Assignment 2

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1 EXERCISE 1: THE PARTIAL TRACE

We saw in the lectures that given a multi-partite state, we obtain the state of a subsystem by applying the partial trace to the other systems.

1. Compute the trace of the following states by applying the trace formula $Tr(A) = \sum_i \langle i| A |i\rangle$ from the lectures:

   (a) $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

   (b) $\Lambda = \frac{I}{3}I + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)$ (here, I is the identity matrix).
   Which of the states is normalized correctly? (*) (4 points)

   Answer: $\xi = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$

   $Tr(\xi) = \langle 0|\xi|0\rangle + \langle 1|\xi|1\rangle = \frac{1}{2}(\langle 0|0\rangle\langle 0|0\rangle - i\langle text|0\rangle\langle 1|0\rangle + i\langle 0|1\rangle\langle 0|0\rangle + \langle 0|1\rangle\langle 1|0\rangle + \langle 1|0\rangle\langle 0|1\rangle - i\langle 1|0\rangle\langle 1|1\rangle + i\langle 1|1\rangle\langle 0|1\rangle + \langle 1|1\rangle\langle 1|1\rangle) = \frac{1}{2}(1+1) = 1$

   $\Lambda = \frac{I}{3}I + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|) = \frac{1}{3}|0\rangle\langle 0| + \frac{1}{6}|0\rangle\langle 1| + \frac{1}{6}|1\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$

   $Tr(\Lambda) = \langle 0|\Lambda|0\rangle + \langle 1|\Lambda|1\rangle$
   $= \frac{1}{3}\langle 0|0\rangle\langle 0|0\rangle + \frac{1}{6}\langle 0|0\rangle\langle 1|0\rangle + \frac{1}{6}\langle 0|1\rangle\langle 0|0\rangle + \frac{1}{3}\langle 0|1\rangle\langle 1|0\rangle + \frac{1}{3}\langle 1|0\rangle\langle 0|1\rangle + \frac{1}{6}\langle 1|0\rangle\langle 1|1\rangle + \frac{1}{6}\langle 1|1\rangle\langle 0|1\rangle + \frac{1}{3}\langle 1|1\rangle\langle 1|1\rangle$
   $= \frac{1}{3}(1+1)$
   $= \frac{2}{3}$

   So ξ is normalized correctly.

2. In the last assignment we found the probability to find a state $|\gamma\rangle$ in another state $|\delta\rangle$ to be $p = |\langle\gamma|\delta\rangle|^2$. Show here that this expression coincides with $\mathrm{Tr}(\gamma\delta)$, for $\gamma = |\gamma\rangle\langle\gamma|$ and $\delta = |\delta\rangle\langle\delta|$. Hint: Apply the trace formula $\mathrm{Tr}(A) = \sum_i \langle i|A|i\rangle$ from the lectures. Use a suitable basis of your choice. (4 points)

   Answer: $\mathrm{Tr}(\gamma\delta)$
   $= \mathrm{Tr}(|\gamma\rangle\langle\gamma|\,|\delta\rangle\langle\delta|)$
   $= \mathrm{Tr}(|\gamma\rangle\langle\gamma|\delta\rangle\langle\delta|)$
   $= \mathrm{Tr}(\langle\gamma|\delta\rangle|\gamma\rangle\langle\delta|)$
   $= \langle\gamma|\delta\rangle\,\mathrm{Tr}(|\gamma\rangle\langle\delta|)$
   $= \langle\gamma|\delta\rangle\sum_i\langle i|\,|\gamma\rangle\langle\delta|\,|i\rangle$
   $= \langle\gamma|\delta\rangle(\langle 0|\,|\gamma\rangle\langle\delta|\,|0\rangle + \langle 1|\,|\gamma\rangle\langle\delta|\,|1\rangle)$
   $= \langle\gamma|\delta\rangle(\langle 0|\gamma\rangle\langle\delta|0\rangle + \langle 1|\gamma\rangle\langle\delta|1\rangle)$
   $= \langle\gamma|\delta\rangle(\langle\gamma|0\rangle\langle 0|\delta\rangle + \langle\gamma|1\rangle\langle 1|\delta\rangle)$
   $= \langle\gamma|\delta\rangle(\langle\gamma|\delta\rangle)$
   $= |\langle\gamma|\delta\rangle|^2$

   So, $\mathrm{Tr}(\gamma\delta) = |\langle\gamma|\delta\rangle|^2 = p$.

