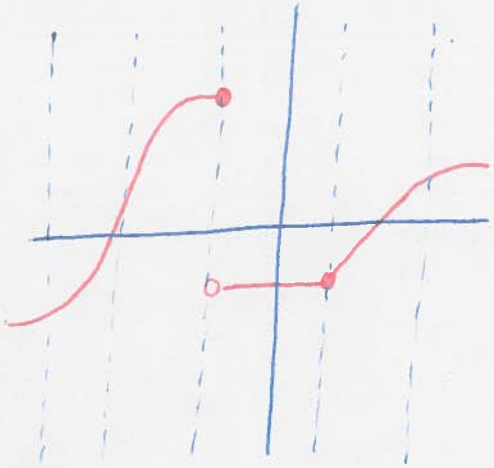
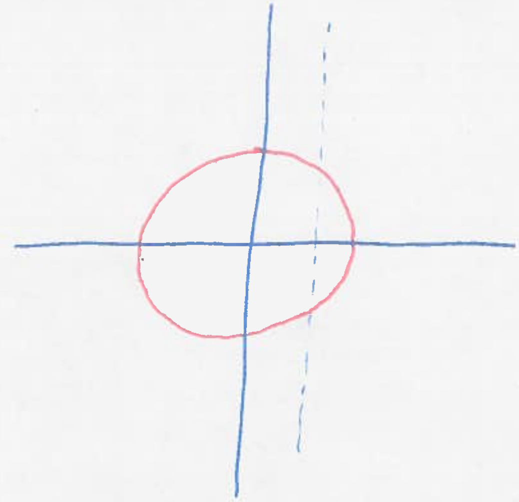


THIS IS THE GRAPH OF A FUNCTION. NO VERTICAL LINE INTERSECTS THE CURVE MORE THAN ONCE.



A CIRCLE IS NOT A FUNCTION. ITS GRAPH FAILS THE VERTICAL LINE TEST.

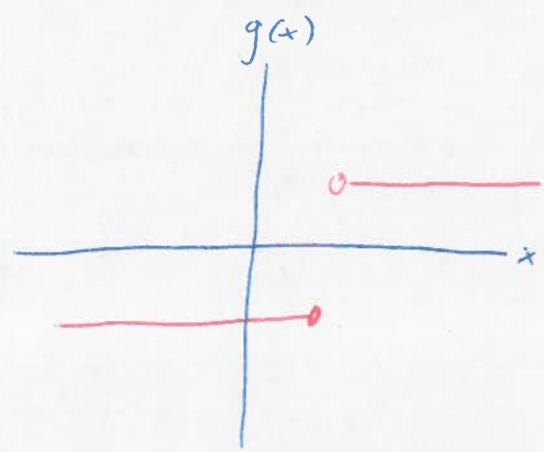
FORMALLY, THE GRAPH OF $f: A \subseteq \mathbb{R} \to \mathbb{R}$ IS THE SET OF POINTS

$$G = \{ (x,y) \in \mathbb{R}^2 : y = f(x),\ x \in A \}.$$

CONTINUITY

A FUNCTION IS CONTINUOUS IF ITS GRAPH CAN BE DRAWN WITHOUT HAVING TO LIFT THE PEN OFF THE PAPER. THE FORMAL DEFINITION INVOLVES LIMITS AND WILL NOT BE PRESENTED HERE.

CONTINUOUS FUNCTION



DISCONTINUOUS FUNCTION

ALL POLYNOMIALS ARE CONTINUOUS FUNCTIONS. ALL SUMS AND PRODUCTS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS FUNCTIONS

A FUNCTION LIKE $f(x) = \begin{cases} -1, & x < 1 \\ 1, & x > 1 \end{cases}$ IS CALLED A PIECEWISE FUNCTION, AND IT IS PIECEWISE CONTINUOUS (EACH PIECE IS A CONTINUOUS FUNCTION).

MONOTONICITY

A FUNCTION f IS MONOTONICALLY INCREASING ON ITS DOMAIN $A \subseteq \mathbb{R}$ IF $\forall x_1, x_2 \in A$,

$$x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2).$$

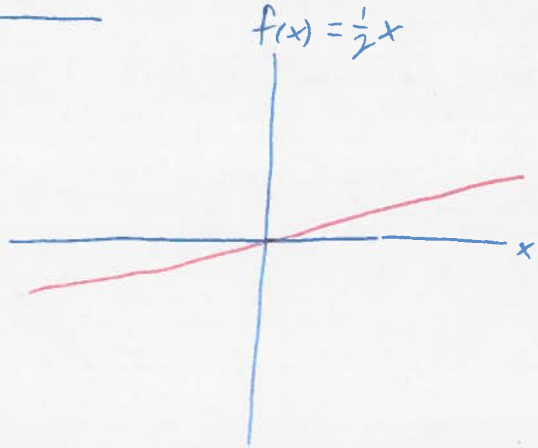
IF THE EQUALITY PART IS NOT ALLOWED (SO $f(x_1) > f(x_2)$), THEN f IS STRICTLY MONOTONICALLY INCREASING.

A FUNCTION f IS (STRICTLY) MONOTONICALLY DECREASING ON ITS DOMAIN $A \subseteq \mathbb{R}$ IF $\forall x_1, x_2 \in A$,

$$x_1 > x_2 \Rightarrow f(x_1) \underset{(<)}{\leq} f(x_2).$$

NOTE: A FUNCTION NEED NOT BE DEFINED EVERYWHERE NOR CONTINUOUS FOR IT TO BE MONOTONE.

EXAMPLES

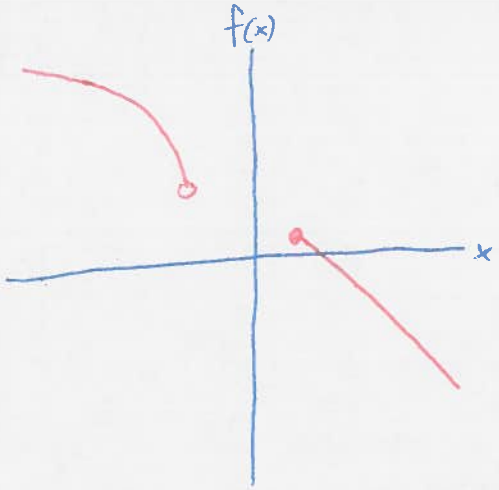


f IS STRICTLY INCREASING: LET $x_1 > x_2$.

$$f(x_1) > f(x_2) \Leftrightarrow \frac{1}{2}x_1 > \frac{1}{2}x_2$$

$$\Leftrightarrow x_1 > x_2 \quad \text{TRUE}.$$

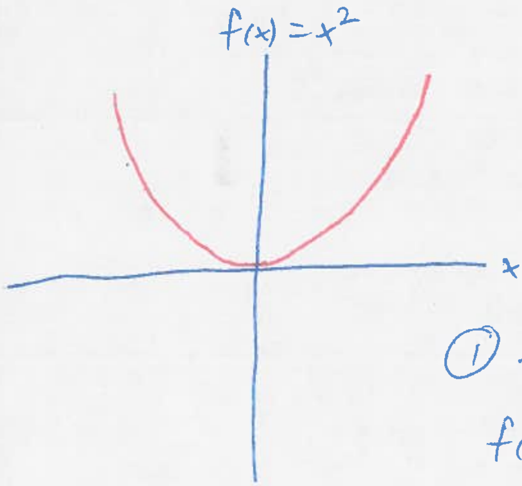
$$\therefore f(x_1) > f(x_2) \;\forall x_1, x_2 \in \mathbb{R}.$$



f IS STRICTLY DECREASING, EVEN THOUGH THERE IS A GAP IN THE DOMAIN AND f IS NOT CONTINUOUS.

