MATH 223

MATHEMATICS FOR IT

PRELIMINARIES

SET: A SET IS A COLLECTION OF ELEMENTS.

Ex: $\{2, 4, 9\}$ IS A SET OF WHOLE NUMBERS.

$\{$BOB, SUSAN$\}$ IS A SET OF NAMES.

$\{$10:15am, 1:05pm, 4:22pm, 12:00am$\}$ IS A SET OF TIMES.

THE ORDER OF THE ELEMENTS DOES NOT MATTER.

$$\{2,4,9\} = \{4,9,2\}$$

THESE ARE EXAMPLES OF FINITE SETS. SOME INFINITE SETS ARE

$\mathbb{N} = \{1, 2, 3, \ldots\}$ THE SET OF NATURAL NUMBERS.

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ THE SET OF INTEGERS.

$\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$ THE SET OF RATIONAL NUMBERS.

"SUCH THAT"

$\mathbb{I}$ THE SET OF IRRATIONAL NUMBERS. Ex: $\pi, \sqrt{2}, -\sqrt[3]{7}, e$.

$\mathbb{R}$ THE SET OF REAL NUMBERS. $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$

"UNION"

FUNCTION: A FUNCTION IS A MAP OR A RULE THAT ASSIGNS TO EACH ELEMENT OF A SET A AN ELEMENT OF ANOTHER SET B. WE WRITE $f: A \to B$, TO MEAN "FUNCTION f GOES FROM A TO B".

SOMETIMES, A FUNCTION IS DEFINED BY LISTING EACH ELEMENT:

$f:\{2,4,9\} \to \mathbb{Z}$, $f(2)=1$, $f(4)=100$, $f(9)=-5$.

SOMETIMES, AN ALGEBRAIC EXPRESSION IS USED:

$f: \mathbb{Q} \to \mathbb{Q}$, $f(x) = \frac{x}{2}$. THIS DESCRIBES f FOR EVERY ELEMENT OF $\mathbb{Q}$.

$f(0) = \frac{0}{2} = 0$, $f(\frac{3}{4}) = \frac{3/4}{2} = \frac{3}{8}$, $f(-20) = \frac{-20}{2} = -10$, ETC.

FOR $f: A \to B$, A IS CALLED THE <u>DOMAIN</u> OF f. WE WRITE $\operatorname{dom} f = A$.

B IS CALLED THE <u>CODOMAIN</u> OF f.

IF <u>EVERY</u> ELEMENT OF B CAN BE OBTAINED BY $f(x)$ FOR SOME $x \in A$, ("IN")

THEN B IS THE <u>RANGE</u> OF f. WE WRITE $\operatorname{ran} f = B$.

<u>SEQUENCE</u>: A SEQUENCE IS A FUNCTION WHOSE DOMAIN IS $\mathbb{N}$.

SEQUENCES CAN BE WRITTEN IN SET NOTATION: (ORDER MATTERS)

$\{2,4,6,8,\ldots\}$; $\{-1,1,-1,1,-1,\ldots\}$; $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\}$;

OR IN FUNCTION NOTATION:

$f(n) = 2n$; $f(n) = (-1)^n$; $f(n) = \frac{1}{n}$

OR IN ELEMENT NOTATION:

$a_n = 2n$; $a_n = (-1)^n$; $a_n = \frac{1}{n}$

WITH ELEMENT NOTATION, THE SUBINDEX INDICATES THE ELEMENT NUMBER. FOR $a_n = 2n$, WE HAVE

$a_1 = 2 \cdot 1 = 2$

$a_2 = 2 \cdot 2 = 4$

$a_3 = 2 \cdot 3 = 6$, ETC. THIS MAKES IT EASY TO FIND ANY ELEMENT.

$$a_{325} = 2 \cdot 325 = 650.$$

OFTEN IN APPLICATIONS, WE WANT TO ADD UP ALL THE ELEMENTS OF A SEQUENCE. THIS SUM IS CALLED A <u>SERIES</u>, REPRESENTED BY <u>SIGMA NOTATION</u>:

$$\sum_{i=1}^{10} a_i = a_1 + a_2 + a_3 + \dots + a_{10}.$$

$$\sum_{j=4}^{k} a_j = a_4 + a_5 + a_6 + \dots + a_k.$$

THESE ARE <u>FINITE SUMS</u> OR <u>FINITE SERIES</u>. THE i IS A DUMMY INDEX; RELABELING DOES NOT CHANGE THE SUM.

$$\sum_{i=1}^{10} a_i = \sum_{j=1}^{10} a_j.$$

NOTE THAT THERE ARE 10 TERMS IN $\sum_{i=1}^{10} a_i$. IN GENERAL, HOW MANY TERMS DOES $\sum_{i=m}^{n} a_i$ HAVE?

