

Assignment 1

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September 24, 2023

EXERCISE 1: COMPLEX NUMBERS

1. Assume $x = a_1 + b_1i$, $x = a_2 + b_2i$, $x = a_3 + b_3i$,

we have $x + y + z = a_1 + a_2 + a_3 + (b_1 + b_2 + b_3)i$,

To a complex number $n = a + bi$ we have $n^* = a - bi$,

so

$$|x|^2 = x \cdot x^* = a_1^2 - b_1^2(i)^2 = a_1^2 - b_1^2 \cdot (-1) = a_1^2 + b_1^2,$$

x^*y

$$= (a_1 - b_1i) \cdot (a_2 + b_2i)$$

$$= a_1a_2 + a_1b_2i - a_2b_1i - b_1b_2i^2$$

$$= a_1a_2 + b_1b_2 + (a_1b_2 - a_2b_1)i,$$

$$Re(x^*y) = a_1a_2 + b_1b_2,$$

similarly, we have

$$|y|^2 = a_2^2 + b_2^2,$$

$$|z|^2 = a_3^2 + b_3^2,$$

$$Re(y^*z) = a_2a_3 + b_2b_3$$

$$Re(x^*z) = a_1a_3 + b_1b_3$$

and

$$(x + y + z)^* = a_1 + a_2 + a_3 - (b_1 + b_2 + b_3)i,$$

$$|x + y + z|^2$$

$$= (x + y + z) \cdot (x + y + z)^*$$

$$= ((a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i) \cdot ((a_1 + a_2 + a_3) - (b_1 + b_2 + b_3)i)$$

$$= ((a_1 + a_2 + a_3)^2 - (b_1 + b_2 + b_3)^2 \cdot (-1))$$

$$= ((a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2)$$

$$= a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 + 2a_1a_3 + 2a_2a_3 + b_1^2 + b_2^2 + b_3^2 + 2b_1b_2 + 2b_1b_3 + 2b_2b_3$$

$$= a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2 + 2(a_1a_2 + b_1b_2 + a_2a_3 + b_2b_3 + a_1a_3 + b_1b_3)$$

$$= |x|^2 + |y|^2 + |z|^2 + 2[Re(x^*y) + Re(y^*z) + Re(x^*z)]$$

This shows that

$$|x + y + z|^2 = |x|^2 + |y|^2 + |z|^2 + 2[Re(x^*y) + Re(y^*z) + Re(x^*z)]$$

$$\begin{aligned}
2. \quad & (i+2)(3-4i)/(2-i) \\
&= (3i-4i^2+2*3-2*4i)/(2-i) \\
&= (3i-4 \times (-1)+2*3-2*4i)/(2-i) \\
&= (3i+4+6-8i)/(2-i) \\
&= (10-5i)/(2-i) \\
&= 5(2-i)/(2-i) \\
&= 5 \\
3. \quad & (i-4)/(2i-3) \\
&= [(i-4)(2i+3)]/[(2i-3)(2i+3)] \\
&= (2i^2+3i-8i-4*3)/((2i)^2-3*3) \\
&= [2 \times (-1)+3i-8i-4*3]/[4 \times (-1)-3*3] \\
&= (-2-5i-12)/(-4-9) \\
&= (-14-5i)/(-13) \\
&= [(-1)(14+5i)]/(-1 \times 13) \\
&= (14+5i)/13 \\
&= 14/13 + (5/13)i
\end{aligned}$$

so, the real part is $14/13$ and imaginary part is $5/13$.

$$\begin{aligned}
4. \quad & i^{33} \\
&= i^{32}i \\
&= i^{2 \times 16}i \\
&= (i^2)^{16}i \\
&= (-1)^{16}i \\
&= i
\end{aligned}$$

so, the absolute value of i^{33} is $|i|$

$$\begin{aligned}
|i| &= |0+i| = \sqrt{|0+i|^2} = \sqrt{(0+i)(0+i)^*} = \sqrt{(0+i)(0-i)} = \sqrt{-i^2} = \\
&= \sqrt{-(-1)} = \sqrt{1} = 1
\end{aligned}$$

