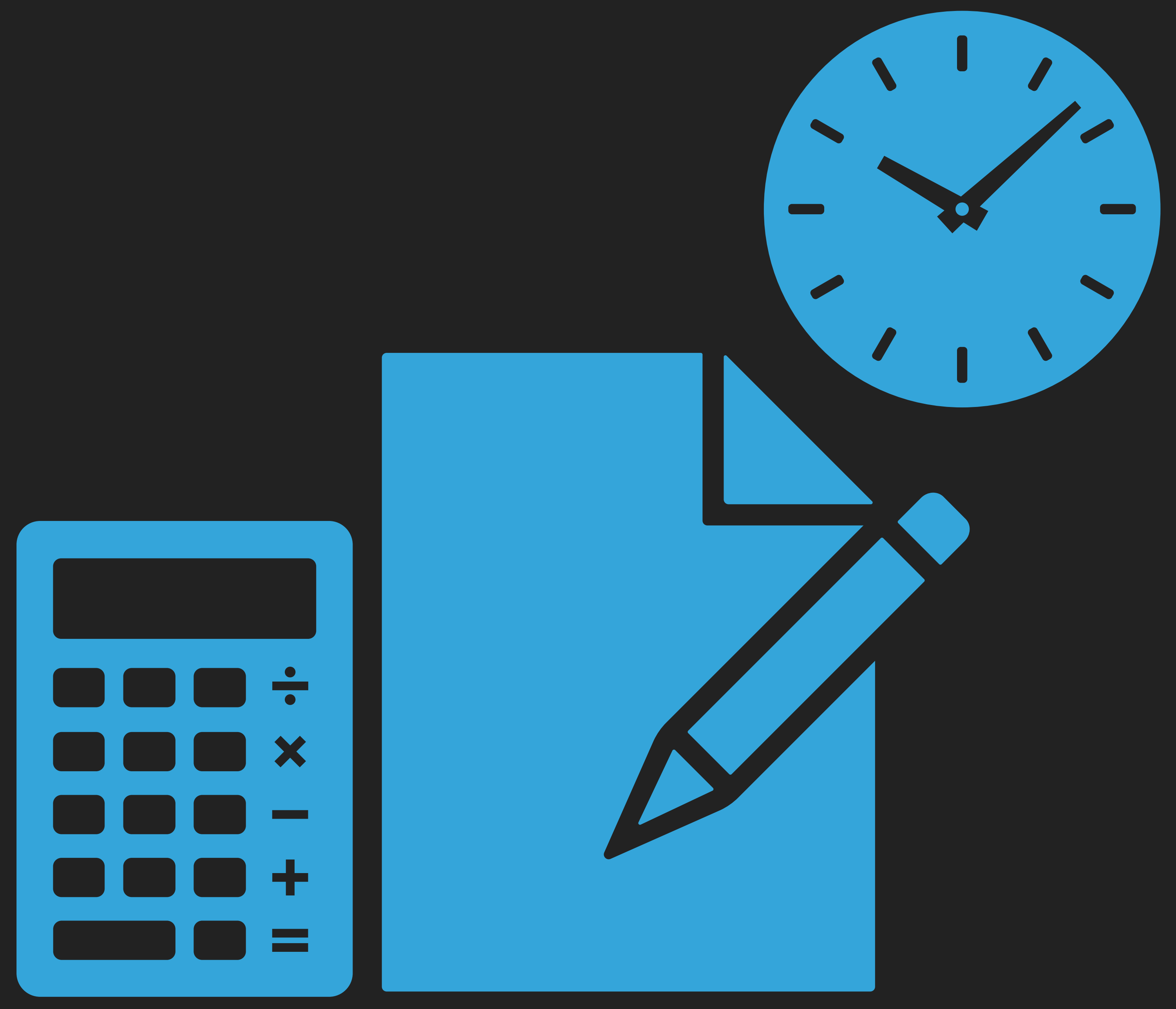
深宵教室 - DSE M2 模擬試題解答

2023

2023

- Section A
- Section B



2023 - SECTION A

Q1.) Assume $(2-3x)^5(x+\frac{a}{x})^2 \equiv A + \frac{160x}{3} + Bx^2 + \ldots$, *where, a, A, and B are constant*

Find a and B .

