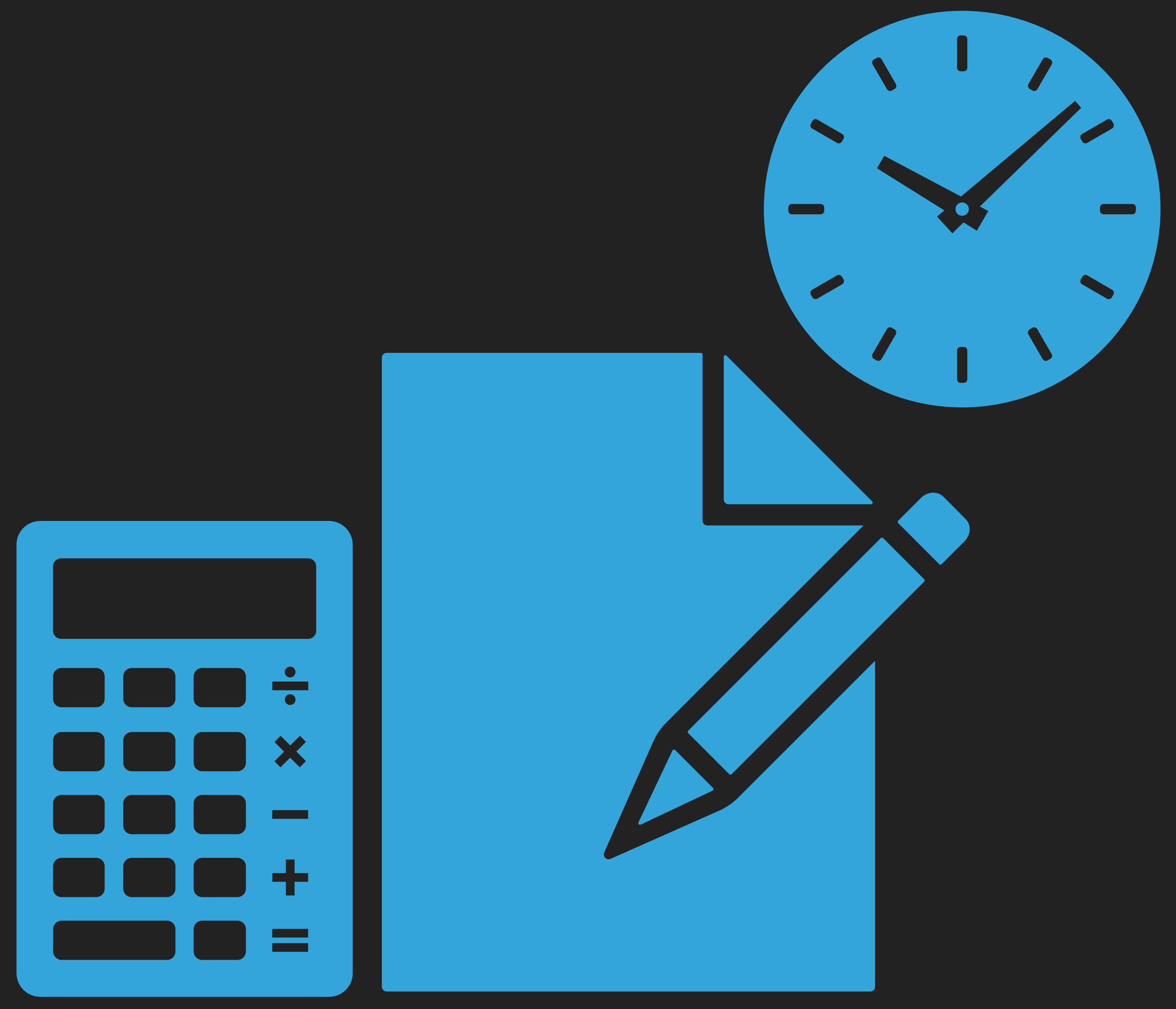
深宵教室 - DSE M1 模擬試題解答

2021

此為參考2021試題之模擬試題，原版請另行購買

2021

- Section A
- Section B



2021 - SECTION A

Q1.) Let X be discrete random vaiable, where a and p are constant, $p > 0$

k	-1	0	1	2	3	4
$P(X=k)$	a	0.15	0.15	b	0.05	0.25

Given that $E(5X+1) = 10$, *find* a, b *and* $P(X > 0 \mid X \le 2)$.

