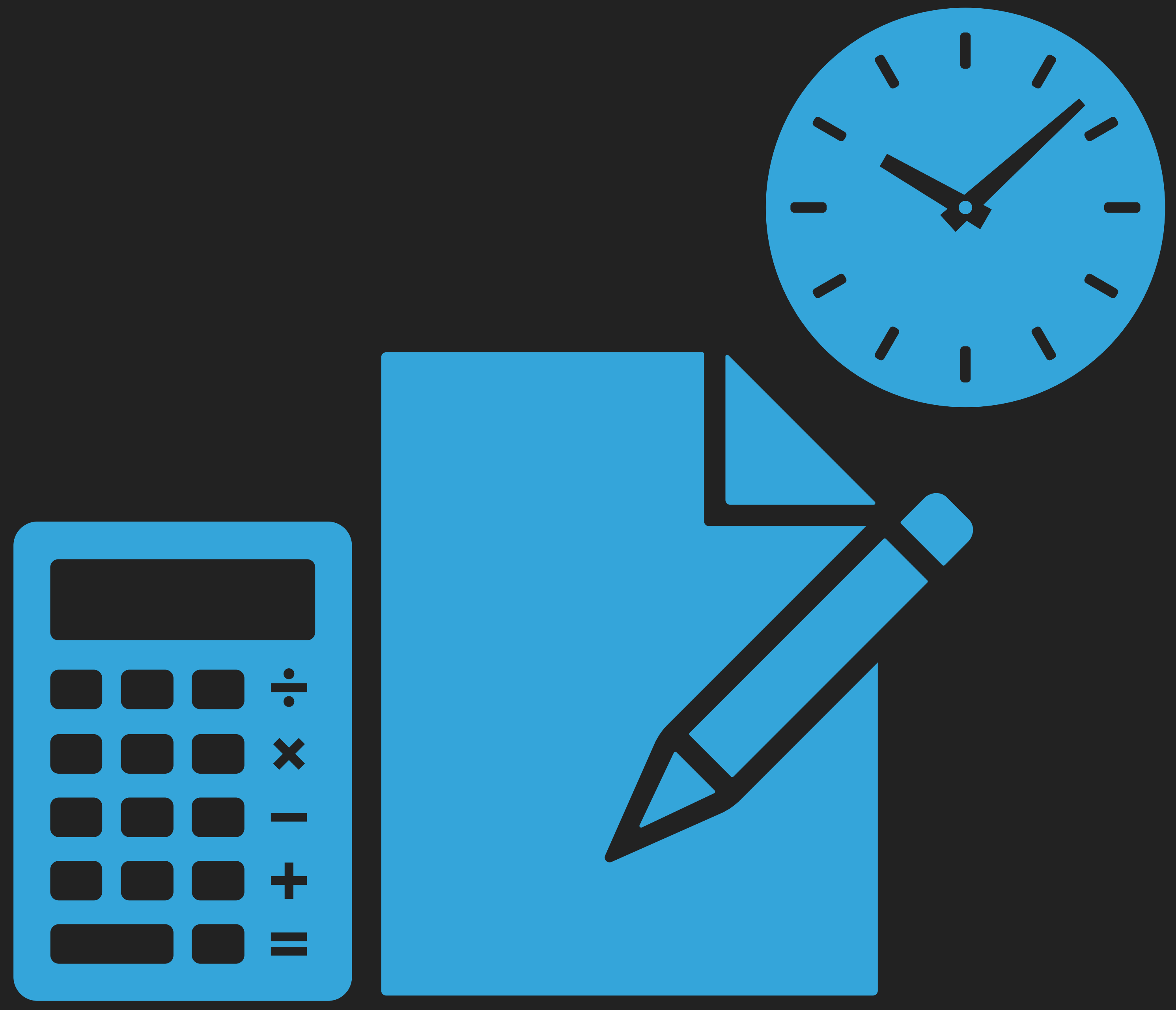
深宵教室 - DSE M2 模擬試題解答

2019

此為參考2019試題之模擬試題，原版請另行購買

2019

- Section A
- Section B



2019 - SECTION A

$Q1.)\ f(x) = 10x(7+3x^2)^{-1}.\ f'(1) = ?\ \ (By\ First\ Principles)$

* 參考課程 3.1 及 3.2

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}(10(h+1)(7+3(1+h)^2)^{-1} - 1)$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{4h-3h^2}{3h^2+6h+10}$$

$$= \lim_{h\to 0} \frac{4-3h}{3h^2+6h+10}$$

$$= \frac{2}{5}$$

* 微分定義

2019 - SECTION A

Q2.)

$$Let\ P(x) = \begin{vmatrix} x+\lambda & 1 & 2 \\ 0 & (x+\lambda)^2 & 3 \\ 4 & 5 & (x+\lambda)^3 \end{vmatrix},\ \lambda \in \mathbb{R},\ where\ the\ coefficient\ of\ x^3 = 160$$

$Find\ \lambda\ and\ P'(0)$

* 參考課程 1.1, 3.2 及 4.9

$$P(x) = \boxed{(x+\lambda)^6} + 12 - 8(x+\lambda)^2 - 15(x+\lambda)$$

$$By\ consider\ the\ coefficient\ of\ x^3:\ C^6_3\lambda^3 = 160 \rightarrow 20\lambda^3 = 160$$

$$\rightarrow \lambda = 2$$

$$Hence,\ P'(x) = 6(x+2)^5 - 16(x+2)^2 - 15$$

$$\rightarrow P'(0) = 145$$

* 用類似交叉相乘相加減方式計算

* 只須要留意 $\boxed{}$ 去搵 x^3 coefficient

$$* \ C^n_r = \frac{n!}{r!(n-r)!}$$

2019 - SECTION A

Q3.) There is a container with $580cm^3$ *of water. The water is leaking at a rate* $-2t$ *where* $t =$ *number of hours. Is there still of water in the container after 24 hours?*
Given that the volume of water, $V = h^2 + 24h$*, where* $h =$ *depth of water (cm)*
Find the rate of change of h against t, when $t = 18$

