MATH221: Mathematics for Computer Science

Outline Solutions to Tutorial Sheet Week 3

Autumn 2017

1. (i) Note 1: "$\vee$" is only False when both parts are False.
Note 2: "$\Longrightarrow$" is only False when the first part is True and the second is False.

$$
\begin{array}{ccccccccl}
(P & \Longrightarrow & Q) & \vee & (P & \Longrightarrow & \sim Q) & & \\
 & & & \text{F} & & & & & \\
 & \text{F} & & & & \text{F} & & & \text{(Note 1)} \\
\text{T} & & \mathbf{F} & & \text{T} & & \text{F} & & \text{(Note 2)} \\
 & & & & & & & \mathbf{T} & \text{(Negation)}
\end{array}
$$

Therefore, we have a contradiction for Q. Thus the statement is a tautology.

(ii) Note 1: "$\vee$" is only False when both parts are False.
Note 2: "$\Longrightarrow$" is only False when the first part is True and the second is False.

$$
\begin{array}{cccccccccl}
\sim & (P & \Longrightarrow & Q) & \vee & (Q & \Longrightarrow & P) & & \\
 & & & & \text{F} & & & & & \\
\text{F} & & & & & & \text{F} & & & \text{(Note 1)} \\
 & & \text{T} & & & & & & & \text{(Negation)} \\
 & & & & & \text{T} & & \text{F} & & \text{(Note 2)} \\
 & \text{F} & & \text{T} & & & & & & \text{(Transferring values across)}
\end{array}
$$

There is no contradiction. Therefore, the statement is NOT a tautology.

(iii) Note 1: "$\vee$" is only False when both parts are False.
Note 2: "$\Longrightarrow$" is only False when the first part is True and the second is False.
Note 3: "$\wedge$" is only True when both parts are True.

$$
\begin{array}{cccccccccl}
(P & \wedge & Q) & \Longrightarrow & (\sim R & \vee & (P & \Longrightarrow & Q)) & \\
 & & & \text{F} & & & & & & \\
 & \text{T} & & & & \text{F} & & & & \text{(Note 2)} \\
\text{T} & & \mathbf{T} & & \text{F} & & & \text{F} & & \text{(Notes 3 \& 1)} \\
 & & & & & & \text{T} & & \mathbf{F} & \text{(Note 2)}
\end{array}
$$

We have a contradiction for Q. Thus the statement is a tautology.

2. (i)

$$
\begin{aligned}
\big((P \wedge Q) \Longrightarrow R\big) &\equiv \big(\sim (P \wedge Q) \vee R\big) && \text{[Thm 1.4.2, 6]} \\
&\equiv \big((\sim P \vee \sim Q) \vee R\big) && \text{[DeMorgan's Laws]} \\
&\equiv (\sim P \vee \sim Q \vee R) && \text{[Thm 1.4.2, 2]}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\big(P \Longrightarrow (P \vee Q)\big) &\equiv \big(\sim P \vee (P \vee Q)\big) && \text{[Thm 1.4.2, 6]} \\
&\equiv (\sim P \vee P \vee Q) && \text{[Thm 1.4.2, 2]}
\end{aligned}
$$

3. (i) Firstly, $(P \Longrightarrow Q) \equiv (\sim P \vee Q)$ [Thm 1.4.2, 6]

Thus,

$$
\begin{aligned}
\sim (P \Longrightarrow Q) &\equiv \sim (\sim P \vee Q) && \\
&\equiv (\sim\sim P \wedge \sim Q) && \text{[DeMorgan's Laws, Thm 1.4.2, 5]} \\
&\equiv (P \wedge \sim Q) && \text{[Thm 1.4.2, 4]}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\big((P \wedge \sim Q) \Longrightarrow R\big) &\equiv \big(\sim (P \wedge \sim Q) \vee R\big) && \text{[Thm 1.4.2, 6]} \\
&\equiv \big((\sim P \vee \sim\sim Q) \vee R\big) && \text{[DeMorgan's Laws, Thm 1.4.2, 5]} \\
&\equiv \big((\sim P \vee Q) \vee R\big) && \text{[Thm 1.4.2, 4]} \\
&\equiv \big(\sim P \vee (Q \vee R)\big) && \text{[Thm 1.4.2, 2]} \\
&\equiv \big(P \Longrightarrow (Q \vee R)\big) && \text{[Thm 1.4.2, 6]}
\end{aligned}
$$

