CSE 386D: Methods of Applied Mathematics II	**Spring 2025**

Problem Set 6

Student Name: Noah Reef

Problem 7.9

Let $\Omega \subseteq \mathbb{R}^d$ be bounded and suppose that $f_j \in H^2(\Omega)$ is such that $f_j \rightharpoonup f$ in $H^1(\Omega)$ and $D^\alpha f_j \rightharpoonup g_\alpha$ in $L^2(\Omega)$ for all $|\alpha| = 2$. Without loss of generality, we will assume that Ω is Lipschitz, since we can always find a bounded extension operator from Ω to a Lipschitz domain and the results will still hold.

Since $f_j \in H^2(\Omega)$, we have that $||f_j||_{H^2(\Omega)} < \infty$ and thus we have that there exists a subsequence $\{f_{j_k}\}_{k=1}^\infty$ such that $f_{j_k} \to f_{H^2}$ in $H^2(\Omega)$. Additionally, we have that since $f_j \rightharpoonup f$ in $H^1(\Omega)$, then we have that $||f_j||_{H^1(\Omega)}$ is bounded and thus there exists a subsequence $\{f_{j_k}\}_{k=1}^\infty$ such that $f_{j_k} \rightharpoonup f$ in $H^1(\Omega)$. Since $H^2 \hookrightarrow H^1$ and we have that the weak limit is unique, we get that $f_{j_k} \rightharpoonup f$ in $H^2(\Omega)$. Thus we have that

$$\langle f_{j_k}, h\rangle_{H^1} = \sum_{|\alpha|\leq 2} \langle D^\alpha f_{j_k}, D^\alpha h\rangle_{L^2} \to \sum_{|\alpha|\leq 2} \langle D^\alpha f, D^\alpha h\rangle_{L^2} = \langle f, h\rangle_{H^2}$$

which implies that

$$\langle D^\alpha f_{j_k}, D^\alpha h\rangle_{L^2} \to \langle D^\alpha f, h\rangle_{L^2}$$

but since $D^\alpha f_{j_k} \rightharpoonup g_\alpha$ in $L^2(\Omega)$, we have that $g_\alpha = D^\alpha f$ and hence $f \in H^2(\Omega)$. Additionally since $H^2(\Omega) \hookrightarrow H^1(\Omega)$, we have that $f_j \rightharpoonup f$ in $H^2(\Omega)$ and thus there exists a subsequence $f_{j_k} \to f$ in $H^1(\Omega)$.

Problem 7.10

Suppose that $\Omega \subset \mathbb{R}^d$ is bounded with a Lipschitz boundary and $f_j \rightharpoonup f$ and $g_j \rightharpoonup g$ in $H^1(\Omega)$. Note that for $\varphi \in \mathcal{D}(\Omega)$, we have that

$$\begin{aligned}
\langle \nabla(f_j g_j), \varphi\rangle &= \int_\Omega g_j \nabla f_j \cdot \varphi\, dx + \int_\Omega f_j \nabla g_j \cdot \varphi\, dx \\
&= \int_\Omega (g_j - g)\nabla f_j \cdot \varphi\, dx + \int_\Omega g\nabla f_j \cdot \varphi\, dx + \int_\Omega (f_j - f)\nabla g_j \cdot \varphi\, dx + \int_\Omega f\nabla g_j \cdot \varphi\, dx \\
&= \langle (g_j - g)\nabla f_j, \varphi\rangle_{L^2(\Omega)} + \langle g\nabla f_j, \varphi\rangle_{L^2(\Omega)} + \langle (f_j - f)\nabla g_j, \varphi\rangle_{L^2(\Omega)} + \langle f\nabla g_j, \varphi\rangle_{L^2(\Omega)}
\end{aligned}$$

then by Corollary 7.23, we have that there exists $f_{j_k} \to f$ and $g_{j_k} \to g$ in $L^2(\Omega)$ and thus by using the subsequences above we get that

$$\langle (g_{j_k} - g)\nabla f_{j_k}, \varphi\rangle_{L^2(\Omega)} + \langle g\nabla f_{j_k}, \varphi\rangle_{L^2(\Omega)} + \langle (f_{j_k} - f)\nabla g_{j_k}, \varphi\rangle_{L^2(\Omega)} + \langle f\nabla g_{j_k}, \varphi\rangle_{L^2(\Omega)}$$

which becomes $\langle g\nabla f_{j_k} + f\nabla g_{j_k}, \varphi\rangle_{L^2(\Omega)} \to \langle \nabla(fg), \varphi\rangle$ and hence we have that there is a subsequence such that $\nabla(f_{j_k} g_{j_k}) \to \nabla(fg)$ in $L^2(\Omega)$. To have the sequence weakly converge in $L^p(\Omega)$, we require that the sequence is uniformly bounded in $L^p(\Omega)$ and so we first note that by Holder's Inequality, we have that

$$\begin{aligned} ||f_j \nabla g_j + g_j \nabla f_j||_{L^p} &\leq ||f_j \nabla g_j||_{L^p} + ||g_j \nabla f_j||_{L^p} \\ &\leq ||f_j||_{L^q} \, ||\nabla g_j||_{L^2} + ||g_j||_{L^q} \, ||\nabla f_j||_{L^2} \end{aligned}$$

where $1/p = 1/q + 1/2$. Note that since $f_j, g_j \in H^1(\Omega)$ we get that the sequence $||\nabla g_j||_{L^2}$ and $||\nabla f_j||_{L^2}$ are bounded.

