MAT 4800 Homework # 4

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Problem #1

Part a

Suppose we are given the following function,

$$f(x_1, x_2) = 5x_1^2 + 2x_1x_2 + x_2^2 - x_1 + 2x_2 + 3$$

on $D = \mathbb{R}^2$.

Here we will compute the Hessian as

$$Hf = \begin{bmatrix} 10 & 2 \\ 2 & 2 \end{bmatrix}$$

by the principal minors approach we see that,

$$\begin{aligned} \Delta_1 &= 10 > 0 \\ \Delta_2 &= 20 - 4 = 16 > 0 \end{aligned}$$

thus the Hessian is Positive Definite and thus $f(x_1, x_2)$ is convex, namely strictly convex by *Theorem 2.3.7*.

Part b

Suppose we are given the following function,

$$f(x_1, x_2) = \frac{x_1^2}{2} + \frac{3x_2^2}{2} + \sqrt{3}x_1x_2$$

on $D = \mathbb{R}^2$.

$$Hf = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}$$

by the principal minors approach we see that,

$$\Delta_1 = 1 > 0$$
$$\Delta_2 = 0$$

thus the Hessian is Positive semi-definite and thus $f(x_1, x_2)$ is convex by *Theorem 2.3.7*.

Part e

Suppose we are given the following function,

$$f(x_1, x_2) = c_1 x_1 + \frac{c_2}{x_1} + c_3 x_2 + \frac{c_4}{x_2}$$

on $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ where c_i is a positive number for $i = 1, 2, 3, 4$.

$$Hf = \begin{bmatrix} \frac{2c_2}{x_1^3} & 0 \\ 0 & \frac{2c_4}{x_2^3} \end{bmatrix}$$

by the principal minors approach we see that,

$$\Delta_1 = \frac{2c_2}{x_1^3} > 0$$
$$\Delta_2 = \frac{4c_2 c_4}{x_1^3 x_2^3} > 0$$

thus the Hessian is Positive Definite and thus $f(x_1, x_2)$ is convex, namely strictly convex by *Theorem 2.3.7*.

Problem #2

Suppose we have the function $f(\vec{x})$ defined on the set,

$$D = \{\vec{x} \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\}$$

by,

$$f(\vec{x}) = (x_1)^{r_1} + (x_2)^{r_2} + (x_2)^{r_2}$$

where $r_i > 0$ for $i = 1, 2, 3$

we compute the Hessian Matrix as,

$$Hf = \begin{bmatrix} (r_1 - 1)r_1 x_1^{r_1 - 2} & 0 & 0 \\ 0 & (r_2 - 1)r_2 x_2^{r_2 - 2} & 0 \\ 0 & 0 & (r_3 - 1)r_3 x_3^{r_3 - 2} \end{bmatrix}$$

such that,

$$d_1 = (r_1 - 1)r_1 x_1^{r_1 - 2}$$
$$d_2 = (r_2 - 1)r_2 x_2^{r_2 - 2}$$
$$d_3 = (r_3 - 1)r_3 x_3^{r_3 - 2}$$

Part a

Here if we assume that $r_i > 1$, then note by construction that

$$d_1 > 0$$
$$d_2 > 0$$
$$d_3 > 0$$

which means that the Hessian is **Positive Definite** and thus $f(x_1, x_2)$ is convex, namely strictly convex by *Theorem 2.3.7.*

Part b

Here if we instead assume that $r_i < 1$, then note by construction that

$$d_1 < 0$$
$$d_2 < 0$$
$$d_3 < 0$$

which means that the Hessian is **Negative Definite** and thus $f(x_1, x_2, x_3)$ is concave, namely strictly concave by *Theorem 2.3.7.*

Problem #3

Suppose we are given the function,

$$f(x_1, x_2, x_3) = x_1^{-q} + x_2^{-r} + x_3^{-s}$$

defined on $D = \{(x_1, x_2, x_3) : x_1 > 0, x_2 > 0, x_3 > 0\}$ for $q, r, s \in \mathbb{N}$. Here we first compute the Hessian matrix as,

$$Hf = \begin{bmatrix} -(-q-1)qx_1^{-q-2} & 0 & 0 \\ 0 & -(-r-1)rx_2^{-r-2} & 0 \\ 0 & 0 & -(-s-1)sx_3^{-s-2} \end{bmatrix}$$

such that,

$$\begin{aligned} d_1 &= -(-q-1)qx_1^{-q-2} = (q+1)qx_1^{-q-2} > 0 \\ d_2 &= -(-r-1)rx_2^{-r-2} = (r+1)rx_2^{-r-2} > 0 \\ d_3 &= -(-s-1)sx_3^{-s-2} = (s+1)sx_3^{-s-2} > 0 \end{aligned}$$

Thus the Hessian is Positive Definite and thus $f(x_1, x_2, x_3)$ is convex, namely strictly convex by *Theorem 2.3.7*.

Problem #4

Suppose we are given the following function,

$$f(x_1, x_2, x_3) = x_1 \ln x_1 + x_2 \ln x_2 + x_3 \ln x_3$$

on $x_1, x_2, x_3 > 0$. Here we compute the Hessian as,

$$Hf = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$$

such that,

$$\begin{aligned} d_1 &= \frac{1}{x} > 0 \\ d_2 &= \frac{1}{y} > 0 \\ d_3 &= \frac{1}{z} > 0 \end{aligned}$$

Thus the Hessian is Positive Definite and thus $f(x_1, x_2, x_3)$ is convex, namely strictly convex by *Theorem 2.3.7*.

