Assignment 2

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October 15, 2023

1 EXERCISE 1: THE PARTIAL TRACE

We saw in the lectures that given a multi-partite state, we obtain the state of a subsystem by applying the partial trace to the other systems.

1. Compute the trace of the following states by applying the trace formula $Tr(A) = \sum_i \langle i| A |i\rangle$ from the lectures:

   (a) $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

   (b) $\Lambda = \frac{I}{3}I + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$ (here, I is the identity matrix).
   Which of the states is normalized correctly? (*) (4 points)

   Answer: $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

   $= \frac{1}{2}(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - i\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + i\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix})$

   $= \frac{1}{2}(\begin{bmatrix}1 & -i\\i & 1\end{bmatrix})$

   $= \begin{bmatrix}\frac{1}{2} & -\frac{i}{2}\\\frac{i}{2} & \frac{1}{2}\end{bmatrix}$

   $Tr(\xi) = |0\rangle\xi\langle 0| + |1\rangle\xi\langle 1|$

   $= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{i}{2}\\\frac{i}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{i}{2}\\\frac{i}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$

   $= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\\frac{i}{2}\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-\frac{i}{2}\\\frac{1}{2}\end{bmatrix}$

   $= \frac{1}{2} + \frac{1}{2}$

   $= 1$

   $\Lambda = \frac{I}{3}I + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$

   $= \frac{I}{3} + \frac{1}{6}(\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix})$

   $= \frac{I}{3} + \frac{1}{6}(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix})$

   $= \frac{I}{3} + \frac{1}{6}(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix})$

   $= \begin{bmatrix}0 & \frac{1}{3}\\\frac{1}{3} & 0\end{bmatrix} + \begin{bmatrix}0 & \frac{1}{6}\\\frac{1}{6} & 0\end{bmatrix}$

   $= \begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\\frac{1}{6} & \frac{1}{3}\end{bmatrix}$

   $Tr(\Lambda) = |0\rangle\Lambda\langle 0| + |1\rangle\Lambda\langle 1|$

   $= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\\frac{1}{6} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\\frac{1}{6} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$

   $= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{3}\\\frac{1}{6}\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{6}\\\frac{1}{3}\end{bmatrix}$

$= \frac{1}{3} + \frac{1}{3}$
$= \frac{2}{3}$
So ξ is normalized correctly.

2. In the last assignment we found the probability to find a state $|\gamma\rangle$ in another state $|\delta\rangle$ to be $p = |\langle\gamma|\delta\rangle|^2$. Show here that this expression coincides with $\mathrm{Tr}(\gamma\delta)$, for $\gamma = |\gamma\rangle\langle\gamma|$ and $\delta = |\delta\rangle\langle\delta|$. Hint: Apply the trace formula $\mathrm{Tr}(A) = \sum_i \langle i|A|i\rangle$ from the lectures. Use a suitable basis of your choice. (4 points)

   Answer: $\mathrm{Tr}(\gamma\delta)$
   $= \mathrm{Tr}(|\gamma\rangle\langle\gamma|\,|\delta\rangle\langle\delta|)$
   $= \mathrm{Tr}(|\gamma\rangle\langle\gamma|\delta\rangle\langle\delta|)$
   $= \mathrm{Tr}(\langle\gamma|\delta\rangle|\gamma\rangle\langle\delta|)$
   $= \langle\gamma|\delta\rangle\,\mathrm{Tr}(|\gamma\rangle\langle\delta|)$
   $= \langle\gamma|\delta\rangle\sum_i \langle i|\,|\gamma\rangle\langle\delta|\,|i\rangle$
   $= \langle\gamma|\delta\rangle\,(\langle 0|\,|\gamma\rangle\langle\delta|\,|0\rangle + \langle 1|\,|\gamma\rangle\langle\delta|\,|1\rangle)$
   $= \langle\gamma|\delta\rangle\,(\langle 0|\gamma\rangle\langle\delta|0\rangle + \langle 1|\gamma\rangle\langle\delta|1\rangle)$
   $= \langle\gamma|\delta\rangle\,(\langle\gamma|0\rangle\langle 0|\delta\rangle + \langle\gamma|1\rangle\langle 1|\delta\rangle)$
   $= \langle\gamma|\delta\rangle\,(\langle\gamma|\delta\rangle)$
   $= |\langle\gamma|\delta\rangle|^2$
   So, $\mathrm{Tr}(\gamma\delta) = |\langle\gamma|\delta\rangle|^2 = p$.