* 參考課程 4.1, 4.2, 4.3 及 4.4

$$a + 0.15 + 0.15 + b + 0.05 + 0.25 = 1 \rightarrow a + b = 0.4 - (1)$$

$$E(X) = -1 \cdot a + 0 \cdot 0.15 + 1 \cdot 0.15 + 2 \cdot b + 3 \cdot 0.05 + 4 \cdot 0.25$$

$$= 2b - a + 1.3$$

$$\text{Given that } E(5X+1) = 10 \rightarrow 5E(X) + 1 = 10 \rightarrow 2b - a = 0.5 - (2)$$

$$(1) + (2) : 3b = 0.9 \rightarrow b = 0.3 \text{ and } a = 0.1$$

$$P(X > 0 \mid X \le 2) = \frac{P(0 < X \le 2)}{P(X \le 2)} = \frac{0.15 + 0.3}{0.1 + 0.15 + 0.15 + 0.3} = \frac{9}{14}$$

* 機率總加 = 1

* $E(X) = \sum_{i=1}^{n} k_i P(X = k_i)$

* $E(aX + b) = aE(X) + b$

* 條件概率

2021 - SECTION A

Q2.) The probability for a person has disease is 0.12. The probability a test shows a person has disease is 0.97 if that person has disease. The probability the test shows a person has no disease is 0.89 if that person has no disease. Find the probability;

a.) The test shows the correct result.

b.) The test shows a person has disease.

c.) A person actually has disease if the test shows he has disease.

* 參考課程 4.2

a.) Let D be the event a person has disease.

T be the event the test shows a person has disease.

$$\text{The required probability} = P(T|D)P(D) + P(\bar{T}|\bar{D})P(\bar{D})$$
$$= 0.97 \cdot 0.12 + 0.89 \cdot (1 - 0.12)$$
$$= 0.8996$$

b.) $$\text{The required probability, } P(T) = P(T|D)P(D) + P(T|\bar{D})P(\bar{D})$$
$$= 0.97 \cdot 0.12 + (1 - 0.89) \cdot (1 - 0.12)$$
$$= 0.2132$$

* P(Not A) = 1 - P(A)

CONT'D

2021 - SECTION A

$$c.)\ The\ required\ probability = P(D|T) = \frac{P(D\cap T)}{P(T)} = \frac{P(T|D)P(D)}{P(T)}$$

$$= \frac{0.97\cdot 0.12}{0.2132} = 0.5460\ (to\ 4\ d.p.)$$

* 條件概率

2021 - SECTION A

Q3.) There are 10 questions. For each question, Peter has 0.8 chance knows how to answer and 0.1 chance answer wrongly if he knows how to answer. He answer wrongly if he does not know the answer. He passes if he answers 8 or more questions correctly

a.) The probability Peter passes.

b.) The probability he knows how to do all questions and passes

c.) The probability he knows how to do all questions given that he passes

* 參考課程 4.2 及 4.4

a.) The probability he answer correctly for a question, $p_c = 0.8 \cdot (1 - 0.1)$

$= 0.72$

The required probability, $p_t = C_8^{10} p_c^8 (1 - p_c)^2 + C_9^{10} p_c^9 (1 - p_c) + p_c^{10}$

$= 0.4378$ (*to 4 d.p.*)

b.) The required probability, $p_{kp} = 0.8^{10}[C_8^{10} 0.9^8 0.1^2 + C_9^{10} 0.9^9 0.1 + 0.9^{10}]$

$= 0.0998$ (*to 4 d.p.*)

c.) The required probability $= \dfrac{p_{kt}}{p_t} = 0.2280$ (*to 4 d.p.*)

* P(Not A) = 1 - P(A)

* $P(X = k) = C_k^n p^k (1 - p)^{n-k}$

* 條件概率

2021 - SECTION A

$Q4.)$ *Peter wants to estimate the proportion p of the Youtuber in school. Given that there are 28 Youtubers in 40 randomly selected students in school.*

a.) Find the 90 % confidence interval for p.

b.) Using the above sample proportion find the least sample size for the width of 99 % confidence interval does not exceed 0.1.

