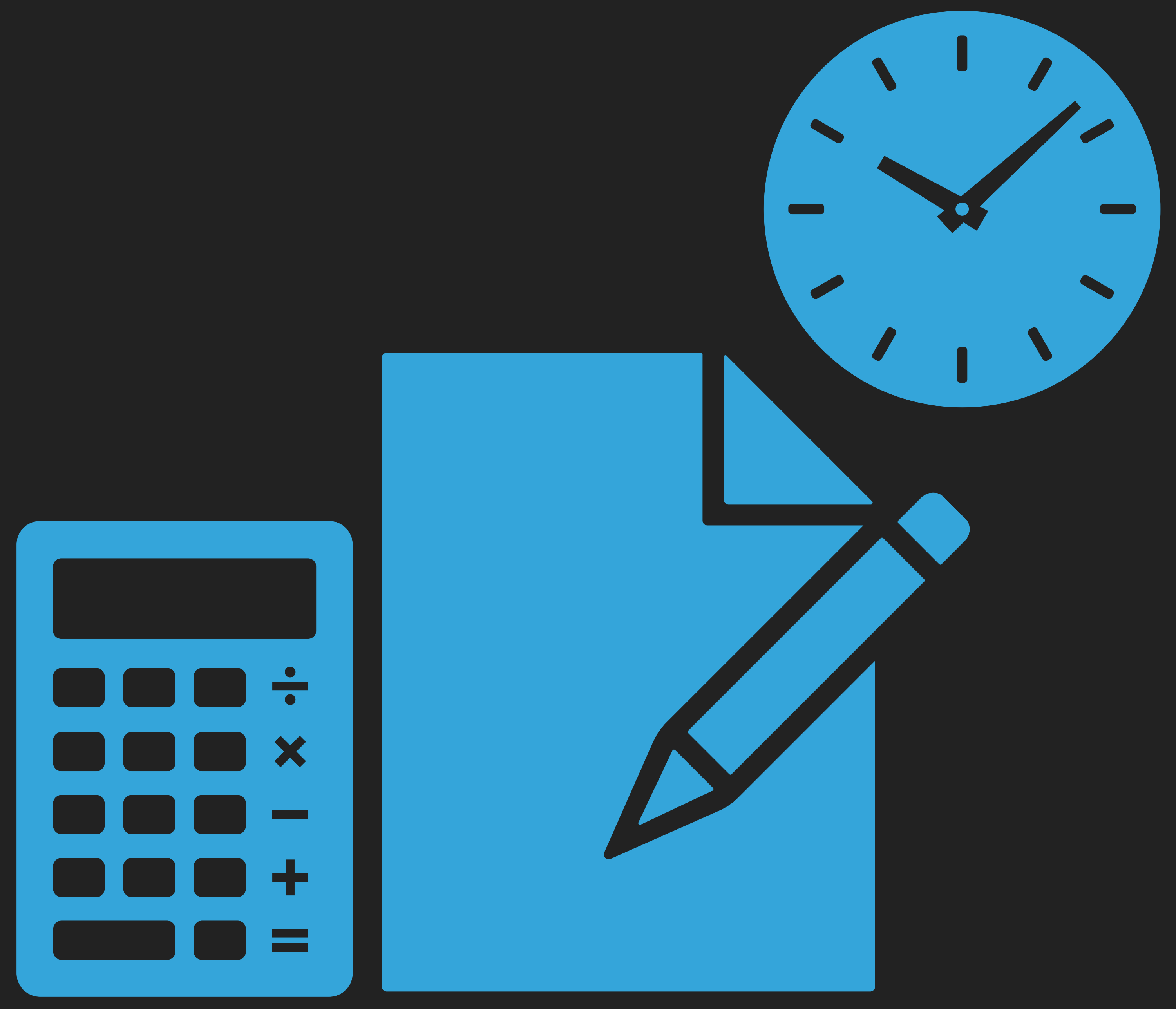
深宵教室 - DSE M1 模擬試題解答

2016

此為參考2016試題之模擬試題，原版請另行購買

2016

- Section A
- Section B



2016 - SECTION A

$Q1.)$ *Let A and B be the event such that* $P(A) = 0.4$, $P(B) = 0.7$, *and* $P(B|A) = 0.5$

a.) Are A and B independent? Explain your answer.

b.) Find $P(A \cup B)$.

