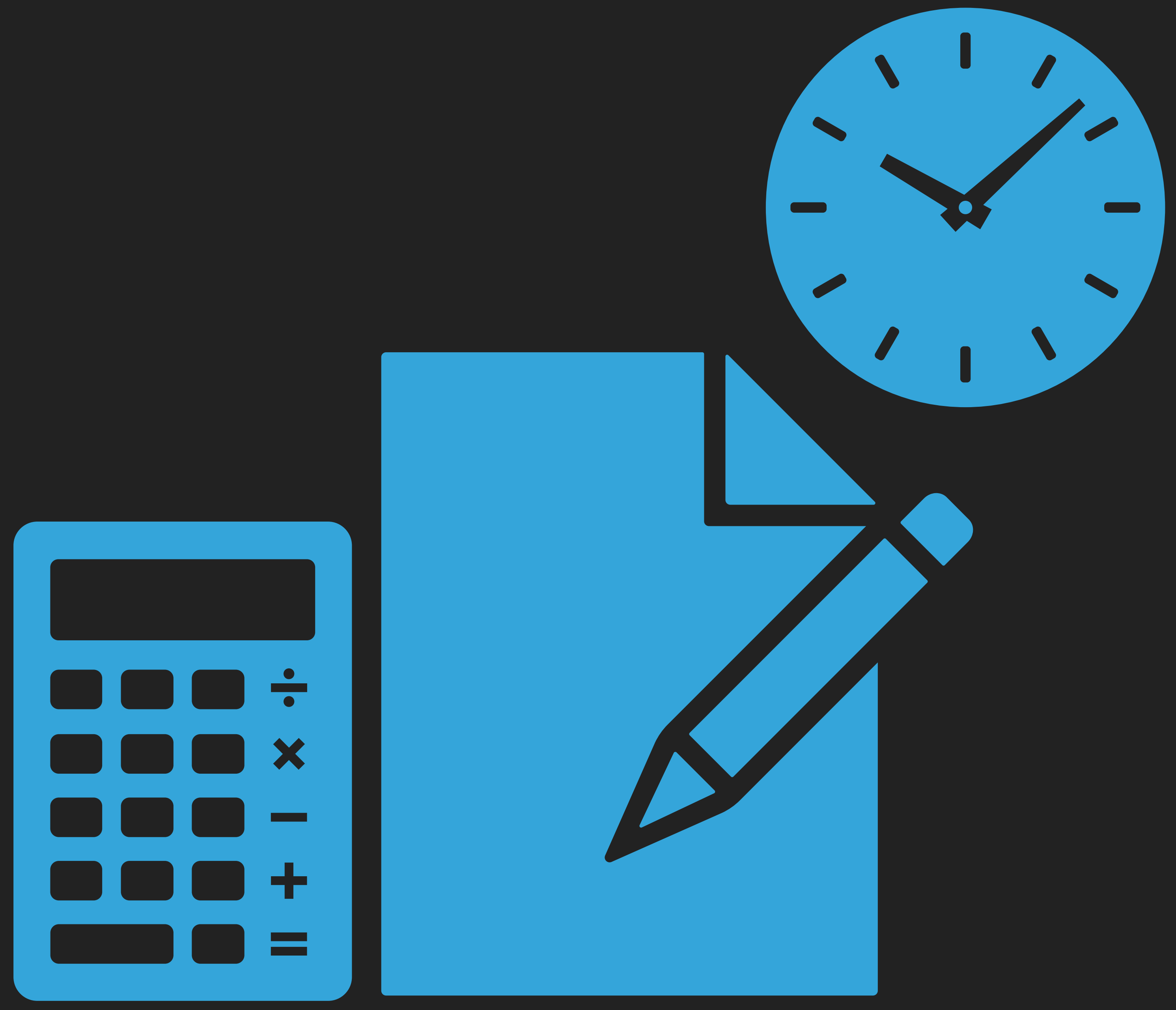
深宵教室 - DSE M2 模擬試題解答

2020

2020

- Section A
- Section B



2020 - SECTION A

$Q1.)\ (1+kx)^9(1-x)^4 = A + Bx - 3x^2 + \ldots, \quad k = ?$

* 參考課程 1.1

$$(1+kx)^9(1-x)^4 \equiv \left(\sum_{r=0}^{9} C_r^9 (kx)^r\right)\left(\sum_{r=0}^{4} C_r^4 (-x)^r\right)$$

By compare coefficient of x^2

$$-3 = (C_0^9)(C_2^4(-1)^2) + (C_1^9 k)(C_1^4(-1)) + (C_2^9 k^2)(C_0^4)$$

$$\rightarrow -3 = 6 + (9k)(-4) + (36k^2) = 6 - 36k + 36k^2$$

$$\rightarrow (2k-1)^2 = 0$$

$$\therefore k = \frac{1}{2}$$

* Binomial Expansion

$* \ C_r^n = \dfrac{n!}{r!(n-r)!} \rightarrow C_1^n = n \text{ and } C_2^n = \dfrac{n(n-1)}{2}$

2020 - SECTION A

$Q2.)\ f(x) = \dfrac{x}{\sqrt{2+x}}$, where $x > -2$. $f'(2) = ?$ (By First Principles)

