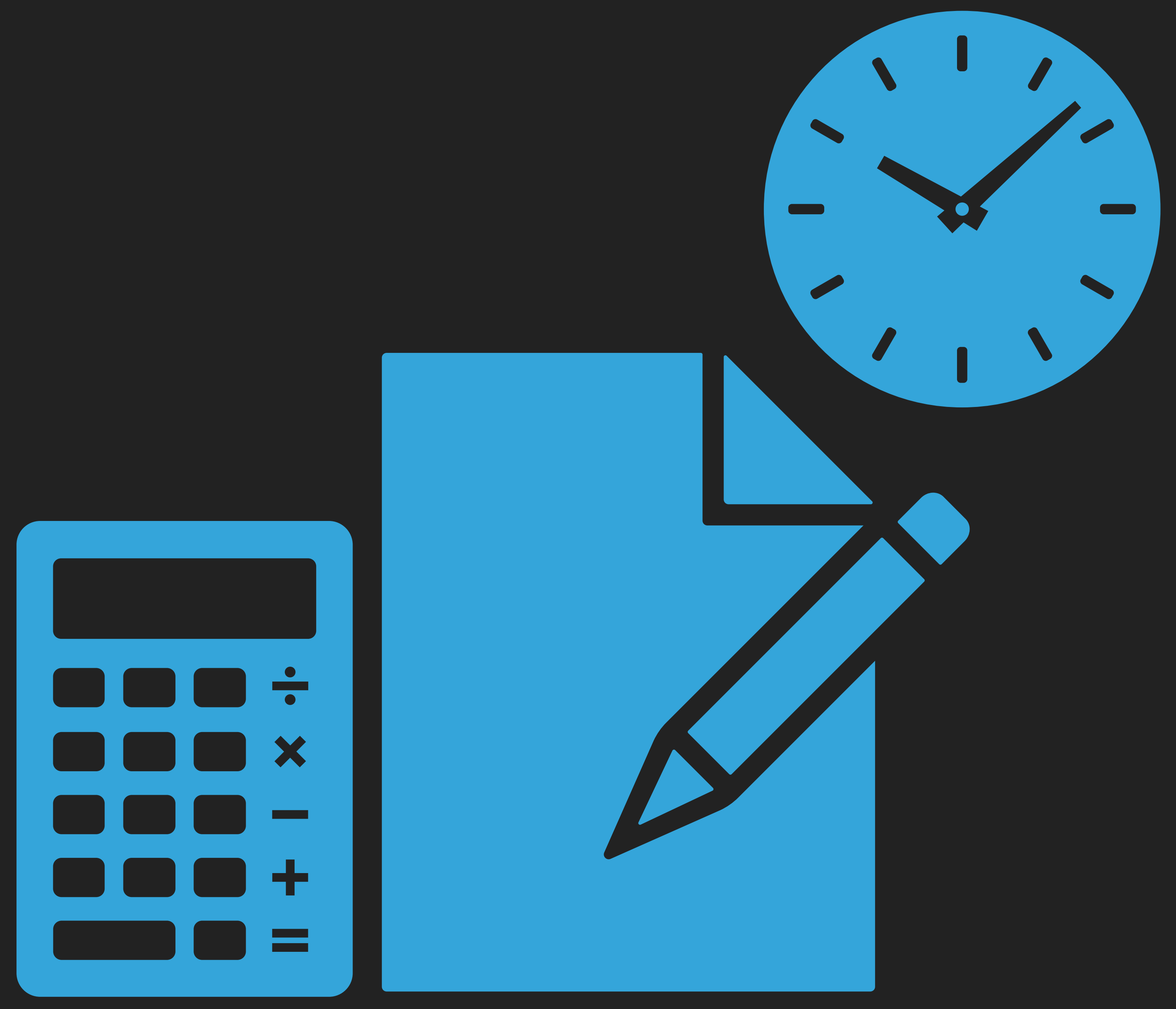
深宵教室 - DSE M1 模擬試題解答

2013

此為參考2013試題之模擬試題，原版請另行購買

2013

- Section A
- Section B



2013 - SECTION A

Q1.) The coefficient of x^2 *of* $(e^{ax}+e^{-ax})^4$ *is 2.* $a=?$

* 參考課程 1.1 及 3.2

$$(e^{ax}+e^{-ax})^4=\sum_{r=0}^{4}C_r^4(e^{rax})(e^{-(4-r)ax})=\sum_{r=0}^{4}C_r^4e^{(2r-4)ax}$$

