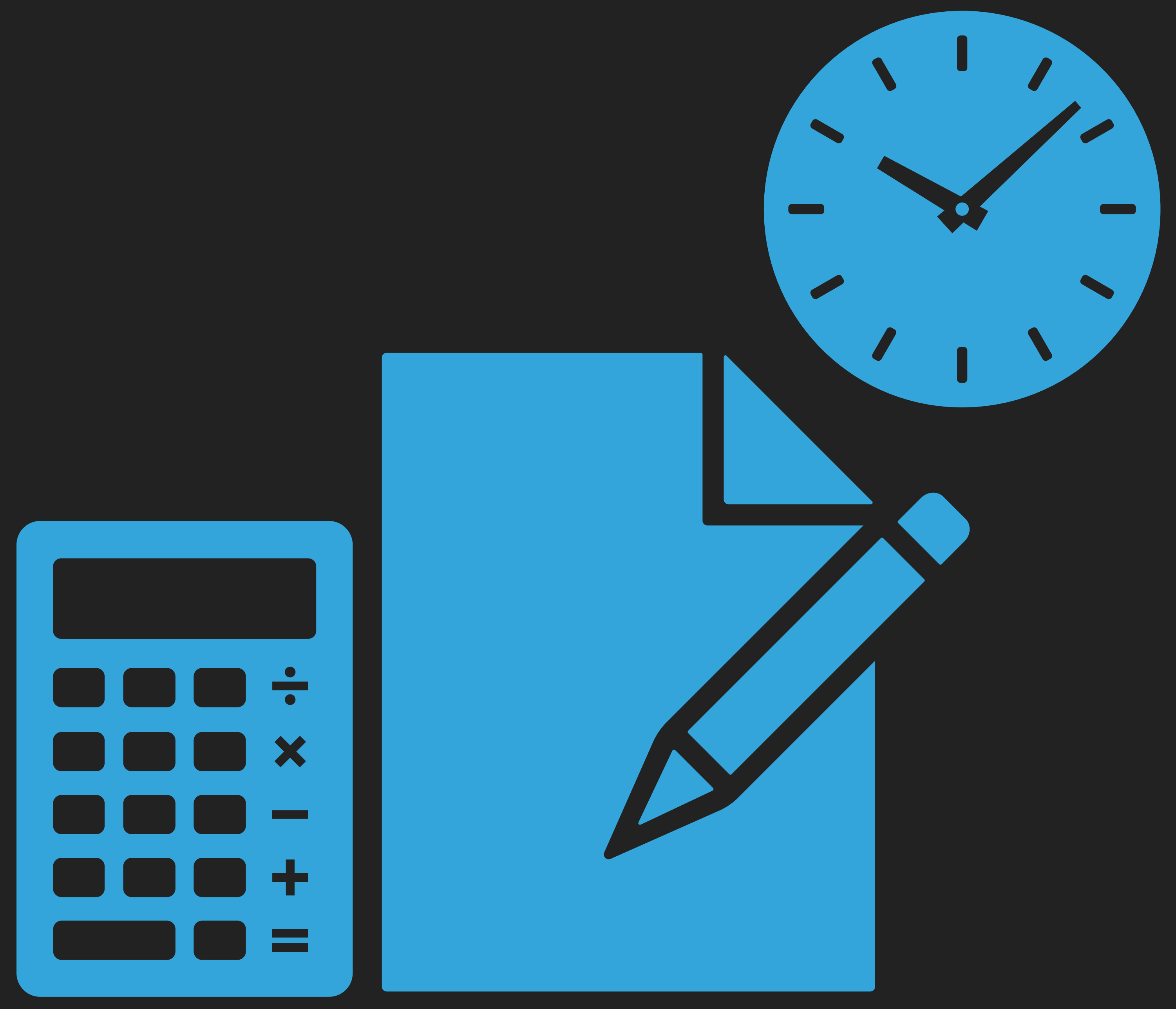
深宵教室 - DSE M2 模擬試題解答

2022

此為參考2022試題之模擬試題，原版請另行購買

2022

- Section A
- Section B



2022 - SECTION A

Q1.) $f(x)=\dfrac{1}{\sqrt{5x+4}}$, $f'(1)=?$ (By First Principles)

* 參考課程 3.1 及 3.2

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{5h+9}}-\frac{1}{\sqrt{9}}\right)=\lim_{h\to 0}\frac{3-\sqrt{5h+9}}{3h\sqrt{5h+9}}\cdot\frac{3+\sqrt{5h+9}}{3+\sqrt{5h+9}}$$

$$=\lim_{h\to 0}\frac{9-(5h+9)}{3h\sqrt{5h+9}(3+\sqrt{5h+9})}=\lim_{h\to 0}\frac{-5\cancel{h}}{3\cancel{h}\sqrt{5h+9}(3+\sqrt{5h+9})}$$

$$=-\frac{5}{54}$$

* 微分定義

* $(a+b)(a-b)=a^2-b^2$

2022 - SECTION A

Q2.) $Solve\ \dfrac{tan\theta}{1-cot\theta}+\dfrac{cot\theta}{1-tan\theta}=5,\ for\ \dfrac{\pi}{4}<\theta<\dfrac{\pi}{2}$

* 參考課程 2.1, 2.2 及 2.3

方法1

$$\rightarrow \frac{tan\theta(1-tan\theta)+cot\theta(1-cot\theta)}{(1-cot\theta)(1-tan\theta)}=5$$

$$\rightarrow tan\theta(1-tan\theta)+\frac{1}{tan\theta}(1-\frac{1}{tan\theta})=5(1-\frac{1}{tan\theta})(1-tan\theta)$$

$$\rightarrow tan\theta(1-tan\theta)-\frac{1-tan\theta}{tan^2\theta}=-\frac{5(1-tan\theta)^2}{tan\theta}$$

$$\rightarrow (1-tan\theta)(\frac{tan^3\theta-1}{tan^2\theta})+\frac{5(1-tan\theta)^2}{tan\theta}=0$$

$$\rightarrow (1-tan\theta)(\frac{(tan\theta-1)(tan^2\theta+tan\theta+1)}{tan^2\theta})+\frac{5(1-tan\theta)^2}{tan\theta}=0$$

