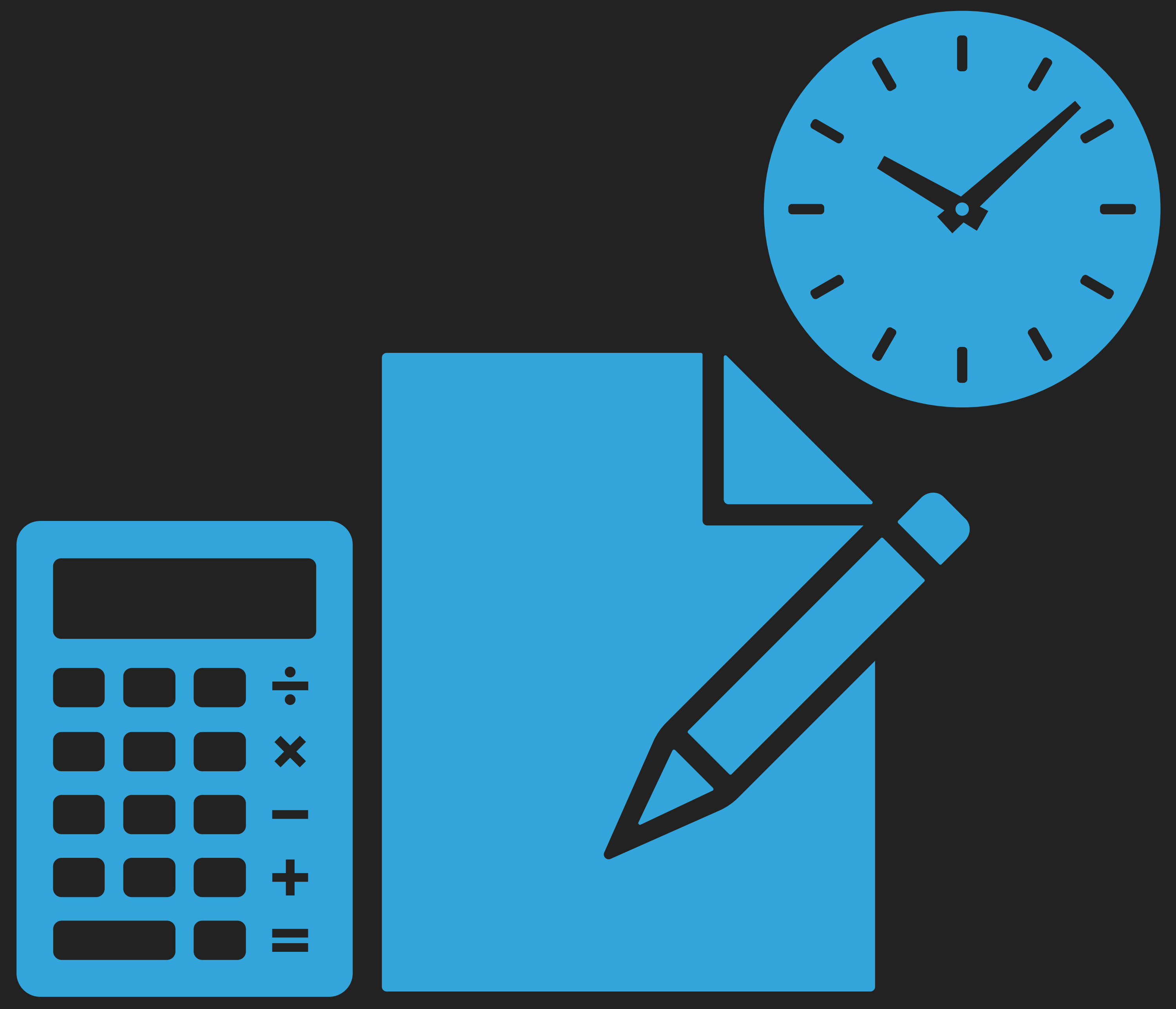
深宵教室 - DSE M2 模擬試題解答

2012

此為參考2012試題之模擬試題，原版請另行購買

2012

- Section A
- Section B



2012 - SECTION A

Q1.) $f(x) = e^{2x}$. $f'(0) = ?$ *(By First Principles)*

* 參考課程 1.1, 3.1 及 3.2

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}(e^{2h} - 1)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(1 + \sum_{r=1}^{\infty}\frac{(2h)^r}{r!} - 1\right)$$

$$= \lim_{h\to 0}\sum_{r=1}^{\infty}\frac{(2h)^{r-1}}{r!} = \lim_{h\to 0} 2 + \sum_{r=2}^{\infty}\cancel{\frac{(2h)^{r-1}}{r!}}^{\,0}$$

$$= 2$$

* 微分定義

* $e^x = \sum_{r=0}^{\infty}\frac{x^r}{r!}$

* 將第一項抽出並改變首項

2012 - SECTION A

$Q2.)\ (1+ax)^n = 1 + 6x + 16x^2 + \ldots,\ \ a = ?\ \ and\ \ n = ?$

* 參考課程 1.1

$$(1+ax)^n \equiv \sum_{r=0}^{n} C_r^n (ax)^r$$

By compare coefficient of x *and* x^2

$$\begin{cases} C_1^n a = na = 6 & \text{———— (1)} \\ C_2^n a^2 = \dfrac{n(n-1)a^2}{2} = 16 & \text{———— (2)} \end{cases}$$

$$In\ (2):\ \ (na)^2 - (na)a = 2(16) = 32 \rightarrow a = \frac{2}{3}$$

$$\therefore n = \frac{6(3)}{2} = 9\ and\ a = \frac{2}{3}$$

*** Binomial Expansion**

$$* \ C_r^n = \frac{n!}{r!(n-r)!} \rightarrow C_1^n = n\ and\ C_2^n = \frac{n(n-1)}{2}$$

2012 - SECTION A

$Q3.)\ Prove\ 1x2 + 2x5 + \ldots + n(3n-1) = n^2(n+1),\ \ \forall n \in \mathbb{Z}^+$

* 參考課程 1.1 及 1.2

方法1

$$Let\ P(n):\ \sum_{r=1}^{n} r(3r-1) = n^2(n+1)\ \forall n \in \mathbb{Z}^+$$

* 先 Let Statement

$$For\ P(1):\ L.H.S. = 2 = R.H.S.$$

* 証明 P(1) is true

$$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+,\ then\ P(k+1):$$

* 假設 P(k) is true. 証明 P(k+1) is true

$$L.H.S. = \sum_{r=1}^{k+1} r(3r-1) = \boxed{\sum_{r=1}^{k} r(3r-1)} + (k+1)(3k+2)$$

* 將未項抽出並改變未項

$$= \boxed{k^2(k+1)} + (k+1)(3k+2) = (k+1)^2(k+2)$$

$$= R.H.S.$$

$$\therefore P(k+1)\ is\ true\ if\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+$$

$$i.e.\ By\ M.I.,\ P(n)\ is\ true,\ \forall n \in \mathbb{Z}^+$$

* 寫結論

CONT'D

2012 - SECTION A

方法2

$$\sum_{r=1}^{n} r(3r-1)=\sum_{r=1}^{n}(3r^2-r)$$

$$=3\sum_{r=1}^{n} r^2-\sum_{r=1}^{n} r$$

$$=\frac{3n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}(2n+1-1)$$

$$=n^2(n+1)$$

* Summation 可析開做加減及抽常數

* $1+2+\ldots+n=\dfrac{n(n+1)}{2}$

* $1^2+2^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

2012 - SECTION A

$Q4.)\ \int \frac{x+1}{x}dx = ?\ and\ \int \frac{x^3}{x^2+1}dx = ?$

* 參考課程 3.6 及 3.7

$$\int \frac{x+1}{x}dx = dx = \int 1 + \frac{1}{x}dx$$

$$= x + lnx + C,\ \ C\ is\ constant$$

方法1

$$\int \frac{x^3}{x^2-1}dx = \int \frac{x^2 \boxed{xdx}}{x^2-1} = \int \frac{x^2}{x^2-1}\boxed{\frac{1}{2}d(x^2-1)}$$

$$= \frac{1}{2}\int \frac{x^2-1+1}{x^2-1}d(x^2-1)$$

$$= \frac{1}{2}((x^2-1)) + ln(x^2-1)) + C,\ \ C\ is\ constant$$

* $\boxed{}$ 積分三寶: Substitution 可代入

* 用上面 result, x 變咗 x^2-1

CONT'D

2012 - SECTION A

方法2

$$\int \frac{x^3}{x^2-1}dx = \int \frac{x(x^2-1)+x}{x^2-1}dx = \int xdx + \int \frac{x}{x^2-1}dx$$

