CSE 386D: Methods of Applied Mathematics II **Spring 2025**

Problem Set 10

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Problem 8.14

Part a

Let $\mathcal{H} = H_0^1(\Omega)$ and $H = V_n$ for some n, and define

$$\begin{aligned} B(u_n, v_n) &= \langle \nabla u_n, \nabla v_n \rangle_{L^2(\Omega)} + \langle u_n, v_n \rangle_{L^2(\Omega)} \\ F(v_n) &= \langle f, v_n \rangle_{L^2(\Omega)} \end{aligned}$$

Note that,

$$|F(v_n)| = \left| \langle f, v_n \rangle_{L^2(\Omega)} \right| \leq ||f||_{L^2(\Omega)} \, ||v_n||_{L^2(\Omega)} \leq C_p \, ||f||_{H^1(\Omega)} \, ||v_n||_{H^1(\Omega)}$$

thus F is continuous. Then we see that,

$$\begin{aligned} |B(u_n, v_n)| &= \left| \langle \nabla u_n, \nabla v_n \rangle_{L^2(\Omega)} + \langle u_n, v_n \rangle_{L^2(\Omega)} \right| \\ &\leq \left| \langle \nabla u_n, \nabla v_n \rangle_{L^2(\Omega)} \right| + \left| \langle u_n, v_n \rangle_{L^2(\Omega)} \right| \\ &\leq ||\nabla u_n||_{L^2(\Omega)} \, ||\nabla v_n||_{L^2(\Omega)} + ||u_n||_{L^2(\Omega)} \, ||v_n||_{L^2(\Omega)} \\ &\leq (C_p + 1) \, ||u_n||_{H^1(\Omega)} \, ||v_n||_{H^1(\Omega)} \end{aligned}$$

thus B is continous. Similarly, we have that

$$\begin{aligned} B(u_n, u_n) &= \langle \nabla u_n, \nabla u_n \rangle_{L^2(\Omega)} + \langle u_n, u_n \rangle_{L^2(\Omega)} \\ &\geq ||\nabla u_n||^2_{L^2(\Omega)} + ||u_n||^2_{L^2(\Omega)} \\ &\geq ||u_n||^2_{H^1(\Omega)} \end{aligned}$$

and hence B si coercive. Thus by the Lax-Milgram theorem, we have that there exists a unique solution $u_n \in V_n$ such that

$$B(u_n, v_n) = F(v_n) \quad \forall v_n \in V_n$$

and we see that

$$|B(u_n, u_n)| = ||u_n||^2_{H^1(\Omega)} = | \langle f, u_n \rangle_{L^2(\Omega)} | \leq ||f||_{L^2(\Omega)} \, ||u_n||_{L^2(\Omega)}$$

dividing by $||u_n||_{L^2(\Omega)}$ gives us

$$||u_n||_{H^1(\Omega)} \leq ||f||_{L^2(\Omega)}$$

Part b

From part a, we have that u_n is uniformly bounded in $H^1(\Omega)$, thus by the Banach-Alaoglu theorem we have that there exists a subsequence $u_n \rightharpoonup u$ in $H^1(\Omega)$. Note that $V = \bigcup_{n=1}^{\infty} V_n$ is dense in $H^1(\Omega)$, and hence $\bar{V} = H^1(\Omega)$. Since the variational problem, for each n, of finding $u_n \in V_n$ such that

$$B(u_n, v_n) = F(v_n) \quad \forall v_n \in V_n$$

has a unique solution. We have that the same problem posed on $H^1(\Omega)$, also has a unique solution, that is

$$B(u, v) = F(v) \quad \forall v \in H^1(\Omega)$$

has a unique solution $u^* \in H^1(\Omega)$. For each $v \in H^1(\Omega)$, there exists a sequence $v_n \to v$ where $v_n \in V_n$, and hence we have that since $u_n \rightharpoonup u$ we get that

$$B(u_n, v_n) = F(v_n) \to B(u, v) = F(v) \quad \forall v \in H^1(\Omega)$$

which implies that u is a solution to the variational problem posed on $H^1(\Omega)$, and hence $u = u^*$.

Part c

Note that

$$||u - u_n||_{H^1(\Omega)} \le \frac{M}{\gamma} \inf_{v_n \in V_n} ||u - v_n||_{H^1(\Omega)}$$

where M and γ are the continuity and coercivity constants of B respectively. Note that since $V_1 \subseteq V_2 \subseteq \dots$, we get that

$$\inf_{v_n \in V_n} ||u - v_n||_{H^1(\Omega)} \ge \inf_{v_n \in V_{n+1}} ||u - v_n||_{H^1(\Omega)}$$

for all n. Additionally since $V_n \to V$ as $n \to \infty$, which as stated above is dense in $H^1(\Omega)$, we get that

$$\inf_{v_n \in V_n} ||u - v_n||_{H^1(\Omega)} \to 0 \quad \text{as } n \to \infty$$

Thus we have that

$$||u - u_n||_{H^1(\Omega)} \le \frac{M}{\gamma} \inf_{v_n \in V_n} ||u - v_n||_{H^1(\Omega)} \to 0$$

monotically as $n \to \infty$.

Part d

Recall that the variational formulation of the problem is given by

$$\langle \nabla u_n, \nabla v_n \rangle_{L^2(\Omega)} + \langle u_n, v_n \rangle_{L^2(\Omega)} = \langle f, v_n \rangle_{L^2(\Omega)} \quad \forall v_n \in V_n$$

and then for the $\langle \nabla u_n, \nabla v_n \rangle_{L^2(\Omega)}$ term, we have that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega (\nabla^2 u) v \, dx + \int_{\partial\Omega} (\nabla u \cdot \nu) v \, d\sigma(x)$$

which implies that $u \in H^2(\Omega)$ and we get that

$$-\int_\Omega (\nabla^2 u)v \ dx + \int_{\partial\Omega} (\nabla u \cdot \nu)v \, d\sigma(x) + \int_\Omega uv \, dx = \int_\Omega fv \, dx$$

which holds for all $v \in H^1(\Omega)$, and hence we have that $\nabla u \cdot \nu = 0$ on $\partial\Omega$.

Problem 8.17

Notice that I_h is well-defined since for any $\{v(x_j)\}_{j=1}^{n-1}$ there exists only one line that passes through the points $(x_j, v(x_j))$ for $j = 1, \ldots, n-1$. We can see that I_h is linear since Since $\Omega = (0,1)$ is bounded then we have by the Sobolev Embedding theorem that $H_0^1(0,1) \hookrightarrow C_B^0(0,1)$, that means that there exists $C > 0$ such that for all $u \in H_0^1(0,1)$

$$||u||_{C_B^0(0,1)} \leq C \, ||u||_{H_0^1(0,1)}$$

so then for $v \in H_0^1(0,1)$ we have that

$$||\mathcal{I}_h v||_{C_B^0(0,1)} = ||v|| \leq C \, ||\mathcal{I}_h v||_{H_0^1(0,1)} \leq C \, ||v||_{H_0^1(0,1)}$$

and hence we have that $\mathcal{I}_h$ is continuous.