MAT 4800 Homework # 1

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Problem #1

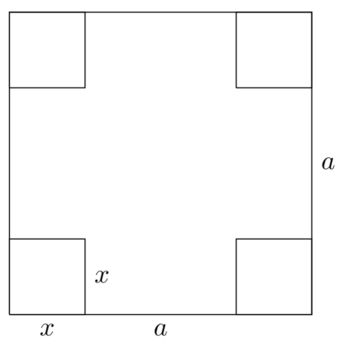

Figure 1: $a \times a$ Square Paper

Part a

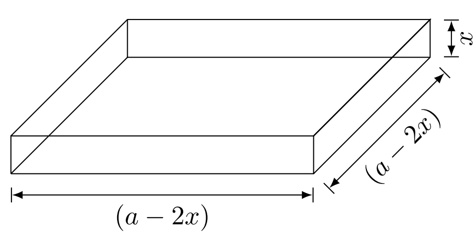

Figure 2: Open-Top Box

Part b

Here we can denote the volume of **Figure 2** by

$$V(x) = (a - 2x)(a - 2x)(x) = x(a - 2x)^2 \tag{1}$$

Part c

Here we see from (1),

$$\begin{aligned} V'(x) &= (a-2x)^2 - 4x(a-2x) \\ &= 12x^2 - 8ax + a^2 = (a-2x)(a-6x) = 0 \end{aligned}$$

we get $x = \frac{a}{2}, \frac{a}{6}$, then

$$V''(x) = -8(a-3x)$$

and

$$\begin{aligned} V''\left(\frac{a}{2}\right) &= 4a \\ V''\left(\frac{a}{6}\right) &= -4a \end{aligned}$$

since $V''\left(\frac{a}{6}\right) < 0$ when $x = \frac{a}{6}$, then when $x = \frac{a}{6}$ the volume of **Figure 2** maximized.

Part d

For the maximum volume of **Figure 2** we get,

$$V\left(\frac{a}{6}\right) = \frac{2}{27}a^3$$

Part e

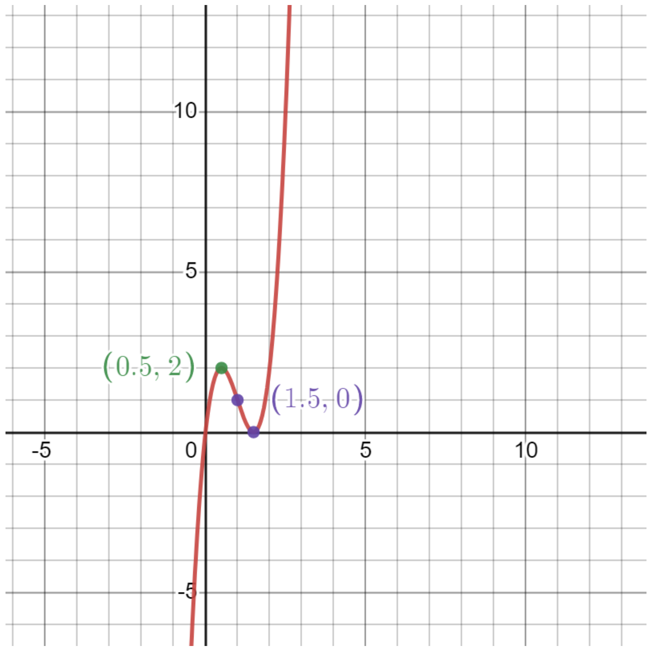

Figure 3: Graph of V(x) with $a = 3$

Here we see that the maximum for $a = 3$ is $x = \frac{1}{2}$ with $V(1/2) = 2$.