* 參考課程 3.1 及 3.2

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2+h}{\sqrt{4+h}} - 1\right) = \lim_{h\to 0} \frac{(2+h) - \sqrt{4+h}}{h\sqrt{4+h}}$$

$$= \lim_{h\to 0} \frac{[(2+h) - \sqrt{4+h}][(2+h) + \sqrt{4+h}]}{h\sqrt{4+h}[(2+h) + \sqrt{4+h}]}$$

$$= \lim_{h\to 0} \frac{(2+h)^2 - (4+h)}{h\sqrt{4+h}[(2+h) + \sqrt{4+h}]} = \lim_{h\to 0} \frac{\cancel{h}(h+3)}{\cancel{h}\sqrt{4+h}(2+h+\sqrt{4+h})}$$

$$= \frac{3}{8}$$

* 微分定義

* 無中生有整個 h 出來

* $(a+b)(a-b) = a^2 - b^2$

2020 - SECTION A

$Q3.)\ For\ x \neq \frac{k\pi}{6}\ where\ k = \pm 1, \pm 2, ...$

$a.)\ Prove\ tanx tan(\frac{\pi}{3} - x) tan(\frac{\pi}{3} + x) = tan3x$

$b.)\ Prove\ tan55^o tan65^o tan75^o = tan85^o$

* 參考課程 1.2, 2.1, 2.2 及 2.3

$a.)\ Consider,\ tanx tan(\frac{\pi}{3} - x) tan(\frac{\pi}{3} + x) = tan3x$

$$\leftrightarrow tanx \frac{\sqrt{3} - tanx}{1 + \sqrt{3}tanx} \frac{\sqrt{3} + tanx}{1 - \sqrt{3}tanx} = \frac{tan2x + tanx}{1 - tan2xtanx}$$

$$\leftrightarrow tanx \frac{3 - tan^2x}{1 - 3tan^2x} = \frac{\frac{2tanx}{1 - tan^2x} + tanx}{1 - \frac{2tanx}{1 - tan^2x} tanx}$$

* 雙向推論

* tan 複角公式

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$

$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$

$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$

$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$

* $(a + b)(a - b) = a^2 - b^2$

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$$\leftrightarrow tanx\frac{3-tan^2x}{1-3tan^2x}=\frac{tanx(2+1-tan^2x)}{1-tan^2x-2tan^2x}$$

$$\leftrightarrow \frac{tanx(3-tan^2x)}{1-3tan^2x}=\frac{tanx(3-tan^2x)}{1-3tan^2x}$$

$\therefore$ *Prove is complete*

b.) Using a.) result with $x=5^o$

$$tan(5^o)tan(55^o)tan(65^o)=tan(15^o)$$

$$\rightarrow \frac{1}{tan(85^o)}tan(55^o)tan(65^o)=\frac{1}{tan(75^o)}$$

$$\rightarrow tan(55^o)tan(65^o)tan(75^o)=tan(85^o)$$

* $tan(90^o-x)=\dfrac{1}{tanx}$

2020 - SECTION A

Q4.) Find the volume of the solid generated by revolving the region, $0 \le x \le 1$, *along* $x-axis$

$$G: \quad y = 4x(1-x^2)^{\frac{1}{4}}$$

* 參考課程 3.8, 3.10 及 3.12

$$The\ volume = \pi\int_0^1 y^2dx = \pi\int_0^1 16x^2\sqrt{1-x^2}\,dx$$

$$= \pi\int_0^{\frac{\pi}{2}} 16sin^2\theta\boxed{\sqrt{(1-sin^2\theta)}}cos\theta d\theta$$

$$= 16\pi\int_0^{\frac{\pi}{2}} sin^2\theta\boxed{cos\theta}cos\theta d\theta = 16\pi\int_0^{\frac{\pi}{2}} \boxed{sin^2\theta cos^2\theta}d\theta$$

$$= 16\pi\int_0^{\frac{\pi}{2}} \boxed{\frac{1}{4}sin^2 2\theta}d\theta$$

* 旋轉體積 (x-axis)= $\pi\int_a^b [f(x)]^2dx$

* 利用三角代入, $x = sin\theta$

* $1 - sin^2\theta = cos^2\theta$

* 定積分代入要改範圍

* sin 雙角公式

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$$= 4\pi\int_0^{\frac{\pi}{2}} sin^2 2\theta d\theta = 4\pi\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - cos4\theta)d\theta$$

$$= 2\pi\int_0^{\frac{\pi}{2}} d\theta - 2\pi\cancelto{0}{\int_0^{\frac{\pi}{2}} cos4\theta d\theta}$$

$$= \pi^2\ sq.unit$$

* cos 雙角公式

* cos4x $(0 \to \frac{\pi}{2})$ 面積互相抵消

2020 - SECTION A

$Q5.)\ Prove\ \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n\cdot(n+1)\cdot(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)},\ \forall n\in\mathbb{Z}^+$

*** 參考課程 1.1 及 1.2**

方法1

$$Let\ P(n):\ \sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{n(n+3)}{4(n+1)(n+2)}\ \forall n\in\mathbb{Z}^+$$

* 先 Let Statement

$$For\ P(1):\ L.H.S.=\frac{1}{6}=R.H.S.$$

* 証明 P(1) is true

$$Assume\ P(k)\ is\ true\ \exists k\in\mathbb{Z}^+,\ then\ P(k+1):$$

* 假設 P(k) is true. 証明 P(k+1) is true

$$\begin{aligned} L.H.S. &= \sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)} \\ &= \sum_{r=1}^{k}\frac{1}{r(r+1)(r+2)}+\frac{1}{(k+1)(k+2)(k+3)} \end{aligned}$$

* 將末項抽出並改變末項

CONT'D

2020 - SECTION A

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)} = \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

* 按 RHS 想要的結果, 用長除試 k+1

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$= R.H.S.$$

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

i.e. By M.I., $P(n)$ *is true,* $\forall n \in \mathbb{Z}^+$

* 寫結論

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2020 - SECTION A

方法2

$$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\sum_{r=1}^{n}\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}$$

$$=\sum_{r=1}^{n}\frac{1}{2r}-\sum_{r=1}^{n}\frac{1}{r+1}+\sum_{r=1}^{n}\frac{1}{2(r+2)}$$

$$=\frac{1}{2}\left(\sum_{r=1}^{n}\frac{1}{r}-\sum_{r=1}^{n}\frac{1}{r+1}\right)+\frac{1}{2}\left(\sum_{r=1}^{n}\frac{1}{r+2}-\sum_{r=1}^{n}\frac{1}{r+1}\right)$$

$$=\frac{1}{2}\left(\sum_{r=1}^{n}\frac{1}{r}-\sum_{r=2}^{n+1}\frac{1}{r}\right)+\frac{1}{2}\left(\sum_{r=3}^{n+2}\frac{1}{r}-\sum_{r=2}^{n+1}\frac{1}{r}\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{n+1}\right)+\frac{1}{2}\left(\frac{1}{n+2}-\frac{1}{2}\right)=\frac{n(n+3)}{4(n+1)(n+2)}$$

