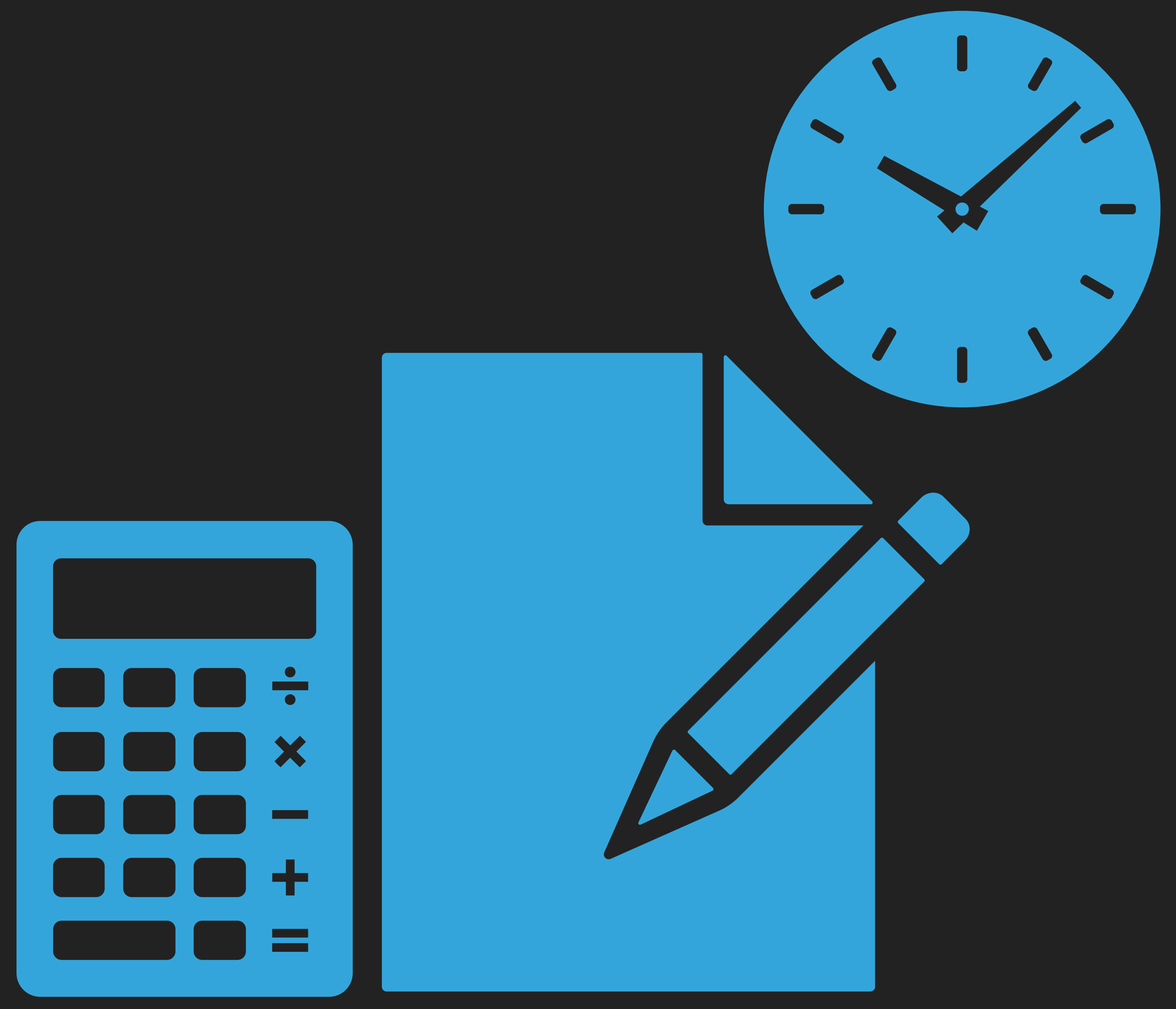
深宵教室 - DSE M2 模擬試題解答

2015

此為參考2015試題之模擬試題，原版請另行購買

2015

- Section A
- Section B



2015 - SECTION A

Q1.) $f(x) = (x^5 + 4)$. $f'(x) = ?$ (By First Principles)

* 參考課程 1.1, 3.1 及 3.2

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\boxed{(x+h)^5} - x^5\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\boxed{h^5 + 5h^4x + 10h^3x^2 + 10h^2x^3 + 5hx^4 + \cancel{x^5}} - \cancel{x^5}\right)$$

$$= 5x^4$$

* 微分定義

* $\boxed{}$ *Binomial Expansion*

2015 - SECTION A

Q2.) Let $f(x) = xsinx + cosx$, find k, $k \in \mathbb{R}$, such that $xf''(x) + kf'(x) + xf(x) \equiv 0$

* 參考課程 3.2

$$f(x) = xsinx + cosx \rightarrow f'(x) = sinx + xcosx - sinx = xcosx$$
$$\rightarrow f''(x) = cosx - xsinx$$

$$Then,\ xf''(x) + kf'(x) + xf(x) \equiv 0$$
$$\rightarrow x(cosx - xsinx) + kxcosx + x(xsinx + cosx) \equiv 0$$
$$\rightarrow (k+2)xcosx \equiv 0$$
$$\rightarrow k = -2$$

* Product rule

* Product rule

2015 - SECTION A

$Q3.)\ \int_1^9 \frac{1}{\sqrt{x}e^{2\sqrt{x}}}dx = ?$

* 參考課程 3.10

$$\int_1^9 \frac{1}{\sqrt{x}e^{2\sqrt{x}}}dx = \int_1^9 e^{-2\sqrt{x}}\boxed{\frac{1}{\sqrt{x}}dx}$$

$$= \int_1^9 e^{-2\sqrt{x}}\boxed{d(2\sqrt{x})}$$

$$= [-e^{-2\sqrt{x}}]_1^9$$

$$= e^{-2} - e^{-6}$$

* 積分三寶: 積分代入法

2015 - SECTION A

$Q4.)$ *The slope of equation of a curve is* $9x^2 lnx$*, and the curve passes through* $(1, 4)$.
The equation of curve $= ?$

