Formal Logic

- **Logic** is a language for reasoning.
- An argument is a sequence of statements aimed at demonstrating the truth of an assertion.
    - The assertion at the end of the sequence is called the **conclusion**, and the preceding statements are called **premises**.
- We are interested in whether a **statement** is true or false, and in determining truth/falsehood of statements from other statements.
- **Statement** (or **proposition**)**:** a sentence that is true or false, but not both.
- Much of Mathematics is about *proving* a statement is true, or *demonstrating* a statement is false.

Logical Connectives

- **Connectives** are key words/symbols that connect two or more simple statements to form new, longer ones.
- We use *p*, *q*, *r*, ... to denote simple statements (**statement variables**) i.e. *p*: I need to work hard in MATH221.
- There are 5 Connectives:

Name	**Example**	**Definition**
Negation	$\sim p$	NOT *p*
Disjunction	$p \vee q$	*p* OR *q*
Conjunction	$p \wedge q$	*p* AND *q*
Conditional	$p \Rightarrow q$ ($p \rightarrow q$)	*p* IMPLIES *q*
Biconditional	$p \Leftrightarrow q$ ($p \leftrightarrow q$)	*p* IF AND ONLY IF *q*

- **Compound statement:** an expression of simple statements and connectives.
    - Each simple statement has a truth value T for true and F for false.
- The truth value of a compound statement is determined by logic, using the simple statement values and the connectives.
- We do this by constructing *truth tables*.

Negation

- If *p* is a statement variable, then “NOT *p*”, denoted by ~*p*, has the opposite value.
    - If *p* is true, ~*p* is false.
    - If *p* is false. ~*p* is true.

Definition

If *p* is a statement variable, the **negation** of *p* is “not *p*” or “it is not the case that *p*” and is denoted ~*p*. It has the opposite truth value from *p*: if *p* is true, ~*p* is false; if *p* is false, ~*p* is true.

- The truth values for **negation** are summarised in a table.

p	**~*p***
T	F
F	T

- NOTE:
  - The truth table above tells us that for any statement *p*, exactly *one* of *p* and ~*p* is true.
  - This gives us 2 options for proving *p* is true:
    - Show it directly
    - Show indirectly by proving ~*p* is false **(proof by contradiction)**

Priority

- In expressions that include the symbol ~ as well as $\wedge$ or $\vee$, the **order of operations** specifies that ~ is performed first.
- $\sim p \vee q$ means $(\sim p) \vee q$, which is different from $\sim(p \vee q)$
- In logical expressions, as in ordinary algebraic expressions, the order of operations can be overridden through the uses of parentheses.

Conjunction

- If *p* and *q* are statement variables, the conjunction is "*p* AND *q*", denoted by $p \wedge q$.
  - If *p* AND *q* are both true, then $p \wedge q$ is true.
  - Otherwise, $p \wedge q$ is false.

> **Definition**
>
> If *p* and *q* are statement variables, the **conjunction** of *p* and *q* is "*p* and *q*,' denoted by $p \wedge q$. It is true when, and only when, both *p* and *q* are true. If either p or q is false, or if both are false, $p \wedge q$ is false.

- The truth table for conjunction is:

p	***q***	$\boldsymbol{p \wedge q}$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

- If p and q are statement variables, the disjunction is "p OR q", denoted by $p \vee q$.
  - If p and q are both false, $p \vee q$ is false.
  - Otherwise, $p \vee q$ is true.

Definition

If p and q are statement variables, the **disjunction** of p and q is "p or q," denoted by $p \vee q$. It is true when either p is true, or q is true, or both p and q are true; it is false only when both p and q are false.

- The truth table for disjunction is:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- NOTE:
  - The word "OR" can be used in an exclusive sense, i.e. p OR q but not both.

Exclusive OR

- The **exclusive or** statement is sometimes denoted by $\otimes$
- It can also be represented by AND/OR/NOT symbols. i.e. $p \otimes q$ = "p OR q, but NOT BOTH"
- Note that when *or* is used in its exclusive sense, the statement "p or q" means "p or q but not both" or "p or q and not both p and q," which can be denoted as: $(p \vee q) \wedge \sim(p \wedge q)$.
- The truth table for exclusive or is:

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Conditionals

- When you make a logical inference or deduction, you reason from a *hypothesis* to a *conclusion.*
- The statement has the form "if something is true, then something else is true."
- If *p* and *q* are statement variables, the **conditional** of *q* by *p* is "If *p*, then *q*" or "*p* IMPLIES *q*", denoted by *p* => *q*.
  - If *p* is true and *q* is false, then *p* => *q* is false.
  - Otherwise, *p* => *q* is true
  - *p* is the **hypothesis** (**antecedent**)
  - *q* is the **conclusion** (**consequent**)
- Conditionals take priority over conjunctions and disjunctions.

