Assignment 1

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EXERCISE 1: COMPLEX NUMBERS

1. Assume $x = a_1 + b_1 i$, $x = a_2 + b_2 i$, $x = a_3 + b_3 i$,

   we have $x + y + z = a_1 + a_2 + a_3 + (b_1 + b_2 + b_3)i$,

   To a complex number $n = a + bi$ we have $n^* = a - bi$,

   so

   $|x|^2 = x \cdot x^* = a_1^2 - b_1^2 (i)^2 = a_1^2 - b_1^2 \cdot (-1) = a_1^2 + b_1^2$,

   $x^* y$
   $= (a_1 - b_1 i) \cdot (a_2 + b_2 i)$
   $= a_1 a_2 + a_1 b_2 i - a_2 b_1 i - b_1 b_2 i^2$
   $= a_1 a_2 + b_1 b_2 + (a_1 b_2 - a_2 b_1)i$,

   $Re(x^* y) = a_1 a_2 + b_1 b_2$,

   similarly, we have

   $|y|^2 = a_2^2 + b_2^2$,

   $|z|^2 = a_3^2 + b_3^2$,

   $Re(y^* z) = a_2 a_3 + b_2 b_3$

   $Re(x^* z) = a_1 a_3 + b_1 b_3$

   and

   $(x + y + z)^* = a_1 + a_2 + a_3 - (b_1 + b_2 + b_3)i$,

   $|x + y + z|^2$
   $= (x + y + z) \cdot (x + y + z)^*$
   $= ((a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i) \cdot ((a_1 + a_2 + a_3) - (b_1 + b_2 + b_3)i)$
   $= ((a_1 + a_2 + a_3)^2 - (b_1 + b_2 + b_3)^2 \cdot (-1))$
   $= ((a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2)$
   $= a_1^2 + a_2^2 + a_3^2 + 2a_1 a_2 + 2a_1 a_3 + 2a_2 a_3 + b_1^2 + b_2^2 + b_3^2 + 2b_1 b_2 + 2b_1 b_3 + 2b_2 b_3$
   $= a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2 + 2(a_1 a_2 + b_1 b_2 + a_2 a_3 + b_2 b_3 + a_1 a_3 + b_1 b_3)$
   $= |x|^2 + |y|^2 + |z|^2 + 2[Re(x^* y) + Re(y^* z) + Re(x^* z)]$

   This shows that

   $|x + y + z|^2 = |x|^2 + |y|^2 + |z|^2 + 2[Re(x^* y) + Re(y^* z) + Re(x^* z)]$

2. $(i+2)(3-4i)/(2-i)$
$= (3i-4i^2+2*3-2*4i)/(2-i)$
$= (3i-4\times(-1)+2*3-2*4i)/(2-i)$
$= (3i+4+6-8i)/(2-i)$
$= (10-5i)/(2-i)$
$= 5(2-i)/(2-i)$
$= 5$

3. $(i-4)/(2i-3)$
$= [(i-4)(2i+3)]/[(2i-3)(2i+3)]$
$= (2i^2+3i-8i-4*3)/((2i)^2-3*3)$
$= [2\times(-1)+3i-8i-4*3]/[4\times(-1)-3*3]$
$= (-2-5i-12)/(-4-9)$
$= (-14-5i)/(-13)$
$= [(-1)(14+5i)]/(-1\times 13)$
$= (14+5i)/13$
$= 14/13+(5/13)i$

   so, the real part is 14/13 and imaginary pary is 5/13.

4. i^{33}
$= i^{32}i$
$= i^{2\times 16}i$
$= (i^2)^{16}i$
$= (-1)^{16}i$
$= i$

   so, the absolute value of i^{33} is $|i|$
   $|i| = |0+i| = \sqrt{|0+i|^2} = \sqrt{(0+i)(0+i)^*} = \sqrt{(0+i)(0-i)} = \sqrt{-i^2} = \sqrt{-(-1)} = \sqrt{1} = 1$