$$=\sum_{r=0}^{4}C_r^4\left(1+(2r-4)ax+\frac{(2r-4)^2a^2x^2}{2}+\dots\right)$$

The coefficient of $x^2=2$

$$\rightarrow \sum_{r=0}^{4}C_r^4\frac{(2r-4)^2a^2}{2}=2$$

$$\rightarrow 32a^2=2$$

$$\rightarrow a=\pm\frac{1}{4}$$

* $(a+b)^n=\sum_{r=0}^{n}C_r^n a^r b^{n-r}$

* $e^x=\sum_{r=0}^{\infty}\frac{x^r}{r!}$

2013 - SECTION A

Q2.) Given that $p = 8 - \dfrac{2.1}{\sqrt{t+4}}, t > 0$ *(t in year) and* $C = 2^p$ *unit*

Find the rate of change of C when $t = 5$.

* 參考課程 2.2, 2.3 及 2.4

$$Consider,\ C = 2^p \rightarrow lnC = pln2 \rightarrow C' = 2^p ln2,\ hence$$

$$\boxed{\frac{dC}{dt}} = \boxed{\frac{d2^p}{dp}\cdot\frac{dp}{dt}} = 2^p ln2 \cdot \frac{0.5\cdot 2.1}{(t+4)^{\frac{3}{2}}}$$

$$Where,\ p(5) = 7.3,\ hence$$

$$\frac{dC}{dt}\Big|_{t=5} = 2^{7.3} ln2 \cdot \frac{0.5\cdot 2.1}{(5+4)^{\frac{3}{2}}} = 4.2479\ (to\ 4\ d.p.)$$

$\therefore$ *The rate of change of C* $= 4.2479$ *unit/yr*

* Implicit 微分法

* Chain rule

2013 - SECTION A

Q3.) Let the curve $C : y = x(x-2)^{\frac{1}{3}}$, and a straight line L that passes through $(0,0)$ // to the tangent to C at $x = 3$. Find the equation of L and the area bounded by C and L.

* 參考課程 2.2, 2.3, 2.4 及 2.9

$$L: \ y - 0 = \boxed{y'(3)}(x-0) \rightarrow y = \boxed{[(x-2)^{\frac{1}{3}} + \frac{1}{3}x(x-2)^{\frac{-2}{3}}]\,|_{x=3}}\,x$$

$$\rightarrow y = 2x$$

To find the intercept of L and C, consider, $2x = x(x-2)^{\frac{1}{3}}$

$$\rightarrow 8x^3 = x^3(x-2)$$

$$\rightarrow x^3(10-x) = 0$$

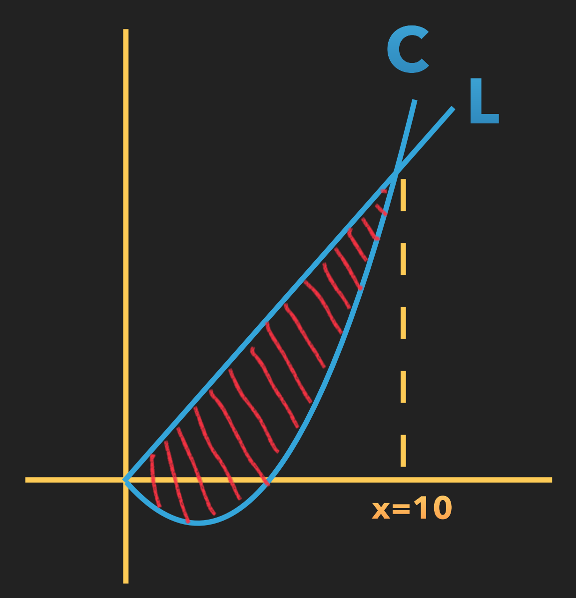
Hence the x coordinate of the intercepts of C and L $= 0, 10$

