CSE 389D: Mathematical Modelling **Spring 2025**

Problem Set 1

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Problem 1

Problem 2

Part a

$$L(x, \dot{x}, t) = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Part b

Using the Euler-lagrange equation, we have:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = -kx - m\ddot{x} = 0$$

solving the differential equation above gives the general solution,

$$x(t) = c_2 \sin(\omega t) + c_1 \cos(\omega t)$$

where $\omega = \sqrt{\frac{k}{m}}$. Then we see with boundary equations that

$$\begin{aligned} x(0) &= c_1 = x_1 \\ x(\tau) &= c_2 \sin(\omega\tau) + x_1 \cos(\omega\tau) = x_2 \end{aligned}$$

solving for c_2 gives

$$c_2 = \frac{x_2 - x_1 \cos(\omega\tau)}{\sin(\omega\tau)}$$

since $\tau \neq \frac{n\pi}{\omega}$, we have that $\sin(\omega\tau) \neq 0$.

Part c

To solve for the action S, we have that

$$\begin{aligned} S &= \int_0^\tau L(x, \dot{x}, t)dt \\ &= \int_0^\tau \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) dt \\ &= \frac{1}{2}m \int_0^\tau \dot{x}^2 - \omega^2 x^2 \, dt \end{aligned}$$

Now notice that

$$\int_0^\tau \dot{x}^2\,dt = x\dot{x}\Big|_0^\tau - \int_0^\tau x\ddot{x}\,dt$$

and since $\ddot{x} = -\omega^2 x$, we have that

$$\int_0^\tau \dot{x}^2\,dt = x\dot{x}\Big|_0^\tau + \omega^2\int_0^\tau x^2\,dt$$

Thus, we have that

$$S = \frac{1}{2}m\left(x\dot{x}\Big|_0^\tau + \omega^2\int_0^\tau x^2\,dt - \omega^2\int_0^\tau x^2\,dt\right)$$

thus

$$S = \frac{1}{2}m\left(x\dot{x}\Big|_0^\tau\right)$$

to solve the inside term we have that

$$x\dot{x} = \left(c_2\sin(\omega t) + x_1\cos(\omega t)\right)\left(\omega c_2\cos(\omega t) - x_1\omega\sin(\omega t)\right)$$

now evaluating at τ and 0 gives

$$\begin{aligned}
x\dot{x}\Big|_0^\tau &= \left(c_2\sin(\omega\tau) + x_1\cos(\omega\tau)\right)\left(\omega c_2\cos(\omega\tau) - x_1\omega\sin(\omega\tau)\right) - x_1\omega c_2\\
&= \left(\frac{x_2 - x_1\cos(\omega\tau)}{\sin(\omega\tau)}\sin(\omega\tau) + x_1\cos(\omega\tau)\right)\left(\frac{\omega x_2\cos(\omega\tau) - \omega x_1\cos^2(\omega\tau)}{\sin(\omega\tau)} - x_1\omega\sin(\omega\tau)\right) - \dots\\
&= \frac{\omega x_2^2\cos(\omega\tau) - \omega x_2 x_1}{\sin(\omega\tau)} - \frac{\omega x_1 x_2 - \omega x_1^2\cos(\omega\tau)}{\sin(\omega\tau)}\\
&= \frac{\omega}{\sin(\omega\tau)}\left(-2x_1x_2 + (x_1^2 + x_2^2)\cos(\omega\tau)\right)
\end{aligned}$$

thus

$$S = \frac{m\omega}{2\sin(\omega\tau)}\left(-2x_1x_2 + (x_1^2 + x_2^2)\cos(\omega\tau)\right)$$

Part d

To write the action $S(x,\tau)$, we set $x_2 = x$ and $x_1 = 0$ to get

$$S(x,\tau) = \frac{m\omega}{2\sin(\omega\tau)}\left(-2x + x^2\cos(\omega\tau)\right)$$

to solve for the action of a silicon atom, with mass $m_{\mathrm{Si}} = 4.66\times10^{-26}$kg, vibrating at frequecy $\omega = 15$THz and amplitude $x = a_0$ and a time $\tau = \pi/(4\omega)$