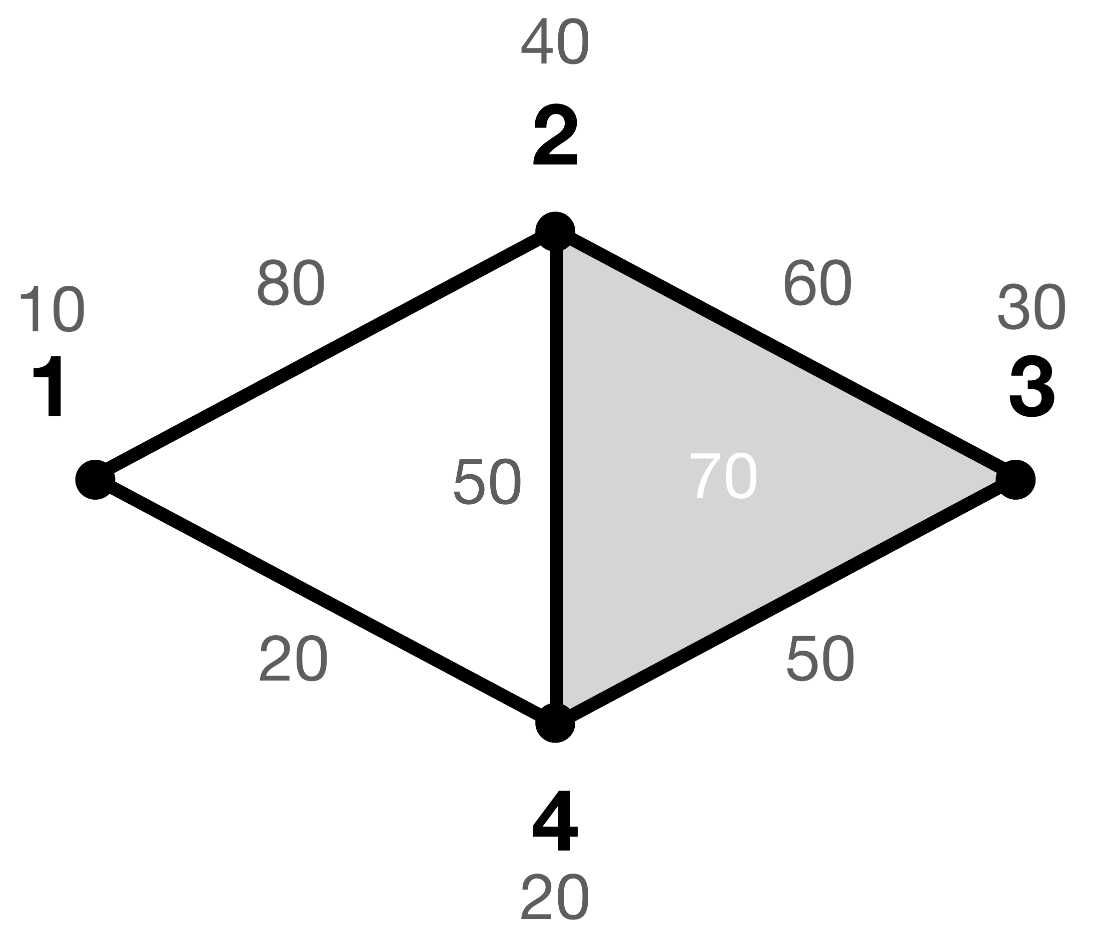
Filtration function







Reordering w.r.t. filtration function



$\mathbf{B} =$

	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
0										
1			1							1
4			1			1	1			
14										
3							1	1		
2						1		1		1
24									1	
34									1	
23									1	
12										

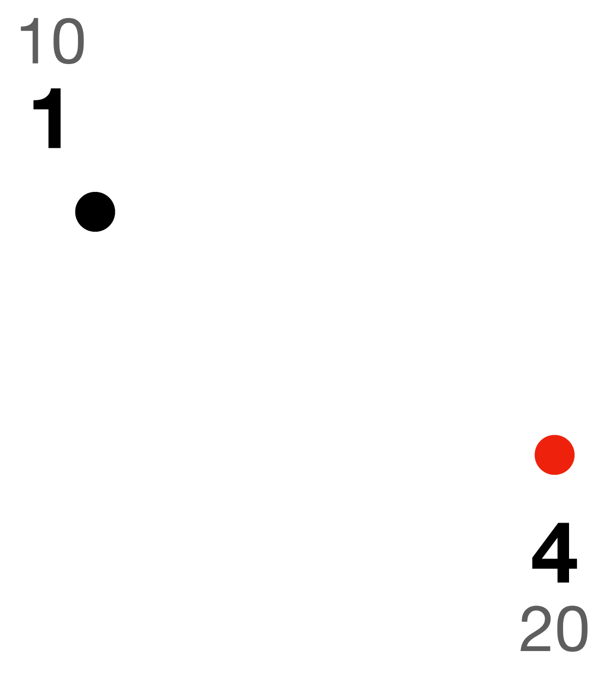
Filtration function provides order on simplices, therefore on columns and rows of the filtration matrix. Ties are broken, first by simplex dimension, second by lexicographic order given by order on vertices.

Boundary matrix reduction

	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1			
14										
3							1	1		
2						1		1		1
24									1	
34									1	
23									1	
12										

t=10

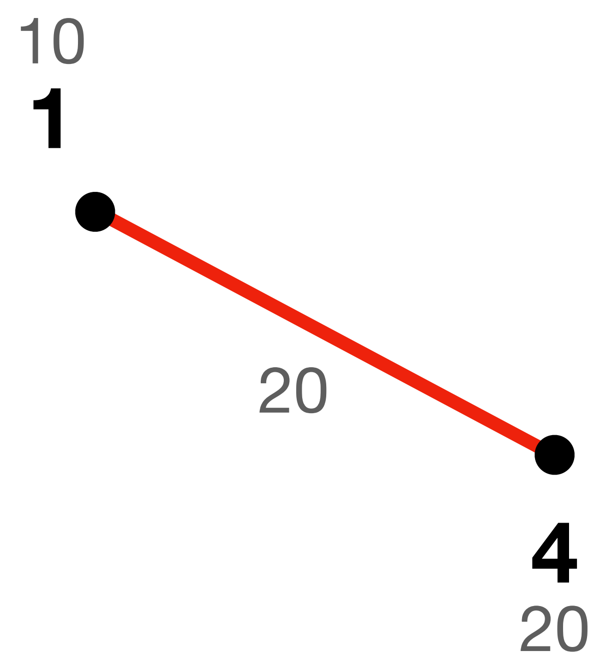
Boundary matrix reduction



	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1			
14										
3							1	1		
2						1		1		1
24									1	
34									1	
23									1	
12										