3. Consider the bipartite state $|\phi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i\,|01\rangle_{AB} - |11\rangle_{AB})$. Write the density operator $\rho_{AB} = |\phi\rangle_{AB}\langle\phi|_{AB}$ explicitly in matrix form. (*) (4 points)

   Answer: ρ_{AB}
   $= |\phi\rangle_{AB}\langle\phi|_{AB}$
   $= \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i\,|01\rangle_{AB} - |11\rangle_{AB})(\langle 00|_{AB} - i\,\langle 01|_{AB} - \langle 11|_{AB})$
   $= \frac{1}{3}(|00\rangle\langle 00| - i\,|00\rangle\langle 01| - |00\rangle\langle 11| + i\,|01\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 11| - |11\rangle\langle 00| - |11\rangle\langle 01| + |11\rangle\langle 11|)$
   $= \frac{1}{3}(\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} - i\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix} + i\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{bmatrix} -$
   $\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix})$
   $= \frac{1}{3}\begin{bmatrix}1&i&0&-1\\-i&1&0&1\\0&0&0&0\\-1&i&0&1\end{bmatrix}$
   $= \begin{bmatrix}\frac{1}{3}&\frac{i}{3}&0&-\frac{1}{3}\\-\frac{i}{3}&\frac{1}{3}&0&\frac{1}{3}\\0&0&0&0\\-\frac{1}{3}&\frac{i}{3}&0&\frac{1}{3}\end{bmatrix}$

4. Let σ_{AB} be a general 2-qubit state. The 4×4 matrix describing σ_{AB} can be split into four sub-matrices of size 2×2 (upper left block, upper right block, lower left block, lower right block). Show that the reduced single qubit state $\sigma_B = \mathrm{Tr}_A(\sigma_{AB})$ is described by a 2×2 matrix that is the sum of the upper left block and lower right block matrices of σ_{AB}. Hint: start from $\sigma_{AB} = \sigma_{00}|00\rangle\langle 00| + \sigma_{01}|00\rangle\langle 01| + \ldots + \sigma_{33}|11\rangle\langle 11|$ and compute the partial trace in bra-ket notation. Write σ_{AB} as a matrix and compare. (8 points)

   Answer: $\sigma_{AB} = \sigma_{00}\,|00\rangle\langle 00| + \sigma_{01}\,|00\rangle\langle 01| + \ldots + \sigma_{33}\,|11\rangle\langle 11|$

   $\sigma_B =$
   $= \mathrm{Tr}_A(\sigma_{AB})$

$= \sum_i \langle i|_A \sigma_{AB} |i\rangle_A$
$= \langle 0|_A \sigma_{AB} |0\rangle_A + \langle 1|_A \sigma_{AB} |1\rangle_A$
$= \langle 0|_A (\sigma_{00} |00\rangle_{AB} \langle 00| + \sigma_{01} |00\rangle_{AB} \langle 01|_{AB} + ... + \sigma_{33} |11\rangle_{AB} \langle 11|_{AB}) |0\rangle + \langle 1| (\sigma_{00} |00\rangle_{AB} \langle 00|_{AB} + \sigma_{01} |00\rangle_{AB} \langle 01|_{AB} + ... + \sigma_{33} |11\rangle_{AB} \langle 11|_{AB}) |1\rangle_A$
$= \sigma_{00} |0\rangle \langle 0| + \sigma_{01} |0\rangle \langle 1| + \sigma_{10} |1\rangle \langle 0| + \sigma_{11} |1\rangle \langle 1| + \sigma_{22} |0\rangle \langle 0| + \sigma_{23} |0\rangle \langle 1| + \sigma_{32} |1\rangle \langle 0| + \sigma_{33} |1\rangle \langle 1|$
$= (\sigma_{00} + \sigma_{22}) |0\rangle \langle 0| + (\sigma_{01} + \sigma_{23}) |0\rangle \langle 1| + (\sigma_{10} + \sigma_{32}) |1\rangle \langle 0| + (\sigma_{12} + \sigma_{33}) |1\rangle \langle 1|$

This is the sum of the upper left block and lower right block matrices of σ_{AB}.

