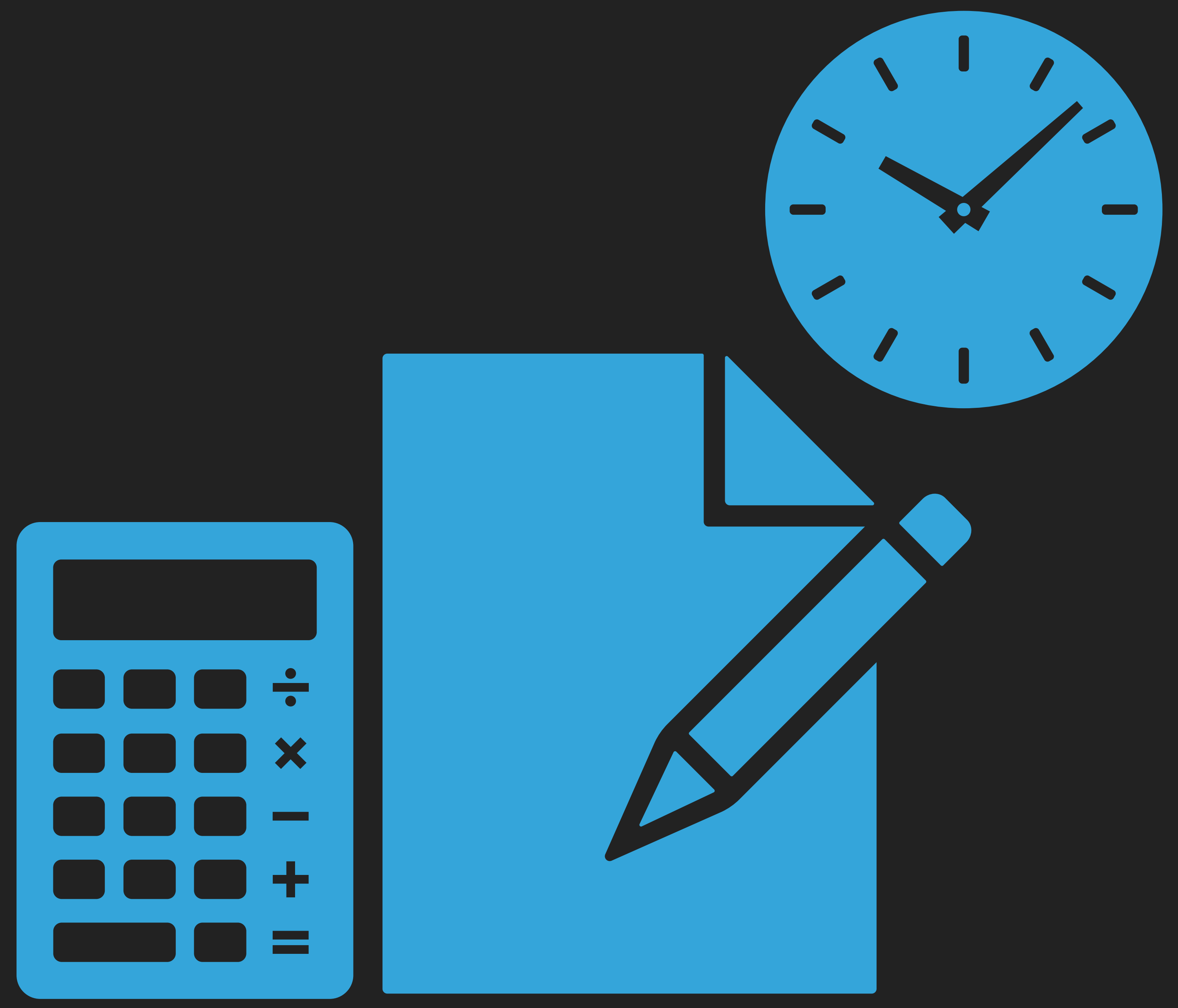
深宵教室 - DSE M1 模擬試題解答

2015

此為參考2015試題之模擬試題，原版請另行購買

2015

- Section A
- Section B



2015 - SECTION A

Q1.) Let X be discrete random vaiable with $E(X) = 5.64$ and

k	2	3	5	7	9
$P(X = k)$	0.08	0.15	a	0.45	b

Find a, b and $Var(6 - 5X)$

* 參考課程 4.1, 4.3 及 4.4

$$\text{Given that} \sum_{i=1}^{5} k_i P(X = k_i) = 5.64 \rightarrow 5a + 9b = 1.88 \quad - \quad (1)$$

