CSE 386D: Methods of Applied Mathematics II	**Spring 2025**

Problem Set 7

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Problem 7.12

Part a

Consider the integral,

$$\begin{aligned}\int_{\mathbb{R}^d}(1+|\xi|^2)^{-s}\,d\xi &= \int_{S_1(0)}\int_0^\infty (1+r^2)^{-s}r^{d-1}\,dr\,d\omega. \\ &\le \omega_d\int_0^\infty r^(-2s)r^{d-1}\,dr \\ &= \omega_d\int_0^\infty r^{d-2s-1}\,dr\end{aligned}$$

and since $s > d/2$, we have that $d-2s-1 < d-d-1 < -1$. Thus, the integral converges.

Part b

Let $\phi \in \mathcal{S}$, then we see for $x \in \mathbb{R}^d$ that

$$\phi(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\hat{\phi}(\xi)\,d\xi = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ix\cdot\xi}(1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}\hat{\phi}(\xi)\,d\xi.$$

and get that

$$|\phi(x)|^2 \le (2\pi)^{-d}\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{\phi}(\xi)|^2\,d\xi\int_{\mathbb{R}^d}(1+|\xi|^2)^{-s}\,d\xi = C\,||u||_{H^s}^2.$$

where

$$C = (2\pi)^{-d}\int_{\mathbb{R}^d}(1+|\xi|^2)^{-s}\,d\xi.$$

Part c

Note that $\mathcal{S}$ is dense in $H^s(\mathbb{R}^d)$, and hence for any $\phi \in H^s(\mathbb{R}^d)$, there exists a sequence of $\phi_n \in \mathcal{S}$ such that $\phi_n \to \phi$ in $H^s(\mathbb{R}^d)$. Additionally for each $\phi_n \in \mathcal{S}$ we have the result above that

$$||\phi_n||_{L^\infty(\mathbb{R}^d)} \le C\,||\phi_n||_{H^s(\mathbb{R}^d)}.$$

and hence we have that

$$\phi_{L^\infty(\mathbb{R}^d)} \le C\,||\phi||_{H^s(\mathbb{R}^d)}.$$

for $\phi \in H^s(\mathbb{R}^d)$. This proves the goal result that $H^s(\mathbb{R}^d) \hookrightarrow C_B^0(\mathbb{R}^d)$.

Problem 7.13

Part a

Let $f \in H^1(\mathbb{R}^d)$, then we see for $0 \leq s \leq 1$ we get that

$$\begin{aligned} ||f||^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1+|\xi|^2)^2 |\hat{f}(\xi)|^2 \, d\xi &= \int_{\mathbb{R}^d} \left((1+|\xi|^2)|\hat{f}(\xi)|^2\right)^s \left(|\hat{f}(\xi)|^2\right)^{1-s} \, d\xi \\ &\leq \left(\int_{\mathbb{R}^d} (1+|\xi|^2)|\hat{f}(\xi)|^2 \, d\xi\right)^s \left(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, d\xi\right)^{1-s} \\ &= ||f||^{2s}_{H^1(\mathbb{R}^d)} \, ||f||^{2(1-s)}_{L^2(\mathbb{R}^d)} \end{aligned}$$

In general, we have that if $f \in H^r(\mathbb{R}^d)$ and $0 \leq s \leq 1$, we get that

$$\begin{aligned} ||f||^2_{H^{rs}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1+|\xi|^2)^{rs} |\hat{f}(\xi)|^2 \, d\xi &= \int_{\mathbb{R}^d} \left((1+|\xi|^2)^r|\hat{f}(\xi)|^2\right)^s \left(|\hat{f}(\xi)|^2\right)^{1-s} \, d\xi \\ &\leq \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^r|\hat{f}(\xi)|^2 \, d\xi\right)^s \left(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, d\xi\right)^{1-s} \\ &= ||f||^{2s}_{H^r(\mathbb{R}^d)} \, ||f||^{2(1-s)}_{L^2(\mathbb{R}^d)} \end{aligned}$$

Part b

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. We can show this by using the trace theorem and showing that

$$||f||_{L^2(\partial\Omega)} = ||\gamma_0 f||_{H^0(\partial\Omega)} \leq C \, ||f||_{H^{1/2}(\Omega)} \leq ||f||^{1/2}_{H^1(\Omega)} \, ||f||^{1/2}_{L^2(\Omega)}$$