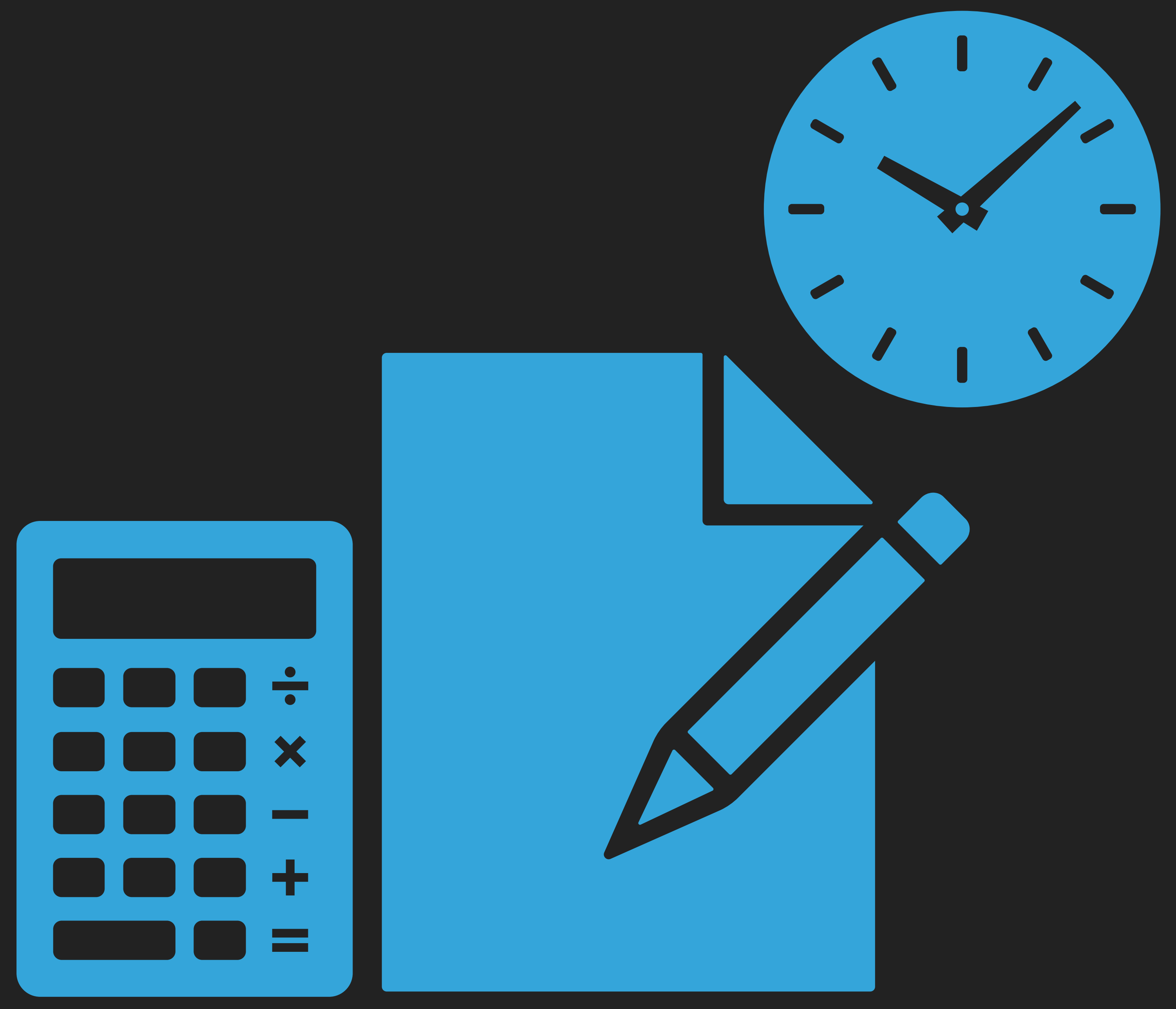
深宵教室 - DSE M2 模擬試題解答

2018

2018

- Section A
- Section B



2018 - SECTION A

$Q1.)\ f(x) = (x^2 - 1)e^x.\ \ f'(1) = ?\ \ (By\ First\ Principles)$

* 參考課程 3.1 及 3.2

$$f'(x) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}((1+h)^2 - 1)e^{1+h}$$

$$= \lim_{h \to 0} \frac{1}{h} h(h+2)e^{1+h}$$

$$= 2e$$

* 微分定義

* $a^2 - b^2 = (a+b)(a-b)$

2018 - SECTION A

Q2.) Find the coefficient of x^3 *of* $(1+3)^5(x-\dfrac{4}{x})^2$

* 參考課程 1.1

$$(x+3)^5(x-\frac{4}{x})^2 \equiv (\sum_{r=0}^{5} C_r^5 3^{5-r} x^r)(x^2-8+\frac{16}{x^2})$$

$$\text{The coefficient of } x^3 = C_1^5 3^4 - C_3^5 3^2(8) + C_5^5(16)$$

$$= -299$$

*** Binomial Expansion**

$$* \ C_r^n = \frac{n!}{r!(n-r)!}$$

2018 - SECTION A

Q3.) For $0 \le x \le \frac{\pi}{2}$, *Find x if* $cot(x+\frac{4\pi}{9}) = 3cot(x+\frac{5\pi}{18})$

