MATH 221

- FORMAL LOGIC: TRUTH TABLES
  LOGICAL EQUIVALENCE
  PREDICATE LOGIC
- MODULAR ARITHMETIC
- SET THEORY: SET-BUILDER NOTATION
  INDUCTION
  FUNCTIONS
- INTRO. TO COMBINATORICS, PROBABILITY, STATISTICS.

1 LOGIC

LOGIC IS A LANGUAGE FOR REASONING. WE ARE INTERESTED IN WHETHER A STATEMENT IS TRUE OR FALSE, AND IN DETERMINING TRUTH/FALSEHOOD OF STATEMENTS FROM OTHER STATEMENTS.

DEF: A STATEMENT IS A SENTENCE THAT IS TRUE OR FALSE, BUT NOT BOTH.

WHICH ARE STATEMENTS?

a) THERE ARE 10 PEOPLE IN THE CLASS.

b) IS IT LUNCHTIME?

c) $3+4=7$.

d) $x<2$.

e) THERE EXISTS x SUCH THAT $x<2$.

f) THIS SENTENCE IS FALSE.

MUCH OF MATHEMATICS IS ABOUT PROVING A STATEMENT IS TRUE, OR DEMONSTRATING A STATEMENT IS FALSE.
(SHOWING)

Ex: a) SHOW THAT THE STATEMENT, "IF $x^2=9$, THEN $x=1$ OR $x=-1$" IS FALSE.

b) PROVE THAT THE STATEMENT, "IF $x^2=9$, THEN $x=3$ OR $x=-3$" IS TRUE

LOGICAL CONNECTIVES

CONNECTIVES ARE KEY WORDS/SYMBOLS THAT CONNECT TWO OR MORE SIMPLE STATEMENTS TO FORM NEW, LONGER ONES.

WE USE $p, q, r, \ldots$ TO DENOTE SIMPLE STATEMENTS (STATEMENT VARIABLES)

p: I LOVE MATH 221.

THERE ARE 5 CONNECTIVES:

1) NEGATION: $\sim p$. "NOT p"
2) DISJUNCTION: $p \vee q$. "p OR q"
3) CONJUNCTION: $p \wedge q$. "p AND q"
4) CONDITIONAL: $p \rightarrow q$ $(p \Rightarrow q)$ "p IMPLIES q"
5) BICONDITIONAL: $p \leftrightarrow q$ $(p \Longleftrightarrow q)$ "p IF AND ONLY IF q"

AN EXPRESSION OF SIMPLE STATEMENTS AND CONNECTIVES IS CALLED A <u>COMPOUND STATEMENT</u>. EACH SIMPLE STATEMENT HAS A <u>TRUTH VALUE</u> T FOR TRUE, F FOR FALSE. THE TRUTH VALUE OF A COMPOUND STATEMENT IS DETERMINED BY LOGIC, USING THE SIMPLE STATEMENT VALUES AND THE CONNECTIVES. WE DO THIS BY CONSTRUCTING <u>TRUTH TABLES</u>.

TRUTH TABLES

1) NEGATION: IF p IS A STATEMENT VARIABLE, THEN "NOT p", DENOTED BY $\sim p$, HAS THE OPPOSITE VALUE.

IF p IS TRUE, $\sim p$ IS FALSE.

IF p IS FALSE, $\sim p$ IS TRUE.

EX: p: IT IS RAINING NOW. $\sim p$:

q: MATH 221 IS NOT FUN. $\sim q$:

r: $x > 2$ OR $x < 2$. $\sim r$:

CAN $\sim r$ BE SIMPLIFIED?

THE TRUTH VALUES FOR NEGATION ARE SUMMARIZED IN A TABLE:

p	$\sim p$
T	
F	

NOTE: THE TRUTH TABLE ABOVE TELLS US THAT FOR ANY STATEMENT p, EXACTLY ONE OF p AND $\sim p$ IS TRUE. THIS GIVES US 2 OPTIONS FOR PROVING p IS TRUE: EITHER SHOW IT DIRECTLY, OR SHOW IT INDIRECTLY BY PROVING $\sim p$ IS FALSE (PROOF BY CONTRADICTION).