$$= \int xdx + \frac{1}{2}\int \frac{1}{x+1} + \frac{1}{x-1}dx$$

$$= \frac{1}{2}(x^2 + ln(x+1) + ln(x-1)) + C, \quad C \text{ is constant}$$

* 積分三寶: Partial Fraction

$$Let \ \frac{x}{x^2-1} \equiv \frac{A}{x+1} + \frac{B}{x-1}$$

$$\rightarrow x \equiv A(x-1) + B(x+1)$$

$$\rightarrow A = B = \frac{1}{2}$$

2012 - SECTION A

$$Q5.)\ y = \frac{x^2+x+1}{x+1},\ \ min\,.\,pt(s)\,.\,?\ and\ asymptote(s)\ ?$$

* 參考課程 3.2 及 3.5

$$y = \frac{x(x+1)+1}{x+1} = x + \frac{1}{x+1} \rightarrow \frac{dy}{dx} = 1 - \frac{1}{(x+1)^2}$$

$$Let\ x_0 \in \mathbb{R}\ such\ that\ \frac{dy}{dx}\Big|_{x=x_0} = 0$$

$$\rightarrow 1 - \frac{1}{(x_0+1)^2} = 0 \rightarrow (x_0+1)^2 - 1 = 0$$

$$\rightarrow x_0(x_0+2) = 0 \rightarrow x_0 = 0\ or\ x_0 = -2$$

* 搵 turning point = 搵 x_0 使度 $y'(x_0)=0$

* $a^2 - b^2 = (a+b)(a-b)$

CONT'D

2012 – SECTION A

	x < -2	x = -2	-2 < x < 0	x = 0	x > 0
y′	+	0	-	0	+
y	Inc.		Dec.		Inc.

$\therefore$ *The local min . pt .* $= (0, 1)$

Vertical Asymptote : $x = -1$

Horizontal Asymptote : No Horizontal Asymptotes

Oblique Asymptote : $y = x$

*** 利用表格計算 turning point 附近上升定下降**

$f'(x) > 0 \rightarrow Increasing$

$f'(x) < 0 \rightarrow Decreasing$

*** x 係幾多, 分母係零**

$* \; Find \; \lim_{x \to \infty} y$

$* \; Find \; m \; and \; c \; such \; that \; \lim_{x \to \infty} [y - (mx + c)] = 0$

$$y = x + \frac{1}{x+1} \rightarrow y - x = \frac{1}{x+1}$$

$$\rightarrow \lim_{x \to \infty} (y - x) = 0$$

2012 - SECTION A

Q6.) In Figure 1, water is being poured into glass .
Let h be the depth of water inside the glass at time t s

a.) Show the volume V cm^3 of water at time t s

$$V = \frac{\pi}{300}(h^3 + 90h^2 + 2700h)$$

b.) V is increasing at $7\pi\ cm^3s^{-1}$, when $h = 5$
the rate of increasing of depth of water = ?