3. Consider the bipartite state $|\phi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i|01\rangle_{AB} - |11\rangle_{AB})$. Write the density operator $\rho_{AB} = |\phi\rangle_{AB}\langle\phi|_{AB}$ explicitly in matrix form. (*) (4 points)

Answer: $\phi_{AB} = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + i\,|01\rangle_{AB} - |11\rangle_{AB})$

$= \frac{1}{\sqrt{3}}(\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + i\begin{bmatrix}0\\1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\0\\0\\1\end{bmatrix})$

$= \frac{1}{\sqrt{3}}\begin{bmatrix}1\\i\\0\\-1\end{bmatrix}$

ρ_{AB}

$= |\phi\rangle_{AB}\langle\phi|_{AB}$

$= \frac{1}{3}\begin{bmatrix}1\\i\\0\\-1\end{bmatrix}\begin{bmatrix}1 & -i & 0 & -1\end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix}1 & -i & 0 & -1\\ i & 1 & 0 & -i\\ 0 & 0 & 0 & 0\\ -1 & -i & 0 & 1\end{bmatrix}$

$= \begin{bmatrix}\frac{1}{3} & -\frac{i}{3} & 0 & -\frac{1}{3}\\ \frac{i}{3} & \frac{1}{3} & 0 & -\frac{i}{3}\\ 0 & 0 & 0 & 0\\ -\frac{1}{3} & \frac{i}{3} & 0 & \frac{1}{3}\end{bmatrix}$

4. Let σ_{AB} be a general 2-qubit state. The 4×4 matrix describing σ_{AB} can be split into four sub-matrices of size 2×2 (upper left block, upper right block, lower left block, lower right block). Show that the reduced single qubit state $\sigma_B = \mathrm{Tr}_A(\sigma_{AB})$ is described by a 2×2 matrix that is the sum of the upper left block and lower right block matrices of σ_{AB}. Hint: start from $\sigma_{AB} = \sigma_{00}|00\rangle\langle00| + \sigma_{01}|00\rangle\langle01| + \ldots + \sigma_{33}|11\rangle\langle11|$ and compute the partial trace in bra-ket notation. Write σ_{AB} as a matrix and compare. (8 points)

   Answer: $\sigma_{AB} = \sigma_{00}\,|00\rangle\,\langle00| + \sigma_{01}\,|00\rangle\,\langle01| + \ldots + \sigma_{33}\,|11\rangle\,\langle11|$

   $\sigma_B =$
   $= \mathrm{Tr}_A(\sigma_{AB})$
   $= \sum_i \langle i|_A\,\sigma_{AB}\,|i\rangle_A$
   $= \langle0|_A\,\sigma_{AB}\,|0\rangle_A + \langle1|_A\,\sigma_{AB}\,|1\rangle_A$
   $= \langle0|_A\,(\sigma_{00}\,|00\rangle_{AB}\,\langle00| + \sigma_{01}\,|00\rangle_{AB}\,\langle01|_{AB} + \ldots + \sigma_{33}\,|11\rangle_{AB}\,\langle11|_{AB})\,|0\rangle + \langle1|\,(\sigma_{00}\,|00\rangle_{AB}\,\langle00|_{AB} + \sigma_{01}\,|00\rangle_{AB}\,\langle01|_{AB} + \ldots + \sigma_{33}\,|11\rangle_{AB}\,\langle11|_{AB})\,|1\rangle_A$
   $= \sigma_{00}\,|0\rangle\,\langle0| + \sigma_{01}\,|0\rangle\,\langle1| + \sigma_{10}\,|1\rangle\,\langle0| + \sigma_{11}\,|1\rangle\,\langle1| + \sigma_{22}\,|0\rangle\,\langle0| + \sigma_{23}\,|0\rangle\,\langle1| + \sigma_{32}\,|1\rangle\,\langle0| + \sigma_{33}\,|1\rangle\,\langle1|$
   $= (\sigma_{00}+\sigma_{22})\,|0\rangle\,\langle0| + (\sigma_{01}+\sigma_{23})\,|0\rangle\,\langle1| + (\sigma_{10}+\sigma_{32})\,|1\rangle\,\langle0| + (\sigma_{12}+\sigma_{33})\,|1\rangle\,\langle1|$

