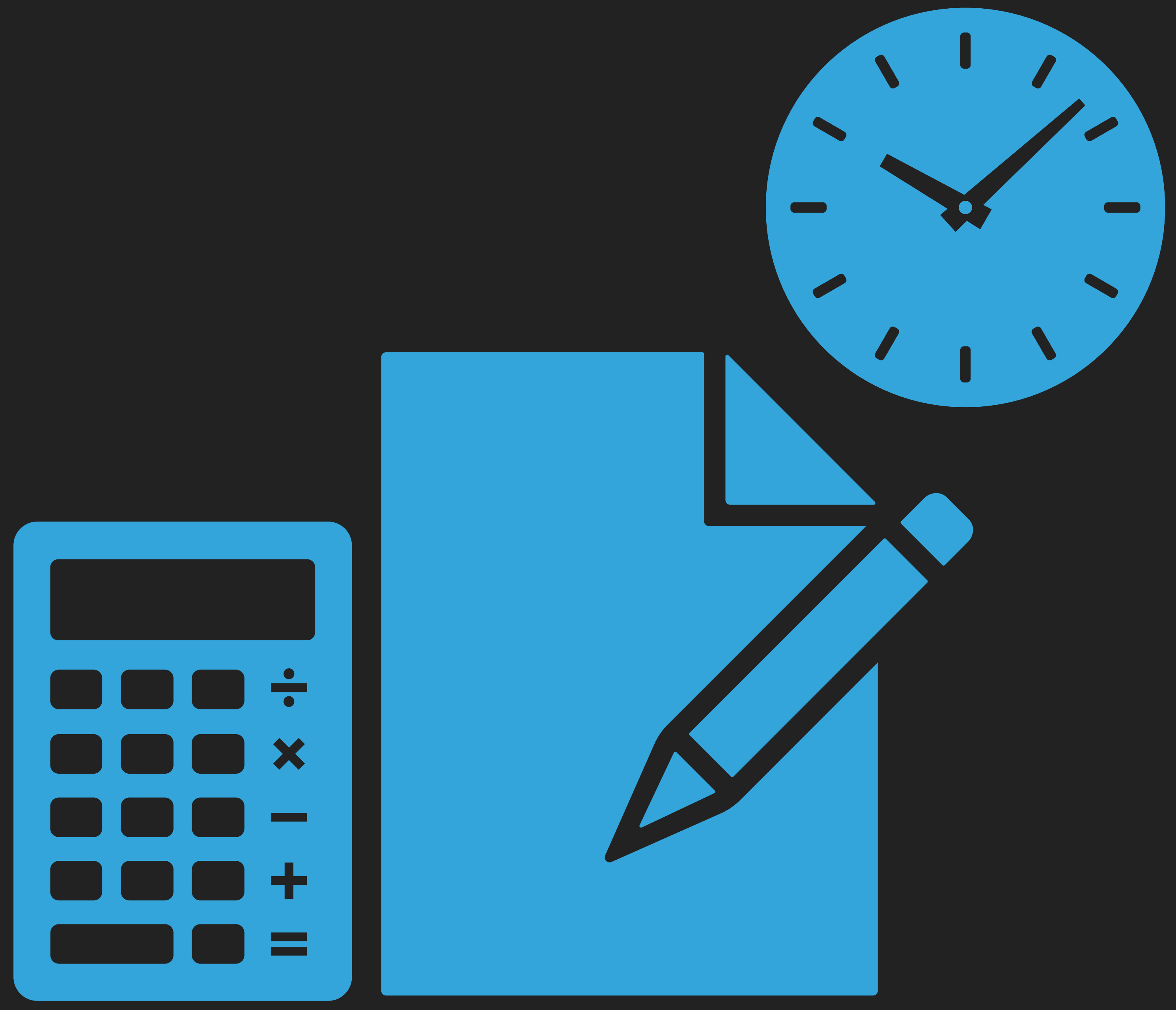
深宵教室 - DSE M2 模擬試題解答

2017

此為參考2017試題之模擬試題，原版請另行購買

2017

- Section A
- Section B



2017 - SECTION A

Q1.) $f(x) = sec6x$. $f'(x) = ?$ (By First Principles)

* 參考課程 2.2, 3.1 及 3.2

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}(sec(6x+6h) - sec6x)$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{cos6x - cos(6x+6h)}{cos(6x+6h)cos6x}$$

$$= \lim_{h\to 0} \frac{2sin(6x+3h)sin3h}{hcos(6x+6h)cos6x}$$

$$= \lim_{h\to 0} \frac{6sin(6x+3h)}{hcos(6x+6h)cos6x} \lim_{h\to 0} \frac{sin3h}{3h}$$

$$= \frac{6sin(6x+0)}{cos(6x+0)cos6x}(1) = 6tan6xsec6x$$

* 微分定義

* Sum to product

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

* lim 可乘除

* $\lim_{h\to 0} \dfrac{sinh}{h} = 1$

2017 - SECTION A

$Q2.)\ (1+ax)^8 = \lambda_0 + \lambda_1 x + \ldots + \lambda_8 x^8,\ (b+x)^9 = \mu_0 + \mu_1 x + \ldots + \mu_9 x^9,\ a, b \in \mathbb{R}$

