Relations and Functions

Cartesian Product

- Let A, B be sets, $a \in A, b \in B$.
- An **ordered pair** (a, b) is a pair of elements with the property.

$$(a, b) = (c, d) \Leftrightarrow a = c \wedge b = d$$

- NOTE: The open internal $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ uses the same notation, but context makes it clear.
- The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) with $a \in A, b \in B$.

$$A \times B = \{(a, b): a \in A \wedge b \in B\}$$

Exercise:
Let $A = B = \mathbb{R}$. What is $A \times B$?

$A \times B = \mathbb{R}^2$

Exercise:
Let $A = \{3\}, B = \{2, 3\}$. What is $A \times B$?

$A \times B = \{(3,2), (3,3)\}$

Exercise:
Let $A = \{x, y\}, B = \{1, 2, 3\}, C = \{a, b\}$. What are $A \times B$ and $(A \times B) \times C$?

$A \times B = \{(x,1), (x,2), (x,3), (y,1), (y,2), (y,3)\}$

$(A \times B) \times C = \{(x,1,a), (x,1,b), (x,2,a), (x,2,b), (x,3,a), (x,3,b),$
$(y,1,a), (y,1,b), (y,2,a), (y,2,b), (y,3,a), (y,3,b)\}$

Exercise:
Let $A = \{1, 2\}, B = \{\pi, e\}$. Is $A \times B = B \times A$?

No.

Relations

- We say that R is a **(binary) relation** from A to B if R is a subset of $A \times B$.
- If $R \subseteq A \times A$, then R is called a **relation of A.**
- We say that a is related to b by R if $(a, b) \in R$.
- This is denoted by aRb.

Exercise:

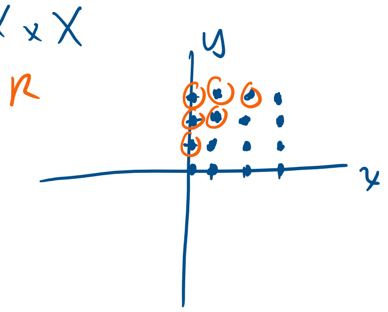
Let $X = \{0, 1, 2, 3\}, R = \{(x, y): \exists\, z \in \mathbb{N} \ni x + z = y\}$

a) What is an easier way of expressing R?
b) List all the elements of R.
c) Sketch $X \times X$ and circle the elements of R.

a) $\{(x,y): y - x \in \mathbb{N}\}$; $\{(x,y): y > x\}$

b) $R = \{(0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}$

c) $X \times X$



Exercise:

Let R *on* $\mathbb{Z}\backslash\{0\}$ be given by $R = \{(x, y): \exists\, z \in \mathbb{Z} \ni xz = y\}$.

a) Describe the relation R.
b) True or false?

a) It's the set $\{(x,y): x$ is a divisor of $y\}$

b) $(2,-4) \in R \longrightarrow 2 \mid -4$ ✓ T

$-3R0 \longrightarrow -3 \mid 0$ × F

$(3,5) \in R \longrightarrow z = \frac{y}{x}, \; z \in \mathbb{Z}$

$= \frac{5}{3}, \; z \in \mathbb{Z}$ □ ✓ T

Exercise:

Let R on $\mathbb{Z}$ be given by $R = \{(m,n) : m - n \text{ } is \text{ } even\}$

a) Give another description of R.
b) Which are elements of R?
   a. $(0,3)$
   b. $(-5,-6)$
   c. $(2,-11)$
   d. $(17,1)$
c) Prove that n $odd \Rightarrow nR1$.

a) $\{(m,n)$: m and n have the same parity$\}$

i.e. $4-2=2$, 2 is even

$2-4=-2$, -2 is also even

b) $(17,1)$

c) Let $m=1$, and $n=3$

$1-3=-2$

-2 is even

$\therefore n$ odd $\Rightarrow nR1$

Union and Intersection of Relations

- Relations are sets, so the set operations apply.

