CSE 386D: Methods of Applied Mathematics II **Spring 2025**

Problem Set 11

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Problem 9.8

Part a

Let $X = C^0([0,T])$ and define the operator $G : X \to X$ by

$$G(u) = u_0 + \int_0^t \cos(u(s)) - u(s)\, ds$$

Then we see that,

$$\begin{aligned}
||G(u) - G(v)||_{L^\infty} &= \sup_{0\le t\le T} \left| \int_0^t [\cos(u(s)) - \cos(v(s))] + [u(s) - v(s)]\, ds \right| \\
&\le \sup_{0\le t\le T} \left| \int_0^t [\cos(u(s)) - \cos(v(s))]\, ds \right| + \sup_{0\le t\le T} \left| \int_0^t [u(s) - v(s)]\, ds \right| \\
&\le \sup_{0\le t\le T} \int_0^t |\cos(u(s)) - \cos(v(s))|\, ds + \sup_{0\le t\le T} \int_0^t |u(s) - v(s)|\, ds \\
&\le \sup_{0\le t\le T} \int_0^t |u(s) - v(s)|\, ds + \sup_{0\le t\le T} \int_0^t |u(s) - v(s)|\, ds \\
&\le 2T\, ||u - v||_{L^\infty}
\end{aligned}$$

so then by taking $T < 1/2$ we have that by the Contraction Mapping Theorem that G has a unique fixed point u. We can iterate this process to extend the solution uniquely to any $T > 0$.

Part b

Problem 9.9

Suppose we have the following differential equation

$$\begin{cases} -u_{xx} + u - \epsilon u^2 = f(x) & \text{for } x \in (0, +\infty) \\ u(0) = u(+\infty) = 0 \end{cases}$$

Let $\mathcal{L} : C^2((0,\infty)) \to C^2((0,\infty))$ be the operator defined by

$$\mathcal{L}(u) = -u_{xx} + u$$

Then we have that there exists a Green's Function g such that

$$G(u) = u(x) = \int_0^\infty g(x,y)\left[f(y) + \epsilon u(y)^2\right]\,dy$$

Then we have that

$$\begin{aligned}
||G(u) - G(v)||_{L^\infty} &= \sup_{0\le x<\infty}\left|\int_0^\infty g(x,y)\left[f(y) + \epsilon u(y)^2 - f(y) - \epsilon v(y)^2\right]\,dy\right| \\
&\le \sup_{0\le x<\infty}\int_0^\infty |g(x,y)|\left|u(y)^2 - v(y)^2\right|\,dy \\
&\le \sup_{0\le x<\infty}\int_0^\infty |g(x,y)|\,|u(y) - v(y)|\,|u(y) + v(y)|\,dy \\
&\le \epsilon\,||u+v||_{L^\infty}\,||u-v||_{L^\infty}\sup_{0\le x<\infty}\int_0^\infty |g(x,y)|\,dy \\
&\le \epsilon 2RM\,||u-v||_{L^\infty}
\end{aligned}$$

Then we see that for $\epsilon < \frac{1}{2RM}$ we have that by the Contraction Mapping Theorem that G has a unique fixed point u.

Problem 9.10

Suppose we have the following differential equation

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} - \epsilon u^3 = f, & -\infty < x < \infty, t > 0 \\ u(x,0) = g(x) \end{cases}$$

Note that we can rewrite the above as

$$(1 - \partial_x^2)u_t = f + \epsilon u^3 = h$$

then by taking the Fourier Transform we have that

$$(1+\xi^2)\hat{u}_t = \hat{h}$$

and then we see that it can be formally deduced that

$$u_t = \tilde{\kappa} * h = \tilde{\kappa} * (f + \epsilon u^3)$$

where

$$\tilde{\kappa} = \sqrt{2\pi}\mathcal{F}^{-1}\left(\frac{1}{1+\xi^2}\right) = \frac{1}{2}e^{-|x|}$$

Now by letting $k = -\tilde{k}_x \in L^1(\mathbb{R})$ we have that

$$u_t(x,t) = \kappa * (f + \epsilon u^3)$$

Now by using the Fundemental Theorem of Calculus we get that

$$G(u) = u(x,t) = g(x) + \int_0^t \kappa * (f + \epsilon u^3)\, dt$$

To show that G is a contraction map we see that

$$||G(u) - G(v)||_{L^\infty} = \sup_{(x,t)\in\mathbb{R}\times[0,T]} \left| \int_0^t \int \kappa * (\epsilon u^3 + \epsilon v^3)\, d \right|$$

Problem 9.12

Part a

We see that $H : X \times \mathbb{R} \to Y$ defined by $H(x,\epsilon) = F(x) + \epsilon G(x)$ is C^1 in a neighborhood around $(x_0, 0)$ since $DH(x_0, 0) = DF(x_0) = 0$. Then we have by the Implicit Function Theorem that there exists a unique mapping $g \in C^1$ such that $\epsilon = g(x, y)$. This means the we have that $H(x_0, g(x_0, 0)) = 0$