Combinatorics

Sequences and Words

- A **sequence** is an ordered list of objects, with repetitions of the same objects allowed (as opposed to a set).
- The objects of a sequence are called **terms**.
- A sequence may be finite:

  $$(1, 2, 3, 4); (a, b, \dots, z);$$

  Or infinite

  $$(2, 4, 6, \dots); (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$$

- The order matters;
- $(1, 2, 3)$ is a different sequence than $(3, 2, 1)$.
- If all terms of a sequence are from a set U, the sequence is a **sequence in U** or a **U-sequence**.
- For example, $(1, 2, 3)$ is a sequence in $\mathbb{N}$.
  - It's also a sequence in $\{0, 1, 2, 3, 4\}$, in $\mathbb{Q}$, in $\mathbb{Z}$, and in $\mathbb{R}$.
- A sequence can also be called a **word** in the alphabet U.
- The sequence $(t_1, t_2, \dots, t_k)$ with n_i possible values for each t_i. Then:

  $$s = n_1 n_2 \dots n_k$$

Corollary:
- Let $|A| = n$.
- Then there are n^k sequences of length k in A.

Exercise:
How many 3-letter words can be formed with the English alphabet?

Term 1 2 3

$\underline{26} \cdot \underline{26} \cdot \underline{26} = 17576$. The words are AAA, AAB, AAC, ..., ZZZ

↑
26 letter options in alphabet

Permutations

- A sequence in which all terms are distinct is called a **permutation.**
- If $|S| = n$, a sequence of length $k \leq n$ of all distinct objects is called a **permutation of n objects taken k at a time.**
- If $k = n$, we just say **permutations of n objects.**

Exercise:

Let $S = \{1, 2, 3, 4, 5, 6\}$. The following words in S are permutations of 6 objects taken 3 at a time.

$$s_1 = \{1, 4, 6\}, s_2 = \{3, 2, 4\}, s_3 = \{5, 3, 1\}$$

The following words in S are permutations of 6 objects.

$$s_4 = \{1, 2, 3, 4, 5, 6\}, s_5 = \{1, 3, 2, 6, 4, 5\}, s_6 = \{6, 5, 1, 3, 2, 4\}$$

- There are $P_k^n = \frac{n!}{(n-k)!}$ Permutations of n objects taken k at a time.
- Notice that $\frac{(n!)}{(n-k)!} = \frac{1 \cdot 2 \cdot \ldots \cdot n}{1 \cdot 2 \cdot \ldots \cdot k} = n(n-1) \ldots (n-k+1)$.
- This has a shorter notation called **"falling factorial"** $n^{\underline{k}}$, which is also used for $k > n$.
- When $k \leq n$, we have:

$$n^{\underline{k}} = \frac{n!}{(n-k)!}, \text{ and when } k > n, n^{\underline{k}} = 0$$

Exercise:

Let $n = 7, k = 10$.

$$7^{\underline{10}} = 7 \cdot 6 \cdot \ldots \cdot 1 \cdot 0 \cdot (-1) \cdot (-2) = 0$$

Theorem:

- For all $n, k \in \mathbb{N}$, there are $n^{\underline{k}}$ permutations of n objects taken k at a time.

Proof:

If $k > n$, there is no way to permute n objects k at a time, so the answer must be zero.
If $k \leq n$, there are n choices for the 1st element, then $(n-1)$ choices for the 2nd, etc.
So the possibilities are:

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = n^{\underline{k}} = \frac{n!}{(n-k)!} \blacksquare$$

Corollary:

- For all $n \in \mathbb{N}$, there are $n!$ permutations of n objects.

Counting Strategies

- Consider the problem, "how many sequences satisfy a certain set of properties?"
- We use counting strategy to answer this question methodically.
- For a sequence of length k, use k empty slots:

$$\underset{1}{\_\_}\ \underset{2}{\_\_}\ \underset{3}{\_\_}\ \cdots\ \underset{k}{\_\_}$$

- Fill each slot one at a time, with the number of possible values for each term, given the restrictions of the properties.

$$\underset{1}{\underline{n_1}}\ \underset{2}{\underline{n_2}}\ \underset{3}{\underline{n_3}}\ \cdots\ \underset{k}{\underline{n_k}}$$

- By multiplication rule, there are $n_1 n_2 n_3 \dots n_k$ possible sequences.

Exercise:
There are 2 highways from Brisbane to Sydney, and 3 highways from Sydney to Adelaide. How many different round trips from Brisbane to Adelaide via Sydney are there? How many are there without taking the same highway twice?