* $cot\theta=\dfrac{1}{tan\theta}$

* $a^3-b^3\equiv(a-b)(a^2+ab+b^2)$

CONT'D

2022 - SECTION A

$$\rightarrow (1-tan\theta)^2(-1-\frac{1}{tan^2\theta}+\frac{4}{tan\theta})=0$$

$$\rightarrow tan\theta = 1 \ or \ \frac{1}{tan^2\theta}-\frac{4}{tan\theta}+1=0$$

$$\rightarrow \theta=\frac{\pi}{4} \ (rejected) \ or \ tan^2\theta-4tan\theta+1=0$$

$$\rightarrow tan\theta = \frac{4\pm\sqrt{4^2-4(1)(1)}}{2}=2+\sqrt{3}$$

$$(rejected \ 2-\sqrt{3}, \frac{\pi}{4}<\theta<\frac{\pi}{2}\rightarrow tan\theta>1)$$

$$\rightarrow \theta = tan^{-1}(2+\sqrt{3})$$

方法2

$$\rightarrow \frac{sin^2\theta}{cos\theta(sin\theta-cos\theta)}+\frac{cos^2\theta}{sin\theta(cos\theta-sin\theta)}=5$$

* 留意角度範圍

* 二次方程根公式

* $cot\theta=\frac{cos\theta}{sin\theta}, tan\theta=\frac{sin\theta}{cos\theta}$

CONT'D

2022 - SECTION A

$$\rightarrow \frac{sin^3\theta - cos^3\theta}{cos\theta sin\theta(sin\theta - cos\theta)} = 5$$

$$\rightarrow (sin\theta - cos\theta)(sin^2\theta + sin\theta cos\theta + cos^2\theta) = 5cos\theta sin\theta(sin\theta - cos\theta)$$

$$\rightarrow (sin\theta - cos\theta)(1 + sin\theta cos\theta - 5cos\theta sin\theta) = 0$$

$$\rightarrow tan\theta = 1\ (rejected)\ or\ 4sin\theta cos\theta = 1 \rightarrow 2sin2\theta = 1 \rightarrow sin2\theta = 0.5$$

$$\rightarrow 2\theta = \pi - \frac{\pi}{6}$$

$$(\because \frac{\pi}{4} < \theta < \frac{\pi}{2} \rightarrow tan\theta > 1\ and\ \frac{\pi}{2} < 2\theta < \pi)$$

$$\rightarrow \theta = \frac{5\pi}{12}$$

* $a^3 - b^3 \equiv (a-b)(a^2 + ab + b^2)$
* $sin^2\theta + cos^2\theta \equiv 1$
* sin 複角公式
* 留意角度範圍

2022 - SECTION A

$$Q3.)\ Prove\ \sum_{r=1}^{2n}(-1)^r r^2 = n(2n+1), \forall n \in \mathbb{Z}^+$$

* 參考課程 1.1 及 1.2

方法1

$$Let\ P(n):\ \sum_{r=1}^{2n}(-1)^r r^2 = n(2n+1)\ \forall n \in \mathbb{Z}^+$$

* 先 Let Statement

$$For\ P(1):\ L.H.S. = 3 = R.H.S.$$

* 証明 P(1) is true

$$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+,\ then\ P(k+1):$$

* 假設 P(k) is true. 証明 P(k+1) is true

$$L.H.S. = \sum_{r=1}^{2(k+1)}(-1)^r r^2$$

$$= \sum_{r=1}^{2k}(-1)^r r^2 + (-1)^{2k+1}(2k+1)^2 + (-1)^{2k+2}(2k+2)^2$$

* 將未項抽出並改變未項

$$= k(2k+1) - (2k+1)^2 + (2k+2)^2$$

CONT'D

2022 - SECTION A

$$= -(2k+1)(k+1)+4(k+1)^2$$

$$= (k+1)(2k+3) = R.H.S.$$

$$\therefore P(k+1) \text{ is true if } P(k) \text{ is true } \exists k \in \mathbb{Z}^+$$

$$i.e. \text{ By } M.I., P(n) \text{ is true}, \forall n \in \mathbb{Z}^+$$

* 寫結論

方法2

$$(r+1)^3 - r^3 = (r+1)^2 + r(r+1) + r^2 = 3r^2 + 3r + 1$$

$$\rightarrow \sum_{r=1}^{n} (r+1)^3 - \sum_{r=1}^{n} r^3 = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + n$$

$$\rightarrow \sum_{r=2}^{n+1} r^3 - \sum_{r=1}^{n} r^3 = 3\sum_{r=1}^{n} r^2 + \frac{3n(n+1)}{2} + n$$

$$\rightarrow \sum_{r=2}^{n} r^3 + (n+1)^3 - 1 - \sum_{r=2}^{n} r^3 = 3\sum_{r=1}^{n} r^2 + \frac{n(3n+5)}{2}$$

* $a^3 - b^3 \equiv (a-b)(a^2+ab+b^2)$

* 等差數列之和

* 透過改變首未項改變公項

CONT'D

2022 - SECTION A

$$\rightarrow (n+1)^3 - 1 = 3\sum_{r=1}^{n} r^2 + \frac{n(3n+5)}{2}$$

$$\rightarrow 3\sum_{r=1}^{n} r^2 = n((n+1)^2 + (n+1) + 1) - \frac{n(3n+5)}{2}$$

$$\rightarrow \sum_{r=1}^{n} r^2 = \frac{n(2n+1)(n+1)}{6}$$

$$Hence,\ \sum_{r=1}^{2n} (-1)^r r^2 = 2(2^2 + 4^2 + \ldots + (2n)^2) - \sum_{r=1}^{2n} r^2$$

$$= 8\sum_{r=1}^{n} r^2 - \sum_{r=1}^{2n} r^2 = \frac{8n(2n+1)(n+1)}{6} - \frac{2n(4n+1)(2n+1)}{6}$$

$$= n(2n+1)$$

$* \ a^3 - b^3 \equiv (a-b)(a^2 + ab + b^2)$

2022 - SECTION A

$Q4.)\ Define\ C : y = f(x), f(x) = (7x - 2x^2)e^{-x}$

$Find\ the\ number\ of\ pt.\ of\ inflexion\ of\ curve\ C.$

* 參考課程 3.3 及 3.5

$$
\begin{aligned}
f(x) = (7x - 2x^2)e^{-x} \rightarrow f'(x) &= (7 - 4x)e^{-x} - (7x - 2x^2)e^{-x} \\
&= (2x^2 - 11x + 7)e^{-x} \\
\rightarrow f''(x) &= (4x - 11)e^{-x} - (2x^2 - 11x + 7)e^{-x} \\
&= -(2x^2 - 15x + 18)e^{-x}
\end{aligned}
$$

$For\ f''(x_0) = 0,\ 2x_0^2 - 15x_0 + 18 = 0\ \ -\ \ (*)$

$\because \Delta = 15^2 - 4(2)(18) = -81 < 0$

$\therefore There\ is\ no\ real\ solution\ for\ (*)$

$Hence,\ there\ is\ no\ pt.\ of\ inflexions\ for\ C$

* Product rule

* Product rule

* 搵 pt. of inflexion = 搵 x_0 使度 $f''(x_0)=0$

2022 - SECTION A

$Q5.)$ Let $(a+x)^n = \sum_{k=0}^{n} \mu_k x^k$ and $(bx-1)^n = \sum_{k=0}^{n} \lambda_k x^k$, where, a, b are constant and $n \in \mathbb{Z}^+$