$\lambda_2 : \mu_7 = 7 : 4,\ and\ \lambda_1 + \mu_8 + 6 = 0,\ a = ?$

* 參考課程 1.1

$$(1+ax)^8 \equiv \sum_{r=0}^{8} C_r^8 a^r x^r, \quad (b+x)^9 \equiv \sum_{r=0}^{9} C_r^9 b^{9-r} x^r$$

$$Hence,\ \lambda_r = C_r^8 a^r,\ \mu_r = C_r^9 b^{9-r}$$

$$\begin{cases} C_2^8 a^2 : C_7^9 b^2 = 7 : 4 & \text{——— (1)} \\ C_1^8 a + C_8^9 b + 6 = 0 & \text{——— (2)} \end{cases}$$

$$In\ (2):\ 8a + 9b = -6 \rightarrow b = \frac{-6-8a}{9} \quad \text{——— (3)}$$

$$Put\ (3)\ into\ (1):\ 4C_2^8 a^2 = 7C_7^9 \left(\frac{-6-8a}{9}\right)^2$$

$$\rightarrow 7a^2 + 24a + 9 = 0 \rightarrow a = \frac{-3}{7}\ or\ a = -3$$

*** Binomial Expansion**

$$* \ C_r^n = \frac{n!}{r!(n-r)!}$$

$$* \ C_r^n = \frac{n!}{r!(n-r)!}$$

2017 - SECTION A

Q3.) There is a point P lying on AB such that AP : PB = 3 : 2.

$$OA = 45,\ OB = 20 \text{ and } \cos\angle AOB = \frac{1}{4},\ OP = ?$$

* 參考課程 4.2 及 4.3

方法1

$$\overrightarrow{OP} = \frac{2}{5}\overrightarrow{OA} + \frac{3}{5}\overrightarrow{OB}$$

$$\text{Hence, } |\overrightarrow{OP}|^2 = \left(\frac{2}{5}\overrightarrow{OA} + \frac{3}{5}\overrightarrow{OB}\right)\cdot\left(\frac{2}{5}\overrightarrow{OA} + \frac{3}{5}\overrightarrow{OB}\right)$$

$$= \frac{4}{25}|\overrightarrow{OA}|^2 + \frac{12}{25}(\overrightarrow{OA}\cdot\overrightarrow{OB}) + \frac{9}{25}|\overrightarrow{OB}|^2$$

$$= \frac{4}{25}|\overrightarrow{OA}|^2 + \frac{12}{25}|\overrightarrow{OA}||\overrightarrow{OB}|\cos\angle AOB + \frac{9}{25}|\overrightarrow{OB}|^2$$

$$= 576 \ \rightarrow OP = 24 \text{ unit}$$

* 分割公式

* $|\vec{a}|^2 = \vec{a}\cdot\vec{a}$

* Dot product 可以拆括號

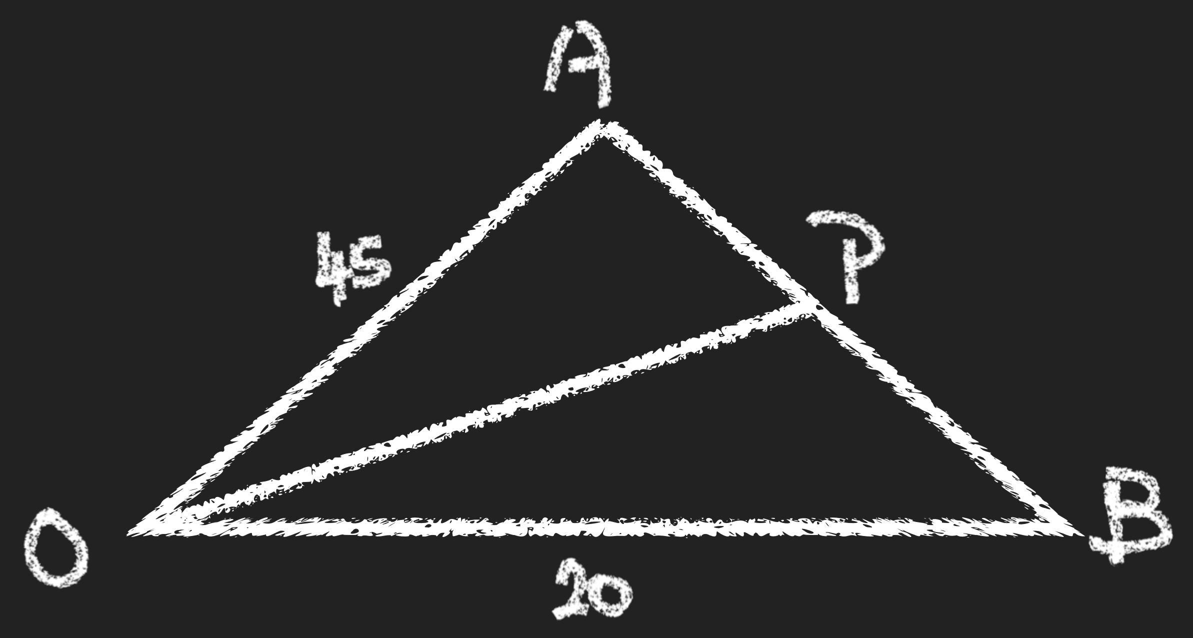
* $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

CONT'D

2017 - SECTION A

方法2

Consider the following figure :



$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB\cos\angle AOB$$
$$= 1975$$

* Cosine law 响 OAB

$$\cos\angle ABO = \frac{AB^2 + OB^2 - OA^2}{2AB \cdot OB}$$

$$\text{Then, } OP^2 = PB^2 + OB^2 - 2PB \cdot OB\cos\angle ABO$$

* Cosine law 响 OPB

$$\rightarrow OP^2 = (\frac{2AB}{5})^2 + OB^2 - 2\frac{\cancel{2AB}}{5} \cdot \cancel{OB}(\frac{AB^2 + OB^2 - OA^2}{\cancel{2AB \cdot OB}})$$

$$\rightarrow OP^2 = \frac{-6}{25}AB^2 + \frac{3}{5}OB^2 + \frac{2}{5}OA^2 = 576$$

$$\rightarrow OP = 24 \text{ unit}$$

2017 - SECTION A

Q4.) Find the area bounded by $y = x^2e^{-x}$, x – axis and $x = 6$.

