MAT 4800 Homework # 3

Noah Reef

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Problem #1

Part a

Suppose we are given the following matrix,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

whose corresponding characteristic polynomial is,

$$(1 - \lambda)(3 - \lambda)(5 - \lambda) = 0$$

which implies $\lambda = 1, 3, 5 > 0$. Thus by *Theorem 1.5.1*, then A is **Positive Definite**.

Part b

Suppose we are given the following matrix,

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

whose corresponding characteristic polynomial is,

$$(-1 - \lambda)(-3 - \lambda)(-2 - \lambda) = 0$$

which implies $\lambda = -1, -3, -2 < 0$. Thus by *Theorem 1.5.1*, then A is **Negative Definite**.

Part c

Suppose we are given the following matrix,

$$A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

whose corresponding characteristic polynomial is,

$$(7-\lambda)(-8-\lambda)(5-\lambda) = 0$$

which implies $\lambda = 7, -8, 5$. Thus by *Theorem 1.5.1*, then A is **Indefinite**.

Part d

Suppose we are given the following matrix,

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{pmatrix}$$

whose principal minors are

$$\begin{aligned} &\Delta_1 = 3 > 0 \\ &\Delta_2 = 15 - 1 = 14 > 0 \\ &\Delta_3 = 3(35-9) - (7-6) + 2(3-10) = 63 > 0 \end{aligned}$$

since $\Delta_{1,2}, \Delta_3 > 0$. Thus by *Theorem 1.3.3*, then A is **Definite Positive**.

Part e

Suppose we are given the following matrix,

$$A = \begin{pmatrix} -4 & 0 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & -5 \end{pmatrix}$$

whose principal minors are

$$\begin{aligned} &\Delta_1 = -4 < 0 \\ &\Delta_2 = 12 > 0 \\ &\Delta_3 = -4(15-4) + (0-(-3)) = -41 < 0 \end{aligned}$$

since $(-1)^k \Delta_k > 0$. Thus by *Theorem 1.3.3*, then A is **Definite Negative**.

Problem #2

Part a

Suppose we are given the following matrix,

$$A = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}$$

Then

$$Q_A(\vec{x}) = x_1(-x_1 + 2x_2) + x_2(2x_1 + 3x_2) = -x_1^2 + 4x_1x_2 + 3x_2^2$$

Part b

Suppose we are given the following matrix,

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 0 \end{pmatrix}$$

Then

$$Q_A(\vec{x}) = x_1(2x_1 - 3x_2) + x_2(-3x_1) = 2x_1^2 - 6x_1x_2$$

Part c

Suppose we are given the following matrix,

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

Then

$$Q_A(\vec{x}) = x_1(x_1 - x_2) + x_2(-x_1 - 2x_2 + 2x_3) + x_3(2x_2 + 3x_3)$$

Problem #3

Part a

Suppose we are given the quadratic form of

$$Q_A(\vec{x}) = 3x_1^2 - x_1x_2 + 2x_2^2$$

results from the matrix

$$A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}$$

thus,

Part b

Suppose we are given the quadratic form of

$$Q_A(\vec{x}) = x_1^2 + 2x_2^2 - 3x_3^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3$$

results from the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 6 \\ -4 & 0 & -3 \end{pmatrix}$$

Part c

Suppose we are given the quadratic form of

$$Q_A(\vec{x}) = 2x_1^2 - 4x_3^2 + x_1x_2 - x_2x_3$$

results from the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -4 \end{pmatrix}$$

Problem #4

Suppose that $f(\vec{x})$ is defined on $\mathbb{R}^3$ by

$$f(\vec{x}) = c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_4x_1x_2 + c_5x_1x_3 + c_6x_2x_3$$

to show that the Quadratic Form of $f(\vec{x})$ is associated with $\frac{1}{2}Hf$, we first compute $\nabla f(\vec{x})$,

$$\nabla f(\vec{x}) = \begin{bmatrix} 2c_1x_1 + c_4x_2 + c_5x_3 \\ 2c_2x_2 + c_4x_1 + c_6x_3 \\ 2c_3x_3 + c_5x_1 + c_6x_2 \end{bmatrix}$$

then we compute the Hessian as

$$Hf = \begin{bmatrix} 2c_1 & c_4 & c_5 \\ c_4 & 2c_2 & c_6 \\ c_5 & c_6 & 2c_3 \end{bmatrix}$$

$$\frac{1}{2}Hf = \begin{bmatrix} c_1 & \frac{1}{2}c_4 & \frac{1}{2}c_5 \\ \frac{1}{2}c_4 & c_2 & \frac{1}{2}c_6 \\ \frac{1}{2}c_5 & \frac{1}{2}c_6 & c_3 \end{bmatrix}$$

