MATH221: Mathematics for Computer Science

Outline Solutions to Tutorial Sheet Week 4

Autumn 2017

1 (i) We let $P =$ "I go to the movies", $Q =$ "I carry my phone" and $R =$ "I carry my 3D glasses". Then the argument's logical form is

$$P \implies (Q \vee R).$$
$$Q \wedge \sim R.$$
$$\text{Therefore, } P.$$

We make the following truth table:

P	Q	R	$P \implies (Q \vee R)$	$Q \wedge \sim R$	P
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	T	F	T
T	F	F	F	F	T
F	T	T	T	F	F
F	**T**	**F**	**T**	**T**	**F**
F	F	T	T	F	F
F	F	F	T	F	F

We see that the bolded row means the argument is **not** valid.

(ii) We let $P =$ "I buy a new bike", $Q =$ "I buy a used car" and $R =$ "I need a loan". Then the argument's logical form is

$$P \vee Q.$$
$$(P \wedge Q) \implies R.$$
$$Q \wedge \sim R.$$
$$\text{Therefore, } \sim P.$$

We make the following truth table:

P	Q	R	$P \vee Q$	$(P \wedge Q) \implies R$	$Q \wedge \sim R$	$\sim P$
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	T	T	F	F
T	F	F	T	T	F	F
F	T	T	T	T	F	T
F	T	F	T	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	F	T

We see from the table that the argument is valid.

2 (i) Applying the Rule of Modus Ponens, we know that Caz will phone.

(ii) Applying the Law of Syllogism, we know the final conclusion is as follows. Therefore, if $x^2 - 3x + 2 = 0$ then $x = 2$ or $x = 1$.

(iii) Applying the Rule of Modus Ponens, we know that $y = \left(\sqrt{y}\right)^2$ is positive or zero.

3 (i) Disprove the statement $\forall n \in \mathbb{N},\ n^2 + n + 29$ is prime. Let $n = 29$. Then

$$n^2 + n + 29 = 29^2 + 29 + 29 = 29(29 + 1 + 1) = (29)(31).$$

In this case, $n^2 + n + 29$ is not prime, and thus, we have a counterexample. Therefore, it is false to say "$\forall n \in \mathbb{N},\ n^2 + n + 29$ is prime."

(ii) Prove the statement $\exists x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\ xy \neq 1$. Let $x = 0$ and let $y \in \mathbb{Q}$. Then $xy = 0 \neq 1$. Thus, the statement is true.

(iii) Disprove the statement $\forall a, b \in \mathbb{R},\ (a + b)^2 = a^2 + b^2$. Let $a = b = 1$. Then $(a + b)^2 = (1 + 1)^2 = 4$ and $a^2 + b^2 = 1^2 + 1^2 = 2 \neq (a + b)^2$ Thus, we have a counterexample. Therefore, it is false to say that $\forall a, b \in \mathbb{R},\ (a + b)^2 = a^2 + b^2$.

(iv) Let O represent the set of all odd integers. Disprove the statement $\forall n, m \in O$, the average of n and m is odd. Let $n = 1$ and $m = 3$. Then the average is $\frac{n+m}{2} = \frac{1+3}{2} = 2$, which is not odd. Thus, we have a counterexample. Therefore, it is false to say that the average of any two odd integers is odd.

4 (i) The statement is a Universal statement and therefore requires a general proof. One example is not enough to prove a Universal statement.

(ii) The mistake is in the use of the definitions of odd and even numbers. When using an existential statement on two separate occasions, you should not use the same variable; that is, if we use k for defining n as an odd integer ($n = 2k+1$ for some $k \in \mathbb{Z}$), then we must use a different letter for defining m as an even integer (eg. $m = 2q$ for some $q \in \mathbb{Z}$).

5 (i) *Forward:* ???
Backward:

$$x^2 - 2x + 1 \geq 0 \Longrightarrow x^2 + 1 \geq 2x$$
$$(x - 1)^2 \geq 0 \Longrightarrow x^2 - 2x + 1 \geq 0.$$

But we know that $(x - 1)^2 \geq 0$ for all $x \in \mathbb{R}$.
Proof: We know that $(x - 1)^2 \geq 0$ for all $x \in \mathbb{R}$. Thus, $x^2 - 2x + 1 \geq 0 \Longrightarrow x^2 + 1 \geq 2x$ (adding $2x$ to both sides).

(ii) *Forward:* n is odd $\Longrightarrow\ \exists p \in \mathbb{N}, n = 2p + 1$. Thus, $n^2 = 4p^2 + 4p + 1$.
Backward: To prove n^2 is odd, we must have $n^2 = 2k + 1$ for some $k \in \mathbb{N}$; i.e., $\exists k \in \mathbb{N}, n^2 = 2k + 1 \Longrightarrow n^2$ is odd.
Proof: n is odd $\Longrightarrow \exists p \in \mathbb{N},\ n = 2p + 1 \Longrightarrow n^2 = 4p^2 + 4p + 1 \Longrightarrow n^2 = 2(2p^2 + 2p) + 1 \Longrightarrow n^2$ is odd. Therefore, if n is odd then n^2 is odd.

