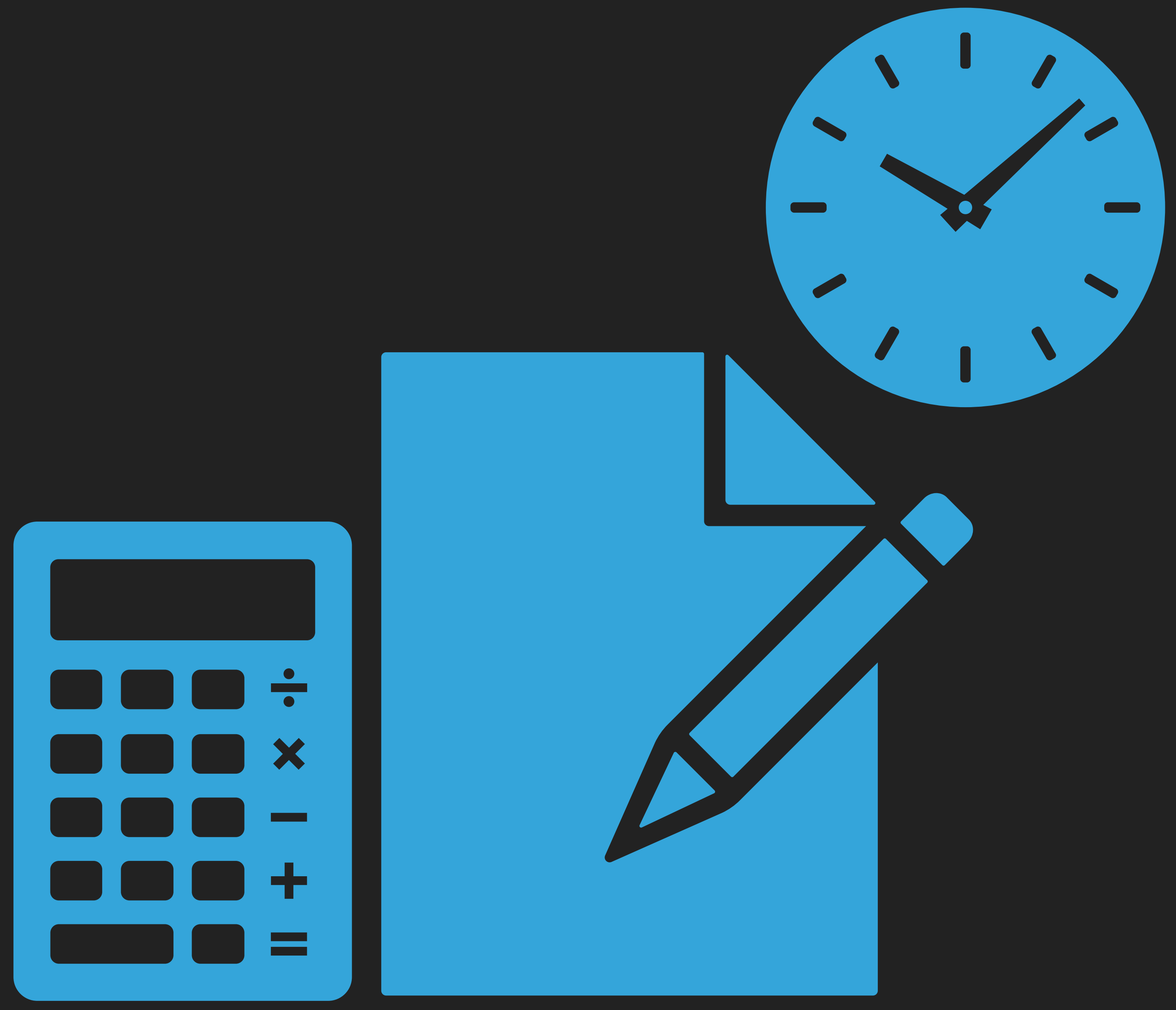
深宵教室 - DSE M1 模擬試題解答

2014

此為參考2014試題之模擬試題，原版請另行購買

2014

- Section A
- Section B



2014 - SECTION A

Q1.) The volume of a spherical balloon reduces at a constant rate $100cm^3/s$.
Find the rate of change of the radius of the ballon at the radius $= 10cm$.

