MATH221 Mathematics for Computer Science

Tutorial Sheet Week 5

Autumn 2017

In the following induction questions, write out *full* arguments, like those in the examples from lectures. That is, call the assertion being made "CLAIM(n)", perform Steps 0, 1 and 2, then wrap up the argument with a concluding sentence ("Therefore, by induction, ..."). Also, in Step 2, write down *in full* what CLAIM(k) and CLAIM($k+1$) say, respectively.

1. Prove by mathematical induction that $1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

2. Prove by mathematical induction that $1^3+2^3+\cdots+n^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{N}$.

3. Evaluate

   (i) $\displaystyle\sum_{i=1}^{5}(2i-5)$ (ii) $\displaystyle\sum_{j=-2}^{2} 2^j$ (iii) $\displaystyle\sum_{k=0}^{3} \frac{k!}{2}$ (iv) $\displaystyle\sum_{\ell=0}^{99} \frac{(-1)^\ell}{3}$

4. (i) Express the sum $2+6+10+\cdots+(4n-2)$ using sigma notation.

   (ii) Prove by induction that $2+6+10+\cdots+(4n-2) = 2n^2$ for all $n \in \mathbb{N}$.

5. Consider the statement $2^n \geq n^2$. Test its correctness for a range of values of n, make a conjecture about the range of values for which it is true, and then use a suitable form of mathematical induction to prove it.

6. Prove by induction that $n! > 2^n$ for $n \geq 4$.

7. Here is a small example illustrating how mathematical induction may be used in computer science.

   Assume that `X` and `Y` have been declared as integer variables, and that initially `X` has value x and `Y` has value y. Consider the following fragment of C or C++ code:

```
while (X != 0)
{
    X = X - 1
    Y = Y + 1
}
```

For $n \in \mathbb{N}$, let CLAIM(n) be the statement "before the nth loop iteration, `X+Y` has the value $x + y$." Prove by induction that CLAIM(n) is true for all n.

Deduce that *if* the loop terminates, then `Y` will have value $x + y$.

8. Prove by mathematical induction CLAIM(n): "$n + 1 < n$" for all n. However, you find yourself in a bit of a rush, and decide to skip the *Basis step* for an induction proof. Start with the induction step, and observe why the Basis step is actually really, really important.