t=20

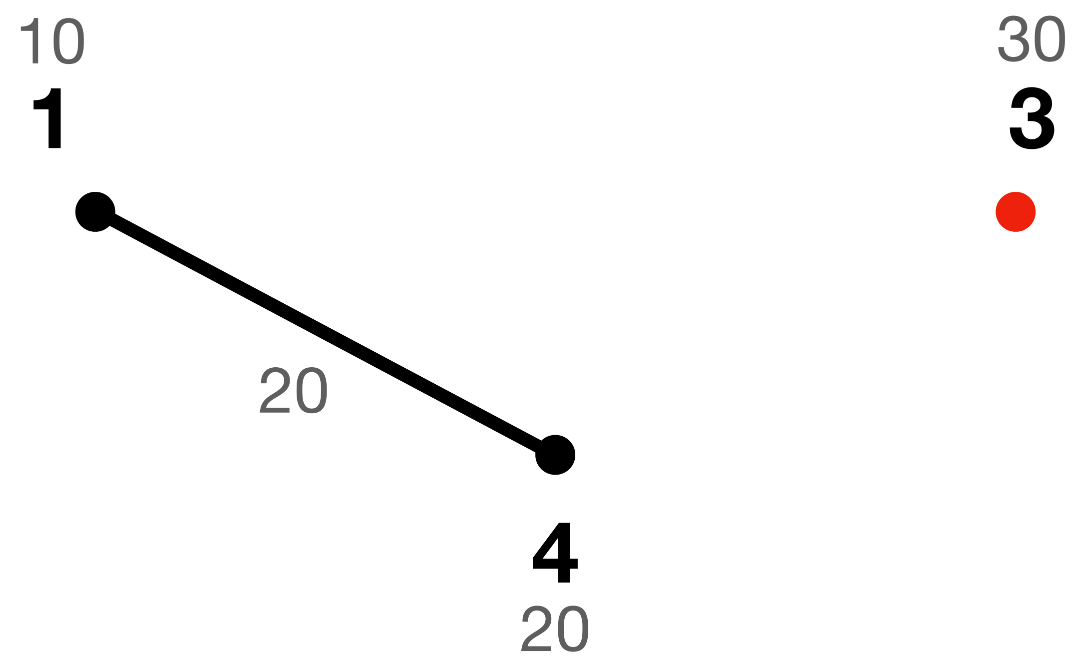
Boundary matrix reduction



	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1			
14										
3							1	1		
2						1		1		1
24									1	
34									1	
23									1	
12										

t=20

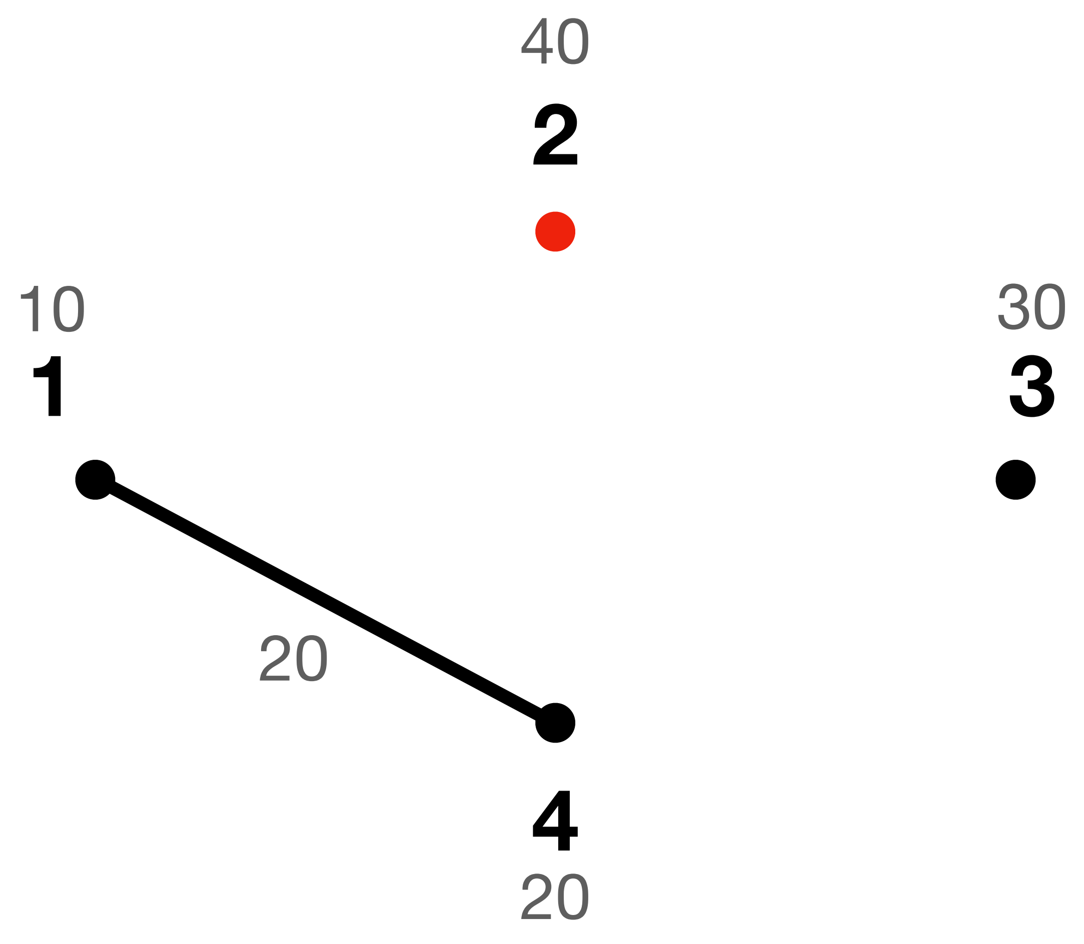
Boundary matrix reduction



	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			**1**			1	1			
14										
3							1	1		
2						1		1		1
24									1	
34									1	
23									1	
12										

