Math 300: Homework 3

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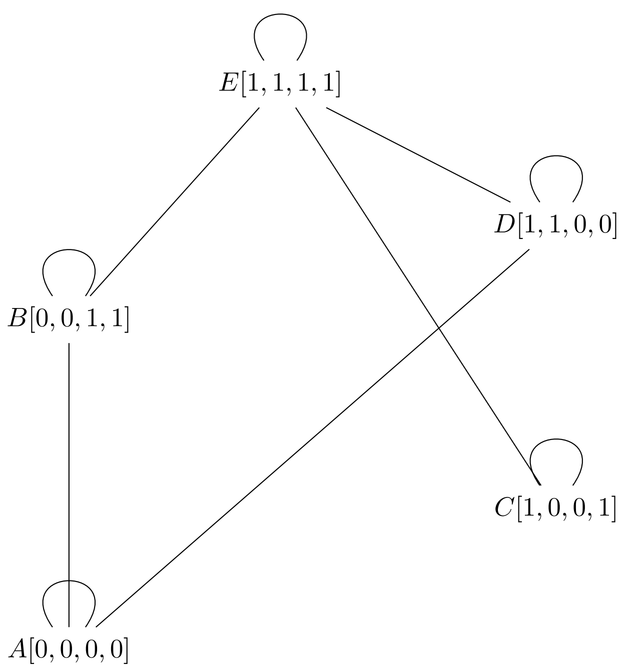
Please solve the following problems and typeset your results in LaTeX. Please type the problem statement at the beginning of each problem and number the problems according to the same enumeration here. The assignment is out of 100 points with 125 points available. Late work is not accepted. If you use AI cite the model, date used, and the prompt. You are also expected to explain how any results generated by AI work in your own words.

1. (50 points) Create a Markov chain on the 2×10 tilings that has the uniform distribution as its stationary distribution (the rotational walk from class). Verify this using the adjacency matrix, and empirically using the total variational distance and a sample. Do not use MCMC.

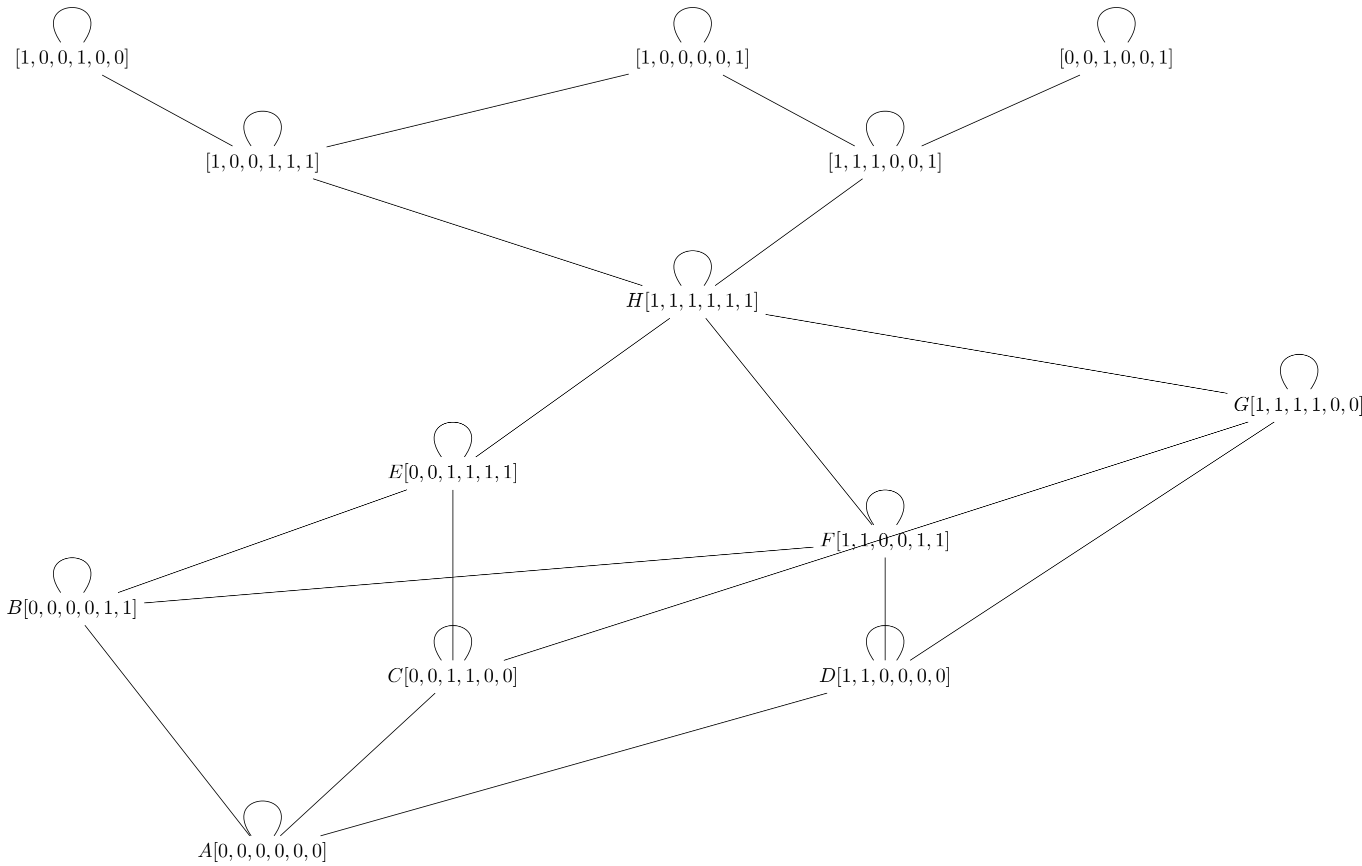
   (Hint: include loops, and make the edges undirected. This is too many edges to do by hand, so you should find a way to generate the Markov chain using code. The hint to the next problem may be helpful as well)

