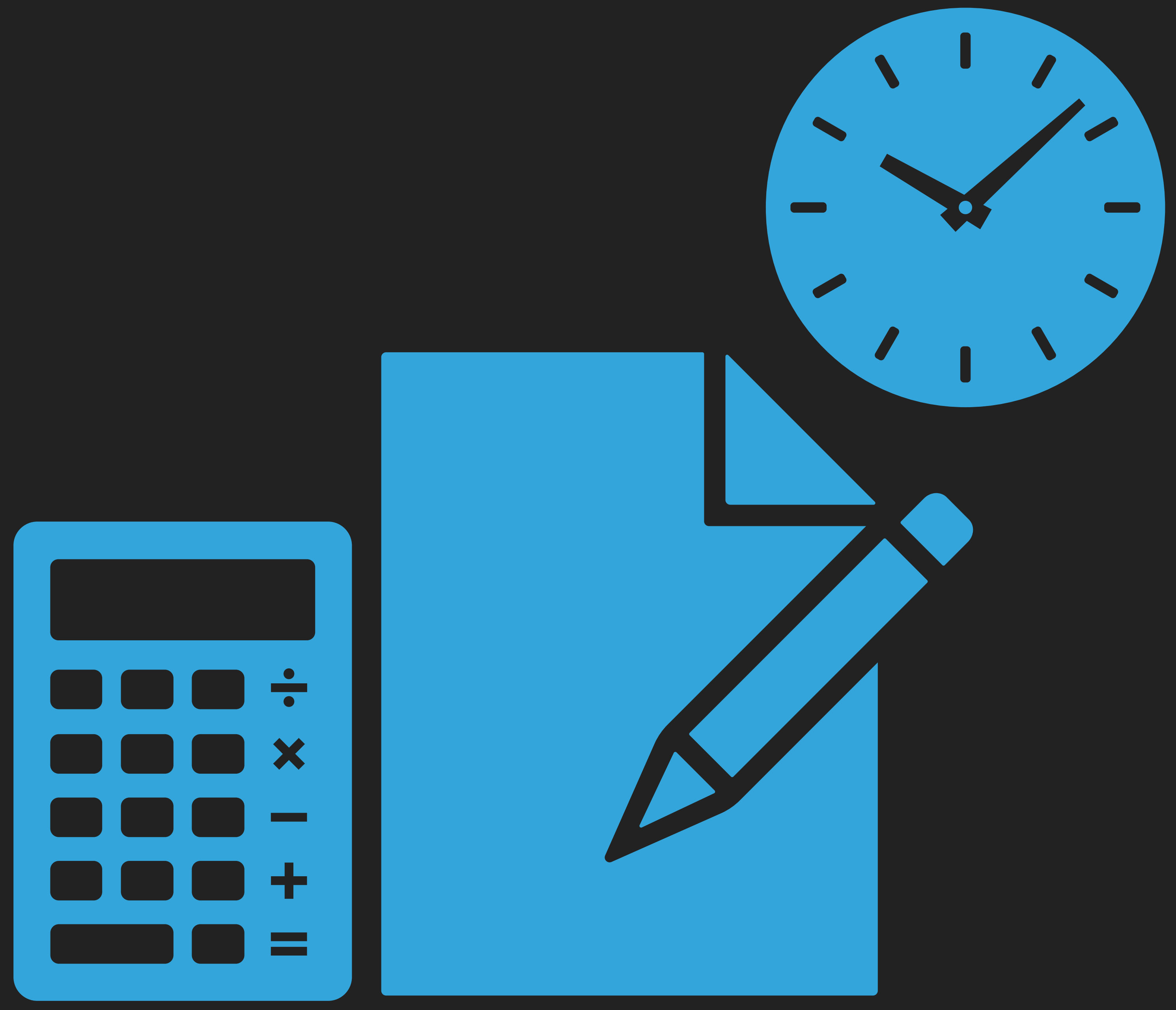
深宵教室 - DSE M2 模擬試題解答

2021

此為參考2021試題之模擬試題，原版請另行購買

2021

- Section A
- Section B



2021 - SECTION A

Q1.) $f(x) = \dfrac{1}{3x^2+4}$, $f'(x) = ?$ (By First Principles)

* 參考課程 3.1 及 3.2

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{3(x+h)^2+4}-\frac{1}{3x^2+4}\right) = \lim_{h\to 0}\frac{(3x^2+4)-(3(x+h)^2+4)}{h(3x^2+4)(3(x+h)^2+4)}$$

$$= \lim_{h\to 0}\frac{3(x^2-(x+h)^2)}{h(3x^2+4)(3(x+h)^2+4)} = \lim_{h\to 0}\frac{3(-\cancel{h})(2x+h)}{\cancel{h}(3x^2+4)(3(x+h)^2+4)}$$

$$= \lim_{h\to 0}\frac{-3(2x+h)}{(3x^2+4)(3(x+h)^2+4)}$$

$$= -\frac{6x}{(3x^2+4)^2}$$

* 微分定義

* $(a+b)(a-b) = a^2-b^2$

2021 - SECTION A

Q2.) Prove $\sum_{r=1}^{n}(3r^5+r^3)=\frac{n^3(n+1)^3}{2}, \forall n \in \mathbb{Z}^+$