* 參考課程 3.10 及 3.11

$$\text{The area, } A = \int_0^6 x^2 e^{-x}dx = \int_0^6 x^2 d(-e^{-x})$$

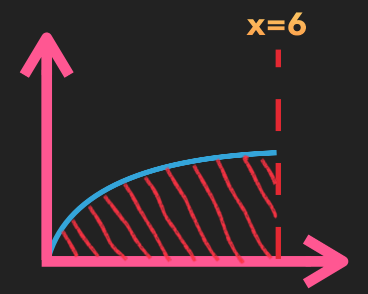
$$= [-x^2e^{-x}]_0^6 + \int_0^6 2xe^{-x}dx$$

$$= [-x^2e^{-x}]_0^6 + \int_0^6 2xd(-e^{-x})$$

$$= [-x^2e^{-x}]_0^6 + [-2xe^{-x}]_0^6 + \int_0^6 2e^{-x}dx$$

$$= [-x^2e^{-x} - 2xe^{-x} - 2e^{-x}]_0^6 = 2 - 50e^{-6}\ sq.unit$$

* 用手 Sketch 了解要搵的面積



* 積分三寶: Integration by part

2017 - SECTION A

$$Q5.)\quad \begin{cases} x+2y-z=11 \\ 3x+8y-11z=49 \\ 2x+3y+hz=k \end{cases} \text{—— (E)} \qquad h,k\in\mathbb{R}$$

a.) *The range of h and z = ? if (E) has unique solution*

b.) $h=?,\ k=?$ *and the solution if (E) has infinite many solution*

* 參考課程 4.7

$$(E):\left(\begin{array}{ccc|c} 1 & 2 & -1 & 11 \\ 3 & 8 & -11 & 49 \\ 2 & 3 & h & k \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 11 \\ 0 & 2 & -8 & 16 \\ 0 & -1 & h+2 & k-22 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 11 \\ 0 & 2 & -8 & 16 \\ 0 & 0 & 2h-4 & 2k-28 \end{array}\right)$$

a.) *(E) has unique solution* $\rightarrow h\neq 2,\ and\ z=\dfrac{k-14}{h-2}$

* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

* 如果 ■ 不等如 0, 有唯一答案

* 如果 ■ = 0, ■ =0, 有直線答案

CONT'D

2017 - SECTION A

$b.)\ (E)\ has\ infinite\ many\ solution \rightarrow h = 2,\ and\ k = 14$

$$(E): \left(\begin{array}{ccc|c} 1 & 2 & -1 & 11 \\ 0 & 2 & -8 & 16 \end{array}\right)$$

$Let\ z = t,\ t \in \mathbb{R}$

$(x, y, z) = (-3(1+t), 4(2+t), t)$

* 三條公式剩返兩條

2017 - SECTION A

Q6.) Consider a inverted right circular cone container, with the depth of water is increasing at a rate $= \dfrac{3}{\pi}\ cms^{-1}$. *Find the rate of change of the wet curved surface area when the volume* $= 96\pi\ cm^3$, *and the container has radius* $= 15cm$ *and height* $= 20cm$

* 參考課程 3.3 及 3.4

a.) Let A be the curved wet surface area

h be the depth of water

V be the volume of water

$$\text{Then, } \frac{A}{\pi(15)(\sqrt{15^2+20^2})} = \left(\frac{h}{20}\right)^2 \rightarrow A = \frac{15}{16}\pi h^2$$

$$\frac{V}{\frac{1}{3}\pi(15)^2 20} = \left(\frac{h}{20}\right)^3 \rightarrow V = \frac{3}{16}\pi h^3$$

* 先 let 符號方便計算

* 數學 Core 課程: 相似圖形特性

* 曲面表面面積 $= \pi rl$

* 圓錐體體積 $= \frac{1}{3}\pi r^2 h$

CONT'D

2017 - SECTION A

$$Hence,\ \frac{dA}{dt} = \frac{15}{16}\pi 2(h)\frac{dh}{dt}$$

$$When\ V = 96\pi \to 96\pi = \frac{3}{16}\pi h^3 \to h = 8$$

$$\therefore \frac{dA}{dt}\bigg|_{V=96\pi} = \frac{15}{8}\pi(8)\frac{3}{\pi} = 45$$

$$i.e.\ The\ rate\ of\ change\ of\ the\ curved\ wet\ area = 45\ cm^2 s^{-1}$$

* Implicit 微分法

2017 - SECTION A

$$Q7.)\ Solve\ \frac{cos3x+sin3x}{cosx-sinx}=2,\ where\ \frac{\pi}{4}<x<\frac{\pi}{2}$$

