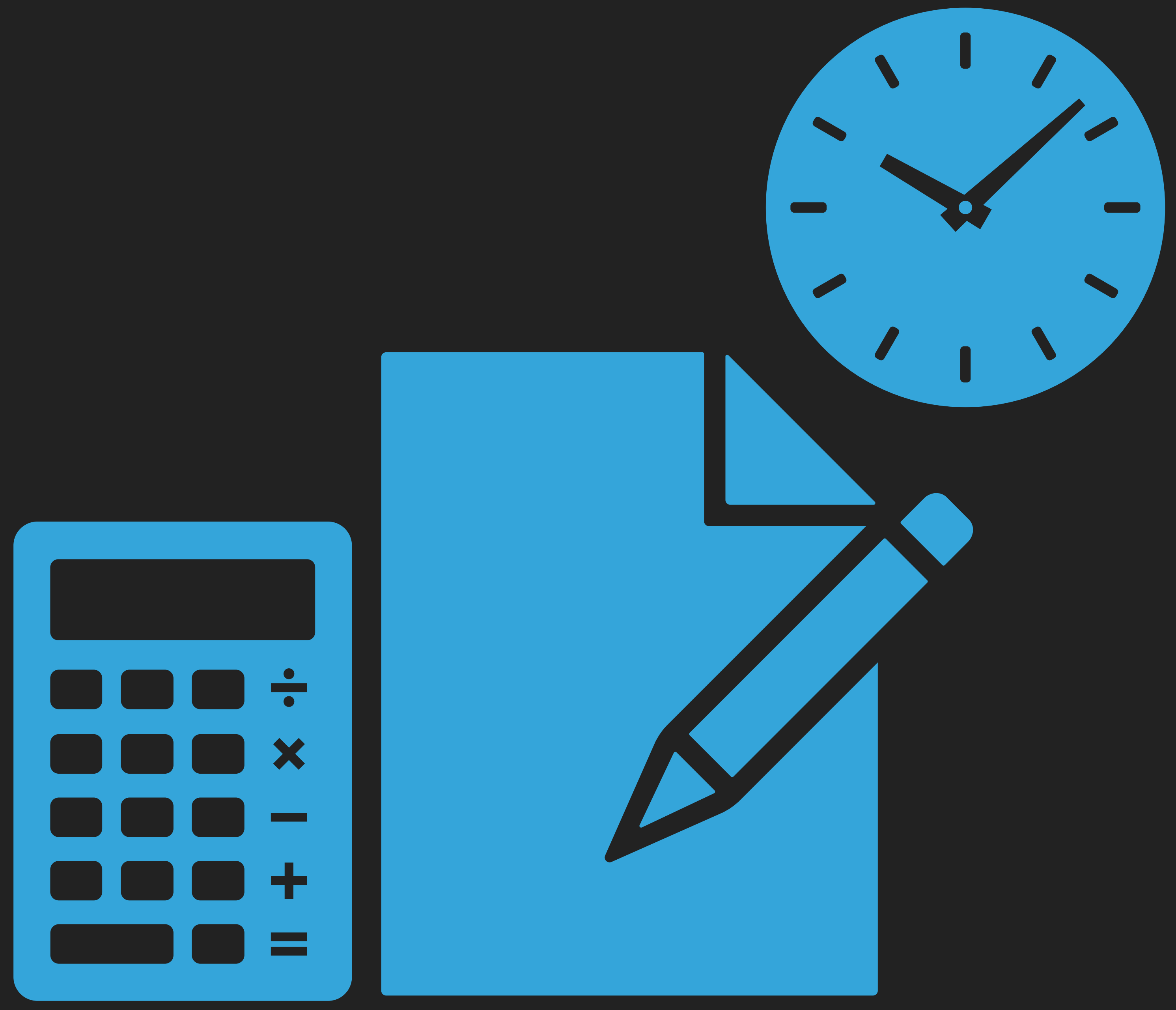
深宵教室 - DSE M1 模擬試題解答

2022

此為參考2022試題之模擬試題，原版請另行購買

2022

- Section A
- Section B



2022 - SECTION A

Q1.) Let X be discrete random vaiable, where a and b are constant

k	0	4	6
$P(X=k)$	0.1	a	b

Let $\bar{X}$ be the mean of 225 independent random selection of X

Given that $E(X)=4.6$, find a, b and $P(\bar{X}>4.75)$ by the central limit theorem.