* 參考課程 1.1

$$(2-3x)^5(x+\frac{a}{x})^2 \equiv (\sum_{r=0}^{5} C_r^5 2^{5-r}(-3)^r x^r)(x^2+2a+a^2x^{-2})$$

$$\text{The coefficient of } x = C_1^5(2)^4(-3)(2a) + C_3^5(2)^2(-3)^3a^2 = \frac{160}{3}$$

$$\rightarrow -(\cancel{5})(3^2)(2^{\cancel{5}\,2})a - (\cancel{5})(3^4)(\cancel{2^3})a^2 = (\cancel{5})(2^{\cancel{5}\,2})$$

$$\rightarrow 81a^2 + 36a + 4 = 0 \rightarrow a = -\frac{2}{9}$$

$$\text{The coefficient of } x^2 = 2^5 + C_2^5(2^3)(-3)^2(2a) + C_4^5(2)(-3)^4a^2$$

$$\rightarrow B = -248$$

* Binomial Expansion

2023 - SECTION A

$Q2.)\ f(x) = -x\sin x,\ f'(\frac{\pi}{2}) = ?\ \ (By\ First\ Principles)$

* 參考課程 2.1, 3.1 及 3.2

$$f'(\frac{\pi}{2}) = \lim_{h\to 0}\frac{f(\frac{\pi}{2}+h)-f(\frac{\pi}{2})}{h} = \lim_{h\to 0}\frac{1}{h}\cdot[-(\frac{\pi}{2}+h)\sin(\frac{\pi}{2}+h)+\frac{\pi}{2}\sin\frac{\pi}{2}]$$

$$= \lim_{h\to 0}\frac{1}{h}\cdot[-(\frac{\pi}{2}+h)\cosh+\frac{\pi}{2}] = \lim_{h\to 0}\frac{\pi}{2h}(1-\cosh) - \lim_{h\to 0}\cosh$$

$$= \lim_{h\to 0}\frac{\pi}{2h}(1-(1-2\sin^2\frac{h}{2})) - 1 = \lim_{h\to 0}\frac{\pi}{2}\frac{2}{h}\sin\frac{h}{2}\cdot\sin\frac{h}{2} - 1$$

$$= -1$$

1

1

0

* 微分定義

* sin 複角公式

$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$	$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$	$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$	$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A\cos B=\sin(A+B)+\sin(A-B)$	$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A\cos B=\cos(A+B)+\cos(A-B)$	
$2\sin A\sin B=\cos(A-B)-\cos(A+B)$	

* cos 雙角公式

2023 - SECTION A

Q3.) Find $\int_0^{\frac{\pi}{4}} \frac{11sinx + 7cosx}{3sinx + cosx}dx$

* 參考課程 3.6, 3.7 及 3.10

方法1

$$\int_0^{\frac{\pi}{4}} \frac{11sinx + 7cosx}{3sinx + cosx}dx = \int_0^{\frac{\pi}{4}} \frac{4(3sinx + cosx) + (3cosx - sinx)}{3sinx + cosx}dx$$

$$= \int_0^{\frac{\pi}{4}} 4 + \frac{3cosx - sinx}{3sinx + cosx}dx = \pi + \int_0^{\frac{\pi}{4}} \frac{3cosx - sinx}{3sinx + cosx}dx$$

$$= \pi + \int_0^{\frac{\pi}{4}} \frac{d(3sinx + cosx)}{3sinx + cosx} = \pi + [ln(3sinx + cosx)]_0^{\frac{\pi}{4}}$$

$$= \pi + \frac{3ln2}{2}$$

* 積分代入法

CONT'D

2023 - SECTION A

方法2

$$Let\ I = \int_0^{\frac{\pi}{4}} \frac{11sinx + 7cosx}{3sinx + cosx}dx = \int_0^{\frac{\pi}{4}} \frac{11tanx + 7}{3tanx + 1}dx,\ let\ t = tanx$$

$$Then,\ dt = sec^2x dx = (1 + tan^2x)dx \rightarrow dx = \frac{dt}{1 + t^2}$$

$$\rightarrow I = \int_0^1 \frac{11t + 7}{(t^2 + 1)(3t + 1)}dt = \int_0^1 \frac{At + B}{t^2 + 1} + \frac{C}{3t + 1}dt$$

$where\ A,\ B\ and\ C\ are\ constant.\ To\ Find\ A,\ B\ and\ C,\ consider$

$$(At + B)(3t + 1) + C(t^2 + 1) \equiv 11t + 7$$

$$\rightarrow (3A + C)t^2 + (A + 3B)t + (B + C) \equiv 11t + 7$$

$$\rightarrow A = -1,\ B = 4,\ C = 3$$

$$Hence, I = \int_0^1 \frac{4 - t}{t^2 + 1} + \frac{3}{3t + 1}dt$$

* 積分代入法, t Method
* $sec^2\theta = 1 + tan^2\theta$
* 積分代入法, 要改範圍
* 用 Partial Fraction
* 3A+C=0, A+3B=11,B+C=7

CONT'D

2023 - SECTION A

$$= \int_0^1 \frac{4dt}{t^2+1} - \int_0^1 \frac{tdt}{t^2+1} + \int_0^1 \frac{3dt}{3t+1}$$

$$= 4\int_0^{\frac{\pi}{4}} dx - \frac{1}{2}\int_0^1 \frac{d(t^2+1)}{t^2+1} + [ln(3t+1)]_0^1$$

$$= \pi - \frac{1}{2}[ln(t^2+1)]_0^1 + [ln(3t+1)]_0^1$$

$$= \pi + \frac{3ln2}{2}$$

* = dx
* 積分代入法

2023 - SECTION A

Q4.) Solve $sec^3x - 6sec^2x + 8 = 0$, *where* $\frac{\pi}{2} < x < \frac{3\pi}{2}$