4. (i) $$\begin{aligned} P \implies (Q \lor P) &\equiv \big(\sim P \lor (Q \lor P)\big) && \text{[Thm 1.4.2, 6]}\\ &\equiv (\sim P \lor Q \lor P) && \text{[Thm 1.4.2, 2]}\\ &\equiv (\sim P \lor P \lor Q) && \text{[Thm 1.4.2, 1]} \end{aligned}$$

However, $\sim P \lor P$ is a tautology, therefore, by the Conclusion, this statement is also a tautology, that is, $P \implies (Q \lor P)$ is a tautology.

(ii) $$\begin{aligned} &(P \land Q) \implies (\sim R \lor (P \implies Q))\\ &\equiv \Big(\sim (P \land Q) \lor \big(\sim R \lor (P \implies Q)\big)\Big) && \text{[Thm 1.4.2, 6]}\\ &\equiv \big((\sim P \lor \sim Q) \lor \sim R \lor (\sim P \lor Q)\big) && \text{[Thm 1.4.2, 5, 2 and 6]}\\ &\equiv (\sim P \lor \sim Q \lor \sim R \lor \sim P \lor Q) && \text{[Thm 1.4.2, 2]}\\ &\equiv (Q \lor \sim Q \lor \sim P \ \sim R) && \text{[Thm 1.4.2, 1]} \end{aligned}$$

As in part (i), we have a tautology as $Q \lor \sim Q$ is a tautology.

5. (i) The proposition is true:
If x is a positive integer, then $x^2 \leq 3 \implies x \leq \sqrt{3}$.
Now $\sqrt{3} \approx 1.7$ and so $x = 1$.

(ii) The proposition is false. You should have tried proving it using DeMorgan's Laws and failed.
Now, find values of x and y that make the statement false.
Let $x = 0$ and $y = 1$. $\quad \sim (x > 1) \lor \sim (y \leq 0)$ is True.
$(x \leq 1) \land (y > 0)$ is also True.
Thus, $\quad \sim \big((x \leq 1) \land (y > 0)\big)$ is False and the proposition is False.

6. (i) (a) $$\begin{aligned} \big(\sim (x > 1) \implies \sim (y \leq 0)\big) &\equiv \big(\sim\sim (x > 1) \lor \sim (y \leq 0)\big) && \text{[Thm 1.4.2, 6]}\\ &\equiv \big((x > 1) \lor (y > 0)\big) && \text{[Thm 1.4.2, 4]} \end{aligned}$$

(b) $$\begin{aligned} \big((y \leq 0) \implies (x > 1)\big) &\equiv \big(\sim (y \leq 0) \lor (x > 1) && \text{[Thm 1.4.2, 6]}\\ &\equiv \big((y > 0) \lor (x > 1)\big) \end{aligned}$$

The statements are the same!

(ii) $$\begin{aligned} \sim \big(\sim (P \lor Q) \land \sim Q\big) &\equiv \big(\sim\sim (P \lor Q) \lor \sim\sim Q\big) && \text{[DeMorgan's Laws]}\\ &\equiv \big((P \lor Q) \lor Q\big) && \text{[Thm 1.4.2, 4]}\\ &\equiv (P \lor Q) && \text{[Thm 1.4.2, 2]} \end{aligned}$$

7. (i) Every real number that is not zero is either positive or negative.
The statement is true.

(ii) The square root of every natural number is also a natural number.
The statement is false (consider $n = 2$).

(iii) Every student in MATH221 can correctly solve at least one assigned problem.
Lecturers are yet to work out if this is true or false!

8. (i) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \ \big(xy = 0 \implies (x = 0 \land y = 0)\big)$
The statement is false (consider $x = 1$ and $y = 0$).

(ii) $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, \ x \leq y$
The statement is true.