* 參考課程 2.2 及 2.3

方法1

$$Let\ A_1sin(B_1+3x)\equiv cos3x+sin3x$$
$$A_2sin(B_2-x)\equiv cosx-sinx$$

$$\rightarrow A_1sinB_1cos3x+A_1cosB_1sin3x\equiv cos3x+sin3x$$
$$A_2sinB_2cosx-A_2cosB_2sinx\equiv cosx-sinx$$

$$\rightarrow \begin{cases} A_1sinB_1=1 & \text{———— (1)} \\ A_1cosB_1=1 & \text{———— (2)} \end{cases} \quad \begin{cases} A_2sinB_2=1 & \text{———— (3)} \\ A_2cosB_2=1 & \text{———— (4)} \end{cases}$$

$$By\ consider\ (1)^2+(2)^2,\ (3)^2+(4)^2,\ A_1=A_2=\sqrt{2}$$

$$By\ consider\ \frac{(1)}{(2)},\ \frac{(3)}{(4)},\ B_1=B_2=\frac{\pi}{4}$$

* 利用 Compare coefficient 改變成Asin(B+x)

* sin 複角公式

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$
$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$
$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$
$$2\sin A\cos B=\sin(A+B)+\sin(A-B)$$
$$2\cos A\cos B=\cos(A+B)+\cos(A-B)$$
$$2\sin A\sin B=\cos(A-B)-\cos(A+B)$$
$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

CONT'D

2017 - SECTION A

$$Hence,\ (E):\ \frac{cos3x+sin3x}{cosx-sinx}=2 \rightarrow \frac{sin(\frac{\pi}{4}+3x)}{sin(\frac{\pi}{4}-x)}=2$$

$$\rightarrow \frac{sin(\pi-(\frac{3\pi}{4}-3x))}{sin(\frac{\pi}{4}-x)}=2$$

$$\rightarrow \frac{sin3(\frac{\pi}{4}-x)}{sin(\frac{\pi}{4}-x)}=2$$

$$Let\ y=\frac{\pi}{4}-x,\ (\because \frac{\pi}{4}<x<\frac{\pi}{2}\rightarrow -\frac{\pi}{4}<y<0)$$

$$Then,\ (E):\ sin3y=2siny \rightarrow sin3y-siny-siny=0$$

$$\rightarrow 2cos2ysiny-siny=0$$

$$\rightarrow siny(2cos2y-1)=0$$

$$\therefore siny=0\ (rejected)\ or\ cos2y=\frac{1}{2}$$

* $sin(\pi-\theta)=sin\theta$

* **Sum to product**

$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$

$2\sin A\sin B=\cos(A-B)-\cos(A+B)$

$\sin A+\sin B=2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A-\sin B=2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A+\cos B=2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A-\cos B=-2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

CONT'D

2017 - SECTION A

$$\because -\frac{\pi}{4} < y < 0 \rightarrow -\frac{\pi}{2} < 2y < 0$$

$$\therefore 2y = -\frac{\pi}{3} \rightarrow y = -\frac{\pi}{6} \rightarrow x = \frac{5\pi}{12}$$

方法2

$$\frac{cos3x + sin3x}{cosx - sinx} = 2 \rightarrow cos(2x + x) + sin(2x + x) = 2(cosx - sinx)$$

$$\rightarrow cos2xcosx - sin2xsinx + sin2xcosx + cos2xsinx = 2(cosx - sinx)$$

$$\rightarrow (2cos^2x - 1)cosx + (1 - 2sin^2x)sinx = (2 - sin2x)(cosx - sinx)$$

$$\rightarrow 2(cos^3x - sin^3x) = (3 - sin2x)(cosx - sinx)$$

$$\rightarrow 2(cosx - sinx)(1 + sinxcosx) = (3 - sin2x)(cosx - sinx)$$

$$\rightarrow (cosx - sinx)(2 + 2sinxcosx) = (3 - sin2x)(cosx - sinx)$$

$$\rightarrow (cosx - sinx)(2 + sin2x) = (3 - sin2x)(cosx - sinx)$$

$$\rightarrow (cosx - sinx)(2sin2x - 1) = 0$$

* sin 複角公式
* cos 複角公式

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

* cos 雙角公式
* $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
* sin 雙角公式

CONT'D

2017 - SECTION A

$$\rightarrow cosx = sinx \ or \ sin2x = \frac{1}{2}$$

$$\rightarrow tanx = 0 \ (rejected) \ or \ 2x = \pi - \frac{\pi}{6} \quad (\because \frac{\pi}{4} < x < \frac{\pi}{2} \rightarrow \frac{\pi}{2} < 2x < \pi)$$

$$\rightarrow x = \frac{5\pi}{12}$$

* 留意角度範圍

2017 - SECTION A

Q8.) Suppose a curve $\Gamma : y = f(x), x \in \mathbb{R}^+$, *given that* Γ *passes through* $P = (e^3, 7)$ *and*

$$f'(x) = \frac{2}{x}lnx, \forall\ x > 0, find$$

a.) Equation of tangent to Γ *at* P

b.) Equation of Γ

c.) Point(s) of inflexion of Γ

* 參考課程 3.4, 3.5, 3.6 及 3.7

a.) The equation of tangent to Γ *at* P, L_1 :

$$\frac{y-7}{x-e^3} = f'(e^3) \rightarrow y - 7 = 6e^{-3}(x - e^3)$$

$$\rightarrow y = 6e^{-3}x + 1$$

$$b.)\ f(x) = \int f'(x)dx = \int \frac{2}{x}lnxdx$$

* Point-slope form

* 積分係類似微分逆函數

CONT'D

2017 - SECTION A

$$= \int 2lnx \frac{1}{x} dx = \int 2lnx\, d(lnx) = (lnx)^2 + C, where\ C\ is\ constant$$