LINEAR FUNCTIONS: FOR ANY $a > 0$, THE FUNCTION $f(x) = ax + b$ IS STRICTLY INCREASING AND THE FUNCTION $g(x) = -ax + b$ IS STRICTLY DECREASING.

☆ WHAT ABOUT THE CASE $a = 0$? ☆



A QUADRATIC FUNCTION IS NEITHER INCREASING NOR DECREASING. FOR EXAMPLE WITH $f(x) = x^2$,

① $x_1 = 0, x_2 = -1$

$f(x_1) = 0, f(x_2) = 1$

$\Rightarrow f(x_1) < f(x_2)$

② $x_1 = 1, x_2 = 0$

$f(x_1) = 1, f(x_2) = 0$

$\Rightarrow f(x_1) > f(x_2)$.

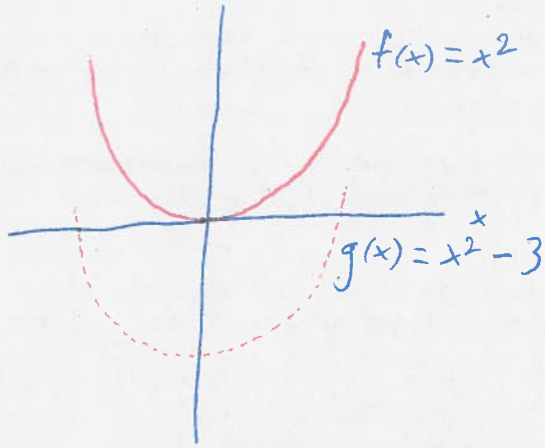
DILATION, TRANSLATION AND REFLECTION OF FUNCTIONS

GIVEN THE GRAPH OF f, THERE ARE 5 BASIC WAYS TO CHANGE THE SHAPE OR THE POSITION.

1) VERTICAL TRANSLATION: $\boxed{g(x) = f(x) + k.}$

IF $k > 0$, THEN g IS f SHIFTED UP BY k UNITS.

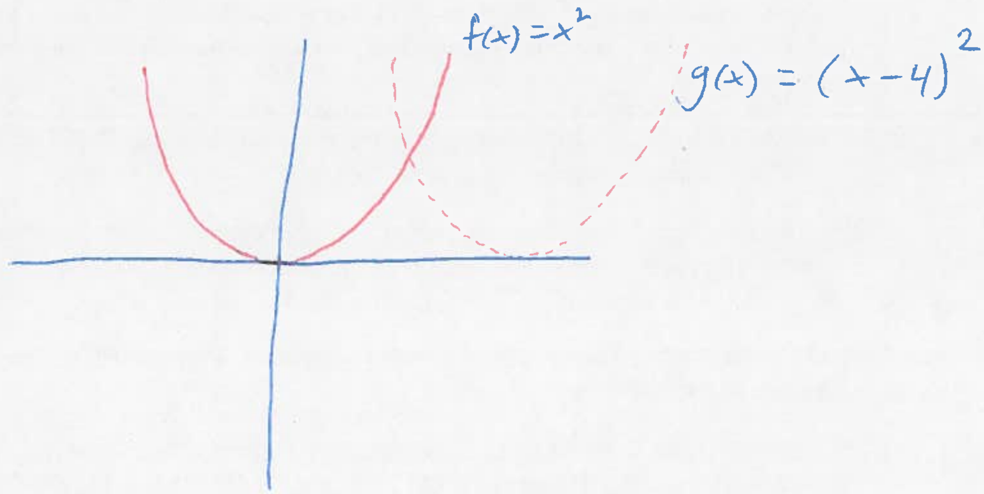
IF $k < 0$, THEN g IS f SHIFTED DOWN BY k UNITS.



2) HORIZONTAL TRANSLATION: $\boxed{g(x) = f(x+k).}$

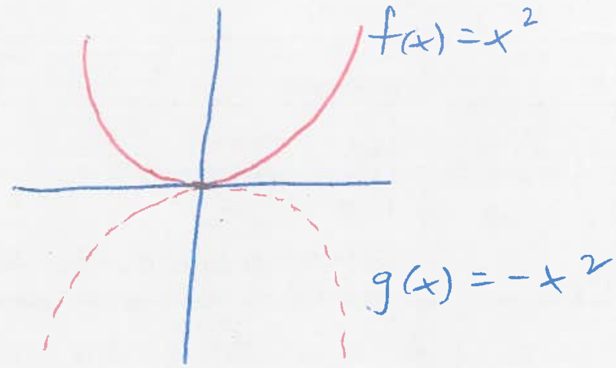
IF $k > 0$, THEN g IS f SHIFTED LEFT BY k UNITS.

IF $k < 0$, THEN g IS f SHIFTED RIGHT BY k UNITS.



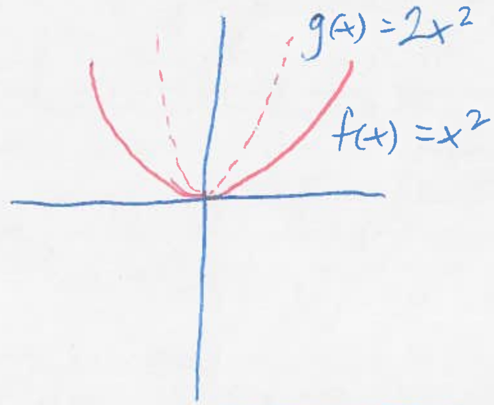
3) REFLECTION: $\boxed{g(x) = -f(x)}$

g IS f TURNED UPSIDE-DOWN.



