CSE 386D: Methods of Applied Mathematics II **Spring 2025**

Problem Set 9

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Problem 8.2

Let B be a bilinear map that satisfies the conditions of the Generalized Lax-Milgram Theorem. Additionally let $x_{0,1}, x_{0,2} \in \mathcal{X}$ be such that $X + x_{0,1} = X + x_{0,2}$, then we see that there exists unique u_1 and u_2 such that for $F \in Y^*$ we get that

$$B(u_1, v) = F(v) \quad \text{and} \quad B(u_2, v) = F(v)$$

which implies that

$$B(u_1, v) - B(u_2, v) = B(u_1 - u_2, v) = 0$$

then we see that by setting $w = u_1 - u_2 \neq 0$ and a rescaling arguement that we have that

$$0 = \sup_{||v||=1} B(w, v) > \inf_{||w||=1} \sup_{||v||=1} B(w, v) > 0$$

which is a contradiction, thus $w = 0$ and hence $u_1 = u_2$. This implies that for the Dirichlet Boundary Problem, we have that for the boundary condition that $H_0^1(\Omega) + u_{D,1} = H_0^1(\Omega) + u_{D,2}$ then the solution u is the same for both problems.

Problem 8.5

Suppose that $\Omega \subset \mathbb{R}^d$ is a smooth, bounded, connected domain. Additionally let

$$H := \left\{ u \in H^2(\Omega) : \int_\Omega u(x)\, dx = 0 \text{ and } \nabla u \cdot v = 0 \text{ on } \partial\Omega \right\}$$

Here we can see that see that H is a closed linear subspace of $H^2(\Omega)$ since for $u, v \in H$ and $\alpha \in \mathbb{R}$ we have that

$$\int_\Omega (u + \alpha v)\, dx = \int_\Omega u\, dx + \alpha \int_\Omega v\, dx = 0$$

and

$$\nabla(u + \alpha v) \cdot v = \nabla u \cdot v + \alpha \nabla v \cdot v = 0$$

additionally for any converging sequence $u_n \to u$ for $u_n \in H$ we have that

$$0 = \int_\Omega u_n(x)\, dx \to \int_\Omega u(x)\, dx$$

and

$$0 = \nabla u_n \cdot v \to \nabla u \cdot v$$

thus H is closed in $H^2(\Omega)$ and hence a hilbert space with inner product $\langle \cdot, \cdot \rangle_{H^2(\Omega)}$. Note that

$$||u||^2_{H^1(\Omega)} = ||u||^2_{L^2} + ||\nabla u||^2_{L^2} \leq (1 + C_p^2)\, ||\nabla u||^2_{L^2}$$

then by IBP we have that,

$$||\nabla u||^2_{L^2(\Omega)} = \int_\Omega |\nabla u|^2 \, dx = -\int_\Omega u \Delta u \, dx + \int_{\partial\Omega} u \nabla u \cdot \nu \, dx$$

then by the boundary condition we have that the boundary term goes away and we are left with

$$||\nabla u||^2_{L^2(\Omega)} = -\int_\Omega u \Delta u \, dx \leq ||u||_{L^2(\Omega)} \, ||\Delta u||_{L^2(\Omega)} \leq C_p \, ||\nabla u||_{L^2(\Omega)} \, ||\Delta u||_{L^2(\Omega)}$$

then we get that

$$||u||^2_{H^1(\Omega)} \leq (1 + C_p^2) C_p \, ||\nabla u||_{L^2(\Omega)} \, ||\Delta u||_{L^2(\Omega)}$$

which implies that

$$||u||_{H^1(\Omega)} \leq (1 + C_p^2) C_p \, ||\Delta u||_{L^2(\Omega)} \leq (1 + C_p^2) C_p \sum_{|\alpha|=2} ||D^\alpha u||_{L^2(\Omega)}$$