t=30

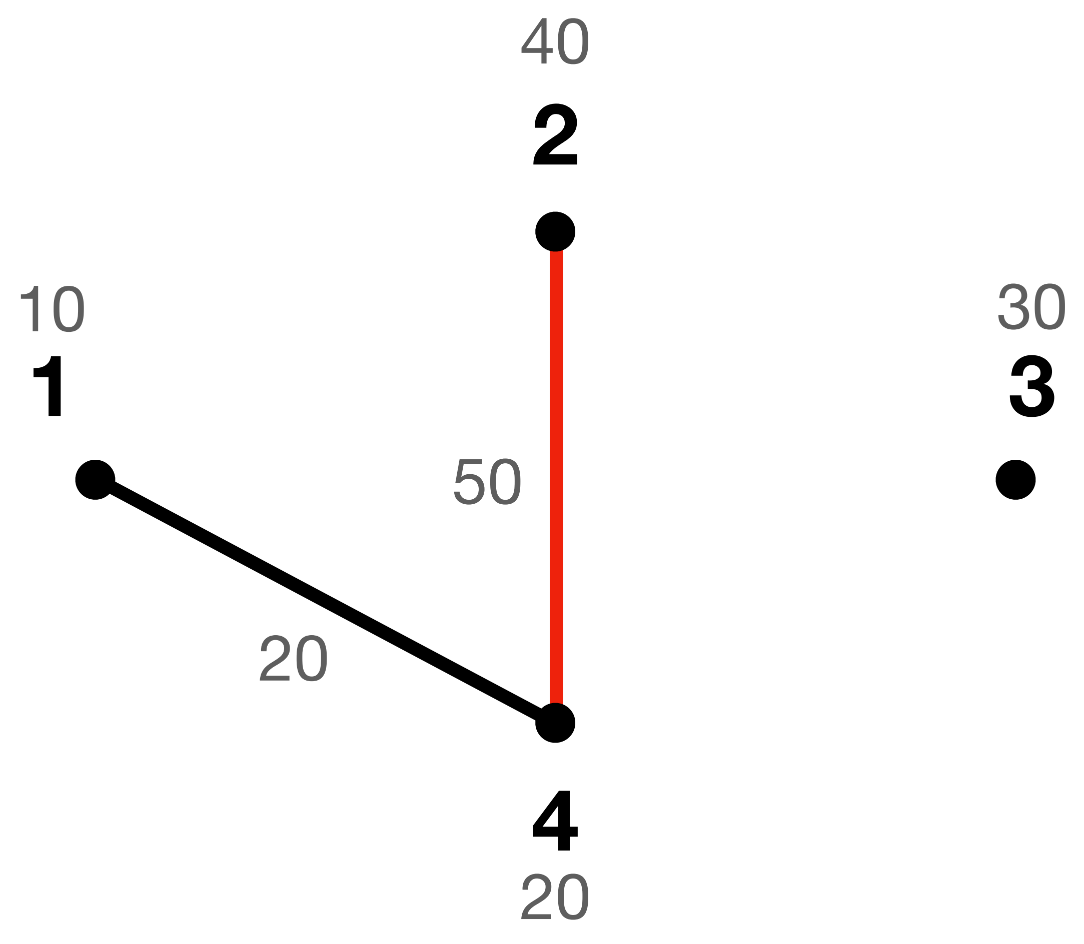
Boundary matrix reduction



	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1			
14										
3							1	1		
2						1		1		1
24									1	
34									1	
23									1	
12										

t=40

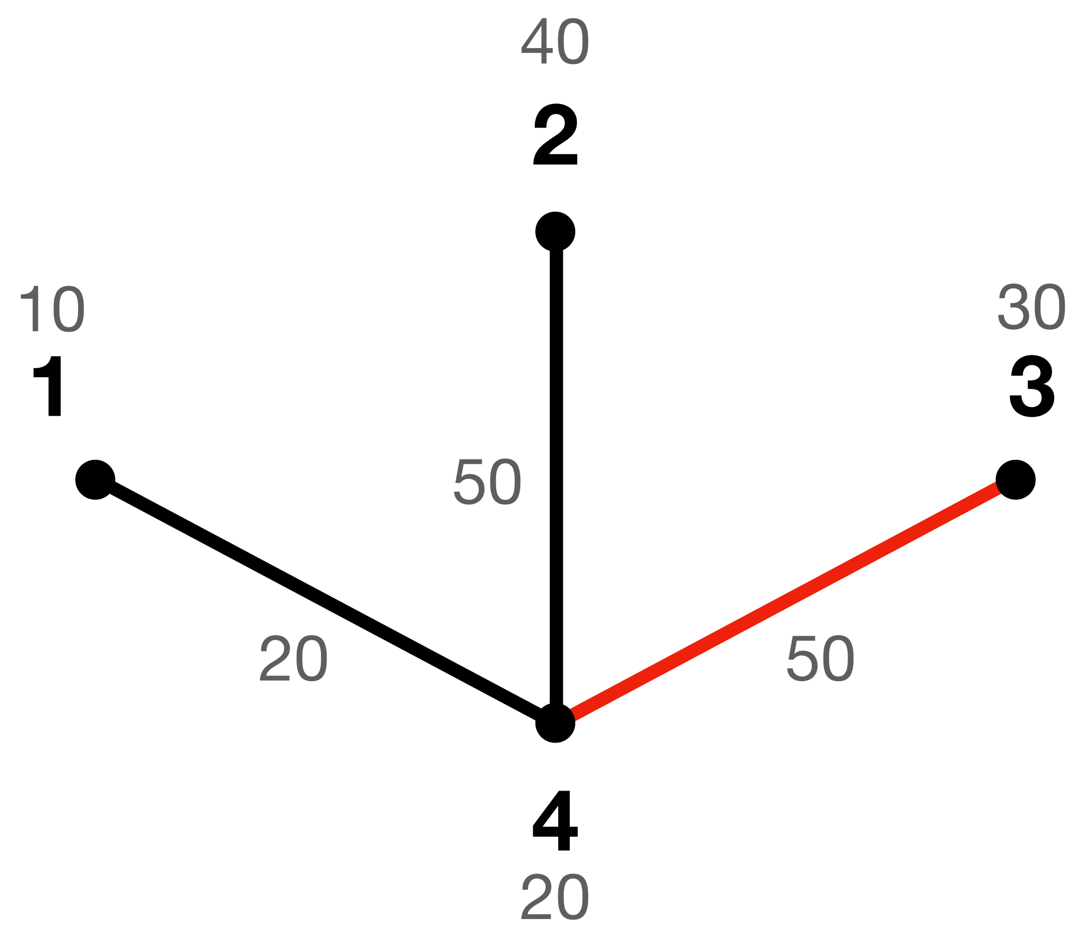
Boundary matrix reduction



	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1			
14										
3							1	1		
2						1		1		1
24									1	
34									1	
23									1	
12										