* 參考課程 3.4 及 3.9

Let $V(t)$ *be the volume of water at time* t
$h(t)$ *be the depth of water at time* t

Given that $V'(t) = -2t$ *and* $V(0) = 580$

$$V(t) = \int -2t\,dt = -t^2 + C, \text{ where } C \text{ is constant}$$

* 積分係類似微分逆函數

$$\because V(0) = 580 \rightarrow C = 580$$

$$\therefore V(24) = -(24)^2 + 580 = 4 > 0$$

i.e. There is still water inside the container after 24 hours

CONT'D

2019 - SECTION A

$$Given\ that\ V(t) = [h(t)]^2 + 24h(t)$$

$$V'(t) = (2h(t) + 24)h'(t)$$

$$Also,\ V(18) = [h(18)]^2 + 24h(18) \rightarrow -18^2 + 580 = [h(18)]^2 + 24h(18)$$

$$\rightarrow [h(18)]^2 + 24h(18) - 256 = 0$$

$$\rightarrow h(18) = 8\ or\ -32\ (rejected)$$

$$\therefore V'(18) = 2(8+12)h'(18) \rightarrow h'(18) = -0.9$$

$$i.e.\ The\ rate\ of\ change\ of\ the\ depth\ of\ water = -0.9\ cm\ hr^{-1}$$

*** Implicit 微分法**

*** Chain rule**

2019 – SECTION A

Q4.) Find the revolving volume along x – axis bounded by $\Gamma: \ y = x^{-\frac{1}{2}} \ln x$ *(where* $0 < x < 99$*)*
x – axis, and the vertical line passing throught max . pt . of Γ

* 參考課程 3.4 及 3.12

$$y = x^{-\frac{1}{2}} \ln x \rightarrow \sqrt{x} y = \ln x \rightarrow \frac{y}{2\sqrt{x}} + \sqrt{x}\frac{dy}{dx} = \frac{1}{x}$$

* 用 Implicit 微分

$$Let \ x_0 \in \mathbb{R} \ such \ that \ \frac{dy}{dx}\Big|_{x=x_0} = 0 \rightarrow \frac{\ln x_0}{2} = 1 \rightarrow x_0 = e^2$$

* 搵 turning point = 搵 x_0 使度 $y'(x_0)=0$

	$0 < x < e^2$	$x = e^2$	$e^2 < x < 99$
y′	+	0	-
y	Up.		Down.

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow Increasing$
$f'(x) < 0 \rightarrow Decreasing$

$$\therefore The \ max . \ point = (e^2, 2e^{-1})$$

$$\because when \ x = 1, y = 0, \ \therefore The \ volume, V = \pi \int_1^{e^2} y^2 dx$$

CONT'D

2019 - SECTION A

$$V = \pi \int_1^{e^2} \frac{(ln x)^2}{\boxed{x}} \boxed{dx} = \pi \int_1^{e^2} (ln x)^2 \boxed{d(ln x)}$$

$$= \pi [\frac{1}{3}(ln x)^3]_1^{e^2} = \frac{8\pi}{3}\ cu.\ unit$$

* 積分三寶: 積分代入

2019 - SECTION A

$$Q5.)\ Prove\ \sum_{r=n}^{2n}\frac{1}{r(r+1)}=\frac{n+1}{n(2n+1)},\ \forall n\in\mathbb{Z}^+,\ Hence,\ \sum_{r=50}^{200}\frac{1}{r(r+1)}=?$$

* 參考課程 1.1 及 1.2

方法1

$$Let\ P(n):\ \sum_{r=n}^{2n}\frac{1}{r(r+1)}=\frac{n+1}{n(2n+1)}\ \forall n\in\mathbb{Z}^+$$

* 先 Let Statement

$$For\ P(1):L.H.S.=\frac{1}{(1)(2)}+\frac{1}{(2)(3)}=\frac{2}{3}=R.H.S.$$

* 証明 P(1) is true

$$Assume\ P(k)\ is\ true\ \exists k\in\mathbb{Z}^+,\ then\ P(k+1):$$

* 假設 P(k) is true. 証明 P(k+1) is true

$$L.H.S.=\sum_{r=k+1}^{2k+2}\frac{1}{r(r+1)}$$

$$=\sum_{r=k+1}^{2k}\frac{1}{r(r+1)}+\frac{1}{(2k+1)(2k+2)}+\frac{1}{(2k+2)(2k+3)}$$

* 將末項抽出並改變末項

CONT'D

2019 – SECTION A

$$= -\frac{1}{k(k+1)} + \frac{1}{k(k+1)} + \sum_{r=k+1}^{2k} \frac{1}{r(r+1)} + \frac{2}{(k+1)(k+3)}$$

$$= \boxed{\sum_{r=k}^{2k} \frac{1}{r(r+1)}} + \frac{2}{(2k+1)(2k+3)} - \frac{1}{k(k+1)}$$

$$= \frac{k+1}{k(2k+1)} - \frac{1}{k(k+1)} + \frac{2}{(2k+1)(2k+3)}$$

$$= \frac{k}{(k+1)(2k+1)} + \frac{2}{(2k+1)(2k+3)} = \frac{k+2}{(k+1)(2k+3)} = R.H.S.$$

