Assignment 2

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1 EXERCISE 1: THE PARTIAL TRACE

We saw in the lectures that given a multi-partite state, we obtain the state of a subsystem by applying the partial trace to the other systems.

1. Compute the trace of the following states by applying the trace formula $Tr(A) = \sum_i \langle i| A |i\rangle$ from the lectures:

   (a) $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

   (b) $\Lambda = \frac{I}{3}I + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$ (here, I is the identity matrix).
   Which of the states is normalized correctly? (*) (4 points)

   Answer: $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$
   $= \frac{1}{2}(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - i\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + i\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix})$
   $= \frac{1}{2}(\begin{bmatrix}1 & -i\\i & 1\end{bmatrix})$
   $= \begin{bmatrix}\frac{1}{2} & -\frac{i}{2}\\\frac{i}{2} & \frac{1}{2}\end{bmatrix}$
   $Tr(\xi) = |0\rangle\xi\langle 0| + |1\rangle\xi\langle 1|$
   $= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{i}{2}\\\frac{i}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{i}{2}\\\frac{i}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$
   $= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\\frac{i}{2}\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-\frac{i}{2}\\\frac{1}{2}\end{bmatrix}$
   $= \frac{1}{2} + \frac{1}{2}$
   $= 1$
   $\Lambda = \frac{I}{3}I + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$
   $= \frac{I}{3} + \frac{1}{6}(\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix})$
   $= \frac{I}{3} + \frac{1}{6}(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix})$
   $= \frac{I}{3} + \frac{1}{6}(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix})$
   $= \begin{bmatrix}0 & \frac{1}{3}\\\frac{1}{3} & 0\end{bmatrix} + \begin{bmatrix}0 & \frac{1}{6}\\\frac{1}{6} & 0\end{bmatrix}$
   $= \begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\\frac{1}{6} & \frac{1}{3}\end{bmatrix}$
   $Tr(\Lambda) = |0\rangle\Lambda\langle 0| + |1\rangle\Lambda\langle 1|$
   $= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\\frac{1}{6} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\\frac{1}{6} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$
   $= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{3}\\\frac{1}{6}\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{6}\\\frac{1}{3}\end{bmatrix}$

$= \frac{1}{3} + \frac{1}{3}$
$= \frac{2}{3}$

So ξ is normalized correctly.

2. In the last assignment we found the probability to find a state $|\gamma\rangle$ in another state $|\delta\rangle$ to be $p = |\langle\gamma|\delta\rangle|^2$. Show here that this expression coincides with $\mathrm{Tr}(\gamma\delta)$, for $\gamma = |\gamma\rangle\langle\gamma|$ and $\delta = |\delta\rangle\langle\delta|$. Hint: Apply the trace formula $\mathrm{Tr}(A) = \sum_i \langle i|A|i\rangle$ from the lectures. Use a suitable basis of your choice. (4 points)

   Answer: $\mathrm{Tr}(\gamma\delta)$
   $= \mathrm{Tr}(|\gamma\rangle\langle\gamma|\,|\delta\rangle\langle\delta|)$
   $= \mathrm{Tr}(|\gamma\rangle\langle\gamma|\delta\rangle\langle\delta|)$
   $= \mathrm{Tr}(\langle\gamma|\delta\rangle|\gamma\rangle\langle\delta|)$
   $= \langle\gamma|\delta\rangle\,\mathrm{Tr}(|\gamma\rangle\langle\delta|)$
   $= \langle\gamma|\delta\rangle\sum_i \langle i|\,|\gamma\rangle\langle\delta|\,|i\rangle$
   $= \langle\gamma|\delta\rangle(\langle 0|\,|\gamma\rangle\langle\delta|\,|0\rangle + \langle 1|\,|\gamma\rangle\langle\delta|\,|1\rangle)$
   $= \langle\gamma|\delta\rangle(\langle 0|\gamma\rangle\langle\delta|0\rangle + \langle 1|\gamma\rangle\langle\delta|1\rangle)$
   $= \langle\gamma|\delta\rangle(\langle\gamma|0\rangle\langle 0|\delta\rangle + \langle\gamma|1\rangle\langle 1|\delta\rangle)$
   $= \langle\gamma|\delta\rangle(\langle\gamma|\delta\rangle)$
   $= |\langle\gamma|\delta\rangle|^2$

   So, $\mathrm{Tr}(\gamma\delta) = |\langle\gamma|\delta\rangle|^2 = p$.

