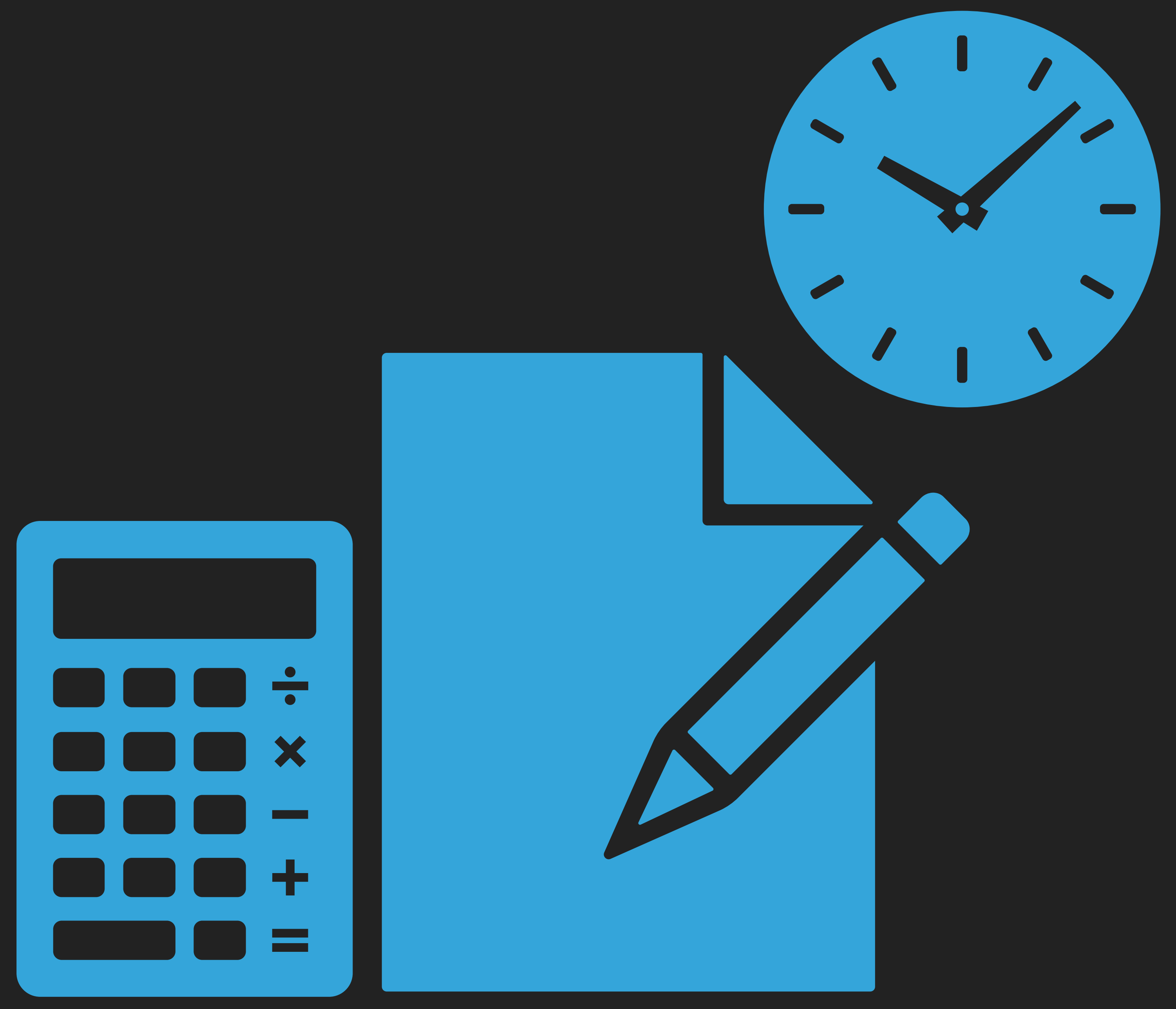
深宵教室 - DSE M1 模擬試題解答

2017

此為參考2017試題之模擬試題，原版請另行購買

2017

- Section A
- Section B



2017 - SECTION A

Q1.) Let X be discrete random vaiable

k	0	2	4	5	8	9
$P(X=k)$	a^2	0.16	0.18	0.3	a	0.12

where a is a constant .
Find a, E(X) and Var(2 − 3X)