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

$i.e.$ *By* $M.I.$, $P(n)$ *is true*, $\forall n \in \mathbb{Z}^+$

* 加頭項並改變頭項

* 寫結論

CONT'D

2019 - SECTION A

方法2

$$\sum_{r=n}^{2n}\frac{1}{r(r+1)}=\sum_{r=n}^{2n}\left(\frac{1}{r}-\frac{1}{r+1}\right)=\sum_{r=n}^{2n}\frac{1}{r}-\sum_{r=n}^{2n}\frac{1}{r+1}$$

$$=\sum_{r=n}^{2n}\frac{1}{r}-\sum_{r=n+1}^{2n+1}\frac{1}{r}=\frac{1}{n}+\sum_{r=n+1}^{2n}\frac{1}{r}-\sum_{r=n+1}^{2n}\frac{1}{r}-\frac{1}{2n+1}$$

$$=\frac{1}{n}-\frac{1}{2n+1}=\frac{n+1}{n(2n+1)}$$

$$Hence,\ \sum_{r=50}^{200}\frac{1}{r(r+1)}=\sum_{r=50}^{100}\frac{1}{r(r+1)}+\sum_{r=101}^{200}\frac{1}{r(r+1)}$$

$$=\sum_{r=50}^{100}\frac{1}{r(r+1)}+\sum_{r=100}^{200}\frac{1}{r(r+1)}-\frac{1}{100(101)}$$

* Summation 可析開做加減
* 透過改變首未項改變公項
* 透過抽首尾項改變首未項
* 拆開加
* 加頭項並改變頭項

CONT'D

2019 – SECTION A

$$= \frac{51}{50(101)} + \frac{101}{100(201)} - \frac{1}{100(101)}$$

$$= \frac{2(51)(201) + 101(101) + 201}{100(101)(201)}$$

$$= \frac{151}{10050}$$

* 用以上結果

2019 - SECTION A

Q6.) $\begin{cases} x-2y-2z=\beta \\ 5x+\alpha y+\alpha z=5\beta \\ 7x+(\alpha-3)y+(2\alpha+1)z=8\beta \end{cases}$ —— (E) $\alpha, \beta \in \mathbb{R}$

a.) *The range of α and $y = ?$ if (E) has unique solution*

b.) *$\beta = ?$ if (E) is inconsistent and $\alpha = -4$*

* 參考課程 4.7

$$(E): \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 5 & \alpha & \alpha & 5\beta \\ 7 & \alpha-3 & 2\alpha+1 & 8\beta \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & \alpha+10 & \alpha+10 & 0 \\ 0 & \alpha+11 & 2\alpha+15 & \beta \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & \alpha+10 & \alpha+10 & 0 \\ 0 & 0 & A & B \end{array}\right)$$

where $A = (\alpha+4)(\alpha+10)$

$B = \beta(\alpha+10)$

* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \to \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

* 如果 | | 不等如 0, 有唯一答案

CONT'D

2019 - SECTION A

$a.)\ If\ (E)\ has\ unique\ solution,\ (\alpha+10)^2(\alpha+4)\neq 0$

$\rightarrow \alpha \neq -10\ and\ -4$

$$Then,\ y=-z=-\frac{B}{A}=\frac{\beta}{\alpha+4}$$

$b.)\ If\ (E)\ is\ inconsistent\ with\ \alpha=-4,\ B\neq 0$

$\rightarrow \beta \neq 0$

* 如果要沒有答案, B 必須不等如零

2019 - SECTION A

$Q7.)\ \int e^x \sin\pi x\,dx = ?,\ \int_0^3 e^{3-x}\sin\pi x\,dx = ?$

* 參考課程 3.7 及 3.10

$$Let\ I_1 = \int e^x \sin\pi x\,dx = \int e^x d\left(\frac{-1}{\pi}\cos\pi x\right)$$

$$= \frac{-e^x\cos\pi x}{\pi} + \int \frac{1}{\pi}e^x \cos\pi x\,dx = \frac{-e^x\cos\pi x}{\pi} + \int \frac{1}{\pi}e^x d\left(\frac{1}{\pi}\sin\pi x\right)$$

$$= \frac{-e^x\cos\pi x}{\pi} + \frac{e^x\sin\pi x}{\pi^2} - \frac{1}{\pi^2}\int e^x \sin\pi x\,dx$$

$$\therefore \left(1+\frac{1}{\pi^2}\right)I_1 = \frac{-e^x\cos\pi x}{\pi} + \frac{e^x\sin\pi x}{\pi^2} \rightarrow I_1 = \frac{e^x\sin\pi x - \pi e^x\cos\pi x}{\pi^2+1} + C$$

$, where\ C\ is\ constant$

* 積分三寶: Integration by part
* 積分三寶: Integration by part
* = I_1

CONT'D

2019 - SECTION A

$$Let\ I_2 = \int_0^3 e^{3-x} sin\pi x dx = \int_0^{-3} e^{3+u} sin(-\pi u) d(-u),\ where\ u = -x$$

$$= -e^3 \int_{-3}^{0} e^u sin\pi u du = -e^3 \left[\frac{e^x sin\pi x - \pi e^x cos\pi x}{\pi^2 + 1}\right]_{-3}^{0}$$

$$= \frac{\pi(e^3+1)}{\pi^2+1}$$

* 定積分代入要改範圍

* 負數定積分範圍上下倒轉

* $sin(-x) = -sinx$

2019 - SECTION A

$Q8.)$ *Suppose a curve* $\Gamma : y = f(x),\ x \in \mathbb{R}^+$, *given that* Γ *passes through* $P = (1, 3)$ *and*

$$f'(x) = \frac{2x^2 - 7x + 8}{x},\ \forall\ x > 0,\ find$$

a.) *Equation of* Γ, *and prove* $f(x)$ *is increasing function*

b.) *Point(s) of inflexion of* Γ

* 參考課程 3.4, 3.5, 及 3.6

$$a.)\ Given\ that\ f'(x) = \frac{2x^2 - 7x + 8}{x} = \frac{2(x - \frac{7}{4})^2 + \frac{15}{8}}{x}$$

