

Assignment 2

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1 EXERCISE 1: THE PARTIAL TRACE

We saw in the lectures that given a multi-partite state, we obtain the state of a subsystem by applying the partial trace to the other systems.

1. Compute the trace of the following states by applying the trace formula $Tr(A) = \sum_i \langle i| A |i\rangle$ from the lectures:

(a) $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

(b) $\Lambda = \frac{I}{3} + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$ (here, I is the identity matrix).

Which of the states is normalized correctly? (*) (4 points)

Answer: $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

$$= \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

$$Tr(\xi) = |0\rangle\langle 0| \xi |0\rangle + |1\rangle\langle 1| \xi |1\rangle$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{i}{2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{i}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

$$\Lambda = \frac{I}{3} + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$= \frac{I}{3} + \frac{1}{6} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= \frac{I}{3} + \frac{1}{6} \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

$$= \frac{I}{3} + \frac{1}{6} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{6} \\ \frac{1}{6} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

$$Tr(\Lambda) = |0\rangle\langle 0| \Lambda |0\rangle + |1\rangle\langle 1| \Lambda |1\rangle$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \end{bmatrix}$$

$$= \frac{1}{3} + \frac{1}{3}$$

$$= \frac{2}{3}$$

So ξ is normalized correctly.

2. In the last assignment we found the probability to find a state $|\gamma\rangle$ in another state $|\delta\rangle$ to be $p = |\langle\gamma|\delta\rangle|^2$. Show here that this expression coincides with $\text{Tr}(\gamma\delta)$, for $\gamma = |\gamma\rangle\langle\gamma|$ and $\delta = |\delta\rangle\langle\delta|$. Hint: Apply the trace formula $\text{Tr}(A) = \sum_i \langle i|A|i\rangle$ from the lectures. Use a suitable basis of your choice. (4 points)

$$\begin{aligned} \text{Answer: } \text{Tr}(\gamma\delta) &= \text{Tr}(|\gamma\rangle\langle\gamma||\delta\rangle\langle\delta|) \\ &= \text{Tr}(|\gamma\rangle\langle\gamma|\delta\rangle\langle\delta|) \\ &= \text{Tr}(\langle\gamma|\delta\rangle|\gamma\rangle\langle\delta|) \\ &= \langle\gamma|\delta\rangle \text{Tr}(|\gamma\rangle\langle\delta|) \\ &= \langle\gamma|\delta\rangle \sum_i \langle i||\gamma\rangle\langle\delta||i\rangle \\ &= \langle\gamma|\delta\rangle (\langle 0||\gamma\rangle\langle\delta||0\rangle + \langle 1||\gamma\rangle\langle\delta||1\rangle) \\ &= \langle\gamma|\delta\rangle (\langle 0|\gamma\rangle\langle\delta|0\rangle + \langle 1|\gamma\rangle\langle\delta|1\rangle) \\ &= \langle\gamma|\delta\rangle (\langle\gamma|0\rangle\langle 0|\delta\rangle + \langle\gamma|1\rangle\langle 1|\delta\rangle) \\ &= \langle\gamma|\delta\rangle (\langle\gamma|\delta\rangle) \\ &= |\langle\gamma|\delta\rangle|^2 \end{aligned}$$

$$\text{So, } \text{Tr}(\gamma\delta) = |\langle\gamma|\delta\rangle|^2 = p.$$

3. Consider the bipartite state $|\phi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i|01\rangle_{AB} - |11\rangle_{AB})$. Write the density operator $\rho_{AB} = |\phi\rangle_{AB}\langle\phi|_{AB}$ explicitly in matrix form. (*) (4 points)

$$\begin{aligned} \text{Answer: } \rho_{AB} &= |\phi\rangle_{AB}\langle\phi|_{AB} \\ &= \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i|01\rangle_{AB} - |11\rangle_{AB})(\langle 00|_{AB} - i\langle 01|_{AB} - \langle 11|_{AB}) \\ &= \frac{1}{3}(|00\rangle\langle 00| - i|00\rangle\langle 01| - |00\rangle\langle 11| + i|01\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 11| - |11\rangle\langle 00| - |11\rangle\langle 01| + |11\rangle\langle 11|) \\ &= \frac{1}{3} \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} - \right. \\ &\quad \left. \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \\ &= \frac{1}{3} \begin{bmatrix} 1 & i & 0 & -1 \\ -i & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & i & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{i}{3} & 0 & -\frac{1}{3} \\ -\frac{i}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{i}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{aligned}$$