$$\text{The bounded area} = \int_0^{10} 2x - x(x-2)^{\frac{1}{3}}dx$$

$$= [x^2]_0^{10} - \int_0^{10} x(x-2)^{\frac{1}{3}}dx$$

* 直線方程, 點斜式

* Chain rule



CONT'D

2013 - SECTION A

$$= 100 - \int_{-2}^{8} (u+2)u^{\frac{1}{3}}du$$

$$= 100 - \int_{-2}^{8} u^{\frac{4}{3}} + 2u^{\frac{1}{3}}du$$

$$= 100 - [\frac{3}{7}u^{\frac{7}{3}} + \frac{3}{2}u^{\frac{4}{3}}]_{-2}^{8}$$

$$= 22.7628\ unit^2\ (to\ 4\ d.p.)$$

* 積分代入法, u=x-2
* 定積分代入要改範圍

2013 - SECTION A

Q4.) Let $y = \dfrac{8(1-ae^{-bx})}{1+ae^{-bx}}$. $a, b \in \mathbb{Z}^+$ and $x \geq 0$. The following table shows their relationship.

x	4	6	8	10
y	6.26	6.75	7.11	7.37

Let $u = ae^{-bx}$. Express $\ln u$ in term of linear function of x.
Hence, find a and b. (to 2 d.p.)

* 參考課程 1.1 及 3.1

$$\ln u = \ln(ae^{-bx}) \rightarrow \ln u = \ln a - bx$$

$$\text{Hence, } y = \frac{8(1-ae^{-bx})}{1+ae^{-bx}} = \frac{8(1-u)}{1+u} \rightarrow u = \frac{8-y}{8+y}$$

Then, consider the following table

x	4	6	8	10
$\ln u$	-2.1036	-2.4681	-2.8319	-3.1945

* ln(AB)=lnA+lnB

CONT'D

2013 - SECTION A

$$-b = \frac{4(-77.8232) - 28(-10.5981)}{4(216) - (28)^2} = -0.181825$$

$$\rightarrow \ b = 0.18 \ (to \ 2 \ d.p.)$$

$$lna = \frac{(-10.5981)(216) - 28(-77.8232)}{4(216) - (28)^2} = -1.37675$$

$$\rightarrow \ a = 0.25 \ (to \ 2 \ d.p.)$$

$$* \ m = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

$$* \ c = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

$[y_i = lnu_i]$

2013 - SECTION A

Q5.) Find $\int_1^e lnxdx$.

* 參考課程 2.8

$$Let\ I = \int_1^e lnxdx = [xlnx]_1^e - \int_1^e xd[lnx]$$

$$= e - \int_1^e dx = e - [e-1]$$

$$= 1$$

* 積分三寶: Integration by part

2013 - SECTION A

Q6.) In a size of 120 random sample, there are 75 people do not like government. Let p be the proportion of the people dislike government. Find the 90 % confidence interval of p.

* 參考課程 4.7

Let p_s be the proportion of the sample.

$$\text{Given that } p_S = \frac{75}{120} = 0.625$$

$$\text{For } 90\% \; C.I., \; P\left(-\alpha < \boxed{\frac{p_s - p}{\sqrt{p_s(1-p_s)/36}}} < \alpha\right) = 90\% \rightarrow \alpha = 1.625$$

$$\rightarrow p = \left(p_s - 1.625 \cdot \frac{\sqrt{p_s(1-p_s)}}{6}, \; p_s + 1.625 \cdot \frac{\sqrt{p_s(1-p_s)}}{6}\right)$$

$$\rightarrow p = (0.5523, 0.6977) \; (\text{to } 4 \; d.p.)$$

$* \; \blacksquare \; B(120, p) \rightarrow N\left(p, \frac{p(1-p)}{120}\right)$

* 當樣本足夠大, 可用樣本標準差

2013 - SECTION A

Q7.) Let X and Y be 2 independent discrete random vaiable with $E(Y) = 2.4$ *and*

k	0	1	3	5	7
$P(X=k)$	0.2	0.3	0.3	0.1	0.1

k	1	2	4	m
$P(Y=k)$	0.4	0.3	0.2	0.1

Let A be the event $X+Y \le 2$ *and B be the event* $X = 0$

Find P(A) and hence show A and B are not independent.

