Assignment 1: Basics

GENERAL INFORMATION ABOUT THE ASSIGNMENT

The assignment needs to be handed in online. You can either write the solutions out by hand and submit a PDF of the scans, or just write solutions in a program of your choice (word/latex/lyx for example). Exercises marked with a star (*) can be solved either analytically or numerically. If you solve them numerically, please attach the codes and solutions to your PDF. Exercises without a star need to be solved analytically. At the end of the sheet you find two extra exercises, marked with a †. These exercises are **not compulsory** and targeted for extra practise (they do **not contribute points**). The assignments need to be handed in individually (no group work or copying allowed). The due date of Assignment 1 is 24th September 2023.

EXERCISE 1: COMPLEX NUMBERS (19 POINTS)

We saw in the lectures that a complex number x can be written as $x = x_1 + ix_2$. The imaginary unit i satisfies $i^2 = -1$.

1. Let x, y and z be complex numbers. Show, that the modulus squared $|x+y+z|^2$ can be written as
$$|x+y+z|^2 = |x|^2 + |y|^2 + |z|^2 + 2\left[\mathrm{Re}(x^*y) + \mathrm{Re}(x^*z) + \mathrm{Re}(y^*z)\right] \ , \tag{1}$$
where Re is the real part of the number in brackets. *(3 points)*
2. Simplify the expression $(i+2)(3-4i)/(2-i)$. Derive your answer step-by-step. *(3 points)*
3. Compute the real and the imaginary part of $(i-4)/(2i-3)$ *(3 points)*
4. Simplify the expression i^{33}. What is the absolute value of the expression? *(2 points)*
5. We saw in the lecture that for two probabilities p_1, p_2, we always have $(p_1+p_2) \geq p_1$ and $(p_1+p_2) \geq p_2$. *i)* Find two complex numbers c_1, c_2 such that $|c_1+c_2|^2 \geq |c_1|^2$ and $|c_1+c_2|^2 \ngeq |c_2|^2$. *ii)* Is it possible to find two complex numbers which satisfy $|c_1+c_2|^2 \ngeq |c_1|^2$ and $|c_1+c_2|^2 \ngeq |c_2|^2$ (if yes, write an example down)? *(4 points)*
6. For two real-valued vectors $\vec{r}_1, \vec{r}_2 \in \mathbb{R}^d$, the inner product is defined as $\langle \vec{r}_1, \vec{r}_2 \rangle = \vec{r}_1^T \vec{r}_2$, where $\vec{r}_1^T$ is the transpose vector. How do we define the inner product for two complex vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{C}^d$? Explain your definition and check that it satisfies all the properties of an inner product. *(4 points)*

EXERCISE 2: THE TENSOR PRODUCT (*) (18 POINTS)

We saw in the lectures that the tensor product allows us to compose states of two systems, which we denote by A and B in the following. Write down the following composite systems in the basis $\mathcal{B}_1 = \{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$.

1. System A is in the state $|0\rangle_A$ and B in $|1\rangle_B$. *(1 point)*
2. System A is in $|-\rangle_A$ and B in $|+\rangle_B$. (remember: $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ and $|-\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$). *(2 points)*

Next, write the following states in the basis and $\mathcal{B}_2 = \{|+\rangle \otimes |+\rangle, |+\rangle \otimes |-\rangle, |-\rangle \otimes |+\rangle, |-\rangle \otimes |-\rangle\}$

3. System A is in $|0\rangle_A$ and B in $|-\rangle_B$. *(3 points)*
4. System A is in the state $|1\rangle_A$ and B in $|1\rangle_B$. *(3 points)*

Finally, show that

5. Show that the state $|\Phi^+\rangle = 1/\sqrt{2}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)$ cannot be written as a tensor product of two individual states on A and B. *(4 points)*
6. Show that $|\Phi^-\rangle = 1/\sqrt{2}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$ in basis $\mathcal{B}_1$ is equal to $-|\Phi^-\rangle$ in basis $\mathcal{B}_2$. *(5 points)*

EXERCISE 3: OVERLAPS OF STATES (23 POINTS)

In the lectures we saw that overlaps of states are important quantities: for instance, the overlap $\langle\psi|\psi\rangle$ of a state $|\psi\rangle$ with itself tells us if the state has the correct normalization.

1. Show that $\langle\psi|\psi\rangle = \|\psi\|_2^2$, meaning that it is the square of the norm of $|\psi\rangle$. *(3 points)*
2. Compute the norms of the following states: $\frac{1}{3}|-\rangle$, $\frac{1}{\sqrt{2}}(i|0\rangle - |1\rangle)$, $(\frac{2}{5}|0\rangle + \frac{3}{5}|1\rangle)$. *(6 points)*
3. Which of the state have the correct normalization? Re-normalize the ones with wrong normalization and write them down. *(4 points)*

We also found that the overlap between two states $|\phi\rangle$ and $|\psi\rangle$ relates to the probability p of finding state $|\phi\rangle$ to be in state $|\psi\rangle$.