t=50

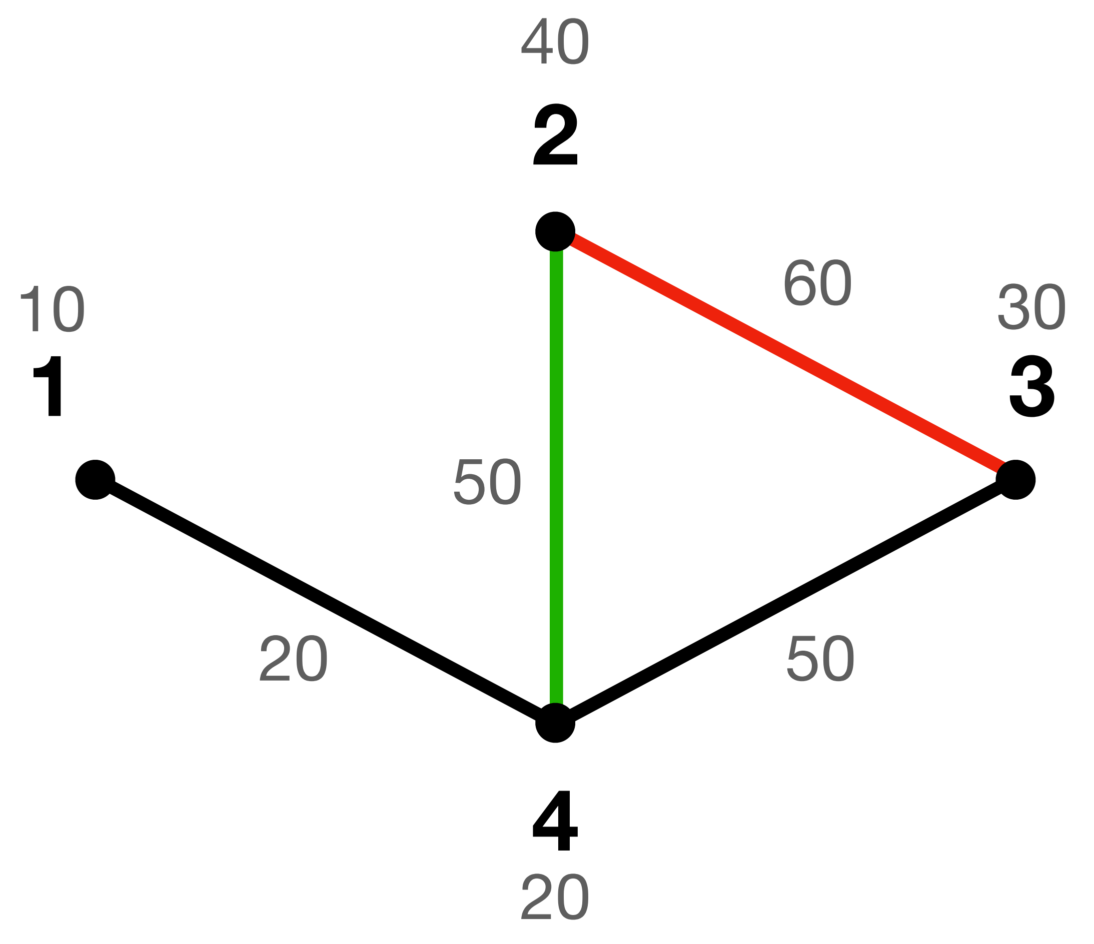
Boundary matrix reduction



	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1			
14										
3							1	1		
2						1		1		1
24									1	
34									1	
23									1	
12										

t=50

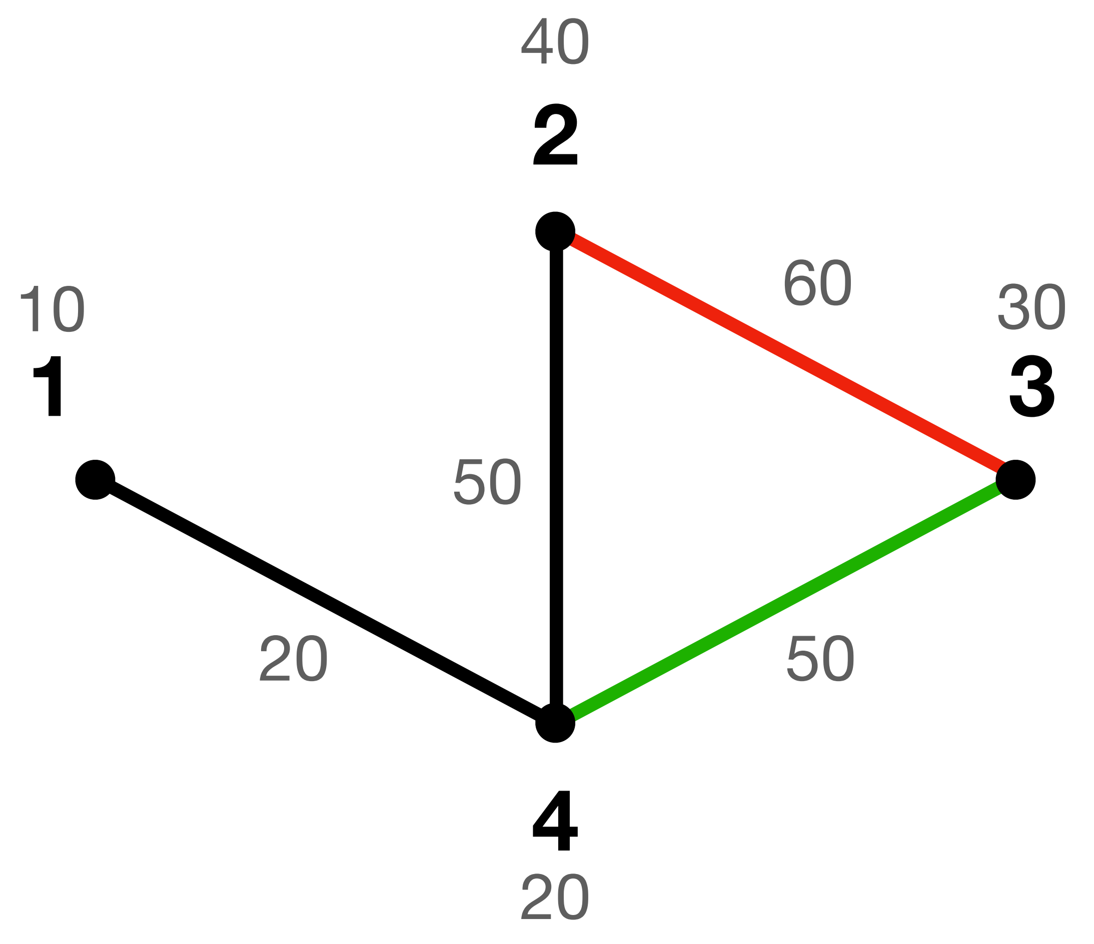
Boundary matrix reduction



	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			**1**			1	1			
14										
3							**1**	1		
2						**1**		1		1
24									1	
34									1	
23									1	
12										

