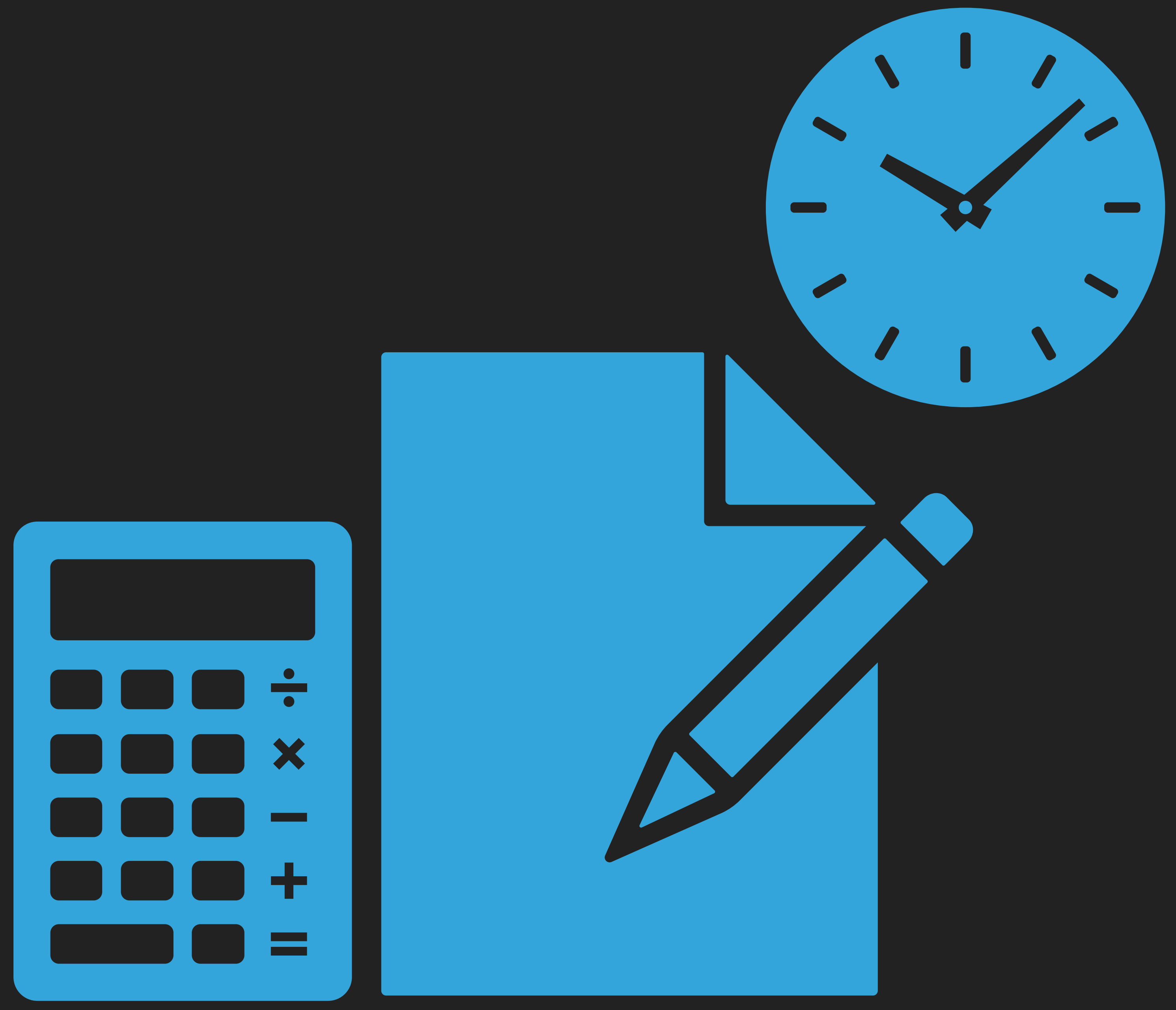
深宵教室 - DSE M2 模擬試題解答

2013

此為參考2013試題之模擬試題，原版請另行購買

2013

- Section A
- Section B



2013 - SECTION A

$Q1.)\ f(x) = sin2x\ .\ \ f'(x) = ?\ \ (By\ First\ Principles)$

* 參考課程 2.2, 3.1 及 3.2

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}(sin(2x+2h) - sin2x)$$

$$= \lim_{h\to 0}\frac{2}{h}cos(2x+h)sinh$$

$$= \lim_{h\to 0} cos(2x+h)\lim_{h\to 0}\frac{sinh}{h} = cos(2x+0)(1)$$

$$= 2cos2x$$

* 微分定義

* Sum to product

$$\sin(A\pm B) = \sin A\cos B \pm \cos A \sin B$$

$$\cos(A\pm B) = \cos A\cos B \mp \sin A \sin B$$

$$\tan(A\pm B) = \frac{\tan A \pm \tan B}{1\mp \tan A\tan B}$$

$$2\sin A\cos B = \sin(A+B)+\sin(A-B)$$

$$2\cos A\cos B = \cos(A+B)+\cos(A-B)$$

$$2\sin A\sin B = \cos(A-B)-\cos(A+B)$$

$$\sin A+\sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A-\sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A-\cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

* lim 可乘除

* $\lim_{h\to 0}\frac{sinh}{h} = 1$

2013 - SECTION A

$Q2.)\ (1+ax)^n = 1 - 20x + 180x^2 + \ldots,\quad a = ?\ \ and\ \ n = ?$

* 參考課程 1.1

$$(1+ax)^n \equiv \sum_{r=0}^{n} C_r^n (ax)^r$$

$By\ compare\ coefficient\ of\ x\ and\ x^2$

$$\begin{cases} C_1^n a = na = -20 & \text{———— (1)} \\ C_2^n a^2 = \dfrac{n(n-1)a^2}{2} = 180 & \text{———— (2)} \end{cases}$$

$$In\ (2):\ (na)^2 - (na)a = 2(180) = 360 \rightarrow a = -2$$

$$\therefore n = \frac{-20}{-2} = 10\ and\ a = -2$$

*** Binomial Expansion**

$$*\ C_r^n = \frac{n!}{r!(n-r)!} \rightarrow C_1^n = n\ and\ C_2^n = \frac{n(n-1)}{2}$$

2013 – SECTION A

$Q3.)\ Prove\ 1+\frac{1}{1\cdot 4}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{4n+1}{3n+1},\ \forall n\in\mathbb{Z}^+$

* 參考課程 1.1 及 1.2

方法1

$$Let\ P(n):\ 1+\sum_{r=1}^{n}\frac{1}{(3r-2)(3r+1)}=\frac{4n+1}{3n+1}\ \forall n\in\mathbb{Z}^+$$

* 先 Let Statement

$$For\ P(1):\quad L.H.S.=\frac{5}{4}=R.H.S.$$

* 證明 P(1) is true

$$Assume\ P(k)\ is\ true\ \exists k\in\mathbb{Z}^+,\ then\ P(k+1):$$

* 假設 P(k) is true. 證明 P(k+1) is true

$$L.H.S.=1+\sum_{r=1}^{k+1}\frac{1}{(3r-2)(3r+1)}$$

$$=1+\sum_{r=1}^{k}\frac{1}{(3r-2)(3r+1)}+\frac{1}{(3k+1)(3k+4)}$$

* 將末項抽出並改變未項

CONT'D

2013 - SECTION A

$$= 1 + \sum_{r=1}^{k} \frac{1}{(3r-2)(3r+1)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{4k+1}{3k+1} + \frac{1}{(3k+1)(3k+4)} = \frac{4k+5}{3k+4}$$

$$= R.H.S.$$

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

i.e. By M.I., $P(n)$ *is true,* $\forall n \in \mathbb{Z}^+$

方法2

$$1 + \sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)} = 1 + \sum_{r=1}^{n} \left(\frac{1}{3(3r-2)} - \frac{1}{3(3r+1)}\right)$$