* 參考課程 4.1, 4.3 及 4.4

$$a^2 + 0.16 + 0.18 + 0.3 + a + 0.12 = 1$$

$$\rightarrow a^2 + a - 0.24 = 0 \rightarrow a = -1.2 \text{ (rejected) or } 0.2 \rightarrow a = 0.2$$

$$E(X) = \sum_{i=1}^{6} k_i P(X=k_i) = 5.22 \text{ and } E(X^2) = \sum_{i=1}^{6} k_i^2 P(X=k_i) = 33.54$$

$$Var(2-3X) = 3^2 Var(X) = 9(E(X^2) - [E(X)]^2) = 56.6244$$

* 機率總加 = 1

* 0 < 機率 < 1

* $E(X) = \sum_{i=1}^{n} k_i P(X=k_i)$

* $Var(aX+b) = a^2 Var(X)$

* $Var(X) = E(X^2) - [E(X)]^2$

2017 - SECTION A

Q2.) Let A and B be the event such that $P(A) = 0.2$, $P(B') = 0.7$, *and* $P(A \mid B) = 0.6$ *where* B' *is the complementary event of* B

a.) Are A and B independent? Explain your answer.

b.) Are A and B mutually exclusive? Explain your answer.

* 參考課程 4.1 及 4.2

$a.)\ P(A \cap B) = P(A \mid B)[1 - P(B')] = 0.18$

$P(A)P(B) = 0.06 \neq P(A \cap B)$

$\therefore$ *A and B are not independent*

$b.)\ P(A \cap B) \neq 0$

$\therefore$ *A and B are not mutually exclusive*

* ■ P(A & B)=P(A|B)P(B)=P(B|A)P(A)

* 如果 A 同 B independent, P(A & B)=P(A)P(B)

* 如果 A 同 B mutually exclusive, P(A & B)=0

2017 - SECTION A

Q3.) The weight of cat (in kg) follows $N(\mu, \sigma^2)$. Given that the percentage of cats being lighter than 1.83kg is same as that being heavier than 3.43kg. Also, 89.04 % cats weight between 1.83kg and 3.43kg. If 9 cats are selected randomly, find the probability the mean of their lies between 2.5kg and 3.1kg.

* 參考課程 4.5

Let X be the random variable of the weight of a cat

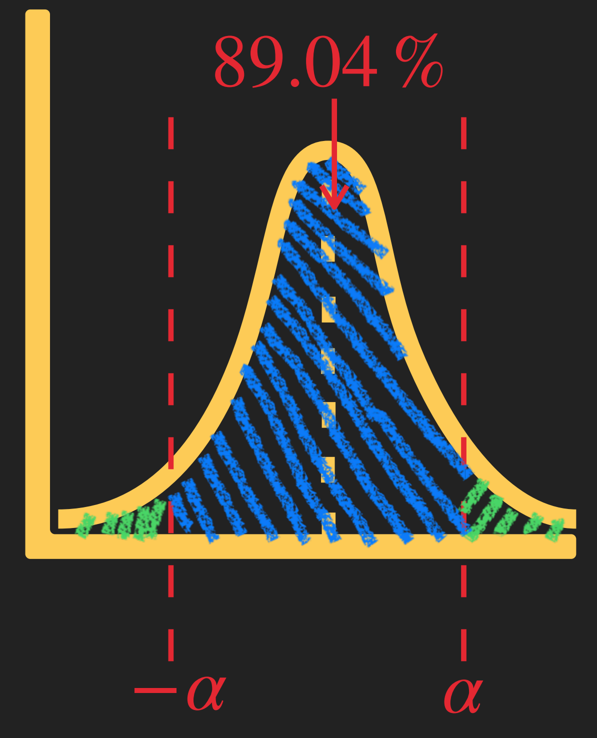
$\bar{X}$ be the sample mean of the selected cats' weight

$$\text{Given, } P(X < 1.83) = P(X > 3.43)$$

$$\rightarrow -\frac{1.83-\mu}{\sigma} = \frac{3.43-\mu}{\sigma} \rightarrow \mu = 2.63$$