5. i. For complex number $c_1 = a_1 + b_1i$ and $c_2 = a_2 + b_2i$, we have

$$|c_1|^2 = a_1^2 + b_1^2$$

$$|c_2|^2 = a_2^2 + b_2^2$$

$$|c_1 + c_2|^2 = (a_1 + a_2)^2 + (b_1 + b_2)^2 = a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2$$

so we need to find a_1, a_2, b_1, b_2 that makes

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 \geq a_1^2 + b_1^2$$

and

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 < a_2^2 + b_2^2$$

so we have

$$a_2^2 + 2a_1a_2 + b_2^2 + 2b_1b_2 \geq 0$$

and

$$a_1^2 + 2a_1a_2 + b_1^2 + 2b_1b_2 < 0$$

so, we need $2a_1a_2 + 2b_1b_2 \geq -(a_2^2 + b_2^2)$ and $2a_1a_2 + 2b_1b_2 < -(a_1^2 + b_1^2)$,
which means $-(a_2^2 + b_2^2) \leq 2a_1a_2 + 2b_1b_2 < -(a_1^2 + b_1^2)$,

Through observing, it is easy to find $a_1 = -1, a_2 = 3, b_1 = 1, b_2 = -3$
makes $|c_1 + c_2|^2 \geq |c_1|^2$ and $|c_1 + c_2|^2 < |c_2|^2$.

ii. Yes. For $c_1 = -1 + 2i, c_2 = 2 - i, c_1 + c_2 = 1 + i$

$$|c_1 + c_2|^2 = 2$$

$$|c_1|^2 = 5$$

$$|c_2|^2 = 5$$

in this case, two complex numbers satisfy $|c_1 + c_2|^2 \leq |c_1|^2$ and $|c_1 + c_2|^2 \leq |c_2|^2$.

6. Assume both \vec{v}_1 and \vec{v}_2 has a length of n .

$$\vec{v}_1 = (\psi_{10}, \psi_{11}, \psi_{12}, \psi_{13}, \dots, \psi_{1n})^T,$$

$$\vec{v}_2 = (\psi_{20}, \psi_{21}, \psi_{22}, \psi_{23}, \dots, \psi_{2n})^T.$$

For real vectors \vec{r}_1 and \vec{r}_2 , we have $\langle \vec{r}_1, \vec{r}_2 \rangle = \vec{r}_1^T \vec{r}_2$.

Similarly, we can define the inner product of \vec{v}_1, \vec{v}_2 that

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1^T \vec{v}_2$$

where \vec{v}_1^T is the transpose of \vec{v}_1 .

This means

$$\langle \vec{v}_1, \vec{v}_2 \rangle = (\psi_{10}^*, \psi_{11}^*, \psi_{12}^*, \psi_{13}^*, \dots, \psi_{1n}^*)(\psi_{20}, \psi_{21}, \psi_{22}, \psi_{23}, \dots, \psi_{2n})^T$$

so,

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \sum_{i=0}^{n-1} \psi_{1i}^* \psi_{2i}$$

The properties of an inner product \langle, \rangle are as followed[1].