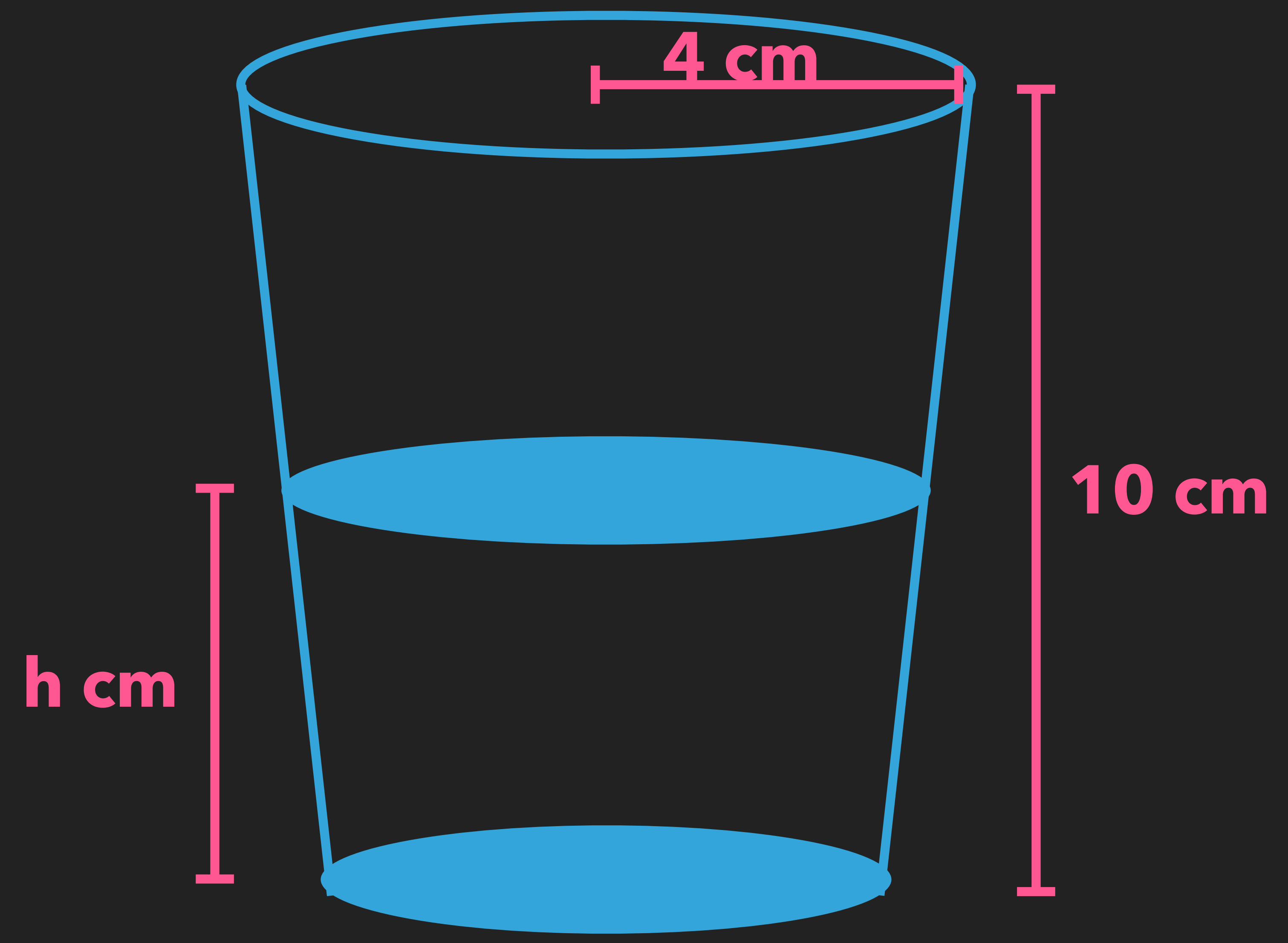


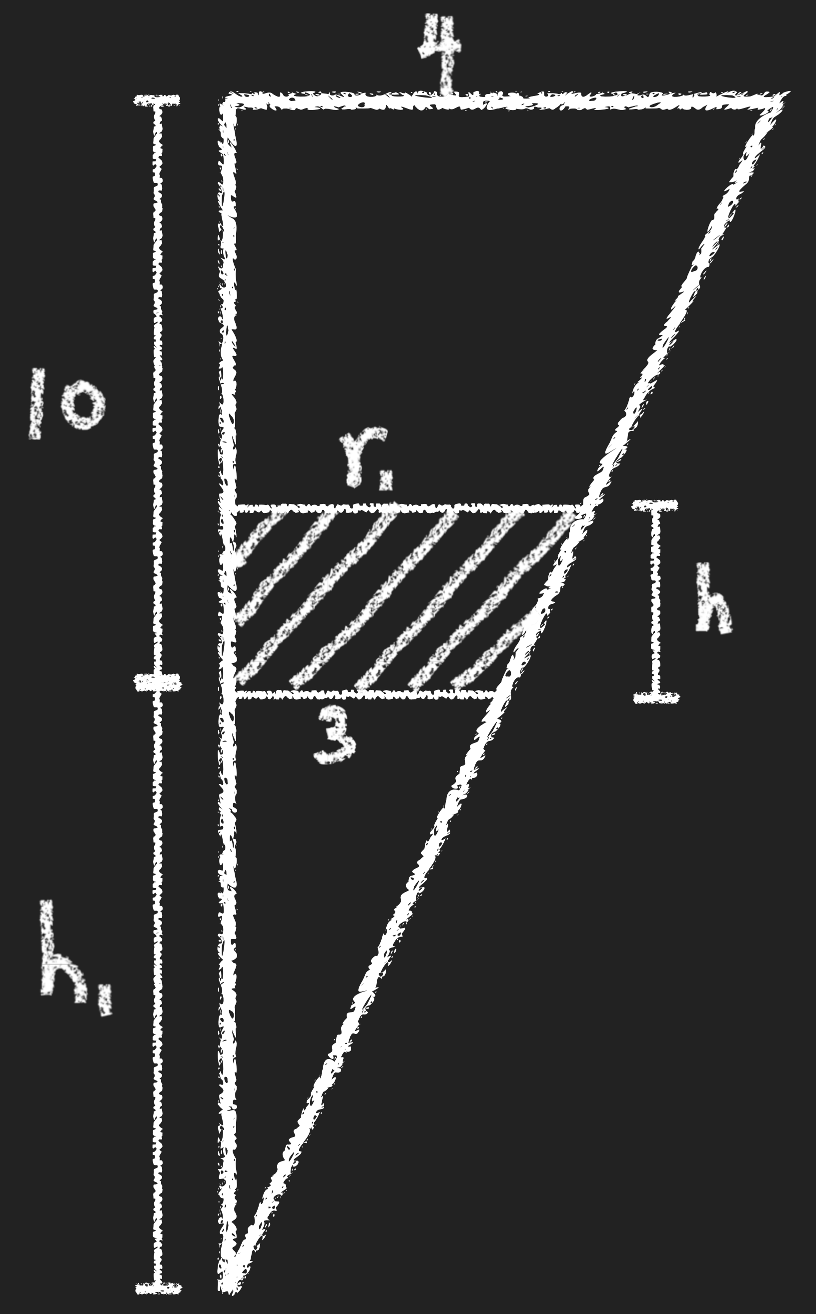
Figure 1

* 參考課程 3.3 及 3.4

CONT'D

2012 - SECTION A

$a.)$ *Consider the similar* Δ *in the graph :*



$$\frac{4}{10+h_1}=\frac{3}{h_1}\rightarrow h_1=30$$

$$\frac{r_1}{h+h_1}=\frac{3}{h_1}\rightarrow r_1=\frac{30+h}{10}$$

$$\therefore V=\frac{1}{3}\pi r_1^2(h_1+h)-\frac{1}{3}\pi(3)^2h_1$$

$$=\frac{1}{3}\pi(\frac{30+h}{10})^2(30+h)-\frac{1}{3}\pi(3)^2(30)$$

$$=\frac{\pi}{300}[(30+h)^3-27000]$$

$$=\frac{\pi}{300}(h^3+90h^2+2700h)$$

* 基本 Core 立體計算

* $a^3-b^3=(a-b)(a^2+ab+b^2)$

CONT'D

2012 - SECTION A

b.) Consider :

$$\frac{dV}{dt} = \frac{\pi}{300}\frac{d}{dt}(h^3 + 90h^2 + 2700h)$$

$$\rightarrow \frac{dV}{dt} = \frac{\pi}{300}\frac{d}{dh}(h^3 + 90h^2 + 2700h)\frac{dh}{dt}$$

$$\rightarrow \frac{dV}{dt} = \frac{\pi}{300}(3h^2 + 180h + 2700)\frac{dh}{dt}$$

$$\rightarrow \cancelto{7\pi}{\frac{dV}{dt}}\Big|_{h=5} = \frac{\pi}{300}(3(5)^2 + 180(5) + 2700)\frac{dh}{dt}\Big|_{h=5}$$

$$\rightarrow \frac{dh}{dt}\Big|_{h=5} = \frac{4}{7}$$

$$\therefore \textit{The rate of increase of depth } = \frac{4}{7}\ cm\ s^{-1}$$

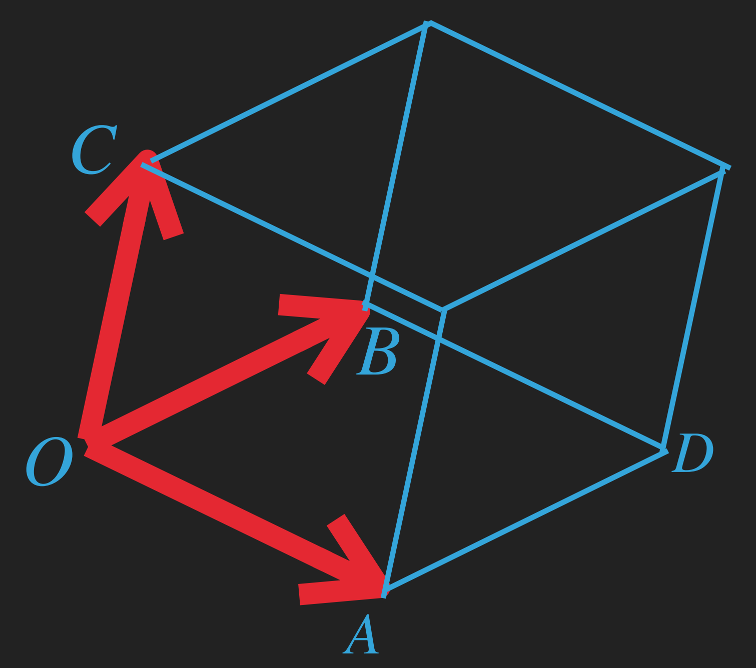
* Implicit 微分法

* Chain Rule

* 結論及單位

2012 - SECTION A

Q7.)



$$\overrightarrow{OA} = 6\hat{i} + 2\hat{j} - \hat{k}$$
$$\overrightarrow{OB} = 2\hat{i} + \hat{j}$$
$$\overrightarrow{OC} = 5\hat{i} - \hat{j} + 2\hat{k}$$

a.) *The area OADB = ?*

b.) *The distance between point C and the plane OADB = ?*

* 參考課程 3.3 及 3.4

$$a.)\ The\ area\ OADB = |\overrightarrow{OA}\ x\ \overrightarrow{OB}| = |\hat{i} - 2\hat{j} + 2\hat{k}|$$
$$= \sqrt{1^2 + 2^2 + 2^2} = 3\ sq.\ unit.$$

$$b.)\ The\ distance = \frac{(\overrightarrow{OA}\ x\ \overrightarrow{OB}) \cdot \overrightarrow{OC}}{The\ area\ OADB}$$
$$= \frac{(\hat{i} - 2\hat{j} + 2\hat{k}) \cdot (5\hat{i} - \hat{j} + 2\hat{k})}{3} = \frac{11}{3}\ unit.$$

$$* \vec{a}\ x\ \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$*(\vec{a}\ x\ \vec{b}) \cdot \vec{c}$ =平行六面體體積

2012 - SECTION A

Q8.) Solve

$$\begin{cases} x + y + z = 0 \\ 2x - y + 5z = 6 \\ x - y + \lambda z = 4 \end{cases}$$

* 參考課程 4.7

方法1

Consider :

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 5 & 6 \\ 1 & -1 & \lambda & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 3 & 6 \\ 0 & -2 & \lambda - 1 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & \boxed{3\lambda - 9} & 0 \end{array}\right)$$

* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \to \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

* 如果 ▇ = 0, 直線答案, 否則唯一答案

CONT'D

2012 - SECTION A

For $3\lambda - 9 = 0$,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 5 & 6 \\ 1 & -1 & \lambda & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

* 三條公式變兩條

Let $z = t,\ t \in \mathbb{R}$

$(x, y, z) = (2 - 2t,\ t - 2,\ t)$

For $3\lambda - 9 \neq 0$, $\rightarrow z = 0$

$(x, y, z) = (2,\ -2,\ 0)$

方法2

Solve

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 5 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 0 & 6 & 6 \end{array}\right)$$