* 參考課程 2.5 及 2.7

Let r be the radius of the balloon .

V(r) be the function volume of the balloon of r

$$\boxed{V(r) = \frac{4}{3}\pi r^3} \to \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \to \frac{dV}{dt}\Big|_{r=10} = 400\pi \frac{dr}{dt}\Big|_{r=10}$$

$$\to \frac{dr}{dt}\Big|_{r=10} = \frac{\boxed{-100}}{400\pi} = -\frac{1}{4\pi}\ cm/s$$

* 球體體積 = 4/3 π (半徑)³

* 減少為負值

2014 - SECTION A

Q2.) Given that $f(x)=\dfrac{x^x}{(2x+13)^6}$, $x>1$. *Show that* $f(x)$ *is increasing function.*

* 參考課程 2.5, 2.6 及 2.7

$$ln[f(x)] = \boxed{xlnx} - 6ln(2x+13)$$

$$\rightarrow \frac{f'(x)}{f(x)} = \boxed{lnx+1} - \frac{12}{2x+13} \rightarrow f'(x) = f(x)[lnx + \frac{2x+1}{2x+13}]$$

$$For\ x>1, f(x)>0\ and\ lnx + \frac{2x+1}{2x+13} > 0$$

$$Hence, f'(x)>0\ for\ x>1$$

$$\therefore f(x)\ is\ the\ increasing\ function\ for\ x>1$$

* ln(A/B) = lnA - lnB

* $\ln A^n = n\ln A$

* ln微分法

* $\boxed{}$ Product rule

* $f'(x)>0 \rightarrow increasing$

2014 - SECTION A

Q3.) Let the curve $C: y = f(x)$ *and* $f'(x) = (2x - \frac{1}{x})^3$. *Given that a point* $P = (1, 5)$ *lie on* C

Find the equation of tangent to C at P and the equation of C.

* 參考課程 1.1, 2.4 及 2.7

Let L be the tangent to C at P

$L: \ y - 5 = f'(1)(x - 1) \rightarrow y - 5 = x - 1 \rightarrow y = x + 4$

* 直線方程, 點斜式

$$f'(x) = (2x)^3 - 3(2x)^2(\frac{1}{x}) + 3(2x)(\frac{1}{x})^2 - (\frac{1}{x})^3$$

$$= 8x^3 - 12x + 6x^{-1} - x^{-3}$$

* $(a+b)^n = \sum_{r=0}^{n} C_r^n a^r b^{n-r}$

$$\rightarrow f(x) = \int 8x^3 - 12x + 6x^{-1} - x^{-3} dx = 2x^4 - 6x^2 + 6lnx + \frac{1}{2x^2} + C$$

* 積分係類似微分逆函數

where C is constant

Given that $f(1) = 5 \rightarrow C = \frac{17}{2}$

Hence, $C: \ y = 2x^4 - 6x^2 + 6lnx + \frac{1}{2x^2} + \frac{17}{2}$

2014 - SECTION A

Q4.) Find $\displaystyle\int_1^3 \frac{x^2+3x+9}{x^2+4x+11}dx$.

* 參考課程 2.8

$$Let\ I = \int_1^3 \frac{(x^2+4x+11)-(x+2)}{x^2+4x+11}dx$$

$$= \int_1^3 1 - \frac{x+2}{x^2+4x+11}dx = [x]_1^3 - \frac{1}{2}\int_{16}^{32} u^{-1}du$$

$$where\ u = x^2+4x+11 \rightarrow du = 2(x+2)dx$$

$$= 2 - \frac{1}{2}ln\frac{32}{16} = 2 - \frac{1}{2}ln2$$

* 積分三寶: Partial fraction

* 積分三寶: 積分代入, u=x²+4x+11

* 定積分代入要改範圍

2014 - SECTION A

Q5.) The initial population of country is 8 million. Given that the rate of change of population

$$= \frac{t\sqrt{9-t^2}}{3}, (0 \le t \le 3), \textit{ where } t \textit{ is the time in year. Find the population in term of } t.$$

