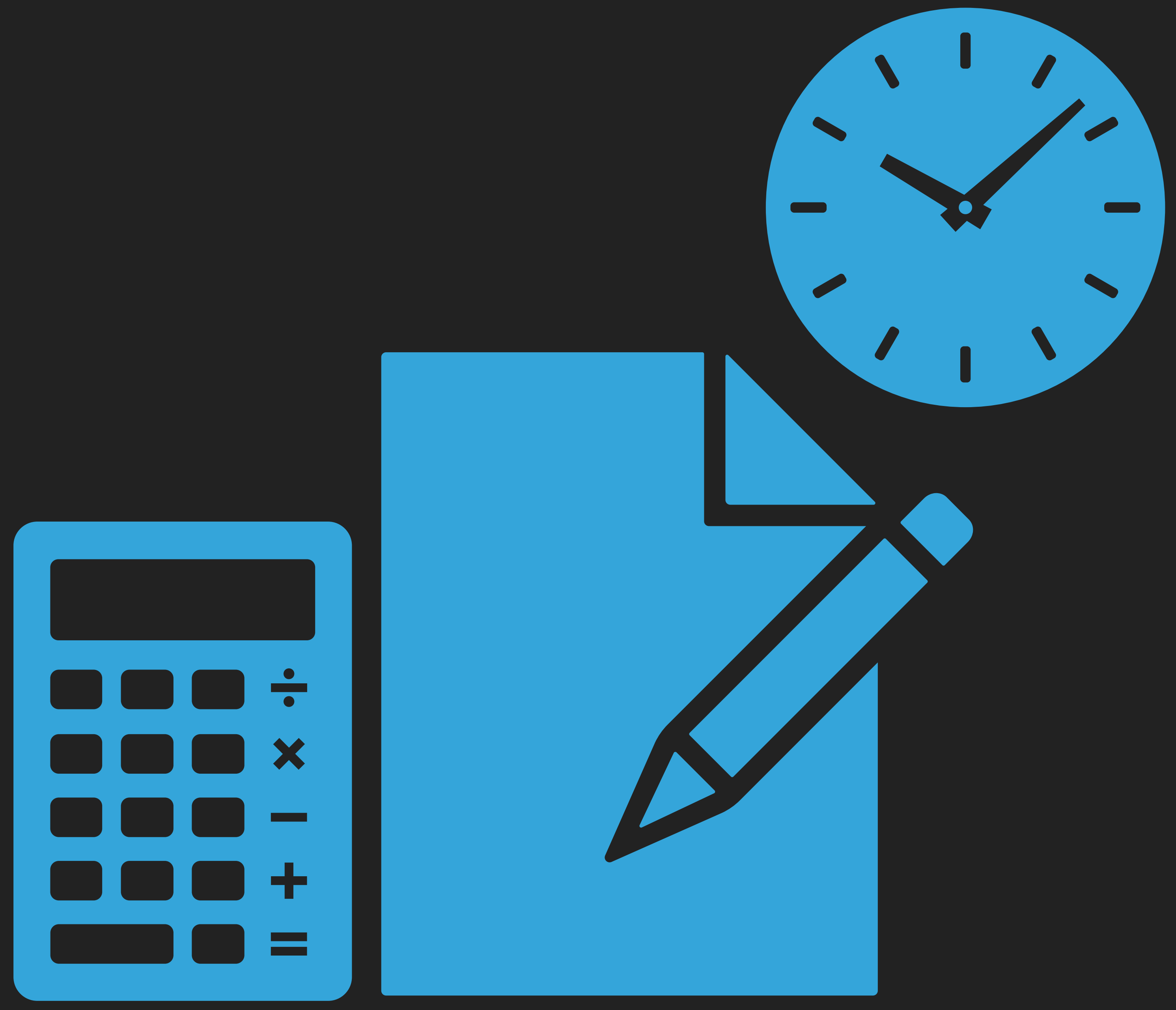
深宵教室 - DSE M1 模擬試題解答

2019

2019

- Section A
- Section B



2019 - SECTION A

Q1.) Let X be discrete random vaiable, where b is a constant.

k	8	11	b	27	32
$P(X=k)$	0.2	0.1	0.3	0.3	0.1

Given that $Var(X)=66$, find b, $E(3X+5)$ and $Var(3X+5)$.