Exercise:

Let R_1, R_2 on $\mathbb{R}$ by given by $R_1 = \{(x,y) : x = y\}, R_2 = \{(x,y) : x = -y\}$.
Write expressions for $R_1 \cup R_2$ and $R_1 \cap R_2$.

$R_1 \cup R_2 = \{(x,y) : x = y \vee x = -y\}$

$R_1 \cap R_2 = \varnothing$

Definition (Domain and Range)

- Let R be a relation from A to B.
- The **domain** of R and the **range** of R, denoted respectively by $domR$ and $ranR$, are defined:

$$dom\ R = \{x : \exists\ y \ni xRy\}$$
$$ran\ R = \{y : \exists\ x \ni xRy\}$$

- Note that $dom\ R \subseteq A$ and $ran\ R \subseteq B$.

Exercise:

Let $A = \{0, 1, 2, 3\}, R = \{(0,0), (0,1), (0,2), (3,0)\}$. Write $dom\ R$ and $ran\ R$.

$dom\ R = \{0, 3\}$

$ran\ R = \{0, 1, 2\}$

Exercise:

Find domain and range of R on $\mathbb{Z} \times \mathbb{Q}, R = \{(x, y): x \neq 0 \wedge y = \frac{1}{x}\}$.

$dom\ R = \mathbb{Z} \setminus \{0\}$

$ran\ R = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup \{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\}$

Exercise:

Find domain and range of R on $\mathbb{Z}, R = \{(x, y) : xy \neq 0\}$.

$dom\ R = \{x : x \neq 0\}$

$ran\ R = \{y : y \neq 0\}$

The Inverse of a Relation

- If R is on $A \times B$, then a relation R^{-1}on $B \times A$ can be defined by interchanging the elements of the ordered pairs of R.

Definition:

- Let R be on $A \times B$. The inverse relation of R is:

$$R^{-1} = \{(y, x) \in B \times A : (x, y) \in R\}$$

- Note that $dom\ R^{-1} = ran\ R$ and $ran\ R^{-1} = dom\ R$.

Exercise:

Let $A = \{a, b, c\}, B = \{1, 2, 3, 4\}, R = \{(a, 1), (b, 2), (c, 3), (a, 4)\}$. Find R^{-1}.

$R^{-1} = \{(1, a), (2, b), (3, c), (4, a)\}$

Exercise:

Define R on $\mathbb{N}$ by $R = \{(x, y) : y = 2x\}$. Write 3 elements of R and 3 elements of R^{-1}. Write a definition of R^{-1}.

$R = \{(1, 2), (2, 4), (3, 6), \ldots\}$

$R^{-1} = \{(2, 1), (4, 2), (6, 3), \ldots\}$

$R^{-1} = \{(y, x) : x = 2y\}$

Exercise:

The identity relation on $\mathbb{R}$ is $R = \{(x, x) : x \in \mathbb{R}\}$. What is R^{-1}?

$R = R^{-1}$

Properties of Relations

- Let R be a relation on A. Then:
  1) R is **reflexive** on A IFF $\forall x \in A, (x, x) \in R$.
  2) R is **symmetric** on A IFF $\forall\, x, y \in A, (x, y) \in R \Rightarrow (y, x) \in R$.
  3) R is **transitive** on A IFF $\forall\, x, y, z \in A, (x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$.

Exercise:

Which properties do the following relations satisfy?

a) On $\mathbb{N}$, $R = \{(x, y) : x \textit{ is a factor of } y\}$ - reflexive, transitive
b) On $\mathbb{R}$, the identity relation – all 3
c) On $\mathbb{Z}$, $R = \{(x, y) : x < y\}$ - transitive
d) On $\mathbb{R}$, $R = \{(x, y) : y = x^2\}$ – None
e) On the set of all people, $R = \{(x, y) : x \textit{ is in the family of } y\}$ – symmetric, reflexive
f) On the set of all people, $R = \{(x, y) : x \textit{ loves } y\}$- reflexive

Equivalence Relations

Definition

- Let R be a relation on A. Then R is an equivalence relation of A IFF R is *reflexive*, *symmetric*, and *transitive* on A.

