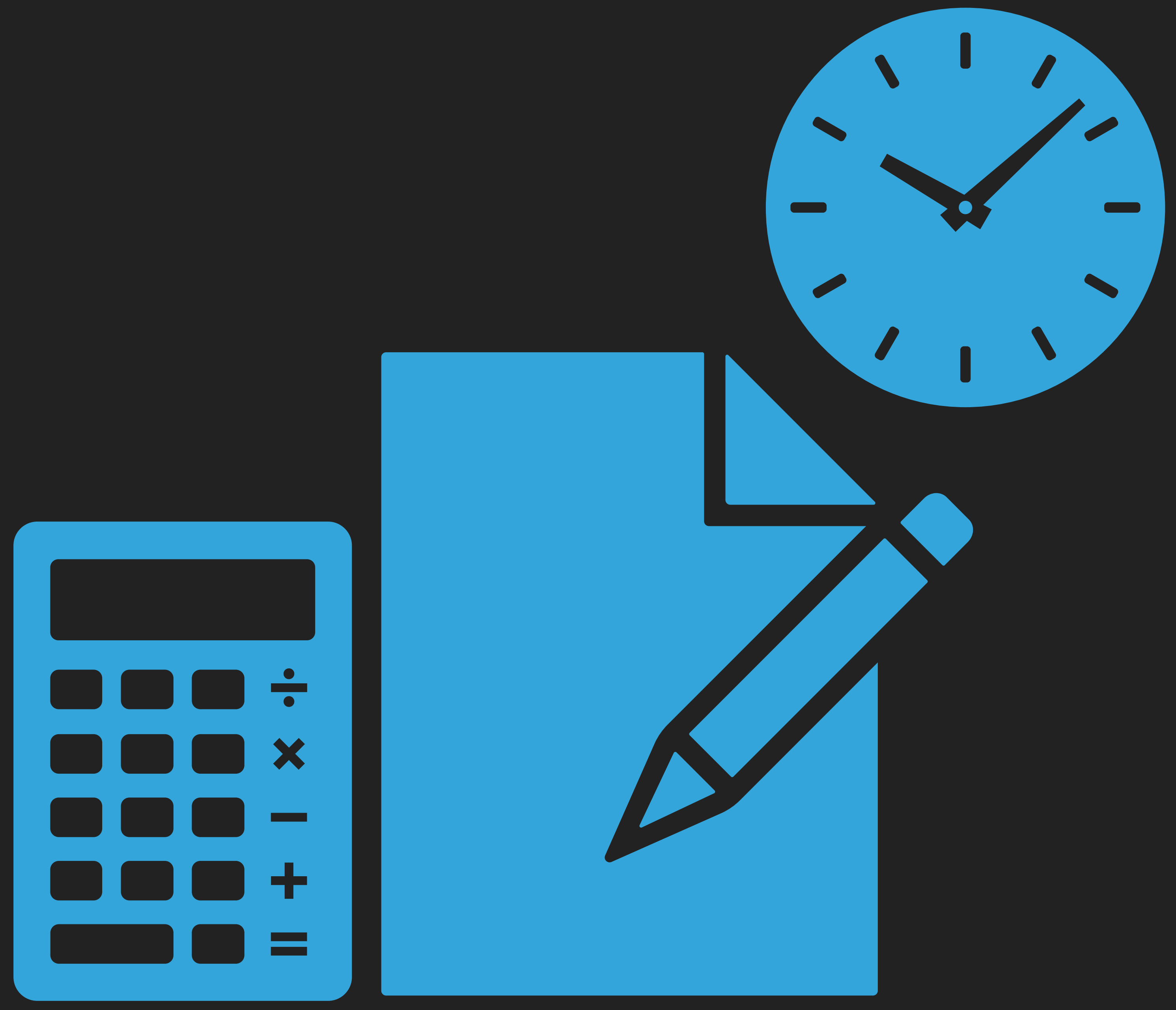
深宵教室 - DSE M2 模擬試題解答

2016

此為參考2016試題之模擬試題，原版請另行購買

2016

- Section A
- Section B



2016 - SECTION A

Q1.) Let $(x+5)^4(1-2x^{-1})^3 \equiv A + Bx + Cx^2 + \ldots,\ A = ?$

*** 參考課程 1.1**

$$(x+5)^4(1-2x^{-1})^3 \equiv [\sum_{r=0}^{4} C_r^4 5^{4-r} x^r][\sum_{r=0}^{3} C_r^3 (-2)^r x^{-r}],$$

