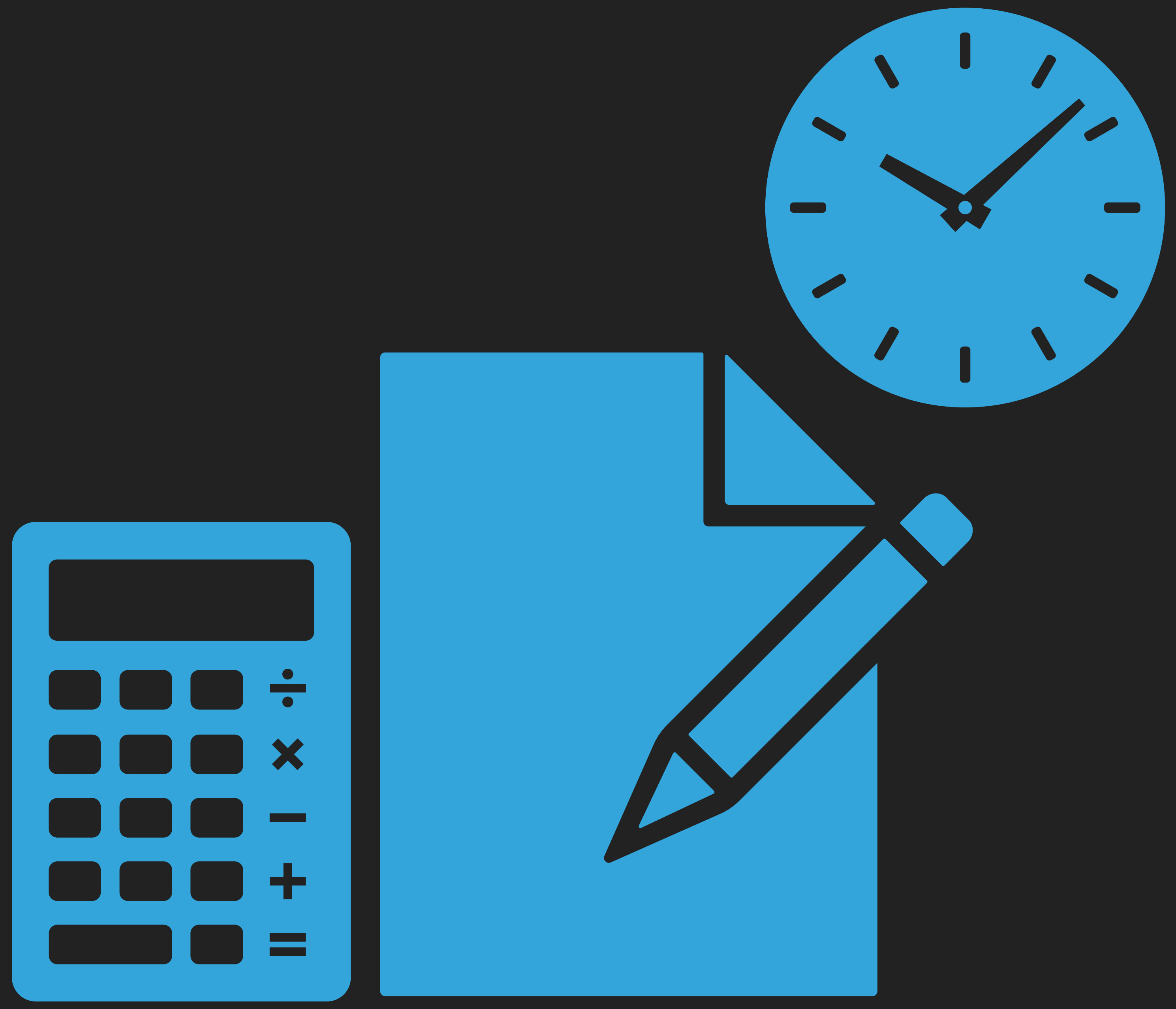
深宵教室 - DSE M2 模擬試題解答

2014

2014

- Section A
- Section B



2014 - SECTION A

$Q1.)\ (1-4x)^2(1+x)^n = 1 + x + Ax^2 + \ldots,\ \ n = ?\ \ and\ \ A = ?$

* 參考課程 1.1

$$(1-4x)^2(1+x)^n \equiv (1-8x+16x^2)\left(\sum_{r=0}^{n} C_r^n x^r\right)$$

By compare coefficient of x and x^2

$$\begin{cases} 1 = C_1^n(1) - (8) = n - 8 & \text{———— (1)} \\ A = C_2^n - 8C_1^n + 16 & \text{———— (2)} \end{cases}$$

$$\therefore n = 9\ and\ A = C_2^9 - 8C_1^9 + 16 = -20$$

*** Binomial Expansion**

$$* \ C_r^n = \frac{n!}{r!(n-r)!} \rightarrow C_1^n = n\ and\ C_2^n = \frac{n(n-1)}{2}$$

2014 - SECTION A

Q2.) $f(x) = x^3 - 3x$. $f'(x) = ?$ (By First Principles), When will $f(x)$ be decreasing?