   To get a better feeling for the problem at hand, I decided to make graphs of the problem with smaller n.

   This is what a 2×4 graph should look like:



   And here is a 2×6:




```python
import sympy as sp
from sympy import fibonacci
from sympy import eye
import numpy as np
import math


def find_tilings(n):
  tilings = []

  stack = [(0, [])] # [(1, [1]), (3, [0,0,1])]

  i = 0

  while stack:
    i, tiling = stack.pop()

    if i == n:
      tilings.append(tiling)

    if i + 1 <= n:
      stack.append((i + 1, tiling + [1]))

    if i + 2 <= n:
      stack.append((i + 2, tiling + [0,0]))

  return tilings


def check_adj(tiling1, tiling2):
  differences = []
  for i in range(len(tiling1)):
    if tiling1[i] != tiling2[i]:
```

```python
        differnences.append(i)
    if len(differnences) == 2:
      j, k = differnences
      if abs(j - k) == 1:
        return True
    return False


  def adj_matrix(tilings):
    n = len(tilings)
    adj = np.identity(n, dtype=int)
    for i in range(n):
      for j in range(i + 1, n):
        if check_adj(tilings[i], tilings[j]):
          adj[i][j] = 1
          adj[j][i] = 1
    return adj

  def find_prob_matrix(adj_matrix, n):

    k = len(adj_matrix)


    prob_matrix = np.zeros((k, k), dtype=float)
    for i in range(k):
      for j in range(k):
        if adj_matrix[i][j] == 1 and i != j:
          prob_matrix[i][j] = 1 / n
    for i in range(n):
      prob_matrix[i][i] = 1 - (np.sum(prob_matrix[i]))

    return prob_matrix


  def find_markov_chain_2byn(n):
    tilings = find_tilings(n)
    adj = adj_matrix(tilings)
    prob_matrix = find_prob_matrix(adj, len(tilings))
    return prob_matrix



  mark_chain = find_markov_chain_2byn(10)

  print("-----------------------------------------Originating Markov Chain
  ------------------------------------------")
  print(mark_chain)

  steady_state = np.linalg.matrix_power(mark_chain, 10000)

  print("-----------------------------------------------Steady State
  -------------------------------------------------")
  print(steady_state)

```

This prints out:

```
-----------------------------------------Originating Markov Chain-----------------------------------------
[[0.94382022 0.01123596 0.         ... 0.         0.         0.        ]
 [0.01123596 0.94382022 0.         ... 0.         0.         0.        ]
 [0.         0.         0.95505618 ... 0.         0.         0.        ]
 ...
 [0.         0.         0.         ... 0.92134831 0.         0.01123596]
 [0.         0.         0.         ... 0.         0.91011236 0.01123596]
 [0.         0.         0.         ... 0.01123596 0.01123596 0.8988764 ]]
```

```
----------------------------------------------Steady State-----------------------------------------------
[[0.01123596 0.01123596 0.01123596 ... 0.01123596 0.01123596 0.01123596]
 [0.01123596 0.01123596 0.01123596 ... 0.01123596 0.01123596 0.01123596]
 [0.01123596 0.01123596 0.01123596 ... 0.01123596 0.01123596 0.01123596]
 ...
 [0.01123596 0.01123596 0.01123596 ... 0.01123596 0.01123596 0.01123596]
 [0.01123596 0.01123596 0.01123596 ... 0.01123596 0.01123596 0.01123596]
 [0.01123596 0.01123596 0.01123596 ... 0.01123596 0.01123596 0.01123596]]
```

