

Assignment 1

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EXERCISE 1: COMPLEX NUMBERS

1. Assume $x = a_1 + b_1i$, $x = a_2 + b_2i$, $x = a_3 + b_3i$,

we have $x + y + z = a_1 + a_2 + a_3 + (b_1 + b_2 + b_3)i$,

To a complex number $n = a + bi$ we have $n^* = a - bi$,

so

$$|x|^2 = x \cdot x^* = a_1^2 - b_1^2(i)^2 = a_1^2 - b_1^2 \cdot (-1) = a_1^2 + b_1^2,$$

x^*y

$$= (a_1 - b_1i) \cdot (a_2 + b_2i)$$

$$= a_1a_2 + a_1b_2i - a_2b_1i - b_1b_2i^2$$

$$= a_1a_2 + b_1b_2 + (a_1b_2 - a_2b_1)i,$$

$$Re(x^*y) = a_1a_2 + b_1b_2,$$

similarly, we have

$$|y|^2 = a_2^2 + b_2^2,$$

$$|z|^2 = a_3^2 + b_3^2,$$

$$Re(y^*z) = a_2a_3 + b_2b_3$$

$$Re(x^*z) = a_1a_3 + b_1b_3$$

and

$$(x + y + z)^* = a_1 + a_2 + a_3 - (b_1 + b_2 + b_3)i,$$

$$|x + y + z|^2$$

$$= (x + y + z) \cdot (x + y + z)^*$$

$$= ((a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i) \cdot ((a_1 + a_2 + a_3) - (b_1 + b_2 + b_3)i)$$

$$= ((a_1 + a_2 + a_3)^2 - (b_1 + b_2 + b_3)^2 \cdot (-1))$$

$$= ((a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2)$$

$$= a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 + 2a_1a_3 + 2a_2a_3 + b_1^2 + b_2^2 + b_3^2 + 2b_1b_2 + 2b_1b_3 + 2b_2b_3$$

$$= a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2 + 2(a_1a_2 + b_1b_2 + a_2a_3 + b_2b_3 + a_1a_3 + b_1b_3)$$

$$= |x|^2 + |y|^2 + |z|^2 + 2[Re(x^*y) + Re(y^*z) + Re(x^*z)]$$

This shows that

$$|x + y + z|^2 = |x|^2 + |y|^2 + |z|^2 + 2[Re(x^*y) + Re(y^*z) + Re(x^*z)]$$

$$\begin{aligned}
2. \quad & (i+2)(3-4i)/(2-i) \\
&= (3i-4i^2+2*3-2*4i)/(2-i) \\
&= (3i-4 \times (-1)+2*3-2*4i)/(2-i) \\
&= (3i+4+6-8i)/(2-i) \\
&= (10-5i)/(2-i) \\
&= 5(2-i)/(2-i) \\
&= 5 \\
3. \quad & (i-4)/(2i-3) \\
&= [(i-4)(2i+3)]/[(2i-3)(2i+3)] \\
&= (2i^2+3i-8i-4*3)/((2i)^2-3*3) \\
&= [2 \times (-1)+3i-8i-4*3]/[4 \times (-1)-3*3] \\
&= (-2-5i-12)/(-4-9) \\
&= (-14-5i)/(-13) \\
&= [(-1)(14+5i)]/(-1 \times 13) \\
&= (14+5i)/13 \\
&= 14/13+5/13i
\end{aligned}$$

so, the real part is $14/13$ and imaginary part is $5/13$.

$$\begin{aligned}
4. \quad & i^{33} \\
&= i^{32}i \\
&= i^{2 \times 16}i \\
&= (i^2)^{16}i \\
&= (-1)^{16}i \\
&= i
\end{aligned}$$

so, the absolute value of i^{33} is $|i|$

$$\begin{aligned}
|i| &= |0+i| = \sqrt{|0+i|^2} = \sqrt{(0+i)(0+i)^*} = \sqrt{(0+i)(0-i)} = \sqrt{-i^2} = \\
&= \sqrt{-(-1)} = \sqrt{1} = 1
\end{aligned}$$