5. i. For complex number $c_1 = a_1 + b_1 i$ and $c_2 = a_2 + b_2 i$, we have

$$|c_1|^2 = a_1^2 + b_1^2$$

$$|c_2|^2 = a_2^2 + b_2^2$$

$$|c_1+c_2|^2 = (a_1+a_2)^2 + (b_1+b_2)^2 = a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2$$

so we need to find a_1, a_2, b_1, b_2 that makes

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 \geq a_1^2 + b_1^2$$

and

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 < a_2^2 + b_2^2$$

so we have

$$a_2^2 + 2a_1a_2 + b_2^2 + 2b_1b_2 \geq 0$$

and

$$a_1^2 + 2a_1a_2 + b_1^2 + 2b_1b_2 < 0$$

so, we need $2a_1a_2+2b_1b_2 \geq -(a_2^2+b_2^2)$ and $2a_1a_2+2b_1b_2 < -(a_1^2+b_1^2)$, which means $-(a_2^2+b_2^2) \leq 2a_1a_2+2b_1b_2 < -(a_1^2+b_1^2)$,
Through observing, it is easy to find $a_1 = -1, a_2 = 3, b_1 = 1, b_2 = -3$ makes $|c_1+c_2|^2 \geq |c_1|^2$ and $|c_1+c_2|^2 < |c_2|^2$.

ii. Yes. For $c_1 = -1+2i, c_2 = 2-i$, $c_1+c_2 = 1+i$
$|c_1+c_2|^2 == 2$
$|c_1|^2 = 5$
$|c_2|^2 = 5$
in this case, two complex numbers satisfy $|c_1+c_2|^2 \leq |c_1|^2$ and $|c_1+c_2|^2 \leq |c_2|^2$.

6. Assume both $\vec{v}_1$ and $\vec{v}_2$ has a length of n.
$\vec{v}_1 = (\psi_{10}, \psi_{11}, \psi_{12}, \psi_{13}, ..., \psi_{1n})^T$,
$\vec{v}_2 = (\psi_{20}, \psi_{21}, \psi_{22}, \psi_{23}, ..., \psi_{2n})^T$.
For real vectors $\vec{r}_1$ and $\vec{r}_2$, we have $\langle \vec{r}_1, \vec{r}_2 \rangle = \vec{r}_1^T \vec{r}_2$.
Similarly, we can define the inner product of $\vec{v}_1, \vec{v}_2$ that

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1^T \vec{v}_2$$

where $\vec{v}_1^T$ is the transpose of $\vec{v}_1$.
This means

$$\langle \vec{v}_1, \vec{v}_2 \rangle = = (\psi_{10}^*, \psi_{11}^*, \psi_{12}^*, \psi_{13}^*, ..., \psi_{1n}^*)(\psi_{20}, \psi_{21}, \psi_{22}, \psi_{23}, ..., \psi_{2n})^T$$

so,

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \sum_{i=0}^{n-1} \psi_{1i}\psi_{2i}$$

The properties of an inner product $\langle , \rangle$ are as followed[1].

(a) **Linearity:** $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$
(b) **Symmetric Property:** $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (For complex vectors, the inner product doesn't satisfy this property.)
(c) **Positive Definite Property:** For any $\mathbf{u} \in \mathbf{V}$, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$; and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$;

For complex vectors $\vec{v1}, \vec{v_2}, \vec{v_3}$, all of them have a length of n.
$\langle a\vec{v_1} + b\vec{v_2}, \vec{v_3} \rangle$
$= \sum_{i=0}^{n-1} (a\psi_{1i}^* + b\psi_{2i}^*)(\psi_{3i})$
$= \sum_{i=0}^{n-1} (a\psi_{1i}^*)(\psi_{3i}) + \sum_{i=0}^{n-1} (b\psi_{2i}^*)(\psi_{3i})$
$= a\sum_{i=0}^{n-1} (\psi_{1i}^*)(\psi_{3i}) + b\sum_{i=0}^{n-1} (\psi_{2i}^*)(\psi_{3i})$
$= a\langle \vec{v_1}, \vec{v_3} \rangle + b\langle \vec{v_2}, \vec{v_3} \rangle$
This proves the linearity.

Also, we have

$\langle \vec{v_1}, \vec{v_2} \rangle$
$= \sum_{i=0}^{n-1} \psi_{1i}^* \psi_{2i}$
$\langle \vec{v_2}, \vec{v_1} \rangle$
$= \sum_{i=0}^{n-1} \psi_{2i}^* \psi_{1i}$

Because $\sum_{i=0}^{n-1} \psi_{1i}^* \psi_{2i}$ doesn't equal to $\sum_{i=0}^{n-1} \psi_{2i}^* \psi_{1i}$, the inner product of complex numbers doesn't have the symmetric property.