t=60

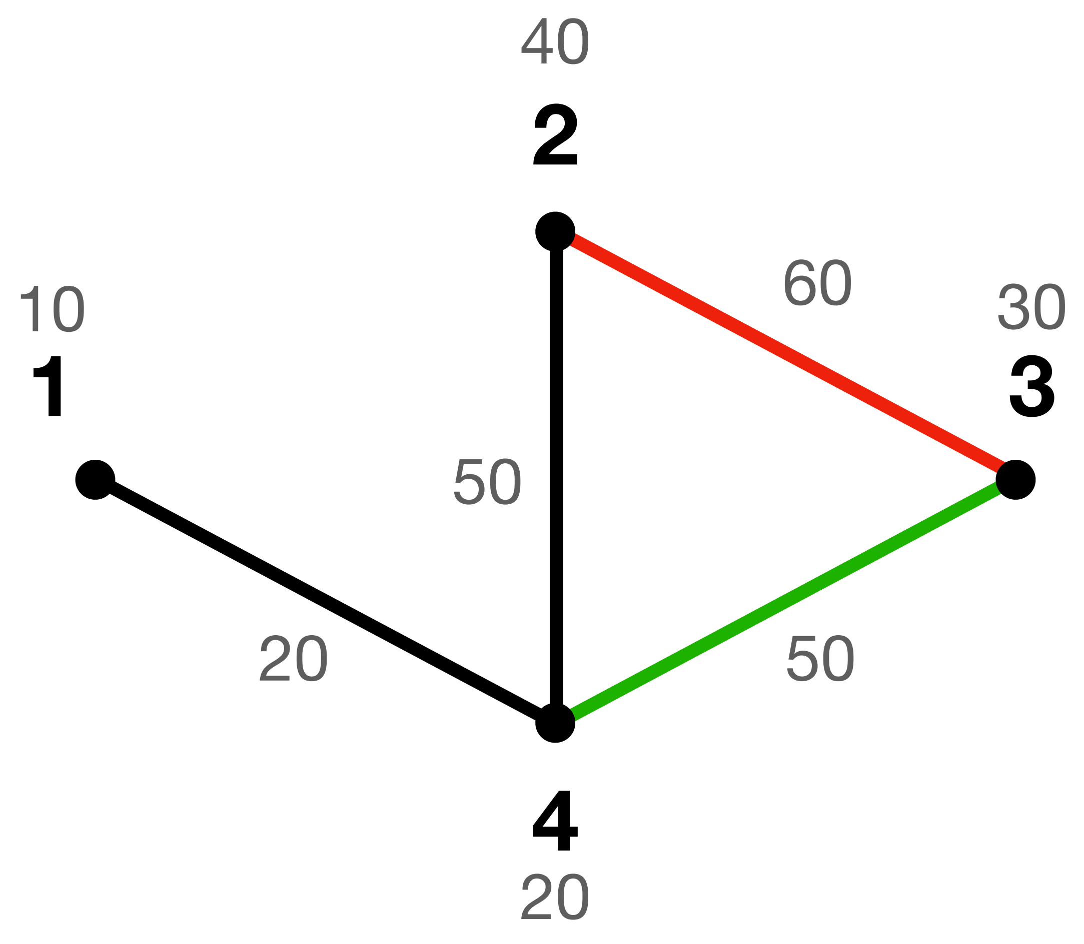
Boundary matrix reduction



								[23+24]		
	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1	1		
14										
3							1	1		
2						1				1
24									1	
34									1	
23									1	
12										

t=60

Boundary matrix reduction

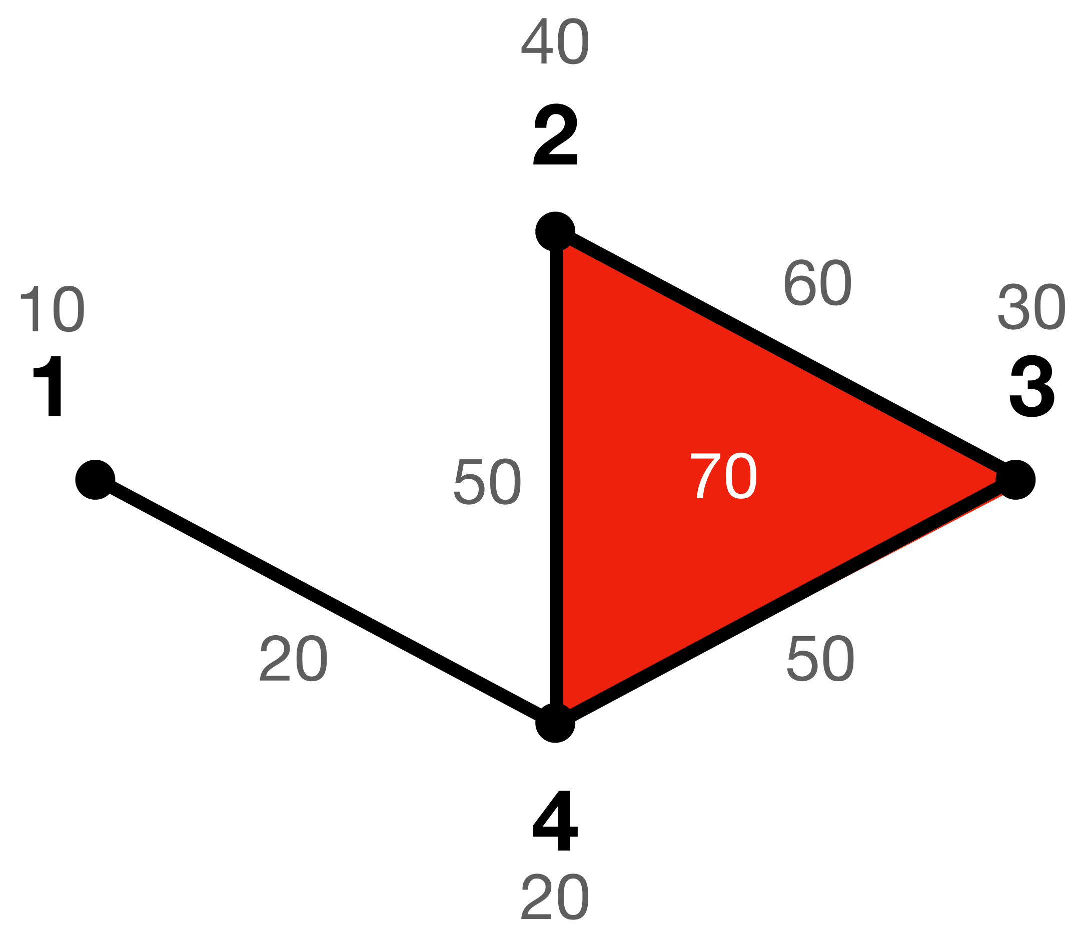


[23+24+34]

	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1			
14										
3							1			
2						1				1
24									1	
34									1	
23									1	
12										