* 參考課程 4.1 及 4.2

$$a.)\ P(A \cap B) = P(B|A)P(A) = 0.2$$

$$P(A)P(B) = 0.28 \neq P(A \cap B)$$

$$\therefore A \text{ and } B \text{ are not independent}$$

$$b.)\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.9$$

* P(A & B)=P(A|B)P(B)=P(B|A)P(A)
* 如果 A 同 B independent, P(A & B)=P(A)P(B)

2016 – SECTION A

Q2.) A fair dice is thrown one by one for three times.

a.) Given that the sum of 3 dices number = 7, find the probability the number 1 is thrown exactly 2 times.

b.) The dice replaced with a fair dice with 1 – 5 numbers only. Will the probability of a.) change? Explain your answer.

* 參考課程 4.2 及 4.4

$$a.)\ P(sum = 7) = C^3_2\left(\frac{1}{6}\right)^2\frac{1}{6} + P^3_3\frac{1}{6}\frac{1}{6}\frac{1}{6} + C^3_2\frac{1}{6}\left(\frac{1}{6}\right)^2 + C^3_2\frac{1}{6}\left(\frac{1}{6}\right)^2 = \frac{5}{72}$$

* 骰子點數 1, 1, 5

* 骰子點數 1, 2, 4

* 骰子點數 1, 3, 3

* 骰子點數 2, 2, 3

$$P(exact\ 2\ dices = 1\,|\,sum = 7) = \frac{\frac{1}{72}}{\frac{5}{72}} = \frac{1}{5}$$

* 條件概率

$$b.)\ P(sum = 7) = \frac{3}{25},\ P(exact\ 2\ dices = 1\,|\,sum = 7) = \frac{\frac{3}{125}}{\frac{3}{25}} = \frac{1}{5}$$

$\therefore$ *The probability will not change.*

2016 – SECTION A

$Q3.)$ *A mall open at* $9:00am$. *Tne number of vistors entering to the mall in a minute follows* $Po(1.8)$.

a.) Find the variance of the number of vistors entering the mall in a minute.

b.) Find the probability 3 vistors enter the mall in the first two minutes after opening.

c.) Only one door opened at $9:00am$. *If in any 2 consecutive minutes, are at least 4 vistors entering in each minute, the second door will open. Find the probability, after 3 minutes the mall opens, the second door opens.*

* 參考課程 4.4

Let X be the random variable of the number of vistors entering mall

$a.)\ Var(X) = 1.8$

$b.)\ P(3 \text{ vistors in } 2 \text{ minutes}) = 2(P(X=0)P(X=3) + P(X=1)P(X=2))$

$$= 2\left(\boxed{e^{-1.8}} \cdot \boxed{e^{-1.8}\frac{(1.8)^3}{3!}} + \boxed{e^{-1.8}(1.8)} \cdot \boxed{e^{-1.8}\frac{(1.8)^2}{2!}}\right)$$

$= 0.2125$ (to 4 d.p.)

$*\ X \sim Po(\lambda),\ Var(X) = \lambda$

$*\ \blacksquare\ P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

CONT'D

2016 - SECTION A

$b.)\ P(X \geq 4) = 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3)$

$$= 1 - e^{-1.8}\left(1 + 1.8 + \frac{1.8^2}{2!} + \frac{1.8^3}{3!}\right) = 0.10871$$

$$P(Second\ door\ open) = (1 - P(X \geq 4))P(X \geq 4)^2$$

$$= 0.0105\ (to\ 4\ d.p.)$$

* P(Not A) = 1-P(A)
* 頭 1 分鐘無多過 4 人
* 連續 2 分鐘多過 4 人

2016 – SECTION A

Q4.) There are many packs of balls and each pack contains 100 balls. Let p be the population proportion of balls that is red in a pack.

a.) A pack of balls is randomly selected, 64 balls is red. Find the 95 % confidence interval for p.

b.) Given that the proportion of balls in red in these packs of balls follows $N(p, 0.05^2)$. Find the least sample size to be taken such that the width of 90 % confidence interval for p is less than 0.04.