*** 參考課程 1.1, 3.1, 3.2 及 3.4**

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left((x+h)^3 - x^3 - 3(x+h) + 3x\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(h((x+h)^2 + (x+h)x + x^2) - 3h\right) \\
&= \lim_{h\to 0} \left((x+h)^2 + x(x+h) + x^2 - 3\right) = 3x^2 - 3
\end{aligned}
$$

When $-1 < x < 1, f'(x) < 0$

$\therefore f(x)$ is decreasing when $-1 < x < 1$,

*** 微分定義**

* 🟦 $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

* $f'(x) > 0 \rightarrow$ Increasing
$f'(x) < 0 \rightarrow$ Decreasing

2014 - SECTION A

Q3.) Find the equation of tangent of $xlny + y = 2$ *at the* $y-$*interception .*

* 參考課程 3.2 及 3.4

$$The\ y - interception = 2$$

$$Consider : xlny + y = 2 \rightarrow \frac{d}{dx}(xlny + y) = 0$$

$$\rightarrow \left(lny + \frac{x}{y}\frac{dy}{dx}\right) + \frac{dy}{dx} = 0$$

$$\rightarrow \left(ln2 + \frac{0}{2}\frac{dy}{dx}\Big|_{(x,y)=(0,2)}\right) + \frac{dy}{dx}\Big|_{(x,y)=(0,2)} = 0$$

$$\rightarrow \frac{dy}{dx}\Big|_{(x,y)=(0,2)} = -\,ln2$$

$\therefore$ *the equation of tangent :* $y = -(ln2)x + 2$

* Implicit 微分法

* 🟦 Product Rule + Chain Rule

* y=mx+c

2014 - SECTION A

$$Q4.)\ if\ x = 2y + siny, \frac{d^2y}{dx^2} = ?$$

* 參考課程 3.2 及 3.3

方法1

$$\frac{dx}{dy} = 2 + cosy \rightarrow \frac{dy}{dx} = \frac{1}{2 + cosy}$$

$$\rightarrow \frac{dy^2}{dx^2} = \left(\frac{-1}{(2 + cosy)^2}\right)(-siny)\frac{dy}{dx}$$

$$\rightarrow \frac{dy^2}{dx^2} = \frac{siny}{(2 + cosy)^3}$$

方法2

$$1 = 2\frac{dy}{dx} + cosy\frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{1}{(2 + cosy)} \rightarrow \frac{dy^2}{dx^2} = \frac{siny}{(2 + cosy)^3}$$

* 先計 $\frac{dx}{dy}$ 然後用 $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

* 🟦 用 Chain rule

* Implicit 微分法

2014 - SECTION A

Q5.) $\int \frac{1}{\sqrt{9-x}}dx = ?$ and $\int \frac{1}{\sqrt{9-x^2}}dx = ?$, for $-3 < x < 3$

* 參考課程 3.6 及 3.8

$$\int \frac{dx}{\sqrt{9-x}} = \int \frac{-d(9-x)}{\sqrt{9-x}} = -2\sqrt{9-x} + C, \text{ where } C \text{ is constant}$$

$$\text{Let } I = \int \frac{dx}{\sqrt{9-x^2}} \text{ and } x = 3sin\theta \ (-3 < x < 3 \rightarrow -1 < sin\theta < 1)$$

$$I = \int \frac{3cos\theta d\theta}{\sqrt{9-9sin^2\theta}} = \int \frac{3cos\theta}{3cos\theta}d\theta = \theta + C$$

$$= sin^{-1}\frac{x}{3} + C, \text{ where } C \text{ is constant}$$

* 🟦 可代入

* 用三角代入法, **Let** $x = 3sin\theta$

* 🟥 利用 $cos^2\theta = 1 - sin^2\theta$

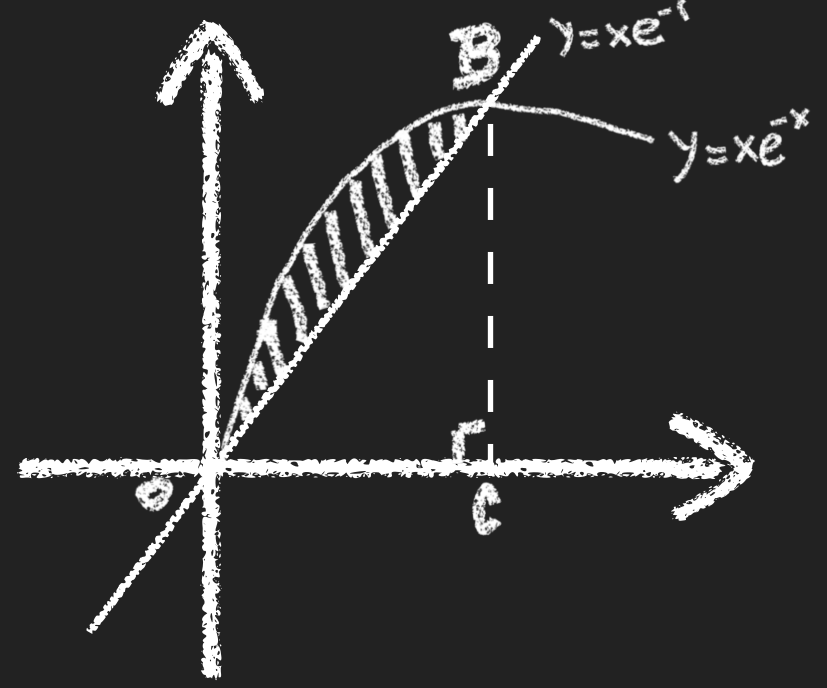
2014 - SECTION A

Q6.) *Find the area of region bounded by* $C_1 : y = xe^{-x}$ *and* $C_2 : y = \frac{x}{e}$

* 參考課程 3.10 及 3.11

The interception of C_1 *and* C_2 *are* $A = (0, 0)$ *and* $B = (1, 1)$

From the figure :



$$\begin{aligned}
\text{The bounded area} &= \int_0^1 xe^{-x}dx - \Delta OBC\ \text{Area} \\
&= \int_0^1 xd(-e^{-x}) - \frac{1}{2}(1)(e^{-1}) \\
&= [-xe^{-x}]_0^1 + \int_0^1 e^{-x}d(x) - \frac{e^{-1}}{2} \\
&= [-e^{-x}(x+1)]_0^1 - \frac{e^{-1}}{2} = 1 - \frac{5e^{-1}}{2}\ sq.\ unit
\end{aligned}$$

* 利用基本幾何面積計算

* 面積大減細

* 🟦 積分三寶: Integration by part

2014 - SECTION A

Q7.)

