

Assignment 2

Wang Dingrui

October 15, 2023

1 EXERCISE 1: THE PARTIAL TRACE

We saw in the lectures that given a multi-partite state, we obtain the state of a subsystem by applying the partial trace to the other systems.

1. Compute the trace of the following states by applying the trace formula $\text{Tr}(A) = \sum_i \langle i| A |i\rangle$ from the lectures:

(a) $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

(b) $\Lambda = \frac{I}{3} + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$ (here, I is the identity matrix).

Which of the states is normalized correctly? (*) (4 points)

Answer: $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

$$\text{Tr}(\xi) = \langle 0|\xi|0\rangle + \langle 1|\xi|1\rangle = \frac{1}{2}(\langle 0|0\rangle\langle 0|0\rangle - i\langle 0|1\rangle\langle 0|0\rangle + i\langle 0|1\rangle\langle 0|0\rangle + \langle 0|1\rangle\langle 1|0\rangle + \langle 1|0\rangle\langle 0|1\rangle - i\langle 1|0\rangle\langle 1|1\rangle + i\langle 1|1\rangle\langle 0|1\rangle + \langle 1|1\rangle\langle 1|1\rangle) = \frac{1}{2}(1 + 1) = 1$$

$$\Lambda = \frac{I}{3} + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|) = \frac{1}{3}|0\rangle\langle 0| + \frac{1}{6}|0\rangle\langle 1| + \frac{1}{6}|1\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$$

$$\begin{aligned}\text{Tr}(\Lambda) &= \langle 0|\Lambda|0\rangle + \langle 1|\Lambda|1\rangle \\ &= \frac{1}{3}\langle 0|0\rangle\langle 0|0\rangle + \frac{1}{6}\langle 0|0\rangle\langle 1|0\rangle + \frac{1}{6}\langle 0|1\rangle\langle 0|0\rangle + \frac{1}{3}\langle 0|1\rangle\langle 1|0\rangle + \frac{1}{3}\langle 1|0\rangle\langle 0|1\rangle + \frac{1}{6}\langle 1|0\rangle\langle 1|1\rangle + \frac{1}{6}\langle 1|1\rangle\langle 0|1\rangle + \frac{1}{3}\langle 1|1\rangle\langle 1|1\rangle \\ &= \frac{1}{3}(1 + 1) \\ &= \frac{2}{3}\end{aligned}$$

So ξ is normalized correctly.

2. In the last assignment we found the probability to find a state $|\gamma\rangle$ in another state $|\delta\rangle$ to be $p = |\langle\gamma|\delta\rangle|^2$. Show here that this expression coincides with $\text{Tr}(\gamma\delta)$, for $\gamma = |\gamma\rangle\langle\gamma|$ and $\delta = |\delta\rangle\langle\delta|$. Hint: Apply the trace formula $\text{Tr}(A) = \sum_i \langle i|A|i\rangle$ from the lectures. Use a suitable basis of your choice. (4 points)

$$\begin{aligned}\text{Answer: } \text{Tr}(\gamma\delta) &= \text{Tr}(|\gamma\rangle\langle\gamma||\delta\rangle\langle\delta|) \\ &= \text{Tr}(|\gamma\rangle\langle\gamma|\delta\rangle\langle\delta|) \\ &= \text{Tr}(\langle\gamma|\delta\rangle|\gamma\rangle\langle\delta|) \\ &= \langle\gamma|\delta\rangle\text{Tr}(|\gamma\rangle\langle\delta|) \\ &= \langle\gamma|\delta\rangle\sum_i \langle i||\gamma\rangle\langle\delta||i\rangle \\ &= \langle\gamma|\delta\rangle(\langle 0||\gamma\rangle\langle\delta||0\rangle + \langle 1||\gamma\rangle\langle\delta||1\rangle) \\ &= \langle\gamma|\delta\rangle(\langle 0|\gamma\rangle\langle\delta|0\rangle + \langle 1|\gamma\rangle\langle\delta|1\rangle) \\ &= \langle\gamma|\delta\rangle(\langle\gamma|0\rangle\langle 0|\delta\rangle + \langle\gamma|1\rangle\langle 1|\delta\rangle) \\ &= \langle\gamma|\delta\rangle\langle\gamma|\delta\rangle \\ &= |\langle\gamma|\delta\rangle|^2\end{aligned}$$