* 參考課程 4.1, 4.2, 4.3 及 4.4

$$\text{Given that } \sum_{i=1}^{4} k_i P(Y=k_i) = 2.4 \to m = 6$$

$$\text{Also, } P(A) = P(X+Y \le 2) = P(X=0)P(Y=1) + P(X=0)P(Y=2)$$
$$+P(X=1)P(Y=1)$$
$$= 0.2 \cdot 0.4 + 0.2 \cdot 0.3 + 0.3 \cdot 0.4 = 0.26$$

$$\text{Consider, } P(A \cap B) = P(X=0)P(Y=1) + P(X=0)P(Y=2) = 0.14$$
$$P(A)P(B) = (0.26)(0.2) = 0.052 \ne P(A \cap B)$$

$\therefore$ *A and B are not independent*

* $E(X) = \sum_{i=1}^{n} k_i P(X=k_i)$

* 如果獨立事件, P(A & B)=P(A)P(B)

2013 - SECTION A

Q8.) In a game, a player is selected from a team to shoot three basketball. If at least one ball in the net, the team wins the game. Given that there are 2 players, A and B, in the team. The probability that player A is selected to shoot = 0.7 while the propability of player A and player B hit the target are 0.6 and 0.5 respectively.

a.) Find the probability the team wins the game.

b.) Find the probability player B is selected given that the team do not win the game.

* 參考課程 4.2

Let W be the event of the team wins the game

$$a.)\ P(W) = (0.7)(1-(1-0.6)^3) + (0.3)(1-(1-0.5)^3)$$

$$= 0.9177$$

$$b.)\ P(Player\ B\ is\ selected \mid \bar{W}) = \frac{P(Player\ B\ is\ selected\ and\ \bar{W})}{P(\bar{W})}$$

$$= \frac{0.3 \cdot 0.5^3}{1-0.9177} = 0.4557$$

* P(最少一次中)=1-P(三次都唔中)

* P(A|B)=P(A & B)/P(B)

2013 – SECTION A

Q9.) *The lifetime of a randomly selected machine is assumed to be normally distributed with mean* μ *and the standard deviation 5000 hours. Given that 96.41 % of a machine will have a lifttime shorter than 39000 hours.*

a.) *Find* μ

b.) *Find the probability of the mean of a 100 sample lies between 30200 and 30800 hours.*

c.) *Find the least sample size such that the sample mean exceeding 28500 is at least 0.985.*

* 參考課程 4.5 及 4.6

a.) *Let X be the ramdom variable of lifetime of the selected machine*

$\bar{X}_n$ *be the sample mean of n size*

$$P(X < 39000) = P\left(Z < \frac{39000 - \mu}{5000}\right) = 0.9641$$

From normal distribution table, $\dfrac{39000 - \mu}{5000} = 1.8$

$\rightarrow \mu = 30000$ *hours*

* 先計算 Z ~ N(0,1), 再對表

CONT'D

2013 - SECTION A

$$b.)\ P(30200 < \bar{X}_{100} < 30800) = P\left(\frac{30200-\mu}{\frac{5000}{\sqrt{100}}} < Z < \frac{30800-\mu}{\frac{5000}{\sqrt{100}}}\right)$$

$$= P(0.4 < Z < 1.6)$$

$$= 0.4452 - 0.1554$$

$$= 0.2898$$

$$c.)\ Consider,\ P(\bar{X}_n > 28500) = P\left(Z > \frac{28500-\mu}{\frac{5000}{\sqrt{n}}}\right) \geq 0.985$$

$$From\ normal\ distribution\ table,\ -\frac{1500\sqrt{n}}{5000} \geq -2.17$$

$$\rightarrow n \geq 52.3211$$

$\therefore$ *the least sample size is 53*

* 先計算 Z ~ N(0,1), 再對表

2013 - SECTION B

Q10.) *Let $N(t)$ (in thousand) be the number of population of a place at time t (in month). Given that the birth rate and the death rate of the population at time t are $10ln(t^2+16)$ and $10ln(3t+10)$ respectively. $N(t) > 0$ and $N(0) = 8$. By using trapezoidal rule with 4 sub – interval to estimate $N(t)$, do you agree the poplution will die out at $t = 4$? Explain your answer.*