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ Prove } A^{n+1} = 2^n A$$

$$\text{Hence, } A^2 = 2A \rightarrow A^2A^{-1} = 2AA^{-1} \rightarrow A = 2I, \text{ Where is the problem?}$$

* 參考課程 **1.1, 1.2, 4.8 及 4.10**

方法1

$$\text{Let } B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow A = I + B, \ B^{2k} = I, B^{2k+1} = B, k \in \mathbb{Z}^+$$

$$A^{n+1} = (I+B)^{n+1} = C_0^{n+1} I + \sum_{r=1}^{n+1} C_r^{n+1} B^r$$

* 🟦 $B^2 = I, B^3 = B^2B = B$

CONT'D

2014 - SECTION A

$$Consider,\ (1+x)^{n+1} \equiv \sum_{r=0}^{n+1} C_r^{n+1} x^r$$

$$For\ x=1,\ S_1 = \sum_{r=0}^{n+1} C_r^{n+1} = 2^{n+1},\quad For\ x=-1,\ S_2 = \sum_{r=0}^{n+1} C_r^{n+1}(-1)^r = 0$$

$$A^{n+1} = (C_0^{n+1} + C_2^{n+1} + \ldots)I + (C_1^{n+1} + C_3^{n+1} + \ldots)B$$

$$= \frac{1}{2}(S_1+S_2)I + \frac{1}{2}(S_1-S_2)B = 2^n(I+B) = 2^n A$$

* Binomial Expansion

* 🟦 指數雙數 $B^{2k}=I$

* 🟥 指數單數 $B^{2k+1}=B$

方法2

$$Let\ P(n):\ A^{n+1} = 2^n A,\ \forall n \in \mathbb{Z}^+ and\ 0$$

$$For\ P(0):\ L.H.S. = A = R.H.S.$$

$$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+,\ then\ P(k+1):$$

* 先 Let Statement

* 証明 P(0) is true

CONT'D

2014 - SECTION A

$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+\ and\ 0\ then\ P(k+1):$

* 假設 P(k) is true. 証明 P(k+1) is true

$$L.H.S. = A^{k+2} = A^{k+1}A = 2^kA^2$$

$$= 2^k\begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 1\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{pmatrix} = 2^k\begin{pmatrix}2 & 0 & 2\\ 0 & 4 & 0\\ 2 & 0 & 2\end{pmatrix}$$

$$= 2^{k+1}A = R.H.S.$$

$\therefore P(k+1)\ is\ true\ if\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+\ and\ 0$

$i.e.\ By\ M.I.,\ P(n)\ is\ true,\ \forall n \in \mathbb{Z}^+\ and\ 0$

* 寫結論

$\because |A| = 0 \rightarrow A^{-1}\ does\ not\ exist$

$i.e.\ the\ statement\ is\ incorrect$

2014 - SECTION A

Q8.) $\overrightarrow{OA} = -\hat{i} + 2\hat{j} + 2\hat{k}, \overrightarrow{OB} = \hat{i} - \hat{j} + 2\hat{k}, \overrightarrow{OC} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

The volume of the tetrahedron OABC = ?, the acute angle between OAB and OC = ?

* 參考課程 4.4 及 4.5

$$\overrightarrow{OA}\ x\ \overrightarrow{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 6\hat{i} + 4\hat{j} - \hat{k}$$

$$The\ volume = \left|\frac{1}{6}(\overrightarrow{OA}\ x\ \overrightarrow{OB}) \cdot \overrightarrow{OC}\right| = \frac{1}{6}(6\hat{i} + 4\hat{j} - \hat{k}) \cdot (2\hat{i} - 3\hat{j} + 6\hat{k})$$

$$= 1\ cu\ .\ unit$$

$$Let\ the\ angle\ be\ \theta,\ sin\theta = \frac{\boxed{The\ height\ of\ tetrahedron}}{|\overrightarrow{OC}|}$$

$$= \boxed{\frac{6}{|\overrightarrow{OA}x\overrightarrow{OB}|}}\ \frac{1}{|\overrightarrow{OC}|} \rightarrow \theta = sin^{-1}\left(\frac{6\sqrt{53}}{371}\right)$$

* 3x3 Determinant 用差叉相減

* 四面體體積=1/6 x 平行六面體體積

* 🟦 $height = \dfrac{3\ x\ Volume}{Area = \frac{1}{2}|\overrightarrow{OA}x\overrightarrow{OB}|}$

2014 – SECTION A

Q9.) Mary has \$100 to buy three types of pen (A, B, C) with \$0.5, \$3 and \$5 respectively
She spend exactly \$100 to buy total number 100 pens with m A pen, n B pen and k C pen
Is there is only one combination of (m, n, k)? Please explain .

