MAT 4800 Homework # 5

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Problem #1

Here we are asked to find the “largest circular cylinder that can be inscribed in a sphere of a given radius" to do this we first note that the volume of cylinder is denoted as,

$$V_c = \pi r_c^2 h_c$$

where r_c is the radius of the cylinder and h_c is the height of the cylinder. Now supposing that circular cylinder is inscribed in a sphere with fixed radius r_0, our goal is now to,

$$\max V_c = \pi r_c^2 h_c$$

subject to:

$$r_c^2 + \left(\frac{h_c}{2}\right)^2 = r_0^2$$

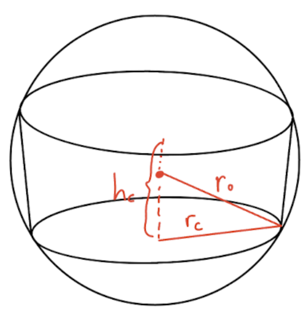

Figure 1: Cylinder Inscribed in Sphere with fixed radius r_0

Note that we can rewrite the above constraint as,

$$\frac{2}{3}\left(\frac{r_c^2}{2}\right)+\frac{1}{3}\left(\frac{h_c}{2}\right)^2=\frac{1}{3}r_0^2$$

then by the A-G inequality we get,

$$\left(\left(\frac{r_c^2}{2}\right)^{2/3}\left(\frac{h_c}{2}\right)^{2/3}\right)\leq\frac{2}{3}\left(\frac{r_c^2}{2}\right)+\frac{1}{3}\left(\frac{h_c}{2}\right)^2=\frac{1}{3}r_0^2$$

with,

$$\left(\left(\frac{r_c^2}{2}\right)^{2/3}\left(\frac{h_c}{2}\right)^{2/3}\right)=\left(\frac{1}{4}r_c^2h_c\right)^{2/3}=\left(\frac{1}{4\pi}V_c\right)^{2/3}\leq\frac{1}{3}r_0^2$$

or,

$$V_c\leq\frac{4\pi r_0^3}{3\sqrt{3}}$$

note that we attain equality when,

$$\frac{r_c^2}{2}=\frac{h_c^2}{4}$$

and get,

$$r_c=\sqrt{\frac{2}{3}}r_0$$

and,

$$h_c=\frac{2}{\sqrt{3}}r_0$$

to get,

$$V_c=\frac{4\pi r_0^3}{3\sqrt{3}}$$

Problem #2

Here we will solve again the problem above using the standard calculus approach. First note the formula above for the volume a cylinder inscribed in a sphere with a fixed radius as,

$$V_c = 2\pi r_c^2 \sqrt{(r_0^2 - r_c^2)}$$

next we take the derivative as,

$$\frac{d}{dr_c} V_c = 4\pi r_c \sqrt{(r_0^2 - r_c^2)} - 2\pi r_c^3 (r_0^2 - r_c^2)^{-1/2}$$

and setting it equal to 0,

$$4\pi r_c \sqrt{(r_0^2 - r_c^2)} - 2\pi r_c^3 (r_0^2 - r_c^2)^{-1/2} = 0$$

$$2(r_0^2 - r_c^2) = r_c^2$$

$$r_c = \sqrt{\frac{2}{3}} r_0$$

and plugging into h_c to find,

$$h_c = \frac{2}{\sqrt{3}} r_0$$

we also find the volume of the inscribed circular cylinder as,

$$V_c = \frac{4\pi r_0^3}{3\sqrt{3}}$$

which is the same as above in **Problem #1**

Problem #3

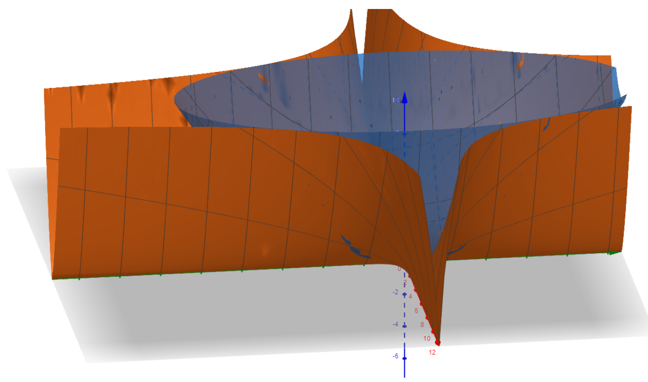

Figure 2: Plot of Summation (Blue) and Product (Orange)

Here we see that the product is below the summation, which agrees with our AG inequality.