$$> 0,\ for\ x \in \mathbb{R}^+$$

$$\therefore f(x)\ is\ increasing\ function\ for\ x \in \mathbb{R}^+$$

$$f(x) = \int f'(x)dx = \int 2x - 7 + 8x^{-1}dx = x^2 - 7x + 8\ln x + C$$

$$,\ where\ C\ is\ constant$$

* **Completing Square**

$*\ f'(x) > 0 \rightarrow Increasing$

* 積分係類似微分逆函數

CONT'D

2019 - SECTION A

$\because \Gamma$ *passes through* $P \rightarrow f(1) = 3$

$\therefore C = 9$

$i.e. f(x) = x^2 - 7x + 8lnx + 9$

$$b.)\ f'(x) = \frac{2x^2 - 7x + 8}{x} \rightarrow xf'(x) = 2x^2 - 7x + 8$$

$$\rightarrow f'(x) + xf''(x) = 4x - 7$$

$$Let\ x_0 \in \mathbb{R}^+\ such\ that\ f''(x_0) = 0 \rightarrow f'(x_0) = 4x_0 - 7$$

$$\rightarrow 2x_0^2 - 7x_0 + 8 = 4x_0^2 - 7x_0$$

$$\rightarrow x_0 = 2\ or\ -2\ (rejected)$$

$\therefore$ *The point of inflexion* $= (2, 8ln2 - 1)$

* 利用題目資料搵 C

* 用 Implicit 微分法

* 搵 pt. of inflexion = 搵 x_0
使度 $f''(x_0)=0$

2019 - SECTION B

Q9.) Assume a curve $\Gamma: y = \frac{1}{3}\sqrt{12 - x^2}$, where $0 < x < 2\sqrt{3}$

$C: \ y = \sqrt{4 - x^2}$, where $0 < x < 2$

Let L be the tangent to Γ at $x = 3$ and the tangent to C

a.) Find the equation of L and the point of contact of L at C

b.) Find the area of region bounded by Γ, L and C

* 參考課程 3.4, 3.8 及 3.11

$$a.)\ L: \frac{dy}{dx}\bigg|_{x=3} = \frac{y - \frac{1}{3}\sqrt{12 - 3^2}}{x - 3} \rightarrow \frac{-1(3)}{3\sqrt{12 - 3^2}} = \frac{3y - \sqrt{3}}{3(x - 3)}$$

$$\rightarrow \frac{-1}{\sqrt{3}} = \frac{3y - \sqrt{3}}{3(x - 3)} \rightarrow 3y + \sqrt{3}x - 4\sqrt{3} = 0$$

* 利用微分搵 tangent slope
* 利用 point-slope form 搵 equation

CONT'D

2019 - SECTION B

Let the point of contact of L at C = (x_0, y_0)

For C :

$$\frac{dy}{dx} = -\frac{x}{\sqrt{12-x^2}} \rightarrow \frac{dy}{dx}\Big|_{x=x_0} = -\frac{x_0}{y_0} = \text{The slope of } L$$

$$\rightarrow -\frac{x_0}{y_0} = -\frac{1}{\sqrt{3}} \rightarrow y_0 = \sqrt{3}x_0 \quad - \quad (1)$$

$$\text{Also, } 3y_0 + \sqrt{3}x_0 - 4\sqrt{3} = 0 \quad - (2)$$

$$\text{Put (1) into (2) : } 3\sqrt{3}x_0 + \sqrt{3}x_0 = 4\sqrt{3} \rightarrow x_0 = 1$$

$$\therefore (x_0, y_0) = (1, \sqrt{3})$$

b.) Let the interception of Γ *and C be* (x_1, y_1)*, then*

$$(E) : \begin{cases} 9y_1^2 = 12 - x_1^2 & -(1) \\ y_1^2 = 4 - x_1^2 & -(2) \end{cases}$$