* 參考課程 3.9

Let $f(x)$ *be the equation of the curve .*

$$f'(x) = 9x^2 lnx$$

$$\rightarrow f(x) = 9\int \boxed{x^2} lnx\boxed{dx} = 9\int lnx\boxed{d(\frac{1}{3}x^3)}$$

$$\rightarrow f(x) = \boxed{3x^3 lnx - 3\int x^2 dx}$$

$$= 3x^3 lnx - x^3 + C, \text{ where } C \text{ is constant}$$

$\because$ *The curve passes throught* $(1, 4) \rightarrow f(1) = 4 \rightarrow C = 5$

$i.e.\ f(x) = 3x^3 lnx - x^3 + 5$

* 積分三寶: Integration by part

2015 - SECTION A

Q5.) Solve

$$\begin{cases} x+y+z=2 \\ 2x+3y-3z=4 \\ 3x+2y+kz=6 \end{cases}$$

* 參考課程 4.7

方法1

Consider :

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & -3 & 4 \\ 3 & 2 & k & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & -1 & k-3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & \boxed{k-8} & 0 \end{array}\right)$$

* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \to \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

* 如果 ■ = 0, 直線答案, 否則唯一答案

CONT'D

2015 – SECTION A

$For\ k = 8$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & -3 & 4 \\ 3 & 2 & k & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

* 三條公式變兩條

$Let\ z = t,\ t \in \mathbb{R}$

$(x, y, z) = (2 - 6t, 5t, t)$

$For\ k \neq 8, \rightarrow z = 0$

$(x, y, z) = (2, 0, 0)$

方法2

$Solve$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & -3 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -5 & 0 \end{array}\right)$$

* 先解頭兩條得直線答案

$Let\ z = t,\ t \in \mathbb{R} \rightarrow (x, y, z) = (2 - 6t, 5t, t)$

CONT'D

2015 - SECTION A

$Substitute\ (x, y, z) = (2-6t, 5t, t)\ into\ \ 3x + 2y + kz = 6$

$\rightarrow 3(2-6t) + 2(5t) + kt = 6$

$\rightarrow (k-8)t = 0$

$For\ k = 8,\ t\ can\ be\ any\ real\ number$

$(x, y, z) = (2-6t, 5t, t)$

$For\ k \neq 8, \rightarrow t = 0$

$(x, y, z) = (2, 0, 0)$

* 將直線答案代入第三式

2015 - SECTION A

Q6.)

$$Let\ \ A=\begin{pmatrix}-1 & a & b\\ -a & -1 & -8\\ -b & 8 & -1\end{pmatrix}, a,b\in\mathbb{R}$$

$a.)\ If\ M^T=-M,\ show\ that\ |M|=0\ for\ M\ is\ 3x3\ matrix$

$b.)\ Prove\ A^3+I_3\ is\ singular$

* 參考課程 4.8 及 4.9

$$a.)\ M^T=-M\rightarrow |M^T|=|-M|\rightarrow |M|=(-1)^3|M|$$

$$\rightarrow |M|=0$$

$$b.)\ A^3+I_3=(A+I_3)(A^2-A+I_3)$$

$$\rightarrow |A^3+I_3|=|A+I_3||A^2-A+I_3|$$

方法1

$$Consider,\ (A+I_3)^T=\begin{pmatrix}0 & -a & -b\\ a & 0 & 8\\ b & -8 & 0\end{pmatrix}=-(A+I_3)$$

- * $|M^T|=|M|$
- * $|kM|=k^n|M|$, 如果 M 係 nxn 矩陣
- * 因為 AI=IA, 所以可以用類似恆等式 $a^3+b^3=(a+b)(a^2-ab+b^2)$

CONT'D

2015 - SECTION A

$\therefore$ *from a.)* $|A + I_3| = 0 \rightarrow |A^3 + I_3| = 0$

$i.e.\ A^3 + I_3$ *is singular*

* 如果 |M| 係等如零, M 係 singular matrix

方法2

$$|A + I_3| = \begin{vmatrix} 0 & a & b \\ -a & 0 & -8 \\ -b & 8 & 0 \end{vmatrix} = -a\begin{vmatrix} -a & -8 \\ -b & 0 \end{vmatrix} + b\begin{vmatrix} -a & 0 \\ -b & 8 \end{vmatrix}$$

$$= 8ab - 8ab = 0$$

* 揾一個 row / column 做擴展

$\therefore |A + I_3| = 0 \rightarrow |A^3 + I_3| = 0$

$i.e.\ A^3 + I_3$ *is singular*

* 如果 |M| 係等如零, M 係 singular matrix

CONT'D

2015 - SECTION A

Q7.) $Let\ f(x) = sin^4x + cos^4x$

$Solve\ 8f(x) = 7,\ where\ 0 < x < \frac{\pi}{2}$

* 參考課程 2.1, 2.2 及 2.3

方法1

$$f(x) = (cos^2x + sin^2x)^2 - 2sin^2xcos^2x$$

$$= 1 - 2\left(\frac{sin^2 2x}{4}\right) = 1 - \frac{sin^2 2x}{2} = 1 - \frac{1 - cos4x}{4}$$

$$= \frac{3}{4} + \frac{cos4x}{4}$$

$$Hence,\ 8f(x) = 7 \rightarrow 2(3 + cos4x) = 7 \rightarrow cos4x = \frac{1}{2}$$

$$\rightarrow 4x = \frac{\pi}{3}\ or\ 4x = 2\pi - \frac{\pi}{3} \rightarrow x = \frac{\pi}{12}\ or\ x = \frac{5\pi}{12}$$