$$A = (C_0^4 5^4)(C_0^3) + (C_1^4 5^3)(C_1^3(-2)) + (C_2^4 5^2)(C_2^3(-2)^2) + (C_3^4 5)(C_3^3(-2)^3)$$

$$= (625)(1) + (500)(-6) + (150)(12) + (20)(-8)$$

$$= -735$$

*** Binomial Expansion**

$* \ C_r^n = \dfrac{n!}{r!(n-r)!}$

2016 - SECTION A

$Q2.)\ f(x) = \sqrt{3x^{-1}}$. $f'(x) = ?$ $(By\ First\ Principles)$

* 參考課程 3.1 及 3.2

$$f'(0) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sqrt{3}}{\sqrt{x+h}} - \frac{\sqrt{3}}{\sqrt{x}}\right) = \sqrt{3}\lim_{h\to 0}\frac{1}{h}\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}\right)$$

$$= \sqrt{3}\lim_{h\to 0}\frac{1}{h}\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right)$$

$$= \sqrt{3}\lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\right) = \frac{-\sqrt{3}}{2x\sqrt{x}}$$

$$= -\frac{\sqrt{3}}{2}x^{-\frac{3}{2}}$$

* 微分定義

* lim 抽常數

* $(a+b)(a-b) = a^2 - b^2$

2016 - SECTION A

Q3.) Let $P=(u,y_0)$ moving on the curve $C: y=2e^x\ (x>0)$. There is a horizontal line passing throught P and cut $y-$axis at Q. Given that OQ increases at a constant rate 6 unit s^{-1}. When $u=4$, the change of rate of the area of $\Delta OPQ = ?$

* 參考課程 3.2, 3.3 及 3.4

$$\text{The area of } \Delta OPQ, A=\frac{1}{2}u(2e^u)=ue^u$$

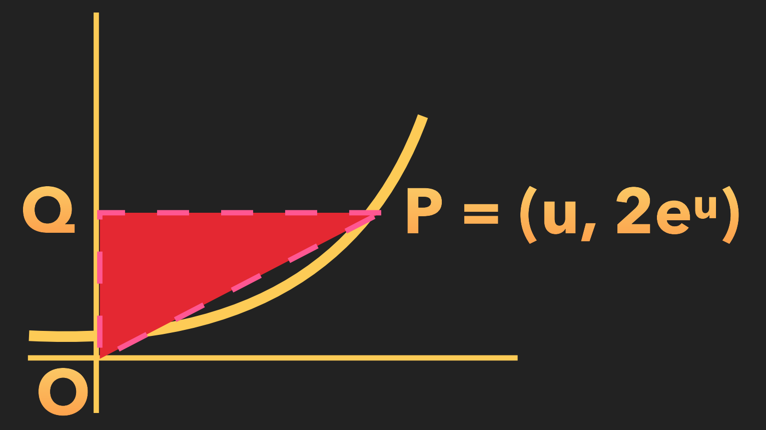
$$\rightarrow \frac{dA}{dt}=\boxed{\frac{d(ue^u)}{du}\frac{du}{dt}}=\boxed{(e^u+ue^u)\frac{du}{dt}}$$

$$\because OQ=2e^u \rightarrow \frac{d(OQ)}{dt}=2e^u\frac{du}{dt}\rightarrow 6=2e^u\frac{du}{dt}\rightarrow\frac{du}{dt}=3e^{-u}$$

$$\therefore \frac{dA}{dt}\Big|_{u=4}=3e^{-4}(e^4+4e^4)=15$$

i.e. The of change of area of $\Delta OPQ = 15\ unit^2 s^{-1}$

* 畫圖了解面積



* Implicit 微分法

* Chain rule + Product rule

2016 – SECTION A

Q4.) Consider a curve $G: y = \dfrac{2x^2+x+1}{x-1}$

a.) The asymptote(s) of $G = ?$

b.) The slope of the normal to G *at* $(2,1)$

* 參考課程 3.4 及 3.5

a.) $$y = \frac{2x^2+x+1}{x-1} = \frac{(x-1)(2x+3)+4}{x-1} = 2x+3+\frac{4}{x-1}$$

Vertical Asymptote : $x = 1$

Horizontal Asymptote : No Horizontal Asymptotes

Oblique Asymptote : $y = 2x+3$

b.) *The slope of tangent at* $(2,1) = \left.\dfrac{dy}{dx}\right|_{x=2} = 2 - \dfrac{4}{(2-1)^2}$