* 參考課程 2.2 及 2.3

$$sec^3x - 6sec^2x + 8 = 0 \rightarrow 1 - 6cosx + 8cos^3x = 0$$

$$\rightarrow 8cosx\left(\frac{1}{2}(cos2x + 1)\right) - 6cosx + 1 = 0$$

$$\rightarrow 4cosxcos2x - 2cosx + 1 = 0 \rightarrow 2cos3x + 2cosx - 2cosx + 1 = 0$$

$$\rightarrow cos3x = -\frac{1}{2},\ where\ \frac{\pi}{2} < x < \frac{3\pi}{2} \rightarrow \frac{3\pi}{2} < 3x < \frac{9\pi}{2} = 4\pi + \frac{\pi}{2}$$

$$\rightarrow 3x = 3\pi - \frac{\pi}{3}\ or\ 3\pi + \frac{\pi}{3} \rightarrow x = \pi - \frac{\pi}{9}\ or\ \pi + \frac{\pi}{9}$$

$$\rightarrow x = \frac{8\pi}{9}\ or\ \frac{10\pi}{9}$$

* 兩邊乘 cos³x

* cos 雙角公式

* Product 轉 Sum 公式

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A\cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A\sin B = \cos(A-B) - \cos(A+B)$	

* 留意角度範圍

2023 - SECTION A

Q5.) Given that A is a 2x2 matrix such that $A^2 + A + I_2 = 0$. Please explain if $(A^{1000} + (A^{-1})^{2000})^{-1}$ is in form of $\alpha I_2 + \beta A$, where α and β are constant

* 參考課程 4.9 及 4.10

Based on information, $A \neq I_2 \rightarrow (A - I_2)(A^2 + A + I_2) = 0$

$\rightarrow A^3 - I_2 = 0 \rightarrow A^3 = I_2 \rightarrow A \cdot A^2 = I_2 \rightarrow A^{-1} = A^2$

Hence, $(A^{1000} + (A^{-1})^{2000})^{-1} = (A^{1000} + A^{4000})^{-1}$

$= (A^{1000}(I_2 + A^{3000}))^{-1}$

$= (A^{1000}(I_2 + I_2))^{-1} = (2A^{1000})^{-1} = \frac{1}{2}(A^{1000})^{-1}$

$= \frac{1}{2}(A^{-1})^{1000} = \frac{1}{2}A^{2000} = \frac{1}{2}A^{3 \cdot 666 + 2} = \frac{1}{2}A^2 = -\frac{1}{2}I_2 - \frac{1}{2}A$

$\therefore \ (A^{1000} + (A^{-1})^{2000})^{-1} = \alpha I_2 + \beta A$, *where* $\alpha = \beta = -\frac{1}{2}$

* 如果 AB=BA, 可以用平時恆等式

* $a^3 - b^3 \equiv (a - b)(a^2 + ab + b^2)$

* 如果 AB=I, A^{-1}=B

* $(kA)^{-1} = \frac{1}{k}A^{-1}$

2023 - SECTION A

Q6.) Consider a metal sphere with radius = 10cm put into an empty cylindrical container with radius = 11cm and the height = 20cm. Then, water is added to this container with constant rate $1cm^3s^{-1}$. Let h be the depth of water after t seconds. Find the max. value of $\frac{dh}{dt}$

* 參考課程 3.4 及 3.12

Let V_1 = *the volume of metal sphere immersed into water with depth* = h

Consider $x^2+(y-10)^2=10^2$ *represent the metal sphere geometry*

$$V_1=\pi\int_0^h x^2dy=\pi\int_0^h 10^2-(y-10)^2dy=\pi[10^2y-\frac{(y-10)^3}{3}]_0^h$$

$$=\pi(10^2h-\frac{(h-10)^3}{3}+\frac{(-10)^3}{3})$$

* Disk method

CONT'D

2023 - SECTION A

$Hence, the\ volume\ of\ water\ at\ depth = h,\ V$

$$= \pi(11)^2 h - V_1 = \pi((11^2 - 10^2)h + \frac{(h-10)^3}{3} - \frac{(-10)^3}{3})$$

$$Hence, \frac{dV}{dt} = \pi(21\frac{dh}{dt} + (h-10)^2\frac{dh}{dt})$$

* 兩邊微分

$$\rightarrow 1 = \frac{dh}{dt}\pi(21 + (h-10)^2)$$

$$\rightarrow \pi\frac{dh}{dt} = \frac{1}{(h-10)^2 + 21},\ given\ that\ 0 \le h \le 20$$

$Obviously, when\ h = 10,\ (h-10)^2 + 21\ obtain\ min.value = 21$

* 二次方程頂點

$$\therefore the\ max.value\ of\ \frac{dh}{dt} = \frac{1}{21\pi} cm{\cdot}s^{-1}$$

2023 - SECTION A

Q7.) Let $C: y = f(x)$, where $-2 < x < 2$. Given that the tangent at any point at $y = f(x)$ is $\dfrac{k-3x}{\sqrt{4-x^2}}$, where k is a constant. Given that $(0,0)$ lies on C

a.) Find the equation of C

b.) Suppose C has a turning point. Find the range of k and pt. of inflexion.