NOTE: TO CHANGE PRIORITY, USE PARENTHESES.

$\sim p \vee q$ MEANS $(\sim p) \vee q$, WHICH IS DIFFERENT FROM $\sim(p \vee q)$.

2) CONJUNCTION: IF p AND q ARE STATEMENT VARIABLES, THE CONJUNCTION IS "p AND q", DENOTED BY $p \wedge q$. IF p AND q ARE BOTH TRUE, THEN $p \wedge q$ IS TRUE. OTHERWISE, $p \wedge q$ IS FALSE.

EX: $p: x < 2$. $q: x > -1$. $p \wedge q$:

p: IT'S HOT. q: IT'S SUNNY. $p \wedge q$:

THE TRUTH TABLE FOR CONJUNCTION IS:

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

EX: WRITE THE TRUTH VALUE.

a) $3 < 5 \wedge 6 > \tilde{\pi}$

b) $3 > 5 \wedge 6 > \tilde{\pi}$

c) $1 = 2 \wedge 4 = 7$

EX: WRITE USING THE CONNECTIVE $\wedge$.

a) I LIKE ROCK AND ROLL.

b) TIGERS ARE BIG AND STRONG.

3) DISJUNCTION: IF p AND q ARE STATEMENT VARIABLES, THE DISJUNCTION IS "p OR q", DENOTED BY $p \vee q$. IF p AND q ARE BOTH FALSE, $p \vee q$ IS FALSE. OTHERWISE, $p \vee q$ IS TRUE.

EX: WRITE USING THE CONNECTIVE $\vee$.

a) I TAKE THE BUS OR TRAIN TO SCHOOL.

b) $|x| < 1$.

THE TRUTH TABLE FOR DISJUNCTION IS:

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	

NOTE: THE WORD "OR" CAN BE USED IN AN EXCLUSIVE SENSE, i.e. p OR q BUT NOT BOTH. CONSIDER THE DIFFERENCE IN MEANING:

COFFEE OR TEA?   MILK OR SUGAR?

THE EXCLUSIVE OR STATEMENT IS SOMETIMES DENOTED BY $\oplus$, BUT IT CAN BE REPRESENTED BY AND/OR/NOT SYMBOLS:

$p \oplus q$ = "p OR q, BUT NOT BOTH" =

EXERCISE: MAKE A TRUTH TABLE FOR $\oplus$.

TO MAKE A TRUTH TABLE FOR COMPOUND STATEMENTS (LIKE ⊕), WRITE THE VARIABLES, THEN THE BASIC COMBINATIONS, THEN MORE COMPLEX COMBINATIONS. IF THERE ARE n VARIABLES, THERE ARE 2^n ROWS. USE AS MANY COLUMNS AS YOU NEED.

EX: COMPLETE THE TRUTH TABLES.

a) $p \vee \sim p$

b) $\sim p \wedge q$

c) $(p \wedge q) \vee \sim r$

4) CONDITIONAL: WHEN YOU MAKE A LOGICAL INFERENCE OR DEDUCTION, YOU REASON FROM A HYPOTHESIS TO A CONCLUSION. THE STATEMENT HAS THE FORM "IF SOMETHING IS TRUE, THEN SOMETHING ELSE IS TRUE."

IF p AND q ARE STATEMENT VARIABLES, THE CONDITIONAL OF q BY p IS "IF p, THEN q" OR "p IMPLIES q", DENOTED BY $p \Rightarrow q$. IF p IS TRUE AND q IS FALSE, THEN $p \Rightarrow q$ IS FALSE. OTHERWISE, $p \Rightarrow q$ IS TRUE. p IS THE HYPOTHESIS (ANTECEDENT), q IS THE CONCLUSION (CONSEQUENT).

EX:

a) p: I WORK HARD. q: I DO WELL. $p \Rightarrow q$:

b) p: $x = 2$. q: $x^2 = 4$. $p \Rightarrow q$:

IS $q \Rightarrow p$ TRUE?