5. i. For complex number $c_1 = a_1 + b_1i$ and $c_2 = a_2 + b_2i$, we have

$$|c_1|^2 = a_1^2 + b_1^2$$

$$|c_2|^2 = a_2^2 + b_2^2$$

$$|c_1 + c_2|^2 = (a_1 + a_2)^2 + (b_1 + b_2)^2 = a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2$$

so we need to find a_1, a_2, b_1, b_2 that makes

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 \geq a_1^2 + b_1^2$$

and

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 < a_2^2 + b_2^2$$

so we have

$$a_2^2 + 2a_1a_2 + b_2^2 + 2b_1b_2 \geq 0$$

and

$$a_1^2 + 2a_1a_2 + b_1^2 + 2b_1b_2 < 0$$

so, we need $2a_1a_2 + 2b_1b_2 \geq -(a_2^2 + b_2^2)$ and $2a_1a_2 + 2b_1b_2 < -(a_1^2 + b_1^2)$, which means $-(a_2^2 + b_2^2) \leq 2a_1a_2 + 2b_1b_2 < -(a_1^2 + b_1^2)$,

Through observing, it is easy to find $a_1 = -1, a_2 = 3, b_1 = 1, b_2 = -3$ makes $|c_1 + c_2|^2 \geq |c_1|^2$ and $|c_1 + c_2|^2 < |c_2|^2$.

ii. To make $|c_1 + c_2|^2 < |c_1|^2$ and $|c_1 + c_2|^2 < |c_2|^2$, we need to make

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 < a_1^2 + b_1^2$$

and

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 < a_2^2 + b_2^2$$

so, we have

$$a_2^2 + 2a_1a_2 + b_2^2 + 2b_1b_2 < 0$$

and

$$a_1^2 + 2a_1a_2 + b_1^2 + 2b_1b_2 < 0$$

I don't think we can find a_1, a_2, b_1, b_2 that can make $|c_1 + c_2|^2 < |c_1|^2$ and $|c_1 + c_2|^2 < |c_2|^2$ true.

6. Assume both \vec{v}_1 and \vec{v}_2 has a length of n .

$$\vec{v}_1 = (\psi_{10}, \psi_{11}, \psi_{12}, \psi_{13}, \dots, \psi_{1n})^T,$$

$$\vec{v}_2 = (\psi_{20}, \psi_{21}, \psi_{22}, \psi_{23}, \dots, \psi_{2n})^T.$$

For real vectors \vec{r}_1 and \vec{r}_2 , we have $\langle \vec{r}_1, \vec{r}_2 \rangle = \vec{r}_1^T \vec{r}_2$.

Similarly, we can define the inner product of \vec{v}_1, \vec{v}_2 that

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1^T \vec{v}_2$$

where \vec{v}_1^T is the transpose of \vec{v}_1 .

This means

$$\langle \vec{v}_1, \vec{v}_2 \rangle = (\psi_{10}, \psi_{11}, \psi_{12}, \psi_{13}, \dots, \psi_{1n})(\psi_{20}, \psi_{21}, \psi_{22}, \psi_{23}, \dots, \psi_{2n})^T$$

so,

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \sum_{i=0}^{n-1} \psi_{1i} \psi_{2i}$$

The properties of an inner product \langle, \rangle are as followed[1].

- (a) **Linearity:** $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$
- (b) **Symmetric Property:** $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
- (c) **Positive Definite Property:** For any $\mathbf{u} \in \mathbf{V}$, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$; and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$;

For complex vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, all of them have a length of n.

$$\begin{aligned}
& \langle a\vec{v}_1 + b\vec{v}_2, \vec{v}_3 \rangle \\
&= \sum_{i=0}^{n-1} (a\psi_{1i} + b\psi_{2i})(\psi_{3i}) \\
&= \sum_{i=0}^{n-1} (a\psi_{1i})(\psi_{3i}) + \sum_{i=0}^{n-1} (b\psi_{2i})(\psi_{3i}) \\
&= a \sum_{i=0}^{n-1} (\psi_{1i})(\psi_{3i}) + b \sum_{i=0}^{n-1} (\psi_{2i})(\psi_{3i}) \\
&= a\langle \vec{v}_1, \vec{v}_3 \rangle + b\langle \vec{v}_2, \vec{v}_3 \rangle
\end{aligned}$$

This proves the linearity.