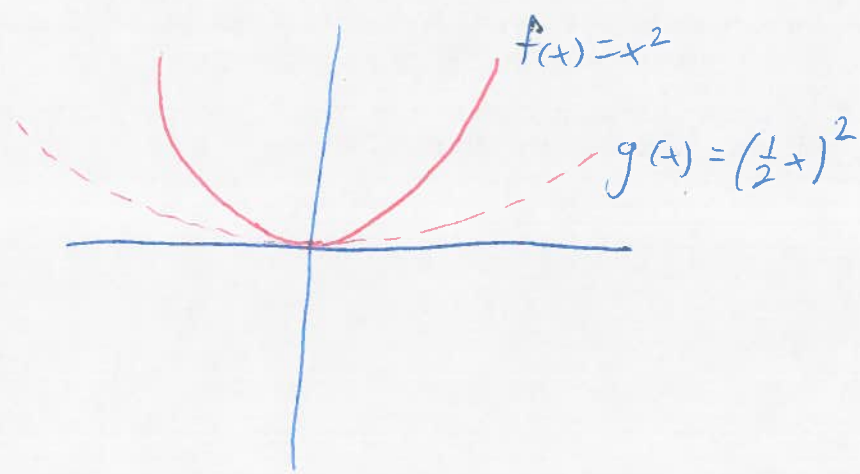
4) VERTICAL DILATION: $\boxed{g(x) = k f(x)}$

IF $|k| > 1$, THEN g IS f STRETCHED VERTICALLY BY FACTOR k.
IF $|k| < 1$, THEN g IS f COMPRESSED VERTICALLY BY FACTOR k.



5) HORIZONTAL DILATION: $\boxed{g(x) = f(kx)}$

IF $|k| > 1$, THEN g IS f COMPRESSED HORIZONTALLY BY FACTOR k.
IF $|k| < 1$, THEN g IS f STRETCHED HORIZONTALLY BY FACTOR k.



COST-PROFIT ANALYSIS

PROFIT = REVENUE − COSTS $\qquad P = R - C$

COSTS = FIXED COSTS + VARIABLE COSTS $\qquad C = C_F + C_V$

VARIABLE COSTS = (# UNITS)(COST PER UNIT) $\qquad C_V = qu$

REVENUE = (# UNITS)(PRICE PER UNIT) $\qquad R = qp$

$$
\begin{aligned}
\Rightarrow P &= R - C \\
&= qp - (C_F + C_V) \\
&= qp - (C_F + qu) \\
&= (p-u)q - C_F . \qquad (1)
\end{aligned}
$$

SOME OBVIOUS DEDUCTIONS CAN BE MADE:

- FIXED COSTS SHOULD BE REDUCED AS MUCH AS POSSIBLE.
- FOR $P > 0$, WE NEED $p > u$. LARGER $p-u$ MEANS MORE PROFIT.

NOTE: THIS BASIC MODEL DOES NOT CONSIDER ALL BUSINESS MATTERS, SUCH AS STORAGE COST, UNSOLD UNITS, ETC.

$P > 0 \Rightarrow$ PROFIT. $P < 0 \Rightarrow$ LOSS. $P = 0$ IS THE BREAK-EVEN POINT.

THE <u>CONTRIBUTION MARGIN</u> PER UNIT IS $p-u$. TOTAL CONTRIBUTION MARGIN IS $(p-u)q$. BY (1), $(p-u)q = P + C_F$.

THE <u>CONTRIBUTION RATE</u> IS $\dfrac{p-u}{p}$.

<u>EX</u>: IN A SUITCASE COMPANY, EACH SUITCASE IS PRODUCED FOR \$80 AND SOLD FOR \$100. THE CONTRIBUTION MARGIN IS $100 - 80 = \$20$, AND THE CONTRIBUTION RATE IS $\frac{20}{100} = 0.2$, OR 20%.

EXAMPLE: BREAK-EVEN ANALYSIS.

THE PRICE OF EACH CELL PHONE IS \$200. THE FIXED COSTS FOR THE PRODUCTION PERIOD IS \$3000 AND THE PRODUCTION COST PER UNIT IS \$40. HOW MANY UNITS MUST BE SOLD IN THIS PERIOD IN ORDER TO BREAK EVEN?

A: FROM ①, $P = 0 \Rightarrow (p-u)q = C_F \Rightarrow q = \dfrac{C_F}{p-u}$.

$$q = \frac{3000}{200-40} = 18.75.$$

REMEMBER THAT ONLY WHOLE UNITS MAKE SENSE! YOU CANNOT SELL THREE QUARTERS OF A CELL PHONE.

18 UNITS IS NOT ENOUGH TO BREAK EVEN, SO 19 UNITS MUST BE SOLD.

EXAMPLE: CONTRIBUTION MARGIN.

A BUSINESS HAS FIXED COSTS \$2500 OVER A PERIOD. IF IT SELLS 100 UNITS OF PRODUCT IN THE PERIOD AND BREAKS EVEN, WHAT IS THE CONTRIBUTION MARGIN?

A: $P = 0 \Rightarrow (p-u)q = C_F \Rightarrow p - u = \dfrac{C_F}{q} = \dfrac{2500}{100} = \$25$.

THE DERIVATIVE OF A SINGLE-VARIABLE FUNCTION

THE DERIVATIVE OF A FUNCTION AT A POINT IS THE INSTANTANEOUS RATE OF CHANGE OF THE FUNCTION AT THAT POINT. IT INDICATES HOW FAST THE FUNCTION IS INCREASING/DECREASING AT THE MOMENT.

EX: A CONSTANT FUNCTION DOES NOT CHANGE, SO ITS DERIVATIVE IS ZERO AT ALL POINTS.

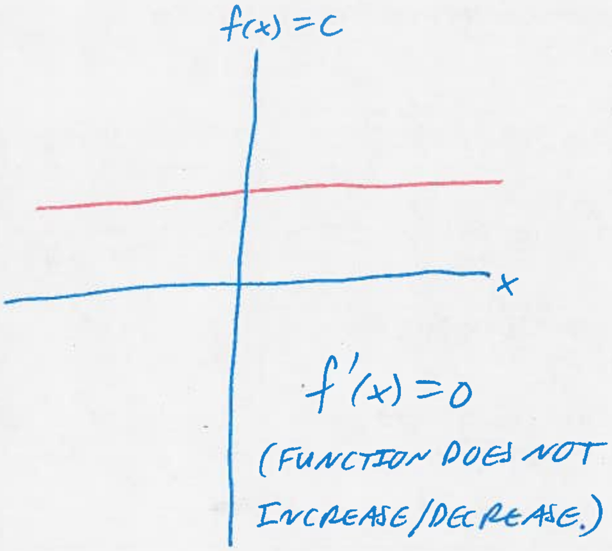
$f(x) = c \Rightarrow f'(x) = 0$. "f PRIME OF x EQUALS ZERO."