Problem 8.8

Let $f \in L^2(\mathbb{R}^d)$, our goal is to show there exists a unique solution $u \in H^1(\mathbb{R}^d)$ for

$$-\Delta u + u = f$$

To derive the variational problem, we consider $v \in \mathcal{D}(\mathbb{R}^d)$ and take the integral

$$\int_{\mathbb{R}^d} (-\Delta u + u) v \, dx = -\int_{\mathbb{R}^d} \nabla \cdot (\nabla u) v \, dx + \int_{\mathbb{R}^d} uv \, dx = \int_{\mathbb{R}^d} fv \, dx$$

Then for the left-most integral we get the following by the Divergence Theorem

$$-\int_{\mathbb{R}^d} \nabla \cdot (\nabla u) v \, dx = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx - \int_{\partial\mathbb{R}^d} \nabla u \cdot nv \, dx$$

which holds since $\mathcal{D}(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$ and hence can consider $v \in H^1(\mathbb{R}^d)$. Note that the boundary term goes away and hence we have that

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^d} uv \, dx = \int_{\mathbb{R}^d} fv \, dx$$

thus by letting

$$B(u, v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^d} uv \, dx$$

and

$$F(v) = \int_{\mathbb{R}^d} fv \, dx$$

then we see that

$$|B(u,v)| \leq ||\nabla u||_{L^2} ||\nabla v||_{L^2} + ||u||_{L^2} ||v||_{L^2} \leq 2 ||u||_{H^1} ||v||_{H^1}$$

and for $v \neq 0$ we have that

$$B(v,v) = \int_{\mathbb{R}^d} |\nabla v|^2 \, dx + \int_{\mathbb{R}^d} |v|^2 \, dx \geq \int_{\mathbb{R}^d} |\nabla v|^2 \, dx = ||\nabla v||_{L^2}^2 \geq (1/C^2) ||v||_{H^1}^2$$

and hence by Lax-Milgram we have that there exists a unique solution $u \in H^1(\mathbb{R}^d)$ such that

$$B(u,v) = F(v)$$

for all $v \in H^1(\mathbb{R}^d)$.

Problem 8.9

Consider the following boundary value problem for $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$ such that

$$-u_{xx} + e^y u = f, \quad \text{for } (x,y) \in (0,1)^2$$
$$u(0,y) = 0, u(1,y) = \cos(y), \quad \text{for } y \in (0,1)$$

Let,

$$V_o = \{v \in H^1((0,1)^2) : v(0,y) = 0 \text{ and } v(1,y) = 0 \text{ a.e. } y \in (0,1)\}$$

if we let $u_D = x\cos(y)$ we have that $u = \tilde{u} + u_D$, where $\tilde{u} \in V_o$. Then we have that for $v \in V_o$ we get

$$-\int_{(0,1)} (\tilde{u}_{xx}) v \, dx + \int_{(0,1)} e^y (\tilde{u} + u_D) v \, dx = \int_{(0,1)} f v \, dx$$

then by integration by parts we have

$$-\int_{(0,1)} \tilde{u}_{xx} v \, dx = \int_{(0,1)} \tilde{u}_x v_x \, dx$$

thus we have that

$$\int_{(0,1)^2} \tilde{u}_x v_x \, dx + \int_{(0,1)^2} e^y \tilde{u} v \, dx = \int_{(0,1)^2} f v - e^y u_D v \, dx$$

Then we have that,

$$B(x,y) = \int_{(0,1)^2} \tilde{u}_x v_x \, dx + \int_{(0,1)^2} e^y \tilde{u} v \, dx$$

we get that

$$|B(u,v)| \leq ||\tilde{u}_x||_{L^2} ||v_x||_{L^2} + ||e^y||_{L^\infty} ||\tilde{u}||_{L^2} ||v||_{L^2} \leq C ||u||_{H^1} ||v||_{H^1}$$

and

$$B(u,u) = \int_{(0,1)^2} |u_x|^2 \, dx + \int_{(0,1)^2} e^y |u|^2 \, dx \geq \int_{(0,1)^2} |u_x|^2 \, dx = ||u_x||_{L^2}^2 \geq (1/C^2) ||u||_{H^1}^2$$

then by the Lax-Milgram Theorem we have that there exists a unique solution $u \in H^1((0,1)^2)$ such that

$$B(u,v) = \int_{(0,1)^2} f v \, dx$$

for all $v \in V_o$.