* $(a+b)^2 = a^2 + 2ab + b^2$

$\rightarrow a^2 + b^2 = (a+b)^2 - 2ab$

* sin 雙角公式

* cos 雙角公式

* $0 < x < \frac{\pi}{2} \rightarrow 0 < 4x < 2\pi$

* cos 1 及 4 象限為正數

CONT'D

2015 - SECTION A

方法2

$$f(x) = (1 - sin^2x)^2 + sin^4x = 1 - 2sin^2x + 2sin^4x$$

$$Hence,\ 8f(x) = 7 \rightarrow 8f(x) - 7 = 0 \rightarrow 16(sin^2x)^2 - 16sin^2x + 1 = 0$$

$$sin^2x = \frac{16 \pm \sqrt{16^2 - 4(16)}}{2(16)} = \frac{2 \pm \sqrt{3}}{4} \rightarrow \frac{1 - cos2x}{2} = \frac{2 \pm \sqrt{3}}{4}$$

$$\rightarrow cos2x = \frac{\pm\sqrt{3}}{2} \rightarrow 2x = \frac{\pi}{6}\ or\ 2x = \pi - \frac{\pi}{6} \rightarrow x = \frac{\pi}{12}\ or\ x = \frac{5\pi}{12}$$

* 二次公式 (sin^2x)

$* \ 0 < sin^2x < 1$

* cos 雙角公式

$* \ 0 < x < \frac{\pi}{2} \rightarrow 0 < 2x < \pi$

2015 - SECTION A

Q8.) Prove $sin\frac{x}{2}\sum_{r=1}^{n}cos(rx) = sin\frac{nx}{2}cos\frac{(n+1)x}{2}$, $\forall n \in \mathbb{Z}^+$, Hence, $\sum_{r=1}^{567}cos\frac{r\pi}{7} = ?$

* 參考課程 1.1, 1.2 及 2.2

方法1

Let $P(n): \ sin\frac{x}{2}\sum_{r=1}^{n}cos(rx) = sin\frac{nx}{2}cos\frac{(n+1)x}{2}, \ \forall n \in \mathbb{Z}^+$

* 先 Let Statement

For $P(1): L.H.S. = sin\frac{x}{2}cosx = R.H.S.$

* 証明 P(1) is true

Assume $P(k)$ is true $\exists k \in \mathbb{Z}^+$, then $P(k+1)$:

* 假設 P(k) is true. 証明 P(k+1) is true

$$L.H.S. = sin\frac{x}{2}\sum_{r=1}^{k+1}cos(rx) = sin\frac{x}{2}\sum_{r=1}^{k}cos(rx) + sin\frac{x}{2}cos(k+1)x$$

* 將未項抽出並改變未項

$$= sin\frac{kx}{2}cos\frac{(k+1)x}{2} + sin\frac{x}{2}cos(k+1)x$$

CONT'D

2015 - SECTION A

$$= \sin\frac{kx}{2}\cos\frac{(k+1)x}{2} + \sin\frac{x}{2}\cos(k+1)x$$

$$= \frac{1}{2}\left[\cancel{\sin\frac{(2k+1)x}{2}} + \sin\frac{-x}{2} + \sin\frac{(2k+3)x}{2} + \cancel{\sin\frac{-(2k+1)x}{2}}\right]$$

$$= \frac{1}{2}\left[\sin\frac{(2k+3)x}{2} - \sin\frac{x}{2}\right] = \sin\frac{(k+1)x}{2}\cos\frac{(k+2)x}{2}$$

$$= R.H.S.$$

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

i.e. By M.I., $P(n)$ *is true,* $\forall n \in \mathbb{Z}^+$

* **Product to Sum**

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

* $\sin(-\theta) = -\sin\theta$

* **Sum to Product**

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

* **寫結論**

CONT'D

2015 - SECTION A

方法2

$$\sin\frac{x}{2}\sum_{r=1}^{n}\cos(rx)=\sum_{r=1}^{n}\boxed{\sin\frac{x}{2}\cos(rx)}$$

$$=\frac{1}{2}\sum_{r=1}^{n}\left(\boxed{\sin\frac{(2r+1)x}{2}-\sin\frac{(2r-1)x}{2}}\right)$$

$$=\frac{1}{2}\sum_{r=1}^{n}\sin\frac{(2r+1)x}{2}-\frac{1}{2}\sum_{r=1}^{n}\sin\frac{(2r-1)x}{2}$$

$$=\frac{1}{2}\sum_{r=2}^{n+1}\sin\frac{(2r-1)x}{2}-\frac{1}{2}\sum_{r=1}^{n}\sin\frac{(2r-1)x}{2}$$

$$=\frac{1}{2}\left[\cancel{\sum_{r=2}^{n}\sin\frac{(2r-1)x}{2}}+\sin\frac{(2n+1)x}{2}-\sin\frac{x}{2}-\cancel{\sum_{r=2}^{n}\sin\frac{(2r-1)x}{2}}\right]$$

* Product to Sum

$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$

$2\sin A\sin B=\cos(A-B)-\cos(A+B)$

$\sin A+\sin B=2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A-\sin B=2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A+\cos B=2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A-\cos B=-2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

* Summation 可析開做加減
* 透過改變首末項改變公項
* 透過抽首尾項改變首末項

CONT'D

2015 - SECTION A

$$= \frac{1}{2}\left[\sin\frac{(2n+1)x}{2} - \sin\frac{x}{2}\right] = \sin\frac{nx}{2}\cos\frac{(n+1)x}{2}$$

$$Then,\ \sin\frac{\pi}{14}\sum_{r=1}^{567}\cos\frac{r\pi}{7} = \sin\frac{567\pi}{14}\cos\frac{568\pi}{14}$$

