CSE 386D NOTES

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1 Normed Linear Spaces and Banach Spaces

1.1 Basic Concepts and Definitions

Definition 1.1. Let X be a V.S. over $\mathbb{F}$. X is a NLS if X has a norm, a map $\|\cdot\| : X \to \mathbb{R}^+$ such that for $x, y \in X$ and $\lambda \in$ such that

1. $\|\lambda x\| = |\lambda| \|x\|$
2. $\|x\| = 0 \iff x = 0$
3. $\|x + y\| \leq \|x\| + \|y\|$

Proposition 1.1. Let $(X, \|\cdot\|)$ be a NLS. Then X is a metric space if we define a metric d on X by

$$d(x, y) = \|x - y\|$$

Proposition 1.2. In a NLS X, the operations addition $+ : X \times X \to X$, scalar multiplication, $\cdot : \mathbb{F} \times X \to X$, and the norm, $\|\cdot\| : X \to \mathbb{R}$ are continuous

Definition 1.2. Let (X, d) be a metric space, then $\{x_n\}_{n=1}^{\infty} \subseteq X$ is called *Cauchy*, if for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m, n \geq N$ we have that

$$d(x_n, x_m) \leq \epsilon$$

Definition 1.3. A metric space is called *complete*, if every cauchy sequence in the space converges to a point in the space. A complete NLS is called a *Banach Space*

Proposition 1.3. If X and Y are NLS's and $T : X \to Y$ is linear, then T is bounded iff there is $C > 0$ such that

$$\|Tx\|_Y \leq C \|x\|_X$$

for all $x \in X$.

2 The Fourier Transform

2.1 The $L^1(\mathbb{R}^d)$ Theory

If $\xi \in \mathbb{R}^d$, the function

$$\varphi_\xi(x) = e^{-ix\cdot\xi} = \cos(x \cdot \xi) - i \sin(x \cdot \xi)$$

for $x \in \mathbb{R}^d$ is a plane wave in the direction ξ. Its period in the jth direction is $1\pi/\xi_j$.

Proposition 2.1. For such φ we have the following:

1. $|\varphi_\xi| = 1$ and $\bar{\varphi_\xi} = \varphi_{-\xi}$ for any $\xi \in \mathbb{R}^d$
2. $\varphi_\xi(x+y) = \varphi_\xi(x)\varphi_\xi(y)$ for any $x, y, \xi \in \mathbb{R}^d$
3. $-\Delta\varphi_\xi = |\xi|^2\varphi_\xi$ for any $\xi \in \mathbb{R}^d$

Principle 2.2. If $f \in L^1(\mathbb{R}^d)$, the Fourier transform of f is

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$$

Proposition 2.3. The Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathcal{R}^d)$$

is a bounded linear operator, and

$$\left\|\hat{f}\right\|_{L^\infty(\mathcal{R}^d)} \leq (2\pi)^{-d/2}\,\|f\|_{L^1(\mathbb{R}^d)}$$

Proposition 2.4. If $f \in L^1(\mathbb{R}^d)$ and τ_y is a translation by y, then

1. $\mathcal{F}(\tau_y f)(\xi) = e^{-iy\cdot\xi}\hat{f}(\xi)$ for all $y \in \mathbb{R}^d$.
2. $\mathcal{F}(e^{ix\cdot y}f)(\xi) = \tau_y\hat{f}(\xi)$ for all $y \in \mathbb{R}^d$
3. if $r > 0$ is given,
$$\mathcal{F}(f(rx))(\xi) = r^{-d}\hat{f}(r^{-1}\xi)$$
4. $\hat{\bar{f}}(\xi) = \overline{\hat{f}(-\xi)}$

Principle 2.5. A continuous function f on $\mathbb{R}^d$ is said to vanish at infinity if for any $\epsilon > 0$ there is $K \subset\subset \mathbb{R}^d$ such that

$$|f(x)| < \epsilon$$

for $x \notin K$, The subspace of all such continuous functions is denoted

$$C_v(\mathbb{R}^d) = \{f \in C^0(\mathbb{R}^d) : f \text{ vanishes at } \infty\}$$

Theorem 2.6. The space $C_v(\mathbb{R}^d)$ is a closed linear subspace of $L^\infty(\mathbb{R}^d)$

Theorem 2.7 (Riemann-Lebesgue Lemma)**.** The Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}^d) \to C_v(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$$

Then for $f \in L^1(\mathbb{R}^d)$

$$\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0 \quad \text{and} \quad \hat{f} \in C^0(\mathbb{R}^d)$$