* 先解頭兩條得直線答案

Let $z = t,\ t \in \mathbb{R} \rightarrow (x, y, z) = (2 - 2t,\ t - 2,\ t)$

CONT'D

2012 - SECTION A

$$Substitute\ (x, y, z) = (2 - 2t,\ t - 2,\ t)\ into\ \ x - y + \lambda z = 4$$

$$\rightarrow (2 - 2t) - (t - 2) + \lambda t = 4$$

$$\rightarrow (\lambda - 3)t = 0$$

$$For\ \lambda \neq 3, \rightarrow z = 0$$

$$(x, y, z) = (2,\ -2,\ 0)$$

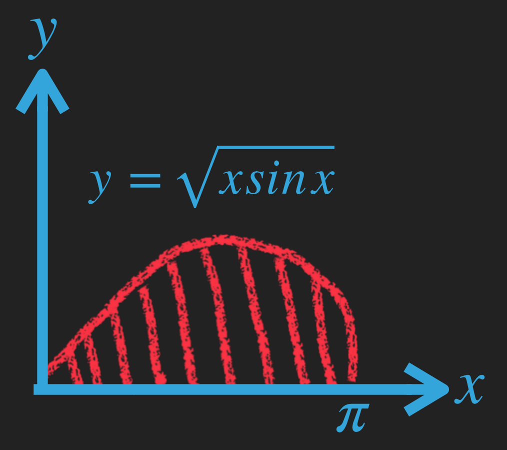
$$For\ \lambda = 3,$$

$$(x, y, z) = (2 - 2t,\ t - 2,\ t)$$

* 將直線答案代入第三式

2012 - SECTION A

Q9.) Find the volume of the solid generated by revolving the region along x − axis



* 參考課程 3.10 及 3.12

$$The\ volume = \pi\int_0^\pi y^2dx = \pi\int_0^\pi x\,sinxdx$$

$$= \pi\int_0^\pi x\,d(-cosx) = \pi[-xcosx]_0^\pi + \cancelto{0}{\int_0^\pi cosxdx}$$

$$= \pi^2\ cu.unit.$$

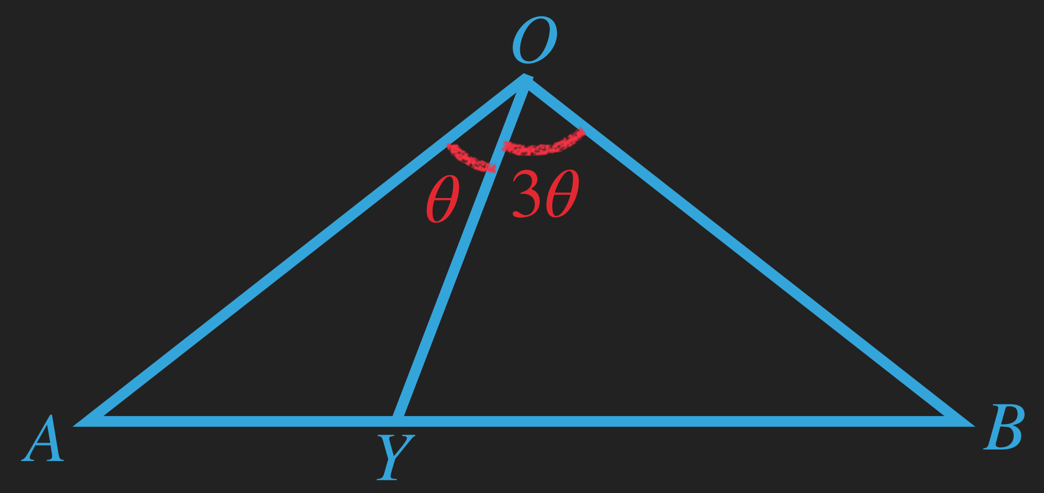
* 旋轉體積 (x-axis)= $\pi\int_a^b [f(x)]^2dx$

* 積分三寶: Integration By Part

* cosx $(0 \rightarrow \pi)$ 面積互相抵消

2012 – SECTION A

Q10.) In the following figure, $OA = OB$, $AB = 1$, $AY = y$



a.) Prove $y = \frac{1}{4}sec^2\theta$

b.) Find the range of y

* 參考課程 2.2 及 2.3

a.) Consider ΔOAY and ΔOBY, by sine law :

$$\frac{y}{sin\theta} = \frac{OA}{sin\angle OYA} \qquad \frac{1-y}{sin3\theta} = \frac{OB}{sin(\pi - \angle OYA)} = \frac{OA}{sin\angle OYA}$$

$$\therefore \frac{1-y}{sin3\theta} = \frac{y}{sin\theta} \rightarrow (1-y)sin\theta = ysin3\theta \rightarrow y(sin3\theta + sin\theta) = sin\theta$$

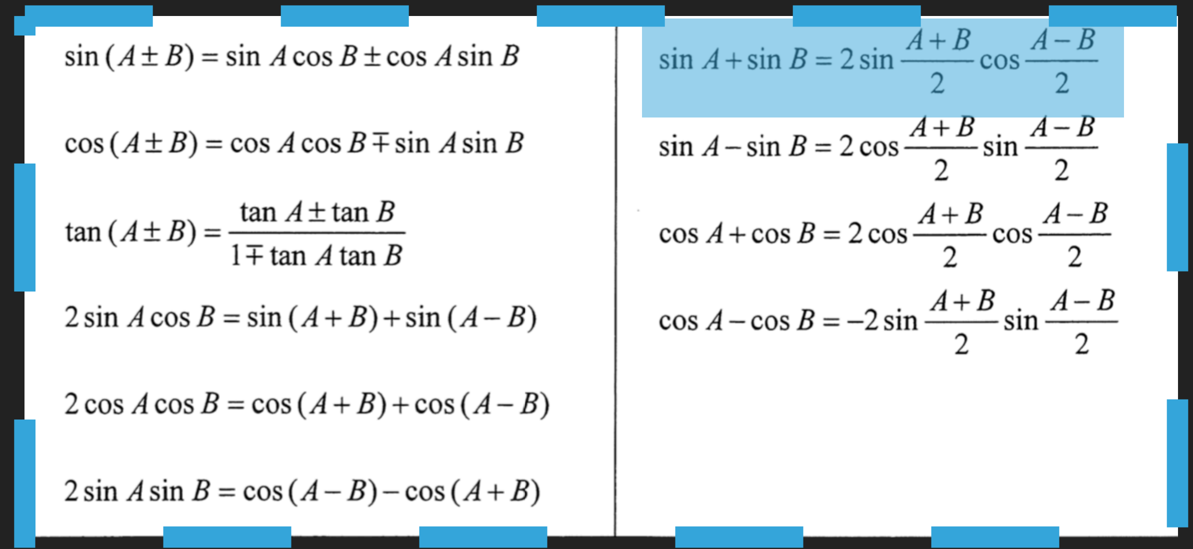
$$\rightarrow y(2sin2\theta cos\theta) = sin\theta \rightarrow 2y(2sin\theta cos^2\theta) = sin\theta$$

$$\rightarrow sin\theta(4ycos^2\theta - 1) = 0 \rightarrow y = \frac{1}{4}sec^2\theta$$

$(\because sin\theta \neq 0 \text{ for } 0 < 4\theta < \pi)$

* Sine law 應用

* Sum to product 及 sin 雙角公式



$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

CONT'D

2012 - SECTION A

$$b.)\ \because 0 < 4\theta < \pi \rightarrow 0 < \theta < \frac{\pi}{4}\ and\ cos\theta\ is\ decreasing\ for\ 0 < \theta < \frac{\pi}{4}$$

$$\therefore cos\frac{\pi}{4} < cos\theta < cos(0) \rightarrow \boxed{sec^2(0) < sec^2\theta < sec^2\frac{\pi}{4}}$$

$$\rightarrow \frac{1}{4}sec^2(0) < \frac{1}{4}sec^2\theta < \frac{1}{4}sec^2\frac{\pi}{4}$$

$$\rightarrow \frac{1}{4} < y < \frac{1}{2}$$

* 注意角範圍

* $\boxed{\ }$ $a > b \rightarrow \frac{1}{b} > \frac{1}{a}$

2012 – SECTION B

Q11.)

a.) *Solve* $\begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix} = 0 \quad -(*)$

b.) *Let* x_1 *and* x_2 *be the solution of* $(*)$ $(x_1 < x_2)$, *also;*

Let $P = \begin{pmatrix} a & c \\ b & 1 \end{pmatrix}$ *with* $|P| = 1$, *given that;*

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = x_1\begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = x_2\begin{pmatrix} c \\ 1 \end{pmatrix}$$