$\therefore$ *The slope of normal at* $(2,1) = \dfrac{1}{2}$

* 利用長除法 f(x)=(x-1)Q(x)+R

* x 係幾多, 分母係零

* *Find* $\lim\limits_{x\to\infty} y$

* *Find m and c such that* $\lim\limits_{x\to\infty}[y-(mx+c)] = 0$

$$y-(2x+3) = \frac{4}{x-1}$$

$$\to \lim_{x\to\infty}(y-(2x+3)) = 0$$

* Tangent 斜率 x Normal 斜率 = -1

2016 - SECTION A

Q5.) Prove $\sum_{r=1}^{n}(-1)^r r^2 = \frac{(-1)^n n(n+1)}{2},\ \forall n \in \mathbb{Z}^+$

* 參考課程 1.1 及 1.2

方法1

$$Let\ P(n):\ \sum_{r=1}^{n}(-1)^r r^2 = \frac{(-1)^n n(n+1)}{2}\ \forall n \in \mathbb{Z}^+$$

* 先 Let Statement

$$For\ P(1):\ L.H.S. = -1 = R.H.S.$$

* 証明 P(1) is true

$$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+,\ then\ P(k+1):$$

* 假設 P(k) is true. 証明 P(k+1) is true

$$L.H.S. = \sum_{r=1}^{k+1}(-1)^r r^2 = \sum_{r=1}^{k}(-1)^r r^2 + (-1)^{k+1}(k+1)^2$$

* 將末項抽出並改變末項

$$= \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2$$

CONT'D

2016 - SECTION A

$$= \frac{(-1)^{k+1}(k+1)}{2}(-k+2k+2) = \frac{(-1)^{k+1}(k+1)(k+2)}{2}$$

$$= R.H.S.$$

$\therefore P(k+1)$ *is true if* $P(k)$ *is true* $\exists k \in \mathbb{Z}^+$

i.e. By M.I., $P(n)$ *is true,* $\forall n \in \mathbb{Z}^+$

* 寫結論

方法2

Let $S_1(n) = \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, $S_2(n) = \sum_{r=1}^{n} (-1)^r r^2$

For n is even,

$$S_2(n) = 2 \cdot 2^2 S_1(\frac{n}{2}) - S_1(n)$$

$$= \frac{1}{3}n(n+1)(n+2) - \frac{1}{6}n(n+1)(2n+1)$$

* $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

* $-1^2 + 2^2 - 3^2 + \ldots + n^2$

$= 2(2^2 + 4^2 + \ldots) - (1^2 + 2^2 + 3^2 + \ldots)$

$= 2 \cdot 2^2(1^2 + 2^2 + \ldots + (\frac{n}{2})^2) - S_1(n)$

$= 2 \cdot 2^2 S_1(\frac{n}{2}) - S_1(n)$

CONT'D

2016 - SECTION A

$$= \frac{n(n+1)}{2}$$

For n is odd,

$$S_2(n) = 2 \cdot 2^2 S_1(\frac{n-1}{2}) - S_1(n)$$

$$= \frac{1}{3}(n-1)(n)(n+1) - \frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{-n(n+1)}{2}$$

$$i.e.\ S_2(n) = \frac{(-1)^n n(n+1)}{2}$$

$$* \quad -1^2 + 2^2 - 3^2 + \ldots + n^2$$

$$= 2(2^2 + 4^2 + \ldots) - (1^2 + 2^2 + 3^2 + \ldots)$$

$$= 2 \cdot 2^2(1^2 + 2^2 + \ldots + (\frac{n-1}{2})^2) - S_1(n)$$

$$= 2 \cdot 2^2 S_1(\frac{n-1}{2}) - S_1(n)$$

2016 - SECTION A

Q6.) By expressing $\cos 3\theta$ into $\cos\theta$, find $\cos\frac{3\pi}{5}$ in surd form