Proposition 2.8. If $f, g \in L^1(\mathbb{R}^d)$, then

1. $\int \mathcal{F}(f)g = \int f\mathcal{F}(g)$
2. $f * g \in L^1(\mathbb{R}^d)$ and $\mathcal{F}(f * g) = (2\pi)^{d/2}\mathcal{F}(f)\mathcal{F}(g)$

Theorem 2.9 (Generalized Young's Inequality)**.** Suppose $K(x, y)$ is measurable of $\mathbb{R}^d \times \mathbb{R}^d$ and there is some $C > 0$ such that

$$\int |K(x, y)|\, dx \le C \quad \text{and} \quad \int |K(x, y)|\, dy \le C$$

for almost every $x, y \in \mathbb{R}^d$, respectively. Define the operator T by

$$Tf(x) = \int K(x, y)f(y)\, dy$$

If $1 \le p \le \infty$, then $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is a bounded linear operator with operator norm $\|T\| \le C$.

Proposition 2.10 (Young's Inequality)**.** If $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$, then $f * g \in L^p(\mathbb{R}^d)$ and

$$\|f * g\|_p \le \|f\|_p \|g\|_1$$

Theorem 2.11 (Paley-Wiener Theorem)**.** If $f \in C_0^\infty(\mathbb{R}^d)$, then $\mathcal{F}(f)$ extend to an entire holomorphic function on $\mathbb{C}^d$.

2.2 The Schwartz Space Theory

Principle 2.12. The Schwartz space or the space of functions of rapid decrease is defined as

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} |x^\alpha Dd^\beta f(x)| < \infty \text{ for all } \alpha, \beta\}$$

Proposition 2.13. One has that

$$C_0^\infty(\mathbb{R}^d) \subsetneq \mathcal{S} \subsetneq L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$$

thus also $\mathcal{S} \subset L^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$.

Principle 2.14. Given $n = 0, 1, 2, \dots$ we define for $\phi \in \mathcal{S}$

$$\rho_n(\phi) = \sup_{|\alpha| \leq n} \sup_x (1 + |x|^2)^{n/2} |D^\alpha \phi(x)|$$

Proposition 2.15. The Schawartz class $\mathcal{S}$ is a complete metric space where the $\{\rho_n\}_{n=0}^\infty$ generate its topology through the metric

$$d(\phi_1, \phi_2) = \sum_{n=0}^\infty 2^{-n} \frac{\rho_n(\phi_1 - \phi_2)}{1 + \rho_n(\phi_1 - \phi_2)}$$

3 Sobolev Spaces

Definition 3.1 (Sobolev Spaces)**.** Let $\Omega \subset^d$ be a domain, $1 \leq p \leq \infty$, and $m \geq 0$ be an integer. The *Sobolev space* of m derivatives in $L^p(\Omega)$ is

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq m\}$$

with the norm, for $f \in W^{m,p}(\Omega)$, given by

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p}$$

and,

$$\|f\|_{W^{m,\infty}(\Omega)} = \sup_{|\alpha| \leq m} \|D^\alpha f\|_{L^\infty(\Omega)}$$

Proposition 3.1. We have the following properties of Sobolev spaces:

1. $\|\cdot\|_{W^{m,p}(\Omega)}$ is a norm on $W^{m,p}(\Omega)$
2. $W^{0,p}(\Omega) = L^p(\Omega)$
3. $W^{m,p}(\Omega) \hookrightarrow W^{k,p}(\Omega)$ for all $m \geq k \geq 0$

Proposition 3.2. The space $W^{m,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$.

Proposition 3.3. The space $W^{m,p}(\Omega)$ is separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$.

Definition 3.2. We denote the mth order Sobolev space in $L^2(\Omega)$ by

$$H^m(\Omega) = W^{m,2}(\Omega)$$

Proposition 3.4. The space $H^m(\Omega) = W^{m,2}(\Omega)$ is a separable Hilber space with the inner product

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha v\rangle_{L^2(\Omega)}$$

where

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)\, dx$$

Theorem 3.5. If $1 \leq p < \infty$ then

$$\{f \in C^\infty(\Omega) : \|f\|_{W^{m,p}(\Omega)} < \infty\} = C^\infty(\Omega) \cap W^{m,p}(\Omega)$$

is dense in $W^{m,p}(\Omega)$.

Lemma 3.6. Suppose that $1 \leq p < \infty$ and $\varphi \in C_0^\infty({}^d)$ is an approximate identity supported in the unit ball about the origin (i.e. $\varphi \geq 0, \int \varphi(x)\, dx = 1, \mathrm{supp}(\varphi) \subset B_1(0)$, and $\varphi_\epsilon(x) = \epsilon^{-d}\varphi(\epsilon^{-1}x)$ for $\epsilon > 0$).

References