* 參考課程 2.7 及 2.8

Let $x(t)$ be the population of a country (in million)

方法1

$$x'(t) = \frac{t\sqrt{9-t^2}}{3} \rightarrow x(t) = \int \frac{t\sqrt{9-t^2}}{3} dt$$

Let $u = 9 - t^2$, $du = -2tdt$

$$x(t) = -\frac{1}{6}\int \sqrt{u}du = -\frac{1}{9}(9-t^2)^{\frac{3}{2}} + C, \textit{ where } C \textit{ is constant.}$$

$$\because x(0) = 8 \rightarrow C = 11$$

$$\therefore x(t) = -\frac{1}{9}(9-t^2)^{\frac{3}{2}} + 11$$

* 積分係類似微分逆函數

* 積分三寶: 積分代入, u=9-t²

CONT'D

2014 - SECTION A

方法2

$$x(t) = x(0) + \int_0^t \frac{u\sqrt{9-u^2}}{3}du$$

$$Let\ z = 9 - u^2,\ dz = -2udu$$

$$x(t) = 8 - \frac{1}{6}\int_9^{9-t^2} \sqrt{z}dz = 8 - [\frac{1}{9}(z)^{\frac{3}{2}}]_9^{9-t^2}$$

$$= -\frac{1}{9}(9-t^2)^{\frac{3}{2}} + 11$$

* x(t)=x(0) + 由 0 到 t 的差

* 積分三寶: 積分代入, u=9-t²

* 定積分代入要改範圍

2014 - SECTION A

Q6.) Let X be discrete random vaiable with $E(X) = 3.4$ *and*

k	a	0	4	6
$P(X=k)$	0.1	0.2	0.3	0.4

a.) Find a and $Var(3-4X)$

a.) Let G and H be the event of $X < 4$ *and* $X \geq -1$ *respectively. Find* $P(G \cap H)$.

* 參考課程 4.1, 4.3 及 4.4

a.) Given that $\sum_{i=1}^{4} k_i P(X = k_i) = 3.4 \rightarrow a = -2$

$Var(3-4X) = 4^2 Var(X) = 16(E(X^2) - [E(X)]^2)$

$= 16(\sum_{i=1}^{4} k_i^2 P(X = k_i) - (3.4)^2) = 128.64$

b.) $P(G \cap H) = P(-1 \leq X < 4)$

$= P(X = 0) = 0.2$

* $E(X) = \sum_{i=1}^{n} k_i P(X = k_i)$

* $Var(aX + b) = a^2 Var(X)$

* $Var(X) = E(X^2) - [E(X)]^2$

2014 - SECTION A

Q7.) Let A and B be the event such that $P(A|B) = 0.4$, $P(A \cup B) = 0.45$, and $P(B') = 0.75$ where B' is the complementary event of B

a.) Find $P(A \cap B)$ and $P(A)$

b.) Are A and B independent? Explain your answer.

* 參考課程 4.1 及 4.2

$$a.)\ P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow 0.4 = \frac{P(A \cap B)}{1 - P(B')} \rightarrow P(A \cap B) = 0.1$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\rightarrow P(A) = P(A \cup B) - [1 - P(B')] + P(A \cap B) = 0.3$$

$$b.)\ \because P(A|B) \neq P(A)$$

$$\therefore A \text{ and } B \text{ are not independent.}$$

* P(Not B)=1-P(B)

* 如果獨立事件, P(A|B)=P(A)

2014 – SECTION A

Q8.) In school, it is known that 4 % student wearing glasses and there are 2 % male students wearing glasses. Among the not wearing glasses students, $\frac{2}{3}$ of them are female. One student is randomly selected from the school.

a.) Find the probability the student is female and not wearing glasses.

b.) Find the probability the student is male.

c.) If the student is female, find the probability the student is not wearing glasses.

