Assignment 1

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EXERCISE 1: COMPLEX NUMBERS

1. Assume $x = a_1 + b_1 i$, $x = a_2 + b_2 i$, $x = a_3 + b_3 i$,

   we have $x + y + z = a_1 + a_2 + a_3 + (b_1 + b_2 + b_3)i$,

   To a complex number $n = a + bi$ we have $n^* = a - bi$,

   so

   $|x|^2 = x \cdot x^* = a_1^2 - b_1^2(i)^2 = a_1^2 - b_1^2 \cdot (-1) = a_1^2 + b_1^2$,

   x^*y
   $= (a_1 - b_1 i) \cdot (a_2 + b_2 i)$
   $= a_1a_2 + a_1b_2 i - a_2b_1 i - b_1b_2 i^2$
   $= a_1a_2 + b_1b_2 + (a_1b_2 - a_2b_1)i$,

   $Re(x^*y) = a_1a_2 + b_1b_2$,

   similarly, we have

   $|y|^2 = a_2^2 + b_2^2$,

   $|z|^2 = a_3^2 + b_3^2$,

   $Re(y^*z) = a_2a_3 + b_2b_3$

   $Re(x^*z) = a_1a_3 + b_1b_3$

   and

   $(x + y + z)^* = a_1 + a_2 + a_3 - (b_1 + b_2 + b_3)i$,

   $|x + y + z|^2$
   $= (x + y + z) \cdot (x + y + z)^*$
   $= ((a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i) \cdot ((a_1 + a_2 + a_3) - (b_1 + b_2 + b_3)i)$
   $= ((a_1 + a_2 + a_3)^2 - (b_1 + b_2 + b_3)^2 \cdot (-1))$
   $= ((a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2)$
   $= a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 + 2a_1a_3 + 2a_2a_3 + b_1^2 + b_2^2 + b_3^2 + 2b_1b_2 + 2b_1b_3 + 2b_2b_3$
   $= a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2 + 2(a_1a_2 + b_1b_2 + a_2a_3 + b_2b_3 + a_1a_3 + b_1b_3)$
   $= |x|^2 + |y|^2 + |z|^2 + 2[Re(x^*y) + Re(y^*z) + Re(x^*z)]$

   This shows that

   $|x + y + z|^2 = |x|^2 + |y|^2 + |z|^2 + 2[Re(x^*y) + Re(y^*z) + Re(x^*z)]$

2. $(i+2)(3-4i)/(2-i)$
$= (3i - 4i^2 + 2*3 - 2*4i)/(2-i)$
$= (3i - 4 \times (-1) + 2*3 - 2*4i)/(2-i)$
$= (3i + 4 + 6 - 8i)/(2-i)$
$= (10 - 5i)/(2-i)$
$= 5(2-i)/(2-i)$
$= 5$

3. $(i-4)/(2i-3)$
$= [(i-4)(2i+3)]/[(2i-3)(2i+3)]$
$= (2i^2 + 3i - 8i - 4*3)/((2i)^2 - 3*3)$
$= [2 \times (-1) + 3i - 8i - 4*3]/[4 \times (-1) - 3*3]$
$= (-2 - 5i - 12)/(-4-9)$
$= (-14 - 5i)/(-13)$
$= [(-1)(14+5i)]/(-1 \times 13)$
$= (14+5i)/13$
$= 14/13 + 5/13i$

so, the real part is 14/13 and imaginary pary is 5/13.

4. i^{33}
$= i^{32} i$
$= i^{2\times16} i$
$= (i^2)^{16} i$
$= (-1)^{16} i$
$= i$

so, the absolute value of i^{33} is $|i|$
$|i| = |0+i| = \sqrt{|0+i|^2} = \sqrt{(0+i)(0+i)^*} = \sqrt{(0+i)(0-i)} = \sqrt{-i^2} = \sqrt{-(-1)} = \sqrt{1} = 1$

5. i. For complex number $c_1 = a_1 + b_1 i$ and $c_2 = a_2 + b_2 i$, we have

$$|c_1|^2 = a_1^2 + b_1^2$$

$$|c_2|^2 = a_2^2 + b_2^2$$

$$|c_1 + c2|^2 = (a_1 + a_2)^2 + (b_1 + b_2)^2 = a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2$$

so we need to find a_1, a_2, b_1, b_2 that makes

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 \geq a_1^2 + b_1^2$$

and

$$a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 < a_2^2 + b_2^2$$

so we have

$$a_2^2 + 2a_1a_2 + b_2^2 + 2b_1b_2 \geq 0$$

and

$$a_1^2 + 2a_1a_2 + b_1^2 + 2b_1b_2 < 0$$

so, we need $2a_1a_2+2b_1b_2 \geq -(a_2^2+b_2^2)$ and $2a_1a_2+2b_1b_2 < -(a_1^2+b_1^2)$, which means $-(a_2^2+b_2^2) \leq 2a_1a_2+2b_1b_2 < -(a_1^2+b_1^2)$,
Through observing, it is easy to find $a_1 = -1, a_2 = 3, b_1 = 1, b2 = -3$ makes $|c_1+c_2|^2 \geq |c_1|^2$ and $|c_1+c_2|^2 < |c_2|^2$.