5. Take the state ρ_{AB} from 3. and compute the reduced state ρ_B, both from the matrix itself (using your result from 4.) and in bra-ket notation. (*) (6 points)

   Answer: $\rho_{AB} = \begin{bmatrix} \frac{1}{3} & \frac{i}{3} & 0 & -\frac{1}{3} \\ -\frac{i}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{i}{3} & 0 & \frac{1}{3} \end{bmatrix}$

   $\rho_B = (\rho_{00} + \rho_{21}) |0\rangle \langle 0| + (\rho_{02} + \rho_{23}) |0\rangle \langle 1| + (\rho_{10} + \rho_{31}) |1\rangle \langle 0| + (\rho_{12} + \rho_{33}) |1\rangle \langle 1|$
   $= (\frac{1}{3} + 0) |0\rangle \langle 0| + (0 + 0) |0\rangle \langle 1| + (-\frac{i}{3} + \frac{i}{3}) |1\rangle \langle 0| + (\frac{1}{3} + \frac{1}{3}) |1\rangle \langle 1|$
   $= \frac{1}{3} |0\rangle \langle 0| + 0 |0\rangle \langle 1| + 0 |1\rangle \langle 0| + \frac{2}{3} |1\rangle \langle 1|$
   $= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$

2 EXERCISE 2: MEASUREMENTS AND REDUCED STATES

Let us look at the density operator $\rho_{AB} = |\Phi^-\rangle\langle\Phi^-|$, with $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(\langle 01| - \langle 10|)$.

1. Compute explicitly the post-measurement state after we performed a projective measurement $M = \{|0\rangle \langle 0|, |1\rangle \langle 1|\}$ on ρ_{AB} on system B, given that the outcome was 0. Compute the same for the measurement $M = \{|+\rangle\langle +|, |-\rangle\langle -|\}$, for outcome $-$. (Hint: Using Exercise 2.6 from Assignment 1 might help (not necessary though).) (8 points)

   Answer: $M_0 = |0\rangle \langle 0|$

   Observe $|0\rangle$ with probability $p_0 = Tr(M_0\rho) = Tr(|0\rangle \langle 0| \otimes I\rho) = Tr(|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-|) = \langle 0|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |0\rangle \otimes I + \langle 1|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |1\rangle \otimes I = \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |0\rangle \otimes I$

   $\rho'_0 = \frac{M_0 \rho M_0^\dagger}{Tr(M_0\rho)}$
   $= \frac{|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |0\rangle \langle 0| \otimes I}{\langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |0\rangle \otimes I}$
   $= |00\rangle \langle 00|$
   $= |0\rangle \langle 0| \otimes |0\rangle \langle 0|$

   For system B, $\rho'_{0B} = |0\rangle \langle 0|$

   $M_1 = |-\rangle \langle -|$

   Observe $|-\rangle$ with probability $p_1 = Tr(M_1\rho) = Tr(|-\rangle \langle -| \otimes I\rho) = Tr(|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-|) = \langle +|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-| |+\rangle \otimes I + \langle -|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-| |-\rangle \otimes I = \langle +| \otimes I |\Phi^-\rangle \langle \Phi^-| |+\rangle \otimes I$

   $\rho'_1 = \frac{M_1 \rho M_1^\dagger}{Tr(M_1\rho)}$
   $= \frac{|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-| |-\rangle \langle -| \otimes I}{\langle +| \otimes I |\Phi^-\rangle \langle \Phi^-| |+\rangle \otimes I}$
   $= |--\rangle \langle --|$
   $= |-\rangle \langle -| \otimes |-\rangle \langle -|$

