Problem Set 3

Student Name: Noah Reef

Problem 3.1

Part a

Recall that the angular momentum operator of a particle is given by

$$\hat{L} = \hat{r} \times \hat{p} = (\hat{r}_y \hat{p}_z - \hat{r}_z \hat{p}_y)u_x + (\hat{r}_z \hat{p}_x - \hat{r}_x \hat{p}_z)u_y + (\hat{r}_x \hat{p}_y - \hat{r}_y \hat{p}_x)u_z$$

then we have that

$$
\begin{aligned}
[\hat{L}_x, \hat{L}_y] &= [\hat{r}_y \hat{p}_z - \hat{r}_z \hat{p}_y, \hat{r}_z \hat{p}_x - \hat{r}_x \hat{p}_z] \\
&= [\hat{r}_y \hat{p}_z, \hat{r}_z \hat{p}_x] - [\hat{r}_y \hat{p}_z, \hat{r}_x \hat{p}_z] - [\hat{r}_z \hat{p}_y, \hat{r}_z \hat{p}_x] + [\hat{r}_z \hat{p}_y, \hat{r}_x \hat{p}_z] \\
&= [\hat{r}_y \hat{p}_z, \hat{r}_z]\hat{p}_x + \hat{r}_z[\hat{r}_y \hat{p}_z, \hat{p}_x] - [\hat{r}_y \hat{p}_z, \hat{r}_x]\hat{p}_z \\
&\quad - \hat{r}_x[\hat{r}_y \hat{p}_z, \hat{p}_z] - [\hat{r}_z \hat{p}_y, \hat{r}_z]\hat{p}_x - \hat{r}_z[\hat{r}_z \hat{p}_y, \hat{p}_x] + [\hat{r}_z \hat{p}_y, \hat{r}_x]\hat{p}_z + \hat{r}_x[\hat{r}_z \hat{p}_y, \hat{p}_z]
\end{aligned}
$$

then we see that

$$
\begin{aligned}
[\hat{r}_y \hat{p}_z, \hat{r}_z] &= \hat{r}_y[\hat{p}_z, \hat{r}_z] + [\hat{r}_y, \hat{r}_z]\hat{p}_z = -i\hbar \hat{r}_y \\
[\hat{r}_y \hat{p}_z, \hat{p}_x] &= \hat{r}_y[\hat{p}_z, \hat{p}_x] + [r_y, \hat{p}_x]\hat{p}_y = 0 \\
[\hat{r}_y \hat{p}_z, \hat{r}_x] &= 0 \\
[\hat{r}_y \hat{p}_z, \hat{p}_z] &= 0 \\
[\hat{r}_z \hat{p}_y, \hat{r}_z] &= 0 \\
[\hat{r}_z \hat{p}_y, \hat{p}_x] &= 0 \\
[\hat{r}_z \hat{p}_y, \hat{r}_x] &= 0 \\
[\hat{r}_z \hat{p}_y, \hat{p}_z] &= i\hbar \hat{p}_y
\end{aligned}
$$

and hence

$$[\hat{L}_x, \hat{L}_y] = -i\hbar \hat{r}_y \hat{p}_x + i\hbar \hat{r}_x \hat{p}_y = i\hbar \hat{L}_z$$

as desired.

Part b

We have that

$$
\begin{aligned}
[\hat{L}_z, \hat{L}^2] &= [\hat{L}_z, \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2] = [\hat{L}_z, \hat{L}_x^2] + [\hat{L}_z, \hat{L}_y^2] + [\hat{L}_z, \hat{L}_z^2] \\
&= [\hat{L}_z, \hat{L}_x]\hat{L}_x + \hat{L}_x[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_y]\hat{L}_y + \hat{L}_y[\hat{L}_z, \hat{L}_y] + [\hat{L}_z, \hat{L}_z]\hat{L}_z + \hat{L}_z[\hat{L}_z, \hat{L}_z] \\
&= 2i\hbar \hat{L}_x \hat{L}_y - 2i\hbar \hat{L}_y \hat{L}_x = 0
\end{aligned}
$$

Part c

Suppose that $|\epsilon\rangle$ is an eigenstate of $\hat{L}_z$, then we have that

$$\hat{L}_z \, |\epsilon\rangle = \epsilon \, |\epsilon\rangle$$

and notice that from part b, we showed that $[\hat{L}_z, \hat{L}^2] = 0$ and hence we have that

$$\hat{L}^2 \hat{L}_z \, |\epsilon\rangle = \epsilon \hat{L}^2 \, |\epsilon\rangle = \hat{L}_z \hat{L}^2 \, |\epsilon\rangle$$

an hence $\hat{L}^2 \, |\epsilon\rangle = K \, |\epsilon\rangle$ and hence $|\epsilon\rangle$ is also an eigenstate of $\hat{L}^2$.

