CSE 389D: Mathematical Modeling **Spring 2025**

Problem Set 2

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Problem 2.1

Part a

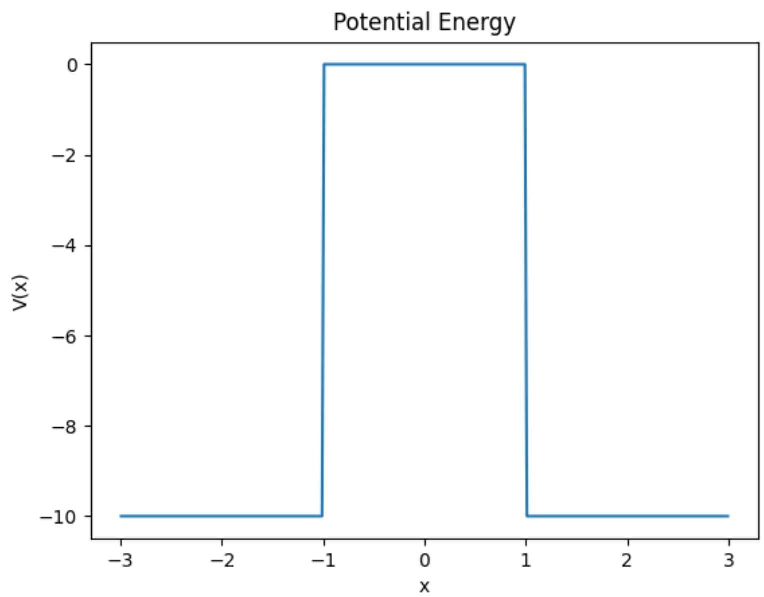

Figure 2.1. Sketch of $V(x)$ for $a = 1$,$b = 3$, and $V_0 = 10$

The time-independent Schrodinger's equation is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\phi(x)}{dx^2} + V(x)\phi(x) = E\phi(x)$$

Part b

If we consider the bounded state where $-V_0 < E < 0$, and $0 < x < a$, then we get that $V(x) = 0$ and

$$-\frac{\hbar^2}{2m}\phi''(x) = E\phi(x) \implies \phi''(x) = \frac{2m|E|}{\hbar^2}\phi(x)$$

solving the differential equation above gives the general solution,

$$\phi(x) = Ae^{\kappa x} + Be^{-\kappa x}$$

where

$$\kappa = \sqrt{\frac{2m|E|}{\hbar^2}}$$

however since it is symmetric we get that

$$\phi(x) = Ae^{\kappa x} + Ae^{-\kappa x} = 2A\cosh(\kappa x)$$

now if we consider the case where $a \leq x \leq b$, we have that $V(x) - V_0$ and get

$$-\frac{\hbar^2}{2m}\phi''(x) - V_0\phi(x) = E\phi(x) \implies \phi''(x) = -\frac{2m(E+V_0)}{\hbar^2}\phi(x)$$

solving the above differential equation yields the following general solution

$$\phi(x) = C\cos(\xi(b-x)) + D\sin(\xi(b-x))$$

where

$$\xi = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

then since $V(b) = \infty$ we require that $\phi(b) = 0$ and hence

$$\phi(x) = D\sin(\xi(b-x))$$

now $x = a$ we have that

$$2A\cosh(\kappa a) = D\sin(\xi(b-a))$$

taking the derivatives

$$2A\kappa\sinh(\kappa a) = -D\xi\cos(\xi(b-a))$$

then dividing both equations yields

$$\frac{\kappa\sinh(\kappa a)}{\cosh(\kappa a)} = -\frac{\xi\cos(\xi(b-a))}{\sin(\xi(b-a))} \implies \kappa\tanh(\kappa a) = -\xi\cot(\xi(b-a)) = -\xi\frac{1+\tan(\xi a)\tan(\xi b)}{\tan(\xi b) - \tan(\xi a)}$$

thus

$$\kappa\tanh(\kappa a) + \xi\frac{1+\tan(\xi a)\tan(\xi b)}{\tan(\xi b) - \tan(\xi a)} = 0$$