   For system B, $\rho'_{1B} = |-\rangle \langle -|$

2. What is the post measurement state if the measurements above were destructive and yielded the same outcomes as in part 1.? (2 points)

   Answer:

   For the outcome 0, $\rho''_0 = Tr(\rho'_{0B}) = Tr(|0\rangle \langle 0|) = 1$

For the outcome -, $\rho_1'' = Tr(\rho_{1B}') = Tr(|-\rangle\langle -|) = 1$

3. What is the reduced state ρ_A if we take state ρ_{AB} and trace over system B in basis $\{|0\rangle, |1\rangle\}$? What if we trace in $\{|+\rangle, |-\rangle\}$? Hint: for the second computation it could be useful to use the basis invariance of the trace operation. (6 points)

   Answer: $\rho_{AB} = |\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|) = \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)$
   $\rho_A = Tr_B(\rho_{AB}) = \frac{1}{2}(\langle 0| \otimes \langle 0| \rho_{AB} |0\rangle \otimes |0\rangle + \langle 1| \otimes \langle 1| \rho_{AB} |1\rangle \otimes 1)$
   $= \frac{1}{2}(\langle 0|\otimes\langle 0| (|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) |0\rangle\otimes|0\rangle + \langle 1|\otimes\langle 1| (|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) |1\rangle \otimes |1\rangle)$
   $= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

   In the basis $\{|+\rangle, |-\rangle\}$, $\Phi = \frac{1}{2}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$

   $\rho_A = Tr_B$

4. Consider now the $2n$-qubit state $\rho = |\Psi_n\rangle\langle\Psi_n|$, with $|\Psi_n\rangle = \frac{1}{\sqrt{2^n}} \sum_i |i\rangle_A \otimes |i\rangle_B$. What is the reduced n-qubit state on Alice's side? (Hint: look at the state in vector form and factorize it in a smart way.) (8 points)

   Answer: $\rho = \frac{1}{2^n} \sum_i |i\rangle_A \otimes |i\rangle_B \sum_j \langle j|_A \langle j|_B$
   $= \frac{1}{2^n} \sum_i \sum_j |i\rangle_A \otimes |i\rangle_B \langle j|_A \otimes \langle j|_B$

   $Tr_B(\rho) = \frac{1}{2^n} \sum_k \sum_i \sum_j \langle k| |i\rangle \langle j| \otimes |i\rangle \langle j| |k\rangle$

   If $i \neq k$, $\langle k| |i\rangle = 0$, if $i \neq j$, $\langle i| |j\rangle = 0$, if $j \neq k$, $\langle j| |k\rangle = 0$

   $Tr_B(\rho) = \frac{1}{2^n} \sum_i \langle i| |i\rangle \otimes |i\rangle \langle i| \otimes \langle i| |k\rangle$
   $= \frac{1}{2^n} \sum_i |i\rangle\langle i|$
   $= \frac{I_n}{2^n}$

5. What is the reduced state on the first k qubits of A and B (i.e. tracing out the last $n-k$ qubits on each side)?(4 points)

   Answer: $\rho_{0tok} = Tr_{\rho_{kton}}(\rho_{AB}) = \sum_{m=0}^{n-k} \langle m| |\Psi_n\rangle \langle\Psi_n| |m\rangle$
   $= \frac{1}{2^n} \sum_{m=0}^{n-k} \sum_i \sum_j \langle m| |i\rangle_A \otimes |i\rangle_B \langle j|_A \otimes \langle j|_B |m\rangle$
   if $m \neq i$ or $m \neq j$, $\langle m| |i\rangle = 0$ or $\langle m| |j\rangle = 0$

   So, $\rho_{0tok} = \frac{1}{2^n} \sum_{i=0}^{n-k} |i\rangle\langle i| = \frac{I_{n-k}}{2^n}$

3 EXERCISE 3: EVOLUTIONS AND KRAUS OPERATORS

1. Consider the following channel C. It maps the classical state $|0\rangle$ to the state $|0\rangle$ with probability $(1-p)$ and to $|1\rangle$ with probability p. Symmetrically, the state $|1\rangle$ is mapped to the state $|1\rangle$ with probability $(1-p)$ and to $|0\rangle$ with probability p. Find a Kraus operator representation of the channel and show that your choice is valid, and that it maps the classical states $|0\rangle$ and $|1\rangle$ correctly. (4 points)