$$\sum_{i=1}^{5} P(X = k_i) = 1 \rightarrow a + b = 0.32 \quad - \quad (2)$$

$$\text{By } (1) - 5x(2) \rightarrow a = 0.25 \text{ and } b = 0.07$$

$$Var(6 - 5X) = 5^2 Var(X) = 25(E(X^2) - [E(X)]^2)$$

$$= 25(\sum_{i=1}^{5} k_i^2 P(X = k_i) - 5.64^2) = 95.76$$

* $E(X) = \sum_{i=1}^{n} k_i P(X = k_i)$

* 機率之和 = 1

* 用消去法整走 a 搵 b, 再代入搵 a

* $Var(aX + b) = a^2 Var(X)$

* $Var(X) = E(X^2) - [E(X)]^2$

2015 – SECTION A

Q2.) Let A and B be the event such that $P(A) = 0.3$, $P(B) = 0.28$, and $P(B'|A') = 0.6$ where A' and B' are the complementary event of A and B respectively.

a.) Find $P(A' \cap B')$ and $P(A' \cap B)$

b.) Are A and B mutually exclusive? Explain your answer.

* 參考課程 4.1 及 4.2

$$a.)\ P(A' \cap B') = P(B'|A')P(A') = P(B'|A')(1 - P(A)) = 0.42$$

$$P(A' \cap B) = P(B|A')P(A') = (1 - P(B'|A'))(1 - P(A)) = 0.28$$

$$b.)\ \because P(A \cap B) = P(A|B)P(B) = [1 - P(A'|B)]P(B)$$

$$= [1 - \frac{P(A' \cap B)}{P(B)}]P(B) = P(B) - P(A' \cap B)$$

$$= 0$$

$\therefore$ A and B are mutually exclusive

* P(A & B)=P(A|B)P(B)=P(B|A)P(A)
* P(Not A)=1-P(A)

* 如果mutually exclusive, P(A & B)=0

2015 - SECTION A

Q3.) A bag contains 2 cards printing a question 'Are you male?' and 5 cards printing a question 'Are you female?'. A randomly selected person from a group of people draws 1 card from the bag and answer either 'Yes' or 'No' based on the question. Given the probability of a randomly selected person is male $= p$

a.) Express, in term of p, the probability of the randomly selected person answer 'Yes'.

b.) Given that 50 out of 91 people answer 'Yes'.
Find p and the probability of the selected person is felmale given that her answer is 'No'.

* 參考課程 4.1 及 4.2

Let M be the event of the selected person is male
F be the event of the selected person is female
m be the event of the selected card is 'Are you male?'
f be the event of the selected card is 'Are you female?'

$$a.)\ P(Answer\ Yes) = P(M|m)P(m) + P(F|f)P(f)$$
$$= p\frac{2}{7} + (1-p)\frac{5}{7} = \frac{5-3p}{7}$$

* P(A)=P(A|B$_1$)P(B$_1$)+P(A|B$_2$)P(B$_2$)+...

* P(Not A) = 1-P(A)

CONT'D

2015 - SECTION A

$$b.)\ \frac{50}{91} = \frac{5-3p}{7} \rightarrow p = \frac{5}{13}$$

$$P(F \mid Answer\ No) = \frac{P(F \cap Answer\ No)}{P(Anser\ No)}$$

$$= \frac{P(F \mid m)P(m)}{1 - P(Anser\ Yes)}$$

$$= \frac{(1-\frac{5}{13})\frac{2}{7}}{\frac{41}{91}} = \frac{16}{41}$$

* 條件概率

* P(Not A) = 1-P(A)

2015 - SECTION A

Q4.) A shop have a promotion with reward points card (3 – points and 7 – points) for each goods. Given that 75 % goods contain 3 – points cards while the rest contain 7 – points card. Customer gets 20 or more points can have discount. Peter buy 4 number of goods and open them one by one.

a.) Find the probability of the first 7 – points card when the 4^{th} good are opened.

b.) Find the probability Peter have discount.

c.) Given that Peter has discount, find the probability he gets 7 – points card at 4^{th} good opened.