(iii) Let n, m be any two odd integers.
Forward: n is odd $\Longrightarrow \exists p \in \mathbb{Z}, n = 2p + 1$; m is odd $\Longrightarrow \exists q \in \mathbb{Z}, m = 2q + 1$ Thus, $n + m = (2p + 1) + (2q + 1) = 2p + 2q + 2$.
Backward: To prove $n + m$ is even, we must have $n + m = 2k$ for some $k \in \mathbb{Z}$; i.e., $\exists k \in \mathbb{N}, n + m = 2k \Longrightarrow n + m$ is even.
Proof: Let n, m be odd integers $\Longrightarrow \exists p, q \in \mathbb{Z},\ (n = 2p + 1 \land m = 2q + 1) \Longrightarrow n + m = 2p + 2q + 2 \Longrightarrow n + m = 2(p + q + 1) \Longrightarrow n + m$

is even, as $p+q+1 \in \mathbb{Z}$. Therefore, the sum of any two odd integers is even.

(iv) Let ABC be a triangle, with angles A, B and C.
Forward: We are given that the sum of two angles is equal to the third angle. That is, $A+B=C$. We know that $A+B+C=180°$. Combining these statements, we have $C+C=180°$.
Backward: To prove ABC is a right angled triangle, we must show one angle is $90°$.
Proof: Let ABC be a triangle such that $A+B=C$. Now $A+B+C=180° \implies C+C=2C=180° \implies C=90° \implies ABC$ is a right angled triangle. Therefore, is the sum of two angles of a triangle is equal to the third angle, then the triangle is a right angled triangle.

6 (i) *Forward:* $x<0 \implies ???$
Backward: $x^2-4x+4>4 \implies (x-2)^2>4$, and $x^2-4x>0 \implies x^2-4x+4>4$, and $x(x-4)>0 \implies x^2-4x>0$ **STOP!** If you continue with "backward" steps, you face a "choice" (i.e. $x<0$ and $x<4$ **or** $x>0$ and $x>4$): this is the step that is not "reversible". But the choice has been made for us because x is negative.
Forward: $x<0 \implies x-4<0$, so $x<0$ and $x-4<0 \implies x(x-4)>0$ —hence forwards and backwards steps are linked.
Proof: We know that $x<0$ and so $x-4<0$ as well. Therefore, $x(x-4)>0$, that is, $x^2-4x>0$. Adding 4 to both sides, we have $x^2-4x+4>4 \implies (x-2)^2>4$.

(ii) Suppose there is a smallest positive real number, x. Then $\frac{x}{2}$ is also a positive real number, and $\frac{x}{2}<x$. However, this is a contradiction, as we said x is the smallest positive real number. Therefore, there is no smallest positive real number.

7 Let n^2 be odd. Suppose that n is an even number. Then $n=2p$ for some $p \in \mathbb{N}$. Now, $n^2=(2p)^2=4p^2=2(2p^2)$. Thus, n^2 is even. Therefore, we have a contradiction, and so n must be odd.

8 (i) To prove the statement, the easiest thing is to prove the result for each value of x. *Case 1:* If $x=4$, then $x^2-3x+21=25 \neq 4$. *Case 2:* If $x=5$, then $x^2-3x+21=31 \neq 5$. *Case 3:* If $x=6$, then $x^2-3x+21=39 \neq 6$. Thus, if $x=4$, 5, or 6, then $x^2-3x+21 \neq x$.

(ii) $\forall x \in \mathbb{Z},\ (x \neq 0 \implies 2^x+3 \neq 4) \equiv \ \forall x \in \mathbb{Z}, \big((x<0) \vee \ (x>0) \implies \ 2^x+3 \neq 4\big)$. This statement requires a proof by cases. In each case, we prove $2^x+3 \neq 4$. *Case 1:* Let $x>0$. Then $2^x>1$ and so $2^x+3>4$.Therefore, $2^x+3 \neq 4$. *Case 2:* Let $x<0$.Then $0<2^x<1$ and so $2^x+3<4$. Therefore, $2^x+3 \neq 4$. Thus, if $x \neq 0$, then $2^x+3 \neq 4$.

9 (i) least element 0; greatest element 1; not well-ordered (e.g., subset $(0,1)$ has no least element)

(ii) least element 0; no greatest element; not well-ordered, as in (i)

(iii) least element $1-\frac{1}{2}=\frac{1}{2}$; no greatest element (there are elements arbitrarily close to 1, but 1 is not in the set); well-ordered

(iv) no least element; no greatest element; not well-ordered.