This code uses 4 functions in order to give us our desired result of a uniform steady state associated with our markov chain. I will go over each function:

- `find_tilings` is the function used to generate all possible tilings of a $2 \times n$ board. It builds up valid tilings using a stack-based approach, where partial candidates are expanded until full tilings are formed. Completed tilings are stored in the `tilings` list.
- `check_adj` determines whether one tiling can be transformed into another via a single valid move. In graph-theoretic terms, this corresponds to the existence of an edge between two vertices. A valid move exists if and only if exactly two entries differ between the two tilings, and those differing entries are adjacent. The function is built based on these two rules.
- `adj_matrix` constructs the adjacency matrix for the tiling graph. It does this by iterating over all pairs of tilings and using `check_adj` to determine if an edge should exist between them. If so, the corresponding entry in the adjacency matrix is updated.
- `make_prob_matrix` builds the transition matrix for the Markov chain. The transition probabilities are assigned by focusing on the edges that actually change the current tiling. To normalize the probabilities, we compute the maximum degree of any vertex in the tiling graph — this occurs in the state composed entirely of vertical dominoes. For example, in the case of a 2×3 board, starting with all vertical dominoes allows two possible moves: converting the two leftmost or the two rightmost vertical dominoes into horizontal ones. Those resulting states, however, each only allow a transition back to the original vertical configuration. The graph for the 2×4 case further illustrates this structure and supports the idea that the vertical-only tiling typically has the highest number of transitions.

2. (50 points) Consider the score function $s : X \to \mathbb{R}$ where $s(t)$ is equal to the number over vertical dominoes in a tiling of the 2×10 board with dominoes. Implement MCMC on these tilings. Compute the stationary distribution of the resulting Markov chain, and estimate its mixing time empirically ($\epsilon = 1/4$).

   (Hint: we can represent these tilings using a binary string: letting 0 mark a column that's covered by a vertical domino and 1 mark a column that's covered by a horizontal domino. For example, the string $[0, 0, 1, 1, 0]$ is a tiling of the 2×5 board with two horizontal dominoes, a pair of columns covered by horizontal dominoes and ending with a vertical domino. Notice that 1s need to come in adjacent pairs for the string to represent a tiling)

```python
import sympy as sp
from sympy import fibonacci
from sympy import eye
import numpy as np
import math
from collections import Counter

rng = np.random.default_rng(54)


def score(tiling):
  return 1 + np.sum(tiling)


def find_tilings(n):
  tilings = []
```

```
    stack = [(0, [])]

    i = 0

    while stack:
      i, tiling = stack.pop()

      if i == n:
        tilings.append(tiling)

      if i + 1 <= n:
        stack.append((i + 1, tiling + [1])) # 0 represents a vertical domino

      if i + 2 <= n:
        stack.append((i + 2, tiling + [0,0])) # 1 represents a horizontal domino pair

    return tilings


  def check_adj(tiling1, tiling2):
    differnences = []
    for i in range(len(tiling1)):
      if tiling1[i] != tiling2[i]:
        differnences.append(i)
    if len(differnences) == 2:
      j, k = differnences
      if abs(j - k) == 1:
        return True
    return False


  def adj_matrix(tilings):
    n = len(tilings)
    adj = np.identity(n, dtype=int)
    for i in range(n):
      for j in range(i + 1, n):
        if check_adj(tilings[i], tilings[j]):
          adj[i][j] = 1
          adj[j][i] = 1
    return adj

  def find_prob_matrix(adj_matrix, n):

    k = len(adj_matrix)


    prob_matrix = np.zeros((k, k), dtype=float)
    for i in range(k):
      for j in range(k):
        if adj_matrix[i][j] == 1 and i != j:
          prob_matrix[i][j] = 1 / n
    for i in range(n):
      prob_matrix[i][i] = 1 - (np.sum(prob_matrix[i]))

    return prob_matrix


  def find_markov_chain_2byn(n):
    tilings = find_tilings(n)
    adj = adj_matrix(tilings)
    prob_matrix = find_prob_matrix(adj, len(tilings))
    return prob_matrix


  def metropolis_hastings(starting_state, score_function, num_steps, A, tilings):
      current_state = starting_state
      for i in range(num_steps):
```

```
        # Propose a move to next_state using the transition probabilities from A
        next_state = rng.choice(len(A), p=A[current_state])

        # Compute acceptance probability based on Metropolis-Hastings correction
        # Add a small epsilon to avoid division by zero
        eps = 1e-14
        acceptance_ratio = (score_function(tilings[next_state]) / (score_function(tilings[current_state])+ eps)
            * A[next_state, current_state] / (A[current_state, next_state] + eps))
        # Accept the move with probability = min(1, acceptance_ratio)
        if rng.random() < min(1, acceptance_ratio):
            current_state = next_state  # Move to the proposed state
    return current_state

def tv(mu, nu):
    return 0.5 * np.abs(mu-nu).sum()
#----------------------------------------------------------------------
N = 100
s = 1000
#----------------------------------------------------------------------
tiles = find_tilings(10)

mark_chain = find_markov_chain_2byn(10)

worst_case_tv = 1
p = 0

scores = np.array([score(i) for i in tiles], dtype=float)
target = scores / np.sum(scores)

target = np.array(target)


while worst_case_tv > 0.25:

  p+=1
  print(p)
  if p % 2 == 1:
    N += 70
  if p % 2 == 0:
    s += 20

  new_mark_chain = []

  for i in range(len(tiles)):
    sample = [metropolis_hastings(i, score, N, mark_chain, tiles) for _ in range(s)]
    counts = Counter(sample)
    prob = [counts[k]/ s for k in range(len(tiles))]
    new_mark_chain.append(prob)

  #for i in range(len(new_mark_chain)):
    #print(new_mark_chain[i])

  dists = []


  for i, row in enumerate(new_mark_chain):
      dists.append(tv(np.array(row), target))

  worst_case_tv = np.max(dists)

  print("Worst-case TV: ", worst_case_tv)
  print("N =", N)
  print("s =", s)