Then computing the quadratic form,

$$\begin{aligned} Q_{\frac{1}{2}Hf}(\vec{x}) &= x_1(c_1x_1 + \frac{1}{2}c_4x_2 + \frac{1}{2}c_5x_3) + x_2(\frac{1}{2}c_4x_1 + c_2x_2 + \frac{1}{2}c_6x_3) + x_3(\frac{1}{2}c_5x_1 + \frac{1}{2}c_6x_2 + c_3x_3) \\ &= c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_4x_1x_2 + c_5x_1x_3 + c_6x_2x_3 \end{aligned}$$

Expanding out the Quadratic form for $\mathbb{R}^4$, we have

$$f(\vec{x}) = c_1x_1^2+c_2x_2^2+c_3x_3^2+c_4x_4^2+c_5x_1x_2+c_6x_1x_3+c_7x_2x_3+c_8x_1x_4+c_9x_2x_4+c_{10}x_3x_4$$

giving us,

$$\nabla f(\vec{x}) = \begin{bmatrix} 2c_1x_1 + c_5x_2 + c_6x_3 + c_8x_4 \\ 2c_2x_2 + c_5x_1 + c_7x_3 + c_9x_4 \\ 2c_3x_3 + c_6x_1 + c_7x_2 + c_{10}x_4 \\ 2c_4x_4 + c_8x_1 + c_9x_2 + c_{10}x_3 \end{bmatrix}$$

$$Hf = \begin{bmatrix} 2c_1 & c_5 & c_6 & c_8 \\ c_5 & 2c_2 & c_7 & c_9 \\ c_6 & c_7 & 2c_3 & c_{10} \\ c_8 & c_9 & c_{10} & 2c_4 \end{bmatrix}$$

$$\frac{1}{2}Hf = \begin{bmatrix} c_1 & \frac{1}{2}c_5 & \frac{1}{2}c_6 & \frac{1}{2}c_8 \\ \frac{1}{2}c_5 & c_2 & \frac{1}{2}c_7 & \frac{1}{2}c_9 \\ \frac{1}{2}c_6 & \frac{1}{2}c_7 & c_3 & \frac{1}{2}c_{10} \\ \frac{1}{2}c_8 & \frac{1}{2}c_9 & \frac{1}{2}c_{10} & c_4 \end{bmatrix}$$

and,

$$Q_{\frac{1}{2}Hf}(\vec{x}) = c_1x_1^2+c_2x_2^2+c_3x_3^2+c_4x_4^2+c_5x_1x_2+c_6x_1x_3+c_7x_2x_3+c_8x_1x_4+c_9x_2x_4+c_{10}x_3x_4$$

Problem #5

Suppose you are given the following,

$$f(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3$$

First we can compute $\nabla f(x_1, x_2, x_3)$,

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 6x_1 + 2x_2 + 2x_3 \\ 4x_2 + 2x_1 + 2x_3 \\ 4x_3 + 2x_2 + 2x_1 \end{bmatrix}$$

then we compute the Hessian,

$$Hf = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Now that we have computed the Hessian, now we compute the Principle Minors as

$$\begin{aligned} &\Delta_1 = 6 > 0 \\ &\Delta_2 = 24 - 4 = 20 > 0 \\ &\Delta_3 = 56 > 0 \end{aligned}$$

Thus the Hessian is **Positive Definite** and thus the critical points are *Strict Global Minimizers*

Problem #6

Suppose we are given the following function,

$$f(x_1, x_2) = x_1^5 - x_1 x_2^6$$

we first compute $\nabla f(x_1, x_2)$ to get,

$$\nabla f(x_1, x_2) = \begin{bmatrix} 5x_1^4 - x_2^6 \\ -6x_1 x_2^5 \end{bmatrix}$$

which we see by plugging in $(0, 0)$ we get $\nabla f(0, 0) = 0$, thus $(0, 0)$ is a critical point of $f(x_1, x_2)$. But we see that

$$f(x_1, 0) = x_1^5$$

which is positive for $x_1 > 0$, negative for $x_1 < 0$, and 0 at $x_1 = 0$, thus $(0, 0)$ is neither a minimum or maximum.

Problem #7

Part a

Suppose we are given the following function,

$$f(x_1, x_2) = x_1^3 - 3ax_1x_2 + x_2^3$$

Here we see that,

$$\lim_{x_1 \to \infty} f(x_1, 0) = +\infty$$

Tells us that $f(x_1, x_2)$ has no global maximimizers.

Part b

We see that by computing the gradient as,

$$\nabla f(x_1, x_2) = \begin{bmatrix} 3x_1^2 - 3ax_2 \\ -3ax_1 + 3x_2^2 \end{bmatrix}$$

and,

$$Hf = \begin{bmatrix} 6x_1 & -3a \\ -3a & 6x_2 \end{bmatrix}$$

to find the critical points of f we compute,

$$\begin{aligned} 3x_1^2 - 3ax_2 &= 0 \\ -3ax_1 + 3x_2^2 &= 0 \end{aligned}$$

Solving the above system we see that the two critical points are $(0, 0)$ and (a, a).