* 參考課程 4.1, 4.3 及 4.4

$$E(X) = 8\cdot 0.2 + 11\cdot 0.1 + b\cdot 0.3 + 27\cdot 0.3 + 32\cdot 0.1 = 14 + 0.3b$$

$$E(X^2) = 8^2\cdot 0.2 + 11^2\cdot 0.1 + b^2\cdot 0.3 + 27^2\cdot 0.3 + 32^2\cdot 0.1 = 346 + 0.3b^2$$

$$Var(X) = E(X^2) - [E(X)]^2 = 346 + 0.3b^2 - (14+0.3b)^2 = 66$$

$$\to 0.21b^2 - 8.4b + 84 = 0$$

$$\to 0.21(b-20)^2 = 0$$

$$\to b = 20 \ (repeated)$$

$$E(3X+5) = 3E(X) + 5 \text{ and } Var(3X+5) = 3^2 Var(X)$$

$$= 65 \qquad\qquad = 594$$

* $E(X) = \sum_{i=1}^{n} k_i P(X=k_i)$

* $Var(X) = E(X^2) - [E(X)]^2$

* $a^2 - 2ab + b^2 \equiv (a-b)^2$

* $E(aX+b) = aE(X) + b$

* $Var(aX+b) = a^2 Var(X)$

2019 - SECTION A

Q2.) Let A and B be the event such that $P(A' \cap B) = 0.12$ and $P(B'|A') = 2P(A)$ where X' is the complementary event of X, Find P(A) and P(B) if A and B are independent.

* 參考課程 4.1 及 4.2

$$P(A' \cap B) = P(B|A')P(A') = [1 - P(B'|A')][1 - P(A)]$$

$$\rightarrow 0.12 = [1 - 2P(A)][1 - P(A)] = 1 - 3P(A) + 2P(A)^2$$

$$\rightarrow 0.88 - 3P(A) + 2P(A)^2 = 0$$

$$\rightarrow P(A) = 0.4 \text{ or } 1.1 \ (rejected) = 0.4$$

$$\because A \text{ and } B \text{ are independent} \rightarrow A' \text{ and } B \text{ are independent}$$

$$\therefore P(A' \cap B) = P(A')P(B) = [1 - P(A)]P(B)$$

$$\rightarrow 0.12 = 0.6P(B) \rightarrow P(B) = 0.2$$

* P(Not A) = 1 - P(A)
* P(A & B)=P(A|B)P(B)=P(B|A)P(A)

* P(A) 係 0 同 1 之間

* 如果獨立事件, P(A & B)=P(A)P(B)

2019 - SECTION A

Q3.) Given that the probability of winning a big prize is 0.2 for each draw. When the big prize is won, the draw will stop on that day and re – start next day.

a.) Find the mean and variance of the number of draw for winning big prize in a day.

b.) Are the probability of winning a big prize and not winning a big prize within 4 draw in a day equal? Explain your answer.

c.) Find the probability of not winning a big prize within 4 draws in each day for 5 days.

* 參考課程 4.4

a.) Let X be the random variable of the number of draws for winning in a day $\rightarrow X \sim G(0.2)$

$$E(X) = \frac{1}{0.2} = 5 \text{ and } Var(X) = \frac{1-0.2}{0.2^2} = 20$$

b.) $$P(X \le 4) = 0.2 \cdot \boxed{\sum_{k=0}^{3} (1-0.2)^k} = 0.2 \cdot \boxed{\frac{1-0.8^4}{0.2}} = 0.5904$$

$$\therefore P(X > 4) = 1 - P(X \le 4) \neq P(X \le 4)$$

c.) The probability $= [1 - P(X \le 4)]^5 = 0.0115$ (to 4 d.p.)

* $X \sim G(p)$, $E(X) = \frac{1}{p}$, $Var(X) = \frac{1-p}{p^2}$

* ▇ 等比數列之和

2019 - SECTION A

Q4.) In each month, there is 0.35 *chance for products have discount. If the products have discount, they have* 0.7 *chance to make profit, otherwise, they have* 0.28 *chance to make profit in a month. Find the probability :*

a.) The products make profit in a certain moth.

b.) The products have discount if they make profit in a certain month.

c.) The products make profit in at least 2 months throughout a year.

* 參考課程 4.2 及 4.4

a.) Let D be the event of products have discount in a month.