* 參考課程 2.2 及 2.3

$$cot(x+\frac{4\pi}{9}) = 3cot(x+\frac{5\pi}{18})$$

$$\rightarrow sin(x+\frac{5\pi}{18})cos(x+\frac{4\pi}{9}) = 3cos(x+\frac{5\pi}{18})sin(x+\frac{4\pi}{9})$$

$$\rightarrow \frac{sin(2x+\frac{13\pi}{18}) + sin\frac{-3\pi}{18}}{2} = 3\frac{sin(2x+\frac{13\pi}{18}) + sin\frac{3\pi}{18}}{2}$$

$$\rightarrow 2sin(2x+\frac{13\pi}{18}) = -2 \rightarrow sin(2x+\frac{13\pi}{18}) = -1$$

$$\because 0 \le x \le \frac{\pi}{2} \rightarrow \frac{13\pi}{18} \le 2x+\frac{13\pi}{18} \le \frac{31\pi}{18}$$

* Product to sum

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A\cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A\sin B = \cos(A-B) - \cos(A+B)$	

* 注意角度範圍

CONT'D

2018 - SECTION A

$$\therefore 2x + \frac{13\pi}{18} = \frac{3\pi}{2} \rightarrow x = \frac{7\pi}{18}$$

* sin 270^0 = -1

2018 - SECTION A

Q4.) Find the area bounded by $y = x(5^{2x})$, $x - axis$ *and* $x = 1$.

* 參考課程 3.10 及 3.11

$$\text{The area, } A = \int_0^1 x(5^{2x})dx, \text{ Let } u = 5^{2x}$$

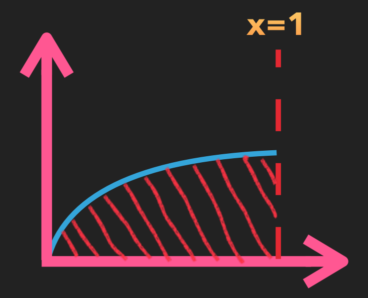
$$\rightarrow \ln u = 2x\ln 5 \rightarrow \frac{du}{u} = 2\ln 5 dx$$

$$\therefore A = \frac{1}{(2\ln 5)^2}\int_1^{25} \frac{u\ln u}{u}du = \frac{1}{(\ln 25)^2}\int_1^{25} \ln u\, du$$

$$= \frac{1}{(\ln 25)^2}\left([u\ln u]_1^{25} - \int_1^{25} u \cdot \frac{1}{u}du\right)$$

$$= \frac{25}{\ln 25} - \frac{24}{(\ln 25)^2} \; sq.unit$$

* 用手 Sketch 了解要搵的面積



* 定積分代入要改範圍

* 積分三寶: Integration by part

2018 - SECTION A

Q5.) The slope of tangent to Γ is $15x^3\sqrt{1+x^2}$, The y − interception of Γ $= 2$.
The equation of $\Gamma = ?$

* 參考課程 3.6, 3.7, 3.8 及 3.9

方法1

$$Let\ \Gamma : y = f(x),\ where\ f'(x) = 15x^3\sqrt{1+x^2}\ and\ f(0) = 2$$

$$Hence, f(x) = \int f'(x)dx$$

$$f(x) = 15\int x^3\sqrt{1+x^2}dx = \frac{15}{2}\int x^2\sqrt{1+x^2}\,\boxed{2xdx}$$

$$= \frac{15}{2}\int [(x^2+1)-1]\sqrt{1+x^2}\,\boxed{d(1+x^2)}$$

$$= \frac{15}{2}[\int (x^2+1)^{\frac{3}{2}}d(1+x^2) - \int \sqrt{1+x^2}d(1+x^2)]$$

$$= 3(x^2+1)^{\frac{5}{2}} - 5(1+x^2)^{\frac{3}{2}} + C,\ where\ C\ is\ constant$$