$$\text{Also, } P\left(Z < \boxed{\frac{3.43-\mu}{\sigma}}\right) = \frac{0.8904}{2} \rightarrow \frac{0.8}{\sigma} = 1.6 \rightarrow \sigma = 0.5$$

$$\therefore P(2.5 < \bar{X} < 3.1) = P\left(\boxed{\frac{2.5-\mu}{\frac{\sigma}{\sqrt{9}}}} < Z < \boxed{\frac{3.1-\mu}{\frac{\sigma}{\sqrt{9}}}}\right) = 77.99\,\%$$



* 先計算 Z ~ N(0,1), 再對表

2017 - SECTION A

Q4.) Peter plays a game. In each trial game, his winning probability = 0.6. He plays the game until he win one trial game.

a.) Find the probability he stops playing at the 4^{th} trial in the game.

b.) Given that he cannot win a game in k trial, the probability of he stops playing within 10^{th} trial in the game is greater than 0.95. Find the greater value of k.

c.) In each trial game, he has to pay \$15. Find the expected amount of money he has to pay for winning the game.

* 參考課程 4.2 及 4.4

Let X be the random variable of the number of trial games he play

Then, $X \sim G(0.6)$

a.) $P(X = 4) = (1 - 0.6)^3(0.6) = 0.0384$

b.) $P(X < 10 \mid k \text{ trials not winning}) > 0.95$

$$\rightarrow \frac{P(X < 10 \ \cap \ k \text{ trials not winning})}{P(k \text{ trials not winning})} > 0.95$$

* $P(X = k) = (1 - p)^{k-1}p$

* 條件概率

CONT'D

2017 - SECTION A

$$\rightarrow \frac{\boxed{\sum_{i=k+1}^{10} P(X=i)}}{P(k\ trials\ not\ winning)} > 0.95$$

$$\rightarrow \frac{0.6\cdot 0.4^k \boxed{\sum_{i=0}^{9-k}(0.4)^i}}{0.4^k} > 0.95$$

$$\rightarrow 0.6\cdot \boxed{\frac{1-0.4^{10-k}}{1-0.4}} > 0.95 \rightarrow 0.4^{10-k} < 0.05$$

$$\rightarrow (10-k)ln0.4 < ln0.95 \rightarrow k < 6.73$$

$$\therefore\ the\ greatest\ vaklue\ of\ k = 6$$

$$c.)\ The\ expected\ amount\ of\ money = \$15\cdot\frac{1}{0.6} = \$25$$

* 佢係第 k+1 或 k+2 或 ... 10 次完

* 等比數列之和

* $X \sim G(p),\ E(X) = \frac{1}{p}$

2017 - SECTION A

Q5.) Let $f(x) = (5-x)^4(1+e^{3x})^2$. *Find the coefficient of* x^2 *of* $f(x)$

* 參考課程 1.1 及 3.2

$$(5-x)^4(1+e^{3x})^2 = (5^4 + C_1^4 5^3(-x) + C_2^4 5^2(-x)^2 + \ldots)(1+2e^{3x}+e^{6x})$$

$$= (625 - 500x + 150x^2 + \ldots)(1 + 2e^{3x} + e^{6x})$$

$$= (625 - 500x + 150x^2 + \ldots)\left(1 + 2\left(1 + (3x) + \frac{(3x)^2}{2!} + \ldots\right) + \left(1 + (6x) + \frac{(6x)^2}{2!} + \ldots\right)\right)$$

$$= (625 - 500x + 150x^2 + \ldots)(4 + 12x + 27x^2 + \ldots)$$

$$\therefore \text{The coefficient of } x^2 = (625)(27) - (500)(12) + (150)(4)$$

$$= 11475$$

* $(a+b)^n = \sum_{r=0}^{n} C_r^n a^r b^{n-r}$

* $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

* $C_r^n = \frac{n!}{r!(n-r)!}$

$\rightarrow C_1^n = n$ *and* $C_2^n = \frac{n(n-1)}{2}$

2017 - SECTION A

Q6.) Let $f(x) = 4x^3 + mx^2 + nx + 615$, *where, m and n are constant. Given that* $(6, -33)$ *is a turning point of* $y = f(x)$. *Find the min. and max. value(s) of* $f(x)$.

