FINAL PROJECT

Incompressible Navier-Stokes for Lid-Driven Cavity using Streamfunction-Vorticity Method

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Contents

1 Introduction

The lid-driven cavity flow has long served as a canonical benchmark for the numerical solution of the incompressible Navier–Stokes equations. Despite its geometric simplicity, the problem exhibits complex flow features such as strong recirculation, corner vortices, and increasing sensitivity to numerical discretization as the Reynolds number increases. Consequently, it has been widely used to assess the accuracy, stability, and efficiency of numerical methods for incompressible flows. The problem can be formulated as solving the Incompressible Navier-Stokes,

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u} \tag{2}$$

with $u(0, y, t) = u(x, 0, t) = u(L, y, t) = 0$, $v(0, y, t) = v(x, 0, t) = v(L, y, t) = v(x, L, t) = 0$, and $u(x, L, t) = 1$.

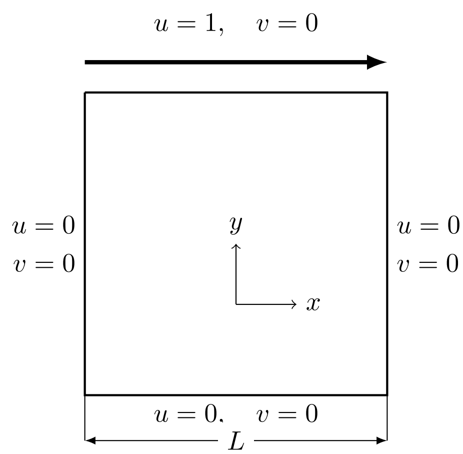

Figure 1: Lid-Driven Cavity

The depiction of the problem can be seen in Figure 1.

2 Related Work

Many benchmark studies compare results to [1], who utilized a multi-grid strategy with a strongly implicit method to discretize the streamfunction and vorticity Navier-Stokes equations. They use grid meshes up to 257×257 number of grid points and Reynolds numbers up to $10,000$.

Other numerical approaches, such as in [2] used a p-type finite-element scheme on 257×257 fine element mesh and yielded highly accurate solutions for Reynolds numbers up to $12,500$.

Further advances in solution accuracy were achieved using spectral methods. In [3] they use Chebyshev spectral discretizations, obtaining highly accurate spectral solutions for the cavity flow with a maximum grid mesh of $N = 160$ for Reynolds numbers up to $9,000$.

In [4] they applied the Block Implicit Multigrid Method (BIMM) to the SMART and QUICK discretizations, yielding results on a 1024×1024 grid mesh for $\mathrm{Re} \leq 1,000$.

The streamfunction-vorticity formulation provides an elegant approach to solving the incompressible Navier-Stokes equations by eliminating the pressure term altogether, automatically satisfying the continuity equation through the streamfunction definition.

3 Problem Formulation

We can express the following problem component-wise, taking $\boldsymbol{u} = (u, v)$, and yield

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

$$\frac{\partial u}{\partial t} + \left(\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4}$$

$$\frac{\partial v}{\partial t} + \left(\frac{\partial v^2}{\partial y} + \frac{\partial uv}{\partial x}\right) = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{5}$$

3.1 Streamfunction-Vorticity Formulation

The streamfunction-vorticity formulation provides an elegant approach to solving incompressible flow problems by introducing two derived variables that eliminate the pressure term and automatically satisfy continuity. We define the streamfunction ψ and vorticity ω as:

$$\psi : \quad \frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u \tag{6}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{7}$$

The streamfunction is defined such that the continuity equation is automatically satisfied. By taking the derivative of the v-momentum equation with respect to x and subtracting the derivative of the u-momentum equation with respect to y, the pressure term cancels out, yielding the vorticity transport equation:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{8}$$

Expressing vorticity in terms of the streamfunction gives the streamfunction Poisson equation:

$$\omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \tag{9}$$

This formulation reduces the problem to two equations with two unknowns (ψ and ω), with no explicit pressure coupling.