* 參考課程 4.5 及 4.7

a.) Let $p_s = 0.64$

The 95 % C.I. for p $= \left(p_s - 1.96\sqrt{\dfrac{p_s(1-p_s)}{100}},\ p_s + 1.96\sqrt{\dfrac{p_s(1-p_s)}{100}}\right)$

$= (0.54592, 0.73408)$

* 95% 置信區間

b.) Let n be the required sample size.

CONT'D

2016 - SECTION A

$$\text{The width of } 90\% \ C.I. \text{ for } p = 2 \cdot 1.645 \cdot \frac{0.05}{\sqrt{n}} < 0.04$$

$$\rightarrow n > 16.91$$

$\therefore$ *the least sample size* $= 17$ *packs* .

* 90% 置信區間

2016 - SECTION A

Q5.) The coefficient of x of $e^{kx}(1+2x)^7 = 8$. Find the coefficient of x^2

* 參考課程 1.1 及 3.2

$$e^{kx}(1+2x)^7 = \left(1 + kx + \frac{1}{2}(kx)^2 + \ldots\right)\left(1 + C_1^7(2x) + C_2^7(2x)^2 + \ldots\right)$$

$$\text{The coefficient of } x = k + C_1^7 2 = 8 \rightarrow k = -6$$

$$\text{The coefficient of } x^2 = C_2^7 2^2 + kC_1^7 2 + 0.5k^2 = 84 - 84 + 18$$

$$= 18$$

$* \quad e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

$* \quad (a+b)^n = \sum_{r=0}^{n} C_r^n a^r b^{n-r}$

$* \quad C_r^n = \frac{n!}{r!(n-r)!}$

$\rightarrow C_1^n = n \text{ and } C_2^n = \frac{n(n-1)}{2}$

2016 - SECTION A

$Q6.)\ Let\ the\ curve\ C : y = f(x), f(x) = 3^{2x} - 10(3^x) + 9$

$Find\ the\ area\ \ of\ the\ region\ bounded\ by\ C\ and\ x - axis\ .$

* 參考課程 2.8 及 2.9

$Let\ u = 3^x, f(x) = u^2 - 10u + 9$

$To\ find\ the\ x - interception\ of\ C, consider$

$u^2 - 10u + 9 = 0 \rightarrow (u-9)(u-1) = 0 \rightarrow u = 9\ or\ u = 1$

$\rightarrow x = 2\ or\ 0$

$\therefore\ The\ x - interception\ are\ 0\ and\ 2$

$The\ area, A = |\int_0^2 f(x)dx|, Let\ u = 3^x \rightarrow lnu = xln3 \rightarrow \dfrac{du}{uln3} = dx$

$\rightarrow A = |\dfrac{1}{ln3}\int_1^9 (u - 10 + 9u^{-1})du| = |\dfrac{1}{ln3}[\dfrac{u^2}{2} - 10u + 9lnu]_1^9|$

$= 40[ln3]^{-1} - 18\ unit^2$

* 二次方程解

* $(x-a)(x-b) \equiv x^2 - (a+b) + ab$

* 面積 =| C 的定積分|

* 積分三寶: 積分代入法

* 定積分代入要改範圍

2016 - SECTION A

Q7.) Let the curve $C: y = f(x)$ *and* $f(x) = (2x+8)^{\frac{3}{2}} + 3x^2$*, where,* $x > -4$*. Are there 2 tangents to C that these 2 tangents are // L :* $6x + y + 4 = 0$*? Explain your answer.*

* 參考課程 2.3 及 2.4

$$f(x) = \boxed{(2x+8)^{\frac{3}{2}}} + 3x^2 \rightarrow f'(x) = \boxed{3(2x+8)^{\frac{1}{2}}} + 6x$$

* 用 Chain rule

Assume the tangent touch at $P(x_0, f(x_0))$*, and* $f'(x_0) = -6$

* 兩條線互相平行, 斜率一樣

$$\rightarrow 3(2x_0+8)^{\frac{1}{2}} + 6x = -6 \rightarrow \boxed{(2x_0+8)^{\frac{1}{2}} = -2(x_0+1)}$$

$$\rightarrow x_0 + 4 = 2(x_0+1)^2$$

$$\rightarrow 2x_0^2 + 3x_0 - 2 = 0$$

$$\rightarrow (x_0+2)(2x_0-1) = 0$$

$$\rightarrow x_0 = -2 \text{ or } \boxed{0.5 \text{ (rejected)}}$$

* 開方 (2x+8) 變負數

$\therefore$ *There is only one tangent to C at* $x = -2$ *that // L*

2016 - SECTION A

Q8.) Let $f(x) = x^{-1}[\ln x]^2$. Let α, β be the roots of $f'(x) = 0$, where $\alpha > \beta$

a.) Find α and β.

b.) Find $\int_{\beta}^{\alpha} f(x)dx$.