Exercise:

Prove or disprove that the identity relation on $\mathbb{R}$ is an equivalence relation.

Reflexive: by definition of the identity relation on $\mathbb{R}$,
$xRx \ \forall x \in \mathbb{R}$

Symmetric: $\forall\, x, y \in \mathbb{R},\ xRy \Rightarrow x = y \Rightarrow yRx$

Transitive: $\forall\, x, y, z \in \mathbb{R},\ xRy \Rightarrow x = y$
$yRz \Rightarrow y = z$
$\Rightarrow x = z$

$\therefore (x, z) \in R$

$\therefore$ The identity relation on $\mathbb{R}$ is an equivalence relation because it satisfies reflexivity, symmetric, and transitivity.

Exercise:

On $\mathbb{Z}$, prove that $R = \{(a, b) : a \equiv b \ (mod\ n)\}$ is an equivalence relation.

$a \equiv b \ (mod\ n)$

$n \mid (b-a)$

For example, suppose $x = 13$, $(13, 13) \in R$

$n \mid (13-13) = n \mid 0$, $\frac{0}{n} = 0$, $0 \in \mathbb{Z}$

Reflexive ✓

$n \mid (b-a)$, $n \mid (a-b) \Rightarrow n \mid (-1)(a-b) \Rightarrow n \mid (b-a)$

Symmetric ✓

If (a, b) and $(b, c) \in R$, then prove $(a, c) \in R$

$(a, b) \in R \Rightarrow n \mid (b-a) \Rightarrow b-a = np$, $p \in \mathbb{Z}$

$(b, c) \in R \Rightarrow n \mid (c-b) \Rightarrow c-b = nq$, $q \in \mathbb{Z}$

$(a, c) \in R \Rightarrow n \mid (c-a) \Rightarrow c-a \underbrace{+b-b}_{\text{add zero}} = \underbrace{(c-b)+(b-a)}_{\text{rearrange}} = np + nq$

$= n(p+q)$

transitive ✓

- To disprove an equivalence relation, you only need to show that one of the properties does not hold.

Exercise:

On $\mathbb{Z}$, prove that $R = \{(a, b) : ab = 0\}$ is not an equivalence relation.

Given $a = 2$, $2 \cdot 2 \neq 0$, $\therefore (a, a) \notin R$ $\therefore$ NOT reflexive

Equivalence Classes

Definition

- Let R be an equivalence relation on A. For each $a \in A$, the **equivalence class** of a, denoted $[a]$, is the set:

  $[a] = \{x \in A : xRa\}$

- Equivalence classes have the following properties:
  1) For any $a, b \in A$, we have either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.
  2) All distinct equivalence classes form a **partition** of A
     a. The *union* of all classes is A, and the *intersection* of any 2 classes is empty.

Exercise:

Let $A = \{0,1,2\}, R = \{(0,0),\ (1,\underline{1}),\ (2,2),\ (0,1),\ (1,\underline{0})\}$. Find [0], [1], [2].

$[0] = \{0,1\}$

$[1] = \{1,0\}$

$[2] = \{2\}$

Exercise:

What do the equivalence classes of the identity relation on $\mathbb{R}$ look like?

$[1] = \{1\}$

$[\frac{2}{5}] = \{\frac{2}{5}\}$

Exercise:

Let $R\ on\ \mathbb{Z}$ be defined by $R = \{(a,b) : a \equiv b\ (mod\ 3)\}$. Find [0], [1], [2].

$3 \mid (b-0) \Rightarrow [0] = \{\ldots, -3, 0, 3, 6, \ldots\}$

$3 \mid (b-1) \Rightarrow [1] = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$

$3 \mid (b-2) \Rightarrow [2] = \{\ldots, -7, -4, -1, 2, 5, \ldots\}$