Set Theory

- A **set** is a loosely defined collection of items called **elements.**
- Sets are completely determined by their elements, i.e. two sets with exactly the same elements are the same set.
- The order in which elements are listed is irrelevant, and elements may be listed more than once without changing the set.
- Examples:

$$\{1,3\} = \{3,1\} = \{3,3,1,3,1,1\}$$

The collection of all people in this room is a set.
The collection of your favourite songs is a set.
The collection of all real numbers $\mathbb{R}$ is a set.

- Sets come from a **universe** of elements $\mathcal{U}$.
- For example, the set of even numbers comes from the universe $\mathbb{Z}$.
- Sets can be contained in other sets and can be finite or infinite.

$$\{1,2,3\}; \{Susan, Robert\}; \{0,\{0\},1,\{0,1\}\}; \{2,4,6,\dots\}; \{2,4,6,\dots,30\}$$

- Some important sets of numbers are:
  - $\mathbb{N} = \{1,2,3,\dots\}$ (NATURAL)
  - $\mathbb{Z} = \{\dots,-2,-1,0,1,3,\dots\}$ (INTEGER)
  - $\mathbb{Q} = \{\frac{a}{b}: a,b \in \mathbb{Z}, b \neq 0\}$ (RATIONAL)
  - $\mathbb{R} = Set\ of\ all\ real\ numbers$ (RATIONAL AND IRRATIONAL)
- A set can be defined by a property of elements of a bigger set.
- Given a set S, define a set T by:

$$T = \{x \in S : p(x)\}$$

All elements of S that satisfy p.

Example:
The set $\mathrm{x} \in \mathbb{R}: -2 < \mathrm{x} \leq 5$ is the set of all real numbers between -2 and 5, not including -2. This set is an internal, which can be denoted as $(-2,5]$.

Exercise:
The set $\{\mathrm{x} \in \mathbb{Z}: -2 < \mathrm{x} \leq 5\}$ can be rewritten how?

A: $\{-1,0,1,2,3,4,5\}$ or $(-2,5]$

Exercise:
The set $\{\mathrm{x} \in \mathbb{R} : \mathrm{x}^3 = \mathrm{x}$ can be rewritten how?

A: $\{-1,0,1\}$

- The **empty set** is the set with no elements, denoted by $\emptyset$.
- It can be represented in different ways:

$$\{x \in \mathbb{N} : x \neq x\}; \{x \in \mathbb{R} : 3 < x < 2\}$$

- A set is **finite** if $\exists n \in \mathbb{N}$ such that there is a one-to-one correspondence with the set $\{1, 2, \dots, n\}$.
- For a set S of this size, we write $|S| = n$ and say that S has **cardinality** n.
  - NOTE: $|\emptyset| = 0$
- A set that is not finite is said to be **infinite**.

Subsets

Definition:

- Let A and B be sets.
- We say A is a subset of B, written $A \subseteq B$, IFF every element of A is also an element of B.

Definition – Subsets:

$$\boldsymbol{A \subseteq B \Leftrightarrow \forall x, x \in A \Rightarrow x \in B}$$

Supersets

Definition:

- If A is a subset of B.
- Then B is called a superset of A.
- We also say that A is contained in B, and that B contains A.
- If at least one element of A is not in B, then A is not a subset of B.

Definition – Supersets:

$$\boldsymbol{A \nsubseteq B \Leftrightarrow \exists x \ni x \in A \wedge x \notin B}$$

Exercise:

Decide true or false.

a) $\{1, 2\} \subseteq \{1, 2, 3\}$ T

b) $\{0, 2\} \subseteq \{1, 2, 3\}$ F

c) $-1 \in \{x \in \mathbb{N} : x^2 = 1$ F (-1, not element of $\mathbb{N}$)

d) $\{1\} \in \{x \in \mathbb{N} : x^2 = 1\}$ F (Not an element)

e) *For all sets* $A, \emptyset \subseteq A$ T

Proper Subsets

Definition:

- A subset $A \subset B$ is **proper** if $\exists x \in B \ni x \notin A$.
  - We write $A \subset B$
  - For example, $\{1\} \subseteq \{1,2\}$ and $\{1,2\} \subseteq \{1,2\}$, but $2 \in \{1,2\}$ and $2 \notin \{1\}$, so actually $\{1\} \subset \{1,2\}$.

> *Definition – Proper Subsets:*
>
> $\boldsymbol{A \subset B, \exists\, x \in B \ni x \notin A}$

Exercise:

Order the sets $\mathbb{R}, \mathbb{N}, \mathbb{Q}, \emptyset, \mathbb{Z}$ in terms of subsets. Are any of these proper subsets?