*** 參考課程 2.2, 2.3 及 2.8**

$$a.)\ f'(x) = 0 \rightarrow -x^{-2}[\ln x]^2 + x^{-1} \cdot \boxed{2\ln x \cdot x^{-1}} = 0$$

$$\rightarrow -[\ln x]^2 + 2\ln x = 0, \text{ where } x \neq 0$$

$$\rightarrow \ln x(2 - \ln x) = 0 \rightarrow x = 1 \text{ or } x = e^2$$

$$\therefore \alpha = e^2 \text{ and } \beta = 1$$

$$b.)\ I = \int_1^{e^2} \frac{[\ln x]^2}{x} dx, \text{ Let } u = \ln x \rightarrow du = \frac{dx}{x}$$

$$\text{Then, } I = \int_0^2 u^2 du = [\frac{u^3}{3}]_0^2 = \frac{8}{3}$$

*** 用 Product rule**

*** ▇ 用 Chain rule**

*** 積分三寶: 積分代入**

*** 定積分代入要改範圍**

2016 – SECTION B

Q9.) X and Y are 2 companys with the same numbers of staffs. The daily resting times (in minutes) of staffs in each company follows normal distribution. In company X, 0.6 % of staffs rest less than 40 minutes while 1.5 % rest more than 70 minutes. In company Y, 1.5 % of staffs rest less than 48 minutes while 1.7 % rest more than 72 minutes.

a.) Which company has less staffs resting more than 60 minutes daily? Explain your answer.

b.) For the answer of a.), find the probability the 4^{th} randomly selected staff is the 2^{nd} one who rest more than 60 mintues.

c.) Staff rest T minutes or more will be penalty. Find the least integral value of T such that no more than 10 % of staffs are penalty in each company.

* 參考課程 4.4 及 4.5

a.) Let $\bar{T}_X$ be the random variable of the daily resting times of staffs in company $X \sim N(\mu_X, \sigma_X^2)$

$\bar{T}_Y$ be the random variable of the daily resting times of staffs in company $Y \sim N(\mu_Y, \sigma_Y^2)$

Given that $P(T_X < 40) = 0.006$ and $P(T_X > 70) = 0.015$

* 先設未知數方便運算

CONT'D

2016 – SECTION B

$$\rightarrow P(Z < \boxed{\frac{40-\mu_X}{\sigma_X}}) = 0.006 \text{ and } P(Z > \boxed{\frac{70-\mu_X}{\sigma_X}}) = 0.015$$

$$\rightarrow \frac{40-\mu_X}{\sigma_X} = -2.51 \quad - \quad (1) \text{ and } \frac{70-\mu_X}{\sigma_X} = 2.17 \quad - \quad (2)$$

$$\frac{(1)}{(2)} : \frac{40-\mu_X}{70-\mu_x} = \frac{-2.51}{2.17} \rightarrow \mu_X = 56.08974 \text{ and } \sigma_X = 6.41025$$

$$Also,\ P(T_Y < 48) = 0.015 \text{ and } P(T_Y > 72) = 0.017$$

$$\rightarrow P(Z < \boxed{\frac{48-\mu_Y}{\sigma_Y}}) = 0.015 \text{ and } P(Z > \boxed{\frac{72-\mu_Y}{\sigma_Y}}) = 0.017$$

$$\rightarrow \frac{48-\mu_Y}{\sigma_Y} = -2.17 \quad - \quad (1) \text{ and } \frac{72-\mu_Y}{\sigma_Y} = 2.12 \quad - \quad (2)$$

$$\frac{(1)}{(2)} : \frac{48-\mu_X}{72-\mu_x} = \frac{-2.17}{2.12} \rightarrow \mu_Y = 60.13986 \text{ and } \sigma_Y = 5.59441$$