* 參考課程 2.3 及 2.4

$f'(x) = 12x^2 + 2mx + n$

Given that $f(6) = -33$ *and* $f'(6) = 0$

$$\begin{cases} -1512 = 36m + 6n \text{ —— } (1) \\ -432 = 12m + n \text{ —— } (2) \end{cases}$$

$(1) - 6(2) : m = -30 \rightarrow n = -72$

Hence, consider $f'(x) = 0 \rightarrow 12x^2 - 60x - 72 = 0$

$\rightarrow x^2 - 5x - 6 = 0 \rightarrow x = 6 \text{ or } -1$

	x < -1	x = -1	-1 < x < 6	x = 6	x > 6
f'(x)	+	0	-	0	+
f(x)	Inc.		Dec.		Inc.

$\therefore$ *The min. value* $= -33$ *and the max. value* $= f(-1) = 653$

* 搵 turning point = 搵 t_0 使度 $f'(t_0)=0$

* 消去法消去 n 搵 m ,再代 (1) 式搵 n

* 利用表格計算 turning point 附近上升定下降

$f'(x) > 0 \rightarrow$ *Increasing*

$f'(x) < 0 \rightarrow$ *Decreasing*

2017 - SECTION A

Q7.) Let the curve $C: y = f(x)$ and $f(x) = x(x-2)^{-\frac{1}{2}}$, where, $x > 2$. Given a tangent to C passing through $(9, 0)$. Find the slope of the tangent.

* 參考課程 2.3 及 2.4

$$f(x) = \boxed{x(x-2)^{-\frac{1}{2}}} \rightarrow f'(x) = \boxed{(x-2)^{-\frac{1}{2}} - \frac{x(x-2)^{-\frac{3}{2}}}{2}}$$

Let the tangent to C at point $P = (x_o, y_o)$, where $x_0, y_0 \in \mathbb{R}$

$$y_0 = f(x_0) \quad - \quad (1) \text{ and } \boxed{\frac{y_0 - 0}{x_0 - 9} = f'(x_0)} \quad - \quad (2)$$

$$\rightarrow f(x_0) = f'(x_0)(x_0 - 9) \rightarrow x_0\cancel{(x_0-2)^{-\frac{1}{2}}} = \cancel{(x_0-2)^{-\frac{1}{2}}}[1 - \frac{x_0(x_0-2)^{-1}}{2}](x_0 - 9)$$

$$\rightarrow 2x_0(x_0 - 2) = (x_0 - 4)(x_0 - 9)$$

$$\rightarrow x_0^2 + 9x_0 - 36 = 0 \rightarrow x_0 = -12(\text{rejected}, x_0 > 2) \text{ or } 3$$

$$\therefore \text{The slope of tangent} = f'(3) = -0.5$$

* 用 Product rule

* 微分計算斜率

* 代 (1) 入 (2) 搵 x_0

2017 - SECTION A

Q8.) Let the curve $C: y = f(x), f(x) = \frac{1}{x} ln \frac{e}{x}$, for all $x > 0$

Find the area of the region bounded by C, x − axis, $x = 1$ and $x = e^2$.

* 參考課程 2.8 及 2.9

To find the x − interception of C, consider $f(x) = 0 \rightarrow x = e$

$\therefore$ The x − interception is e

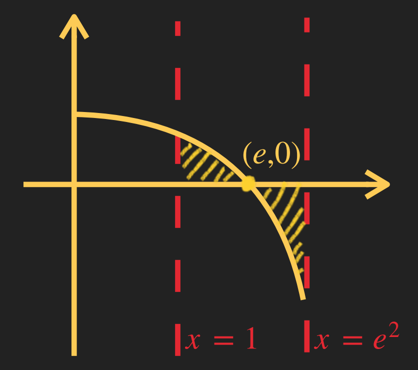
$$Let\ I = \int f(x)dx = \int \frac{1}{x} ln \frac{e}{x} dx = \int \frac{1}{x}[1 - lnx]dx = \int (1 - lnx)d(lnx)$$

$$= (lnx - \frac{[lnx]^2}{2}) + C, \text{ wher } C \text{ is constant}$$

$$\therefore \text{The area} = \left| \int_e^{e^2} f(x)dx \right| + \left| \int_1^e f(x)dx \right| = 1\ unit^2$$