3. Consider the bipartite state $|\phi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i\,|01\rangle_{AB} - |11\rangle_{AB})$. Write the density operator $\rho_{AB} = |\phi\rangle_{AB}\langle\phi|_{AB}$ explicitly in matrix form. (*) (4 points)

   Answer: ρ_{AB}
   $= |\phi\rangle_{AB}\langle\phi|_{AB}$
   $= \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i\,|01\rangle_{AB} - |11\rangle_{AB})(\langle 00|_{AB} - i\,\langle 01|_{AB} - \langle 11|_{AB})$
   $= \frac{1}{3}(|00\rangle\langle 00| - i\,|00\rangle\langle 01| - |00\rangle\langle 11| + i\,|01\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 11| - |11\rangle\langle 00| - |11\rangle\langle 01| + |11\rangle\langle 11|)$

$$= \frac{1}{3}(\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} - i\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix} + i\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{bmatrix} -$$
$$\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix})$$
$$= \frac{1}{3}\begin{bmatrix}1&i&0&-1\\-i&1&0&1\\0&0&0&0\\-1&i&0&1\end{bmatrix}$$
$$= \begin{bmatrix}\frac{1}{3}&\frac{i}{3}&0&-\frac{1}{3}\\-\frac{i}{3}&\frac{1}{3}&0&\frac{1}{3}\\0&0&0&0\\-\frac{1}{3}&\frac{i}{3}&0&\frac{1}{3}\end{bmatrix}$$

4. Let σ_{AB} be a general 2-qubit state. The 4×4 matrix describing σ_{AB} can be split into four sub-matrices of size 2×2 (upper left block, upper right block, lower left block, lower right block). Show that the reduced single qubit state $\sigma_B = \mathrm{Tr}_A(\sigma_{AB})$ is described by a 2×2 matrix that is the sum of the upper left block and lower right block matrices of σ_{AB}. Hint: start from $\sigma_{AB} = \sigma_{00}|00\rangle\langle 00| + \sigma_{01}|00\rangle\langle 01| + \ldots + \sigma_{33}|11\rangle\langle 11|$ and compute the partial trace in bra-ket notation. Write σ_{AB} as a matrix and compare. (8 points)

   Answer: $\sigma_{AB} = \sigma_{00}\,|00\rangle\langle 00| + \sigma_{01}\,|00\rangle\langle 01| + \ldots + \sigma_{33}\,|11\rangle\langle 11|$