Given $\mu_2 = -10$, $\lambda_0 = \mu_0$ and $\lambda_1 = 2\mu_1$, find a, b and n

* 參考課程 1.1

$$(a+x)^n \equiv \sum_{k=0}^{n} C_k^n a^{n-k} x^k \text{ and } (bx-1)^n \equiv \sum_{k=0}^{n} C_k^n (-1)^{n-k} (bx)^k$$

$$\rightarrow \sum_{k=0}^{n} \mu_k x^k \equiv \sum_{k=0}^{n} C_k^n a^{n-k} x^k \text{ and } \sum_{k=0}^{n} \lambda_k x^k \equiv \sum_{k=0}^{n} C_k^n (-1)^{n-k} (bx)^k$$

$$\rightarrow \mu_k = C_k^n a^{n-k} \text{ and } \lambda_k = C_k^n (-1)^{n-k} b^k$$

$$\text{Since } \mu_2 = -10 \rightarrow C_2^n a^{n-2} = -10 \rightarrow \frac{n(n-1)a^{n-2}}{2} = -10$$

Obviously, $a < 0$ while n has to be odd number to produce

a negnative number -10

*** Binomial Expansion**

$$* \; C_r^n = \frac{n!}{r!(n-r)!} \rightarrow C_1^n = n \text{ and } C_2^n = \frac{n(n-1)}{2}$$

CONT'D

2022 - SECTION A

$Also\ \lambda_0 = \mu_0\ and\ \lambda_1 = 2\mu_1$

$\rightarrow a^n = (-1)^n\ and\ na^{n-1} = n(-1)^{n-1}b$

$\rightarrow a^n = -1\ and\ a^{n-1} = b\ (\because n\ is\ odd)$

$\rightarrow a = -1\ and\ b = 1$

$$Then\ \frac{n(n-1)(-1)}{2} = -10 \rightarrow n^2 - n - 20 = 0$$

$\rightarrow (n-5)(n+4) = 0 \rightarrow n = 5\ or -4\ (rejected)$

$\rightarrow n = 5$

$* \ C_r^n = \dfrac{n!}{r!(n-r)!} \rightarrow C_1^n = n\ and\ C_2^n = \dfrac{n(n-1)}{2}$

2022 - SECTION A

Q6.) Given the slope of tangent of curve G to any point (x, y) is $\dfrac{2x+1}{x^2+2x+5}$, *if* $(-3, ln2)$ *lies on G, does G passes through* $(-1, \dfrac{-\pi}{8})$? *Explain your answer.*

* 參考課程 3.8 及 3.9

$$Let\ G: y=f(x) \rightarrow f'(x)=\frac{2x+1}{x^2+2x+5} \rightarrow f(x)=\int \frac{2x+1}{x^2+2x+5}dx$$

$$\rightarrow f(x)=\int \frac{2x+2}{x^2+2x+5}dx-\int \frac{1}{x^2+2x+5}dx=I_1-I_2$$

$$where\ I_1=\int \frac{d(x^2+2x+5)}{x^2+2x+5}=ln|x^2+2x+5|$$

$$I_2=\int \frac{dx}{(x+1)^2+4}=\int \frac{d(2tan\theta-1)}{4tan^2\theta+4},\ (where\ 2tan\theta=x+1)$$

* 微分計算 tangent 斜率
* 積分類似微分逆函數
* 積分代入法
* 利用三角代入法, *let* $x+1=2tan\theta$

CONT'D

2022 - SECTION A

$$= \int \frac{2sec^2\theta d\theta}{4(tan^2\theta + 1)} = \int \frac{2sec^2\theta d\theta}{4sec^2\theta} = \int \frac{d\theta}{2} = \frac{\theta}{2}$$

$$Hence, f(x) = ln|x^2 + 2x + 5| - \frac{1}{2}tan^{-1}(\frac{x+1}{2}) + A, where\ A\ is\ a\ constant$$

$$Given, f(-3) = ln2 \rightarrow A = ln2 - ln|9 - 6 + 5| + \frac{1}{2}tan^{-1}(-1)$$

$$= -2ln2 - \frac{\pi}{8}$$

$$Consider, f(-1) = ln|1 - 2 + 5| + \frac{1}{2}tan^{-1}(0) + A = 2ln2 + A = \frac{-\pi}{8}$$

$$\therefore G\ passes\ through\ (-1, \frac{-\pi}{8})$$

$* \quad tan^2\theta + 1 = sec^2\theta$

2022 - SECTION A

$Q7.)$ Given $C: y = ln(x+2),\ x > 0$. $P\ (h, k)$ is a moving point on C. Let L be the tangent of C to P, and A be the area of the bounded region of C, L and $y-axis$. Given $h = 3^{-t}$, $t =$ the time in second. Find A in term of h and the rate of change of A when $t = 1$.

* 參考課程 3.2, 3.3, 3.4, 3.9. 3.10 及 3.11

$$Let\ f(x) = ln(x+2) \rightarrow f'(x) = (x+2)^{-1}$$

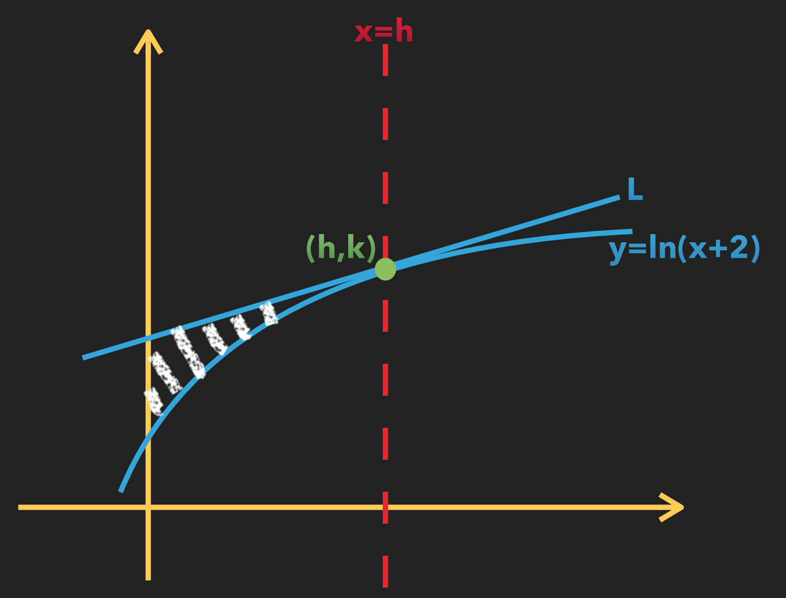
$$L: \frac{y-k}{x-h} = f'(h) \rightarrow y = f'(h)(x-h) + ln(h+2)$$

