Predicate Logic

- The connectives ~, $\wedge$, $\vee$, =>, and <=> are not enough to prove or disprove all types of logical statements.
- For example:
  - All UOW math courses are fun,
  - Math 221 is a UOW math course,
  - Therefore, Math 221 is fun.
- This is correct, but we cannot determine its validity with the tools we have so far.
- We need to be able to manage words such as "all" and "some."
- **Predicate:** a sentence that contains a finite number of variables and becomes a statement when values are substituted.
- The **domain** of a variable is the set of all possible values it can be.
- The **truth set** is the subset of the domain that makes the predicate true.
- Predicates of one value are denoted: **p(x)**, **q(x)**
- Notation:

Symbol	Name	Example
$\mathbb{R}$	Set of all **real numbers**	Can be integers, fractions, etc…
$\mathbb{Q}$	Set of all **rational numbers**	Can be written as a fraction.
$\mathbb{Z}$	Set of **integers**	Whole numbers … -2, -1, 0, 1, 2, …
$\mathbb{N}$	Set of **natural numbers**	Counting numbers … 1, 2, 3, …
$\in$	**Contention**	"Is contained in," "belongs to," "is a member of…"
$\forall$	**Universal quantifier**	"For all"
$\exists$	**Existential quantifier**	"There exists"
$\ni$	**Such that**	

Exercise:

The predicate p(x): "x is a positive integer strictly less than 5" with $dom_p = \mathbb{Z}$ has truth set {…, -2, -1, 0, 1, 2, 3, 4}.

$dom_p = \mathbb{Z}$ has a truth set $\{1, 2, 3, 4\}$

Exercise:

The predicate q(x): "$x^2 > x$" with $dom_q = \mathbb{R}$ has truth set.

$$\{x : x^2 > x\} = \{x : x < -1 \text{ OR } x > 1\}$$
$$= \{x : |x| > 1\}$$
$$= (-\infty, -1) \cup (1, \infty)$$

The Universal Quantifier ∀

- One way to change a predicate into a statement is to assign values to the variables.
- Another way is to add quantifiers.

Exercise:

a) "All humans are mortal."
b) "All real numbers have a nonnegative square."

a) $\forall$ Humans x, x is mortal
b) $\forall x \in \mathbb{R}$, $x^2 \geq 0$

- **Universal statement:** has the form, $\forall\, x \in D$, $p(x)$.
- It is true IFF (if and only if) $p(x)$ is true for every $x \in D$ if at least one $x \in D$ can be found that makes $p(x)$ false, the statement is false.
- Such an x is called a counterexample (contrapositive).

Exercise:

$\forall\, x \in \mathbb{R}$, $x^2 > x$.

This is false if $x = \frac{1}{2}$ (counterexample)

Exercise:

Write using ∀.

a) All dogs are animals
b) Every integer greater than zero has a prime factor.

a) $\forall$ dogs $x \in \{\text{animals}\}$
b) $\forall\, \mathbb{Z}\ x \in \{\text{PRIMES}\} \ni x > 0$

Exercise:

Let D = {1, 2, 3, 4, 5}.

a) Show that the statement $\forall\, x \in D$, $x^2 \geq x$ is true.
b) Show that the statement $x \in D$, $\frac{1}{x}$ is false.

a) $x^2 \geq x$, $1^2 \geq 1$, $2^2 \geq 2$, $\ldots$, $5^2 \geq 5$ $\square$
This is true.

b) If $x = 1$, $\frac{1}{1^2} \not< \frac{1}{1}$ $\square$
This is false.

The Existential Quantifier ∃

Exercise:

a) "There is a cat in my house."
b) "There are integers m and n such that m + n = mn."

a) ∃ A cat $x \ni x$ is in my house.

b) $\exists\, m, n \in \mathbb{Z} \ni m + n = mn$

- **Existential statement:** has the form $\exists\, x \in D \ni p(x)$.
- It is true IFF (if and only if) $p(x)$ is true for at least one $x \in D$.
- It is false IFF $p(x)$ is false for all $x \in D$.

Exercise:

Write using ∃.

a) There exists a real number whose square is negative.
b) Some person is a vegetarian.

a) $\exists\, x \in \mathbb{R} \ni x^2 < 0$

b) ∃ Person $x \ni x$ is vegetarian

Exercise:

Show that the statement "$\exists\, m \in \mathbb{Z} \ni m^2 = m$" is true.