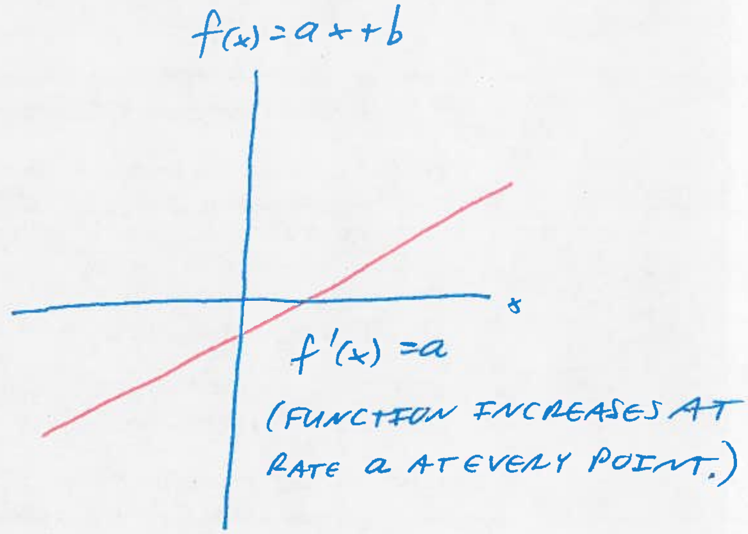
ON THE GRAPH OF A FUNCTION, THE DERIVATIVE IS THE SLOPE OF THE TANGENT LINE AT A POINT.

EX: A LINEAR FUNCTION CHANGES AT A CONSTANT RATE EVERYWHERE.

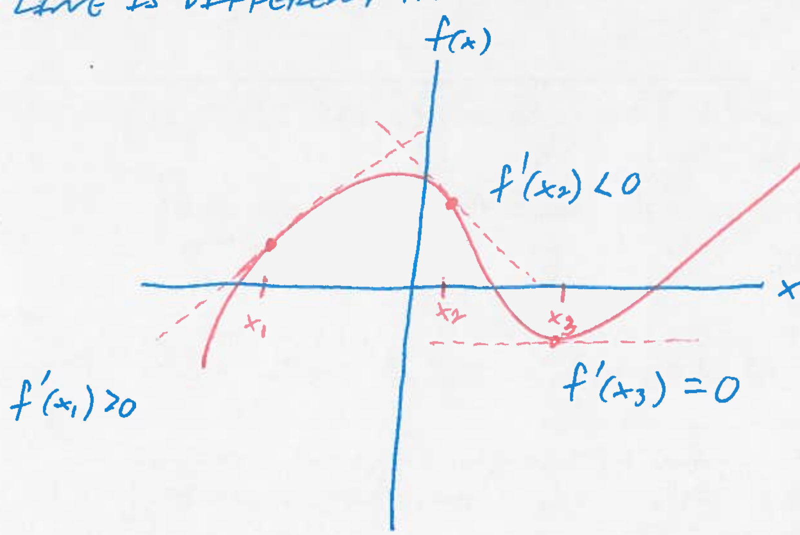
ITS DERIVATIVE EQUALS THE SLOPE OF THE LINE.

$$f(x) = ax + b \Rightarrow f'(x) = a.$$

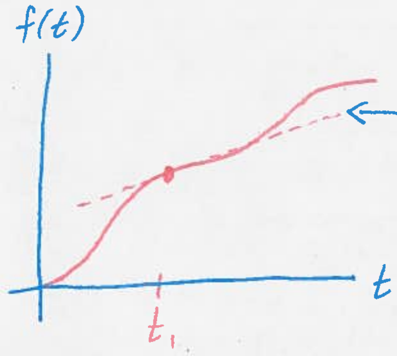




IN GENERAL, $f'(x)$ IS ANOTHER FUNCTION OF x, BECAUSE THE SLOPE OF THE TANGENT LINE IS DIFFERENT AT DIFFERENT POINTS.



EX: IF f IS THE FUNCTION OF DISTANCE TRAVELLED BY A CAR OVER TIME, f' IS THE CHANGE IN DISTANCE AT ANY MOMENT IN TIME. THIS IS THE VELOCITY OF THE CAR.



THE SLOPE OF THIS LINE IS THE VELOCITY OF THE CAR AT TIME t_1.

WE CAN ALSO CONSIDER THE RATE OF CHANGE OF THE VELOCITY FUNCTION BY FINDING THE DERIVATIVE OF $f'(t)$, WRITTEN $f''(t)$. THIS IS THE ACCELERATION OF THE CAR, AND IS CALLED THE SECOND DERIVATIVE OF f.

ANOTHER NOTATION COMMONLY USED: $f'(x) = \frac{d}{dx} f(x)$.

PROPERTIES

1) CONSTANT MULTIPLE RULE:
FOR ANY $k \in \mathbb{R}$, $\frac{d}{dx}[k f(x)] = k \frac{d}{dx} f(x)$

2) SUM RULE: $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

3) MONOMIAL RULE: LET $f_p(x) = x^p$, THEN $f_p'(x) = p x^{p-1}$.

USING THESE PROPERTIES, WE CAN DIFFERENTIATE (FIND THE DERIVATIVE OF) ANY POLYNOMIAL FUNCTION.

EXAMPLES

1) $f(x) = x^2 \Rightarrow f'(x) = 2x$.

2) $g(x) = -\frac{1}{2} x^3 + 4x^4 \Rightarrow g'(x) = \frac{d}{dx}\left(\frac{-1}{2} x^3\right) + \frac{d}{dx}(4x^4) = \frac{-1}{2} \frac{d}{dx} x^3 + 4 \frac{d}{dx} x^4$

$$= \frac{-3}{2} x^2 + 16x^3.$$

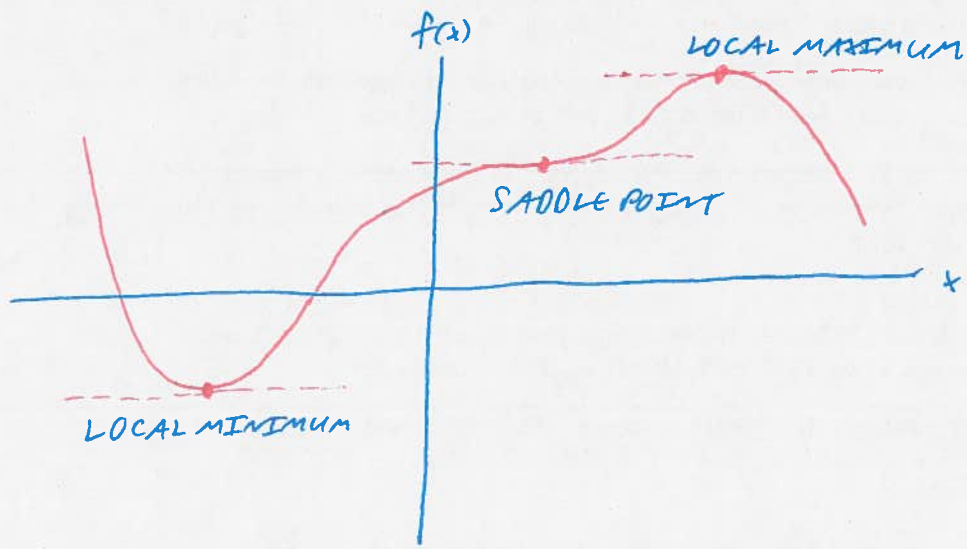
3) $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \Rightarrow$

$p'(x) = a_1 + 2a_2 x + \dots + n a_n x^{n-1}$.