- (a) **Linearity:** $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$
- (b) **Symmetric Property:** $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (For complex vectors, the inner product doesn't satisfy this property.)
- (c) **Positive Definite Property:** For any $\mathbf{u} \in \mathbf{V}$, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$; and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$;

For complex vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, all of them have a length of n .

$$\begin{aligned} & \langle a\vec{v}_1 + b\vec{v}_2, \vec{v}_3 \rangle \\ &= \sum_{i=0}^{n-1} (a\psi_{1i}^* + b\psi_{2i}^*)(\psi_{3i}) \\ &= \sum_{i=0}^{n-1} (a\psi_{1i}^*)(\psi_{3i}) + \sum_{i=0}^{n-1} (b\psi_{2i}^*)(\psi_{3i}) \\ &= a \sum_{i=0}^{n-1} (\psi_{1i}^*)(\psi_{3i}) + b \sum_{i=0}^{n-1} (\psi_{2i}^*)(\psi_{3i}) \\ &= a\langle \vec{v}_1, \vec{v}_3 \rangle + b\langle \vec{v}_2, \vec{v}_3 \rangle \end{aligned}$$

This proves the linearity.

Also, we have

$$\begin{aligned}\langle \vec{v}_1, \vec{v}_2 \rangle &= \sum_{i=0}^{n-1} \psi_{1i}^* \psi_{2i} \\ \langle \vec{v}_2, \vec{v}_1 \rangle &= \sum_{i=0}^{n-1} \psi_{2i}^* \psi_{1i}\end{aligned}$$

Because $\sum_{i=0}^{n-1} \psi_{1i}^* \psi_{2i}$ doesn't equal to $\sum_{i=0}^{n-1} \psi_{2i}^* \psi_{1i}$, the inner product of complex numbers doesn't have the symmetric property.

For any complex vector \vec{v}_1 , $\langle \vec{v}_1, \vec{v}_1 \rangle = \sum_{i=0}^{n-1} \psi_{1i}^* \psi_{1i} = \sum_{i=0}^{n-1} |\psi_{1i}|^2$. For any complex number $\psi = a + bi$, we have $|\psi|^2 = a^2 + b^2 \geq 0$,

so $\langle \vec{v}_1, \vec{v}_1 \rangle = \sum_{i=0}^{n-1} |\psi_{1i}|^2 \geq 0$ and \vec{v}_1 is a complex vector, so $|\vec{v}_1| \neq 0$.

This proves the positive definite property.

So, it satisfies the properties of Linearity and Positive Definite of an inner product, but doesn't satisfy a property of Symmetric.

EXERCISE 2: THE TENSOR PRODUCT

$$\begin{aligned}1. & |0\rangle_A \otimes |1\rangle_B \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 0 \\ 1 \times 1 \\ 0 \times 0 \\ 0 \times 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= 0(|0\rangle \otimes |0\rangle) + 1(|0\rangle \otimes |1\rangle) + 0(|1\rangle \otimes |0\rangle) + 0(|1\rangle \otimes |1\rangle)\end{aligned}$$

$$\begin{aligned}2. & |+\rangle_A \otimes |-\rangle_B \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \\
&= \frac{1}{2}(|0\rangle \otimes |0\rangle) - \frac{1}{2}(|0\rangle \otimes |1\rangle) + \frac{1}{2}(|1\rangle \otimes |0\rangle) - \frac{1}{2}(|1\rangle \otimes |1\rangle)
\end{aligned}$$

$$\begin{aligned}
3. & |0\rangle_A \otimes |-\rangle_B \\
&= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \otimes |-\rangle \\
&= \frac{1}{\sqrt{2}}|+\rangle \otimes |-\rangle + \frac{1}{\sqrt{2}}|-\rangle \otimes |-\rangle \\
&= 0 \times |+\rangle \otimes |+\rangle + \frac{1}{\sqrt{2}}|+\rangle \otimes |-\rangle + 0 \times |-\rangle \otimes |+\rangle + \frac{1}{\sqrt{2}}|-\rangle \otimes |-\rangle
\end{aligned}$$

$$\begin{aligned}
4. & |1\rangle_A \otimes |1\rangle_B \\
&= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \\
&= \frac{1}{2}(|+\rangle \otimes |+\rangle) - \frac{1}{2}(|+\rangle \otimes |-\rangle) - \frac{1}{2}(|-\rangle \otimes |+\rangle) + \frac{1}{2}(|-\rangle \otimes |-\rangle)
\end{aligned}$$

$$\begin{aligned}
5. & \text{We have } |\Phi^+\rangle \\
&= \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B) \\
&= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\
&= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right) \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\text{For } A = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}, \text{ we have } A \otimes B = \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{pmatrix}.$$