* 寫結論

* **Partial fraction**

$$Let\ \frac{1}{(3r-2)(3r+1)} \equiv \frac{A}{3r-2} + \frac{B}{3r+1}$$

$$\rightarrow 1 \equiv A(3r+1) + B(3r-2)$$

$$\rightarrow A = \frac{1}{3},\quad B = -\frac{1}{3}$$

CONT'D

2013 - SECTION A

$$= 1 + \sum_{r=1}^{n} \frac{1}{3(3r-2)} - \sum_{r=1}^{n} \frac{1}{3(3r+1)}$$

$$= 1 + \frac{1}{3} + \sum_{r=2}^{n} \frac{1}{3(3r-2)} - \sum_{r=1}^{n-1} \frac{1}{3(3r-2)} - \frac{1}{3(3n+1)}$$

$$= \frac{4}{3} + \sum_{r=1}^{n-1} \frac{1}{3(3r+1)} - \sum_{r=1}^{n-1} \frac{1}{3(3r+1)} - \frac{1}{3(3n+1)}$$

$$= \frac{4(3n+1)-1}{3(3n+1)} = \frac{4n+1}{3n+1}$$

* Summation 可析開做加減

* 透過抽首尾項改變首末項

* 透過改變首末項改變公項

2013 - SECTION A

Q4.) The curve Γ passing through $(1, e)$ and the slope at $(x, y) = e^x - 1$

a.) $\Gamma = ?$

b.) Equation of tangent of Γ at the $y -$ interception of Γ

* 參考課程 3.4 及 3.9

a.) Let $\Gamma : y = f(x)$, where $f'(x) = e^x - 1$ and $f(1) = e$

$$f'(x) = e^x - 1 \rightarrow f(x) = \int (e^x - 1)dx$$

$$\rightarrow f(x) = e^x - x + c, \text{ where } C \text{ is constant}$$

$$\because f(1) = e \rightarrow C = 1 \quad \therefore f(x) = e^x - x + 1$$

b.) The $y -$ interception of $\Gamma = (0, f(0)) = (0, 2)$

The equation of tangent at $(0, 2)$, $T: \ y = f'(0)x + 2$

$$y = 2$$

* 積分是微分逆函數

* 直線方程 y=mx+c

2013 - SECTION A

$Q5.)\ Sketch\ y = f(x),\ where\ f(x) = \dfrac{3 - 3x^2}{3 + x^2}$

* 參考課程 3.2 及 3.5

$$y = \frac{12 - 3(3 + x^2)}{3 + x^2} = \boxed{\frac{12}{3 + x^2}} - 3 \rightarrow \frac{dy}{dx} = \boxed{-\frac{24x}{(3 + x^2)^2}}$$

$$Let\ x_0 \in \mathbb{R}\ such\ that\ \frac{dy}{dx}\Big|_{x=x_0} = 0 \rightarrow x_0 = 0$$

	x < 0	x = 0	x > 0
y′	+	0	-
y	Inc.		Dec.

$\therefore The\ local\ max.pt. = (0,\ 1)$

* ▇ 用 Chain rule

* 搵 turning point = 搵 x_0 使度 $y'(x_0)=0$

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow Increasing$

$f'(x) < 0 \rightarrow Decreasing$

CONT'D

2013 – SECTION A

$$\frac{dy}{dx} = \boxed{-\frac{24x}{(3+x^2)^2}} \rightarrow \frac{d^2y}{dx^2} = \boxed{-\frac{24}{(3+x^2)^2} + \frac{96x^2}{(3+x^2)^3}}$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{-24(3+x^2)+96x^2}{(3+x^2)^3} = \frac{72(x^2-1)}{(3+x^2)^3}$$

$$Let\ x_0 \in \mathbb{R}\ such\ that\ \frac{d^2y}{dx^2}\Big|_{x=x_0} = 0 \rightarrow x_0 = \pm 1$$

	x < -1	x = -1	-1 < x < 1	x = 1	x > 1
y''	+	0	-	0	+
y	Up.		Down.		Up.