ANSWER: $n - m + 1$.

TO ADD UP AN INFINITE SEQUENCE, WE WRITE

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots$$

THIS IS CALLED AN INFINITE SUM OR A SERIES.

EXAMPLES

$$\sum_{i=1}^{\infty} i = 1+2+3+\ldots = \infty$$

$$\sum_{i=1}^{\infty} (-1)^i = -1+1-1+1-\ldots = ?$$

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots = \frac{\pi^2}{6}$$

ALGEBRA OF FINITE SUMS

FOR A FINITE SUM, THERE ARE ADDITION AND SCALAR MULTIPLICATION PROPERTIES:

$$\sum_{i=j}^{j+k} (a_i + b_i) = \sum_{i=j}^{j+k} a_i + \sum_{i=j}^{j+k} b_i$$

$$\sum_{i=j}^{j+k} \mu a_i = \mu \sum_{i=j}^{j+k} a_i \quad \text{FOR ANY } \mu \in \mathbb{R}.$$

$\boxed{\text{EXERCISE}}$: UNDERSTAND WHY THESE ARE TRUE BY EXPANDING EACH SIDE.

NOTE: THESE ARE NOT NECESSARILY TRUE FOR INFINITE SUMS.

ARITHMETIC SERIES

AN ARITHMETIC SEQUENCE (OR ARITHMETIC PROGRESSION) IS A SEQUENCE WHERE EACH SUCCESSIVE TERM DIFFERS FROM THE PREVIOUS ONE BY A FIXED AMOUNT.

Ex: $\{-6,-4,-2,0,2,4,\ldots\}$ THE COMMON DIFFERENCE IS 2.

$\{\frac{7}{3}, \frac{6}{3}, \frac{5}{3}, \frac{4}{3}, \ldots\}$ THE COMMON DIFFERENCE IS $-\frac{1}{3}$.

THE COMMON DIFFERENCE IS $\boxed{d = a_{i+1} - a_i}$ FOR ANY $i \in \mathbb{N}$. IF THE FIRST TERM IS DENOTED a, THEN THE SEQUENCE IS

$$\{a, a+d, a+2d, a+3d, \ldots\}.$$

THE n^{th} TERM OF AN ARITHMETIC SEQUENCE IS $\boxed{T_n = a+(n-1)d}$.

EXERCISE: GIVEN THESE FORMULAS, CONFIRM THAT $T_{n+1} - T_n = d$.

EXAMPLES:

1) DOES $\{3,7,12,18,25,\ldots\}$ DESCRIBE AN ARITHMETIC SEQUENCE?

A: CHECK IF THERE IS A COMMON DIFFERENCE.

$$a_2 - a_1 = 7-3 = 4$$
$$a_3 - a_2 = 12-7 = 5.$$

NO COMMON DIFFERENCE, SO THE SEQUENCE IS NOT ARITHMETIC.

2) IN THE ARITHMETIC SEQUENCE WITH $a=8$ AND $d=-3$, WHAT IS T_{21}?

A: $T_{21} = a+(21-1)d = 8+20(-3) = -52.$

3) AN ARITHMETIC SEQUENCE HAS $T_5 = 16$ AND $T_2 = 4$. FIND a AND d.

A: $T_5 = a + (5-1)d = a + 4d = 16$ ①

$T_2 = a + (2-1)d = a + d = 4$ ②

① − ②: $\begin{array}{r} a + 4d = 16 \\ -a - d = -4 \\ \hline 3d = 12 \end{array} \Rightarrow \boxed{d = 4}$

SUBSTITUTE: $a + 4 \cdot 4 = 16 \Rightarrow \boxed{a = 0}$

SUMMING ARITHMETIC SERIES

THE SUM OF THE FIRST n TERMS OF A SERIES IS DENOTED S_n AND CALLED THE n^{th} PARTIAL SUM OF THE SERIES.

EX: $\{0, 1, 2, 3, 4, \ldots\}$

$S_1 = \sum_{i=1}^{1} a_i = 0$ (FIRST PARTIAL SUM)

$S_2 = \sum_{i=1}^{2} a_i = 0 + 1 = 1$ (SECOND PARTIAL SUM)

$S_3 = \sum_{i=1}^{3} a_i = 0 + 1 + 2 = 3$ (THIRD PARTIAL SUM)

FOR AN ARITHMETIC SERIES, THERE IS A PATTERN THAT WE CAN USE TO MAKE A FORMULA.