t=60

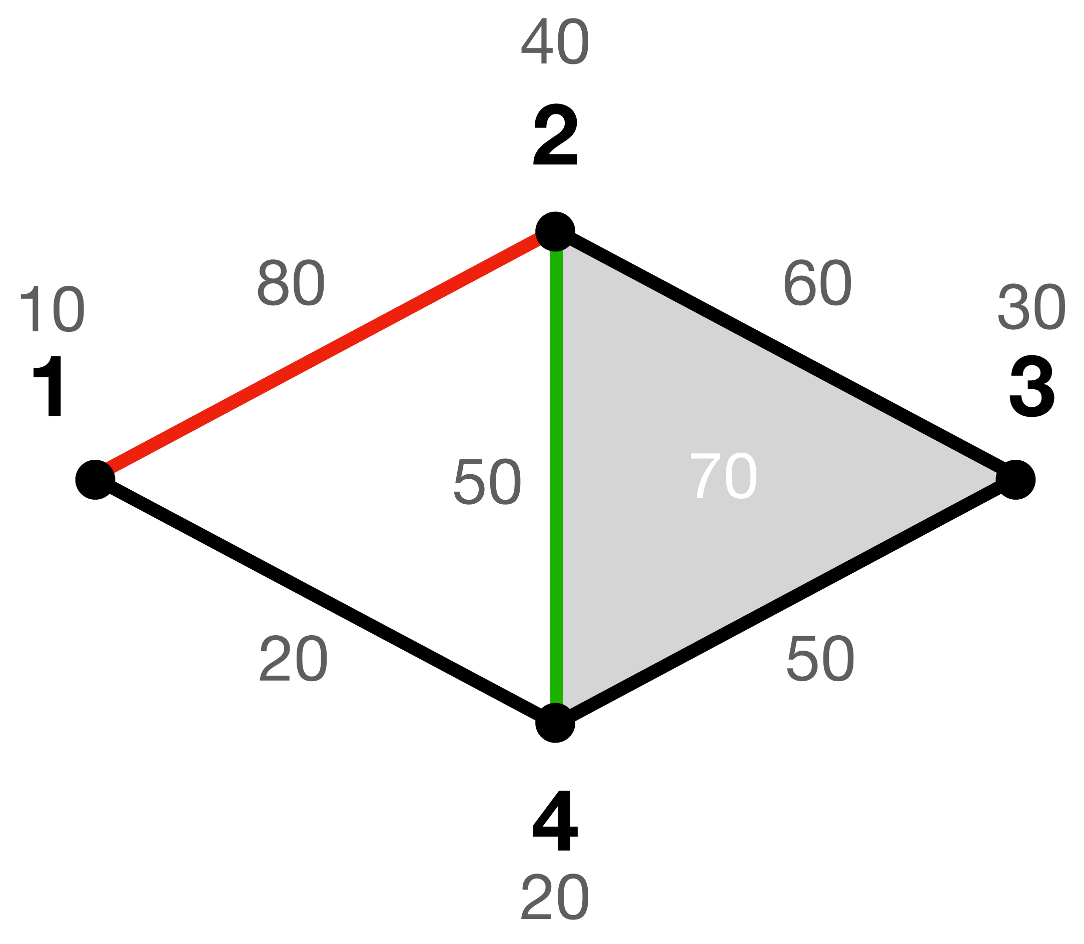
Boundary matrix reduction



	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1			
14										
3							1			
2						1				1
24									1	
34									1	
23									1	
12										

t=70

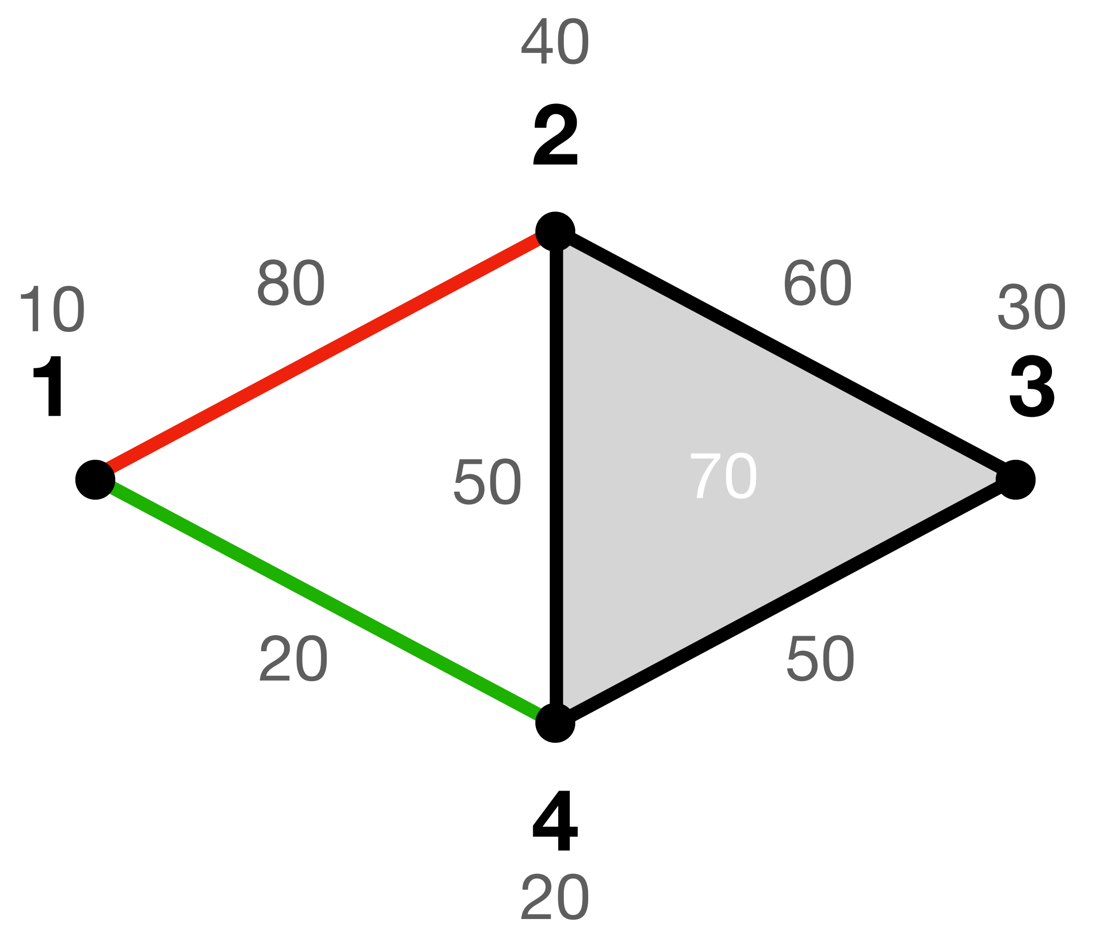
Boundary matrix reduction



	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1			
14										
3							1			
2						1				1
24									1	
34									1	
23									1	
12										