Problem 8.11

Let $\Omega = [0,1]^d$ and define

$$H^1_\#(\Omega) = \{u \in H^1_{\text{loc}}(\mathbb{R}^d) : u \text{ is periodic of period 1 in each direction and } \int_\Omega u\,dx = 0\}$$

and consider the problem of finding a periodic solution $u \in H^1_\#(\Omega)$ such that

$$-\Delta u = f \quad \text{for } x \in \Omega$$

where $f \in L^2(\Omega)$.

Part a

If $v \in H^1(\mathbb{R}^d)$ is periodic with period 1 in each direction, we then expect that

$$v(x + ke_i) = v(x) \quad \text{for } k \in \mathbb{Z}, i = 1, \dots, d$$

where e_i is the i^{th} standard basis vector in $\mathbb{R}^d$.

Part b

We can show that $H^1_\#(\Omega)$ is a closed subspace of $H^1(\Omega)$ by showing that it is closed under addition and scalar multiplication. Let $u, v \in H^1_\#(\Omega)$ and $\alpha \in \mathbb{R}$, then we have that

$$\int_\Omega (u + \alpha v)\,dx = \int_\Omega u\,dx + \alpha \int_\Omega v\,dx = 0$$

and

$$u(x + ke_i) = u(x) \quad \text{and} \quad v(x + ke_i) = v(x) \quad \text{for } k \in \mathbb{Z}, i = 1, \dots, d$$

lastly if we consider a converging sequence $u_n \to u$ for $u_n \in H^1_\#(\Omega)$ then we have that

$$0 = \int_\Omega u_n(x)\,dx \to \int_\Omega u(x)\,dx$$

and

$$u_n(x + ke_i) = u_n(x) \to u(x + ke_i) \quad \text{for } k \in \mathbb{Z}, i = 1, \dots, d$$

thus we have that $H^1_\#(\Omega)$ is a closed subspace of $H^1(\Omega)$, and hence a Hilbert space under the same inner product.

Part c

Let $v \in H^1_\#(\Omega)$, then we have that

$$\int_\Omega -\Delta u v\,dx = \int_\Omega f v\,dx$$

then by IBP we have that

$$-\int_\Omega (\Delta u)v\,dx = \int_\Omega \nabla u \cdot \nabla v\,dx - \int_{\partial\Omega} v(\nabla u \cdot \nu)\,dx$$

note that for the boundary term we have that

$$\int_{\partial\Omega} v(\nabla u \cdot \nu)\,dx = \sum_{i=1}^{d} \int_{\partial\Omega^+} v(\nabla u \cdot e_i)\,dx + \int_{\partial\Omega^-} v(\nabla u \cdot -e_i)\,dx$$

which cancel out since v is periodic with period 1 in each direction, thus we have that

$$-\int_\Omega (\Delta u)v\,dx = \int_\Omega \nabla u \cdot \nabla v\,dx$$

and hence

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega fv\,dx$$

then by setting

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx$$

we get that

$$|B(u,v)| \le ||\nabla u||_{L^2}\,||\nabla v||_{L^2} \le C\,||u||_{H^1}\,||v||_{H^1}$$

and

$$B(u,u) = \int_\Omega |\nabla u|^2\,dx \ge (1/C^2)\,||u||_{H^1}^2$$

then by the Lax-Milgram Theorem we have that there exists a unique solution $u \in H^1_\#(\Omega)$ such that

$$B(u,v) = \int_\Omega fv\,dx$$

for all $v \in H^1_\#(\Omega)$.