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Advanced Signal & Systems

CT Convolution $(f * g) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau$ DT Convolution $(f*g)[n] = \sum_{m=-\infty}^{\infty} f[m] g[n-m]$ Convolution Theorem $\mathcal{F}\{f*g\} = k \cdot \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}$ LTI System, $h(t)$ is impulse response, then $y(t) = (x*h)(t)$ Exponentials as eigen functions $Hf = \lambda f$ where $Hf = \int_{-\infty}^{\infty} h(\tau) A e^{s(t-\tau)} d\tau$ and $\lambda = H(s)$ and $f = Ae^{s(t-\tau)}$ Spectral Density: Energy $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{x}(f)|^2 df$ Energy Spectral Density: $S_{xx}(f) = |\hat{x}(f)|$ Power $P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$ Power Spectral Density $S_{xx}(\omega) = \lim_{T\to\infty} E\left[|\hat{x}_T(\omega)|^2\right] = E\left[\frac{1}{T}\int_0^T x^*(t) e^{i\omega t} dt \int_0^T x(t') e^{-i\omega t'} dt'\right]$ ~~Wiener~~ Wiener-Khinchin Theorem $S_{xx}(\omega) = \int_{-\infty}^{\infty} \gamma(\tau) e^{-i\omega\tau} d\tau$ where $\gamma(\tau)$ is auto-correlation function. Power-bandlimited $P_{\text{bandlimited}} = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} S_{xx}(\omega) d\omega$ Cross-spectral Density $S_{xy}(\omega) = \lim_{T\to\infty} E\left[(F_x^T(\omega))^*(F_y^T(\omega))\right] = \int_{-\infty}^{\infty} R_{xy}(t) e^{-j\omega t}$ where $R_{xy}(t)$ is the cross-correlation of $x(t)$ & $y(t)$.

Periodogram - non-parametric estimate of spectral density for signal. Auto-correlation $R_{ff}(\tau) = (f * g_{-1}(\bar{f}))(\tau) = \int_{-\infty}^{\infty} f(u+\tau) \bar{f}(u) du$ $\mathcal{I}\, g_{-1}(f)(u) = f(-u)$ Discrete Auto-correlation $R_{yy}(l) = \sum_{n\in\mathbb{Z}} y(n) y^*(n-l)$.

Auto-correlation for wide-sense-stationary random processes: $R_{ff}(\tau) = E[f(t) f^*(t-\tau)]$. Autocorrelation is an even function. Cross-correlation $(f \star g)(\tau) = \int_{-\infty}^{\infty} f^*(t) g(t+\tau) dt$ Discrete Cross-correlation $(f\star g)[n] = \sum_{m=-\infty}^{\infty} f^*[m] g[m+n]$ CC is <u>NOT</u> commutative. Frequency Response ~~Transfer~~ function $\hat{h}(\omega) = \frac{R_{xy}(\omega)}{R_{xx}(\omega)}$ Auto-covariance $C_{xx}(t,s) = E[(X_t - \mu_t)(X_s - \mu_s)]$ where s & t are two time periods. Auto-correlation $R(s,t) = C_{xx}(s,t)/\sigma_s \sigma_t$ Correlation is the normalized version of covariance.

Cross-Spectrum: Let γ_{xx}, γ_{yy} be auto-covariance, γ_{xy} be cross-covariance, Γ_{xy} be cross-spectrum, then $\Gamma_{xy}(f) = \mathcal{F}\{\gamma_{xy}\}(f) = \sum_{\tau=-\infty}^{\infty} \gamma_{xy}(\tau) e^{-2\pi i \tau f}$ Decomposition into real & imaginary $\Gamma_{xy}(f) = \Lambda_{xy}(f) + i\Psi_{xy}(f)$ Decomposition into amplitude & phase $\Gamma_{xy}(f) = A_{xy}(f)\, e^{i\phi_{xy}(f)}$ Spectral Coherence $C_{xy}(f) = \frac{|G_{xy}(f)|^2}{G_{xx}(f) G_{yy}(f)}$ where $G_{xy}(f)$ is cross-spectral density and $G_{xx}(f)$, $G_{yy}(f)$ are auto power spectral density.

$h(t) = \mathcal{F}^{-1}\{H(s)\}$, $h(t)$ is impulse response, $H(s)$ is transfer function, including frequency response & phase response.

Volterra Series - model response of a non-linear system, is time-invariant but has memory.
$y(t) = h_0 + \sum_{n=1}^{N} \int_a^b \cdots \int_a^b h_n(\tau_1, \dots, \tau_n) \prod_{j=1}^{n} x(t-\tau_j)\, d\tau_j$ where $h_n(\tau_1, \dots, \tau_n)$ is the n^{th} order Volterra kernel

Cross-correlation is used to estimate Volterra kernel. Power transfer function. $\hat{h}(\omega) = \frac{R_{xy}(\omega)}{R_{xx}(\omega)}$, $\hat{H}_\omega = A_\omega^{-1} \cdot \vec{C}_\omega$
A_ω is stimulus auto-correlation matrix and C_ω is the cross-correlation betw. spike trains & the stimulus amplitudes.

Reverse correlation. Gaussian White Noise Stimulus: $\langle c(t_1) c(t_2)\rangle = \sigma^2 \delta(t_1 - t_2)$ where $\langle\cdot\rangle$ is the mean value and $\langle c(t)\rangle = 0$. Has flat power spectral density. Stationarity $R_{cc}(\tau) = \langle c(t) c(t+\tau)\rangle$ depends only on time diff $\tau = t_2 - t_1$ Ergodicity $R_{cc}(\tau) = \frac{1}{T}\int_0^T c_0(t) c_0(t+\tau) dt$. Derivation of reverse correlation (transfer function)
Firing rate modulation (output): $f(t) = \langle x(t) - f_{mean}\rangle_c$ where $x(t) = \sum_{i=1}^{N} \delta(t - t_i)$ is spike train response to single stimulus, $c(t)$, and $\langle\cdot\rangle_c$ is avg of repeated presentation of $c(t)$. $f(t) = \langle x(t) - f_{mean}\rangle_c = \int w_t(t - t_0) c(t_0) dt_0$
Multiply both sides by $c(t)$, we get $\int w_t(t+\tau-t_0)\langle c(t) c(t_0)\rangle dt_0 = \int w_t(t+\tau-t_0)\sigma^2 \delta(t-t_0) dt_0 = \sigma^2 w_t(\tau)$
Input is $c(t)$.

Notice that $\langle (x(t+\tau) - f_{mean})\, c(t)\rangle = \langle x(t+\tau)\, c(t)\rangle$, then $\langle x(t+\tau)c(t)\rangle = \frac{1}{T}\int_0^T x(t+\tau)c(t)\,dt$

$\frac{1}{T}\int_0^T x(t+\tau)c(t)\,dt = \frac{1}{T}\int_0^T \sum_{i=1}^{N} \delta(t+\tau-t_i)\,c(t)\,dt = \frac{1}{T}\sum_{i=1}^{N} c(t_i-\tau) = f_{mean}\,\frac{1}{N}\sum_{i=1}^{N} c(t_i-\tau)$, N is different stimulus or trial

Summing up, the transfer function $\omega_t(\tau) = \frac{f_{mean}}{\sigma^2}\left(\frac{1}{N}\sum_{i=1}^{N} c(t_i-\tau)\right)$