t=80

Boundary matrix reduction

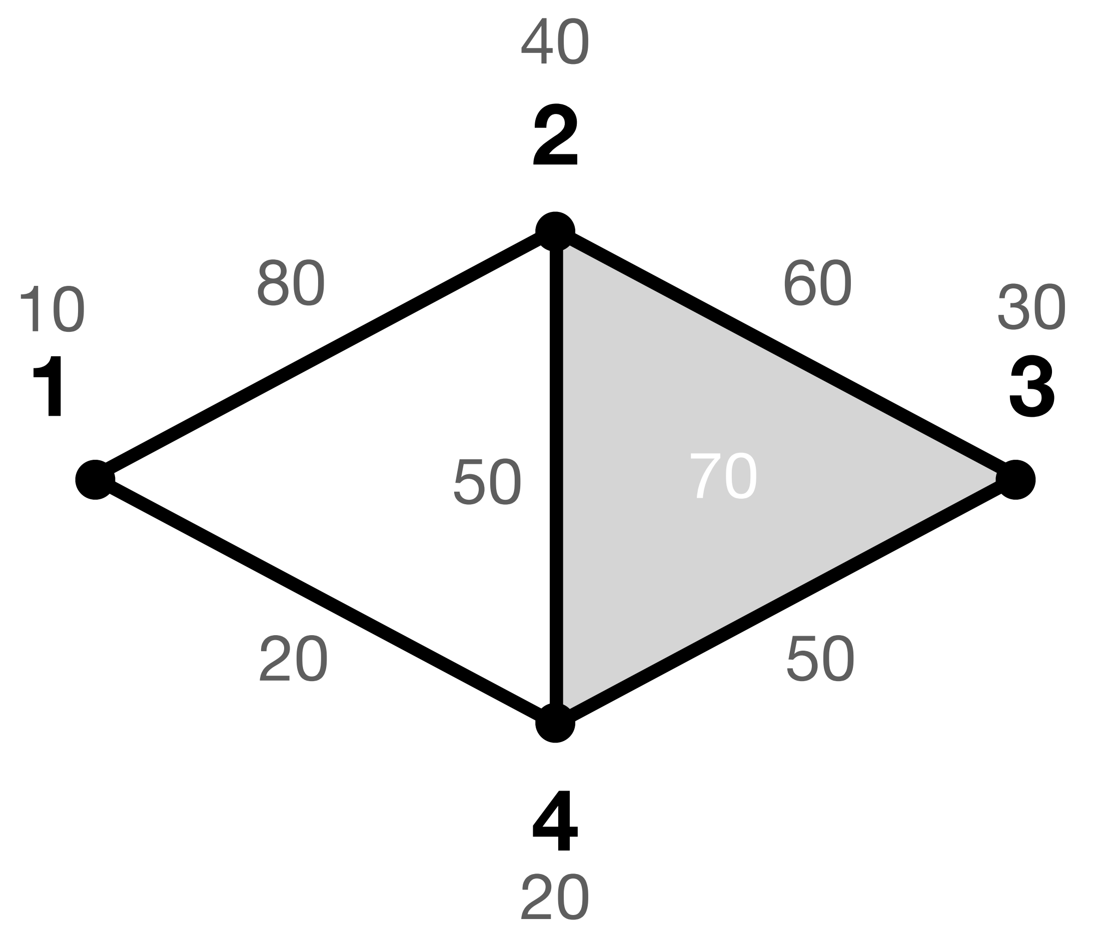


[12+14]

	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							1
4			1			1	1			1
14										
3							1			
2						1				
24									1	
34									1	
23									1	
12										

t=80

Boundary matrix reduction



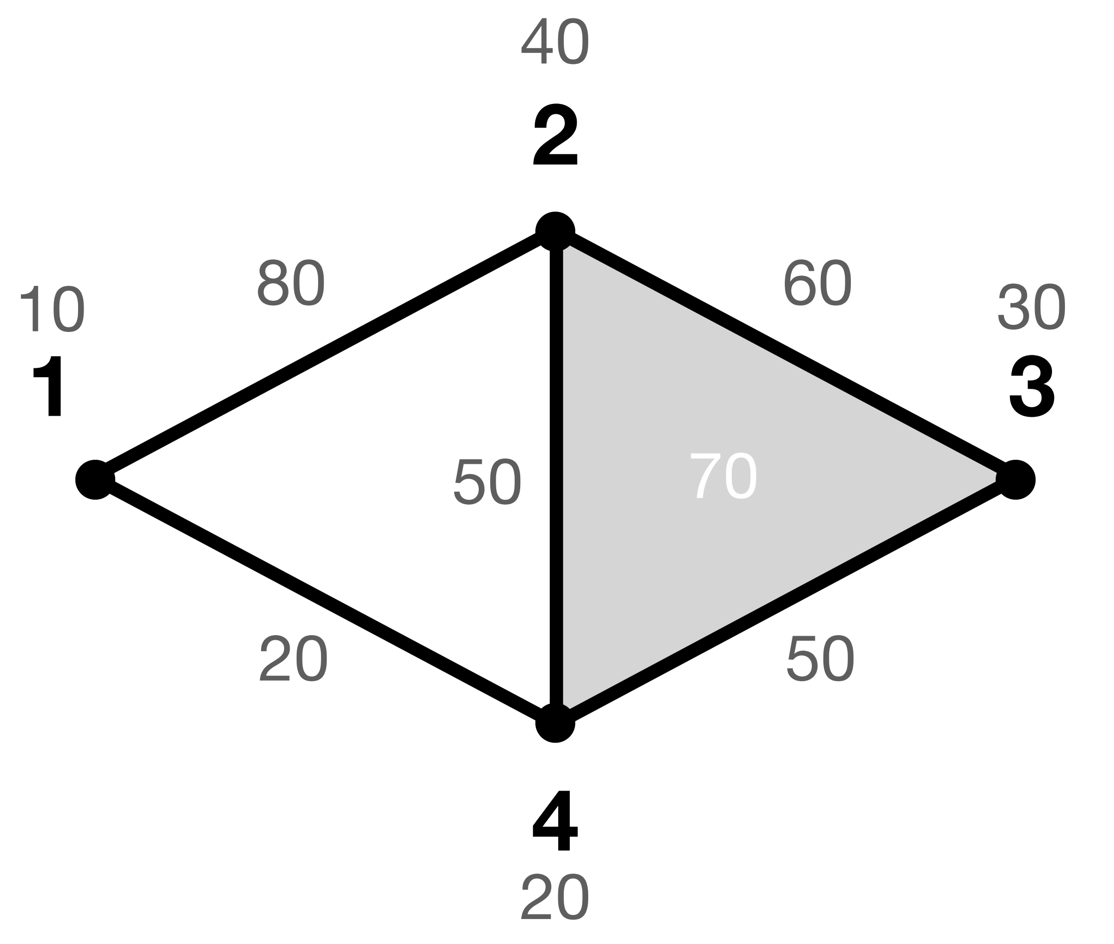
$\mathbf{R} =$

[12+14+24]

	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							
4			**1**			1	1			
14										
3							**1**			
2						**1**				
24									1	
34									1	
23									**1**	
12										