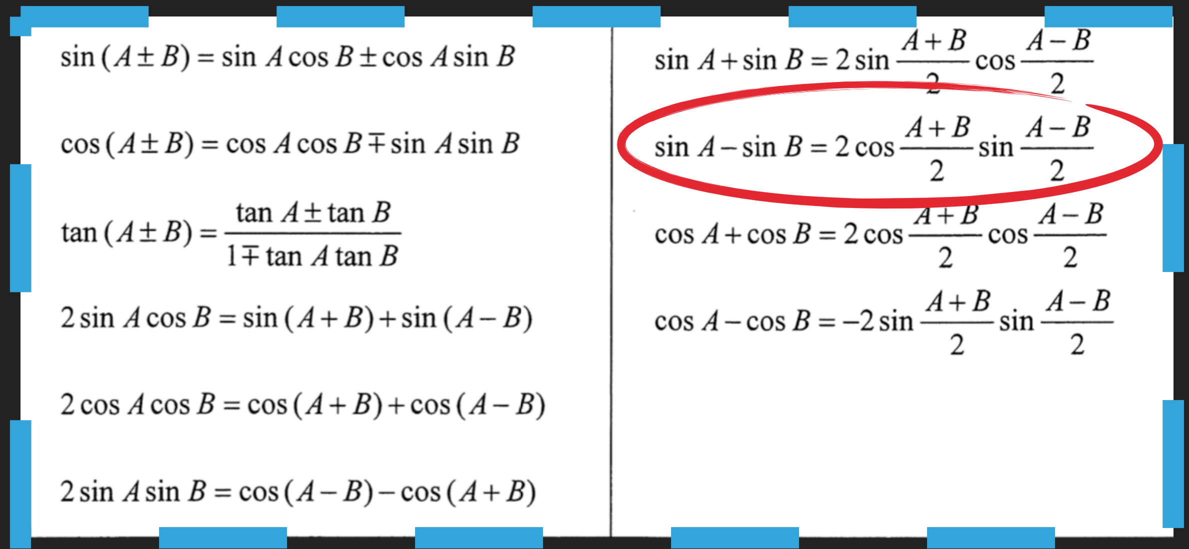
$$= \sin\left(40\pi+\frac{\pi}{2}\right)\cos\left(40\pi+\frac{4\pi}{7}\right)$$

$$= \overset{1}{\sin\frac{\pi}{2}}\cos\frac{4\pi}{7} = \cos\left(\frac{\pi}{2}+\frac{\pi}{14}\right)$$

$$= \cos\frac{\pi}{2}\cos\frac{\pi}{14} - \sin\frac{\pi}{2}\sin\frac{\pi}{14} = -\sin\frac{\pi}{14}$$

$$\therefore \sum_{r=1}^{567}\cos\frac{r\pi}{7} = -1$$

* Sum to Product



$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$

$2\sin A\sin B=\cos(A-B)-\cos(A+B)$

$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

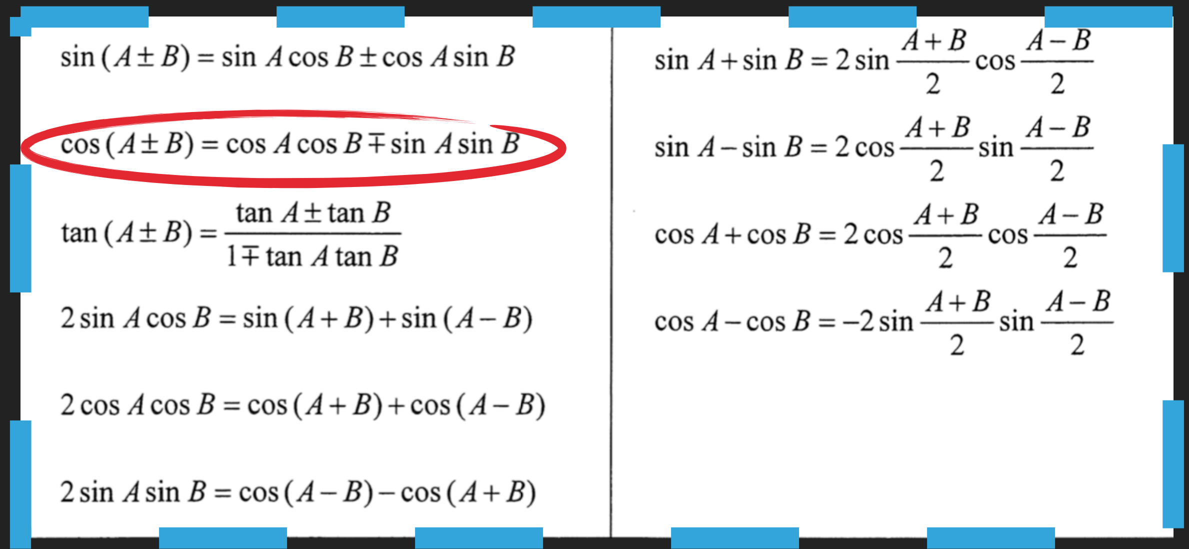
$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

* $\sin(2n\pi + theta) = \sin\theta$

* $\cos(2n\pi + theta) = \cos\theta$

* cos 複角公式



$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$

$2\sin A\sin B=\cos(A-B)-\cos(A+B)$

$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

2015 – SECTION B

Q9.)

$$f(x) = \frac{x^2+12}{x-2},\ where\ x \neq 2$$

a.) Find the local max. point and local min. point of $y = f(x)$

b.) Find all asymptote(s) of $y = f(x)$

c.) Find the area of the region bounded by $y = f(x)$ *and* $y = 14$

* 參考課程 3.5 及 3.11

$$a.)\ f(x) = \frac{(x-2)(x+2)+16}{x-2} = x+2+\frac{16}{x-2}$$

$$\rightarrow f'(x) = 1 - \frac{16}{(x-2)^2}$$

$$Let\ x_0 \in \mathbb{R}\ such\ that\ f'(x_0) = 0 \rightarrow 1 - \frac{16}{(x-2)^2} = 0$$