* 參考課程 2.1 及 2.2

$$\cos 3\theta = \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$

$$= (2\cos^2\theta - 1)\cos\theta - 2\cos\theta\sin\theta\sin\theta$$

$$= (2\cos^2\theta - 1)\cos\theta - 2\cos\theta(1-\cos^2\theta)$$

$$= 4\cos^3\theta - 3\cos\theta$$

$$\rightarrow \cos 3\theta + \cos 2\theta = 4\cos^3\theta - 3\cos\theta + \cos 2\theta$$

$$\rightarrow 2\cos\frac{5\theta}{2}\cos\frac{\theta}{2} = 4\cos^3\theta - 3\cos\theta + 2\cos^2\theta - 1 \quad \text{———— (*)}$$

$$\text{Put } \theta = \frac{3\pi}{5} \text{ into (*)},\ 0 = 4\cos^3\frac{3\pi}{5} + 2\cos^2\frac{3\pi}{5} - 3\cos\frac{3\pi}{5} - 1$$

* cos 複角公式

$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$

$2\sin A\sin B=\cos(A-B)-\cos(A+B)$

$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

* cos 雙角公式

* sin 雙角公式

* Sum to product

$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$

$2\sin A\sin B=\cos(A-B)-\cos(A+B)$

$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

CONT'D

2016 - SECTION A

$$Put\ x = cos\frac{3\pi}{5}\ into\ (*),\ 0 = 4x^3 + 2x^2 - 3x - 1$$

$$Let\ f(x) = 4x^3 + 2x^2 - 3x - 1$$

$$\because f(-1) = 0$$

$$\therefore f(x) = (x+1)(4x^2 - 2x - 1)$$

$$For\ f(x) = 0 \rightarrow x = -1\ or\ x = \frac{2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$$

$$\because \frac{\pi}{2} < \frac{3\pi}{5} < \pi \rightarrow 0 < cos\frac{3\pi}{5} < -1$$

$$\therefore x = \frac{2 - \sqrt{20}}{8}$$

$$i.e.\ cos\frac{3\pi}{5} = \frac{1 - \sqrt{5}}{4}$$

* 用 Remainder theorem 搵 root

* 用長除搵商數做 Factorization

* 留意角度範圍

CONT'D

2016 - SECTION A

Q7.) Find the volume of solid revolving $y-axis$ bounded $y = 48$, $x-axis$, $y-asix$ and

$\Gamma : y = 4x^2 - 4x$

* 參考課程 3.10 及 3.12

方法1

$$Consider,\ y = 4x^2 - 4x = 4x^2 - 4x + 1 - 1 = (2x-1)^2 - 1$$

$$Hence,\ x = \frac{1+\sqrt{y+1}}{2}$$

$$The\ volume = \pi \int_0^{48} \frac{(1+\boxed{\sqrt{y+1}})^2}{4} \boxed{dy}$$

$$= \frac{\pi}{4} \int_{\boxed{1}}^{\boxed{49}} (1+\boxed{\sqrt{u}})^2 \boxed{du} = \frac{\pi}{4} \int_1^{49} (1 + 2\sqrt{u} + u)du$$

* ■ 積分代入法, u=y+1, 範圍要改

CONT'D

2016 - SECTION A

$$= \frac{\pi}{4}[u + \frac{4}{3}u^{\frac{3}{2}} + 2u^2]_1^{49} = 426\pi \ sq.\ unit$$

方法2

$$The\ volume,\ V = \pi \int_0^{48} [f^{-1}(y)]^2 dy,\ where\ f(x) = 4x^2 - 4x$$

$$= \pi[y[f^{-1}(y)]^2]_0^{48} - \pi \int_0^{48} y\, d([f^{-1}(y)]^2)$$

* 積分三法寶: Integration by part

$$For\ y = 0 \to f^{-1}(y) = 0\ or\ 1,\ for\ y = 48 \to f^{-1}(y) = -3\ or\ 4$$

$$Reject\ f^{-1}(y) = -3\ and\ 0\ (\because Out\ of\ bounded\ region)$$

$$Hence,\ let\ x = f^{-1}(y)$$

* 積分代入法, x=f⁻¹(y), 範圍要改

$$V = 768\pi - \pi \int_1^4 f(x)\, d(x^2) = 768\pi - \pi \int_1^4 2x(4x^2 - 4x)dx$$

$$= 768\pi - \pi[2x^4 - \frac{8}{3}x^3]_1^4 = 426\pi \ sq.\ unit$$

2016 - SECTION A

Q8.) Consider,

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

a.) Find A^n *and* $(A^{-1})^n, \forall n \in \mathbb{Z}^+$

b.) Find $B^n, \forall n \in \mathbb{Z}^+$

* 參考課程 4.8, 4.10 及 4.11

$a.)\ \because |A| = 1 > 0 \rightarrow A^{-1}$ *exists*

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

* 用 Row Reduction 搵逆矩陣

* R2=R2-R1

CONT'D

2016 - SECTION A

方法1

$$Let\ P(n):\ A^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}\ \forall n \in \mathbb{Z}^+$$