Problem 3.2

Part a

Suppose we have the Hamilton operator of a certain quantum system in Dirac's notation

$$\hat{H} = E_0 = \sum_{n=1}^{\infty} n^2 \, |n\rangle \, \langle n|$$

then we see that

$$
\begin{aligned}
\hat{H} \, |m\rangle &= E_0 \sum_{n=1}^{\infty} n^2 \, |n\rangle \, \langle n| \, |m\rangle \\
&= E_0 \sum_{n=1}^{\infty} n^2 \, |n\rangle \, \delta_{nm} \\
&= E_0 m^2 \, |m\rangle
\end{aligned}
$$

thus $|m\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue $E_0 m^2$, where m is a positive integer.

Part b

We can define a square root operator of $\hat{H}$ as

$$\hat{H}^{1/2} = \sqrt{E_0} \sum_{n=1}^{\infty} n \, |n\rangle \, \langle n|$$

since each of the $|n\rangle$ are orthogonal, we have that each of the off-diagonal terms are zero and hence we are left with the diagonal terms and hence get the form above.

Part c

Note that using the form of $\hat{H}^{1/2}$ above, we can see that

$$\hat{H}^{1/2} = \sqrt{E_0} \sum_{n=1}^{\infty} (-1)^{mn} n \, |n\rangle \, \langle n|$$

for any choice $m \in \mathbb{Z}$, are also square-root operators of $\hat{H}$, and thus there are infinitely many square-root operators of $\hat{H}$.

Part d

Looking at the eigenvalue spectrum of the Hamiltonian in part a, we see that this system describes a particle in the one-dimensional infinite square well potential

$$V(x) = \begin{cases} -V_0 & |x| < a \\ \infty & |x| > a \end{cases}$$

the square-root operator, $\hat{H}$, is associated with the magnitude of the momentum of the particle.

Problem 3.3

Part a

Suppose that we have the one-dimensional quartic anharmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \kappa\frac{m^2\omega^3}{\hbar}\hat{x}^4$$

recall that

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)$$

and we compute that

$$\begin{aligned}
\hat{a}^\dagger\hat{a} &= \frac{m\omega}{2\hbar}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right) \\
&= \frac{m\omega}{2\hbar}\left(\hat{x}^2 + i\frac{\hat{x}\hat{p}}{m\omega} - i\frac{\hat{p}\hat{x}}{m\omega} + \frac{\hat{p}^2}{m^2\omega^2}\right) \\
&= \frac{m\omega}{2\hbar}\left(\hat{x}^2 + i\frac{[\hat{x},\hat{p}]}{m\omega} + \frac{\hat{p}^2}{m^2\omega^2}\right) \\
&= \frac{m\omega}{2\hbar}\left(\hat{x}^2 - \frac{\hbar}{m\omega} + \frac{\hat{p}^2}{m^2\omega^2}\right) \\
&= -\frac{1}{2} + \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{\hat{p}^2}{2\hbar m\omega}
\end{aligned}$$

and

$$\hat{a} + \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}2\hat{x}$$

then we see that

$$\frac{1}{2} + \hat{a}^\dagger \hat{a} + \frac{\kappa}{4}(\hat{a} + \hat{a}^\dagger)^4 = \frac{1}{2} + \left(-\frac{1}{2} + \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{\hat{p}^2}{2\hbar m\omega} \right) + \frac{\kappa}{4}\left(\frac{4m^2\omega^2}{\hbar^2}\hat{x}^4 \right)$$

$$= \left(\frac{m\omega}{2\hbar}\hat{x}^2 + \frac{\hat{p}}{2\hbar m\omega} + \frac{\kappa m^2\omega^2}{\hbar^2}\hat{x}^4 \right)$$

$$= \frac{1}{\hbar\omega}\left(\frac{m\omega^2}{2}\hat{x}^2 + \frac{\hat{p}}{2m} + \kappa\frac{m^2\omega^3}{\hbar}\hat{x}^4 \right)$$

$$= \frac{\hat{H}}{\hbar\omega}$$

Part b

Consider

$$(\hat{a} + \hat{a}^\dagger)^4 = (\hat{a} + \hat{a}^\dagger)^2(\hat{a} + \hat{a}^\dagger)^2$$