If $|\Phi^+\rangle$ can be written as $A \otimes B$, then

$$\begin{aligned}
a_0 b_0 &= 0 \\
a_0 b_1 &= \frac{1}{\sqrt{2}} \\
a_1 b_0 &= \frac{1}{\sqrt{2}} \\
a_1 b_1 &= 0.
\end{aligned}$$

To make $a_0 b_0 = 0$, either $a_0 = 0$ or $b_0 = 0$ should be true.

If any of them is true, then $a_0 b_1 = \frac{1}{\sqrt{2}}$ and $a_1 b_0 = \frac{1}{\sqrt{2}}$ cannot be true in the same time.

So $|\Phi^+\rangle$ can not be written as $A \otimes B$.

6. We have $|0\rangle|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $|1\rangle|1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

We also have $|+\rangle|-\rangle = \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle)$

and $|-\rangle|+\rangle = \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle)$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$$

So, $-|\Phi^-\rangle$

$$= \frac{1}{\sqrt{2}}(|1\rangle_A \otimes |0\rangle_B - |0\rangle_A \otimes |1\rangle_B)$$

$$= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle) - \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle) \right]$$

$$= \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle)$$

So $|\Phi^-\rangle$ in basis \mathcal{B}_1 is equal to $-|\Phi^-\rangle$ in basis \mathcal{B}_2 .

OVERLAPS OF STATES

1. For $|\psi\rangle = \begin{pmatrix} c_0 \\ c_1 \\ \dots \\ c_{n-1} \end{pmatrix}$ where $c_k = a_k + b_k i$, we know

$$\|\psi\|_2^2 = \sum_{i=0}^{n-1} |c_i|^2 = \sum_{i=0}^{n-1} a_i^2 + b_i^2.$$

now,

$$\langle\psi|\psi\rangle$$

$$= (c_0^*, c_1^*, \dots, c_{n-1}^*) \begin{pmatrix} c_0 \\ c_1 \\ \dots \\ c_{n-1} \end{pmatrix}$$

$$= \sum_{i=0}^{n-1} c_i^* c_i \text{imaginary}$$

$$= \sum_{i=0}^{n-1} a_i^2 + b_i^2$$

$$= \|\psi\|_2^2$$

$$\text{So, } \langle\psi|\psi\rangle = \|\psi\|_2^2$$

2. (a) For $|\psi_1\rangle = \frac{1}{3}|- \rangle$,

$$\begin{aligned} & \|\psi_1\|^2 \\ &= \langle \psi_1 | \psi_1 \rangle \\ &= \frac{1}{9} \langle - | - \rangle \\ &= \frac{1}{9} [\frac{1}{2} \langle 0|0 \rangle + (-1)^2 \frac{1}{2} \langle 1|1 \rangle] \\ &= \frac{1}{9} (\frac{1}{2} + \frac{1}{2}) \\ &= \frac{1}{9} \end{aligned}$$

so,

$$\|\psi_1\| = \sqrt{\|\psi_1\|^2} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

- (b) For $|\psi_2\rangle = \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)$

$$\begin{aligned} & \|\psi_2\|^2 \\ &= \frac{1}{2} \times -(i^2 \times \langle 0|0 \rangle) + \frac{1}{2} \langle 1|1 \rangle \\ &= \frac{1}{2} * 1 + \frac{1}{2} * 1 \\ &= 1 \end{aligned}$$

$$\text{So, } \|\psi_2\| = \sqrt{1} = 1$$

- (c) $\|\frac{2}{5}|0\rangle + \frac{3}{5}|1\rangle\|$

$$\begin{aligned} &= \sqrt{\frac{2}{5} \times \frac{2}{5} \langle 0|1 \rangle + \frac{2}{5} \times \frac{3}{5} \langle 0|0 \rangle + \frac{2}{5} \times \frac{3}{5} \langle 1|1 \rangle + \frac{3}{5} \times \frac{3}{5} \langle 1|1 \rangle} \\ &= \sqrt{\frac{4}{25} \times 1 + \frac{9}{25} \times 1} \\ &= \sqrt{\frac{13}{25}} \\ &= \frac{\sqrt{13}}{5} \end{aligned}$$