* 參考課程 1.1 及 1.2

$$Let\ P(n):\ \sum_{r=1}^{n}(3r^5+r^3)=\frac{n^3(n+1)^3}{2}\ \forall n \in \mathbb{Z}^+$$

* 先 Let Statement

$$For\ P(1):\ L.H.S.=4=R.H.S.$$

* 証明 P(1) is true

$$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+,\ then\ P(k+1):$$

* 假設 P(k) is true. 証明 P(k+1) is true

$$L.H.S.=\sum_{r=1}^{k+1}(3r^5+r^3)$$

$$=\sum_{r=1}^{k}(3r^5+r^3)+3(k+1)^5+(k+1)^3$$

* 將末項抽出並改變末項

CONT'D

2021 - SECTION A

$$= \frac{k^3(k+1)^3}{2} + \frac{6(k+1)^5 + 2(k+1)^3}{2}$$

$$= \frac{(k+1)^3(k^3 + 6(k+1)^2 + 2)}{2}$$

$$= \frac{(k+1)^3(k^3 + 6k^2 + 12k + 8)}{2}$$

$$= \frac{(k+1)^3(k+2)^3}{2} = R.H.S.$$

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

i.e. By M.I., $P(n)$ *is true,* $\forall n \in \mathbb{Z}^+$

* 寫結論

2021 - SECTION A

$Q3.)\ (1-4x)^n = A + Bx + 240x^2 + \dots,\quad n = ?$

$Hence,\ (1-4x)^n(1+\frac{2}{x})^5 = A + Bx + Cx^3 + kx^4 \dots,\quad k = ?$

* 參考課程 1.1

$$(1-4x)^n \equiv \left(\sum_{r=0}^{n} C_r^n(-4x)^r\right)$$

$By\ compare\ coefficient\ of\ x^2$

$$240 = C_2^n(-4)^2 = 16\frac{n(n-1)}{2}$$

$$\rightarrow n^2 - n - 20 = 0 \rightarrow n = 6\ or\ n = -5\ (rejected)$$

$$\therefore n = 6$$

$$Then,\ (1-4x)^6(1+\frac{2}{x})^5 = \frac{1}{x^5}(1-4x)^6(x+2)^5$$

* **Binomial Expansion**

$* \ C_r^n = \frac{n!}{r!(n-r)!} \rightarrow C_1^n = n\ and\ C_2^n = \frac{n(n-1)}{2}$

* ■ 要 $x^{5+4=9}$ 的 **coefficient**

CONT'D

2021 - SECTION A

$$(1-4x)^6\left(1+\frac{2}{x}\right)^5 = \frac{1}{x^5}\left(\sum_{r}^{6} C_r^6(-4x)^r\right)\left(\sum_{r}^{5} C_r^5 2^{5-r}x^r\right)$$

$$\begin{aligned} \textit{Coefficient of } x^4 &= (C_4^6(-4)^4)(C_5^5) + (C_5^6(-4)^5)(C_4^5(2)^1) + (C_6^6(-4)^6)(C_3^5(2)^2) \\ &= 3840 - 61440 + 163840 \\ &= 106240 \end{aligned}$$

*** Binomial Expansion**

2021 - SECTION A

$Q4.)\ a.)\ Prove\ cos2x + cos4x + cos6x = 4cosxcos2xcos3x - 1$

$b.)\ Find\ x,\ for\ 0 \leq x \leq \frac{\pi}{2}\ if\ cos2x + cos8x + cos12x = -1$

* 參考課程 1.2, 2.1, 2.2 及 2.3

$$a.)\ Consider,\ cos2x + cos4x + cos6x - 4cosxcos2xcos3x = -1$$

$$\leftrightarrow 2cos3xcosx + cos6x - 4cosxcos2xcos3x = -1$$

$$\leftrightarrow cos6x + 2cosxcos3x(1 - 2cos2x) = -1$$

$$\leftrightarrow 2cos^2 3x - 1 + 2cosxcos3x(1 - 2cos2x) = -1$$

$$\leftrightarrow 2cos3x(cos3x + cosx - 2cos2xcosx) = 0$$

$$\leftrightarrow 2cos3x(2cos2xcosx - 2cos2xcosx) = 0$$

$$\leftrightarrow 0 = 0$$

$\therefore\ Prove\ is\ complete$

$$b.)\ cos2x + cos8x + cos12x = -1,\ (from\ a.)$$

$$\rightarrow 4cos2xcos4xcos6x - 1 = -1$$

* 雙向推論

* cos 複角公式

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

* cos 雙角公式

CONT'D

2021 - SECTION A

$$\rightarrow cos2xcos4xcos6x = 0$$

$$\rightarrow cos2x = 0 \ or \ cos4x = 0 \ or \ cos6x = 0$$

$$\because 0 \leq x \leq \frac{\pi}{2}$$

$$Hence, 0 \leq 2x \leq \pi, cos2x = 0 \rightarrow 2x = \frac{\pi}{2} \rightarrow x = \frac{\pi}{4}$$

$$0 \leq 4x \leq 2\pi, cos4x = 0 \rightarrow 4x = \frac{\pi}{2} \ or \ \frac{3\pi}{2} \rightarrow x = \frac{\pi}{8} \ or \ \frac{3\pi}{8}$$

$$0 \leq 6x \leq 3\pi, cos6x = 0 \rightarrow 6x = \frac{\pi}{2} \ or \ \frac{3\pi}{2} \ or \ \frac{5\pi}{2} \rightarrow x = \frac{\pi}{12} \ or \ \frac{3\pi}{12} \ or \ \frac{5\pi}{12}$$

$$\therefore x = \frac{\pi}{12} \ or \ \frac{\pi}{8} \ or \ \frac{\pi}{4} \ or \ \frac{3\pi}{8} \ or \ \frac{5\pi}{12}$$