Find P *and* $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}^{12}$

* 參考課程 4.9, 4.10 及 4.11

CONT'D

2012 - SECTION B

$$a.)\ \begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix} = 0 \quad \rightarrow (1-x)(3-x)-8=0$$

$$\rightarrow x^2-4x-5=0$$

$$\rightarrow x=-1 \ or\ x=5$$

*** 2x2 Determinant 計算=差叉相減**

$$b.)\ x_1=-1 \ and\ x_2=5$$

* x_1, x_2 *are eigenvalue*

*** (a, b) 係對應 x_1 的 Eigenvector**

*** (c, 1) 係對應 x_2 的 Eigenvector**

To find a, b, solve;

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -1\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow a+2b=0$$

$$\rightarrow a=-2b$$

To find c solve;

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = 5\begin{pmatrix} c \\ 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow c=1$$

CONT'D

2012 - SECTION B

$$\therefore P = \begin{pmatrix} -2b & 1 \\ b & 1 \end{pmatrix}, \text{ given that } |P| = 1 \rightarrow b = -\frac{1}{3}$$

$$i.e.\ P = \begin{pmatrix} \frac{2}{3} & 1 \\ -\frac{1}{3} & 1 \end{pmatrix} \rightarrow \boxed{P^{-1} = \begin{pmatrix} 1 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}}$$

$$\text{Then, let } A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \text{ we have;}$$

$$AP = P\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \rightarrow P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

* 用 Row Deduction 或 Adj. Matrix

CONT'D

2012 - SECTION B

$$\to P^{-1}A^{12}P = \begin{pmatrix} 1 & 0 \\ 0 & 5^{12} \end{pmatrix} \to A^{12} = P\begin{pmatrix} 1 & 0 \\ 0 & 5^{12} \end{pmatrix}P^{-1}$$

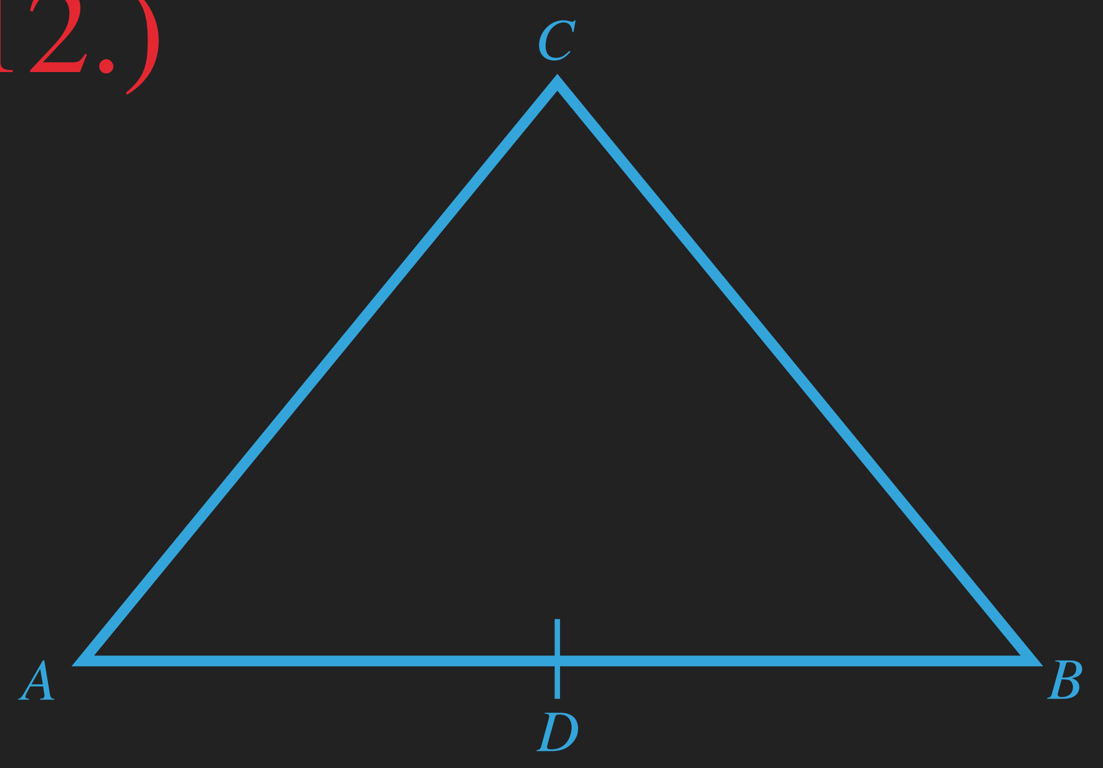
$$\to A^{12} = \begin{pmatrix} \frac{2}{3} & 1 \\ -\frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5^{12} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$\to A^{12} = \frac{1}{3}\begin{pmatrix} 5^{12}+2 & 2\cdot 5^{12}-2 \\ 5^{12}-1 & 2\cdot 5+1^{12} \end{pmatrix}$$

$* \ (P^{-1}AP)^n = P^{-1}A^nP$

2012 – SECTION B

Q12.)



$AD = DB$, Let O is centroid and G is circumcenter of ΔABC

$\vec{a} = \overrightarrow{OA} \quad \vec{b} = \overrightarrow{OB} \quad \vec{c} = \overrightarrow{OC}$

a.) Find $\overrightarrow{AG}$ in term of $\vec{a}$, $\vec{b}$, and $\vec{c}$

b.) Assume CE is the altitude of ΔABC. Extend OG meet CE at F

Find $FG : GO$ and prove F is the orthocenter of ΔABC

* 參考課程 4.3

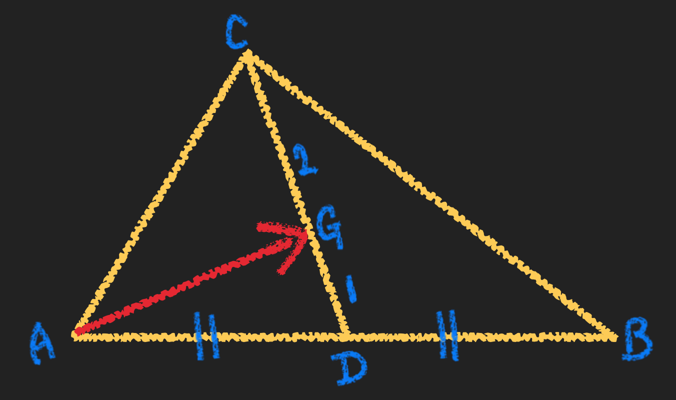
CONT'D

2012 - SECTION B

a.) $\because G$ is centroid $\rightarrow CG : GD = 2 : 1$

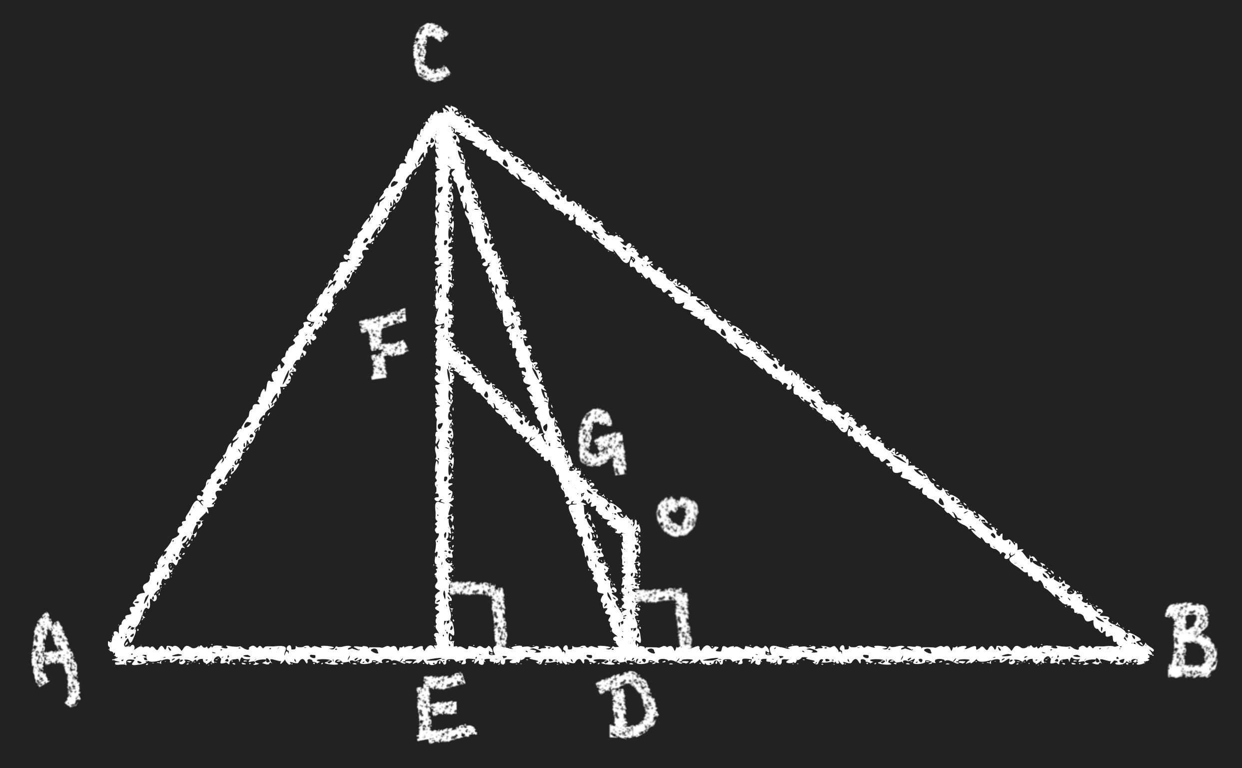
$$\therefore \overrightarrow{AG} = \frac{2\overrightarrow{AD} + \overrightarrow{AC}}{3} = \frac{2(\overrightarrow{OD} - \overrightarrow{OA}) + (\overrightarrow{OC} - \overrightarrow{OA})}{3}$$