Part f

To find the inflection points of $V(x)$ we compute the following,

$$V''(x) = -8(a - 3x) = 0$$

we get $x = \frac{a}{3}$ is the inflection point of $V(x)$. Then for $a = 3$, $x = 1$ is the inflection point with $V(x) = 1$

Question #2

Part a

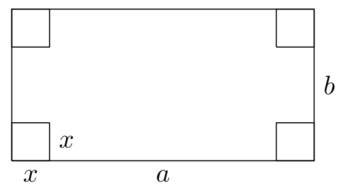

Figure 4: $a \times b$ Square Paper

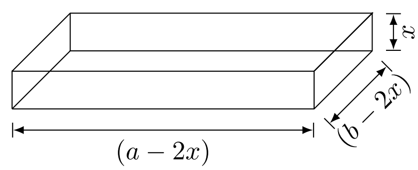

Figure 5: Open-Top Box

Part b

Here we can express the volume of **Figure 5** as,

$$V(x) = x(a - 2x)(b - 2x)$$

Part c

To find the maximum volume of **Figure 5** we first compute,

$$\begin{aligned} V'(x) &= ab - 4ax - 4bx + 12x^2 \\ &= 12x^2 - 4x(a + b) + ab = 0 \end{aligned}$$

and by the *quadratic formula*, we get

$$\begin{aligned} x_1 &= \frac{1}{6}\left(\sqrt{a^2 - ab + b^2} + a + b\right) \\ x_2 &= \frac{1}{6}\left(-\sqrt{a^2 - ab + b^2} + a + b\right) \end{aligned}$$

Next we will find $V''(x)$ by,

$$V''(x) = -4(a + b - 6x)$$

and plug-in the above roots,

$$V''(x_1) = 4\sqrt{a^2 - ab + b^2}$$
$$V''(x_2) = -4\sqrt{a^2 - ab + b^2}$$

Thus since $V''(x_2) < 0$ we see that at $x_2 = \frac{1}{6}\left(-\sqrt{a^2 - ab + b^2} + a + b\right)$ that $V(x)$ achieves a maximum.

Part d

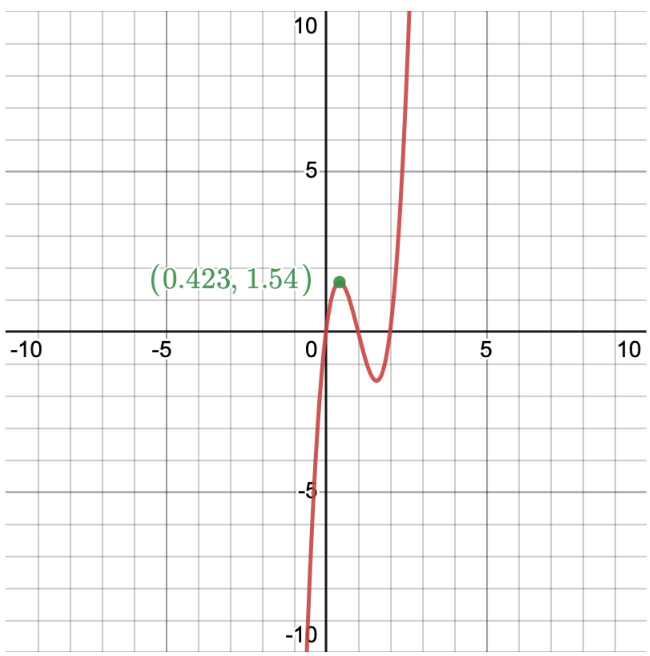

Figure 6: $V(x)$ with $a = 4$ and $b = 2$

Question #3

Part a

Here from the example in $2D$ we will let $a = 8\text{in}$ and $b = 6\text{in}$ for our paper.

Part b

Here we see that the maximum volume is found as,

$$V\left(\frac{1}{6}(14 - 2\sqrt{13})\text{in}\right) = \frac{8}{27}(35 + 13\sqrt{13})\text{in}^3 \approx 24.258\text{in}^3$$

Part c

Below is a physical box with $x = \frac{1}{6}(14 - 2\sqrt{13})$in

Figure 7: 8in ×6in Open-Top Box (best attempt)