

Problem Set 4

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Recall that the spin operator is given by $\hat{S} = \frac{1}{2}\hbar\sigma$ where $\sigma = \sigma_x u_x + \sigma_y u_y + \sigma_z u_z$. Then we have that the expectation of the spin operator is

$$\langle \Psi | \hat{S} | \Psi \rangle = \frac{\hbar}{2} [\langle \Psi | \sigma_x | \Psi \rangle u_x + \langle \Psi | \sigma_y | \Psi \rangle u_y + \langle \Psi | \sigma_z | \Psi \rangle u_z]$$

and we have that

$$\begin{aligned} \langle \Psi | \sigma_x | \Psi \rangle &= [u^*(r) \quad d^*(r)] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} u(r) \\ d(r) \end{bmatrix} = u^*(r)d(r) + d^*(r)u(r) \\ &= \text{Re}(u^*d) + i \text{Im}(u^*d) + \text{Re}(u^*d) - i \text{Im}(u^*d) \\ &= 2 \text{Re}(u^*d) \end{aligned}$$

and

$$\begin{aligned} \langle \Psi | \sigma_y | \Psi \rangle &= [u^*(r) \quad d^*(r)] \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{bmatrix} u(r) \\ d(r) \end{bmatrix} = -iu^*(r)d(r) + id^*(r)u(r) \\ &= -i \text{Re}(u^*d) + \text{Im}(u^*d) + i \text{Re}(u^*d) + \text{Im}(u^*d) \\ &= 2 \text{Im}(u^*d) \end{aligned}$$

lastly,

$$\begin{aligned} \langle \Psi | \sigma_z | \Psi \rangle &= [u^*(r) \quad d^*(r)] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} u(r) \\ d(r) \end{bmatrix} = u^*(r)u(r) - d^*(r)d(r) \\ &= |u|^2 - |d|^2 \end{aligned}$$

and hence then expectation value of the spin operator on this spinsor is given by

$$\langle \Psi | \hat{S} | \Psi \rangle = \frac{\hbar}{2} [2 \text{Re}(u^*d)u_x + 2 \text{Im}(u^*d)u_y + (|u|^2 - |d|^2)u_z]$$

Part b

We compute the norm of $\langle \Psi | \hat{S} | \Psi \rangle$ as

$$\begin{aligned}
 |\langle \Psi | \hat{S} | \Psi \rangle| &= \frac{\hbar}{2} \sqrt{4 \operatorname{Re}(u^* d)^2 + 4 \operatorname{Im}(u^* d)^2 + |u|^4 - 2|u|^2|d|^2 + |d|^4} \\
 &= \frac{\hbar}{2} \sqrt{4|u^* d|^2 + |u|^4 - 2|u|^2|d|^2 + |d|^4} \\
 &= \frac{\hbar}{2} \sqrt{4|u|^2|d|^2 + |u|^4 - 2|u|^2|d|^2 + |d|^4} \\
 &= \frac{\hbar}{2} \sqrt{|u|^4 + 2|u|^2|d|^2 + |d|^4} \\
 &= \frac{\hbar}{2} \sqrt{(|u|^2 + |d|^2)^2} \\
 &= \frac{\hbar}{2} \sqrt{|\Psi|^2} = \frac{\hbar}{2}
 \end{aligned}$$

Part c

Consider

$$\exp\left(i\varphi \frac{\hat{S}_z}{\hbar}\right) = \sum_{n=0}^{\infty} \frac{\left(i\varphi \frac{\hat{S}_z}{\hbar}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\varphi}{2}\right)^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^n$$

note that for n even we have that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and remains unchanged for n odd, thus we can write the above as

$$\exp\left(i\varphi \frac{\hat{S}_z}{\hbar}\right) = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\varphi}{2}\right)^n & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i\varphi}{2}\right)^n \end{pmatrix} = \begin{pmatrix} \exp\left(\frac{i\varphi}{2}\right) & 0 \\ 0 & \exp\left(-\frac{i\varphi}{2}\right) \end{pmatrix}$$