* 參考課程 2.7, 2.8 及 3.3

$$N'(t) = 10ln(t^2+16) - 10ln(3t+10)$$

* 人口增長率 = 出生率 - 死亡率

Consider, $N(4) - N(0) = \int_0^4 N'(t)dt$, and let $I = \int_0^4 f(x)dx$

* 定積分定義

where $f(x) = ln(x^2+16) - ln(3x+10)$

By trapezoidal, $I \approx \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)]$

$$= -0.8246$$

* 計算梯形面積的加總

Also $f(x) = ln(x^2+16) - ln(3x+10)$

CONT'D

2013 – SECTION B

$$\to f'(x) = \boxed{\frac{2x}{x^2+16}} - \frac{3}{3x+10}$$

$$\to f''(x) = \boxed{\frac{2}{x^2+16} - 2x \cdot \frac{2x}{(x^2+16)^2}} + \frac{9}{(3x+10)^2}$$

$$\to f''(x) = \frac{(32-2x^2)}{(x^2+16)^2} + \frac{9}{(3x+10)^2}$$

$$\because \text{for } 0 \le x \le 4, (32-2x^2) \ge 0 \to f''(x) > 0$$

$$\therefore I \text{ is over} - \text{estimated}.$$

$$Hence, N(4) - N(0) = 10\int_0^4 f(t)dt < 10(-0.8246)$$

$$\to N(4) < N(0) - 8.246 = -0.246 < 0$$

i.e. The population will die out at $t = 4$

* 用 **Product rule**

* **個 f(x) 係 concave upward**

2013 - SECTION B

Q11.) Assume P(t) and C(t) be the energy produced and consumed by a city within period [0,t]
where $t > 0$ *(in year). Given that* $P'(t) = 4(4 - e^{\frac{-t}{5}})$ *and* $C'(t) = 9(2 - e^{\frac{-t}{10}})$.
Let $R(t) = P(t) - C(t)$, *with* $t \geq 0$

a.) Find t when $R'(t) = 0$ *and show* $R'(t)$ *is decreasing function with* $t \geq 0$

b.) Find the total energy produced for $R'(t) > 0$

c.) For $t \in [5, \infty]$, *the energy production is improved by Q(t), given that*

$$Q'(t) = \frac{(t+1)[ln(t^2+2t+3)]^3}{t^2+2t+3} + 9$$

Find the total energy produced for the first 3 years after the improvement.

* 參考課程 2.3, 2.4, 2.6 及 2.7

CONT'D

2013 - SECTION B

$$a.)\ R'(t) = 0 \rightarrow P'(t) = C'(t)$$

$$\rightarrow 4(4 - e^{\frac{-t}{5}}) = 9(2 - e^{\frac{-t}{10}})$$

$$\rightarrow 16 - 4(e^{\frac{-t}{10}})^2 = 18 - 9e^{\frac{-t}{10}}$$

$$\rightarrow 4(e^{\frac{-t}{10}})^2 - 9(e^{\frac{-t}{10}}) + 2 = 0$$

$$\rightarrow (4e^{\frac{-t}{10}} - 1)(e^{\frac{-t}{10}} - 2) = 0$$

* 二次方程, 十字相乘

$$\rightarrow t = 10ln4 \ or \ t = -10ln2 \ (rejected \ \because t \geq 0)$$

$$i.e.\ t = 10ln4$$

$$Consider,\ R''(t) = P''(t) - C''(t) = \frac{4}{5}e^{\frac{-t}{5}} - \frac{9}{10}e^{\frac{-t}{10}}$$

$$= \frac{e^{\frac{-t}{10}}}{10}(8e^{\frac{-t}{10}} - 9)$$

$$for\ t \geq 0,\ e^{\frac{-t}{10}} > 0\ and\ (8e^{\frac{-t}{10}} - 9) < 0,\ to\ R''(t) < 0$$