Let $m = 1$, $m \in \mathbb{Z}$

$1^2 = 1$

∴ This statement is true

Exercise:

Let E = {5, 6, …, 10}. Show that the statement ""∃ m ∈ E ∋ m2 = m" is false.

$5^2 \neq 5,\ 6^2 \neq 6,\ 7^2 \neq 7, \ldots, 10^2 \neq 10$

∴ This is false

Exercise:

Rewrite using informal language.

a) $\forall x \in \mathbb{R}, x^2 \geq 0$

b) $\exists m \in \mathbb{Z} \ni m^2 = m$

c) $\forall$ Students s, $\exists$ Math subject y $\ni$ s likes y

d) $\forall x \in \mathbb{R}, x^2 \neq -1$

a) For all real numbers, the square of that number is a positive int.

b) There exists a number in the set of integers where its square is equal to itself.

c) For all students, there exists a math subject where the student likes that math subject.

d) For all real numbers, the square of that number cannot equal -1.

Negation of Quantifiers

- Consider the statement, "all mathematicians wear glasses."
  - What is the negation of the statement?
  - It is natural to think it's "no mathematician wears glasses," but that's not correct.
- The negation is: "there exists a mathematician who does not wear glasses."
- If just one counterexample can be found, the original statement is false.

Negation of a Universal Statement

- The negation of the statement:

  $\forall$ **x** $\in$ **D, p(x)**

- This is logically equivalent to the statement:

  $\forall$ **x** $\in$ **D, ~p(x)**

- Symbolically:

  ~($\forall$ **x** $\in$ **D, p(x))** $\equiv$ $\forall$ **x** $\in$ **D, ~p(x)**

Exercise:

Write negations:

a) No computer hacker is over 40
b) ∀ Primes p, p is odd
c) ∀ People x, if x is blonde, then x has blue eyes.

a) There exists a computer hacker over 40.
b) There exists a prime number that is even
$\exists p \in \{PRIMES\} \ni p = 2m$
c) $\exists$ person $x \ni x$ is not blonde and x has blue eyes

- Consider the statement "some fish breathe air."
- What is the negation of this statement?
- It is, "no fish breathes air."
- You might think it should be "some fish do not breathe air," but this and the original statement can both be true at the same time.

Negation of Existential Statements

- The negation of the statement:

$\exists$ **x** $\in$ **D** $\ni$ **p(x)**

- This is logically equivalent to the statement:

$\exists$ **x** $\in$ **D** $\ni$ **~p(x)**

- Symbolically:

~($\exists$ **x** $\in$ **D** $\ni$ **p(x))** $\equiv$ $\forall$ **x** $\in$ **D** $\ni$ **~p(x)**

Exercise:

Write negations.

a) $\exists$ A triangle whose sum of angles is 200 degrees.
b) There is a woman who is 120 years old.
c) $\exists x \in \mathbb{R} \ni x^2 = -1$

a) $\forall x \in \{triangles\} \ni$ Sum of angles $\neq 200°$
b) $\forall$ woman $x \in \{Women\}$, $x \not\ni$ 120 years old
c) $\forall x \in \mathbb{R}, x^2 \neq -1$

Summary

- The negation of “all are” is “at least one is not.”
- The negation of “at least one sis” is “all are not.”

Exercise:

Write negations and decide which statements are true.

a) $\exists\, x \in \mathbb{R} \ni 3x = 1$

b) $\forall\, \varepsilon \in \mathbb{R}, \forall\, x \in \mathbb{Z}, \exists\, y \in \mathbb{Q} \ni \varepsilon > 0 \Rightarrow |x - y| < \varepsilon$

a) $\sim(\exists x \in \mathbb{R} \ni 3x = 1) \equiv \forall x \in \mathbb{R},\ 3x \neq 1$

b) $\forall \varepsilon \in \mathbb{R}, \forall x \in \mathbb{Z}, \exists y \in \mathbb{Q} \ni \varepsilon > 0 \Rightarrow |x-y| < \varepsilon$

$\equiv \exists \varepsilon \in \mathbb{R}, \exists x \in \mathbb{Z} \ni \forall y \in \mathbb{Q}, \varepsilon > 0 \wedge |x-y| \geq \varepsilon$