3. $|\psi_2\rangle = \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)$ is the correct normalization.

$$\text{The renormalized state } |\psi_1\rangle' = \frac{|\psi_1\rangle}{\frac{1}{3}} = |- \rangle$$

$$\text{The renormalized state } |\psi_3\rangle' = \frac{|\psi_3\rangle}{\frac{\sqrt{13}}{5}} = \frac{2}{\sqrt{13}}|0\rangle + \frac{3}{\sqrt{13}}|1\rangle$$

4. State $\psi = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.

The probability p_1 to find $|\psi\rangle$ in state $|1\rangle$ is:

$$p_1 = |\langle 1|\psi\rangle|^2 = |(\frac{1}{\sqrt{2}})\langle 1|1\rangle|^2 = \frac{1}{2}$$

The probability p_2 to find $|\psi\rangle$ in state $|- \rangle$ is:

$$p_2 = |\langle -|\psi\rangle|^2 = |\langle -|+\rangle|^2 = |\frac{1}{2}(\langle 0|0\rangle - \langle 1|1\rangle)|^2 = 0$$

5. The probability p to find $|\psi\rangle = \frac{1}{\sqrt{2}}(i|0\rangle - |1\rangle)$ in state $|+\rangle$ is:

$$p = |\langle +|\psi\rangle|^2 = |\frac{1}{2}(i\langle 0|0\rangle - \langle 1|1\rangle)|^2 = \frac{1}{4}|-1+i|^2 = \frac{1}{4}(1+1) = \frac{1}{2}$$

6. The probability p of output $|\psi\rangle$ in the state $|+\rangle$ is:

$$\begin{aligned}
 p &= |\langle\phi|\psi\rangle|^2 = \left| \left[\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|) \right] \left(\frac{2}{\sqrt{5}}|0\rangle + i\frac{1}{\sqrt{5}}|1\rangle \right) \right|^2 \\
 &= \left| \frac{2}{\sqrt{10}}\langle 0|0\rangle + 0 + 0 + i\frac{1}{\sqrt{10}}\langle 1|1\rangle \right|^2 \\
 &= \left| \frac{2}{\sqrt{10}} + i\frac{1}{\sqrt{10}} \right|^2 \\
 &= \frac{4}{10} + \frac{1}{10} \\
 &= \frac{1}{2} \\
 \frac{1}{2} &= 50\% > 45\%
 \end{aligned}$$

So, we accept this state.

DENSITY OPERATORS

1. For $|\psi\rangle = |+\rangle_A \otimes |0\rangle_B$ The density operator is $|\psi\rangle\langle\psi|$

$$\begin{aligned}
 |+\rangle_A \otimes |0\rangle_B &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle) \\
 &= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right) \\
 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}
 \end{aligned}$$

The density matrix

$$\begin{aligned}
 \rho &= |+\rangle_A \otimes |0\rangle_B \langle +|_A \otimes \langle 0|_B \\
 &= \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

2. $|\psi\rangle = \frac{\sqrt{3}}{2}(|0\rangle_A \otimes |1\rangle_B) + \frac{1}{2}(|1\rangle \otimes |0\rangle_B)$

The density operator is $|\psi\rangle\langle\psi|$

$$|\psi\rangle = \frac{\sqrt{3}}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