* 利用微分搵 tangent slope

* Tangent 個 slope 就係 L 個 slope

* Pt. of contact 就係 L 上的一點

* 先搵所有 curves 相交點

CONT'D

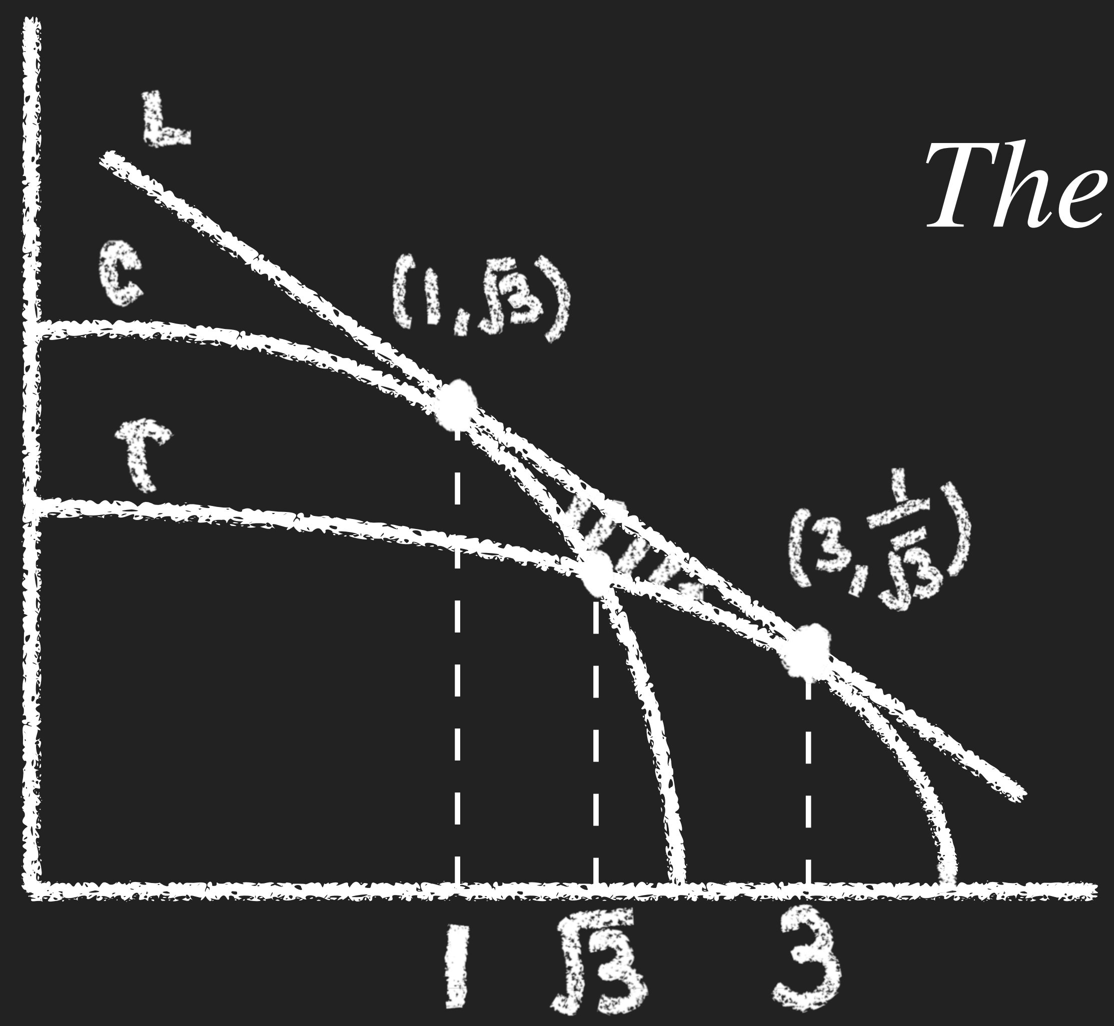
2019 - SECTION B

$$(1)-9x(2):-24+8x_1^2=0 \rightarrow x_1=\sqrt{3}\ or\ -\sqrt{3}\ (rejected)$$

$$Then,\ y_1=1\ or\ -1\ (Not\ consider\ as\ out\ of\ bounded\ region)$$

$$Hence,\ (x_1,y_1)=(\sqrt{3},1)$$

$$Consider\ the\ following\ graph:$$



$$The\ area,\ A=\boxed{\frac{(3-1)(\sqrt{3}+\frac{1}{\sqrt{3}})}{2}}-I_1-I_2$$

$$=\frac{4}{\sqrt{3}}-I_1-I_2$$

$$where,\ I_1=\int_1^{\sqrt{3}}\sqrt{4-x^2}dx$$

$$I_2=\int_{\sqrt{3}}^{3}\frac{1}{3}\sqrt{12-x^2}dx$$

* 畫圖明白範圍

* 盡量計算基本幾何圖形面積(梯形)

* 面積互相加減

CONT'D

2019 - SECTION B

$$Consider,\ I = \int \sqrt{a^2 - x^2}dx = \int \sqrt{a^2 - a^2 sin^2\theta}\, acos\theta d\theta$$

$$,\ where\ x = asin\theta$$

$$= \int acos\theta acos\theta d\theta = a^2 \int cos^2\theta d\theta = a^2 \int \frac{1}{2}(cos2\theta + 1)d\theta$$

$$= \frac{a^2}{2}(\frac{1}{2}sin2\theta + \theta) + C,\ where\ C\ is\ constant$$

$$Hence,\ I_1 = [sin2\theta + 2\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3},\ I_2 = [sin2\theta + 2\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3}$$

$$where\ put\ a = 2\ and\ 2\sqrt{3}\ respectively$$

$$\therefore A = \frac{4}{\sqrt{3}} - \frac{2\pi}{3} = \frac{4\sqrt{3} - 2\pi}{3}\ sq.unit$$

* 先計算一個 common
* 利用三角代入法, $let\ x = asin\theta$
* $1 - sin^2\theta = cos^2\theta$
* cos 雙角公式
* 定積分代入要改範圍

2019 - SECTION B

$Q10.)$

$$a.)\int_0^{\frac{\pi}{4}}\frac{1}{2+cos2x}dx = ? \qquad b.)\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{sin2x}{(2+cos2x)^2}ln(1+e^x)dx = ?$$

* 參考課程 2.2, 3.8 及 3.10

$$a.)\ Let\ I_1 = \int_0^{\frac{\pi}{4}}\frac{1}{2+cos2x}dx$$

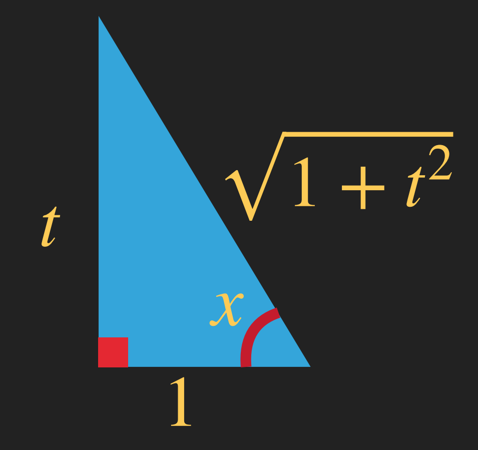
方法1

$$Let\ t = tanx \rightarrow dt = sec^2xdx$$

$$we\ have\ cos2x = cos^2x - sin^2x = \frac{1-t^2}{1+t^2}$$

$$\therefore I_1 = \int_0^1 \frac{\frac{1}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}dt = \int_0^1\frac{1}{3+t^2}dt$$

