

# Assignment 2

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## 1 EXERCISE 1: THE PARTIAL TRACE

We saw in the lectures that given a multi-partite state, we obtain the state of a subsystem by applying the partial trace to the other systems.

1. Compute the trace of the following states by applying the trace formula  $Tr(A) = \sum_i \langle i| A |i\rangle$  from the lectures:

(a)  $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

(b)  $\Lambda = \frac{I}{3} + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$  (here,  $I$  is the identity matrix).

Which of the states is normalized correctly? (\*) (4 points)

Answer:  $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

$$= \frac{1}{2} \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

$$Tr(\xi) = |0\rangle\langle 0| \xi |0\rangle + |1\rangle\langle 1| \xi |1\rangle$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{i}{2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{i}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

$$\Lambda = \frac{I}{3} + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$= \frac{I}{3} + \frac{1}{6} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= \frac{I}{3} + \frac{1}{6} \left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

$$= \frac{I}{3} + \frac{1}{6} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{6} \\ \frac{1}{6} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

$$Tr(\Lambda) = |0\rangle\langle 0| \Lambda |0\rangle + |1\rangle\langle 1| \Lambda |1\rangle$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \end{bmatrix}$$

$$= \frac{1}{3} + \frac{1}{3}$$

$$= \frac{2}{3}$$

So  $\xi$  is normalized correctly.

2. In the last assignment we found the probability to find a state  $|\gamma\rangle$  in another state  $|\delta\rangle$  to be  $p = |\langle\gamma|\delta\rangle|^2$ . Show here that this expression coincides with  $\text{Tr}(\gamma\delta)$ , for  $\gamma = |\gamma\rangle\langle\gamma|$  and  $\delta = |\delta\rangle\langle\delta|$ . Hint: Apply the trace formula  $\text{Tr}(A) = \sum_i \langle i|A|i\rangle$  from the lectures. Use a suitable basis of your choice. (4 points)

$$\begin{aligned} \text{Answer: } \text{Tr}(\gamma\delta) &= \text{Tr}(|\gamma\rangle\langle\gamma||\delta\rangle\langle\delta|) \\ &= \text{Tr}(|\gamma\rangle\langle\gamma|\delta\rangle\langle\delta|) \\ &= \text{Tr}(\langle\gamma|\delta\rangle|\gamma\rangle\langle\delta|) \\ &= \langle\gamma|\delta\rangle \text{Tr}(|\gamma\rangle\langle\delta|) \\ &= \langle\gamma|\delta\rangle \sum_i \langle i||\gamma\rangle\langle\delta||i\rangle \\ &= \langle\gamma|\delta\rangle (\langle 0||\gamma\rangle\langle\delta||0\rangle + \langle 1||\gamma\rangle\langle\delta||1\rangle) \\ &= \langle\gamma|\delta\rangle (\langle 0|\gamma\rangle\langle\delta|0\rangle + \langle 1|\gamma\rangle\langle\delta|1\rangle) \\ &= \langle\gamma|\delta\rangle (\langle\gamma|0\rangle\langle 0|\delta\rangle + \langle\gamma|1\rangle\langle 1|\delta\rangle) \\ &= \langle\gamma|\delta\rangle (\langle\gamma|\delta\rangle) \\ &= |\langle\gamma|\delta\rangle|^2 \end{aligned}$$

$$\text{So, } \text{Tr}(\gamma\delta) = |\langle\gamma|\delta\rangle|^2 = p.$$

3. Consider the bipartite state  $|\phi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i|01\rangle_{AB} - |11\rangle_{AB})$ . Write the density operator  $\rho_{AB} = |\phi\rangle_{AB}\langle\phi|_{AB}$  explicitly in matrix form. (\*) (4 points)

$$\begin{aligned} \text{Answer: } \rho_{AB} &= |\phi\rangle_{AB}\langle\phi|_{AB} \\ &= \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i|01\rangle_{AB} - |11\rangle_{AB})(\langle 00|_{AB} - i\langle 01|_{AB} - \langle 11|_{AB}) \\ &= \frac{1}{3}(|00\rangle\langle 00| - i|00\rangle\langle 01| - |00\rangle\langle 11| + i|01\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 11| - |11\rangle\langle 00| - |11\rangle\langle 01| + |11\rangle\langle 11|) \\ &= \frac{1}{3} \left( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} - \right. \\ &\quad \left. \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \\ &= \frac{1}{3} \begin{bmatrix} 1 & i & 0 & -1 \\ -i & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & i & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{i}{3} & 0 & -\frac{1}{3} \\ -\frac{i}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{i}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{aligned}$$