Then for

$$(\hat{a} + \hat{a}^\dagger)^2 = (\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2})$$

$$= (\hat{a}^2 + 1 + 2\hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2})$$

$$= (\hat{a}^2 + 1 + 2\hat{N} + \hat{a}^{\dagger 2})$$

thus

$$(\hat{a} + \hat{a}^\dagger)^4 = (\hat{a}^2 + \hat{a}^{\dagger 2})^2 + (\hat{a}^2 + \hat{a}^{\dagger 2})(1 + 2\hat{N}) + (1 + 2\hat{N})(\hat{a}^2 + \hat{a}^{\dagger 2}) + (1 + 2\hat{N})^2$$

and we have that

$$(\hat{a}^2 + \hat{a}^{\dagger 2}) = \hat{a}^4 + \hat{a}^2\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{a}^2 + \hat{a}^{\dagger 2}$$

$$(\hat{a}^2 + \hat{a}^{\dagger 2})(1 + 2\hat{N}) = \hat{a}^2 + 2\hat{a}^2\hat{N} + \hat{a}^{\dagger 2} + 2\hat{a}^{\dagger 2}\hat{N}$$

$$(1 + 2\hat{N})(\hat{a}^2 + \hat{a}^{\dagger 2}) = \hat{a}^2 + 2\hat{N}\hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{N}\hat{a}^{\dagger 2}$$

$$(1 + 2\hat{N})^2 = 1 + 4\hat{N} + 4\hat{N}^2$$

and hence

$$(\hat{a}+\hat{a}^\dagger)^4 = \hat{a}^4 + \hat{a}^{\dagger 2} + 2(\hat{a}^2 + \hat{a}^{\dagger 2}) + 2\left[\hat{N}(\hat{a}^2 + \hat{a}^{\dagger 2}) + (\hat{a}^2 + \hat{a}^{\dagger 2})\hat{N} \right] + \hat{a}^2\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{a}^2 + 4\hat{N} + 4\hat{N}^2 + 1$$

then we have

$$\hat{a}^2\hat{a}^{\dagger 2} = 2 + 3\hat{N} + \hat{N}^2$$

$$\hat{a}^{\dagger 2}\hat{a}^2 = \hat{N}^2 - \hat{N}$$

and so we have that

$$(\hat{a} + \hat{a}^\dagger)^4 = \hat{a}^4 + \hat{a}^{\dagger 4} + 2(\hat{a}^2 + \hat{a}^{\dagger 2}) + 2\left[\hat{N}(\hat{a}^2 + \hat{a}^{\dagger 2}) + (\hat{a}^2 + \hat{a}^{\dagger 2})\hat{N} \right] + 3 + 6\hat{N} + 6\hat{N}^2$$

Part c

Let $\kappa = 0$, and let $|n\rangle$ be an eigenstate of $\hat{H}$, then we have that

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \implies \hat{a}^2\,|n\rangle = \sqrt{n(n-1)}\,|n-2\rangle$$

and thus

$$\langle n|\,\hat{a}^2\,|n\rangle = \sqrt{n(n-1)}\,\langle n|n-2\rangle = 0$$

since $|n\rangle$ and $|n-2\rangle$ are orthogonal. Now we consider,

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \implies \hat{a}^{\dagger^2}\,|n\rangle = \sqrt{(n+1)(n+2)}\,|n+2\rangle$$

then we have that

$$\langle n|\,\hat{a}^{\dagger^2}\,|n\rangle = \sqrt{(n+1)(n+2)}\,\langle n|n+2\rangle = 0$$

since $|n\rangle$ and $|n+2\rangle$ are orthogonal. Lastly we have that

$$\langle n|\,\hat{a}^4\,|n\rangle = \sqrt{n(n-1)(n-2)(n-3)}\,\langle n|n-4\rangle = 0$$
$$\langle n|\,\hat{a}^{\dagger^4}\,|n\rangle = \sqrt{(n+1)(n+2)(n+3)(n+4)}\,\langle n|n+4\rangle = 0$$