$$\text{So, } \text{Tr}(\gamma\delta) = |\langle\gamma|\delta\rangle|^2 = p.$$

3. Consider the bipartite state $|\phi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i|01\rangle_{AB} - |11\rangle_{AB})$. Write the density operator $\rho_{AB} = |\phi\rangle_{AB}\langle\phi|_{AB}$ explicitly in matrix form. (*) (4 points)

Answer: $\phi_{AB} = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i|01\rangle_{AB} - |11\rangle_{AB})$

$$= \frac{1}{\sqrt{3}} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ i \\ 0 \\ -1 \end{bmatrix}$$

$$\rho_{AB}$$

$$= |\phi\rangle_{AB} \langle \phi|_{AB}$$

$$= \frac{1}{3} \begin{bmatrix} 1 \\ i \\ 0 \\ -1 \end{bmatrix} [1 \quad -i \quad 0 \quad -1]$$

$$= \frac{1}{3} \begin{bmatrix} 1 & -i & 0 & -1 \\ i & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{i}{3} & 0 & -\frac{1}{3} \\ \frac{i}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

4. Let σ_{AB} be a general 2-qubit state. The 4×4 matrix describing σ_{AB} can be split into four sub-matrices of size 2×2 (upper left block, upper right block, lower left block, lower right block). Show that the reduced single qubit state $\sigma_B = \text{Tr}_A(\sigma_{AB})$ is described by a 2×2 matrix that is the sum of the upper left block and lower right block matrices of σ_{AB} . Hint: start from $\sigma_{AB} = \sigma_{00}|00\rangle\langle 00| + \sigma_{01}|00\rangle\langle 01| + \dots + \sigma_{33}|11\rangle\langle 11|$ and compute the partial trace in bra-ket notation. Write σ_{AB} as a matrix and compare. (8 points)

Answer: $\sigma_{AB} = \sigma_{00}|00\rangle\langle 00| + \sigma_{01}|00\rangle\langle 01| + \dots + \sigma_{33}|11\rangle\langle 11|$

$$\sigma_B =$$

$$= \text{Tr}_A(\sigma_{AB})$$

$$= \sum_i \langle i|_A \sigma_{AB} |i\rangle_A$$

$$= \langle 0|_A \sigma_{AB} |0\rangle_A + \langle 1|_A \sigma_{AB} |1\rangle_A$$

$$= \langle 0|_A (\sigma_{00}|00\rangle_{AB} \langle 00| + \sigma_{01}|00\rangle_{AB} \langle 01|_{AB} + \dots + \sigma_{33}|11\rangle_{AB} \langle 11|_{AB}) |0\rangle + \langle 1| (\sigma_{00}|00\rangle_{AB} \langle 00|_{AB} + \sigma_{01}|00\rangle_{AB} \langle 01|_{AB} + \dots + \sigma_{33}|11\rangle_{AB} \langle 11|_{AB}) |1\rangle_A$$

$$= \sigma_{00}|0\rangle\langle 0| + \sigma_{01}|0\rangle\langle 1| + \sigma_{10}|1\rangle\langle 0| + \sigma_{11}|1\rangle\langle 1| + \sigma_{22}|0\rangle\langle 0| + \sigma_{23}|0\rangle\langle 1| + \sigma_{32}|1\rangle\langle 0| + \sigma_{33}|1\rangle\langle 1|$$

$$= (\sigma_{00} + \sigma_{22})|0\rangle\langle 0| + (\sigma_{01} + \sigma_{23})|0\rangle\langle 1| + (\sigma_{10} + \sigma_{32})|1\rangle\langle 0| + (\sigma_{11} + \sigma_{33})|1\rangle\langle 1|$$

This is the sum of the upper left block and lower right block matrices of σ_{AB} .