4. Let  $\sigma_{AB}$  be a general 2-qubit state. The  $4 \times 4$  matrix describing  $\sigma_{AB}$  can be split into four sub-matrices of size  $2 \times 2$  (upper left block, upper right block, lower left block, lower right block). Show that the reduced single qubit state  $\sigma_B = \text{Tr}_A(\sigma_{AB})$  is described by a  $2 \times 2$  matrix that is the sum of the upper left block and lower right block matrices of  $\sigma_{AB}$ . Hint: start from  $\sigma_{AB} = \sigma_{00}|00\rangle\langle 00| + \sigma_{01}|00\rangle\langle 01| + \dots + \sigma_{33}|11\rangle\langle 11|$  and compute the partial trace in bra-ket notation. Write  $\sigma_{AB}$  as a matrix and compare. (8 points)

$$\text{Answer: } \sigma_{AB} = \sigma_{00}|00\rangle\langle 00| + \sigma_{01}|00\rangle\langle 01| + \dots + \sigma_{33}|11\rangle\langle 11|$$

$$\begin{aligned} \sigma_B &= \\ &= \text{Tr}_A(\sigma_{AB}) \end{aligned}$$

$$\begin{aligned}
&= \sum_i \langle i | \sigma_{AB} | i \rangle \\
&= \langle 0 | \sigma_{AB} | 0 \rangle + \langle 1 | \sigma_{AB} | 1 \rangle \\
&= \langle 0 | (\sigma_{00} | 00 \rangle \langle 00 | + \sigma_{01} | 00 \rangle \langle 01 | + \dots + \sigma_{33} | 11 \rangle \langle 11 |) | 0 \rangle + \langle 1 | (\sigma_{00} | 00 \rangle \langle 00 | + \sigma_{01} | 00 \rangle \langle 01 | + \dots + \sigma_{33} | 11 \rangle \langle 11 |) | 1 \rangle \\
&= \sigma_{00} \langle 0 | 00 \rangle \langle 00 | 0 \rangle + \sigma_{01} \langle 0 | 00 \rangle \langle 01 | 0 \rangle + \sigma_{02} \langle 0 | 00 \rangle \langle 10 | 0 \rangle + \sigma_{03} \langle 0 | 00 \rangle \langle 11 | 0 \rangle + \sigma_{10} \langle 0 | 01 \rangle \langle 00 | 0 \rangle + \sigma_{11} \langle 0 | 01 \rangle \langle 01 | 0 \rangle + \\
&\quad \sigma_{12} \langle 0 | 01 \rangle \langle 10 | 0 \rangle + \sigma_{13} \langle 0 | 01 \rangle \langle 11 | 0 \rangle + \sigma_{20} \langle 0 | 10 \rangle \langle 00 | 0 \rangle + \sigma_{21} \langle 0 | 10 \rangle \langle 01 | 0 \rangle + \sigma_{22} \langle 0 | 10 \rangle \langle 10 | 0 \rangle + \sigma_{23} \langle 0 | 10 \rangle \langle 11 | 0 \rangle + \\
&\quad \sigma_{30} \langle 0 | 11 \rangle \langle 00 | 0 \rangle + \sigma_{31} \langle 0 | 11 \rangle \langle 01 | 0 \rangle + \sigma_{32} \langle 0 | 11 \rangle \langle 10 | 0 \rangle + \sigma_{33} \langle 0 | 11 \rangle \langle 11 | 0 \rangle + \sigma_{00} \langle 1 | 00 \rangle \langle 00 | 1 \rangle + \sigma_{01} \langle 1 | 00 \rangle \langle 01 | 1 \rangle + \\
&\quad \sigma_{02} \langle 1 | 00 \rangle \langle 10 | 1 \rangle + \sigma_{03} \langle 1 | 00 \rangle \langle 11 | 1 \rangle + \sigma_{10} \langle 1 | 01 \rangle \langle 00 | 1 \rangle + \sigma_{11} \langle 1 | 01 \rangle \langle 01 | 1 \rangle + \sigma_{12} \langle 1 | 01 \rangle \langle 10 | 1 \rangle + \sigma_{13} \langle 1 | 01 \rangle \langle 11 | 1 \rangle + \\
&\quad \sigma_{20} \langle 1 | 10 \rangle \langle 00 | 1 \rangle + \sigma_{21} \langle 1 | 10 \rangle \langle 01 | 1 \rangle + \sigma_{22} \langle 1 | 10 \rangle \langle 10 | 1 \rangle + \sigma_{23} \langle 1 | 10 \rangle \langle 11 | 1 \rangle + \sigma_{30} \langle 1 | 11 \rangle \langle 00 | 1 \rangle + \sigma_{31} \langle 1 | 11 \rangle \langle 01 | 1 \rangle + \\
&\quad \sigma_{32} \langle 1 | 11 \rangle \langle 10 | 1 \rangle + \sigma_{33} \langle 1 | 11 \rangle \langle 11 | 1 \rangle \\
&= \sigma_{00} | 0 \rangle \langle 0 | + \sigma_{02} | 0 \rangle \langle 1 | + \sigma_{10} | 1 \rangle \langle 0 | + \sigma_{12} | 1 \rangle \langle 1 | + \sigma_{21} | 0 \rangle \langle 0 | + \sigma_{23} | 0 \rangle \langle 1 | + \sigma_{31} | 1 \rangle \langle 0 | + \sigma_{33} | 1 \rangle \langle 1 | \\
&= (\sigma_{00} + \sigma_{21}) | 0 \rangle \langle 0 | + (\sigma_{02} + \sigma_{23}) | 0 \rangle \langle 1 | + (\sigma_{10} + \sigma_{31}) | 1 \rangle \langle 0 | + (\sigma_{12} + \sigma_{33}) | 1 \rangle \langle 1 |
\end{aligned}$$