* 參考課程 4.6 及 4.7

$$a.)\ \text{The sample proportion, } p_s = \frac{28}{40} = 0.7$$

$$\text{The } 90\%\ C.I.\ \text{for } p = \left(p_s - \boxed{1.65}\sqrt{\frac{p_s(1-p_s)}{40}},\ p_s + \boxed{1.65}\sqrt{\frac{p_s(1-p_s)}{40}}\right)$$

$$= (0.5804,\ 0.8196)\ (to\ 4\ d.p.)$$

$b.)$ *Let n be the required sample size*

$$2 \cdot \boxed{2.575} \cdot \sqrt{\frac{p_s(1-p_s)}{n}} < 0.1 \rightarrow n > 556.9725$$

$\therefore$ *The least sample size is 557*

* 90% 置信區間

* 99% 置信區間

2021 - SECTION A

Q5.) Let the curve $C: y = f(x), f(x) = e^{-x^{\frac{1}{3}}}$

Find the area of the region bounded by C, $x-axis$, $y-axis$ *and the straight line* $x = 8$

* 參考課程 2.8 及 2.9

$$\text{The area, } A = \left|\int_0^8 e^{-x^{\frac{1}{3}}}dx\right|, \text{ Let } u = -x^{\frac{1}{3}} \to du = -\frac{1}{3}x^{-\frac{2}{3}}dx$$

$$\to -3u^2du = dx$$

$$\text{Hence, } A = \left|\int_0^{-2} -3u^2e^udu\right| = 3\left|\int_0^{-2} u^2d(e^u)\right|$$

$$= 3\left|[u^2e^u]_0^{-2} - \int_0^{-2} 2ue^udu\right| = 3\left|4e^{-2} - 2\int_0^{-2} ud(e^u)\right|$$

$$= 6\left|2e^{-2} - [ue^u]_0^{-2} + \int_0^{-2} e^udu\right| = 6\left|4e^{-2} + [e^u]_0^{-2}\right|$$

$$= 6(1 - 5e^{-2})\ unit^2$$

* 積分三寶: 積分代入法

* 定積分代入要改範圍

* Integration by part

2020 - SECTION A

Q6.) Let $f(x) = e^{-6x}(1 - kx^2)^5$. *If the coefficient of* x^4 *of* $f(x) = -26$, $k = ?$

* 參考課程 1.1 及 3.2

$$f(x) = \left(1 - 6x + \frac{(6x)^2}{2!} - \frac{(6x)^3}{3!} + \frac{(6x)^4}{4!} + \ldots\right)\left(1 - 5kx^2 + 10k^2x^4 - \ldots\right)$$

The coefficient of $x^4 = 10k^2 - 5k(18) + 54 = -26$

$\rightarrow k^2 - 9k + 8 = 0$

$\rightarrow k = 1 \text{ or } 8$

* $(a+b)^n = \sum_{r=0}^{n} C_r^n a^r b^{n-r}$

* $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

* $C_r^n = \frac{n!}{r!(n-r)!}$

$\rightarrow C_1^n = n \text{ and } C_2^n = \frac{n(n-1)}{2}$

2021 - SECTION A

$Q6.)\ Let\ f(x) = e^x(x^3 - x + 2)^{-1}, for\ 0 \leq x \leq 5$

$Find\ the\ greatest\ and\ the\ least\ value\ of\ f(x)$

* 參考課程 2.2, 2.4, 2.8 及 2.9

$a.)\ f'(x) = e^x(x^3 - x + 2)^{-1} - e^x(3x^2 - 1)(x^3 - x + 2)^{-2}$

$Let\ x_0 \in [0, 5]\ such\ that\ f'(x_0) = 0$

$\rightarrow (x_0^3 - x_0 + 2) - (3x_0^2 - 1) = 0$

$\rightarrow x_0^3 - 3x_0^2 - x_0 + 3 = 0 \rightarrow x_0(x_0^2 - 1) - 3(x_0^2 - 1) = 0$

$\rightarrow (x_0 - 3)(x_0 - 1)(x_0 + 1) = 0$

$\rightarrow x_0 = 1\ or\ 3\ or\ -1\ (rejected)$

	0 < x < 1	x = 1	1 < x < 3	x = 3	3 < x < 5
f'(x)	+	0	-	0	+
f(x)	Inc.		Dec.		Inc.