   Answer: $\{E_0, E_1, E_2, E_3\} = \{\sqrt{1-p}\,|0\rangle\langle 0|, \sqrt{p}\,|0\rangle\langle 1|, \sqrt{p}\,|1\rangle\langle 0|, \sqrt{1-p}\,|1\rangle\langle 1|\ \}$

   $\sum_i E_i^\dagger E_i = (1-p)\,|0\rangle\langle 0|0\rangle\langle 0| + p\,|1\rangle\langle 0|0\rangle\langle 1| + (1-p)\,|0\rangle\langle 1| |1\rangle\langle 0| + p\,|1\rangle\langle 1| |1\rangle\langle 1|$
   $= |0\rangle\langle 0| + |1\rangle\langle 1| = I$

   $C(\rho) = \sum_i E_i \rho E_i^\dagger$

   $C(|0\rangle\langle 0|) = (1-p)\,|0\rangle\langle 0| |0\rangle\langle 0| |0\rangle\langle 0| + p\,|1\rangle\langle 0| |0\rangle\langle 0| |0\rangle\langle 1| + p\,|0\rangle\langle 1| |0\rangle\langle 0| |1\rangle\langle 0| + (1-p)\,|1\rangle\langle 1| |0\rangle\langle 0| |1\rangle\langle 1|$
   $= (1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|$

   $C(|1\rangle\langle 1|) = (1-p)\,|0\rangle\langle 0| |1\rangle\langle 1| |0\rangle\langle 0| + p\,|1\rangle\langle 0| |1\rangle\langle 1| |0\rangle\langle 1| + p\,|0\rangle\langle 1| |1\rangle\langle 1| |1\rangle\langle 0| + (1-p)\,|1\rangle\langle 1| |1\rangle\langle 1| |1\rangle\langle 1|$
   $= p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$

2. Apply the classical channel C to a general quantum state $\rho = \sum_{i,j} \alpha_{i,j} |i\rangle \langle j|$ (using the Kraus operators found above) and demonstrate that the off-diagonal terms vanish.

   $C(\rho) = \sum_k E_k \rho E_k^\dagger$
   $= \sum_k E_k \sum_{i,j} \alpha_{i,j} |i\rangle \langle j| E_k^\dagger$
   $= (1-p) |0\rangle \langle 0| \sum_{i,j} \alpha_{i,j} |i\rangle \langle j| |0\rangle \langle 0| + p |0\rangle \langle 1| \sum_{i,j} \alpha_{i,j} |i\rangle \langle j| |1\rangle \langle 0| + p |1\rangle \langle 0| \sum_{i,j} \alpha_{i,j} |i\rangle \langle j| |0\rangle \langle 1| + (1-p) |1\rangle \langle 1| \sum_{i,j} \alpha_{i,j} |i\rangle \langle j| |1\rangle \langle 1|$

   If $i \neq k$ or $j \neq k$, $\langle k|i\rangle \langle j|k\rangle = 0$

   $C(\rho) = (1-p)\alpha_{00} |0\rangle \langle 0| + p\alpha_{11} |0\rangle \langle 0| + p\alpha_{00} |1\rangle \langle 1| + (1-p)\alpha_{11} |1\rangle \langle 1|$
   $= [(1-p)\alpha_{00} + p\alpha_{11}] |0\rangle \langle 0| + [(1-p)\alpha_{11} + p\alpha_{00}] |1\rangle \langle 1|$

   This equation shows that the off-diagonal terms vanish.