This is the sum of the upper left block and lower right block matrices of  $\sigma_{AB}$ .

5. Take the state  $\rho_{AB}$  from 3. and compute the reduced state  $\rho_B$ , both from the matrix itself (using your result from 4.) and in bra-ket notation. (\*) (6 points)

$$\text{Answer: } \rho_{AB} = \begin{bmatrix} \frac{1}{3} & \frac{i}{3} & 0 & -\frac{1}{3} \\ -\frac{i}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{i}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

$$\begin{aligned}
\rho_B &= (\rho_{00} + \rho_{21}) | 0 \rangle \langle 0 | + (\rho_{02} + \rho_{23}) | 0 \rangle \langle 1 | + (\rho_{10} + \rho_{31}) | 1 \rangle \langle 0 | + (\rho_{12} + \rho_{33}) | 1 \rangle \langle 1 | \\
&= (\frac{1}{3} + 0) | 0 \rangle \langle 0 | + (0 + 0) | 0 \rangle \langle 1 | + (-\frac{i}{3} + \frac{i}{3}) | 1 \rangle \langle 0 | + (\frac{1}{3} + \frac{1}{3}) | 1 \rangle \langle 1 | \\
&= \frac{1}{3} | 0 \rangle \langle 0 | + 0 | 0 \rangle \langle 1 | + 0 | 1 \rangle \langle 0 | + \frac{2}{3} | 1 \rangle \langle 1 | \\
&= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}
\end{aligned}$$

## 2 EXERCISE 2: MEASUREMENTS AND REDUCED STATES

Let us look at the density operator  $\rho_{AB} = |\Phi^-\rangle\langle\Phi^-|$ , with  $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ .

1. Compute explicitly the post-measurement state after we performed a projective measurement  $M = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$  on  $\rho_{AB}$  on system  $B$ , given that the outcome was 0. Compute the same for the measurement  $M = \{|+\rangle\langle +|, |-\rangle\langle -|\}$ , for outcome  $-$ . (Hint: Using Exercise 2.6 from Assignment 1 might help (not necessary though).) (8 points)

