MATH221 Mathematics for Computer Science

Outline Solutions to Tutorial Sheet Week 11

Autumn 2017

1. (i) $f = \{(1, a), (1, b)\}$ is not a function as the domain is not A and it takes two different values at 1.

(ii) $f = \{(1, a), (2, b), (3, c), (4, a)\}$ is a function which is not one-to-one as $f(1) = f(4) = a$ and is not onto as the values d, e are not taken.

(iii) $f = \{(1, a), (2, b), (3, c), (4, d)\}$ is a function which is one-to-one as $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Note that no function $f \subseteq A \times B$ can be onto.

(iv) $f = \{(a, 1), (b, 2), (c, 3), (d, 4), (e, 1)\}$ is a function which is onto as all values in A are taken. Note that no function $f \subset B \times A$ can be one-to-one by the pigeonhole principle.

(iv) $f = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$ is a function which is one-to-one and onto but is not the identity function (in fact it is a permutation).

(vi) $R = \{(1, 2)\}$ is not reflexive, symmetric or transitive.

(vii) $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3)\}$ is reflexive but not symmetric and not transitive as $(1, 2), (2, 3) \in R_1$ but $(1, 3) \notin R_1$. $R_2 = \{(1, 2), (2, 1)\}$ is symmetric, but not reflexive and not transitive as $(1, 2), (2, 1) \in R_2$ but $(1, 1) \notin R_2$. $R_3 = \{(1, 1)(1, 3)\}$ is not reflexive and not symmetric but is transitive as $(1, 1), (1, 3) \in R_3$ and $(1, 3) \in R_3$.

(viii) $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (3, 4), (4, 3), (3, 1), (1, 3)\}$ is reflexive and symmetric but is not transitive as $(4, 3), (3, 1) \in R_1$ but $(4, 1) \notin R_1$. $R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)(3, 4)\}$ is relexive and transitive but not symmetric. $R_3 = \{(2, 2)\}$ is symmetric and transitive but is not reflexive.

(ix) $R = A \times A$ is reflexive, symmetric and transitive. For any $x \in A$ we have $[x] = A$.

2. (i) Suppose that $f(x_1) = f(x_2)$. Then

$$x_1^2 + 1 = x_2^2 + 1 \iff x_1^2 = x_2^2 \iff x_1 = x_2 \vee x_1 = -x_2.$$

Since the domain of f is $[0, \infty)$, f is one-to-one. Note that there is no $x \in [0, \infty)$ for which $f(x) = x^2 + 1 = 0$. Therefore, f is not onto. If, instead we define $f : [0, \infty) \to \underline{[1, \infty)}$ then f is one-to-one and onto and so $f^{-1} : [1, \infty) \to [0, \infty)$ can be defined by $f^{-1}(y) = \sqrt{y^2 - 1}$.

(ii) Since $-1, 1$ belong to the domain of f and $f(1) = f(-1) = 1$, f is not one-to-one. For each $y \geq 0$, $\sqrt[4]{y} \in \mathbb{R}$, satisfies $f(\sqrt[4]{y}) = y$. Hence f is onto. If, instead we define $f : [0, \infty) \to [0, \infty)$ then f is one-to-one and onto and so $f^{-1} : [0, \infty) \to [0, \infty)$ can be defined by $f^{-1}(y) = \sqrt[4]{y}$.

(iii) The function is one-to-one as the graph of f satisfies the horizontal line test (also $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$). The function is onto as every horizontal line meets the graph (also given $y \in \mathbb{R}$ there is $x = \sqrt[3]{y}$ such that $f(x) = y$). Hence the function has an inverse $f : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(y) = \sqrt[3]{y}$.

(iv) Let $x_1, x_2 \in (0, 1)$ such that $f(x_1) = f(x_2)$. Then we have

$$\frac{x_1}{1 - x_1} = \frac{x_2}{1 - x_2} \iff x_1(1 - x_2) = x_2(1 - x_1) \iff x_1 - x_1x_2 = x_2 - x_1x_2 \iff x_1 = x_2.$$

Hence f is one-to-one. To show that f is onto, observe that for each $y \in (0, \infty)$, $x := \dfrac{y}{1 + y} \in (0, 1)$, the domain of f, such that

$$f(x) = f\left(\frac{y}{1 + y}\right) = \frac{\frac{y}{1+y}}{1 - \frac{y}{1+y}} = y.$$

Since f is one-to-one and onto it has a unique inverse. Now $f = \{(x, x/(1 - x)), x \in (0, 1)\} \subset (0, 1) \times (0, \infty)$, so $f^{-1} = \{(x/(1 - x), x) : x \in (0, 1)\}$. Let $y = x/(1 - x)$ then $y \in (0, \infty)$

and $x = y/(1+y)$, hence we may write $f^{-1}\{(y, y/(1+y)) : y \in (0,\infty)\}$. Also observe that the function $g : (0,\infty) \to (0,1)$ satisfies

$$(g \circ f)(x) = g(f(x)) = \frac{f(x)}{1+f(x)} = \frac{\frac{x}{1-x}}{1+\frac{x}{1-x}} = x, \quad \forall x \in (0,1).$$

Hence g is the inverse of f.

3.

(i) The function $\cos : \mathbb{R} \to \mathbb{R}$ is not one-to-one as $\cos(0) = \cos(2\pi)$ and not onto as the value 2 is not taken. If we restrict the domain to $[0,\pi]$ then $\cos : [0,\pi] \to [-1,1]$ is one-to-one (passes horizontal line test) and onto (every horizontal line cuts the graph of $\sin$, see also right-angled triangle argument in (ii) below, suitably adapted for $\cos$) and hence the inverse $\arccos : [-1,1] \to [0,\pi]$ can be defined.

(ii) The function $\tan : \mathbb{R} \to \mathbb{R}$ is not one-to-one as $\tan(0) = \tan(2\pi)$ but it is onto (given $y \in (0,\infty)$ create a suitable right-angled triangle with adjacent side length 1 and the opposite side length y, then fix picture appropriately for $y \in (-\infty, 0]$). If we restrict the domain to $(-\pi/2, \pi/2)$then $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is one-to-one and onto and hence the inverse $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ can be defined.

(iii) The function $\exp : \mathbb{R} \to \mathbb{R}$ is not onto as the value -1 is not taken. If we restrict the range to $(0,\infty)$ then $\exp : \mathbb{R} \to (0,\infty)$ is one-to-one (passes horizontal line test) and onto (every horizonal line cuts the graph of $\exp$) and hence the inverse $\ln : (0,\infty) \to \mathbb{R}$ can be defined.