DEF: THE CARTESIAN PRODUCT OF A AND B, DENOTED BY $A \times B$, IS THE SET OF ALL ORDERED PAIRS (a,b) WITH $a \in A$, $b \in B$:

$$A \times B = \{(a,b) : a \in A \wedge b \in B\}.$$

EX: LET $A = B = \mathbb{R}$. WHAT IS $A \times B$?

EX: LET $A = \{3\}$, $B = \{2,3\}$. WHAT IS $A \times B$?

EX: LET $A = \{x,y\}$, $B = \{1,2,3\}$, $C = \{a,b\}$. WHAT ARE $A \times B$ AND $(A \times B) \times C$?

EX: LET $A = \{1,2\}$, $B = \{\pi, e\}$. IS $A \times B = B \times A$?

RELATIONS

DEF: WE SAY THAT R IS A (BINARY) RELATION FROM A TO B IF R IS A SUBSET OF $A \times B$. IF $R \subseteq A \times A$, THEN R IS CALLED A RELATION ON A. WE SAY THAT a IS RELATED TO b BY R IF $(a,b) \in R$. THIS IS DENOTED BY aRb.

EX: LET $X = \{0,1,2,3\}$, $R = \{(x,y) : \exists z \in \mathbb{N} \ni x + z = y\}$

a) WHAT IS AN EASIER WAY OF EXPRESSING R?

$\{(x,y) : y - x \in \mathbb{N}\}$; $\{(x,y) : y > x\}$.

b) LIST ALL THE ELEMENTS OF R.

c) SKETCH $X \times X$ AND CIRCLE THE ELEMENTS OF R.

Ex: LET R ON $\mathbb{Z}\setminus\{0\}$ BE GIVEN BY

$$R=\{(x,y): \exists z\in\mathbb{Z} \ni xz=y\}.$$

a) DESCRIBE THE RELATION R.

IT'S THE SET $\{(x,y): x$ IS A DIVISOR OF $y\}$.

b) TRUE OR FALSE?

$(2,-4)\in R$

$-3 R 0$

$(3,5)\in R$

Ex: LET R ON $\mathbb{Z}$ BE GIVEN BY

$$R=\{(m,n): m-n \text{ IS EVEN}\}$$

a) GIVE ANOTHER DESCRIPTION OF R.

$R=\{(m,n): m$ AND n HAVE THE SAME PARITY.$\}$

b) WHICH ARE ELEMENTS OF R?

$(0,3)$ $(-5,-6)$ $(2,-11)$ $(17,1)$

c) PROVE THAT n odd $\Rightarrow nR1$.

UNION AND INTERSECTION OF RELATIONS

RELATIONS ARE SETS, SO THE SET OPERATORS APPLY.

EX: LET R_1, R_2 ON $\mathbb{R}$ BE GIVEN BY

$$R_1 = \{(x,y) : x = y\}, \ R_2 = \{(x,y) : x = -y\}.$$

WRITE EXPRESSIONS FOR $R_1 \cup R_2$ AND $R_1 \cap R_2$.

DEF: LET R BE A RELATION FROM A TO B. THE DOMAIN OF R AND THE RANGE OF R, DENOTED RESPECTIVELY BY $\operatorname{dom} R$ AND $\operatorname{ran} R$, ARE DEFINED

$$\operatorname{dom} R = \{x : \exists y \ni xRy\};$$

$$\operatorname{ran} R = \{y : \exists x \ni xRy\}.$$

NOTE THAT $\operatorname{dom} R \subseteq A$ AND $\operatorname{ran} R \subseteq B$.

EX: LET $A = \{0,1,2,3\}$, $R = \{(0,0), (0,1), (0,2), (3,0)\}$. WRITE $\operatorname{dom} R$ AND $\operatorname{ran} R$.

EX: FIND DOMAIN AND RANGE OF R ON $\mathbb{Z} \times \mathbb{Q}$, $R = \{(x,y) : x \neq 0 \wedge y = \frac{1}{x}\}$

EX: FIND DOMAIN AND RANGE OF R ON $\mathbb{Z}$, $R = \{(x,y) : xy \neq 0\}$.

THE INVERSE OF A RELATION

IF R IS ON $A \times B$, THEN A RELATION R^{-1} ON $B \times A$ CAN BE DEFINED BY INTERCHANGING THE ELEMENTS OF THE ORDERED PAIRS OF R.

DEF: LET R BE ON $A \times B$. THE INVERSE RELATION OF R IS

$$R^{-1} = \{(y,x) \in B \times A : (x,y) \in R\}.$$

NOTE THAT $\operatorname{dom} R^{-1} = \operatorname{ran} R$ AND $\operatorname{ran} R^{-1} = \operatorname{dom} R$.

<u>EX</u>: LET $A=\{a,b,c\}$, $B=\{1,2,3,4\}$, $R=\{(a,1),(b,2),(c,3),(a,4)\}$

THEN $R^{-1}=$

<u>EX</u>: DEFINE R ON $\mathbb{N}$ BY $R=\{(x,y): y=2x\}$. WRITE 3 ELEMENTS OF R AND 3 OF R^{-1}. WRITE A DEFINITION OF R^{-1}.

<u>EX</u>: THE IDENTITY RELATION ON $\mathbb{R}$ IS $R=\{(x,x): x\in\mathbb{R}\}$. WHAT IS R^{-1}?

PROPERTIES OF RELATIONS

LET R BE A RELATION ON A. THEN

1) R IS <u>REFLEXIVE</u> ON A IFF $\forall x\in A$, $(x,x)\in R$;

2) R IS <u>SYMMETRIC</u> ON A IFF $\forall x,y\in A$, $(x,y)\in R \Rightarrow (y,x)\in R$.