```

I decided to try out something different because this was giving me **around 45 minutes of runtime**

to execute, and the error is more likely from the amount of samples I am getting instead of the steps I am taking, so I found a different technique.

```python
import sympy as sp
from sympy import fibonacci
from sympy import eye
import numpy as np
import math
from collections import Counter

rng = np.random.default_rng(54)


def score(tiling):
  return 1 + np.sum(tiling)


def find_tilings(n):
  tilings = []

  stack = [(0, [])]

  i = 0

  while stack:
    i, tiling = stack.pop()

    if i == n:
      tilings.append(tiling)

    if i + 1 <= n:
      stack.append((i + 1, tiling + [1])) # 0 represents a vertical domino

    if i + 2 <= n:
      stack.append((i + 2, tiling + [0,0])) # 1 represents a horizontal domino pair

  return tilings


def check_adj(tiling1, tiling2):
  differnences = []
  for i in range(len(tiling1)):
    if tiling1[i] != tiling2[i]:
      differnences.append(i)
  if len(differnences) == 2:
    j, k = differnences
    if abs(j - k) == 1:
      return True
  return False


def adj_matrix(tilings):
  n = len(tilings)
  adj = np.identity(n, dtype=int)
  for i in range(n):
    for j in range(i + 1, n):
      if check_adj(tilings[i], tilings[j]):
        adj[i][j] = 1
        adj[j][i] = 1
  return adj

def find_prob_matrix(adj_matrix, n):

  k = len(adj_matrix)


  prob_matrix = np.zeros((k, k), dtype=float)
  for i in range(k):
```

```
    for j in range(k):
      if adj_matrix[i][j] == 1 and i != j:
        prob_matrix[i][j] = 1 / n
  for i in range(n):
    prob_matrix[i][i] = 1 - (np.sum(prob_matrix[i]))

  return prob_matrix


def find_markov_chain_2byn(n):
  tilings = find_tilings(n)
  adj = adj_matrix(tilings)
  prob_matrix = find_prob_matrix(adj, len(tilings))
  return prob_matrix

def build_propose_matrix(adj):
  n = len(adj)
  P = np.zeros((n,n), dtype=float)
  for i in range(n):
    for j in range(n):
      if adj[i][j] == 1:
        P[i][j] = 1 / np.sum(adj[i])
  return P

def metropolis_hastings_matrix(score_function, prop, tilings, adj):

  n = len(tilings)
  mh = np.zeros((n,n))
  scores = np.array([score_function(tilings[i]) for i in range(n)], dtype=float)

  for i in range(n):
      for j in range(n):
          if adj[i][j] == 1 and i != j:
                epsilon = 1e-9
                alpha = min(1, (scores[j] / (scores[i] + epsilon)) *
                               (adj[j][i] / (adj[i][j] + epsilon)))
                mh[i][j] = prop[i][j] * alpha
      mh[i][i] = 1 - np.sum(mh[i])
  return mh

def tv(mu, nu):
  return 0.5 * np.abs(mu-nu).sum()

tiles = find_tilings(10)

scores = np.array([score(i) for i in tiles], dtype=float)
target = scores / np.sum(scores)

target = np.array(target)

proposal = build_propose_matrix(adj_matrix(tiles))
mh = metropolis_hastings_matrix(score, proposal, tiles, adj_matrix(tiles))

worst_case_tv = 1
k = 1

while worst_case_tv > 0.25:
  k += 1

  proposal_k = np.linalg.matrix_power(mh, k)

  dists = []

  for i, row in enumerate(proposal_k):
      dists.append(tv(np.array(row), target))

  worst_case_tv = np.max(dists)