* 搵 turning point = 搵 x_0 使度 $f'(x_0)=0$

CONT'D

2015 – SECTION B

$$\to (x-2)^2 = 16 \to x-2 = \pm 4 \to x = -2 \ or\ x = 6$$

	x < -2	x = -2	-2 < x < 6	x = 6	x > 6
f′(x)	+	0	-	0	+
f(x)	Inc.		Dec.		Inc.

$$\therefore The\ local\ max.pt. = (-2, f(-2)) = (-2, -4)$$

$$The\ local\ min.pt. = (6, f(6)) = (6, 12)$$

b.) $Vertical\ Asymptote:\ x = 2$

$Horizontal\ Asymptote:\ No\ horizontal\ asymptotes$

$Oblique\ Asymptote:\ y = x + 2$

c.) $Solve\ (E)\ to\ find\ The\ interception\ of\ y = f(x)\ and\ y = 14$

$$(E): \begin{cases} y = f(x) \\ y = 14 \end{cases} \to x^2 - 14x + 40 = 0 \to x = 4\ or\ 10$$

*** 利用表格計算 turning point 附近上升定下降**

$f'(x) > 0 \to Increasing$

$f'(x) < 0 \to Decreasing$

*** x 係幾多, 分母係零**

$* Find \lim_{x\to\infty} y$

$* Find\ m\ and\ c\ such\ that \lim_{x\to\infty} [y - (mx + c)] = 0$

$$y - (x+2) = \frac{16}{x-2} \to \lim_{x\to\infty} (y - (x+2)) = 0$$

CONT'D

2015 - SECTION B

$\therefore$ *The area of the bounded region* $= 14(10-4) - \int_4^{10} f(x)dx$

$= 84 - \int_4^{10} (x + 2 - \frac{16}{x-2})dx$

$= 84 - [\frac{1}{2}(x+2)^2 - 16ln(x-2)]_4^{10}$

$= 84 - (72 + 16ln8 - 18 - 16ln2)$

$= 30 - 16ln4 \ sq.unit$

* 利用基本幾何(長方形)面積計算

* 面積大減細

* 確保面積為正數

2015 – SECTION B

$Q10.)$ $In\ \Delta OAB,\ P = mid-point\ of\ OA,\ Q\ is\ a\ point\ on\ AB\ such\ that\ AQ : QB = 1 : 2$

$R\ is\ a\ point\ on\ OB\ such\ that\ OR : RB = 3 : 1,\ PR\ and\ OQ\ intersect\ at\ C$

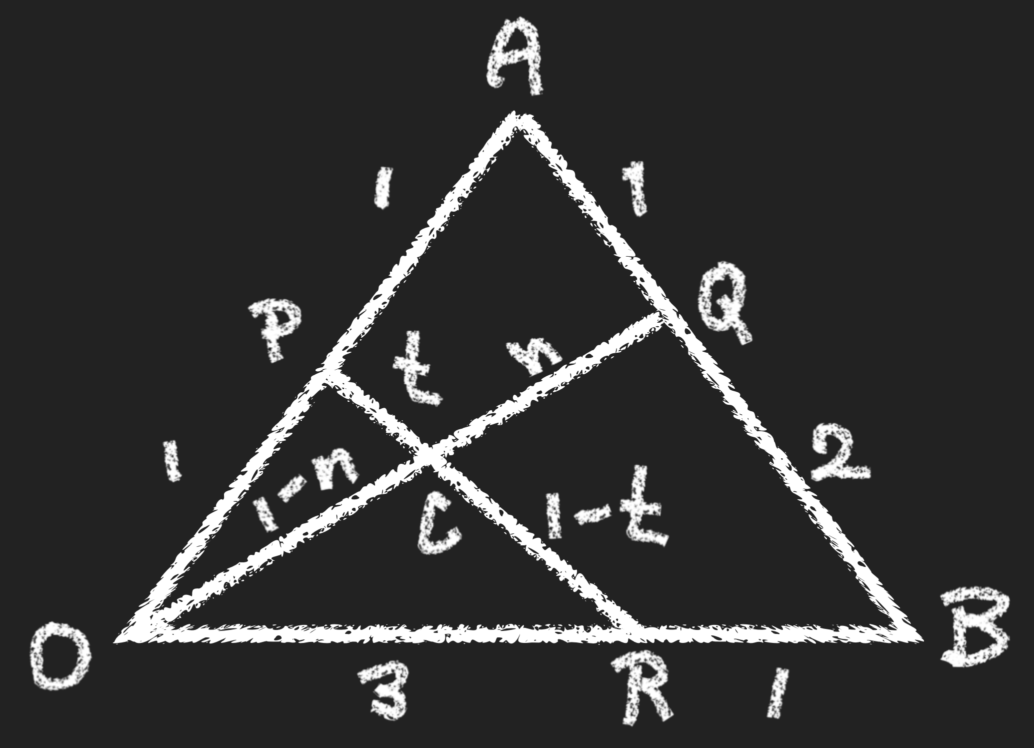
$a.)\ Find\ PC : PR\ and\ CQ : OQ$

$b.)\ Suppose\ \overrightarrow{OA} = 20\hat{i} - 6\hat{j} - 12\hat{k},\ \overrightarrow{OB} = 16(\hat{i} - \hat{j}),\ and\ \overrightarrow{OD} = \hat{i} + 3\hat{j} - 6\hat{k}$

$The\ area\ of\ \Delta OAB = ?\ and\ the\ volume\ of\ the\ tetrahedron\ ABCD = ?$