* 留意角度範圍

2021 - SECTION A

$$Q5.)\ \ Define\ C : y = f(x), f(x) = \frac{x^3 - x^2 - 2x + 3}{(x-1)^2}, where\ x \neq 1$$

a.) *Find the asymptote(s)*

b.) *Find the number of pt. of inflexion.*

* 參考課程 3.3 及 3.5

a.) *Vertical Asymptote :* $x = 1$

* x 係幾多, 分母係零

Horizontal Asymptote : No horizontal asymoptote

* *Find* $\lim_{x\to\infty} y$

$$Consider, f(x) = \frac{x^2(x-1) - 2(x-1) + 1}{(x-1)^2}$$

$$= \frac{(x-1)(x^2-2)+1}{(x-1)^2}$$

$$= \frac{(x-1)([(x-1)+1]^2 - 2) + 1}{(x-1)^2}$$

CONT'D

2021 - SECTION A

$$= \frac{(x-1)([(x-1)+1]^2-2)+1}{(x-1)^2}$$

$$= \frac{(x-1)((x-1)^2+2(x-1)-1)+1}{(x-1)^2}$$

$$= \frac{(x-1)^3+2(x-1)^2-(x-1)+1}{(x-1)^2}$$

$$= (x-1)+2-\frac{1}{(x-1)}+\frac{1}{(x-1)^2}$$

$$Hence,\ \lim_{x\to\infty}[f(x)-(x+1)]=0$$

$$Oblique\ Asymptote:\ \ y=x+1$$

$$b.)\ f(x)=(x+1)-\frac{1}{x-1}+\frac{1}{(x-1)^2}$$

$$*\ Find\ m\ and\ c\ such\ that\ \lim_{x\to\infty}[y-(mx+c)]=0$$

CONT'D

2021 – SECTION A

$$\rightarrow f'(x) = 1 + \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}$$

$$\rightarrow f''(x) = -\frac{2}{(x-1)^3} + \frac{6}{(x-1)^4}$$

$$\rightarrow f''(x) = -\frac{2(x-4)}{(x-1)^4}$$

$For\ f''(x_0) = 0,\ there\ is\ only\ one\ solution\ x_0 = 4$

	x < 4	x = 4	x > 4
y''	+	0	-
y	Up.		Down.

$Hence,\ there\ is\ only\ one\ pt.\ of\ inflexion = (4, f(4))$

* 搵 pt. of inflexion = 搵 x_0 使度 $y''(x_0)=0$

* 利用表格計算 pt. of inflexion 附近情況

$f''(x) > 0 \rightarrow Concave\ upward$

$f''(x) < 0 \rightarrow Concave\ downward$

2021 - SECTION A

Q6.) Let a curve be $\Gamma : y = f(x), f(x) = e^{2x-6}$, L be the normal to Γ at $(3,1)$. Find the area of region bounded by Γ, L, and $x = c$, where $c = x -$ inteception of L.

* 參考課程 3.9, 3.10 及 3.11

$$L: \ y - 1 = \boxed{-\frac{1}{f'(3)}}(x-3) \rightarrow y = -\frac{1}{2e^{2(3)-6}}(x-3)+1$$

$$\rightarrow 2y = -x+5$$

$$Hence,\ c = 5$$

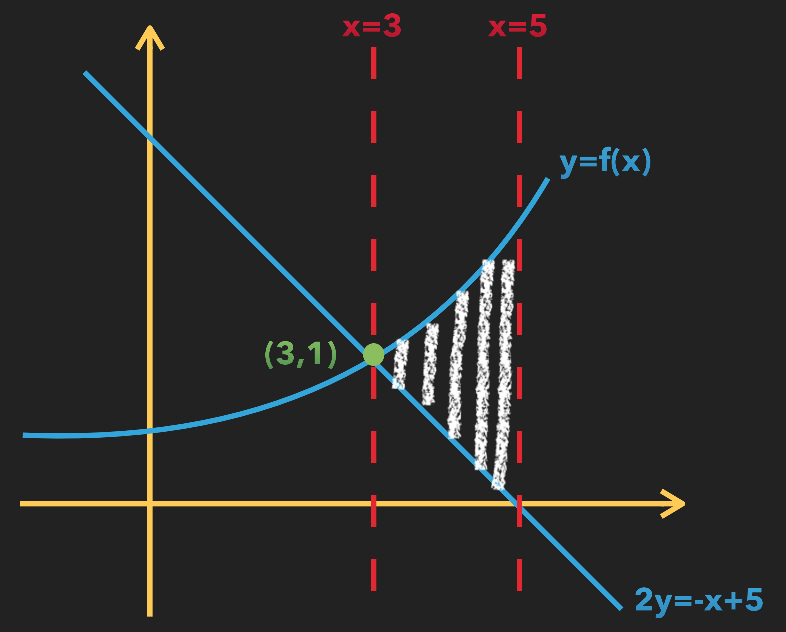
$$\therefore \ The\ bounded\ area = \int_3^5 f(x)dx - \frac{(5-3)(1)}{2}$$