* 參考課程 4.1 及 4.2

Let M be the event of the selected student is male

F be the event of the selected student is female

G be the event of the selected student is wearing glasses

$$\text{Given that } P(G) = 0.04,\ P(G|M) = 0.02,\ P(F|\bar{G}) = \frac{2}{3}$$

$$a.)\ P(F \cap \bar{G}) = P(F|\bar{G})P(\bar{G}) = \frac{2}{3}(1 - 0.04) = 0.64$$

* 有 (1-0.04) 學生無眼鏡
無帶眼鏡中有 2/3 係女仔

CONT'D

2014 - SECTION A

$$b.)\ P(\bar{G}) = P(\bar{G}\,|\,F)P(F) + P(\bar{G}\,|\,M)P(M)$$

$$\rightarrow 1 - P(G) = P(\bar{G} \cap F) + (1 - P(G\,|\,M))P(M)$$

$$\rightarrow 0.96 = 0.64 + 0.98P(M)$$

$$\rightarrow P(M) = \frac{16}{49}$$

$$c.)\ P(\bar{G}\,|\,F) = \frac{P(\bar{G} \cap F)}{P(F)} = \frac{0.64}{1 - P(M)} = \frac{784}{825}$$

* P(A)=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+...
* P(Not A) = 1-P(A)

2014 - SECTION A

9.)

a.) 200 students had been selected and found out that 80 of them had private teacher. Let p be the proportion of the students who had private teacher. Fina an approximate 95 % confident interval for p.

b.) A private school randomly selected n students to join their class. The probability the student will join is 0.85. Let X be the proportion of n invited students joining the class. Find the max. of n such that the probability of more than 100 invited students will join the class is less than 0.05.

* 參考課程 4.5, 4.6 及 4.7

a.) Let p_s be the proportion of the random sample of 200, $p_s = 0.4$

For 95 % C.I., $P(-\alpha < \boxed{\dfrac{p_s - p}{\sqrt{p_s(1-p_s)/200}}} < \alpha) = 90\% \rightarrow \alpha = 1.96$

$$\rightarrow p = \left(p_s - 1.625 \cdot \frac{\sqrt{p_s(1-p_s)}}{\sqrt{200}},\ p_s + 1.625 \cdot \frac{\sqrt{p_s(1-p_s)}}{\sqrt{200}}\right)$$

$\rightarrow p = (0.3321, 0.4679)$ (to 4 d.p.)

* ■ $B(200, p) \rightarrow N(p, \dfrac{p(1-p)}{200})$

* 當樣本足夠大, 可用樣本標準差

CONT'D

2014 - SECTION A

b.) By Normal Distribution Approximation,

$$X \sim N(0.85, \frac{0.85(1-0.85)}{n}) \sim N(0.85, \frac{0.1275}{n})$$

$$Then,\ P(X > \frac{100}{n}) < 0.05 \rightarrow P(Z > \frac{\frac{100}{n} - 0.85}{\sqrt{\frac{0.1275}{n}}}) < 0.05$$

$$From\ normal\ distribution\ table, \frac{\frac{100}{n} - 0.85}{\sqrt{\frac{0.1275}{n}}} > 1.645$$

$$\rightarrow 0.85(\sqrt{n})^2 + 1.645\sqrt{0.1275}\sqrt{n} - 100 < 0$$

$$\rightarrow 0 < \sqrt{n} < 10.5065 \rightarrow 0 < n < 110.4\ (\because\ n > 0)$$

$$\therefore The\ max.\ number = 110$$

* 當樣本足夠大, 二元分佈可用常態分佈

* 先計算 Z ~ N(0,1), 再對表

* 二次不等式

2014 - SECTION B

$Q10.)$ *Let the curve* $C : y = f(x)$, *where* $f(x) = \dfrac{lnx}{x^2}$

a.) Find the bounded area of C and the x − axis from x = 1 to x = 2

b.) By the trapezoidal rule with suitable interval, show that

$$\frac{ln(1.1)}{1.1^2} + \frac{ln(1.2)}{1.2^2} + \frac{ln(1.3)}{1.3^2} + \ldots + \frac{ln(1.9)}{1.9^2} < 5 - \frac{41}{8}ln2$$