For any complex vector $\vec{v_1}$, $\langle \vec{v_1}, \vec{v_1} \rangle = \sum_{i=0}^{n-1} \psi_{1i}^* \psi_{1i} = \sum_{i=0}^{n-1} |\psi_{1i}|^2$. For any complex number $\psi = a + bi$, we have $|\psi|^2 = a^2 + b^2 \geq 0$,

so $\langle \vec{v_1}, \vec{v_1} \rangle = \sum_{i=0}^{n-1} |\psi_{1i}|^2 \geq 0$ and $\vec{v_1}$ is a complex vector, so $|\vec{v_1}| \neq 0$.

This proves the positive definite property.

So, it satisfies the properties of Linearity and Positive Definite of an inner product, but doesn't satisfy a property of Symmetric.

EXERCISE 2: THE TENSOR PRODUCT

1. $|0\rangle_A \otimes |1\rangle_B$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 1 \times 0 \\ 1 \times 1 \\ 0 \times 0 \\ 0 \times 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$
$= 0(|0\rangle \otimes |0\rangle) + 1(|0\rangle \otimes |1\rangle) + 0(|1\rangle \otimes |0\rangle) + 0(|1\rangle \otimes |1\rangle)$

2. $|+\rangle_A \otimes |-\rangle_B$
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$
$= \frac{1}{2}(|0\rangle \otimes |0\rangle) - \frac{1}{2}(|0\rangle \otimes |1\rangle) + \frac{1}{2}(|1\rangle \otimes |0\rangle) - \frac{1}{2}(|1\rangle \otimes |1\rangle)$

3. $|0\rangle_A \otimes |-\rangle_B$
   $= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \otimes |-\rangle$
   $= \frac{1}{\sqrt{2}} |+\rangle \otimes |-\rangle + \frac{1}{\sqrt{2}} |-\rangle \otimes |-\rangle$
   $= 0 \times |+\rangle \otimes |+\rangle + \frac{1}{\sqrt{2}} |+\rangle \otimes |-\rangle + 0 \times |-\rangle \otimes |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \otimes |-\rangle$

4. $|1\rangle_A \otimes |1\rangle_B$
   $= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$
   $= \frac{1}{2}(|+\rangle \otimes |+\rangle) - \frac{1}{2}(|+\rangle \otimes |-\rangle) - \frac{1}{2}(|-\rangle \otimes |+\rangle) + \frac{1}{2}(|-\rangle \otimes |-\rangle)$

5. We have $|\Phi^+\rangle$
   $= \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)$
   $$= \frac{1}{\sqrt{2}}(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix})$$
   $$= \frac{1}{\sqrt{2}}(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix})$$
   $$= \frac{1}{\sqrt{2}}(\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix})$$
   For $A = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$ and $B = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$, we have $A \otimes B = \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{pmatrix}$.

   If $|\Phi^+\rangle$ can be written as $A \otimes B$, then

   $a_0 b_0 = 0$
   $a_0 b_1 = \frac{1}{\sqrt{2}}$
   $a_1 b_0 = \frac{1}{\sqrt{2}}$
   $a_1 b_1 = 0.$

   To make $a_0 b_0 = 0$, either $a_0 = 0$ or $b_0 = 0$ should be true.

   If any of them is true, then $a_0 b_1 = \frac{1}{\sqrt{2}}$ and $a_1 b_0 = \frac{1}{\sqrt{2}}$ cannot be true in the same time.

So $|\Phi^+\rangle$ can not be written as $A \otimes B$.

6. We have $|0\rangle|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $|1\rangle|1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

   We also have $|+\rangle|-\rangle = \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle)$

   and $|-\rangle|+\rangle = \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle)$

   $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$

   So, $-|\Phi^-\rangle$

   $= \frac{1}{\sqrt{2}}(|1\rangle_A \otimes |0\rangle_B - |0\rangle_A \otimes |1\rangle_B)$

   $= \frac{1}{\sqrt{2}}(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix})$

   $= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

   $= \frac{1}{\sqrt{2}}[\frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle) - \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle)]$

   $= \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle)$

   So $|\Phi^-\rangle$ in basis $\mathcal{B}_1$ is equal to $-|\Phi^-\rangle$ in basis $\mathcal{B}_2$.