* 積分係類似微分逆函數

* 積分三寶: 積分代入

CONT'D

2018 - SECTION A

$$f(0)=2 \rightarrow C=4$$

$$\therefore f(x)=3(x^2+1)^{\frac{5}{2}}-5(x^2+1)^{\frac{3}{2}}+4$$

方法2

$$Consider,\ I=15\int x^3\sqrt{1+x^2}dx,\ let\ x=tan\theta$$

$$Then,\ I=15\int tan^3\theta\sqrt{1+tan^2\theta}sec^2\theta d\theta$$

$$=15\int tan^3\theta sec^3\theta d\theta=15\int tan^2\theta sec^2\theta d(sec\theta)$$

$$=15\int(sec^2\theta-1)sec^2\theta d(sec\theta)=15\int sec^4\theta-sec^2\theta d(sec\theta)$$

$$=3sec^5\theta-5sec^3\theta+C,\ where\ C\ is\ constant$$

$$=3(x^2+1)^{\frac{5}{2}}-5(1+x^2)^{\frac{3}{2}}+4\quad(\because f(0)=2\rightarrow C=4)$$

$$\therefore f(x)=3(x^2+1)^{\frac{5}{2}}-5(x^2+1)^{\frac{3}{2}}+4$$

* 利用三角代入, $x=tan\theta$

* $tan^2\theta+1=sec^2\theta$

* $tan\theta sec\theta d\theta=d(sec\theta)$

2018 - SECTION A

$$Q6.)\ Prove\ \sum_{r=1}^{n} r(r+4) = \frac{n(n+1)(2n+13)}{6},\ \forall n \in \mathbb{Z}^+,\ Hence,\ \sum_{r=333}^{555} \left(\frac{r}{112}\right)\left(\frac{r+4}{223}\right) = ?$$

* 參考課程 1.1 及 1.2

方法1

$$Let\ P(n):\ \sum_{r=1}^{n} r(r+4) = \frac{n(n+1)(2n+13)}{6}\ \forall n \in \mathbb{Z}^+$$

* 先 Let Statement

$$For\ P(1):\ L.H.S. = 5 = R.H.S.$$

* 証明 P(1) is true

$$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+,\ then\ P(k+1):$$

* 假設 P(k) is true. 証明 P(k+1) is true

$$L.H.S. = \sum_{r=1}^{k+1} r(r+4) = \sum_{r=1}^{k} r(r+4) + (k+1)(k+5)$$

* 將未項抽出並改變未項

$$= \frac{k(k+1)(2k+13)}{6} + (k+1)(k+5)$$

CONT'D

2018 – SECTION A

$$= \frac{(k+1)[k(2k+13)+6(k+5)]}{6} = \frac{(k+1)(k+2)(2k+15)}{6}$$

$$= R.H.S.$$

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

i.e. By M.I., P(n) is true, $\forall n \in \mathbb{Z}^+$

方法2

$$\sum_{r=1}^{n} r(r+4) = \sum_{r=1}^{n} (r^2+4r)$$

$$= \sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r = \frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+13)}{6}$$

* 寫結論

* Summation 可柝開做加減及抽常數

* $1+2+\ldots+n = \frac{n(n+1)}{2}$

* $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

CONT'D

2018 – SECTION A

$$\sum_{r=333}^{555}(\frac{r}{112})(\frac{r+4}{223}) = \sum_{r=1}^{555}(\frac{r}{112})(\frac{r+4}{223}) - \sum_{r=1}^{332}(\frac{r}{112})(\frac{r+4}{223})$$

$$= \frac{1}{112 \cdot 223}[\sum_{r=1}^{555} r(r+4) - \sum_{r=1}^{332} r(r+4)]$$

$$= \frac{1}{112 \cdot 223}(\frac{555 \cdot 556 \cdot 1123}{6} - \frac{332 \cdot 333 \cdot 677}{6})$$

$$= 1813$$

* 長減細

* Summation 抽常數

* 用題目結果

2018 – SECTION A

Q7.) Consider,

$$M = \begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}, X = \begin{pmatrix} a & 6a \\ b & c \end{pmatrix}, a, b, c \in \mathbb{R} \text{ and } \neq 0, \text{ with } MX = XM$$

$$(X^T)^{-1} = ?, \text{ in term of } a$$

* 參考課程 4.8 及 4.10

$$MX = XM \rightarrow \begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} a & 6a \\ b & c \end{pmatrix} = \begin{pmatrix} a & 6a \\ b & c \end{pmatrix}\begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 7a+3b & 42a+3c \\ -a+5b & -6a+5c \end{pmatrix} = \begin{pmatrix} a & 33a \\ 7b-c & 3b+5c \end{pmatrix}$$

* 兩個矩陣相同, 所有元素相同

$$\therefore 7a + 3b = a \rightarrow b = -2a$$

$$42a + 3c = 33a \rightarrow c = -3a$$

$$Hence, X = a\begin{pmatrix} 1 & 6 \\ -2 & -3 \end{pmatrix} \rightarrow |X| = 9a^2 > 0 \rightarrow X^{-1} \text{ exists}$$