* 先計算 Z ~ N(0,1), 再對表

* 兩式相除再將答案代入 (1) 式

* 兩式相除再將答案代入 (1) 式

CONT'D

2016 – SECTION B

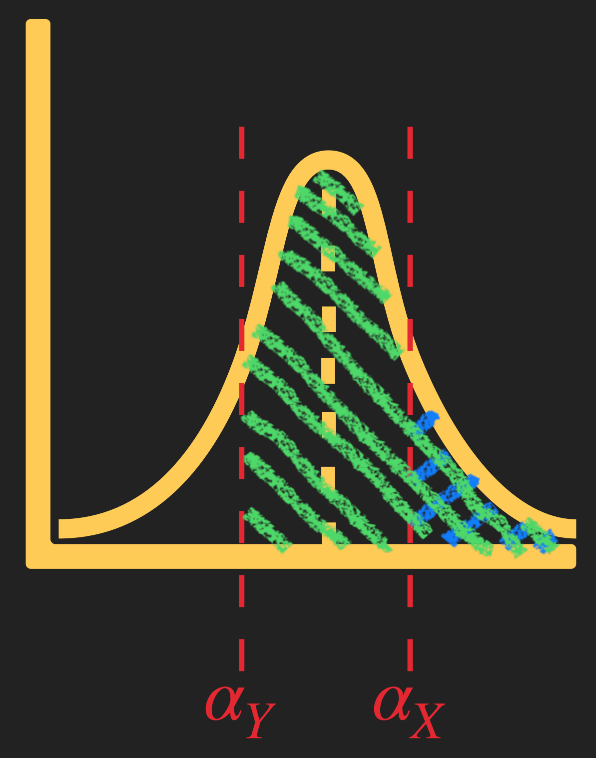
Then, consider $P(T_X > 60)$ *and* $P(T_Y > 60)$

$\because \mu_X < 60 < \mu_Y \rightarrow P(T_X > 60) = P(Z > \alpha_X),\ \alpha_X > 0$

$\rightarrow P(T_Y > 60) = P(Z > \alpha_Y),\ \alpha_Y < 0$

Hence, $P(Z > \alpha_X) < P(Z > \alpha_Y)$

i.e. Company X has less staffs resting more than 60 minutes daily.



b.) $P(T_X > 60) = P(Z > 0.61) = 0.2709$

The probability $= C_1^3 P(T_X > 60)[1 - P(T_X > 60)]^2 \cdot P(T_X > 60)$

$= 0.1170$ *(to 4 d.p.)*

* 頭三個有一個 > 60

* 第四個 > 60

c.) Solve, $P(T_X > T) \leq 0.1$ *and* $P(T_Y > T) \leq 0.1$

$\rightarrow P(Z < \frac{T - \mu_X}{\sigma_X}) \leq 0.1$ *and* $P(Z > \frac{T - \mu_Y}{\sigma_Y}) \leq 0.1$

$\rightarrow \frac{T - \mu_X}{\sigma_X} \geq 1.29$ *and* $\frac{T - \mu_Y}{\sigma_Y} \geq 1.29$

* 先計算 Z ~ N(0,1), 再對表

CONT'D

2016 – SECTION B

$\rightarrow T \geq 1.29\sigma_X + \mu_X \text{ and } T \geq 1.29\sigma_Y + \mu_Y$

$\rightarrow T \geq 64.35 \text{ and } T \geq 67.35 \rightarrow T \geq 67.35$

$\therefore \textit{The least integral value of } T = 68 \textit{ minutes}$

* And case

2016 – SECTION B

$Q10.)$ *Peter arrive at MTR at* $8:20am$. *A train arrives at* $8:30am$ *and the next train arrives at* $9:30am$. *The probability that Peter takes the train is* 0.9 *each time. If Peter takes the train at* $8:30am$. *The probability for him to be late is* 0.1. *If he takes the train at* $9:30am$. *The probability for him to be late is* 0.4. *He will be late if he cannot takes both of trains.*

a.) *Find the probability Peter takes the train on or before* $9:30am$ *on a certain day.*

b.) *Find the probability Peter is late on a certain day*

c.) *Find the probability Peter is late* 2 *times in* 6 *days.*

d.) *There a* 7 *people, including Peter, waiting for bus If Peter is late, he will go to Town* A. *otherwise, go to Town* B. *The probability for other* 6 *people to go to Town* A *and* B *are* 0.7 *and* 0.3 *respectively. Given that no people will takes the bus after these* 7 *people on the bus.*

i.) *Find the probability* 7 *people go to the same Town.*

ii.) *Find the probability exact* 3 *people go to the Town* B.