OVERLAPS OF STATES

1. For $|\psi\rangle = \begin{pmatrix} c_0 \\ c_1 \\ ... \\ c_{n-1} \end{pmatrix}$ where $c_k = a_k + b_k i$, we know

   $\|\psi\|_2^2 = \sum_{i=0}^{n-1} |c_i|^2 = \sum_{i=0}^{n-1} a_i^2 + b_i^2$.

   now,

   $\langle\psi|\psi\rangle$

   $= \left(c_0^*, c_1^*, ..., c_{n-1}^*\right) \begin{pmatrix} c_0 \\ c_1 \\ ... \\ c_{n-1} \end{pmatrix}$

   $= \sum_{i=0}^{n-1} c_i^* c_i maginary$

   $= \sum_{i=0}^{n-1} a_i^2 + b_i^2$

   $= \|\psi\|_2^2$

   So, $\langle\psi|\psi\rangle = \|\psi\|_2^2$

2. (a) For $|\psi_1\rangle = \frac{1}{3}|-\rangle$,
$\|\psi_1\|^2$
$= \langle\psi_1|\psi_1\rangle$
$= \frac{1}{9}\langle -|-\rangle$
$= \frac{1}{9}[\frac{1}{2}\langle 0|0\rangle + (-1)^2\frac{1}{2}\langle 1|1\rangle]$
$= \frac{1}{9}(\frac{1}{2}+\frac{1}{2})$
$= \frac{1}{9}$
so,
$\|\psi_1\| = \sqrt{\|\psi_1\|^2} = \sqrt{\frac{1}{9}} = \frac{1}{3}$

(b) For $|\psi_2\rangle = \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)$
$\|\psi_2\|^2$
$= \frac{1}{2}\times -(i^2 \times \langle 0|0\rangle) + \frac{1}{2}\langle 1|1\rangle$
$= \frac{1}{2}*1 + \frac{1}{2}*1$
$= 1$
So, $\|\psi_2\| = \sqrt{1} = 1$

(c) $\|\frac{2}{5}|0\rangle + \frac{3}{5}|1\rangle\|$
$= \sqrt{\frac{2}{5}\times\frac{2}{5}\langle 0|1\rangle + \frac{2}{5}\times\frac{3}{5}\langle 0|0\rangle + \frac{2}{5}\times\frac{3}{5}\langle 1|1\rangle + \frac{3}{5}\times\frac{3}{5}\langle 1|1\rangle}$
$= \sqrt{\frac{4}{25}\times 1 + \frac{9}{25}\times 1}$
$= \sqrt{\frac{13}{25}}$
$= \frac{\sqrt{13}}{5}$

3. $|\psi_2\rangle = \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)$ is the correct normalization.

The renormalized state $|\psi_1\rangle' = \frac{|\psi_1\rangle}{\frac{1}{3}} = |-\rangle$

The renormalized state $|\psi_3\rangle' = \frac{|\psi_3\rangle}{\frac{\sqrt{13}}{5}} = \frac{2}{\sqrt{13}}|0\rangle + \frac{3}{\sqrt{13}}|1\rangle$

4. State $\psi = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.

The probability p_1 to find $|\psi\rangle$ in state $|1\rangle$ is:

$$p_1 = |\langle 1|\psi\rangle|^2 = |(\frac{1}{\sqrt{2}})\langle 1|1\rangle|^2 = \frac{1}{2}$$

The probability p_2 to find $|\psi\rangle$ in state $|-\rangle$ is:

$$p_2 = |\langle -|\psi\rangle|^2 = |\langle -|+\rangle|^2 = |\frac{1}{2}(\langle 0|0\rangle - \langle 1|1\rangle)|^2 = 0$$

5. The probability p to find $|\psi\rangle = \frac{1}{\sqrt{2}}(i|0\rangle - |1\rangle)$ in state $|+\rangle$ is:

$p = |\langle +|\psi\rangle|^2 = |\frac{1}{2}(i\langle 0|0\rangle - \langle 1|1\rangle)|^2 = \frac{1}{4}|-1+i|^2 = \frac{1}{4}(1+1) = \frac{1}{2}$

6. The probability p of output $|\psi\rangle$ in the state $|+\rangle$ is:
   $p = |\langle\phi|\psi\rangle|^2 = |[\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)](\frac{2}{\sqrt{5}}|0\rangle + i\frac{1}{\sqrt{5}}|1\rangle)|^2$
   $= |\frac{2}{\sqrt{10}}\langle 0|0\rangle + 0 + 0 + i\frac{1}{\sqrt{10}}\langle 1|1\rangle|^2$
   $= |\frac{2}{\sqrt{10}} + i\frac{1}{\sqrt{10}}|^2$
   $= \frac{4}{10} + \frac{1}{10}$
   $= \frac{1}{2}$
   $\frac{1}{2} = 50\% > 45\%$
   So, we accept this state.