3) R IS <u>TRANSITIVE</u> ON A IFF $\forall x,y,z\in A$, $(x,y)\in R \wedge (y,z)\in R \Rightarrow (x,z)\in R$.

<u>EX</u>: WHICH PROPERTIES DO THE FOLLOWING RELATIONS SATISFY?

a) ON $\mathbb{N}$, $R=\{(x,y): x \text{ IS A FACTOR OF } y\}$.

b) ON $\mathbb{R}$, THE IDENTITY RELATION.

c) ON $\mathbb{Z}$, $R=\{(x,y): x<y\}$.

d) ON $\mathbb{R}$, $R=\{(x,y): y=x^2\}$.

e) ON THE SET OF ALL PEOPLE, $R=\{(x,y): x \text{ IS IN THE FAMILY OF } y\}$.

f) ON THE SET OF ALL PEOPLE, $R=\{(x,y): x \text{ LOVES } y\}$.

EQUIVALENCE RELATIONS

DEF: LET R BE A RELATION ON A. THEN R IS AN EQUIVALENCE RELATION ON A IFF R IS REFLEXIVE, SYMMETRIC AND TRANSITIVE ON A.

EX: PROVE OR DISPROVE THAT THE IDENTITY RELATION ON $\mathbb{R}$ IS AN EQUIVALENCE RELATION.

EX: ON $\mathbb{Z}$, PROVE THAT $R = \{(a,b) : a \equiv b \pmod{n}\}$ IS AN EQUIVALENCE RELATION.

TO DISPROVE AN EQUIVALENCE RELATION, YOU ONLY NEED TO SHOW THAT ONE OF THE PROPERTIES DOES NOT HOLD.

EX: ON $\mathbb{Z}$, PROVE THAT $R = \{(a,b) : ab = 0\}$ IS NOT AN EQUIVALENCE RELATION.

EQUIVALENCE CLASSES

DEF: LET R BE AN EQUIVALENCE RELATION ON A. FOR EACH $a \in A$, THE EQUIVALENCE CLASS OF a, DENOTED BY $[a]$, IS THE SET $[a] = \{x \in A : x R a\}$. EQUIVALENCE CLASSES HAVE THE FOLLOWING PROPERTIES.

1) FOR ANY $a, b \in A$, WE HAVE EITHER $[a] = [b]$ OR $[a] \cap [b] = \phi$.

2) ALL DISTINCT EQUIVALENCE CLASSES FORM A PARTITION OF A: THE UNION OF ALL CLASSES IS A, AND THE INTERSECTION OF ANY 2 CLASSES IS EMPTY.

EX: LET $A=\{0,1,2\}$, $R=\{(0,0),(1,1),(2,2),(0,1),(1,0)\}$. FIND $[0],[1],[2]$.

EX: WHAT DO THE EQUIVALENCE CLASSES OF THE IDENTITY RELATION ON $\mathbb{R}$ LOOK LIKE?

EX: LET R ON $\mathbb{Z}$ BE DEFINED BY $R=\{(a,b): a \equiv b \pmod 3\}$. FIND $[0],[1],[2]$.

FUNCTIONS

DEF: A RELATION F FROM A TO B IS A FUNCTION FROM A TO B IFF

1) $\operatorname{dom} F = A$, AND
2) FOR EACH $x \in A$ THERE IS AT MOST ONE $y \in B$ SUCH THAT $(x,y) \in F$.

THEN B IS THE CODOMAIN OF F.

A FUNCTION FROM A TO B IS DENOTED BY $f: A \to B$. THE EQUATION $y=f(x)$ MEANS $(x,y)\in f$. IN THAT CASE, y IS THE IMAGE OF x UNDER f.

RELATIONS ON $\mathbb{R}$ CAN BE PLOTTED BY DRAWING ALL THE POINTS. SUCH RELATIONS ARE FUNCTIONS IF THEY SATISFY THE VERTICAL LINE TEST: EVERY VERTICAL LINE CUTS THE GRAPH AT MOST ONCE.

EX: SKETCH THE RELATIONS AND DETERMINE WHICH ARE FUNCTIONS.

a) ON $\mathbb{R}$, $R=\{(x,y): y=x^2\}$
b) ON $\mathbb{R}$, $R=\{(x,y): x=y^2\}$
c) ON $\mathbb{R}_+ = \{x\in\mathbb{R}: x\geq 0\}$, $R=\{(x,y): x=y^2\}$
d) ON $\mathbb{R}$, $R=\{(x,y): y=\sqrt{x}\}$.

EX: WHICH ARE FUNCTIONS?

a) THE IDENTITY RELATION ON $A=\{1,5,10\}$.

b) $A=\{2,4,6\}$, $B=\{1,3,5\}$, R ON $A\times B$, $R=\{(x,y): x+1=y\}$.

c) ON $\mathbb{Z}$, $F=\{(x,y): x+1=y\}$

d) ON $\mathbb{R}$, $R=\{(x,y): y=1\}$.

DEF: LET $f: A \to B$ BE A FUNCTION. WE SAY THAT f IS ONE-TO-ONE (INJECTIVE) IFF FOR ALL $x_1, x_2 \in A$,

$$f(x_1)=f(x_2) \Rightarrow x_1=x_2.$$

THAT IS, EACH ELEMENT OF THE RANGE IS THE IMAGE OF ONLY ONE ELEMENT OF THE DOMAIN.

A ONE-TO-ONE FUNCTION SATISFIES THE HORIZONTAL LINE TEST.