(iii) $\exists$ student S in MATH221, $\forall$ lecturer's jokes j, S hasn't laughed at j
Caz hopes this is false.

9. Let H be the set of all people (human beings).

(i)
$$\begin{aligned} P &\equiv \exists p \in H,\ \forall q \in H,\ p \text{ loves } q. \\ \sim P &\equiv \sim (\exists p \in H,\ \forall q \in H,\ p \text{ loves } q). \\ &\equiv \forall p \in H,\ \sim (\forall q \in H,\ p \text{ loves } q). \\ &\equiv \forall p \in H,\ \exists q \in H,\ p \text{ doesn't love } q. \end{aligned}$$

In a nice world, P is true!

(ii)
$$\begin{aligned} P &\equiv \forall p \in H,\ \forall q \in H,\ p \text{ loves } q. \\ \sim P &\equiv \sim (\forall p \in H,\ \forall q \in H,\ p \text{ loves } q). \\ &\equiv \exists p \in H,\ \sim (\forall q \in H,\ p \text{ loves } q). \\ &\equiv \exists p \in H,\ \exists q \in H,\ p \text{ doesn't love } q. \end{aligned}$$

In a perfect world, P is true!

(iii)
$$\begin{aligned} P &\equiv \exists p \in H,\ \exists q \in H,\ p \text{ loves } q. \\ \sim P &\equiv \sim (\exists p \in H,\ \exists q \in H,\ p \text{ loves } q). \\ &\equiv \forall p \in H,\ \sim (\exists q \in H,\ p \text{ loves } q). \\ &\equiv \forall p \in H,\ \forall q \in H,\ p \text{ doesn't love } q. \end{aligned}$$

P is definitely true!

(iv)
$$\begin{aligned} P &\equiv \forall p \in H,\ \exists q \in H,\ p \text{ loves } q. \\ \sim P &\equiv \sim (\forall p \in H,\ \exists q \in H,\ p\text{loves } q). \\ &\equiv \exists p \in H,\ \sim (\exists q \in H,\ p \text{ loves } q). \\ &\equiv \exists p \in H,\ \forall q \in H,\ p \text{ doesn't love } q. \end{aligned}$$

In our world, P is probably true!

(v)
$$\begin{aligned} P &\equiv \forall x \in \mathbb{Q},\ x \in \mathbb{Z} \\ \sim P &\equiv \sim (\forall x \in \mathbb{Q},\ x \in \mathbb{Z}) \\ &\equiv \exists x \in \mathbb{Q},\ x \notin \mathbb{Z} \end{aligned}$$

$\sim P$ is true.

(vi)
$$\begin{aligned} P &\equiv \sim (\forall n \in \mathbb{N},\ \exists p \in \mathbb{N},\ n = 2p) \\ &\equiv \exists n \in \mathbb{N},\ \sim (\exists p \in \mathbb{N},\ n = 2p) \\ &\equiv \exists n \in \mathbb{N},\ \forall p \in \mathbb{N},\ n \neq 2p \\ \sim P &\equiv \sim\sim (\forall n \in \mathbb{N},\ \exists p \in \mathbb{N},\ n = 2p) \\ &\equiv \forall n \in \mathbb{N},\ \exists p \in \mathbb{N},\ n = 2p \end{aligned}$$

P is true.

(vii)
$$\begin{aligned} P &\equiv \exists n \in \mathbb{N},\ n \text{ is not prime.} \\ \sim P &\equiv \sim (\exists n \in \mathbb{N},\ n \text{ is not prime}) \\ &\equiv \forall n \in \mathbb{N}, n \text{ is prime.} \end{aligned}$$

P is true.

(viii)
$$\begin{aligned} P &\equiv \forall \text{ triangle } T,\ T \text{ is a right triangle.} \\ \sim P &\equiv \sim (\forall \text{ triangle } T,\ T \text{ is a right triangle} \\ &\equiv \exists \text{ triangle } T,\ T \text{ is not a right triangle.} \end{aligned}$$

$\sim P$ is true.

10. (i) $\forall x \in \mathbb{R},\ (x > 1 \implies x > 0)$
This statement is **true**.
Clearly, $0 < 1 < x$, so $x > 0$.