   This is the sum of the upper left block and lower right block matrices of σ_{AB}.

5. Take the state ρ_{AB} from 3. and compute the reduced state ρ_B, both from the matrix itself (using your result from 4.) and in bra-ket notation. (*) (6 points)

   Answer: $\begin{bmatrix}\frac{1}{3} & -\frac{i}{3} & 0 & -\frac{1}{3}\\ \frac{i}{3} & \frac{1}{3} & 0 & -\frac{i}{3}\\ 0 & 0 & 0 & 0\\ -\frac{1}{3} & \frac{i}{3} & 0 & \frac{1}{3}\end{bmatrix}$

   $\rho_B = (\sigma_{00}+\sigma_{22})\,|0\rangle\,\langle0| + (\sigma_{01}+\sigma_{23})\,|0\rangle\,\langle1| + (\sigma_{10}+\sigma_{32})\,|1\rangle\,\langle0| + (\sigma_{12}+\sigma_{33})\,|1\rangle\,\langle1|$
   $= \frac{1}{3}\,|0\rangle\,\langle0| - \frac{i}{3}\,|0\rangle\,\langle1| + \frac{i}{3}\,|1\rangle\,\langle0| + 0\,|1\rangle\,\langle1|$
   $= \begin{bmatrix}\frac{1}{3} & -\frac{i}{3}\\ \frac{i}{3} & \frac{2}{3}\end{bmatrix}$

2 EXERCISE 2: MEASUREMENTS AND REDUCED STATES

Let us look at the density operator $\rho_{AB} = |\Phi^-\rangle\langle\Phi^-|$, with $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

1. Compute explicitly the post-measurement state after we performed a projective measurement $M = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on ρ_{AB} on system B, given that the outcome was 0. Compute the same for the measurement $M = \{|+\rangle\langle +|, |-\rangle\langle -|\}$, for outcome $-$. (Hint: Using Exercise 2.6 from Assignment 1 might help (not necessary though).) (8 points)

   Answer: $M_0 = |0\rangle\langle 0|$

   Observe $|0\rangle$ with probability $p_0 = Tr(M_0\rho) = Tr(I \otimes |0\rangle\langle 0|\rho) = Tr(I \otimes |0\rangle\langle 0| |\Phi^-\rangle\langle\Phi^-|) = Tr(I \otimes |0\rangle\langle 0| \frac{1}{2}(|01\rangle\langle 10| - |01\rangle\langle 01| - |10\rangle\langle 10| + |10\rangle\langle 01|)) = Tr(\frac{1}{2}(-|10\rangle\langle 10| + |10\rangle\langle 01|)) = -\frac{1}{2}$

   $\rho'_0 = \frac{M_0\rho M_0^\dagger}{Tr(M_0\rho)}$
   $= \frac{I\otimes|0\rangle\langle 0||\Phi^-\rangle\langle\Phi^-|I\otimes|0\rangle\langle 0|}{Tr(\frac{1}{2}(-|10\rangle\langle 10|+|10\rangle\langle 01|))}$
   $= \frac{\frac{1}{2}(-|10\rangle\langle 10|+|10\rangle\langle 01|)I\otimes|0\rangle\langle 0|}{-\frac{1}{2}}$
   $= |10\rangle\langle 10|$