* 計Determinant 不等如零証明逆矩陣存在

* $|aA_{nxn}| = a^2|A_{nxn}|$

CONT'D

2018 - SECTION A

To find X^{-1}, *consider*;

$$\left(\begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ 0 & 9 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ 0 & 1 & \frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -\frac{3}{9} & -\frac{6}{9} \\ 0 & 1 & \frac{2}{9} & \frac{1}{9} \end{array}\right) \quad \text{Hence, } X^{-1} = \frac{1}{9a}\begin{pmatrix} -3 & -6 \\ 2 & 1 \end{pmatrix}$$

$$\therefore (X^T)^{-1} = (X^{-1})^T$$

$$= \frac{1}{9a}\begin{pmatrix} -3 & 2 \\ -6 & 1 \end{pmatrix}$$

* 用 Row Deduction 搵逆矩陣
* R2=R2+2xR1
* R2=R2/9
* R1=R1-6xR2
* $(aA)^{-1} = \frac{1}{a}A^{-1}$
* $(X^T)^{-1} = (X^{-1})^T$

2018 – SECTION A

$Q8.)$ *Assume* $f(x) = \dfrac{A}{x^2 - 4x + 7}$, *A is constant and the extreme value of* $f(x)$ *is 4*

$a.)\ f'(x) = ?$

$b.)$ *Find the asymptote(s) of the graph* $y = f(x)$

$c.)$ *Find the pt. of inflexion*

* 參考課程 3.3 及 3.5

$$a.)\ f(x) = \frac{A}{(x-2)^2 + 3} \rightarrow \text{The extreme value} = \frac{A}{3} = 2 \rightarrow A = 12$$

* 用 Completing Square 搵 Extreme

$$\text{Hence, } f(x) = \frac{12}{x^2 - 4x + 7} \rightarrow (x^2 - 4x + 7)f(x) = 12$$

$$\rightarrow (x^2 - 4x + 7)f'(x) + (2x - 4)f(x) = 0$$

* 用 Implicit 微分法

$$\rightarrow f'(x) = -\frac{24(x-2)}{(x^2 - 4x + 7)^2}$$

CONT'D

2018 – SECTION A

$b.)$ *Vertical Asymptote : No vertical asymptote*

Horizontal Asymptote : $y = 0$

Oblique Asymptote : $y = 0$

$c.)$ *Since* $(x^2 - 4x + 7)f'(x) + (2x - 4)f(x) = 0$

$\rightarrow (x^2 - 4x + 7)f''(x) + 2(2x - 4)f'(x) + 2f(x) = 0$

Let $x_0 \in \mathbb{R}$ *such that* $f''(x_0) = 0$

$\rightarrow \cancel{(x_0^2 - 4x_0 + 7)f''(x_0)} + 4(x_0 - 2)f'(x_0) + 2f(x_0) = 0$ 0

$$\rightarrow -\frac{96(x_0 - 2)^2}{(x_0^2 - 4x_0 + 7)^2} + \frac{24}{x_0^2 - 4x_0 + 7} = 0$$

$\rightarrow (x_0^2 - 4x_0 + 7) - 4(x_0 - 2)^2 = 0 \rightarrow x_0^2 - 4x_0 + 3 = 0$

$\rightarrow x_0 = 1$ *or* 3

$\therefore$ *Pt . of inflexion* $= (1, f(1)), (3, f(3)) = (1, 3), (3, 3)$

* x 係幾多, 分母係零

* *Find* $\lim_{x \to \infty} y$

* *Find m and c such that* $\lim_{x \to \infty} [y - (mx + c)] = 0$

$\rightarrow \lim_{x \to \infty} (y - (0)) = 0$

* 繼續用 Implicit 微分法

* 搵 pt of inflexion = 搵 x_0 使度 $f''(x_0)=0$

2018 – SECTION B

Q9.) Assume a curve $\Gamma : y = \ln\sqrt{x}$, *where* $x > 1$. *P is a moving point on* $\Gamma = (r, \ln\sqrt{r})$.
The normal to Γ *at P cut the* x – *axis at Q. The vertical line passing throught P cut* x – *axis at R. Let the area of* ΔPQR *be A.*

a.) Find greatest value of A

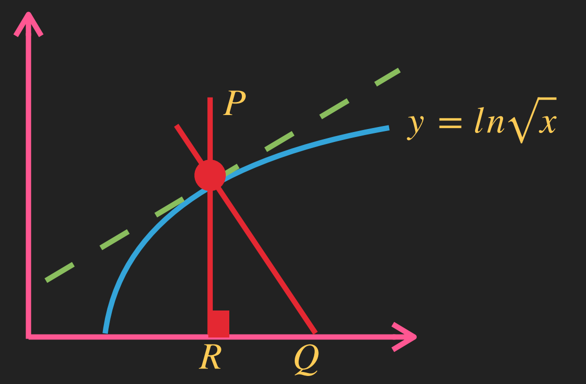
b.) If OP increase at a rate $< 32e^2$ *unit* min^{-1}, *will A increase at a rate lower than 2* $unit^2 min^{-1}$ *when* $r = e$.