$\emptyset \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ - They are all proper

Exercise:

True or false? Let $A = \{1,2,3\}$.

a) $A \subset A$ F
b) $\emptyset \in A$ F
c) $\emptyset \subseteq A$ T
d) $\{\emptyset\} \subseteq A$ F
e) $2 \in A$ T
f) $\{2\} \in A$ F
g) $2 \subseteq A$ F
h) $\{2\} \subseteq A$ T
i) $\{2\} \subseteq \{\{1\},\{2\}\}$ F
j) $\{2\} \in \{\{1\},\{2\}\}$ T

Definition:

- Let A and B be sets.
- We say A **equals** B, written $A = B$, if and only if, A contains B and B contains A.

> *Definition – Set Equality*
>
> $\boldsymbol{A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A}$

- To prove two sets are equal, prove the two contentions, $A \subseteq B$ and $B \subseteq A$.

Exercise:

Prove that $A = \{n \in \mathbb{N} : n \text{ is even}\}$ and $B = \{n \in \mathbb{N} : n^2 \text{ is even}\}$ are equal.

($\subseteq$) Let $n \in A$. Then $n = 2k$ for some $k \in \mathbb{N}$.

$n^2 = n \cdot n = (2k)(2k) = 2(2k^2), 2k^2 \in \mathbb{N} \Rightarrow n^2 \text{is even} \Rightarrow n \in B.$

($\supseteq$) Let $n \in B$, so n^2 is even.

Suppose that n is odd, $n = 2k + 1$ for some $k \in \mathbb{N}$.

Then $n^2 = (2k+1)(2k+1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is a *contradiction*

Hench, n is even and $n \in A$.

$\therefore B \subseteq A$

$\therefore A = B$

Exercise:

Define:

$A = \{n \in \mathbb{Z} : n = 2p, p \in \mathbb{Z}\}$

$B = \{n \in \mathbb{Z} : n \text{ is even}\}$

$C = \{n \in \mathbb{Z} : n = 2q - 2, q \in \mathbb{Z}\}$

$D = \{k \in \mathbb{Z} : k = 3r + 1, r \in \mathbb{Z}\}$

a) Is $A = B$? T
b) Is $A = D$? F
c) Is $A = C$? T

Proof by case:

($\subseteq$): Let $p = 0$, $p \in \mathbb{Z}$

$n = 2(0) = 0$, $n \in \mathbb{Z}$,

($\supseteq$): Let $r = 0$, $r \in \mathbb{Z}$

$k = 3(0) + 1 = 1$, $k \notin A$

Operations on Sets

- Let A, B be subsets pf a universe $\mathcal{U}$/

1) The **union** of A and B,written $A \cup B$, is the set of all elements that are in A or in B.

$$A \cup B = \{x \in \mathcal{U} : x \in A \vee x \in B\}$$

2) The **intersection** of A and B, written $A \cap B$, is the set of all elements that are in A and in B.

$$A \cap B = \{x \in \mathcal{U} : x \in A \wedge x \in B\}$$

3) The **complement** of A, written $\overline{A}$ or $\mathcal{U} \backslash A$, is the set of all elements that are not in A.

$$\overline{A} = \mathcal{U} \backslash A = \{x \in \mathcal{U} : x \notin A\}$$

4) The **difference** of B minus A, written $B - A$, is the set of all elements that are in B and not in A.

$$B - A = \{x \in \mathcal{U} : x \in B \wedge x \notin A\}$$

Power Set

Definition:

- The **power set** of a universe $\mathcal{U}$, denoted by $P(\mathcal{U})$, is the set of all subsets of $\mathcal{U}$.

Exercise:

Let $A = \{1, 2, 3\}$.

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$$

- If $|A| = n$, then $|P(A)| = 2^n$
- The operations of set theory are equivalent to their counterpart connectives of logic, as follows:

Set Operation	Name	Connective
$\setminus$	**Complement**	$\sim$
$\cup$	**Union**	$\vee$
$\cap$	**Intersection**	$\wedge$
$\subseteq$	**Subset**	$\Rightarrow$
$=$	**Equality**	$\Leftrightarrow$

Exercise:

Let $\mathcal{U} = \mathbb{Z}$. Write down $\overline{A}$ for the following.

a) $A = \{1, 2, 3\}$

$\overline{A} = \{x \in \mathbb{Z} : 1 < x > 3\}$ or $\{x \in \mathbb{Z} : x \neq 1, 2, 3\}$

b) $A = \{x \in \mathbb{Z} : x \textit{ is even}\}$

$\overline{A} = \{x \in \mathbb{Z} : x \text{ is odd}\}$

c) $A = \{x \in \mathbb{Z} : x > 0 \vee x < 0\}$

$A = \{0\}$

Exercise:

Let $\mathcal{U} \in \mathbb{R}$. Write down $A \cup B$ for the following, and $A \cap B$.

a) $A = \{1\}, B = \{2\}$

$A \cup B = \{1, 2\}$

$A \cap B = \emptyset$

b) A is the set of even integers, B is the set of odd integers.

