CSE 386D: Methods of Applied Mathematics II **Spring 2025**

Problem Set 4

Student Name: Noah Reef

Problem 7.1

Suppose that $f \in H^1(\mathbb{R}^d)$. Then we see that

$$||f||_{H^1} = ||f||_{W^{1,2}} = \left\{||f||_{L^2}^2 + ||Df||_{L^2}^2\right\}^{1/2}$$

recall that

$$||f||_{L^2} = \left|\left|\hat{f}\right|\right|_{L^2}$$

and

$$||Df||_{L^2} = \left|\left|\widehat{Df}\right|\right|_{L^2} = \left|\left|i\xi\hat{f}\right|\right|_{L^2} = |\xi|\left|\left|\hat{f}\right|\right|_{L^2}$$

Thus we see that

$$\left\{||f||_{L^2}^2 + ||Df||_{L^2}^2\right\}^{1/2} = \left\{\left|\left|\hat{f}\right|\right|_{L^2}^2 + |\xi|^2\left|\left|\hat{f}\right|\right|_{L^2}^2\right\}^{1/2} = \int_{\mathbb{R}^d}(1+|\xi|^2)|\hat{f}(\xi)|^2 d\xi$$

and in general for $f \in H^k(\mathbb{R}^d)$ we have that

$$||f||_{H^k}^2 = \left\{||f||_{L^2}^2 + \sum_{j=1}^{k}\left|\left|D^j f\right|\right|_{L^2}^2\right\}^{1/2} = \int_{\mathbb{R}^d}\sum_{|\alpha|\leq k}|\xi^\alpha|^2|\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^d}(1+|\xi|^2)^k|\hat{f}(\xi)|^2 d\xi$$

Problem 7.3

Let $f \in H^1(\mathbb{R}^d)$ and we define

$$\delta_0(f) = f(0)$$

note that for $d = 1$ we have that $H^1(\mathbb{R})$ is continously embedded in $C^0(\mathbb{R})$ and hence there exists a C such that

$$||f||_{C^0} \leq C\,||f||_{H^1}$$

and from the fundemental theorem of calculus, we get that

$$f(x) - f(0) = \int_0^x f'(t)dt$$

which then, by using Cauchy-Schwartz, gives us

$$|f(0)| \leq ||f||_{C^0} \leq C\,||f||_{H^1}$$

that is δ_0 is a bounded linear functional on $H^1(\mathbb{R})$ and hence in $(H^1(\mathbb{R}))^*$. However for $d \geq 2$ we consider the sequence $f_n(x) = \phi(nx)$ where $\phi \in C_0^\infty(\mathbb{R}^d)$ and $\phi(0) = 1$. Then we see that

$$||f_n||_{H^1}^2 = ||f_n||_{W^{1,2}} = \left\{||f_n||_{L^2}^2 + ||Df_n||_{L^2}^2\right\} = \frac{1}{n^d}\,||\phi||_{L^2}^2 + \frac{n^2}{n^d}\,||\nabla\phi||_{L^2}^2 \leq Cn^{1-\frac{d}{2}}$$

then we see that as $n \to \infty$ we have that $||f_n||_{H^1} \to 0$ but $\delta_0(f_n) = 1$ and hence δ_0 is not a bounded linear functional on $H^1(\mathbb{R}^d)$ for $d \geq 2$.

Problem 7.7

Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$.

Part a

Note that by the fundamental theorem of calculus we have that

$$u(\tau_y\phi) - u(\tau_0\phi) = \int_0^1 \frac{d}{dt}u(\tau_{ty}\phi)dt$$

and we see that by applying the usual chain rule we get

$$\frac{d}{dt}\tau_{ty}\phi = \sum_{j=1}^d y_j\partial_j\phi(\tau_{ty}x)$$

and hence we see that

$$\frac{d}{dt}u(\tau_{ty}\phi) = \sum_{j=1}^d y_j\partial_j u(\tau_{ty}\phi)$$

and finally we get that

$$u(\tau_y\phi) - u(\tau_0\phi) = \int_0^1 \sum_{j=1}^d y_j\partial_j u(\tau_{ty}\phi)dt = \sum_{j=1}^d y_j \int_0^1 \partial_j u(\tau_{ty}\phi)dt$$

Part b

Let $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ and hence $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and hence $f \in \mathcal{D}'(\mathbb{R}^d)$. Then we see that

$$\langle\tau_{-y}f, \phi\rangle - \langle f, \phi\rangle = \sum_{j=1}^d y_j \int_0^1 \partial_j f(\tau_{ty}\phi)dt$$

which is equivalent to

$$\sum_{j=1}^d y_j \int_0^1 \int_{\mathbb{R}^d} \partial_j f(\tau_{ty}x)\phi(x)dxdt = \int_0^1 y\cdot\nabla f(x+ty)\,dt\phi(x)dx$$

which implies that

$$f(x+y) - f(x) = \int_0^1 y\cdot\nabla f(x+ty)\,dt$$

Part c

From b we have that

$$|f(x+y)-f(x)|\left|\int_0^1 y\cdot\nabla f(x+ty)\,dt\right| \le |y|\int_0^1 |\nabla f(x+ty)|\,dt$$

then for any fixed ball $B_R(0)$ if $f\in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ then by taking $L_{R,f}=\sup_{x\in B_R(0)}|\nabla f(x)|$ we see that

$$|f(x+y)-f(x)|\le |y|L_{R,f}$$

and hence f is locally Lipschitz continuous and hence $W^{1,1}_{\text{loc}}(\mathbb{R}^d)\subseteq C^{0,1}_{\text{loc}}(\mathbb{R}^d)$.

