Assignment 1

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EXERCISE 1: COMPLEX NUMBERS

1. Assume $x = a_1 + b_1 i$, $x = a_2 + b_2 i$, $x = a_3 + b_3 i$,

   we have $x + y + z = a_1 + a_2 + a_3 + (b_1 + b_2 + b_3)i$,

   To a complex number $n = a + bi$ we have $n^* = a - bi$,

   so

   $|x|^2 = x \cdot x^* = a_1^2 - b_1^2 (i)^2 = a_1^2 - b_1^2 \cdot (-1) = a_1^2 + b_1^2$,

   $x^* y$
   $= (a_1 - b_1 i) \cdot (a_2 + b_2 i)$
   $= a_1 a_2 + a_1 b_2 i - a_2 b_1 i - b_1 b_2 i^2$
   $= a_1 a_2 + b_1 b_2 + (a_1 b_2 - a_2 b_1)i$,

   $Re(x^* y) = a_1 a_2 + b_1 b_2$,

   similarly, we have

   $|y|^2 = a_2^2 + b_2^2$,

   $|z|^2 = a_3^2 + b_3^2$,

   $Re(y^* z) = a_2 a_3 + b_2 b_3$

   $Re(x^* z) = a_1 a_3 + b_1 b_3$

   and

   $(x + y + z)^* = a_1 + a_2 + a_3 - (b_1 + b_2 + b_3)i$,

   $|x + y + z|^2$
   $= (x + y + z) \cdot (x + y + z)^*$
   $= ((a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i) \cdot ((a_1 + a_2 + a_3) - (b_1 + b_2 + b_3)i)$
   $= ((a_1 + a_2 + a_3)^2 - (b_1 + b_2 + b_3)^2 \cdot (-1))$
   $= ((a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2)$
   $= a_1^2 + a_2^2 + a_3^2 + 2a_1 a_2 + 2a_1 a_3 + 2a_2 a_3 + b_1^2 + b_2^2 + b_3^2 + 2b_1 b_2 + 2b_1 b_3 + 2b_2 b_3$
   $= a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2 + 2(a_1 a_2 + b_1 b_2 + a_2 a_3 + b_2 b_3 + a_1 a_3 + b_1 b_3)$
   $= |x|^2 + |y|^2 + |z|^2 + 2[Re(x^* y) + Re(y^* z) + Re(x^* z)]$

   This shows that

   $|x + y + z|^2 = |x|^2 + |y|^2 + |z|^2 + 2[Re(x^* y) + Re(y^* z) + Re(x^* z)]$

2. $(i+2)(3-4i)/(2-i)$
   $= (3i - 4i^2 + 2*3 - 2*4i)/(2-i)$
   $= (3i - 4\times(-1) + 2*3 - 2*4i)/(2-i)$
   $= (3i + 4 + 6 - 8i)/(2-i)$
   $= (10 - 5i)/(2-i)$
   $= 5(2-i)/(2-i)$
   $= 5$

3. $(i-4)/(2i-3)$
   $= [(i-4)(2i+3)]/[(2i-3)(2i+3)]$
   $= (2i^2 + 3i - 8i - 4*3)/((2i)^2 - 3*3)$
   $= [2\times(-1) + 3i - 8i - 4*3]/[4\times(-1) - 3*3]$
   $= (-2 - 5i - 12)/(-4-9)$
   $= (-14 - 5i)/(-13)$
   $= [(-1)(14+5i)]/(-1\times 13)$
   $= (14+5i)/13$
   $= 14/13 + 5/13i$

   so, the real part is 14/13 and imaginary pary is 5/13.

4. i^{33}
   $= i^{32}i$
   $= i^{2\times 16}i$
   $= (i^2)^{16}i$
   $= (-1)^{16}i$
   $= i$

   so, the absolute value of i^{33} is $|i|$
   $|i| = |0+i| = \sqrt{|0+i|^2} = \sqrt{(0+i)(0+i)^*} = \sqrt{(0+i)(0-i)} = \sqrt{-i^2} = \sqrt{-(-1)} = \sqrt{1} = 1$

5. i. For complex number $c_1 = a_1 + b_1 i$ and $c_2 = a_2 + b_2 i$, we have

   $$|c_1|^2 = a_1^2 + b_1^2$$

   $$|c_2|^2 = a_2^2 + b_2^2$$

   $$|c_1 + c2|^2 = (a_1+a_2)^2 + (b_1+b_2)^2 = a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2$$

   so we need to find a_1, a_2, b_1, b_2 that makes

   $$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 \geq a_1^2 + b_1^2$$

   and

   $$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 < a_2^2 + b_2^2$$

   so we have

   $$a_2^2 + 2a_1a_2 + b_2^2 + 2b_1b_2 \geq 0$$

   and

   $$a_1^2 + 2a_1a_2 + b_1^2 + 2b_1b_2 < 0$$

so, we need $2a_1a_2+2b_1b_2 \geq -(a_2^2+b_2^2)$ and $2a_1a_2+2b_1b_2 < -(a_1^2+b_1^2)$, which means $-(a_2^2+b_2^2) \leq 2a_1a_2+2b_1b_2 < -(a_1^2+b_1^2)$,
Through observing, it is easy to find $a_1 = -1, a_2 = 3, b_1 = 1, b2 = -3$ makes $|c_1+c_2|^2 \geq |c_1|^2$ and $|c_1+c_2|^2 < |c_2|^2$.

ii. To make $|c_1+c_2|^2 < |c_1|^2$ and $|c_1+c_2|^2 < |c_2|^2$, we need to make

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 < a_1^2 + b_1^2$$

and

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 < a_2^2 + b_2^2$$

so, we have

$$a_2^2 + 2a_1a_2 + b_2^2 + 2b_1b_2 < 0$$

and

$$a_1^2 + 2a_1a_2 + b_1^2 + 2b_1b_2 < 0$$

I don't think we can find a_1, a_2, b_1, b_2 that can make $|c_1+c_2|^2 < |c_1|^2$ and $|c_1+c_2|^2 < |c_2|^2$ true.