Problem Set 5: Maximum Likelihood Estimation

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1.

$$\ell(x \mid \theta) = \ln\left[\prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}\right]$$
$$\ell(x \mid \theta) = \sum_{i=1}^{n}\left[\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \left(\frac{(x_i-\mu)^2}{2\sigma^2}\right)\right]$$
$$\ell(x \mid \theta) = n\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}$$

$$\frac{\partial}{\partial\sigma^2}\left[n\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right] = 0$$
$$\frac{\partial}{\partial\sigma^2}\left[n\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)\right] - \frac{\partial}{\partial\sigma^2}\left[\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right] = 0$$

$$\sigma^2(\mathbf{x}) = \frac{-n}{2\sigma^2} + \frac{1}{2}\sigma^4\sum_{i=1}^{n}(x_i-\mu)^2$$
$$\sigma^2(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

2.

$$
\begin{aligned}
\ell(x \mid \theta) &= \ln\left[\prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}\right] \\
\ell(x \mid \theta) &= \sum_{i=1}^{n}\left[\ln(\theta^{x_i}) + \ln(1-\theta^{1-x_i})\right] \\
\ell(x \mid \theta) &= \ln\theta \sum_{i=1}^{n} x_i + \ln(1-\theta)(n - \sum_{i=1}^{n} x_i) \\
&= \frac{\partial}{\partial\theta} \ln\left[n\bar{x}\ln(\theta) + n(1-\bar{x})\ln(1-\theta)\right] = 0 \\
&= n\left(\frac{\bar{x}}{\theta} - \frac{1-\bar{x}}{1-\theta}\right) = 0 \\
\hat{\theta}(\mathbf{x}) &= \hat{\sigma}^2
\end{aligned}
$$