4. Let σ_{AB} be a general 2-qubit state. The 4×4 matrix describing σ_{AB} can be split into four sub-matrices of size 2×2 (upper left block, upper right block, lower left block, lower right block). Show that the reduced single qubit state $\sigma_B = \text{Tr}_A(\sigma_{AB})$ is described by a 2×2 matrix that is the sum of the upper left block and lower right block matrices of σ_{AB} . Hint: start from $\sigma_{AB} = \sigma_{00}|00\rangle\langle 00| + \sigma_{01}|00\rangle\langle 01| + \dots + \sigma_{33}|11\rangle\langle 11|$ and compute the partial trace in bra-ket notation. Write σ_{AB} as a matrix and compare. (8 points)

$$\text{Answer: } \sigma_{AB} = \sigma_{00}|00\rangle\langle 00| + \sigma_{01}|00\rangle\langle 01| + \dots + \sigma_{33}|11\rangle\langle 11|$$

$$\begin{aligned} \sigma_B &= \\ &= \text{Tr}_A(\sigma_{AB}) \end{aligned}$$

$$\begin{aligned}
&= \sum_i \langle i|_A \sigma_{AB} |i\rangle_A \\
&= \langle 0|_A \sigma_{AB} |0\rangle_A + \langle 1|_A \sigma_{AB} |1\rangle_A \\
&= \langle 0|_A (\sigma_{00} |00\rangle_{AB} \langle 00| + \sigma_{01} |00\rangle_{AB} \langle 01|_{AB} + \dots + \sigma_{33} |11\rangle_{AB} \langle 11|_{AB}) |0\rangle + \langle 1| (\sigma_{00} |00\rangle_{AB} \langle 00|_{AB} + \sigma_{01} |00\rangle_{AB} \langle 01|_{AB} + \dots + \sigma_{33} |11\rangle_{AB} \langle 11|_{AB}) |1\rangle_A \\
&= \sigma_{00} |0\rangle \langle 0| + \sigma_{01} |0\rangle \langle 1| + \sigma_{10} |1\rangle \langle 0| + \sigma_{11} |1\rangle \langle 1| + \sigma_{22} |0\rangle \langle 0| + \sigma_{23} |0\rangle \langle 1| + \sigma_{32} |1\rangle \langle 0| + \sigma_{33} |1\rangle \langle 1| \\
&= (\sigma_{00} + \sigma_{22}) |0\rangle \langle 0| + (\sigma_{01} + \sigma_{23}) |0\rangle \langle 1| + (\sigma_{10} + \sigma_{32}) |1\rangle \langle 0| + (\sigma_{12} + \sigma_{33}) |1\rangle \langle 1|
\end{aligned}$$

This is the sum of the upper left block and lower right block matrices of σ_{AB} .

5. Take the state ρ_{AB} from 3. and compute the reduced state ρ_B , both from the matrix itself (using your result from 4.) and in bra-ket notation. (*) (6 points)

$$\text{Answer: } \rho_{AB} = \begin{bmatrix} \frac{1}{3} & \frac{i}{3} & 0 & -\frac{1}{3} \\ -\frac{i}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{i}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

$$\begin{aligned}
\rho_B &= (\rho_{00} + \rho_{21}) |0\rangle \langle 0| + (\rho_{02} + \rho_{23}) |0\rangle \langle 1| + (\rho_{10} + \rho_{31}) |1\rangle \langle 0| + (\rho_{12} + \rho_{33}) |1\rangle \langle 1| \\
&= (\frac{1}{3} + 0) |0\rangle \langle 0| + (0 + 0) |0\rangle \langle 1| + (-\frac{i}{3} + \frac{i}{3}) |1\rangle \langle 0| + (\frac{1}{3} + \frac{1}{3}) |1\rangle \langle 1| \\
&= \frac{1}{3} |0\rangle \langle 0| + 0 |0\rangle \langle 1| + 0 |1\rangle \langle 0| + \frac{2}{3} |1\rangle \langle 1| \\
&= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}
\end{aligned}$$