EX: LET $A=\{0,1,2,3\}$, $f: \mathcal{P}(A) \to \mathbb{N}$,

$f(A_i)$ IS THE NUMBER OF ELEMENTS IN A_i.

PROVE OR DISPROVE THAT f IS ONE-TO-ONE.

EX: WHICH ARE ONE-TO-ONE?

a) ON $A=\{1,2,3\}$, $F=\{(1,2),(2,3),(3,1)\}$.

b) ON $A=\{1,2,3\}$, $F=\{(1,2),(2,1),(3,1)\}$.

c) ON $\mathbb{Z}$, $F=\{(x,y): y=2x\}$

d) ON $\mathbb{Z}\setminus\{0\} \to \mathbb{R}$, $F=\{(x,y): y=\sqrt{x^2-1}\}$.

DEF: A FUNCTION $f: A \to B$ IS ONTO (SURJECTIVE) IFF $\operatorname{ran} f = B$.

THAT IS, FOR ALL $y \in B$, THERE EXISTS $x \in A$ SUCH THAT $f(x) = y$.

EX: LET $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d\}$. WHICH ARE ONTO?

a) $f: A \to B$, $f = \{(1,a), (2,c), (3,c), (4,d), (5,d)\}$.

b) $f: A \to B$, $f = \{(1,a), (2,b), (3,c), (4,d), (5,a)\}$.

c) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 4x - 1$.

d) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = 4x - 1$.

THM: THE INVERSE OF A FUNCTION f, WRITTEN f^{-1}, IS ALSO A FUNCTION IFF f IS ONE-TO-ONE AND ONTO (BIJECTIVE).

EX: SKETCH $f: \mathbb{R}_+ \to \mathbb{R}$, $f = \{(x, y) : y = x^2\}$. FIND AND SKETCH f^{-1}. IS f^{-1} A FUNCTION?

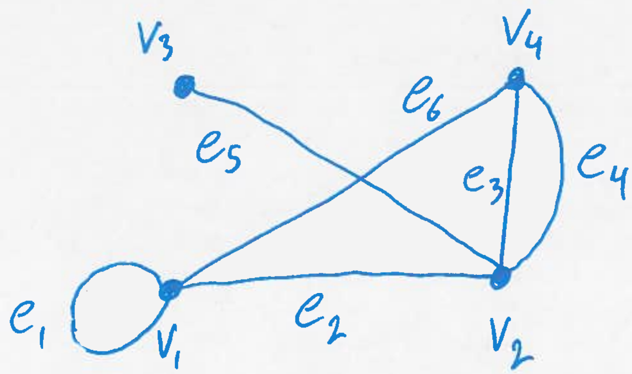
GRAPH THEORY

MANY REAL-WORLD PROBLEMS CONCERN OBJECTS AND RELATIONS, eg PEOPLE WITH FRIENDSHIPS, CITIES CONNECTED BY HIGHWAYS, WEB PAGES LINKED TO OTHERS, ETC. THE MATHEMATICAL ABSTRACTION OF THESE SITUATIONS IS THE STUDY OF GRAPH THEORY. A GRAPH IS A COLLECTION OF POINTS AND CURVES.

DEF: A **GRAPH** G CONSISTS OF A PAIR OF FINITE SETS: A NONEMPTY SET V OF **VERTICES** AND A SET E OF **EDGES**, WHERE EACH EDGE IS ASSOCIATED TO A SUBSET OF V OF EITHER 1 OR 2 VERTICES, CALLED THE **ENDPOINTS** OF THE EDGE.

- AN EDGE WITH JUST ONE ENDPOINT IS CALLED A **LOOP**.
- 2 EDGES WITH THE SAME ENDPOINTS ARE CALLED **PARALLEL EDGES**.
- AN EDGE IS SAID TO **CONNECT** ITS END POINTS AND BE **INCIDENT** ON EACH ENDPOINT. A VERTEX ON WHICH NO EDGES ARE INCIDENT IS CALLED **ISOLATED**. 2 VERTICES CONNECTED BY AN EDGE ARE CALLED **ADJACENT**.

EX: WRITE DOWN V AND E FOR THE FOLLOWING GRAPH. LIST ANY LOOPS AND PARALLEL EDGES.



EX: DRAW A GRAPH THAT HAS 5 VERTEX INCLUDING 1 ISOLATED, 1 LOOP AND 1 PAIR OF PARALLEL EDGES.

A SIMPLE GRAPH IS ONE THAT DOES NOT HAVE LOOPS NOR PARALLEL EDGES.

EX: DRAW A SIMPLE GRAPH WITH $V = \{u, v, w, x\}$ AND 2 EDGES, ONE OF WHICH HAS ENDPOINTS u AND v.

A COMPLETE GRAPH ON n VERTICES, DENOTED BY K_n, IS A SIMPLE GRAPH WITH n VERTICES WHOSE EDGE SET CONTAINS EXACTLY ONE EDGE FOR EVERY PAIR OF DISTINCT VERTICES.

EX: DRAW K_1, K_2, K_3, K_4, K_5.

A COMPLETE BIPARTITE GRAPH ON (m, n) VERTICES, DENOTED BY $K_{m,n}$, IS A SIMPLE GRAPH WITH $V = \{v_1, \dots, v_m, w_1, \dots, w_n\}$ SUCH THAT FOR ALL $1 \leq i, k \leq m$ AND ALL $1 \leq j, \ell \leq n$, WE HAVE

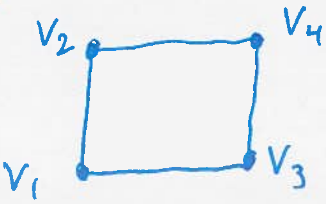
1) AN EDGE FROM EACH v_i TO EACH w_j;
2) NO EDGE FROM ANY v_i TO ANY OTHER v_k;
3) NO EDGE FROM ANY w_j TO ANY OTHER w_ℓ.