letting $v = \xi b$ and $a = \gamma b$ then we get that

$$\kappa\tanh(\kappa a) + (v/b)\frac{1+\tan(\gamma v)\tan(v)}{\tan(v) - \tan(\gamma v)} = 0$$

note that if we define

$$S = \frac{b\sqrt{2mV_0}}{\hbar}$$

then we have that

$$\begin{aligned}\kappa = \sqrt{\frac{2m|E|}{\hbar^2}} = \sqrt{\frac{2m(V_0 - (E+V_0))}{\hbar^2}} &= \sqrt{\frac{2mV_0}{\hbar^2} - \xi^2} \\ &= \sqrt{\frac{2mV_0}{\hbar^2} - \frac{v^2}{b^2}} \\ &= \frac{1}{b}\sqrt{S^2 - v^2}\end{aligned}$$

therefore we have that

$$\sqrt{S^2 - v^2}\tanh(\gamma\sqrt{S^2 - v^2}) - v\frac{1 + \tan(\gamma v)\tan(v)}{\tan(\gamma v) - \tan(v)} = 0$$

and hence our eigenvalues are given by

$$E + V_0 = \frac{\hbar^2\xi^2}{2m} = \frac{\hbar^2 v^2}{2mb^2} \implies \frac{E}{V_0} = -1 + \frac{v^2}{S^2}$$

Part c

Solving for v from the equation below we get

$$\begin{aligned} v_1 &= 9.87725 \\ v_2 &= 6.84406 \\ v_3 &= 3.46525 \end{aligned}$$

plugging these values into the equation

$$\frac{E}{V_0} = -1 + \frac{v^2}{S^2}$$

yields

$$\begin{aligned} \frac{E}{V_0} &= -1 + \frac{(9.87725)^2}{S^2} \approx -0.0244 \\ \frac{E}{V_0} &= -1 + \frac{(6.84406)^2}{S^2} \approx -0.5316 \\ \frac{E}{V_0} &= -1 + \frac{(3.46525)^2}{S^2} \approx -0.8799 \end{aligned}$$

as the eigenvalues of the even bounded state problem.

Part d

Part e

To show the odd parity of the wave function we will get a similar result as in part b. However we notice that since the wave is odd we we get that in the case $|x| < a$ the wave function is given by

$$\phi(x) = Ae^{\kappa x} - Ae^{-\kappa x} = 2A\sinh(\kappa x)$$

and in the case $a < x < b$ we have that

$$\phi(x) = D\sin(\xi(b - x))$$

as before. Then we have that

$$\begin{aligned} 2A\sinh(\kappa a) &= D\sin(\xi(b - a)) \\ 2A\kappa\cosh(\kappa a) &= -D\xi\cos(\xi(b - a)) \end{aligned}$$

then dividing the two equations yields

$$\frac{\kappa \cosh(\kappa a)}{\sinh(\kappa a)} = -\frac{\xi \cos(\xi(b-a))}{\sin(\xi(b-a))} \implies \kappa \coth(\kappa a) = -\xi \cot(\xi(b-a))$$

then by doing similar algebra and subsitutions as in part b we get that

$$\frac{\sqrt{S^2 - v^2}}{\tanh(\gamma\sqrt{S^2 - v^2})} - v\frac{\tan(\gamma v)\tan(v) + 1}{\tan(\gamma v) - \tan(v)} = 0$$

and hence the eigenvalues are given by

$$\frac{E}{V_0} = -1 + \frac{v^2}{S^2}$$

as desired.

Part f

Solving for v from the equation below, using $S = 10$ and $\gamma = 0.2$ we get that

$$v_1 = 6.96192$$
$$v_2 = 3.49913$$

plugging these values into the equation gives the following eigenvalues

$$\frac{E}{V_0} = -1 + \frac{(6.96192)^2}{S^2} \approx -0.5153$$
$$\frac{E}{V_0} = -1 + \frac{(3.49913)^2}{S^2} \approx -0.8775$$

Part g

The parity of the ground-state wavefunction is even.

Part h