5. Take the state ρ_{AB} from 3. and compute the reduced state ρ_B , both from the matrix itself (using your result from 4.) and in bra-ket notation. (*) (6 points)

Answer: $\rho_{AB} = \begin{bmatrix} \frac{1}{3} & -\frac{i}{3} & 0 & -\frac{1}{3} \\ \frac{i}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$

$$\rho_B = (\sigma_{00} + \sigma_{22})|0\rangle\langle 0| + (\sigma_{01} + \sigma_{23})|0\rangle\langle 1| + (\sigma_{10} + \sigma_{32})|1\rangle\langle 0| + (\sigma_{11} + \sigma_{33})|1\rangle\langle 1|$$

$$= \frac{1}{3}|0\rangle\langle 0| - \frac{i}{3}|0\rangle\langle 1| + \frac{i}{3}|1\rangle\langle 0| + 0|1\rangle\langle 1|$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{i}{3} \\ \frac{i}{3} & \frac{1}{3} \end{bmatrix}$$

2 EXERCISE 2: MEASUREMENTS AND REDUCED STATES

Let us look at the density operator $\rho_{AB} = |\Phi^-\rangle\langle\Phi^-|$, with $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

1. Compute explicitly the post-measurement state after we performed a projective measurement $M = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on ρ_{AB} on system B , given that the outcome was 0. Compute the same for the measurement $M = \{|+\rangle\langle +|, |-\rangle\langle -|\}$, for outcome $-$. (Hint: Using Exercise 2.6 from Assignment 1 might help (not necessary though).) (8 points)

Answer: $M_0 = |0\rangle\langle 0|$

Observe $|0\rangle$ with probability $p_0 = \text{Tr}(M_0\rho) = \text{Tr}(I \otimes |0\rangle\langle 0| \rho) = \text{Tr}(I \otimes |0\rangle\langle 0| |\Phi^-\rangle\langle\Phi^-|) = \text{Tr}(I \otimes |0\rangle\langle 0| \frac{1}{2}(|01\rangle\langle 10| - |01\rangle\langle 01| - |10\rangle\langle 10| + |10\rangle\langle 01|)) = \text{Tr}(\frac{1}{2}(-|10\rangle\langle 10| + |10\rangle\langle 01|)) = -\frac{1}{2}$

$$\begin{aligned}\rho'_0 &= \frac{M_0\rho M_0^\dagger}{\text{Tr}(M_0\rho)} \\ &= \frac{I \otimes |0\rangle\langle 0| |\Phi^-\rangle\langle\Phi^-| I \otimes |0\rangle\langle 0|}{\text{Tr}(\frac{1}{2}(-|10\rangle\langle 10| + |10\rangle\langle 01|))} \\ &= \frac{\frac{1}{2}(-|10\rangle\langle 10| + |10\rangle\langle 01|) I \otimes |0\rangle\langle 0|}{-\frac{1}{2}} \\ &= |10\rangle\langle 10|\end{aligned}$$

For system B, $\rho'_{0B} = |0\rangle\langle 0|$

$$M_1 = |-\rangle\langle -|$$

Observe $|-\rangle$ with probability $p_1 = \text{Tr}(M_1\rho) = \text{Tr}(|-\rangle\langle -| \otimes I \rho) = \text{Tr}(|-\rangle\langle -| \otimes I |\Phi^-\rangle\langle\Phi^-|) = \langle +|-\rangle\langle -| \otimes I |\Phi^-\rangle\langle\Phi^-| |+\rangle \otimes I + \langle -|-\rangle\langle -| \otimes I |\Phi^-\rangle\langle\Phi^-| |-\rangle \otimes I = \langle +| \otimes I |\Phi^-\rangle\langle\Phi^-| |+\rangle \otimes I$

$$\begin{aligned}\rho'_1 &= \frac{M_1\rho M_1^\dagger}{\text{Tr}(M_1\rho)} \\ &= \frac{|-\rangle\langle -| \otimes I |\Phi^-\rangle\langle\Phi^-| |-\rangle\langle -| \otimes I}{\langle +| \otimes I |\Phi^-\rangle\langle\Phi^-| |+\rangle \otimes I} \\ &= |--\rangle\langle --| \\ &= |-\rangle\langle -| \otimes |-\rangle\langle -|\end{aligned}$$