* 參考課程 2.2, 2.3, 2.8, 2.9 及 3.3

$$a.)\ The\ bounded\ area,\ A = \int_1^2 \frac{lnx}{x^2}dx,\ Let\ u = lnx \rightarrow du = \frac{dx}{x}$$

$$\rightarrow A = \int_0^{ln2} ue^{-u}du = \int_0^{ln2} ud(-e^{-u}) = [-ue^{-u}]_0^{ln2} + \int_0^{ln2} e^{-u}du$$

$$= [-ue^{-u} - e^{-u}]_0^{ln2} = \frac{1 - ln2}{2}$$

* 積分三寶: 積分代入, u=lnx

* 定積分代入要改範圍

* Integration by part

CONT'D

2014 - SECTION B

$$By\ trapezoidal\ rule\ with\ interval = 0.1,$$

$$A \approx \frac{0.1}{2}[f(1) + 2f(1.1) + \ldots + 2f(1.9) + f(2)]$$

$$Consider,\ f(x) = \frac{lnx}{x^2} \to f'(x) = \boxed{\frac{1}{x^3} - \frac{2lnx}{x^3}} = \frac{1 - 2lnx}{x^3}$$

$$\to f''(x) = \boxed{\frac{-2}{x^4} - \frac{3(1 - 2lnx)}{x^4}} = \frac{6lnx - 5}{x^4} < 0,\ for\ 1 \le x \le 2$$

$$\therefore A\ is\ under - estimated\ by\ the\ trapezoidal\ rule$$

$$Hence,\ \frac{0.1}{2}[f(1) + 2f(1.1) + \ldots + 2f(1.9) + f(2)] < \frac{1 - ln2}{2}$$

$$\to 0.2[f(1.1) + \ldots + f(1.9)] < 1 - ln2 - 0.1f(2)$$

$$\to \frac{ln(1.1)}{1.1^2} + \frac{ln(1.2)}{1.2^2} + \frac{ln(1.3)}{1.3^2} + \ldots + \frac{ln(1.9)}{1.9^2} < 5 - \frac{41}{8}ln2$$

* 計算梯形面積的加總

* $\boxed{\ }$ 用 Product rule

* 個 f(x) 係 concave downward

2014 - SECTION B

$Q11.)$ $\quad Given\ that\ y(t) = \dfrac{340}{2+e^{-t}-2e^{-2t}},\ t \geq 0$

$a.)\ Will\ y > 171?\ Explain\ your\ answer.$

$b.)\ Find\ the\ greates\ ane\ least\ value\ of\ y(t).$

$c.)\ Given\ y(\alpha) = y(3-\alpha).\ Find\ \alpha\ for\ 0 \leq \alpha < 3-\alpha$

* 參考課程 1.1, 2.1 及 2.4

$$a.)\ \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{340}{2+\boxed{e^{-t}}-2\boxed{e^{-2t}}} = \frac{340}{2} = 170,\ \therefore y(t) < 171$$

$$b.)\ y'(t) = \frac{-340(-e^{-t}+4e^{-2t})}{(2+e^{-t}-2e^{-2t})^2}$$

$Let\ t_0\ such\ that\ y'(t_0) = 0.\ t_0 \in \mathbb{R}^+$

$Then,\ 4e^{-2t_0} - e^{-t_0} = 0 \to 4e^{-t_0} - 1 = 0 \to t_0 = ln4$

* t 越大, e^{-t} 越接近 0

* Chain Rule

* 搵 turning point = 搵 t_0 使度 $y'(t_0)=0$

CONT'D

2014 - SECTION B

	$0 < t < t_0$	$t = t_0$	$t > t_0$
y'	-	0	+
y	Dec.		Inc.