* $ln \frac{A}{B} = lnA - lnB$

* 積分三寶: 積分代入法



* 須要確保面積係正數

2017 – SECTION B

Q9.) The daily spent on study of the students follow $N(\mu, 0.4^2)$ in hour .

a.) A sample of 40 students is randomly selected their time on study is recorded below :

Time spent (x hrs)	$0.5 < x \leq 1.0$	$1.0 < x \leq 1.5$	$1.5 < x \leq 2.0$	$2.0 < x \leq 2.5$	$2.5 < x \leq 3.0$
Number of students	11	13	8	5	3

Find 90 % confidence interval for μ .

b.) Find the least sample size for the width of 97 % confidence interval of μ is at most 0.3

c.) Given $\mu = 1.48$. If student spend more than 2 hours to study, he need to attend extra class .

i.) Find the probability a randomly selected student need to attend extra class .

ii.) A sample 15 students is randomly selected and their study time are examined one by one . Given that more than 1 student in the sample have to attend extra class, find the probability that the 10^{th} student is the 2^{nd} student need to attend extra class .

* 參考課程 4.2, 4.4 及 4.5

a.) The sample mean, $\bar{X}$

$$= \frac{11 \cdot 0.75 + 13 \cdot 1.25 + 8 \cdot 1.75 + 5 \cdot 2.25 + 3 \cdot 2.75}{40} = 1.45$$

* 平均值 = 數據加總 / 總數

CONT'D

2017 - SECTION B

$$The\ 90\,\%\ C.I.\ for\ \mu = \left(1.45 - 1.645 \cdot \frac{0.4}{\sqrt{40}},\ 1.45 + 1.645 \cdot \frac{0.4}{\sqrt{40}}\right)$$

$$= (1.3460,\ 1.5540)\ (to\ 4\ d.p.)$$

$b.)$ *Let the sample size be* n.

$$The\ width\ of\ 97\,\%\ C.I.\ of\ \mu = 2 \cdot 2.17 \cdot \frac{0.4}{\sqrt{n}} < 0.3$$

$$\rightarrow n > 33.4855$$

$$\therefore\ The\ least\ sample\ size = 34$$

$c.)$ *Let* E *be the event for a student need to attend extra class.*

$$P(E) = P\left(Z > \frac{2 - 1.48}{0.4}\right) = P(Z > 1.3) = 0.0968$$

$$d.)\ The\ probability = \frac{C_1^9 P(E)P(\bar{E})^8 \cdot P(E)}{1 - P(\bar{E})^{15} - C_1^{15} P(E)P(\bar{E})^{14}} = 0.0861\ (to\ 4\ d.p.)$$

- * 90% 置信區間
- * 97% 置信區間
- * 先計算 Z ~ N(0,1), 再對表
- * 條件概率
- * 頭 9 個有一個要上堂
- * 第十個要上堂
- * 多過一個要上堂

2017 – SECTION B

Q10.) *In a shop, there are* 45 % , 20 % , 10 % *of customers each get one cash coupon of* \$50, \$100 *and* \$200 *respectively in a transaction . Assume the number of transaction per minutes follow* $Po(2)$.

a.) *Find the probability there are at most* 4 *transactions in a certain minute .*

b.) *Find the probability there is exactly* 3 *transactions with total coupon value* \$200 *are issued in a certain minute .*

c.) *Find the probability if there is exactly* 4 *transactions, the total coupon value* \$200 *are issued in a certain minute .*

d.) *Given that there is at most* 4 *transaction in a certain minute . Find the probability the total coupon value* \$200 *are issued in this minute .*