For system B, $\rho'_{1B} = |-\rangle\langle -|$

2. What is the post measurement state if the measurements above were destructive and yielded the same outcomes as in part 1.? (2 points)

Answer:

$$\text{For the outcome 0, } \rho''_0 = \frac{M_0\rho'_0 M_0^\dagger}{\text{Tr}(|0\rangle\langle 0| \rho'_0)} = \frac{|0\rangle\langle 0| \rho'_0 |0\rangle\langle 0|}{1} = |0\rangle\langle 0|$$

$$\text{For the outcome -, } \rho''_1 = \frac{M_1\rho'_1 M_1^\dagger}{\text{Tr}(|-\rangle\langle -| \rho'_1)} = \frac{|-\rangle\langle -| \rho'_1 |-\rangle\langle -|}{1} = |-\rangle\langle -|$$

3. What is the reduced state ρ_A if we take state ρ_{AB} and trace over system B in basis $\{|0\rangle, |1\rangle\}$? What if we trace in $\{|+\rangle, |-\rangle\}$? Hint: for the second computation it could be useful to use the basis invariance of the trace operation. (6 points)

Answer: $\rho_{AB} = |\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 10| - \langle 01|) = \frac{1}{2}(|01\rangle\langle 10| - |01\rangle\langle 01| - |10\rangle\langle 10| + |10\rangle\langle 01|)$

$$\rho_A = \text{Tr}_B(\rho_{AB}) = \langle 0|_B \rho_{AB} |0\rangle_B + \langle 1|_B \rho_{AB} |1\rangle_B = \frac{1}{2}|0\rangle\langle 0| - \frac{1}{2}|1\rangle\langle 1|$$

In the basis $\{|+\rangle, |-\rangle\}$, $\Phi = \frac{1}{2}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$

$$\begin{aligned}\rho_A &= \text{Tr}_B(\rho_{AB}) = \langle +|_B \rho_{AB} |+\rangle_B + \langle -|_B \rho_{AB} |-\rangle_B \\ &= \frac{1}{2}(\langle 0|_B + \langle 1|_B) \rho_{AB} (|0\rangle_B + |1\rangle_B) + \frac{1}{2}(\langle 0|_B - \langle 1|_B) \rho_{AB} (|0\rangle_B - |1\rangle_B) \\ &= \frac{1}{2}(\langle 0|_B \rho_{AB} |0\rangle_B + \langle 0|_B \rho_{AB} |1\rangle_B + \langle 1|_B \rho_{AB} |0\rangle_B + \langle 1|_B \rho_{AB} |1\rangle_B + \langle 0|_B \rho_{AB} |0\rangle_B - \langle 0|_B \rho_{AB} |1\rangle_B - \langle 1|_B \rho_{AB} |0\rangle_B + \langle 1|_B \rho_{AB} |1\rangle_B) \\ &= \frac{1}{2}(2\langle 0|_B \rho_{AB} |0\rangle_B + 2\langle 1|_B \rho_{AB} |1\rangle_B) \\ &= \langle 0|_B \rho_{AB} |0\rangle_B + \langle 1|_B \rho_{AB} |1\rangle_B \\ &= \frac{1}{2}|0\rangle\langle 0| - \frac{1}{2}|1\rangle\langle 1|\end{aligned}$$

