CSE 386D NOTES

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Contents

1 The Fourier Transform

1.1 The $L^1(\mathbb{R}^d)$ Theory

If $\xi \in \mathbb{R}^d$, the function

$$\varphi_\xi(x) = e^{-ix\cdot\xi} = \cos(x\cdot\xi) - i\sin(x\cdot\xi)$$

for $x \in \mathbb{R}^d$ is a plane wave in the direction ξ. Its period in the jth direction is $1\pi/\xi_j$.

Proposition 1.1. For such φ we have the following:

1. $|\varphi_\xi| = 1$ and $\bar{\varphi}_\xi = \varphi_{-\xi}$ for any $\xi \in \mathbb{R}^d$
2. $\varphi_\xi(x+y) = \varphi_\xi(x)\varphi_\xi(y)$ for any $x, y, \xi \in \mathbb{R}^d$
3. $-\Delta\varphi_\xi = |\xi|^2\varphi_\xi$ for any $\xi \in \mathbb{R}^d$

Principle 1.2. If $f \in L^1(\mathbb{R}^d)$, the Fourier transform of f is

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$$

Proposition 1.3. The Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathcal{R}^d)$$

is a bounded linear operator, and

$$\left\|\hat{f}\right\|_{L^\infty(\mathcal{R}^d)} \leq (2\pi)^{-d/2}\left\|f\right\|_{L^1(\mathbb{R}^d)}$$

Proposition 1.4. If $f \in L^1(\mathbb{R}^d)$ and τ_y is a translation by y, then

1. $\mathcal{F}(\tau_y f)(\xi) = e^{-iy\cdot\xi}\hat{f}(\xi)$ for all $y \in \mathbb{R}^d$.
2. $\mathcal{F}(e^{ix\cdot y}f)(\xi) = \tau_y\hat{f}(\xi)$ for all $y \in \mathbb{R}^d$
3. if $r > 0$ is given,
$$\mathcal{F}(f(rx))(\xi) = r^{-d}\hat{f}(r^{-1}\xi)$$
4. $\hat{\bar{f}}(\xi) = \overline{\hat{f}(-\xi)}$

Principle 1.5. A continuous function f on $\mathbb{R}^d$ is said to vanish at infinity if for any $\epsilon > 0$ there is $K \subset\subset \mathbb{R}^d$ such that

$$|f(x)| < \epsilon$$

for $x \notin K$, The subspace of all such continuous functions is denoted

$$C_v(\mathbb{R}^d) = \{f \in C^0(\mathbb{R}^d) : f \text{ vanishes at } \infty\}$$

Theorem 1.6. The space $C_v(\mathbb{R}^d)$ is a closed linear subspace of $L^\infty(\mathbb{R}^d)$

Theorem 1.7 (Riemann-Lebesgue Lemma)**.** The Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}^d) \to C_v(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$$

Then for $f \in L^1(\mathbb{R}^d)$

$$\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0 \quad \text{ and } \quad \hat{f} \in C^0(\mathbb{R}^d)$$

Proposition 1.8. If $f, g \in L^1(\mathbb{R}^d)$, then

1. $\int \mathcal{F}(f)g = \int f\mathcal{F}(g)$
2. $f * g \in L^1(\mathbb{R}^d)$ and $\mathcal{F}(f * g) = (2\pi)^{d/2}\mathcal{F}(f)\mathcal{F}(g)$

Theorem 1.9 (Generalized Young's Inequality)**.** Suppose $K(x, y)$ is measurable of $\mathbb{R}^d \times \mathbb{R}^d$ and there is some $C > 0$ such that

$$\int |K(x,y)|\, dx \leq C \quad \text{ and } \quad \int |K(x,y)|\, dy \leq C$$

for almost every $x, y \in \mathbb{R}^d$, respectively. Define the operator T by

$$Tf(x) = \int K(x,y)f(y)\, dy$$

If $1 \leq p \leq \infty$, then $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is a bounded linear operator with operator norm $\|T\| \leq C$.

Proposition 1.10 (Young's Inequality)**.** If $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$, then $f * g \in L^p(\mathbb{R}^d)$ and

$$\|f * g\|_p \leq \|f\|_p \|g\|_1$$

Theorem 1.11 (Paley-Wiener Theorem)**.** If $f \in C_0^\infty(\mathbb{R}^d)$, then $\mathcal{F}(f)$ extend to an entire holomorphic function on $\mathbb{C}^d$.

1.2 The Schwartz Space Theory

Principle 1.12. The Schwartz space or the space of functions of rapid decrease is defined as

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |x^\alpha D d^\beta f(x)| < \infty \text{ for all } \alpha, \beta\}$$

Proposition 1.13. One has that

$$C_0^\infty(\mathbb{R}^d) \subsetneqq \mathcal{S} \subsetneqq L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$$

thus also $\mathcal{S} \subset L^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$.

Principle 1.14. Given $n = 0, 1, 2, \ldots$ we define for $\phi \in \mathcal{S}$

$$\rho_n(\phi) = \sup_{|\alpha| \leq n} \sup_x (1 + |x|^2)^{n/2} |D^\alpha \phi(x)|$$

Proposition 1.15. The Schawartz class $\mathcal{S}$ is a complete metric space where the $\{\rho_n\}_{n=0}^\infty$ generate its topology through the metric

$$d(\phi_1, \phi_2) = \sum_{n=0}^\infty 2^{-n} \frac{\rho_n(\phi_1 - \phi_2)}{1 + \rho_n(\phi_1 - \phi_2)}$$

References