CSE 386D: MoAM II **Spring 2025**

Problem Set 1

Student Name: Noah Reef

Problem 6.1

To find the Fourier transform of $f(x) = e^{-|x|}$ for $x \in \mathbb{R}$, we compute

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{x(1-i\xi)}\, dx + \int_{0}^{\infty} e^{-x(1+i\xi)}\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\xi} e^{(x(1-i\xi))}\Big|_{-\infty}^{0} - \frac{1}{1+i\xi} e^{-x(1+i\xi)}\Big|_{0}^{\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{(1-i\xi)(1+i\xi)} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+\xi^2} \right)
\end{aligned}$$

Problem 6.2

We will consider the parameteric curve defined as

$$\begin{aligned}
\gamma_1 &= t & -R \le t \le R \\
\gamma_2 &= R - it & 0 \le t \le \frac{\xi}{2a} \\
\gamma_3 &= -t - i\frac{\xi}{2a} & -R \le t \le R \\
\gamma_4 &= -R + it & -\frac{\xi}{2a} \le t \le 0
\end{aligned}$$

and by Cauchy's Integral theorem we have that

$$\begin{aligned}
0 = &\int_{-R}^{R} e^{-at^2 - i\xi t}\, dt + i \int_{0}^{\xi/2a} e^{-a(R-it)^2 - i\xi(R-it)}\, dt \\
&- \int_{-R}^{R} e^{-a(-t-i\xi/2a)^2 - i(-t-i\xi/2a)}\, dt - i \int_{-\xi/2a}^{0} e^{-a(-R+it)^2 - i\xi(-R+it)}\, dt
\end{aligned}$$

Then we see that the second and fourth terms go to zero since,

$$\left| i \int_{0}^{\xi/2a} e^{-a(R-it)^2 - i\xi(R-it)}\, dt \right| + \left| i \int_{-\xi/2a}^{0} e^{-a(-R+it)^2 - i\xi(-R+it)}\, dt \right| \le 2\frac{\xi}{2a} e^{-aR^2} \to 0$$

and we see for the third term that

$$\lim_{R\to\infty}\int_{-R}^{R} e^{-a(-t-i\xi/2a)^2-i(-t-i\xi/2a)}\,dt = \sqrt{\frac{\pi}{a}}e^{-\xi^2/(4a)}$$

and hence

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-a|x|^2}e^{-ix\xi}\,dx = \sqrt{\frac{\pi}{a}}\frac{e^{-\xi^2/(4a)}}{\sqrt{2\pi}}$$

Problem 6.4

Suppose the $f \in L^1(\mathbb{R}^d)$ and $f(x) = g(|x|)$ for some g, then we see that

$$\begin{aligned}
\hat{f}(\xi) &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} g(|x|)e^{-ix\cdot\xi}\,dx \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} g(|x|)e^{-i|x||\xi|\cos(\theta)}\,dx \\
&= (2\pi)^{-d/2}\int_0^\infty\int_{\omega_d} g(r)e^{-ir|\xi|\cos(\theta)}r^d\,dr\,d\theta \\
&= h(|\xi|)
\end{aligned}$$

Problem 6.11

Suppose that $\mathcal{F} : L^1(\mathbb{R}^d) \to C_v(\mathbb{R}^d)$ was onto. Then we have that $\mathcal{F}$ is an onto,injective, bounded, linear map and by the open mapping theorem has a bounded inverse. Let f_n and f_1 be the characteristic functions on the intervals $[-n, n]$ and $[-1, 1]$ respectively. Then we see that $f_n, f_1 \in L^1(\mathbb{R}^d)$ and that $f_n * f_1 \in C_v(\mathbb{R}^d)$ by Exercise 6. By properties of the Fourier transform we have that

$$\mathcal{F}(f_n * f_1) = (2\pi)^{d/2}\mathcal{F}(f_n)\mathcal{F}(f_1)$$

where

$$\mathcal{F}(f_1) = \prod_{j=1}^{d}\sqrt{\frac{2}{\pi}}\frac{\sin(\xi_j)}{\xi_j}$$

$$\mathcal{F}(f_n) = \prod_{j=1}^{d}\sqrt{\frac{2}{\pi}}\frac{\sin(n\xi_j)}{n\xi_j}$$

hence

$$g_n = \mathcal{F}(f_n * f_1) = (2\pi)^{d/2}\prod_{j=1}^{d}\frac{2}{\pi}\frac{\sin(\xi_j)\sin(n\xi_j)}{n\xi_j^2}$$

but we can clearly see that $g_{nL^\infty} \to \infty$ as $n \to \infty$. Additionally recall that

$$\mathcal{F}^{-1}$$