* 參考課程 4.1, 4.3, 4.4 及 4.6

$$0.1+a+b=1 \rightarrow a+b=0.9 - (1)$$

$$E(X)=0\cdot 0.1+4\cdot a+6\cdot b=4.6 \rightarrow 2a+3b=2.3 - (2)$$

$$(2)-2(1):\quad b=0.5 \text{ and } a=0.4$$

$$E(X^2)=0\cdot 0.1+4^2\cdot a+6^2\cdot b=24.4$$

$$Var(X)=E(X^2)-[E(X)]^2=3.24$$

By the central limit theorem, $\bar{X}\sim N(4.6,\dfrac{3.24}{225})\sim N(4.6,0.12^2)$

$$P(\bar{X}>4.75)=P(Z>\frac{4.75-4.6}{0.12})=P(Z>1.25)=0.1056$$

* 機率總加 = 1
* $E(X)=\sum_{i=1}^{n}k_iP(X=k_i)$
* $Var(X)=E(X^2)-[E(X)]^2$
* 中央極限定理 - 當樣本數足夠多, 樣本平均值會趨向常態分佈
* 先計算 Z ~ N(0,1), 再對表

2022 - SECTION A

Q2.) Let X and Y be discrete random vaiable, where $Y = 200 - 4X$.
Given that $E(X) = 8.8$ *and* $Var(Y) = 144$
a.) Find $Var(X)$ *and* $E(Y)$.
b.) Can Y follows Poisson Distribution? Explain your answer.
c.) Can X follows Binominal Distribution? Explain your answer.

* 參考課程 4.3 及 4.4

a.) $E(Y) = 200 - 4E(X) = 200 - 4(8.8) = 164.8$

$Var(Y) = 4^2 Var(X) \rightarrow Var(X) = 0.0625(144) = 9$

b.) Assume $Y \sim Po(\lambda)$, $E(Y) = \lambda$ and $Var(Y) = \lambda$

However, $E(Y) \neq Var(Y) \rightarrow$ Contradiction exists.

$\therefore$ Y cannot follow Poisson Distribution

c.) Assume $X \sim B(n, p)$, where $0 \leq p \leq 1$. Then $E(X) = np \rightarrow np = 8.8$

$Var(X) = np(1-p) \rightarrow 9 = 8.8(1-p) \rightarrow p \approx -0.02272$

However, $p < 0 \rightarrow$ Contradiction exists.

$\therefore$ X cannot follow Binomial Distribution

* $E(aX + b) = aE(X) + b$

* $Var(aX + b) = a^2 Var(X)$

* 如果 Y~Po(a), E(Y)=Var(Y)=a

* 如果 X~B(n,p), E(X)=np, Var(X)=np(1-p)

2022 - SECTION A

Q3.) Let A and B be the event such that $P(A'|B) = 5P(A|B)$ *and* $P(A \cap B') = P(A \cap B) + 0.45$
where X' *is the complementary event of* X

a.) Find $P(A)$ *in term of* $P(B)$.

b.) Are A and B independent events? Explain your answer.

c.) Given C is an event with $P(C) = 0.6$. *Are A and C mutually exclusive? Explain your answer.*

* 參考課程 4.1 及 4.2

$$a.)\ P(A'|B) = 5P(A|B) \rightarrow 1 - P(A|B) = 5P(A|B) \rightarrow P(A|B) = \frac{1}{6}$$

$$\text{Also, } P(A \cap B') = P(A \cap B) + 0.45$$

$$\rightarrow P(B'|A)P(A) = P(A \cap B) + 0.45$$

$$\rightarrow [1 - P(B|A)]P(A) = P(A \cap B) + 0.45$$

$$\rightarrow P(A) = 2P(A \cap B) + 0.45 \rightarrow P(A) = 2P(A|B)P(B) + 0.45$$

$$= \frac{P(B)}{3} + 0.45$$

$$b.)\ \text{Assume A and B are independent} \rightarrow P(A|B) = P(A)$$

* P(Not A) = 1 - P(A)