$$= \left[\frac{e^{2x-6}}{2}\right]_3^5 - 1$$

$$= \frac{e^4-3}{2}\ sq.unit$$

* $\boxed{}$ Normal 與 tangent 互相垂直 斜率相乘 = -1

* Core 課程中直線方程: 點斜式



* 大面積減三角形面積

CONT'D

2021 - SECTION A

Q7.) Find the volume of the solid generated by revolving the region, $0 \le x \le 1$, *along* $x-axis$

$$G: \; y = \sqrt{x} ln(x^2+1)$$

* 參考課程 3.8, 3.10 及 3.12

$$The\ volume = \pi \int_0^1 y^2 dx = \pi \int_0^1 x[ln(x^2+1)]^2 dx$$

$$= \pi \int_0^1 [ln(x^2+1)]^2 d(\frac{1}{2}(x^2+1))$$

$$= \frac{\pi}{2}\int_1^2 [lnu]^2 du = \frac{\pi}{2}[u[lnu]^2]_1^2 - \frac{\pi}{2}\int_1^2 u \cdot 2lnu \cdot \frac{1}{u} du$$

$$= \pi[ln2]^2 - \pi \int_1^2 lnudu = \pi(ln2)^2 - \pi[ulnu]_1^2 + \pi \int_1^2 du$$

$$= \pi[(ln2)^2 - 2ln2 + 1]\ cu.unit$$

* 旋轉體積 (x-axis)= $\pi \int_a^b [f(x)]^2 dx$
* 用 Integration by part
* 用代入法, Let u=x²+1
* 定積分代入要改範圍
* 再用 Integration by part

2021 - SECTION A

Q8.)

$$(E)\begin{cases} x+(d-1)y+(d+3)z=4-d \\ 2x+(d+2)y-z=2d-5 \\ 3x+(d+4)y+5z=2 \end{cases}$$

Find d and solve (E), for (E) has inifinity many solution, $\exists d \in \mathbb{R}$

Is there a real solution for (E) satisfying $x(y+2z)=3$*? Explain your answer.*

* 參考課程 4.7

$$(E): AX=B$$

$$\text{where } A=\begin{pmatrix} 1 & d-1 & d+3 \\ 2 & d+2 & -1 \\ 3 & d+4 & 5 \end{pmatrix}, X=(x\ y\ z)^T, B=(4-d\ 2d-5\ 2)^T$$

$$\because\ (E) \text{ has many solution} \rightarrow |A|=0$$

* 如果 (E) 無唯一答案, |A|=0

CONT'D

2021 - SECTION A

$$\rightarrow \begin{vmatrix} 1 & d-1 & d+3 \\ 2 & d+2 & -1 \\ 3 & d+4 & 5 \end{vmatrix} = 0 \rightarrow \begin{vmatrix} 1 & d-1 & d+3 \\ 2 & d+2 & -1 \\ 1 & 2 & 6 \end{vmatrix} = 0$$

* R3=R3-R2
* C2=C2-C3

$$\rightarrow \begin{vmatrix} 1 & -4 & d+3 \\ 2 & d+3 & -1 \\ 1 & -4 & 6 \end{vmatrix} = 0 \rightarrow \begin{vmatrix} 0 & 0 & d-3 \\ 2 & d+3 & -1 \\ 1 & -4 & 6 \end{vmatrix} = 0$$

* R1=R1-R3
* R2=R2-2R3

$$\rightarrow \begin{vmatrix} 0 & 0 & d-3 \\ 0 & d+11 & -13 \\ 1 & -4 & 6 \end{vmatrix} = 0 \rightarrow -(d+11)(d-3) = 0 \rightarrow d = -11 \text{ or } 3$$

* |A|=對角線相乘 x -1

$For\ d = -11,$

* 要 check 係咪 no solution

$$(E) \left(\begin{array}{ccc|c} 1 & -12 & -8 & 15 \\ 2 & -9 & -1 & -27 \\ 3 & -7 & 5 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -12 & -8 & 15 \\ 0 & 15 & 15 & -57 \\ 0 & 29 & 29 & -43 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -12 & -8 & 15 \\ 0 & 15 & 15 & -57 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

CONT'D

2021 - SECTION A

$\therefore (E)\ is\ inconsistent\ for\ d = -11$

* 最後出 0x+0y+0z=1

$For\ d = 3$

$$(E)\left(\begin{array}{ccc|c}1 & 2 & 6 & 1\\ 2 & 5 & -1 & 1\\ 3 & 7 & 5 & 2\end{array}\right)\sim\left(\begin{array}{ccc|c}1 & -12 & -8 & 1\\ 0 & 1 & -13 & -1\\ 0 & 1 & -13 & -1\end{array}\right)\sim\left(\begin{array}{ccc|c}1 & -12 & -8 & 1\\ 0 & 1 & -13 & -1\\ 0 & 0 & 0 & 0\end{array}\right)$$