$p \Rightarrow q$ CAN BE READ IN MANY WAYS:

P IMPLIES q.	q WHENEVER p
IF p, THEN q	p IS SUFFICIENT FOR q
q IF p	q IS NECESSARY FOR p
q PROVIDED p	p ONLY IF q

EX: WRITE USING CONNECTIVES: "IF $x^2=4$, THEN $x=2$ OR $x=-2$.

THE TRUTH TABLE FOR CONDITIONAL IS!

p	q	$p \Rightarrow q$
T	T	
T	F	
F	T	
F	F	

NOTE: WHY IS $p \Rightarrow q$ TRUE WHEN p IS FALSE? IF A STATEMENT CANNOT BE SAID TO BE FALSE, THEN IT IS TRUE. IF p IS FALSE, THEN WE CANNOT SAY THAT $p \Rightarrow q$ IS FALSE, SO IT IS TRUE!

CONSIDER THE CLAIM, "IF IT RAINS, THEN I WILL GO HOME."

ONLY IF IT RAINS CAN WE MAKE A JUDGEMENT ON ITS TRUTH. IF IT DOESN'T RAIN, THEN REGARDLESS OF WHETHER OR NOT I GO HOME, WE CANNOT CLAIM THAT THE STATEMENT IS FALSE. SO IT IS TRUE.

5) BICONDITIONAL: A BICONDITIONAL STATEMENT HAS THE FORM "p IF AND ONLY IF q" OR "p IFF q". IT'S TRUE ONLY IF BOTH VARIABLES HAVE THE SAME VALUE. IT IS DENOTED BY

$p \Leftrightarrow q$, AND IS READ

p IFF q | p IS EQUIVALENT TO q

p IMPLIES AND IS IMPLIED BY q | p IS NECESSARY AND SUFFICIENT FOR q

EX: a) $p: x^3 = -8$. $q: x = -2$. $p \Leftrightarrow q$:

(NOTICE THAT $p \Leftrightarrow q$ MEANS $(p \Rightarrow q) \wedge (q \Rightarrow p)$.)

b) WRITE USING CONNECTIVES: "MICHAEL IS A BACHELOR IF AND ONLY IF HE IS MALE AND NEVER MARRIED."

THE TRUTH TABLE FOR BICONDITIONAL IS:

p	q	$p \Leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

EX: WRITE THE TRUTH VALUE.

a) $x^2 = 1 \Leftrightarrow (x = 1 \vee x = -1)$.

b) I GET WET $\Leftrightarrow$ IT IS RAINING.

EX: COMPLETE THE TABLE.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T				
T	F				
F	T				
F	F				

NOTICE THAT THE LAST TWO COLUMNS ARE THE SAME. THIS MEANS THAT $p \Leftrightarrow q$ AND $(p \Rightarrow q) \wedge (q \Rightarrow p)$ ARE LOGICALLY EQUIVALENT.

MAIN CONNECTIVES

WHEN BUILDING COMPOUND STATEMENTS, USE PARENTHESES TO AVOID AMBIGUITY. THE MAIN CONNECTIVE IS THE ONE THAT BINDS THE WHOLE STATEMENT TOGETHER. WE MUST KNOW THE RANKING OF ALL CONNECTIVES IN A STATEMENT.

EX: WHAT IS THE MAIN CONNECTIVE?

a) $(p \vee \sim q) \Rightarrow (p \wedge r)$

b) $p \Rightarrow [q \Rightarrow (r \vee \sim r)]$

c) $\sim [(p \wedge q) \vee (\sim p \wedge q)]$

TAUTOLOGY AND FALLACY

A TAUTOLOGY IS A COMPOUND STATEMENT THAT IS ALWAYS TRUE, FOR ALL VALUES OF THE BASIC STATEMENTS. (eg. $p \vee \sim p$)

A FALLACY IS A COMPOUND STATEMENT THAT IS ALWAYS FALSE, FOR ALL VALUES OF THE BASIC STATEMENTS. (eg. $p \wedge \sim p$)

ANY STATEMENT THAT IS NEITHER TAUTOLOGY NOR FALLACY IS CALLED CONTINGENT OR INTERMEDIATE.