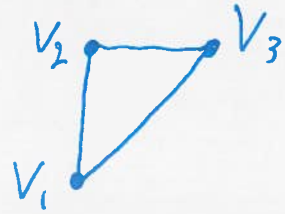
EX: DRAW $K_{3,2}$, $K_{3,3}$.

A SIMPLE GRAPH IS BIPARTITE IF THERE EXIST $U \subseteq V$ AND $W \subseteq V$ SUCH THAT

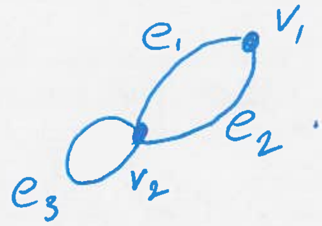
1) $U \cup W = V$ AND $U \cap W = \emptyset$;
2) EVERY EDGE CONNECTS A VERTEX OF U WITH A VERTEX OF W.

EX: WHICH IS BIPARTITE?

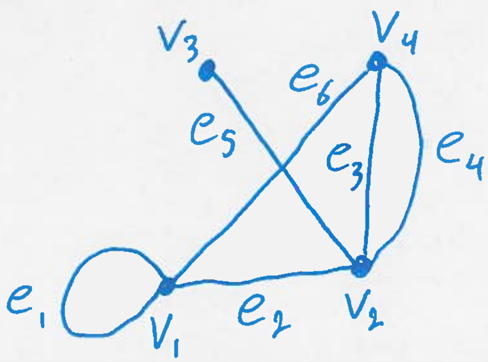




A GRAPH H IS A <u>SUBGRAPH</u> OF A GRAPH G IF EVERY VERTEX IN H IS IN G, EVERY EDGE IN H IS IN G, AND EVERY EDGE IN H HAS THE SAME ENDPOINTS IN G.

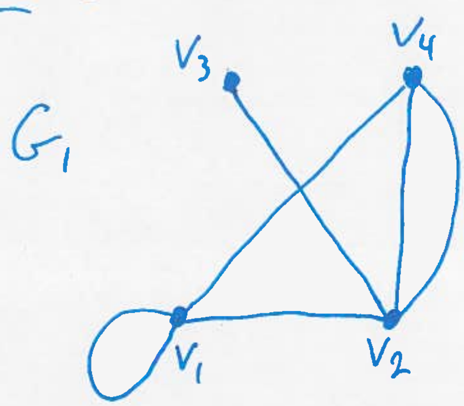
<u>EX</u>: DRAW ALL THE SUBGRAPHS OF .

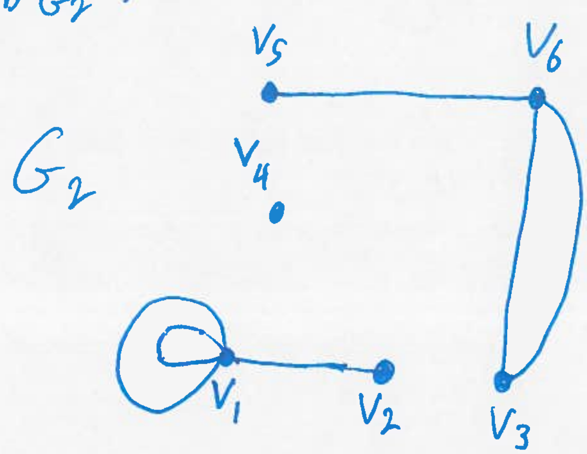
<u>EX</u>: DRAW 2 SUBGRAPHS CONTAINING 2 VERTICES EACH, ONE SIMPLE AND ONE NOT.



<u>DEF</u>: LET G BE A GRAPH, $v \in V$. THE <u>DEGREE</u> OF v, DENOTED BY $\delta(v)$, IS THE NUMBER OF EDGES INCIDENT ON v (WITH LOOPS COUNTED TWICE). THE DEGREE OF G IS THE SUM OF DEGREES OF ALL $v \in V$.

<u>EX</u>: FIND THE DEGREE OF G_1 AND G_2:





EX: DRAW GRAPHS WITH $|V|=4$ AND VERTICES OF DEGREE

a) 1, 1, 3, 3;

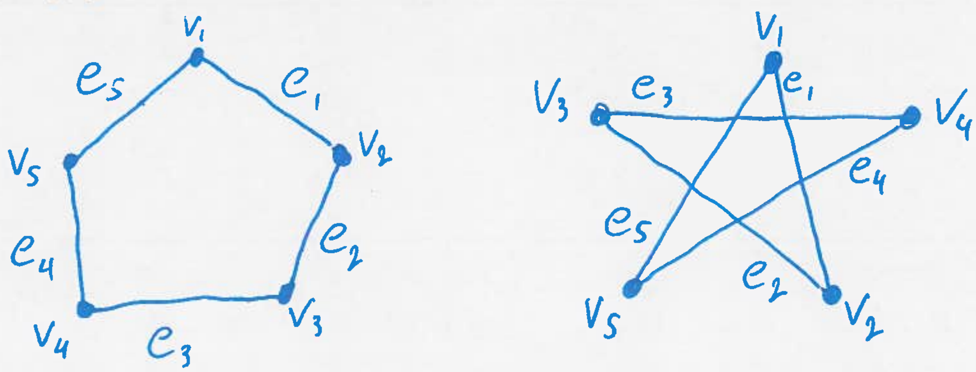
b) 1, 1, 2, 3.