4. Consider now the $2n$ -qubit state $\rho = |\Psi_n\rangle\langle\Psi_n|$, with $|\Psi_n\rangle = \frac{1}{\sqrt{2^n}} \sum_i |i\rangle_A \otimes |i\rangle_B$. What is the reduced n -qubit state on Alice's side? (Hint: look at the state in vector form and factorize it in a smart way.) (8 points)

$$\begin{aligned}\text{Answer: } \rho &= \frac{1}{2^n} \sum_i |i\rangle_A \otimes |i\rangle_B \langle i|_A \langle i|_B \\ &= \frac{1}{2^n} \sum_i |i\rangle_A \otimes |i\rangle_B \langle i|_A \otimes \langle i|_B \\ \text{Tr}_B(\rho) &= \frac{1}{2^n} \sum_k \sum_i \langle k| |i\rangle \langle i| \otimes |i\rangle \langle i| |k\rangle \\ \text{If } i \neq k, \langle k| |i\rangle &= 0 \\ \text{Tr}_B(\rho) &= \frac{1}{2^n} \sum_i \langle i| |i\rangle \otimes |i\rangle \langle i| \otimes \langle i| |i\rangle \\ &= \frac{1}{2^n} \sum_i |i\rangle \langle i| \\ &= \frac{I_n}{2^n}\end{aligned}$$

5. What is the reduced state on the first k qubits of A and B (i.e. tracing out the last $n - k$ qubits on each side)? (4 points)

$$\begin{aligned}\text{Answer: } \rho_{0 \rightarrow k-1} &= \text{Tr}_{\rho_{k \rightarrow n-1}}(\rho_{AB}) = \sum_{m=k}^{n-1} \langle m| |\Psi_n\rangle \langle \Psi_n| |m\rangle \\ &= \frac{1}{2^n} \sum_{m=k}^{n-1} \sum_i \langle m| |i\rangle_A \otimes |i\rangle_B \langle i|_A \otimes \langle i|_B |m\rangle \\ \text{if } m \neq i, \langle m| |i\rangle &= 0. \\ \text{So, } \rho_{0 \rightarrow k} &= \frac{1}{2^n} \sum_{i=k}^{n-1} |i\rangle \langle i| = \frac{I_{n-k-1}}{2^n}\end{aligned}$$

3 EXERCISE 3: EVOLUTIONS AND KRAUS OPERATORS

1. Consider the following channel C . It maps the classical state $|0\rangle$ to the state $|0\rangle$ with probability $(1-p)$ and to $|1\rangle$ with probability p . Symmetrically, the state $|1\rangle$ is mapped to the state $|1\rangle$ with probability $(1-p)$ and to $|0\rangle$ with probability p . Find a Kraus operator representation of the channel and show that your choice is valid, and that it maps the classical states $|0\rangle$ and $|1\rangle$ correctly. (4 points)

$$\begin{aligned}\text{Answer: } \{E_0, E_1, E_2, E_3\} &= \{\sqrt{1-p}|0\rangle\langle 0|, \sqrt{p}|0\rangle\langle 1|, \sqrt{p}|1\rangle\langle 0|, \sqrt{1-p}|1\rangle\langle 1|\} \\ \sum_i E_i^\dagger E_i &= (1-p)|0\rangle\langle 0| + p|1\rangle\langle 0| + (1-p)|0\rangle\langle 1| + p|1\rangle\langle 1| = I \\ C(\rho) &= \sum_i E_i \rho E_i^\dagger \\ C(|0\rangle\langle 0|) &= (1-p)|0\rangle\langle 0| + p|1\rangle\langle 0| \\ C(|1\rangle\langle 1|) &= (1-p)|0\rangle\langle 1| + p|1\rangle\langle 1| \\ \text{Assume a coherent state } \rho &= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)\end{aligned}$$

So, this Kraus operator representation of the channel is valid, and it maps the classical states $|0\rangle$ and $|1\rangle$ correctly. $C(\rho) = \sum_i E_i \rho E_i^\dagger = \frac{1}{2}((1-p)|0\rangle\langle 0| + p|0\rangle\langle 0| + p|1\rangle\langle 1| + (1-p)|1\rangle\langle 1|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

