CSE 386D: MoAM II **Spring 2025**

Problem Set 1

Student Name: Noah Reef

Problem 6.1

To find the Fourier transform of $f(x) = e^{-|x|}$ for $x \in \mathbb{R}$, we compute

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{x(1-i\xi)}\, dx + \int_{0}^{\infty} e^{-x(1+i\xi)}\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\xi} e^{(x(1-i\xi))}\Big|_{-\infty}^{0} - \frac{1}{1+i\xi} e^{-x(1+i\xi)}\Big|_{0}^{\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{(1-i\xi)(1+i\xi)} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+\xi^2} \right)
\end{aligned}$$

Problem 6.2

To find the Fourier transform of $f(x) = e^{-a|x|^2}$ with $a > 0$ and where $x \in \mathbb{R}$, we compute

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|^2} e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2 - ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2 - ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a(x+(i\xi)/2a)^2 - \xi^2/(4a)}\, dx \\
&= \frac{e^{-\xi^2/(4a)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a(x+(i\xi)/2a)^2}\, dx \\
&= \frac{e^{-\xi^2/(4a)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-au^2}\, du \\
&= \frac{e^{-\xi^2/(4a)}}{\sqrt{2\pi}} \sqrt{\frac{\pi}{a}}
\end{aligned}$$

Problem 6.4

Suppose the $f \in L^1(\mathbb{R}^d)$ and $f(x) = g(|x|)$ for some g, then we see that

$$\begin{aligned}
\hat{f}(\xi) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, dx \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(|x|) e^{-ix\cdot\xi}\, dx \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(|x|) e^{-i|x||\xi|\cos(\theta)}\, dx \\
&= (2\pi)^{-d/2} \int_0^\infty \int_{\omega_d} g(r) e^{-ir|\xi|\cos(\theta)} r^d\, dr\, d\theta \\
&= h(|\xi|)
\end{aligned}$$