In the case of $d \geq 3$, we have that for $p^* = 2d/(d-2)$ that $H^1(\Omega) \hookrightarrow L^{p^*}(\Omega)$. Then by taking $q = p^*$ we have that

$$\frac{1}{p} = \frac{d-2}{2d} + \frac{1}{2} = \frac{d-2+2d}{2d} = \frac{d-1}{d}$$

and thus for $d \geq 3$ we have weak convergence in $L^{d/(d-1)}(\Omega)$. Next for $d = 2$, we have that $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and thus we have that $q = \infty$ and thus we have that

$$\frac{1}{p} = 0 + \frac{1}{2} = \frac{1}{2}$$

and hence we have weak convergence in $L^2(\Omega)$.

Problem 7.11

Part a

Since $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary, and $\{u_j\} \subseteq H^{2+\epsilon}(\Omega) = W^{2+\epsilon,2}(\Omega)$, we have that there exists a subsequence $\{u_{j_k}\} \subseteq W^{2,q}(\Omega)$ for $q < 2 < 2d/(d-2\epsilon)$ that converges. Thus by the Rellich-Kondrachov Theorem we have that $W^{2,q}(\Omega) \hookrightarrow W^{2,2}(\Omega) = H^2(\Omega)$ and hence we have that there exists a subsequence $\{u_{j_k}\} \subseteq H^2(\Omega)$ that converges.

Part b

To find such q and $s \geq 0$ such that we have $u_{j_k} \to u$ in $W^{s,q}(\Omega)$ we need that $W^{2+\epsilon,2}(\Omega) \rightrightarrows W^{s,q}(\Omega)$. This means that $s + m = 2 + \epsilon$ with $q < 2d/(d-2m)$, thus we have

$$q < \frac{2d}{d - 2(2+\epsilon - s)} \implies s < 2 + \epsilon + \frac{d}{q} - \frac{d}{2}$$

and we have that

Part c

Suppose we have a subsequence $|u_{j_k}|^r \nabla u_{j_k} \to |u|^r \nabla u$ in $L^2(\Omega)$ for some $r \geq 1$. Then we see that

$$\begin{aligned} |||u_{j_k}|^r \nabla u_{j_k} - |u|^r \nabla u||_{L^2} &= ||(|u_{j_k}|^r - |u|^r)\nabla u_{j_k} - |u|^r(\nabla u - \nabla u_{j_k})||_{L^2} \\ &\leq ||(|u_{j_k}|^r - |u|^r)\nabla u_{j_k}||_{L^2} + |||u|^r(\nabla u - \nabla u_{j_k})||_{L^2} \end{aligned}$$

then by Holder's Inequality we have that

$$||(|u_{j_k}|^r - |u|^r)\nabla u_{j_k}||_{L^2} + |||u|^r(\nabla u - \nabla u_{j_k})||_{L^2} \leq |||u_{j_k}|^r - |u|^r||_{L^\infty} ||\nabla u_{j_k}||_{L^2} + |||u|^r||_{L^\infty} ||\nabla u - \nabla u_{j_k}||_{L^2}$$

since $[u_{j_k}]$ is bounded in $W^{2,2}(\Omega)$, we have that $||\nabla u_{j_k}||_{L^2}$ is bounded, similarly

$$||\nabla u - \nabla u_{j_k}||_{L^2} \to 0$$

and thus we need to check if $|||u_{j_k}|^r - |u|^r||_{L^\infty} \to 0$, which requires that $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. Note we have the following inequality,

$$s < 2 + \frac{d}{q} - \frac{d}{2}$$

and thus for $d \leq 4$ choosing $s = 0$ and $q = \infty$ satisfies that above inequality and hence $|||u_{j_k}|^r - |u|^r||_{L^\infty} \to 0$ and thus we have that $|u_{j_k}|^r \nabla u_{j_k} \to |u|^r \nabla u$ in $L^2(\Omega)$ for any $r \geq 1$. For $d > 4$ we could have instead applied Holder's as

$$||(|u_{j_k}|^r - |u|^r)\nabla u_{j_k}||_{L^2} + |||u|^r(\nabla u - \nabla u_{j_k})||_{L^2} \leq |||u_{j_k}|^r - |u|^r||_{L^s} ||\nabla u_{j_k}||_{L^t} + |||u|^r||_{L^s} ||\nabla u - \nabla u_{j_k}||_{L^t}$$

and here we make the choice that $s = \frac{2d}{d-4}$ and $t = d/2$ to see that $H^2(\Omega) \hookrightarrow L^t(\Omega)$ and thus we have that ∇u_{j_k} converges in $L^t(\Omega)$. Then lastly

$$|||u_{j_k}|^r - |u|^r||_{L^s} \to 0$$

which we can achieve for $r \leq \frac{2}{d-4}$