$When\ t = ln4,\ y(t)\ in\ min\ .$

$Hence,\ the\ least\ value\ of\ y(t) = y(ln4) = 160$

$the\ greatest\ value\ of\ y(t) = y(0) = 340$

$$c.)\ y(\alpha) = y(3-\alpha) \rightarrow e^{-\alpha} - 2e^{-2\alpha} = e^{-(3-\alpha)} - 2e^{-2(3-\alpha)}$$

$$Let\ u = e^{-\alpha},\ then,\ u - 2u^2 = \frac{e^{-3}}{u} - 2\frac{e^{-6}}{u^2}$$

$$\rightarrow u^3 - 2u^4 = ue^{-3} - 2e^{-6}$$

$$\rightarrow u(u^2 - e^{-3}) = 2((u^2)^2 - (e^{-3})^2)$$

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow Increasing$

$f'(x) < 0 \rightarrow Decreasing$

* t 係跌近由 0 去 ln4
t 越大, y(t) 接近 170
所以 y(0) 係最大值

CONT'D

2014 - SECTION B

$$\rightarrow u(u^2 - e^{-3}) = 2(u^2 - e^{-3})(u^2 + e^{-3})$$

$$\rightarrow (u^2 - e^{-3})(2u^2 - u + 2e^{-3}) = 0$$

$$\rightarrow u^2 = e^{-3} \text{ or } 2u^2 - u + 2e^{-3} = 0$$

$$\rightarrow u^2 = e^{-3} \text{ or } u = \frac{1 \pm \sqrt{1 - 4(2)(2e^{-3})}}{4}$$

$$Given,\ 0 \leq \alpha < 3 - \alpha \rightarrow 0 \leq \alpha < \frac{3}{2}$$

$$\rightarrow e^{-\frac{3}{2}} < u \leq 1$$

$$\rightarrow e^{-3} < u^2 \leq 1$$

$$Hence\ u = \frac{1 + \sqrt{1 - 16e^{-3}}}{4} \rightarrow t = -ln(\frac{1 + \sqrt{1 - 16e^{-3}}}{4})$$

$*$ $a^2 - b^2 = (a + b)(a - b)$

2014 – SECTION B

Q12.) *The delivering time* $X \sim N(\mu, \sigma^2)$ *of an order recieved by a shop in minutes. It is known that* 27.43 % *of the deliver longer than* 25 *minutes and* 51.6 % *of that fall between* 3.5 *minutes of* μ.

a.) $\sigma = ?$ *and* $\mu = ?$

b.) *Suppose* 200 *orders recieved in a day and a discount will be given if the delivery time of a order longer than k minutes. Assume indenpendence among delivery times of different order. Find the min. integral value of k such the expected number of discount given out is at most* 5 *in a day.*

c.) *After improvement, the shop have the delivery time* $Y \sim N(\theta, 4.7^2)$ *in minutes. A random sample* 12 *in size is recorded :* 22 15 18 21 22 31 20 16 21 19 23 24

i.) *Construct* 90 % *confidence interval for* θ.

ii.) *Another n size random sample orders are selected. Find min. value of n such that the probability of the sample mean falls within* 3 *minutes of* θ *is greater than* 0.99

* 參考課程 4.4, 4.5, 4.6 及 4.7

CONT'D

2014 – SECTION B

$a.)\ Given,\ P(X > 25) = 0.2743\ and\ P(\mu - 3.5 < X < \mu + 3.5) = 0.516$

$$\rightarrow P(Z > \frac{25-\mu}{\sigma}) = 0.2743\ and\ P(\frac{-3.5}{\sigma} < Z < \frac{3.5}{\sigma}) = 0.516$$

$$\rightarrow P(Z > \frac{25-\mu}{\sigma}) = 0.2743\ and\ P(0 < Z < \frac{3.5}{\sigma}) = \frac{0.516}{2}$$

$$\rightarrow \frac{25-\mu}{\sigma} = 0.6\ and\ \frac{3.5}{\sigma} = 0.7 \rightarrow \mu = 22\ and\ \sigma = 5$$