* f'(t)<0, f(t) 係 decreasing function

$$\therefore R'(t)\ is\ decreasing$$

2013 - SECTION B

$$b.)\ The\ energy\ produced = \int_0^{10ln4} R'(t)dt$$

$$= \int_0^{10ln4} 16 - 4e^{\frac{-t}{5}} - 18 + 9e^{\frac{-t}{10}}dt$$

$$= [-2t + 20e^{\frac{-t}{5}} - 90e^{\frac{-t}{10}}]_0^{10ln4}$$

$$= 21.0241\ (to\ 4\ d.p.)$$

$$c.)\ The\ energy\ produced,\ E = \int_5^8 Q'(t)dt$$

$$= \int_5^8 (\frac{(t+1)[ln(t^2+2t+3)]^3}{t^2+2t+3} + 9)dt$$

* R'(t) decreasing, 當 t=10ln4, R'(t)=0

2013 - SECTION B

$$Let\ u = ln(t^2 + 2t + 3) \rightarrow du = \frac{2t+2}{t^2+2t+3}dt$$

$$Hence,\ E = \frac{1}{2}\int_{ln38}^{ln83} u^3 du + [9t]_5^8$$

$$= \frac{1}{8}[u^4]_{ln38}^{ln83} + 27$$

$$= 52.7730\ (to\ 4\ d.p.)$$

* 積分代入法 u=ln(t²+2t+3)

* 定積分代入要改範圍

2013 – SECTION B

Q12.) *Given that the glycemic level of a children in HK is normally distributed with mean* $= \mu$ *and standard deviation* $= \sigma$. *Given that a* 95 % *confidence interval for* $\mu = (4.596, 5.044)$ *from a random sample of* 49 *children.*

a.) $\sigma = ?$ *and the sample mean* $= ?$

b.) *Another sample of* 15 *children is randomly selected. Their glycemic level is recorded:*

3.6	3.8	3.9	4.3	4.3	4.5	4.8	5.0
5.1	5.2	5.3	5.5	5.8	6.0	6.4	

2 *samples are then combined.*

Construct a 99 % *confidence interval for* μ *for the combined sample*

c.) *Given that* $\mu = 4.8$. *The glycemic level is classified to be low, normal and high if a child glycemic level is respectively at most* 5.2, *within* 5.2 *and* 6.2, *and at least* 6.2.

i.) *Find the probability the glycemic level of a ramdomly selected child in HK is low.*

ii.) *For a randomly selected* 20 *children in HK, find the probability more than* 17 *children has low glycemic level and at least* 1 *is normal.*

* 參考課程 4.4, 4.5, 4.6 及 4.7

CONT'D

2013 – SECTION B

a.) Let $\bar{X}_n$ be the sample mean of n size.

For, $n = 49$, given that 95 % C.I. for μ, $2 \cdot 1.96 \cdot \dfrac{\sigma}{\sqrt{49}} = 5.044 - 4.596$

$\rightarrow \sigma = 0.8$

Hence, $\bar{X}_{49} + 1.96 \cdot \dfrac{0.8}{\sqrt{49}} = 5.044 \rightarrow \bar{X}_{49} = 4.82$

b.) The combined sample mean, $\bar{X}_{64} = \dfrac{1}{64}\left[49 \cdot 4.82 + \sum_{i=1}^{15} X_i\right]$.

where X_i is the i^{th} data of the 15 size of the sample.