   For system B, $\rho'_{0B} = |0\rangle\langle 0|$

   $M_1 = |-\rangle\langle -|$

   Observe $|-\rangle$ with probability $p_1 = Tr(M_1\rho) = Tr(I \otimes |-\rangle\langle -|\rho) = Tr(I \otimes |-\rangle\langle -| |\Phi^-\rangle\langle\Phi^-|) = Tr(I \otimes |-\rangle\langle +| \frac{1}{2}(|+-\rangle\langle -+| - |+-\rangle\langle +-| - |-+\rangle\langle -+| + |-+\rangle\langle +-|))$
   $= \frac{1}{2}Tr(|+-\rangle\langle -+| - |+-\rangle\langle +-|) = -\frac{1}{2}$

   $\rho'_1 = \frac{M_1\rho M_1^\dagger}{Tr(M_1\rho)}$
   $= \frac{(|+-\rangle\langle -+|-|+-\rangle\langle +-|)I\otimes|-\rangle\langle -|}{\langle +|\otimes I|\Phi^-\rangle\langle\Phi^-||+\rangle\otimes I}$
   $= |+-\rangle\langle +-|$
   $= |+\rangle\langle -| \otimes |+\rangle\langle -|$

   For system B, $\rho'_{1B} = |-\rangle\langle -|$

2. What is the post measurement state if the measurements above were destructive and yielded the same outcomes as in part 1.? (2 points)

   Answer:

   For the outcome 0, $\rho''_0 = \frac{M_0\rho'_0 M_0^\dagger}{Tr(M_0\rho_0)} = \frac{M_0|10\rangle\langle 10|M_0^\dagger}{Tr(M_0|10\rangle\langle 10|)} = \frac{I\otimes|0\rangle\langle 0||10\rangle\langle 10|I\otimes|0\rangle\langle 0|}{Tr(I\otimes|0\rangle\langle 0||10\rangle\langle 10|)} = |10\rangle\langle 10|$

   For the outcome -, $\rho''_- = \frac{M_1\rho'_1 M_1^\dagger}{Tr(M_1\rho_1)} = \frac{M_1|+-\rangle\langle +-|M_1^\dagger}{Tr(M_1|+-\rangle\langle +-|)} = \frac{I\otimes|-\rangle\langle -||+-\rangle\langle +-|I\otimes|-\rangle\langle -|}{Tr(I\otimes|-\rangle\langle -||+-\rangle\langle +-|)} = |+-\rangle\langle +-|$

3. What is the reduced state ρ_A if we take state ρ_{AB} and trace over system B in basis $\{|0\rangle, |1\rangle\}$? What if we trace in $\{|+\rangle, |-\rangle\}$? Hint: for the second computation it could be useful to use the basis invariance of the trace operation. (6 points)

   Answer: $\rho_{AB} = |\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 10| - \langle 01|) = \frac{1}{2}(|01\rangle\langle 10| - |01\rangle\langle 01| - |10\rangle\langle 10| + |10\rangle\langle 01|)$

   $\rho_A = Tr_B(\rho_{AB}) = \langle 0|_B \rho_{AB} |0\rangle_B + \langle 1|_B \rho_{AB} |1\rangle_B = -\frac{1}{2}|0\rangle\langle 0| - \frac{1}{2}|1\rangle\langle 1|$

   In the basis $\{|+\rangle, |-\rangle\}$, $\Phi = \frac{1}{2}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$