* P(A & B)=P(A|B)P(B)=P(B|A)P(A)

* 如果獨立事件, P(A|B)=P(A)

CONT'D

2022 - SECTION A

$$\rightarrow \frac{1}{6} = \frac{P(B)}{3} + 0.45 \rightarrow P(B) \approx -0.85 < 0$$

$\text{However, } P(B) > 0 \rightarrow \text{Contradiction exists}$

$\therefore A \text{ and } B \text{ are not independent events}$

$c.) \text{ Assume } A \text{ and } C \text{ are mutually exclusive} \rightarrow P(A \cup C) = P(A) + P(C)$

$$\rightarrow P(A \cup C) = \frac{P(B)}{3} + 0.45 + 0.6 = \frac{P(B)}{3} + 1.05 > 1$$

$\text{However, } P(A \cup C) < 1 \rightarrow \text{Contradiction exists}$

$\therefore A \text{ and } C \text{ are not mutually exclusive events}$

* 機率係0同1之間

* 互拆事件, 沒有重疊地方
  P(A or B) = P(A) + P(B)
* 機率係0同1之間

2022 - SECTION A

Q4.) Let T hours be the time spent by a person complete a survey. It is known that the standard deviation of T is 0.4. To estimate the mean of T, 100 people is randomly selected to complete the survey. The total time spent by them is 150 hours.

a.) Find the 95 % confidence interval for the mean of T.

b.) Some data had been deleted. What is the affect for the width of the confidence interval in a.).

* 參考課程 4.6 及 4.7

a.) The sample mean of T, $\bar{T} = \dfrac{150}{100} = 1.5$

$$\text{The } 95\,\%\ C.I. = \left(\bar{T} - \boxed{1.96}\frac{0.4}{\sqrt{100}}, \bar{T} + \boxed{1.96}\frac{0.4}{\sqrt{100}}\right)$$

$$= (1.4216, 1.5784)$$

$$b.)\ \text{The width of } C.I. \text{ for n size} = 2 \cdot 1.96 \cdot \frac{0.4}{\sqrt{n}} = \frac{1.568}{\sqrt{n}}$$

$\therefore$ *The with of C.I. will increase when n decrease.*

* ■ 95% 置信區間

2022 - SECTION A

Q5.) Let $y = 64e^{-kx}$, where $k > 0$. If the coefficient of x^2 of $\sqrt{y}(1-2x)^5 = 449$

Find the slope of the graph of the linear function of lny

* 參考課程 1.1, 3.1 及 3.2

$$lny = ln(64e^{-kx}) = ln64 - kx$$

$$Also, y = 8e^{-\frac{kx}{2}}(1-2x)^5$$

$$= 8\left(1 - \frac{kx}{2} + \frac{(\frac{kx}{2})^2}{2!} + \ldots\right)\left(1 - C_1^5(2x) + C_2^5(2x)^2 + \ldots\right)$$

$$The\ coefficient\ of\ x^2 = 8(40 + 5k + \frac{k^2}{8}) = 449$$

$$\to k^2 + 40k - 129 = 0$$

$$\to k = 3\ or\ -43\ (rejected)$$

$\therefore$ *The slope of the graph of lny* $= -3$

* ln(AB) = lnA + lnB

* $(a+b)^n = \sum_{r=0}^{n} C_r^n a^r b^{n-r}$

* $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

* $C_r^n = \frac{n!}{r!(n-r)!}$

$\to C_1^n = n\ and\ C_2^n = \frac{n(n-1)}{2}$

* 直線方程, y = (斜率)x + (y-intercept)

2022 - SECTION A

Q6.) Let the curve $C: y = f(x)$, $f(x) = (9-4x^2)(6+2x^2)^{-1}$. If the tangent L to C passing through point $A(3, -2)$, find the equation of L.