B be the event of products have profit in a month.

$$P(B) = 0.35 \cdot 0.7 + (1 - 0.35) \cdot 0.28 = 0.427$$

$$b.)\ P(D|B) = \frac{P(D \cap B)}{P(B)} = \frac{0.35 \cdot 0.7}{0.427} = 0.5738\ (to\ 4\ d.p.)$$

c.) Let X be the random varible of the number of months have profit in a year. $X \sim B(12, P(B))$

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - P(\bar{B})^{12} - C_1^{12} P(B) P(\bar{B})^{11}$$

$$= 0.9875\ (to\ 4\ d.p.)$$

* P(Not A) = 1 - P(A)

* 條件概率

* $P(X = k) = C_k^n p^k (1 - p)^{n-k}$

2019 - SECTION A

Q5.) Let $f(x) = (6 - x)(x + 3)^{-1}$, *for* $x > -3$.

a.) *Prove* $f(x)$ *is decreasing and find* $\lim_{x \to \infty} f(x)$.

b.) *Find the area of the region bounded by* $C : y = f(x)$, $x-$*axis, and* $y-$*axis*.

* 參考課程 2.1, 2.2, 2.4, 2.8 及 2.9

a.) $f(x) = \dfrac{6 - x}{x + 3} = \dfrac{9}{x + 3} - 1 \to f'(x) = -\dfrac{9}{(x + 3)^2} < 0$, *for* $x > -3$

$\therefore f(x)$ *is decreasing function*

$\lim_{x \to \infty} f(x) = -1$

b.) *Obviously, the* $x-$*interception* $= 6$

The area $= \displaystyle\int_0^6 \frac{9}{x + 3} - 1 dx = [9ln(x + 3) - x]_0^6$

$= 9ln3 - 6$ *unit*^{2}

* f'(x) 細過 0, f(x) 係遞減函數

* 分母越大, 個數趨向 0

2019 - SECTION A

Q6.) Let $f(x) = e^{-18x}(1+4x)^n$. *If the coefficient of* $x^2 = -38$, *find* n, *for* $n \in \mathbb{Z}^+$.

* 參考課程 1.1 及 3.2

$$e^{-18x}(1+4x)^n = \left(1 - 18x + \frac{(-18x)^2}{2!} + \ldots\right)\left(1 + C_1^n(4x) + C_2^n(4x)^2 + \ldots\right)$$

Given, the coefficient of $x^2 = -38$

$$\rightarrow \frac{18^2}{2} - 18C_1^n(4) + C_2^n(4^2) = -38$$

$$\rightarrow 162 - 72n + 8n(n-1) = -38$$

$$\rightarrow n^2 - 10n + 25 = 0$$

$$\rightarrow n = 5 \text{ (repeated)}$$

* $(a+b)^n = \sum_{r=0}^{n} C_r^n a^r b^{n-r}$

* $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

* $C_r^n = \frac{n!}{r!(n-r)!}$

$\rightarrow C_1^n = n$ *and* $C_2^n = \frac{n(n-1)}{2}$

2019 - SECTION A

Q7.) Let the curve $C : y = f(x)$ and $f(x) = (x-2)\sqrt{3x+6} - 8x$, where $x > -2$.
Are there 2 horizontal tangents to C? Explain your answer.

* 參考課程 2.3 及 2.4

$$f(x) = (x-2)(3x+6)^{\frac{1}{2}} - 8x$$

$$\rightarrow f'(x) = (3x+6)^{\frac{1}{2}} + \frac{3}{2}(x-2)(3x+6)^{\frac{-1}{2}} - 8 = \frac{3(3x+2)}{2\sqrt{3x+6}} - 8$$

Assume $f'(x_0) = 0$, *for* $x_0 > -2$

$$0 = \frac{3(3x_0+2)}{2\sqrt{3x_0+6}} - 8 \rightarrow 16\sqrt{3x_0+6} = 3(3x_0+6-4)$$

$$\rightarrow 3(\sqrt{3x_0+6})^2 - 16\sqrt{3x_0+6} - 12 = 0$$

$$\rightarrow \sqrt{3x_0+6} = 6 \text{ or } -\frac{2}{3} \text{ (rejected)}$$

$\because$ *There is only one solution for* x_o

$\therefore$ *There is only one horizontal tangent to C*

* 用 Product rule

* 搵 turning point = 搵 x_0 使度 $f'(x_0)=0$

* 開方需為正數

2019 - SECTION A

Q8.) Let $h(x) = x7^{-x}$, for all $x \in \mathbb{R}$. Given that $h'(x) = 0$ has only one real root, α.

Find α and $\int_0^\alpha h(x)dx$ in term of e.