   $\rho_A = Tr_B(\rho_{AB}) = \langle +|_B \rho_{AB} |+\rangle_B + \langle -|_B \rho_{AB} |-\rangle_B$
   $= \frac{1}{2}((\langle 0|_B + \langle 1|_B)\rho_{AB}(|0\rangle_B + |1\rangle_B)) + \frac{1}{2}((\langle 0|_B - \langle 1|_B)\rho_{AB}(|0\rangle_B - |1\rangle_B))$
   $= \frac{1}{2}(\langle 0|_B \rho_{AB} |0\rangle_B + \langle 0|_B \rho_{AB} |1\rangle_B + \langle 1|_B \rho_{AB} |0\rangle_B + \langle 1|_B \rho_{AB} |1\rangle_B + \langle 0|_B \rho_{AB} |0\rangle_B - \langle 0|_B \rho_{AB} |1\rangle_B - \langle 1|_B \rho_{AB} |0\rangle_B + \langle 1|_B \rho_{AB} |1\rangle_B)$
   $= \frac{1}{2}(2\langle 0|_B \rho_{AB} |0\rangle_B + 2\langle 1|_B \rho_{AB} |1\rangle_B)$
   $= \langle 0|_B \rho_{AB} |0\rangle_B + \langle 1|_B \rho_{AB} |1\rangle_B$
   $= -\frac{1}{2}|0\rangle\langle 0| - \frac{1}{2}|1\rangle\langle 1|$

4. Consider now the $2n$-qubit state $\rho = |\Psi_n\rangle\langle\Psi_n|$, with $|\Psi_n\rangle = \frac{1}{\sqrt{2^n}}\sum_i |i\rangle_A \otimes |i\rangle_B$. What is the reduced n-qubit state on Alice's side? (Hint: look at the state in vector form and factorize it in a smart way.) (8 points)

   Answer: $\rho = \frac{1}{2^n}\sum_i |i\rangle_A \otimes |i\rangle_B \langle i|_A \langle i|_B$
   $= \frac{1}{2^n}\sum_i |i\rangle_A \otimes |i\rangle_B \langle i|_A \otimes \langle i|_B$
   $Tr_B(\rho) = \frac{1}{2^n}\sum_k \sum_i \langle k| |i\rangle \langle i| \otimes |i\rangle \langle i| |k\rangle$
   If $i \neq k$, $\langle k| |i\rangle = 0$
   $Tr_B(\rho) = \frac{1}{2^n}\sum_i \langle i| |i\rangle \otimes |i\rangle \langle i| \otimes \langle i| |i\rangle$
   $= \frac{1}{2^n}\sum_i |i\rangle \langle i|$
   $= \frac{I_n}{2^n}$

5. What is the reduced state on the first k qubits of A and B (i.e. tracing out the last $n-k$ qubits on each side)?(4 points)

   Answer: $\rho_{0\to k-1} = Tr_{\rho_{k\to n-1}}(\rho_{AB}) = \sum_{m=k}^{n-1} \langle m| |\Psi_n\rangle \langle\Psi_n| |m\rangle$
   $= \frac{1}{2^n}\sum_{m=k}^{n-1}\sum_i \langle m| |i\rangle_A \otimes |i\rangle_B \langle i|_A \otimes \langle i|_B |m\rangle$
   if $m \neq i$, $\langle m| |i\rangle = 0$.
   So, $\rho_{0\to k} = \frac{1}{2^n}\sum_{i=k}^{n-1} |i\rangle\langle i| = \frac{I_{n-k-1}}{2^n}$

3 EXERCISE 3: EVOLUTIONS AND KRAUS OPERATORS

1. Consider the following channel C. It maps the classical state $|0\rangle$ to the state $|0\rangle$ with probability $(1-p)$ and to $|1\rangle$ with probability p. Symmetrically, the state $|1\rangle$ is mapped to the state $|1\rangle$ with probability $(1-p)$ and to $|0\rangle$ with probability p. Find a Kraus operator representation of the channel and show that your choice is valid, and that it maps the classical states $|0\rangle$ and $|1\rangle$ correctly. (4 points)

