Fourier Series $e^{st} = x(t)$ Eigenfunction of LTI systems $H(s)$ eigenvalue. Euler's relation $\cos(4\pi t) = \frac{1}{2}\left[e^{j4\pi t} + e^{-j4\pi t}\right]$ $\sin(4\pi t) = \frac{1}{2j}\left[e^{j4\pi t} - e^{-j4\pi t}\right]$ Integral $\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos((k-n)\omega_0 t)\,dt + j\underbrace{\int_0^T \sin((k-n)\omega_0 t)\,dt}_{0} = \begin{cases} T, & k=n \\ 0, & k\neq n \end{cases}$

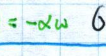
Synthesis Equation $x(t) = \sum_{k=-\infty}^{\infty} a_k \boxed{e^{jk\omega_0 t}}$ Analysis Equation $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$ & $a_0 = \frac{1}{T}\int_T x(t)\,dt$. Important Integral for square wave $\frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = \frac{-1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{\sin k\omega_0 T_1}{k\pi}$ $\left(\omega_0 = \frac{2\pi}{T}\right)$. 3 conditions. 1) Absolutely integrable $\int_T |x(t)|\,dt < \infty$

2) finite # of maxima & minima. 3) finite # of discontinuities Linearity $c(t) = Ax_1(t) + Bx_2(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k$ If $x(t)$ is real, then $a_{-k} = a_k^*$ Time Shift $x(t) \leftrightarrow a_k$ $x(t-t_0) \longleftrightarrow a_k e^{-jk\omega_0 t_0}$ Time Reversal $y(t) = x(-t)$, $b_m = a_{-m}$ Time Scaling $y(t) = x(\alpha t)$, $\sum_{k=-\infty}^{\infty} a_k e^{jk(\omega_0\alpha)t}$ Multiplication $x(t) \leftrightarrow a_k$, $y(t) \leftrightarrow b_k$, $x(t)\cdot y(t) \longleftrightarrow h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$ Parseval's Relation $\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$ DT Fourier Series Representation $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ finite series. Finite Geometric Series $\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha=1 \\ \frac{1-\alpha^N}{1-\alpha}, & \alpha\neq 1 \end{cases}$ DT Fourier Series Pair. Synthesis $x[n] = \sum_{k=\langle N\rangle} a_k \boxed{e^{jk\omega_0 n}}$ Analysis $a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n}$

First Difference of Sequence $y[n] = x[n] - x[n-1] \xleftrightarrow{F.S.} b_k = \left(1 - e^{-jk(2\pi/N)}\right)a_k$

Fourier Transform Synthesis $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$ Analysis $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ The Fourier coefficients of a periodic signal are proportional to samples of the Fourier transform of one period of the periodic signal. Square pulse $X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = \frac{2\sin(\omega T_1)}{\omega}$ Square pulse (frequency) $x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{\sin(Wt)}{\pi t}$ Uncertainty Principle. As width in frequency becomes smaller, width in time becomes larger – w/ a constant product. Conjugate symmetry $X(-j\omega) = X^*(j\omega) \Rightarrow |X(-j\omega)| = |X(j\omega)|$ (Even) $\angle X(-j\omega) = -\angle X(j\omega)$ (odd) $\mathrm{Re}\{X(-j\omega)\} = \mathrm{Re}\{X(j\omega)\}$ (Even) $\mathrm{Im}\{X(-j\omega)\} = -\mathrm{Im}\{X(j\omega)\}$ (odd) Synthesis (Mag & Phase) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|e^{j\angle X(j\omega)}e^{j\omega t}\,d\omega$

