CSE 386D: Methods of Applied Mathematics II	Spring 2025

Problem Set 4

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Problem 7.2

Suppose that $f \in H_0^1(0,1)$ then we have that, there exists $f_k \in C_0^\infty(0,1)$ such that $f_k \to f$ and $f_k' \to f'$. Then we have that $f_k(0) = f_k(1) = 0$ and by the fundamental theorem of calculus we have that

$$f_k(x) = f_k(x) - f_k(0) = \int_0^x f_k'(t)\,dt$$

and so

$$\begin{aligned}
||f||^2_{L^2(0,1)} \leftarrow ||f_k||^2_{L^2(0,1)} = \int_0^1 |f_k(x)|^2\,dx &= \int_0^1 \left|\int_0^x f_k'(t)\,dt\right|^2\,dx \\
&\leq \int_0^1 \left(\int_0^x 1^2\,dt\right)\left(\int_0^x |f_k'(t)|^2\,dt\right)\,dx \\
&\leq \frac{1}{2}\,||f_k'||^2_{L^2(0,1)} \int_0^1 \int_0^x 1\,dt\,dx \\
&= ||f_k'||^2_{L^2(0,1)} \to \frac{1}{2}\,||f'||^2_{L^2(0,1)}
\end{aligned}$$

thus we get that

$$||f||_{L^2(0,1)} \leq \frac{1}{\sqrt{2}}\,||f'||_{L^2(0,1)}$$

Note that similarly that if $f \in \{g \in H^1(0,1) : \int_0^1 g(x)\,dx = 0\}$, then we have that

$$\int_0^1 f(x)\,dx = 0$$

and hence by the Intermediate Value Theorem we have that there exists a $f(c)$ such that $f(c) = 0$. Then we get that

$$f(x) = f(x) - f(c) = \int_c^x f'(t)\,dt$$

and so we have that

$$||f||^2_{L^2(0,1)} = \int_0^1 |f(x)|^2\,dx = \int_0^1 \left|\int_c^x f'(t)\,dt\right|^2\,dx \leq ||f'||^2_{L^2(0,1)} \int_0^1 (x-c)\,dx = \frac{1-2c}{2}\,||f'||^2_{L^2(0,1)}$$

and get that

$$||f||_{L^2(0,1)} \leq \sqrt{\frac{1-2c}{2}}\,||f'||_{L^2(0,1)}$$

Problem 7.4

Recall that $C^\infty(0,1) \cap H^1(0,1)$ is dense in $H^1(0,1)$ that is for $f \in H^1(0,1)$, there exists $f_k \in C^\infty(0,1) \cap H^1(0,1)$ such that $f_k \to f$ and $f_k' \to f'$. Then we have that

$$f_k(x) = f_k(x_0) + \int_{x_0}^{x} f_k'(t)\,dt$$

and choose x_0 such that is satisfies the mean value theorem, that is

$$f_k(x_0) = \int_0^1 f_k(t)\,dt$$

then we have that

$$\begin{aligned}|f_k(x)| \leq |f_k(x_0)| + \int_{x_0}^{x} |f_k'(t)|\,dt &\leq \int_0^1 |f_k(t)|\,dt + \int_0^1 |f_k'(t)|\,dt \\ &\leq C(||f_k||_{L^2(0,1)} + ||f_k'||_{L^2(0,1)}) \to C(||f||_{L^2(0,1)} + ||f'||_{L^2(0,1)})\end{aligned}$$

where the last inequality follows from Cauchy-Schwartz. Then we have that

$$||f||_{L^\infty(0,1)} \leq C(||f||_{L^2(0,1)} + ||f'||_{L^2(0,1)}) = C\,||f||_{H^1(0,1)}$$

thus $H^1(0,1)$ is continously imbedded in $C_B(0,1)$.