NOTE THAT THE NEGATION OF A TAUTOLOGY IS A FALLACY, AND VICE VERSA.

EX: SHOW THAT FOR ANY STATEMENT p, $p \vee \sim p$ IS A TAUTOLOGY AND $p \wedge \sim p$ IS A FALLACY.

EX: DETERMINE WHETHER $\sim[(\sim p \wedge q) \wedge p]$ IS A TAUTOLOGY, FALLACY OR CONTINGENT STATEMENT.

"QUICK" METHOD OF IDENTIFYING TAUTOLOGIES

WITH TRUTH TABLES, 2^n ROWS ARE REQUIRED. THIS GETS BIG AND IMPRACTICAL QUICKLY (4 STATEMENTS $\Rightarrow$ 16 ROWS, 5 STATEMENTS $\Rightarrow$ 32 ROWS, ... SO THERE IS A QUICKER METHOD WE WILL SEE NOW.

NOTE: TRUTH TABLES ARE RELIABLE; IT'S NOT EASY TO MAKE MISTAKES. THE QUICK METHOD CAN BE MORE DIFFICULT IN THAT RESPECT.

IT RELIES ON THE FACT THAT IF F CAN OCCUR UNDER THE MAIN CONNECTIVE, THEN THE STATEMENT IS NOT A TAUTOLOGY. IF F IS NOT POSSIBLE, IT IS A TAUTOLOGY. THE METHOD IS:

ASSUME THE MAIN CONNECTIVE YIELDS F, THEN WORK BACKWARDS TO SEE IF A VALID COMBINATION OF VALUES EXISTS.

<u>EX</u>: IS $(p \wedge q) \Rightarrow (r \wedge s)$ A TAUTOLOGY?

PLACE AN F UNDER THE MAIN CONNECTIVE:

$$(p \wedge q) \underset{F}{\Rightarrow} (r \wedge s)$$

REMEMBER THE CONDITIONAL TABLE: FOR THIS TO HAPPEN, $p \wedge q$ MUST BE TRUE AND $r \wedge s$ MUST BE FALSE.

THEREFORE, p IS TRUE, q IS TRUE, AND EITHER r OR s MUST BE FALSE. SINCE THESE ARE PERFECTLY VALID VALUES FOR p, q, r, s, WE HAVE THAT $(p \wedge q) \Rightarrow (r \wedge s)$ IS <u>NOT</u> A TAUTOLOGY.

(eg. $p = T$, $q = T$, $r = T$, $s = F$ YIELDS $(p \wedge q) \Rightarrow (r \wedge s) = F$.)

WE SAVED OURSELVES 16 ROWS OF TRUTH TABLE.

<u>EX</u>: IS $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$ A TAUTOLOGY?

$$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \underset{F}{\Rightarrow} (p \Rightarrow r)$$

THEN $(p \Rightarrow q) \wedge (q \Rightarrow r)$ IS TRUE, AND $(p \Rightarrow r)$ IS FALSE.

SINCE $(p \Rightarrow r)$ IS FALSE, WE MUST HAVE $p = T$ AND $r = F$.

NOW SINCE $(p \Rightarrow q) \wedge (q \Rightarrow r)$ IS TRUE, WE HAVE $p \Rightarrow q$ IS TRUE AND $q \Rightarrow r$ IS TRUE. $p \Rightarrow q$ IS TRUE AND $p = T$, SO $q = T$. BUT $q \Rightarrow r$ IS TRUE AND $q = T$, SO $r = T$, A CONTRADICTION. THEREFORE, THERE IS NO WAY TO MAKE THE MAIN CONNECTIVE FALSE. THE STATEMENT IS A TAUTOLOGY.

<u>EXERCISE</u>: MAKE THE TRUTH TABLE FOR THIS STATEMENT, AND VERIFY THAT THE LAST COLUMN IS ALL T.

THIS METHOD WORKS FOR PROVING FALLACIES AS WELL.

<u>EX</u>: IS $\sim[(p \Rightarrow q) \Rightarrow (\sim p \vee q)]$ A FALLACY?