*** 利用 Partial Fraction**

$$* \ Let\ \frac{1}{r(r+1)(r+2)}\equiv\frac{A}{r}+\frac{B}{(r+1)}+\frac{C}{(r+2)}$$

$$\rightarrow 1\equiv A(r+1)(r+2)+Br(r+2)+Cr(r+1)$$

分別代r=0, -1, -2 得 A, B, C 答案

*** 透過改變首末項改變公項**

*** 透過抽首尾項改變首末項**

$$\sum_{r=1}^{n}\frac{1}{r}-\sum_{r=2}^{n+1}\frac{1}{r}=1+\cancel{\sum_{r=2}^{n}\frac{1}{r}}-\cancel{\sum_{r=2}^{n}\frac{1}{r}}-\frac{1}{n+1}$$

$$\sum_{r=3}^{n+2}\frac{1}{r}-\sum_{r=2}^{n+1}\frac{1}{r}=\cancel{\sum_{r=3}^{n+1}\frac{1}{r}}+\frac{1}{n+2}-\frac{1}{2}-\cancel{\sum_{r=3}^{n+1}\frac{1}{r}}$$

2020 - SECTION A

Q6.) *Given a curve* $C_1: \ y = 2^{x-1}, x > 0$ *and* $P = (u, v)$ *move along* C_1. *Let C be the circle with OP be the diameter such that C area increase at a rate* $= 5\pi \ unit^2 s^{-1}$

a.) *Does* $S = u^2 + v^2$ *have a constant rate of change?*

b.) *Let* $C_2 : y = 2^x, x > 0$, *be a curve that the vertical line passing through P and intersect at* C_2 *at point Q. Find the rate of change of the area* ΔOPQ *when* $u = 2$

* 參考課程 3.3 及 3.4

a.) $S = \boxed{u^2 + v^2} = \boxed{OP^2} = \frac{4}{\pi}[\pi(\frac{OP}{2})^2] = \frac{4A}{\pi}$, *where* $A =$ *the area of C*

* 畢氏定理

$$\therefore \frac{dS}{dt} = \frac{4}{\pi}\frac{dA}{dt} = \frac{4}{\pi}(5\pi) = 20$$

$$i.e. \frac{dS}{dt} = constant$$

b.) *Let the area* ΔOPQ *be* A_1 *and* $Q = (u, 2^u)$

$$\because v = 2^{u-1} \rightarrow 2^u = 2v \rightarrow Q = (u, 2v)$$

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$$S = u^2 + v^2 \rightarrow \frac{dS}{dt} = 2u\frac{du}{dt} + 2v\frac{dv}{dt} = 20$$

$$\rightarrow u\frac{du}{dt} + v\frac{dv}{dt} = 10$$

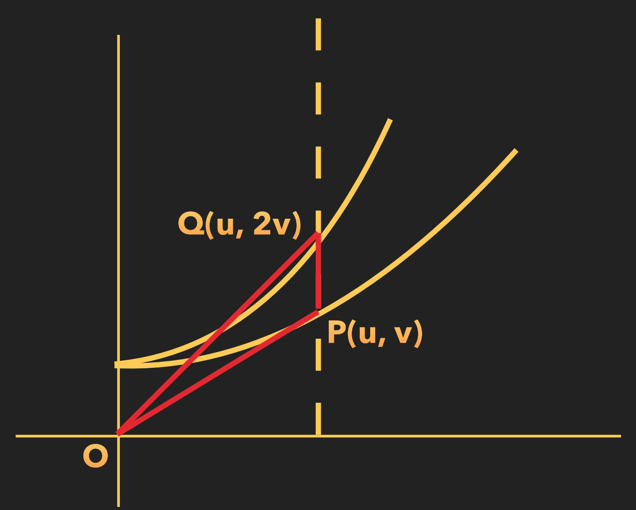
$$\rightarrow 2\frac{du}{dt}|_{u=2} + 2^{2-1}\frac{dv}{dt}|_{u=2} = 10$$

$$\rightarrow \frac{du}{dt}|_{u=2} + \frac{dv}{dt}|_{u=2} = 5$$

$$Also, A_1 = \frac{uxPQ}{2} = \frac{u(2v-v)}{2} = \frac{uv}{2} = \frac{1}{4}[(u+v)^2 - (u^2+v^2)]$$

$$\rightarrow \frac{dA_1}{dt} = \frac{1}{4}[2(u+v)(\frac{du}{dt} + \frac{dv}{dt}) - \frac{dS}{dt}]$$

* Implicit 微分法



* $(a+b)^2 = a^2 + 2ab + b^2$

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2020 - SECTION A

$$\rightarrow \frac{dA_1}{dt}|_{u=2} = \frac{1}{4}[2(2+2)(\frac{du}{dt}|_{u=2} + \frac{dv}{dt}|_{u=2}) - 20]$$

$$= \frac{1}{4}[2(4)(5) - 20] = 5\ unit^2 s^{-1}$$

* 用上述計算結果

2020 - SECTION A

Q7.) Let a curve be $\Gamma : y = f(x)$, where Γ passes through $(1, 2)$ and $f'(x) = -2x + 8, \forall x \in \mathbb{R}$

Let L be a tangent line to Γ at P and passes through $(5, 14)$ with negative slope