* 參考課程 3.3 及 3.4

a.) Let $Q = (x_0, 0)$, *and the slope PQ* $= m$

$$-\frac{1}{m} = \frac{dy}{dx}\Big|_{x=r} = \frac{1}{2r},\ m = \frac{\ln\sqrt{r} - 0}{x_0 - r}$$

$$\rightarrow -2r = \frac{\frac{1}{2}\ln r}{x_0 - r} \rightarrow x_0 = \frac{4r^2 + \ln r}{4r}$$



* 微分計 tangent Slope, normal 同 tangent 互相垂直

CONT'D

2018 - SECTION B

$$A = \frac{1}{2}PR \cdot QR = \frac{1}{2}ln\sqrt{r} \cdot (\frac{4r^2 + lnr}{4r} - r) = \boxed{\frac{(lnr)^2}{16r}}$$

$$\frac{dA}{dr} = \boxed{-\frac{(lnr)^2}{16r^2} + \frac{2lnr}{16r^2}} = \frac{lnr(2 - lnr)}{16r^2}$$

$$Let\ r_0 \in \mathbb{R}, r_0 > 1\ such\ that\ \frac{dA}{dr}|_{r=r_0} = 0 \rightarrow lnr_0 = 0\ or\ 2$$
$$\rightarrow r_0 = e^2$$

	$1 < r < e^2$	$r = e^2$	$r > e^2$
A′	+	0	-
A	Up.		Down.

$$\therefore The\ local\ max.\ point = (e^2, \frac{1}{4e^2})$$

$$i.e.\ The\ greatest\ area = \frac{1}{4e^2}\ sq.\ unit$$

* Product rule

* 搵 turning point = 搵 r_0 使度 $A'(r_0)=0$

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow Increasing$

$f'(x) < 0 \rightarrow Decreasing$

CONT'D

2018 - SECTION B

$$b.)\ When\ r = e \rightarrow\ P = (e, \frac{1}{2}) \rightarrow OP = \sqrt{e^2 + \frac{1}{4}} = \frac{\sqrt{4e^2+1}}{2}$$

$$Also,\ OP^2 = r^2 + \frac{(lnr)^2}{4} \rightarrow 2OP\frac{dOP}{dt} = (2r + \frac{lnr}{2r})\frac{dr}{dt}$$

$$\rightarrow \sqrt{4e^2+1}\frac{dOP}{dt}|_{r=e} = (\frac{4e^2+1}{2e})\frac{dr}{dt}|_{r=e}$$

$$\rightarrow \frac{dr}{dt}|_{r=e} = \frac{2e}{\sqrt{4e^2+1}}\frac{dOP}{dt}|_{r=e} < \frac{2e(32e^2)}{\sqrt{4e^2}} = 32e^2$$

$$Then,\ \frac{dA}{dt}|_{r=e} = \frac{1}{16e^2}\frac{dr}{dt}|_{r=e} < \frac{(32e^2)}{16e^2} = 2$$

$\therefore A\ increase\ at\ a\ rate\ lower\ than\ 2\ unit^2 min^{-1}$

*** Implicit 微分法**

*** Chain rule**

$* \ \frac{1}{a+1} < \frac{1}{a}$

2018 - SECTION B

$Q10.)\ Let\ f(x) = sin^4x,\ C : y = \sqrt{x}sin^2x\ (\pi < x < 2\pi)$

$a.)\ \int_0^{\pi} f(x)dx = ?\quad b.)\ \int_0^{\pi} xf(x)dx = ?\quad c.)\ The\ revolving\ volume\ of\ C\ along\ x - axis = ?$

* 參考課程 2.2, 3.7, 3.8, 3.10 及 3.12

$a.)\ Let\ I_1 = \int_0^{\pi} f(x)dx$

方法1

$$I_1 = \int_0^{\pi} sin^3x\,sinxdx = \int_0^{\pi} sin^3x\,d(-cosx)$$

$$= [-cosxsin^3x]_0^{\pi} + \int_0^{\pi} cosxd(sin^3x) = 3\int_0^{\pi} sin^2x\,cos^2x\,dx$$

$$= 3\int_0^{\pi} sin^2x(1 - sin^2x)dx = 3\int_0^{\pi} sin^2xdx - 3I_1$$

* 積分三寶: Integration by part

* $cos^2x + sin^2x \equiv 1$

CONT'D

2018 - SECTION B

$$\rightarrow 4I_1 = 3\int_0^{\pi} sin^2x dx = 3\int_0^{\pi} \frac{1}{2}(1 - cos2x) dx$$

$$\rightarrow 8I_1 = 3\int_0^{\pi} dx - 3\int_0^{\pi} cos2xdx \xrightarrow{} 0 \rightarrow I_1 = \frac{3\pi}{8}$$