Definition

IF *p* and *q* are statement variables, the **conditional** of *q* by *p* is "if *p*, then *q*" or "*p* implies *q*," denoted by *p* => *q*. If *p* is true and *q* is false, then *p* => *q* is false. Otherwise, *p* => *q* is true.

- The truth table for conditionals is:

p	***q***	***p* => *q***
T	T	T
T	F	F
F	T	T
F	F	T

- NOTE:
  - Why is *p* => *q* true when *p* is false.
  - If a statement cannot be said to be false, then it is true.
  - If *p* is false, then we cannot say that *p* => *q* is false, it is true.
  - Consider the claim, "if it rains, then I will go home."
    - Only if it rains can we make a judgement on its truth.
    - If it doesn't rain, then regardless of whether or not I go home, we cannot claim that the statement is false. So, it is true.

Exercise:

Write using connectives: "If $x^2 = 4$, then x = 2 or x = -2."

$p: x^2 = 4$

$q: x = 2$

$r: x = -2$

$p \Rightarrow (q \vee r)$

Biconditionals

- A **biconditional** statement has the form "*p* if and only if *q*" or "*p* IFF *q*."
- It's true only if both variables have the same value.
- It is denoted by $p \Leftrightarrow q$, and is read
  - *p* IFF *q*
  - *p* is EQUIVALENT to *q*
  - *p* IMPLIES AND IS IMPLIED by *q*
  - *p* is NECESSARY AND SUFFICIENT for *q*
- The truth table for biconditionals is:

p	**q**	**p <=> q**
T	T	T
T	F	F
F	T	F
F	F	T

Exercise:

a) *p*: $x^3 = -8$, *q*: $x = -2$, $p \Leftrightarrow q$.

p	q	p => q	q => p	(p => q) ∧ (q => p)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

b) Write using connectives: "Michael is a bachelor if and only if he is male and never married."

p: Michael is a bachelor

q: Michael is a male

r: Michael is never married

$p \Leftrightarrow (q \wedge r)$

EXERCISE:
Complete the table

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	**T**	T	T	T	T
T	**F**	F	T	F	F
F	**T**	T	F	F	F
F	**F**	T	T	T	T

- Notice that the last two columns are the same.
- This means that $p \Leftrightarrow q$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$ are **logically equivalent**.
- NOTE:
  - Notice that $p \Leftrightarrow q$ means $(p \Rightarrow q) \wedge (q \Rightarrow p)$

Main Connectives

- When building compound statements, use parentheses to avoid ambiguity.
- The **main connective** is the one that binds the whole statement together.
- We must know the ranking of all connectives in a statement.
- E.g.

$(p \vee \sim q) \Rightarrow (p \wedge q)$

Main Connective

$\sim[(p \wedge q) \vee (\sim p \wedge q)]$

Tautology and Fallacy

- A **tautology** is a compound statement that is always true, for all values of the basic statements
  - E.g. $p \vee \sim p$
- A **fallacy** is a compound statement that is always false, for all values of the basic statements
  - E.g. $p \wedge \sim p$
- Any statement that is neither a tautology nor a fallacy is called **contingent** or **intermediate**.
- Note that the negative of a tautology is a fallacy, and vice versa.

Definition

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement**.

A **contradiction** (**fallacy**) is a statement that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is a **contradictory statement**.

Exercise:

Show that for any statement *p*, $p \vee {\sim}p$ is a tautology and $p \wedge {\sim}p$ is a fallacy.

p	${\sim}p$	$p \vee {\sim}p$	$p \wedge {\sim}p$
T	F	T	F
F	T	T	F

Exercise:

Determine whether ${\sim}[({\sim}p \wedge q) \wedge p]$ is a tautology, fallacy, or contingent statement.

p	q	${\sim}p$	${\sim}p \wedge q$	${\sim}[({\sim}p \wedge q) \wedge p]$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	T

It is a tautlogy.