* Product rule

* 搵 turning point = 搵 x_0 使度 $f'(x_0)=0$

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow Increasing$

$f'(x) < 0 \rightarrow Decreasing$

CONT'D

2021 – SECTION A

$$Hence, the\ local\ max.\ valve = f(1) = \frac{e}{2}$$

$$the\ local\ min.\ valve = f(3) = \frac{e^3}{26}$$

$$Also, for\ the\ boundary\ case, f(0) = \frac{1}{2}\ and\ f(5) = \frac{e^5}{122}$$

$$\therefore\ The\ greatest\ value = \frac{e}{2}$$

$$The\ least\ value = \frac{1}{2}$$

* 搵極大極細值要考慮 Boundary Case

2021 - SECTION A

Q8.) *Let the curve* $C : y = f(x)$, $f'(x) = 2^{kx}(1+2^{kx})^{-1}$, *where k is constant, Given that a straight line* $8x - 9y + 10 = 0$ *touchs C at point* $A(1, h)$, *where h is a constant. Find* $f(x)$.

* 參考課程 2.4, 2.5, 2.6 及 2.7

$$\text{The slope of the tangent to } C \text{ at } A = \frac{8}{9} = f'(1) \rightarrow \frac{8}{9} = \frac{2^k}{1+2^k}$$

$$\rightarrow 2^k = 8 \rightarrow k = 3$$

* 微分計算切線斜率

$$\text{Since } A \text{ lies on the straight line } 8x - 9y + 10 = 0,$$

$$\rightarrow 8(1) - 9(h) + 10 = 0 \rightarrow h = 2$$

$$\text{Then, } f(x) = \int \frac{\boxed{2^{3x}}}{1+2^{3x}} \boxed{dx}, \text{ Let } u = 2^{3x} \rightarrow \ln u = 3x\ln 2 \rightarrow du = 3u\ln 2 \cdot dx$$

* 積分三寶: 積分代入

$$\rightarrow f(x) = \boxed{\frac{1}{3\ln 2}} \int \frac{\boxed{du}}{1+u} = \frac{\ln(1+u)}{3\ln 2} + C, \text{ where } C \text{ is constant}$$

$$\text{Since, } f(1) = 2 \rightarrow C = 2 - \frac{\ln 9}{3\ln 2} \rightarrow f(x) = 2 + \frac{\ln(1+2^{3x}) - \ln 9}{3\ln 2}$$

2021 – SECTION B

Q9.) A batch of oranges are classified base on their weight, which follows $N(200, \sigma^2)$ *in gram.*

Weight (x g)	$x < 180$	$180 \le x \le 230$	$x > 230$
Classification	Small	Medium	Large

Given that there are 21.19 % of the oranges are Small

a.) Find the percentage of the orange is Medium.

b.) The orange are inspected one by one. Find the probability the 4^{th} *orange inspected is the* 2^{nd} *Large orange.*

c.) 5 number of orange are randomly selected :

i.) Find the probability there are exact 1 Large and 2 small in the selection

ii.) Given that there is exact 1 Large orange in the selection, find the probability there is at least 2 small orange in the selection.

* 參考課程 4.2, 4.4 及 4.5

a.) Let S, M, L be the event of the selected orange is Small, Medium and Large respectively.

$$\text{Given } P(S) = P\left(Z < \frac{180-200}{\sigma}\right) = 0.2119 \rightarrow \frac{20}{\sigma} = 0.8 \rightarrow \sigma = 25$$

* 先計算 Z ~ N(0,1), 再對表

CONT'D

2021 - SECTION B

$$P(M) = P\left(\frac{180-200}{25} \le Z \le \frac{230-200}{25}\right) = P(-0.8 \le Z \le 1.2)$$
$$= 0.673$$

$$b.)\ P(L) = 1 - P(S) - P(M) = 0.1151$$

$$The\ required\ probability = C_1^3 P(L)[1-P(L)]^2 \cdot P(L)$$
$$= 0.0311\ (to\ 4\ .d\ .p)$$

$$ci.)\ The\ required\ probability,\ p_1 = C_1^5 \cdot P(L) \cdot [C_2^4 P(M)^2 \cdot P(S)^2]$$
$$= 0.0702\ (to\ 4\ .d\ .p)$$

$$ii.)\ The\ required\ probability = \frac{1-p_2-p_3}{C_1^5 P(L)[1-P(L)]^4}$$

$$where\ p_2 = C_1^5 P(L)P(M)^4\ and\ p_3 = C_1^5 P(L)[C_1^4 P(S)P(M)^3]$$

$$= 0.2441\ (to\ 4\ .d\ .p)$$

* 先計算 Z ~ N(0,1), 再對表
* $P(X=k) = C_k^n p^k (1-p)^{n-k}$
  3 個有 1 個係 Large
* 第 4 個係 Large
* P(not A) = 1 - P(A)
* 1大 2中 2小
* 條件概率
* 1大
* 1大 4中 0小
* 1大 3中 1小