```

```python
    print("Worst-case TV: ", worst_case_tv)
    print("k =", k)

```

This prints out:

```
Worst-case TV:  0.8703339882121808
k = 2
Worst-case TV:  0.7137618112306409
k = 3
Worst-case TV:  0.6060090453568564
k = 4
Worst-case TV:  0.5302084143471535
k = 5
Worst-case TV:  0.47007711794291546
k = 6
Worst-case TV:  0.4164449144387715
k = 7
Worst-case TV:  0.3699776825119781
k = 8
Worst-case TV:  0.33062445807747753
k = 9
Worst-case TV:  0.29593294007304666
k = 10
Worst-case TV:  0.26551618645951264
k = 11
Worst-case TV:  0.23890557788211902
k = 12
```

This goes off the fact that our acceptance ratio, $\alpha = \min\left(1, \frac{s(z)}{s(y)} \frac{A_{z,y}}{A_{y,z}}\right)$, can be treated the same as a probability. Since β is picked uniformly from 0 to 1, and the next state is picked from having $\beta < \alpha$, that's the same thing as saying that the probability of going from our current state to the next state is α. The matrix implementation goes right off this logic, but instead of going down a far path, we are just looking at what happens at the first step. Our proposal matrix can be read as "the probability of PROPOSING going from state x to state y", and once we multiply by the acceptance then that gives us the full probability of going from state x to y. In other words, α is conditional.

3. (25 points) Consider the elf-climbing-stairs problem from homework 1 problem 2. Find the Ordinary and Exponential generating functions for this problem. See if sympy fps can compute a formula for the coeficients in both cases.

   Ordinary:

$$F_n = F_{n-1} + F_{n-2} + F_{n-4}$$

$$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 6$$

$$F_n z^n = F_{n-1} z^n + F_{n-2} z^n + F_{n-4} z^n$$

$$\sum_{n=5}^{\infty} F_n z^n = \sum_{n=5}^{\infty} F_{n-1} z^n + \sum_{n=5}^{\infty} F_{n-2} z^n + \sum_{n=5}^{\infty} F_{n-4} z^n$$

$$f(z) - 6z^4 - 3z^3 - 2z^2 - z = z(f(z) - 3z^3 - 2z^2 - z) + z^2(f(z) - 2z^2 - z) + z^4(f(z))$$

$$f(z) - 6z^4 - 3z^3 - 2z^2 - z = zf(z) - 3z^4 - 2z^3 - z^2 + z^2 f(z) - 2z^4 - z^3 + z^4 f(z)$$

$$f(z) - zf(z) - z^2 f(z) - z^4 f(z) = z^4 + z^2 + z$$

$$f(z) = \frac{z^4 + z^2 + z}{1 - z - z^2 - z^4}$$

The next part is wrong after reviewing, but thought I'd keep it in:
Exponential:

$$F_n = F_{n-1} + F_{n-2} + F_{n-4}$$

$$F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3$$

$$\sum_{n\geq 4} F_n \frac{z^n}{n!} = \sum_{n\geq 4} F_{n-1} \frac{z^n}{n!} + \sum_{n\geq 4} F_{n-2} \frac{z^n}{n!} + \sum_{n\geq 4} F_{n-4} \frac{z^n}{n!}$$

$$F(z) - (F_0 + F_1 z + F_2 \frac{z^2}{2} + F_3 \frac{z^3}{3!}) = z(F(z) - F_0 - F_1 z - F_2 \frac{z^2}{2}) + \frac{z^2}{2}(F(z) - F_0 - F_1 z) + \frac{z^4}{4!} F(z)$$

$$F(z) - A(z) = zF(z) - zB(z) + \frac{z^2}{2} F(z) - \frac{z^2}{2} C(z) + \frac{z^4}{4!} F(z)$$
$$F(z) - zF(z) - \frac{z^2}{2} F(z) - \frac{z^4}{4!} F(z) = A(z) - zB(z) - \frac{z^2}{2} C(z)$$
$$F(z) = \frac{A(z) - zB(z) - \frac{z^2}{2} C(z)}{1 - z - \frac{z^2}{2} - \frac{z^4}{24}}$$

