

Statistics 305/605: Introduction to Biostatistical Methods for Health Sciences

Chapter 18, part 1 (including Demo): Simple Linear-Regression
Models

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Response and Explanatory Variables

- ▶ In simple linear regression,
 - ▶ The **response** variable, Y , measures the outcome.
 - ▶ The **explanatory** variable(s), X , are there to explain the outcome.

Example

- ▶ Recall the study of head circumference in 100 infants with birth weight less than 1500g.
 - ▶ Variables included head circumference (cm) and gestational age (weeks), among others.

```
uu <- url("http://people.stat.sfu.ca/~jgraham/Teaching/S305_17/Data/lbwt.csv")
lbwt <- read.csv(uu)
head(lbwt)
```

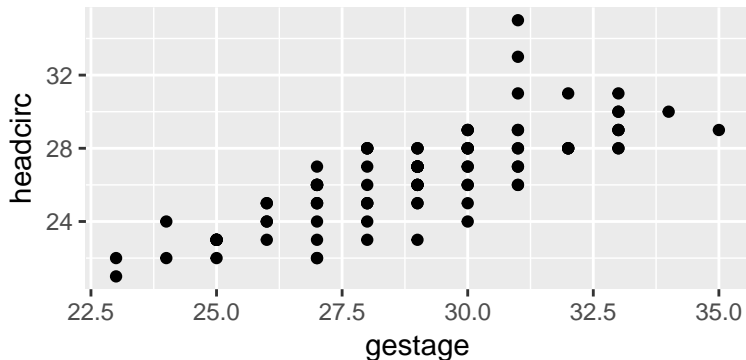
```
##   headcirc length gestage birthwt momage toxemia
## 1      27     41      29    1360     37      0
## 2      29     40     31    1490     34      0
## 3      30     38     33    1490     32      0
## 4      28     38     31    1180     37      0
## 5      29     38     30    1200     29      1
## 6      23     32     25     680     19      0
```

- ▶ Let's view **head circumference** (headcirc) as the response variable, Y , with observed measurements denoted y .
- ▶ Use **gestational age** (gestage) as an explanatory variable, X , with observed values x .

Scatterplot of the Low Birthweight Data

- There appears to be a linear relationship between Y and X :

```
library(ggplot2)
ggplot(lbwt,aes(x=gestage,y=headcirc)) + geom_point()
```



Linear Regression

- ▶ If we have response and explanatory variables, we may summarize a linear relationship by a **regression line** through the scatterplot.
- ▶ The regression line describes how the average value of Y changes as X changes.
 - ▶ Specifically, the line models the **population mean** of Y given that $X = x$.
- ▶ We use the method of least squares to fit or estimate the line from our sample of data.
- ▶ Under modelling assumptions, we can:
 - ▶ infer the slope of the regression line in the population from the slope fitted in our sample, and
 - ▶ make predictions from the model we have fitted to our data.
- ▶ Model assumptions are checked *after* the model is fit to our sample of data.

Model Overview

- ▶ The components of the statistical model are:
 1. the linear predictor,
 2. normal error terms,
 3. constant SD.

Linear Model:

$$Y = \underbrace{\alpha + \beta X}_{\substack{\text{1. linear} \\ \text{predictor} \\ \text{(systematic} \\ \text{component)}}} + \underbrace{\varepsilon}_{\substack{\text{2. Normal errors} \\ \varepsilon \sim N(0, \sigma^2) \\ \text{(random} \\ \text{component)}}}$$

3. Constant sd.
 $\text{sd}(\varepsilon) = \sigma$,
regardless of value
of linear predictor

- ▶ Will discuss each component.
- ▶ In addition, we assume that the observations are **independent**.

Linear Predictor

- ▶ When there is a linear relationship between Y and X , the conditional mean of Y given $X = x$ in the population, denoted $\mu_{Y|X}$, is modelled by a line:

$$\mu_{Y|X} = \alpha + \beta x,$$

- ▶ Think of $\mu_{Y|X}$ as the population mean value of Y for all data with $X = x$.
- ▶ β is the change in $\mu_{Y|X}$ for a one-unit increase in x .