2021 – SECTION B

$Q10.)$ *The number of commerical email recieved each hour follows* $Po(1.3)$ *while the number of non – commerical email recieved each hour follows* $Po(0.9)$

a.) *Find the probability the number of non – commerical email recieved for a certain hour is fewer than 3.*

b.) *Find the probability the number of commerical email recieved for 6 hours is 5.*

c.) *Find the probability of email received for a certain hour is 2.*

d.) *Find the probability 2 emails are non – commerical if the number of emails recieved in a certain hour is 2.*

e.) *Given the number of emails recieved in a certain hour is fewer than 3, find the probability there is no commerical email in that hour.*

* 參考課程 4.2 及 4.4

a.) *Let X be the random variable of the number of non – commerical email recieved in a certain hour*

Y be the random variable of the number of commerical email recieved in a certain hour

$$P(X < 3) = e^{-0.9} + e^{-0.9}(0.9) + \frac{e^{-0.9}(0.9)^2}{2!} = 0.9371\ (to\ 4\ .d\ .p)$$

* $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$

CONT'D

2021 - SECTION B

$$b.)\ P(6Y=5)=\boxed{\frac{e^{-6\cdot 1.3}(6\cdot 1.3)^5}{5!}}=0.0986\ (to\ 4\ .d\ .p)$$

$$c.)\ P(X+Y=2)=\boxed{\frac{e^{-(0.9+1.3)}(0.9+1.3)^2}{2!}}=0.2681\ (to\ 4\ .d\ .p)$$

$$d.)\ P(X=2\,|\,X+Y=2)=\frac{P(X=2\ \cap\ Y=0)}{P(X+Y=2)}=\frac{\boxed{\frac{e^{-0.9}(0.9)^2}{2!}}\cdot\boxed{e^{-1.3}}}{P(X+Y=2)}$$

$$=0.1674\ (to\ 4\ .d\ .p)$$

$$e.)\ P(Y=0\,|\,X+Y<3)=\frac{P(X<3\ \cap\ Y=0)}{P(X+Y<3)}=\frac{P(X<3)\cdot\boxed{e^{-1.3}}}{\boxed{e^{-2.2}[1+2.2+\frac{2.2^2}{2!}]}}$$

$$=0.4101\ (to\ 4\ .d\ .p)$$

* $6Y \sim Po(6\times 1.3)$
* $X+Y \sim Po(0.9+1.3)$
* 條件概率
* $P(X=k)=\dfrac{e^{-\lambda}\lambda^k}{k!}$
* 條件概率

2021 - SECTION B

Q11.) Let $f(x) = \sqrt{\dfrac{x}{2-x}}$, $J = \int_0^{0.5} f(x)dx$, *and* $K = \int_{0.5}^{1} f(x)dx$ *. Given that* $\int_0^1 f(x)dx = \dfrac{\pi - 2}{2}$

where $0 \le x \le 1$

a.) Find $f''(x)$

b.) Using trapezoidal rule with 5 sub − intervals, estimate J *.*

c.) Using the result of b.), estimate K

d.) Is $\dfrac{J}{K} < 0.44$*? Explain your answer .*

* 參考課程 2.2, 2.3, 3.2 及 3.3

$$a.)\ f(x) = \left(\frac{2}{2-x} - 1\right)^{\frac{1}{2}} \rightarrow [f(x)]^2 = \frac{2}{2-x} - 1 \rightarrow 2f(x)f'(x) = \frac{2}{(2-x)^2}$$

* Implicit 微分法

$$\rightarrow f'(x) = \frac{1}{(2-x)^2} \cdot \left(\frac{2-x}{x}\right)^{\frac{1}{2}} = x^{-\frac{1}{2}}(2-x)^{-\frac{3}{2}}$$

$$\rightarrow f''(x) = -\frac{1}{2}x^{-\frac{3}{2}}(2-x)^{-\frac{3}{2}} + \frac{3}{2}x^{-\frac{1}{2}}(2-x)^{-\frac{5}{2}} = \frac{2x-1}{x^{\frac{3}{2}}(2-x)^{\frac{5}{2}}}$$