$$\rightarrow y = \frac{x-h}{h+2} + ln(h+2)$$

$$A = \int_0^h \frac{x-h}{h+2} + ln(h+2) - ln(x+2)dx$$

$$= [\frac{0.5x^2 - hx}{h+2} + xln(h+2)]_0^h - \int_0^h ln(x+2)dx$$

* 用微分計斜率並用點斜計算直線方程



CONT'D

2022 - SECTION A

$$= [\frac{0.5x^2 - hx}{h+2} + xln(h+2)]_0^h - \int_0^h ln(x+2)dx$$

$$= [\frac{-0.5h^2}{h+2} + hln(h+2)] - [xln(x+2)]_0^h + \int_0^h \frac{xdx}{x+2}$$

$$= \frac{-0.5h^2}{h+2} + hln(h+2) - hln(h+2) + \int_0^h (1 - \frac{2}{x+2})dx$$

$$= \frac{-h^2}{2h+4} + h - 2ln(h+2) + 2ln2 = \frac{h^2+4h}{2h+4} - 2ln(h+2) + 2ln2$$

$$Then, \frac{dA}{dt} = [-\frac{2(h^2+4h)}{(2h+4)^2} + \frac{2h+4}{2h+4} - \frac{2}{h+2}] \cdot \frac{dh}{dt}$$

$$= [1 - \frac{2h(h+4)}{(2h+4)^2} - \frac{2}{h+2}] \cdot \frac{dh}{dt}$$

* 用 Integration by part

* Product rule + Chain rule

CONT'D

2022 - SECTION A

$$where\ h = 3^{-t} \rightarrow \boxed{lnh = -tln3 \rightarrow \frac{dh}{dt} = -hln3}$$

$$Hence, \frac{dA}{dt}|_{t=1} = [1 - \frac{\frac{2}{3}(\frac{1}{3} + 4)}{(\frac{2}{3} + 4)^2} - \frac{2}{\frac{1}{3} + 2}] \cdot (-\frac{ln3}{3})$$

$$= -\frac{ln3}{294}\ sq.unit/s$$

* $\blacksquare$ 用 ln 微分法

2022 - SECTION A

Q8.)

$$(E)\begin{cases} ax+2y-z=4k \\ -x+ay+2z=4 \\ 2x-y+az=k^2 \end{cases} \quad \textit{where } a, k \textit{ are constant}$$

a.) Assume (E) has unique solution, express y in yerm of a and k .

b.) Solve (E), for (E) has inifinity many solution .

* 參考課程 4.7

a.)

$$(E) \sim \left(\begin{array}{ccc|c} a & 2 & -1 & 4k \\ -1 & a & 2 & 4 \\ 2 & -1 & a & k^2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & a & -1 & k^2 \\ -1 & 2 & a & 4 \\ a & -1 & 2 & 4k \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & a & -1 & k^2 \\ 0 & a+4 & 2a-1 & 8+k^2 \\ 0 & -2-a^2 & 4+a & 8k-ak^2 \end{array}\right)$$

* 公式轉位, 方便計算
* 因計算 y, 將相關列移位
* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \to \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

CONT'D

2022 - SECTION A

$$\sim \left(\begin{array}{ccc|c} 2 & a & -1 & k^2 \\ 0 & a+4 & 2a-1 & 8+k^2 \\ 0 & 0 & A & B \end{array}\right)$$

$$\begin{aligned} where\ A &= (a+4)^2 + (2a-1)(a^2+2) \\ &= 2(a^3+6a+7) \\ B &= (8k-ak^2)(a+4) + (a^2+2)(k^2+8) \\ &= 2(4a^2+(4k-2k^2)a+(k^2+16k+8)) \end{aligned}$$

$$y = \frac{B}{A} = \frac{4a^2+(4k-2k^2)a+(k^2+16k+8)}{a^3+6a+7}$$

* y 列已移到未位

b.) If (E) has infinite many solution, $A = 0 - (1)$ *and* $B = 0 - (2)$

* 如果有無限答案, **A** 同 **B**要係 **0**

$\rightarrow (1) : a^3 + 6a + 7 = 0$

$\rightarrow a^3 + 6a + 6 + 1 = 0$

CONT'D

2022 - SECTION A

$$\to a^3+1+6(a+1)=0 \to (a+1)(a^2-a+1)+6(a+1)=0$$

$$\to (a+1)(a^2-a+7)=0$$

$$\to a=-1 \text{ or } a^2-a+7=0 \ (\Delta=-27<0, \text{no real solution})$$

$$Hence,\ (2): 4(1)+(4k-2k^2)(-1)+(k^2+16k+8)=0$$

$$\to k^2+4k+4=0 \to (k+2)^2=0 \to k=-2$$

$$Then,\ (E) \sim \left(\begin{array}{ccc|c} 2 & -1 & -1 & 4 \\ 0 & 1 & -1 & 4 \end{array}\right)$$

$$Let\ y=t,\ t\in\mathbb{R} \to z=4+t \text{ and } x=4+t$$

$$\therefore (x,y,z)=(4+t,\ t,\ 4+t),\ \text{where } t\in\mathbb{R}$$

* $a^3+b^3 \equiv (a+b)(a^2-ab+b^2)$

* 利用判別式証明無實根

* $a^2+2ab+b^2 \equiv (a+b)^2$

* 三條公式剩返兩條

2022 - SECTION B

Q9.) Given $f(x) = (x^2 + 3x)(x-1)^{-1}, x \neq 1$. Denote the curve $H : y = f(x)$.

a.) Find the asymptote(s) of H

b.) Find the max. and min. points of H

c.) Sketch H.

d.) Let R be the region bounded by H and $y = 10$. Find the volume of the solid of revolutoion generated by revolving R about the axis $y = 10$

* 參考課程 3.5 及 3.12

a.) Vertical Asymptote : $x = 1$

Horizontal Asymptote : No Horizontal Asymptote

Oblique Asymptote : $y = x + 4$

b.) $f(x) = (x+4) + \dfrac{4}{x-1}$

$$\to f'(x) = 1 - \frac{4}{(x-1)^2} = \frac{(x-1)^2 - 4}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2}$$

* x 係幾多, 分母係零

* Find $\lim_{x\to\infty} y$

* Find m and c such that $\lim_{x\to\infty} [y - (mx + c)] = 0$

$$\frac{x^2+3x}{x-1} = \frac{(x^2+3x-4)+4}{x-1} = \frac{(x-1)(x+4)+4}{x-1}$$

$$= (x+4) + \frac{4}{x-1}$$

$$\therefore \lim_{x\to\infty} [y - (x+4)] = 0$$

* $(a+b)(a-b) = a^2 - b^2$

CONT'D

2022 - SECTION B

	x < -1	x = -1	-1 < x < 3	x = 3	x > 3
f′(x)	+	0	-	0	+
f(x)	Inc.		Dec.		Inc.