Normal Errors, Constant SD

- ▶ Observed values of y will not fall perfectly along a line.
- ▶ Deviations of the y 's from the line are called errors.
- ▶ Write $y = \alpha + \beta x + \epsilon$ where ϵ is the error term.
- ▶ Errors are assumed to be normally distributed with mean zero and SD $\sigma_{y|x}$.
- ▶ The SD of the error terms is assumed to be constant for all x ; i.e. $\sigma_{y|x} = \sigma_y$

Model Summary

- ▶ We can summarize the model assumptions by saying that:
 1. the (X, Y) pairs are independent;
 - ▶ i.e., for individual i with measurements (X_i, Y_i) and a different individual j with measurements (X_j, Y_j) , knowing i 's measurements tells us nothing about what j 's are, and vice versa.
 2. conditional on $X = x$, the outcome Y has a normal distribution $N(\mu_{y|x}, \sigma_{y|x})$, with
 - ▶ mean $\mu_{y|x} = \alpha + \beta x$, and
 - ▶ SD $\sigma_{y|x}$ being the same for all x , so that $\sigma_{y|x} = \sigma_y$.

Fitting the Model

- ▶ Goal: Let's use the observed data on the n individuals — $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ — to fit the model

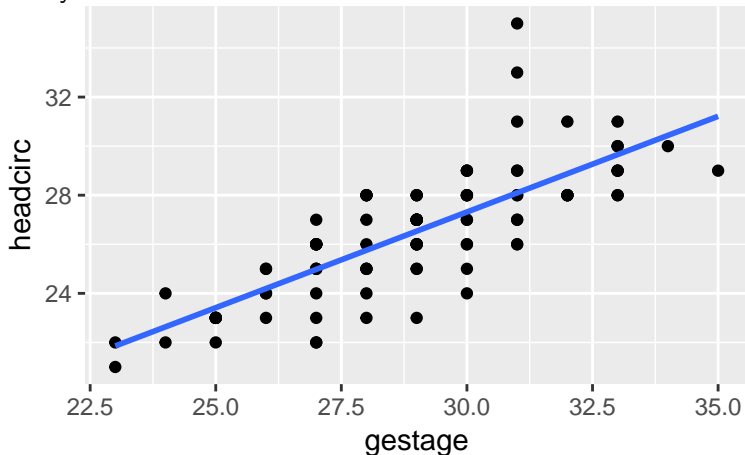
$$\hat{y}_i = \hat{\alpha} + \hat{\beta}x_i,$$

where \hat{y}_i is the **predicted** or **fitted value** of Y for $X = x_i$.

- ▶ Idea: Try all possible $\hat{\alpha}$ and $\hat{\beta}$, until we find the line that fits the data the “best” in the sense that the \hat{y} 's are as close to the y 's as possible.
- ▶ Need to explore the criteria for “best” ...

Vertical Distance

- ▶ Here is a plot of the data from the low-birth-weight babies study:



- ▶ By comparing y to \hat{y} , we are measuring the vertical distance between points in the scatterplot and the regression line.

Vertical Distance

- ▶ Question: How should we summarize vertical distances between the points, y , and the regression line, \hat{y} ?
- ▶ We will discuss the method that minimizes the sum of squared distances, or least squares.
- ▶ There are many visual demonstrations of the least squares idea on the internet; e.g.,
<http://www.dangoldstein.com/regression.html>
 - ▶ The sum of squared distances between the y 's and their \hat{y} 's is summarized by the blue square in this demo.
 - ▶ To minimize the sum of squared distances, try clicking the buttons for
 - ▶ – slope, + slope,
 - ▶ – intercept, + intercept.
 - ▶ Then click “Fit and lock” to see the line that minimizes the sum of squares.