$$= \frac{2(\frac{\vec{b}}{2} - \vec{a}) + (\vec{c} - \vec{a})}{3} = \frac{\vec{b} + \vec{c} - 2\vec{a}}{3}$$



* 分割定理

b.) Consider the following graph :



$\angle CGF = \angle OGD$ (vert. opp. $\angle$)

$\angle FCG = 90^o - \angle CDE$ ($\angle$s sum of ΔCED)

$\angle ODG = 90^o - \angle CDE = \angle FCG$

$\angle GFC = \angle DOG$ ($\angle$s sum of Δ)

$\therefore \Delta CFG \sim \Delta DOG$ (AAA)

i.e. $FG : GO = CG : GD = 2 : 1$

* Core 相似三角形証明及特性

CONT'D

2012 - SECTION B

$$\overrightarrow{AG} = \frac{2\overrightarrow{AO} + \overrightarrow{AF}}{3} \rightarrow \frac{\vec{b} + \vec{c} - 2\vec{a}}{3} = \frac{\overrightarrow{AF} - 2\vec{a}}{3}$$

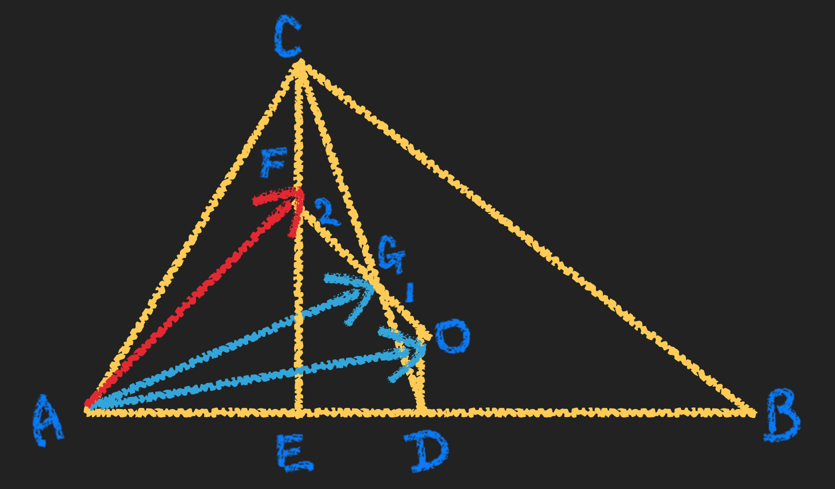
$$\rightarrow \overrightarrow{AF} = \vec{b} + \vec{c}$$

Given that F lies on CE and CE ⊥ AB

$$Also\ \overrightarrow{AF} \cdot \overrightarrow{BC} = (\vec{b} + \vec{c}) \cdot (\vec{c} - \vec{b}) = |\vec{c}|^2 - |\vec{b}|^2$$
$$= 0$$

$\therefore AF \perp BC$

i.e. F is orthocenter of ΔABC



*** 分割定理**

* $\vec{b} \cdot \vec{b} = |\vec{b}|^2$

*** O 係 circumcenter, OB=OC=半徑**

* $\vec{a} \cdot \vec{b} = 0 \rightarrow \vec{a} \perp \vec{b}$

2012 – SECTION B

Q13.)

$$a.)\ For\ \ -\frac{\pi}{2} < u < \frac{\pi}{2}\ and\ \ -\frac{\pi}{2} < v < \frac{\pi}{2},$$

$$tanu = \frac{-1 + cos\frac{2\pi}{5}}{sin\frac{2\pi}{5}} \qquad tanv = \frac{1 + cos\frac{2\pi}{5}}{sin\frac{2\pi}{5}}$$

$$Show\ u = -\frac{\pi}{5}\ and\ v = ?$$

$$b.)\int_{-1}^{1}\frac{sin\frac{2\pi}{5}}{x^2 + 2xcos\frac{2\pi}{5} + 1}dx = ?\ and\ \int_{-1}^{1}\frac{sin\frac{7\pi}{5}}{x^2 + 2xcos\frac{7\pi}{5} + 1}dx = ?$$

* 參考課程 2.2, 3.8, 3.10 及 3.11

CONT'D

2012 - SECTION B

方法1

$$a.)\ tanu = \frac{-1+(1-2sin^2\frac{\pi}{5})}{2cos\frac{\pi}{5}sin\frac{\pi}{5}}$$

$$\rightarrow tanu = -tan\frac{\pi}{5}$$

$$\rightarrow tanu = tan(-\frac{\pi}{5})$$

$$\rightarrow u = -\frac{\pi}{5}$$

$$tanv = \frac{1+(2cos^2\frac{\pi}{5}-1)}{2cos\frac{\pi}{5}sin\frac{\pi}{5}}$$

$$\rightarrow tanv = cot\frac{\pi}{5}$$

$$\rightarrow tanv = tan(\frac{\pi}{2}-\frac{\pi}{5})$$

$$\rightarrow v = \frac{3\pi}{10}$$

方法2

$$a.)\ Consider,\ tan(-\frac{\pi}{5}) = \frac{-1+cos\frac{2\pi}{5}}{sin\frac{2\pi}{5}}$$

*** sin 及 cos 雙角公式**

* $tan(\frac{\pi}{2}-\theta) = \frac{1}{tan\theta} = cot\theta$

* $tan(-\theta) = -tan\theta$

*** 雙向推論**

CONT'D

2012 - SECTION B

$$\Leftrightarrow sin(-\frac{\pi}{5})sin\frac{2\pi}{5} = -cos(-\frac{\pi}{5}) + cos(-\frac{\pi}{5})cos\frac{2\pi}{5}$$

$$\Leftrightarrow \frac{1}{2}(cos(-\frac{3\pi}{5}) - cos\frac{\pi}{5}) = -cos\frac{\pi}{5} + \frac{1}{2}(cos\frac{\pi}{5} + cos(-\frac{3\pi}{5}))$$

$$\Leftrightarrow \frac{1}{2}(cos(-\frac{3\pi}{5}) - cos\frac{\pi}{5}) = \frac{1}{2}(cos(-\frac{3\pi}{5}) - cos\frac{\pi}{5})$$

Prove is complete, $\therefore u = -\frac{\pi}{5}$

* **Product to sum**

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

* $cos(-\theta) = cos\theta$

CONT'D

2012 - SECTION B

$$\tan v = \frac{1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}} \rightarrow \tan v = \frac{2+(-1+\cos\frac{2\pi}{5})}{\sin\frac{2\pi}{5}}$$

$$\rightarrow \tan v = \frac{2}{\sin\frac{2\pi}{5}} + \tan u \rightarrow \tan v = \frac{2}{2\sin\frac{\pi}{5}\cos\frac{\pi}{5}} - \tan\frac{\pi}{5}$$

$$\rightarrow \tan v = \frac{1}{\sin\frac{\pi}{5}\cos\frac{\pi}{5}} - \frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}} \rightarrow \tan v = \frac{1-\sin^2\frac{\pi}{5}}{\sin\frac{\pi}{5}\cos\frac{\pi}{5}} = \cot\frac{\pi}{5}$$

$$\rightarrow \tan v = \tan(\frac{\pi}{2}-\frac{\pi}{5}) \rightarrow v = \frac{3\pi}{10}$$