$\therefore (E)\ has\ many\ solution\ for\ d = 3$

* 最後出 0x+0y+0z=0

$$(E)\sim\left(\begin{array}{ccc|c}1 & -12 & -8 & 1\\ 0 & 1 & -13 & -1\end{array}\right)$$

* 三條公式變兩條

$$Let\ x = t, \forall t \in \mathbb{R},\quad (x, y, z) = \left(t, \frac{7-13t}{32}, \frac{3-t}{32}\right), \forall t \in \mathbb{R}$$

* x 做 t 方便下面計算

$$Then,\ x(y+2z) = 3 \rightarrow \frac{t}{32}(7 - 13t + 6 - 2t) = 3$$

$$\rightarrow 15t^2 - 13t + 96 = 0\ \ -(*)$$

CONT'D

2021 - SECTION A

$\because \Delta = 13^2 - 4(15)(96) = -5591 < 0$

$\therefore (*)$ *has no real root*

i.e. There is no real solution for (E) satisfying $x(y + 2z) = 3$?

* 判別式決定有無根

2021 - SECTION B

$$Q9.)\ a.)\ \int sec^3xdx = ?,\ for\ -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$b.)\ Assume\ g(x) + g(-x) = 1\ and\ h(x) = h(-x),\ \forall x \in \mathbb{R},\ show\ that$$

$$\int_{-a}^{a} g(x)h(x)dx = \int_{0}^{a} h(x)dx$$

$$c.)\ \int_{-1}^{1} \frac{3^x x^2}{(3^x + 3^{-x})\sqrt{x^2+1}}dx = ?$$

* 參考課程 2.2, 3.8, 3.9, 3.10 及 3.11

$$a.)\ \int sec^3xdx = \int secx\,sec^2xdx = \int secx\,d(tanx)$$

$$= secxtanx - \int secx\,tan^2xdx = secxtanx - \int secx(sec^2x - 1)dx$$

* 用 Integration by part

* $tan^2x = sec^2x - 1$

CONT'D

2021 - SECTION B

$$\rightarrow \int sec^3xdx = secxtanx - \int sec^3xdx + \int secxdx$$

$$\rightarrow 2\int sec^3xdx = secxtanx + \int secxdx$$

$$\rightarrow 2\int sec^3xdx = secxtanx + \int secx\frac{secx + tanx}{secx + tanx}dx$$

$$\rightarrow 2\int sec^3xdx = secxtanx + \int \frac{1}{secx + tanx}d(secx + tanx)$$

$$\rightarrow \int sec^3xdx = \frac{1}{2}(secxtanx + ln\,|\,secx + tanx\,|\,) + C$$

$$, where\ C\ is\ constant$$

* 無中生有, 典型 secx 積分做法

CONT'D

2021 - SECTION B

$$b.)\ Let\ I_1 = \int_{-a}^{a} g(x)h(x)dx$$

$$Then,\ by\ substitution\ with\ u = -x, I_1 = \int_{a}^{-a} g(-u)h(-u)d(-u)$$

$$\rightarrow I_1 = \int_{-a}^{a} (1 - g(x))h(x)dx = \int_{-a}^{a} h(x)dx - I_1$$

$$\rightarrow 2I_1 = \int_{0}^{a} h(x)dx + \int_{-a}^{0} h(x)dx = \int_{0}^{a} h(x)dx + \int_{a}^{0} h(-x)d(-x)$$

$$\rightarrow 2I_1 = \int_{0}^{a} h(x)dx + \int_{0}^{a} h(x)dx$$

$$\rightarrow I_1 = \int_{0}^{a} h(x)dx$$

* 用代入法, Let u=-x
* 定積分代入要改範圍
* 負數定積分範圍上下倒轉
* 定積分範圍可以析開分開積

CONT'D

2021 - SECTION B

$$c.)\ By\ using\ b.),\ g(x) = \frac{3^x}{3^x + 3^{-x}},\ and\ h(x) = \frac{x^2}{\sqrt{x^2+1}}\ with\ a = 1$$

$$Obviously,\ g(x) + g(-x) = 1\ and\ h(x) = h(-x)$$

$$Hence,\ \int_{-1}^{1} g(x)h(x)dx = \int_0^1 \frac{x \cdot x}{\sqrt{x^2+1}}dx = \int_0^1 x d(\sqrt{x^2+1})$$

$$= [x\sqrt{x^2+1}]_0^1 - \int_0^1 \sqrt{x^2+1}dx$$

$$= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sqrt{tan^2y+1}\,sec^2y dy = \sqrt{2} - \int_0^{\frac{\pi}{4}} secy \cdot sec^2y dy$$

$$= \sqrt{2} - \frac{1}{2}[\sqrt{2} + ln(\sqrt{2}+1)] = \frac{1}{2}[\sqrt{2} - ln(\sqrt{2}+1)]$$