A: LET'S SEE IF IT'S POSSIBLE TO OBTAIN A T:

$$\underset{T}{\sim}[(p \Rightarrow q) \Rightarrow (\sim p \vee q)]$$

$$(p \Rightarrow q) \underset{F}{\Rightarrow} (\sim p \vee q)$$

$$\underset{T}{p \Rightarrow q} \quad \text{AND} \quad \underset{F}{\sim p \vee q}$$

$$\underset{F}{\sim p} \quad \text{AND} \quad \underset{F}{q}$$

$$p = T \quad \text{AND} \quad q = F$$

$T \Rightarrow F$ IS <u>FALSE</u>, SO IT'S NOT POSSIBLE. THE STATEMENT IS A FALLACY.

LOGICAL EQUIVALENCE

TWO STATEMENTS ARE CALLED LOGICALLY EQUIVALENT IFF THEY HAVE IDENTICAL TRUTH TABLES. THE LOGICAL EQUIVALENCE OF p AND q IS DENOTED BY $p \equiv q$. p AND q ARE LOGICALLY EQUIVALENT IFF $p \Leftrightarrow q$ IS A TAUTOLOGY.

EX: IS $p \equiv \sim(\sim p)$?

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

↑ IDENTICAL, ↑ SO YES, $p \equiv \sim(\sim p)$.

SUBSTITUTION OF EQUIVALENCE

WE CAN MAKE SUBSTITUTIONS IN STATEMENTS, USING EQUIVALENT EXPRESSIONS. THERE ARE 2 RULES.

1) RULE OF SUBSTITUTION: IF IN A TAUTOLOGY ALL OCCURRENCES OF A VARIABLE ARE REPLACED BY THE SAME STATEMENT, THE RESULT IS ANOTHER TAUTOLOGY.

   EX: $p \vee \sim p$ IS A TAUTOLOGY, SO $q \vee \sim q$ IS AS WELL, AND

   $[(p \vee q) \Rightarrow r] \vee \sim [(p \vee q) \Rightarrow r]$ IS AS WELL.

2) RULE OF SUBSTITUTION OF EQUIVALENCE: IF IN A TAUTOLOGY WE REPLACE ANY PART OF A STATEMENT BY A STATEMENT EQUIVALENT TO THAT PART, THE RESULT IS ANOTHER TAUTOLOGY.

EX: $p \equiv \sim(\sim p)$, SO THE TAUTOLOGY $p \vee \sim p$ CAN BE WRITTEN $\sim(\sim p) \vee \sim p$ AND IS STILL A TAUTOLOGY.

THIS KIND OF SUBSTITUTION OFTEN HAPPENS IN ALGEBRA. FOR EXAMPLE, WE KNOW THAT $\sin^2 x + \cos^2 x = 1$, SO THE EXPRESSION $\frac{1-\sin^2 x}{\cos x}$ CAN BE SIMPLIFIED:

$$\frac{1-\sin^2 x}{\cos x} = \frac{\cos^2 x}{\cos x} = \cos x, \ \forall x \ni \cos x \neq 0.$$

EX: $p \Rightarrow$ IS LOGICALLY EQUIVALENT TO $\sim p \vee q$. $q \Rightarrow (p \Rightarrow q)$ IS A TAUTOLOGY. PROVE THAT $s \Rightarrow (\sim r \vee s)$ IS A TAUTOLOGY.

A: $q \Rightarrow (p \Rightarrow q)$

$q \Rightarrow (r \Rightarrow q)$ (SUBSTITUTE r FOR p)

$s \Rightarrow (r \Rightarrow s)$ (SUBSTITUTE s FOR q)

$s \Rightarrow (\sim r \vee s)$ (EQUIVALENCE SUBSTITUTE $\sim r \vee s$ FOR $r \Rightarrow s$) $\square$

LOGICAL EQUIVALENCE LAWS

1) COMMUTATIVE LAWS

a) $p \vee q \equiv q \vee p$

b) $p \wedge q \equiv q \wedge p$

c) $p \Leftrightarrow q \equiv q \Leftrightarrow p$

2) ASSOCIATIVE LAWS

a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

c) $(p \Leftrightarrow q) \Leftrightarrow r \equiv p \Leftrightarrow (q \Leftrightarrow r)$

