Problem Set 3: Deterministic Finite Automaton

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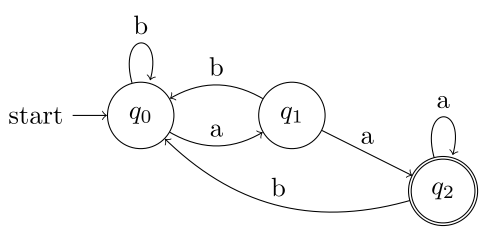
Critical Thinking

1. (a) $(Q = \{q_0, q_1, q_2\},\ \Sigma = \{a, b\},\ \delta,\ q_0,\ F = \{q_2\})$

$\delta =$

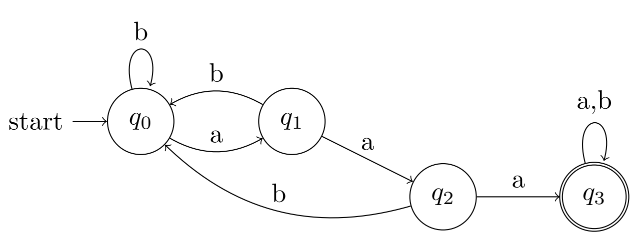
	a	b
q_0	q_1	q_0
q_1	q_2	q_0
q_2	q_2	q_0



(b) $(Q = \{q_0, q_1, q_2, q_3\},\ \Sigma = \{a, b\},\ \delta,\ q_0,\ F = \{q_3\})$

$\delta =$

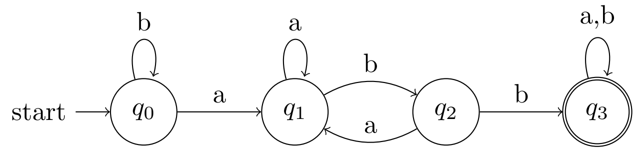
	a	b
q_0	q_1	q_0
q_1	q_2	q_0
q_2	q_3	q_0
q_3	q_3	q_3



(c) ($Q = \{q_0, q_1, q_2, q_3\}$, $\Sigma = \{a, b\}$, δ, q_0, $F = \{q_3\}$)

$\delta =$

	a	b
q_0	q_1	q_0
q_1	q_1	q_2
q_2	q_1	q_3
q_3	q_3	q_3



2. **Proof**

Let M_1 recognize A, where $M_1 = \big(Q_1, \Sigma, \delta_1, q_1, F_1\big)$,
M_2 recognize B, where $M_2 = \big(Q_2, \Sigma, \delta_2, q_2, F_2\big)$, and
M_3 recognize C, where $M_3 = \big(Q_3, \Sigma, \delta_3, q_3, F_3\big)$.

We shall construct M to recognize $A \cup B \cup C$, where $M = \big(Q, \Sigma, \delta, q, F\big)$

(a) $Q = \{(q_1, q_2, q_3) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \text{ and } q_3 \in Q_3\}$. This is the Cartesian Product of sets Q_1, Q_2, and Q_3: $Q_1 \times Q_2 \times Q_3$, making up the set of all pairs of states where the first is from Q_1, the second is from Q_2, and the last is from Q_3.

(b) The alphabet, Σ, is the same for M_1, M_2, and M_3. Thus the alphabet for M is as follows, $\Sigma = \{a, b\}$, as it is for the other machines.

(c) The transition function, δ, shall be defined now. For each tuple, (q_1, q_2, q_3), $\in Q$ and $a \in \Sigma$, let

$$\delta((q_1, q_2, q_3), a) = (\delta_1(q_1, a), \delta_2(q_2, a), \delta_3(q_3, a))$$

Now we can clearly see that δ receives a pair of states from M_1, M_2, and M_3 which more simply put is just a state of M.

(d) q_0 is the pair of the start states of M_1, M_2, and M_3: (q_1, q_2, q_3).