$\rightarrow \bar{X}_{64} = 4.83875$

The 99 % C.I. for $\mu = \left(\bar{X}_{64} - 2.575 \cdot \dfrac{0.8}{\sqrt{64}}, \bar{X}_{64} + 2.575 \cdot \dfrac{0.8}{\sqrt{64}}\right)$

$= (4.58125, 5.09625)$

* 95% 置信區間

* 總和=總數 x 平均值

* 99% 置信區間

CONT'D

2013 - SECTION B

ci.) Let L be the event of selected child had low glycemic level

N be the event of selected child had normal glycemic level

H be the event of selected child had high glycemic level

$$P(L) = P\left(Z \le \frac{5.2-4.8}{0.8}\right) = P(Z \le 0.5) = 0.6915$$

$$ii)\ P(N) = P\left(\frac{5.2-4.8}{0.8} < Z < \frac{6.2-4.8}{0.8}\right) = 0.2684$$

$$P(H) = 1 - P(L) - P(N) = 0.0401$$

$$The\ probability = C^{20}_{19}[P(L)]^{19}P(N) + C^{20}_{18}[P(L)]^{18}P(N)P(H) + C^{20}_{18}[P(L)]^{18}[P(N)]^2$$

$$= 0.0281\ (to\ 4\ d.p.)$$

* 先計算 Z ~ N(0,1), 再對表
* P(Not A) = 1- P(A)
* 19 個係 L, 1 個係 N
* 18 個係 L, 1 個係 N, 1 個係 H
* 18 個係 L, 2 個係 N

2013 – SECTION B

Q13.) A pump supplier provide maintenance service for every pumps in a city. Given that the numbers of pump breakdown in a month follows Possion Distribution with mean = 1.9. The supplier monthly maintance service is unacceplable if there are more than 2 breakdown for a pump of the total 15 pumps in a city in a month. Given that the monthly number breakdown of a pump is independent.

a.) Find the probability of unacceptable monthly maintance service for a randomly selected pump in the city.

b.) For a certain pump, find the probability June of 2014 is 3^{rd} month in 2014 the monthly maintenance service is found unacceptable.

c.) Find the expected total number of pumps for a city to be unacceptable in a year.

d.) Since 2015, warning will be issued for each pump if the maintanence is unacceptable for 3 consecutive months. 1 warning will be issued given that no warning issued before. Find the probability 3 or more warning will be issued on or before April 2015.

* 參考課程 4.3 及 4.4

CONT'D

2013 – SECTION B

a.) *Let* $X \sim Po(1.9)$ *be the number of pump breakbreak*

$\bar{A}$ *be the event of unacceptable service in a month for a pump*

$$P(\bar{A}) = 1 - P(X=0) - P(X=1) - P(X=2)$$

$$= 1 - e^{-1.9} - e^{-1.9}(1.9) - \frac{1}{2}e^{-1.9}(1.9)^2 = 0.2963\ (to\ 4\ d.p.)$$

b.) *The probability* $= C_2^5[P(\bar{A})]^2(1 - P(\bar{A}))^3 \cdot P(\bar{A})$

$= 0.0906\ (to\ 4\ d.p.)$

c.) *Assume* $Y \sim B(15, 0.2963)$

The expected total number of unacceptable pumps in a year

$= 12 \cdot E(Y) = 12 \cdot 15 \cdot 0.2963 = 53.3303\ (to\ 4\ d.p.)$

d.) *Let W be the event of warning issued for a randomly selected pump on or before April 2015.*

* **P(Not A) = 1 - P(A)**
* $P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$
* **頭5個月有2次 unacceptable** $P(Y=k) = C_k^n p^k(1-p)^{n-k}$
* **6月第3次 unacceptable**
* **E(X+Y)=E(X)+E(Y)**

CONT'D

2013 - SECTION B

$$P(W) = [P(\bar{A})]^3 + (1 - P(\bar{A}))[P(\bar{A})]^3 = 0.0443$$

$$\begin{aligned}The\ probability &= 1 - C_0^{15}[1 - P(W)]^{15} - C_1^{15}P(W)[1 - P(W)]^{14} \\ &\quad - C_2^{15}[P(W)]^2[1 - P(W)]^{13}\end{aligned}$$

$$= 0.0265\ (to\ 4\ d.p.)$$

* 1月到3月連續3次 unacceptable
* 2月到4月連續3次 unacceptable
* P(W>2) = 1-P(0W)-P(1W)-P(2W)