$$\therefore (-1, f(-1)) \text{ is local max . pt .}$$

$$(3, f(3)) \text{ is local min . pt .}$$

$$i.e. \text{ The local max . pt .} = (-1, 1)$$

$$\text{The local min . pt .} = (3, 9)$$

$$c.)\ Also, f'(x) = 1 - \frac{4}{(x-1)^2} \rightarrow f''(x) = \frac{8}{(x-1)^3}$$

$$\text{For } x < 1, f''(x) < 0 \rightarrow f(x) \text{ is concave downward}$$

$$\text{For } x > 1, f''(x) > 0 \rightarrow f(x) \text{ is concave upward}$$

$$\text{The } y - \text{intercept} = 0$$

$$\text{The } x - \text{intercept} = -3 \text{ and } 0$$

* 利用表格計算 turning pt. 附近情況

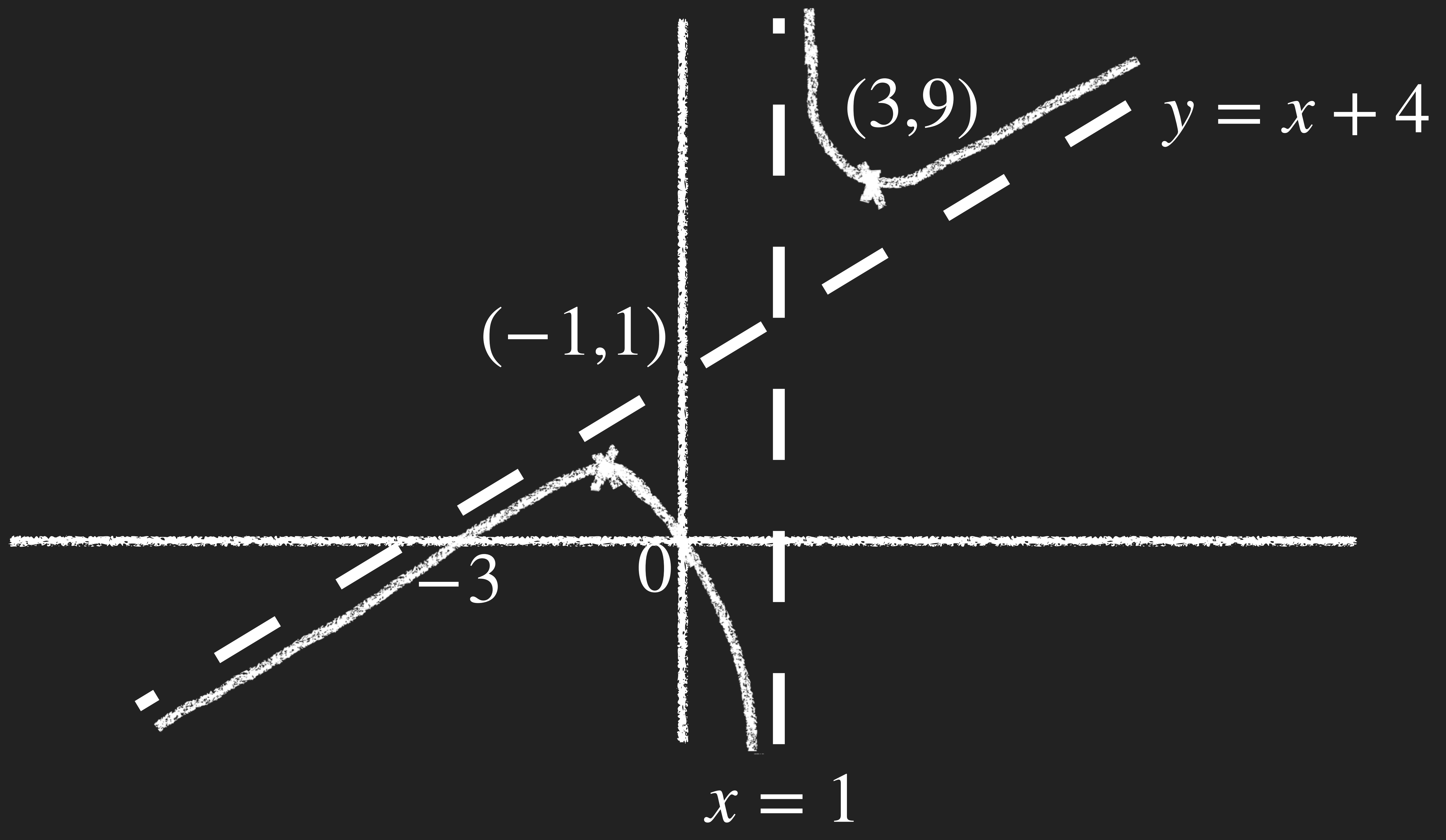
$f'(x) > 0 \rightarrow increasing$

$f'(x) < 0 \rightarrow decreasing$

* 利用 f′′(x) 找出何時凹位向上及向下

CONT'D

2022 - SECTION B



$d.)\ Consider,\ f(x) = 10 \rightarrow x^2 + 3x = 10x - 10 \rightarrow x^2 - 7x + 10 = 0$

$\rightarrow (x-5)(x-2) = 0 \rightarrow x = 2\ or\ x = 5$

$$Volume = \pi \int_2^5 [10 - (x + 4 + \frac{4}{x-1})]^2 dx$$

* 十字相乘

CONT'D

2022 - SECTION B

$$Volume = \pi \int_1^4 [10 - (u + 5 + \frac{4}{u})]^2 du, \text{ where } u = x - 1$$

$$= \pi \int_1^4 [(5-u) - \frac{4}{u})]^2 du$$

$$= \pi \int_1^4 (5-u)^2 du - 2\pi \int_1^4 (5-u)\frac{4}{u} du + \pi \int_1^4 \frac{16}{u^2} du$$

$$= \pi[\frac{-(5-u)^3}{3}]_1^4 - 8\pi \int_1^4 (\frac{5}{u} - 1) du + 16\pi[\frac{-1}{u}]_1^4$$

$$= 21\pi - 8\pi[5lnu - u]_1^4 + 12\pi$$

$$= \pi[57 - 80ln2] \; cu . unit$$

* 利用 Disk Method

* 積分代入法, 定積要改範圍

* $(a-b)^2 = a^2 - 2ab + b^2$

2022 - SECTION B

$Q10.)\ Let\ g(x) = cos^2xcos2x$

$a.)\ Find\ \int_0^{\pi} xg(x)dx$

$b.)\ Find\ \int_{-\pi}^{2\pi} xg(x)dx$

* 參考課程 2.2, 3.8, 3.9 及 3.10

$$a.)\ Let\ I = \int_0^{\pi} xg(x)dx = \int_{\pi}^{0} (\pi - x)g(\pi - x)d(-x)$$

$$= \int_0^{\pi} (\pi - x)g(x)dx = \pi\int_0^{\pi} g(x)dx - I$$

$$\rightarrow 2I = \pi\int_0^{\pi} cos^2xcos2xdx = \pi\int_0^{\pi} cos^2xd(\frac{sin2x}{2})$$

