CSE 386D: MoAM II **Spring 2025**

Problem Set 1

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Problem 6.1

To find the Fourier transform of $f(x) = e^{-|x|}$ for $x \in \mathbb{R}$, we compute

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{x(1-i\xi)}\, dx + \int_{0}^{\infty} e^{-x(1+i\xi)}\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\xi} e^{(x(1-i\xi))}\Big|_{-\infty}^{0} - \frac{1}{1+i\xi} e^{-x(1+i\xi)}\Big|_{0}^{\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{(1-i\xi)(1+i\xi)} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+\xi^2} \right)
\end{aligned}$$

Problem 6.2

We will consider the parameteric curve defined as

$$\begin{aligned}
\gamma_1 &= t & -R \le t \le R \\
\gamma_2 &= R - it & 0 \le t \le \frac{\xi}{2a} \\
\gamma_3 &= -t - i\frac{\xi}{2a} & -R \le t \le R \\
\gamma_4 &= -R + it & -\frac{\xi}{2a} \le t \le 0
\end{aligned}$$

and by Cauchy's Integral theorem we have that

$$\begin{aligned}
0 = &\int_{-R}^{R} e^{-at^2 - i\xi t}\, dt + i \int_{0}^{\xi/2a} e^{-a(R-it)^2 - i\xi(R-it)}\, dt \\
&- \int_{-R}^{R} e^{-a(-t-i\xi/2a)^2 - i(-t-i\xi/2a)}\, dt - i \int_{-\xi/2a}^{0} e^{-a(-R+it)^2 - i\xi(-R+it)}\, dt
\end{aligned}$$

Then we see that the second and fourth terms go to zero since,

$$\left| i \int_{0}^{\xi/2a} e^{-a(R-it)^2 - i\xi(R-it)}\, dt \right| + \left| i \int_{-\xi/2a}^{0} e^{-a(-R+it)^2 - i\xi(-R+it)}\, dt \right| \le 2\frac{\xi}{2a} e^{-aR^2} \to 0$$

and we see for the third term that

$$\lim_{R\to\infty}\int_{-R}^{R} e^{-a(-t-i\xi/2a)^2-i(-t-i\xi/2a)}\,dt = \sqrt{\frac{\pi}{a}}e^{-\xi^2/(4a)}$$

and hence

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-a|x|^2}e^{-ix\xi}\,dx = \sqrt{\frac{\pi}{a}}\frac{e^{-\xi^2/(4a)}}{\sqrt{2\pi}}$$

Problem 6.4

Suppose the $f \in L^1(\mathbb{R}^d)$ and $f(x) = g(|x|)$ for some g, then we see that

$$\begin{aligned}
\hat{f}(\xi) &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} g(|x|)e^{-ix\cdot\xi}\,dx \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} g(|x|)e^{-i|x||\xi|\cos(\theta)}\,dx \\
&= (2\pi)^{-d/2}\int_0^\infty\int_{\omega_d} g(r)e^{-ir|\xi|\cos(\theta)}r^d\,dr\,d\theta \\
&= h(|\xi|)
\end{aligned}$$

Problem 6.11

Consider $\mathcal{F} : L^1(\mathbb{R}^d)\ to C_v(\mathbb{R}^d)$. Recall that $\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ is a bounded linear map by Proposition 6.2 and we know that $C_v(\mathbb{R}^d)$ is a closed linear subspace of $L^\infty(\mathbb{R}^d)$ by Proposition 6.4, hence $\mathcal{F} : L^1(\mathbb{R}^d) \to C_v(\mathbb{R}^d)$ is a bounded linear map. Next we note that if $f, g \in L^1(\mathbb{R}^d)$ such that $\mathcal{F}(f) = \mathcal{F}(g)$ for all $\xi \in \mathbb{R}^d$, then we have that $\mathcal{F}(f - g) = 0$ and hence $f - g = 0$ and so $f = g$, thus $\mathcal{F}$ is injective. Now suppose that $\mathcal{F}$ is surjective, then we have by the Open Mapping Theorem, that $\mathcal{F}^{-1}$ is bounded.

Next suppose we have the characteristic functions $f_n, f_1 \in L^1(\mathbb{R}^d)$ and consider $f_n * f_1 \in$

$L^1(\mathbb{R}^d)$, then we see that

$$\begin{aligned}
f_n * f_1 &= \int_{R^d} f_n(x-y) f_1(y)\, dy \\
&= \int_{[-1,1]^d} f_n(x-y)\, dy \\
&= \int_{[x-1,x+1]^d} f_n(z) \dot{d}z \\
&= \int_{[x-1,0]^d} f_n(z)\, dz + \int_{[0,x+1]^d} f_n(z)\, dz && (\in C_v(\mathbb{R}^d) \text{ By Exercise 5}) \\
&= \begin{cases} x+n+1 & x \in [-n-1, -n+1] \\ 2 & x \in [-n+1, n-1] \\ n+1-x & x \in [n-1, n+1] \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

then we see that as $n \to \infty$ we have that $f_n * f_1 \to 2$. Note that

$$\mathcal{F}^{-1}(f_n * f_1) = (2\pi)^{-d/2} \frac{\sin(nx)\sin(x)}{x^2}$$

where C is some constant. However as $n \to \infty$ we see that

$$\begin{aligned}
\left\|\mathcal{F}^{-1}[f_n * f_1]\right\|_{L^1} &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{|\sin(nx)\sin(x)|}{|x^2|}\, dx \\
&\geq (2\pi)^{-d/2} \int_{[0,1]^d} \frac{|\sin(nx)\sin(x)|}{|x^2|}\, dx \\
&\geq (2\pi)^{-d/2} \frac{2}{\pi} \int_{[0,1]^d} \left|\frac{\sin(nx)}{x}\right| dx \to \infty
\end{aligned}$$

which contradicts our assumption that $\mathcal{F}^{-1}$ is bounded and hence $\mathcal{F}$ is not surjective. Since $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a bounded linear map that is one-to-one and onto, such that $C_0^\infty(\mathbb{R}^d) \subsetneqq \mathcal{S} \subsetneqq L^1(\mathbb{R}^d)$ then we see that for $\phi \in C_0^\infty(\mathbb{R}^d)$ we have that $\mathcal{F}^{-1}(\phi)$ exists and is in $L^1(\mathbb{R}^d)$. All there is left to show is that $C_0^\infty(\mathbb{R}^d)$ is dense in $C_v(\mathbb{R}^d)$. Consider $g \in C_v(\mathbb{R}^d)$,letting φ_ϵ be an approximation to the identity, defined as

$$\varphi_\epsilon(x) = \epsilon^{-d} \varphi(x/\epsilon)$$

with $\epsilon = 1/n$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$, then we have that $g * \varphi_\epsilon \in C_0^\infty(\mathbb{R}^d)$ and $g * \varphi_\epsilon \to g$ in $C_v(\mathbb{R}^d)$ as $n \to \infty$ and hence $C_0^\infty(\mathbb{R}^d)$ is dense in $C_v(\mathbb{R}^d)$.