MAT 4800 Homework # 2

Noah Reef

Spring 2023

Problem #1

Suppose we are given,

$$f(a,b) = \sum_{i=1}^{n} (a + bx_i - y_i)^2$$

to find the critical point we compute,

$$\nabla f(a,b) = \begin{bmatrix} 2\sum_{i=1}^{n}(a + bx_i - y_i) \\ 2\sum_{i=1}^{n}(a + bx_i - y_i)x_i \end{bmatrix}$$

checking (a,b) such that $\nabla f = \vec{0}$ occurs when,

$$\begin{aligned} \sum_{i=1}^{n}(a + bx_i - y_i) &= 0 \\ (a + bx_1 - y_1) + (a + bx_2 - y_2) + \cdots + (a + bx_n - y_n) &= 0 \\ na + b(x_1 + x_2 + \cdots + x_n) - (y_1 + y_2 + \cdots + y_n) &= 0 \\ a = \frac{(y_1 + y_2 + \cdots + y_n) - b(x_1 + x_2 + \cdots + x_n)}{n} &= \bar{y} - b\bar{x} \end{aligned}$$

thus

$$a = \bar{y} - b\bar{x}$$

to solve for b, we do the following,

$$\begin{aligned}
\sum_{i=1}^{n}(a+bx_i-y_i)x_i &= 0\\
ax_1+bx_1^2-y_1x_1+ax_2+bx_2^2-y_2x_2+\cdots+ax_n+bx_n^2-y_nx_n &= 0\\
a(x_1+x_2+\cdots+x_n)+b(x_1^2+x_2^2+\cdots+x_n^2)-(y_1x_1+x_2y_2+\cdots+x_ny_n) &= 0\\
a\sum_{i=1}^{n}x_i+b\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}y_ix_i &= 0
\end{aligned}$$

now replacing $a=\bar{y}-b\bar{x}$,

$$\begin{aligned}
\left(\frac{1}{n}\sum_{i=1}^{n}y_i-b\frac{1}{n}\sum_{i=1}^{n}x_i\right)\sum_{i=1}^{n}x_i+b\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}y_ix_i &= 0\\
n\bar{x}\bar{y}-b\left(n\bar{x}^2-\sum_{i=1}^{n}x_i^2\right)-\sum_{i=1}^{n}y_ix_i &= 0
\end{aligned}$$

re-arranging terms,

$$b=\frac{n\bar{x}\bar{y}-\sum_{i=1}^{n}y_ix_i}{n\bar{x}^2-\sum_{i=1}^{n}x_i^2}$$

Thus $f(a,b)$ has a critical point at,

$$\begin{aligned}
a^* &= \bar{y}-b^*\bar{x}\\
b^* &= \frac{n\bar{x}\bar{y}-\sum_{i=1}^{n}y_ix_i}{n\bar{x}^2-\sum_{i=1}^{n}x_i^2}
\end{aligned}$$

Problem #2

To show that (a^*,b^*) is a global minimizer for $f(a,b)$ we will first the Hessian as,

$$Hf(a,b)=\begin{bmatrix}2n & 2\sum_{i=1}^{n}x_i\\ 2\sum_{i=1}^{n}x_i & 2\sum_{i=1}^{n}x_i^2\end{bmatrix}=\begin{bmatrix}2n & 2n\bar{x}\\ 2n\bar{x} & 2\sum_{i=1}^{n}x_i^2\end{bmatrix}$$

Since the Hessian is Symmetric, we check

$$\begin{aligned} 2n &> 0 \\ \det(Hf) = 4n \sum_{i=1}^{n} x_i^2 + 4 \left(\sum_{i=1}^{n} x_i \right)^2 &> 0 \end{aligned}$$

for $\vec{x} \neq \vec{0}$. Thus Hf is a **Definite Positive** matrix by *Theorem 1.3.2*, and hence (a^*, b^*) is a global minimizer.