DENSITY OPERATORS

1. For $|\psi\rangle = |+\rangle_A \otimes |0\rangle$ The density operator is $|\psi\rangle\langle\psi|$
   $|+\rangle_A \otimes |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle)$
   $$= \frac{1}{\sqrt{2}}(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix})$$
   $$== \frac{1}{\sqrt{2}}(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix})$$
   The density matrix
   ρ
   $= |+\rangle_A \otimes |0\rangle_B \langle +|_A \otimes \langle 0|_B$
   $$= \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$$
   $$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
   $$= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. $|\psi\rangle = \frac{\sqrt{3}}{2}(|0\rangle_A \otimes |1\rangle_B) + \frac{1}{2}(|1\rangle \otimes |0\rangle_B)$
   The density operator is $|\psi\rangle\langle\psi|$

$$|\psi\rangle = \frac{\sqrt{3}}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

The density matrix

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{\sqrt{3}}{4} & 0 \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3. For $|\psi\rangle = \frac{1}{\sqrt{2}}(|000...0\rangle + |111...1\rangle) = \frac{1}{\sqrt{2}}(\begin{pmatrix} 1 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ ... \\ 1 \end{pmatrix}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix}$

   The density operator is $|\psi\rangle\langle\psi|$

   The density matrix $\rho = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & ... & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$

4. $|i\rangle = \begin{pmatrix} 0 \\ 0 \\ ... \\ 1 \\ 0 \\ ... \\ 0 \end{pmatrix}$,the $i+1$th number in the vector is 1, others are 0.

   So, $|i\rangle \otimes |i\rangle == \begin{pmatrix} 0 \\ 0 \\ ... \\ |i\rangle \\ 0 \\ ... \\ 0 \end{pmatrix}$, above $|i\rangle$ there are $i \times n$ zeros.

let $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_i (|i\rangle \otimes |i\rangle) = \frac{1}{\sqrt{2^n}} \begin{pmatrix} |0\rangle \\ |1\rangle \\ ... \\ |k\rangle \\ ... \\ ... \\ |2^n - 1\rangle \end{pmatrix}$

the density operator is $|\psi\rangle \langle\psi|$

the density matrix

$\rho =$

$$= \frac{1}{2^n} \begin{pmatrix} |0\rangle \\ |1\rangle \\ ... \\ |k\rangle \\ ... \\ ... \\ |2^n - 1\rangle \end{pmatrix} (\langle 0| , \langle 1| , ..., \langle k| , ..., ..., \langle 2^n - 1|)$$

$$= \frac{1}{2^n} \begin{pmatrix} |0\rangle\langle 0| & |1\rangle\langle 0| & ... & |k\rangle\langle 0| & ... & |2^n-1\rangle\langle 0| \\ |0\rangle\langle 1| & |1\rangle\langle 1| & ... & |k\rangle\langle 1| & ... & |2^n-1\rangle\langle 1| \\ ... & ... & ... & ... & ... & ... \\ |0\rangle\langle k| & |1\rangle\langle k| & ... & |k\rangle\langle k| & ... & |2^n-1\rangle\langle k| \\ ... & ... & ... & ... & ... & ... \\ |0\rangle\langle 2^n-1| & |1\rangle\langle 2^n-1| & ... & |k\rangle\langle 2^n-1| & ... & |2^n-1\rangle\langle 2^n-1| \end{pmatrix}$$

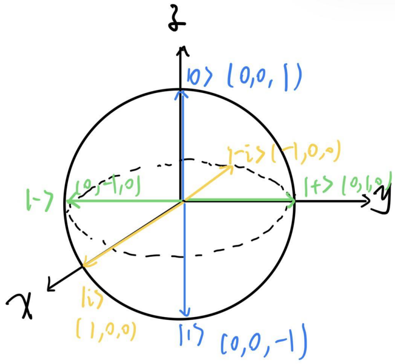
Amng them, $|i\rangle \langle j| = \begin{pmatrix} 0 & 0 & ... & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & \\ 0 & 0 & ... & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & \\ 0 & 0 & ... & 0 & ... & 0 & 0 \end{pmatrix}$, the $i + 1$th row and $j + 1$th column is 1, other elements are 0.