* 參考課程 2.2, 2.3 及 2.8

$$Let\ y(x) = 7^{-x},\ then,\ ln[y(x)] = -xln7 \rightarrow y'(x) = -y(x)ln7$$

$$Hence,\ h(x) = xy(x) \rightarrow h'(x) = y(x) - xy(x)ln7 = y(x)[1 - xln7]$$

$$Given,\ h'(\alpha) = 0 \rightarrow 1 - \alpha ln7 = 0 \rightarrow \alpha = \frac{1}{ln7}$$

$$Let\ I = \int_0^\alpha xy(x)dx = -\frac{1}{ln7}\int_0^\alpha xd[y(x)],\ where\ d[y(x)] = -y(x)ln7dx$$

$$= -\frac{1}{ln7}[xy(x)]_0^\alpha + \frac{1}{ln7}\int_0^\alpha y(x)dx = -\frac{\alpha y(\alpha)}{ln7} - \frac{1}{(ln7)^2}\int_0^\alpha y'(x)dx$$

$$= -\frac{y(\alpha)}{(ln7)^2} - \frac{y(\alpha) - y(0)}{(ln7)^2}$$

* Implicit 微分法

* 積分三寶: 積分代入法

* 積分三寶: Integration by part

CONT'D

2019 - SECTION A

$$\because ln[y(\alpha)] = -\alpha ln7 = -\frac{1}{ln7}\cdot ln7 = -1 \rightarrow y(\alpha) = e^{-1}$$

$$\therefore I = -\frac{e^{-1}}{(ln7)^2} - \frac{e^{-1}-1}{(ln7)^2} = \frac{1-2e^{-1}}{(ln7)^2}$$

* ln 係 e 的逆函數

2019 - SECTION B

Q9.) The number of matches won by a basketball team in a season follows $Po(3)$. *The points scored by the team in a match follows* $N(66, 10^2)$.

a.) Find the probability the team wins fewer than 6 matches in a certain season.

b.) Find the probability the team scores higher than 70 points in a certain match.

c.) The team receives bouns if the team wins a match and scores more than 70 points in that match for more than 2 matches in a season. Find the probability :

i.) The team wins exactly 3 matches and is awarded a bouns in a certain season.

ii.) The team is awarded a bouns if the team wins exactly 4 matches in a certain season.

iii.) The team is awarded a bouns if the team wins fewer than 6 matches in a certain season.

*** 參考課程 4.2, 4.4 及 4.5**

a.) Let X be random variable of number of winning matches of the team in certain season.
Y be random variable of scores of the team in a certain match

$$P(X < 6) = \boxed{e^{-3}\left[1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!}\right]} = 0.9161 \ (to\ 4\ d.p.)$$

* $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

CONT'D

2019 - SECTION B

$$b.)\ P(Y>70)=P\left(Z>\frac{70-66}{10}\right)=P(Z>0.4)=0.3446$$

* 先計算 Z ~ N(0,1), 再對表

$ci.)$ *Let B be the event of the team is awarded a bouns in a certain season.*

$$P(B\cap X=3)=P(X=3)[P(Y>70)]^3=\frac{e^{-3}3^3}{3!}(0.3446)^3$$

$$=0.0092\ (to\ 4\ d.p.)$$

* 三場都贏並三場取 70 分以上
* $P(X=k)=\dfrac{e^{-\lambda}\lambda^k}{k!}$

$$ii.)\ P(B\,|\,X=4)=\frac{P(B\cap X=4)}{P(X=4)}$$

$$=\frac{\cancel{P(X=4)}([P(Y>70)]^4+C^4_3[P(Y>70)]^3[1-P(Y>70)])}{\cancel{P(X=4)}}$$

$$=0.1214\ (to\ 4\ d.p.)$$

* 條件概率
* 四場都贏並四場取 70 分以上
* $P(X=k)=C^n_k p^k(1-p)^{n-k}$ 四場都贏並三場取 70 分以上

CONT'D

2019 - SECTION B

$$ii.)\ P(B\,|\,X<6)=\frac{P(B\cap X<6)}{P(X<6)}$$

$$=\frac{P(B\cap X=3)+P(B\cap X=4)+P(B\cap X=5)}{P(X<6)}$$

$$Let\ p=P(Y>70)$$

$$Then,\ P(B\cap X=5)=P(X=5)[p^5+C^5_4p^4(1-p)+C^5_3p^3(1-p)]$$

$$=0.02287$$

$$\therefore P(B\,|\,X<6)=\frac{0.0092+\frac{e^{-3}(3)^4}{4!}\cdot 0.1214+0.02287}{0.9161}$$

$$=0.0572\ (to\ 4\ d.p.)$$

* 條件概率

* 五場都贏並五場取 70 分以上

* 五場都贏並四場取 70 分以上

* 五場都贏並三場取 70 分以上

* $P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$

2019 – SECTION B

Q10.) The gas consumption in a month of each family in HK follows $N(\mu, 4^2)$ *in* m^3.