* 參考課程 4.2 及 4.4

a.) The probability $= (0.75)^3(0.25) = \dfrac{27}{256}$

b.) The probability $= 1-(0.75)^4 - C_1^4(0.25)(0.75)^3 = \dfrac{67}{256}$

c.) The probability $= \dfrac{(1-(0.75)^3)(0.25)}{\frac{67}{256}} = \dfrac{37}{67}$

* 係四張咭, 最小要有兩張 7 分積分咭

* 條件概率
* 係三張咭, 最小要有一張 7 分積分咭
* 第四張係 7 分積分咭

2015 - SECTION A

Q5.) Find the coefficient of x^2 of $e^{-4x}(2+x)^5$

* 參考課程 1.1 及 3.2

$$\boxed{e^{-4x}}\boxed{(2+x)^5} = (\boxed{1 - 4x + \frac{1}{2}(-4x)^2 + \ldots})(\boxed{2^5 + C_1^5 2^4 x + C_2^5 2^3 x^2 + \ldots})$$

$$The\ coefficient\ of\ x^2 = C_2^5 2^3 - 4C_1^5 2^4 + 8 \cdot 2^5 = 80 - 320 + 256$$

$$= 16$$

* $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

* $(a+b)^n = \sum_{r=0}^{n} C_r^n a^r b^{n-r}$

* $C_r^n = \frac{n!}{r!(n-r)!}$

$\rightarrow C_1^n = n\ and\ C_2^n = \frac{n(n-1)}{2}$

2015 - SECTION A

Q6.) Let the curve $C_1 : y = e^{2x} + e^4$, and $C_2 : y = e^{x+3} + e^{x+1}$

Find the area of the region bounded by C_1 and C_2.

* 參考課程 2.8 及 2.9

Let $u = e^x$, $C_1 : y = u^2 + e^4$ and $C_2 : y = (e^3 + e)u$

To find the intercept of C_1 and C_2, consider,

$$u^2 + e^4 = (e^3 + e)u \rightarrow u^2 - (e^3 + e)u + e^4 = 0$$

$$\rightarrow (u - e^3)(u - e) = 0 \rightarrow x = 3 \text{ or } 1$$

$\therefore$ The x − coordination of the intercept are 1 and 3

$$\text{The area} = \left| \int_1^3 e^{x+3} + e^{x+1} - e^{2x} - e^4 dx \right|$$

$$= \left| \left[e^{x+3} + e^{x+1} - \frac{e^{2x}}{2} - e^4 x \right]_1^3 \right| = \frac{e^2(e^4 - 4e^2 - 1)}{2} \text{ unit}^2$$

* 二次方程解

* $(x-a)(x-b) \equiv x^2 - (a+b) + ab$

* 面積 =| (C_2-C_1)的定積分|

2015 - SECTION A

Q7.) Let the curve $C : y = f(x)$ *and* $f(x) = x\sqrt{2x^2+1}$. *Find 2 equtaion of tangent to C such that these 2 tangents are* $\perp L : 3x + 17y = 0$.

* 參考課程 2.3 及 2.4

$$f(x) = x\sqrt{2x^2+1} \rightarrow [f(x)]^2 = 2x^4 + x^2$$

$$\rightarrow 2f(x)f'(x) = 8x^4 + 2x$$

$$\rightarrow f'(x) = \frac{4x^2+1}{\sqrt{2x^2+1}}$$

* Implicit 微分法

$$Assume\ the\ tangent\ touch\ at\ P(x_0, f(x_0)),\ and\ f'(x_0) = \frac{17}{3}$$

* 兩條線互相垂直, 斜率相乘 = -1

$$\rightarrow \frac{4x_0^2+1}{\sqrt{2x_0^2+1}} = \frac{17}{3} \rightarrow 9(4x_0^2+1)^2 = 289(2x_0^2+1)$$

$$\rightarrow 144x_0^4 - 506x_0^2 - 280 = 0 \rightarrow x_0^2 = 4\ or\ x_0^2 = -\frac{70}{144}\ (rejected)$$

CONT'D

2015 - SECTION A

$$\rightarrow x_0 = \pm 2$$

Hence there are 2 *tangents to* C *touch at* $(2, f(2))$ *and* $(-2, f(-2))$

The equation of the tangents are :

$$y - f(2) = \frac{17}{3}(x-2) \text{ and } y - f(-2) = \frac{17}{3}(x+2)$$

$$\rightarrow 3(y-6) = 17(x-2) \text{ and } 3(y+6) = 17(x+2)$$

$$\rightarrow 17x - 3y - 16 = 0 \text{ and } 17x - 3y + 16 = 0$$

* 直線方程, 點斜式

2015 - SECTION A

Q8.) Find $\int x^5 ln(x^2+1)dx$.