“Quick” Method of Identifying Tautologies

- With truth tables, 2^n rows are required.
  - This gets big and impractical quickly (i.e. 4 statements requires 16 rows, 5 statements requires 32 rows, …)
- There is a quicker method we can use:
- NOTE:
  - Truth tables are reliable; it’s not easy to make mistakes. The quick method can be more difficult in that respect.
- It relies on the fact that if a false can occur under the main connective, then the statement is not a tautology.
- If a false is not possible, it is a tautology.
- They method is:
  - Assume the main connective yields a false, then work backwards to see if a valid combination of values exists.
  - Firstly, place an F under the main connective.

$$(p \wedge q) \Rightarrow (r \wedge s)$$

F

- Remember the conditional table: for this to happen, $p \wedge q$ must be true and $r \wedge s$ must be false.

$$\underbrace{\underbrace{(p \wedge q)}_{T} \Rightarrow \underbrace{(r \wedge s)}_{F}}_{F}$$

∴ p: T, q: T r/s = ?

e.g. p: T, q: T, r: T, s: F,

the main connective is validly false

- Since these are perfectly valid values for p, q, r, s, we have that $(p \wedge q) \Rightarrow (r \wedge s)$ is *not* a tautology.

Exercise:

a) Is [(p => q) ∧ (q => r)] => (p => r) a tautology?

F

T F

p: T, r: F?

T T

p: T, q: T q: T, r: T?

There is no possible combination to make the main connective False.
Therefore, this is a tautology

b) Make the truth table for this statement, and verify that the last column is all T.

p	***q***	***r***	***p* => *q***	***q* => *r***	***p* => *r***	**[(*p* => *q*) ∧ (*q* => *r*)] => (*p* => *r*)**
T	**T**	**T**	T	T	T	T
T	**T**	**F**	T	F	F	F T F
T	**F**	**T**	F	T	T	F T T
T	**F**	**F**	F	T	F	T
F	**T**	**T**	T	T	T	T
F	**T**	**F**	T	F	T	F T T
F	**F**	**T**	T	T	T	T
F	**F**	**F**	T	T	T	T

Logical Equivalence

- Two statements are called **logically equivalent** if, and only if, they have identical truth tables.
- The logical equivalence of p and q is denoted by $p \equiv q$.
- p and q are logically equivalent IFF p <=> q is a tautology.

Exercise:

Is $p \equiv \sim(\sim p)$?

p	**$\sim p$**	**$\sim(\sim p)$**
T	F	T
F	T	F

↑ ↑

These are identical,
So, yes they are equivalent

Substitution of Equivalence

- We can make substitutions in statements, using equivalence expressions.
- There are 2 rules:

Rule of Substitution

- If in a tautology all occurrences of a variable are replaced by the same statement, the result is another tautology:

Example:

$p \vee \sim p$ is a tautology, so $q \vee \sim q$ is as well, and $[(p \vee q) \Rightarrow r] \vee \sim[(p \vee q) \Rightarrow r]$ is as well.

Rule of Substitution of Equivalence

- If in a tautology we replace any part of a statement by a statement equivalent to that part, the result is another tautology.

Example:

$p \equiv \sim(\sim p)$, so the tautology $p \vee \sim p$ can be written $\sim(\sim p) \vee \sim p$.

Exercise:

$p \Rightarrow q$ is logically equivalent to $\sim p \vee q$. $q \Rightarrow (p \Rightarrow q)$ is a tautology.
Prove that $s \Rightarrow (\sim r \vee s)$ is a tautology.

$q \Rightarrow (p \Rightarrow q)$

- Substitute q for s

$s \Rightarrow (p \Rightarrow s)$

- Substitute r for p

$s \Rightarrow (r \Rightarrow s)$

- Since $p \Rightarrow q$ is logically equivalent to $\sim p \vee q$, we can substitute $r \Rightarrow s$ for $\sim r \vee s$

$s \Rightarrow (\sim r \vee s)$

Truth Tables Cheat Sheet

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditionals

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditionals

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Equivalence Laws

Commutative Laws

1. $p \vee q \equiv q \vee p$
2. $p \wedge q \equiv q \wedge p$
3. $p \Leftrightarrow q \equiv q \Leftrightarrow p$