$$For\ P(1):\ L.H.S. = A = R.H.S.$$

$$Assume\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+,\ then\ P(k+1):$$

$$L.H.S. = A^{k+1} = A^k A = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = A^{k+1} = R.H.S.$$

$$\therefore P(k+1)\ is\ true\ if\ P(k)\ is\ true\ \exists k \in \mathbb{Z}^+$$

$$i.e.\ By\ M.I.,\ P(n)\ is\ true,\ \forall n \in \mathbb{Z}^+$$

方法2

$$Let\ A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ such\ that\ A = I_2 + A_1$$

$$A_1^2 = 0 \rightarrow A_1^n = 0,\ for\ \forall n \in \mathbb{Z}^+\ and\ n > 1$$

* 先試A², A³ 搵 Pattern, 再用M.I. 証明
* 先 Let Statement
* 証明 P(1) is true
* 假設 P(k) is true. 証明 P(k+1) is true
* 寫結論

CONT'D

2016 - SECTION A

$$Then, A^n = (I_2 + A_1)^n = I_2 + \sum_{r=1}^{n} C_r^n A_1^r = I_2 + nA_1$$

$$= \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

* 因為 $IA_1=A_1I$, 所以可以用 Binomial expansion

方法1

$$Similarly, by\ M.I.,\ (A^{-1})^n = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$$

* 好似 A^n 咁做多似M.I.

方法2

$$Similarly, consider\ A^{-1} = I_2 - A_1 \rightarrow (A^{-1})^n = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$$

* 好似 A^n 咁做多似 Binomial expansion

方法3

$$(A^{-1})^n = (A^n)^{-1}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ n & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -n & 1 \end{array}\right) \rightarrow (A^{-1})^n = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$$

* $(AB)^{-1}=B^{-1}A^{-1}$

* 用 Row Reduction 搵逆矩陣
* R2=R2-nxR1

CONT'D

2016 - SECTION A

$b.)$

方法1

$$B^n = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} B^{n-2} = \begin{pmatrix} 1 & 0 \\ 1+2 & 2^2 \end{pmatrix} B^{n-2}$$

$$= \begin{pmatrix} 1 & 0 \\ 1+2+2^2 & 2^3 \end{pmatrix} B^{n-3} \ldots = \begin{pmatrix} 1 & 0 \\ \sum_{r=0}^{n-1} 2^r & 2^n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$$

方法2

$$ABA^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\rightarrow B^n = A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$$

* Core 裏面 Sum of G.P. (r=2)

* Eigenvalue 與 eigenvector 應用

CONT'D

2016 - SECTION A

方法3

$$Let\ B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},\ such\ that\ B = I_2 + B_1$$

$$B_1^2 = B_1 \to B_1^n = B_1,\ for\ \forall n \in \mathbb{Z}^+$$

$$Then,\ B^n = (I_2 + B_1)^n = I_2 + \sum_{r=1}^{n} C_r^n B_1^r = I_2 + \left(\sum_{r=1}^{n} C_r^n\right) B$$

$$= I_2 + (2^n - 1)B = \begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$$

* 因為 $IB_1=B_1I$, 所以可以用 Binomial expansion

* $(x+1)^n \equiv \sum_{r=0}^{n} C_r^n x^r,\ Put\ x = 1 \to \sum_{r=1}^{n} C_r^n = 2^n - 1$

2016 - SECTION B

Q9.) Let $f(x) = x^3 + ax^2 + bx + 5, x \in \mathbb{R}$, There is a curve $C: \ y = f(x)$ such that $(-1, 10)$ is one of a turning point.

a.) Find a and b

b.) Find all local min. and local max. points

c.) Find pt. of inflexion

d.) Find the area of the bounded region by C and the tangent of C at $(-1, 10)$

* 參考課程 3.4, 3.5, 3.10 及 3.11

a.)

$$(-1, 2) \text{ is a turning pt.} \rightarrow \begin{cases} f(-1) = 10 \\ f'(-1) = 0 \end{cases}$$

$$\rightarrow \begin{cases} a - b = 6 \\ -2a + b = -3 \end{cases}$$

* $f'(x_0)=0$, $(x_0,f(x_0))$ 就係 turning point

CONT'D

2016 - SECTION B

$\rightarrow (a, b) = (-3, -9)$

$b.)\ f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$

$Let\ x_0 \in \mathbb{R}\ such\ that\ f'(x_0) = 0 \rightarrow x_0 = -1\ or\ 3$

	x < -1	x = -1	-1 < x < 3	x = 3	x > 3
f'(x)	+	0	-	0	+
f(x)	Inc.		Dec.		Inc.