A: 1st problem: $\underset{B-S}{\underline{2}} \cdot \underset{S-A}{\underline{3}} \cdot \underset{A-S}{\underline{3}} \cdot \underset{S-B}{\underline{2}} = 36$

2nd problem: $\underset{B-S}{\underline{2}} \cdot \underset{S-A}{\underline{3}} \cdot \underset{A-S}{\underline{2}} \cdot \underset{S-B}{\underline{1}} = 12$

↳ one less to avoid same highway.

- You don't necessarily have to start with the 1st position.
- Start where it's most convenient.

Exercise:
How many 5-digit odd numbers with no repeated digits are there?

$\underline{8} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5}$

↳ Last digit must be odd to be an odd number. Start here with options 1, 3, 5, 7, 9

↳ Next Cannot be one chosen from digit 5

↳ Can't include 1st digit chosen but can include last.

- Sometimes, we need to break a problem up into subproblems.

Exercise:
How many 5-digit even numbers with no repeated digits are there?

① If last digit is 0

$\underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{1}$

② If last digit is not 0

$\underline{8}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{4}$

There's the same restriction on digit 5, but the restriction on digit 1 is different if digit 5 is zero

Required Adjacency

- For a required adjacency, treat the adjacency as a single object, then multiple by the number of arrangements of the adjacency.

Exercise:
Three single people and a married couple are to be seated in a row of chairs. In how many ways can it be done such that the spouses sit together?

$\underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 24$ total arrangements with $n = 4$ because H-W must sit together.

$24 \cdot 2 = 48$

The permutations of required adjacency i.e. (W-H), (H-W)

Forbidden Adjacency

- For a forbidden adjacency, calculate it as a required adjacency, and then subtract from the total possible arrangements.

Exercise:
In how many ways can you align a cow, a goat, a fox, and a chicken such that the fox and the chicken are not next to each other.

Required adjacency (CH-FX): $\underline{3} \cdot \underline{2} \cdot \underline{1} = 6$

$\underline{6} \cdot \underline{2} = 12$

CH-FX FX-CH

Total permutations w/out adjacency: $\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 24$

Forbidden adjacency: $24 - 12 = 12$ $\square$

Binomial Coefficients

- Recall the power set of x: $P(x) = \{A : A \subseteq X\}$.

  $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

- Another notation for $P(X)$ is 2^X. This is because of the following.

Theorem:

- Let $|X| = n \in \mathbb{N}U\{0\}$
- Then X has 2^n subsets, i.e. $|P(X)| = 2^{|X|}$

Proof:

With induction.

a) Let $n = 0$, then $X = \emptyset$, and $P(X) = \emptyset$, so $|P(X)| = 1 = 2^0$

b) Let $k \in \mathbb{N}$, suppose $|X| = k$ and $|P(X)| = 2^k$. Define

$$Y = X \cup \{y\} = \{x_1, x_2, \dots, x_k, y\}.$$

The subsets of Y are those that contain y, and those that do not. Those that do not are exactly the subsets of X, of which there are 2^k. Those that do contain y are of the form $Z \cup \{y\}$, where $Z \in P(X)$, so there are exactly 2^k of those too. Therefore, $|Y| = 2^k + 2^k = 2^{k+1}$.

$\therefore |P(X)| = 2^{|X|}\ \forall X\ finite\ \blacksquare$

- Let $|X| = n \in \mathbb{N} \cup \{0\}$. For every $k \in \mathbb{N} \cup \{0\}$, we denote by $\binom{n}{k}$ the number of subsets of X with k elements.

$$\binom{n}{k} = |\{A : A \subseteq X\ and\ |A| = k\}|$$

- The symbol $\binom{n}{k}$ Is read **"n choose k"** or **"the k^{th} BINOMIAL COEFFICIENT of order n"**
- Some $\binom{n}{k}$ are obvious:

  $\binom{n}{0} = 1$, since the only subset of cardinality 0 is $\emptyset$.
  $\binom{n}{n} = 1$, since X is the only subset of X with n elements.
  If $k > n$, then $\binom{n}{k} = 0$, as it's impossible to have a subset of X with cardinality larger than that of X.

Theorem:

- For all $n, k \in \mathbb{N} \cup \{0\}$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!}$$

Proof:

For $k > n$, we've seen that $n^{\underline{k}} = 0$, and $\binom{n}{k} = 0$.

Let $k \leq n$. Recall that the number of permutations of n objects taken k at a time is $P_k^n = \frac{n!}{(n-k)!}$.