The density matrix

$$\begin{aligned} \rho = |\psi\rangle \langle\psi| &= \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{\sqrt{3}}{4} & 0 \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$3. \text{ For } |\psi\rangle = \frac{1}{\sqrt{2}}(|000\dots 0\rangle + |111\dots 1\rangle) = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}$$

The density operator is $|\psi\rangle \langle\psi|$

$$\text{The density matrix } \rho = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$4. |i\rangle = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \text{ the } i+1\text{th number in the vector is 1, others are 0.}$$

$$\text{So, } |i\rangle \otimes |i\rangle = \begin{pmatrix} 0 \\ 0 \\ \dots \\ |i\rangle \\ 0 \\ \dots \\ 0 \end{pmatrix}, \text{ above } |i\rangle \text{ there are } i \times n \text{ zeros.}$$

$$\text{let } |\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_i (|i\rangle \otimes |i\rangle) = \frac{1}{\sqrt{2^n}} \begin{pmatrix} |0\rangle \\ |1\rangle \\ \dots \\ |k\rangle \\ \dots \\ |2^n - 1\rangle \end{pmatrix}$$

the density operator is $|\psi\rangle \langle\psi|$

the density matrix

$$\begin{aligned} \rho &= \\ &= \frac{1}{2^n} \begin{pmatrix} |0\rangle \\ |1\rangle \\ \dots \\ |k\rangle \\ \dots \\ |2^n - 1\rangle \end{pmatrix} (\langle 0|, \langle 1|, \dots, \langle k|, \dots, \dots, \langle 2^n - 1|) \\ &= \frac{1}{2^n} \begin{pmatrix} |0\rangle \langle 0| & |1\rangle \langle 0| & \dots & |k\rangle \langle 0| & \dots & |2^n - 1\rangle \langle 0| \\ |0\rangle \langle 1| & |1\rangle \langle 1| & \dots & |k\rangle \langle 1| & \dots & |2^n - 1\rangle \langle 1| \\ \dots & \dots & \dots & \dots & \dots & \dots \\ |0\rangle \langle k| & |1\rangle \langle k| & \dots & |k\rangle \langle k| & \dots & |2^n - 1\rangle \langle k| \\ \dots & \dots & \dots & \dots & \dots & \dots \\ |0\rangle \langle 2^n - 1| & |1\rangle \langle 2^n - 1| & \dots & |k\rangle \langle 2^n - 1| & \dots & |2^n - 1\rangle \langle 2^n - 1| \end{pmatrix} \\ \text{Amng them, } |i\rangle \langle j| &= \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \end{pmatrix}, \text{ the } i + 1\text{th row} \end{aligned}$$

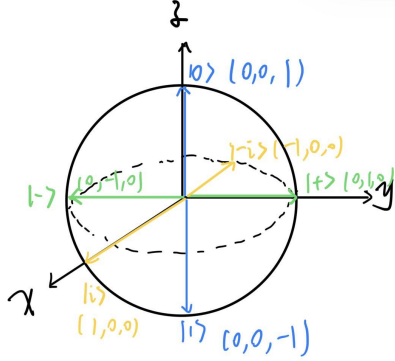
and $j + 1\text{th}$ column is 1, other elements are 0.

THE BLOCH SPHERE

1. (a) For $|0\rangle$, $\vec{r} = (0, 0, 1)^T$
- (b) For $|1\rangle$, $\vec{r} = (0, 0, -1)^T$
- (c) For $|+\rangle$, $\vec{r} = (0, 1, 0)^T$
- (d) For $|-\rangle$, $\vec{r} = (0, -1, 0)^T$
- (e) For $|i\rangle$, $\vec{r} = (1, 0, 0)^T$
- (f) For $|-i\rangle$, $\vec{r} = (-1, 0, 0)^T$

$$\frac{I}{2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(I + O)$$

So, $\vec{r}\vec{\sigma} = O$



Assume $\vec{r} = (a, b, c)^T$

$$\begin{aligned} \vec{r}\vec{\sigma} &= (a, b, c)^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & -bi \\ bi & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \\ &= \begin{pmatrix} c & a - bi \\ a + bi & -c \end{pmatrix} \end{aligned}$$

$$\text{So, } \begin{pmatrix} c & a - bi \\ a + bi & -c \end{pmatrix} = O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which means $a = b = c = 0$ and $\vec{r} = (0, 0, 0)^T$