IN SUMMATION NOTATION, $p(x) = \sum_{k=0}^{n} a_k x^k \Rightarrow p'(x) = \sum_{k=0}^{n} k a_k x^{k-1}$.

NOTICE THAT IF $f'(x_0) > 0$, THEN f IS INCREASING AT x_0, AND IF $f'(x_0) < 0$, THEN f IS DECREASING AT x_0.

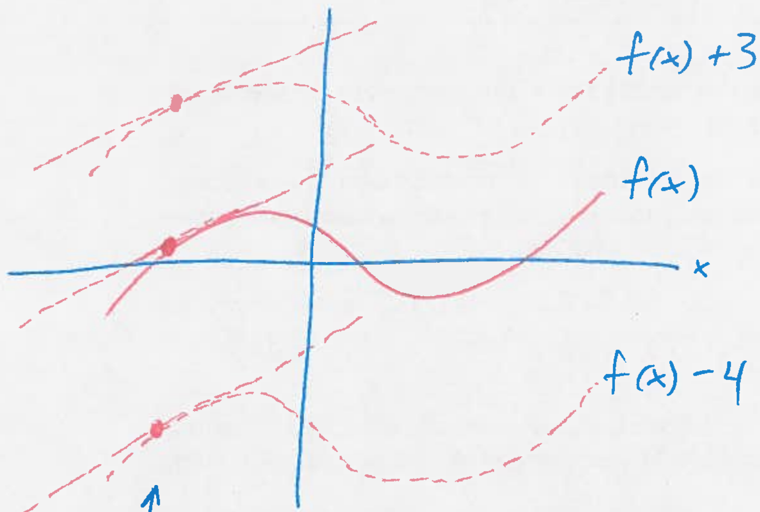
IF $f'(x_0) = 0$, THEN x_0 IS A SPECIAL POINT CALLED A STATIONARY POINT, WHERE THE TANGENT LINE IS HORIZONTAL AND THE FUNCTION IS NEITHER INCREASING NOR DECREASING. A STATIONARY POINT IDENTIFIES A LOCAL MINIMUM VALUE OF f, A LOCAL MAXIMUM OF f, OR A SADDLE POINT OF f.



INTEGRATION / ANTIDIFFERENTIATION

INTEGRATION IS THE OPPOSITE PROCESS OF DIFFERENTIATION. YOU CAN THINK OF IT AS FINDING $f(x)$ WHEN YOU ARE GIVEN $f'(x)$. THE INTEGRATION SYMBOL IS $\int$, SO $\int f'(x)\,dx = f(x)$. THE INTEGRAL OF f IS WRITTEN $\int f(x)\,dx$, WHERE THE "dx" CAN BE INTERPRETED AS "WITH RESPECT TO x".

AN ARBITRARY CONSTANT IS ADDED TO THE SOLUTION OF ANY INTEGRAL. THIS IS BECAUSE ADDING A CONSTANT SHIFTS THE GRAPH UP OR DOWN, BUT DOES NOT CHANGE THE DERIVATIVE ANYWHERE.



DERIVATIVES ARE ALL THE SAME.

SOME SIMPLE INTEGRALS, WORKING BACKWARD FROM DERIVATIVES:

1) WE KNOW THAT $f(x) = c \Rightarrow f'(x) = \emptyset$, SO

$$\int \emptyset \, dx = C, \text{ FOR ANY } C \in \mathbb{R}.$$

2) WE KNOW THAT $f(x) = ax + b \Rightarrow f'(x) = a$, SO

$$\int a \, dx = ax + C, \text{ FOR ANY } C \in \mathbb{R} \text{ (INCLUDING } C = b).$$

3) WE KNOW THAT $f(x) = x^p \Rightarrow f'(x) = px^{p-1}$. SO FOR ANY MONOMIAL, RELABELING $p \to p+1$, WE HAVE $\int x^p \, dx = \frac{1}{p+1} x^{p+1} + C, \; C \in \mathbb{R}$.

PROPERTIES

1) CONSTANT MULTIPLE RULE:

FOR ANY $k \in \mathbb{R}$, $\int k f(x) \, dx = k \int f(x) \, dx$.

2) SUM RULE:

$$\int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx$$

SO FOR ANY POLYNOMIAL $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$, WE HAVE

$$\int f(x)\,dx = C + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots + \frac{a_n}{n+1} x^{n+1}, \quad C \in \mathbb{R}.$$

NOTE: GIVEN A FUNCTION f, THE ANTIDERIVATIVE $\int f(x)\,dx$ IS A <u>FAMILY</u> OF FUNCTIONS, INDEXED BY THE ARBITRARY CONSTANT.

<u>EX</u>: LET $f(x) = 3x^2 - 2x + 5$.

$$\begin{aligned}
\int f(x)\,dx &= \int (3x^2 - 2x + 5)\,dx \\
&= \int 3x^2\,dx + \int -2x\,dx + \int 5\,dx \\
&= 3\int x^2\,dx - 2\int x\,dx + 5\int dx \\
&= 3\left[\tfrac{1}{3}x^3 + C_1\right] - 2\left[\tfrac{1}{2}x^2 + C_2\right] + 5\left[x + C_3\right] \\
&= x^3 - x^2 + 5x + (3C_1 - 2C_2 + 5C_3) \\
&= x^3 - x^2 + 5x + C, \quad C \in \mathbb{R}.
\end{aligned}$$

WE NEED WRITE ONLY ONE CONSTANT OF INTEGRATION, $3C_1 - 2C_2 + 5C_3 = C$.

SO $x^3 - x^2 + 5x$ IS AN ANTIDERIVATIVE OF $3x^2 - 2x + 5$ (CHOOSING $C = 0$), BUT SO IS $x^3 - x^2 + 5x + 1$, AND

$x^3 - x^2 + 5x - 2\pi$, AND

$x^3 - x^2 + 5x + \frac{7}{4}, \dots$

<u>DEFINITE INTEGRALS</u>

BECAUSE THE CONSTANT C CAN BE ANY NUMBER, $\int f(x)\,dx$ IS CALLED AN <u>INDEFINITE INTEGRAL</u>. THE RESULT IS A FAMILY OF FUNCTIONS. THERE IS ALSO A <u>DEFINITE INTEGRAL</u> $\int_a^b f(x)\,dx$, WHICH IS ANY ANTIDERIVATIVE EVALUATED AT b, MINUS THE SAME ANTIDERIVATIVE EVALUATED AT a. THE RESULT IS A NUMBER. WRITTEN WITH DERIVATIVE NOTATION, WE GET THE <u>FUNDAMENTAL THEOREM OF CALCULUS</u>:

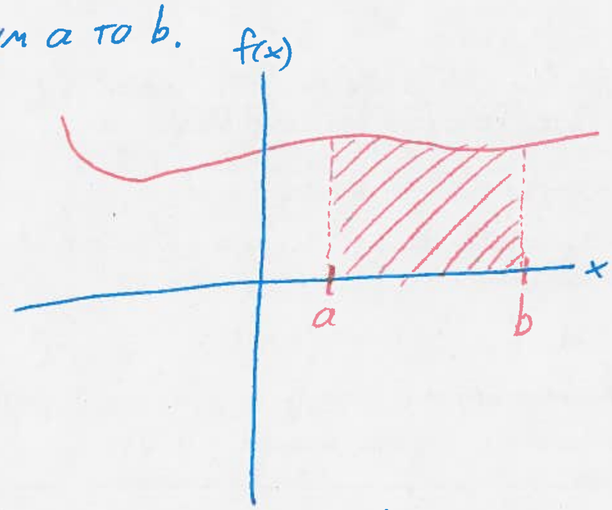
$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

EX: $\int_0^1 (2x+3)\,dx = (x^2+3x)\Big|_0^1 = (1^2+3\cdot 1) - (0^2+3\cdot 0) = 4.$

NOTICE THAT WE DO NOT NORMALLY USE THE CONSTANT C IN DEFINITE INTEGRALS, BECAUSE IT ALWAYS CANCELS:

$$\int_0^1 (2x+3)\,dx = (x^2+3x+C)\Big|_0^1 = (1^2+3\cdot 1+C) - (0^2+3\cdot 0+C) = 4.$$

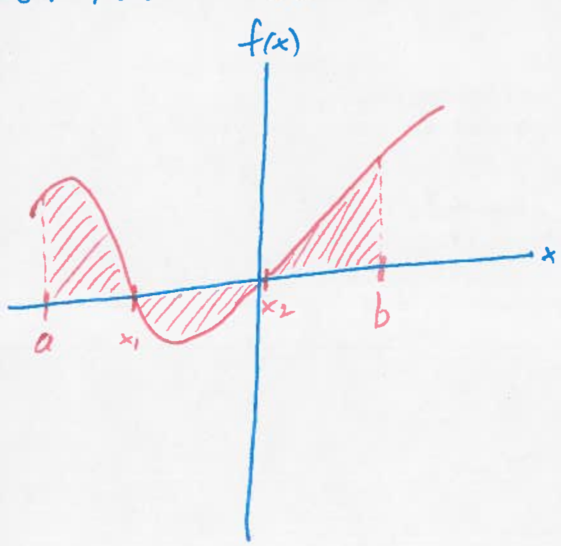
GRAPHICALLY, THE DEFINITE INTEGRAL IS THE AREA UNDER THE CURVE OF THE FUNCTION FROM a TO b.



THE SHADED AREA IS $\int_a^b f(x)\,dx.$

SO FOR THE PREVIOUS EXAMPLE $\int_0^1 (2x+3)\,dx$, THE AREA UNDER THE LINE $y=2x+3$ FROM $x=0$ TO $x=1$ IS 4.

IF f CHANGES SIGN (i.e. DROPS BELOW THE x-AXIS) FROM a TO b, THE NEGATIVE PORTION IS SUBTRACTED, SO THE DEFINITE INTEGRAL GIVES THE NET POSITIVE AREA.



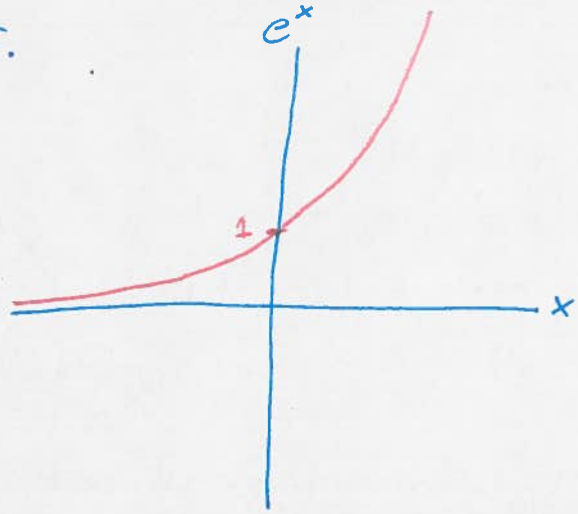
$$\int_a^b f(x)\,dx = f(b) - f(a)$$
$$= \int_a^{x_1} f(x)\,dx - \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{b} f(x)\,dx.$$

THE EXPONENTIAL FUNCTION

THIS IS A FUNCTION $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$, WHERE THE NUMBER

$e \approx 2.71828$. e IS AN IRRATIONAL NUMBER; IT CANNOT BE EXPRESSED AS

A RATIO OF INTEGERS.



THIS FUNCTION HAS SOME SPECIAL PROPERTIES, FOR EXAMPLE $\frac{d}{dx} e^x = e^x$!

AT EVERY POINT, THE GROWTH RATE OF THE FUNCTION IS THE SAME AS THE FUNCTION VALUE.

IT IS A MONOTONICALLY INCREASING FUNCTION, WITH

$\lim_{x \to -\infty} e^x = 0$ (HORIZONTAL ASYMPTOTE)

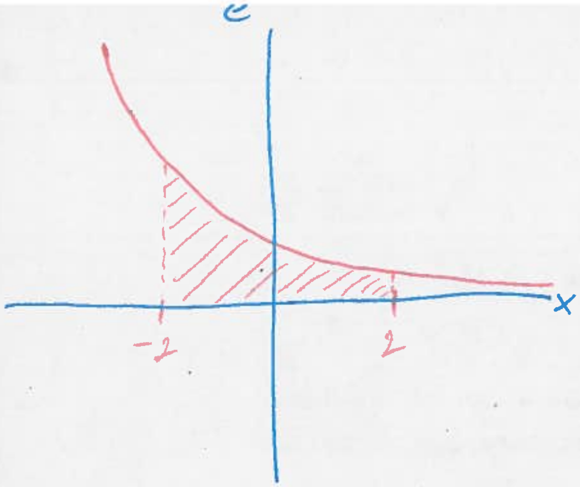
$e^0 = 1$

$\lim_{x \to \infty} e^x = \infty$.

$\int e^x dx = e^x + C$, AND BY USING POWER LAWS WE CAN GET

$\frac{d}{dx} e^{ax+b} = a e^{ax+b}$ AND $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$.

EX: FIND THE AREA UNDER $f(x) = e^{-x}$ FROM $x = -2$ TO $x = 2$.