* 參考課程 4.2 及 4.4

a.) *Let N be the random variable of the number of transaction in a minute .*

$$P(N \le 4) = e^{-2}(1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}) = 7e^{-2} = 0.9473 \ (to\ 4\ d.p.)$$

b.) *Let X be the random variable of the coupon value for each transaction*

* $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

CONT'D

2017 - SECTION B

$$Let\ T_x\ be\ the\ event\ of\ the\ total\ coupon\ value = \$x$$

$$P(T_{200} \cap N = 3) = P(N = 3)[C_2^3 P(X = 0)^2 P(X = 200)$$
$$+ C_2^3 P(X = 0) P(X = 100)^2 + C_2^3 P(X = 100) P(X = 50)^2]$$
$$= \frac{e^{-2} 2^3}{3!} \cdot 3(0.25^2 \cdot 0.1 + 0.25 \cdot 0.2^2 + 0.45^2 \cdot 0.2)$$
$$= 0.0307\ (to\ 4\ d.p.)$$

$$c.)\ P(T_{200} | N = 4) = C_1^4 P(X = 200) P(X = 0)^3 + C_2^4 P(X = 100)^2 P(X = 0)^2$$
$$+ P_3^4 P(X = 0) P(X = 50)^2 P(X = 100) + P(X = 50)^4$$
$$= 0.1838\ (to\ 4\ d.p.)$$

$$d.)\ P(T_{200} | N \leq 4) = \frac{P(T_{200} \cap N \leq 4)}{P(N \leq 4)}$$

* P(Not A) = 1 - P(A)

* $P(X = k) = \dfrac{e^{-\lambda} \lambda^k}{k!}$

* 問題係如果係 4 個交易

* 條件概率

CONT'D

2017 - SECTION B

$$= \frac{P(T_{200} \cap N = 1) + P(T_{200} \cap N = 2) + P(T_{200} \cap N = 3) + P(T_{200} \cap N = 4)}{P(N \leq 4)}$$

$$where\ P(T_{200} \cap N = 1) = \boxed{e^{-2}} \cdot 2 \cdot \boxed{0.1}$$

$$P(T_{200} \cap N = 2) = \boxed{\frac{e^{-2}2^2}{2!}} \cdot [\boxed{(0.2)^2} + \boxed{C_1^2(0.1)(0.25)}]$$

$$P(T_{200} \cap N = 3) = 0.0307$$

$$P(T_{200} \cap N = 4) = \boxed{\frac{e^{-2}2^4}{4!}} \cdot 0.1838$$

$$= 0.1042\ (to\ 4\ d.p.)$$

* 一張 200

* 兩張 100

* $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

2017 - SECTION B

$Q11.)$ Let $f(x) = x^{-1}e^{0.1x}$ and $I = \int_{0.5}^{1} f(x)dx$. There are two method to estimate I.

$M1$: By the trapezoidal rule with 5 sub – interval

$M2$: By replacing $e^{0.1x} = 1 + 0.1x + 0.005x^2$

a.) Find the estimation based on $M1$ and $M2$.

b.) Determine if over – estimation or under – estimation for $M1$ and $M2$

c.) Is $|I - 0.746| < 0.002$? Explain your answer.

* 參考課程 2.2, 2.3, 3.2 及 3.3

a.) Let I_1 be the estimation by $M1$

I_2 be the estimation by $M2$

$$I_1 = \frac{1-0.5}{5\cdot 2}[f(0.5) + 2f(0.6) + 2f(0.7) + 2f(0.8) + 2f(0.9) + f(1)]$$

$$= 0.7476 \ (to\ 4\ d.p.)$$

* 計算梯形面積的加總

CONT'D

2017 - SECTION B

$$I_2 = \int_{0.5}^{1} x^{-1}(1 + 0.1x + 0.005x^2)dx = [lnx + 0.1x + 0.0025x^2]_{0.5}^{1}$$

$$= 0.7450\ (to\ 4\ d.p.)$$

$$b.)\ Consider\ f'(x) = -x^{-2}e^{0.1x} + x^{-1}\cdot 0.1\cdot e^{0.1x}$$

$$= f(x)[-x^{-1} + 0.1]$$

$$\rightarrow f''(x) = f(x)x^{-2} + f'(x)[-x^{-1} + 0.1]$$

$$= f(x)x^{-2} + f(x)[-x^{-1} + 0.1]^2$$

$$\because f(x) > 0,\ where\ 0.5 \le x \le 1$$

$$\therefore f''(x) > 0,\ where\ 0.5 \le x \le 1$$

$$i.e.\ I_1\ is\ over - estimated$$

$$\because e^{0.1x} = \sum_{k=0}^{\infty}\frac{(0.1x)^k}{k!} = 1 + 0.1x + 0.005x^2 + \sum_{k=3}^{\infty}\frac{(0.1x)^k}{k!}$$