* 參考課程 2.2, 2.3 及 2.4

$Let\ L\ touchs\ C\ at\ point\ B(a, f(a))$

$Consider,\ f(x) = (9-4x^2)(6+2x^2)^{-1} \rightarrow (6+2x^2)f(x) = 9-4x^2$

$\rightarrow 4xf(x) + (6+2x^2)f'(x) = -8x \rightarrow f'(a) = \dfrac{-4a(2+f(a))}{6+2a^2}$

$Then\ \ the\ slope\ of\ L = f'(a) = \dfrac{f(a)+2}{a-3} \rightarrow \dfrac{-4a(2+f(a))}{6+2a^2} = \dfrac{f(a)+2}{a-3}$

$\rightarrow 6(a^2-2a+1)(2+f(a)) = 0 \rightarrow 6(a-1)^2(\dfrac{21}{6+2a^2}) = 0 \rightarrow a = 1$

$Then\ \ the\ slope\ of\ L = \dfrac{\frac{5}{8}+2}{1-3} = -\dfrac{21}{16}$

$\rightarrow The\ equation\ of\ L:\ \ 21x + 16y - 31 = 0$

* Product rule

* 微分計算切線斜率

* $(a-b)^2 \equiv a^2 - 2ab + b^2$

2022 - SECTION A

Q7.) $Let\ f'(x) = x^{\beta}3^{\sqrt{x}}$, *where* β *is constant and* $x > 0$. *Given that* $f'(9) = 2f'(4)$ *and* $f(4) = 0$

Find $f(9)$

* 參考課程 2.8 及 2.9

$$f'(9) = 2f'(4) \rightarrow 9^{\beta}3^{3} = 2 \cdot 4^{\beta}3^{2} \rightarrow 3^{2\beta+1} = 2^{2\beta+1} \rightarrow \boxed{(\frac{3}{2})^{2\beta+1} = 1}$$

$$\rightarrow \boxed{2\beta + 1 = 0} \rightarrow \beta = -0.5$$

$$Then,\ f(9) - f(4) = \int_{4}^{9} \frac{3^{\sqrt{x}}}{\sqrt{x}}dx \rightarrow f(9) = \int_{4}^{9} \boxed{\frac{3^{\sqrt{x}}}{\sqrt{x}}dx}$$

$$Let\ y = 3^{\sqrt{x}} \rightarrow lny = \sqrt{x}ln3 \rightarrow dy = \frac{3^{\sqrt{x}}ln3}{2\sqrt{x}}dx$$

$$\therefore f(9) = \boxed{\frac{2}{ln3}}\int_{9}^{27} \boxed{dy} = \frac{36}{ln3}$$

* $A^0 = 1$

* 積分三寶: 積分代入

* 定積分代入要改範圍

2022 - SECTION A

Q8.) Let $f(x) = ax^8 - 152x^5 - 4320x^2$, where a is a constant. Given that $f(x)$ attains its minimum value when $x = -2$. Find the greatest and the least value of $f(x)$

* 參考課程 2.2, 2.4, 2.8 及 2.9

a.) $f'(x) = 8ax^7 - 760x^4 - 8640x \rightarrow 8x(ax^6 - 95x^3 - 1080)$

Let $x_0 \in \mathbb{R}$ such that $f'(x_0) = 0$

$\rightarrow x_0 = 0$ or $ax_0^6 - 95x_0^3 - 1080 = 0$

Given that $f'(-2) = 0 \rightarrow a = 5$, hence,

$\rightarrow x_0 = 0$ or $x_0^6 - 19x_0^3 - 216 = 0 \rightarrow (x_0^3 + 8)(x_0^3 - 27) = 0$

$\rightarrow x_0 = 0$ or $x_0 = -2$ or $x_0 = 3$

	x < -2	x = -2	-2 < x < 0	x = 0	0 < x < 3	x = 3	x > 3
f′(x)	-	0	+	0	-	0	+
f(x)	Dec.		Inc.		Dec.		Inc.