Find Γ, P and the equation of normal to Γ at P

*** 參考課程 3.4, 3.6 及 3.9**

$$f'(x) = -2x + 8 \to f(x) = \int (-2x+8)\, dx$$

$$\to f(x) = -x^2 + 8x + C, \text{ where } C \text{ is constant}$$

$$\because f(1) = 2, \therefore C = -5, \quad i.e.\ \Gamma : y = -x^2 + 8x - 5$$

$$\text{Let } P = (a, b) \text{ and } L:\ y - 14 = f'(a)(x-5), \text{ where } f'(a) < 0$$

$$\text{Hence, } b = -2a^2 + 18a - 26 \text{ ——— } (1)$$

$$\text{Also, } P \text{ lies on } \Gamma, \text{ such that } b = -a^2 + 8a - 5 \text{ ——— } (2)$$

$$(1) - (2) : 0 = -a^2 + 10a - 21 \to a = 3\ (f'(3) > 0, \text{ rejected}) \text{ or } 7$$

* 積分係類似微分的逆函數

* Core 課程中直線方程: 點斜式

CONT'D

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$$\therefore \ P = (7, 2)$$

$$\textit{The equation of normal}: \ y - 14 = \boxed{-\frac{1}{f'(7)}}(x - 5)$$

$$x - 6y + 5 = 0$$

* Normal 與 tangent 互相垂直
斜率相乘 = -1

2020 - SECTION A

Q8.)

$$P=\begin{pmatrix}-5 & -2\\ 15 & 6\end{pmatrix} \quad R=\begin{pmatrix}6 & 2\\ -15 & -5\end{pmatrix}$$

$$Prove\ (aP+bR)^n=a^nP+b^nR,\ where\ a,b\in\mathbb{R},\forall n\in\mathbb{Z}^+$$

* 參考課程 1.2, 4.8, 4.9 及 4.10

方法1

$$Let\ P(n):\ (aP+bR)^n=a^nP+b^nR,\ \forall n\in\mathbb{Z}^+$$

$$For\ P(1):\ L.H.S.=aP+bR=R.H.S.$$

$$Assume\ P(k)\ is\ true\ \exists k\in\mathbb{Z}^+,\ then\ P(k+1):$$

$$L.H.S.=(aP+bR)^{k+1}=(aP+bR)^k(aP+bR)$$
$$=(a^kP+b^kR)(aP+bR)$$
$$=(a^{k+1}P^2+a^kbPR+ab^kRP+b^{k+1}R^2)$$
$$=(a^{k+1}P+a^kb0+ab^k0+b^{k+1}R)=a^{k+1}P+b^{k+1}R$$

* 先 Let Statement

* 証明 P(1) is true

* 假設 P(k) is true. 証明 P(k+1) is true

CONT'D

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$$= R.H.S.$$

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

i.e. By M.I., $P(n)$ *is true,* $\forall n \in \mathbb{Z}^+$

* 寫結論

方法2

$\because \; P^2 = P \rightarrow P^n = P,\ R^2 = R \rightarrow R^n = R,\ PR = RP = 0$

$$\therefore (aP + bR)^n = \sum_{r=0}^{n} C_r^n (aP)^r (bR)^{n-r}$$

$$= a^n P^n + \cancel{\sum_{r=1}^{n-1} C_r^n (aP)^r (bR)^{n-r}}^{\,0} + b^n R^n$$

$$= a^n P + b^n R$$

* 因為 PR=RP, 所以可以用恆等式

2020 - SECTION B

$Q9.)$ $a.)$ $Sketch\ C : y = f(x), f(x) = \dfrac{(x+4)^3}{(x-4)^2}, where\ x \neq 4$

$b.)$ $Find\ the\ bounded\ area\ between\ x - axis, y - axis\ and\ C$

* 參考課程 3.5, 3.10 及 3.11

$The\ x - interception = -4, The\ y - interception = 4$

$$y = \frac{(x+4)^3}{(x-4)^2} \rightarrow \boxed{(x-4)^2 y} = (x+4)^3$$

$$\rightarrow \boxed{2(x-4)y + (x-4)^2\frac{dy}{dx}} = 3(x+4)^2$$

$$\rightarrow \frac{2(x+4)^3}{x-4} + (x-4)^2\frac{dy}{dx} = 3(x+4)^2$$

$$\rightarrow \frac{dy}{dx} = \frac{(x+4)^2(x-20)}{(x-4)^3}$$

* 用 Implicit 微分法

* 用 Product rule

CONT'D

2020 – SECTION B

$$Let\ x_0 \in \mathbb{R}\ such\ that\ \frac{dy}{dx}\Big|_{x=x_0} = 0 \rightarrow x_0 = -4\ or\ 20$$

	x < -4	x = -4	-4 < x < 4	4 < x <20	x = 20	x > 20
y′	+	0	+	-	0	+
y	Up.		Up.	Down.		Up.

$$\therefore (-4, 0)\ is\ stagnation\ point$$

$$(20, 54)\ is\ local\ min.\ point$$

$$Also, \frac{dy}{dx} = \frac{(x+4)^2(x-20)}{(x-4)^3} \rightarrow (x-4)^3\frac{dy}{dx} = (x+4)^2(x-20)$$

$$\rightarrow 3(x-4)^2\frac{dy}{dx} + (x-4)^3\frac{dy^2}{d^2x} = 2(x+4)(x-20) + (x+4)^2$$

$$\rightarrow \frac{dy^2}{d^2x} = \frac{(x+4)[(x+4)(x-4) - (x+20)(x-20)]}{(x-4)^4}$$

* 搵 turning point = 搵 x_0 使度 $y'(x_0)=0$
* 利用表格計算 turning point 附近上升定下降

  $f'(x) > 0 \rightarrow Increasing$

  $f'(x) < 0 \rightarrow Decreasing$
* x 响 4 附近(分母)對 f′(x) 都有正負影响
* 用 Implicit 微分法
* 用 Product rule

CONT'D

2020 - SECTION B

$$\rightarrow \frac{d^2y}{dx^2} = \frac{384(x+4)}{(x-4)^4}$$

$$Let\ x_0 \in \mathbb{R}\ such\ that\ \frac{d^2y}{dx^2}\Big|_{x=x_0} = 0 \rightarrow x_0 = -4$$

	x < -4	x = -4	x > -4
y''	-	0	+
y	Down.		Up.