3) DISTRIBUTIVE LAWS

a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

c) $p \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$

d) $p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$

4) DOUBLE NEGATION LAW

$$\sim\sim p \equiv p$$

5) DEMORGAN'S LAWS

a) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

b) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

6) IMPLICATION LAWS

a) $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

b) $p \Rightarrow q \equiv \sim p \vee q$

c) $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

d) $\sim(p \Rightarrow q) \equiv p \wedge \sim q$

Ex: TO UNDERSTAND DEMORGAN'S LAWS, WRITE NEGATIONS OF THESE:

1) JOHN IS 6 FEET TALL AND HE WEIGHS AT LEAST 200 POUNDS.

2) THE BUS WAS LATE OR TOM'S WATCH WAS SLOW.

EX: PROVE DEMORGAN'S LAWS USING TRUTH TABLES.

EX: IS $(p \wedge \sim q) \wedge (\sim p \vee q)$ A TAUTOLOGY OR A FALLACY?

EX: IS $(p \Leftrightarrow q) \Leftrightarrow (\sim p \Leftrightarrow q)$ A TAUTOLOGY OR A FALLACY?

EX: PROVE THE EQUIVALENCE $(p \Rightarrow q) \Rightarrow r \equiv [(\sim p \Rightarrow r) \wedge (q \Rightarrow r)]$.

A: LET ✪ BE THE TAUTOLOGY $s \Rightarrow t \equiv \sim s \vee t$.

$$
\begin{aligned}
(p \Rightarrow q) \Rightarrow r &\equiv (\sim p \vee q) \Rightarrow r && ✪ \\
&\equiv \sim(\sim p \vee q) \vee r && ✪ \\
&\equiv (\sim\sim p \wedge \sim q) \vee r && \text{DEMORGAN} \\
&\equiv (p \wedge \sim q) \vee r && \text{DOUBLE NEGATIVE} \\
&\equiv (p \vee r) \wedge (\sim q \vee r) && \text{DISTRIBUTIVE} \\
&\equiv (\sim\sim p \vee r) \wedge (\sim q \vee r) && \text{DOUBLE NEGATIVE} \\
&\equiv (\sim p \Rightarrow r) \wedge (q \Rightarrow r) && ✪ \text{ TWICE.}
\end{aligned}
$$

$\square$

PREDICATE LOGIC

THE CONNECTIVES $\sim, \wedge, \vee, \Rightarrow, \Leftrightarrow$ ARE NOT ENOUGH TO PROVE OR DISPROVE ALL TYPES OF LOGICAL STATEMENTS. FOR EXAMPLE, THE ARGUMENT:

1) ALL UOW MATH COURSES ARE FUN,
2) MATH 121 IS A UOW MATH COURSE,
3) THEREFORE, MATH 121 IS FUN,

IS CORRECT, BUT WE CANNOT DETERMINE ITS VALIDITY WITH THE TOOLS WE HAVE SO FAR. WE NEED TO BE ABLE TO MANAGE WORDS SUCH AS "ALL" AND "SOME."

A **PREDICATE** IS A SENTENCE THAT CONTAINS A FINITE NUMBER OF VARIABLES AND BECOMES A STATEMENT WHEN VALUES ARE SUBSTITUTED. THE **DOMAIN** OF A VARIABLE IS THE SET OF ALL POSSIBLE VALUES IT CAN BE. THE **TRUTH SET** IS THE SUBSET OF THE DOMAIN THAT MAKES THE PREDICATE TRUE. PREDICATES OF ONE VARIABLE ARE DENOTED BY $p(x), q(x)$, ETC.

NOTATION

$\mathbb{R}$: THE SET OF ALL REAL NUMBERS.

$\mathbb{Q}$: THE SET OF RATIONAL NUMBERS (CAN BE WRITTEN AS A FRACTION).