CONT'D

2021 - SECTION B

$a.)\ Let\ J_1\ be\ the\ estimation\ of\ J$

$$J_1 = \frac{0.5-0}{5\cdot 2}[f(0)+2f(0.1)+2f(0.2)+2f(0.3)+2f(0.4)+f(0.5)]$$

$$= 0.1772\ (to\ 4\ d.p.)$$

* 計算梯形面積的加總

$$b.)\ J+K=\frac{\pi-2}{2} \rightarrow K=\frac{\pi-2}{2}-J \approx 0.3936\ (to\ 4\ d.p.)$$

* 定積分範圍加減

$c.)\ f''(x)<0,\ for\ 0<x<0.5 \rightarrow J_1\ is\ under-estimated\ of\ J,\ J>J_1$

$$Let\ A=\frac{\pi-2}{2},\ \frac{J}{K}=\frac{J}{A-J}=\frac{A}{A-J}-1>\frac{A}{A-J_1}-1$$

$$\approx 0.45 > 0.44$$

$$\therefore \frac{J}{K}<0.44\ is\ not\ true$$

* 個 f(x) 係凹口向下

2020 – SECTION B

$Q12.)$ *Given that* $\dfrac{dV}{dt} = \sqrt{t+1}\sqrt{3-\sqrt{t+1}}, 0 \le t \le 7$

where V is the water volume in a tank in m^3

t is the number of minutes elapsed since the tank is injected with water

The tank is empty at $t = 0$ *and the rate of change attains maximum when* $t = T$.

a.) *Find T and the exact value of V when* $t = T$

b.) *Given that the tank is an inverted right circular cone of height* $= 1m$ *and base radius* $= 6m$. *Find the rate of change of the depth of water in tank when* $t = T$.

* 參考課程 2.2, 2.3, 2.4, 2.8 及 2.9

$$a.)\ Let\ y = \sqrt{t+1} \rightarrow y^2 = t+1 \rightarrow \frac{dy}{dt} = \frac{1}{2y}$$

$$Hence, \frac{dV}{dt} = y\sqrt{3-y} \rightarrow \frac{dV^2}{d^2t} = \boxed{\frac{dy}{dt}[\sqrt{3-y} - \frac{y}{2\sqrt{3-y}}]}$$

$$= \frac{3(2-y)}{4y\sqrt{3-y}}$$

* Implicit 微分法

* Product rule + Chain rule

CONT'D

2021 - SECTION B

$$Assume\ V''(t_0) = 0 \rightarrow y(t_0) = 2 \rightarrow t_0 = 3$$

	0 < t < 3	t = 3	3 < t < 7
V''(t)	+	0	-
V'(t)	Inc.		Dec.

$$\therefore V'(t)\ attains\ greatest\ value\ when\ t = 3 \rightarrow T = 3$$

$$Then\ V(3) = V(0) + \int_0^3 V'(t)dt \rightarrow Let\ y = \sqrt{t+1} \rightarrow 2ydy = dt$$

$$= \int_1^2 2y^2\sqrt{3-y}dy\ \ Let\ z = 3 - y \rightarrow dz = -dy$$

$$= 2\int_1^2 (3-z)^2\sqrt{z}dz = 2\int_1^2 9z^{\frac{1}{2}} - 6z^{\frac{3}{2}} + z^{\frac{5}{2}}dz$$

$$= 2[6z^{\frac{3}{2}} - \frac{12}{5}z^{\frac{5}{2}} + \frac{2}{7}z^{\frac{7}{2}}]_1^2 = \frac{328\sqrt{2} - 272}{35}$$

* 搵 turning point = 搵 t_0 使度 $V''(t_0)=0$

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow Increasing$

$f'(x) < 0 \rightarrow Decreasing$

* 積分三寶: 積分代入法

* 定積分代入要改範圍

* 負數倒轉範圍

CONT'D

2021 - SECTION B

b.) Let $h(t)$ be the depth of water in the tank at time t

$$\text{Given that } \left(\frac{h(t)}{1}\right)^3 = \frac{V(t)}{\frac{1}{3}\pi(6)^2(1)} \rightarrow V(t) = 12\pi[h(t)]^3$$

$$\rightarrow V'(t) = 36\pi[h(t)]^2 h'(t)$$

$$\text{Hence, } h'(3) = \frac{V'(3)}{36\pi[h(3)]^2} = \frac{V'(3)}{36\pi\left(\frac{V(3)}{12\pi}\right)^{\frac{2}{3}}}$$

$$= 0.0640\ m/min\ (to\ 4\ d.p.)$$

* 相似立體, 體積比 = (邊比)³

* 錐體體積 = (1/3) x 底面積 x 高