   Answer: $\{E_0, E_1, E_2, E_3\} = \{\sqrt{1-p}\,|0\rangle\langle 0|, \sqrt{p}\,|0\rangle\langle 1|, \sqrt{p}\,|1\rangle\langle 0|, \sqrt{1-p}\,|1\rangle\langle 1|\ \}$
   $\sum_i E_i^\dagger E_i = (1-p)\,|0\rangle\langle 0|0\rangle\langle 0| + p\,|1\rangle\langle 0|0\rangle\langle 1| + (1-p)\,|0\rangle\langle 1| |1\rangle\langle 0| + p\,|1\rangle\langle 1| |1\rangle\langle 1|$
   $= |0\rangle\langle 0| + |1\rangle\langle 1| = I$
   $C(\rho) = \sum_i E_i \rho E_i^\dagger$
   $C(|0\rangle\langle 0|) = (1-p)\,|0\rangle\langle 0| |0\rangle\langle 0| |0\rangle\langle 0| + p\,|1\rangle\langle 0| |0\rangle\langle 0| |0\rangle\langle 1| + p\,|0\rangle\langle 1| |0\rangle\langle 0| |1\rangle\langle 0| + (1-p)\,|1\rangle\langle 1| |0\rangle\langle 0| |1\rangle\langle 1|$
   $= (1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|$
   $C(|1\rangle\langle 1|) = (1-p)\,|0\rangle\langle 0| |1\rangle\langle 1| |0\rangle\langle 0| + p\,|1\rangle\langle 0| |1\rangle\langle 1| |0\rangle\langle 1| + p\,|0\rangle\langle 1| |1\rangle\langle 1| |1\rangle\langle 0| + (1-p)\,|1\rangle\langle 1| |1\rangle\langle 1| |1\rangle\langle 1|$
   $= p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$
   Assume a coherent state $\rho = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$
   $C(\rho) = \sum_i E_i \rho E_i^\dagger = \frac{1}{2}((1-p)\,|0\rangle\langle 0| + p\,|0\rangle\langle 0| + p\,|1\rangle\langle 1| + (1-p)\,|1\rangle\langle 1|) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$
   So, this Kraus operator representation of the channel is valid, and it maps the classical states $|0\rangle$ and $|1\rangle$ correctly.

2. Apply the classical channel C to a general quantum state $\rho = \sum_{i,j}\alpha_{i,j}|i\rangle\langle j|$ (using the Kraus operators found above) and demonstrate that the off-diagonal terms vanish.

   $C(\rho) = \sum_k E_k \rho E_k^\dagger$
   $= \sum_k E_k \sum_{i,j}\alpha_{i,j}|i\rangle\langle j| E_k^\dagger$
   $= (1-p)\,|0\rangle\langle 0| \sum_{i,j}\alpha_{i,j}|i\rangle\langle j| |0\rangle\langle 0| + p\,|0\rangle\langle 1| \sum_{i,j}\alpha_{i,j}|i\rangle\langle j| |1\rangle\langle 0| + p\,|1\rangle\langle 0| \sum_{i,j}\alpha_{i,j}|i\rangle\langle j| |0\rangle\langle 1| + (1-p)\,|1\rangle\langle 1| \sum_{i,j}\alpha_{i,j}|i\rangle\langle j| |1\rangle\langle 1|$
   If $i \neq k$ or $j \neq k$, $\langle k|i\rangle\langle j|k\rangle = 0$
   $C(\rho) = (1-p)\alpha_{00}|0\rangle\langle 0| + p\alpha_{11}|0\rangle\langle 0| + p\alpha_{00}|1\rangle\langle 1| + (1-p)\alpha_{11}|1\rangle\langle 1|$
   $= [(1-p)\alpha_{00} + p\alpha_{11}]\,|0\rangle\langle 0| + [(1-p)\alpha_{11} + p\alpha_{00}]\,|1\rangle\langle 1|$
   This equation shows that the off-diagonal terms vanish.

