reinforcement

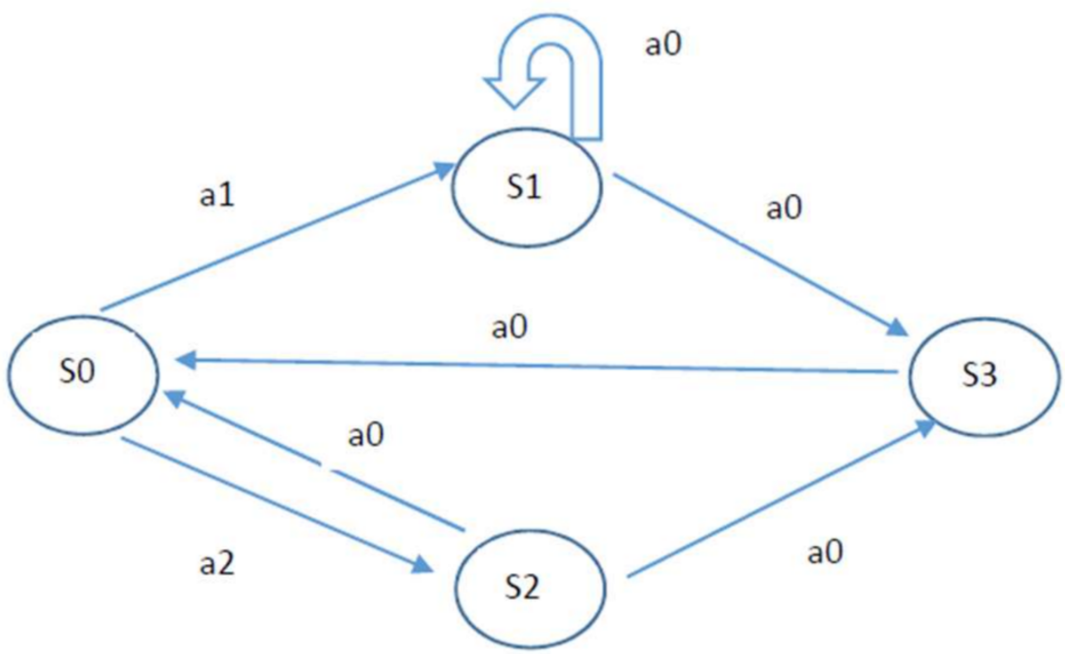
October 20, 2025

1 Reinforcement Learning

1.1 By Noël Jumin and Louis Maurin

1.1.1 October 9, 2025

[1]:
```
import numpy as np
```



We define the transition matrices as follows:

$$T(S, a0, S') = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-x & 0 & x \\ 1-y & 0 & 0 & y \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$T(S, a1, S') = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T(S, a2, S') = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With the reward vector:

$$R = [0, 0, 1, 10]$$

2 Q1

We can so write the possible policies as:

$$\pi_1 : \begin{cases} S_0 \to a_1 \\ S_1 \to a_0 \\ S_2 \to a_0 \\ S_3 \to a_0 \end{cases} \qquad \pi_2 : \begin{cases} S_0 \to a_2 \\ S_1 \to a_0 \\ S_2 \to a_0 \\ S_3 \to a_0 \end{cases}$$

3 Q2

We can write the Bellman equations for the optimal value function (V^*) as follows:

$$V^*(S) = R(S) + \gamma \max_a \sum_{S'} T(S, a, S') V^*(S')$$

Where (γ) is the discount factor.

To solve for (V^*), we can set up the equations for each state:

1. For state 0:
$$V^*(S_0) = R(S_0) + \gamma \max(V^*(S_1), V^*(S_2))$$
$$V^*(S_0) = \gamma \max(V^*(S_1), V^*(S_2))$$

2. For state 1:
$$V^*(S_1) = R(S_1) + \gamma((1-x)V^*(S_1) + xV^*(S_3))$$
$$V^*(S_1) = \gamma((1-x)V^*(S_1) + xV^*(S_3))$$

3. For state 2:
$$V^*(S_2) = R(S_2) + \gamma \max((1-y)V^*(S_0) + yV^*(S_3))$$
$$V^*(S_2) = 1 + \gamma((1-y)V^*(S_0) + yV^*(S_3))$$

4. For state 3:
$$V^*(S_3) = R(S_3) + \gamma \max(V^*(S_0))V^*(S_3) = 10 + \gamma V^*(S_0)$$

4 Q3

Now that we have these equations, we could determine a value for x, that for all $\gamma \in [0, 1[$, and $y \in [0, 1]$, $\pi^*(S_0) = a_2$.