Plugging in conditions:

$$A(z) = z + z^2 + \frac{1}{2} z^3$$
$$B(z) = z + z^2$$
$$C(z) = z$$

$$A(z) - zB(z) - \frac{z^2}{2} C(z) = z + z^2 + \frac{1}{2} z^3 - z(z + z^2) - \frac{z^2}{2} z = z - z^3$$

$$F(z) = \frac{z - z^3}{1 - z - \frac{z^2}{2} - \frac{z^4}{24}}$$

Using OGF to find F_n:

```python
import sympy as sp

z = sp.symbols('z')
F = (z**4 + z**2 + z) / (1 - z - z**2 - z**4)

series = sp.series(F, z, 0, 20).removeO()

# Extract coefficients using the .coeff() method of the series expansion
coeffs = [series.coeff(z, n) for n in range(20)]

print("------------------Ordinary------------------")

for n, a_n in enumerate(coeffs):
    print(f"F_{n} =", a_n)

print("------------------Ordinary------------------")
```

Prints out:

```
------------------Ordinary------------------
F_0 = 0
F_1 = 1
F_2 = 2
F_3 = 3
F_4 = 6
F_5 = 10
F_6 = 18
F_7 = 31
F_8 = 55
F_9 = 96
F_10 = 169
F_11 = 296
F_12 = 520
F_13 = 912
F_14 = 1601
F_15 = 2809
F_16 = 4930
F_17 = 8651
F_18 = 15182
F_19 = 26642
------------------Ordinary------------------
```

Using EGF to find F_n:

```python
import sympy as sp

z = sp.symbols('z')
F = (z - z**3)/(1 - z - (z**2/sp.factorial(2)) - (z**4/sp.factorial(4)))

series_expansion = sp.series(F, z, 0, 20).removeO()
# Extract coefficients using the .coeff() method of the series expansion
coeffs = [series_expansion.coeff(z, n) for n in range(20)]

print("------------------Exponential------------------")

for n, a_n in enumerate(coeffs):
    print(f"F_{n} =", a_n)

print("------------------Exponential------------------")
```

Prints:

```
-----------------Exponential-----------------
F_0 = 0
F_1 = 1
F_2 = 1
F_3 = 1/2
F_4 = 1
F_5 = 31/24
F_6 = 11/6
F_7 = 5/2
F_8 = 83/24
F_9 = 2743/576
F_10 = 1261/192
F_11 = 10429/1152
F_12 = 7189/576
F_13 = 237853/13824
F_14 = 10247/432
F_15 = 452045/13824
F_16 = 311593/6912
F_17 = 20618857/331776
F_18 = 28424993/331776
F_19 = 26124311/221184
-----------------Exponential-----------------
```

The EGF doesn't do the right result, so lets try a different way:

Exponential:

$$F_n = F_{n-1} + F_{n-2} + F_{n-4}$$

$$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 6$$

$$z^{n-4}\frac{F_n}{(n-4)!} = z^{n-4}\frac{F_{n-1}}{(n-4)!} + z^{n-4}\frac{F_{n-2}}{(n-4)!} + z^{n-4}\frac{F_{n-4}}{(n-4)!}$$

$$z^{n-4}n(n-1)(n-2)(n-3)\frac{F_n}{(n)!} = z^{n-4}(n-1)(n-2)(n-3)\frac{F_{n-1}}{(n-1)!} + z^{n-4}(n-2)(n-3)\frac{F_{n-2}}{(n-2)!} + z^{n-4}\frac{F_{n-4}}{(n-4)!}$$

$$\sum_{n\geq 5} z^{n-4}n(n-1)(n-2)(n-3)\frac{F_n}{(n)!} = \sum_{n\geq 5} z^{n-4}(n-1)(n-2)(n-3)\frac{F_{n-1}}{(n-1)!} + \sum_{n\geq 5} z^{n-4}(n-2)(n-3)\frac{F_{n-2}}{(n-2)!} + \sum_{n\geq 5} z^{n-4}\frac{F_{n-4}}{(n-4)!}$$

$$F^{(4)}(z) = F^{(3)}(z) + F''(z) + F(z)$$

$$F(z) = F^{(4)}(z) - F^{(3)}(z) - F''(z)$$

Assume solution $F = e^{cx}$, which gives us the polynomial:

$$0 = c^4 - c^3 - c^2 - 1$$

Because of laziness, I solve using sympy:

```
import sympy as sp
from sympy import simplify

c = sp.Symbol('c')

char = c**4 - c**3 - c**2 - 1


for i, root in enumerate(sp.solve(char, c)):
    approx = root.evalf(20)
    print(f"Root {i+1}: {approx}")