e.) *Given that exact* 3 *people go to Town* B. *Find the probability Peter is late.*

* 參考課程 4.2 及 4.4

CONT'D

2016 - SECTION B

$a.)\ The\ probability = 0.9 + (0.1)(0.9) = 0.99$

$b.)\ The\ probability = (1 - 0.99) + (0.9)(0.1) + (0.1)(0.9)(0.4)$

$= 0.136$

$c.)\ The\ probability = C_2^6(0.136)^2(1 - 0.136)^4 = 0.1546\ (to\ 4\ d.p.)$

$di.)\ The\ probability = (0.7)^6(0.136) + (0.3)^6(1 - 0.136)$

$= 0.0166\ (to\ 4\ d.p.)$

$ii.)\ The\ probability = C_2^6(0.3)^2(0.7)^4(1 - 0.136) + C_3^6(0.3)^3(0.7)^3(0.136)$

$= 0.3052\ (to\ 4\ d.p.)$

$$ii.)\ The\ probability = \frac{C_3^6(0.3)^3(0.7)^3(0.136)}{C_2^6(0.3)^2(0.7)^4(1 - 0.136) + C_3^6(0.3)^3(0.7)^3(0.136)}$$

$= 0.0825\ (to\ 4\ d.p.)$

- * 上第一班火車
- * 上第二班火車
- * 兩班都上唔到宜遲到
- * 上第一班火車宜遲到
- * 上第二班火車宜遲到
- * $X \sim B(n, p)$, $P(X = k) = C_k^n p^k(1 - p)^{n-k}$
- * 6 人去 Town A, Peter 遲到
- * 6 人去 Town B, Peter 無遲到
- * 2 人去 Town B, Peter 無遲到
- * 3 人去 Town B, Peter 遲到

* 條件概率 P(A|B)=P(A & B)/P(B)

2016 – SECTION B

Q11.) Given that the rate of change of population (unit per day) of country A and B are :

$$f(t) = \ln(t^2 - 8t + 95) \text{ and } g(t) = \frac{t+8}{\sqrt{t+3}} \text{ respectively, where } 0 \le t \le 12 \text{ measured in day}$$

a.) By the trapezoidal rule with 4 sub – interval, estimate the total population of country A from $t = 0$ to $t = 12$. Determine if the estimation is over – estimated .

b.) Find the total population of country B from $t = 0$ to $t = 12$. Determined if the difference of the poplution of country A and B from $t = 0$ to $t = 12$ exceed 2.

* 參考課程 2.2, 2.3, 2.8, 2.9 及 3.3

$$a.)\ \text{The estimation of } P_A = \int_0^{12} f(t)dt,\ I$$

$$= \frac{12-0}{4 \cdot 2}[f(0) + 2f(3) + 2f(6) + 2f(9) + f(12)]$$

$$= 54.6109 \text{ unit (to 4 } d.p.)$$

* 計算梯形面積的加總

CONT'D

2016 – SECTION B

$$Consider,\ f(t) = ln(t^2 - 8t + 95) \rightarrow f'(t) = \boxed{\frac{2t-8}{t^2-8t+95}}$$

$$\rightarrow f''(t) = \boxed{\frac{2}{t^2-8t+95} - \frac{(2t-8)^2}{(t^2-8t+95)^2}} = \frac{-2((t-4)^2-79)}{(t^2-8t+95)^2}$$

$$f''(t) > 0,\ for\ 0 \le t \le 12 \rightarrow -63 \le (t-4)^2 - 79 \le -15$$

$$\therefore I\ is\ over-estimated\ by\ the\ trapezoidal\ rule$$

$$b.)\ The\ total\ population\ of\ country\ B,\ P_B = \int_0^{12} \frac{t+8}{\sqrt{t+3}}dt$$

$$= \int_0^{12} \boxed{\frac{(t+3)+5}{\sqrt{t+3}}}dt = \int_0^{12} \boxed{\sqrt{t+3} + \frac{5}{\sqrt{t+3}}}dt$$

$$= [\frac{2(t+3)^{\frac{3}{2}}}{3} + 10\sqrt{t+3}]_0^{12} = 56.6751\ unit\ (to\ 4\ d.p.)$$