* **sin 雙角公式**
* $\tan(-\theta) = -\tan\theta$
* $\tan(\frac{\pi}{2}-\theta) = \frac{1}{\tan\theta} = \cot\theta$

CONT'D

2012 - SECTION B

$$b.)\ Let\ a = cos\frac{2\pi}{5},\ and\ b = sin\frac{2\pi}{5},\ then\ a^2 + b^2 = 1$$

$$Let\ I = \int_{-1}^{1} \frac{sin\frac{2\pi}{5}}{x^2 + 2xcos\frac{2\pi}{5} + 1}dx = \int_{-1}^{1} \frac{b}{x^2 + 2ax + 1}dx$$

$$= \int_{-1}^{1} \frac{b}{\boxed{x^2 + 2ax + a^2} + b^2}dx = \int_{-1}^{1} \frac{b}{\boxed{(x+a)^2} + b^2}dx$$

$$Let\ x + a = btan\theta \to dx = bsec^2\theta d\theta$$

$$I = \int_{A}^{B} \frac{b^2sec^2\theta}{\boxed{b^2tan^2\theta + b^2}}d\theta = \int_{A}^{B} \frac{sec^2\theta}{\boxed{sec^2\theta}}d\theta = [\theta]_A^B,\ where$$

$$A = tan^{-1}(\frac{-1+a}{b}),\ B = tan^{-1}(\frac{1+a}{b})$$

$*$ $(a+b)^2 = a^2 + 2ab + b^2$

$*$ $\int \frac{dx}{x^2 + Bx + C} \to \int \frac{dx}{(x+h)^2 + k}$

利用三角代入, $x + h = \sqrt{k}tan\theta$

$*$ $tan^2\theta + 1 = sec^2\theta$

* 定積分代入要改範圍

CONT'D

2012 - SECTION B

$$A = -\frac{\pi}{5}, B = \frac{3\pi}{10}, (from\ a.)\ result)$$

$$\therefore I = B - A = \frac{\pi}{2}$$

方法1

$$Let\ J = \int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2 + 2x\cos\frac{7\pi}{5} + 1} dx = \int_{-1}^{1} \frac{\sin(\pi + \frac{2\pi}{5})}{x^2 + 2x\cos(\pi + \frac{2\pi}{5}) + 1} dx$$

$$= \int_{-1}^{1} \frac{-\sin\frac{2\pi}{5}}{x^2 - 2x\cos\frac{2\pi}{5} + 1} dx = \int_{1}^{-1} \frac{\sin\frac{2\pi}{5}}{u^2 + 2u\cos\frac{2\pi}{5} + 1} du = -I$$

$$\therefore J = -\frac{\pi}{2}\ (from\ above\ result)$$

* $\sin(\pi + \theta) = -\sin\theta$

* $\cos(\pi + \theta) = -\cos\theta$

* 代 u=-x

* 定積分代入要改範圍

* $\int_a^b f(x)dx = -\int_b^a f(x)dx$

CONT'D

2012 - SECTION B

方法2

$$Similarly,\ Let\ a = cos\frac{7\pi}{5},\ and\ b = sin\frac{7\pi}{5},\ then\ a^2 + b^2 = 1$$

$$Let\ J = \int_{-1}^{1} \frac{sin\frac{7\pi}{5}}{x^2 + 2xcos\frac{7\pi}{5} + 1}dx = B' - A',\ where$$

$$A' = \frac{-1 + cos\frac{7\pi}{5}}{sin\frac{7\pi}{5}} = \frac{-1 + cos(\pi + \frac{2\pi}{5})}{sin(\pi + \frac{2\pi}{5})} = \frac{-1 - cos\frac{2\pi}{5}}{-sin\frac{2\pi}{5}} = B$$

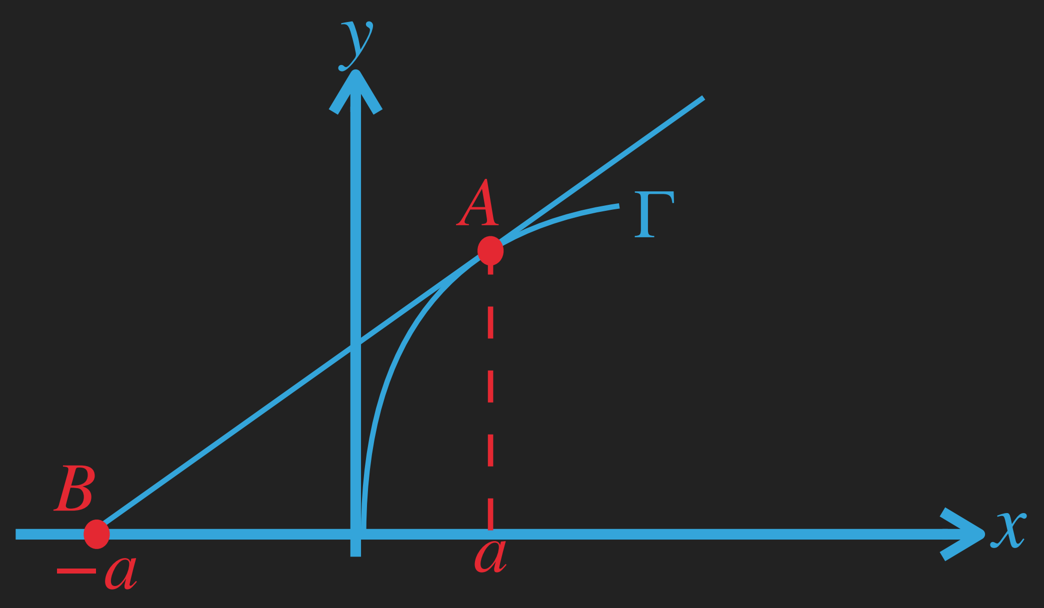
$$B' = \frac{1 + cos\frac{7\pi}{5}}{sin\frac{7\pi}{5}} = \frac{1 + cos(\pi + \frac{2\pi}{5})}{sin(\pi + \frac{2\pi}{5})} = \frac{1 - cos\frac{2\pi}{5}}{-sin\frac{2\pi}{5}} = A$$

$$\therefore J = A - B = -\frac{\pi}{2}$$

* $sin(\pi + \theta) = -sin\theta$
* $cos(\pi + \theta) = -cos\theta$

2012 – SECTION B

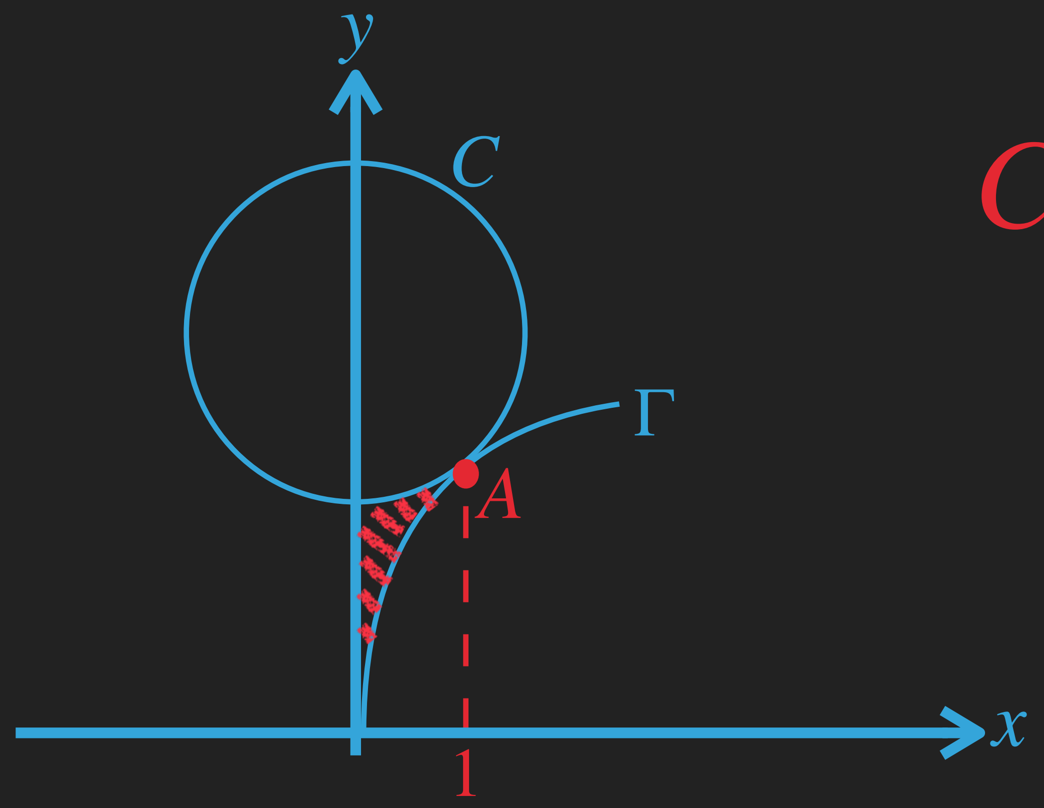
*Q*14.)