```

```
Root 1: -1.0000000000000000000
Root 2: 0.12256116687665361998 - 0.74486176661974423659*I
Root 3: 0.12256116687665361998 + 0.74486176661974423659*I
Root 4: 1.7548776662466927600
```

This gave us the solution:

$$F(z) = C_1 e^{-z} + C_2 e^{1.754882z} + e^{-0.127561z} \left(C_3 \cos(0.74482z) + C_4 \sin(0.74482z) \right)$$

Solving for constants, we know that:

$$F(0) = 1, F'(0) = 1, F''(0) = 2, F'''(0) = 3$$

Now solving this in python:

```
import sympy as sp
from sympy import simplify

sp.init_printing()

def taylor_coeff_extractor(f, n):
    return sp.diff(f, z, n).subs(z, 0).evalf()


z, C1, C2, C3, C4 = sp.symbols('z C1 C2 C3 C4')

c = sp.Symbol('c')

char = c**4 - c**3 - c**2 - 1

roots = sp.solve(char, c)
for i, root in enumerate(roots):
    approx = root.evalf(20)
    print(f"Root {i+1}: {approx}")

r1 = -1
r2 = roots[3]
a = -sp.re(roots[1])
b = sp.im(roots[2])


F = C1 * sp.exp(r1 * z) + C2 * sp.exp(r2 * z) + sp.exp(a * z) * (C3 * sp.
cos(b * z) + C4 * sp.sin(b * z))

```

```python
            F0 = F.subs(z, 0)
            F1 = sp.diff(F, z).subs(z, 0)
            F2 = sp.diff(F, z, 2).subs(z, 0)
            F3 = sp.diff(F, z, 3).subs(z, 0)

            rhs = [
                1.0,
                1.0,
                2.0,
                3.0
            ]

            solutions = sp.solve([
                sp.Eq(F0, rhs[0]),
                sp.Eq(F1, rhs[1]),
                sp.Eq(F2, rhs[2]),
                sp.Eq(F3, rhs[3])
            ], (C1, C2, C3, C4), dict=True, simplify = False, rational=True)

            for sol in solutions:
                for const, val in sol.items():
                    print(f"{const}     {sp.N(val, 20)}")


            print(solutions)

            C_1 = solutions[0][C1]
            C_2 = solutions[0][C2]
            C_3 = solutions[0][C3]
            C_4 = solutions[0][C4]

            F = (
                C_1 * sp.exp(-z)
                + C_2 * sp.exp(r2 * z)
                + sp.exp(a * z)
                * (C_3 * sp.cos(b * z) + C_4 * sp.sin(b * z))
            )

            L = 20



            for i in range(L):
                print(f"F_{i} = {taylor_coeff_extractor(F, i)}")

```

```
Root 1: -1.0000000000000000000
Root 2: 0.12256116687665361998 - 0.74486176661974423659*I
Root 3: 0.12256116687665361998 + 0.74486176661974423659*I
Root 4: 1.7548776662466927600
C1  0.24108659628668698843
C2  0.61172506094208698600
C3  0.14718834277122602557
C4  0.24920533220423107566
```

This gave us the following results:
$F_0 = 1.00000000000000$
$F_1 = 1.00000000000000$
$F_2 = 2.00000000000000$
$F_3 = 3.00000000000000$
$F_4 = 6.12973491647861$
$F_5 = 9.95552814794549$

$F_6 = 18.0540368515238$
$F_7 = 31.1164892773244$
$F_8 = 55.2918454710500$
$F_9 = 96.3049959799187$
$F_{10} = 169.670103431716$
$F_{11} = 297.120420921190$
$F_{12} = 522.064347146673$
$F_{13} = 915.477752040893$
$F_{14} = 1607.22541707811$
$F_{15} = 2819.82719580671$
$F_{16} = 4949.10854604653$
$F_{17} = 8684.41350163872$
$F_{18} = 15240.7522574926$
$F_{19} = 26744.9917757199$

Our equation is:

$$F(z) \approx 0.24108659628668577257e^{-z} + 0.61172506094208733973e^{1.754882z}$$
$$+ e^{-0.127561z}\big(0.14718834277122691545\cos(0.74482z) + 0.24920533220423130016\sin(0.74482z)\big)$$

The error above (all F_n should be integers) is caused by our approximations of the roots which are accumulating more and more error overtime, but it roughly approximates the recurrence overtime.

