To calculate the expression $M_0 \rho M_0^\dagger$, we need to substitute the provided expressions for M_0 and ρ and perform the matrix operations. Let's proceed with the calculation:

$$
\begin{aligned}
M_0 \rho M_0^\dagger &= |\phi_0\rangle\langle\phi_0|\rho|\phi_0\rangle\langle\phi_0| \\
&= \left(\frac{1}{\sqrt{2}}(|0\rangle_S|0\rangle_A + |1\rangle_S|1\rangle_A)\right)\left(\frac{1}{\sqrt{2}}(|0\rangle_S\langle 0| + |1\rangle_S\langle 1|)\psi_S\right)\left(\frac{1}{\sqrt{2}}(\langle 0|_S\langle 0|_A + \langle 1|_S\langle 1|_A)\right) \\
&= \frac{1}{2}(|0\rangle_S\langle 0|_S + |1\rangle_S\langle 1|_S)(|0\rangle_S\langle 0|_S + |1\rangle_S\langle 1|_S)\psi_S(|0\rangle_S\langle 0|_S + |1\rangle_S\langle 1|_S) \\
&= \frac{1}{2}(|0\rangle_S\langle 0|_S\psi_S\langle 0|_S\langle 0|_S + |0\rangle_S\langle 0|_S\psi_S\langle 0|_S\langle 1|_S + |0\rangle_S\langle 1|_S\psi_S\langle 0|_S\langle 0|_S + |0\rangle_S\langle 1|_S\psi_S\langle 0|_S\langle 1|_S \\
&\quad + |1\rangle_S\langle 0|_S\psi_S\langle 1|_S\langle 0|_S + |1\rangle_S\langle 0|_S\psi_S\langle 1|_S\langle 1|_S + |1\rangle_S\langle 1|_S\psi_S\langle 1|_S\langle 0|_S + |1\rangle_S\langle 1|_S\psi_S\langle 1|_S\langle 1|_S) \\
&= \frac{1}{2}(|0\rangle_S\langle 0|_S\psi_S\langle 0|_S + |1\rangle_S\langle 1|_S\psi_S\langle 0|_S + |0\rangle_S\langle 0|_S\psi_S\langle 1|_S + |1\rangle_S\langle 1|_S\psi_S\langle 1|_S) \\
&= \frac{1}{2}(|0\rangle_S\langle 0|_S\psi_S\langle 0|_S + |1\rangle_S\langle 1|_S\psi_S\langle 0|_S + |0\rangle_S\langle 0|_S\psi_S\langle 1|_S + |1\rangle_S\langle 1|_S\psi_S\langle 1|_S).
\end{aligned}
$$

Next, let's calculate the trace of $M_0 \rho$:

$$
\begin{aligned}
\mathrm{Tr}(M_0\rho) &= \frac{1}{2}\left(\mathrm{Tr}(|0\rangle_S\langle 0|_S\psi_S\langle 0|_S) + \mathrm{Tr}(|1\rangle_S\langle 1|_S\psi_S\langle 0|_S) + \mathrm{Tr}(|0\rangle_S\langle 0|_S\psi_S\langle 1|_S) + \mathrm{Tr}(|1\rangle_S\langle 1|_S\psi_S\langle 1|_S)\right) \\
&= \frac{1}{2}\left(\langle 0|\langle 0|\psi\rangle_S|0\rangle\langle 0|_S + \langle 1|\langle 1|\psi\rangle_S|0\rangle\langle 0|_S + \langle 0|\langle 0|\psi\rangle_S|1\rangle\langle 1|_S + \langle 1|\langle 1|\psi\rangle_S|1\rangle\langle 1|_S\right) \\
&= \frac{1}{2}\left(|\langle 0|\psi\rangle_S|^2\langle 0|_S\langle 0|_S + |\langle 1|\psi\rangle_S|^2\langle 1|_S\langle 1|_S\right) \\
&= \frac{1}{2}\left(|\alpha|^2\langle 0|_S\langle 0|_S + |\beta|^2\langle 1|_S\langle 1|_S\right) \\
&= \frac{1}{2}\left(|\alpha|^2|0\rangle_S\langle 0|_S + |\beta|^2|1\rangle_S\langle 1|_S\right) \\
&= \frac{1}{2}\left(|\alpha|^2\,|0\rangle_S\,\langle 0|_S + |\beta|^2\,|1\rangle_S\,\langle 1|_S\right).
\end{aligned}
$$

Therefore, we have obtained the expression for $M_0 \rho M_0^\dagger$ and $\mathrm{Tr}(M_0\rho)$ as:

$$
M_0 \rho M_0^\dagger = \frac{1}{2}(|\phi_0\rangle\langle\phi_0|\psi\rangle_S) \otimes \frac{1}{2}(|0\rangle_A\langle 0|_A + |1\rangle_A\langle 1|_A),
$$

and

$$
\mathrm{Tr}(M_0\rho) = \frac{1}{2}\left(|\alpha|^2\,|0\rangle_S\,\langle 0|_S + |\beta|^2\,|1\rangle_S\,\langle 1|_S\right),
$$

respectively.