* 積分三寶: 積分代入

$\because \Gamma$ *passes through* $P \rightarrow f(e^3) = 7$

$\therefore C = -2$

$i.e. f(x) = (lnx)^2 - 2$

* 利用題目資料搵 C

$$c.)\ f'(x) = \frac{2}{x} lnx \rightarrow xf'(x) = 2lnx \rightarrow f'(x) + xf''(x) = \frac{2}{x}$$

$$Let\ x_0 \in \mathbb{R}^+ \ such\ that\ f''(x_0) = 0 \rightarrow f'(x_0) = \frac{2}{x_0} \rightarrow 2lnx_0 = 2$$

$$\rightarrow x_0 = e$$

$\therefore$ *The point of inflexion* $= (e, -1)$

* 用 Implicit 微分法

* 搵 pt. of inflexion = 搵 x_0 使度 $f''(x_0)=0$

2017 - SECTION B

$Q9.)$ *Assume a curve* $\Gamma : y = \dfrac{x^2 - 5x}{x+4}$, *where* $x \neq -4$

a.) *Find the asymptote(s) of* Γ

b.) *Find local max. and min. points of* Γ

c.) *Find volume of solid bounded by* Γ *and* $x-$*axis revloving by the* $x-$*axis*

* 參考課程 3.5 及 3.12

a.) *Vertical Asymptote :* $x = -4$

Horizontal Asymptote : No Horizontal Asymptote

$$y = \frac{(x+4)(x-9)+36}{x+4} = (x-9) + \frac{36}{x+4}$$

Oblique Asymptote : $y = x - 9$

* x 係幾多, 分母係零

* *Find* $\lim\limits_{x\to\infty} y$

* *Find m and c such that* $\lim\limits_{x\to\infty} [y - (mx+c)] = 0$

$$y = (x-9) + \frac{36}{x+4} \to y - (x-9) = \frac{36}{x+4}$$

$$\to \lim_{x\to\infty} (y - (x-9)) = 0$$

CONT'D

2017 - SECTION B

b.)

$$\frac{dy}{dx} = 1 - \frac{36}{(x+4)^2} = \frac{(x+4)^2 - 36}{(x+4)^2} = \frac{(x-2)(x+10)}{(x+4)^2}$$

$$Let\ x_0 \in \mathbb{R}\ such\ that\ \frac{dy}{dx}\Big|_{x=x_0} = 0 \rightarrow x_0 = -10\ or\ 2$$

	x < -10	x = -10	-10 < x < 2	x = 2	x > 2
y′	+	0	-	0	+
y	Up.		Down.		Up.

$\therefore The\ local\ min.\ point = (2, -1)$

$The\ local\ max.\ point = (-10, -25)$

* $a^2 - b^2 = (a+b)(a-b)$

* 搵 turning point = 搵 x_0 使度 $y'(x_0)=0$

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow Increasing$

$f'(x) < 0 \rightarrow Decreasing$

CONT'D

2017 - SECTION B

$c.)$

$$The\ x-interception\ points\ are\ (0,0)\ and\ (5,0)$$

$$\therefore\ The\ volume,\ V=\pi\int_0^5 y^2dx=\pi\int_0^5[(x-9)+\frac{36}{x+4}]^2dx$$

$$=\pi\int_0^5(x-9)^2+\frac{72(x-9)}{x+4}+\frac{36^2}{(x+4)^2}dx$$

$$=\pi[\frac{(x-9)^3}{3}-\frac{36^2}{(x+4)}]_0^5+72\pi\int_0^5\frac{(x+4)-13}{x+4}dx$$

$$=\pi[\frac{(x-9)^3}{3}-\frac{36^2}{(x+4)}+72x-936ln(x+4)]_0^5$$

$$=\pi(\frac{2285}{3}+1872ln\frac{2}{3})\ cu.unit$$

* 先搵旋轉範圍

* 積分可加減

2017 - SECTION B

Q10.) In ΔABC, D is mid – point of AC. E is a point lying on BC such that $BE : EC = 1 : r$.

There is a point F forming a straight line with AB and DE, such that $DE : EF = 1 : 10$

Given that,

$$\overrightarrow{OA} = 2\hat{i} + 3\hat{j} - 2\hat{k}$$
$$\overrightarrow{OB} = 4\hat{i} + 4\hat{j} - \hat{k}$$
$$\overrightarrow{OC} = 8\hat{i} - 3\hat{j} - 2\hat{k}$$
$$\overrightarrow{OP} = 3\hat{i} + 10\hat{j} - 4\hat{k}$$