$A \cup B = \mathbb{Z}$

$A \cap B = \emptyset$

c) $A = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$ and $B = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$

$A \cup B = [0, 3]$

$A \cap B = [1, 2]$

Exercise:

Prove or disprove: $(A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cup B \subseteq C$.

A: $[(A \Rightarrow C) \wedge (B \Rightarrow C)] \Rightarrow [(A \vee B) \Rightarrow C]$

Use a truth table, or suppose the main connective is false.

$[(A \Rightarrow C) \wedge (B \Rightarrow C)] \Rightarrow [(A \vee B) \Rightarrow C]$

F

T F

T T T F

C : T ? A : T C : F ?

The contradiction is C cannot be false for LHS to be true.

$\therefore (A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cup B \subseteq C$ $\square$

Exercise:

Prove or disprove: $(A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cap B \subseteq C$

A: $[(A \Rightarrow C) \wedge (B \Rightarrow C)] \Rightarrow [(A \cap B) \Rightarrow C]$

F

T F

T T T F

C : T ? A : T, B : T C : F ? (!)

$\therefore (A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cap B$

Exercise:

Let $\mathcal{U} = \mathbb{R}, A = \{1, 2, 3\}, B = \{2\}, C = \{2, 3, 4\}. D = [0,1]$.

a) $A - C =$ $\{1\}$

b) $B - C =$ $\emptyset$

c) $D - B =$ D

d) $D - A =$ $[0, 1)$

e) $A - D =$ $\{2, 3\}$

Disjoints

Definition:

- The sets A and B are **disjoints** if $A \cap B = \emptyset$.

Exercise:

Let $\mathcal{U} = \mathbb{R}$. Write down some sets that are disjoint to the following.

a) $A = \{x \in \mathbb{Z} : x \text{ is even}\}$ $B = \{x \in \mathbb{Z} : x \text{ is odd}\}$

b) $A = \{x \in \mathbb{R} : x^2 - 5x + 6 \geq 0\}$ $B = \{x \in \mathbb{R} : x < 0\}$

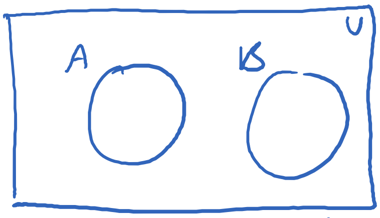
c) $A = \mathbb{Q}$ $B = \mathbb{R} \setminus \mathbb{Q}$

Definition (Addition Rule):

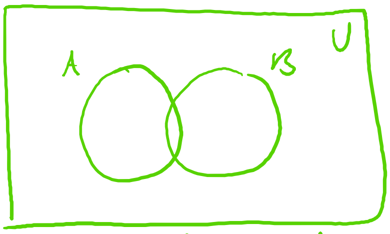
- Let A, B be finite, disjoint sets.
- Then $A \cup B$ is finite and $|A \cup B| = |A| + |B|$.

Venn Diagrams

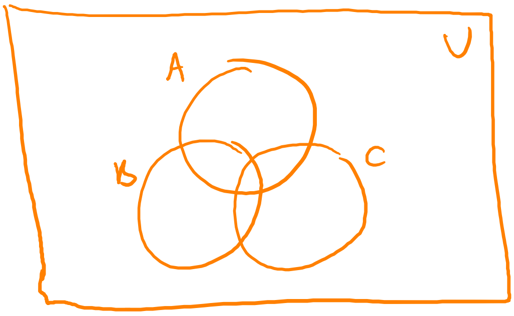
- If we represent sets as regions in the plane, then the relationships among sets can be represented by drawing called **Venn diagrams**.



2 disjoint sets



2 intersecting sets - $A \cap B$



3 intersecting sets - $A \cap B \cap C$

Algebra on Sets

- There are many rules that govern set theory and the relationships among sets.
- All the following statements can be proved using the definitions we have seen so far.

