CSE 386D: Methods of Applied Mathematics II	**Spring 2025**

Problem Set 9

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Problem 8.2

Let B be a bilinear map that satisfies the conditions of the Generalized Lax-Milgram Theorem. Additionally let $x_{0,1}, x_{0,2} \in \mathcal{X}$ be such that $X + x_{0,1} = X + x_{0,2}$, then we see that there exists unique u_1 and u_2 such that for $F \in Y^*$ we get that

$$B(u_1, v) = F(v) \quad \text{and} \quad B(u_2, v) = F(v)$$

which implies that

$$B(u_1, v) - B(u_2, v) = B(u_1 - u_2, v) = 0$$

then we see that by setting $w = u_1 - u_2 \neq 0$ and a rescaling arguement that we have that

$$0 = \sup_{||v||=1} B(w, v) > \inf_{||w||=1} \sup_{||v||=1} B(w, v) > 0$$

which is a contradiction, thus $w = 0$ and hence $u_1 = u_2$. This implies that for the Dirichlet Boundary Problem, we have that for the boundary condition that $H_0^1(\Omega) + u_{D,1} = H_0^1(\Omega) + u_{D,2}$ then the solution u is the same for both problems.

Problem 8.5

Suppose that $\Omega \subset \mathbb{R}^d$ is a smooth, bounded, connected domain. Additionally let

$$H := \left\{ u \in H^2(\Omega) : \int_\Omega u(x)\, dx = 0 \text{ and } \nabla u \cdot v = 0 \text{ on } \partial\Omega \right\}$$

Note that

$$||u||_{H^1(\Omega)}^2 = ||u||_{L^2}^2 + ||\nabla u||_{L^2}^2 \leq (1 + C_p^2)\, ||\nabla u||_{L^2}^2$$

then by IBP we have that,

$$||\nabla u||_{L^2(\Omega)}^2 = \int_\Omega |\nabla u|^2\, dx = -\int_\Omega u \Delta u\, dx + \int_{\partial\Omega} u \nabla u \cdot \nu\, dx$$

then by the boundary condition we have that the boundary term goes away and we are left with

$$||\nabla u||_{L^2(\Omega)}^2 = -\int_\Omega u \Delta u\, dx \leq ||u||_{L^2(\Omega)}\, ||\Delta u||_{L^2(\Omega)} \leq C_p\, ||\nabla u||_{L^2(\Omega)}\, ||\Delta u||_{L^2(\Omega)}$$

then we get that

$$||u||_{H^1(\Omega)}^2 \leq (1 + C_p^2) C_p\, ||\nabla u||_{L^2(\Omega)}\, ||\Delta u||_{L^2(\Omega)}$$

which implies that

$$||u||_{H^1(\Omega)} \leq (1 + C_p^2) C_p\, ||\Delta u||_{L^2(\Omega)} \leq (1 + C_p^2) C_p \sum_{|\alpha|=2} ||D^\alpha u||_{L^2(\Omega)}$$

Problem 8.8

Let $f \in L^2(\mathbb{R}^d)$, our goal is to show there exists a unique solution $u \in H^1(\mathbb{R}^d)$ for

$$-\Delta u + u = f$$

To derive the variational problem, we consider $v \in \mathcal{D}(\mathbb{R}^d)$ and take the integral

$$\int_{\mathbb{R}^d} (-\Delta u + u) v \, dx = -\int_{\mathbb{R}^d} \nabla \cdot (\nabla u) v \, dx + \int_{\mathbb{R}^d} uv \, dx = \int_{\mathbb{R}^d} fv \, dx$$

Then for the left-most integral we get the following by the Divergence Theorem

$$-\int_{\mathbb{R}^d} \nabla \cdot (\nabla u) v \, dx = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx - \int_{\partial \mathbb{R}^d} \nabla u \cdot nv \, dx$$

which holds since $\mathcal{D}(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$ and hence can consider $v \in H^1(\mathbb{R}^d)$. Note that the boundary term goes away and hence we have that

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^d} uv \, dx = \int_{\mathbb{R}^d} fv \, dx$$

thus by letting

$$B(u, v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^d} uv \, dx$$

and

$$F(v) = \int_{\mathbb{R}^d} fv \, dx$$

then we see that

$$|B(u, v)| \leq ||\nabla u||_{L^2} \, ||\nabla v||_{L^2} + ||u||_{L^2} \, ||v||_{L^2} \leq 2 \, ||u||_{H^1} \, ||v||_{H^1}$$

and for $v \neq 0$ we have that

$$B(v, v) = \int_{\mathbb{R}^d} |\nabla v|^2 \, dx + \int_{\mathbb{R}^d} |v|^2 \, dx \geq \int_{\mathbb{R}^d} |\nabla v|^2 \, dx = ||\nabla v||_{L^2}^2 \geq (1/C^2) \, ||v||_{H^1}^2$$

and hence by Lax-Milgram we have that there exists a unique solution $u \in H^1(\mathbb{R}^d)$ such that

$$B(u, v) = F(v)$$

for all $v \in H^1(\mathbb{R}^d)$.

Problem 8.9

Consider the following boundary value problem for $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{aligned} -u_{xx} + e^y u = f, &\quad \text{for} (x, y) \in (0, 1)^2 \\ u(0, y) = 0, u(1, y) = \cos(y), &\quad \text{for} y \in (0, 1) \end{aligned}$$

First we will define $g \in H^1((0,1)^2)$ such that

$$g(x,y) = x\cos(y)$$

then we have that there exists $\tilde{u} \in V_0 := \{u \in H^1((0,1)^2) : u(0,y) = u(1,y) = 0\}$ such that $u = \tilde{u} + g$ and hence we can rewrite the above PDE as

$$-(\tilde{u}_{xx} + g_{xx}) + e^y(\tilde{u} + g) = f$$

note that $g_{xx} = 0$ and hence we have that the above reduces to,

$$-\tilde{u}_{xx} + e^y\tilde{u} = f - e^y g$$

multiplying by $v \in \mathcal{D}((0,1)^2)$ and integrating we have that

$$\int_{(0,1)^2} (-\tilde{u}_{xx} + e^y\tilde{u})v\,dx = \int_{(0,1)^2} (f - e^y g)v\,dx$$

which we can expand as

$$-\int_{(0,1)^2} \tilde{u}_{xx}v\,dx + \int_{(0,1)^2} e^y\tilde{u}v\,dx = \int_{(0,1)^2} fv\,dx - \int_{(0,1)^2} e^y gv\,dx$$