a.) $r = ?$

b.) Prove B, C, D, F are concyclic

c.) The volume of tetrahedron $ABPQ$, where Q is the circumcenter of ΔBCF

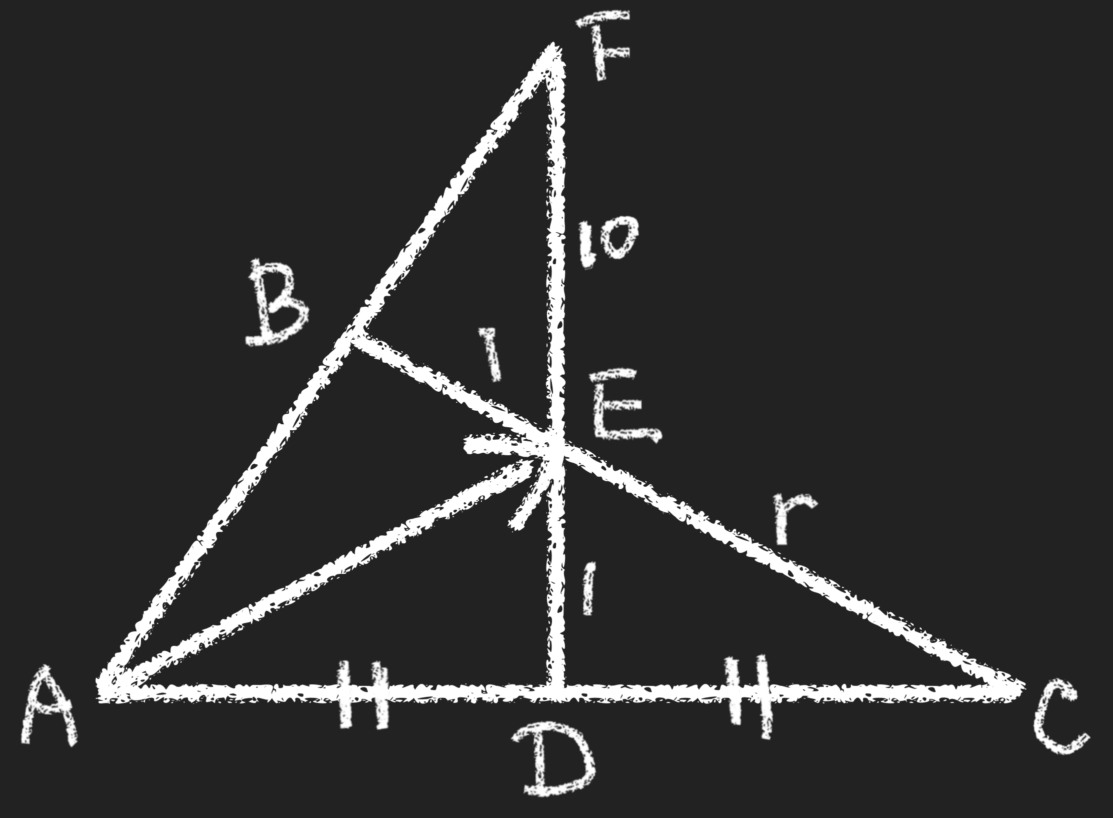
* 參考課程 4.4 及 4.5

CONT'D

2017 - SECTION B

$a.)$ *Consider the following graph :*

$Let\ AF = nAB,\ n \in \mathbb{R}$



$$\overrightarrow{AE} = \frac{\overrightarrow{AC}}{r+1} + \frac{r\overrightarrow{AB}}{r+1}$$

$$\overrightarrow{AE} = \frac{10\overrightarrow{AD}}{11} + \frac{\overrightarrow{AF}}{11} = \frac{5\overrightarrow{AC}}{11} + \frac{n\overrightarrow{AB}}{11}$$

$$\therefore \frac{1}{r+1} = \frac{5}{11}\ and\ \frac{r}{r+1} = \frac{n}{11} \rightarrow r = 1.2\ and\ n = 6$$

$b.)$

方法1

$$Consider,\ \overrightarrow{AB} \cdot \overrightarrow{BC} = (\overrightarrow{OB} - \overrightarrow{OA}) \cdot (\overrightarrow{OC} - \overrightarrow{OB})$$
$$= (2\hat{i} + \hat{j} + \hat{k}) \cdot (4\hat{i} - 7\hat{j} - \hat{k}) = 0$$
$$\rightarrow AB \perp BC$$

* 用兩種方法表達一支 Vector
* 分割定理在 BE:EC
* 分割定理在 DE:EF
* $A\vec{a} + B\vec{b} = C\vec{a} + D\vec{b} \rightarrow A = C\ and\ B = D$
* $\vec{a} \cdot \vec{b} = 0 \rightarrow \vec{a} \perp \vec{b}$

CONT'D

2017 - SECTION B

$$\text{Consider, } \overrightarrow{DF} \cdot \overrightarrow{AC} = (\overrightarrow{AF} - \overrightarrow{AD}) \cdot (\overrightarrow{OC} - \overrightarrow{OA})$$
$$= (6\overrightarrow{AB} - \frac{1}{2}\overrightarrow{AC}) \cdot (\overrightarrow{OC} - \overrightarrow{OA})$$
$$= (9\hat{i} + 9\hat{j} + 6\hat{k}) \cdot (6\hat{i} - 6\hat{j}) = 0$$
$$\rightarrow DF \perp AC$$

* $\vec{a} \cdot \vec{b} = 0 \rightarrow \vec{a} \perp \vec{b}$

$$\therefore \angle ABC = \angle FDC = \frac{\pi}{2}$$

i.e. B, C, D, F are concyclic (converse of ∠s in the same segment)