* 參考課程 4.3 及 4.5

$a.)\ Consider\ the\ following\ graph$



$Let\ PC : PR = t : 1\ and\ CQ : OQ = n : 1$

$$\overrightarrow{OQ} = \frac{1}{3}(2\overrightarrow{OA} + \overrightarrow{OB})$$

* 應用在 AB 的分割定理

$$\overrightarrow{OC} = (1-t)\overrightarrow{OP} + t\overrightarrow{OR} = \frac{1-t}{2}\overrightarrow{OA} + \frac{3t}{4}\overrightarrow{OB}$$

* 應用在 PR 的分割定理

CONT'D

2015 - SECTION B

$$\rightarrow (1-n)\overrightarrow{OQ} = = \frac{1-t}{2}\overrightarrow{OA} + \frac{3t}{4}\overrightarrow{OB}$$

$$\rightarrow \overrightarrow{OQ} = = \frac{1-t}{2(1-n)}\overrightarrow{OA} + \frac{3t}{4(1-n)}\overrightarrow{OB}$$

$$\therefore \begin{cases} \dfrac{1-t}{2(1-n)} = \dfrac{2}{3} \\ \dfrac{3t}{4(1-n)} = \dfrac{1}{3} \end{cases} \rightarrow \begin{cases} 3t - 4n = -1 \\ 9t + 4n = 4 \end{cases}$$

$$i.e.\ t = \frac{1}{4}, n = \frac{7}{16}$$

$$PC : PR = 1 : 4\ and\ CQ : OQ = 7 : 16$$

$$b.)\ The\ area\ \Delta OAB = \frac{1}{2}|\overrightarrow{OA}x\overrightarrow{OB}|$$

*** 用兩條式表達一支 Vector**

$* A\vec{a} + B\vec{b} = C\vec{a} + D\vec{b} \rightarrow A = C\ and\ B = D$

*** Cross Product 大小計緊平行四邊形面積一半為三角形面積**

CONT'D

2015 - SECTION B

$$\overrightarrow{OA}x\overrightarrow{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 20 & -6 & -12 \\ 16 & -16 & 0 \end{vmatrix} = 16\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 20 & -6 & -12 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 16\begin{vmatrix} \hat{i} & \hat{i}+\hat{j} & \hat{k} \\ 20 & 14 & -12 \\ 1 & 0 & 0 \end{vmatrix} = 16\begin{vmatrix} \hat{i}+\hat{j} & \hat{k} \\ 14 & -12 \end{vmatrix}$$

$$= -32(6\hat{i}+6\hat{j}+7\hat{k})$$

$$\therefore \text{The area } \Delta OAB = 16\sqrt{6^2+6^2+7^2} = 176\ sq.\ unit$$

* Determinent 可以抽常數
* C2=C2+C1
* 做R3 expansion

方法1

$$\text{From a.) } \overrightarrow{OC} = \frac{3}{8}\overrightarrow{OA} + \frac{3}{16}\overrightarrow{OB}$$

CONT'D

2015 - SECTION B

$$\text{The volume} = \frac{1}{6}(\overrightarrow{CA} x \overrightarrow{CB}) \cdot \overrightarrow{CD}$$

$$= \frac{1}{6}((\overrightarrow{OA} - \overrightarrow{OC}) x (\overrightarrow{OB} - \overrightarrow{OC})) \cdot (\overrightarrow{OD} - \overrightarrow{OC})$$

$$= \frac{1}{6}((\frac{5}{8}\overrightarrow{OA} - \frac{3}{16}\overrightarrow{OB}) x (\frac{13}{16}\overrightarrow{OB} - \frac{3}{8}\overrightarrow{OA})) \cdot (\overrightarrow{OD} - \overrightarrow{OC})$$

$$= \frac{1}{6}(\frac{7}{16}\overrightarrow{OA} x \overrightarrow{OB}) \cdot (\overrightarrow{OD} - \overrightarrow{OC})$$

$$= \frac{7}{96}[(\overrightarrow{OA} x \overrightarrow{OB}) \cdot \overrightarrow{OD} - \boxed{\cancel{(\overrightarrow{OA} x \overrightarrow{OB}) \cdot \overrightarrow{OC}}}^{0}]$$

$$= \frac{7}{96}(-32(6\hat{i} + 6\hat{j} + 7\hat{k}) \cdot (\hat{i} + 3\hat{j} - 6\hat{k}))$$

$$= 42 \ cu \ unit$$

* 四面體體積=1/6 平行六面體體積

* Cross product 可以拆括號

$*\ \vec{a} x \vec{b} = -\vec{b} x \vec{a},\ \vec{a} x \vec{a} = 0$

* Dot product 可以拆括號

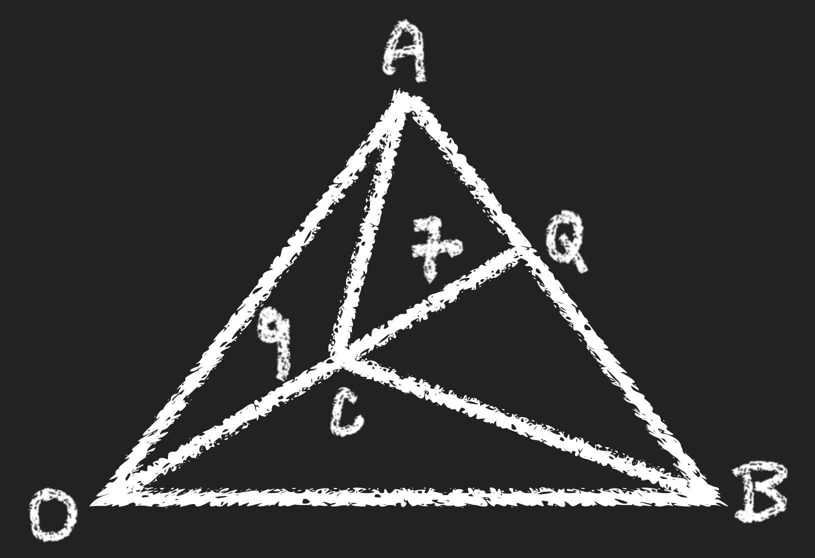
* $\boxed{}$ O, A, B, C 係同一個平面, 無體積

CONT'D

2015 – SECTION B

方法2

$$\text{The volume} = \frac{1}{3}x(\text{Area of } \Delta ABC)x(\text{Height from } D \text{ to plane } ABC)$$



$$= \frac{1}{3}x(\frac{7}{16}\text{Area of } \Delta OAB)x(\text{Height from } D \text{ to plane } OAB)$$