4 Numerical Formulation

The streamfunction-vorticity approach provides a straightforward numerical solution strategy. Starting with an initial velocity field, we solve the vorticity transport equation explicitly, then solve the streamfunction Poisson equation. Once the streamfunction is obtained, we compute the velocity field and march forward in time.

4.1 Discretization

We use second-order accurate central difference discretization for all spatial derivatives and first-order forward Euler for time integration (FTCS scheme). The discretized equations are:

$$\frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n}}{\Delta t} = -u_{i,j}^{n}\delta_x \omega_{i,j}^{n} - v_{i,j}^{n}\delta_y \omega_{i,j}^{n} + \frac{1}{\mathrm{Re}}\left(\delta_{xx}(\omega_{i,j}^{n}) + \delta_{yy}(\omega_{i,j}^{n})\right) \tag{10}$$

$$\delta_{xx}(\psi_{i,j}^{n+1}) + \delta_{yy}(\psi_{i,j}^{n+1}) = -\omega_{i,j}^{n+1} \tag{11}$$

where δ_x and δ_{xx} denote first and second-order central difference operators:

$$\delta_x(\phi_{i,j}) = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h_x}, \quad \delta_{xx}(\phi_{i,j}) = \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{h_x^2}$$

Note that we consider that the steady-state is achieved when $||\Delta\omega|| \leq \epsilon$ for some error tolerance ϵ. In our experiment we set the error tolerance to be $\epsilon = 10^{-5}$.

4.2 Peudo-Code

Streamfunction-Vorticity Solver for Lid-Driven Cavity

1. Initialize velocity field
2. Derive initial ψ and ω fields
3. Select time step according to stability criteria
4. While not converged:
5. Solve vorticity transport equation explicitly to obtain ω^{n+1}
6. Apply boundary conditions on ω
7. Solve streamfunction Poisson equation for ψ^{n+1}
8. Compute velocity field
9. Check convergence
10. End While

5 Boundary Conditions

For the lid-driven cavity problem, we impose velocity boundary conditions: $u = 1, v = 0$ at the top lid, and $u = 0, v = 0$ (no-slip) at the other three walls.

In the streamfunction-vorticity formulation, we must derive boundary conditions for ψ and ω from the velocity conditions:

5.1 Streamfunction Boundary Conditions

Using four of the velocity boundary conditions, we determine that the streamfunction must be constant along each wall. We set $\psi = 0$ on all walls for convenience.

5.2 Vorticity Boundary Conditions

The remaining four velocity conditions (normal derivatives of ψ) are used to derive vorticity boundary conditions. At each wall, we use the Poisson equation for ψ with ghost cells. For example, at the bottom wall ($y = 0$):

$$\omega_{i,1}^{n+1} = -\left(\frac{\psi_{i+1,1}^n - 2\psi_{i,1}^n + \psi_{i-1,1}^n}{h_x^2} + \frac{\psi_{i,2}^n - 2\psi_{i,1}^n + \psi_{i,0}^n}{h_y^2}\right)$$

Using the boundary condition $\left.\frac{\partial \psi}{\partial y}\right|_{y=0} = u|_{y=0} = 0$, we obtain:

$$\frac{\psi_{i,2}^n - \psi_{i,0}^n}{2h_y} = 0 \implies \psi_{i,0}^n = \psi_{i,2}^n$$

Substituting this into the vorticity expression yields:

$$\omega_{i,1}^{n+1} = 2\left(\frac{\psi_{i,1}^n - \psi_{i,2}^n}{h_y^2}\right)$$

Similar expressions are derived for all walls. Note that this approach uses ψ values from the previous time step, which is valid for steady-state problems since the values converge to the correct solution.

6 Results

6.1 Velocity Profiles

6.2 Vorticity Contours

6.3 Streamlines

6.4 Convergence Analysis

7 Conclusion

References

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