* 利用 T-method, $let\ t = tanx$



$$cosx = \frac{1}{\sqrt{1+t^2}}$$

$$sinx = \frac{t}{\sqrt{1+t^2}}$$

* cos 雙角公式

* 定積分代入要改範圍

CONT'D

2019 - SECTION B

$$= \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}sec^2\theta d\theta}{3+3tan^2\theta}, \text{ where } t = \sqrt{3}tan\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}sec^2\theta d\theta}{3sec^2\theta} = \frac{1}{\sqrt{3}}[\theta]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}\pi}{18}$$

方法2

$$I_1 = \int_0^{\frac{\pi}{4}} \frac{1}{2+cos2x}dx = \int_0^{\frac{\pi}{4}} \frac{1}{2+2cos^2x-1}dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{cos^2x}}{2+\frac{1}{cos^2x}}dx = \int_0^{\frac{\pi}{4}} \frac{sec^2xdx}{2+sec^2x} = \int_0^{\frac{\pi}{4}} \frac{d(tanx)}{3+tan^2x}$$

$$= \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}sec^2\theta d\theta}{3+3tan^2\theta} = \frac{\sqrt{3}\pi}{18} \quad , \text{where } tanx = \sqrt{3}tan\theta$$

* 利用三角代入法, *let* $t = \sqrt{3}tan\theta$
* 定積分代入要改範圍
* $tan^2\theta + 1 = sec^2\theta$
* cos 雙角公式
* 積分三寶: 積分代入
* 利用三角代入法, *let* $tanx = \sqrt{3}tan\theta$
* 定積分代入要改範圍

CONT'D

2019 - SECTION B

$$b.)\ Let\ I_2 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x)ln(1+e^x)dx,\ where\ f(x) = \frac{sin2x}{(2+cos2x)^2}$$

$$Then,\ I_2 = \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} f(-u)ln(1+e^{-u})d(-u),\ where\ u = -x$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} -f(u)ln(\frac{e^u+1}{e^u})du,\quad \because f(-u) = \frac{sin(-2u)}{(2+cos(-2u))^2} = -f(u)$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} -f(u)[ln(e^u+1)-u]du \rightarrow 2I_2 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} uf(u)du$$

$\because uf(u)\ is\ even\ function$

$$\therefore 2I_2 = 2\int_0^{\frac{\pi}{4}} uf(u)du \rightarrow I_2 = \int_0^{\frac{\pi}{4}} uf(u)du$$

* 定積分代入要改範圍

* 負數定積分範圍上下倒轉

* $sin(-x) = -sinx$

* $cos(-x) = cosx$

* ln(A/B) = lnA - lnB

* Even function 由 -a 積到 a

= 兩倍由 0 積到 a

CONT'D

2019 - SECTION B

$$\rightarrow I_2 = \int_0^{\frac{\pi}{4}} \frac{u\sin 2u du}{(2+\cos 2u)^2} = \int_0^{\frac{\pi}{4}} u d\left(\frac{1}{2}\frac{1}{2+\cos 2u}\right)$$

$$= \left[\frac{u}{2(2+\cos 2u)}\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2u} du$$

$$= \frac{\pi}{16} - \frac{1}{2}\frac{\sqrt{3}\pi}{18} = \left(\frac{1}{16} - \frac{\sqrt{3}}{36}\right)\pi$$

* 積分三寶: Integration by part
* 用a.) 結果

2019 - SECTION B

Q11.)

$$M = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$$

$a.)\ Prove\ 6M^n = (1-(-5)^n)M + (5+(-5)^n)I_2, \forall n \in \mathbb{Z}^+$

$b.)\ Is\ there\ exist\ a\ matrix\ A\ and\ B\ such\ that\ (M^n)^{-1} = A + \dfrac{1}{(-5)^n}B, \forall n \in \mathbb{Z}^+?$

*** 參考課程 1.2, 4.9, 4.10 及 4.11**

$a.)$

方法1

$Let\ P(n):\ 6M^n = (1-(-5)^n)M + (5+(-5)^n)I_2, \forall n \in \mathbb{Z}^+$

$For\ P(1):\ L.H.S. = 6M = R.H.S.$

$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+, then\ P(k+1):$

$$\begin{aligned} L.H.S. &= 6M^{k+1} = 6M^kM \\ &= [(1-(-5)^k)M + (5+(-5)^k)I_2]M \end{aligned}$$

*** 先 Let Statement**

*** 証明 P(1) is true**

*** 假設 P(k) is true. 証明 P(k+1) is true**

CONT'D

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$$= [(1-(-5)^k)M^2 + (5+(-5)^k)M$$

$$\text{where } M^2 = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} = \begin{pmatrix} -3 & -28 \\ 4 & 29 \end{pmatrix}$$

$$\text{Consider } P(2): \ 6M^2 = -24M + 30I_2 \leftrightarrow M^2 = -4M + 5I_2$$

$$\leftrightarrow M^2 = \begin{pmatrix} -3 & -28 \\ 4 & 29 \end{pmatrix}$$

$$\therefore \ P(2) \text{ is true} \rightarrow M^2 = -4M + 5I_2$$

$$\text{Hence } (1-(-5)^k)M^2 + (5+(-5)^k)M$$

$$= (1-(-5)^k)(-4M+5I_2) + (5+(-5)^k)M$$

$$= (-4+4(-5)^k + 5 + (-5)^k)M + (5+(-5)^{k+1})I_2$$

$$= (1-(-5)^{k+1})M + (5+(-5)^{k+1})I_2 = R.H.S.$$

$$\therefore P(k+1) \text{ is true if } P(k) \text{ is true } \exists k \in \mathbb{Z}^+$$

$$i.e. \text{ By } M.I., P(n) \text{ is true}, \forall n \in \mathbb{Z}^+$$

* 諗辨法將 M² 寫成 M 同 I 關係
透過証明 P(2) 去証實關係

* 寫結論

CONT'D

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方法2

$$Consider\ (E):\left(\begin{array}{cc|c}2-\lambda & 7 & 0\\ -1 & -6-\lambda & 0\end{array}\right),\ has\ non-trival\ solution$$