$$\therefore The\ pt.\ of\ inflexion = (-4, 0)$$

$$Vertical\ Asymptote:\ x = 4$$

$$Horizontal\ Asymptote:\ No\ horizontal\ asymoptote$$

*** 搵 pt. of inflexion = 搵 x_0 使度 $y''(x_0)=0$**

*** 利用表格計算 pt. of inflexion 附近情況**

$f''(x) > 0 \rightarrow Concave\ upward$

$f''(x) < 0 \rightarrow Concave\ downward$

*** x 係幾多, 分母係零**

* $Find\ \lim_{x\to\infty} y$

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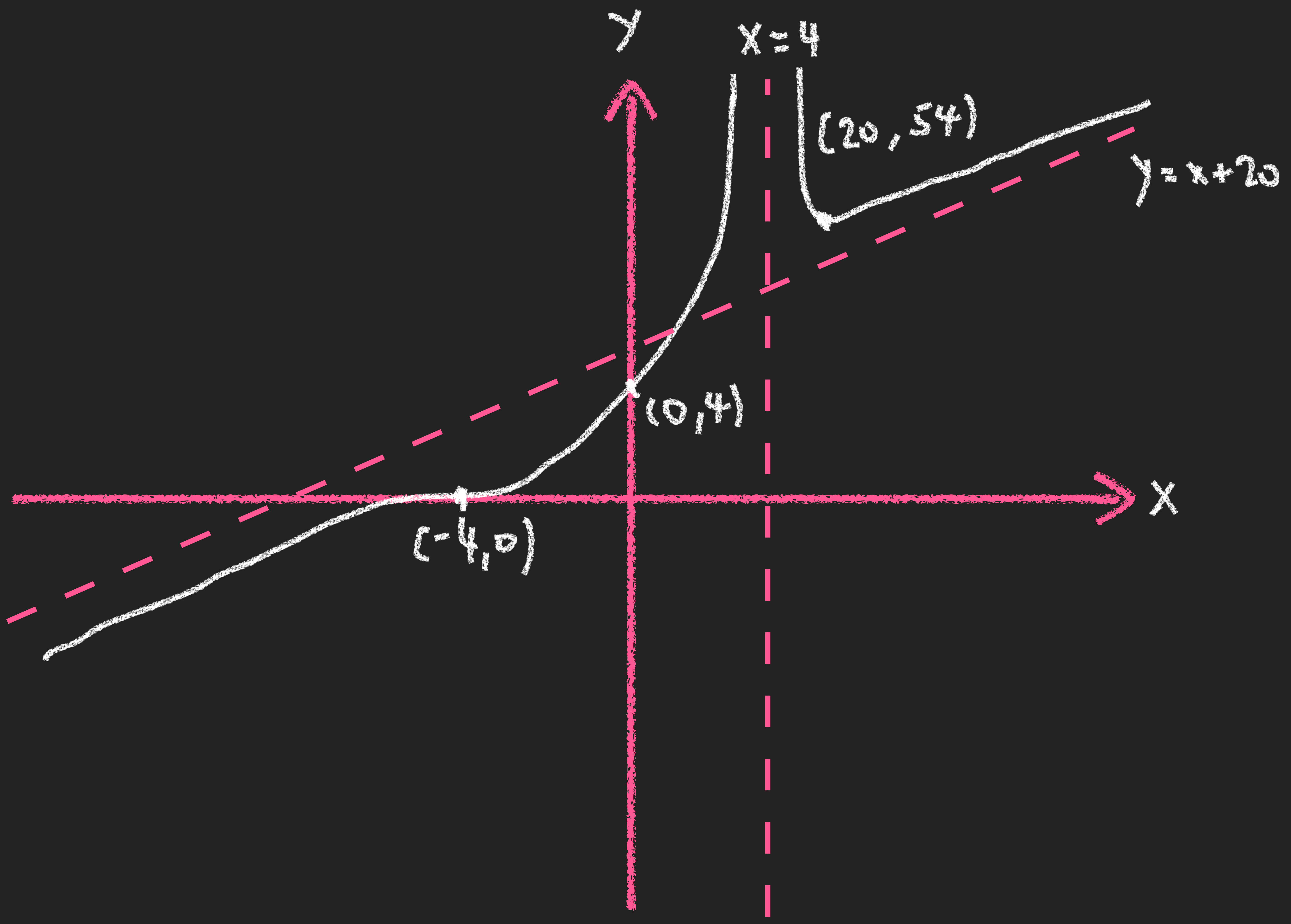
2020 - SECTION B

$$\textit{Oblique Asymptote}: \quad y = x + 20$$



$* \textit{ Find } m \textit{ and } c \textit{ such that } \lim_{x\to\infty}[y-(mx+c)] = 0$

$$y = \frac{[(x-4)+8]^3}{(x-4)^2}$$

$$\to y = \frac{(x-4)^3 + 24(x-4)^2 + 192(x-4) + 512}{(x-4)^2}$$

$$\to y = x - 4 + 24 + \frac{192}{(x-4)} + \frac{512}{(x-4)^2}$$

$$\to \lim_{x\to\infty}(y-(x+20)) = 0$$

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2020 - SECTION B

$$The\ area = \int_{-4}^{0} y dx = \int_{-4}^{0} \frac{(x+4)^3}{(x-4)^2} dx$$

$$= \int_{-8}^{-4} \frac{(u+8)^3}{u^2} du = \int_{-8}^{-4} \left(u + 24 + \boxed{\frac{192}{u}} + \frac{512}{u^2}\right) du$$

$$= \left[\frac{u^2}{2} + 24u + \boxed{192 ln|u|}) - \frac{512}{u}\right]_{-8}^{-4}$$

$$= 136 - 192 ln2\ sq.unit$$

* 利用定積分計算面積

* 利用積分代入法, let u=x-4

* 定積分代入要改範圍

* ■ 1/x 積分 = ln|x| 確保正數

2020 – SECTION B

$Q10.)$

$$a.)\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} ln(cotx-1)dx = ? \qquad b.)\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{xcsc^2x}{cotx-1}dx = ?$$