Problem #4

Here we want to find the smallest radius r such that a circular cylinder of volume 8 cubic units can be inscribed in a a sphere of radius r, that is we want to do the following,

$$\min r^2 = r_c^2 + \left(\frac{h_c}{2}\right)^2$$

subject to:

$$\pi r_c^2 h_c = 8$$

note that we can write the following,

$$\frac{2}{3}\left(\frac{r_c^2}{2}\right) + \frac{1}{3}\left(\frac{h_c}{2}\right)^2 = \frac{1}{3}r^2$$

then by the A-G inequality we get,

$$\left(\left(\frac{r_c^2}{2}\right)^{2/3}\left(\frac{h_c}{2}\right)^{2/3}\right) \leq \frac{2}{3}\left(\frac{r_c^2}{2}\right) + \frac{1}{3}\left(\frac{h_c}{2}\right)^2 = \frac{1}{3}r^2$$

with,

$$\left(\left(\frac{r_c^2}{2}\right)^{2/3}\left(\frac{h_c}{2}\right)^{2/3}\right) = \left(\frac{1}{4}r_c^2 h_c\right)^{2/3} = \left(\frac{8}{4\pi}\right)^{2/3} \leq \frac{1}{3}r^2$$

or,

$$\sqrt{3}\sqrt[3]{\frac{2}{\pi}} = \frac{2\sqrt{3}}{\sqrt[3]{4\pi}} \leq r$$

thus the smallest r is when,

$$r = \frac{2\sqrt{3}}{\sqrt[3]{4\pi}}$$

Problem #5

Here we can see that, following in similar steps as in **Problem #1** we get,

$$V_c = \frac{4\pi r^3}{3\sqrt{3}}$$

as the volume of a cylinder, note that however if we let $V_c = 8$ be the largest possible cylinder to be inscribed in our Sphere we can find the smallest possible radius to allow this to be,

$$8 = \frac{4\pi r^3}{3\sqrt{3}}$$

to get,

$$r = \frac{2\sqrt{3}}{\sqrt[3]{4\pi}}$$

which is the same as we expected above.

Problem #6

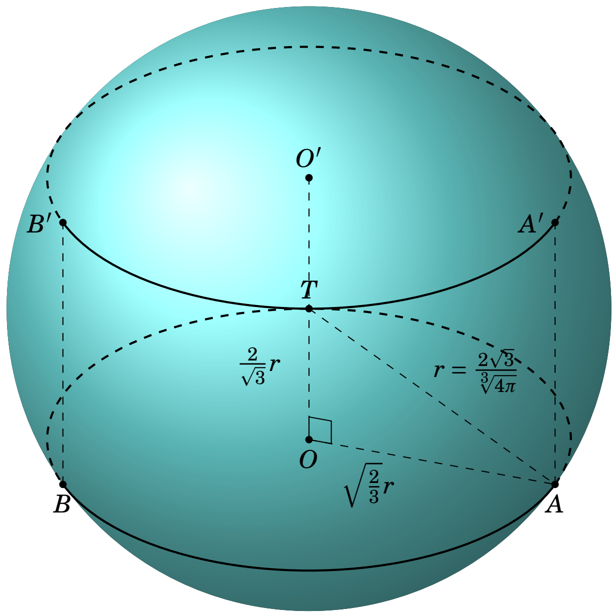

Figure 3: Cylinder Inscribed in Sphere