* 參考課程 3.8 及 3.9

a.) $$f'(x) = \frac{k-3x}{\sqrt{4-x^2}} \rightarrow f(x) = \int \frac{k-3x}{\sqrt{4-x^2}}dx$$

$$\text{Let } x = 2\sin\theta \rightarrow dx = 2\cos\theta d\theta$$

$$\rightarrow f(x) = \int \frac{k-6\sin\theta}{\sqrt{4-4\sin^2\theta}} 2\cos\theta d\theta = \int \frac{k-6\sin\theta}{2\cos\theta} 2\cos\theta d\theta$$

* 積分類似微分逆函數

* 利用三角代入法, let $x = 2\sin\theta$

* $1 - \sin^2\theta = \cos^2\theta$

CONT'D

2023 - SECTION A

$$= \int (k - 6sin\theta)d\theta = k\theta + 6cos\theta + C, \, where \; C \; is \; constant \, .$$

$$= ksin^{-1}(\frac{x}{2}) + 6\sqrt{1 - \frac{x^2}{4}} + C$$

$$Given \; that \; f(0) = 0, \, C = -6 \rightarrow f(x) = ksin^{-1}(\frac{x}{2}) + 3\sqrt{4 - x^2} - 6$$

$$b.) \; Since, \, there \; is \; a \; turning \; point \rightarrow f'(x) = 0 \; has \; one \; solution$$

$$if \; f'(x) = 0 \rightarrow x = \frac{k}{3} \; and \; -2 < x < 2$$

$$i.e. -6 < k < 6$$

$$f'(x) = \frac{k - 3x}{\sqrt{4 - x^2}} \rightarrow (4 - x^2)[f'(x)]^2 = (k - 3x)^2$$

* $\cos^2 x = 1 - \sin^2 x$

CONT'D

2023 - SECTION A

$$\to 2(4-x^2)f'(x)f''(x) - 2x[f'(x)]^2 = 2(k-3x)(-3)$$

$$\to (4-x^2)f'(x)f''(x) = x[f'(x)]^2 - 3(k-3x)$$

$$\to (4-x^2)f'(x)f''(x) = x\left(\frac{(k-3x)^2}{4-x^2}\right) - 3(k-3x)$$

$$\to (4-x^2)^2 f'(x)f''(x) = (k-3x)(x(k-3x) - 3(4-x^2))$$

$$\to (4-x^2)^2 \frac{k-3x}{\sqrt{4-x^2}} f''(x) = (k-3x)(kx-12)$$

$$\to f''(x) = (kx-12)(4-x^2)^{-\frac{3}{2}}$$

$$Assume\ there\ is\ pt.\ of\ inflexion,\ then\ f''(x) = 0 \to x = \frac{12}{k}$$

$$Then,\ -6 < k < 6 \to \frac{12}{k} < -\frac{12}{6}\ or\ \frac{12}{k} > \frac{12}{6} \to x < -2\ or\ x > 2$$

$$Contradiction\ exists,\ There\ is\ no\ pt.\ of\ inflexion$$

* Implicit 微分法

* 利用 f''(x) =0, 搵 pt. of inflexion

2023 - SECTION A

$Q8.)$ Prove $\sin\theta\sum_{r=1}^{n}\sin 2r\theta = \sin n\theta\sin(n+1)\theta, \forall n\in\mathbb{Z}^+$, hence simplify $\sum_{k=1}^{111}\sin\frac{k\pi}{11}\cos\frac{k\pi}{11}$

* 參考課程 1.1, 1.2 及 2.2

方法1

$$Let\ P(n):\ \sin\theta\sum_{r=1}^{n}\sin 2r\theta = \sin(n+1)\theta\sin\theta\ \forall n\in\mathbb{Z}^+$$

$$For\ P(1):\ L.H.S. = \sin\theta\sin 2\theta = R.H.S.$$

$$Assume\ P(k)\ is\ true\ \exists k\in\mathbb{Z}^+,\ then\ P(k+1):$$

$$L.H.S. = \sin\theta\sum_{r=1}^{k+1}\sin 2r\theta = \sin\theta\sum_{r=1}^{k}\sin 2r\theta + \sin\theta\boxed{\sin 2(k+1)\theta}$$

$$= \sin k\theta\sin(k+1)\theta + \sin\theta\cdot\boxed{2\sin(k+1)\theta\cos(k+1)\theta}$$