$\therefore The\ local\ max.\ pt. = (-1, 10)$

$The\ local\ min.\ pt. = (3, -22)$

$c.)\ f''(x) = 6x - 6 = 6(x-1)$

$Let\ x_0 \in \mathbb{R}\ such\ that\ f''(x_0) = 0 \rightarrow x_0 = 1$

$\therefore The\ pt.\ of\ inflexion = (1, -6)$

* 搵 turning point = 搵 x_0 使度 $f'(x_0)=0$

* 利用表格計算 turning point 附近上升定下降

  $f'(x) > 0 \rightarrow Increasing$

  $f'(x) < 0 \rightarrow Decreasing$

* 搵 pt. of inflexion = 搵 x_0 使度 $f''(x_0)=0$

CONT'D

2016 – SECTION B

$d.)\ \because (-1, 10)\ is\ turning\ pt.$

$\therefore The\ tangent\ at\ (-1, 10):\ \ y = 10$

* 所有 turning point 的 slope 係等如零

$Consider,\ f(x) = 10\ to\ find\ all\ interception(s)$

$\rightarrow f(x) - 10 = 0,\ with\ x = -1\ is\ one\ of\ the\ solution$
$and\ (-1, 10)\ touches\ at\ y = 10$

* 切線有 repeated root

$\rightarrow (x+1)^2(x+A) = 0,\ where\ A \in \mathbb{R}$

$By\ compare\ coefficient\ of\ constant:$

$$f(x) - 10 \equiv (x-1)^2(x+A)$$

$\rightarrow A = -5,\ Hence,\ the\ interception\ are\ x = -1\ and\ 5$

$$The\ area = \int_{-1}^{5} 10 - f(x)dx = 108\ sq.unit$$

* 面積大減細

2016 - SECTION B

$$Q10.)\ \int_0^{\frac{\pi}{4}} ln(1+tan\theta)d\theta = ?\ and\ \int_0^{\frac{\pi}{4}} \frac{xsec^2x}{1+tanx}dx = ?$$

* 參考課程 3.10

$$Let\ I_1 = \int_0^{\frac{\pi}{4}} ln(1+tan\theta)d\theta,\ I_2 = \int_0^{\frac{\pi}{4}} \frac{xsec^2x}{1+tanx}dx$$

$$I_1 = -\int_{\frac{\pi}{4}}^{0} ln(1+tan(\frac{\pi}{4}-x))dx,\ where\ x = \frac{\pi}{4}-\theta$$

$$= \int_0^{\frac{\pi}{4}} ln(1+\frac{1-tanx}{1+tanx})dx = \int_0^{\frac{\pi}{4}} ln(\frac{2}{1+tanx})dx$$

$$= \int_0^{\frac{\pi}{4}} ln2dx - I_1 \rightarrow 2I_1 = \frac{\pi}{4}ln2 \rightarrow I_1 = \frac{\pi}{8}ln2$$

* 定積分代入要改範圍

* 面積有正負，正變負範圍上下倒轉

* tan 複角公式

$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A\pm B) = \frac{\tan A \pm \tan B}{1\mp \tan A\tan B}$

$2\sin A\cos B = \sin(A+B)+\sin(A-B)$

$2\cos A\cos B = \cos(A+B)+\cos(A-B)$

$2\sin A\sin B = \cos(A-B)-\cos(A+B)$

$\sin A+\sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A-\sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A-\cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

* ln(A/B) = lnA-lnB

CONT'D

2016 - SECTION B

$$I_2 = \int_0^{\frac{\pi}{4}} \frac{x\sec^2 x dx}{1+tanx} = \int_0^{\frac{\pi}{4}} \frac{xd(tanx)}{1+tanx} = \int_0^{\frac{\pi}{4}} x\frac{d(1+tanx)}{1+tanx}$$

$$= \int_0^{\frac{\pi}{4}} xd(ln(1+tanx)) = [xln(1+tanx)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} ln(1+tanx)dx$$

$$= \frac{\pi}{4}ln2 - \frac{\pi}{8}ln2 = \frac{\pi}{8}ln2$$

* 積分三寶: 積分代入

* 可用 Integration by part

2016 – SECTION B

*Q*11.)