$$\therefore I = I_2 + \int_{0.5}^{1} x^{-1}\sum_{k=3}^{\infty}\frac{(0.1)^k}{k!}dx \rightarrow I_2\ is\ under - estimated$$

* 用 Tayler expansion 响 e^x

* Product rule

* 個 f(x) 係 concave upward

* $e^x = \sum_{r=0}^{\infty}\frac{x^r}{r!}$

CONT'D

2017 - SECTION B

$c.)\ I_2 < I < I_1 \rightarrow I_2 - 0.746 < I - 0.746 < I_1 - 0.746$

$\rightarrow -0.001 < I - 0.746 < 0.0016$

$\rightarrow |I - 0.746| < 0.0016$

$\therefore |I - 0.746| < 0.002$

* I_1 係估大左, I_2 係估細左

2017 – SECTION B

Q12.) The number of fishes in river is by the following equation :

$$x = 4 + \frac{3k}{2^{\lambda t} - k}, \text{ where } x \text{ is the number of fishes in thousand, } t \geq 0 \text{ in years, } \lambda, k > 0$$

a.) Find $(x-4)(x-1)$ *in term of* λ, k *and* t.

b.) Is the numner of fishes in river does not lies between 1000 *and* 4000*? Explain your answer.*

c.) Given that $\dfrac{dx}{dt} = \dfrac{-\ln 2}{24}(x-4)(x-1)$

i.) Find λ.

ii.) For each following condition (1) *and* (2)*, find* k *and determine if the fishes in the river will eventually become extinct. When will they become extinct if they will extinct? If not, estimate the number of fishes in the long run.*

(1) *When* $t = 0$, $x = 0.8$

(2) *When* $t = 0$, $x = 7$

* 參考課程 2.1, 2.2, 2.3 及 2.4

CONT'D

2017 - SECTION B

$$a.)\ (x-4)(x-1)=\frac{3k}{2^{\lambda t}-k}\cdot\frac{3\cdot 2^{\lambda t}}{2^{\lambda t}-k}=\frac{9k\cdot 2^{\lambda t}}{(2^{\lambda t}-k)^2}$$

$b.)$ *Obviously,* $(x-4)(x-1)>0$ *for all* $t\geq 0$ $(\because k>0)$

$i.e.\ 1<x$ *or* $x>4$

$\therefore$ *The numner of fishes in river does not lies between* 1000 *and* 4000

$$ci.)\ x=4+\frac{3k}{\boxed{2^{\lambda t}}-k}\rightarrow\frac{dx}{dt}=\frac{-3k\cdot\boxed{2^{\lambda t}ln2}\cdot\lambda}{(2^{\lambda t}-k)^2}$$

$$\rightarrow\frac{-ln2}{24}(x-4)(x-1)=\frac{-3k\cdot 2^{\lambda t}ln2\cdot\lambda}{(2^{\lambda t}-k)^2}$$

$$\rightarrow\frac{1}{24}\frac{9\cancel{k\cdot 2^{\lambda t}}}{\cancel{(2^{\lambda t}-k)^2}}=\frac{3\cancel{k\cdot 2^{\lambda t}}\cdot\lambda}{\cancel{(2^{\lambda t}-k)^2}}$$

$$\rightarrow\lambda=0.125$$

* 用表格証明不等式

* Chain rule

* 假設 y=2ˣ

$$lny=xln2\rightarrow\frac{y'}{y}=ln2\rightarrow y'=2^x ln2$$

CONT'D

2017 - SECTION B

$$ii.)\ For\ condition\ (1),\ x(0) = 4 + \frac{3k}{1-k} = 0.8 \rightarrow k = 16$$