DTFT Discrete-Time Fourier Series. $x[n]$ is periodic w/ period = N, fundamental frequency $\Omega_s = \frac{2\pi}{N}$, then there are $k = 0, \dots, N-1$ (N coefficients), which are periodic. As $N \to \infty$, $f_s \to \infty$, and envelope remains the same, $\to$ aperiodic DTFT. Let $\tilde{x}[n]$ be periodic signal w/ $x[n] = \tilde{x}[n]$ for $|n| < N/2$. These Series: $\tilde{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{2\pi} X(k\Omega_0)e^{jk\Omega_0 n}\Omega_0$, $X(k\Omega_0) = Na_k = \sum_{n=-N/2}^{N/2} \underbrace{\tilde{x}[n]}_{x[n]} e^{-jk\Omega_0 n}$ As $N \to \infty$, transform $x[n] = \frac{1}{2\pi}\int_{T=2\pi} X(\Omega)e^{j\Omega n}\,d\Omega$, $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ Fourier series coefficients of $\tilde{x}[n]$ equal $\frac{1}{N}$ times samples of F.T. of one period. For both C.T. & D.T., if $x(t)$ is real, then $\mathrm{Re}\{X(j\omega)\}$ and $|X(j\omega)|$ are even and $\mathrm{Im}\{X(j\omega)\}$ and $\angle X(j\omega)$ are odd. $\tilde{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{N} X(e^{jk\omega_0})e^{jk\omega_0 n}$ Ex $\mathcal{F}\left\{x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}\right\}$ is periodic, $N = 2\pi$, $= X(e^{j\omega}) = \frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$ and $\mathcal{F}\left\{x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}\right\}$ is not periodic, $X(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$. Eigenfunctions. CT. $x(t) = e^{j\omega_0 t} \longleftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$ DT $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = 2\pi\sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)$ DTFT of periodic signals $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k\delta\left(\omega - \frac{2\pi k}{N}\right)$ $(-1)^n = e^{j\pi n}$

Time and Frequency C.T. Parseval's $\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} \frac{1}{2\pi}|X(j\omega)|^2\,d\omega$ D.T. Parseval's $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{2\pi} |X(e^{j\omega})|^2\,d\omega$ Phase does not impact energy, only shape. Linear phase, shift in time. $H(j\omega) = e^{-j\omega\alpha}$ $|H(j\omega)| = 1$, $\angle H(j\omega) = -\alpha\omega$ Group Delay $\tau(\omega) = -\frac{d}{d\omega}(\angle H(j\omega))$ for small band of f. Ideal LP Filter Step Response $s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$ where $h(\tau) =$  τ

Sampling Sampling theorem. 1) Band-limited. 2) $f_s \geq 2B$. Impulse Sampling Train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, $x_p(t) = x(t)\cdot p(t) \longleftrightarrow X_p(j\omega) = \frac{1}{2\pi}X(j\omega) * P(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$ Zero-Order Hold, use LP Filter. Reconstruction $h(t) = \frac{T\sin(\omega_c t)}{\pi t}$, $x_r(t) = x_p(t) * h(t)$, $\omega_M < \omega_c < (\omega_s - \omega_M)$, using low-pass filter $x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{T\sin[\omega_c(t - nT)]}{\pi(t - nT)}$

Summation 1) $\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1-a}$ 2) $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ 3) $\sum_{k=0}^{\infty} ka^k = \frac{a}{(1-a)^2}$

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Signals & Systems

CT FT Properties Linearity $ax(t)+by(t) \leftrightarrow aX(j\omega)+bY(j\omega)$ Time Shifting $x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$ Freq. shifting $e^{j\omega_0 t}x(t) \leftrightarrow X(j(\omega-\omega_0))$
Conjugation $x^*(t) \leftrightarrow X^*(-j\omega)$ Time Reversal $x(-t) \leftrightarrow X(-j\omega)$ Time & Freq. Scaling $x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$ Conv $x(t)*y(t) \leftrightarrow X(j\omega)Y(j\omega)$
Mult. $x(t)y(t) \leftrightarrow \frac{1}{2\pi}X(j\omega)*Y(j\omega)$ Diff. $\frac{d}{dt}x(t) \leftrightarrow j\omega X(j\omega)$ Int. $\int_{-\infty}^{t}x(t)dt \leftrightarrow \frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$ Diff. Freq. $tx(t) \leftrightarrow j\frac{d}{d\omega}X(j\omega)$

1) $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega-k\omega_0)$ 2) $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$, $a_1=1$, $a_k=0$ 3) $\cos(\omega_0 t) \leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$, $a_1=a_{-1}=\frac{1}{2}$, $a_k=0$

4) $\sin(\omega_0 t) \leftrightarrow \frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$, $a_1=-a_{-1}=\frac{1}{2j}$ 5) $x(t)=1 \leftrightarrow 2\pi\delta(\omega)$ 6) $\sum_{n=-\infty}^{\infty}\delta(t-nT) \leftrightarrow \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$, $a_k=\frac{1}{T}$