* 要 check g(x) 同 h(x) 是否符合

* 用 Integration by part

* 用三角代入法, x=tany

* $tan^2x + 1 = sec^2x$

2021 - SECTION B

$Q10.)$ *Given a curve* $C_1: \ y = \sqrt{x^2+36}$, $C_2: y = -\sqrt{(20-x)^2+16}$, *Assume* $P=(u,v)$ *moving along curve* C_1, *with a vertical line passing through* P *cuts* C_2 *at* Q *Meanwhile, there is a horizontal line passing* P *cut at* $y-axis$ *at* R *and a horizontal line passing through* Q *cut* $y-axis$ *at* S. *When* $u=a$, PQ *attains a minimum value.*

a.) Find a. *Is the area of* $PQRS$ *minimum when* $u=a$? *Explain your answer.*

b.) OP *increases at a rate 28 unit per min. Find the rate of change of the perimeter of* $PQRS$ *when* $u=a$ *and* $O=(0,0)$

* 參考課程 3.3 及 3.4

$$a.)\ PQ = \sqrt{u^2+36} + \sqrt{(20-u)^2+16},\ where\ 0<u<20$$

$$\rightarrow \frac{dPQ}{du} = \frac{u}{\sqrt{u^2+36}} - \frac{(20-u)}{\sqrt{(20-u)^2+16}}$$

$$\rightarrow 0 = \frac{a}{\sqrt{a^2+36}} - \frac{(20-a)}{\sqrt{(20-a)^2+16}}$$

* a=turning pt., y'(a)=0

CONT'D

2021 - SECTION B

$$\rightarrow a\sqrt{(20-a)^2+16} = (20-a)\sqrt{a^2+36}$$

$$\rightarrow a^2[(20-a)^2+16] = (20-a)^2(a^2+36)$$

$$\rightarrow 16a^2 = 36(20-a)^2 \rightarrow 16a^2 - 36(20-a)^2 = 0$$

$$\rightarrow [4a+6(20-a)][4a-6(20-a)] = 0$$

$$\rightarrow a = 12 \text{ or } 60 \ (\text{rejected} \because \text{ out of range})$$

	0 < u <12	u =12	12 < u < 20
PQ'	-	0	+
PQ	Decreasing		Increasing

$\therefore a = 12$

$$\text{The area of } PQRS,\ A = u \times PQ \rightarrow \frac{dA}{du} = \frac{d(PQ)}{du} + PQ$$

* $(a+b)(a-b) = a^2 - b^2$

*** 利用表格計算 turning pt. 附近情況**

$f'(x) > 0 \rightarrow \text{increasing}$

$f'(x) < 0 \rightarrow \text{decreasing}$

CONT'D

2021 - SECTION B

$$\rightarrow \frac{dA}{du}\Big|_{u=a} = PQ(a) > 0$$

$\therefore$ *The area of PQRS is not minimum when* $u = a$

* a 唔係 turning pt.

The perimeter of PQRS, $L = 2(u + PQ)$

where, $OP^2 = u^2 + (u^2 + 36) = 2(u^2 + 18)$

$$\rightarrow 2OP \cdot \frac{dOP}{dt} = 4\frac{du}{dt} \rightarrow u\frac{du}{dt} = 14OP \rightarrow \frac{du}{dt}\Big|_{u=12} = \frac{14(18)}{12} = 21$$

$$Hence, \frac{dL}{dt} = 2\left(\frac{du}{dt} + \boxed{\frac{dPQ}{dt}}\right) = 2\left(\frac{du}{dt} + \boxed{\frac{dPQ}{du}\frac{du}{dt}}\right)$$

$$\frac{dL}{dt}\Big|_{u=12} = 2\frac{du}{dt}\Big|_{u=12}\left(1 + \cancel{\frac{dPQ}{du}}\Big|_{u=12}\right) = 2\frac{du}{dt}\Big|_{u=12} = 42 \text{ unit per min}$$

* Chain Rule

2021 - SECTION B

Q11.) Define $P = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$, where $\frac{\pi}{2} < \theta < \pi$, $A = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbb{R}$

a.) Find PAP^{-1}

b.) Let $B = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$, Find θ such that $PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda, \mu \in \mathbb{R}$

Hence, find B^n and $(B^{-1})^{555}$, $\forall n \in \mathbb{Z}^+$

* 參考課程 2.1, 2.2, 2.3, 4.9, 4.10 及 4.11

a.) $P^{-1} = \frac{1}{|P|}\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}^T = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$

* 用 adj. matrix 搵 P^{-1}

$\therefore PAP^{-1} = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}\begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$