$$\therefore The\ pt.\ of\ inflexion = (-1, 0)\ and\ (1, 0)$$

*** 用 Product rule + Chain rule**

*** 搵 pt. of inflexion = 搵 x_0 使度 $y''(x_0)=0$**

*** 利用表格計算 pt. of inflexion 附近情況**

$f''(x) > 0 \rightarrow Concave\ upward$

$f''(x) < 0 \rightarrow Concave\ downward$

CONT'D

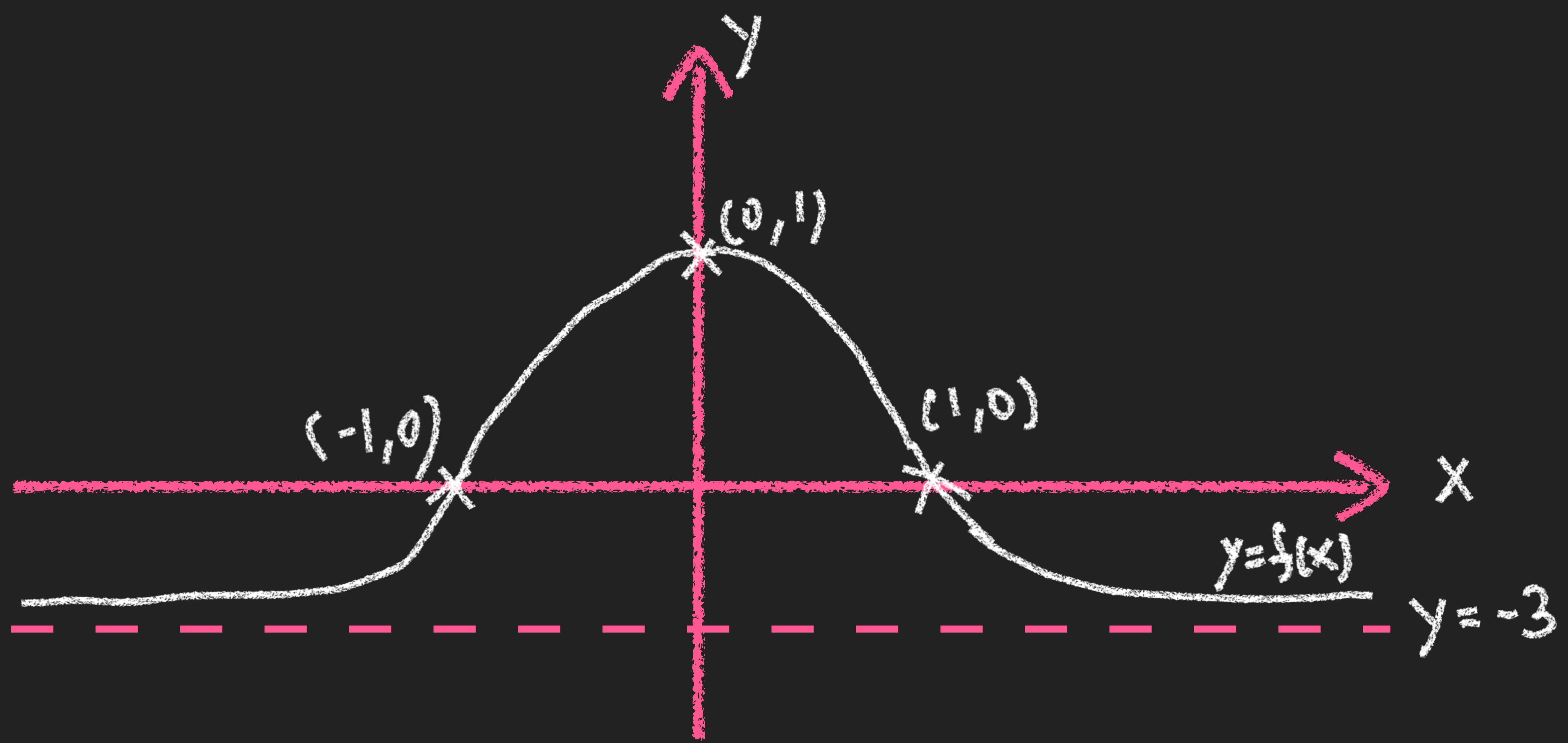
2013 – SECTION A

Vertical Asymptote : *No Vertical Asymptotes*

Horizontal Asymptote : $y = -3$

Oblique Asymptote : $y = -3$

$f(x)$ *is even function* $(f(-x) = f(x))$



* x 係幾多, 分母係零

* *Find* $\lim_{x\to\infty} y$

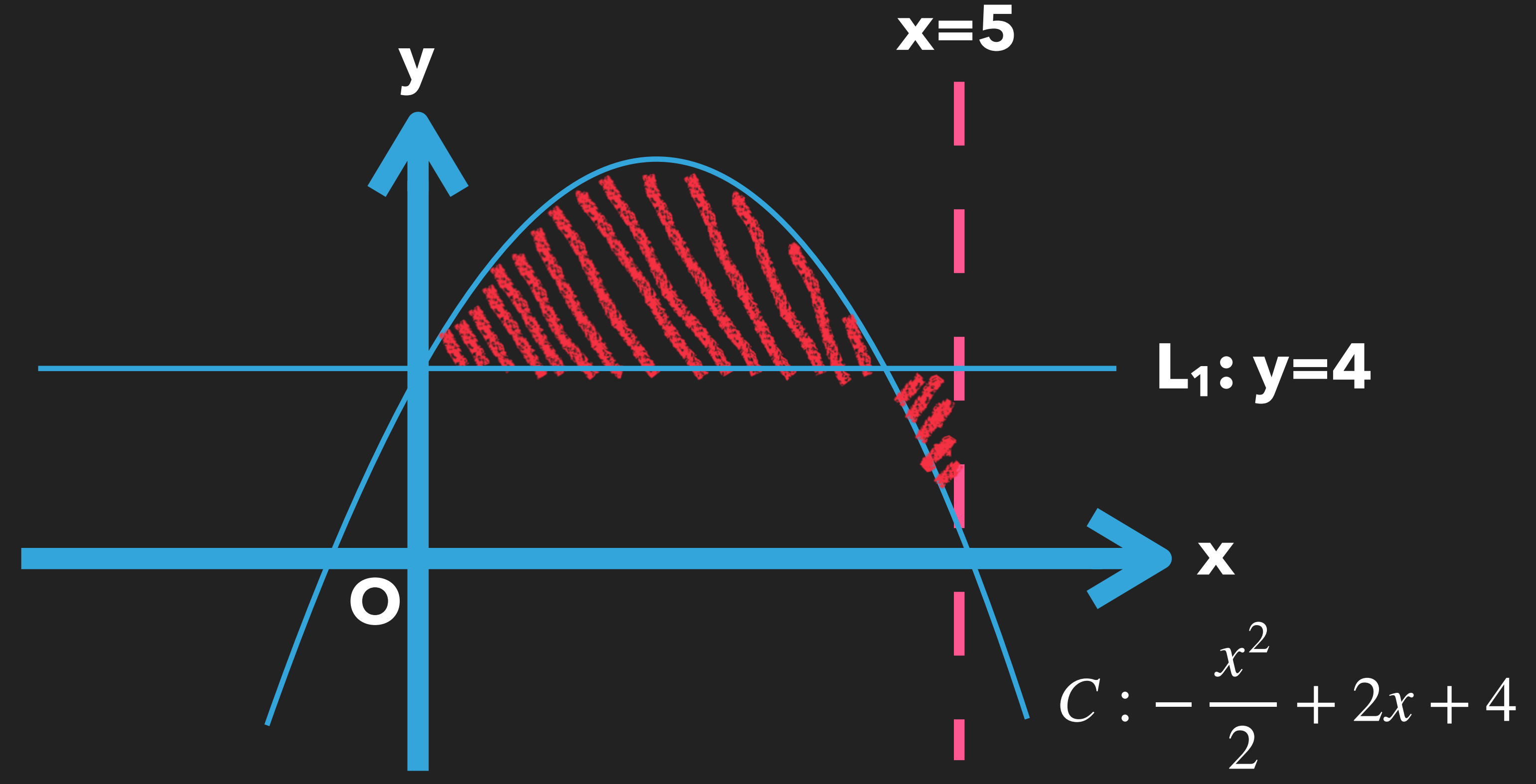
* *Find m and c such that* $\lim_{x\to\infty} [y - (mx + c)] = 0$

$$y = -3 + \frac{12}{3 + x^2} \to y - (-3) = \frac{12}{3 + x^2}$$

$$\to \lim_{x\to\infty} (y - (-3)) = 0$$

2013 - SECTION A

Q6.)



a.) Find the area of shaded region

b.) Find the volume of solid of revolution when the shaded region is revolved at $y = L_1$

* 參考課程 3.11 及 3.12

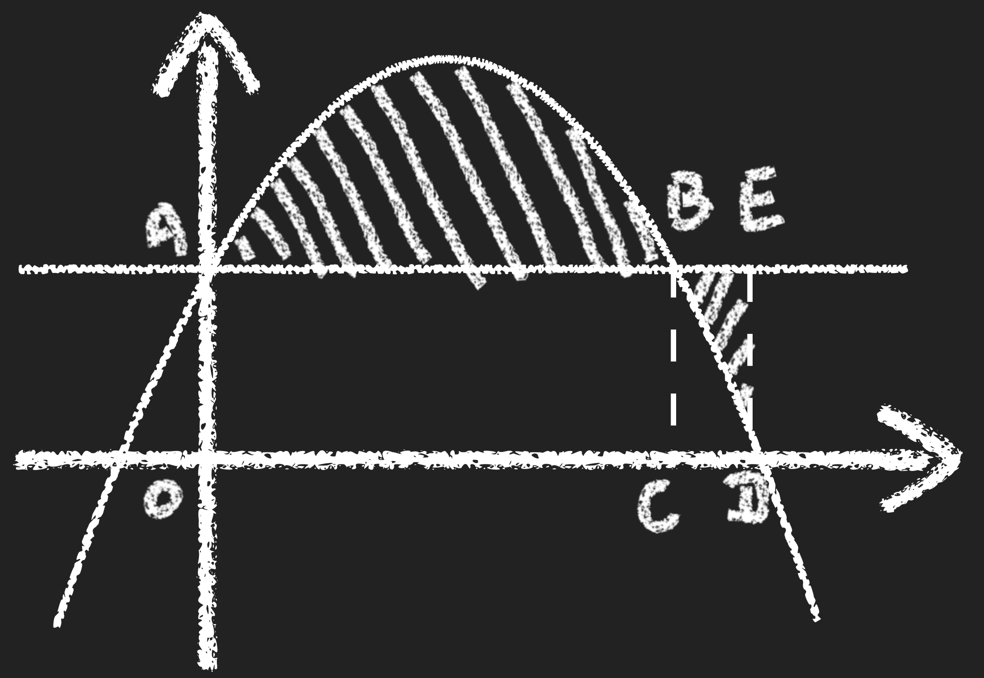
The interception of C and L_1 are $A = (0, 0)$ and $B = (4, 4)$

$$\text{Let } f(x) = -\frac{x^2}{2} + 2x + 4$$

CONT'D

2013 – SECTION A

a.) Consider the following figure



$$\begin{aligned} \text{The shaded area} &= \left(\int_0^4 f(x)dx - OABC\ Area\right) \\ &\quad + \left(BECD\ Area - \int_4^5 f(x)dx\right) \\ &= \left[-\frac{x^3}{6} + x^2 + 4x\right]_0^4 - (4)(4) \\ &\quad + (4)(5-4) - \left[-\frac{x^3}{6} + x^2 + 4x\right]_4^5 \\ &= \frac{13}{2}\ sq\ .\ unit \end{aligned}$$