a.) A random sample of 16 families is selected. Their consumption in a month as shown below: 17 17 18 19 19 20 20 21 21 21 22 23 23 23 24 24

Find a 95% confidence interval for μ.

b.) Find the least sample size to be taken such that the width of a 99.5% confidence interval for μ *is less than 3.*

c.) Suppose $\mu = 20$. *The family is defined as 'Normal' if their gas consumption in a month lies between* $18m^3$ *and* $23m^3$.

i.) Find the percentage of 'Normal' family in HK.

ii.) The family in HK is randomly selected one by one until there are 3 'Normal' families selected. Given that there are more than 6 families selected, find the probability there are 9 families are selected.

* 參考課程 **4.2, 4.4, 4.6** 及 **4.7**

a.) The sample mean, $\bar{X} = \dfrac{2(17+19+20+24)+18+22+3(21+23)}{16}$

* 平均值 = 數據總和 / 總數

CONT'D

2019 - SECTION B

$\rightarrow \bar{X} = 20.75$

$\therefore \text{The } 95\,\%\ C.I. = (\bar{X} - 1.96 \cdot \frac{4}{\sqrt{16}}, \bar{X} + 1.96 \cdot \frac{4}{\sqrt{16}})$

$= (18.79, 22.71)$

* 95% 置信區間

b.) *Let the required sample size be n.*

$\text{The width of } 99.5\,\%\ C.I. = 2 \cdot 2.807 \cdot \frac{4}{\sqrt{n}} < 3 \rightarrow n > 56.0302$

$\therefore$ *The least sample size* $= 57.$

* 99.5% 置信區間

c.) *The percentage* $= P(\frac{18-20}{4} < Z < \frac{23-20}{4}) = P(-2 < Z < 0.75)$

$= 46.49\,\%$

* 先計算 Z ~ N(0,1), 再對表

d.) *The percentage of more than 6 families selected,* p_1

$= C_2^6 0.4649^2(1-0.4649)^4 + C_1^6 0.4649(1-0.4629)^5 + (1-0.4649)^6$

* 頭6次只有2個Normal
* 頭6次只有1個Normal
* 頭6次只有0個Normal

CONT'D

2019 - SECTION B

$$\textit{The percentage of } 3^{rd} \textit{ 'Normal' family at } 9^{th} \textit{ selected family, } p_2$$

$$= C_2^8(0.4649)^2(1-0.4649)^6 \cdot (0.469)$$

$$\textit{The probability} = \frac{p_2}{p_1} = 0.1604 \ (to\ 4\ d.p.)$$

* 8個有2個Normal
* 第9個係Normal

* 條件概率

2019 - SECTION B

Q11.) Let $f(t) = 2t\ln(t^2 + 4)$ and $I = \int_0^4 f(t)dx$. There are two method to estimate I.

$M1$: By the trapezoidal rule with 4 sub − interval

$M2$: By replacing $f(t) = 4\ln(2e^t + 1)(e^{-t} + 2)^{-1}$

a.) Find the estimation based on $M1$ and $M2$.

b.) Determine if over − estimation or under − estimation for $M1$

c.) Is I exceeds 30 % of the sum of I and the estimation of $M2$? Explain your answer.

* 參考課程 2.2, 2.3, 3.2 及 3.3

a.) Let I_1 be the estimation by $M1$

I_2 be the estimation by $M2$

$$I_1 = \frac{4-0}{4\cdot 2}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)]$$

$$= 38.9093 \ (to\ 4\ d.p.)$$

* 計算梯形面積的加總

CONT'D

2019 - SECTION B

$$I_2 = \int_0^4 \frac{4ln(2e^t+1)}{e^{-t}+2}dt, \ Let\ u = ln(2e^t+1) \rightarrow du = \frac{2e^t dt}{2e^t+1}$$

$$\rightarrow du = \frac{2dt}{2+e^{-t}}$$

$$\rightarrow I_2 = \int_{ln3}^{ln(2e^4+1)} 2udu \rightarrow I_2 = [ln(2e^4+1)]^2 - [ln3]^2$$

$$b.)\ Consider\ f'(t) = 2ln(t^2+4) + \frac{4t^2}{t^2+4} = 2ln(t^2+4) + 4 - \frac{16}{t^2+4}$$