* Core 圓形証証明方法

方法2

$$\text{Consider, } |\overrightarrow{AB}| = |\overrightarrow{OB} - \overrightarrow{OA}| = \sqrt{6}$$
$$|\overrightarrow{AC}| = |\overrightarrow{OC} - \overrightarrow{OA}| = 6\sqrt{2}$$

$$\angle FAD = \angle CBA \ (\text{Common } \angle)$$

CONT'D

2017 - SECTION B

$$\frac{AF}{AC}=\frac{6AB}{AC}=\sqrt{3} \qquad \frac{AD}{AB}=\frac{\frac{1}{2}AC}{AB}=\sqrt{3}$$

$$\Delta AFD \sim \Delta ACB \ (ratio\ of\ 2\ sides,\ inc.\ \angle)$$

$$\angle AFD = \angle ACB \ (Prop.\ of\ \sim \Delta)$$

$$i.e.\ B, C, D, F\ are\ concyclic\ (converse\ of\ \angle s\ in\ the\ same\ segment)$$

c.) $\because \overrightarrow{AD}\cdot\overrightarrow{DE}=0 \rightarrow \angle ADE=\frac{\pi}{2}$

$\therefore FC$ *is the diameter of the circle passing through* D, F, C ($\angle$ *in semi circle*)

$i.e.\ Q$ *is a mid – point of* FC

$$\overrightarrow{AQ}=\frac{1}{2}(\overrightarrow{AF}+\overrightarrow{AC})=3\overrightarrow{AB}+\frac{1}{2}\overrightarrow{AC}=9\hat{i}+3\hat{k}$$

* Core 相似三角形証明

* Core 圓形証証明方法

* Core 圓形特性

* 分割定理在 FQ:QC

CONT'D

2017 – SECTION B

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \hat{i} + 7\hat{j} - 2\hat{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\hat{i} + \hat{j} + \hat{k}$$

$$(\overrightarrow{AB}\ x\ \overrightarrow{AP}) \cdot \overrightarrow{AQ} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 7 & -2 \\ 9 & 0 & 3 \end{vmatrix} = 9\begin{vmatrix} 1 & 1 \\ 7 & -2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & 7 \end{vmatrix} = -42$$

$$\therefore \text{The volume of tetrahedron } ABPQ = \frac{1}{6}(42) = 7\ cu\,.\,unit$$

* 用 Determinant 計算體積

* 四面體體積=1/6平行六面體體積

2017 - SECTION B

Q11.)

$$a.)\ \int_0^{\frac{\pi}{4}} \frac{1}{sin2x + cos2x + 2}dx = ?$$

$$b.)\ \int_0^{\frac{\pi}{4}} \frac{8sin2x + 9}{sin2x + cos2x + 2}dx = ?$$

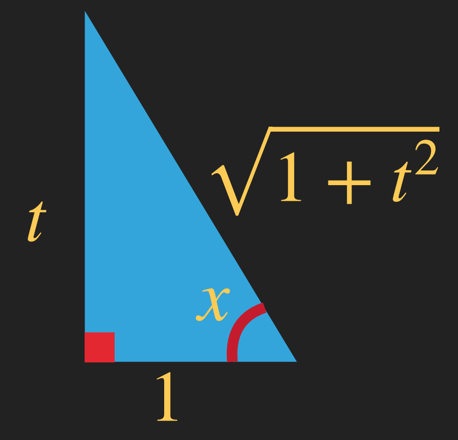
* 參考課程 2.2, 3.8 及 3.10

$$a.)\ Let\ I_1 = \int_0^{\frac{\pi}{4}} \frac{1}{sin2x + cos2x + 2}dx,\ Let\ t = tanx \rightarrow dt = sec^2xdx$$

$$we\ have,\ sin2x = 2sinxcosx = \frac{2t}{1+t^2}$$

$$cos2x = cos^2x - sin^2x = \frac{1-t^2}{1+t^2}$$

* 利用 T-method, $let\ t = tanx$



$$cosx = \frac{1}{\sqrt{1+t^2}}$$

$$sinx = \frac{t}{\sqrt{1+t^2}}$$

* sin 雙角公式, cos 雙角公式

CONT'D

2017 - SECTION B

$$Hence\ I_1 = \int_0^1 \frac{\frac{1}{1+t^2}dt}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 2} = \int_0^1 \frac{dt}{t^2+2t+3} = \int_0^1 \frac{dt}{(t+1)^2+2}$$

$$Let\ t+1 = \sqrt{2}tan\theta \rightarrow dt = \sqrt{2}sec^2\theta d\theta$$

$$Let\ \alpha = tan^{-1}\frac{\sqrt{2}}{2},\ \beta = tan^{-1}\sqrt{2}$$

$$Hence\ I_1 = \int_\alpha^\beta \frac{\sqrt{2}sec^2\theta d\theta}{2tan\theta^2+2} = \int_\alpha^\beta \frac{\sqrt{2}sec^2\theta d\theta}{2sec^2\theta} = \frac{\sqrt{2}}{2}(\beta-\alpha)$$

$$Consider,\ tan(\beta-\alpha) = \frac{tan\beta - tan\alpha}{1+tan\beta tan\alpha} = \frac{\sqrt{2}-\frac{\sqrt{2}}{2}}{1+\sqrt{2}\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{4}$$

$$\therefore\ I_1 = \frac{\sqrt{2}}{2}tan^{-1}\frac{\sqrt{2}}{4}$$

* 定積分代入要改範圍

* 利用三角代入法, $let\ t+1 = \sqrt{2}tan\theta$

* 定積分代入要改範圍

* $tan^2\theta + 1 = sec^2\theta$

* tan 複角公式

$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

CONT'D

2017 - SECTION B

$$Let\ I_2 = \int_0^{\frac{\pi}{4}} \frac{8sin2x + 9}{sin2x + cos2x + 2}dx$$

$$= 8\int_0^{\frac{\pi}{4}} \frac{sin2x + 1}{sin2x + cos2x + 2}dx + \int_0^{\frac{\pi}{4}} \frac{1}{sin2x + cos2x + 2}dx$$