So, $\frac{I}{2}$ is located in $(0, 0, 0)$, which is the centre of the Bloch ball.

$$\begin{aligned} 2. \quad (a) \quad \rho_1 &= \frac{1}{2}(I + \vec{r}\vec{\sigma}) \\ &= \frac{1}{2}(I + (0, -\frac{1}{3}, 0)\vec{\sigma}) \\ &= \frac{1}{2}(I + \begin{pmatrix} 0 & \frac{1}{3}i \\ -\frac{1}{3}i & 0 \end{pmatrix}) \\ &= \frac{1}{2}(\begin{pmatrix} 1 & \frac{1}{3}i \\ -\frac{1}{3}i & 1 \end{pmatrix}) \\ &= (\begin{pmatrix} \frac{1}{2} & \frac{1}{6}i \\ -\frac{1}{6}i & \frac{1}{2} \end{pmatrix}) \\ \text{Because } \rho_1^2 &= \begin{pmatrix} \frac{1}{2} & \frac{1}{6}i \\ -\frac{1}{6}i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{6}i \\ -\frac{1}{6}i & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{11}{36} & 0 \\ 0 & \frac{11}{36} \end{pmatrix}, \\ \text{Tr}(\rho_1^2) &= \frac{11}{18} \neq 1, \rho_1 \text{ is not a pure state.} \end{aligned}$$

$$\begin{aligned} (b) \quad \rho_2 &= \frac{1}{2}(I + \vec{r}\vec{\sigma}) \\ &= \frac{1}{2}(I + (-\frac{1}{2}, \frac{1}{2}, 0)\vec{\sigma}) \\ &= \frac{1}{2}(I + \begin{pmatrix} 0 & \frac{-1-i}{\sqrt{2}}i \\ -\frac{1+i}{\sqrt{2}}i & 0 \end{pmatrix}) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left(\begin{pmatrix} 1 & \frac{-1-i}{\sqrt{2}}i \\ \frac{-1+i}{\sqrt{2}}i & 1 \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} \frac{1}{2} & \frac{-1-i}{2\sqrt{2}}i \\ \frac{-1+i}{2\sqrt{2}}i & \frac{1}{2} \end{pmatrix} \right) \\
\rho_2^2 &= \begin{pmatrix} \frac{1}{2} & \frac{-1-i}{2\sqrt{2}}i \\ \frac{-1+i}{2\sqrt{2}}i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{-1-i}{2\sqrt{2}}i \\ \frac{-1+i}{2\sqrt{2}}i & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{-1-i}{2\sqrt{2}}i \\ \frac{-1+i}{2\sqrt{2}}i & \frac{1}{2} \end{pmatrix} = \\
&\rho_2,
\end{aligned}$$

Because $Tr(\rho_2^2) = 1$, ρ_2 is nota pure state.

3. The surface of the Bloch sphere.

PROOF:

Assume a pure state $|\psi\rangle$'s Bloch vector $\vec{r} = (a, b, c)^T$.