Part d

$$\langle n|\,\hat{N}\hat{a}^2\,|n\rangle = \sqrt{n(n-1)}\,\langle n|\,\hat{N}\,|n-2\rangle = (n-2)\sqrt{n(n-1)}\,\langle n|n-2\rangle = 0$$
$$\langle n|\,\hat{N}\hat{a}^{\dagger^2}\,|n\rangle = \sqrt{(n+1)(n+2)}\,\langle n|\,\hat{N}\,|n+2\rangle = (n+2)\sqrt{(n+1)(n+2)}\,\langle n|n+2\rangle = 0$$
$$\langle n|\,\hat{a}^2\hat{N}\,|a\rangle = n\,\langle n|\,\hat{a}^2\,|n\rangle = n\sqrt{n(n-1)}\,\langle n|n-2\rangle = 0$$
$$\langle n|\,\hat{a}^{\dagger^2}\hat{N}\,|a\rangle = n\,\langle n|\,\hat{a}^{\dagger^2}\,|n\rangle = n\sqrt{(n+1)(n+2)}\,\langle n|n+2\rangle = 0$$

Part e

$$\langle n|\,\hat{H}\,|n\rangle = \hbar\omega\left(\frac{1}{2}\,\langle n|n\rangle + \langle n|\,\hat{N}\,|n\rangle + \frac{\kappa}{4}\,\langle n|\,(\hat{a}+\hat{a}^\dagger)^4\,|n\rangle\right)$$

note that

$$\langle n|n\rangle = 1$$
$$\langle n|\,\hat{N}\,|n\rangle = n\,\langle n|n\rangle = n$$
$$\langle n|\,(\hat{a}+\hat{a}^\dagger)\,|n\rangle = \langle n|\,\hat{a}^4 + \hat{a}^{\dagger^4} + 2(\hat{a}^2 + \hat{a}^{\dagger^2}) + 2\left[\hat{N}(\hat{a}^2 + \hat{a}^{\dagger^2}) + (\hat{a}^2 + \hat{a}^{\dagger^2})\hat{N}\right] + 3 + 6\hat{N} + 6\hat{N}^2\,|n\rangle$$

and we have that the above reduces to

$$\langle n|\,(\hat{a}+\hat{a}^\dagger)\,|n\rangle = 3\,\langle n|n\rangle + 6\,\langle n|\,\hat{N}\,|n\rangle + 6\,\langle n|\,\hat{N}^2\,|n\rangle = 3 + 6n + 6n^2$$

and hence

$$\langle n|\,\hat{H}\,|n\rangle = \hbar\omega\left(\frac{1}{2} + n\right) + \frac{\kappa\hbar\omega}{4}\left(2 + 6n + 6n^2\right)$$
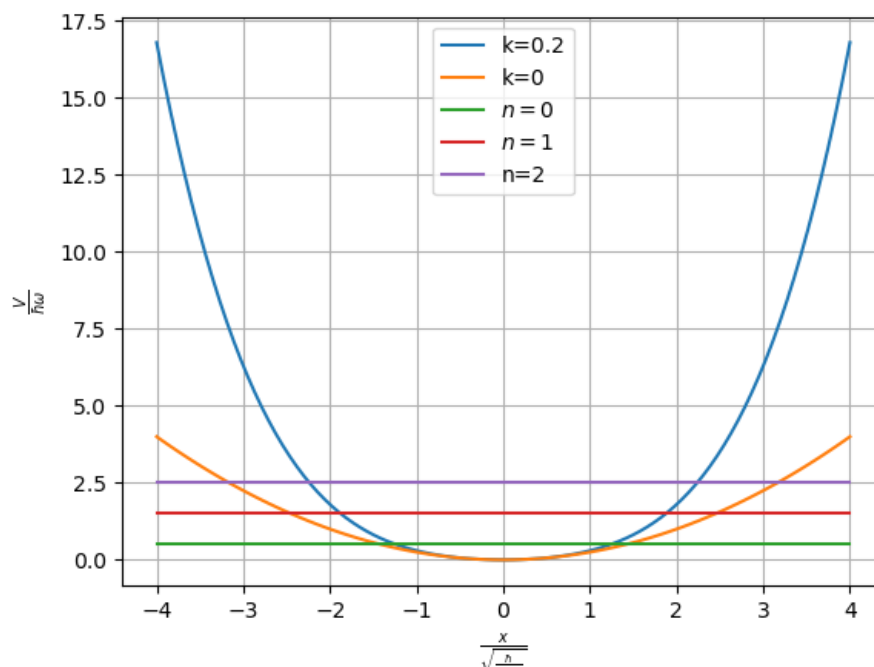
Part f & g

Figure 3.1. $\frac{V}{\hbar\omega}$ vs $x/(\sqrt{\hbar/2m\omega})$