This number can be obtained by taking all $\binom{n}{k}$ combinations of k elements and ordering the elements in each combination, which can be done in P_k^k ways. Thus,

$$P_k^n = \binom{n}{k} P_k^k \Rightarrow \binom{n}{k} = \frac{P_k^n}{P_k^k} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{k!\,(n-k)!} = \frac{n^{\underline{k}}}{k!}$$

- The symbol $\binom{n}{k}$ is also denoted by C_k^n, the number of combinations of n objects taken k at a time.

Exercise:

How many different poker hands are there?

A: There are 5 cards in a poker hand.
Order is not important.
They are taken from a deck of 52 cards.

$$\binom{52}{5} = \frac{52!}{5!\,(52-5)!} = 2{,}598{,}960 \text{ poker hands}$$

Theorem:

- For all $n, k \in \mathbb{N} \cup \{0\} \ni 0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$

Proof:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)![n-(n-k)!]} = \binom{n}{n-k}$$

Theorem:

- For all $n, k \in \mathbb{N} \cup \{0\} \ni 0 \leq k \leq n$,
  a) $\binom{n}{0} = 1$
  b) $\binom{0}{k} = 0$
  c) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Proof:

?

The Binomial Theorem

- Motivation:
  - In how many ways can 3 red marbles and 4 blue marbles be arranged in a row? (Or a more practical example: how many binary words are there with 3 zeros and 4 ones?). The multiplication rule isn't very helpful here; there are too many cases. However, considering the 7 slots:

    1 2 3 4 5 6 7

    Notice that once you choose slots for the red marbles, the placement of the blue ones is automatic. So the question is, how many ways are there to choose 3 of the 7 slots? We know the answer is $\binom{7}{3} = 35$. Similarly, if you choose 4 slots for the blue marbles first, there are $\binom{7}{4} = 35$ ways to do it. The answer is the same, because $\binom{7}{3} = \binom{7}{4} = \frac{7!}{3!4!}$

Theorem:

- The number of words of length n consisting of n_1 letters of one sort, and $n_2 = n - n_1$, letters of a second sort is:

$$\binom{n}{n_1} = \binom{n}{n_2} = \frac{(n_1 + n_2)!}{n_1!\, n_2!}$$

- Consider the binomial expansion

$$(x + y)^2 = xx = xy + yx + yy$$

  which is the sum of all words of length 2 in the alphabet $\{x, y\}$.
  Similarly,

$$(x + y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

  is the sum of all words of length 3 in the alphabet $\{x, y\}$.
  By simplifying, we get the familiar formulae:

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

- The binomial theorem below is a formula for the coefficients of binomial expansion to any power in $\mathbb{N}$.

Theorem (Binomial Theorem):

- For all $n \in \mathbb{N} \cup \{0\}$,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof:

The case $n = 0$ is easily verified by hand. For $n \in \mathbb{N}$, the expansion of $(x+y)^n$ is (before simplification) the sum of all 2^n words of length n in the alphabet $\{x, y\}$.
The number of such words that consist of k $x's$ and $(n-k)$ $y's$ is $\binom{n}{k}$ by the previous theorem.

The binomial theorem as written gives the expansion in ascending powers of x:

$$(x+y)^n = y^n + n \cdot xy^{n-1} + \binom{n}{2} x^2 y^{n-2} + \binom{n}{3} x^3 y^{n-3} + \cdots + n \cdot x^{n-1} y + x^n$$

Equivalently, it can be written in reverse:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = x^n + n \cdot x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + n \cdot xy^{n-1} + y^n$$

We can substitute values for x and y to obtain identities.

Exercise:
Let $x = y = 1$. Then the binomial theorem gives:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Exercise:
Let $x = -1, y = 1$. Then the binomial theorem gives:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0;$$
$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots;$$
$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k}$$

- Sometimes, a useful trick is to use the fact that $x = x \cdot 1$.

Exercise:
Simplify $\sum_{k=0}^{n} \binom{n}{k} a^k$.

A:

$$\sum_{k=0}^{n} \binom{n}{k} a^k = \sum_{k=0}^{n} \binom{n}{k} a^k \cdot 1^{n-k} = (a+1)^n$$

Exercise:

Simplify $\sum_{k=1}^{17}(-1)^k\binom{17}{k}13^{17-k}$

A:

$$\sum_{k=1}^{17}\binom{17}{k}13^{17-k}(-1)^k = \sum_{k-0}^{17}\binom{17}{k}13^{17-k}(-1)^k - \binom{17}{0}13^{17-0}(-1)^0$$

$$= (13-1)^{17} - 1\cdot 13^{17}\cdot 1$$

$$= 12^{17} - 13^{17}$$