* 利用 eigenvalue, eigenvector

$$\to \begin{vmatrix}2-\lambda & 7\\ -1 & -6-\lambda\end{vmatrix}=0\to(\lambda-2)(\lambda+6)+7=0$$

* (E) 有非零答案, 必須係直線答案

$$\to \lambda^2+4\lambda-5=0\ \to \lambda=-5\ or\ 1$$

$$When\ \lambda=-5,\ (-1,1)\ is\ one\ of\ the\ solution$$

$$When\ \lambda=1,\ (-7,1)\ is\ one\ of\ the\ solution$$

* 解 (E) 搵各自 eigenvector

$$Let\ P=\begin{pmatrix}-1 & -7\\ 1 & 1\end{pmatrix},\ P^{-1}=\frac{1}{6}\begin{pmatrix}1 & 7\\ -1 & -1\end{pmatrix}$$

* 用 row deduction 或 adj.matrix

$$Then,\ P^{-1}MP=\begin{pmatrix}-5 & 0\\ 0 & 1\end{pmatrix}\to M^n=P\begin{pmatrix}(-5)^n & 0\\ 0 & 1\end{pmatrix}P^{-1}$$

CONT'D

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$$\rightarrow 6M^n = \begin{pmatrix} -1 & -7 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-5)^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -(-5)^n & -7 \\ (-5)^n & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -(-5)^n + 7 & 7(1-(-5)^n) \\ (-5)^n - 1 & 7(-5)^n + 1 \end{pmatrix}$$

$$= \begin{pmatrix} -(-5)^n + 7 & 7(1-(-5)^n) \\ \boxed{(-5)^n - 1} & 7(-5)^n + 1 \end{pmatrix}$$

$$= (1-(-5)^n)\begin{pmatrix} 2 & 7 \\ \boxed{-1} & -6 \end{pmatrix} + (5+(-5)^n)\begin{pmatrix} 1 & 0 \\ \boxed{0} & 1 \end{pmatrix}$$

$$= (1-(-5)^n)M + (5+(-5)^n)I_2$$

* 響 M 裏面係 -1, I 係0

CONT'D

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$$b.)\ 6M^n = (1-(-5)^n)\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} + (5+(-5)^n)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-2(-5)^n+5+(-5)^n & 7-7(-5)^n \\ -1+(-5)^n & -6+6(-5)^n+5+(-5)^n \end{pmatrix}$$

$$= \begin{pmatrix} 7-(-5)^n & 7-7(-5)^n \\ -1+(-5)^n & -1+7(-5)^n \end{pmatrix}$$

$$\rightarrow (6M^n)^{-1} = \frac{1}{6^2|M|^n}\begin{pmatrix} -1+7(-5)^n & -7+7(-5)^n \\ 1-(-5)^n & 7-(-5)^n \end{pmatrix}$$

$$\rightarrow \frac{1}{6}(M^n)^{-1} = \frac{1}{6^2|M|^n}\left[\begin{pmatrix} -1 & -7 \\ 1 & 7 \end{pmatrix} + (-5)^n\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix}\right]$$

* 矩陣加減 = 各自元素做加減

* 用 adj. matrix 搵逆矩陣

* $|6M^n| = 6^2|M|^n$

* $(6M^n)^{-1} = 1/6(M^n)^{-1}$

CONT'D

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$$\rightarrow (M^n)^{-1} = \frac{1}{6(-5)^n}\left[\begin{pmatrix} -1 & -7 \\ 1 & 7 \end{pmatrix} + (-5)^n \begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix}\right]$$

$$= \frac{1}{6}\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix} + \frac{1}{(-5)^n}\frac{1}{6}\begin{pmatrix} -1 & -7 \\ 1 & 7 \end{pmatrix}$$

$$\therefore A = \frac{1}{6}\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix}, \ B = \frac{1}{6}\begin{pmatrix} -1 & -7 \\ 1 & 7 \end{pmatrix}$$

* 2x2 Determinent 用類似交差相乘相加減計算

2019 – SECTION B

Q12.) P, Q, R *lying on the plane* ΔABC *such that* $\overrightarrow{OP} = p\hat{i}, \overrightarrow{OQ} = q\hat{j}$ *and* $\overrightarrow{OR} = r\hat{k}$.