Matrix is called reduced if
all lowest nonzero elements are in unique rows

Boundary matrix reduction

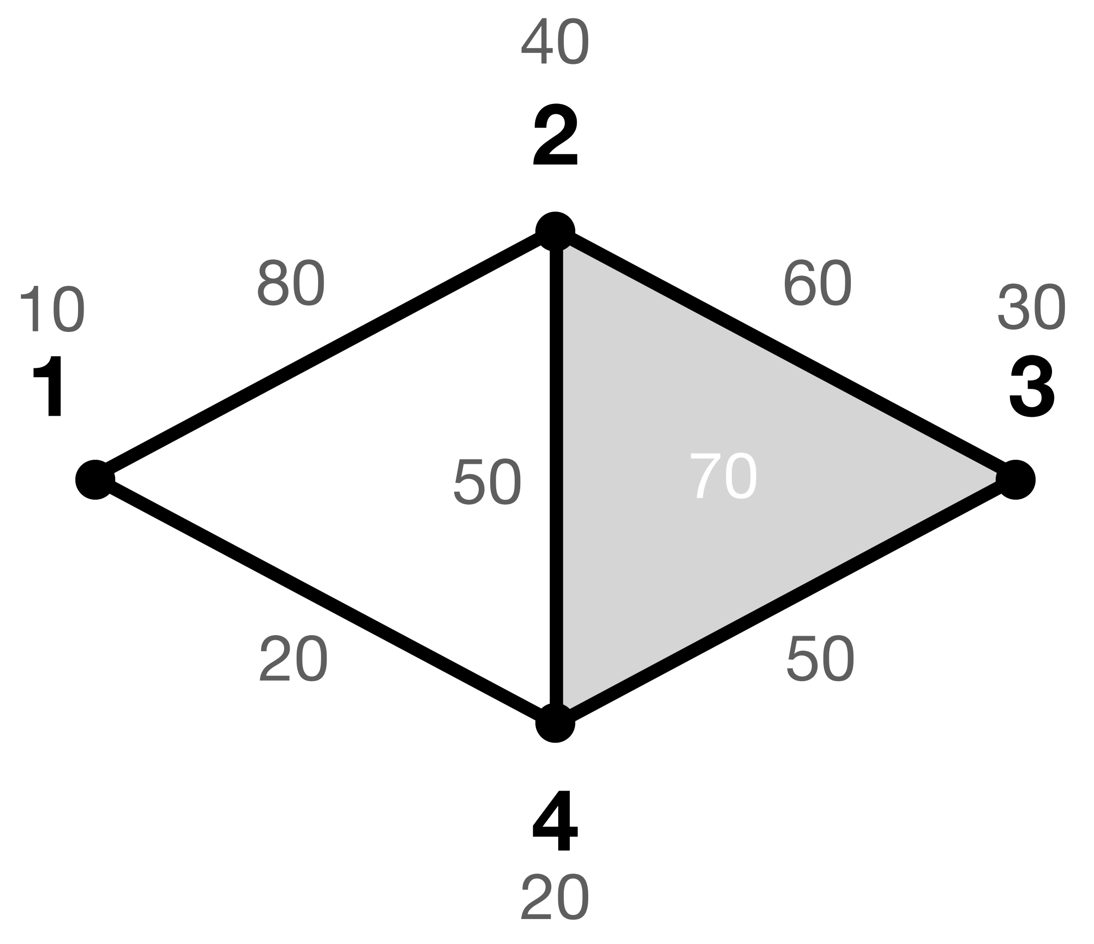


R =

	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							
4			**1**			1	1			
14										
3							**1**			
2						**1**				
24									1	
34									1	
23									**1**	
12										

Matrix is called reduced if
all lowest nonzero elements are in unique rows

Extracting information



$\mathbf{R} =$

	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							
4			1			1	1			
14										
3							1			
2						1				
24									1	
34									1	
23									1	
12										

Essential simplices correspond to unpaired empty columns

Persistence pairing

(4, 14) 0

(2, 24) 0

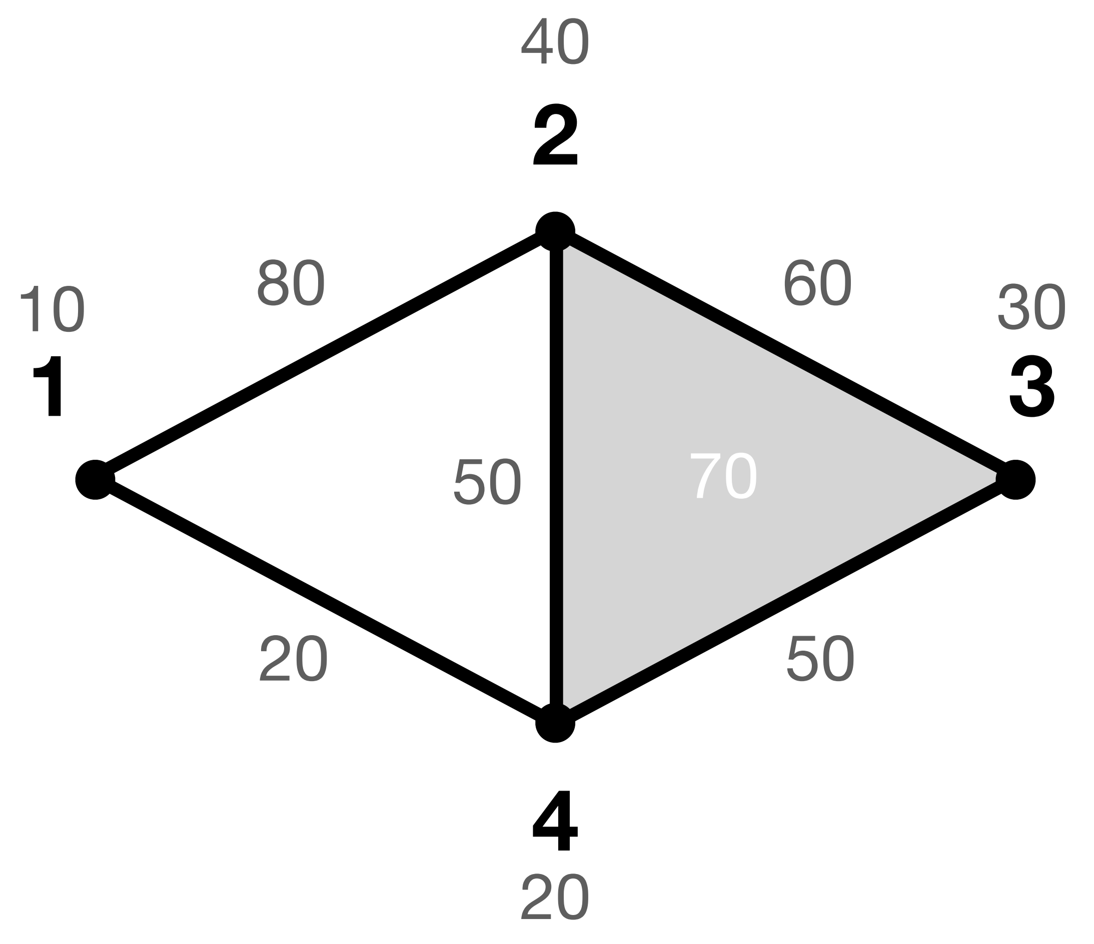
(3, 34) 0

(23, 234) 1

(1, $\underline{\emptyset}$) 0

(12, $\underline{\emptyset}$) 1

Extracting information



R =

	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							
4			1			1	1			
14										
3							1			
2						1				
24									1	
34									1	
23									1	
12										

Persistence pairing [representatives]

(4, 14) 0 [4]
(2, 24) 0 [2]
(3, 34) 0 [3]
(23, 234) 1 [23+24+34]

(1, Ø) 0 [1]
(12, Ø) 1 [12+14+24]

Persistence diagram

(20, 20) 0
(40, 50) 0
(30, 50) 0
(60, 70) 1

(10, Ø) 0
(80, Ø) 1