Assignment 2

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1 EXERCISE 1: THE PARTIAL TRACE

We saw in the lectures that given a multi-partite state, we obtain the state of a subsystem by applying the partial trace to the other systems.

1. Compute the trace of the following states by applying the trace formula $Tr(A) = \sum_i \langle i| A |i\rangle$ T r(A) from the lectures:

   (a) $\xi = \frac{1}{2}(|0\rangle\langle 0| - i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0| + |1\rangle\langle 1|)$

   (b) $\Lambda = \frac{I}{3}I + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$ (here, I is the identity matrix).
   Which of the states is normalized correctly? (*) (4 points)

   Answer: $\xi = \frac{1}{2}(|0\rangle\langle 0| - i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0| + |1\rangle\langle 1|)$

   $= \frac{1}{2}(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$

   $= \frac{1}{2}(\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix})$

   $= \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$

   $Tr(\xi) = |0\rangle\,\xi\,\langle 0| + |1\rangle\,\xi\,\langle 1|$

   $= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   $= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{i}{2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{i}{2} \\ \frac{1}{2} \end{bmatrix}$

   $= \frac{1}{2} + \frac{1}{2}$

   $= 1$

   $\Lambda = \frac{I}{3}I + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$

   $$= \frac{I}{3} + \frac{1}{6}(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}) \\ = \frac{I}{3} + \frac{1}{6}(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) \\ = \frac{I}{3} + \frac{1}{6}(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) \\ = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{6} \\ \frac{1}{6} & 0 \end{bmatrix} \end{bmatrix}$$

   $= \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$ $Tr(\Lambda) = |0\rangle\,\Lambda\,\langle 0| + |1\rangle\,\Lambda\,\langle 1|$

   $= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   $= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \end{bmatrix}$

$= \frac{1}{3} + \frac{1}{3}$
$= \frac{2}{3}$

So ξ is normalized correctly.

2. In the last assignment we found the probability to find a state $|\gamma\rangle$ in another state $|\delta\rangle$ to be $p = |\langle\gamma|\delta\rangle|^2$. Show here that this expression coincides with $\mathrm{Tr}(\gamma\delta)$, for $\gamma = |\gamma\rangle\langle\gamma|$ and $\delta = |\delta\rangle\langle\delta|$. Hint: Apply the trace formula $\mathrm{Tr}(A) = \sum_i \langle i|A|i\rangle$ from the lectures. Use a suitable basis of your choice. (4 points)

   Answer: $\mathrm{Tr}(\gamma\delta)$
   $= \mathrm{Tr}(|\gamma\rangle\langle\gamma|\,|\delta\rangle\langle\delta|)$
   $= \mathrm{Tr}(|\gamma\rangle\langle\gamma|\delta\rangle\langle\delta|)$
   $= \mathrm{Tr}(\langle\gamma|\delta\rangle\,|\gamma\rangle\langle\delta|)$
   $= \langle\gamma|\delta\rangle\,\mathrm{Tr}(|\gamma\rangle\langle\delta|)$
   $= \langle\gamma|\delta\rangle \sum_i \langle i|\,|\gamma\rangle\langle\delta|\,|i\rangle$
   $= \langle\gamma|\delta\rangle\,(\langle 0|\,|\gamma\rangle\langle\delta|\,|0\rangle + \langle 1|\,|\gamma\rangle\langle\delta|\,|1\rangle)$
   $= \langle\gamma|\delta\rangle\,(\langle 0|\gamma\rangle\langle\delta|0\rangle + \langle 1|\gamma\rangle\langle\delta|1\rangle)$
   $= \langle\gamma|\delta\rangle\,(\langle\gamma|0\rangle\langle 0|\delta\rangle + \langle\gamma|1\rangle\langle 1|\delta\rangle)$
   $= \langle\gamma|\delta\rangle\,(\langle\gamma|\delta\rangle)$
   $= |\langle\gamma|\delta\rangle|^2$

   So, $\mathrm{Tr}(\gamma\delta) = |\langle\gamma|\delta\rangle|^2 = p$.

3. Consider the bipartite state $|\phi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i\,|01\rangle_{AB} - |11\rangle_{AB})$. Write the density operator $\rho_{AB} = |\phi\rangle_{AB}\langle\phi|_{AB}$ explicitly in matrix form. (*) (4 points)