Hence, $f(0)$ is max. value, while $f(-2)$ and $f(3)$ is min. value

$\therefore$ The greatest value $= f(0) = 0$, The least value $= f(3) = -43011$

* 搵 turning point = 搵 x_0 使度 $f'(x_0)=0$

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow$ *Increasing*

$f'(x) < 0 \rightarrow$ *Decreasing*

2022 - SECTION B

Q9.) There is a 0.3085 *chance a newborn boy weight above* $3.7kg$*, while* 0.1587 *chance a newborn girl weight above* $3.7kg$*. There is equal chance a newborn to be boy or girl.*

a.) Find the probability the newborn weight above $3.7kg$*.*

b.) Find the probability the newborn is a boy if the newborn weight above $3.7kg$*.*

c.) Let X be the number of newborn per day follows $Po(2.1)$*.*

i.) Find the probability there are 2 exact newborns and their weight greater than $3.7kg$ *in a certain day.*

ii.) Given there is at most 2 newborns and all of their weight greater than $3.7kg$ *in a certain day, find the probability that there is exactly 1 newborn boy on that day.*

iii.) Is the probability of no newborns are of weight $3.7kg$ *or below on a certain day is lower that* 0.2*? Explain your answer.*

* 參考課程 3.1, 4.2 及 4.4

a.) The probability of newborn weight above $3.7kg$, $p_n = 0.5(0.3085 + 0.1587)$

$$= 0.2336$$

* 男+女重量>3.7機率

b.) The required probability $= \dfrac{0.5 \cdot 0.3085}{p_n} = 0.6603 \ (to\ 4\ d.p.)$

* 條件機率

CONT'D

2022 - SECTION B

ci.) $The\ required\ probability,\ p_2 = \dfrac{e^{-2.1}(2.1)^2}{2!} \cdot (0.2336)^2$

$= 0.0147\ (to\ 4\ d.p.)$

ii.) $The\ required\ probability$

$$= \frac{e^{-2.1}(2.1) \cdot 0.5 \cdot 0.3085 + \dfrac{e^{-2.1}(2.1)^2}{2!} \cdot 2(0.5)^2(0.3085)(0.1587)}{e^{-2.1} + e^{-2.1}(2.1) \cdot 0.2336 + p_2}$$

$= 0.2346\ (to\ 4\ d.p.)$

iii.) $The\ required\ probability$

$$= e^{-2.1}\left[1 + (2.1 \cdot 0.2336) + \frac{(2.1 \cdot 0.2336)^2}{2!} + \ldots\right]$$

$$= e^{-2.1} \cdot e^{2.1 \cdot 0.2336} = 0.199999582 < 0.2$$

$\therefore\ The\ required\ probability\ is\ less\ than\ 0.2$

* $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$
* 條件概率
* 1個>3.7kg男
* 1個>3.7kg男 + 1個>3.7kg女
* 0個BB
* 1個>3.7kg BB
* $e^x = \sum_{r=0}^{\infty} \dfrac{x^r}{r!}$

2022 - SECTION B

Q10.) There are 48 students in a running competition. In 1^{st} round, there students are randomly grouped in 8 for each group. The time to finish the running by each studtent follows $N(12.3, 0.5^2)$ in second. Given that the time to finish running by each student are independent.

a.) For a certain student, find the percentage he finishes running more than 12.1s.

b.) For a certain group, find the probability there are at least 6 students finish running more than 12.1s.

c.) Given that John finishes running in 12.1s in a certain group.

i.) Find the probability John finishes running in the 1^{st} place in his group.

ii.) Find the probability John finishes running in the 3^{rd} place in his group.

iii.) For the 1^{st} and 2^{nd} place student in each group can join the next round. Among the 3^{rd} place student in each group, the 4 students who finishes in shorter time join the next round. Find the probability John can join the next round.