2 EXERCISE 2: MEASUREMENTS AND REDUCED STATES

Let us look at the density operator $\rho_{AB} = |\Phi^-\rangle \langle \Phi^-|$, with $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

1. Compute explicitly the post-measurement state after we performed a projective measurement $M = \{|0\rangle \langle 0|, |1\rangle \langle 1|\}$ on ρ_{AB} on system B , given that the outcome was 0. Compute the same for the measurement $M = \{|+\rangle \langle +|, |-\rangle \langle -|\}$, for outcome $-$. (Hint: Using Exercise 2.6 from Assignment 1 might help (not necessary though).) (8 points)

$$\text{Answer: } M_0 = |0\rangle \langle 0|$$

$$\text{Observe } |0\rangle \text{ with probability } p_0 = \text{Tr}(M_0 \rho) = \text{Tr}(|0\rangle \langle 0| \otimes I \rho) = \text{Tr}(|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-|) = \langle 0|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |0\rangle \otimes I + \langle 1|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |1\rangle \otimes I = \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |0\rangle \otimes I$$

$$\begin{aligned}
\rho'_0 &= \frac{M_0 \rho M_0^\dagger}{\text{Tr}(M_0 \rho)} \\
&= \frac{|0\rangle \langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |0\rangle \langle 0| \otimes I}{\langle 0| \otimes I |\Phi^-\rangle \langle \Phi^-| |0\rangle \otimes I} \\
&= |00\rangle \langle 00| \\
&= |0\rangle \langle 0| \otimes |0\rangle \langle 0|
\end{aligned}$$

$$\text{For system B, } \rho'_{0B} = |0\rangle \langle 0|$$

$$M_1 = |-\rangle \langle -|$$

$$\text{Observe } |-\rangle \text{ with probability } p_1 = \text{Tr}(M_1 \rho) = \text{Tr}(|-\rangle \langle -| \otimes I \rho) = \text{Tr}(|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-|) = \langle +|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-| |+\rangle \otimes I + \langle -|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-| |-\rangle \otimes I = \langle +| \otimes I |\Phi^-\rangle \langle \Phi^-| |+\rangle \otimes I$$

$$\begin{aligned}
\rho'_1 &= \frac{M_1 \rho M_1^\dagger}{\text{Tr}(M_1 \rho)} \\
&= \frac{|-\rangle \langle -| \otimes I |\Phi^-\rangle \langle \Phi^-| |-\rangle \langle -| \otimes I}{\langle +| \otimes I |\Phi^-\rangle \langle \Phi^-| |+\rangle \otimes I} \\
&= |--\rangle \langle --| \\
&= |-\rangle \langle -| \otimes |-\rangle \langle -|
\end{aligned}$$

$$\text{For system B, } \rho'_{1B} = |-\rangle \langle -|$$

2. What is the post measurement state if the measurements above were destructive and yielded the same outcomes as in part 1.? (2 points)

Answer:

$$\text{For the outcome 0, } \rho''_0 = \text{Tr}(\rho'_{0B}) = \text{Tr}(|0\rangle \langle 0|) = 1$$

For the outcome $-$, $\rho_1'' = \text{Tr}(\rho_{1B}') = \text{Tr}(|-\rangle\langle -|) = 1$

3. What is the reduced state ρ_A if we take state ρ_{AB} and trace over system B in basis $\{|0\rangle, |1\rangle\}$? What if we trace in $\{|+\rangle, |-\rangle\}$? Hint: for the second computation it could be useful to use the basis invariance of the trace operation. (6 points)

$$\begin{aligned}\text{Answer: } \rho_{AB} &= |\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|) = \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) \\ \rho_A &= \text{Tr}_B(\rho_{AB}) = \frac{1}{2}(\langle 0| \otimes \langle 0| \rho_{AB} |0\rangle \otimes |0\rangle + \langle 1| \otimes \langle 1| \rho_{AB} |1\rangle \otimes |1\rangle) \\ &= \frac{1}{2}(\langle 0| \otimes \langle 0| (|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) |0\rangle \otimes |0\rangle + \langle 1| \otimes \langle 1| (|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) |1\rangle \otimes |1\rangle) \\ &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)\end{aligned}$$