* 參考課程 1.2 及 4.7

Consider,

$$(E):\left(\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 0.5 & 3 & 5 & 100 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 0 & 5 & 9 & 100 \end{array}\right)$$

$$(m, n, k) = (\frac{4}{5}(100 + t), \frac{1}{5}(100 - 9t), t), t \in \mathbb{R}, \textit{and } m, n, k \in \mathbb{Z}^{+}$$

$$\textit{For } t = 0, (m, n, k) = (80, 20, 0)$$
$$\textit{For } t = 10, (m, n, k) = (88, 2, 10)$$

$\therefore$ *There is not only one combination .*

* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \end{array}\right)$$

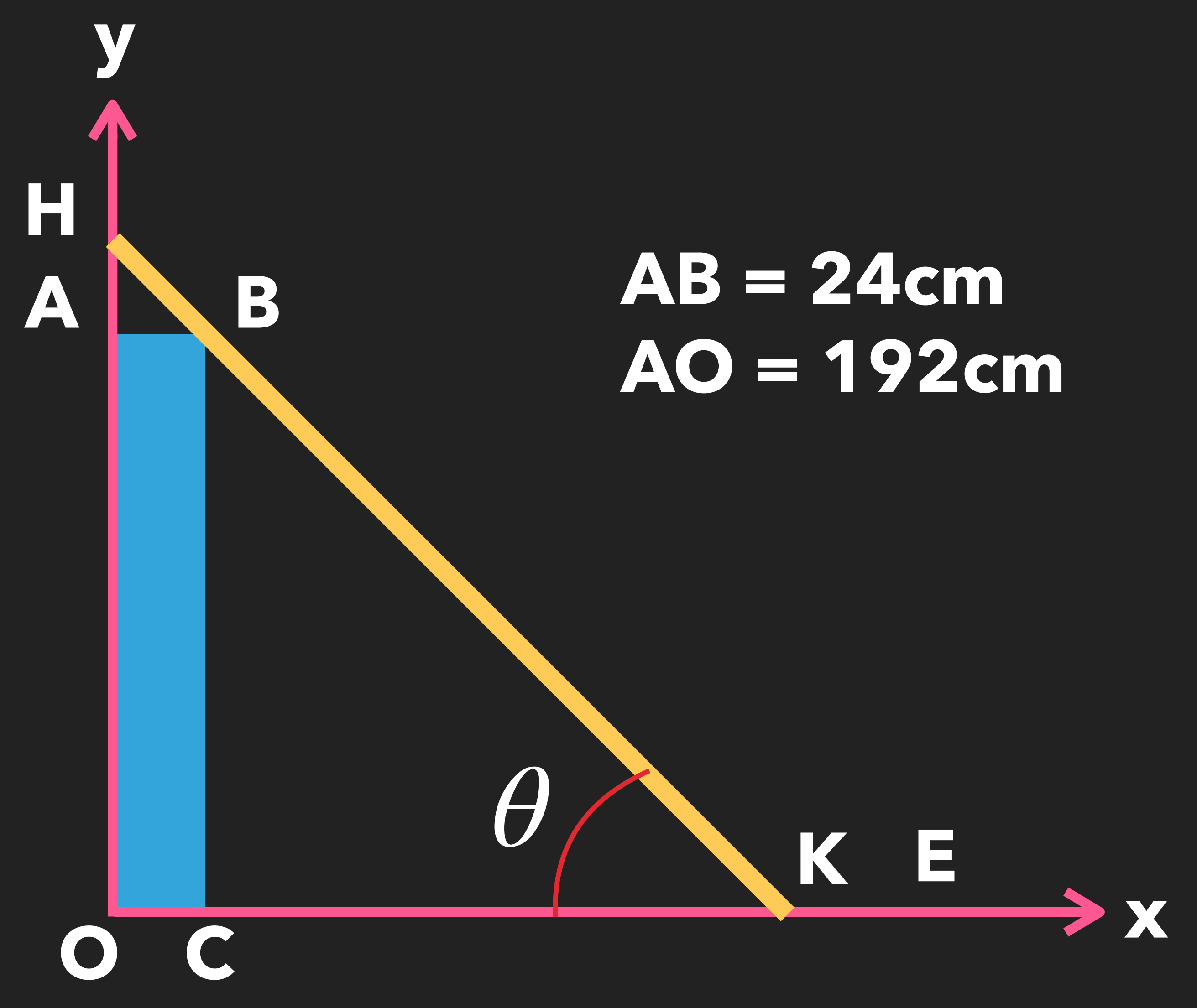
* 舉反例証明錯

2014 - SECTION B

Q10.) Consider the figure at right hand side :

a.) Find the shortest distance of length HK .

b.) Suppose $HK = 270cm$ *and K move toward E along* $x-axis$ *.*
Let xcm and ycm be the horizontal distance of H and K from the $y-axis$ *. Find the rate of change of x when* $CK = 160cm$ *and the rate of change of* $\theta = -0.1\ rad\ s^{-1}$ *.*
Is K moving faster than H? Please explain .



AB = 24cm
AO = 192cm

* 參考課程 3.3 及 3.4

$$\Delta HAB \sim \Delta BCK\ (AAA) \rightarrow \angle HBA = \theta$$

$$Also,\ HK = HB + BK = 24sec\theta + 192csc\theta$$

$$\frac{dHK}{d\theta} = 24sec\theta tan\theta - 192csc\theta cot\theta$$

$$To\ find\ turning\ point,\ solve\ 24sec\theta tan\theta - 192csc\theta cot\theta = 0$$

* Core 基本相似三角形証明

* 搵 turning point 睇下個微分幾時零

CONT'D

2014 - SECTION B

$$\frac{24sin\theta}{cos^2\theta}-\frac{192cos\theta}{sin^2\theta}=0\rightarrow tan^3\theta=8\rightarrow tan\theta=2$$

$$\because 0<\theta<\frac{\pi}{2}\rightarrow\theta=tan^{-1}(2)\ and$$

	$\theta<tan^{-1}(2)$	$\theta=tan^{-1}(2)$	$\theta>tan^{-1}(2)$
HK′	-	0	+
HK	Dec.		Inc.