* 利用基本幾何面積計算

* 面積大減細

CONT'D

2013 - SECTION A

$$b.)\ The\ volume = \pi\int_0^5 (f(x)-4)^2dx$$

$$= \pi\int_0^5 (2x-\frac{x^2}{2})^2dx = \pi\int_0^5 (\frac{x^4}{4}-2x^3+4x^2)dx$$

$$= \pi[\frac{x^5}{20}-\frac{x^4}{2}+\frac{4x^3}{3}]_0^5$$

$$= \frac{125\pi}{12}\ cu\,.\,unit\,.$$

*** 旋轉體積對應 y=a**

$$= \pi\int_{x_0}^{x_1} [f(x)-a]^2dx$$

2013 - SECTION A

$Q7.)$ $Prove\ tanx = \dfrac{sin2x}{1+cos2x}$ $and\ tanx = \dfrac{sin8xcos4xcos2x}{(1+cos8x)(1+cos4x)(1+cos2x)}$

* 參考課程 2.2

$$\frac{sin2x}{1+cos2x} = \frac{2sinxcosx}{1-(2cos^2x-1)} = \frac{sinx}{cosx} = tanx$$

$Then,$

$$\frac{sin8xcos4xcos2x}{(1+cos8x)(1+cos4x)(1+cos2x)} = \left(\frac{sin4x}{\cancel{cos4x}}\right)\frac{\cancel{cos4x}cos2x}{(1+cos4x)(1+cos2x)}$$

$$= \frac{sin4xcos2x}{(1+cos4x)(1+cos2x)}$$

$$= \left(\frac{sin2x}{\cancel{cos2x}}\right)\frac{\cancel{cos2x}}{(1+cos2x)} = tanx$$

* sin 雙角公式

* cos 雙角公式

* $tan4x = \dfrac{sin2(4x)}{1+cos2(4x)}$

* $tan2x = \dfrac{sin2(2x)}{1+cos2(2x)}$

2013 - SECTION A

Q8.) Find k, for $k \neq 0$ *and*

$$\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 1 \\ k & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

* 參考課程 4.7 及 4.8

方法1

$$\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 1 \\ k & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \rightarrow \begin{cases} x + k = 2 & \text{——— (1)} \\ 1 + z = 2 & \text{——— (2)} \\ kx = 1 & \text{——— (3)} \end{cases}$$

$$(1)\ and\ (3)\ give\ \frac{1}{k} + k = 2 \rightarrow k^2 - 2k + 1 = 0 \rightarrow k = 1$$

* 由 Matrix form 轉做多元方程

CONT'D

2013 – SECTION A

方法2

$$Let \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 1 \\ k & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \ be \ \boxed{MX = B}$$

$$M^{-1} = \frac{1}{k^2} \begin{pmatrix} 0 & 0 & k \\ k & 0 & -1 \\ -k & k^2 & 1 \end{pmatrix} \rightarrow \boxed{X = M^{-1}B}$$

$$\rightarrow \begin{pmatrix} x \\ 1 \\ z \end{pmatrix} = \frac{1}{k^2} \begin{pmatrix} 0 & 0 & k \\ k & 0 & -1 \\ -k & k^2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{k^2} \begin{pmatrix} k \\ 2k-1 \\ -2k+2k^2+1 \end{pmatrix}$$

$$\therefore \frac{2k-1}{k^2} = 1 \rightarrow k^2 - 2k + 1 = 0 \rightarrow k = 1$$

* 用 Row Deduction 或 Adj. Matrix 搵逆矩陣

* $MX = B \rightarrow M^{-1}MX = M^{-1}B \rightarrow IX = M^{-1}B$

$\rightarrow X = M^{-1}B$

2013 - SECTION A

Q9.) Find, a, b $(a, b \in \mathbb{R})$ *and solve (E) if (E) has infinite many solution*

$$\begin{cases} x - ay + z = 2 \\ 2x + (1-2a)y + (2-b)z = a+4 \\ 3x + (1-3a)y + (3-ab)z = 4 \end{cases} \text{——— (E)}$$

* 參考課程 4.7

$$(E):\left(\begin{array}{ccc|c} 1 & -a & 1 & 2 \\ 2 & 1-2a & 2-b & a+4 \\ 3 & 1-3a & 3-ab & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -a & 1 & 2 \\ 0 & 1 & -b & a \\ 0 & 1 & -ab & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -a & 1 & 2 \\ 0 & 1 & -b & a \\ 0 & 0 & \boxed{-b(a-1)} & \boxed{-2-a} \end{array}\right)$$

$\because$ (E) has infinite many solution

$\therefore -b(a-1) = 0$ and $-a-2=0$ $\rightarrow a = -2$ and $b = 0$

* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

* 如果 $\boxed{}$ = 0, 得直線答案

CONT'D

2013 - SECTION A

$$\text{Then, } (E) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & -2 \end{array}\right)$$

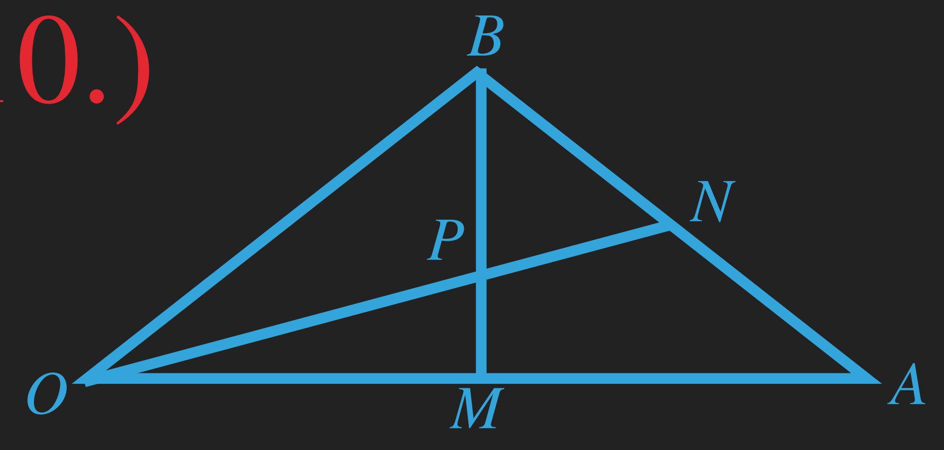
$$\text{Let } z = t,\ t \in \mathbb{R}$$

$$(x, y, z) = (6 - t,\ -2,\ t)$$

* 三條公式剩返兩條

2013 – SECTION A

Q10.)