* 用積分代入法
$Let\ x = \pi - u$

* 負數範圍倒轉

* $cos^2(\pi - x) = cos^2x,\ cos2(\pi - x) = cos2x$

* 用 Integration by part

CONT'D

2022 - SECTION B

$$= \pi([\frac{1}{2}cos^2xsin2x]_0^\pi - \frac{1}{2}\int_0^\pi sin2xd(cos^2x))$$

$$= \frac{\pi}{2}\int_0^\pi sin2x2coxsinxdx = \frac{\pi}{2}\int_0^\pi sin2xsin2xdx = \frac{\pi}{2}\int_0^\pi sin^2 2xdx$$

$$= \frac{\pi}{2}\int_0^\pi \frac{1}{2}(1 - cos4x) = \frac{\pi}{4}(\int_0^\pi dx - \cancel{\int_0^\pi cos4xdx}) = \frac{\pi^2}{4}$$

$$\rightarrow I = \frac{\pi^2}{8}$$

$$b.)\ Let\ I_2 = \int_{-\pi}^{2\pi} xg(x)dx = \int_{\pi}^{2\pi} xg(x)dx + \int_{-\pi}^{\pi} xg(x)dx$$

$$where\ f(x) = xg(x) \rightarrow f(-x) = (-x)g(-x) = -xg(x) = -f(x)$$

$$\therefore f(x)\ is\ an\ odd\ function$$

* sin 雙角公式

* cos 雙角公式

* 面積互相抵消

* 定積分範圍可以拆開計算

CONT'D

2022 - SECTION B

$$Hence, \int_{-\pi}^{\pi} xg(x)dx = 0$$

$$\rightarrow I_2 = \int_{\pi}^{2\pi} xg(x)dx = \int_{0}^{\pi} (x+\pi)g(x+\pi)dx$$

$$where\ g(x+\pi) = cos^2(x+\pi)cos2(x+\pi) = [cosxcos\pi - sixsin\pi]^2 cos2x$$

$$= cos^2xcos2x = g(x)$$

$$\rightarrow I_2 = \int_{0}^{\pi} xg(x)dx - \pi\int_{0}^{\pi} g(x)dx = \frac{\pi^2}{8} + \frac{\pi^2}{4} = \frac{3\pi^2}{8}$$

* 面積互相抵消

* 用積分代入法
$Let\ x = u + \pi$

* cos 複角公式

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

2022 - SECTION B

$Q11.)$ $Define\ A = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}, where\ \theta \neq 2\pi k, for\ all\ k \in \mathbb{Z}, for\ n \in \mathbb{Z}^+$

$a.)\ Simplify\ (I_2 - A)(I_2 + A + A^2 + \ldots + A^n)$

$b.)\ Prove\ A^n = \begin{pmatrix} cosnx & -sinnx \\ sinnx & cosnx \end{pmatrix}$

$c.)\ Find\ (I_2 - A)^{-1}\ and\ hence,\ (I_2 + A + A^2 + \ldots + A^n)$

$d.)\ Use\ the\ above\ result, find:$

$i.)\ cos\dfrac{5\pi}{18} + cos\dfrac{5\pi}{9} + cos\dfrac{5\pi}{6} \ldots + cos25\pi$

$ii.)\ cos^2\dfrac{\pi}{7} + cos^2\dfrac{2\pi}{7} + cos^2\dfrac{3\pi}{7} \ldots + cos^2 7\pi$

* 參考課程 2.1, 2.2, 2.3, 4.9, 4.10 及 4.11

CONT'D

2022 - SECTION B

$$a.)\ Since,\ AI_2 = I_2A \rightarrow (I_2 - A)(I_2 + A + A^2 + \ldots + A^n) = I_2 - A^{n+1}$$

$$b.)\ Let\ P(n) : A^n = \begin{pmatrix} cosn\theta & -sinn\theta \\ sinn\theta & cosn\theta \end{pmatrix}$$

$$For\ P(1):\ L.H.S. = A = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} = R.H.S.$$

$$\therefore P(1)\ is\ true$$

$$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+,\ then\ P(k+1):$$

$$L.H.S. = A^{k+1} = \begin{pmatrix} cosk\theta & -sink\theta \\ sink\theta & cosk\theta \end{pmatrix}\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} cosk\theta cos\theta - sink\theta sin\theta & -cosk\theta sin\theta - sink\theta cos\theta \\ sink\theta cos\theta + cosk\theta sin\theta & cosk\theta cos\theta - sink\theta sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} cos(k+1)\theta & -sin(k+1)\theta \\ sin(k+1)\theta & cos(k+1)\theta \end{pmatrix} = R.H.S.$$

* AB=BA, 可用數字恆等式
* 運用等比數列之和公式
* 先 Let Statement
* 証明 P(1) is true
* 假設 P(k) is true. 証明 P(k+1) is true
* sin 複角公式
* cos 複角公式

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

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$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

CONT'D

2022 - SECTION B

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

i.e. By M.I., $P(n)$ *is true,* $\forall n \in \mathbb{Z}^+$

$$b.)\ I_2 - A = \begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix}$$

$$|I_2 - A| = (1-\cos\theta)^2 + \sin^2\theta = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta$$

$$= 1 - 2\cos\theta + 1 = 2(1-\cos\theta) \neq 0\ (\because \theta \neq 2k\pi)$$

$\therefore (I_2 - A)^{-1}$ *exists*

$$Then,\ adj(I_2 - A) = \begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix}^T = \begin{pmatrix} 1-\cos\theta & -\sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix}$$

$$\rightarrow (I_2 - A)^{-1} = \frac{1}{2(1-\cos\theta)} \begin{pmatrix} 1-\cos\theta & -\sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix}$$