$$\text{Answer: } M_0 = |0\rangle\langle 0|$$

Observe  $|0\rangle$  with probability  $p_0 = \text{Tr}(M_0\rho) = \text{Tr}(|0\rangle\langle 0| \otimes I \rho) = \text{Tr}(|0\rangle\langle 0| \otimes I |\Phi^-\rangle\langle\Phi^-|) = \langle 0 | 0 \rangle \langle 0 | \otimes I |\Phi^-\rangle\langle\Phi^-| | 0 \rangle \otimes I + \langle 1 | 0 \rangle \langle 0 | \otimes I |\Phi^-\rangle\langle\Phi^-| | 1 \rangle \otimes I = \langle 0 | \otimes I |\Phi^-\rangle\langle\Phi^-| | 0 \rangle \otimes I$

$$\begin{aligned}
\rho'_0 &= \frac{M_0 \rho M_0^\dagger}{\text{Tr}(M_0 \rho)} \\
&= \frac{|0\rangle\langle 0| \otimes I |\Phi^-\rangle\langle\Phi^-| | 0 \rangle \otimes I}{\langle 0 | \otimes I |\Phi^-\rangle\langle\Phi^-| | 0 \rangle \otimes I} \\
&= |00\rangle\langle 00| \\
&= |0\rangle\langle 0| \otimes |0\rangle\langle 0|
\end{aligned}$$

$$\text{For system B, } \rho'_{0B} = |0\rangle\langle 0|$$

$$M_1 = |-\rangle\langle -|$$

Observe  $|-\rangle$  with probability  $p_1 = \text{Tr}(M_1\rho) = \text{Tr}(|-\rangle\langle -| \otimes I \rho) = \text{Tr}(|-\rangle\langle -| \otimes I |\Phi^-\rangle\langle\Phi^-|) = \langle + | - \rangle \langle - | \otimes I |\Phi^-\rangle\langle\Phi^-| | + \rangle \otimes I + \langle - | - \rangle \langle - | \otimes I |\Phi^-\rangle\langle\Phi^-| | - \rangle \otimes I = \langle + | \otimes I |\Phi^-\rangle\langle\Phi^-| | + \rangle \otimes I$

$$\begin{aligned}
\rho'_1 &= \frac{M_1 \rho M_1^\dagger}{\text{Tr}(M_1 \rho)} \\
&= \frac{|-\rangle\langle -| \otimes I |\Phi^-\rangle\langle\Phi^-| | - \rangle \otimes I}{\langle + | \otimes I |\Phi^-\rangle\langle\Phi^-| | + \rangle \otimes I} \\
&= |--\rangle\langle --| \\
&= |-\rangle\langle -| \otimes |-\rangle\langle -|
\end{aligned}$$

$$\text{For system B, } \rho'_{1B} = |-\rangle\langle -|$$

2. What is the post measurement state if the measurements above were destructive and yielded the same outcomes as in part 1.? (2 points)

Answer:

For the outcome 0,  $\rho''_0 = Tr(\rho'_{0B}) = Tr(|0\rangle\langle 0|) = 1$

For the outcome -,  $\rho''_1 = Tr(\rho'_{1B}) = Tr(|-\rangle\langle -|) = 1$

3. What is the reduced state  $\rho_A$  if we take state  $\rho_{AB}$  and trace over system B in basis  $\{|0\rangle, |1\rangle\}$ ? What if we trace in  $\{|+\rangle, |-\rangle\}$ ? Hint: for the second computation it could be useful to use the basis invariance of the trace operation. (6 points)

Answer:  $\rho_{AB} = |\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|) = \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)$

$\rho_A = Tr_B(\rho_{AB}) = \frac{1}{2}(\langle 0| \otimes \langle 0| \rho_{AB} |0\rangle \otimes |0\rangle + \langle 1| \otimes \langle 1| \rho_{AB} |1\rangle \otimes |1\rangle)$

$= \frac{1}{2}(\langle 0| \otimes \langle 0| (|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) |0\rangle \otimes |0\rangle + \langle 1| \otimes \langle 1| (|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) |1\rangle \otimes |1\rangle)$

$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

In the basis  $\{|+\rangle, |-\rangle\}$ ,  $\Phi = \frac{1}{2}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$

$\rho_A = Tr_B$

4. Consider now the  $2n$ -qubit state  $\rho = |\Psi_n\rangle\langle\Psi_n|$ , with  $|\Psi_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle_A \otimes |i\rangle_B$ . What is the reduced  $n$ -qubit state on Alice's side? (Hint: look at the state in vector form and factorize it in a smart way.) (8 points)