$$= 8I_3 + I_1$$

$$where\ I_3 = \int_0^{\frac{\pi}{4}} \frac{sin2x + 1}{sin2x + cos2x + 2}dx = \int_{\frac{\pi}{4}}^{0} \frac{(sin2(\frac{\pi}{4} - y) + 1)d(-y)}{sin2(\frac{\pi}{4} - y) + cos2(\frac{\pi}{4} - y) + 2}$$

$$= \int_0^{\frac{\pi}{4}} \frac{cos2y + 1}{cos2y + sin2y + 2}dy$$

$$\rightarrow 2I_3 = \int_0^{\frac{\pi}{4}} \frac{sin2x + 1}{sin2x + cos2x + 2}dx + \int_0^{\frac{\pi}{4}} \frac{cos2x + 1}{sin2x + cos2x + 2}dx = \frac{\pi}{4}$$

* $Let\ y = \frac{\pi}{4} - x \rightarrow dy = -dx$
* 定積分代入要改範圍
* $sin(\frac{\pi}{2} - \theta) = cos\theta$
* $cos(\frac{\pi}{2} - \theta) = sin\theta$
* 定積分負數, 範圍倒轉

CONT'D

2017 - SECTION B

$$Hence\ I_2 = 4(2I_3) + I_1$$

$$= \pi + \frac{\sqrt{2}}{2} tan^{-1} \frac{\sqrt{2}}{4}$$

* 將所有加總得答案

2017 - SECTION B

$Q12.)$

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}, P = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$$

$$a.)\ Prove\ A^n = 3I_2^n + 3^{n-1}n\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \forall n \in \mathbb{Z}^+$$

$$b.)\ Prove\ B^n = 3I_2^n + 3^{n-1}n\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}, by\ considering\ P^{-1}BP, or\ otherwise, \forall n \in \mathbb{Z}^+$$

$$c.)\ Solve\ |A^m - B^m| = 4m^2$$

* 參考課程 1.2, 4.8, 4.9 及 4.11

CONT'D

2017 - SECTION B

$a.)$

方法1

$$Let\ P(n): \ A^n = 3I_2^n + 3^{n-1}n\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \forall n \in \mathbb{Z}^+$$

* 先 Let Statement

$$For\ P(1): \ L.H.S. = A = R.H.S.$$

* 証明 P(1) is true

$$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+,\ then\ P(k+1):$$

* 假設 P(k) is true. 証明 P(k+1) is true

$$\begin{aligned} L.H.S. &= A^{k+1} = A^kA \\ &= [3^kI_2 + 3^{k-1}k\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}][3I_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}] \\ &= 3^{k+1}I_2 + 3^k(k+1)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= R.H.S. \end{aligned}$$

$$\therefore P(k+1)\ is\ true\ if\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+$$

$$i.e.\ By\ M.I.,\ P(n)\ is\ true, \forall n \in \mathbb{Z}^+$$

* 寫結論

CONT'D

2017 - SECTION B

方法2

$$Let\ C=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ then\ A=3I_2+C$$

$$C^2=0\rightarrow C^n=0,\ \forall n\in Z^+\ and\ n>1$$

$$\therefore A^n=(3I_2+C)^n=3^nI_2+\sum_{r=0}^{n}C_r^n3^{n-r}C^r$$

$$=3^nI_2+3^{n-1}C_1^nC=3^nI_2+3^{n-1}nC$$

* 因為 **CI=IC,** 所以可以用 **Binomial Expansion**

$$b.)\ Using\ row\ deduction\ to\ find\ P^{-1}$$

$$\left(\begin{array}{cc|cc}-1&0&1&0\\2&-1&0&1\end{array}\right)\sim\left(\begin{array}{cc|cc}1&0&-1&0\\-2&1&0&-1\end{array}\right)\sim\left(\begin{array}{cc|cc}1&0&-1&0\\0&1&-2&-1\end{array}\right)$$

$$\therefore P^{-1}=\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}\rightarrow P^{-1}BP=\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}\begin{pmatrix}5&1\\-4&1\end{pmatrix}\begin{pmatrix}-1&0\\2&-1\end{pmatrix}$$

CONT'D

2017 - SECTION B

$$\rightarrow P^{-1}BP = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = A$$

$$\rightarrow P^{-1}B^nP = A^n$$

$$\rightarrow B^n = PA^nP^{-1} = P[3^nI_2 + 3^{n-1}nC]P^{-1}$$

$$= 3^nI_2 + 3^{n-1}nPCP^{-1}$$

$$= 3^nI_2 + 3^{n-1}n\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$$

$$c.)\ |A^m - B^m| = 4m^2 \rightarrow \left|3^{m-1}m\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\right)\right| = 4m^2$$

$$\rightarrow (3^{m-1}m)^2\begin{vmatrix} 2 & 0 \\ -4 & -2 \end{vmatrix} = 4m^2 \rightarrow -(3^{m-1}m)^2 = m^2 \quad (no\ solution,\ m > 0)$$

* $(P^{-1}BP)^n = P^{-1}B^nP$

* 矩陣拆括號須不影响次序

* $P^{-1}I_2P = I_2$

* $|kA_{nxn}| = k^n|A_{nxn}|$

* 不可能負數 = 正數