Here we see that,

$$Hf(0, 0) = \begin{bmatrix} 0 & -3a \\ -3a & 0 \end{bmatrix}$$

here we compute the eigenvalues as, $\lambda = \pm 3a$ thus $Hf(0, 0)$ is indefinite and hence $(0, 0)$ is a saddle for all $a \neq 0$, by *Theorem 1.3.7.* Then for (a, a) we get,

$$Hf(a, a) = \begin{bmatrix} 6a & -3a \\ -3a & 6a \end{bmatrix}$$

which we see that,

$$\Delta_1 = 6a$$
$$\Delta_2 = 27a^2$$

If $a > 0$, then $Hf(a, a)$ is Positive Definite and (a, a) is a local minimum. If $a < 0$, then $Hf(a, a)$ is Negative Definite and (a, a) is a local maximum.

Problem #8

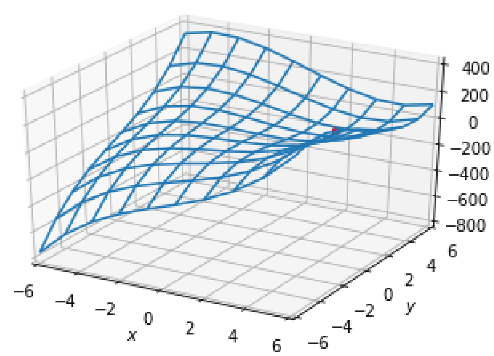

Figure 1: Plot of $f(x, y)$ when $a = 3$

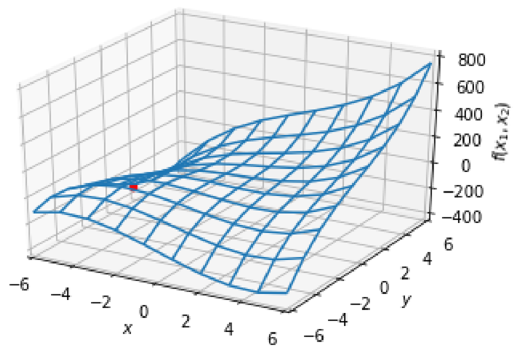

Figure 2: Plot of $f(x, y)$ when $a = -3$

Problem #9

Suppose we are given,

$$f(x, y) = x^3 + e^{3y} - 3xe^y$$

and compute the gradient as,

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 - 3e^y \\ 3e^{3y} - 3xe^y \end{bmatrix}$$

to find the critical points, we compute the following

$$\begin{aligned} 3x^2 - 3e^y &= 0 \\ 3e^{3y} - 3xe^y &= 0 \end{aligned}$$

solving the system above we get the critical point to be $(1, 0)$. Next we compute the Hessian as,

$$Hf = \begin{bmatrix} 6x & -3e^y \\ -3e^y & 9e^{3y} - 3xe^y \end{bmatrix}$$

and computing,

$$Hf(1, 0) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

and see that $Hf(1,0)$ is positive definite since $\Delta_1 = 6 > 0, \Delta_2 = 27 > 0$, thus $(1,0)$ is a local minimizer. However since,

$$\lim_{x_1 \to -\infty} f(x_1, 0) = -\infty$$

We see that $f(x,y)$ has no global minimizers, thus $(1,0)$ cannot be a global minimizer.

Problem #10

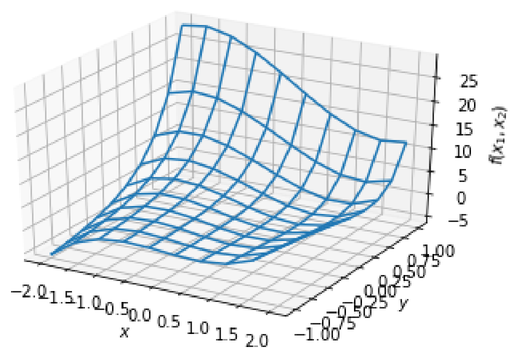

Figure 3: Plot of $f(x,y) = x^3 + e^{3y} - 3xe^y$

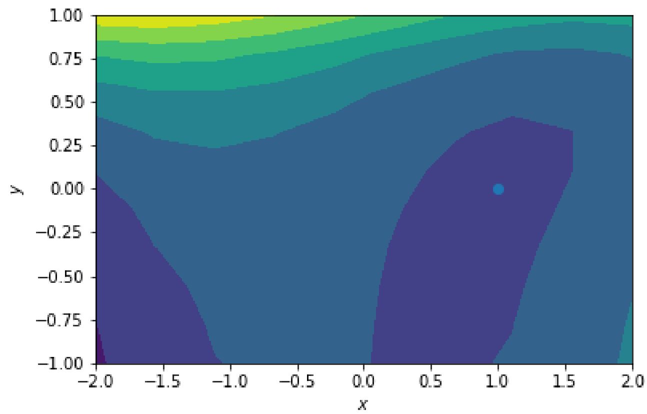

Figure 4: Contour Plot of $f(x, y) = x^3 + e^{3y} - 3xe^y$