4. Given the state $|\psi\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, what is the probability to find $|\psi\rangle$ in state $|\phi\rangle = |1\rangle$? How about $|\phi\rangle = |-\rangle$? *(3 points)*
5. Given the state $|\psi\rangle = \frac{1}{\sqrt{2}}(i|0\rangle - |1\rangle)$, what is the probability to find $|\psi\rangle$ in state $|\phi\rangle = |+\rangle$? *(3 points)*
6. We design an experiment in order to produce the state $|\phi\rangle = |+\rangle$. We reject the output of the experiment if the output state $|\psi\rangle$ is with probability lower than 45% in the desired state $|\phi\rangle$. Given the experiment output $|\psi\rangle = \frac{2}{\sqrt{5}}|0\rangle + i\frac{1}{\sqrt{5}}|1\rangle$, do we accept or reject this state? *(4 points)*

EXERCISE 4: DENSITY OPERATORS (18 POINTS)

Write down **both** the density operators **and** the corresponding density matrix for the following composite systems, using the basis $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$.

1. The 2-qubit state $|+\rangle_A \otimes |0\rangle_B$. *(2 points)*
2. Two qubits which are three times likelier in the state $|0\rangle_A \otimes |1\rangle_B$ than in $|1\rangle_A \otimes |0\rangle_B$. *(4 points)* (*)

For the following two exercises, use the n-qubit extension of basis $\mathcal{B}$.

3. The n-qubit state $1/\sqrt{2}(|00\ldots0\rangle + |11\ldots1\rangle)$. *(4 points)*
4. The maximally entangled state between two systems A and B, where each system has n qubits. The state consists of $2n$ qubits in total (Alice and Bob each have n qubits) and is given by $1/\sqrt{2^n}\sum_i(|i\rangle_A \otimes |i\rangle_B)$ (each state $|i\rangle$ is one of the 2^n possible n-qubit states). *(8 points)*

EXERCISE 5: THE BLOCH SPHERE (22 POINTS)

A qubit can be represented as a vector in a 3-dimensional sphere of radius 1, called Bloch sphere. Each qubit has a position in the sphere, indicated by its **Bloch vector** $\vec{r} = (r_1, r_2, r_3)^T$. Then, any qubit can be written as

$$\begin{aligned} \rho &= \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) \\ r_i &= \mathrm{Tr}(\rho\sigma_i) \ , \end{aligned} \tag{2}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{3}$$

1. Find the Bloch vector for the following 6 pure states. Then, draw them into the Bloch sphere

$$|0\rangle, \quad |1\rangle, \quad |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle), \quad |\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle) \ . \tag{4}$$

Where is the maximally mixed state $\frac{I}{2} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ located? *(6 points)*

2. Find the states corresponding to $\vec{r}_1 = (0, -1/3, 0)^T$ and $\vec{r}_2 = (-1/\sqrt{2}, 1/\sqrt{2}, 0)^T$. Are the states pure or mixed? Prove your claim. *(6 points)* (*)

3. Where are the pure states located in the Bloch sphere? Prove your statement. *(5 points)*

4. Show that the operator $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$ as defined above is indeed a valid density operator for any vector $\vec{r}$ satisfying $|\vec{r}| \leq 1$. *(Hint: check, whether all requirements of a valid density matrix are satisfied. Looking at the eigenvalues of ρ may help.) (5 points)*

EXTRA EXERCISES†

1. Write the operator

$$\left(|x_1\rangle_A \otimes |x_2\rangle_B + |y_1\rangle_A \otimes |y_2\rangle_B\right)\left(\langle x_1|_A \otimes \langle x_2|_B + \langle y_1|_A \otimes \langle y_2|_B\right) \tag{5}$$

as a linear combination of operators of the form $|z\rangle\langle z| \otimes |z'\rangle\langle z'|$. (*Hint:* $\sum_{k=0}^{3} i^k = 1 + i - 1 - i = 0$.)

2. Given a, b, and $c \in \mathbb{R}_+$ such that $a^2 + b^2 + c^2 = 1$, let $|\psi\rangle = a\,|0\rangle + b\,|1\rangle + c\,|2\rangle$ and $|\phi\rangle = \sin\theta\,|0\rangle + \cos\theta\,|2\rangle$ where θ is a parameter with values in $[0, 2\pi)$. Find the value $\theta_{\max}$ for which the overlap between $|\psi\rangle$ and $|\phi\rangle$ is maximised. Also, write the overlap achieved for $\theta = \theta_{\max}$ in terms of a, b, and c only.