$$\therefore\ HK\ is\ shortest\ when\ \theta=\theta_0=tan^{-1}(2)$$

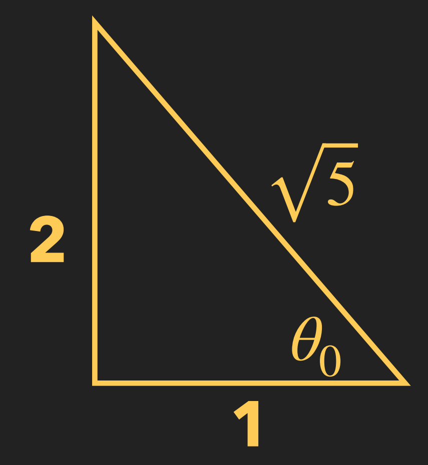
$$\rightarrow sin\theta_0=\frac{2}{\sqrt{5}}\ and\ cos\theta_0=\frac{1}{\sqrt{5}}$$

$$i.e.\ The\ shortest\ HK=24\sqrt{5}+96\sqrt{5}=120\sqrt{5}\ cm$$

* 留意角度範圍響 0-90⁰

* 用表格証明 local min.

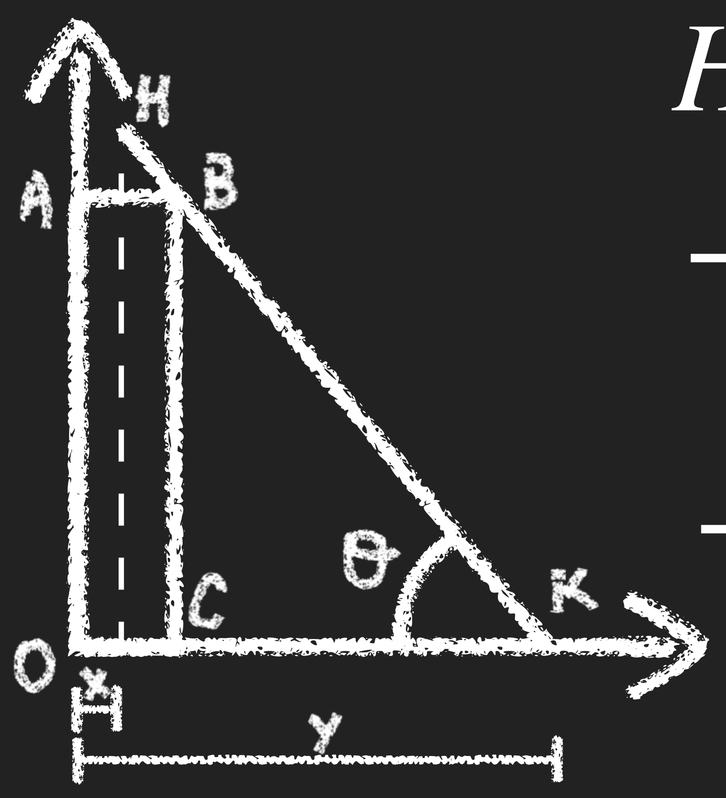
* 利用基本三角搵 sin 同 cos



CONT'D

2014 – SECTION B

Consider the figure :



$$HK = HB + BK = (24 - x)sec\theta + 192csc\theta$$

$$\rightarrow 270 = (24 - x)sec\theta + 192csc\theta$$

$$\rightarrow 0 = -sec\theta\frac{dx}{dt} + (24 - x)sec\theta tan\theta\frac{d\theta}{dt} - 192csc\theta cot\theta\frac{d\theta}{dt}$$

$$\rightarrow 0 = -sec\theta\frac{dx}{dt} + (270 - 192csc\theta)tan\theta\frac{d\theta}{dt} - 192csc\theta cot\theta\frac{d\theta}{dt}$$

$$\rightarrow sec\theta\frac{dx}{dt} = 270tan\theta\frac{d\theta}{dt} - 192csc\theta(cot\theta + tan\theta)\frac{d\theta}{dt}$$

$$\rightarrow \frac{dx}{dt} = (270sin\theta - 192csc^2\theta)\frac{d\theta}{dt}$$

$$\rightarrow \frac{dx}{dt}\Big|_{CK=160} = (270\frac{192}{\sqrt{192^2 + 160^2}} - 192\frac{192^2 + 160^2}{192^2})(-0.1)$$

*** Implicit 微分法**

*** 🟦 Product rule+Chain rule**

CONT'D

2014 - SECTION B

$$\rightarrow \frac{dx}{dt}|_{CK=160} = 11.79\ cm\ s^{-1}\ (to\ .\ 4\ sig\ .\ fig)$$

$$Consider,\ \frac{y-x}{270} = cos\theta \rightarrow \frac{1}{270}(\frac{dy}{dt} - \frac{dx}{dt}) = -\ sin\theta\frac{d\theta}{dt}$$

$$\rightarrow \frac{1}{270}(\frac{dy}{dt} - \frac{dx}{dt}) > 0,\ (0 < \theta < \frac{\pi}{2} \rightarrow sin\theta > 0,\ and\ \frac{d\theta}{dt} < 0)$$