Problem 7.8

Part a

Let $\Omega\subseteq\mathbb{R}^d$ be bounded and contains 0. Let $f(x)=|x|^\alpha$ and $1\le p<d$ and $q>dp/(d-p).Thenifweconsider 1-\frac{d}{p}<\alpha\le-\frac{d}{q}$ since we have that for $f\in W^{1,p}(\Omega)$ we have that

$$||f||_{W^{1,p}}=||f||^p_{L^p}+||Df||^p_{L^p}=\int_0^R r^{\alpha p}r^{d-1}dr+\alpha\int_0^R r^{(\alpha-1)p}r^{d-1}dr$$

which we see that both integrals converge for $\alpha>1-(d/p)$ and hence $f\in W^{1,p}(\Omega)$. Additionally we have that

$$||f||^q_{L^q(\Omega)}=\int_0^R r^{q\alpha}r^{d-1}dr$$

which only converges for $q\alpha+d>0$, however since $\alpha\le -d/q$ we see that $q\alpha+d\le 0$ and hence $f\notin L^q(\Omega)$.

Part b

Since we see that our function f in part a is such that $f\notin L^q(\Omega)$ for $\alpha\le -d/q$ and hence the Dirac mass is in $W^{-s,p}(\Omega)$ for $s>d/p$ and $1\le p<d$.

Part c

Let $\Omega\subseteq\mathbb{R}^d=B_R(0)$ and let $f(x)=\log(\log(4R/|x|))$. Then we see that

$$||f||^p_{W^{1,p}(B_R(0))}=||f||^{p\ p}_{L\ (B_R(0))}+||Df||^{p\ p}_{L\ (B_R(0))}$$

Note that

$$||f||^{p\ p}_{L}(B_R(0))=\int_{B_R(0)}|\log(\log(4R/|x|))|^d\,dx=\int_0^R|\log(\log(4R/r))|^d r^{d-1}dr$$

$$||Df||^{p\ p}_{L}(B_R(0))=\int_{B_R(0)}\left|\frac{1}{|x|\log(4R/|x|)}\right|^d dx=\int_0^R\left|\frac{1}{r\log(4R/r)}\right|^d r^{d-1}dr$$

note that
$$\int_0^R |\log(4R/r)|^d \, r^{d-1} dr \le \int_0^R r^{2d-1} \, dr < \infty$$
and similarly
$$\int_0^R \left| \frac{1}{r\log(4R/r)} \right|^d r^{d-1} dr = \int_0^R \frac{1}{r|\log(4R/r)|^d} \, dr$$
and making the subsitution $r = 4Re^{-u}$ and $dr = -4Re^{-u} du$ we see that

$$\int_0^R \frac{1}{r|\log(4R/r)|^d} \, dr = \int_{u_0}^\infty \frac{1}{4Re^{-u}|u|^d} 4Re^{-u} \, du = \int_{u_0}^\infty \frac{1}{|u|^d} \, du < \infty$$

thus $f \in W^{1,p}(B_R(0))$ for $1 \le p = d$, but $f \notin L^\infty(\Omega)$.

Part d

Let $\Omega = (-1, 1)$ and $u(x) = |x|$, then we see that

$$||u||_{W^{1,\infty}} = \max\{||u||_{L^\infty}, ||Du||_{L^\infty}\}$$

where
$$||u||_{L^\infty} = \max_{x\in\Omega} |u(x)| = 1$$
and
$$||Du||_{L^\infty} = \max_{x\in\Omega} |Du(x)| = 1$$
and hence
$$||u||_{W^{1,\infty}} = 1$$
and thus $u \in W^{1,\infty}(\Omega)$. However we see that if $\{u_k\}_{k=1}^\infty \subseteq C^\infty$ are such that $u_k \to u$ in $C^\infty(\Omega)$, but we see that

$$||u_k - u||_{W^{1,\infty}} = \max\{||u_k - u||_{L^\infty}, ||Du_k - Du||_{L^\infty}\}$$

and we see that
$$||u_k - u||_{L^\infty} = \max_{x\in\Omega} |u_k(x) - u(x)| = 0$$
however since $u_k \in C^\infty$ we see that $\lim_{x\to 0^+} Du_k(x) = 1$ and $\lim_{x\to 0^-} Du_k(x) = -1$ and hence there must exist an N such that for all $k \ge N$ we have that

$$\lim_{x\to 0^+} |Du_k(x) - 1| < \epsilon \quad \text{and} \quad \lim_{x\to 0^-} |Du_k(x) + 1| < \epsilon$$

however this is a contradiction since $Du(x)$ is not continuous at $x = 0$ and hence u_k does not converge to u in $W^{1,\infty}(\Omega)$.