* 參考課程 2.2, 3.8, 3.10 及 3.11

$$a.)\ Let\ I_1 = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} ln(cotx-1)dx$$

$$I_1 = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} ln\left(\frac{cosx-sinx}{sinx}\right)dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} ln(cosx-sinx)-ln(sinx)\ dx$$

$$Let\ I_a = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} ln(cosx-sinx)dx,\ I_b = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} ln(sinx)dx,\ such\ that\ I_1 = I_a - I_b$$

* ln(A/B) = lnA - lnB

CONT'D

2020 – SECTION B

$$where\ I_a = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x - \sin x)dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sqrt{2}\sin\left(\frac{\pi}{4} - x\right)\right)dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{1}{2}\ln 2\,dx + \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4} - x\right)\right)dx$$

$$= \frac{\pi}{24}\ln 2 + \int_{\frac{\pi}{6}}^{\frac{\pi}{12}} \ln(\sin u)d(-u)$$

$$= \frac{\pi}{24}\ln 2 + \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin u)du = \frac{\pi}{24}\ln 2 + I_b$$

$$\therefore I_1 = \frac{\pi}{24}\ln 2 + I_b - I_b = \frac{\pi}{24}\ln 2$$

* 假設 cosx-sinx = Asin(B-x), 搵A同B

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A\cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A\sin B = \cos(A-B) - \cos(A+B)$	

$$\cos x - \sin x \equiv A\sin B\cos x - A\cos B\sin x$$

$$\therefore A\sin B = 1,\ A\cos B = 1$$

$$i.e.\ A = \sqrt{2},\ and\ B = \frac{\pi}{4}$$

* 利用積分代入: $Let\ u = \frac{\pi}{4} - x$

* 定積分代入要改範圍

* 負數定積分範圍上下倒轉

CONT'D

2020 - SECTION B

$$b.)\ Let\ I_2 = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x\csc^2 x}{\cot x - 1}dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} x\frac{\csc^2 x}{\cot x - 1}dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} x\,d(-\ln(\cot x - 1))$$

$$= [-x\ln(\cot x - 1)]_{\frac{\pi}{12}}^{\frac{\pi}{6}} + \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x - 1)dx$$

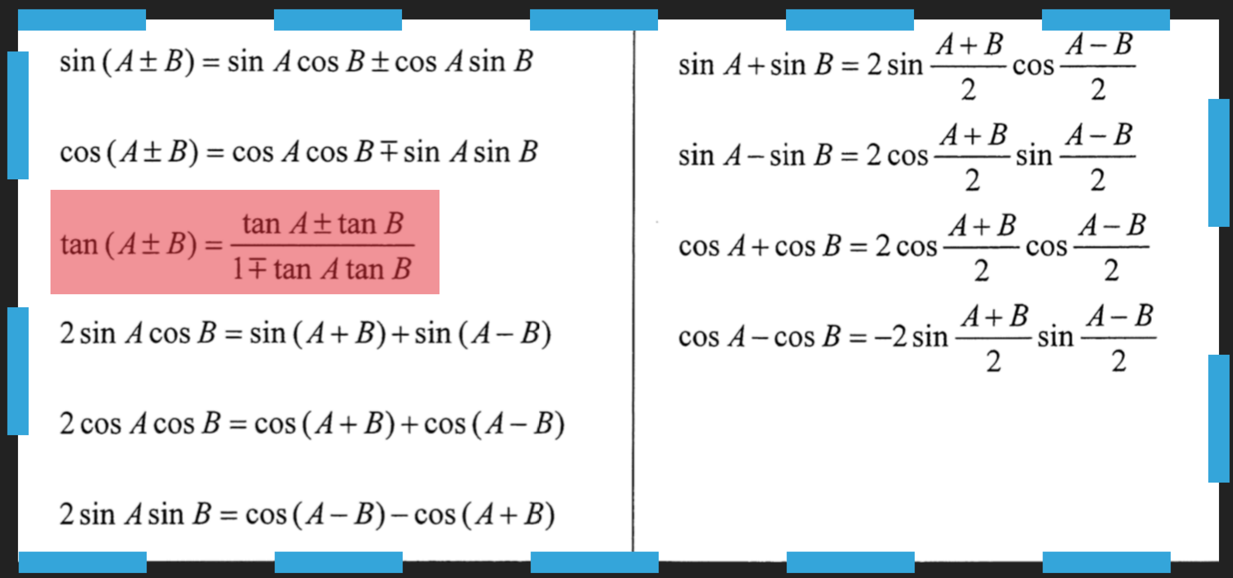
$$= -\frac{\pi}{6}\ln(\sqrt{3} - 1) + \frac{\pi}{12}\ln(\cot\frac{\pi}{12} - 1) + I_1$$

$$Given\ that\ \tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1 - \tan^2\frac{\pi}{12}} \to 1 - \tan^2\frac{\pi}{12} = 2\sqrt{3}\tan\frac{\pi}{12}$$

$$\to \tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} - 1 = 0 \to \tan\frac{\pi}{12} = \frac{-2\sqrt{3} \pm \sqrt{12 + 4}}{2}$$

* 積分三寶: Integration by part

* 利用 tan 複角公式



$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

* Core 二次方程公式解

CONT'D

2020 - SECTION B

$$\rightarrow tan\frac{\pi}{12} = -2-\sqrt{3}\ (rejected)\ or\ 2-\sqrt{3}$$

$$\rightarrow cot\frac{\pi}{12} = \frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}}\cdot\frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{1}$$

$$Hence,\ I_2 = -\frac{\pi}{6}ln(\sqrt{3}-1)+\frac{\pi}{12}ln(\sqrt{3}+1)+\frac{\pi}{24}ln2$$

$$= -\frac{\pi}{12}ln(\sqrt{3}-1)^2+\frac{\pi}{12}ln(\sqrt{3}+1)+\frac{\pi}{12}ln\sqrt{2}$$