$b.)\ Let\ D \sim B(200, P(X > k))\ be\ random\ variable\ of\ the\ number\ of\ discount\ given\ out\ in\ a\ day$

$$Then\ E(D) = 200P(X > k) \le 5 \rightarrow P(X > \frac{k-22}{5}) \le \frac{5}{200}$$

$$\rightarrow \frac{k-22}{5} \ge 1.96 \rightarrow k \ge 31.8$$

$\therefore\ The\ min.\ integral\ value\ of\ k = 32$

* 先計算 Z ~ N(0,1), 再對表

* 常態分佈對稱特性

* 二元分佈, E(X) = np

CONT'D

2014 - SECTION B

$ci.)\ Let\ \bar{Y}_n\ be\ the\ n\ size\ random\ sample$

$$\bar{Y}_{12} = \frac{22+15+18+21+22+31+20+16+21+19+23+24}{12}$$
$$= 21$$

$$\therefore\ The\ 90\,\%\ C.I.\ of\ \theta = (21 - 1.645 \cdot \frac{4.7}{\sqrt{12}},\ 21 + 1.645 \cdot \frac{4.7}{\sqrt{12}})$$

$$= (18.7681,\ 23.2319)\ (to\ 4\ d.p.)$$

$$ii.)\ P(\theta - 3 < \bar{Y}_n < \theta + 3) > 0.99 \rightarrow P(\frac{-3\sqrt{n}}{4.7} < Z < \frac{3\sqrt{n}}{4.7}) > 0.99$$

$$\rightarrow \boxed{P(0 < Z < \frac{3\sqrt{n}}{4.7}) > \frac{0.99}{2}} \rightarrow \frac{3\sqrt{n}}{4.7} > 2.575 \rightarrow n > 16.2745$$

$\therefore\ The\ min.\ value\ of\ n = 17$

* 90% 置信區間

* 常態分佈對我稱特性

2014 – SECTION B

Q13.) The number of delay of a bus in a day follows $Po(4.8)$. *Assume the daily numbers of delays are independent.*

a.) Find the probability of not more than 3 delays in a day

b.) For 3 consecutive days, find the probability there are at most 2 days with not more than 3 delays in each day.

c.) Let B be the event there are more than 5 delays in a day.

i.) Suppose B happen today, find the mean number of $\bar{B}$ *happen between today and next B happen.*

ii.) Find the probability the last of a week is the 3^{rd} *happening of B in that week.*

iii.) Find the probability at least 4 consecutive days B happen in a week.

*** 參考課程 4.3 及 4.4**

a.) Let $X \sim Po(4.8)$ *be the number of delay in a day*

$$P(X \leq 3) = \boxed{e^{-4.8}\left(1 + 4.8 + \frac{4.8^2}{2!} + \frac{4.8^3}{3!}\right)} = 0.2942 \text{ (to 4 d.p.)}$$

* $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

CONT'D

2014 - SECTION B

$b.)\ Let\ Y \sim B(3, P(X \leq 3))\ be\ the\ number\ of\ day\ with\ at\ most\ 3\ delays$

$$P(Y \leq 2) = 1 - [P(X \leq 3)]^3 = 0.9745\ (to\ 4\ d.p.)$$

$ci.)\ Consider\ Z \sim G(P(B))$

$$P(B) = 1 - \boxed{\sum_{i=0}^{5} \frac{e^{-4.8}(4.8)^i}{i!}}$$

$$The\ mean\ number\ of\ days = \boxed{\frac{1}{P(B)} - 1} = 1.8645\ (to\ 4\ d.p.)$$

$$ii.)\ The\ probability = \boxed{C_2^6 P(B)^2[1 - P(B)]^4} \cdot \boxed{P(B)} = 0.1145\ (to\ 4\ d.p.)$$

$$iii.)\ The\ probability = [P(B)]^4 + 3[1 - P(B)][P(B)]^4$$

$$= 0.0438\ (to\ 4\ d.p.)$$

* P(Not A) = 1 - P(A)

* $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$

* 幾何分佈, E(Z)=1/p

* 經過咗幾多日先 B Event 要減番最後果日

* 頭 6 日有兩日 B Event

* 最後一日 B Event