$\{a, a+d, a+2d, a+3d, \ldots, a+(n-1)d\}$

$S_1 = T_1 = a$

$S_2 = T_1 + T_2 = a + (a+d) = 2a + d$

$S_3 = T_1 + T_2 + T_3 = a + (a+d) + (a+2d) = 3a + 3d$

$S_4 = T_1 + T_2 + T_3 + T_4 = a + (a+d) + (a+2d) + (a+3d) = 4a + 6d$

$\vdots$

$$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right].} \quad \text{OR} \quad \boxed{S_n = \frac{n}{2}\left[a + T_n\right].}$$

IMPORTANT: THESE FORMULAS WORK ONLY FOR ARITHMETIC SERIES.

MAKE SURE YOU HAVE A FIRST TERM AND A COMMON DIFFERENCE BEFORE USING THEM.

GEOMETRIC SEQUENCE

IN A GEOMETRIC SEQUENCE, INSTEAD OF A COMMON DIFFERENCE THERE IS A COMMON RATIO, DENOTED r. THE RATIO OF ANY TERM OVER THE PREVIOUS TERM EQUALS r.

Ex: IS $\{1, 2, 4, 8, 16, \ldots\}$ A GEOMETRIC SEQUENCE?

A: CHECK FOR COMMON RATIO.

$$\frac{T_2}{T_1} = \frac{2}{1} = 2; \quad \frac{T_3}{T_2} = \frac{4}{2} = 2; \quad \frac{T_4}{T_3} = \frac{8}{4} = 2; \quad \frac{T_5}{T_4} = \frac{16}{8} = 2.$$

THE SEQUENCE IS GEOMETRIC WITH $r = 2$.

Ex: IS $\{3, 6, 9, 12, \ldots\}$ GEOMETRIC?

$$\frac{T_2}{T_1} = \frac{6}{3} = 2; \quad \frac{T_3}{T_2} = \frac{9}{6} = \frac{3}{2}.$$ NOT GEOMETRIC.

THE DIFFERENCE BETWEEN CONSECUTIVE TERMS OF A GEOMETRIC SEQUENCE IS A FACTOR OF r: $T_2 = rT_1$, $T_3 = rT_2$, ETC.

SO THE n^{th} TERM OF A GEOMETRIC SEQUENCE IS $\boxed{T_n = ar^{n-1}.}$

EX: A GEOMETRIC SEQUENCE HAS $T_2 = 3$ AND $T_4 = 27$. FIND T_{10}.

A: WE FIRST FIND a AND r.

$$T_4 = ar^3 = 27 \; ; \; T_2 = ar = 3$$

$$\frac{T_4}{T_2} = \frac{ar^3}{ar} = \frac{27}{3} \Rightarrow r^2 = 9 \Rightarrow \boxed{\begin{array}{c} r = 3 \\ \text{OR} \\ r = -3 \end{array}}$$

a) IF $r = 3$, THEN $T_2 = a \cdot 3 = 3 \Rightarrow a = 1$.

THEN $T_{10} = ar^9 = 1 \cdot 3^9 = 19683$.

b) IF $r = -3$, THEN $T_2 = a(-3) = 3 \Rightarrow a = -1$.

THEN $T_{10} = ar^9 = -1(-3)^9 = 19683$.

NOTE: THESE ARE TWO DIFFERENT GEOMETRIC SEQUENCES.

SEQUENCE a) IS $\{1, 3, 9, 27, 81, \ldots\}$

SEQUENCE b) IS $\{-1, 3, -9, 27, -81, \ldots\}$

EX: HOW MANY TERMS ARE IN THE GEOMETRIC SEQUENCE $\{2, 4, \ldots, 2048\}$

A: $a = 2$, $r = \frac{4}{2} = 2$.

$T_n = 2048$, WE WANT TO FIND n.

$$T_n = ar^{n-1} = 2 \cdot 2^{n-1} = 2^n = 2048$$

$$n = \log_2 2048 = \frac{\ln 2048}{\ln 2} = 11.$$

THERE ARE 11 TERMS.

SUMMING A GEOMETRIC SEQUENCE.

THE n^{th} PARTIAL SUM IS $S_n = a + ar + ar^2 + \dots + ar^{n-1}$. ①

NOTICE THAT $S_{n+1} = a + ar + \dots + ar^{n-1} + ar^n = S_n + ar^n$. ②

MULTIPLY ① BY r, AND ADD a:

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$
$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n$$
$$a + rS_n = a + ar + ar^2 + ar^3 + \dots + ar^n = S_{n+1} \quad ③$$

WE HAVE 2 DIFFERENT EXPRESSIONS FOR S_{n+1}.