ii. To make $|c_1+c_2|^2 < |c_1|^2$ and $|c_1+c_2|^2 < |c_2|^2$, we need to make

$$a_1^2+a_2^2+2a_1a_2+b_1^2+b_2^2+2b_1b_2 < a_1^2+b_1^2$$

and

$$a_1^2+a_2^2+2a_1a_2+b_1^2+b_2^2+2b_1b_2 < a_2^2+b_2^2$$

so, we have

$$a_2^2+2a_1a_2+b_2^2+2b_1b_2 < 0$$

and

$$a_1^2+2a_1a_2+b_1^2+2b_1b_2 < 0$$

I don't think we can find a_1, a_2, b_1, b_2 that can make $|c_1+c_2|^2 < |c_1|^2$ and $|c_1+c_2|^2 < |c_2|^2$ true.

6. Assume both $\vec{v}_1$ and $\vec{v}_2$ has a length of n.
$\vec{v}_1 = (\psi_{10}, \psi_{11}, \psi_{12}, \psi_{13}, ..., \psi_{1n})^T$,
$\vec{v}_2 = (\psi_{20}, \psi_{21}, \psi_{22}, \psi_{23}, ..., \psi_{2n})^T$.
For real vectors $\vec{r}_1$ and $\vec{r}_2$, we have $\langle \vec{r}_1, \vec{r}_2 \rangle = \vec{r}_1^T \vec{r}_2$.
Similarly, we can define the inner product of $\vec{v}_1, \vec{v}_2$ that

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1^T \vec{v}_2$$

where $\vec{v}_1^T$ is the transpose of $\vec{v}_1$.
This means

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \, = (\psi_{10}, \psi_{11}, \psi_{12}, \psi_{13}, ..., \psi_{1n})(\psi_{20}, \psi_{21}, \psi_{22}, \psi_{23}, ..., \psi_{2n})^T$$

so,

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \sum_{i=0}^{n-1} \psi_{1i}\psi_{2i}$$

The properties of an inner product $\langle , \rangle$ are as followed[1].

(a) **Linearity:** $\langle a\mathbf{u}+b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$
(b) **Symmetric Property:** $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
(c) **Positive Definite Property:** For any $\mathbf{u} \in \mathbf{V}$, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$; and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$;

For complex vectors $\vec{v1}, \vec{v_2}, \vec{v_3}$, all of them have a length of n.

$\langle a\vec{v_1} + b\vec{v_2}, \vec{v_3}\rangle$
$= \sum_{i=0}^{n-1}(a\psi_{1i} + b\psi_{2i})(\psi_{3i})$
$= \sum_{i=0}^{n-1}(a\psi_{1i})(\psi_{3i}) + \sum_{i=0}^{n-1}(b\psi_{2i})(\psi_{3i})$
$= a\sum_{i=0}^{n-1}(\psi_{1i})(\psi_{3i}) + b\sum_{i=0}^{n-1}(\psi_{2i})(\psi_{3i})$
$= a\langle\vec{v_1}, \vec{v_3}\rangle + b\langle\vec{v_2}, \vec{v_3}\rangle$

This proves the linearity.

Also, we have

$\langle\vec{v_1}, \vec{v_2}\rangle$
$= \sum_{i=0}^{n-1}\psi_{1i}\psi_{2i}$
$= \sum_{i=0}^{n-1}\psi_{2i}\psi_{1i}$
$= \langle\vec{v_2}, \vec{v_1}\rangle$ This proves the symmetric property.

For any complex vector $\vec{v_1}$, $\langle\vec{v_1}, \vec{v_1}\rangle = \sum_{i=0}^{n-1}\psi_{1i}^2$. For any complex number $\psi = a + bi$, we have $\psi^2 = a^2 + b^2 \geq 0$,

so $\langle\vec{v_1}, \vec{v_1}\rangle = \sum_{i=0}^{n-1}\psi_{1i}^2 \geq 0$ and $\vec{v_1}$ is a complex vector, so $\vec{v_1} \neq 0$.

This proves the positive definite property.

So, it satisfies all the properties of an inner product.