   $\sigma_B =$
   $= \mathrm{Tr}_A(\sigma_{AB})$

$= \sum_i \langle i| \sigma_{AB} |i\rangle$
$= \langle 0| \sigma_{AB} |0\rangle + \langle 1| \sigma_{AB} |1\rangle$
$= \langle 0| (\sigma_{00} |00\rangle \langle 00| + \sigma_{01} |00\rangle \langle 01| + ... + \sigma_{33} |11\rangle \langle 11|) |0\rangle + \langle 1| (\sigma_{00} |00\rangle \langle 00| + \sigma_{01} |00\rangle \langle 01| + ... + \sigma_{33} |11\rangle \langle 11|) |1\rangle$
$= \sigma_{00} \langle 0|00\rangle \langle 00|0\rangle + \sigma_{01} \langle 0|00\rangle \langle 01|0\rangle + \sigma_{02} \langle 0|00\rangle \langle 10|0\rangle + \sigma_{03} \langle 0|00\rangle \langle 11|0\rangle + \sigma_{10} \langle 0|01\rangle \langle 00|0\rangle + \sigma_{11} \langle 0|01\rangle \langle 01|0\rangle +$
$\sigma_{12} \langle 0|01\rangle \langle 10|0\rangle + \sigma_{13} \langle 0|01\rangle \langle 11|0\rangle + \sigma_{20} \langle 0|10\rangle \langle 00|0\rangle + \sigma_{21} \langle 0|10\rangle \langle 01|0\rangle + \sigma_{22} \langle 0|10\rangle \langle 10|0\rangle + \sigma_{23} \langle 0|10\rangle \langle 11|0\rangle +$
$\sigma_{30} \langle 0|11\rangle \langle 00|0\rangle + \sigma_{31} \langle 0|11\rangle \langle 01|0\rangle + \sigma_{32} \langle 0|11\rangle \langle 10|0\rangle + \sigma_{33} \langle 0|11\rangle \langle 11|0\rangle + \sigma_{00} \langle 1|00\rangle \langle 00|1\rangle + \sigma_{01} \langle 1|00\rangle \langle 01|1\rangle +$
$\sigma_{02} \langle 1|00\rangle \langle 10|1\rangle + \sigma_{03} \langle 1|00\rangle \langle 11|1\rangle + \sigma_{10} \langle 1|01\rangle \langle 00|1\rangle + \sigma_{11} \langle 1|01\rangle \langle 01|1\rangle + \sigma_{12} \langle 1|01\rangle \langle 10|1\rangle + \sigma_{13} \langle 1|01\rangle \langle 11|1\rangle +$
$\sigma_{20} \langle 1|10\rangle \langle 00|1\rangle + \sigma_{21} \langle 1|10\rangle \langle 01|1\rangle + \sigma_{22} \langle 1|10\rangle \langle 10|1\rangle + \sigma_{23} \langle 1|10\rangle \langle 11|1\rangle + \sigma_{30} \langle 1|11\rangle \langle 00|1\rangle + \sigma_{31} \langle 1|11\rangle \langle 01|1\rangle +$
$\sigma_{32} \langle 1|11\rangle \langle 10|1\rangle + \sigma_{33} \langle 1|11\rangle \langle 11|1\rangle$
$= \sigma_{00} |0\rangle \langle 0| + \sigma_{02} |0\rangle \langle 1| + \sigma_{10} |1\rangle \langle 0| + \sigma_{12} |1\rangle \langle 1| + \sigma_{21} |0\rangle \langle 0| + \sigma_{23} |0\rangle \langle 1| + \sigma_{31} |1\rangle \langle 0| + \sigma_{33} |1\rangle \langle 1|$
$= (\sigma_{00} + \sigma_{21}) |0\rangle \langle 0| + (\sigma_{02} + \sigma_{23}) |0\rangle \langle 1| + (\sigma_{10} + \sigma_{31}) |1\rangle \langle 0| + (\sigma_{12} + \sigma_{33}) |1\rangle \langle 1|$

This is the sum of the upper left block and lower right block matrices of σ_{AB}.

5. Take the state ρ_{AB} from 3. and compute the reduced state ρ_B, both from the matrix itself (using your result from 4.) and in bra-ket notation. (*) (6 points)

Answer: $\rho_{AB} = \begin{bmatrix} \frac{1}{3} & \frac{i}{3} & 0 & -\frac{1}{3} \\ -\frac{i}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{i}{3} & 0 & \frac{1}{3} \end{bmatrix}$

$\rho_B = (\rho_{00} + \rho_{21}) |0\rangle \langle 0| + (\rho_{02} + \rho_{23}) |0\rangle \langle 1| + (\rho_{10} + \rho_{31}) |1\rangle \langle 0| + (\rho_{12} + \rho_{33}) |1\rangle \langle 1|$
$= (\frac{1}{3} + 0) |0\rangle \langle 0| + (0 + 0) |0\rangle \langle 1| + (-\frac{i}{3} + \frac{i}{3}) |1\rangle \langle 0| + (\frac{1}{3} + \frac{1}{3}) |1\rangle \langle 1|$
$= \frac{1}{3} |0\rangle \langle 0| + 0 |0\rangle \langle 1| + 0 |1\rangle \langle 0| + \frac{2}{3} |1\rangle \langle 1|$
$= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$

2 EXERCISE 2: MEASUREMENTS AND REDUCED STATES

Let us look at the density operator $\rho_{AB} = |\Phi^-\rangle\langle\Phi^-|$, with $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(\langle 01| - \langle 10|)$.