方法2

$$I_1 = \int_0^{\pi} sin^3x sinxdx = \int_0^{\pi} sin^3x d(-cosx)$$

$$= [-cosxsin^3x]_0^{\pi} + \int_0^{\pi} cosxd(sin^3x) = 3\int_0^{\pi} sin^2xcos^2xdx$$

$$= 3\int_0^{\pi} \frac{1}{4}sin^2 2xdx = 3\int_0^{\pi} \frac{1}{8}(1 - cos4x)dx$$

$$= \frac{3}{8}\int_0^{\pi} dx - \frac{3}{8}\int_0^{\pi} cos4xdx = \frac{3\pi}{8}$$

0

* cos 雙角公式

* 面積互相抵消

* 積分三寶: Integration by part

* sin cos 雙角公式

* 面積互相抵消

CONT'D

2018 – SECTION B

方法3

$$Let\ I_2 = \int_0^{\pi} cos^4 x dx$$

$$Then,\ I_1 + I_2 = \int_0^{\pi} sin^4 x + cos^4 x dx$$

$$= \int_0^{\pi} (sin^2 x + cos^2 x)^2 - \boxed{2sin^2 x cos^2 x} dx$$

$$= \int_0^{\pi} 1 - \boxed{\frac{1}{2} sin^2 2x} dx = \int_0^{\pi} 1 - \boxed{\frac{1}{4}(1 - cos2x)} dx$$

$$= \frac{3}{4}\int_0^{\pi} dx + \frac{1}{4}\boxed{\cancel{\int_0^{\pi} cos2xdx}}^{\,0} = \frac{3\pi}{4} \quad -(1)$$

$* \ (a^2 + b^2)^2 = a^4 + 2a^2b^2 + b^4$

* sin cos 雙角公式

* 面積互相抵消

CONT'D

2018 - SECTION B

$$Also,\ I_1 - I_2 = \int_0^{\pi} sin^4x - cos^4xdx$$

$$= \int_0^{\pi} (sin^2x + cos^2x)\boxed{(sin^2x - cos^2x)}dx$$

$$= \underbrace{\boxed{\int_0^{\pi} -cos2xdx}}_{0} = 0 \quad -(2)$$

$$\therefore (1) + (2) : 2I_1 = \frac{3\pi}{4} \rightarrow I_1 = \frac{3\pi}{8}$$

$$b.)\ Let\ I_3 = \int_0^{\pi} xsin^4xdx$$

方法1

$$I_3 = \int_{\pi}^{0} (\pi - u)sin^4(\pi - u)d(-u),\ where\ u = \pi - x$$

* $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2)$

* cos 雙角公式

* 面積互相抵消

* 定積分代入要改範圍

CONT'D

2018 - SECTION B

$$I_3 = \int_0^{\pi} (\pi - u)\sin^4 u\,du = \pi I_1 - I_3 \rightarrow 2I_3 = \frac{3\pi^2}{8} \rightarrow I_3 = \frac{3\pi^2}{16}$$

方法2

$$\text{Let } I_4 = \int_0^{\pi} x\cos^4 x\,dx,\ I_5 = \int_0^{\pi} x\cos 2x\,dx$$

$$\text{where, } I_5 = \int_0^{\pi} x\,d\left(\frac{1}{2}\sin 2x\right) = \frac{1}{2}\cancelto{0}{\int_0^{\pi} \sin 2x\,dx} = 0$$

$$\text{Hence, } I_3 + I_4 = \frac{3}{4}\int_0^{\pi} x\,dx + \frac{1}{4}\cancelto{0}{\int_0^{\pi} x\cos 2x\,dx} = \frac{3\pi^2}{8} \quad -(1)$$

$$I_3 - I_4 = \int_0^{\pi} -x\cos 2x\,dx = 0 \quad -(2)$$

$$\therefore (1) + (2) : 2I_3 = \frac{3\pi^2}{8} \rightarrow I_1 = \frac{3\pi^2}{16}$$

* 定積分負數, 範圍倒轉
* $\sin(\pi - x) = \sin x$
* 積分三寶: Integration by part
* 面積互相抵消

CONT'D

2018 - SECTION B

$$c.)\ The\ volume,\ V = \pi\int_{\pi}^{2\pi} x\sin^4 x dx = \pi\int_{0}^{\pi}(\pi+u)\sin^4(\pi+u)du$$

$$,\ where\ u = x - \pi$$

$$Hence,\ V = \pi\int_0^{\pi}\pi\sin^4 u du + \pi\int_0^{\pi} u\sin^4 u du$$

$$= \pi^2 I_1 + \pi I_3 = \frac{9\pi^3}{16}\ cu.unit$$

* 定積分代入要改範圍

$* \sin(\pi + x) = -\sin x$

2018 – SECTION B

Q11.)