Problem #5

Suppose we are given the function,

$$f(x_1, x_2) = \sqrt{x_1 x_2}$$

with $x_1, x_2 > 0$. We compute the Hessian as,

$$Hf = \begin{bmatrix} -\frac{y^2}{4(xy)^{3/2}} & \frac{1}{2\sqrt{xy}} - \frac{xy}{4(xy)^{3/2}} \\ \frac{1}{2\sqrt{xy}} - \frac{xy}{4(xy)^{3/2}} & -\frac{x^2}{4(xy)^{3/2}} \end{bmatrix}$$

Note that,

$$\begin{aligned} \Delta_1 &= -\frac{y^2}{4(xy)^{3/2}} < 0 \\ \Delta_2 &= 0 \end{aligned}$$

Here we see that that the Hessian is Negative Semi-definite and thus $f(x_1, x_2, x_3)$ is concave by *Theorem 2.3.7*.

Problem #6

Suppose we are given the function,

$$f(x) = x + \frac{1}{x}$$

for $x > 0$. We first compute the derivative,

$$f'(x) = 1 - \frac{1}{x^2}$$

setting equal to 0, we get

$$0 = 1 - \frac{1}{x^2}$$

yields $x^* = 1$ which is the critical point.

Part a

By computing the second derivative we get

$$f''(x) = \frac{2}{x^3}$$

plugging in x^* to get,

$$f''(1) = \frac{2}{1^3} = 2 > 0$$

Thus see that $x^* = 1$ is a strict minimizer of $f(x)$.

Problem #7

Plotting the $f(x) = x + \frac{1}{x}$ we get,

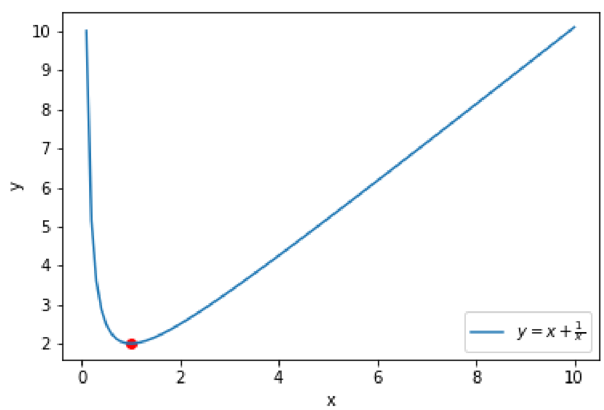

Figure 1: Graph of $f(x) = x + \frac{1}{x}$

Problem #8

Suppose we are given,

$$g(x_1, x_2) = 2x_1^2 + x_2^2 + \frac{1}{2x_1^2 + x_2^2}$$

let $y(x_1, x_2)$ be defined as,

$$y(x_1, x_2) = 2x_1^2 + x_2^2$$

we then rewrite g as,

$$g(x_1, x_2) = y(x_1, x_2) + \frac{1}{y(x_1, x_2)}$$

note that from (a) the above equation has a global minimum at $y(x_1, x_2) = 1$, thus we find the critical points of g lie on $2x_1^2 + x_2^2 = 1$ which are the minimizers of $g(x_1, x_2)$, where $g(x_1, x_2) = 2$.

Problem #9

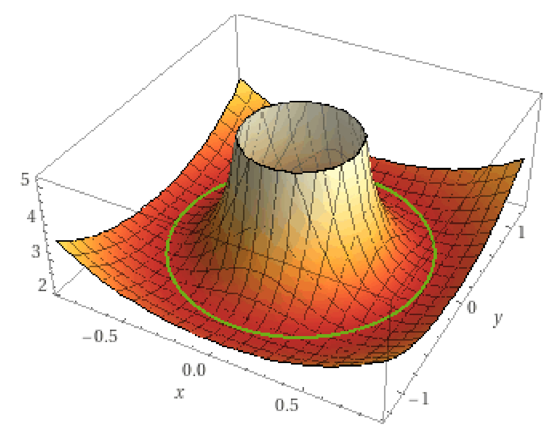

Figure 2: graph of $g(x_1, x_2)$

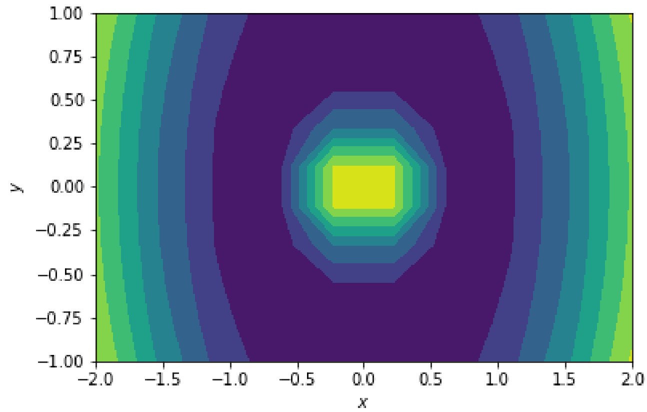

Figure 3: contour of $g(x_1, x_2)$

Problem #10

Suppose we are given,

$$h(x_1, x_2, x_3) = e^{x_1 - x_2 + x_3} + e^{-x_1 + x_2 - x_3}$$

here we let $j(x_1, x_2)$ be defined as,

$$j(x_1, x_2, x_3) = x_1 - x_2 + x_3$$

then we rewrite h as,

$$h(x_1, x_2, x_3) = e^{j(x_1, x_2, x_3)} + e^{-j(x_1, x_2, x_3)}$$

next let $t = e^{j(x_1, x_2, x_3)}$ to get,

$$h(x_1, x_2, x_3) = t + \frac{1}{t}$$

then we see that from (a) we see that the minimizers occur at $t^* = 1$, thus we see,

$$e^{j(x_1, x_2, x_3)=1}$$

which implies that,

$$x_1 - x_2 + x_3 = 0$$

of which the plane defined above is the minimizers of h, with $h(x_1, x_2, x_3) = 2$.