* 參考課程 2.5 及 2.6

$$\int x^5 ln(x^2+1)dx = \int ln(x^2+1)d(\frac{1}{6}x^6)$$

$$= \frac{1}{6}x^6 ln(x^2+1) - \frac{1}{6}\int \frac{x^6}{x^2+1}d(x^2+1)$$

$$= \frac{1}{6}x^6 ln(x^2+1) - \frac{1}{6}\int \frac{[(x^2+1)-1]^3}{x^2+1}d(x^2+1)$$

$$= \frac{1}{6}x^6 ln(x^2+1) - \frac{1}{6}\int u^2 - 3u + 3 - u^{-1}du, \text{ where } u = x^2+1$$

$$= \frac{1}{6}[(x^6+1)ln(x^2+1) - \frac{(x^2+1)^3}{3} + \frac{3(x^2+1)^2}{2} - 3(x^2+1)] + C$$

where C is a constant

* 積分三寶: Integration by part

* 無中生有 $(a+b)^n = \sum_{r=0}^{n} C_r^n a^r b^{n-r}$

2015 – SECTION B

Q9.) Let the speed of truck (in km/hr) passing through a roadblock be $X \sim N(\mu, 16^2)$.

a.) A random sample of 25 trucks is shown below :

Stem (tens)	Leaf (units)
6	0 0 1 1 1 2 2 3 4 4 5 5 6 6 7
7	1 1 2 3 5 5 6
8	3 6 7

Find the 95 % confidence interval for μ

b.) Find the least sample size such that width of 97.5 % confidence interval of $\mu < 9$

c.) Given that $\mu = 66$*. If 12 trucks pass through the roadblock, find the probability that more than 2 trucks travel exceed 90 km/hr.*

* 參考課程 4.4, 4.5, 4.6 及 4.7

a.) Let $\bar{X}_n$ *be the n size random sample*

$$\bar{X}_{25} = \frac{\sum_{i=1}^{25} x_i}{25} = 68.64$$

* 平均值 = 數據加總 / 總數

CONT'D

2015 - SECTION B

$$\therefore \ The\ 95\ \%\ C.I.\ of\ \theta = \left(68.64 - 1.96 \cdot \frac{16}{\sqrt{25}},\ 68.64 + 1.96 \cdot \frac{16}{\sqrt{25}}\right)$$

$$= (62.368,\ 74.912)$$

$$b.)\ For\ \bar{X}_n,\ the\ width\ of\ the\ 97.5\ \%\ C.I.\ for\ \mu = 2 \cdot 2.24 \cdot \frac{16}{\sqrt{n}} < 9$$

$$\rightarrow n > 63.43$$

$$\therefore \ The\ min.\ sample\ size = 64$$

$$c.)\ P(X > 90) = P\left(Z > \frac{90-66}{16}\right) = P(Z > 1.5) = 0.0668$$

$$Let\ p = P(X > 90),\ and\ q = 1 - p$$

$$The\ required\ probability = 1 - C_0^{12} q^{12} - C_1^{12} p q^{11} - C_2^{12} p^2 q^{10}$$

$$= 0.0416$$

* 95% 置信區間

* 97.5% 置信區間

* 先計算 Z ~ N(0,1), 再對表

* 無車超過 90

* 1架車超過 90

* 2架車超過 90

2015 – SECTION B

Q10.) The number of people in a minute purchase for cokes follows $Po(3.2)$. The numbers of cokes purchased by a person shows as below :

Numbers of coke	1	2	3	4	5	6	>7
Probability	0.12	0.7	0.08	0.04	0.03	0.02	0.01

a.) Find the probability fewer than 4 people buy cokes in a minute.

b.) Find the probability the 8^{th} person is the 3^{rd} person buys 2 cokes.

c.) Find the probability exact 3 people buy cokes in a minutes and each of them buys 2 cokes.

d.) Find the probability exact 3 people buy cokes in a minutes and they buy total 6 cokes.

e.) Given that fewer than 4 people buy cokes. Find the probability they buy total 6 cokes.