$$\rightarrow f''(t) = \frac{4t}{t^2+4} + \frac{32t}{(t^2+4)^2} > 0,\ for\ 0 \le t \le 4$$

$$i.e.\ I_1\ is\ over-estimated$$

$$c.)\ Since\ I < I_1,\ (I_1\ is\ over-estimated)$$

* 積分三寶: 積分代入

* 定積分代入要改範圍

* Product rule

* 個 f(t) 係凹口向上

CONT'D

2019 – SECTION B

$$Hence,\ \frac{(I+I_2)-I}{I+I_2}x100\,\% > \frac{I_2}{I_1+I_2}x100\,\% \approx 34.95\,\% > 30\,\%$$

* a < b, 1/a > 1/b

$\therefore I\ exceeds\ 30\,\%\ of\ the\ sum\ of\ I\ and\ the\ estimation\ of\ M2.$

2019 - SECTION B

$Q12.)$ Given that $V = \dfrac{64}{he^{kt}+4}$, where h and $k \in \mathbb{R}$, $t \geq 0$

a.) Express $\ln(\dfrac{64}{V} - 4)$ as a linear function of t

b.) Given that the linear function in a.) passing through $(0,0)$ and $(2,1)$. Find h, k and the valve of V when $V'(t)$ attains its least valve.

c.) Give $S = V^{\frac{2}{3}}$. Find $S'(t)$ when $V'(t)$ attains its least valve. Is it also the least value of $S'(t)$? Explain your answer.

* 參考課程 2.2, 2.3, 2.4 及 3.1

$$a.)\ \ln(\frac{64}{V} - 4) = \boxed{\ln(he^{kt})} = \boxed{\ln h + kt}$$

$b.)$ Given that the function passing through $(0,0)$ and $(2,1)$

$\rightarrow \ln h = 0$ and $1 = \ln h + 2k$

$\rightarrow h = 1$ and $k = 0.5$

* ln(AB) = lnA + lnB

* 直線方程, y = (斜率)x + (y-intercept)

CONT'D

2019 - SECTION B

$$Then\ V = \frac{64}{e^{0.5t}+4} \to \frac{dV}{dt} = \boxed{-\frac{64 \cdot 0.5e^{0.5t}}{(e^{0.5t}+4)^2}}$$

$$\to \frac{dV}{dt} = -32[\frac{e^{0.5t}+4-4}{(e^{0.5t}+4)^2}] = -32[\frac{1}{e^{0.5t}+4} - \frac{4}{(e^{0.5t}+4)^2}]$$

$$\to \frac{d^2V}{dt^2} = -32[\boxed{\frac{-0.5e^{0.5t}}{(e^{0.5t}+4)^2}} + \boxed{\frac{4 \cdot 2 \cdot 0.5e^{0.5t}}{(e^{0.5t}+4)^3}}]$$

$$= \frac{16e^{0.5t}}{(e^{0.5t}+4)^2}[1 - \frac{8}{e^{0.5t}+4}]$$

$$Assume\ \frac{d^2V}{dt^2}|_{t=t_0} = 0 \to 1 = \frac{8}{e^{0.5t_0}+4} \to t_0 = ln16$$

	0 < t < ln16	t = 16	t > ln16
V''(t)	-	0	+
V'(t)	Dec.		Inc.

* Chain rule

* 搵 turning point = 搵 t_0 使度 $V''(t_0)=0$

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \to Increasing$

$f'(x) < 0 \to Decreasing$

CONT'D

2019 - SECTION B

$$Hence,\ when\ t = ln16,\ \frac{dV}{dt}\ attains\ least\ value\ .$$

$$\therefore\ V(ln16) = 8$$

$$c.)\ S = V^{\frac{2}{3}} \rightarrow S'(t) = \frac{2}{3}[V(t)]^{-\frac{1}{3}}V'(t) \rightarrow S'(ln16) = \frac{2}{3(8)^{\frac{1}{3}}}V'(ln16)$$

$$= -\frac{2}{3}$$

$$Also,\ S''(t) = \frac{2}{3}[-\frac{1}{3}[V(t)]^{-\frac{4}{3}}[V'(t)]^2 + [V(t)]^{-\frac{1}{3}}V''(t)]$$

$$\rightarrow S''(ln16) = \frac{2}{3}[-\frac{1}{3}[8]^{-\frac{4}{3}}[-2]^2] \neq 0$$

$$\therefore\ It\ is\ not\ the\ least\ value\ of\ S'(t)$$

* Chain rule