THE HAND SHAKE THEOREM: THE DEGREE OF A GRAPH IS TWICE THE NUMBER OF ITS EDGES.

THIS HOLDS BECAUSE EACH EDGE ALWAYS HAS 2 ENDPOINTS. SO THE DEGREE OF A GRAPH IS ALWAYS EVEN, AND A GRAPH WITH 4 VERTICES OF DEGREE 1, 1, 2, 3 IS IMPOSSIBLE.

ISOMORPHIC GRAPHS

IS THERE ANY DIFFERENCE BETWEEN THESE GRAPHS?

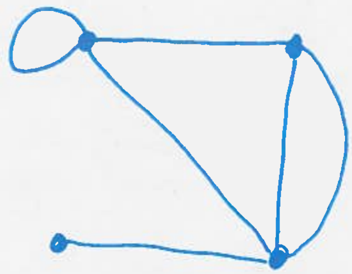


DEF: LET G, G' BE GRAPHS $G=(V,E)$, $G'=(V',E')$. WE SAY G IS ISOMORPHIC TO G' IF THERE EXIST BIJECTIVE FUNCTIONS $f: V \to V'$, $h: E \to E'$ THAT PRESERVE ADJACENCY, i.e.

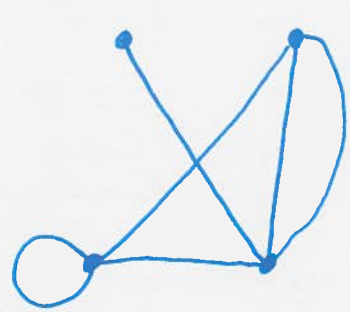
v IS AN ENDPOINT OF $e \iff f(v)$ IS AN ENDPOINT OF $h(e)$.

EX: SHOW THAT G AND G' ARE ISOMORPHIC:

G:



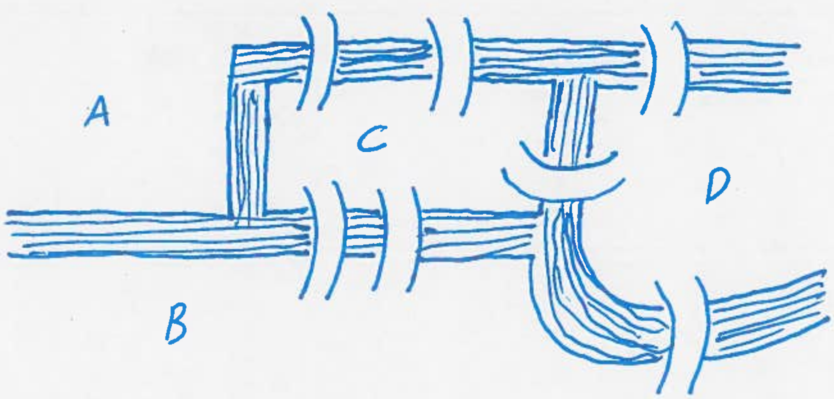
G':



EX: DRAW ALL POSSIBLE GRAPHS (UP TO ISOMORPHISM) WITH $|V|=|E|=2$.

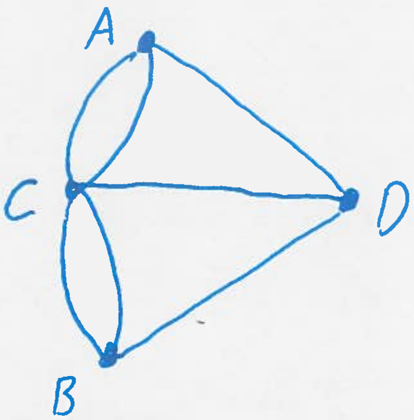
THE KÖNIGSBERG BRIDGE PROBLEM

IN 1736, LEONHARD EULER INTRODUCED GRAPH THEORY BY SOLVING THE FOLLOWING PROBLEM.



IS IT POSSIBLE FOR SOMEONE TO WALK IN KÖNIGSBERG, STARTING AND ENDING AT THE SAME POINT, AND CROSSING EACH OF 7 BRIDGES EXACTLY ONCE?

THIS CAN BE TRANSLATED TO A GRAPH: BRIDGES ARE EDGES AND REGIONS A, B, C, D ARE VERTICES.



IS IT POSSIBLE TO FIND A ROUTE THROUGH THE GRAPH THAT STARTS AND ENDS AT A VERTEX AND TRAVERSES EACH EDGE EXACTLY ONCE?

WALKS, PATHS AND CIRCUITS

- A WALK FROM VERTEX V TO VERTEX W IN G IS A FINITE ALTERNATING SEQUENCE OF ADJACENT VERTICES AND EDGES OF G THAT STARTS AT V AND ENDS AT W:

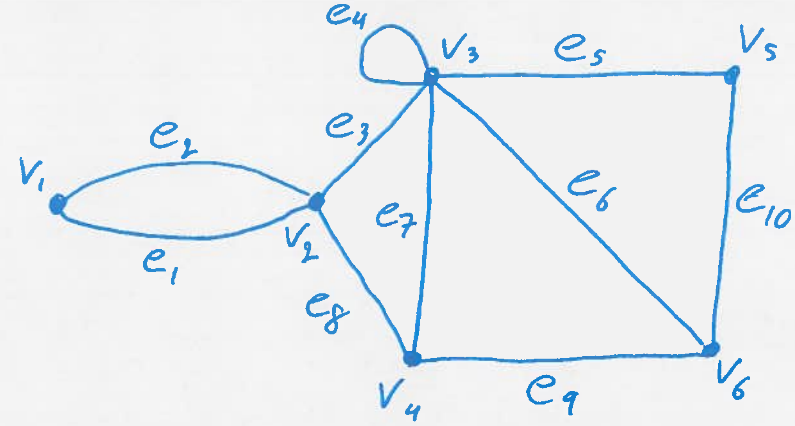
$$V_0 e_1 V_1 e_2 \cdots V_{n-1} e_n V_n, \quad \text{WHERE } V_0 = V \text{ AND } V_n = W.$$

THE LENGTH OF A WALK IS THE NUMBER OF EDGES IN THE SEQUENCE.