3. Can you think of a quantum version of the channel, which operates correctly on the coherent (off-diagonal) terms of ρ? Write it down and show that it is valid and correct. Hint: apply the new set of Kraus operators first to the classical states $|0\rangle$ and $|1\rangle$ to check correctness. Then, apply it to ρ and show that it preserves coherence. (8 points)

   Answer: $\{E_0, E_1\} = \{\sqrt{p}\,|0\rangle\langle 0| + \sqrt{p}\,|1\rangle\langle 1|, \sqrt{1-p}\,|0\rangle\langle 1| + \sqrt{1-p}\,|1\rangle\langle 0|\}$

   $\sum_i E_i^\dagger E_i = (1-p)\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1| + p\,|0\rangle\langle 0| + p\,|1\rangle\langle 1| = I$

   $C(|0\rangle\langle 0|) = E_0\,|0\rangle\langle 0|\,E_0^\dagger + E_1\,|0\rangle\langle 0|\,E_1^\dagger = p\,|1\rangle\langle 1| + (1-p)\,|0\rangle\langle 0|$

   $C(|1\rangle\langle 1|) = E_0\,|1\rangle\langle 1|\,E_0^\dagger + E_1\,|1\rangle\langle 1|\,E_1^\dagger = p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$

   For a classical state the off-diagonal terms vanish.

   For a coherent state $\rho = \alpha_{00}\,|0\rangle\langle 0| + \alpha_{01}\,|0\rangle\langle 1| + \alpha_{10}\,|1\rangle\langle 0| + \alpha_{11}\,|1\rangle\langle 1|$

   $C(\rho) = \sum_i E_i \rho E_i^\dagger = p\alpha_{00}\,|0\rangle\langle 0| + p\alpha\,|1\rangle\langle 1| + (1-p)\alpha_{10}\,|0\rangle\langle 1| + (1-p)\alpha_{01}\,|1\rangle\langle 0|$

   For a quantum state, the off-diagonal terms do not vanish.

   So, this channel is valid and correct and it operates correctly on the coherent (off-diagonal) terms of ρ.

4 EXERCISE 4: THE GATE MODEL

Consider the following circuit diagram:

The initial states are the two qubits $|+\rangle$ and $|-\rangle$. The upper wire is initialized in state $|+\rangle$ and experiences a controlled X gate. The lower wire is initialized in state $|-\rangle$, then first transformed by the Hadamard gate, and afterwards acts as a control state for the controlled X gate. Finally, a measurement M measures each qubit in the computational basis.

1. What is the state of the system before the measurement M? (4 points)

   Answer: $H(|-\rangle) = (\frac{1}{\sqrt{2}}\,|0\rangle\langle 0| + \frac{1}{\sqrt{2}}\,|0\rangle\langle 1| + \frac{1}{\sqrt{2}}\,|1\rangle\langle 0| - \frac{1}{\sqrt{2}}\,|1\rangle\langle 1|)\,|-\rangle$
   $= \frac{1}{2}(|0\rangle + |1\rangle - |0\rangle + |1\rangle)$
   $= |1\rangle$

   So the output of gate H is $|1\rangle$, which means gate X will be applied to the upper wire.

   $X(|+\rangle) = (|0\rangle\langle 1| + |1\rangle\langle 0|)\,|+\rangle$
   $= (|0\rangle\langle 1| + |1\rangle\langle 0|)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
   $= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   So the state before measurement is $|+\rangle \otimes |1\rangle$.

2. Which are the possible measurement outcomes of M and what are their probabilities? (4 points)

   Answer: The possible measurement outcomes of M are $|01\rangle$ and $|11\rangle$.

   $P(|01\rangle) = |\frac{1}{\sqrt{2}}\,\langle 10|\,(|01\rangle + |11\rangle)|^2 = \frac{1}{2}$

   $P(|11\rangle) = |\frac{1}{\sqrt{2}}\,\langle 11|\,|01\rangle + |11\rangle|^2 = \frac{1}{2}$

   The probability of $|01\rangle$ is $\frac{1}{2}$ and the probability of $|11\rangle$ is $\frac{1}{2}$.