* 先 Let Statement

* 証明 P(1) is true

* 假設 P(k) is true. 証明 P(k+1) is true

* 將未項抽出並改變未項

* sin 雙角公式

CONT'D

2023 - SECTION A

$$= sin(k+1)\theta(sink\theta + 2sin\theta cos(k+1)\theta)$$

$$= sin(k+1)\theta(sink\theta + sin(k+2)\theta - sink\theta)$$

$$= sin(k+1)\theta sin(k+2)\theta = R.H.S.$$

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

i.e. By M.I., P(n) is true, $\forall n \in \mathbb{Z}^+$

方法2

$$sin\theta \sum_{r=1}^{n} sin2r\theta = \sum_{r=1}^{n} sin\theta sin2r\theta = \sum_{r=1}^{n} \frac{cos(2r-1)\theta - cos(2r+1)\theta}{2}$$

$$= \frac{1}{2}\left(\sum_{r=1}^{n} cos(2r-1)\theta - \sum_{r=1}^{n} cos(2r+1)\theta\right)$$

$$= \frac{1}{2}\left(cos\theta + \sum_{r=2}^{n} cos(2r-1)\theta - \sum_{r=1}^{n-1} cos(2r+1)\theta - cos(2n+1)\theta\right)$$

* Product 轉 Sum 公式

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

* 寫結論

* Product 轉 Sum 公式

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

* 將首項抽出並改變首項

* 將末項抽出並改變末項

CONT'D

2023 - SECTION A

$$=\frac{1}{2}(cos\theta - cos(2n+1)\theta + \sum_{r=2}^{n} cos(2r-1)\theta - \sum_{r=1}^{n-1} cos(2r+1)\theta)$$

$$=\frac{1}{2}(cos\theta - cos(2n+1)\theta + \cancel{\sum_{r=1}^{n-1} cos(2r+1)\theta} - \cancel{\sum_{r=1}^{n-1} cos(2r+1)\theta})$$

$$= -sin(n+1)\theta sin(-n\theta) \quad = sin(n+1)\theta sinn\theta$$

$$\sum_{k=1}^{111} sin\frac{k\pi}{11}cos\frac{k\pi}{11} = \sum_{k=1}^{111} \frac{sin\frac{2k\pi}{11}}{2} = \frac{1}{2}\frac{sin(112)(\frac{\pi}{11})sin(111)(\frac{\pi}{11})}{sin\frac{\pi}{11}}$$

$$=\frac{1}{2}\frac{sin(110+2)(\frac{\pi}{11})sin(110+1)(\frac{\pi}{11})}{sin\frac{\pi}{11}} = \frac{1}{2}\frac{sin(10\pi+\frac{2\pi}{11})sin(10\pi+\frac{\pi}{11})}{sin\frac{\pi}{11}}$$

$$=\frac{1}{2}sin\frac{2\pi}{11}$$

* 改變首末項改變公項

* Sum 轉 Product 公式

$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$	$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$	$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$	$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A\cos B=\sin(A+B)+\sin(A-B)$	$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A\cos B=\cos(A+B)+\cos(A-B)$	
$2\sin A\sin B=\cos(A-B)-\cos(A+B)$	

* sin 雙角公式

2023 - SECTION B

Q9.) Given $f(x) = xe^{-x^2}$, $x \in \mathbb{R}$. Denote the curve $G: y = f(x)$.

a.) Find the max. and min. points of G

b.) Let L be the tangent of G at $(1, e^{-1})$

i.) Find L, and show that G lies below L when $x \in (0,1)$

ii.) Let R be the region bounded by G, L and y – axis. Find the area of R.

* 參考課程 3.5 及 3.11

a.) $f(x) = xe^{-x^2} \rightarrow f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = e^{-x^2}(1 - 2x^2)$

Obviously, when $x = a$ or $b \rightarrow f'(x) = 0$

where $a = -\dfrac{1}{\sqrt{2}}$ and $b = \dfrac{1}{\sqrt{2}}$

	x < a	x = a	a < x < b	x = b	x > b
f′(x)	-	0	+	0	-
f(x)	Dec.		Inc		Dec.