Part d

Note that

$$\Psi' = \exp\left(i\varphi \frac{\hat{S}_z}{\hbar}\right) \Psi = \begin{pmatrix} \exp\left(\frac{i\varphi}{2}\right) & 0 \\ 0 & \exp\left(-\frac{i\varphi}{2}\right) \end{pmatrix} \begin{bmatrix} u \\ d \end{bmatrix} = \begin{bmatrix} e^{i\varphi/2} u \\ e^{-i\varphi/2} d \end{bmatrix}$$

then we get that

$$\begin{aligned}
 \langle \Psi | \sigma_x | \Psi \rangle &= \begin{bmatrix} e^{-i\varphi/2} u^*(r) & e^{i\varphi/2} d^*(r) \end{bmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} e^{i\varphi/2} u(r) \\ e^{-i\varphi/2} d(r) \end{bmatrix} = e^{-i\varphi} u^*(r) d(r) + e^{i\varphi} d^*(r) u(r) \\
 &= \operatorname{Re}(e^{-i\varphi} u^* d) + i \operatorname{Im}(e^{-i\varphi} u^* d) + \operatorname{Re}(e^{-i\varphi} u^* d) - i \operatorname{Im}(e^{-i\varphi} u^* d) \\
 &= 2 \operatorname{Re}(e^{-i\varphi} u^* d)
 \end{aligned}$$

and

$$\begin{aligned}\langle \Psi | \sigma_y | \Psi \rangle &= [e^{-i\varphi/2}u^*(r) \quad e^{i\varphi/2}d^*(r)] \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{bmatrix} e^{i\varphi/2}u(r) \\ e^{-i\varphi/2}d(r) \end{bmatrix} = -ie^{i\varphi}u^*(r)d(r) + ie^{i\varphi}d^*(r)u(r) \\ &= -i \operatorname{Re}(e^{-i\varphi}u^*d) + \operatorname{Im}(e^{-i\varphi}u^*d) + i \operatorname{Re}(e^{-i\varphi}u^*d) + \operatorname{Im}(e^{-i\varphi}u^*d) \\ &= 2 \operatorname{Im}(e^{-i\varphi}u^*d)\end{aligned}$$

lastly,

$$\begin{aligned}\langle \Psi | \sigma_z | \Psi \rangle &= [e^{-i\varphi/2}u^*(r) \quad e^{i\varphi/2}d^*(r)] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} e^{i\varphi/2}u(r) \\ e^{-i\varphi/2}d(r) \end{bmatrix} = u^*(r)u(r) - d^*(r)d(r) \\ &= |u|^2 - |d|^2\end{aligned}$$

and hence then expectation value of the spin operator on this spensor is given by

$$\langle \Psi' | \hat{S} | \Psi' \rangle = \frac{\hbar}{2} [2 \operatorname{Re}(e^{-i\varphi}u^*d)u_x + 2 \operatorname{Im}(e^{-i\varphi}u^*d)u_y + (|u|^2 - |d|^2)u_z]$$

Part e

Suppose that $u^*d = a + bi$ then $a = \frac{1}{2}S_x$ and $b = \frac{1}{2}S_y$ and we have that

$$e^{-i\varphi}u^*d = (a + bi)(\cos(i\varphi) - i \sin(\varphi)) = a \cos(\varphi) + b \sin(\varphi) + i(b \cos(\varphi) - a \sin(\varphi))$$

thus

$$S'_x = 2 \operatorname{Re}(e^{-i\varphi}u^*d) = S_x \cos(\varphi) + S_y \sin(\varphi)$$

and

$$S'_y = 2 \operatorname{Im}(e^{-i\varphi}u^*d) = S_y \cos(\varphi) - S_x \sin(\varphi)$$

and clearly

$$S'_z = S_z$$

Part f

From the change of coordinates in Part e, we have that $\exp\left(i\varphi \frac{\hat{S}_z}{\hbar}\right)$ is a clockwise rotation about the z -axis by angle φ .