   In the lectures we saw the controlled Z and controlled X gate, where $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. Are the following two circuits the same? How about when we replace Z with X? (6 points)

   Answer: Assume the input of the upper wire is $|\psi\rangle_A$ and the input of the lower wire is $|\psi\rangle_B$.

   When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 0$, for the first circuit, the upper output is $|0\rangle$, and Z gate is not activated, the lower output is $|0\rangle$. For the second circuit, the output of lower wire is $|0\rangle$, the Z gate is not activated, the output of upper wire is $|0\rangle$.

When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 1$, for the first circuit, the upper output is $|0\rangle$, and Z gate is not activated, the lower output is $|1\rangle$. For the second circuit, the output of lower wire is $|1\rangle$, the Z gate is activated, the output of upper wire is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle = |0\rangle$.

When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 0$, for the first circuit, the upper output is $|1\rangle$, and Z gate is activated, the lower output is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle = |0\rangle$. For the second circuit, the output of lower wire is $|0\rangle$, the Z gate is activated, the output of upper wire is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle = -|1\rangle$.

When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 1$, for the first circuit, the upper output is $|1\rangle$, and Z gate is activated, the lower output is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle = |1\rangle$. For the second circuit, the output of lower wire is $|1\rangle$, the Z gate is activated, the output of upper wire is $(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle = -|1\rangle$.

So for gate Z, the two circuits are not the same.

If we replace Z with X, the two circuits are not the same. When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 0$, the output of the upper wire of the first circuit is $|0\rangle$ and the output of the lower wire is $|0\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|0\rangle$. When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 1$, the output of the upper wire of the first circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$. When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 0$, the output of the upper wire of the first circuit is $|1\rangle$ and the output of the lower wire is $|1\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$. When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 1$, the output of the upper wire of the first circuit is $|1\rangle$ and the output of the lower wire is $|0\rangle$. The output of the upper wire of the second circuit is $|0\rangle$ and the output of the lower wire is $|1\rangle$.

So for gate X, the two circuits are not the same.

4. The SWAP gate acts on a quantum state $|i\rangle \otimes |j\rangle$ as SWAP $|ij\rangle = |ji\rangle$. Show that the following circuit implements the SWAP gate (the symbol L is often used in the literature for the X gate). (8 points)

   Answer: Assume the input of the upper wire is $|\psi\rangle_A$ and the input of the lower wire is $|\psi\rangle_B$.

   When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 0$, the output of the upper wire is $|0\rangle$ and the output of the lower wire is $|0\rangle$.

   When $|\psi\rangle_A = 0$, $|\psi\rangle_B = 1$, the output of the upper wire is $|1\rangle$ and the output of the lower wire is $|0\rangle$.

   When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 0$, the output of the upper wire is $|0\rangle$ and the output of the lower wire is $|1\rangle$.

   When $|\psi\rangle_A = 1$, $|\psi\rangle_B = 1$, the output of the upper wire is $|1\rangle$ and the output of the lower wire is $|1\rangle$.

5. Construct a controlled X gate by using Hadamard gates and controlled Z gates (you can use either as often as you like). Draw your circuit diagram. (6 points)

   Answer: the picture is shown as figure 1.

   I designed the circuit because for input ρ, $H(Z(H(\rho))) = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}$$
$$= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$
$$= X$$

   and the upper wire controls the gates.

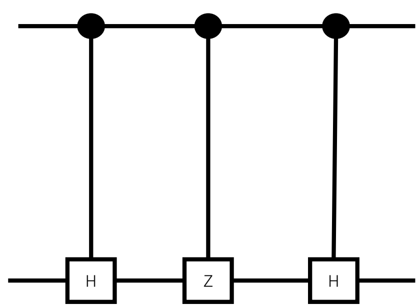

Figure 1: c-X circuit