$$(E):\begin{cases} x+y-z=3 \\ 4x+6y+az=b \\ 5x+(1-a)y+(3a-1)z=b-1 \end{cases}$$

a.) Assume (E) has unique solution, show that $a \neq -2$ *and* $\neq -12$*. Then solve (E)*

b.) Assume $a=2$ *and (E) is consistent, find b and solve (E).*

c.) $\begin{cases} x+y-z=3 \\ 2x+3y-z=7 \\ 5x+3y-7z=13 \end{cases}$ *satisfy* $x^2+y^2-6z^2>14$?

* 參考課程 4.7

CONT'D

2016 - SECTION B

$a.)\ Consider:$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 4 & 6 & a & b \\ 5 & 1-a & 3a-1 & b-1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & a+4 & b-12 \\ 0 & -a-4 & 3a+4 & b-16 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & a+4 & b-12 \\ 0 & 0 & A & B \end{array}\right)$$

$$where\ A = 2(3a+4) - (a+4)(-a-4) = \boxed{(a+2)(a+12)}$$
$$B = 2(b-16) - (a+4)(b-12) = 6b - 12a + ab - 80$$

$$(E)\ has\ unique\ solution \rightarrow A \neq 0 \rightarrow (a+2)(a+12) \neq 0$$
$$\rightarrow a \neq -2\ and\ a \neq -12$$

* 消去法

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

* 如果 $\boxed{}$ 唔等如零先有唯一答案

CONT'D

2016 - SECTION B

$$z = \frac{B}{A}$$

$$y = \frac{1}{2}[(b-12)-(a+4)\frac{B}{A}]$$

$$x = 3 + \frac{B}{A} - \frac{1}{2}[(b-12)-(a+4)\frac{B}{A}]$$

$$\therefore (x, y, z)^T = \begin{pmatrix} \dfrac{3a^2+50a-ab+6b-24}{(a+2)(a+12)} \\ \dfrac{2(ab-10a+8)}{(a+2)(a+12)} \\ \dfrac{6b-12a+ab-80}{(a+2)(a+12)} \end{pmatrix}$$

* 先用三式搵z, 再用二式搵y, 最後一式搵x

CONT'D

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$b.)\ For\ (E)\ is\ consistent \rightarrow B = 0 \rightarrow 2(b-16) + 2(b-12) = 0$

$\rightarrow b = 14$

$Then,\ (E) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & 2 & 2 \end{array}\right)$

$Let\ z = t,\ t \in \mathbb{R}$

$(x, y, z) = (2 + 2t, 1 - t, t)$

$c.)\ When\ a = -2\ and\ b = 14,\ (E)\ become$

$$\begin{cases} x + y - z = 3 \\ 2x + 3y - z = 7 \\ 5x + 3y - 7z = 13 \end{cases} \rightarrow (x, y, z) = (2 + 2t, 1 - t, t),\ t \in \mathbb{R}$$

$Then,\ x^2 + y^2 - 6z^2 - 14 = (2 + 2t)^2 + (1 - t)^2 - 6t^2 - 14 = -(t-3)^2 \le 0$

$\therefore x^2 + y^2 - 6z^2 - 14 \le 0 \rightarrow x^2 + y^2 - 6z^2 > 14\ is\ not\ correct$

* 三條公式剩返兩條

* 用 t 代表 x, y, z

2016 - SECTION B

$Q12.)$ *Assume* $\overrightarrow{OA} = 2(\hat{j} + \hat{k}),\ \overrightarrow{OB} = 4\hat{i} + \hat{j} + \hat{k},\ \overrightarrow{OC} = 2\hat{i} - \hat{j} + 4\hat{k},\ \overrightarrow{OD} = 3\hat{i} + 2\hat{j} + 5\hat{k}$

Also, $\overrightarrow{OP} = \hat{i} + t\hat{j}$, *with* $AP = BP$

a.) $t = ?$

b.) *The angle between CD and plane ABC*

c.) *Let E be the point on the plane ABC, such that DE* $\perp$ *the plane ABC. Let F be a point such that* $\overrightarrow{PF} = \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}$, *What is the geometric relation between D, E, F ?*

