

Problem Set 1

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Problem 6.1

To find the Fourier transform of $f(x) = e^{-|x|}$ for $x \in \mathbb{R}$, we compute

$$\begin{aligned}
 \hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ix\xi} dx \\
 &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^0 e^{x(1-i\xi)} dx + \int_0^{\infty} e^{-x(1+i\xi)} dx \right) \\
 &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\xi} e^{x(1-i\xi)} \Big|_{-\infty}^0 - \frac{1}{1+i\xi} e^{-x(1+i\xi)} \Big|_0^{\infty} \right) \\
 &= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{(1-i\xi)(1+i\xi)} \right) \\
 &= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+\xi^2} \right)
 \end{aligned}$$

Problem 6.2

We will consider the parameteric curve defined as

$$\begin{aligned}
 \gamma_1 &= t & -R \leq t \leq R \\
 \gamma_2 &= R - it & 0 \leq t \leq \frac{\xi}{2a} \\
 \gamma_3 &= -t - i\frac{\xi}{2a} & -R \leq t \leq R \\
 \gamma_4 &= -R + it & -\frac{\xi}{2a} \leq t \leq 0
 \end{aligned}$$

and by Cauchy's Integral theorem we have that

$$\begin{aligned}
 0 &= \int_{-R}^R e^{-at^2 - i\xi t} dt + i \int_0^{\xi/2a} e^{-a(R-it)^2 - i\xi(R-it)} dt \\
 &\quad - \int_{-R}^R e^{-a(-t-i\xi/2a)^2 - i(-t-i\xi/2a)} dt - i \int_{-\xi/2a}^0 e^{-a(-R+it)^2 - i\xi(-R+it)} dt
 \end{aligned}$$

Then we see that the second and fourth terms go to zero since,

$$\left| i \int_0^{\xi/2a} e^{-a(R-it)^2 - i\xi(R-it)} dt \right| + \left| i \int_{-\xi/2a}^0 e^{-a(-R+it)^2 - i\xi(-R+it)} dt \right| \leq 2 \frac{\xi}{2a} e^{-aR^2} \rightarrow 0$$

and we see for the third term that

$$\lim_{R \rightarrow \infty} \int_{-R}^R e^{-a(-t-i\xi/2a)^2 - i(-t-i\xi/2a)} dt = \sqrt{\frac{\pi}{a}} e^{-\xi^2/(4a)}$$

and hence

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-a|x|^2} e^{-ix\xi} dx = \sqrt{\frac{\pi}{a}} \frac{e^{-\xi^2/(4a)}}{\sqrt{2\pi}}$$

Problem 6.4

Suppose the $f \in L^1(\mathbb{R}^d)$ and $f(x) = g(|x|)$ for some g , then we see that

$$\begin{aligned} \hat{f}(\xi) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(|x|) e^{-ix \cdot \xi} dx \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(|x|) e^{-i|x||\xi| \cos(\theta)} dx \\ &= (2\pi)^{-d/2} \int_0^\infty \int_{\omega_d} g(r) e^{-ir|\xi| \cos(\theta)} r^d dr d\theta \\ &= h(|\xi|) \end{aligned}$$

Problem 6.11

Consider $\mathcal{F} : L^1(\mathbb{R}^d) \rightarrow C_v(\mathbb{R}^d)$. Recall that $\mathcal{F} : L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)$ is a bounded linear map by Proposition 6.2 and we know that $C_v(\mathbb{R}^d)$ is a closed linear subspace of $L^\infty(\mathbb{R}^d)$ by Proposition 6.4, hence $\mathcal{F} : L^1(\mathbb{R}^d) \rightarrow C_v(\mathbb{R}^d)$ is a bounded linear map. Next we note that if $f, g \in L^1(\mathbb{R}^d)$ such that $\mathcal{F}(f) = \mathcal{F}(g)$ for all $\xi \in \mathbb{R}^d$, then we have that $\mathcal{F}(f - g) = 0$ and hence $f - g = 0$ and so $f = g$, thus \mathcal{F} is injective. Now suppose that \mathcal{F} is surjective, then we have by the Open Mapping Theorem, that \mathcal{F}^{-1} is bounded.

Next suppose we have the characteristic functions $f_n, f_1 \in L^1(\mathbb{R}^d)$ and consider $f_n * f_1 \in$

$L^1(\mathbb{R}^d)$, then we see that

$$\begin{aligned}
 f_n * f_1 &= \int_{\mathbb{R}^d} f_n(x-y) f_1(y) dy \\
 &= \int_{[-1,1]^d} f_n(x-y) dy \\
 &= \int_{[x-1, x+1]^d} f_n(z) dz \\
 &= \int_{[x-1, 0]^d} f_n(z) dz + \int_{[0, x+1]^d} f_n(z) dz \quad (\in C_v(\mathbb{R}^d) \text{ By Exercise 5}) \\
 &= \begin{cases} x+n+1 & x \in [-n-1, -n+1] \\ 2 & x \in [-n+1, n-1] \\ n+1-x & x \in [n-1, n+1] \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

then we see that as $n \rightarrow \infty$ we have that $f_n * f_1 \rightarrow 2$. Note that

$$\mathcal{F}^{-1}(f_n * f_1) = C \frac{\sin(nx) \sin(x)}{x^2}$$

where C is some constant. However as $n \rightarrow \infty$ we see that

$$\begin{aligned}
 \|\mathcal{F}^{-1}[f_n * f_1]\|_{L^1} &= C \int_{\mathbb{R}^d} \frac{|\sin(nx) \sin(x)|}{|x^2|} dx \\
 &\geq C \int_{[0,1]^d} \frac{|\sin(nx) \sin(x)|}{|x^2|} dx \\
 &\geq \frac{2C}{\pi} \int_{[0,1]^d} \left| \frac{\sin(nx)}{x} \right| dx \rightarrow \infty
 \end{aligned}$$

which contradicts our assumption that \mathcal{F}^{-1} is bounded and hence \mathcal{F} is not surjective.