Theorem:

- Let $\mathcal{U}$ be a set, and A, B, C be element of $P(\mathcal{U})$. Then…
  1) $(A \subseteq B \land B \subseteq) \Rightarrow A \subseteq C$
     $A \subseteq A \cup B; B \subseteq A \cup B$
     $A \cap B \subseteq A; A \cap B \subseteq B$
  2) $A = B \Leftrightarrow (A \subseteq B \land B \subseteq A)$
     $A \subseteq B \Leftrightarrow A \cup B = B$
     $A \subseteq B \Leftrightarrow A \cap B = A$
  3) $A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$
     $A \subseteq B \Rightarrow A \cap C \subseteq B \cap C$
  4) **Commutative Laws:**
     $A \cup B = B \cup A$
     $A \cap B = B \cap A$
  5) **Associative Laws:**
     $(A \cup B) \cup C = A \cup (B \cup C)$
     $(A \cap B) \cap C = A \cap (B \cap C)$
  6) **Distributive Laws:**
     $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
     $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  7) **De Morgan's Laws:**
     $\overline{A \cup B} = \overline{A} \cap \overline{B}$
     $\overline{A \cap B} = \overline{A} \cup \overline{B}$
  8) $\overline{(\overline{A})} = A$
     $A \subseteq B \Leftrightarrow \overline{B} \subseteq \overline{A}$
     $A - B = A \cap \overline{B}$
     $\overline{\mathcal{U}} = \emptyset$
     $\overline{\emptyset} = \mathcal{U}$
  9) $A \cap \mathcal{U} = A; A \cup \emptyset = A$
     $A \cap \emptyset = \emptyset; A \cup \mathcal{U} = \mathcal{U}$
     $A \cap \overline{A} = \emptyset; A \cup \overline{A} = \mathcal{U}$
  10) $(A \subseteq C \land B \subseteq C) \Leftrightarrow (A \cup B) \subseteq C$
     $(A \subseteq B \land A \subseteq C) \Leftrightarrow A \subseteq (B \cap C)$

Exercise:

Prove (7) $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

$(\subseteq)$ Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in \overline{A}$ and $x \in \overline{B}$

$\Rightarrow x \in \overline{A} \cap \overline{B}$

$(\supseteq)$ Let $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin A \cup B$

$\Rightarrow x \in \overline{A \cup B}$

$\therefore \overline{A \cup B} = \overline{A} \cap \overline{B}$

Exercise:

Prove (2) $A \subseteq B \Leftrightarrow A \cup B = B$.

$(\Rightarrow)$ Let $A \subseteq B$

$(\subseteq)$ Let $x \in A \cup B$. Then $x \in A$ or $x \in B$

Since $A \subseteq B$, if $x \in A$ then $x \in B$.

$\Rightarrow x \in B$

$\therefore x \in A \cup B \Rightarrow x \in B$ i.e. $A \cup B \subseteq B$

$(\supseteq)$ Let $x \in B$. Then $x \in A \cup B$

$\therefore B \subseteq A \cup B$

$\therefore A \subseteq B \Rightarrow A \cup B = B$

$(\Leftarrow)$ Let $A \cup B = B$

Let $x \in A$. Then $x \in A \cup B$. But $A \cup B = B$, so $x \in B$

$\Rightarrow A \subseteq B$

$\therefore A \cup B = B \Rightarrow A \subseteq B$

$\therefore A \subseteq B \Leftrightarrow A \cup B = B$

Exercise:

Prove that the difference operator is not commutative.

A: Show that $A-(B-C) \neq (A-B)-C$

① $A-(B-C) = A-(B\cap \overline{C})$ (8) $A-B = A\cap\overline{B}$

$= A\cap \overline{(B\cap\overline{C})}$ (8) $A-B = A\cap\overline{B}$

$= A\cap(\overline{B}\cup C)$ (7) $\overline{A\cap B} = \overline{A}\cup\overline{B}$ (De Morgan's)

$= (A\cap\overline{B})\cup(A\cap C)$ (6) $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$ (Distrib)

② $(A-B)-C = (A\cap\overline{B})-C$ (8)

$= A\cap\overline{B}\cap\overline{C}$ (8)

Let $x\in A\cap C$. Then by ①, $x\in A-(B-C)$, and since $x\in A$ and $x\in C$, we have $x\notin\overline{C}$. So by ②, $x\notin(A-B)-C$

$\therefore A-(B-C)\neq(A-B)-C$

Pairwise Disjoint

Definition:

- The sets $A_1, A_2, \ldots, A_k$ are **pairwise disjoint** if $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Theorem (Extension of Addition Rule):

- Let $A_1, A_2, \ldots, A_k$ be finite, pairwise disjoint sets.
- Then $A_1 \cup A_2 \cup \ldots \cup A_k$ is finite and $|A_1 \cup A_2 \cup \ldots \cup A_k| = |A_1| + |A_2| + \cdots + |A_k|$.