② = ③ : $S_n + ar^n = a + rS_n$

$$S_n(1-r) = a - ar^n$$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}.}$$

EX: FIND S_{10} FOR $\{27, 9, 3, 1, \frac{1}{3}, \dots\}$

A: $a = 27$, $r = \frac{9}{27} = \frac{1}{3}$.

$$S_{10} = \frac{a(1-r^{10})}{1-r} = \frac{27(1-\frac{1}{3^{10}})}{1-\frac{1}{3}} = \frac{27}{2/3}\left(1-\frac{1}{3^{10}}\right)$$

$$= \frac{81}{2}\,\frac{3^{10}-1}{3^{10}} = \frac{3^{10}-1}{2\cdot 3^6} = \boxed{\frac{59048}{1458}}$$

LIMITING SUM OF A GEOMETRIC SERIES

SOMETIMES, EVEN AN INFINITE SERIES HAS A SUM. THE SUM OF A SERIES IS DEFINED AS THE LIMIT OF THE n^{th} PARTIAL SUM AS n GOES TO INFINITY.

CONSIDER THE GEOMETRIC PARTIAL SUM $S_n = \dfrac{a(1-r^n)}{1-r}$.

WHAT HAPPENS AS n INCREASES?

IF $-1 < r < 1$, THEN $r^n \to 0$ AS $n \to \infty$. THEN THE LIMIT EXISTS AND WE WRITE $S = \dfrac{a}{1-r}$. SO AS LONG AS $|r| < 1$, THE GEOMETRIC SERIES HAS AN ANSWER.

$$\boxed{|r| < 1 \Rightarrow \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}}$$

IF $|r| \geq 1$, THEN THE SUM DOES NOT EXIST.

EXAMPLES

1) FIND THE SUM OF THE SERIES DEFINED BY $\{27, 9, 3, 1, \frac{1}{3}, \ldots\}$, IF IT EXISTS.

A: WE FOUND $a = 27$ AND $r = \frac{1}{3}$ BEFORE, SO $|r| < 1$.

$$S = \frac{a}{1-r} = \frac{27}{1-\frac{1}{3}} = \boxed{\frac{81}{2}}.$$

2) FIND THE SUM OF THE SERIES DEFINED BY $\left\{\frac{3}{4}, -1, \frac{4}{3}, \frac{-16}{9}, \frac{64}{27}, \ldots\right\}$, IF IT EXISTS

$$r = \frac{-1}{3/4} = \frac{-4}{3} \Rightarrow |r| = \frac{4}{3} > 1.$$ THE SUM DOES NOT EXIST.

3) EXPRESS $0.4\overline{7}$ AS A PROPER FRACTION.

A: $0.4\overline{7} = 0.47777\ldots = \frac{4}{10} + \underbrace{\frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \ldots}_{\text{GEOMETRIC SERIES!}}$

$a = \frac{7}{100}$, $r = \frac{1}{10} \Rightarrow$ THE SUM EXISTS.

$$S = \frac{7/100}{1 - \frac{1}{10}} = \frac{7}{100} \cdot \frac{10}{9} = \frac{7}{90}.$$

$$0.4\overline{7} = \frac{4}{10} + \frac{7}{90} = \frac{36+7}{90} = \boxed{\frac{43}{90}}.$$

EXAMPLE: ANNUITIES AND PERPETUITIES

ANNUITY: A SEQUENCE OF PAYMENTS, USUALLY OF EQUAL SIZE, MADE AT PERIODIC INTERVALS.

AN ANNUITY IS ORDINARY IF PAYMENTS OCCUR AT THE END OF EACH INTERVAL. IT IS SIMPLE IF THE INTEREST IS CALCULATED OVER THE SAME PERIOD. THE ANNUITY FORMULA IS

$$\boxed{S = R\,\frac{(1+i)^n - 1}{i}}$$

S: FUTURE VALUE (ACCUMULATED VALUE) OF AN ORDINARY SIMPLE ANNUITY.

R: SIZE OF PERIODIC PAYMENT.

i: INTEREST RATE PER CONVERSION PERIOD.

n: NUMBER OF PERIODIC PAYMENTS.

HOW DO WE OBTAIN THIS FORMULA?

$n=1$: FIRST PAYMENT. THERE IS AMOUNT R IN THE ACCOUNT.

$n=2$: A NEW DEPOSIT OF R IS MADE, AND INTEREST ON $n=1$ IS CALCULATED:

NEW AMOUNT IS $(1+i)R+R$.