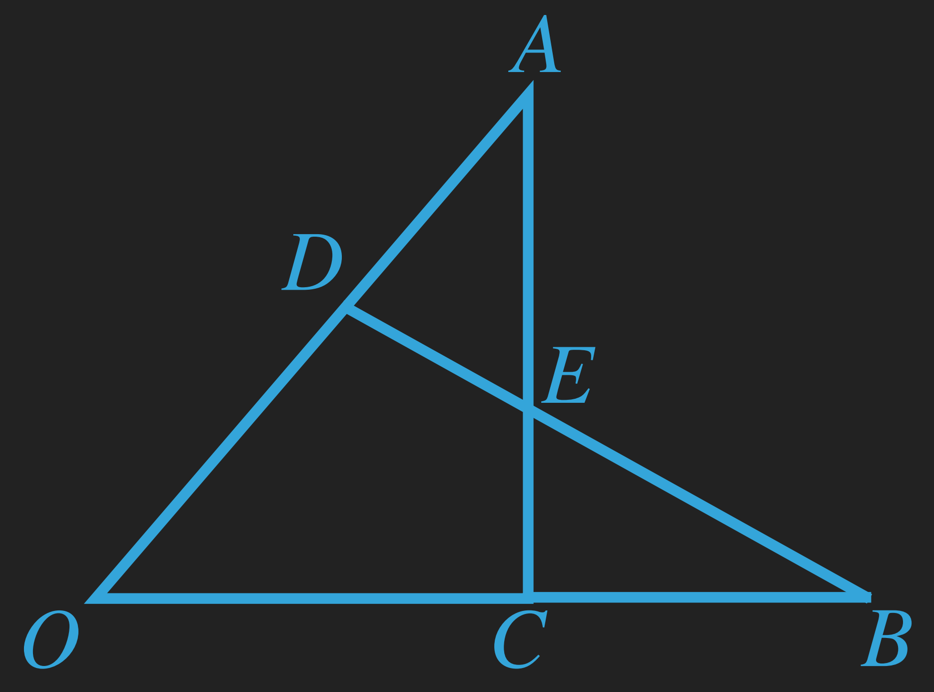
$$\rightarrow \frac{dy}{dt} > \frac{dx}{dt}$$

$\therefore K\ is\ moving\ faster\ than\ H$

*** Implicit 微分法**

2014 – SECTION B

Q11.)



In the figure, $AD : DO = OC : OB = t : (1-t)$

$AE : EC = m : 1,\ DE : EB = 1 : n$, where $m, n, t \in \mathbb{R}$

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

a.) Find m, n in term of t, Hence, prove E is the centroid of ΔAOB if $m = n$

b.) if $OA = 1$, $OB = 2$, and $AC \perp OB$, then BD is always $\perp OA$. Is it correct?

參考課程 4.1 及 4.2

a.) Consider, $\overrightarrow{OE} = \dfrac{n(1-t)\vec{a} + \vec{b}}{n+1} = \dfrac{\vec{a} + mt\vec{b}}{m+1}$

$$\rightarrow \begin{cases} \dfrac{n}{n+1}(1-t) = \dfrac{1}{m+1} & \text{——— (1)} \\ \dfrac{1}{n+1} = \dfrac{m}{m+1}t & \text{——— (2)} \end{cases}$$

* 用兩條式表達一支 Vector

* 🟦 分割定理 DE:EB

* 🟥 分割定理 AE:EC

* $A\vec{a} + B\vec{b} = C\vec{a} + D\vec{b} \rightarrow A = C$ and $B = D$

CONT'D

2014 - SECTION B

$$\frac{(1)}{(2)}:\ n(1-t)=\frac{1}{mt}\rightarrow n=\frac{1}{mt(1-t)}\ and\ m=\frac{1}{nt(1-t)}\ \text{———}\ \textbf{(3)}$$

Put (3) *into* (1) *and* (2) *respectively* :

$$(1):\ \frac{n}{n+1}(1-t)=\frac{1}{\frac{1}{nt(1-t)}+1}$$

$$\rightarrow \frac{1}{n+1}=\frac{t}{1+nt(1-t)}$$

$$\rightarrow nt+t=1+nt-nt^2$$

$$\rightarrow n=\frac{1-t}{t^2}$$

$$(2):\ \frac{1}{\frac{1}{mt(1-t)}+1}=\frac{m}{m+1}t$$

$$\rightarrow \frac{(1-t)}{1+mt(1-t)}=\frac{1}{m+1}$$

$$\rightarrow 1+mt-mt^2=m-mt+1-t$$

$$\rightarrow m=\frac{t}{(1-t)^2}$$

* 透過公式加減乘除
搵 m 同 n 關係

CONT'D

2014 - SECTION B

$$if\ m = n \rightarrow \frac{1-t}{t^2} = \frac{t}{(1-t)^2} \rightarrow (1-t)^3 - t^3 = 0 \rightarrow (1-2t)(1-t+t^2) = 0$$

$$\rightarrow t = \frac{1}{2} \rightarrow OD = DA\ and\ OC = CB$$

$\therefore$ *E is the centroid of ΔAOB*

$$b.)\ \because AC \perp OB,\ (\overrightarrow{OC} - \vec{a}) \cdot \vec{b} = 0 \rightarrow t|\vec{b}|^2 - \vec{a} \cdot \vec{b} = 0 \rightarrow \vec{a} \cdot \vec{b} = 4t$$

$$Consider,\ \overrightarrow{BD} \cdot \overrightarrow{OA} = (\overrightarrow{OD} - \vec{b}) \cdot \vec{a} = (1-t)|\vec{a}|^2 - \vec{a} \cdot \vec{b} = 1 - 5t$$

$$For\ t \neq \frac{1}{5},\ \overrightarrow{BD} \cdot \overrightarrow{OA} \neq 0$$