CONT'D

2021 - SECTION B

$$=\begin{pmatrix}\alpha sin\theta+\beta cos\theta & \beta sin\theta-\alpha cos\theta\\ -\alpha cos\theta+\beta sin\theta & -\beta cos\theta-\alpha sin\theta\end{pmatrix}\begin{pmatrix}sin\theta & -cos\theta\\ cos\theta & sin\theta\end{pmatrix}$$

$$=\begin{pmatrix}-\alpha(cos^2\theta-sin^2\theta)+\beta 2sin\theta cos\theta & -\alpha 2sin\theta cos\theta-\beta(cos^2\theta-sin^2\theta)\\ -\alpha 2sin\theta cos\theta-\beta(cos^2\theta-sin^2\theta) & \alpha(cos^2\theta-sin^2\theta)-\beta 2sin\theta cos\theta\end{pmatrix}$$

$$=\begin{pmatrix}-\alpha cos2\theta+\beta sin2\theta & -\alpha sin2\theta-\beta cos2\theta\\ -\alpha sin2\theta-\beta cos2\theta & \alpha cos2\theta-\beta sin2\theta\end{pmatrix}$$

$$=\begin{pmatrix}-\alpha cos2\theta+\beta sin2\theta & -\alpha sin2\theta-\beta cos2\theta\\ -\alpha sin2\theta-\beta cos2\theta & \alpha cos2\theta-\beta sin2\theta\end{pmatrix}$$

$b.)$ With $\alpha=1,\ \beta=\sqrt{3},$

$$\therefore PBP^{-1}=\begin{pmatrix}-cos2\theta+\sqrt{3}sin2\theta & -sin2\theta-\sqrt{3}cos2\theta\\ -sin2\theta-\sqrt{3}cos2\theta & cos2\theta-\sqrt{3}sin2\theta\end{pmatrix}=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$$

* sin 雙角公式
* cos 雙角公式

CONT'D

2021 - SECTION B

$$Hence,\ -sin2\theta - \sqrt{3}cos2\theta = 0$$

$$\rightarrow -sin2\theta = \sqrt{3}cos2\theta \rightarrow tan2\theta = -\sqrt{3}$$

$$Given\ that\ \frac{\pi}{2} < \theta < \pi \rightarrow \pi < 2\theta < 2\pi$$

$$Hence,\ 2\theta = 2\pi - \frac{\pi}{3} \rightarrow \theta = \frac{5\pi}{6}$$

$$Then,\ PBP^{-1} = \begin{pmatrix} -cos(\frac{5\pi}{3}) + \sqrt{3}sin(\frac{5\pi}{3}) & 0 \\ 0 & cos(\frac{5\pi}{3}) - \sqrt{3}sin(\frac{5\pi}{3}) \end{pmatrix}$$

$$\rightarrow PB^nP^{-1} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}^n \rightarrow B^n = P^{-1}\begin{pmatrix} (-2)^n & 0 \\ 0 & 2^n \end{pmatrix}P$$

* 注意角度範圍

* 答案响第四象限

CONT'D

2021 - SECTION B

$$\rightarrow B^n = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} 2^n \begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

$$\rightarrow B^n = 2^{n-2}\begin{pmatrix} (-1)^n & \sqrt{3} \\ (-1)^{n+1}\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

$$\rightarrow B^n = 2^{n-2}\begin{pmatrix} (-1)^n + 3 & (-1)^{n+1}\sqrt{3} + \sqrt{3} \\ (-1)^{n+1}\sqrt{3} + \sqrt{3} & (-1)^n \cdot 3 + 1 \end{pmatrix}$$

$$B^{555} = 2^{553}\begin{pmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} = 2^{554}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

* 先抽公倍數方便運算

CONT'D

2021 - SECTION B

$$\rightarrow (B^{555})^{-1} = 2^{-554} \cdot \frac{1}{-4}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

$$\rightarrow (B^{-1})^{555} = 2^{-556}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

* $(kA)^{-1}=(1/k)A^{-1}$
* 用 adj. matrix 搵 $(B^{555})^{-1}$

* $(B^{-1})^{555}=(B^{555})^{-1}$

2021 - SECTION B

Q12.) Assume $s, t \in \mathbb{R}$, such that

$$\overrightarrow{OA} = t\hat{i} + 14\hat{j} + s\hat{k},\ \overrightarrow{OB} = 12\hat{i} - s\hat{j} - 2\hat{k}$$

$$\overrightarrow{OC} = (s+2)\hat{i} - 16\hat{j} + 10\hat{k},\ \overrightarrow{OD} = -t\hat{i} + (s+2)\hat{j} + 14\hat{k}$$

Given that $\overrightarrow{AB} \,//\, 5\hat{i} - 4\hat{j} - 2\hat{k}$

a.) Find s, t, the area ΔABC, the volume of tetrahedron ABCD and the shortest distance between point D and the plane ΔABC

b.) Is the circumcenter of ΔABC is the projection of D on the plane ΔABC? Explain your answer.