* 參考課程 4.2, 4.4 及 4.5

a.) Let T be the random varible of the time (s) to finish running for a student

$$P(T > 12.1),\ p = P\left(Z > \frac{12.1 - 12.3}{0.5}\right) = P(Z > -0.4) = 0.6554$$

* 先計算 Z ~ N(0,1) 再對表

CONT'D

2022 - SECTION B

b.) *Let X be the random varible of the number of students finish running more than 12.1s in a certain group .* $X \sim B(8, p)$

$$P(X \geq 6) = P(X=6) + P(X=7) + P(X=8)$$

$$= C_6^8 p^6 (1-p)^2 + C_7^8 p^7 (1-p) + p^8 = 0.4408 \ (to\ 4\ d.p.)$$

ci.) *Let* W_n *be the event of John finish in* n^{th} *place in a group*

T_j *be the time spent by John to complete running*

$$P(W_1 \mid T_j = 12.1) = \frac{P(T_j = 12.1)p^7}{P(T_j = 12.1)} = p^7 = 0.0519 \ (to\ 4\ d.p.)$$

ii.) $$P(W_3 \mid T_j = 12.1) = \frac{P(T_j = 12.1)[C_5^7 p^5 (1-p)^2]}{P(T_j = 12.1)} = 0.3016 \ (to\ 4\ d.p.)$$

iii.) *Let* $$p_1 = P(W_1 \cup W_2 \mid T_j = 12.1) = \frac{P(T_j = 12.1)[p^7 + C_6^7 p^6 (1-p)]}{P(T_j = 12.1)}$$

* $P(X=k) = C_k^n p^k (1-p)^{n-k}$

* 條件概率

* 其他人跑超過 12.1s

* 有5個跑超過 12.1s

* 有6個跑超過 12.1s

CONT'D

2022 - SECTION B

$$= 0.2431293$$

Let Y be the number of students of the 3^{rd} *place among 5 groups finishes running more than* $12.1s$.

The probability the 3^{rd} *student spend more than* $12.1s$ *in a certain group,* q

$$= p^8 + C_7^8 p^7(1-p) + C_6^8 p^6 (1-p)^2 = 0.440775526$$

$$\text{Let } p_2 = P(W_3 \cap Y \geq 2 \mid T_j = 12.1)$$

$$= P(W_3 \mid T_j = 12.1) \cdot [q^5 + C_4^5 q^4 (1-q) + C_3^5 q^3 (1-q)^2 + C_2^5 q^2 (1-q)^3]$$

$$= 0.3016 \cdot 0.729764991 = 0.220097121$$

$$\text{The required probability} = p_1 + p_2 = 0.4632 \ (to\ 4\ d.p.)$$

* 8個跑超過 12.1s
* 7個跑超過 12.1s
* 6個跑超過 12.1s
* 响第三名裏面最少2個跑超過 12.1s

2022 – SECTION B

$Q11.)\ Let\ f(x) = e^x lnx,\ J = \int_2^1 f(x)dx,\ K = \int_1^2 [(x+1)e^x lnx + \frac{1}{x}]dx$

$a.)\ Using\ trapezoidal\ rule\ with\ 5\ sub-intervals,\ estimate\ J.$

$b.)\ Find\ K$

$c.)\ Let\ A\ be\ the\ area\ of\ bounded\ region\ by\ the\ curve\ y = xe^x lnx + x^{-1},\ the\ x-axis,\ x = 1\ and\ x = 2.$

$i.)\ Using\ a.)\ and\ b.),\ estimate\ A.$

$ii.)\ Is\ A > 4?\ Explain\ your\ answer.$

* 參考課程 2.8, 2.9 及 3.3

$a.)\ Let\ J_1\ be\ the\ estimation\ of\ J$

$$J_1 = \frac{2-1}{5 \cdot 2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$

$$= 2.0829\ (to\ 4\ d.p.)$$

$$b.)\ K = \int_1^2 (x+1)e^x lnx dx + \int_1^2 \frac{1}{x}dx = \int_1^2 (x+1)lnx d(e^x) + ln2$$