$\therefore$ *BD is not always $\perp$ OA .*

* 🟦 **No real solution**
$\Delta = 0$

* 🟥 $\vec{a} \cdot \vec{b} = 0 \leftrightarrow \vec{a} \perp \vec{b}$

2014 - SECTION B

$Q12.)$

$$Let\ M=\begin{pmatrix}k-1 & k\\ 1 & 0\end{pmatrix},\ A=\begin{pmatrix}1 & p\\ -1 & 1\end{pmatrix},\ where\ k,p\in\mathbb{R},\ p\neq -1$$

$a.)\ A^{-1}MA=?\ and\ if\ p=k,\ M^n=?$

$b.)\ Consider\ \{x_k\}\ be\ the\ sequence\ such\ that$

$$x_1=0,\ x_2=1\ and\ x_n=x_{n-1}+2x_{n-2},\ n\in\mathbb{Z}, n\geq 3$$

$Find\ x_n\ in\ term\ of\ n$

* 參考課程 4.8, 4.9 及 4.11

$a.)\ |A|=1+p\neq 0, for\ p\neq -1\rightarrow A^{-1}\ does\ exist$

$$\left(\begin{array}{cc|cc}1 & p & 1 & 0\\ -1 & 1 & 0 & 1\end{array}\right)\sim\left(\begin{array}{cc|cc}1 & p & 1 & 0\\ 0 & 1+p & 1 & 1\end{array}\right)\sim\left(\begin{array}{cc|cc}1 & p & 1 & 0\\ 0 & 1 & \frac{1}{1+p} & \frac{1}{1+p}\end{array}\right)$$

* |A| 唔等如零, A^{-1} 先存在

* 用 row deduction 搵 A^{-1}

CONT'D

2014 – SECTION B

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & \dfrac{1}{1+p} & \dfrac{-p}{1+p} \\ 0 & 1 & \dfrac{1}{1+p} & \dfrac{1}{1+p} \end{array}\right) \rightarrow A^{-1} = \frac{1}{1+p}\begin{pmatrix} 1 & -p \\ 1 & 1 \end{pmatrix}$$

$$\therefore A^{-1}MA = \frac{1}{1+p}\begin{pmatrix} 1 & -p \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k-1 & k \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & p \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & k-p \\ 0 & k \end{pmatrix}$$

$$if\ k = p,\ A^{-1}MA = \begin{pmatrix} -1 & 0 \\ 0 & k \end{pmatrix} \rightarrow (A^{-1}MA)^n = \begin{pmatrix} -1 & 0 \\ 0 & k \end{pmatrix}^n$$

$$\rightarrow A^{-1}M^nA = \begin{pmatrix} (-1)^n & 0 \\ 0 & k^n \end{pmatrix} \rightarrow M^n = A\begin{pmatrix} (-1)^n & 0 \\ 0 & k^n \end{pmatrix}A^{-1}$$

$* \ (P^{-1}AP)^n = P^{-1}A^nP$

CONT'D

2014 - SECTION B

$$=\frac{1}{1+k}\begin{pmatrix}(-1)^n+k^{n+1} & (-1)^{n+1}k+k^{n+1}\\ (-1)^{n+1}+k^n & (-1)^n k+k^n\end{pmatrix}$$

b.) The sequence can be expressed into :

$$\begin{pmatrix}x_n\\ x_{n-1}\end{pmatrix}=\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_{n-1}\\ x_{n-2}\end{pmatrix}=\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}^{n-2}\begin{pmatrix}x_2\\ x_1\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}(-1)^{n-2}+2^{n-1}+ & A\\ B & C\end{pmatrix}^{n-2}\begin{pmatrix}1\\ 0\end{pmatrix},\ \text{where } A,\ B \text{ and } C\in\ \mathbb{R}$$