In the basis $\{|+\rangle, |-\rangle\}$, $\Phi = \frac{1}{2}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$

$$\rho_A = \text{Tr}_B$$

4. Consider now the $2n$ -qubit state $\rho = |\Psi_n\rangle\langle\Psi_n|$, with $|\Psi_n\rangle = \frac{1}{\sqrt{2^n}} \sum_i |i\rangle_A \otimes |i\rangle_B$. What is the reduced n -qubit state on Alice's side? (Hint: look at the state in vector form and factorize it in a smart way.) (8 points)

$$\begin{aligned}\text{Answer: } \rho &= \frac{1}{2^n} \sum_i |i\rangle_A \otimes |i\rangle_B \sum_j \langle j|_A \langle j|_B \\ &= \frac{1}{2^n} \sum_i \sum_j |i\rangle_A \otimes |i\rangle_B \langle j|_A \otimes \langle j|_B \\ \text{Tr}_B(\rho) &= \frac{1}{2^n} \sum_k \sum_i \sum_j \langle k|_B \langle i|_B \langle j|_A \otimes |i\rangle_B \langle j|_A |k\rangle \\ \text{If } i \neq k, \langle k|_B |i\rangle_B &= 0, \text{ if } i \neq j, \langle i|_B |j\rangle_B = 0, \text{ if } j \neq k, \langle j|_A |k\rangle_A = 0 \\ \text{Tr}_B(\rho) &= \frac{1}{2^n} \sum_i \langle i|_B |i\rangle_B \otimes |i\rangle_A \langle i|_A \\ &= \frac{1}{2^n} \sum_i |i\rangle_A \langle i|_A \\ &= \frac{I_n}{2^n}\end{aligned}$$

5. What is the reduced state on the first k qubits of A and B (i.e. tracing out the last $n - k$ qubits on each side)? (4 points)

$$\begin{aligned}\text{Answer: } \rho_{0tok} &= \text{Tr}_{\rho_{kton}}(\rho_{AB}) = \sum_{m=0}^{n-k} \langle m|_B |\Psi_n\rangle \langle\Psi_n| |m\rangle_B \\ &= \frac{1}{2^n} \sum_{m=0}^{n-k} \sum_i \sum_j \langle m|_B |i\rangle_B \otimes |i\rangle_A \langle j|_A \otimes \langle j|_B |m\rangle_B \\ \text{if } m \neq i \text{ or } m \neq j, \langle m|_B |i\rangle_B &= 0 \text{ or } \langle m|_B |j\rangle_B = 0 \\ \text{So, } \rho_{0tok} &= \frac{1}{2^n} \sum_{i=0}^{n-k} |i\rangle_A \langle i|_A = \frac{I_{n-k}}{2^n}\end{aligned}$$

3 EXERCISE 3: EVOLUTIONS AND KRAUS OPERATORS

1. Consider the following channel C . It maps the classical state $|0\rangle$ to the state $|0\rangle$ with probability $(1-p)$ and to $|1\rangle$ with probability p . Symmetrically, the state $|1\rangle$ is mapped to the state $|1\rangle$ with probability $(1-p)$ and to $|0\rangle$ with probability p . Find a Kraus operator representation of the channel and show that your choice is valid, and that it maps the classical states $|0\rangle$ and $|1\rangle$ correctly. (4 points)

$$\text{Answer: } \{E_0, E_1, E_2, E_3\} = \{\sqrt{1-p}|0\rangle\langle 0|, \sqrt{p}|0\rangle\langle 1|, \sqrt{p}|1\rangle\langle 0|, \sqrt{1-p}|1\rangle\langle 1|\}$$

$$\begin{aligned}\sum_i E_i^\dagger E_i &= (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1| + (1-p)|0\rangle\langle 1| + p|1\rangle\langle 0| \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| = I\end{aligned}$$

$$C(\rho) = \sum_i E_i \rho E_i^\dagger$$

$$\begin{aligned}C(|0\rangle\langle 0|) &= (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1| \\ &= (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|\end{aligned}$$

$$\begin{aligned}C(|1\rangle\langle 1|) &= (1-p)|0\rangle\langle 1| + p|1\rangle\langle 0| \\ &= p|0\rangle\langle 1| + (1-p)|1\rangle\langle 0|\end{aligned}$$

