CSE 386D: Methods of Applied Mathematics II	**Spring 2025**

Problem Set 10

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Problem 8.14

Part a

Let $\mathcal{H} = H_0^1(\Omega)$ and $H = V_n$ for some n, and define

$$\begin{aligned} B(u_n, v_n) &= \langle \nabla u_n, \nabla v_n \rangle_{L^2(\Omega)} + \langle u_n, v_n \rangle_{L^2(\Omega)} \\ F(v_n) &= \langle f, v_n \rangle_{L^2(\Omega)} \end{aligned}$$

Note that,

$$|F(v_n)| = \left| \langle f, v_n \rangle_{L^2(\Omega)} \right| \leq ||f||_{L^2(\Omega)} \, ||v_n||_{L^2(\Omega)} \leq C_p \, ||f||_{H^1(\Omega)} \, ||v_n||_{H^1(\Omega)}$$

thus F is continuous. Then we see that,

$$\begin{aligned} |B(u_n, v_n)| &= \left| \langle \nabla u_n, \nabla v_n \rangle_{L^2(\Omega)} + \langle u_n, v_n \rangle_{L^2(\Omega)} \right| \\ &\leq \left| \langle \nabla u_n, \nabla v_n \rangle_{L^2(\Omega)} \right| + \left| \langle u_n, v_n \rangle_{L^2(\Omega)} \right| \\ &\leq ||\nabla u_n||_{L^2(\Omega)} \, ||\nabla v_n||_{L^2(\Omega)} + ||u_n||_{L^2(\Omega)} \, ||v_n||_{L^2(\Omega)} \\ &\leq (C_p + 1) \, ||u_n||_{H^1(\Omega)} \, ||v_n||_{H^1(\Omega)} \end{aligned}$$

thus B is continous. Similarly, we have that

$$\begin{aligned} B(u_n, u_n) &= \langle \nabla u_n, \nabla u_n \rangle_{L^2(\Omega)} + \langle u_n, u_n \rangle_{L^2(\Omega)} \\ &\geq ||\nabla u_n||_{L^2(\Omega)}^2 + ||u_n||_{L^2(\Omega)}^2 \\ &\geq ||u_n||_{H^1(\Omega)}^2 \end{aligned}$$

and hence B si coercive. Thus by the Lax-Milgram theorem, we have that there exists a unique solution $u_n \in V_n$ such that

$$B(u_n, v_n) = F(v_n) \quad \forall v_n \in V_n$$

and we see that

$$|B(u_n, u_n)| = ||u_n||_{H^1(\Omega)}^2 = | \langle f, u_n \rangle_{L^2(\Omega)} | \leq ||f||_{L^2(\Omega)} \, ||u_n||_{L^2(\Omega)}$$

dividing by $||u_n||_{L^2(\Omega)}$ gives us

$$||u_n||_{H^1(\Omega)} \leq ||f||_{L^2(\Omega)}$$

Part b

From part a, we have that u_n is uniformly bounded in $H^1(\Omega)$, thus by the Banach-Alaoglu theorem we have that there exists a subsequence $u_n \rightharpoonup u$ in $H^1(\Omega)$. Note that $V = \bigcup_{n=1}^{\infty} V_n$ is dense in $H^1(\Omega)$, and hence $\bar{V} = H^1(\Omega)$. Since the variational problem, for each n, of finding $u_n \in V_n$ such that

$$B(u_n, v_n) = F(v_n) \quad \forall v_n \in V_n$$

has a unique solution. We have that the same problem posed on $H^1(\Omega)$, also has a unique solution, that is

$$B(u, v) = F(v) \quad \forall v \in H^1(\Omega)$$

has a unique solution $u^* \in H^1(\Omega)$. For each $v \in H^1(\Omega)$, there exists a sequence $v_n \to v$ where $v_n \in V_n$, and hence we have that since $u_n \rightharpoonup u$ we get that

$$B(u_n, v_n) = F(v_n) \to B(u, v) = F(v) \quad \forall v \in H^1(\Omega)$$

which implies that u is a solution to the variational problem posed on $H^1(\Omega)$, and hence $u = u^*$.

Part c