*** Product Rule + Chain Rule**

*** 利用表格計算 turning pt. 附近情況**

$f'(x) > 0 \rightarrow$ *increasing*

$f'(x) < 0 \rightarrow$ *decreasing*

CONT'D

2023 – SECTION B

$\therefore (b, f(b))$ *is local max . pt . and* $(a, f(a))$ *is local min . pt .*

$$i.e.\ The\ local\ max.\ pt. = \left(\frac{1}{\sqrt{2}}, \frac{e^{-\frac{1}{2}}}{\sqrt{2}}\right)$$

$$The\ local\ min.\ pt. = \left(-\frac{1}{\sqrt{2}}, -\frac{e^{-\frac{1}{2}}}{\sqrt{2}}\right)$$

b) i.) $The\ slope\ of\ L = f'(1) = -e^{-1}$

$$L:\ y - e^{-1} = -e^{-1}(x-1) \rightarrow L: y = -e^{-1}(x-2)$$

* 直線方程, 點斜式

$$Consider,\ H(x) = y - f(x) \rightarrow H'(x) = -e^{-1} - f'(x) \rightarrow H''(x) = -f''(x)$$

$$Also,\ f'(x) = e^{-x^2}(1-2x^2) \rightarrow f''(x) = (-2x)(1-2x^2)e^{-x^2} + (-4x)e^{-x^2}$$

$$= 2x(2x^2-3)e^{-x^2} < 0,\ 0 < x < 1$$

$*\ f'(x) > 0 \rightarrow increasing$

$$Hence,\ H''(x) > 0 \rightarrow H'(x)\ is\ inc.\ function\ for\ 0 < x < 1$$

CONT'D

2023 - SECTION B

$H'(x) < H'(1) = -e^{-1} - f'(1) = 0$

$\because H'(x) < 0 \rightarrow H(x)$ *is dec . function for* $0 < x < 1$

$\rightarrow H(x) > H(1) = e^{-1} - f(1) = 0 \rightarrow y > f(x)$ *for* $0 < x < 1$

Hence, G lies below L when $x \in (0,1)$

ii.) Let A_1 *be the area bounded by G and* $x -$ *axis,* $x \in (0,1)$

A_2 *be the area bounded by L and* $x -$ *axis,* $x \in (0,1)$

$$A_1 = \int_0^1 xe^{-x^2} dx = \frac{1}{2}\int_0^1 e^{-x^2} d(x^2) = -\frac{1}{2}(e^{-1} - 1)$$

$$A_2 = \frac{(1)(y(0) + y(1))}{2} = \frac{-e^{-1}(0-2) - e^{-1}(1-2)}{2} = \frac{3e^{-1}}{2}$$

The required area $= A_2 - A_1 = 2e^{-1} - \frac{1}{2}$ *unit*^{2}

* $f'(x) < 0 \rightarrow$ *decreasing*

* 積分代入法

* 面積大減細

2023 - SECTION B

Q10.) Let O be the origin, $\overrightarrow{OP} = -2\hat{i} - \hat{k}$ and $\overrightarrow{OQ} = 2\hat{i} - \hat{j} + \hat{k}$. Denote a circle C passes through O, P and Q. Let G be the center of C.

a.) If R lies on PQ and $OR \perp OQ$. Find $\overrightarrow{OR}$.

b.) OR produced meet C at point S. Find $\overrightarrow{OS}$.

c.) Denote B is a projection point of $A = (-6, -22, 2)$ on the plane contains C. Describe the geometric relationship of O, B, G.

* 參考課程 4.4 及 4.5

a.) Let $\vec{p} = \overrightarrow{OP}$, $\vec{q} = \overrightarrow{OQ}$

Then, $\overrightarrow{OR} = \vec{p} + r\overrightarrow{PQ}$, where r is a constant scalar

$= (1 - r)\vec{p} + r\vec{q}$, where $\overrightarrow{PQ} = \vec{q} - \vec{p}$

$\because OR \perp OQ \rightarrow \overrightarrow{OR} \cdot \overrightarrow{OQ} = 0 \rightarrow (1 - r)\vec{p} \cdot \vec{q} + r|\vec{q}|^2 = 0$

* Vector 直線方程

* 兩支Vector互相垂直, 互Dot = 0

CONT'D

2023 - SECTION B

$$(1-r)(-5)+r(6)=0 \to r=\frac{5}{11}$$

$$Hence,\ \overrightarrow{OR}=\frac{6}{11}\vec{p}+\frac{5}{11}\vec{q}=-\frac{1}{11}(2\hat{i}+5\hat{j}+\hat{k})$$

方法1

$$b.)\ \overrightarrow{OS}=s\overrightarrow{OR},\ where\ s\ is\ a\ constant\ scalar$$

$$Then\ PQ\perp PS\ (\angle s\ in\ the\ same\ segment,\ QS)$$

$$\overrightarrow{PQ}\cdot\overrightarrow{PS}=0\to(\vec{q}-\vec{p})\cdot(s(\frac{6}{11}\vec{p}+\frac{5}{11}\vec{q})-\vec{p})=0$$

$$\to(s-11)\vec{q}\cdot\vec{p}+5s|\vec{q}|^2+(11-6s)|\vec{p}|^2=0$$

$$\to(s-11)(-5)+5s(6)+(11-6s)(5)=0\to s=22$$

$$Hence,\ \overrightarrow{OS}=22(\overrightarrow{OR})=-2(2\hat{i}+5\hat{j}+\hat{k})$$