* 共高三角形, 面積比=邊比

$$= \frac{1}{6}\frac{7}{16}(\text{The volume of parallelepiped } OABD)$$

$$= \frac{1}{6}\frac{7}{16}(\overrightarrow{OA}x\overrightarrow{OB})\cdot\overrightarrow{OD}$$

* 四面體體積
=1/6 平行六面體體積

$$= \frac{7}{96}(-32(6\hat{i}+6\hat{j}+7\hat{k})\cdot(\hat{i}+3\hat{j}-6\hat{k}))$$

$$= 42 \text{ cu unit}$$

CONT'D

2015 – SECTION B

方法3

$$\text{From a.) } \overrightarrow{OC} = \frac{3}{8}\overrightarrow{OA} + \frac{3}{16}\overrightarrow{OB}, \text{ hence;}$$

$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = \frac{5}{8}\overrightarrow{OA} - \frac{3}{16}\overrightarrow{OB} = 9.5\hat{i} - 0.75\hat{j} - 7.5\hat{k}$$

$$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \frac{13}{16}\overrightarrow{OB} - \frac{3}{8}\overrightarrow{OA} = 5.5\hat{i} - 10.75\hat{j} + 4.5\hat{k}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \overrightarrow{OD} - \frac{3}{8}\overrightarrow{OA} - \frac{3}{16}\overrightarrow{OB} = -9.5\hat{i} + 8.25\hat{j} - 1.5\hat{k}$$

$$\therefore \text{The volume} = \frac{1}{6}\begin{vmatrix} 9.5 & -0.75 & -7.5 \\ 5.5 & -10.75 & 4.5 \\ -9.5 & 8.25 & -1.5 \end{vmatrix} = 42 \text{ cu unit}$$

* 用 Determinant 方式計算
* 四面體體積
  =1/6 平行六面體體積

2015 - SECTION B

$Q11.)$ *Let* μ, λ *be any real number satisfy* $\mu - \lambda \neq 2$

$$Assume,\ M = \begin{pmatrix} \lambda & 1 \\ \lambda - \mu + 1 & \mu \end{pmatrix},$$

$$A = \frac{1}{\lambda - \mu + 2}(I_2 - \mu I_2 + M)\ and$$

$$B = \frac{1}{\lambda - \mu + 2}(I_2 + \lambda I_2 - M)$$

$a.)$ *Prove* $M^n = (\lambda + 1)^n A + (\mu - 1)^n B, \forall n \in \mathbb{Z}^+$

$b.)$ $\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}^{315} = ?$

* 參考課程 1.2 及 4.8

CONT'D

2015 – SECTION B

$$Let\ P(n):\ \ M^n = (\lambda+1)^n A + (\mu-1)^n B,\ \ \forall n \in \mathbb{Z}^+$$

$$For\ P(1): R.H.S. = (\lambda+1)A + (\mu-1)B$$

$$= \frac{1}{\lambda-\mu+2}[(\lambda+1)(I_2-\mu I_2+M)+(\mu-1)(I_2+\lambda I_2-M)]$$

$$= \frac{\lambda-\mu+2}{\lambda-\mu+2}M = M = L.H.S.$$

$$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+,\ and\ given\ that$$

$$(\lambda-\mu+2)A = (1-\mu)I_2 + M = \begin{pmatrix} 1-\mu+\lambda & 1 \\ \lambda-\mu+1 & 1 \end{pmatrix}$$

$$(\lambda-\mu+2)B = (1+\lambda)I_2 - M = \begin{pmatrix} 1 & -1 \\ -\lambda+\mu-1 & 1+\lambda-\mu \end{pmatrix}$$

* 先 Let Statement

* 証明 P(1) is true

* 假設 P(k) is true. 証明 P(k+1) is true

CONT'D

2015 - SECTION B

$$\text{Hence, we have } A + B = I_2$$

$$(\lambda - \mu + 2)^2 AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow AB = 0$$

$$(\lambda - \mu + 2)^2 BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow BA = 0$$

$$\text{Hence for } P(k+1):$$

$$L.H.S. = M^{k+1} = MM^k$$

$$= [(\lambda+1)A + (\mu-1)B][(\lambda+1)^k A + (\mu-1)^k B]$$

$$= (\lambda+1)^{k+1}A^2 + (\mu-1)^{k+1}B^2$$

$$= (\lambda+1)^{k+1}A(I_2 - B) + (\mu-1)^{k+1}B(I_2 - A)$$

$$= (\lambda+1)^{k+1}A + (\mu-1)^{k+1}B = R.H.S.$$

* 矩陣相加=各自元素相加

* AB 唔一定等於 BA

* AB = BA = 0

* A+B=I

CONT'D

2015 - SECTION B

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

$i.e.$ *By M.I.,* $P(n)$ *is true,* $\forall n \in \mathbb{Z}^+$

$$b.)\ Let\ X = \begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}^{315} = 2^{315}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}^{315}$$