THE BLOCH SPHERE

1. (a) For $|0\rangle$, $\vec{r} = (0, 0, 1)^T$
   (b) For $|1\rangle$, $\vec{r} = (0, 0, -1)^T$
   (c) For $|+\rangle$, $\vec{r} = (0, 1, 0)^T$
   (d) For $|-\rangle$, $\vec{r} = (0, -1, 0)^T$
   (e) For $|i\rangle$, $\vec{r} = (1, 0, 0)^T$
   (f) For $|-i\rangle$, $\vec{r} = (-1, 0, 0)^T$

   $\frac{I}{2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(I + O)$

   So, $\vec{r}\vec{\sigma} = O$



Assume $\vec{r} = (a, b, c)^T$

$\vec{r}\vec{\sigma}$

$= (a, b, c)^T \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$

$= \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & -bi \\ bi & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$

$= \begin{pmatrix} c & a-bi \\ a+bi & -c \end{pmatrix}$

So, $\begin{pmatrix} c & a-bi \\ a+bi & -c \end{pmatrix} = O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

which means $a = b = c = 0$ and $\vec{r} = (0, 0, 0)^T$

So, $\frac{I}{2}$ is located in $(0, 0, 0)$, which is the centre of the Bloch ball.

2. (a) ρ_1

$= \frac{1}{2}(I + \vec{r}\vec{\sigma})$

$= \frac{1}{2}(I + (0, -\frac{1}{3}, 0)\vec{\sigma})$

$= \frac{1}{2}\left(I + \begin{pmatrix} 0 & \frac{1}{3}i \\ -\frac{1}{3}i & 0 \end{pmatrix}\right)$

$== \frac{1}{2}\left(\begin{pmatrix} 1 & \frac{1}{3}i \\ -\frac{1}{3}i & 1 \end{pmatrix}\right)$

$= \left(\begin{pmatrix} \frac{1}{2} & \frac{1}{6}i \\ -\frac{1}{6}i & \frac{1}{2} \end{pmatrix}\right)$

Because $\rho_1^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{6}i \\ -\frac{1}{6}i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{6}i \\ -\frac{1}{6}i & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{11}{36} & 0 \\ 0 & \frac{11}{36} \end{pmatrix}$,

$Tr(\rho_1^2) = \frac{11}{18} \neq 1$, ρ_1 is not a pure state.

(b) ρ_2

$= \frac{1}{2}(I + \vec{r}\vec{\sigma})$

$= \frac{1}{2}(I + (-\frac{1}{2}, \frac{1}{2}, 0)\vec{\sigma})$

$= \frac{1}{2}\left(I + \begin{pmatrix} 0 & \frac{-1-i}{\sqrt{2}}i \\ -\frac{-1+i}{\sqrt{2}}i & 0 \end{pmatrix}\right)$

$== \frac{1}{2}(\begin{pmatrix} 1 & \frac{-1-i}{\sqrt{2}}i \\ \frac{-1+i}{\sqrt{2}}i & 1 \end{pmatrix})$
$= (\begin{pmatrix} \frac{1}{2} & \frac{-1-i}{2\sqrt{2}}i \\ \frac{-1+i}{2\sqrt{2}}i & \frac{1}{2} \end{pmatrix})$
$\rho_2^2 = \begin{pmatrix} \frac{1}{2} & \frac{-1-i}{2\sqrt{2}}i \\ \frac{-1+i}{2\sqrt{2}}i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{-1-i}{2\sqrt{2}}i \\ \frac{-1+i}{2\sqrt{2}}i & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{-1-i}{2\sqrt{2}}i \\ \frac{-1+i}{2\sqrt{2}}i & \frac{1}{2} \end{pmatrix} = \rho_2$,
Because $Tr(\rho_2^2) = 1$, ρ_2 is a pure state.