$$\begin{aligned}
\rho &= \frac{1}{2}(I + \vec{r}\vec{\sigma}) \\
&= \frac{1}{2} \left(I + \begin{pmatrix} c & a-bi \\ a+bi & -c \end{pmatrix} \right) \\
&= \frac{1}{2} \begin{pmatrix} 1+c & a-bi \\ a+bi & 1-c \end{pmatrix} \\
&= \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix} \\
\rho^2 &= \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix} \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{(1+c)^2 + a^2 - (bi)^2}{4} & \frac{(1+c)(a-bi) + (a-bi)(1-c)}{4} \\ \frac{(a+bi)(1+c) + (1-c)(a+bi)}{4} & \frac{a^2 - (bi)^2 + (1-c)^2}{4} \end{pmatrix} \\
Tr(\rho^2) &= \frac{(1+c)^2 + a^2 - (bi)^2}{4} + \frac{a^2 - (bi)^2 + (1-c)^2}{4} \\
&= \frac{(1+c)^2 + a^2 - (bi)^2 + a^2 - (bi)^2 + (1-c)^2}{4} \\
&= \frac{2a^2 + 2b^2 + c^2 + 2c + 1 + c^2 - 2c + 1}{4} \\
&= \frac{2(a^2 + b^2 + c^2) + 2}{4}
\end{aligned}$$

Because $|\psi\rangle$ is a pure state, so $Tr(\rho^2)=1$, which means $\frac{2(a^2+b^2+c^2)+2}{4} = 1$.

So, $a^2 + b^2 + c^2 = 1$, this means $\vec{r} = (a, b, c)^T$ is on the surface of the Bloch sphere, pure state $|\psi\rangle$ is on the surface of the Bloch sphere.

4. The requirements of sensity matrancies are as followed,

- (a) **Positive semi-definite:** all eigenvalues $\lambda_i \geq 0$
- (b) **Hermitian:** $\rho = \rho^\dagger$
- (c) **Normalization:** $Tr(\rho) = 1$

$$\text{For vector } \vec{r}, \rho = \frac{1}{2}(I + \vec{r}\vec{\sigma}) = \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix}$$

Assume the eigenvalues of ρ are $\lambda_1, \dots, \lambda_n$

$$\left| \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix} - \lambda I \right| = 0$$

$$\left| \begin{matrix} \frac{1+c}{2} - \lambda & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} - \lambda \end{matrix} \right| = 0$$

$$\left(\frac{1+c}{2} - \lambda \right) \left(\frac{1-c}{2} - \lambda \right) - \left(\frac{a-bi}{2} \right) \left(\frac{a+bi}{2} \right) = 0$$

$$\frac{1-c^2}{4} - \left(\frac{1+c+1-c}{2} \lambda \right) + \lambda^2 - \frac{a^2+b^2}{4} = 0$$

$$\lambda^2 - \lambda + \frac{1-a^2-b^2-c^2}{4} = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-4 \times \frac{1-a^2-b^2-c^2}{4}}}{2} = \frac{1 \pm \sqrt{a^2+b^2+c^2}}{2} = \frac{1 \pm \sqrt{|\vec{r}|^2}}{2}$$

Because $|\vec{r}| \leq 1$, $\frac{1 \pm \sqrt{|\vec{r}|^2}}{2} \in [0, 1]$, which means $0 \leq \lambda \leq 1$, this proves the **Positive semi-definite**.

$$\rho^\dagger = \begin{pmatrix} \frac{1+c}{2} & \left(\frac{a+bi}{2} \right)^* \\ \left(\frac{a-bi}{2} \right)^* & \frac{1-c}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix} = \rho$$

This proves the **Hermitian**.

The trace of matrix ρ

$$\text{Tr}(\rho) = \lambda_1 + \lambda_2 = \frac{1+\sqrt{|\vec{r}|^2}}{2} + \frac{1-\sqrt{|\vec{r}|^2}}{2} = 1.$$

This proves the **Normalization**.

So ρ is indeed a valid density operator for any vector \vec{r} satisfying $|\vec{r}| \leq 1$.

References

- [1] HKUST Department of Mathematics. "Inner Product Spaces and Orthogonality" Hong Kong University of Science and Technology, n.d., <https://www.math.hkust.edu.hk/~mabfchen/Math111/Week13-14.pdf>.