Use a.) result with $\lambda = 2$, $\mu = 3$, $n = 315$, *then*

$$X = 2^{315}[(2+1)^{315}\frac{1}{1}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + (3-1)^{315}\frac{1}{1}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}]$$

$$= 2^{315}[3^{315}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + 2^{315}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}]$$

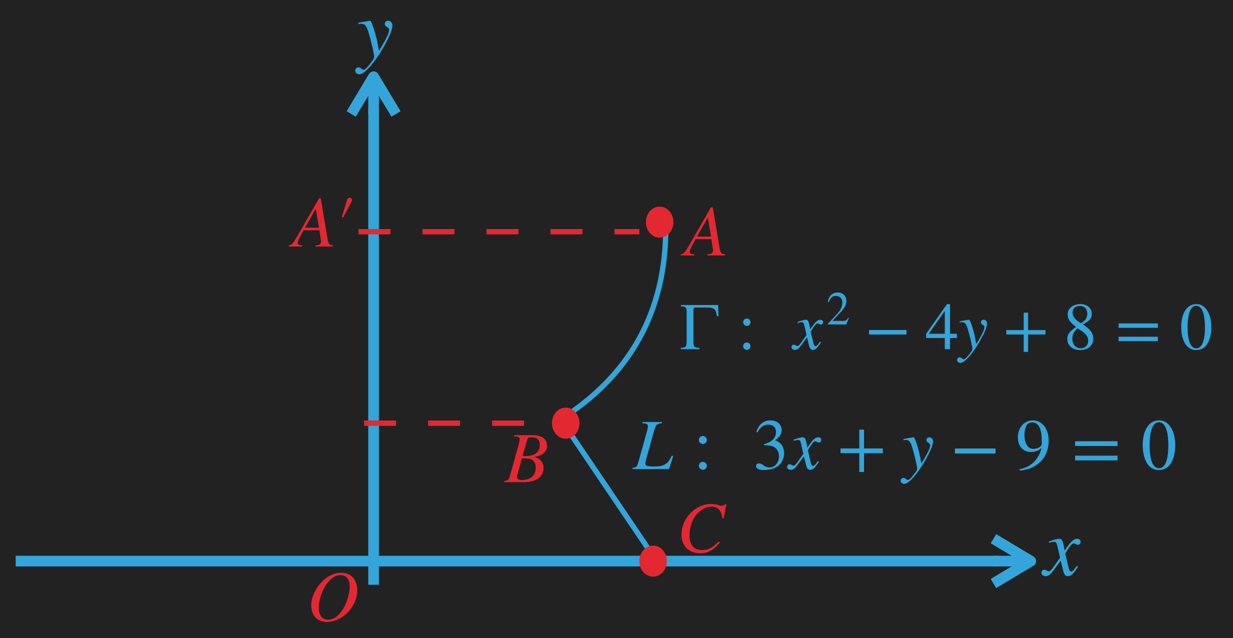
$$= 2^{315}\begin{pmatrix} 2^{315} & 3^{315} - 2^{315} \\ 0 & 3^{315} \end{pmatrix}$$

* 寫結論

* 矩陣抽常數

2015 - SECTION B

$Q12.)$



The cup is formed by revolving the curve Γ *and the line L along y − axis. B lies in first quadrant. Let* $A = (x_0, h)$, *and* $h > 3$

a.) *Find the capacity of the cup in term of h*

b.) *Water is pour into the cup at constant rate* $(24\pi\ cm^3s^{-1})$, *what is the increasing rate of the depth of water when the water volume* $= 35\pi\ cm^3$?

* 參考課程 3.3, 3.4 及 3.12

a.) *To find the coordination of B, we solve (E)*

$$(E) : \begin{cases} x^2 - 4y + 8 = 0 \\ 3x + y - 9 = 0 \end{cases} \rightarrow x^2 + 12x - 28 = 0 \rightarrow x = -14 \text{ or } 2$$

* 先搵線的相交

CONT'D

2015 - SECTION B

$Rejected\ x = -14\ (\because B\ lies\ in\ 1^{st}\ quadrant)$

$\therefore B = (2, 3)$

$To\ find\ the\ coordination\ of\ C,\ we\ put\ y = 0\ into\ L:\ \ 3x + y - 9 = 0$

$\therefore C = (3, 0)$

$$The\ capacity\ of\ cup,\ V = \pi\int_3^h (4y-8)dy + \pi\int_0^3 (3-\frac{y}{3})^2 dy$$

$$= \pi\int_3^h (4y-8)dy + \pi\int_0^3 (3-\frac{y}{3})^2 (-3)d(3-\frac{y}{3})$$

$$= \pi[2y^2 - 8y]_3^h + \pi[-(3-\frac{y}{3})^3]_0^3$$

$$= \pi(2h^2 - 8h + 25)$$

*** 用 disk method**

*** Along y-axis 定積分對應 dy**

$$= \pi\int_a^b [f(y)]^2 dy$$

*** 積分代入法**

CONT'D

2015 - SECTION B

$\text{When the volume of water} = 35\pi \ cm^3, \text{ let the depth of water be } h'$

$\text{Hence, } 35 = 2h'^2 - 8h' + 25 \rightarrow h'^2 - 4h' - 5 = 0$

$\rightarrow h' = 5 \text{ or } -1 \text{ (rejected)}$

$\text{Also, } V = \pi(2h^2 - 8h + 25) \rightarrow \frac{dV}{dt} = \pi(4h - 8)\frac{dh}{dt}$

$\rightarrow 24\pi = \pi(4h' - 8)\frac{dh}{dt}\Big|_{h=h'}$

$\rightarrow \frac{dh}{dt}\Big|_{h=h'} = 2$

$\therefore \text{The increasing rate of depth of water} = 2cms^{-1}$

* 用 implicit 微分法

* Chain rule

* Constant rate

* 單位係 cm s^{-1}