Problem #3

Let our set S of ordered points be

$$S = \{(0,0), (2,1), (1,4)\}$$

next we will compute,

$$\begin{aligned} \bar{x} &= \frac{0+2+1}{3} = 1 \\ \bar{y} &= \frac{0+1+4}{3} = \frac{5}{3} \\ \sum_{i=1}^{3} y_i x_i &= (0(0) + 2(1) + 1(4)) = 6 \\ \sum_{i=1}^{3} x_i^2 &= 0^2 + 2^2 + 1^2 = 5 \end{aligned}$$

then

$$\begin{aligned} a^* &= \bar{y} - b\bar{x} = \frac{5}{3} - \frac{1}{2} = \frac{7}{6} \\ b^* &= \frac{n\bar{x}\bar{y} - \sum_{i=1}^{n} y_i x_i}{n\bar{x}^2 - \sum_{i=1}^{n} x_i^2} = \frac{3(5/3) - 6}{3 - 5} = \frac{-1}{-2} = \frac{1}{2} \end{aligned}$$

therefore $(a^*, b^*) = (7/6, 1/2)$.

Problem #4

Using the ordered points from S and the above (a^*, b^*), we get the following plot

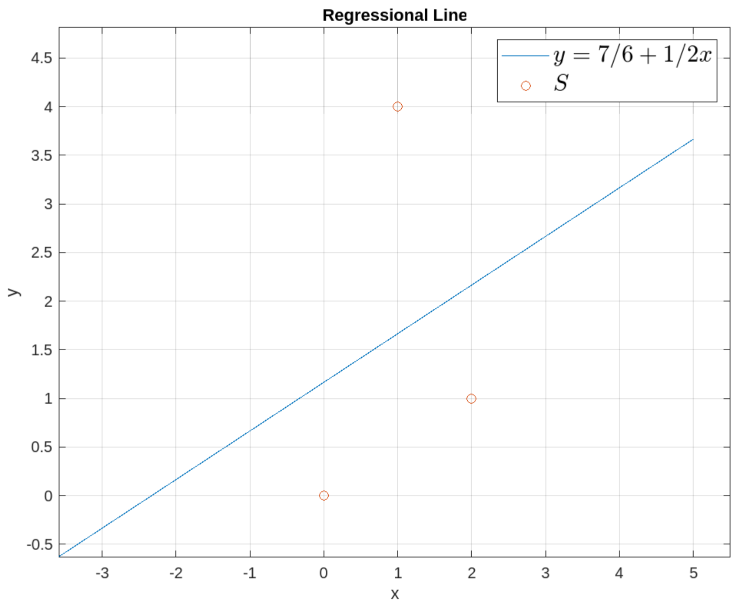

Figure 1: Plot of Regeressional Line $y = \frac{7}{6} + \frac{1}{2}x$

Problem #5

Plotting $f(a, b)$ we get,

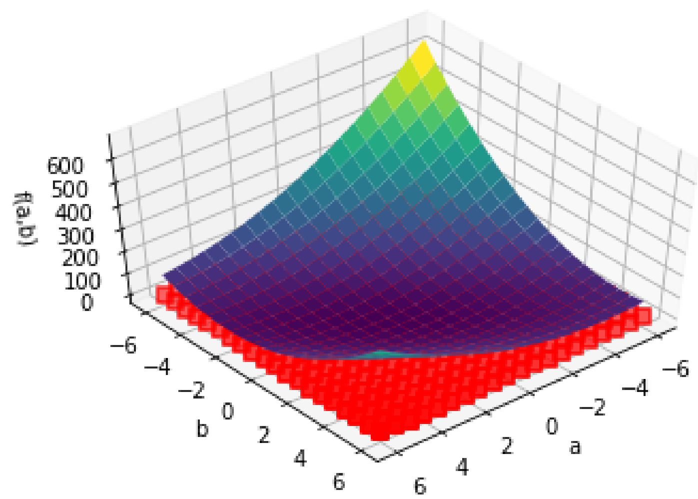

Figure 2: Plot of $f(a, b)$

Where the plane in red denotes $Z = f\left(\frac{7}{6}, \frac{1}{2}\right)$