* 寫結論

* $(a-b)^2 \equiv a^2 - 2ab + b^2$

* $\sin^2 x + \cos^2 x \equiv 1$

* |A| 不等如 0, A^{-1} 存在

* $A^{-1} = 1/|A|\ adj(A)$

CONT'D

2022 - SECTION B

$$= \frac{1}{2\left(2\sin^2\frac{\theta}{2}\right)}\begin{pmatrix} 2\sin^2\frac{\theta}{2} & -2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & 2\sin^2\frac{\theta}{2} \end{pmatrix} = \frac{1}{2\sin\frac{\theta}{2}}\begin{pmatrix} \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \\ \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}$$

$$\because\ (I_2 - A)(I_2 + A + A^2 + \ldots A^n) = I_2 - A^{n+1}$$

$$\rightarrow (I_2 + A + A^2 + \ldots A^n) = (I_2 - A)^{-1}(I_2 - A^{n+1})$$

$$= \frac{1}{2\sin\frac{\theta}{2}}\begin{pmatrix} \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \\ \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1-\cos(n+1)\theta & \sin(n+1)\theta \\ -\sin(n+1)\theta & 1-\cos(n+1)\theta \end{pmatrix}$$

$$= \frac{1}{2\sin\frac{\theta}{2}}\begin{pmatrix} \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \\ \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 2\sin^2\frac{(n+1)\theta}{2} & 2\sin\frac{(n+1)\theta}{2}\cos\frac{(n+1)\theta}{2} \\ -2\sin\frac{(n+1)\theta}{2}\cos\frac{(n+1)\theta}{2} & 2\sin^2\frac{(n+1)\theta}{2} \end{pmatrix}$$

* sin 雙角公式
* cos 雙角公式

CONT'D

2022 - SECTION B

$$= \frac{sin\frac{(n+1)\theta}{2}}{sin\frac{\theta}{2}}\begin{pmatrix} sin\frac{\theta}{2} & -cos\frac{\theta}{2} \\ cos\frac{\theta}{2} & sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix} sin\frac{(n+1)\theta}{2} & cos\frac{(n+1)\theta}{2} \\ -cos\frac{(n+1)\theta}{2} & sin\frac{(n+1)\theta}{2}\end{pmatrix}$$

$$= \frac{sin\frac{(n+1)\theta}{2}}{sin\frac{\theta}{2}}\begin{pmatrix} a & -b \\ b & a\end{pmatrix}$$

$$where\ a = sin\frac{\theta}{2}sin\frac{(n+1)\theta}{2} + cos\frac{\theta}{2}cos\frac{(n+1)\theta}{2} = cos\frac{n\theta}{2}$$

$$b = cos\frac{\theta}{2}sin\frac{(n+1)\theta}{2} - sin\frac{\theta}{2}cos\frac{(n+1)\theta}{2} = sin\frac{n\theta}{2}$$

$$i.e. (I_2 + A + \ldots + A^n) = \frac{sin\frac{(n+1)\theta}{2}}{sin\frac{\theta}{2}}\begin{pmatrix} cos\frac{n\theta}{2} & -sin\frac{n\theta}{2} \\ sin\frac{n\theta}{2} & cos\frac{n\theta}{2}\end{pmatrix}$$

* sin 複角公式
* cos 複角公式

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

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$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

CONT'D

2022 - SECTION B

$$di.)\ A+\ldots+A^n=\frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}\begin{pmatrix}\cos\frac{n\theta}{2} & -\sin\frac{n\theta}{2}\\ \sin\frac{n\theta}{2} & \cos\frac{n\theta}{2}\end{pmatrix}-I_2$$

$$\rightarrow\begin{pmatrix}a_n(\theta) & -b_n(\theta)\\ b_n(\theta) & a_n(\theta)\end{pmatrix}=\frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}\begin{pmatrix}\cos\frac{n\theta}{2} & -\sin\frac{n\theta}{2}\\ \sin\frac{n\theta}{2} & \cos\frac{n\theta}{2}\end{pmatrix}-I_2$$

$$where\ a_n(\theta)=\sum_{r=1}^{n}cosr\theta,\ b_n(\theta)=\sum_{r=1}^{n}sinr\theta$$

$$Consider\ \theta=\frac{5\pi}{18},\ with\ n=18x5=90$$

$$a_{90}(\frac{5\pi}{18})=\sum_{r=1}^{90}cosr\frac{5\pi}{18}=\frac{\sin\frac{(91)(5\pi)}{36}}{\sin\frac{5\pi}{36}}\cancelto{0}{\cos\frac{(90)(5\pi)}{36}}-1=-1$$

* 矩陣加減 = 各自項數做加減

* 矩陣相等 = 各自項數相等

CONT'D

2022 - SECTION B

ii.) $$\sum_{r=1}^{49} \boxed{cos^2\frac{r\pi}{7}} = \sum_{r=1}^{49} \boxed{\frac{1}{2}(cos\frac{2r\pi}{7}+1)} = \frac{1}{2}\sum_{r=1}^{49} cosr\frac{2\pi}{7} + \frac{\boxed{49}}{2}$$

$$= \frac{1}{2}a_{49}(\frac{2\pi}{7}) + \frac{49}{2} = \frac{1}{2}[\frac{sin\frac{(50)(\pi)}{7}}{sin\frac{\pi}{7}}cos\frac{(49)(\pi)}{7} - 1] + \frac{49}{2}$$

$$= \frac{1}{2}[\frac{\boxed{sin(49\pi+\frac{\pi}{7})}}{sin\frac{\pi}{7}}(-1) - 1] + \frac{49}{2} = \frac{1}{2}[\frac{\boxed{-sin\frac{\pi}{7}}}{sin\frac{\pi}{7}}(-1) - 1] + \frac{49}{2}$$

$$= \frac{49}{2}$$

* cos 雙角公式
* 個 1 加左 49 次

* sin 複角公式

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

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$\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$\sin A - \sin B = 2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

$\cos A + \cos B = 2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

2022 - SECTION B

Q12.) Consider ΔABC, with $O = (0, 0)$. Given that D on BC such that AD is the angle bisector of $\angle BAC$. Let $a = BC$, $b = AC$ and $c = AB$

a.) Prove $\overrightarrow{AD} = -\overrightarrow{OA} + \dfrac{b}{b+c}\overrightarrow{OB} + \dfrac{c}{b+c}\overrightarrow{OC}$

b.) Let E lies on AC such that BE is the angle bisector of $\angle ABC$, Given that

$$\overrightarrow{OJ} = \frac{a}{a+b+c}\overrightarrow{OA} + \frac{b}{a+b+c}\overrightarrow{OB} + \frac{c}{a+b+c}\overrightarrow{OC},$$ prove J lies on AD and BE.

c.) Given $\overrightarrow{OA} = 35\hat{i} + 9\hat{j} + \hat{k}$, $\overrightarrow{OB} = 40\hat{i} - 3\hat{j} + \hat{k}$ and $\overrightarrow{OC} = -3\hat{j} + \hat{k}$

Let I be the in – center of ΔABC.

i.) Find $\overrightarrow{OI}$

ii.) Find the radius of the inscribed circle of ΔABC.