* 參考課程 4.3 及 4.4

a.) Let $X \sim Po(3.2)$ be the number of people buy cokes in a minutes

$$P(X<4) = e^{-3.2}\left(1+3.2+\frac{3.2^2}{2!}+\frac{3.2^3}{3!}\right) = 0.6025 \ (to\ 4\ d.p.)$$

* $P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

CONT'D

2015 - SECTION B

$b.)\ The\ probability = C_2^7(0.7)^2(1-0.7)^5 \cdot 0.7 = 0.0175\ (to\ 4\ d.p.)$

$c.)\ The\ probability = P(X=3) \cdot 0.7^3 = 0.0764\ (to\ 4\ d.p.)$

$d.)\ Let\ Y_n\ be\ the\ event\ of\ a\ person\ buys\ n\ cokes$

$Z_n\ be\ the\ event\ of\ exact\ n\ people\ buy\ cokes\ in\ a\ minutes\ and$
$they\ buy\ total\ 6\ cokes.$

$p_n = P(X=n)$

$$P(Z_3) = p_3(C_2^3 P(Y_1)^2 P(Y_4) + P_3^3 P(Y_1)P(Y_2)P(Y_3) + P(Y_2)^3)$$

$$= 0.0857\ (to\ 4\ d.p.)$$

$e.)\ P(Z_1) = p_1 P(Y_6) = 0.002609$

$$P(Z_2) = p_2(2P(Y_1)P(Y_5) + 2P(Y_2)P(Y_4) + P(Y_3)^2) = 0.014526$$

$$P(Total\ 6\ cokes\ \cap X < 4) = P(Z_1) + P(Z_2) + P(Z_3) = 0.102835$$

$$P(Total\ 6\ cokes \mid X < 4) = \frac{0.102835}{0.6025} = 0.1707\ (to\ 4\ d.p.)$$

* 頭 7 個有 2 個買 2 罐可樂
* 第 8 個買 2 罐可樂
* $P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

* 1, 1, 4 罐可樂組合
* 1, 2, 3 罐可樂排序
* 2, 2, 2 罐可樂組合
* 1 個買 6 罐可樂組合
* 2 個買 1, 5 罐可樂組合
* 2 個買 2, 4 罐可樂組合
* 2 個買 3, 3 罐可樂組合

* 條件概率

2015 – SECTION B

Q11.) Given that the rate of change of population (unit per day) of country A and B are :

$$f(t) = \ln(e^t - t) \text{ and } g(t) = \frac{8t}{t+1} \text{ respectively, where } 2 \leq t \leq 12 \text{ measured in day}$$

a.) By the trapezoidal rule with 5 sub – interval, estimate the total population of country A from $t = 2$ to $t = 12$. Determine if the estimation is over – estimated .

b.) Find the exact total population of country B from $t = 2$ to $t = 12$. Determined which country have more poplution from $t = 2$ to $t = 12$.

*** 參考課程 2.2, 2.3, 2.8, 2.9 及 3.3**

$$a.)\ \text{The estimation of } P_A = \int_2^{12} f(t)dt,\ I$$

$$= \frac{12-2}{5 \cdot 2}[f(2) + 2f(4) + 2f(6) + 2f(8) + 2f(10) + f(12)]$$

$$= 69.4959 \text{ unit (to 4 d.p.)}$$

*** 計算梯形面積的加總**

CONT'D

2015 – SECTION B

$$Consider,\ f(t) = ln(e^t - t) \rightarrow f'(t) = \boxed{\frac{e^t - 1}{e^t - t}}$$

$$\rightarrow f''(t) = \boxed{\frac{e^t}{e^t - t} - \frac{(e^t - 1)^2}{(e^t - t)^2}} = \frac{e^t(2 - t) - 1}{(e^t - t)^2} < 0,\ for\ 2 \leq t \leq 12$$

$$\therefore I\ is\ under-estimated\ by\ the\ trapezoidal\ rule$$

$$b.)\ The\ total\ population\ of\ country\ B,\ P_B = \int_2^{12} \frac{8t}{t+1} dt$$

$$= 8\int_2^{12} \boxed{\frac{(t+1) - 1}{t+1}} dt = 8\int_2^{12} \boxed{1 - \frac{1}{t+1}} dt = 8[t - ln(t+1)]_2^{12}$$

$$= 80 - 8ln\frac{13}{3}\ unit \approx 68.2693\ unit$$

$$\because\ The\ under-estimated\ value\ of\ P_A > P_B$$

$$\therefore P_A > P_B$$

* 用 Chain rule

* 用 Product rule

* 個 f(t) 係 concave downward

* 積分三寶: Partial fraction

2015 - SECTION B

$Q12.)$ $$Given\ that\ f(t) = \frac{200}{1+a2^{bt}},\ where\ a\ and\ b \in \mathbb{R},\ t \geq 0$$