* O,S,R 係共線關係

* 兩支Vector互相垂直, 互Dot = 0

CONT'D

2023 - SECTION B

方法2

b.) $\overrightarrow{OS} = s\overrightarrow{OR}$, *where s is a constant scalar*

QS *is a diameter of C* ($\angle s$ *in semi circle*)

$$Hence,\ \overrightarrow{OG} = \frac{1}{2}\overrightarrow{OS} + \frac{1}{2}\vec{q} = \frac{1}{2}(s\overrightarrow{OR} + \vec{q})$$

$$Also,\ |\overrightarrow{OG}|^2 = |\vec{p} - \overrightarrow{OG}|^2 = (\vec{p} - \overrightarrow{OG}) \cdot (\vec{p} - \overrightarrow{OG})$$

$$\rightarrow |\vec{p}|^2 - 2\vec{p} \cdot \overrightarrow{OG} = 0 \rightarrow s\overrightarrow{OR} \cdot \vec{p} + \vec{p} \cdot \vec{q} = |\vec{p}|^2$$

$$\rightarrow \frac{s}{11}(6\vec{p} + 5\vec{q}) \cdot \vec{p} + \vec{p} \cdot \vec{q} = |\vec{p}|^2 \rightarrow s = 22$$

$$Hence,\ \overrightarrow{OS} = 22(\overrightarrow{OR}) = -2(2\hat{i} + 5\hat{j} + \hat{k})$$

* O,S,R 係共線關係
* 直徑圓周角 = 90⁰
* 分割定理
* 圓半徑

CONT'D

2023 - SECTION B

$c.)\ QS$ *is a diameter of* C ($\angle$*s in semi circle*)

$$\text{Hence, } \overrightarrow{OG} = \frac{1}{2}\overrightarrow{OS} + \frac{1}{2}\vec{q} = -\frac{1}{2}(2\hat{i} + 11\hat{j} + \hat{k})$$

$$\text{Consider, } \vec{n_1} = \vec{p} x \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -1 \\ 2 & -1 & 1 \end{vmatrix} = -\hat{i} + 2\hat{k}$$

$$\vec{n_2} = \overrightarrow{OA} x \overrightarrow{OG} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & -22 & 2 \\ -1 & -\frac{11}{2} & -\frac{1}{2} \end{vmatrix} = 22\hat{i} - 5\hat{j} + 11\hat{k}$$

$$\because\ \vec{n_1} \cdot \vec{n_2} = 0 \rightarrow \vec{n_1} \perp \vec{n_2} \rightarrow \text{the plane } OPQG \perp \text{the plane } OAG$$

$$\therefore\ O, B, G \text{ are collinear}$$

* 直徑圓周角 = 90⁰

* 分割定理

* 向右乘相加 - 向左乘相加

2023 - SECTION B

Q11.)

$$\begin{cases} x + ay + (a+1)z = 2 \\ x + (a+4)y + (2a+4)z = b+1 \\ 2x + 3y + 5z = b \end{cases} \text{ —— (E)} \qquad a, b \in \mathbb{R}$$

a.) The range of a if (E) has unique solution

b.) If $a = 1$ and (E) is consistent, find b.

c.) If $a \neq 1$ and (E) is not consistent, find the range of b.

d.) Consider,

$$\begin{cases} x + 2y + 3z = 2 \\ x + 6y + 8z = s+1 \\ 2x + 3y + 5z = s \end{cases} \text{ —— (F)} \qquad s \in \mathbb{R}$$

Is there a real solution of F satisfying $mx + ny + z = -2$, where m and n are constant and independent of s? Explain your answer.

* 參考課程 4.7

CONT'D

2023 - SECTION B

$a.)$

$$(E): \left(\begin{array}{ccc|c} 1 & a & a+1 & 2 \\ 1 & a+4 & 2a+4 & b+1 \\ 2 & 3 & 5 & b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & a & a+1 & 2 \\ 0 & 4 & a+3 & b-1 \\ 0 & 2a-3 & 2a-3 & 4-b \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & a+4 & 2 \\ 0 & 4 & a+3 & b-1 \\ 0 & 0 & A & B \end{array}\right) \quad \text{where } A = (2a-3)(a-1)$$

$$B = 2ab - 2a + b - 13$$

$$\text{if } (E) \text{ has unique solution} \to A \neq 0 \to a \neq \frac{3}{2} \text{ and } a \neq 1$$

$$b.)\ \text{If } a = 1 \text{ and } (E) \text{ is consistent} \to B = 0 \to b = 5$$

$$c.)\ \text{If } a \neq 1 \text{ and } (E) \text{ is inconsistent} \to A = 0 \text{ and } B \neq 0$$

$$\to a = \frac{3}{2} \text{ and } 2ab - 2a + b - 13 \neq 0$$

$$\to b \neq 4$$

* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \to \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