$\overrightarrow{OA} = 2\hat{i}, \overrightarrow{OB} = \hat{i} + 2\hat{j}, OM = MA, and\ AN : NB = 1 : k$

$k = ?\ if\ P, N, M, A\ are\ concyclic$

* 參考課程 4.2 及 4.3

方法1

$$\overrightarrow{OA} \cdot \overrightarrow{MB} = \overrightarrow{OA} \cdot (\overrightarrow{OB} - \overrightarrow{OM}) = \overrightarrow{OA} \cdot (\overrightarrow{OB} - \frac{\overrightarrow{OA}}{2}) = 2\hat{i} \cdot (\hat{i} + 2\hat{j} - \hat{i}) = 0$$

$$\therefore \angle BMA = \frac{\pi}{2} \to \angle PNA = \frac{\pi}{2}\ \ (opp.\angle\ supp.)$$

$$i.e.\ \overrightarrow{ON} \cdot \overrightarrow{BA} = 0 \to \left(\frac{k}{k+1}\overrightarrow{OA} + \frac{1}{k+1}\overrightarrow{OB}\right) \cdot (\overrightarrow{OA} - \overrightarrow{OB}) = 0$$

$$\to ((2k+1)\hat{i} + 2\hat{j}) \cdot (\hat{i} - 2\hat{j}) = 0 \to (2k+1) - 4 = 0$$

$$\to k = \frac{3}{2}$$

* $\vec{a} \cdot \vec{b} = 0 \to \vec{a} \perp \vec{b}$

* 分割公式

CONT'D

2013 - SECTION A

方法2

$$\because \angle ONA = \angle BMA = \frac{\pi}{2}$$
$$\angle BAM = \angle OAN \ (common)$$
$$\angle MBA = \angle NOA \ (\angle s \ sum \ of \ \Delta)$$
$$\therefore \Delta ONA \sim \Delta BMA \ (AAA)$$

$$i.e. \frac{BA}{OA} = \frac{MA}{NA} \to \frac{OB}{OA} = \frac{MA}{\frac{OB}{k+1}} \quad (property \ of \ isos. \ \Delta)$$

$$\to OB^2 = \frac{k+1}{2} OA^2 \to 2|\overrightarrow{OB}|^2 = (k+1)|\overrightarrow{OA}|^2$$

$$\to k = \frac{10}{4} - 1 \to k = \frac{3}{2}$$

* Core 相似三角証明

* 相似三角邊比相等

* ΔOAB 係等腰三角形

  $OM \perp OA \ and \ OM = MA$

  $\therefore OB = BA$

* $|\overrightarrow{OA}|^2 = \overrightarrow{OA} \cdot \overrightarrow{OA} = OA^2$

2013 - SECTION B

$Q11.)$

$$a.)\ Find \int \frac{dx}{\sqrt{x^2-1}},\ for\ x>0,\ Hence\ solve \int_0^1 \frac{2x}{\sqrt{x^4+4x^2+3}}dx$$

$$b.) \int_0^{\frac{\pi}{4}} \frac{tan\theta}{\sqrt{1+2cos^2\theta}}d\theta = ?$$

* 參考課程 3.8 及 3.10

$$a.)\ Let\ I = \int \frac{dx}{\sqrt{x^2-1}},\ let\ x = sec\theta$$

$$I = \int \frac{sec\theta tan\theta d\theta}{\sqrt{\boxed{sec^2\theta - 1}}} = \int \frac{sec\theta \cancel{tan\theta} d\theta}{\boxed{\cancel{tan\theta}}} = \int sec\theta d\theta$$

$$I = \int \frac{\boxed{sec\theta(tan\theta + sec\theta)d\theta}}{(tan\theta + sec\theta)} = \int \frac{\boxed{d(sec\theta + tan\theta)}}{sec\theta + tan\theta}$$

* 利用三角代入, $x = sec\theta$

* $tan^2\theta + 1 = sec^2\theta$

* $d(sec\theta + tan\theta) = sec\theta(tan\theta + sec\theta)d\theta$

CONT'D

2013 - SECTION B

$$= ln|sec\theta + tan\theta| + C, \text{ where } C \text{ is constant}$$

$$= ln|sec\theta + \sqrt{sec^2\theta - 1}| + C = ln(x + \sqrt{x^2 - 1}) + C \quad (x > 0)$$

$$Let\ J = \int_0^1 \frac{2xdx}{\sqrt{x^4 + 4x^2 + 3}} = \int_0^1 \frac{d(x^2 + 2)}{\sqrt{(x^2 + 2)^2 - 1}}$$

$$= [ln((x^2 + 2) + \sqrt{(x^2 + 2)^2 - 1})]_0^1, \text{ from above result}$$

$$= ln(3 + \sqrt{8}) - ln(2 + \sqrt{3}) = ln(\frac{3 + 2\sqrt{2}}{2 + \sqrt{3}})$$

$$= ln(\frac{3 + 2\sqrt{2}}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}}) = ln(6 + 4\sqrt{2} - 3\sqrt{3} - 2\sqrt{6})$$

* $tan^2\theta + 1 = sec^2\theta$

* $d(x^2 + c) = 2xdx$

* $\int \frac{dx}{x^2 + Bx + C} \to \int \frac{dx}{(x + h)^2 + k}$

利用以上結果, $x \to x^2 + 2$

* $lnA - lnB = ln\frac{A}{B}$

* **Rationalization**

CONT'D

2013 - SECTION B

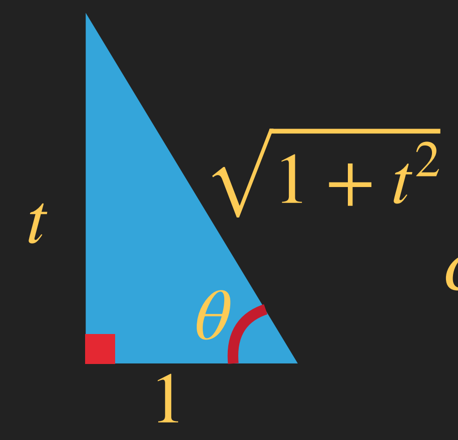
$$Let\ K = \int_0^{\frac{\pi}{4}} \frac{tan\theta}{\sqrt{1 + 2cos^2\theta}} d\theta,\ let\ t = tan\theta \rightarrow dt = sec^2\theta d\theta$$

$$Hence\ K = \int_0^1 \frac{tan\theta cos^2\theta dt}{\sqrt{1 + 2cos^2\theta}} = \int_0^1 \frac{t \cdot \frac{1}{1+t^2}}{\sqrt{1 + 2 \cdot \frac{1}{1+t^2}}} dt$$

$$= \int_0^1 \frac{t}{\sqrt{(1+t^2)(3+t^2)}} dt = \frac{1}{2} \int_0^1 \frac{2t}{\sqrt{t^4 + 4t^2 + 3}} dt$$