$n=3$: NEW DEPOSIT R, AND INTEREST ON $n=2$:

$$(1+i)[(1+i)R+R]+R = R+(1+i)R+(1+i)^2R$$

$n=4$: NEW AMOUNT IS $\quad R+(1+i)R+(1+i)^2R+(1+i)^3R$

AT THE END OF n TOTAL PAYMENT PERIODS, WE HAVE

$$S = R+(1+i)R+(1+i)^2R+\dots+(1+i)^{n-1}R$$
$$= R\left[1+(1+i)+(1+i)^2+\dots+(1+i)^{n-1}\right].$$

THIS IS A FINITE GEOMETRIC SUM WITH $a=1$, $r=1+i$, SO

$$S = R\frac{a(1-r^n)}{1-r} = R\frac{1(1-(1+i)^n)}{1-(1+i)} = \boxed{R\frac{(1+i)^n-1}{i}}.$$

NOTE: A <u>PERPETUITY</u> IS AN ANNUITY WHOSE PAYMENTS CONTINUE INDEFINITELY.

COMBINATORICS

<u>THE FACTORIAL FUNCTION</u>: FOR $n \in \mathbb{N}$, THE NOTATION $n!$ (READ "n FACTORIAL") MEANS $1\cdot 2\cdot 3\cdot\ldots\cdot(n-2)(n-1)n$.

<u>EX:</u> $3! = 1\cdot 2\cdot 3 = 6$; $\quad 7! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 = 5040$.

FOR CONVENIENCE, WE DEFINE $0! = 1$.

NOTICE THAT $n! = n(n-1)! = n(n-1)(n-2)!$, ETC. AND

$$n = \frac{n!}{(n-1)!}$$

PERMUTATIONS AND COMBINATIONS

HOW MANY WAYS CAN WE SELECT r OBJECTS WITH ORDER, FROM A GROUP OF n OBJECTS?

A: FOR THE 1^{st} SELECTED OBJECTS, THERE ARE n POSSIBILITIES. FOR THE 2^{nd}, THERE REMAIN $n-1$ POSSIBILITIES, AND SO ON.

POSITION: $1 \quad 2 \quad 3 \quad \ldots \quad (r-2) \quad (r-1) \quad r$

CHOICES: $n \quad n-1 \quad n-2 \quad \quad n-(r-3) \quad n-(r-2) \quad n-(r-1)$

TOTAL NUMBER OF POSSIBILITIES: $n(n-1)(n-2)\ldots[n-(r-3)][n-(r-2)][n-(r-1)]$. ①

THIS CAN BE EXPRESSED AS A QUOTIENT OF FACTORIALS:

$$\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+3)(n-r+2)(n-r+1)\cancel{(n-r)}\cancel{(n-r-1)}\cancel{(n-r-2)}\ldots\cancel{3\cdot2\cdot1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cancel{(n-r-2)}\ldots\cancel{3\cdot2\cdot1}}$$

$$= n(n-1)(n-2)\ldots(n-r+3)(n-r+2)(n-r+1) = ①.$$

THE NUMBER OF PERMUTATIONS OF n ITEMS CHOSEN r AT A TIME IS

$$\boxed{P_r^n = \frac{n!}{(n-r)!}.}$$

IF ORDER DOES NOT MATTER, THEN THERE ARE FEWER POSSIBILITIES. THERE ARE $r!$ WAYS TO ORDER r OBJECTS, SO THE NUMBER OF COMBINATIONS OF n OBJECTS CHOSEN r AT A TIME IS $\frac{P_r^n}{r!}$:

$$\boxed{C_r^n = \frac{n!}{r!(n-r)!}}$$ "n CHOOSE r."

EXERCISE: VERIFY THAT $C_r^n = C_{n-r}^n$, AND THAT $C_0^n = C_n^n = 1$.

EXERCISE: VERIFY THAT THERE ARE $r!$ WAYS TO ORDER r OBJECTS.

EX: FIND P_2^5 AND C_2^5. WHAT DO THESE NUMBERS MEAN?

A: $P_2^5 = \dfrac{5!}{(5-2)!} = \dfrac{120}{6} = 20.$

THERE ARE 20 DIFFERENT WAYS OF CHOOSING 2 OBJECTS, WITH ORDER, FROM A SET OF 5 OBJECTS.

$$C_2^5 = \frac{5!}{2!(5-2)!} = \frac{120}{2\cdot 6} = 10.$$

THERE ARE 10 DIFFERENT WAYS OF CHOOSING 2 OBJECTS, UNORDERED, FROM A SET OF 5 OBJECTS.