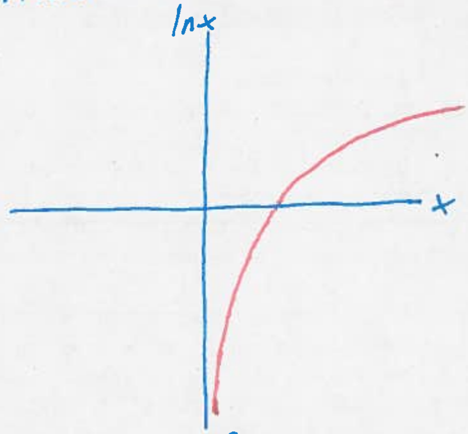
$$\int_{-2}^{2} e^{-x}\,dx = -e^{-x}\Big|_{-2}^{2}$$

$$= -e^{-2} - (-e^{2})$$

$$= e^{2} - \frac{1}{e^{2}} \approx 7.25$$

THE LOGARITHM FUNCTION

THIS IS A FUNCTION $f: \mathbb{R}_+ \to \mathbb{R}$, $f(x) = \ln x$ (PRONOUNCED "LAWN x"), THE LOGARITHM OF BASE e. NOTICE THAT $\operatorname{dom} f = \mathbb{R}_+$; THERE ARE NO LOGARITHMS OF NEGATIVE NUMBERS OR OF ZERO.



$\lim_{x \searrow 0} \ln x = -\infty$; $\ln 1 = 0$; $\lim_{x \to \infty} \ln x = \infty$.

$\ln x$ IS A MONOTONICALLY INCREASING FUNCTION, AND IT'S THE INVERSE FUNCTION OF e^x. THIS MEANS THAT $\ln(e^x) = x$ AND $e^{(\ln x)} = x$.

DERIVATIVE: $\frac{d}{dx} \ln x = \frac{1}{x}$

$$\frac{d}{dx} \ln(ax) = \frac{1}{x}$$

NET PRESENT VALUE (NPV)

NPV IS A METHOD OF EVALUATING WHETHER A BUSINESS SHOULD PROCEED WITH A PROJECT, ESPECIALLY WHEN CASH INFLOWS AND CASH OUTLAYS OCCUR AT DIFFERENT TIMES. THE IDEA IS TO CONVERT ALL CASH AMOUNTS TO PRESENT-DAY VALUES. IF THE DIFFERENCE OF CONVERTED CASH IN AND CASH OUT IS POSITIVE, THE BUSINESS SHOULD PROCEED WITH THE PROJECT. THIS CAN ALSO BE USED TO RANK POSSIBLE PROJECTS.

EX: A FIRM COULD SELL ITS PRODUCT FOR A NET PROFIT OF \$1000 IN A YEAR. THEY WOULD PAY \$900 FOR PRODUCTION EQUIPMENT THAT WOULD HAVE NO VALUE AFTER A YEAR. IF THEY REQUIRE A RATE OF RETURN OF 10%, SHOULD THEY PROCEED?

A: THE CASH OUTLAY IS \$900 IN PRESENT-DAY DOLLARS. THE \$1000 WOULD BE RECEIVED AT THE END OF THE YEAR AND MUST INCLUDE 10% RETURN, SO THE PRESENT-DAY VALUE IS

$$\frac{1000}{1+0.1} = \$909.09.$$

$\therefore NPV = 909.09 - 900 = 9.09 > 0$. THE PROJECT SHOULD PROCEED.

EX: A COMPANY IS OFFERED A PROJECT PROMISING ANNUAL NET RETURNS OF \$36,000 FOR 7 YEARS. THEY WOULD HAVE TO SPEND \$150,000 IMMEDIATELY TO EXPAND THE PLANT. SHOULD THEY ACCEPT THE CONTRACT IF THE REQUIRED RATE OF RETURN IS

i) 12% ii) 15% iii) 18% ?

i) PRESENT-DAY VALUE OF RETURNS:

YEAR 1: $\dfrac{36,000}{1+0.12}$; YEAR 2: $\dfrac{36,000}{(1+0.12)^2}$; ... ; YEAR 7: $\dfrac{36,000}{(1+0.12)^7}$

TOTAL: $36,000\left[\frac{1}{1.12} + \frac{1}{(1.12)^2} + \dots + \frac{1}{(1.12)^7}\right]$.

THIS IS A FINITE GEOMETRIC SERIES WITH $a = \dfrac{36,000}{1.12}$, $r = \dfrac{1}{1.12}$, SO

$$S_7 = \frac{36,000}{1.12}\,\frac{1-\frac{1}{(1.12)^7}}{1-\frac{1}{1.12}} \approx 164,295.$$

$NPV = 164,295 - 150,000 = 4,295 > 0$. ACCEPT THE CONTRACT.

EXERCISE: DO PARTS ii) AND iii).

THIS PATTERN REOCCURS IN SIMILAR SITUATIONS. LET

C = PRESENT UPFRONT COSTS ; PR = PRESENT VALUE RETURNS. THEN

$NPV = PR - C.$

IF PR CONSISTS OF A RETURN OF R EACH PERIOD FOR n PERIODS WITH REQUIRED RATE OF RETURN i, THEN

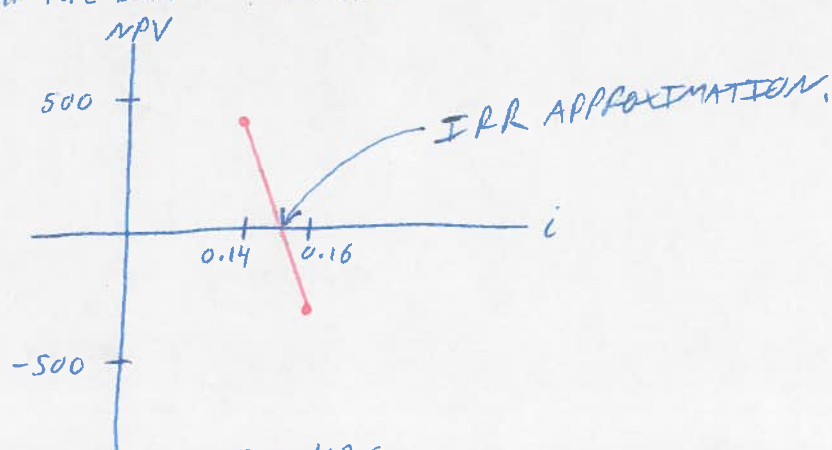
$$NPV = \frac{R}{1+i} \frac{1-\left(\frac{1}{1+i}\right)^n}{1-\frac{1}{1+i}} - C = \frac{R}{i}\left[1-\left(\frac{1}{1+i}\right)^n\right] - C.$$

THE <u>INTERNAL RATE OF RETURN (IRR)</u> IS THE VALUE OF i SUCH THAT $NPV = 0$, i.e. $\frac{R}{i}\left[1-\frac{1}{(1+i)^n}\right] = C$. THIS IS A NONLINEAR EQUATION THAT MAY NOT BE SOLVABLE FOR $n \geq 5$, SO WE FIND AN APPROXIMATE SOLUTION, FOR INSTANCE BY <u>LINEAR INTERPOLATION</u>.