$$= \frac{1}{2} ln(6 + 4\sqrt{2} - 3\sqrt{3} - 2\sqrt{6})$$

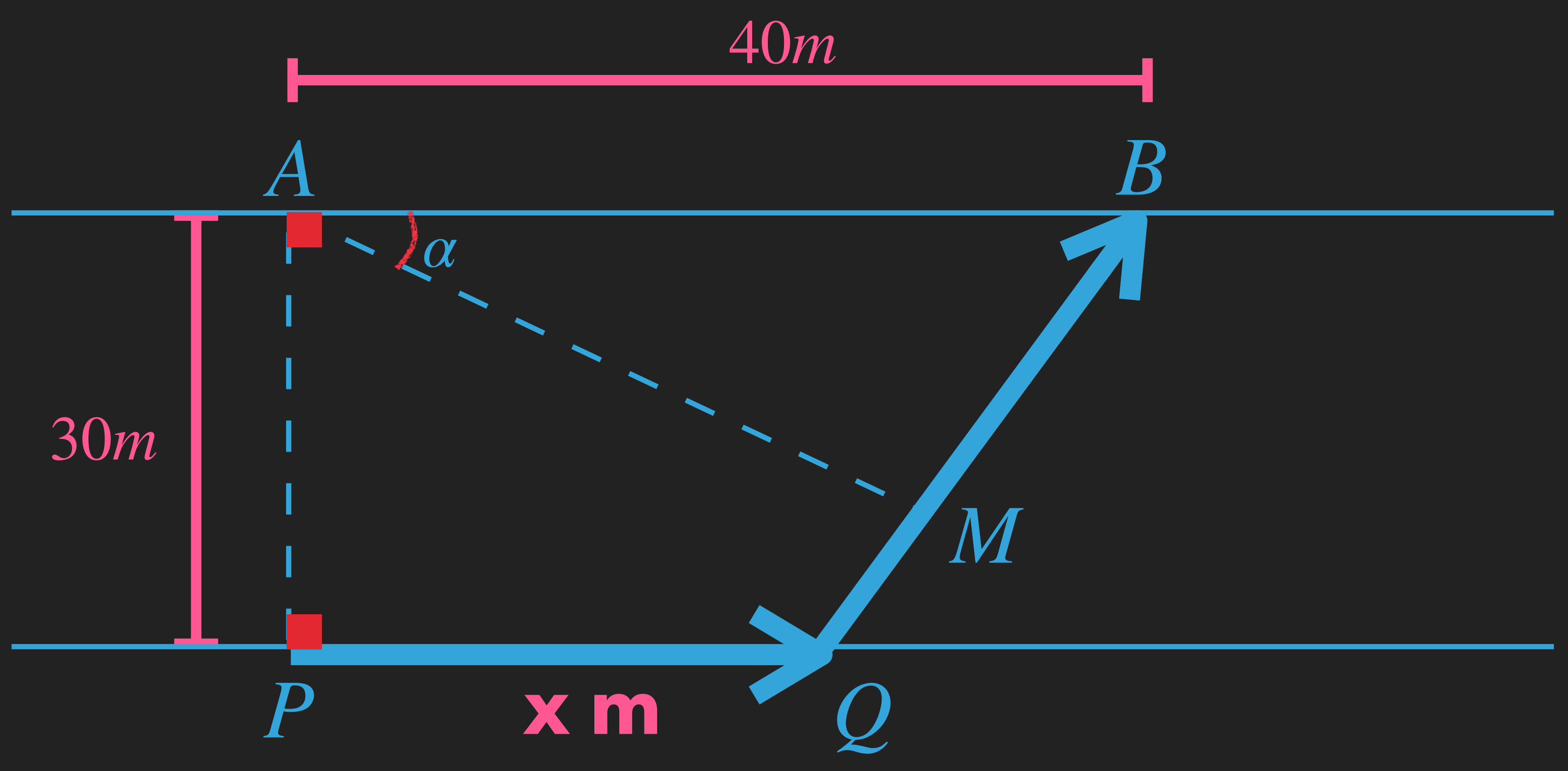
*** 利用 T-method,** $let\ t = tan\theta$



$$cos\theta = \frac{1}{\sqrt{1+t^2}}$$

2013 – SECTION B

*Q*12.)



In the figure, there is a particle moving from P to Q with constant speed $7\ ms^{-1}$. *Then move to B with a constant speed* $1.4\ ms^{-1}$.

Assume the total time travel $= T$ *second and* $PQ = x\ m$

a.) *When T is min .* $QB = ?$

b.) *By proving that* $MB = \dfrac{200\tan\alpha}{\tan\alpha + 2\sqrt{6}}$, *find the rate of change of* α

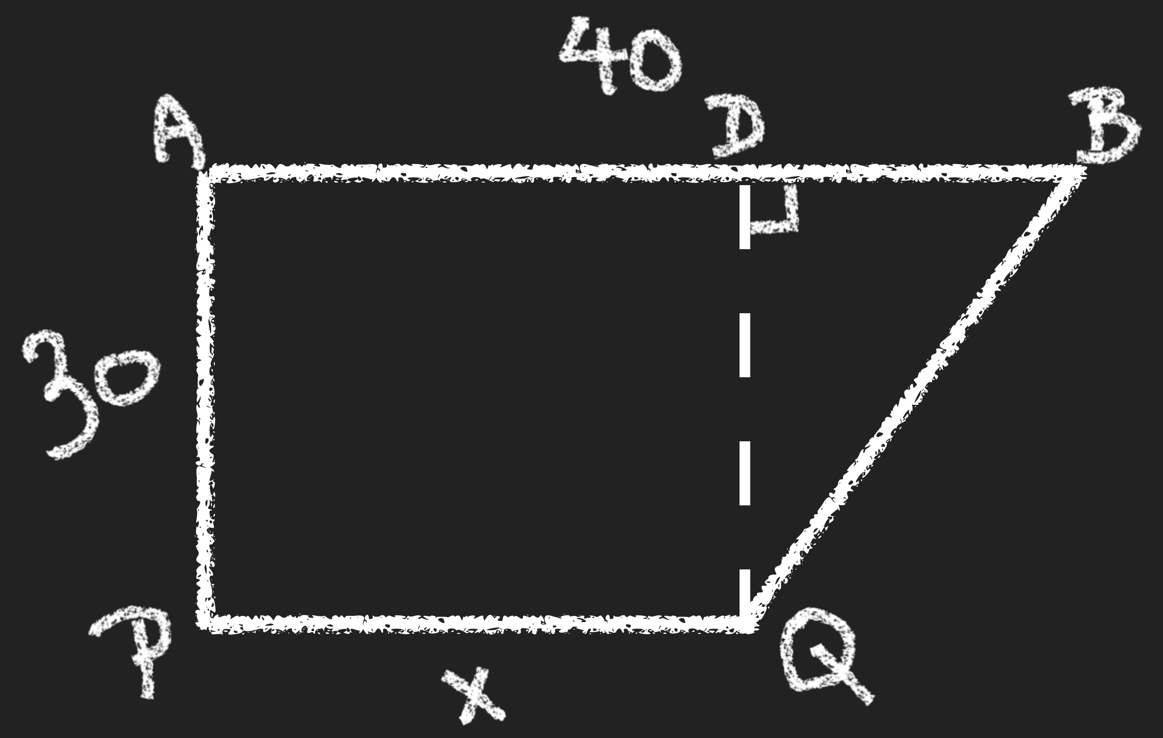
when $\alpha = 0.2$ *radian and T is min .*

* 參考課程 2.1 及 3.4

CONT'D

2013 – SECTION B

a.) Consider the following graph :



$$T = \frac{x}{7} + \frac{QB}{1.4}, \text{ and } QB = \sqrt{DB^2 + MQ^2}$$

$$\rightarrow T = \frac{x}{7} + \frac{\sqrt{(40-x)^2 + 30^2}}{1.4}$$

$$\rightarrow \frac{dT}{dx} = \frac{1}{7} + \frac{1}{2}\frac{2(40-x)(-1)}{1.4\sqrt{(40-x)^2+30^2}}$$

Let (x_0, T_0) be the turning point of T

$$\frac{dT}{dx}\Big|_{x=x_0} = \frac{1}{7} + \frac{1}{2}\frac{2y_0(-1)}{1.4\sqrt{y_0^2+30^2}} = 0, \text{ where } y_0 = 40 - x_0$$

* 時間=距離 / 速度

* Chain rule

CONT'D

2013 – SECTION B

$$\to 2\sqrt{y_0^2+30^2}-10y_0=0$$

$$\to 2\sqrt{y_0^2+30^2}=10y_0 \to 4(y_0^2+30^2)=100y_0^2$$

$$\to 24y_0^2=30^2 \to y_0=\frac{30}{2\sqrt{6}}=\frac{5\sqrt{6}}{2},(\because y_0>0)$$

	x < 40-y_0	x = 40-y_0	x > 40-y_0
T′	-	0	+
T	Dec.		Inc.