* 參考課程 4.4

$$a.)\ AP = BP \rightarrow |\overrightarrow{AP}| = |\overrightarrow{BP}|$$

$$\rightarrow |\overrightarrow{OP} - \overrightarrow{OA}| = |\overrightarrow{OP} - \overrightarrow{OB}|$$

$$\rightarrow |\hat{i} + (t-2)\hat{j} - 2\hat{k}| = |-3\hat{i} + (t-1)\hat{j} - \hat{k}|$$

$$\rightarrow 5 + (t-2)^2 = 10 + (t-1)^2$$

$$\rightarrow t = -1$$

$* \ |a\hat{i} + b\hat{j} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$

CONT'D

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b.) Let $k\hat{n}$ be the normal vector of the plane ABC

$$k\hat{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (\overrightarrow{OB} - \overrightarrow{OA}) \times (\overrightarrow{OC} - \overrightarrow{OA})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & -1 \\ 2 & -3 & 2 \end{vmatrix} = -5\hat{i} - 10\hat{j} - 10\hat{k}$$

$$Hence,\ \hat{n} = -\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$$

The angle between CD and the plane ABC

$$= \left|\frac{\pi}{2} - \frac{\hat{n}\cdot\overrightarrow{CD}}{|\overrightarrow{CD}|}\right| = \left|\frac{\pi}{2} - \hat{n}\cdot\frac{\overrightarrow{OD} - \overrightarrow{OC}}{|\overrightarrow{OD} - \overrightarrow{OC}|}\right|$$

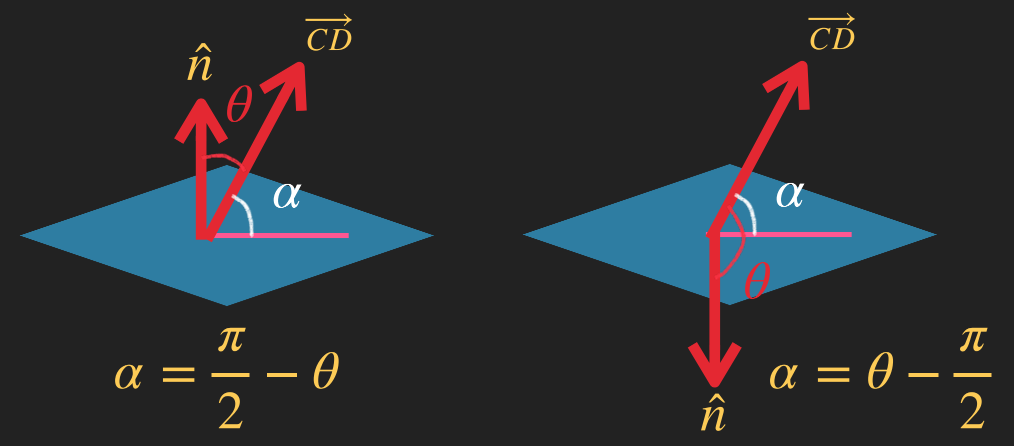
$$= 64.8^{o}\ (to\ 3\ sig.fig.)$$

*** Normal vector 代表一個 plane**

*** 用 cross product 計 normal vector**

* $\vec{a}$ *unit vector* $= \dfrac{\vec{a}}{|\vec{a}|}$

*** 共兩種情況**



$$\therefore\ \alpha = \left|\frac{\pi}{2} - \theta\right|$$

CONT'D

2016 - SECTION B

$c.)\ DE \perp plane\ ABC, \rightarrow\ \overrightarrow{DE} = r\hat{n}, where\ r \in \mathbb{R}$

$$\begin{aligned}\overrightarrow{DF} &= \overrightarrow{DP} + \overrightarrow{PF} = (\overrightarrow{OP} - \overrightarrow{OD}) + (\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}) \\ &= (\overrightarrow{OP} - \overrightarrow{OD}) + (\overrightarrow{OA} - \overrightarrow{OP}) + (\overrightarrow{OB} - \overrightarrow{OP}) + (\overrightarrow{OC} - \overrightarrow{OP}) \\ &= \hat{i} + 2\hat{j} + 2\hat{k} = -3\hat{n}\end{aligned}$$

$\therefore\ D, E, F\ are\ collinear$

* 如果兩支 vector 平行, $\vec{b} = r\vec{a}$
* 一般幾何關係, 離不開垂直, 共線同共面