2. Apply the classical channel C to a general quantum state $\rho = \sum_{i,j} \alpha_{i,j} |i\rangle\langle j|$ (using the Kraus operators found above) and demonstrate that the off-diagonal terms vanish.

$$\begin{aligned}C(\rho) &= \sum_k E_k \rho E_k^\dagger \\ &= \sum_k E_k \sum_{i,j} \alpha_{i,j} |i\rangle\langle j| E_k^\dagger \\ &= (1-p)|0\rangle\langle 0| \sum_{i,j} \alpha_{i,j} |i\rangle\langle j| + p|0\rangle\langle 1| \sum_{i,j} \alpha_{i,j} |i\rangle\langle j| + p|1\rangle\langle 0| \sum_{i,j} \alpha_{i,j} |i\rangle\langle j| + (1-p)|1\rangle\langle 1| \sum_{i,j} \alpha_{i,j} |i\rangle\langle j| \\ \text{If } i \neq k \text{ or } j \neq k, \langle k| |i\rangle \langle j| |k\rangle &= 0 \\ C(\rho) &= (1-p)\alpha_{00}|0\rangle\langle 0| + p\alpha_{00}|0\rangle\langle 1| + (1-p)\alpha_{11}|1\rangle\langle 1| \\ &= [(1-p)\alpha_{00} + p\alpha_{11}]|0\rangle\langle 0| + [(1-p)\alpha_{11} + p\alpha_{00}]|1\rangle\langle 1|\end{aligned}$$

This equation shows that the off-diagonal terms vanish.

3. Can you think of a quantum version of the channel, which operates correctly on the coherent (off-diagonal) terms of ρ ? Write it down and show that it is valid and correct. Hint: apply the new set of Kraus operators first to the classical states $|0\rangle$ and $|1\rangle$ to check correctness. Then, apply it to ρ and show that it preserves coherence. (8 points)

Answer: $\{E_0, E_1\} = \{|0\rangle\langle 1|, |1\rangle\langle 0|\}$

$$\sum_i E_i^\dagger E_i = |1\rangle\langle 0|0\rangle\langle 1| + |0\rangle\langle 1|1\rangle\langle 0| = |1\rangle\langle 1| + |0\rangle\langle 0| = I$$

$$C(|0\rangle\langle 0|) = E_0 |0\rangle\langle 0| E_0^\dagger + E_1 |0\rangle\langle 0| E_1^\dagger = |1\rangle\langle 1|$$

$$C(|1\rangle\langle 1|) = E_0 |1\rangle\langle 1| E_0^\dagger + E_1 |1\rangle\langle 1| E_1^\dagger = |0\rangle\langle 0|$$

For a coherent state $\rho = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$

$$C(\rho) = \sum_i E_i \rho E_i^\dagger = |0\rangle\langle 1| \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) |1\rangle\langle 0| + |1\rangle\langle 0| \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) |1\rangle\langle 1| + |1\rangle\langle 1| \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) |0\rangle\langle 1| + |0\rangle\langle 1| \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) |0\rangle\langle 0|$$

4 EXERCISE 4: THE GATE MODEL

Consider the following circuit diagram:

The initial states are the two qubits $|+\rangle$ and $|-\rangle$. The upper wire is initialized in state $|+\rangle$ and experiences a controlled X gate. The lower wire is initialized in state $|-\rangle$, then first transformed by the Hadamard gate, and afterwards acts as a control state for the controlled X gate. Finally, a measurement M measures each qubit in the computational basis.