Associative Laws

1. $(p \vee q) \vee r \equiv p \vee (q \vee r)$
2. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
3. $(p \Leftrightarrow q) \Leftrightarrow r \equiv p \Leftrightarrow (q \Leftrightarrow r)$

Distributive Laws

1. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (q \vee r)$
2. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
3. $p \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$
4. $p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$

Double Negative Law

1. $\sim(\sim p) \equiv p$

De Morgan's Laws

1. $\sim(p \vee q) \equiv \sim p \wedge \sim q$
2. $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Implication Laws

1. $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$
2. $p \Rightarrow q \equiv \sim p \vee q$
3. $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$
4. $\sim(p \Rightarrow q) \equiv p \wedge \sim q$

Definition

The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Exercise:
To understand De Morgan's Laws, write negatives of these:

a) John is 6 feet tall and he weighs at least 200 pounds

p: John is 6 feet tall
q: John weighs at least 200 pounds

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

John is <u>not</u> 6 feet tall <u>or</u> he weighs <u>less than</u> 200 pounds

b) The bus was late or Tom's watch was slow

p: the bus was late
q: Tom's watch was slow

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

The bus was <u>not</u> late <u>and</u> Tom's watch was <u>not</u> slow

Exercise:
Prove De Morgan's Laws using truth tables.

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	**q**	**$\sim(p \wedge q)$**	**$\sim p \vee \sim q$**
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	**q**	**$\sim(p \vee q)$**	**$\sim p \wedge \sim q$**
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Inequalities and De Morgan's Laws

Textbook Exercise:
Use De Morgan's laws to write the negation of $-1 < x \leq 4$.

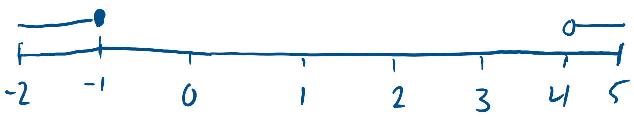
$$-1 < x \leq 4 \equiv (-1 < x) \wedge (x \leq 4)$$

By De Morgan's laws, the negation is:

$$(-1 \nless x) \vee (x \nleq 4)$$

This is equivalent to:

$$(-1 \geq x) \vee (x > 4)$$



x is between those points

- De Morgan's laws are frequently used in writing computer programs.
  - For instance, supposed you the program to delete all files modified outside a certain range of dates, say from date1 through date2 inclusive.
  - You would use the fact that:

    ~(*date1* ≤ *file_modification_date* ≤ *date2*)

    is equivalent to:

    (*file_modification_date* < *date1*) or (*date2* < *file_modification_date*)

Exercise:
Is ($p \wedge \sim q) \wedge (\sim p \vee q$) a tautology or a fallacy?

F

T F

p: T; ~q: T   q: F, ~p: F

Since this combination is possible, it is not a tautalogy.

Exercise:
Is (p <=> q) <=> (~p <=> q) a tautology or a fallacy?

T

T T

p: T ; q: T   ~p: F;

Since q cannot be a true, the main connective cannot be possible. This is a fallacy.

F

T F

① T T → F T ✓

② F F → T F ✓

} Not a tautology

Exercise:

Prove the equivalence $(p \Rightarrow q) \Rightarrow r \equiv [(\sim p \Rightarrow r) \wedge (q \Rightarrow r)]$

p	q	r	(p => q)	(~p => r)	(q => r)	(p => q) => r	[(~p => r) ∧ (q => r)]
T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	F	T	F	F

These are the same

$$(p \Rightarrow q) \Rightarrow r \equiv [(\sim p \Rightarrow r) \wedge (q \Rightarrow r)]$$

Let α be the tautology $s \Rightarrow t \equiv \sim s \vee t$

$(p \Rightarrow q) \Rightarrow r \equiv (\sim p \vee q) \vee r$ (Substitute $s \Rightarrow t$ for $(p \Rightarrow q)$)

$\equiv (\sim\sim p \wedge \sim q) \vee r$ (De Morgan's) $\Rightarrow$ flip $\vee$ to $\wedge$ by adding negation

$\equiv (p \wedge \sim q) \vee r$ (Double negation)

$\equiv (p \vee r) \wedge (\sim q \vee r)$ (Distrib. 1)

$\equiv (p \Rightarrow r) \wedge (q \Rightarrow r)$ α again