$$For\ condition\ (2),\ x(0) = 4 + \frac{3k}{1-k} = 7 \quad \rightarrow k = 0.5$$

$$\because \frac{dx}{dt} = \frac{-ln2}{24}(x-4)(x-1) < 0\ (\because (x-4)(x-1) > 0\ for\ all\ t \geq 0)$$

$$\therefore x\ is\ decreasing\ for\ t\ increasing$$

$$Also,\ \lim_{t\to\infty} x = 4 + \lim_{t\to\infty} \frac{3k}{2^{\lambda t} - k} = 4$$

$$Also,\ the\ vertical\ asymtotes: 2^{0.125t} - k = 0 \rightarrow t = 8log_2 k$$

$$Hence, Condition\ (1),\ the\ vertical\ asymtotes: t = 32$$

$$Condition\ (2),\ the\ vertical\ asymtotes: t = -8$$

$$Hence, Condition\ (1),\ x\ is\ discontinuous\ function\ for\ x \geq 0$$

$$Condition\ (2),\ x\ is\ continuous\ decreasing\ function\ for\ x \geq 0$$

* $f'(x) < 0 \rightarrow Decreasing$

* $t \rightarrow \infty, 2^t \rightarrow \infty, \frac{1}{2^t - k} \rightarrow 0$

*t 係幾多, 分母係零

2017 – SECTION B

$$\text{For condition (1)} : 0 = 4 + \frac{3 \cdot 16}{2^{0.125t} - 16}$$

$$\rightarrow t = 16$$

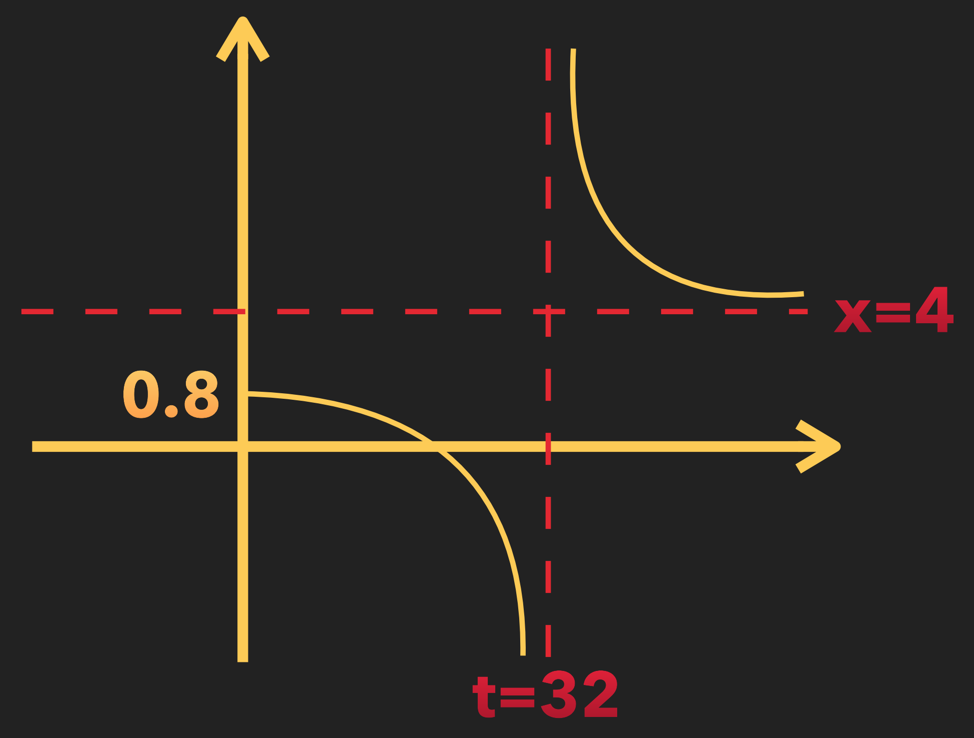
$\therefore$ *Fishes will become extinct when* $t = 16$ *years*

$$\text{For condition (2)} : \text{There is no roots for } x(t) = 0$$

$\therefore$ *Fishes will not become extinct and*

The number of fishes in the long run $= 4000$

* k=16



* k=0.5