1. What is the state of the system before the measurement M ? (4 points)

$$\begin{aligned} \text{Answer: } H(|-\rangle) &= \left(\frac{1}{\sqrt{2}}|0\rangle\langle 0| + \frac{1}{\sqrt{2}}|0\rangle\langle 1| + \frac{1}{\sqrt{2}}|1\rangle\langle 0| - \frac{1}{\sqrt{2}}|1\rangle\langle 1|\right)|-\rangle \\ &= \frac{1}{2}(|0\rangle + |1\rangle - |0\rangle + |1\rangle) \\ &= |1\rangle \end{aligned}$$

So the output of gate H is $|1\rangle$, which means gate X will be applied to the upper wire.

$$\begin{aligned} X(|+\rangle) &= (|0\rangle\langle 1| + |1\rangle\langle 0|)|+\rangle \\ &= (|0\rangle\langle 1| + |1\rangle\langle 0|)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \end{aligned}$$

So the state before measurement is $|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

2. Which are the possible measurement outcomes of M and what are their probabilities? (4 points)

Answer: The possible measurement outcomes of M are $|01\rangle$ and $|11\rangle$.

$$P(|01\rangle) = \left|\frac{1}{\sqrt{2}}\langle 01| |10\rangle + |11\rangle\right|^2 = \frac{1}{2}$$

$$P(|11\rangle) = \left|\frac{1}{\sqrt{2}}\langle 11| |10\rangle + |11\rangle\right|^2 = \frac{1}{2}$$

The probability of $|01\rangle$ is $\frac{1}{2}$ and the probability of $|11\rangle$ is $\frac{1}{2}$.

In the lectures we saw the controlled Z and controlled X gate, where $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. Are the following two circuits the same? How about when we replace Z with X ? (6 points)

Answer: Assume the input of the upper wire is $|\psi\rangle_A$ and the input of the lower wire is $|\psi\rangle_B$.

When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 0$, for the first circuit, the upper output is $|0\rangle$, and Z gate is not activated, the lower output is $|0\rangle$. For the second circuit, the output of lower wire is $|0\rangle$, the Z gate is not activated, the output of upper wire is $|0\rangle$.

When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 1$, for the first circuit, the upper output is $|0\rangle$, and Z gate is not activated, the lower output is $|1\rangle$. For the second circuit, the output of lower wire is $|1\rangle$, the Z gate is activated, the output of upper wire is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle = |0\rangle$.

When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 0$, for the first circuit, the upper output is $|1\rangle$, and Z gate is activated, the lower output is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle = |0\rangle$. For the second circuit, the output of lower wire is $|0\rangle$, the Z gate is activated, the output of upper wire is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle = |1\rangle$.

When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 1$, for the first circuit, the upper output is $|1\rangle$, and Z gate is activated, the lower output is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle = |1\rangle$. For the second circuit, the output of lower wire is $|1\rangle$, the Z gate is activated, the output of upper wire is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle = |1\rangle$.

So for gate Z, the two circuits are the same.

If we replace Z with X, the two circuits are not the same. When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 0$, the output of the upper wire of the first circuit is $|0\rangle$ and the output of the lower wire is $|0\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|0\rangle$. When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 1$, the output of the upper wire of the first circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$. When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 0$, the output of the upper wire of the first circuit is $|1\rangle$ and the output of the lower wire is $|1\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$. When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 1$, the output of the upper wire of the first circuit is $|1\rangle$ and the output of the lower wire is $|0\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$.

So for gate X, the two circuits are not the same.

4. The SWAP gate acts on a quantum state $|i\rangle \otimes |j\rangle$ as $\text{SWAP } |ij\rangle = |ji\rangle$. Show that the following circuit implements the SWAP gate (the symbol L is often used in the literature for the X gate). (8 points)

Answer: Assume the input of the upper wire is $|\psi\rangle_A$ and the input of the lower wire is $|\psi\rangle_B$.

When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 0$, the output of the upper wire is $|0\rangle$ and the output of the lower wire is $|0\rangle$.

When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 1$, the output of the upper wire is $|1\rangle$ and the output of the lower wire is $|0\rangle$.

When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 0$, the output of the upper wire is $|0\rangle$ and the output of the lower wire is $|1\rangle$.

When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 1$, the output of the upper wire is $|1\rangle$ and the output of the lower wire is $|1\rangle$.