$$(E):\begin{cases} x+ay+4(a+1)z=18 \\ 2x+(a-1)y+2(a-1)z=20 \\ x-y-12z=b \end{cases}$$

a.) Assume (E) has unique solution, Find the range of a *. Then solve (E)*

b.) Assume $a=3$ *and (E) is consistent, find* b *and solve (E) .*

c.)

$$(F):\begin{cases} x+3y+16z=18 \\ x+y+2z=10 \\ x-y-12z=s \\ 2x-5y-45z=t \end{cases},\quad \textit{Find } s \textit{ and } t \textit{ when } (F) \textit{ is consistent}$$

* 參考課程 4.7

CONT'D

2018 – SECTION B

$a.)\ Consider:$

$$\left(\begin{array}{ccc|c} 1 & a & 4(a+1) & 18 \\ 2 & a-1 & 2(a-1) & 20 \\ 1 & -1 & -12 & b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & a & 4(a+1) & 18 \\ 0 & a+1 & 6a+10 & 16 \\ 0 & a+1 & 4a+16 & 18-b \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 4(a+1) & 18 \\ 0 & a+1 & 6a+10 & 16 \\ 0 & 0 & A & B \end{array}\right)$$

$where\ A = 6a + 10 - 4a - 16 = 2(a-3)$

$B = b - 2$

$(E)\ has\ unique\ solution \rightarrow 2(a+1)(a-3) \neq 0$

$\rightarrow a \neq -1\ and\ a \neq 3$

* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

* 如果 | | 唔等如零先有唯一答案

CONT'D

2018 - SECTION B

$$z = \frac{B}{A}$$

$$y = \frac{1}{a+1}[16 - (6a+10)\frac{B}{A}]$$

$$x = 18 - 4(a+1)\frac{B}{A} - \frac{a}{a+1}[16 - (6a+10)\frac{B}{A}]$$

$$\therefore (x, y, z)^T = \begin{pmatrix} \frac{a^2b + 10a + ab - 2b - 50}{(a+1)(a-3)} \\ \frac{22a - 3ab - 5b - 38}{(a+1)(a-3)} \\ \frac{b-2}{2(a-3)} \end{pmatrix}$$

* 先用三式搵z, 再用二式搵y, 最後一式搵x

CONT'D

2018 - SECTION B

$b.)\ For\ (E)\ is\ consistent \rightarrow B = 0 \rightarrow b = 2$

$$Then,\ (E) \sim \left(\begin{array}{ccc|c} 1 & 3 & 16 & 18 \\ 0 & 4 & 28 & 16 \end{array}\right)$$

$Let\ z = t,\ t \in \mathbb{R}$

$(x, y, z) = (6 + 5t, 4 - 7t, t)$

$c.)$

$$(F) : \begin{cases} x + 3y + 16z = 18 \\ x + y + 2z = 10 \\ x - y - 12z = s \\ 2x - 5y - 45z = t \end{cases} \rightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & a & 4(a+1) & 18 \\ 2 & a-1 & 2(a-1) & 20 \\ 1 & -1 & -12 & b \end{array}\right) \text{—— } (E) \\ \hline 2x - 5y + 45z = t \text{ —————— } (2) \end{array}$$

$where\ a = 3,\ and\ b = s$

$When\ (F)\ is\ consistent \rightarrow (E)\ is\ consistent \rightarrow s = 2$

$From\ b.)\ result,\ put\ (E)\ solution\ into\ (2) \rightarrow t = -8$

* 三條公式剩返兩條

2018 – SECTION B

Q12.) Given that :

$$\overrightarrow{OA} = 4\hat{i} - 3\hat{j} + \hat{k} \quad \overrightarrow{OB} = -\hat{i} + 3\hat{j} - 3\hat{k} \quad \overrightarrow{OC} = 7\hat{i} - \hat{j} + 5\hat{k} \quad \overrightarrow{OD} = 3\hat{i} - 2\hat{j} - 5\hat{k}$$

a.) The volume of tetrahedron ABCD = ?

b.) $\overrightarrow{DE} = ?$*, where E is D projection on plane ABC*

c.) The angle between ΔBCD and ΔABC

* 參考課程 4.4 及 4.5

a.) The volume of tetrahedron ABCD, $V = \frac{1}{6}(\overrightarrow{AB}\ x\ \overrightarrow{AC}) \cdot \overrightarrow{AD}$

$$= \frac{1}{6}[(-5\hat{i} + 6\hat{j} - 4\hat{k})\ x\ (3\hat{i} + 2\hat{j} + 4\hat{k})] \cdot (-\hat{i} + \hat{j} - 6\hat{k})$$

$$= \frac{1}{6}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 6 & -4 \\ 3 & 2 & 4 \end{vmatrix} \cdot (-\hat{i} + \hat{j} - 6\hat{k})$$