(ii) $\forall x \in \mathbb{R},\ (x > 1 \implies x > 2)$
This statement is **false**.
Let $x = 1.5$. Then $x > 1$ but $x < 2$.

(iii) $\exists x \in \mathbb{R},\ (x > 1 \implies x^2 > x)$
This statement is **true**.
Let $x = 2$.
Then $x > 1$ and $x^2 = 4 > 2 = x$

(iv) $\exists x \in \mathbb{R},\ \left(x > 1 \implies \dfrac{x}{x^2+1} < \dfrac{1}{3}\right)$
This statement is **true**.
Let $x = 3$. Then $x > 1$ and
$\dfrac{x}{x^2+1} = \dfrac{3}{10} < \dfrac{1}{3}$.

(v) $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x^2 + y^2 = 9$
This statement is **false**.
Let $x = 1$ and $y = 1$,
then $x^2 + y^2 = 2 \neq 9$.

(vi) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x^2 < y + 1$
This statement is **true**.
For $x \in \mathbb{R}$, let $y = x^2$.
Then clearly $x^2 < y + 1$.

(vii) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x^2 + y^2 \geq 0$
This statement is **true**.
Let $x = 0$. For each $y \in \mathbb{R}$, $y^2 \geq 0$,
and we have $x^2 + y^2 = y^2 \geq 0$.

(viii) $\exists x \in \mathbb{R},\ \exists y \in \mathbb{R},\ (x < y \implies x^2 < y^2)$
This statement is **true**.
Let $x = 0$ and $y = 1$. Then $x < y$ and
$x^2 = 0 < 1 = y^2$.

11. (i) (a)
$$\begin{aligned} \sim (\forall \varepsilon > 0,\ \exists x \neq 0,\ |x| < \varepsilon) &\equiv \exists \varepsilon > 0,\ \sim (\exists x \neq 0,\ |x| < \varepsilon) \\ &\equiv \exists \varepsilon > 0,\ \forall x \neq 0,\ |x| \geq \varepsilon \end{aligned}$$

(b) The negation of the statement is false.

(c) For any $\varepsilon > 0$, we can take $x = \frac{\varepsilon}{2}$ and we have $x \neq 0$ but $|x| < \varepsilon$.

(ii) (a)
$$\begin{aligned} \sim (\exists y \in \mathbb{R},\ \forall x \in \mathbb{R},\ y < x^2) &\equiv \forall y \in \mathbb{R},\ \sim (\forall x \in \mathbb{R},\ y < x^2) \\ &\equiv \forall y \in \mathbb{R},\ \exists x \in \mathbb{R},\ y \geq x^2 \end{aligned}$$

(b) The negation of the statement is false.

(c) Let $y = -1$. We know $x^2 \geq 0$ for all $x \in \mathbb{R}$, ie. $x^2 > y$

(iii) (a)

$$\begin{aligned}
&\sim\left(\forall y \in \mathbb{R},\ \forall x \in \mathbb{R},\ \left(x < y \implies x < \frac{x+y}{2} < y\right)\right)\\
&\equiv \exists y \in \mathbb{R},\ \sim\left(\forall x \in \mathbb{R},\ \left(x < y \implies x < \frac{x+y}{2} < y\right)\right)\\
&\equiv \exists y \in \mathbb{R},\ \exists x \in \mathbb{R},\ \sim\left(x < y \implies x < \frac{x+y}{2} < y\right)\\
&\equiv \exists y \in \mathbb{R},\ \exists x \in \mathbb{R},\ \left(x < y \ \wedge\ \left(\frac{x+y}{2} \le x \ \vee\ \frac{x+y}{2} \ge y\right)\right)\\
&\equiv \exists y \in \mathbb{R},\ \exists x \in \mathbb{R},\ (x < y \ \wedge\ (y \le x \ \vee\ x \ge y))
\end{aligned}$$

(b) The negation of the statement is false.

(c) Clearly, $x < y \ \wedge\ (y \le x \ \vee\ x \ge y)$ is equivalent to $x < y \ \wedge\ x = y$, which is impossible.