LINEAR INTERPOLATION TAKES SOME DATA POINTS AND FINDS THE BEST-FIT STRAIGHT LINE THAT APPROXIMATES THE FUNCTION. A VERY SIMPLE CASE IS THAT OF 2 DATA POINTS.

<u>Ex</u>: WITH $i = 14\%$, NPV IS \$420, WHILE WITH $i = 16\%$, NPV IS $-$\$280. USE LINEAR INTERPOLATION TO APPROXIMATE IRR.

A: SUPPOSING NPV IS A LINEAR FUNCTION OF i (WHICH IS NOT TRUE), WE DRAW A STRAIGHT LINE THROUGH THE DATA POINTS:



THE SLOPE IS $\frac{-280-420}{0.16-0.14}$, AND ALSO $\frac{0-420}{x-0.14}$. SOLVE FOR x:

$$\frac{-420}{x-0.14} = \frac{-700}{0.02} \Rightarrow x - 0.14 = \frac{-420(0.02)}{-700} \Rightarrow \boxed{15.2\% = x}$$

IRR IS THE RATE OF RETURN AT WHICH THE PROJECT WOULD BREAK EVEN. IF IRR IS HIGH, THEN A LARGE RATE OF RETURN IS REQUIRED TO ENSURE THE PROJECT IS WORTHWHILE.

NVP WITH RESIDUAL VALUE

OFTEN, AN INITIAL INVESTMENT SUCH AS A PLANT OR EQUIPMENT WILL RETAIN SOME VALUE AT THE END OF THE PERIOD. THIS VALUE MUST ALSO BE CONVERTED TO PRESENT VALUE IN CALCULATING NPV.

EX: A PROJECT REQUIRES AN INITIAL INVESTMENT OF $70,000, WHICH HAS A RESIDUAL VALUE OF $15,000 AFTER 6 YEARS. ANNUAL RETURNS ARE $20,000. SHOULD THE PROJECT BE ACCEPTED IF THE REQUIRED RATE OF RETURN IS 16%?

A: PRESENT VALUE OF CASH INCOMING:

$$20{,}000\left[\frac{1}{1.16}+\dots+\left(\frac{1}{1.16}\right)^6\right] = \frac{20{,}000}{1.16}\,\frac{1-\left(\frac{1}{1.16}\right)^6}{1-\frac{1}{1.16}}$$

$$= \frac{20{,}000}{0.16}\left[1-\left(\frac{1}{1.16}\right)^6\right] \approx 73{,}694.72.$$

PRESENT VALUE OF RESIDUAL:

$$\frac{15{,}000}{(1.16)^6} \approx 6156.63$$

$$\therefore NPV = 73{,}694.72 + 6156.63 - 70{,}000 = 9851.35 > 0.$$

THE PROJECT SHOULD PROCEED.

NOTE: THE PRESENT VALUE OF THE RESIDUAL IS LESS THAN THE VALUE AT THE END OF THE PERIOD. NEVERTHELESS, IT CONTRIBUTES TO THE NPV AND MAKES THE i REQUIRED FOR BREAK-EVEN (IRR) SLIGHTLY LOWER.

DATA ANALYSIS

FOR THE STATISTICS PART OF THE COURSE, WE USE THE R SOFTWARE PACKAGE, AVAILABLE FREE AT r-project.org.

OBSERVATIONS ARE USUALLY RECORDED IN A RECTANGULAR ARRAY (SPREADSHEET), WITH ONE ROW FOR EACH OBSERVED UNIT, ONE COLUMN FOR EACH VARIABLE RECORDED.

NAME	GENDER	AGE
HARRY	MALE	20
SALLY	FEMALE	18

DATA CAN BE OF DIFFERENT TYPES: CATEGORICAL (NOMINAL/ORDINAL) OR QUANTITATIVE (INTERVAL/RATIO), DISCRETE OR CONTINUOUS. CATEGORIES MAY BE LABELLED BY TEXT OR NUMBERS.

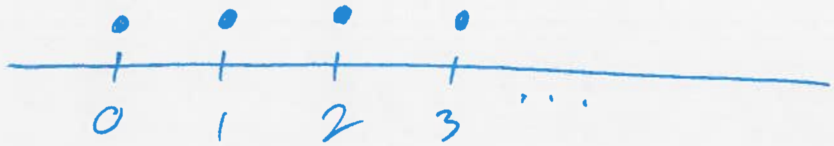
NOMINAL MEASUREMENTS INVOLVE UNORDERED CATEGORIES, eg. GENDER.

ORDINAL MEASUREMENTS INVOLVE ORDERED CATEGORIES, eg. AGE.

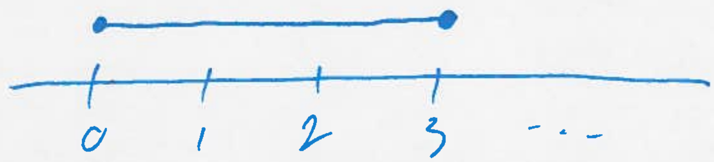
FOR INTERVAL MEASUREMENTS, DIFFERENCES HAVE MEANING BUT RATIOS DO NOT, eg. TEMPERATURE.

FOR RATIO MEASUREMENTS, DIFFERENCES AND RATIOS HAVE MEANING, eg. WEIGHT.

IF ALL POSSIBLE VALUES ARE SEPARATE POINTS ON A NUMBER LINE, THE MEASUREMENT IS <u>DISCRETE</u>. e.g. NUMBER OF EMAILS.



IF VALUES FORM AN INTERVAL ON A NUMBER LINE, THE MEASUREMENT IS <u>CONTINUOUS</u>. e.g. DURATION OF A PHONE CALL.



QUANTITATIVE DATA: SAMPLE MEAN

CONSIDER n DATA VALUES $x_1, \dots, x_n$. THE <u>MEAN</u> IS THE AVERAGE VALUE, DENOTED BY $\bar{x}$:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i .$$

SAMPLE R CODE:

```
x <- c(3,1,4,5,9)
mx <- mean(x)
```

IF DATA VALUES ARE RESCALED BY A <u>LINEAR</u> TRANSFORMATION $x_i \mapsto a x_i + b$, THE MEAN IS RESCALED THE SAME WAY.

<u>EX:</u> $X = \{1, 2, 3\} \Rightarrow \bar{x} = 2$. TRANSFORMING $x_i \mapsto 3x_i + 10$, WE GET $\tilde{X} = \{13, 16, 19\}$, WHOSE MEAN IS 16, AND $3\bar{x} + 10 = 16$.

THIS IS NOT TRUE FOR A NONLINEAR TRANSFORMATION, SUCH AS $x_i \mapsto x_i^2$:

$$\frac{1^2 + 2^2 + 3^2}{3} = \frac{14}{3} \neq \left(\frac{1+2+3}{3}\right)^2 = 4.$$