3. Can you think of a quantum version of the channel, which operates correctly on the coherent (off-diagonal) terms of ρ? Write it down and show that it is valid and correct. Hint: apply the new set of Kraus operators first to the classical states $|0\rangle$ and $|1\rangle$ to check correctness. Then, apply it to ρ and show that it preserves coherence. (8 points)

4 EXERCISE 4: THE GATE MODEL

Consider the following circuit diagram:

The initial states are the two qubits $|+\rangle$ and $|-\rangle$. The upper wire is initialized in state $|+\rangle$ and experiences a controlled X gate. The lower wire is initialized in state $|-\rangle$, then first transformed by the Hadamard gate, and afterwards acts as a control state for the controlled X gate. Finally, a measurement M measures each qubit in the computational basis.

1. What is the state of the system before the measurement M? (4 points)

   Answer: $H(|-\rangle) = (\frac{1}{\sqrt{2}} |0\rangle \langle 0| + \frac{1}{\sqrt{2}} |0\rangle \langle 1| + \frac{1}{\sqrt{2}} |1\rangle \langle 0| - \frac{1}{\sqrt{2}} |1\rangle \langle 1|) |-\rangle$
   $= \frac{1}{2}(|0\rangle + |1\rangle - |0\rangle + |1\rangle)$
   $= |1\rangle$

   So the output of gate H is $|1\rangle$, which means gate X will be applied to the upper wire.

   $X(|+\rangle) = (|0\rangle \langle 1| + |1\rangle \langle 0|) |+\rangle$
   $= (|0\rangle \langle 1| + |1\rangle \langle 0|) \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
   $= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   So the state before measurement is $|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

2. Which are the possible measurement outcomes of M and what are their probabilities? (4 points)

   Answer: The possible measurement outcomes of M are $|01\rangle$ and $|11\rangle$.

   $P(|01\rangle) = |\frac{1}{\sqrt{2}} \langle 01| |10\rangle + |11\rangle|^2 = \frac{1}{2}$

   $P(|11\rangle) = |\frac{1}{\sqrt{2}} \langle 11| |10\rangle + |11\rangle|^2 = \frac{1}{2}$

   The probability of $|01\rangle$ is $\frac{1}{2}$ and the probability of $|11\rangle$ is $\frac{1}{2}$.

   In the lectures we saw the controlled Z and controlled X gate, where $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. Are the following two circuits the same? How about when we replace Z with X? (6 points)

   Answer: Assume the input of the upper wire is $|\psi\rangle_A$ and the input of the lower wire is $|\psi\rangle_B$.

   When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 0$, for the first circuit, the upper output is $|0\rangle$, and Z gate is not activated, the lower output is $|0\rangle$. For the second circuit, the output of lower wire is $|0\rangle$, the Z gate is not activated, the output of upper wire is $|0\rangle$.

When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 1$, for the first circuit, the upper output is $|0\rangle$, and Z gate is not activated, the lower output is $|1\rangle$. For the second circuit, the output of lower wire is $|1\rangle$, the Z gate is activated, the output of upper wire is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle = |0\rangle$.

When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 0$, for the first circuit, the upper output is $|1\rangle$, and Z gate is activated, the lower output is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle = |0\rangle$. For the second circuit, the output of lower wire is $|0\rangle$, the Z gate is activated, the output of upper wire is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle = |1\rangle$.

When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 1$, for the first circuit, the upper output is $|1\rangle$, and Z gate is activated, the lower output is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle = |1\rangle$. For the second circuit, the output of lower wire is $|1\rangle$, the Z gate is activated, the output of upper wire is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle = |1\rangle$.

So for gate Z, the two circuits are the same.

If we replace Z with X, the two circuits are not the same. When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 0$, the output of the upper wire of the first circuit is $|0\rangle$ and the output of the lower wire is $|0\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|0\rangle$. When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 1$, the output of the upper wire of the first circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$. When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 0$, the output of the upper wire of the first circuit is $|1\rangle$ and the output of the lower wire is $|1\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$. When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 1$, the output of the upper wire of the first circuit is $|1\rangle$ and the output of the lower wire is $|0\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$.

So for gate X, the two circuits are not the same.

4. The SWAP gate acts on a quantum state $|i\rangle \otimes |j\rangle$ as SWAP $|ij\rangle = |ji\rangle$. Show that the following circuit implements the SWAP gate (the symbol L is often used in the literature for the X gate). (8 points)