Problem Set 1

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Problem 6.1

To find the Fourier transform of $f(x) = e^{-|x|}$ for $x \in \mathbb{R}$, we compute

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ix\xi} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{x(1-i\xi)} \, dx + \int_{0}^{\infty} e^{-x(1+i\xi)} \, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\xi} e^{(x(1-i\xi))} \Big|_{-\infty}^{0} - \frac{1}{1+i\xi} e^{-x(1+i\xi)} \Big|_{0}^{\infty} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{(1-i\xi)(1+i\xi)} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+\xi^2} \right)$$

Problem 6.2

We will consider the parameteric curve defined as

$$\gamma_1 = t \qquad\qquad\qquad -R \leq t \leq R$$

$$\gamma_2 = R - it \qquad\qquad\qquad 0 \leq t \leq \frac{\xi}{2a}$$

$$\gamma_3 = -t - i\frac{\xi}{2a} \qquad\qquad\qquad -R \leq t \leq R$$

$$\gamma_4 = -R + it \qquad\qquad\qquad -\frac{\xi}{2a} \leq t \leq 0$$

and by Cauchy's Integral theorem we have that

$$0 = \int_{-R}^{R} e^{-at^2 - i\xi t} \, dt + i \int_{0}^{\xi/2a} e^{-a(R-it)^2 - i\xi(R-it)} \, dt$$

$$- \int_{-R}^{R} e^{-a(-t-i\xi/2a)^2 - i(-t-i\xi/2a)} \, dt - i \int_{-\xi/2a}^{0} e^{-a(-R+it)^2 - i\xi(-R+it)} \, dt$$

Then we see that the second and fourth terms go to zero since,

$$\left| i \int_{0}^{\xi/2a} e^{-a(R-it)^2 - i\xi(R-it)} \, dt \right| + \left| i \int_{-\xi/2a}^{0} e^{-a(-R+it)^2 - i\xi(-R+it)} \, dt \right| \leq 2\frac{\xi}{2a} e^{-aR^2} \to 0$$

and we see for the third term that

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-a(-t-i\xi/2a)^2 - i(-t-i\xi/2a)} \, dt = \sqrt{\frac{\pi}{a}} e^{-\xi^2/(4a)}$$

and hence

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-a|x|^2} e^{-ix\xi} \, dx = \sqrt{\frac{\pi}{a}} \frac{e^{-\xi^2/(4a)}}{\sqrt{2\pi}}$$

Problem 6.4

Suppose the $f \in L^1(\mathbb{R}^d)$ and $f(x) = g(|x|)$ for some g, then we see that

$$\begin{aligned}
\hat{f}(\xi) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} \, dx \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(|x|) e^{-ix\cdot\xi} \, dx \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(|x|) e^{-i|x||\xi|\cos(\theta)} \, dx \\
&= (2\pi)^{-d/2} \int_0^\infty \int_{\omega_d} g(r) e^{-ir|\xi|\cos(\theta)} r^d \, dr \, d\theta \\
&= h(|\xi|)
\end{aligned}$$