- A TRAIL IS A WALK THAT DOES NOT CONTAIN A REPEATED EDGE.
- A PATH IS A TRAIL THAT DOES NOT CONTAIN A REPEATED VERTEX.
- A CIRCUIT IS A WALK WHOSE FIRST AND LAST VERTICES ARE THE SAME.
- A SIMPLE CIRCUIT IS A TRAIL WHOSE FIRST AND LAST VERTICES ARE THE SAME.

IF IT IS NOT AMBIGUOUS, A WALK CAN BE DENOTED BY A SEQUENCE OF ONLY VERTICES OR ONLY EDGES.

EX:



ARE THE FOLLOWING WALKS TRAILS, PATHS, CIRCUITS OR SIMPLE CIRCUITS?

1) $V_1 e_1 V_2 e_3 V_3 e_4 V_3 e_5 V_5$

2) $e_1 e_3 e_4 e_4 e_6$

3) $V_2 V_3 V_4 V_6$

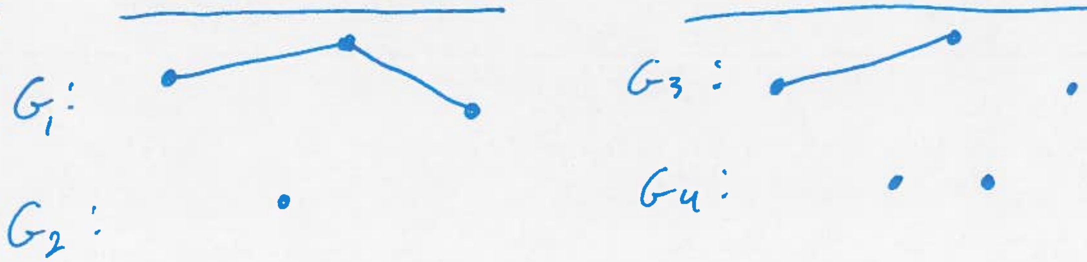
4) $V_2 V_3 V_4 V_2$

5) $V_1 e_1 V_2 e_1 V_1$

6) V_1

DEF: 2 VERTICES V, W IN G ARE <u>CONNECTED</u> IF THERE EXISTS A WALK FROM V TO W. G IS CONNECTED IF THERE IS A WALK BETWEEN EVERY PAIR OF VERTICES. OTHERWISE, G IS <u>DISCONNECTED</u>.

EX: CONNECTED GRAPHS | DISCONNECTED GRAPHS

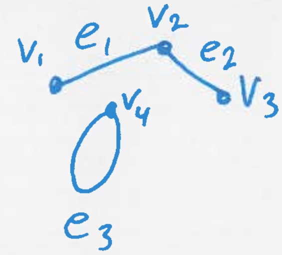


A GRAPH H IS A <u>CONNECTED COMPONENT</u> OF G IF

1) H IS A SUBGRAPH OF G

2) H IS CONNECTED

3) NO CONNECTED SUBGRAPH OF G HAS H AS A SUBGRAPH AND CONTAINS VERTICES OR EDGES THAT ARE OUTSIDE OF H.

EX: G: 

$(\{v_1, v_2, v_3\}, \{e_1, e_2\})$ IS A CONNECTED COMPONENT OF G.

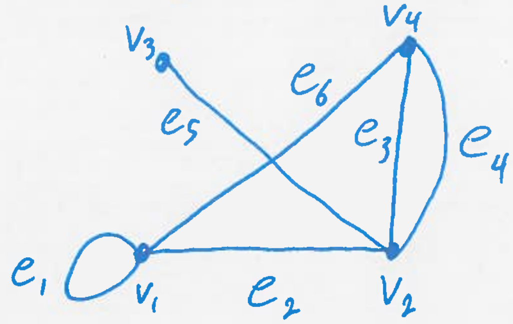
$(\{v_1, v_2\}, \{e_1\})$ IS NOT.

AN EULERIAN CIRCUIT OF G IS A (SIMPLE) CIRCUIT THAT CONTAINS EVERY VERTEX AND EVERY EDGE OF G. IF AN EULERIAN CIRCUIT EXISTS, G IS AN EULERIAN GRAPH. AN EULERIAN PATH FROM v TO w IS A PATH FROM v TO w THAT PASSES THROUGH EVERY VERTEX IN G AT LEAST ONCE, AND EVERY EDGE IN G EXACTLY ONCE.

THM: IF G IS AN EULERIAN GRAPH, THEN EVERY VERTEX OF G HAS EVEN DEGREE. EQUIVALENTLY, IF SOME VERTEX OF G HAS ODD DEGREE, THEN G IS NOT EULERIAN.