* 用 Chain rule

* 用 Product rule

* 個 f(t) 係 concave upward

* 積分三寶: Partial fraction

CONT'D

2016 - SECTION B

$Consider,\ P_B - I \approx 56.6751 - \boxed{54.6109} = 2.0642$

$\because\ I\ is\ over-estimated\ value\ for\ P_A$

$\therefore\ P_B - P_A\ exceed\ 2$

* 估大咗的數字

2016 - SECTION B

Q12.) Given that $f(t) = \dfrac{27}{2 + ate^{bt}}$, where a and $b \in \mathbb{R}$, $t \geq 0$

a.) Express $ln(\dfrac{27 - 2f(t)}{tf(t)})$ as a linear function of t

b.) Given that the linear function in a.) have the $x-intercept = 10ln0.03$ and $slope = -0.1$. Find a and b . Determine if $f(t)$ will be less than 12 for a certain value of t

c.) Describe how $f'(t)$ vary for $0 \leq t \leq 20$.

* 參考課程 2.2, 2.3, 2.4 及 3.1

a.) $ln(\dfrac{27 - 2f(t)}{tf(t)}) = ln(ae^{bt}) = lna + bt$

b.) $slope = b = -0.1$ and $0 = lna - 0.1 \cdot 10ln0.03$

$\rightarrow a = 0.03$ and $b = -0.1$

$f(t) = \dfrac{27}{2 + 0.03t \cdot e^{-0.1t}} \rightarrow f'(t) = -\dfrac{0.81e^{-0.1t}(1 - 0.1t)}{(2 + 0.03t \cdot e^{-0.1t})^2}$

* ln(AB) = lnA + lnB

* 直線方程, y = (斜率)x + (y-intercept)

* Chain rule

CONT'D

2016 - SECTION B

$$Assume\ there\ exist\ t_o,\ t_0 \geq 0\ such\ that\ f'(t_0) = 0\ \rightarrow t_0 = 10$$

	0 < t < 10	t = 10	t > 10
f''(t)	-	0	+
f'(t)	Dec.		Inc.

$$Hence,\ (10, f(10))\ is\ a\ min.\ point.$$

$$\because f(10) > 12$$

$$\therefore f(t)\ will\ not\ be\ less\ than\ 12$$

$$c.)\ f'(t) = -\frac{0.81e^{-0.1t}(1-0.1t)}{(2+0.03t\cdot e^{-0.1t})^2} = -\frac{0.81e^{-0.1t}(1-0.1t)}{[\frac{27}{f(t)}]^2}$$

$$\rightarrow 900e^{0.1t}f'(t) = (0.1t-1)[f(t)]^2$$

$$\rightarrow 900e^{0.1t}[0.1f'(t)+f''(t)] = 0.1[f(t)]^2 + 2(0.1t-1)f(t)f'(t)$$

$$\rightarrow 900e^{0.1t}f''(t) = -90e^{0.1t}f'(t) + 0.1[f(t)]^2 + 2(0.1t-1)f(t)f'(t)$$

* 搵 turning point = 搵 t_0 使度 $f'(t_0)=0$

* 利用表格計算 turning point 附近上升定下降

  $f'(x) > 0 \rightarrow Increasing$

  $f'(x) < 0 \rightarrow Decreasing$

* Implicit 微分法

CONT'D

2016 - SECTION B

$\rightarrow 900e^{0.1t}f''(t) = (2-0.1t)[f(t)]^2 + 2(1-0.1t)f(t)f'(t)$

$\rightarrow f''(t) = \dfrac{(2-0.1t)[f(t)]^2}{900e^{0.1t}} + \dfrac{2(1-0.1t)^2[f(t)]^3}{(900e^{0.1t})^2}$

$\because 0 \le t \le 20, \rightarrow (2-0.1t) \ge 0 \; and \; f(t) > 12 > 0$

$\therefore f''(t) > 0$

i.e. $f'(t)$ is increasing for $0 \le t \le 20$.

$* \;\; f'(t) = \dfrac{(0.1t-1)e^{-0.1t}[f(t)]^2}{900}$

$* \;\; f'(x) > 0 \rightarrow Increasing$