C_r^n IS ALSO DENOTED $\binom{n}{r}$ AND CALLED A BINOMIAL COEFFICIENT, BECAUSE IT APPEARS IN THE BINOMIAL THEOREM:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k.$$

THIS IS THE FAST WAY TO EXPAND ANY BINOMIAL $(a+b)$ TO ANY POWER $n \in \mathbb{N}$. SOME PARTICULARLY USEFUL CASES ARE $n=2$ AND $n=3$:

$$(a+b)^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

BINOMIAL COEFFICIENTS ALSO CAN BE SEEN IN PASCAL'S TRIANGLE:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

EXAMPLES:

$$(x+1)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \dots + \binom{n}{n-1}x + \binom{n}{n}$$

$$= x^n + nx^{n-1} + \frac{n!}{2!(n-2)!}x^{n-2} + \dots + nx + 1.$$

$$(3x-2y)^4 = \binom{4}{0}(3x)^4(-2y)^0 + \binom{4}{1}(3x)^3(-2y)^1 + \binom{4}{2}(3x)^2(-2y)^2 + \binom{4}{3}(3x)(-2y)^3 + \binom{4}{4}(3x)^0(-2y \dots$$

$$= 81x^4 + 4(27x^3)(-2y) + 6(9x^2)(4y^2) + 4(3x)(-8y^3) + 16y^4$$

$$= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4.$$

$$(0.99)^3 = (1-0.01)^3$$

$$= \binom{3}{0}(1)^3(-0.01)^0 + \binom{3}{1}(1)^2(-0.01)^1 + \binom{3}{2}(1)^1(-0.01)^2 + \binom{3}{3}(1)^0(-0.01)^3$$

$$= 1 - 0.03 + 0.0003 - 0.000001$$

$= 0.97$ TO 2 DECIMAL PLACES.

FUNCTIONS

RECALL: A FUNCTION IS A RULE THAT ASSIGNS TO EACH ELEMENT OF A SET A ONE (AND ONLY ONE) ELEMENT OF A SET B.

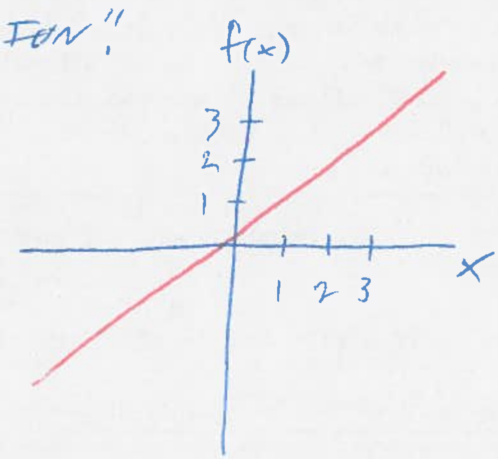
$f: A \to B$, FOR SOME $a \in A$, $f(a) = b \in B$.

A IS THE <u>DOMAIN</u> OF f, B IS THE <u>CODOMAIN</u> OF f.

EXAMPLES:

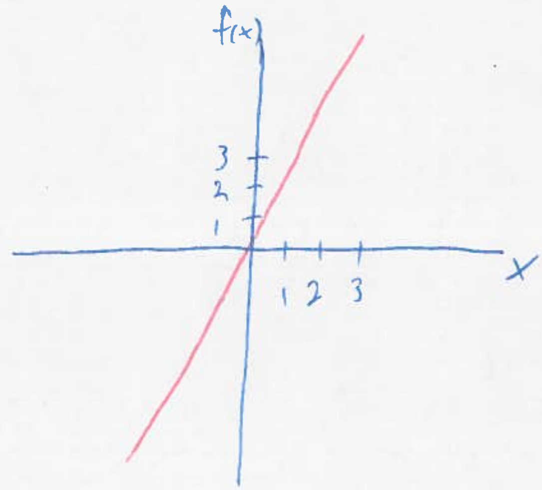
$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$.

FOR EACH $x \in \mathbb{R}$, f DELIVERS THE SAME NUMBER. THIS IS THE "IDENTITY FUNCTION".



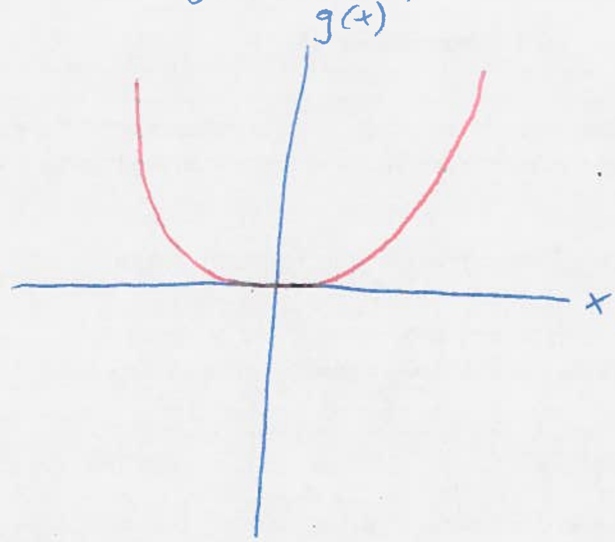
$f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$. EACH INPUT IS MULTIPLIED BY 2.