$\mathbb{Z}$: THE SET OF INTEGERS (WHOLE NUMBERS) $\ldots -2, -1, 0, 1, 2, \ldots$

$\mathbb{N}$: THE SET OF NATURAL NUMBERS $1, 2, 3, \ldots$

$\in$: CONTENTION. "IS CONTAINED IN", "BELONGS TO", "IS A MEMBER OF"

$\forall$: UNIVERSAL QUANTIFIER. "FOR ALL"

$\exists$: EXISTENTIAL QUANTIFIER. "THERE EXISTS"

EX: THE PREDICATE $p(x)$: "x IS A POSITIVE INTEGER STRICTLY LESS THAN 5" WITH $\operatorname{dom} p = \mathbb{Z}$ HAS TRUTH SET $\{\ldots, -2, -1, 0, 1, 2, 3, 4\}$.

EX: THE PREDICATE $q(x)$: "$x^2 > x$" WITH $\operatorname{dom} q = \mathbb{R}$ HAS TRUTH SET

$$\{x : x^2 > x\} = \{x : x < -1 \text{ OR } x > 1\}$$
$$= \{x : |x| > 1\}$$
$$= (-\infty, -1) \cup (1, \infty).$$

THE UNIVERSAL QUANTIFIER $\forall$

ONE WAY TO CHANGE A PREDICATE INTO A STATEMENT IS TO ASSIGN VALUES TO THE VARIABLES. ANOTHER WAY IS TO ADD QUANTIFIERS.

EX: "ALL HUMANS ARE MORTAL."

$\forall$ HUMANS x, x IS MORTAL.

"ALL REAL NUMBERS HAVE A NONNEGATIVE SQUARE."

$\forall x \in \mathbb{R}, x^2 \geq 0.$

A UNIVERSAL STATEMENT HAS THE FORM $\forall x \in D, p(x)$. IT IS TRUE IFF $p(x)$ IS TRUE FOR EVERY $x \in D$. IF AT LEAST ONE $x \in D$ CAN BE FOUND THAT MAKES $p(x)$ FALSE, THE STATEMENT IS FALSE. SUCH AN x IS CALLED A COUNTEREXAMPLE.

EX: $\forall x \in \mathbb{R}, x^2 > x$. FALSE. COUNTEREXAMPLE: $x = \frac{1}{2}$.

EX: WRITE USING $\forall$.

a) ALL DOGS ARE ANIMALS.

b) EVERY INTEGER GREATER THAN ZERO HAS A PRIME FACTOR. (TRUE?)

EX: LET $D = \{1, 2, 3, 4, 5\}$. SHOW THAT THE STATEMENT $\forall x \in D, x^2 \geq x$ IS TRUE

SHOW THAT THE STATEMENT $\forall x \in D, \frac{1}{x^2} < \frac{1}{x}$ IS FALSE.

THE EXISTENTIAL QUANTIFIER $\exists$

EX: "THERE IS A CAT IN MY HOUSE."

$\exists$ A CAT $x \ni x$ IS IN MY HOUSE.

"THERE ARE INTEGERS m AND n SUCH THAT $m+n = mn$."

$\exists\, m, n \in \mathbb{Z} \ni m+n = mn.$

AN **EXISTENTIAL STATEMENT** HAS THE FORM $\exists x \in D \ni p(x)$. IT IS TRUE IFF $p(x)$ IS TRUE FOR AT LEAST ONE $x \in D$. IT IS FALSE IFF $p(x)$ IS FALS FOR ALL $x \in D$.

EX: WRITE USING $\exists$.

a) THERE EXISTS A REAL NUMBER WHOSE SQUARE IS NEGATIVE.

b) SOME PERSON IS A VEGETARIAN.

EX: SHOW THAT THE STATEMENT "$\exists m \in \mathbb{Z} \ni m^2 = m$" IS TRUE.

EX: LET $E = \{5, 6, \ldots, 10\}$. SHOW THAT THE STATEMENT "$\exists m \in E \ni m^2 = m$" IS FALSE.