(e) F is the set of pairs where one of the three values is an accept state from M_1, M_2, or M_3. let

$$F = \{(q_1, q_2, q_3) \mid q_1 \in F_1 \text{ or } q_2 \in F_2 \text{ or } q_3 \in F_3\}$$
or
$$F = (F_1 \times Q_2 \times Q_3) \cup (Q_1 \times F_2 \times Q_3) \cup (Q_1 \times Q_2 \times F_3)$$

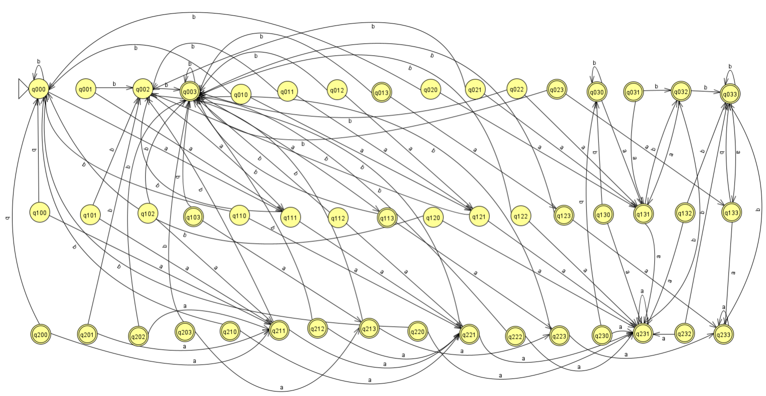
Thus F is a pair of states from M_1, M_2 and M_3 where either $q_1 \in F_1$, $q_2 \in F_2$, or $q_3 \in F_3$.

This concludes the construction of the finite automaton M that recognizes $A \cup B \cup C$. I have just shown that the union of three regular languages is regular, therefore proving that the class of regular languages is closed under the union operation.

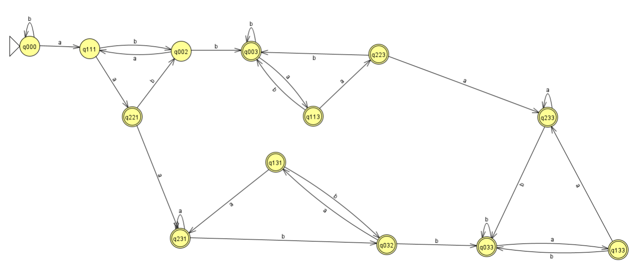
DFA Formal Definition, Problem 2

$M = \Big(Q = \{(q_1, q_2, q_3) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \text{ and } q_3 \in Q_3\}, \Sigma = \{a, b\}, \delta = (\delta_1(q_1, a), \delta_2(q_2, a), \delta_3(q_3, a)), q_0 = q_{000}, F = \{(q_1, q_2, q_3) \mid q_1 \in F_1 \text{ or } q_2 \in F_2 \text{ or } q_3 \in F_3\}\Big)$

DFA Initial Construction:



With Unreachable States Removed:



3. ## Proof

Let M_1 recognize A, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$,
M_2 recognize B, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, and

M_3 recognize C, where $M_3 = \big(Q_3, \Sigma, \delta_3, q_3, F_3\big)$.

We shall construct M to recognize $A \cap B \cap C$, where $M = \big(Q, \Sigma, \delta, q, F\big)$

(a) $Q = \{(q_1, q_2, q_3) \mid q_1 \in Q_1$ and $q_2 \in Q_2$ and $q_3 \in Q_3\}$. This is the Cartesian Product of sets Q_1, Q_2, and Q_3: $Q_1 \times Q_2 \times Q_3$, making up the set of all pairs of states where the first is from Q_1, the second is from Q_2, and the last is from Q_3.

(b) The alphabet, Σ, is the same for M_1, M_2, and M_3. Thus the alphabet for M is as follows, $\Sigma = \{a, b\}$.

(c) The transition function, δ, shall be defined now. For each tuple, (q_1, q_2, q_3), $\in Q$ and $a \in \Sigma$, let

$$\delta((q_1, q_2, q_3), a) = (\delta_1(q_1, a), \delta_2(q_2, a), \delta_3(q_3, a))$$

Now we can clearly see that δ receives a pair of states from M_1, M_2, and M_3 which more simply put is just a state of M.

(d) q_0 is the pair of the start states of M_1, M_2, and M_3: (q_1, q_2, q_3).