$f(1) = 2$, $f(5) = 10$, $f(0) = 0$, $f(-3) = -6$, $f(\frac{4}{5}) = \frac{8}{5}$, $f(\pi) = 2\pi$



$g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$. THIS FUNCTION SQUARES EACH INPUT.

$g\left(\frac{-1}{2}\right) = \frac{1}{4}$, $g(0) = 0$, $g(9) = 81$, ...



THE <u>RANGE</u> OF A FUNCTION IS THE SET OF ALL OUTPUTS. IT COULD BE EQUAL TO THE CODOMAIN, OR A SUBSET.

<u>EX</u>: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$. DOMAIN $\mathbb{R}$, CODOMAIN $\mathbb{R}$, RANGE $\mathbb{R}$.

$g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$. DOMAIN $\mathbb{R}$, CODOMAIN $\mathbb{R}$, RANGE $\mathbb{R}^+ \cup \{0\}$,

$= \{x \in \mathbb{R} : x \geq 0\}$.

WE CAN DEFINE A <u>SEQUENCE OF FUNCTIONS</u>, FOR EXAMPLE $f_p(x) = x^p$, $p \in \mathbb{N} \cup \{0\}$

$f_0(x) = x^0 = 1$, $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$, ...

IF p IS EVEN, THEN $\operatorname{ran} f_p = \mathbb{R}^+ \cup \{0\}$.

IF p IS ODD, THEN $\operatorname{ran} f_p = \mathbb{R}$.

IF p IS ZERO, THEN $\operatorname{ran} f_p = \{1\}$. (AND $\operatorname{dom} f_p = \mathbb{R} \setminus \{0\} = \{x \in \mathbb{R} : x \neq 0\}$.)

NEW FUNCTIONS CAN BE FORMED BY USING ARITHMETIC AND OTHER OPERATIONS ON BASIC FUNCTIONS. THE RESULTING FUNCTIONS MAY HAVE DIFFERENT DOMAINS AND/OR RANGES.

Ex: $f(x) = x$, $g(x) = x$. $\operatorname{dom} f = \operatorname{dom} g = \mathbb{R}$, $\operatorname{ran} f = \operatorname{ran} g = \mathbb{R}$.

$$h(x) = \frac{f(x)}{g(x)} = \frac{x}{x} = 1, \quad x \neq 0.$$

$\operatorname{dom} h = \mathbb{R} \setminus \{0\}$ AND $\operatorname{ran} h = \{1\}$!

POLYNOMIALS

A POLYNOMIAL FUNCTION IS A FINITE, LINEAR COMBINATION OF THE FUNCTIONS $f_p(x) = x^p$, $p \in \mathbb{N} \cup \{0\}$. THEY HAVE THE GENERAL FORM

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n = \sum_{k=0}^{n} a_k x^k,$$

WHERE THE REAL NUMBERS $a_0, a_1, \dots, a_n$ ARE THE COEFFICIENTS.

a_0: CONSTANT COEFFICIENT,

a_1: LINEAR COEFFICIENT,

a_2: QUADRATIC COEFFICIENT, ETC.

THE HIGHEST POWER OF x (THE NUMBER n) IS CALLED THE DEGREE OF THE POLYNOMIAL. UNLESS OTHERWISE STATED, $\operatorname{dom} p = \mathbb{R}$.

EXAMPLES

1) $P_0(x) = 4$. THIS IS A DEGREE-ZERO POLYNOMIAL (CONSTANT FUNCTION). $\operatorname{ran} P_0 = \{4\}$.

2) $p_1(x) = -3 - \frac{7}{4}x$. THIS IS A DEGREE-ONE POLYNOMIAL (LINEAR FUNCTION

$\operatorname{ran} p_1 = \mathbb{R}$.