$$\therefore \textit{when } x_0=40-y_0, T \textit{ is min}.$$

$$i.e.\ QB=\sqrt{y_0^2+30^2}=\frac{25\sqrt{6}}{2}\ m$$

* y_0 = DB > 0

* 利用表格計算 turning point 附近上升定下降 從以決定是否局部最小值

$f'(x)>0 \to \textit{Increasing}$

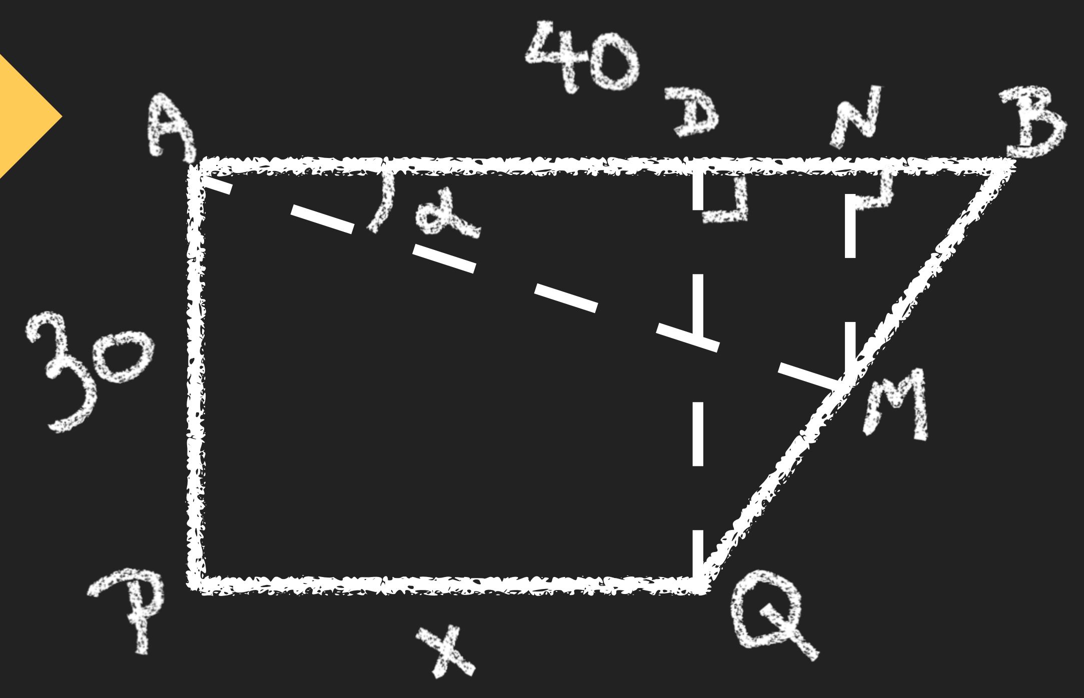
$f'(x)<0 \to \textit{Decreasing}$

CONT'D

2013 – SECTION B

b.) Consider the following graph :

方法1



$$\because \Delta NBM \sim \Delta MBQ$$

$$\frac{MB}{QB} = \frac{MN}{QD} = \frac{NB}{DB}, \quad tan\alpha = \frac{MN}{40 - NB}$$

$$\rightarrow tan\alpha = \frac{\frac{30MB}{QB}}{40 - \frac{DB \cdot MB}{QB}} \rightarrow (40QB - DB \cdot MB)tan\alpha = 30MB$$

$$\rightarrow (500\sqrt{6} - \frac{5\sqrt{6}}{2}MB)tan\alpha = 30MB \rightarrow (200 - MB)tan\alpha = \frac{12}{\sqrt{6}}MB$$

$$\rightarrow MB = \frac{200tan\alpha}{2\sqrt{6} + tan\alpha}$$

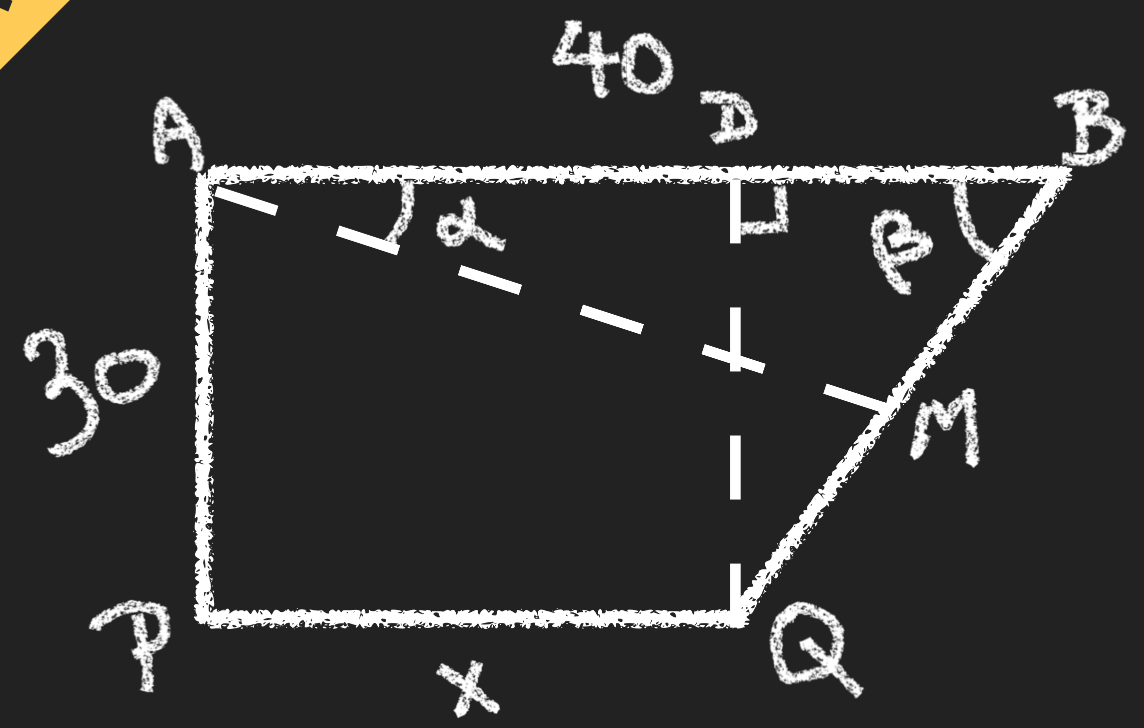
* Core 相似三角形証明 (AAA)

CONT'D

2013 – SECTION B

b.) Consider the following graph :

方法2



Sine law in ΔABM

$$\frac{40}{\sin(\pi-(\alpha+\beta))}=\frac{MB}{\sin\alpha}$$

$$\rightarrow \frac{40}{\sin(\alpha+\beta)}=\frac{MB}{\sin\alpha}$$

$$\rightarrow MB=\frac{40\sin\alpha}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}=\frac{40\tan\alpha}{\tan\alpha\frac{DB}{QB}+\frac{30}{QB}}$$

$$\rightarrow MB=\frac{200\tan\alpha}{2\sqrt{6}+\tan\alpha}$$

* $\sin(180^0-A)=\sin A$ 及複角公式

$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$

$2\sin A\sin B=\cos(A-B)-\cos(A+B)$

$\sin A+\sin B=2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A-\sin B=2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A+\cos B=2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A-\cos B=-2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

CONT'D

2013 - SECTION B

$\rightarrow (2\sqrt{6} + tan\alpha)MB = 200tan\alpha$

$\rightarrow MBsec^2\alpha\frac{d\alpha}{dt} + (2\sqrt{6} + tan\alpha)\boxed{\frac{dMB}{dt}} = 200sec^2\alpha\frac{d\alpha}{dt}$

$With\ \alpha = 0.2,\ MB = 7.94678,\ sec^2\alpha = 1.04109$

$Hence, \frac{d\alpha}{dt}|_{\alpha=0.2} = 0.0357\ (to\ 4\ d.p.)$

$\therefore\ The\ rate\ of\ change\ of\ \alpha\ is\ anticlockwise\ with\ 0.0357\ rad\ s^{-1}$

* Implicit 微分法

* $\boxed{\ }$ Constant speed 1.4 ms^{-1}

2013 – SECTION B

$Q13.)$

$a.)$ *Let M and N be any 2x2 matrix. Prove $tr(MN) = tr(NM)$ where $tr(X)$ is the sum of diagonal of any 2x2 matrix X*