(e) F is the set of pairs where all three states are accept states from M_1, M_2, or M_3. let

$$F = F_1 \times F_2 \times F_3$$

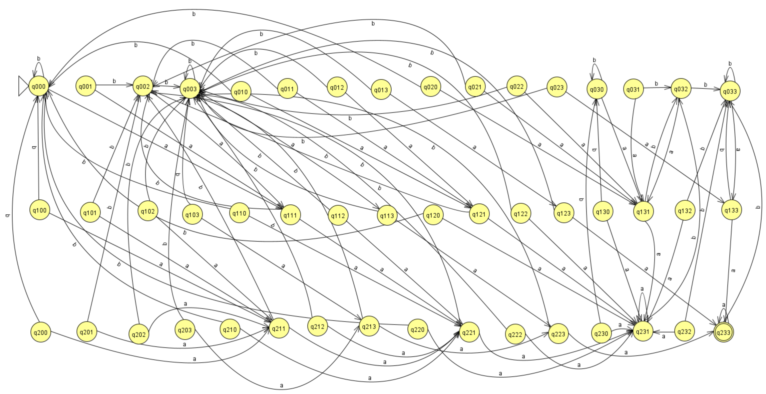
Thus F is a pair of states from M_1, M_2, and M_3 following the form (q_1, q_2, q_3) where $q_1 \in F_1$, $q_2 \in F_2$, and $q_3 \in F_3$. This is the key departure from the proof for closure under union. For intersect to be proven all 3 machines accept states must be used in conjunction to create new accept states for the intersection; Therefore all states in the pair of states that is F must be accept states from F_1, F_2, and F_3.

This concludes the construction of the finite automaton M that recognizes $A \cap B \cap C$. I have just shown that the intersection of three regular languages is regular, therefore proving that the class of regular languages is closed under the intersection operation.

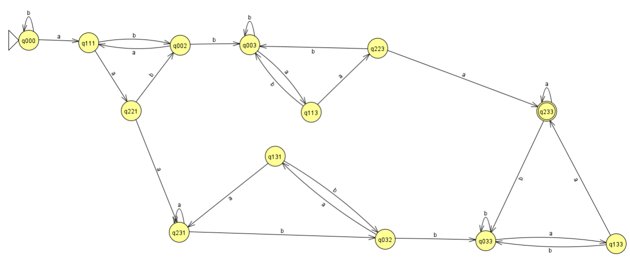
DFA Formal Definition, Problem 3

$M = \Big(Q = \{(q_1, q_2, q_3) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \text{ and } q_3 \in Q_3\}, \Sigma = \{a, b\}, \delta = (\delta_1(q_1, a), \delta_2(q_2, a), \delta_3(q_3, a)), q_0 = q_{000}, F = \{(q_1, q_2, q_3) \mid q_1 \in F_1 \text{ and } q_2 \in F_2 \text{ and } q_3 \in F_3\}\Big)$

DFA Initial Construction:



With Unreachable States Removed:



4. Let $\delta(q, i) = q$ for all $i \in \Sigma, q \in Q$.

   Basis: Let $w_i \in w$. Assume that $w_i \in \Sigma$. Let $|w| = 0$. From our start state q_0, we use the defined transition function.

   $$\delta(q, w) = q$$

   We know that $q \in Q$ because of our definition of the transition function; The transition function must result in a state in Q provided a state in Q and a symbol of Σ. Due to the size of the input string being zero our input for the function would simply be ϵ. This also means there is only one input for w which would make our first transition subsequently our last. It would be as if there were no transition in this case and we would simply stay at the state that we initially input into the function therefore returning a final state q. Thereby proving at the basis that $\hat{\delta}(q, w) = q$ for $|w| = 0$.

   Inductive Step: For $|w| = i$ where $i \geq 1$, assume that the following formula is true: $\hat{\delta}(q, w) = q$. We now want to prove that $\hat{\delta}(q, w) = q$, for $|w| = i+1$.

Since we assume that the formula holds for i we can say that $\hat{\hat{\delta}}(q, w) = q$ for $|w| = i$. This means that we can transition i times to the point where our next input for the transition function is some element $w_{i+1} \in w$ and some q. We already know that $q \in Q$ and $w_{i+1} \in \Sigma$; This is all means that we have the necessary inputs for the function to transition one more time to some state q. We also know that this state q is subsequently the final state as there are no more transitions to be made and we have just read the final symbol of w, w_{i+1} for a string length of $i + 1$. Thereby proving at the Inductive Step that $\hat{\hat{\delta}}(q, w) = q$ for $|w| = i + 1$, where $i \geq 1$.

Since we have concluded at the Basis and Inductive Step that the transition function holds, we can say that for all strings w, $\hat{\hat{\delta}}(q, w) = q$.