Least-Squares Regression

- ▶ We choose the regression line to minimize the squares of the discrepancies $y - \hat{y}$; i.e, to

$$\text{minimize } Q = \sum_{i=1}^n (y_i - \hat{y}_i)^2.$$

- ▶ The line that minimizes Q has

$$\begin{aligned}\hat{\beta} &= r \frac{s_y}{s_x} \\ \hat{\alpha} &= \bar{y} - \hat{\beta} \bar{x},\end{aligned}$$

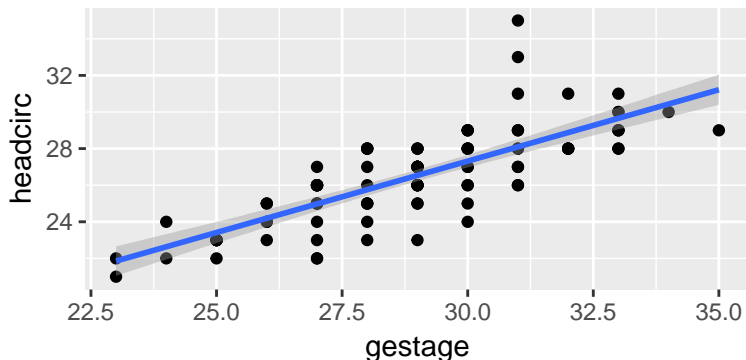
where r, s_y, s_x, \bar{y} and \bar{x} are, respectively:

- ▶ the sample correlation, the sample SD of y , the sample SD of x , the sample mean of y and the sample mean of x .
- ▶ However, we'll use computer software to get the least-squares estimates of the parameters α and β .

Example

- ▶ We can superpose the least-squares regression line onto our initial scatterplot of head circumference vs. gestational age, as follows:

```
ggplot(lbwt, aes(x=gestage, y=headcirc)) + geom_point() +  
  geom_smooth(method="lm")
```



Software Notes

- ▶ overlaying `geom_smooth()` adds a curve to the plot that summarizes the trends and is called a *scatterplot smoother*
 - ▶ the argument `method=lm` specifies that the smoother should be the least squares regression line.
- ▶ The grey shaded region around the regression line is a *point-wise confidence interval* for the population means $\mu_{y|x}$: more on these later.

Fitted Model and Coefficients

- ▶ To fit the model in R, we will use the `lm()` function and put the resulting fitted-model into an R object called `lfit`.

```
lfit <- lm(headcirc ~ gestage, data=lbwt)  
names(lfit)
```

```
## [1] "coefficients" "residuals"      "effects"      "rank"  
## [5] "fitted.values" "assign"         "qr"          "df.residual"  
## [9] "xlevels"      "call"          "terms"       "model"
```

- ▶ Let's see what the fitted coefficients are that estimate the population intercept α and the population slope β .

```
coefficients(lfit)
```

```
## (Intercept)      gestage  
##    3.9142641    0.7800532
```

- ▶ The estimated intercept and slope are $\hat{\alpha} = 3.9$ and $\hat{\beta} = 0.78$.
 - ▶ A one week increase in gestational age is associated with an estimated 0.78cm increase in head circumference.

Software Notes

- ▶ `lm()` is the R function that fits linear models to data by the least-squares method of minimizing the sum of squared vertical distances between the y 's and their \hat{y} 's.
- ▶ `lm()` uses formulas to specify the response and explanatory variables.
 - ▶ e.g., in the call to `lm()`, we specify
`lfit <- lm(headcirc ~ gestage, data=lbwt)`
and the formula being used is `headcirc ~ gestage`
 - ▶ the response variable, `headcirc` is on the left-hand side of the formula, to the left of `~`.
 - ▶ the explanatory variable, `gestage` is on the right-hand side of the formula, to the right of `~`.
- ▶ Extract the fitted coefficients with the `coefficients()` function; i.e.

```
coefficients(lfit)
```

```
## (Intercept)      gestage  
##   3.9142641    0.7800532
```