$b.)$ *Let A, B and C be a 2x2 matrix such that*

$$B^{-1}AB = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } (E): \; C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } \lambda \in \mathbb{R}$$

Show that $\lambda^2 - tr(C)\lambda + |C| = 0$, if (E) has non – trival solution

$tr(A) = ?$, $|A| = ?$ *and* $\lambda = ?$ *if* $C = A$

* 參考課程 4.8, 4.9 及 4.11

CONT'D

2013 - SECTION B

$$a.)\ Let\ M=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\ and\ N=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$$

$$MN=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}=\begin{pmatrix}a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22}\\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}$$

$$MN=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix}b_{11}a_{11}+b_{12}a_{21} & b_{11}a_{12}+b_{12}a_{22}\\ b_{21}a_{11}+b_{22}a_{21} & b_{21}a_{12}+b_{22}a_{22}\end{pmatrix}$$

$$\therefore\ tr(MN)=tr(NM)$$

$$b.)\ C\begin{pmatrix}x\\ y\end{pmatrix}=\lambda\begin{pmatrix}x\\ y\end{pmatrix}\rightarrow(C-\lambda I_2)\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},Let\ C=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$$\because\ there\ is\ non-trival\ solution$$

$$\therefore\ |C-\lambda I_2|=0\rightarrow\begin{vmatrix}a-\lambda & b\\ c & d-\lambda\end{vmatrix}=0\rightarrow(a-\lambda)(d-\lambda)-bc=0$$

* MN 未必等如 NM

* (0, 0) 是唯一答案, 但題目指明有非零答案 (non-trival)

* Determinant=0

CONT'D

2013 - SECTION B

$$\rightarrow \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$$\rightarrow \lambda^2 - tr(C)\lambda + |C| = 0$$

$$tr(B^{-1}AB) = tr(\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}) = 1 + 3 = 4$$

$$\rightarrow tr(ABB^{-1}) = 4, \text{ (using a.) result with } M = B^{-1}, \text{ and } N = AB)$$

$$\rightarrow tr(A) = 4$$

$$\rightarrow |B^{-1}AB| = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3$$

$$\rightarrow |B^{-1}BA| = 3$$

$$\rightarrow |A| = 3$$

$$\lambda = 1 \text{ or } 3$$

* $B^{-1}B = = BB^{-1} = I$

* $|AB| = |BA|$

* *Eigenvalue of A*

2013 – SECTION B

Q14.) *There is a tetrahedron OABC with* $\angle AOB = \angle BOC = \angle COA = \dfrac{\pi}{2}$

Let P be the variable point and D be the fixed point, such that

$$\overrightarrow{AP}\cdot\overrightarrow{BP}+\overrightarrow{BP}\cdot\overrightarrow{CP}+\overrightarrow{CP}\cdot\overrightarrow{AP}=0 \qquad \overrightarrow{OD}=\frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$$

a.) Prove $|\overrightarrow{OP}|^2 = 2\overrightarrow{OP}\cdot\overrightarrow{OD}$*, and P is lies on the sphere with center D*

b.) Assume P_1, P_2 *and* P_3 *are the distinct points on the sphere mentioned above*

Prove P_1, P_2 *and* P_3 *lie on a circle with radius* $= OD$ *if* $\overrightarrow{DP_1}x\overrightarrow{DP_2} = \overrightarrow{DP_2}x\overrightarrow{DP_3}$

* 參考課程 4.4

CONT'D

2013 - SECTION B

$$a.)\ \overrightarrow{AP}\cdot\overrightarrow{BP}+\overrightarrow{BP}\cdot\overrightarrow{CP}+\overrightarrow{CP}\cdot\overrightarrow{AP}=0$$

$$\begin{aligned}\rightarrow\quad &(\overrightarrow{OP}-\overrightarrow{OA})\cdot(\overrightarrow{OP}-\overrightarrow{OB})\\ &+(\overrightarrow{OP}-\overrightarrow{OB})\cdot(\overrightarrow{OP}-\overrightarrow{OC})\\ &+(\overrightarrow{OP}-\overrightarrow{OC})\cdot(\overrightarrow{OP}-\overrightarrow{OA})=0\end{aligned}$$

$$\begin{aligned}\rightarrow\quad &|\overrightarrow{OP}|^2-(\overrightarrow{OA}+\overrightarrow{OB})\cdot\overrightarrow{OP}+\boxed{\overrightarrow{OA}\cdot\overrightarrow{OB}}\\ &+|\overrightarrow{OP}|^2-(\overrightarrow{OB}+\overrightarrow{OC})\cdot\overrightarrow{OP}+\boxed{\overrightarrow{OB}\cdot\overrightarrow{OC}}\\ &+|\overrightarrow{OP}|^2-(\overrightarrow{OC}+\overrightarrow{OA})\cdot\overrightarrow{OP}+\boxed{\overrightarrow{OC}\cdot\overrightarrow{OA}}=0\end{aligned}$$

$$\rightarrow 3|\overrightarrow{OP}|^2-2(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})\cdot\overrightarrow{OP}=0\rightarrow 3|\overrightarrow{OP}|^2=2\cdot 3\overrightarrow{OD}\cdot\overrightarrow{OP}$$

$$\rightarrow |\overrightarrow{OP}|^2=2\overrightarrow{OD}\cdot\overrightarrow{OP}$$

$* \ \boxed{}\ \vec{a}\cdot\vec{b}=0\rightarrow\vec{a}\perp\vec{b}$

CONT'D

2013 – SECTION B

$Consider,\ PD^2 = \boxed{|\overrightarrow{OP} - \overrightarrow{OD}|^2} = \boxed{(\overrightarrow{OP} - \overrightarrow{OD}) \cdot (\overrightarrow{OP} - \overrightarrow{OD})}$

$= \cancel{|\overrightarrow{OP}|^2 - 2\overrightarrow{OP} \cdot \overrightarrow{OD}} + |\overrightarrow{OD}|^2 = OD^2$

$\because PD\ is\ always = OD$

$\therefore\ P\ lies\ on\ the\ sphere\ with\ center\ at\ D\ and\ radius = OD$

$b.)\ Assume\ \overrightarrow{DP_1} x \overrightarrow{DP_2} = \overrightarrow{DP_2} x \overrightarrow{DP_3} = k\hat{n}$

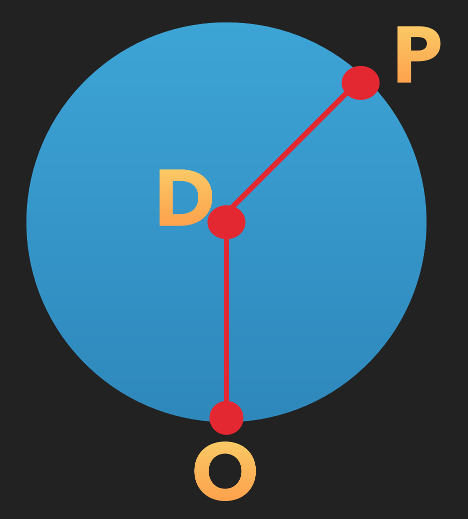
$\rightarrow D,\ P_1,\ P_2\ and\ P_3\ are\ coplaner\ lie\ on\ the\ plane\ with\ normal\ k\hat{n}$

$Also, OP_1 = OP_2 = OP_3 = OD = radius\ of\ a\ sphere$

$\therefore\ P_1,\ P_2\ and\ P_3\ lie\ on\ the\ circle\ with\ center\ at\ D\ and\ radius = OD$

* $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

* $P\ is\ variable\ with\ PD = OD$



* $\vec{a} x \vec{b}$ 計緊平面的

Normal vector