3. The surface of the Bloch sphere.

   PROOF:

   Assume a pure state $|\psi\rangle$'s Bloch vector $\vec{r} = (a, b, c)^T$.
   $\rho = \frac{1}{2}(I + \vec{r}\vec{\sigma})$
   $= \frac{1}{2}(I + \begin{pmatrix} c & a-bi \\ a+bi & -c \end{pmatrix})$
   $= \frac{1}{2}\begin{pmatrix} 1+c & a-bi \\ a+bi & 1-c \end{pmatrix}$
   $= \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix}$
   $\rho^2 = \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix} \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix}$
   $= \begin{pmatrix} \frac{(1+c)^2+a^2-(bi)^2}{4} & \frac{(1+c)(a-bi)+(a-bi)(1-c)}{4} \\ \frac{(a+bi)(1+c)+(1-c)(a+bi)}{4} & \frac{a^2-(bi)^2+(1-c)^2}{4} \end{pmatrix}$
   $Tr(\rho^2)$
   $= \frac{(1+c)^2+a^2-(bi)^2}{4} + \frac{a^2-(bi)^2+(1-c)^2}{4}$
   $= \frac{(1+c)^2+a^2-(bi)^2+a^2-(bi)^2+(1-c)^2}{4}$
   $= \frac{2a^2+2b^2+c^2+2c+1+c^2-2c+1}{4}$
   $= \frac{2(a^2+b^2+c^2)+2}{4}$

   Because $|\psi\rangle$ is a pure state, so $Tr(\rho^2)$=1, which means $\frac{2(a^2+b^2+c^2)+2}{4} = 1$. So, $a^2+b^2+c^2 = 1$, this means $\vec{r} = (a, b, c)^T$ is on the surface of the Bloch sphere, pure state $|\psi\rangle$ is on the surface of the Bloch sphere.

4. The requirements of sensity matracies are as followed,

   (a) **Positive semi-definite:** all eigenvalues $\lambda_i \geq 0$
   (b) **Hermitian:** $\rho = \rho^\dagger$
   (c) **Normalization:** $Tr(\rho) = 1$

   For vector $\vec{r}$, $\rho = \frac{1}{2}(I + \vec{r}\vec{\sigma}) = \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix}$

Assume the eigenvalues of ρ are $\lambda_1, ..., \lambda_n$

$$\left|\begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix} - \lambda I\right| = 0$$

$$\begin{vmatrix} \frac{1+c}{2} - \lambda & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} - \lambda \end{vmatrix} = 0$$

$$(\frac{1+c}{2} - \lambda)(\frac{1-c}{2} - \lambda) - (\frac{a-bi}{2})(\frac{a+bi}{2}) = 0$$

$$\frac{1-c^2}{4} - (\frac{1+c+1-c}{2}\lambda) + \lambda^2 - \frac{a^2+b^2}{4} = 0$$

$$\lambda^2 - \lambda + \frac{1-a^2-b^2-c^2}{4} = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-4\times\frac{1-a^2-b^2-c^2}{4}}}{2} = \frac{1\pm\sqrt{a^2+b^2+c^2}}{2} = \frac{1\pm\sqrt{|\vec{r}|^2}}{2}$$

Because $|\vec{r}| \leq 1$, $\frac{1\pm\sqrt{|\vec{r}|^2}}{2} \in [0,1]$, which means $0 \leq \lambda \leq 1$, this proves the **Positive semi-definite**.

$$\rho^\dagger = \begin{pmatrix} \frac{1+c}{2} & (\frac{a+bi}{2})^* \\ (\frac{a-bi}{2})^* & \frac{1-c}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+c}{2} & \frac{a-bi}{2} \\ \frac{a+bi}{2} & \frac{1-c}{2} \end{pmatrix} = \rho$$

This proves the **Hermitian**.

The trace of matrix ρ

$$Tr(\rho) = \lambda_1 + \lambda_2 = \frac{1+\sqrt{|\vec{r}|^2}}{2} + \frac{1-\sqrt{|\vec{r}|^2}}{2} = 1.$$

This proves the **Normalization**.

So ρ is indeed a valid density operator for any vector $\vec{r}$ satisfing $|\vec{r}| \leq 1$.

References

[1] HKUST Department of Mathematics. "Inner Product Spaces and Orthogonality" Hong Kong University of Science and Technology, n.d., https://www.math.hkust.edu.hk/ mabfchen/Math111/Week13-14.pdf.