$$= \frac{\pi}{12}ln\frac{\sqrt{2}(\sqrt{3}+1)}{2(2-\sqrt{3})} = \frac{\pi}{12}ln\frac{5+3\sqrt{3}}{\sqrt{2}} = \frac{\pi}{12}ln(2+\sqrt{3})^{\frac{3}{2}}$$

$$= \frac{\pi}{8}ln(2+\sqrt{3})$$

* 象限 1, tan 為正數

* $(a+b)(a-b) = a^2 - b^2$

* ln(A/B) = lnA - lnB, lnAB = lnA + lnB

* $$(2+\sqrt{3})^3 = 8+3(4)\sqrt{3}+3(2)3+3\sqrt{3}$$
$$= 26+15\sqrt{3}$$

$$Also,\ (5+3\sqrt{3})^2 = 52+30\sqrt{3}$$
$$= 2(26+15\sqrt{3})$$
$$= 2(2+\sqrt{3})^3$$

$$\therefore 5+3\sqrt{3} = \sqrt{2}(2+\sqrt{3})^{\frac{3}{2}}$$

2020 – SECTION B

Q11.)

a.)

$$(E)\begin{cases} x - y - 2z = 1 \\ x - 2y + hz = k \\ 4x + hy - 7z = 7 \end{cases}$$

Solve (E) for all possible value of h and $k \in \mathbb{R}$

b.)

$$(F)\begin{cases} x - y - 2z = 1 \\ x - 2y + hz = -2 \\ 4x + hy - 7z = 7 \\ 3x^2 + 4y^2 - 7z^2 = 1 \end{cases}$$

$h \in \mathbb{R}$

Are there at least two values of h such that (F) have real solution

* 參考課程 4.7

2020 - SECTION B

$a.)$ *Consider* :

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 1 & -2 & h & k \\ 4 & h & -7 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & -1 & h+2 & k-1 \\ 0 & h+4 & 1 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & -1 & h+2 & k-1 \\ 0 & 0 & \boxed{A} & \boxed{B} \end{array}\right)$$

$$\text{where } A = 1 + (h+2)(h+4)$$

$$= h^2 + 6h + 9 = \boxed{(h+3)^2}$$

$$B = \boxed{3 + (k-1)(h+4)}$$

when $h \neq -3$, *(E) has unique solution*

* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \to \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

* 如果 $\boxed{}$ 不等如 0, 得唯一答案

* 如果 $\boxed{}$ $\boxed{}$ 等如 0, 得直線答案

CONT'D

2020 - SECTION B

$$(x, y, z) = (1 + \frac{B(h+2)}{A} - (k-1) + \frac{2B}{A}, \frac{B(h+2)}{A} - (k-1), \frac{B}{A})$$

* 先用三式搵z, 再用二式搵y, 最後一式搵x

$$\therefore (x, y, z)^T = \begin{pmatrix} \dfrac{h^2+7h+2hk+7k+14}{(h+3)^2} \\ \dfrac{3h-k+7}{(h+3)^2} \\ \dfrac{hk+4k-h-1}{(h+3)^2} \end{pmatrix}$$

$when\ h = -3,$

$k = -2, (E)\ is\ consistent, otherwise\ there\ is\ no\ solution$

$$\textbf{(E)} \sim \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & -1 & -1 & -3 \end{array}\right)$$

* 三條公式剩返兩條

CONT'D

2020 - SECTION B

$Let\ z = t,\ t \in \mathbb{R}$

$(x, y, z) = (4 + t, 3 - t, t)$

$b.)\ (F)\ consist\ (E)\ with\ k = -2\ and\ 3x^2 + 4y^2 - 7z^2 = 1$

$For\ h \neq -3,\ \ (x, y, z) = (\frac{h}{h+3}, \frac{3}{h+3}, -\frac{3}{h+3})$

$Hence,\ 3x^2 + 4y^2 - 7z^2 = 1 \rightarrow \frac{3h^2}{(h+3)^2} + \frac{36}{(h+3)^2} - \frac{63}{(h+3)^2} = 1$

$\rightarrow 3h^2 - 27 = (h+3)^2$

$\rightarrow 3(h+3)(h-3) = (h+3)^2$

$\rightarrow h = 6$

$For\ h = 3,\ \ (x, y, z) = (4 + t, 3 - t, t),\ t \in \mathbb{R}$

$Hence,\ 3x^2 + 4y^2 - 7z^2 = 1 \rightarrow 84 = 1,\ which\ is\ wrong$

$i.e.\ There\ is\ only\ h = 6\ such\ that\ (F)\ has\ real\ solution$

* 先考慮唯一答案

* $(a+b)(a-b) = a^2 - b^2$

* 再考慮直線答案

2020 - SECTION B

$Q12.)$ *There is point R on the line PQ such that* $PR : RQ = 1 : 3.$

Given that $\overrightarrow{OP} = \hat{i} + \hat{j} + 4\hat{k}$ *and* $\overrightarrow{OQ} = 5\hat{i} - 7\hat{j} - 4\hat{k}$

Let $\overrightarrow{ON} = \lambda(\overrightarrow{OP}x\overrightarrow{OR})$ *and* $\overrightarrow{a} = 11\hat{i} + \mu\hat{j} - 10\hat{k}$, *where* $\lambda, \mu \in \mathbb{R}$

a.) If $\overrightarrow{NQ} \,//\, \overrightarrow{a}$, *find* λ *and* μ

b.) Let θ *be the angle between* ΔOPQ *and* ΔNPQ, $\tan\theta = ?$

* 參考課程 4.4 及 4.5

$$a.)\ \overrightarrow{OR} = \frac{3}{4}\overrightarrow{OP} + \frac{1}{4}\overrightarrow{OQ}$$

$$\therefore \overrightarrow{OP}x\overrightarrow{OR} = \overrightarrow{OP}x(\frac{3}{4}\overrightarrow{OP} + \frac{1}{4}\overrightarrow{OQ}) = \frac{1}{4}\overrightarrow{OP}x\overrightarrow{OQ}$$

$$= \frac{1}{4}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 4 \\ 5 & -7 & -4 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 4 \\ 6 & -6 & 0 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} \hat{i} & \hat{i}+\hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 6 & 0 & 0 \end{vmatrix}$$