Problem 7.5

Let $\Omega \subseteq \mathbb{R}^d$ be bounded, and hence $\bar{\Omega}$ compact. Suppose that $\{U_j{}_{j=1}^N$ is a finite collection of open sets such that

$$\bar{\Omega} \subseteq \bigcup_{j=1}^{N} U_j$$

since $\bar{\Omega}$ is compact we have that there exists a finite subcover $\{V_k\}_{k=1}^M$ such that

$$\bar{\Omega} \subseteq \bigcup_{k=1}^{M} V_k \subseteq \bigcup_{j=1}^{N} U_j$$

now let $\psi_k \in C_0^\infty(\Omega)$ such that $0 \leq \psi_k \leq 1$, $\psi_k \equiv 1$ on V_k and $\text{supp}(\psi_k) \subseteq U_{j_k}$, where

$$V_k \subseteq \bigcup_{k} U_{j_k}$$

now let $u \in C_0^\infty(\Omega)$ be such that u maps one-to-one and onto Ω, then we get that

$$\phi_k = \frac{\psi_k}{\sum_{k=1}^M \psi_k u} u \in C_0^\infty(\Omega)$$

then we have that $\text{supp}(\phi_k) \subseteq U_{j_k}$, $\phi_k \subseteq U_{j_k}$, and $\sum_{k=1}^M \phi_k = 1$ and hence we have that $\{\phi_k\}_{k=1}^M$ is a partition of unity subordinate to $\{U_j\}_{j=1}^N$.

Problem 7.6

Let $\Omega \subseteq \mathbb{R}^d$ be a domain and $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of open sets in $\mathbb{R}^d$ that cover Ω, that is

$$\Omega \subseteq \bigcup_{\alpha \in \mathcal{I}} U_\alpha$$

Let S be the set of rational coordinates of Ω and let

$$\mathcal{B} = \{B_r(x) \subseteq \mathbb{R}^d : r \text{ is rational, } x \in S \text{ and } B_r(x) \subseteq U_\alpha \text{ for some } \alpha \in \mathcal{I}\}.$$

Then by assigning an ordering to $B_j = B_{r_j}(x_j)$ we define $\phi_j \in C_0^\infty(\Omega)$ such that $0 \leq \phi_j \leq 1$ and $\phi_j \equiv 1$ on $B_{r_j/2}(x_j)$ and we let $\psi_1 = \phi_1$ and $\psi_j = (1-\phi_1)(1-\phi_2)\dots(1-\phi_{j-1})\phi_j$. Clearly we see that $\psi_j \geq 0$ and by letting $A_k = \prod_{j=1}^k (1-\phi_j)$ with $A_0 = 1$, we get that $\psi_{k-1} = A_k \phi_k$ and

$$A_{k-1} - A_k = A_{k-1} - A_{k-1}(1-\phi_k) = A_{k-1}\phi_k = \psi_k$$

thus

$$\sum_{k=1}^{\infty} \psi_k = \sum_{k=1}^{\infty} A_{k-1} - A_k = A_0 - \lim_{k\to\infty} A_k = 1$$

additionally note that we have that,

$$\Omega \subseteq \bigcup_{j \in \mathcal{J}} B_j$$

Then we see that if $K \subset\subset \Omega$ then we have have that there exists some subset $\{B_{j_k}\}_{k=1}^M$ such that

$$K \subseteq \bigcup_{k=1}^{M} B_{j_k}$$

and hence there exists some finite subcover that covers K, such that

$$K \subseteq \bigcup_{k} B_{r_k/2(x_k)} \subseteq \bigcup_{j \in \mathcal{J}} B_j$$

then for ψ_k where k is greater than the maximum index of the finite subcover, we get that $\psi_k = 0$ and hence ψ_k vanishes for all but a finitely many terms. Lastly since $\text{supp}(\psi_k) \subseteq B_{r_k}(x_k)$ we have that $\text{supp}(\psi_k) \subseteq U_{\alpha_k}$ for some $\alpha_k \in \mathcal{I}$ and hence $\{\psi_k\}_{k=1}^{\infty}$ is a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in \mathcal{I}}$.