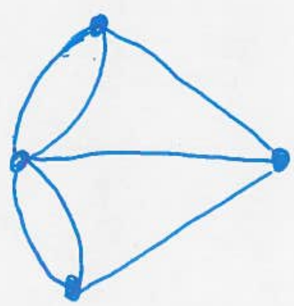
PROOF: G HAS AN EULERIAN CIRCUIT, WHICH USES EACH EDGE EXACTLY ONCE. BEGINNING AT VERTEX v, FOLLOW THE CIRCUIT. AS THE CIRCUIT PASSES THROUGH A VERTEX, IT USES 2 EDGES; ONE ARRIVING TO THE VERTEX AND ONE LEAVING IT. EACH EDGE IS USED ONCE, SO EACH VERTEX USES AN EVEN NUMBER OF INCIDENT EDGE ENDPOINTS. THE STARTING POINT v IS OF EVEN DEGREE AS WELL, SINCE THE CIRCUIT BEGINS BY LEAVING v, THEN USING v AN EVEN NUMBER OF TIMES, THEN ARRIVING AT v. $\square$

EX: DOES THE FOLLOWING HAVE AN EULERIAN CIRCUIT? AN EULERIAN PATH?



EULER'S THEOREM: IN A CONNECTED GRAPH, THE DEGREE OF EVERY VERTEX IS EVEN AND POSITIVE IFF THE GRAPH IS EULERIAN.

BACK TO THE KÖNIGSBERG PROBLEM. DOES THE FOLLOWING GRAPH HAVE AN EULERIAN CIRCUIT?



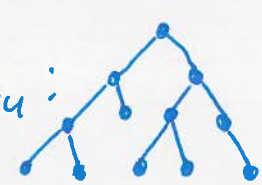
DEF: A GRAPH IS A TREE IF IT IS CONNECTED AND HAS NO CIRCUITS.

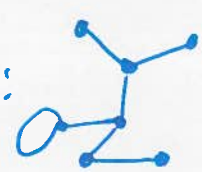
EX: WHICH ARE TREES?

G_1: 

G_2: 

G_3: 

G_4: 

G_5: 

G_6: 

THM: FOR ANY $n \in \mathbb{N}$, A TREE WITH n VERTICES HAS $n-1$ EDGES.

THM: FOR ANY $n \in \mathbb{N}$, IF G IS CONNECTED WITH $|V| = n$ AND $|E| = n-1$, THEN G IS A TREE.

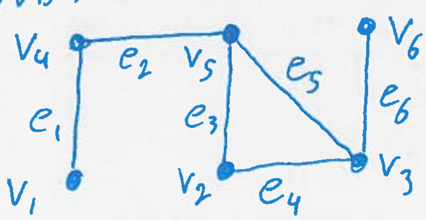
EX: DRAW A TREE WITH 5 VERTICES AND 4 EDGES.

DRAW A GRAPH WITH 5 VERTICES AND 4 EDGES THAT IS NOT A TREE.

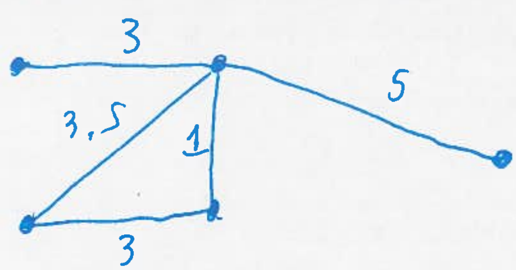
DEF: A SPANNING TREE OF G IS A SUBGRAPH THAT CONTAINS EVERY VERTEX OF G AND IS A TREE.

- EVERY CONNECTED GRAPH HAS A SPANNING TREE.
- ANY 2 SPANNING TREES FOR A GRAPH HAVE THE SAME NUMBER OF EDGES.

EX: FIND ALL SPANNING TREES.



EX: LET THE EDGES REPRESENT PHONE LINES, THE NUMBERS REPRESENT THE COST (IN THOUSANDS) OF INSTALLING THE LINES.



FIND THE SPANNING TREES, AND DETERMINE THE MINIMUM COST OF INSTALLING THE NETWORK.

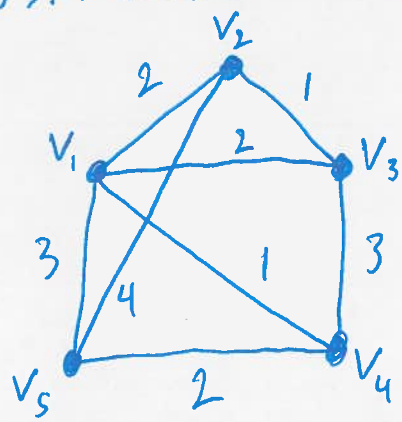
THE PREVIOUS EXAMPLE IS A WEIGHTED GRAPH.

DEF: A WEIGHTED GRAPH IS A GRAPH FOR WHICH EACH EDGE HAS AN ASSOCIATED POSITIVE WEIGHT. THE SUM OF EDGE WEIGHT IS THE WEIGHT OF THE GRAPH.

A MINIMUM SPANNING TREE FOR A CONNECTED, WEIGHTED GRAPH IS A SPANNING TREE THAT HAS THE LEAST POSSIBLE WEIGHT. NOTE THAT MINIMUM SPANNING TREES ARE NOT NECESSARILY UNIQUE. WE USE $w(e)$ AND $w(G)$ FOR THE WEIGHTS OF EDGE e AND GRAPH G.