* 參考課程 4.4 及 4.5

CONT'D

2022 - SECTION B

$a.)\ The\ area\ of\ \Delta ABD : The\ area\ of\ \Delta ACD = AD : CD$

$$\rightarrow \frac{1}{2}ABxADsin\angle BAD : \frac{1}{2}ACxADsin\angle CAD = AD : CD$$

$$\rightarrow c : b = AD : CD\ (\because \angle BAD = \angle CAD)$$

$$\therefore \overrightarrow{AD} = \frac{b}{b+c}\overrightarrow{AB} + \frac{c}{b+c}\overrightarrow{AC}$$

$$= \frac{b}{b+c}(\overrightarrow{OB} - \overrightarrow{OA}) + \frac{c}{b+c}(\overrightarrow{OC} - \overrightarrow{OA})$$

$$= -\overrightarrow{OA} + \frac{b}{b+c}\overrightarrow{OB} + \frac{c}{b+c}\overrightarrow{OC}$$

$b.)\ Consider\ \overrightarrow{AJ} = \overrightarrow{OJ} - \overrightarrow{OA}$

$$= -\frac{b+c}{a+b+c}\overrightarrow{OA} + \frac{b}{a+b+c}\overrightarrow{OB} + \frac{c}{a+b+c}\overrightarrow{OC}$$

* 共高三角形, 面積比=邊比

* 三角形面積 = 1/2 x absinC

* 分割公式

CONT'D

2022 - SECTION B

$$= \frac{b+c}{a+b+c}[-\overrightarrow{OA} + \frac{b}{b+c}\overrightarrow{OB} + \frac{c}{b+c}\overrightarrow{OC}] = \frac{b+c}{a+b+c}\overrightarrow{AD}$$

$\therefore AJ \ // \ AD$ → *J lies on AD*

$$Similarly\ \overrightarrow{BE} = -\overrightarrow{OB} + \frac{a}{a+c}\overrightarrow{OA} + \frac{c}{a+c}\overrightarrow{OC}\ (from\ a.)$$

$$and\ \overrightarrow{BJ} = \overrightarrow{OJ} - \overrightarrow{OB} = \frac{a+c}{a+b+c}[-\overrightarrow{OB} + \frac{a}{a+c}\overrightarrow{OA} + \frac{c}{a+c}\overrightarrow{OC}]$$

$$= \frac{a+c}{a+b+c}\overrightarrow{BE}$$

$\therefore BJ \ // \ BE$ → *J lies on BE*

$$ci.)\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 5\hat{i} - 12\hat{j} \rightarrow c = \boxed{\sqrt{5^2 + 12^2}} = 13$$

* 如果兩支 vector 成比例, 它們平行

* $|A|$ = 開方 $a^2 + b^2 + c^2$

$A = a\hat{i} + b\hat{j} + c\hat{k}$

CONT'D

2022 - SECTION B

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -35\hat{i} - 12\hat{j} \rightarrow b = \boxed{\sqrt{35^2 + 12^2}} = 37$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -40\hat{i} \rightarrow a = 40$$

$$\text{from b.) } \overrightarrow{OI} = \frac{40}{90}\overrightarrow{OA} + \frac{37}{90}\overrightarrow{OB} + \frac{13}{90}\overrightarrow{OC}$$

$$= 32\hat{i} + \frac{7}{3}\hat{j} + \hat{k}$$

$$\text{ii.) The radius, } r = |\overrightarrow{AI}|\boxed{\sin\angle BAI} = |\overrightarrow{AI}| \cdot \boxed{\frac{|\overrightarrow{AI} x \overrightarrow{AB}|}{|\overrightarrow{AI}||\overrightarrow{AB}|}}$$

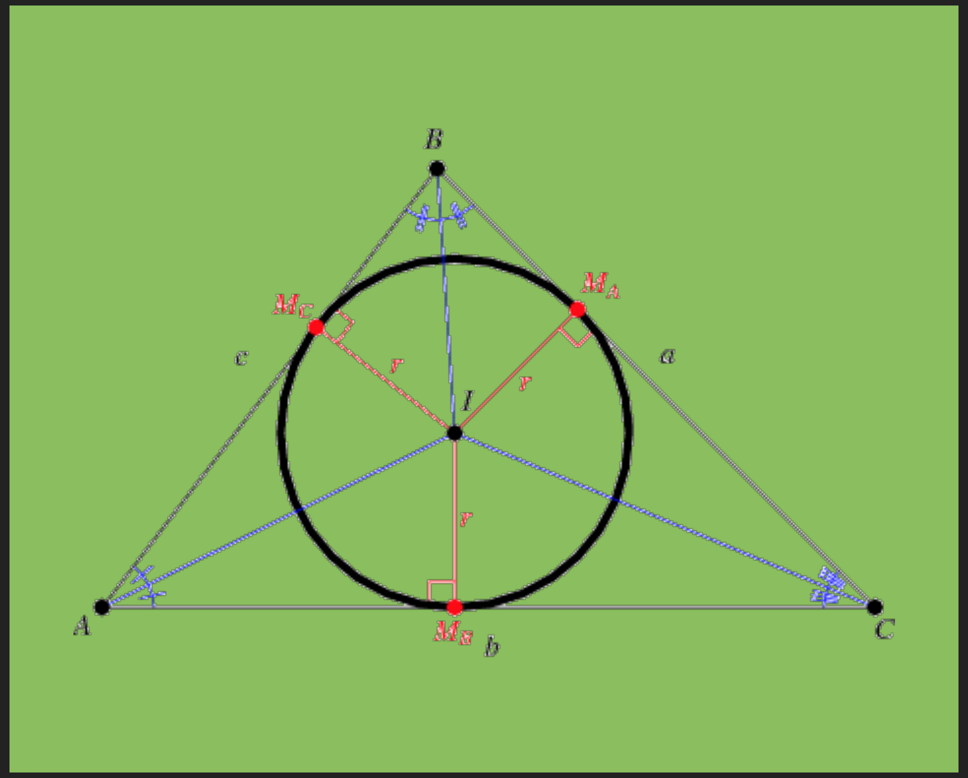
$$= \frac{|\overrightarrow{AI} x \overrightarrow{AB}|}{|\overrightarrow{AB}|} = \frac{1}{13}|(-3\hat{i} - \frac{20}{3}\hat{j})x(5\hat{i} - 12\hat{j})|$$

$$= \frac{1}{13}\frac{208}{3} = \frac{16}{3}$$

* $|A|$ = 開方 $a^2 + b^2 + c^2$

$A = a\hat{i} + b\hat{j} + c\hat{k}$

* 三角形內心



* 三角形面積 = 1/2 |兩支vector 乘積|

= 1/2 absinC

* 乘積可以拆括號
* ixi=0, jxj=0, ixj=k, jxi=-k