* 參考課程 4.4 及 4.5

a.) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = k(5\hat{i} - 4\hat{j} - 2\hat{k})$, where $\exists k \in \mathbb{R}$

$\rightarrow (12-t)\hat{i} - (s+14)\hat{j} - (s+2)\hat{k} = k(5\hat{i} - 4\hat{j} - 2\hat{k})$

* If $\vec{a} \,//\, \vec{b}$, $\vec{a} = k\vec{b}$

CONT'D

2021 - SECTION B

$$\begin{cases} 12 - t = 5k & \text{——— (1)} \\ s + 14 = 4k & \text{——— (2)} \\ s + 2 = 2k & \text{——— (3)} \end{cases}$$

$$\frac{(2)}{(3)} : \frac{s+14}{s+2} = 2 \rightarrow s = 10 \rightarrow In\ (2) :\ k = 6$$

$$(1) : 12 - t = 5(6) \rightarrow t = -18$$

$$The\ area\ \Delta ABC,\ A_1 = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2}|(\overrightarrow{OB} - \overrightarrow{OA}) \times (\overrightarrow{OC} - \overrightarrow{OA})|$$

$$= \frac{1}{2}|(30\hat{i} - 24\hat{j} - 12\hat{k}) \times (30\hat{i} - 30\hat{j})|$$

$$= \frac{1}{2}|-180(2\hat{i} + 2\hat{j} + \hat{k})| \quad = 270\ sq.unit$$

* $A\overrightarrow{i} + B\hat{j} + C\hat{k} = D\overrightarrow{i} + E\hat{j} + F\hat{k}$

  $A = D, B = E, C = F$

* 三角形面積=1/2平行四邊形面積

* $Cross\ Product = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 30 & -24 & -12 \\ 30 & -30 & 0 \end{vmatrix}$

CONT'D

2021 - SECTION B

$$\text{The volume of tetrahedron } ABCD,\ V_1 = \frac{1}{6}\left|(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}\right|$$

$$= \frac{1}{6}\left|[-180(2\hat{i}+2\hat{j}+\hat{k})]\cdot(\overrightarrow{OD}-\overrightarrow{OA})\right|$$

$$= 30(2\hat{i}+2\hat{j}+\hat{k})\cdot(36\hat{i}-2\hat{j}+4\hat{k}) = 2160\ cu.\ unit$$

* 四面體體積 = 1/6平行六面體體積

The shortest distance from D to ΔABC

$$= \text{the height of tetrahedron } ABCD,\ H_1 = \boxed{\frac{3V_1}{A_1}} = 24\ unit$$

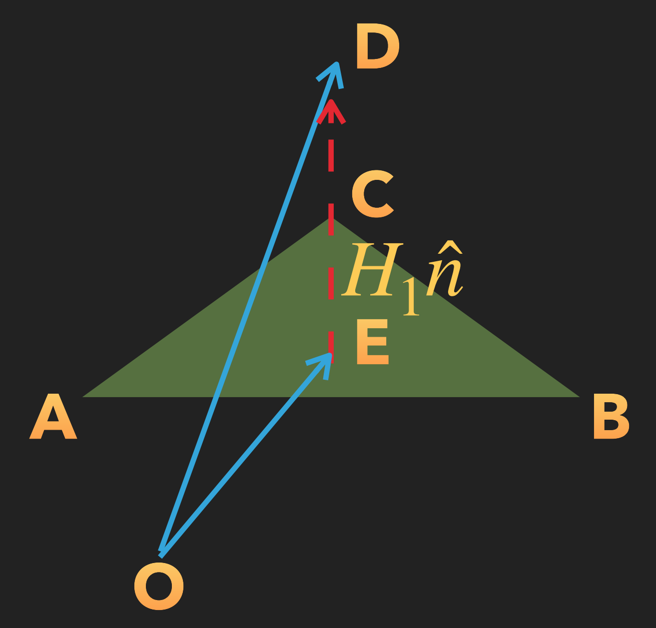
* 四面體體積=1/3 (三角底面積) (高)

b.) Let E = projection of D on the plane ΔABC.

Then $\overrightarrow{ED} = H_1\hat{n}$, *where* $\hat{n}$ = *the unit normal vector of* ΔABC

$$\overrightarrow{OE} = \overrightarrow{OD} - \overrightarrow{ED} = \overrightarrow{OD} - H_1\hat{n}$$

$$\text{Consider, } \overrightarrow{AB}\cdot\overrightarrow{CE} = \overrightarrow{AB}\cdot(\overrightarrow{OD} - H_1\hat{n} - \overrightarrow{OC})$$



CONT'D

2021 - SECTION B

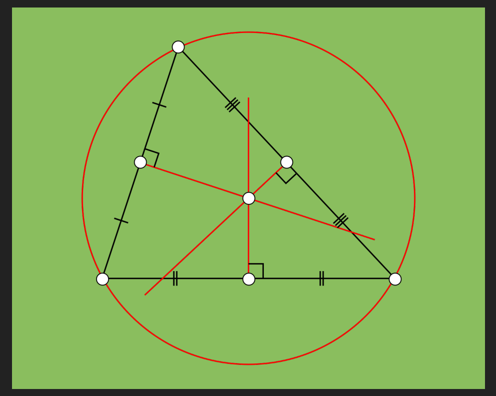
$$= \overrightarrow{AB} \cdot \overrightarrow{CD} = (30\hat{i} - 24\hat{j} - 12\hat{k}) \cdot (6\hat{i} + 28\hat{j} + 4\hat{k})$$
$$= -540 \neq 0$$

$$\therefore \overrightarrow{AB} \textit{ does not } \perp \overrightarrow{CE}$$

i.e. The circumcenter of ΔABC

$\neq$ the projection of D on the plane ΔABC

*** AB 同 normal vector 係互相垂直**
所以佢地 dot product = 0



外心 (Circumcenter)