7) Periodic Square Wave $x(t)=\begin{cases}1, & |t|<T_1\\ 0, & T_1<|t|\le T/2\end{cases} \leftrightarrow \sum_{k=-\infty}^{\infty}\frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega-k\omega_0)$ 8) Non-periodic Square Wave $x(t)=\begin{cases}1, & |t|<T_1\\ 0, & \text{o.w.}\end{cases} \leftrightarrow \frac{2\sin(\omega T_1)}{\omega}$

9) $\frac{\sin Wt}{\pi t} \leftrightarrow X(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$ 10) $\delta(t) \leftrightarrow 1$ 11) $u(t) \leftrightarrow \frac{1}{j\omega}+\pi\delta(\omega)$ 12) $\delta(t-t_0) \leftrightarrow e^{-j\omega t_0}$ 13) $e^{-at}u(t)$, $\Re\{a\}>0 \leftrightarrow \frac{1}{a+j\omega}$

14) $te^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^2}$ 15) $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^n}$

DT FT Properties Linearity $ax[n]+by[n] \leftrightarrow aX(e^{j\omega})+bY(e^{j\omega})$
Time Shift $x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$ Freq. Shift $e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$

Conjugation $x^*[n] \leftrightarrow X^*(e^{-j\omega})$ Time Reversal $x[-n] \leftrightarrow X(e^{-j\omega})$ Time Expansion $x_{(k)}[n]=\begin{cases}x[n/k], & \text{if } n=\text{multiples of } k\\ 0, & \text{o.w.}\end{cases} \leftrightarrow X(e^{jk\omega})$

Convolution $x[n]*y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$ Multiplication $x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Diff. in Time $x[n]-x[n-1] \leftrightarrow (1-e^{-j\omega})X(e^{j\omega})$ Accumulation $\sum_{k=-\infty}^{n}x[k] \leftrightarrow \frac{1}{1-e^{-j\omega}}X(e^{j\omega})+\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$ Diff. in Freq.
$nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$

1) $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \leftrightarrow 2\pi\sum_{k=-\infty}^{\infty}a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$ 2) $e^{j\omega_0 n} \leftrightarrow 2\pi\sum_{l=-\infty}^{\infty}\delta(\omega-\omega_0-2\pi l)$ 5) $x[n]=1 \leftrightarrow 2\pi\sum_{l=-\infty}^{\infty}\delta(\omega-2\pi l)$

3) $\cos[\omega_0 n] \leftrightarrow \pi\sum_{l=-\infty}^{\infty}\{\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)\}$ 4) $\sin[\omega_0 n] \leftrightarrow \frac{\pi}{j}\sum_{l=-\infty}^{\infty}\{\delta(\omega-\omega_0-2\pi l)-\delta(\omega+\omega_0-2\pi l)\}$

6) Periodic Square Wave $x[n]=\begin{cases}1, & |n|\le N_1\\ 0, & N_1<|n|\le N/2\end{cases} \leftrightarrow 2\pi\sum_{k=-\infty}^{\infty}a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$ $a_k=\sin[(2\pi k/N)(N_1+\frac{1}{2})]/[N\sin(2\pi k/2N)]$
$k\ne 0, \pm N, \pm 2N, \ldots$

$a_k=\frac{2N_1+1}{N}$, $k=0,\pm N,\pm 2N,\ldots$ 7) Non-periodic Square Wave $x[n]=\begin{cases}1, & |n|\le N_1\\ 0, & |n|>N_1\end{cases} \leftrightarrow \frac{\sin[\omega(N_1+\frac{1}{2})]}{\sin(\omega/2)}$ 8) $\sum_{k=-\infty}^{\infty}\delta[n-kN] \leftrightarrow \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$

9) $a^n u[n]$, $|a|<1 \leftrightarrow \frac{1}{1-ae^{-j\omega}}$ 10) $\delta[n] \leftrightarrow 1$ 11) $u[n] \leftrightarrow \frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega-2\pi k)$ 12) $\delta[n-n_0] \leftrightarrow e^{-j\omega n_0}$

13) $(n+1)a^n u[n] \leftrightarrow \frac{1}{(1-ae^{-j\omega})^2}$ 14) $\frac{(n+r-1)!}{n!(r-1)!}a^n u[n] \leftrightarrow \frac{1}{(1-ae^{-j\omega})^r}$ $|a|<1$ for 13) and 14).