CSE 386D: Methods of Applied Mathematics II **Spring 2025**

Problem Set 8

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Problem 8.1

Suppose that A is a positive definite matrix, then we have that $x^T Ax \geq 0$ for all $x \in \mathbb{R}^d$ and has equality only when $x = 0$. Then we see that if v is a normalized eigenvector of A with corresponding eigenvalue λ then we have that

$$v^T Av = v^T \lambda v = \lambda \left|\left|v\right|\right|_2^2 = \lambda \geq 0$$

and thus the eigenvalues of A are positive. Now suppose A is symmetric matrix and has positive eigenvalues, then we have that A is full rank and thus for all x we have that

$$x = \sum_{j=1}^{n} \langle x, v_j \rangle v_j$$

where v_j are the eigenvectors of A. Then we have that

$$x^T Ax = \sum_{j=1}^{n} \langle x, v_j \rangle v_j^T A \sum_{k=1}^{n} \langle x, v_k \rangle v_k = \sum_{j=1}^{n} \langle x, v_j \rangle^2 \lambda_j \geq 0$$

and thus A is positive definite.

Problem 8.3

Suppose that $u \in H^2(\Omega)$, for a bounded lipschitz boundary $\Omega \subseteq \mathbb{R}^d$. Then we have the Sobolev Embedding Theorem that for $j \geq 0$ and $m \geq 1$ that

$$W^{m+j,2}(\Omega) \hookrightarrow W^{j,q}(\Omega)$$

for finitely many $q \leq 2d/(d-2m)$. Note that we can rewrite the bound as

$$\frac{2d}{d-2m} = \frac{2d}{d-2(2-j)} = \frac{2d}{d-4+2j}$$

then in order to have the largest possible bound for q, we choose $j = 0$ and have

$$q \leq \frac{2d}{d-4}$$

and thus we have that

$$q^* = \begin{cases} \infty & d < 4 \\ \frac{2d}{d-4} & d \geq 4 \end{cases}$$

Now note that by Holder's inequality we have that

$$\left|\left|cu^2\right|\right|_{L^2} \leq ||c||_{L^p} \left|\left|u^2\right|\right|_{L^{q'}}$$

for $1/p + 1/q' = 1/2$. Note that $||u||_{L^q} < \infty$ for $q' = q/2$ and hence we have

$$\frac{1}{p} + \frac{1}{q'} = \frac{1}{p} + \frac{2}{q} = \frac{1}{p} + \frac{2(d-4)}{2d} = \frac{1}{2}$$

and then we have $p = 2d/(8-d)$. Thus we have that

$$p^* = \begin{cases} 2 & d \leq 4 \\ \frac{2d}{8-d} & 4 < d < 8 \\ \infty & d \geq 8 \end{cases}$$

are the smallest possible values for p such that $c \in L^{p^*}(\Omega)$.

Problem 8.4

Part a

Let $\gamma : H^1(\Omega) \to L^2(\Omega)$ be the trace operator, defined as $\gamma(u) = u|_V$ (which is continuous since V has positive measure), then we see that $\ker(\gamma) = H(\Omega)$ which is a closed subspace of $H^1(\Omega)$. Then equipping the space with the norm and inner production of $H^1(\Omega)$, we retrive that H is a Hilbert space.

Part b

To show that the Poincare inequality, we will assume that contrary, that is, suppose for $n \in \mathbb{N}$ there exists a function $u_n \in H^1(\Omega)$ such that

$$||u_n||_{L^2(\Omega)} > n\,||\nabla u_n||_{L^2(\Omega)}$$

and $||u||_{L^2(\Omega)} = 1$. Then we have that

$$||\nabla u_n||_{L^2(\Omega)} < \frac{1}{n}$$

Since $||v_n||_{L^2(\Omega)} = 1$ and $||\nabla u_n||_{L^2(\Omega)} < \frac{1}{n}$, we have that $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in $H^1(\Omega)$ and thus by the corollary of Rellich-Kondrachov we have that there exists a subsequence of $\{u_{n_k}\}_{k=1}^{\infty}$ such that $u_{n_k} \to u \in L^2(\Omega)$ in $L^2(\Omega)$. Since $||\nabla u_{n_k}|| \to 0$ as $n \to \infty$ we have that $\nabla u = 0$, and hence we have that $v \equiv C$ for some constant C on Ω. Then since each $u_n \in H$ we have that $u_n|_V = 0$ and hence we have that $u|_V = 0$, but since we found that u is a constant on Ω, we get that $u \equiv 0$ on all of Ω. However,

$$||u_n||_{L^2(\Omega)} = 1 \quad \text{but} \quad ||u_n||_{L^2(\Omega)} \to ||u||_{L^2(\Omega)} = 0$$

which is a contradiction and thus the Poincare inequality holds.