3) $p_2(x) = 1 - 2x + x^2$. THIS IS A DEGREE-TWO POLYNOMIAL (QUADRATIC FUNCTI

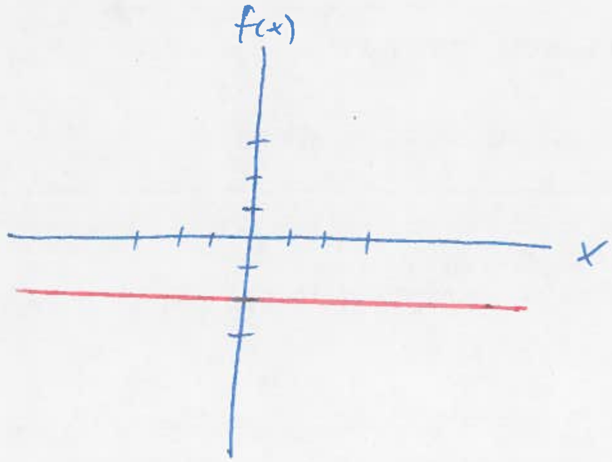
$\operatorname{ran} p_2 = \mathbb{R}^+ \cup \{0\}$. WE SEE THIS BY NOTING THAT $p_2(x) = (1-x)^2 \geq 0 \ \forall x \in \mathbb{R}$

PLOTTING FUNCTIONS

MARKING COORDINATES $(x, f(x))$ IN THE xy-plane, WE OBTAIN THE GRAPH OF FUNCTION $f: A \subseteq \mathbb{R} \to \mathbb{R}$.

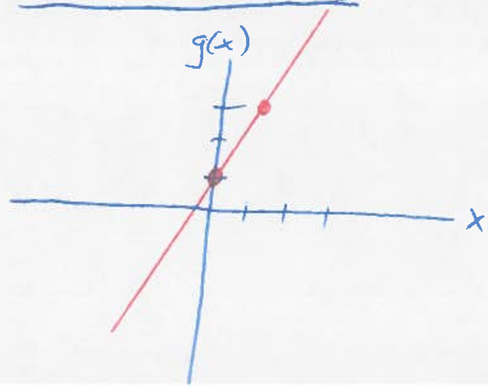
EXAMPLES

1) CONSTANT FUNCTION. $f(x) = -2$, NO MATTER WHAT THE x-VALUE IS, THE y-VALUE IS -2.



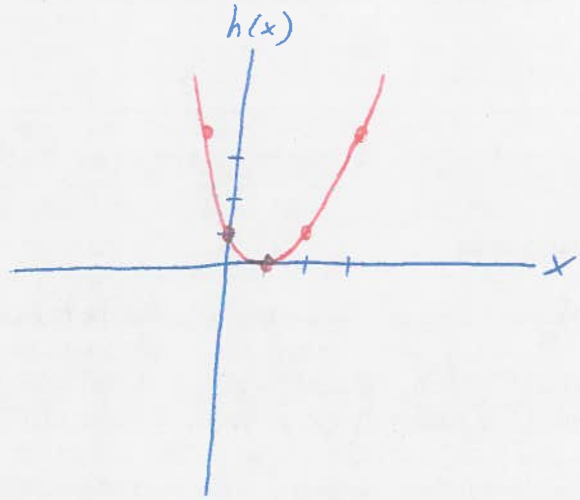
2) LINEAR FUNCTION. $g(x) = 2x + 1$. TWO POINTS IS ENOUGH TO GRAPH ANY LINEAR FUNCTION. A TABLE OF VALUES CAN BE HELPFUL:

x	$g(x)$
0	1
1	3



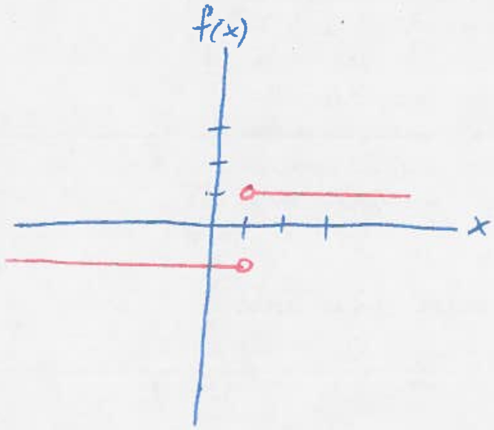
3) QUADRATIC FUNCTION. $h(x) = 1 - 2x + x^2$.

x	$h(x)$
-1	4
0	1
1	0
2	1
3	4



USE AN OPEN CIRCLE TO INDICATE THAT A POINT IS NOT INCLUDED.

EX: $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$, $f(x) = \begin{cases} -1, & x < 1 \\ 1, & x > 1 \end{cases}$



THE VERTICAL LINE TEST DETERMINES WHETHER A GRAPH IS THE GRAPH OF A FUNCTION OR NOT. IF f IS A FUNCTION, THEN EACH $x \in \operatorname{dom} f$ YIELDS ONLY ONE $y \in \operatorname{ran} f$. SO ANY VERTICAL LINE THROUGH THE GRAPH WILL INTERSECT WITH f AT ONLY ONE POINT.