2. Apply the classical channel C to a general quantum state $\rho = \sum_{i,j} \alpha_{i,j} |i\rangle \langle j|$ (using the Kraus operators found above) and demonstrate that the off-diagonal terms vanish.

$$\begin{aligned} C(\rho) &= \sum_k E_k \rho E_k^\dagger \\ &= \sum_k E_k \sum_{i,j} \alpha_{i,j} |i\rangle \langle j| E_k^\dagger \\ &= (1-p) |0\rangle \langle 0| \sum_{i,j} \alpha_{i,j} |i\rangle \langle j| |0\rangle \langle 0| + p |0\rangle \langle 1| \sum_{i,j} \alpha_{i,j} |i\rangle \langle j| |1\rangle \langle 0| + p |1\rangle \langle 0| \sum_{i,j} \alpha_{i,j} |i\rangle \langle j| |0\rangle \langle 1| + (1-p) |1\rangle \langle 1| \sum_{i,j} \alpha_{i,j} |i\rangle \langle j| |1\rangle \langle 1| \end{aligned}$$

If $i \neq k$ or $j \neq k$, $\langle k|i\rangle \langle j|k\rangle = 0$

$$\begin{aligned} C(\rho) &= (1-p) \alpha_{00} |0\rangle \langle 0| + p \alpha_{11} |0\rangle \langle 0| + p \alpha_{00} |1\rangle \langle 1| + (1-p) \alpha_{11} |1\rangle \langle 1| \\ &= [(1-p) \alpha_{00} + p \alpha_{11}] |0\rangle \langle 0| + [(1-p) \alpha_{11} + p \alpha_{00}] |1\rangle \langle 1| \end{aligned}$$

This equation shows that the off-diagonal terms vanish.

3. Can you think of a quantum version of the channel, which operates correctly on the coherent (off-diagonal) terms of ρ ? Write it down and show that it is valid and correct. Hint: apply the new set of Kraus operators first to the classical states $|0\rangle$ and $|1\rangle$ to check correctness. Then, apply it to ρ and show that it preserves coherence. (8 points)

4 EXERCISE 4: THE GATE MODEL

Consider the following circuit diagram:

The initial states are the two qubits $|+\rangle$ and $|-\rangle$. The upper wire is initialized in state $|+\rangle$ and experiences a controlled X gate. The lower wire is initialized in state $|-\rangle$, then first transformed by the Hadamard gate, and afterwards acts as a control state for the controlled X gate. Finally, a measurement M measures each qubit in the computational basis.

1. What is the state of the system before the measurement M ? (4 points)

$$\begin{aligned} \text{Answer: } H(|-\rangle) &= (\frac{1}{\sqrt{2}} |0\rangle \langle 0| + \frac{1}{\sqrt{2}} |0\rangle \langle 1| + \frac{1}{\sqrt{2}} |1\rangle \langle 0| - \frac{1}{\sqrt{2}} |1\rangle \langle 1|) |-\rangle \\ &= \frac{1}{2} (|0\rangle + |1\rangle - \text{ket}0 + \text{ket}1) \\ &= |1\rangle \end{aligned}$$

So the output of gate H is $|1\rangle$, which means gate X will be applied to the upper wire.

$$\begin{aligned} X(|+\rangle) &= (|0\rangle \langle 1| + |1\rangle \langle 0|) |+\rangle \\ &= (|0\rangle \langle 1| + |1\rangle \langle 0|) \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\ &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \end{aligned}$$

So the state before measurement is $|1\rangle \otimes \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$.

2. Which are the possible measurement outcomes of M and what are their probabilities? (4 points)

Answer: The possible measurement outcomes of M are $|01\rangle$ and $|11\rangle$.

$$P(01) = |01\rangle \langle 01| \text{text}$$

The probability of $|01\rangle$ is $\frac{1}{2}$ and the probability of $|11\rangle$ is $\frac{1}{2}$.

In the lectures we saw the controlled Z and controlled X gate, where $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. Are the following two circuits the same? How about when we replace Z with X ? (6 points)