CSE 389D: Mathematical Modelling **Spring 2025**

Problem Set 1

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Problem 1

Problem 2

Part a

$$L(x, \dot{x}, t) = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Part b

Using the Euler-lagrange equation, we have:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = -kx - m\ddot{x} = 0$$

solving the differential equation above gives the general solution,

$$x(t) = c_2 \sin(\omega t) + c_1 \cos(\omega t)$$

where $\omega = \sqrt{\frac{k}{m}}$. Then we see with boundary equations that

$$\begin{aligned} x(0) &= c_1 = x_1 \\ x(\tau) &= c_2 \sin(\omega\tau) + x_1 \cos(\omega\tau) = x_2 \end{aligned}$$

solving for c_2 gives

$$c_2 = \frac{x_2 - x_1 \cos(\omega\tau)}{\sin(\omega\tau)}$$

since $\tau \neq \frac{n\pi}{\omega}$, we have that $\sin(\omega\tau) \neq 0$.

Part c

To solve for the action S, we have that

$$
\begin{aligned}
S &= \int_0^\tau L(x, \dot{x}, t) dt \\
&= \int_0^\tau \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) dt \\
&= \frac{1}{2}m \int_0^\tau \dot{x}^2 - \omega^2 x^2 \, dt \\
&= \frac{1}{2}m \int_0^\tau (G\omega\cos(\omega t) - \omega x_1 \sin(\omega t))^2 - \omega^2(x_1\cos(\omega t) + G\sin(\omega t))^2 \, dt \\
&= \frac{1}{2}m \int_0^\tau (G^2\omega^2 - \omega^2 x_1^2)\cos^2(\omega t) - 4G\omega^2 x_1 \sin(\omega t)\cos(\omega t) + (\omega^2 x_1^2 - G^2\omega^2)\sin^2(\omega t)\, dt \\
&= \frac{1}{2}m(G^2\omega^2 - \omega^2 x_1^2)\left[\frac{2\omega\tau + \sin(2\omega\tau)}{4\omega}\right] + \frac{1}{2}m(\omega^2 x_1^2 - G^2\omega^2)\left[\frac{2\omega\tau - \sin(2\omega\tau)}{4\omega}\right] - 2Gm\omega^2 x_1\left[\frac{\sin^2(\omega\tau)}{2\omega}\right] \\
&= \frac{m}{4\omega}\left[G^2\omega^2\sin(2\omega\tau) - \omega^2 x_1^2 \sin(2\omega\tau) - 4G\omega^2 x_1 \sin^2(\omega\tau)\right]
\end{aligned}
$$

where $G = \frac{x_2 - x_1\cos(\omega\tau)}{\sin(\omega\tau)}$. Now substituting G into the equation above gives

$$
\begin{aligned}
S &= \frac{m}{4\omega}\left[\frac{(x_2 - x_1\cos(\omega\tau))^2}{\sin(\omega\tau)} - \omega^2 x_1^2 \sin(2\omega\tau) - 4(x_2 - x_1\cos(\omega\tau))\omega^2 x_1 \sin(\omega\tau)\right] \\
&= \frac{m}{4\omega}\left[\frac{x_2^2}{\sin(\omega\tau)} - \frac{2x_1x_2(\cos^2(\omega\tau) + 2\omega^2\sin^2(\omega\tau))}{\sin(\omega\tau)} + \frac{x_1^2(\cos^2(\omega\tau) + 2\omega^2\sin^2(\omega\tau))}{\sin(\omega\tau)}\right] \\
&= \frac{m}{4\omega\sin(\omega\tau)}(x_2 - x_1(\cos^2(\omega\tau) + 2\omega^2\sin^2(\omega\tau))x_2)^2
\end{aligned}
$$