$a.)\ Express\ ln(\frac{200}{f(t)} - 1)\ as\ a\ linear\ function\ of\ t$

$b.)\ Given\ that\ the\ linear\ function\ in\ a.)\ have\ the\ y-intercept = ln4\ and\ x-intercept = 4$

$Find\ a\ and\ b\ .$

$c.)\ Describe\ how\ f(t)\ and\ f'(t)\ vary\ for\ 0 \leq t \leq 48.$

* 參考課程 2.2, 2.3, 2.4 及 3.1

$$a.)\ ln(\frac{200}{f(t)} - 1) = ln(a2^{bt}) = lna + ln2^{bt} = (bln2)t + lna$$

$$b.)\ y - intercept = lna = ln4\ and\ slope = \frac{0 - ln4}{4 - 0} = bln2$$

$$\rightarrow a = 4\ and\ b = -0.5$$

$$c.)\ f(t) = \frac{200}{1 + 4 \cdot 2^{-0.5t}} \rightarrow (1 + 4 \cdot 2^{-0.5t})f(t) = 200$$

* ln(AB) = lnA + lnB
* $lnA^n = nlnA$
* 直線方程, y = (斜率)x + (y-intercept)

CONT'D

2015 - SECTION B

$$\rightarrow 4 \cdot (-0.5ln2 \cdot 2^{-0.5t})f(t) + (1 + 4 \cdot 2^{-0.5t})f'(t) = 0$$

$$\rightarrow f'(t) = \frac{(2ln2 \cdot 2^{-0.5t})f(t)}{\frac{200}{f(t)}} = \frac{2^{-0.5t}ln4[f(t)]^2}{200} > 0$$

$\therefore f(t)$ *is increasing for* $0 \leq t \leq 48$

Then, $200f'(t) = 2^{-0.5t}ln4[f(t)]^2$

$$\rightarrow 200f''(t) = (-0.5ln2 \cdot 2^{-0.5t})ln4[f(t)]^2 + 2^{-0.5t}ln4 \cdot 2f(t) \cdot f'(t)$$

$$\rightarrow 200f''(t) = -[ln2]^2 \cdot 2^{-0.5t} \cdot [f(t)]^2 + \frac{[2^{-0.5t}ln4]^2 \cdot [f(t)]^3}{100}$$

Assume there exist $0 \leq t_0 \leq 48$ *such that* $f''(t_0) = 0$

$$Then,\ 0 = -[ln2]^2 \cdot 2^{-0.5t_0} \cdot [f(t_0)]^2 + \frac{[2^{-0.5t_0}]^2 \cdot [ln2]^2 \cdot [f(t_0)]^3}{25}$$

*** Implicit 微分法**

* $y = 2^{-0.5t} \rightarrow lny = (-0.5ln2)t$

$\rightarrow \frac{y'}{y} = -0.5ln2$

$\rightarrow y' = (-0.5ln2)2^{-0.5t}$

* $f(t) = \frac{200}{1 + 4 \cdot 2^{-0.5t}} \rightarrow 1 + 4 \cdot 2^{-0.5t} = \frac{200}{f(t)}$

$f'(t) > 0 \rightarrow$ *Increasing*

*** 搵 turning point = 搵 t_0 使度 $f''(t_0)=0$**

CONT'D

2015 - SECTION B

$$\rightarrow \cancel{[ln2]^2} \cdot \cancel{2^{-0.5t_0}} \cdot \cancel{[f(t_0)]^2} = \frac{[2^{-0.5t_0}]^{\cancel{2}} \cdot \cancel{[ln2]^2} \cdot [f(t_0)]^{\cancel{3}}}{25}$$

$$where\ f(t_0) \neq 0,\ 0 \leq t \leq 48$$

$$\rightarrow 25 = 2^{-0.5t_0} \cdot f(t_0) \rightarrow 25(1 + 4 \cdot 2^{-0.5t_0}) = 200 \cdot 2^{-0.5t_0}$$

$$\rightarrow 2^{-0.5t_0} = 2^{-2} \rightarrow t_0 = 4$$

	0 < t < 4	t = 4	4 < t < 48
f′′(t)	+	0	-
f′(t)	Inc.		Dec.

$\therefore f'(t)\ is\ increasing\ for\ 0 \leq t \leq 4$

$f'(t)\ is\ decreasing\ for\ 4 < t \leq 48$

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow Increasing$

$f'(x) < 0 \rightarrow Decreasing$