CSE 386D NOTES

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Contents

1 The Fourier Transform

1.1 The $L^1(\mathbb{R}^d)$ Theory

If $\xi \in \mathbb{R}^d$, the function

$$\varphi_\xi(x) = e^{-ix\cdot\xi} = \cos(x\cdot\xi) - i\sin(x\cdot\xi)$$

for $x \in \mathbb{R}^d$ is a plane wave in the direction ξ. Its period in the jth direction is $1\pi/\xi_j$.

Proposition 1.1. For such φ we have the following:

1. $|\varphi_\xi| = 1$ and $\bar{\varphi}_\xi = \varphi_{-\xi}$ for any $\xi \in \mathbb{R}^d$
2. $\varphi_\xi(x+y) = \varphi_\xi(x)\varphi_\xi(y)$ for any $x, y, \xi \in \mathbb{R}^d$
3. $-\Delta\varphi_\xi = |\xi|^2\varphi_\xi$ for any $\xi \in \mathbb{R}^d$

Principle 1.2. If $f \in L^1(\mathbb{R}^d)$, the Fourier transform of f is

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$$

Proposition 1.3. The Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathcal{R}^d)$$

is a bounded linear operator, and

$$\left\|\hat{f}\right\|_{L^\infty(\mathcal{R}^d)} \leq (2\pi)^{-d/2}\,\|f\|_{L^1(\mathbb{R}^d)}$$

Proposition 1.4. If $f \in L^1(\mathbb{R}^d)$ and τ_y is a translation by y, then

1. $\mathcal{F}(\tau_y f)(\xi) = e^{-iy\cdot\xi}\hat{f}(\xi)$ for all $y \in \mathbb{R}^d$.
2. $\mathcal{F}(e^{ix\cdot y}f)(\xi) = \tau_y\hat{f}(\xi)$ for all $y \in \mathbb{R}^d$
3. if $r > 0$ is given,
$$\mathcal{F}(f(rx))(\xi) = r^{-d}\hat{f}(r^{-1}\xi)$$
4. $\hat{\hat{f}}(\xi) = \overline{\hat{f}(-\xi)}$

Principle 1.5. A continuous function f on $\mathbb{R}^d$ is said to vanish at infinity if for any $\epsilon > 0$ there is $K \subset\subset \mathbb{R}^d$ such that

$$|f(x)| < \epsilon$$

for $x \notin K$, we define

$$C_v(\mathbb{R}^d) = \{f \in C^0(\mathbb{R}^d) : f \text{ vanishes at } \infty\}$$

Theorem 1.6. The space $C_v(\mathbb{R}^d)$ is a closed linear subspace of $L^\infty(\mathbb{R}^d)$

Theorem 1.7 (Riemann-Lebesgue Lemma)**.** The Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}^d) \to C_v(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$$

The for $f \in L^1(\mathbb{R}^d)$

$$\lim_{|\xi| \to \infty} |f(\xi)| = 0 \quad \text{and} \quad \hat{f} \in C^0(\mathbb{R}^d)$$

References