EXERCISE 2: THE TENSOR PRODUCT

1. $|0\rangle_A \otimes |1\rangle_B$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 1 \times 0 \\ 1 \times 1 \\ 0 \times 0 \\ 0 \times 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

2. $|+\rangle_A \otimes |-\rangle_B$
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

3. $|0\rangle_A \otimes |-\rangle_B$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times \frac{1}{\sqrt{2}} \\ 1 \times -\frac{1}{\sqrt{2}} \\ 0 \times \frac{1}{\sqrt{2}} \\ 0 \times -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{pmatrix}$$

4. $|1\rangle_A \otimes |1\rangle_B$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \times 0 \\ 0 \times 1 \\ 1 \times 0 \\ 1 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

5. We have $|\Phi^+\rangle$
$= \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)$
$= \frac{1}{\sqrt{2}}(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix})$
$= \frac{1}{\sqrt{2}}(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix})$
$= \frac{1}{\sqrt{2}}(\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix})$

For $A = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$ and $B = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$, we have $A \otimes B = \begin{pmatrix} a_0b_0 \\ a_0b_1 \\ a_1b_0 \\ a_1b_1 \end{pmatrix}$.

If $|\Phi^+\rangle$ can be written as $A \otimes B$, then

$a_0b_0 = 0$
$a_0b_1 = 1$
$a_1b_0 = 1$
$a_1b_1 = 0.$

To make $a_0b_0 = 0$, either $a_0 = 0$ or $b_0 = 0$ should be true.

If any of them is true, then $a_0b_1 = 1$ and $a_1b_0 = 1$ cannot be true in the same time.

So $|\Phi^+\rangle$ can not be written as $A \otimes B$.

6. We have $|0\rangle|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $|1\rangle|1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

We also have $|+\rangle|-\rangle = \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle)$
and $|-\rangle|+\rangle = \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle)$
$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$

So, $-|\Phi^-\rangle$
$= \frac{1}{\sqrt{2}}(|1\rangle_A \otimes |0\rangle_B - |0\rangle_A \otimes |1\rangle_B)$

$= \frac{1}{\sqrt{2}}(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix})$
$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
$= \frac{1}{\sqrt{2}}[\frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle) - \frac{1}{2}(|0\rangle|0\rangle - |1\rangle|1\rangle)]$
$= \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle)$
So $|\Phi^-\rangle$ in basis $\mathcal{B}_1$ is equal to $-|\Phi^-\rangle$ in basis $\mathcal{B}_2$.

OVERLAPS OF STATES

1. For $|\psi\rangle = \begin{pmatrix} c_0 \\ c_1 \\ ... \\ c_{n-1} \end{pmatrix}$ where $c_k = a_k + b_k i$, we know
   $\|\psi\|_2^2 = \sum_{i=0}^{n-1} |c_i|^2 = \sum_{i=0}^{n-1} a_i^2 + b_i^2$.
   now,
   $\langle\psi|\psi\rangle$
   $= \left(c_0^*, c_1^*, ..., c_{n-1}^*\right) \begin{pmatrix} c_0 \\ c_1 \\ ... \\ c_{n-1} \end{pmatrix}$
   $= \sum_{i=0}^{n-1} c_i^* c_i maginary$
   $= \sum_{i=0}^{n-1} a_i^2 + b_i^2$
   $= \|\psi\|_2^2$
   So, $\langle\psi|\psi\rangle = \|\psi\|_2^2$
2. (a) For $|\psi_1\rangle = \frac{1}{3}|-\rangle$,
   $\|\psi_1\|^2$
   $= \langle\psi_1|\psi_1\rangle$
   $= \frac{1}{9}\langle -|-\rangle$
   $= \frac{1}{9}[\frac{1}{2}\langle 0|0\rangle + (-1)^2\frac{1}{2}\langle 1|1\rangle]$
   $= \frac{1}{9}(\frac{1}{2} + \frac{1}{2})$
   $= \frac{1}{9}$
   so,
   $\|\psi_1\| = \sqrt{\|\psi_1\|^2} = \sqrt{\frac{1}{9}} = \frac{1}{3}$

   (b) For $|\psi_2\rangle = \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)$
   $\|\psi_2\|^2$
   $= \frac{1}{2} \times -(i^2 \times \langle 0|0\rangle) + \frac{1}{2}\langle 1|1\rangle$

$= \frac{1}{2} * 1 + \frac{1}{2} * 1$
$= 1$
So, $\|\psi_2\| = \sqrt{1} = 1$

References

[1] HKUST Department of Mathematics. "Math111 Week 13-14 Lecture Notes." Hong Kong University of Science and Technology, n.d., https://www.math.hkust.edu.hk/ mabfchen/Math111/Week13-14.pdf.