* 分割定理

* Cross product 可以拆括號

$*\ \overrightarrow{OP}x\overrightarrow{OP} = 0$

* R3=R3+R2

* C2=C2+C1

CONT'D

2020 - SECTION B

$$= \frac{1}{4}\begin{vmatrix} \hat{i} & \hat{i}+\hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 6 & 0 & 0 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} \hat{i} & \hat{i}+\hat{j} & -2(\hat{i}+\hat{j})+\hat{k} \\ 1 & 2 & 0 \\ 6 & 0 & 0 \end{vmatrix}$$

$$= 6\hat{i} + 6\hat{j} - 3\hat{k}$$

$$Hence,\ \overrightarrow{ON} = 3\lambda(2\hat{i} + 2\hat{j} - \hat{k})$$

$$\overrightarrow{NQ} = \overrightarrow{OQ} - \overrightarrow{ON} = (5-6\lambda)\hat{i} - (7+6\lambda)\hat{j} + (3\lambda - 4)\hat{k}$$

$$\because \overrightarrow{NQ} \,//\, \vec{a} \rightarrow \overrightarrow{NQ} = k\vec{a},\ k \in \mathbb{R}$$

$$\therefore \begin{cases} 5 - 6\lambda = 11k & \text{—— (1)} \\ 7 + 6\lambda = -\mu k & \text{—— (2)} \\ 4 - 3\lambda = 10k & \text{—— (3)} \end{cases}$$

$$\frac{(3)}{(1)} : \frac{4-3\lambda}{5-6\lambda} = \frac{10}{11} \rightarrow \lambda = \frac{2}{9} \qquad \frac{(2)}{(1)} : \frac{7+6\lambda}{5-6\lambda} = \frac{-\mu}{11} \rightarrow \mu = -25$$

* C3=C3-2C2

$*\ If\ \vec{a} // \vec{b},\ \vec{a} = k\vec{b}$

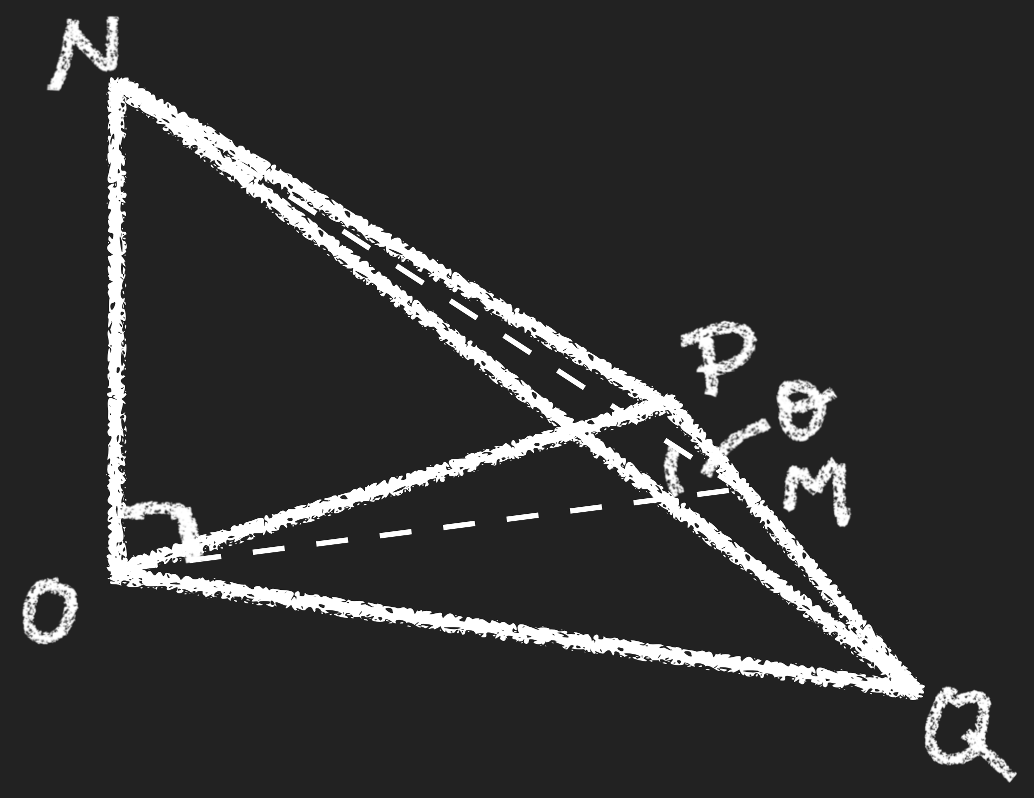
$*\ A\vec{i} + B\hat{j} + C\hat{k} = D\vec{i} + E\hat{j} + F\hat{k}$

$A = D,\ B = E,\ C = F$

CONT'D

2020 – SECTION B

b.) Consider the following :



Let M be the point on PQ such that $OM \perp PQ$

Obviously, ΔOPQ is the projection of ΔNPQ

$\therefore\ NM \perp PQ$

$$i.e.\ \tan\theta = \frac{|\overrightarrow{ON}|}{OM},\ where\ OM x PQ = |\overrightarrow{OP} x \overrightarrow{OQ}|$$

$$\rightarrow \tan\theta = \frac{|\overrightarrow{ON}||\overrightarrow{PQ}|}{|\overrightarrow{OP} x \overrightarrow{OQ}|} = \frac{2 \cdot |4\hat{i} - 8\hat{j} - 8\hat{k}|}{9 \cdot 4} = \frac{2}{3}$$

* $\overrightarrow{ON}$ 係 **OPQ Normal vector**

*** Cross product 大小係 平行四邊形面積**

* $\overrightarrow{ON} = 3\lambda(2\hat{i} + 2\hat{j} - \hat{k})$

$\overrightarrow{OP} x \overrightarrow{OQ} = 4 \cdot 3(2\hat{i} + 2\hat{j} - \hat{k})$