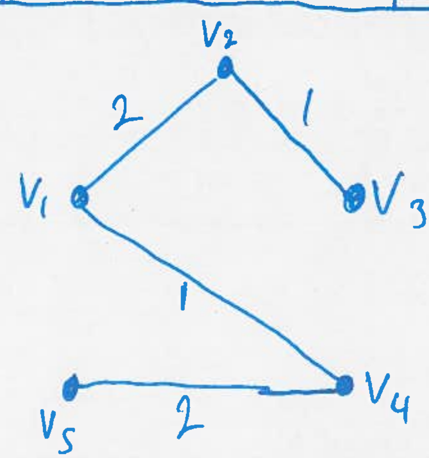
KRUSKAL'S ALGORITHM : TO FIND A MINIMUM SPANNING TREE, THE EDGES ARE EXAMINED IN ORDER OF INCREASING WEIGHT. AT EACH STEP, WE ADD AN EDGE TO WHAT WILL BE THE MINIMUM SPANNING TREE, ONE THAT DOES NOT CREATE A CIRCUIT.

EX: FIND A MINIMUM SPANNING TREE.

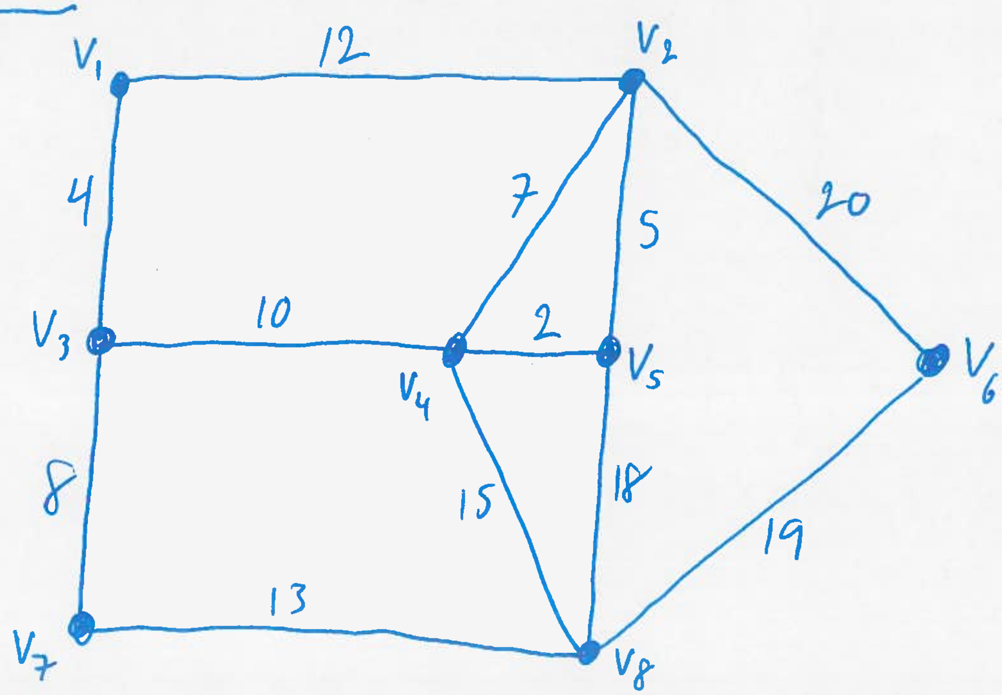


FIRST, PUT THE EDGES IN ORDER BY WEIGHT.

EDGE	WEIGHT	WILL ADDING EDGE MAKE A CIRCUIT?	ACTION	CUMULATIVE WEIGHT
V_2V_3	1	NO	ADD	1
V_1V_4	1	NO	ADD	2
V_1V_2	2	NO	ADD	4
V_1V_3	2	YES	SKIP	4
V_4V_5	2	NO	ADD	6
V_1V_5	3	YES	SKIP	6
V_3V_4	3	YES	SKIP	6
V_2V_5	4	YES	SKIP	6

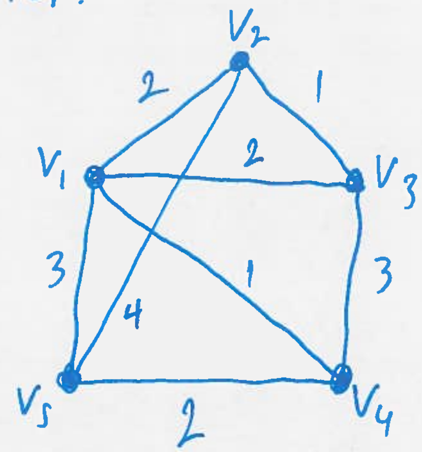


EXERCISE: FIND A MINIMUM SPANNING TREE.



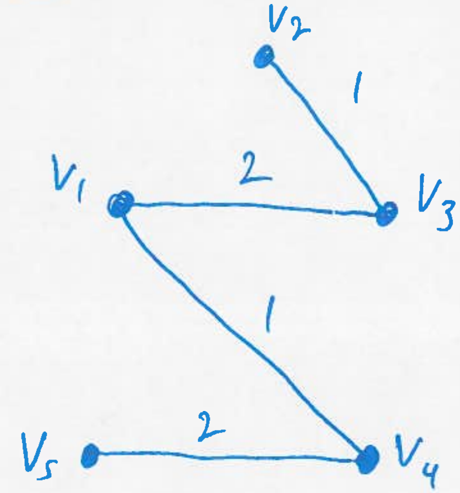
PRIM'S ALGORITHM: BUILD A MINIMUM SPANNING TREE BY CHOOSING A VERTEX AND EXPANDING OUTWARDS ADDING ONE EDGE AND ONE VERTEX AT EACH STEP.

EX:



START WITH (ARBITRARILY) V_1.

VERTEX ADDED	EDGE ADDED	WEIGHT	CUMULATIVE WEIGHT
V_4	V_1V_4	1	1
V_3	V_1V_3	2	3
V_2	V_2V_3	1	4
V_5	V_4V_5	2	6



EXERCISE: REPEAT LAST EXERCISE WITH PRIM'S ALGORITHM.