WE CAN DO BETTER!!!!!! If we wait to evaluate the roots, our results will be way more accurate, which is implemented here:

```python
import sympy as sp
z = sp.symbols('z')
C1, C2, C3, C4 = sp.symbols('C1 C2 C3 C4')

poly = z**4 - z**3 - z**2 - 1

roots = [sp.RootOf(poly, i) for i in range(4)]

F = sum(C*sp.exp(r*z) for C, r in zip((C1, C2, C3, C4), roots))

eqs = [
    sp.Eq(F.subs(z, 0), 1),
    sp.Eq(sp.diff(F, z).subs(z, 0), 1),
    sp.Eq(sp.diff(F, z, 2).subs(z, 0), 2),
    sp.Eq(sp.diff(F, z, 3).subs(z, 0), 3)
]
Cs = sp.solve(eqs, (C1, C2, C3, C4), rational = True)


F = sp.simplify(F.subs(Cs))


F = sp.expand(F)

def taylor_coeff_extractor(f, n):
    coeff = sp.diff(f, z, n).subs(z, 0)
    return coeff.evalf().as_real_imag()[0]

for i in range(20):
  print(f"F_{i} = {int(taylor_coeff_extractor(F, i))}")
```

This prints out:
$F_0 = 1$
$F_1 = 1$
$F_2 = 2$
$F_3 = 3$

$F_4 = 6$
$F_5 = 10$
$F_6 = 18$
$F_7 = 31$
$F_8 = 55$
$F_9 = 96$
$F_{10} = 169$
$F_{11} = 296$
$F_{12} = 520$
$F_{13} = 912$
$F_{14} = 1601$
...
THE EXACT RESULT WE WANT (YAY)!!!!

AI Model Used: OpenAI ChatGPT (GPT-4o) Date Used: July 17, 2025 Prompt(s) Used:

- ”did I find the correct generating functio”
- ”how to convert to derivatives”
- ”why does the index go up by 1 each derivative”
- ”I want the console to actually print it out as latex text”
- ”how to do summation in latex”
- ”solution to F = F”” - F”’ - F””
- ”how to find roots using sympy”
- ”what do these roots mean for my general equation F = ez”
- ”check my work, did I do things wrong? the generating function doesn’t seem to be accurate”
- ”how should I change F”
- ”the roots are [-I, I, 1/2 - sqrt(5)/2, 1/2 + sqrt(5)/2]”
- ”these are my roots, how should I change F now”
- ”how to get imaginary part of 1 + 2*I in python”
- ”this isnt the right sequence”
- ”What do u mean by cancellation must be exact, give me example”
- ”Why is it that when we wait to evaluate the roots we get the exact answer but if we dont we get error”
- ”what is the problem with this monte carlo implementation”
- ”is the steady state 89 rows of score / sum(score) for each entry of each row”
- ”is count a dictionary”
- ”how to find target distribution ”
- ”I want the dist to be less than 1/4 but its 78 right now, what can I fix”
- ”why is i going straight to 89”
- ”is this correct for finding optimal steps t so that we are approximately the same as the stationary”
- ”this is taking ¿ 13 minutes to execute, is this normal”
- ”P[i] = P[i] / np.sum(P[i]) would this work for normalizing”
- ”why does the monte carlo matrix method work exactly the same as the matrix monte carlo method”
- ”sdhow me why they are the same by comparing the steps in each algorithm”
- ”so each state on the graph is a vertice”
- ”what are the ways I can transition from one state to another”
- ”if I define a move by only changing 2 horizontal adjacent dominoes to two vertical and vice versa than these are all the moves I can do correct? this isn’t all the states tho”
- ”will the max degree always equal n”

- ”but E’s degree is 4, not 3? are u not including the edge E $\rightarrow$ E”
- “fibonacci function in python”
- “is there a function inside of sympy that I can just grab the nth fibonacci”
- “how to initialize the identity matrix of size n”
- Uploaded a Markov chain graph and asked: “what states am I missing”
- ”how to make a an if statement that makes while loop go back to top”
- ”how can I remove the list of lists of lists and just make it a list of lists”
- ”whats wwrong, I dont want zero matrix I want identity”
- ”how to do reverse fibonacci in python so u are finding n such that Fn = what we know”
- ”how to do powers of matrix in python”
- ”what am I missing in my version”
- ”do I have to do recursion”