Given that,

$$\overrightarrow{OA} = \hat{i} - 4\hat{j} + 2\hat{k}$$
$$\overrightarrow{OB} = -5\hat{i} - 4\hat{j} + 8\hat{k}$$
$$\overrightarrow{OC} = -5\hat{i} - 12\hat{j} + t\hat{k}, \text{ where } t \text{ is constant}$$
$$\overrightarrow{OE} = \frac{1}{p}\hat{i} + \frac{1}{q}\hat{j} + \frac{1}{r}\hat{k}$$
$$AC = BC$$

a.) *Prove* $pqr \neq 0$.

b.) *Let* D *be the projection of* O *on* ΔABC. $\overrightarrow{OD} = ?$

c.) *The geometric relationship of* D, E *and* O

* 參考課程 1.2, 4.4 及 4.5

CONT'D

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$a.)$ *Assume* $pqr = 0$, *then either* $\overrightarrow{OP}$, $\overrightarrow{OQ}$, *or* $\overrightarrow{OR}$ *is* $\vec{0}$

$\rightarrow$ *O is on the plane* ΔABC ($\because$ *P, Q, R are on* ΔABC)

$$\because AC = BC \rightarrow |\overrightarrow{OC} - \overrightarrow{OA}|^2 = |\overrightarrow{OC} - \overrightarrow{OB}|^2$$

$$\rightarrow 36 + 64 + (t-2)^2 = 64 + (t-8)^2$$

$$\rightarrow t = 2$$

Hence, the volume of OABC pyramid, $V = \dfrac{1}{6}|(k\hat{n}) \cdot \overrightarrow{AO}|$

where the normal of ΔABC, $k\hat{n} = \overrightarrow{AB}\ x\ \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 0 & 6 \\ -6 & -8 & 0 \end{vmatrix}$

$$= 12(4\hat{i} - 3\hat{j} + 4\hat{k})$$

* 利用反証法揾矛盾

* 四面體體積 = 1/6平行六面體體積

* 3x3 矩陣可用類似交叉相乘相加減計算 determinent

CONT'D

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$$\therefore\ V = \frac{1}{6}\cdot\left|12(4\hat{i}-3\hat{j}+4\hat{k})\cdot\boxed{(-\hat{i}+4\hat{j}-2\hat{k})}\right|$$

$$= 2|(-4-12-8)| = 48 > 0$$

$\because V > 0 \rightarrow O$ *is not on the plane* ΔABC

$\therefore$ *contradiction exists*

$i.e.\ pqr \neq 0$

b.) *Let h be the height from O to the plane ABC*

$$V = \frac{1}{3}\left(\frac{1}{2}|\overrightarrow{AB}\ x\ \overrightarrow{AC}|\right)h \rightarrow h = \frac{288}{|\overrightarrow{AB}\ x\ \overrightarrow{AC}|}$$

$$Also,\ \hat{n} = \frac{\overrightarrow{AB}\ x\ \overrightarrow{AC}}{|\overrightarrow{AB}\ x\ \overrightarrow{AC}|} \rightarrow h\hat{n} = \frac{288\ \overrightarrow{AB}\ x\ \overrightarrow{AC}}{|\overrightarrow{AB}\ x\ \overrightarrow{AC}|^2}$$

* ■ 方向倒轉, vector 乘 -1 $\overrightarrow{AO} = -\overrightarrow{OA}$

* 有體積, 四點不在同一平面

* 四面體體積=1/3 (三角底面積) (高)

$* \ \hat{a} = \frac{\overrightarrow{a}}{|\overrightarrow{a}|}$

CONT'D

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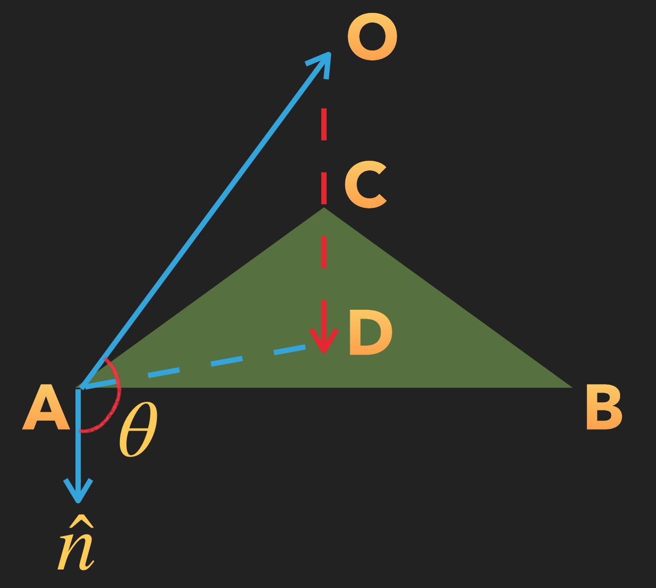
$$\because \hat{n}\cdot\overrightarrow{AO} < 0 \to \textit{the angle between } \hat{n} \textit{ and } \overrightarrow{AO} > \frac{\pi}{2}$$

$$\therefore \overrightarrow{OD} = h\hat{n} = \frac{24}{41}(4\hat{i} - 3\hat{j} + 4\hat{k})$$



* 確保 Normal Vector 同 OD 同方向
$(\theta > 90^o)$

c.) $\because$ *P, Q, R are on the plane* ΔABC

$$\therefore \begin{cases} k\hat{n}\cdot\overrightarrow{PQ} = 0 \to k\hat{n}\cdot(\overrightarrow{OQ} - \overrightarrow{OP}) = 0 \to -3q = 4p \\ k\hat{n}\cdot\overrightarrow{PR} = 0 \to k\hat{n}\cdot(\overrightarrow{OR} - \overrightarrow{OP}) = 0 \to p = r \\ k\hat{n}\cdot\overrightarrow{QR} = 0 \to k\hat{n}\cdot(\overrightarrow{OR} - \overrightarrow{OQ}) = 0 \to -3q = 4r \end{cases}$$

* 所有平面上的 vector 同 normal 互相垂直

$$\textit{Hence, } (p, q, r) = \left(r, \ -\frac{4r}{3}, r\right)$$

$$\overrightarrow{OE} = \frac{1}{4r}(4\hat{i} - 3\hat{j} + 4\hat{k}) = \frac{1}{4r}\frac{41}{24}\overrightarrow{OD} \to OE//OD \ (r \neq 0)$$

* 一支 vector 係另一支 vector 倍數, 兩支 vector 互相平行

i.e. O, D, E are collinear