EXERCISE: REWRITE USING INFORMAL LANGUAGE.

a) $\forall x \in \mathbb{R}, x^2 \geq 0$

b) $\exists m \in \mathbb{Z} \ni m^2 = m$

c) $\forall$ STUDENTS S, $\exists$ MATH SUBJECT $y \ni S$ LIKES y.

d) $\forall x \in \mathbb{R}, x^2 \neq -1$.

NEGATION OF QUANTIFIERS

CONSIDER THE STATEMENT "ALL MATHEMATICIANS WEAR GLASSES." WHAT IS THE NEGATION OF THE STATEMENT? IT IS NATURAL TO THINK IT'S "NO MATHEMATICIAN WEARS GLASSES," BUT THAT'S NOT CORRECT. THE NEGATION IS "THERE EXISTS A MATHEMATICIAN WHO DOESN'T WEAR GLASSES." IF JUST ONE COUNTEREXAMPLE CAN BE FOUND, THE ORIGINAL STATEMENT IS FALSE.

THM: (NEGATION OF A UNIVERSAL STATEMENT).

THE NEGATION OF THE STATEMENT

$$\forall x \in D, P(x)$$

IS LOGICALLY EQUIVALENT TO THE STATEMENT

$$\exists x \in D \ni \sim p(x).$$

SYMBOLICALLY,

$$\sim(\forall x \in D, p(x)) \equiv \exists x \in D \ni \sim p(x).$$

EX: WRITE NEGATIONS.

a) NO COMPUTER HACKER IS OVER 40.

b) $\forall$ PRIMES p, p IS ODD.

c) $\forall$ PEOPLE x, IF x IS BLONDE, THEN x HAS BLUE EYES.

NEGATION OF EXISTENTIAL STATEMENTS

CONSIDER THE STATEMENT "SOME FISH BREATHE AIR." WHAT IS THE NEGATION OF THIS STATEMENT? IT IS "NO FISH BREATHES AIR." YOU MIGHT THINK IT SHOULD BE "SOME FISH DO NOT BREATHE AIR", BUT THIS AND THE ORIGINAL STATEMENT CAN BOTH BE TRUE AT THE SAME TIME.

THM: (NEGATION OF AN EXISTENTIAL STATEMENT).

THE NEGATION OF THE STATEMENT

$$\exists x \in D \ni p(x)$$

IS LOGICALLY EQUIVALENT TO THE STATEMENT

$$\forall x \in D, \sim p(x).$$

SYMBOLICALLY,

$$\sim(\exists x \in D \ni p(x)) \equiv \forall x \in D, \sim p(x).$$

EX: WRITE NEGATIONS.

a) $\exists$ A TRIANGLE WHOSE SUM OF ANGLES IS $200°$.

b) THERE IS A WOMAN WHO IS 120 YEARS OLD.

c) $\exists x \in \mathbb{R} \ni x^2 = -1$.

IN SUMMARY, THE NEGATION OF "ALL ARE" IS "AT LEAST ONE IS NOT". THE NEGATION OF "AT LEAST ONE IS" IS "ALL ARE NOT".

EX: WRITE NEGATIONS AND DECIDE WHICH STATEMENTS ARE TRUE.

a) $\exists x \in \mathbb{R} \ni 3x = 1$.

b) $\forall \varepsilon \in \mathbb{R}, \forall x \in \mathbb{Z}, \exists y \in \mathbb{Q} \ni \varepsilon > 0 \Rightarrow |x - y| < \varepsilon$.

METHODS OF PROOF

ARGUMENTS

CONSIDER THE SEQUENCE OF STATEMENTS.

IF x IS A PIG, THEN x IS PINK.

PEPPA IS A PIG.

THEREFORE, PEPPA IS PINK.

AN ARGUMENT IS A SEQUENCE OF STATEMENTS, ALL BUT THE FINAL OF WHICH ARE CALLED ASSUMPTIONS/PREMISES/HYPOTHESES, AND THE FINAL OF WHICH IS CALLED THE CONCLUSION. THE WORD "THEREFORE" IS NORMALLY PLACED JUST BEFORE THE CONCLUSION.

THE LOGICAL FORM OF THE ABOVE ARGUMENT IS

IF p, THEN q.

p.

THEREFORE, q.