* 計算梯形面積的加總

* 積分三寶: Integration by part

CONT'D

2022 - SECTION B

$$= 3ln2e^2 - \int_1^2 e^x \boxed{d[(x+1)lnx]} + ln2$$

$$= ln2(3e^2+1) - \int_1^2 \boxed{e^x}\boxed{[(x+1)\frac{1}{x} + lnx]}\boxed{dx}$$

$$= ln2(3e^2+1) - \int_1^2 e^x dx - \int_1^2 \frac{e^x}{x}dx - \int_1^2 lnx\boxed{d(e^x)}$$

$$= ln2(3e^2+1) + e - e^2 - \cancel{\int_1^2 \frac{e^x}{x}dx} - [e^x lnx]_1^2 + \cancel{\int_1^2 \frac{e^x}{x}dx}$$

$$= ln2(2e^2+1) + e - e^2$$

$$ci.)\ A = \int_1^2 xe^x lnx + \frac{1}{x}dx = \int_1^2 (x+1)e^x lnx + \frac{1}{x}dx - \int_1^2 e^x lnxdx$$

$$= K - J \approx ln2(2e^2+1) + e - e^2 - J_1 = 4.1829\ (to\ 4\ d.p.)$$

* Product rule

* Integration by part

CONT'D

2022 - SECTION B

$$ii.)\ f(x) = e^x lnx \rightarrow f'(x) = e^x lnx + e^x x^{-1} = f(x) + e^x x^{-1}$$

$$\rightarrow f''(x) = f'(x) + \frac{e^x(x-1)}{x^2} > 0\ for\ 1 \le x \le 2,\ f'(x) > 0$$

$$\therefore J_1\ is\ over-estimated\ J$$

$$Hence,\ J_1 > J \rightarrow K - J_1 < K - J \rightarrow A > K - J_1$$

$$\therefore A > 4$$

* Product rule

* 個 f(x) 係凹口向上

2022 - SECTION B

Q12.) Given that $u = e^{6-2t}$, $N = Ae^{-u}$, *where A is constant*

a.) Find $\frac{du}{dt}$ and $\frac{dN}{dt}$ in term of A and u

b.) Find the polynomial $p(u)$ such that $\frac{d^2N}{dt^2} = Np(u)$

c.) Find the value of t when $\frac{dN}{dt}$ attains its extreme value . Is the extreme value maximum? Explain your answer .

d.) Estimate N in term of A when $t \to \infty$.

* 參考課程 2.1, 2.2, 2.3 及 2.4

$$a.)\ u = e^{6-2t} \to \frac{du}{dt} = \boxed{-2e^{6-2t}} = -2u$$

$$N = Ae^{-u} \to \frac{dN}{dt} = A\boxed{\frac{de^{-u}}{du} \cdot \frac{du}{dt}} = 2Aue^{-u}$$

* Chain rule

CONT'D

2022 – SECTION B

$$b.)\ \frac{dN}{dt} = 2Aue^{-u} \rightarrow \frac{d^2N}{dt^2} = 2A\frac{d(ue^{-u})}{du}\cdot\frac{du}{dt}$$

$$= 2A[e^{-u} - ue^{-u}]\cdot(-2u)$$

$$= -4uAe^{-u}[1-u] = N[4u(u-1)]$$

$$\therefore p(u) = 4u(u-1)$$

$$c.)\ When\ u = 1,\ \frac{d^2N}{dt^2} = 0\ (u > 0, for\ all\ value\ of\ t)$$

$$Hence, for\ \frac{dN}{dt} attains\ its\ extreme\ value \rightarrow e^{6-2t} = 1$$

$$\rightarrow t = 3$$

	t < 3	t = 3	t > 3
N''(t)	+	0	-
N'(t)	Inc.		Dec.

$$\therefore The\ extreme\ value\ is\ max.\ value.$$

* Chain rule
* Product rule

* 搵 turning point = 搵 t_0 使度 $N''(t_0)=0$

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow Increasing$

$f'(x) < 0 \rightarrow Decreasing$

CONT'D

2022 - SECTION B

$d.)\ u \to 0,\ when\ t \to \infty$

$$\therefore \lim_{t\to\infty} N = Ae^{-u_0},\ where\ u_o = \lim_{t\to\infty} u = 0$$

$$i.e.\ \lim_{t\to\infty} N = A$$

* 連續函數, lim f(u) = f(lim u)