* 如果 | | 不等如 0, 有唯一答案

* 如果 consistent, A=0同時B=0

* 如果 inconsistent, A=0同時B不等如0

CONT'D

2023 - SECTION B

d.) *When* $a = 2$ *and* $b = s$, (E) *becomes* (F).

$$z = \frac{B}{A} = 5s - 17, \quad y = \frac{1}{4}(b - 1 - (a + 3)z) = 21 - 6s$$

$$x = 2 - ay - (a + 1)z = 11 - 3s$$

Then, consider $mx + ny + z = -2$

$$\rightarrow m(11 - 3s) + n(21 - 6s) + (5s - 17) = -2$$

$$\rightarrow (11m + 21n - 15) - (3m + 6n - 5)s = 0$$

If $\forall s \in \mathbb{R}, \quad 11m + 21n - 15 = 0$ *and* $3m + 6n - 5 = 0$

$$\rightarrow m = -5, n = \frac{10}{3}$$

$\therefore$ *There is a real solution of F when* $m = -5$ *and* $n = \dfrac{10}{3}$

* a=2, 有唯一答案

* 用消去法

2023 - SECTION B

$Q12.)\ Let\ g(x) = x(ln(1+x))^2$

$a.)\ Find \displaystyle\int_0^{e-1} g(x)dx$

$b.)\ Hence\ find \displaystyle\int_0^{\frac{\pi}{2}} (ln(1+(e-1)cosx))^2 sin2xdx\ and \displaystyle\int_{\frac{\pi}{2}}^{\pi} (ln(1+(e-1)sinx))^2 sin2xdx$

* 參考課程 2.2, 3.8, 3.9 及 3.10

$$a.)\ Let\ I = \int_0^{e-1} g(x)dx,\ let\ y = ln(1+x) \rightarrow x = e^y - 1 \rightarrow dx = e^y dy$$

$$Then\ I = \int_0^1 e^y(e^y - 1)y^2dy = \int_0^1 e^{2y}y^2dy - \int_0^1 e^y y^2 dy$$

$$= \int_0^1 y^2 d\left(\frac{e^{2y}}{2}\right) - \int_0^1 y^2 d(e^y)$$

* 用積分代入法, 範圍要更正

* 用積分代入法

* 用積分代入法

CONT'D

2023 – SECTION B

$$= [\frac{y^2e^{2y}}{2}]_0^1 - \int_0^1 \frac{e^{2y}}{2} d(y^2) - [y^2e^y]_0^1 + \int_0^1 e^y d(y^2)$$

* 用 Integration by part

$$= \frac{e^2}{2} - \int_0^1 ye^{2y}dy - e + \int_0^1 2ye^y dy$$

* 用積分代入法
* 用積分代入法

$$= \frac{e^2}{2} - \int_0^1 y d(\frac{e^{2y}}{2}) - e + \int_0^1 2y d(e^y)$$

$$= \frac{e^2}{2} - [\frac{ye^{2y}}{2}]_0^1 + \int_0^1 \frac{e^{2y}}{2} dy - e + [2ye^y]_0^1 - \int_0^1 2e^y dy$$

* 用 Integration by part

$$= \frac{e^2}{2} - \frac{e^2}{2} + [\frac{e^{2y}}{4}]_0^1 - e + 2e - [2e^y]_0^1$$

$$= \frac{e^2}{4} - e + \frac{7}{4}$$

CONT'D

2023 – SECTION B

$$b.)\ Let\ I = \int_0^{\frac{\pi}{2}} (ln(1+(e-1)cosx))^2 sin2xdx$$

$$J = \int_{\frac{\pi}{2}}^{\pi} (ln(1+(e-1)sinx))^2 sin2xdx$$

$$For\ J,\ let\ y = x - \frac{\pi}{2},\ dy = dx$$

$$J = \int_0^{\frac{\pi}{2}} (ln(1+(e-1)sin(y+\frac{\pi}{2}))^2 sin(2y+\pi)dy$$

$$= -\int_0^{\frac{\pi}{2}} (ln(1+(e-1)cosy)^2 sin2ydy = -I$$

$$For\ I,\ let\ y = (e-1)cosx,\ dy = -(e-1)sinxdx$$

* 用積分代入法, 範圍要更正
* sin 複角公式
* cos 複角公式

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

CONT'D

2023 - SECTION B

$$Also\ I = \int_0^{\frac{\pi}{2}} (ln(1+(e-1)cosx))^2 (2sinxcosx)dx$$

$$= 2\int_{e-1}^{0} \frac{y}{e-1}(ln(1+y))^2(\frac{-dy}{e-1})$$

$$= \frac{2}{(e-1)^2}\int_0^{e-1} y(ln(1+y))^2 dy$$

$$= \frac{e^2-4e+7}{2(e-1)^2}$$

$$J = -\frac{e^2-4e+7}{2(e-1)^2}$$

* sin 雙角公式
* 用積分代入法, 範圍要更正
* 負數定積分範圍上下倒轉