Also, we have

$$\begin{aligned}
& \langle \vec{v}_1, \vec{v}_2 \rangle \\
&= \sum_{i=0}^{n-1} \psi_{1i}\psi_{2i} \\
&= \sum_{i=0}^{n-1} \psi_{2i}\psi_{1i} \\
&= \langle \vec{v}_2, \vec{v}_1 \rangle
\end{aligned}$$

This proves the symmetric property.

For any complex vector \vec{v}_1 , $\langle \vec{v}_1, \vec{v}_1 \rangle = \sum_{i=0}^{n-1} \psi_{1i}^2$. For any complex number $\psi = a + bi$, we have $\psi^2 = a^2 + b^2 \geq 0$,

so $\langle \vec{v}_1, \vec{v}_1 \rangle = \sum_{i=0}^{n-1} \psi_{1i}^2 \geq 0$ and \vec{v}_1 is a complex vector, so $\vec{v}_1 \neq 0$.

This proves the positive definite property.

So, it satisfies all the properties of an inner product.

EXERCISE 2: THE TENSOR PRODUCT

1. $|0\rangle_A \otimes |1\rangle_B$

$$\begin{aligned}
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 1 \times 0 \\ 1 \times 1 \\ 0 \times 0 \\ 0 \times 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$
2. $|+\rangle_A \otimes |-\rangle_B$

$$\begin{aligned}
&= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
3. \quad &|0\rangle_A \otimes |-\rangle_B \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 1 \times \frac{1}{\sqrt{2}} \\ 1 \times -\frac{1}{\sqrt{2}} \\ 0 \times \frac{1}{\sqrt{2}} \\ 0 \times -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
4. \quad &|1\rangle_A \otimes |1\rangle_B \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 \times 0 \\ 0 \times 1 \\ 1 \times 0 \\ 1 \times 1 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} 5. \text{ We have } |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B) \\ &= \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ &= \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

$$\text{For } A = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}, \text{ we have } A \otimes B = \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{pmatrix}.$$

If $|\Phi^+\rangle$ can be written as $A \otimes B$, then

$$\begin{aligned} a_0 b_0 &= 0 \\ a_0 b_1 &= 1 \\ a_1 b_0 &= 1 \\ a_1 b_1 &= 0. \end{aligned}$$

To make $a_0 b_0 = 0$, either $a_0 = 0$ or $b_0 = 0$ should be true.

If any of them is true, then $a_0 b_1 = 1$ and $a_1 b_0 = 1$ cannot be true in the same time.

So $|\Phi^+\rangle$ can not be written as $A \otimes B$.

$$6. \text{ We have } |0\rangle|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } |1\rangle|1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We also have $|+\rangle|-\rangle = \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle)$

and $|-\rangle|+\rangle = \frac{1}{2}(|0\rangle|0\rangle + |1\rangle|1\rangle)$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$$

$$\begin{aligned} \text{So, } -|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|1\rangle_A \otimes |0\rangle_B - |0\rangle_A \otimes |1\rangle_B) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \left[\frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle) - \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle) \right] \\
&= \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle)
\end{aligned}$$

So $|\Phi^-\rangle$ in basis \mathcal{B}_1 is equal to $-|\Phi^-\rangle$ in basis \mathcal{B}_2 .

References

- [1] HKUST Department of Mathematics. "Math111 Week 13-14 Lecture Notes." Hong Kong University of Science and Technology, n.d., <https://www.math.hkust.edu.hk/~mabfchen/Math111/Week13-14.pdf>.