We know that

$$\pi^*(S_0) = \arg\max_a \sum_{S'} T(S_0, a, S')V^*(S')$$

Hence,

$$\pi^*(S_0) = \begin{cases} a_1 & \text{if } V^*(S_1) > V^*(S_2) \\ a_2 & \text{if } V^*(S_1) < V^*(S_2) \end{cases}$$

So we need to find x so that $V^*(S_2) > V^*(S_1)$

To do that, we'll take $x = 0$ and reformulate $V^*(S_1)$ and $V^*(S_2)$ Bellman equations (Taking into account that $R(S)$ is not function of a) :

$$\begin{aligned} V^*(S_1) = \gamma((1-x)V^*(S_1) + xV^*(S_3)) \\ V^*(S_1) = \gamma V^*(S_1) \\ (1-\gamma)V^*(S_1) = 0 \\ V^*(S_1) = 0 \quad \text{since } \gamma \in [0, 1[\end{aligned}$$

$$\begin{aligned} V^*(S_2) = \gamma((1-y)V^*(S_0) + yV^*(S_3)) \\ V^*(S_2) = \gamma((1-y)V^*(S_0) + y\gamma V^*(S_0)) \\ V^*(S_2) = \gamma V^*(S_0)(1-y+y\gamma) \\ V^*(S_2) = \gamma V^*(S_0)(1+y(\gamma-1)) \end{aligned}$$

So $V^*(S_2) > V^*(S_1)$ for any $y \neq 0$ and $x = 0$ is a solution.

5 Q4

We can now try to determine a value for y, that for all $\gamma \in [0, 1[$, and $x > 0$, $\pi^*(S_0) = a_1$.

As of before,

$$\pi^*(S_0) = \begin{cases} a_1 & \text{if } V^*(S_1) > V^*(S_2) \\ a_2 & \text{if } V^*(S_1) < V^*(S_2) \end{cases}$$

Let's suppose such a value of y exists. We call it y_0 and choose $y = y_0$.

We now have that for all $\gamma \in [0, 1[$, and $x > 0$, $\pi^*(S_0) = a_1$ which is equivalent to $V^*(S_1) > V^*(S_2)$.

We can that write,

$$\begin{aligned} V^*(S_0) = \gamma V^*(S_1) \\ V^*(S_3) = 10 + \gamma^2 V^*(S_1) \end{aligned}$$

So,

$$\begin{aligned} V^*(S_1) = \gamma(1-x)V^*(S_1) + \gamma x(10+\gamma^2 V^*(S_1)) \\ V^*(S_1)(1+\gamma(x-1)-x\gamma^3) = 10x\gamma \\ V^*(S_1) = \frac{10x\gamma}{1+\gamma(x-1)-x\gamma^3} \quad (\text{for} \quad 1+\gamma(x-1)-x\gamma^3 \neq 0) \end{aligned}$$

Now using this formula, if $\gamma = 0$ then $1+\gamma(x-1)-x\gamma^3 = 1 \neq 0$ and $V^*(S_1) = 0$.

Actually, we already had from the equations above that. When $\gamma = 0$,

$$\begin{aligned} V^*(S_1) = 0 \\ V^*(S_2) = 1 \end{aligned}$$

So $V^*(S_2) > V^*(S_1)$ and we end up with a contradiction.

So such a value of y doesn't exists.

6 Q5

We'll now implement our value iteration:

[6]:
```
R = [0, 0, 1, 10]
gamma = 0.9
x = 0.25
y = 0.25
tol = 1e-6

# Transition matrices
T_a0 = np.array([
    [0.0, 0.0, 0.0, 0.0],
    [0.0, 1.0 - x, 0.0, x],
    [1.0 - y, 0.0, 0.0, y],
    [1.0, 0.0, 0.0, 0.0],
])
T_a1 = np.array([
    [0.0, 1.0, 0.0, 0.0],
    [0.0, 0.0, 0.0, 0.0],
    [0.0, 0.0, 0.0, 0.0],
    [0.0, 0.0, 0.0, 0.0],
])
T_a2 = np.array([
    [0.0, 0.0, 1.0, 0.0],
    [0.0, 0.0, 0.0, 0.0],
    [0.0, 0.0, 0.0, 0.0],
    [0.0, 0.0, 0.0, 0.0],
])
T = [T_a0, T_a1, T_a2]
```

```python
n_states = 4 # S0, S1, S2, S3
n_actions = len(T) # One action per transition matrix

V = np.zeros(n_states)
delta = np.inf
while delta > tol:
    V_prev = V.copy()
    for s in range(n_states):
        q_vals = [np.dot(T[a][s], V_prev) for a in range(n_actions)]
        V[s] = R[s] + gamma * max(q_vals)
    delta = np.max(np.abs(V - V_prev))

# Extract deterministic optimal policy
action_names = ['a0', 'a1', 'a2']
pi = []
for s in range(n_states):
    q_vals = [np.dot(T[a][s], V) for a in range(n_actions)]
    print(f" S{s}: q = {np.round(q_vals,6)}  -> best = {action_names[int(np.
 ↪argmax(q_vals))]}")
    pi.append(action_names[int(np.argmax(q_vals))])

print("V* =", np.round(V, 6))
print("pi* =", pi)
```

```
 S0: q = [ 0.        15.761813 15.69789 ]  -> best = a1
 S1: q = [17.513126  0.         0.      ]  -> best = a0
 S2: q = [16.33099  0.        0.      ]  -> best = a0
 S3: q = [14.185631  0.         0.      ]  -> best = a0
V* = [14.185631 15.761813 15.69789  22.767067]
pi* = ['a1', 'a0', 'a0', 'a0']
```

To explain what appened here, V^* is the calculation of reward taking in account if it will be immediate reward or not immediate but lead to a big reward.
With Q, we can know from each state, what action take to get the maximum reward and so take the best path.
With this information we know that the best option from S_0 is a1 and a0 for the others.