$$\therefore x_n=\frac{1}{3}((-1)^n+2^{n-1})$$

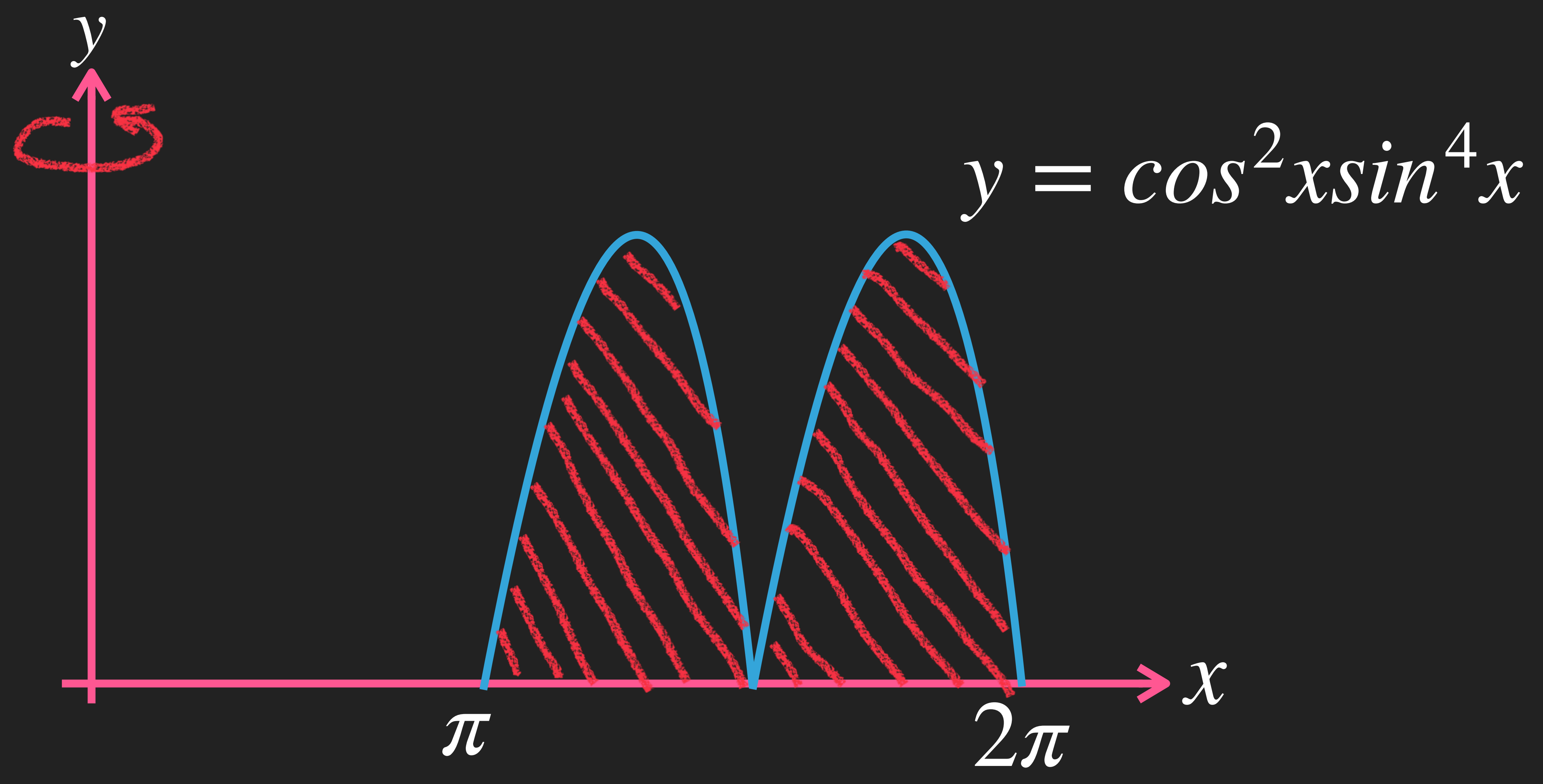
* 🟦 用 A 同 M 但係 k=p=2

* 沒有必要計全部數值係幾多

2014 – SECTION B

Q13.) a.) If $f(x)$ is continuous and $f(k-x)=f(x)$, where $k=$ constant .

Show that $\displaystyle\int_0^k xf(x)dx=\frac{k}{2}\int_0^k f(x)dx$



b.) Find the volume of the solid of revolution of shaded region along $y-axis$

* 參考課程 2.2, 3.10 及 3.12

CONT'D

2014 - SECTION B

a.) Let $I=\int_0^k xf(x)dx$, Let $y=k-x$, then

$$I=\int_k^0 (k-y)f(k-y)(-dy)=\int_0^k (k-y)f(y)dy$$

$$=k\int_0^k f(x)dx-\int_0^k xf(x)dx$$

$$\rightarrow 2I=k\int_0^k f(x)dx \rightarrow I=\frac{k}{2}\int_0^k f(x)dx$$

b.) The volume, $V=2\pi\int_\pi^{2\pi} x\cos^2 x\sin^4 x dx$

$$=2\pi\left(\int_0^{2\pi} x\cos^2 x\sin^4 x dx-\int_0^{\pi} x\cos^2 x\sin^4 x dx\right)$$

* 用積分代入法 y=k-x

* 範圍變更

* 🟦 負積分範圍上下倒轉

* 積分抽常數及積分符號可轉

* 用 Shell Method

* 🟩 積分可加減

CONT'D

2014 - SECTION B

$$Let\ f(x) = cos^2 x sin^4 x,\ f(2\pi - x) = f(x)\ and\ f(\pi - x) = f(x)$$

$$\therefore V = 2\pi\frac{2\pi}{2}\int_0^{2\pi} f(x)dx - 2\pi\frac{\pi}{2}\int_0^{\pi} f(x)dx = 2\pi^2\int_0^{2\pi} f(x)dx - \pi^2\int_0^{\pi} f(x)dx$$

$$Consider,\ \int_0^{n\pi} f(x)dx = \int_0^{n\pi} cos^2 x sin^4 x dx = \int_0^{n\pi} \frac{1}{4}sin^2 2x sin^2 x dx$$

$$= \int_0^{n\pi} \frac{1}{4}sin^2 2x \frac{1}{2}(1 - cos 2x)dx$$

$$= \int_0^{n\pi} \frac{1}{8}sin^2 2x dx - \int_0^{n\pi} \frac{1}{8}sin^2 2x cos 2x dx$$

$$= \int_0^{n\pi} \frac{1}{16}(1 - cos 4x)dx - \int_0^{n\pi} \frac{1}{8}sin^2 2x \frac{1}{2}d(sin 2x)$$

* 可用 a.) result

* 🟦 利用 sin 雙角公式

* 🟥 利用 cos 雙角公式

* 🟩 積分可代入

CONT'D

2014 - SECTION B

$$= \int_0^{n\pi} \frac{1}{16}dx - \frac{1}{16}\cancelto{0}{\int_0^{n\pi} \cos 4x dx} - \frac{1}{16}\cancelto{0}{\int_0^{0} y^2 dy}$$

$$= \frac{n\pi}{16}$$

$$\therefore V = 2\pi^2\left(\frac{2\pi}{16}\right) - \pi^2\left(\frac{\pi}{16}\right) = \frac{3\pi^3}{16}\ cu\,.\,unit$$

* 🟦 $\cos 4x\ (0 \to n\pi)$ 面積互相抵消

* 🟩 沒有面積