1. Compute explicitly the post-measurement state after we performed a projective measurement $M = \{|0\rangle \langle 0|, |1\rangle \langle 1|\}$ on ρ_{AB} on system B, given that the outcome was 0. Compute the same for the measurement $M = \{|+\rangle\langle+|, |-\rangle\langle-|\}$, for outcome $-$. (Hint: Using Exercise 2.6 from Assignment 1 might help (not necessary though).) (8 points)

Answer: $M_0 = |0\rangle \langle 0|$

Observe $|0\rangle$ with probability $p_0 = Tr(M_0\rho) = Tr(|0\rangle \langle 0| \otimes I\rho) = Tr(|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-|) = \langle 0|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |0\rangle \otimes I + \langle 1|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |1\rangle \otimes I = \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |0\rangle \otimes I$

$\rho'_0 = \frac{M_0 \rho M_0^\dagger}{Tr(M_0\rho)}$
$= \frac{|0\rangle\langle 0|\otimes I|\Phi^-\rangle\langle\Phi^-||0\rangle\langle 0|\otimes I}{\langle 0|\otimes I|\Phi^-\rangle\langle\Phi^-||0\rangle\otimes I}$
$= |00\rangle \langle 00|$
$= |0\rangle \langle 0| \otimes |0\rangle \langle 0|$

For system B, $\rho'_{0B} = |0\rangle \langle 0|$

$M_1 = |-\rangle \langle -|$

Observe $|-\rangle$ with probability $p_1 = Tr(M_1\rho) = Tr(|-\rangle \langle -| \otimes I\rho) = Tr(|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-|) = \langle +|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-| |+\rangle \otimes I + \langle -|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-| |-\rangle \otimes I = \langle +| \otimes I |\Phi^-\rangle \langle \Phi^-| |+\rangle \otimes I$

$\rho'_1 = \frac{M_1 \rho M_1^\dagger}{Tr(M_1\rho)}$
$= \frac{|-\rangle\langle -|\otimes I|\Phi^-\rangle\langle\Phi^-||-\rangle\langle -|\otimes I}{\langle +|\otimes I|\Phi^-\rangle\langle\Phi^-||+\rangle\otimes I}$
$= |--\rangle \langle --|$
$= |-\rangle \langle -| \otimes |-\rangle \langle -|$

For system B, $\rho'_{1B} = |-\rangle \langle -|$

2. What is the post measurement state if the measurements above were destructive and yielded the same outcomes as in part 1.? (2 points)

   Answer:

   For the outcome 0, $\rho''_0 = Tr(\rho'_{0B}) = Tr(|0\rangle\langle 0|) = 1$

   For the outcome -, $\rho''_1 = Tr(\rho'_{1B}) = Tr(|-\rangle\langle -|) = 1$

3. What is the reduced state ρ_A if we take state ρ_{AB} and trace over system B in basis $\{|0\rangle, |1\rangle\}$? What if we trace in $\{|+\rangle, |-\rangle\}$? Hint: for the second computation it could be useful to use the basis invariance of the trace operation. (6 points)

   Answer: $\rho_{AB} = |\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|) = \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)$

   $\rho_A = Tr_B(\rho_{AB}) = \frac{1}{2}(\langle 0|\otimes\langle 0|\rho_{AB}|0\rangle\otimes|0\rangle + \langle 1|\otimes\langle 1|\rho_{AB}|1\rangle\otimes 1)$
   $= \frac{1}{2}(\langle 0|\otimes\langle 0|(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)|0\rangle\otimes|0\rangle + \langle 1|\otimes\langle 1|(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)|1\rangle\otimes|1\rangle)$
   $= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

   In the basis $\{|+\rangle, |-\rangle\}$, $\Phi = \frac{1}{2}(|+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle)$

   $\rho_A = Tr_B$

4. Consider now the $2n$-qubit state $\rho = |\Psi_n\rangle\langle\Psi_n|$, with $|\Psi_n\rangle = \frac{1}{\sqrt{2^n}}\sum_i |i\rangle_A \otimes |i\rangle_B$. What is the reduced n-qubit state on Alice's side? (Hint: look at the state in vector form and factorize it in a smart way.) (8 points)

   Answer: $\rho = |\Psi_n\rangle\langle\Psi_n| = \frac{1}{2^n}\sum_i\sum_j |i\rangle_A \otimes |i\rangle_B \langle j|_A \otimes \langle j|_B = \frac{1}{2^n}\sum_i\sum_j |i\rangle_A\langle j|_A \otimes |i\rangle_B\langle j|_B = \frac{1}{2^n}\sum_i |i\rangle_A\langle i|_A \otimes \sum_j |j\rangle_B\langle j|_B$

   $\rho_A = Tr_B(\rho) = \frac{1}{2^n}\sum_i |i\rangle_A\langle i|_A \otimes \sum_j |j\rangle_B\langle j|_B$