* 四面體體積=1/6平行六面體體積

* $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

CONT'D

2018 – SECTION B

$$= \frac{1}{6}(32\hat{i} + 8\hat{j} - 28\hat{k}) \cdot (-\hat{i} + \hat{j} - 6\hat{k}) \quad = 24\ cu.\ unit$$

b.) Let h be the height from D to the plane ABC

$\hat{n}$ be the unit normal vector of the plane ABC

$$V = \frac{1}{3}\left(\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|\right)h \rightarrow h = \frac{144}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$$

$$Also,\ \hat{n} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|} \rightarrow h\hat{n} = \frac{144\,\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|^2}$$

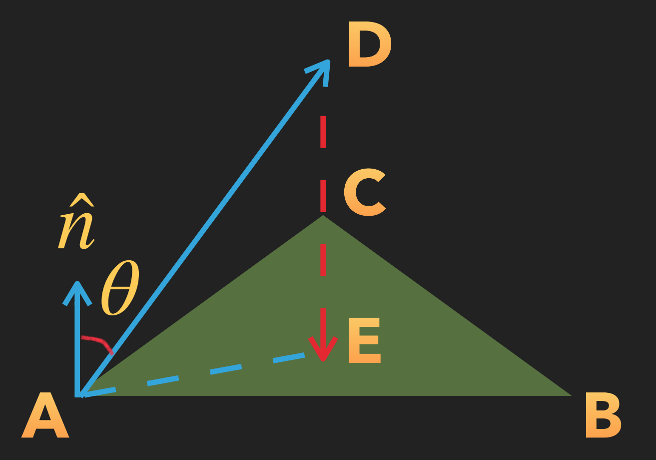
$$= \frac{1}{13}(32\hat{i} + 8\hat{j} - 28\hat{k})$$

$$\because \hat{n} \cdot \overrightarrow{AD} > 0 \rightarrow \text{the angle between } \hat{n} \text{ and } \overrightarrow{AD} < \frac{\pi}{2}$$

$$\therefore \overrightarrow{ED} = h\hat{n} \rightarrow \overrightarrow{DE} = -\frac{1}{13}(32\hat{i} + 8\hat{j} - 28\hat{k})$$

* 四面體體積=1/3 (三角底面積) (高)

* $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$



* 確保 Normal Vector 指住 D 點 $(\theta < 90^o)$

CONT'D

2018 – SECTION B

c.) Let θ be the angle between ΔABC and ΔBCD

方法1

Let F be the point lying on BC, such that;

$$\overrightarrow{DF} = \overrightarrow{DB} + t\overrightarrow{BC}, \text{ where } t \in \mathbb{R}$$

$$if\ DF \perp BC \rightarrow \overrightarrow{DF} \cdot \overrightarrow{BC} = \overrightarrow{DB} \cdot \overrightarrow{BC} + t|\overrightarrow{BC}|^2 = 0$$

$$\rightarrow t = -\frac{\overrightarrow{DB} \cdot \overrightarrow{BC}}{|\overrightarrow{BC}|^2} = \frac{1}{4}$$

$$\therefore \overrightarrow{DF} = \overrightarrow{DB} + \frac{1}{4}\overrightarrow{BC} = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

$$Also,\ \overrightarrow{EF} \cdot \overrightarrow{BC} = (\overrightarrow{DF} - \overrightarrow{DE}) \cdot \overrightarrow{BC} = 0 \rightarrow EF \perp BC$$

$$Hence,\ cos\theta = \frac{\overrightarrow{DF} \cdot \overrightarrow{EF}}{|\overrightarrow{DF}||\overrightarrow{EF}|} \rightarrow \theta = cos^{-1}\left(\frac{3\sqrt{13}}{13}\right)$$

* **Vector 直線方程**

$* \ \vec{a} \perp \vec{b} \rightarrow \vec{a} \cdot \vec{b} = 0$

* **兩平面夾角=兩條垂直在共線的夾角**

CONT'D

2018 – SECTION B

方法2

Let $h_2\hat{n}_2$ *be the normal vector of* ΔBCD

$$h_2\hat{n}_2 = \overrightarrow{BC} \, x \, \overrightarrow{BD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & -4 & 8 \\ 4 & -5 & -2 \end{vmatrix} = 24(2\hat{i} + 2\hat{j} - \hat{k})$$

$$Hence,\ cos\theta = \hat{n} \cdot \hat{n}_2 = \frac{32\hat{i} + 8\hat{j} - 28\hat{k}}{\sqrt{32^2 + 8^2 + 28^2}} \cdot \frac{2\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{2^2 + 2^2 + 1^2}}$$

$$= \frac{27}{9\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\rightarrow \theta = cos^{-1}(\frac{3\sqrt{13}}{13})$$

* 兩平面夾角=兩支 normal 的夾角