$\Gamma : y = kx^p,\ k > 0$ *and* $p > 0$

In the figure, $a > 0$

a.) $p = ?$



C is a circle with radius $= 2$*, center at* $y-$*axis, and touch* Γ *at A*

b.) $k = ?$ *and the area of the shaded region* $= ?$

* 參考課程 3.4, 3.8 及 3.11

CONT'D

2012 – SECTION B

a.) Consider the slope of AB, we have

$$\frac{d(kx^p)}{dx}\Big|_{x=a} = \frac{ka^p - 0}{a-(-a)} \to kpa^{p-1} = \frac{ka^p}{2a}$$

$$\to 2pa^p = a^p \to a^p(2p-1) = 0 \to p = \frac{1}{2}$$

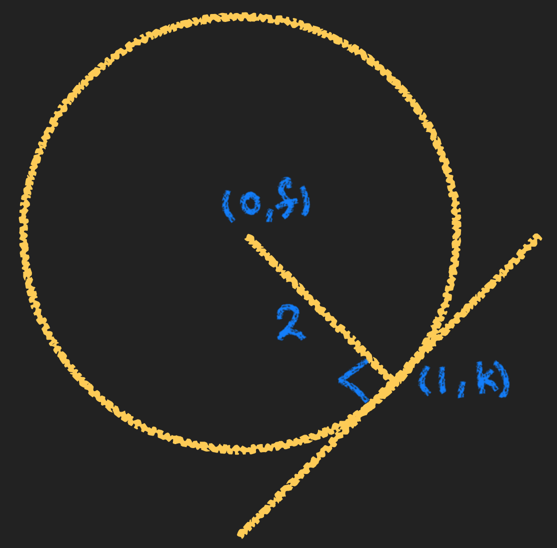
$$\frac{d(kx^p)}{dx}\Big|_{x=a} = \frac{ka^p - 0}{a-(-a)} \to kpa^{p-1} = \frac{ka^p}{2a}$$

b.) Let $C:\ x^2+(y-f)^2 = 4$, *we have*

$$\begin{cases} \dfrac{k-f}{1-0}\cdot\dfrac{k-0}{1-(-1)} = -1 \to k(k-f) = -2 & \text{—————— } \mathbf{(1)} \\ 1+(k-f)^2 = 4 \to (k-f)^2 = 3 & \text{—————— } \mathbf{(2)} \end{cases}$$

* 微分計緊 Slope

* 圓 tangent 垂直圓心



CONT'D

2012 - SECTION B

$$b.)\ In\ (1):\ k^2(k-f)^2=4 \to 3k^2=4$$

$$k=\frac{2\sqrt{3}}{3}\ or\ k=-\frac{2\sqrt{3}}{3}\ (rejected\ for\ k>0)$$

$$Then,\ k(k-f)=-2 \to f=\frac{5\sqrt{3}}{3}$$

方法1

$$C:\ x^2+(y-f)^2=4 \to y=-\sqrt{4-x^2}+f$$

$$The\ shaded\ area=\int_0^1(-\sqrt{4-x^2}+f)dx-\int_0^1 kx^{\frac{1}{2}}dx$$

$$=-\int_0^{\frac{\pi}{6}}\sqrt{4-4sin^2\theta}\cdot 2cos\theta d\theta+[fx-\frac{2k}{3}x^{\frac{3}{2}}]_0^1$$

* 圓心下方取負

* 面積大減細

* 利用三角代入, $x=2sin\theta$

* 定積分代入要改範圍

CONT'D

2012 - SECTION B

$$= -4\int_0^{\frac{\pi}{6}} \cos^2\theta d\theta + \frac{11\sqrt{3}}{9} = -4\int_0^{\frac{\pi}{6}} \frac{1}{2}(\cos 2\theta + 1) d\theta + \frac{11\sqrt{3}}{9}$$

$$= -\int_0^{\frac{\pi}{3}} (\cos t + 1) dt + \frac{11\sqrt{3}}{9}, \text{ where } t = 2\theta$$

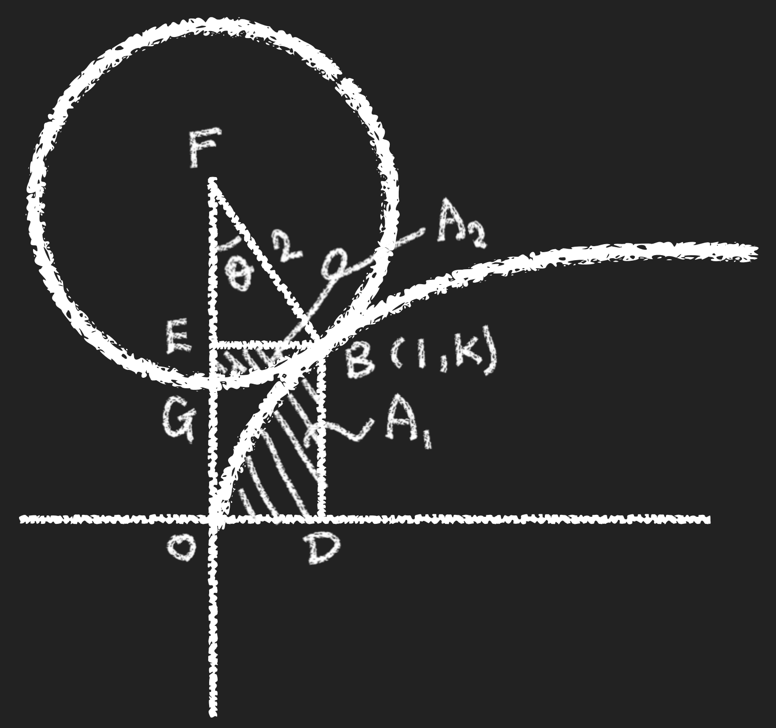
$$= -[\sin t + t]_0^{\frac{\pi}{3}} + \frac{11\sqrt{3}}{9} = \left(\frac{13\sqrt{3}}{18} - \frac{\pi}{3}\right) sq.unit$$

* cos 雙角公式

* 定積分代入要改範圍

方法2

Cosider the graph,



$$EF = f - k, \quad \theta = \sin^{-1}\frac{EB}{FB} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$A_2 = \text{Sector GFB Area} - \Delta EFB \text{ Area}$$

$$= \frac{1}{2}(2)^2\theta - \frac{EFxEB}{2} = \frac{\pi}{3} - \frac{(1)(f-k)}{2} = \frac{\pi}{3} - \frac{3\sqrt{3}}{6}$$

* 利用基本幾何面積

* 扇形面積= $\frac{r^2\theta}{2}$

CONT'D

2012 – SECTION B

$$The\ shaded\ area = Area\ EBOD - A_1 - A_2 = (1)(k) - \int_0^1 kx^{\frac{1}{2}}dx - (\frac{\pi}{3} - \frac{3\sqrt{3}}{6})$$

$$= k - [\frac{2k}{3}x^{\frac{3}{2}}]_0^1 - \frac{\pi}{3} + \frac{3\sqrt{3}}{6}$$

$$= \frac{2\sqrt{3}}{3} + \frac{3\sqrt{3}}{6} - \frac{4\sqrt{3}}{9} - \frac{\pi}{3}$$

$$= (\frac{13\sqrt{3}}{18} - \frac{\pi}{3})\ sq\,.unit$$

* 面積大減細