THEN B IS CALLED A SUPERSET OF A. WE ALSO SAY THAT A IS CONTAINED IN B, AND THAT B CONTAINS A.

IF AT LEAST ONE ELEMENT OF A IS NOT IN B, THEN A IS NOT A SUBSET OF B. SYMBOLICALLY,

$$A \nsubseteq B \iff \exists x \ni x \in A \wedge x \notin B.$$

<u>EXAMPLES</u>: DECIDE TRUE OR FALSE.

a) $\{1,2\} \subseteq \{1,2,3\}$

b) $\{0,2\} \subseteq \{1,2,3\}$

c) $-1 \in \{x \in \mathbb{N} : x^2 = 1\}$

d) $\{1\} \in \{x \in \mathbb{N} : x^2 = 1\}$

e) FOR ALL SETS A, $\phi \subseteq A$.

<u>DEF</u>: A SUBSET $A \subseteq B$ IS <u>PROPER</u> IF $\exists x \in B \ni x \notin A$. WE WRITE $A \subset B$. FOR EXAMPLE, $\{1\} \subseteq \{1,2\}$ AND $\{1,2\} \subseteq \{1,2\}$, BUT $2 \in \{1,2\}$ AND $2 \notin \{1\}$, SO ACTUALLY $\{1\} \subset \{1,2\}$.

<u>EX</u>: ORDER THE SETS $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Q}$, ϕ AND $\mathbb{Z}$ IN TERMS OF SUBSETS. ARE ANY OF THESE PROPER SUBSETS?

EX: TRUE OR FALSE? LET $A = \{1, 2, 3\}$.

a) $A \subset A$

b) $\emptyset \in A$

c) $\emptyset \subseteq A$

d) $\{\emptyset\} \subseteq A$

e) $2 \in A$

f) $\{2\} \in A$

g) $2 \subseteq A$

h) $\{2\} \subseteq A$

i) $\{2\} \subseteq \{\{1\}, \{2\}\}$

j) $\{2\} \in \{\{1\}, \{2\}\}$

DEF: LET A AND B BE SETS. WE SAY A EQUALS B, WRITTEN $A = B$, IFF A CONTAINS B AND B CONTAINS A. SYMBOLICALLY,

$$A = B \iff A \subseteq B \wedge B \subseteq A.$$

TO PROVE TWO SETS ARE EQUAL, PROVE THE TWO CONTENTIONS, $A \subseteq B$ AND $B \subseteq A$.

EX: PROVE THAT $A = \{n \in \mathbb{N} : n \text{ IS EVEN}\}$ AND $B = \{n \in \mathbb{N} : n^2 \text{ IS EVEN}\}$ ARE EQUAL.

($\subseteq$) LET $n \in A$. THEN $n = 2k$ FOR SOME $k \in \mathbb{N}$.
$n^2 = n \cdot n = (2k)(2k) = 2(2k^2)$, $2k^2 \in \mathbb{N}$. $\Rightarrow n^2$ IS EVEN, $\Rightarrow n \in B$.
$\therefore A \subseteq B$.

($\supseteq$) LET $n \in B$, SO n^2 IS EVEN. SUPPOSE THAT n IS ODD, $n = 2k+1$ FOR SOME $k \in \mathbb{N}$. THEN $n^2 = (2k+1)(2k+1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ IS ODD $\overset{\nabla}{\circ}$ HENCE, n IS EVEN AND $n \in A$.
$\therefore B \subseteq A$.

$\therefore A = B$. $\square$

Ex: DEFINE

$$A = \{n \in \mathbb{Z} : n = 2p, p \in \mathbb{Z}\};$$

$$B = \{n \in \mathbb{Z} : n \text{ IS EVEN}\};$$

$$C = \{n \in \mathbb{Z} : n = 2q - 2, q \in \mathbb{Z}\};$$

$$D = \{k \in \mathbb{Z} : k = 3r + 1, r \in \mathbb{Z}\}.$$

a) IS $A = B$? b) IS $A = D$? c) IS $A = C$?

OPERATIONS ON SETS: LET A, B BE SUBSETS OF A UNIVERSE $\mathcal{U}$.

1) THE UNION OF A AND B, WRITTEN $A \cup B$, IS THE SET OF ALL ELEMENTS THAT ARE IN A OR IN B.

$$A \cup B = \{x \in \mathcal{U} : x \in A \vee x \in B\}$$

2) THE INTERSECTION OF A AND B, WRITTEN $A \cap B$, IS THE SET OF ALL ELEMENTS THAT ARE IN A AND IN B.

$$A \cap B = \{x \in \mathcal{U} : x \in A \wedge x \in B\}$$

3) THE COMPLEMENT OF A, WRITTEN $\overline{A}$ OR $\mathcal{U} \setminus A$, IS THE SET OF ALL ELEMENTS THAT ARE NOT IN A.

$$\overline{A} = \mathcal{U} \setminus A = \{x \in \mathcal{U} : x \notin A\}$$

4) THE DIFFERENCE B MINUS A, WRITTEN $B - A$, IS THE SET OF ALL ELEMENTS THAT ARE IN B AND NOT IN A.

$$B - A = \{x \in \mathcal{U} : x \in B \wedge x \notin A\}$$

THE POWER SET OF A UNIVERSE $\mathcal{U}$, DENOTED BY $P(\mathcal{U})$, IS THE SET OF ALL SUBSETS OF $\mathcal{U}$.

EX: LET $A = \{1,2,3\}$. THEN

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, A\}.$$

IF $|A| = n$, THEN $|P(A)| = 2^n$.

THE OPERATIONS OF SET THEORY ARE EQUIVALENT TO THEIR COUNTERPART CONNECTIVES OF LOGIC, AS FOLLOWS.

SET OPERATION	NAME	CONNECTIVE
$\setminus$	COMPLEMENT	$\sim$
$\cup$	UNION	$\vee$
$\cap$	INTERSECTION	$\wedge$
$\subseteq$	SUBSET	$\Rightarrow$
$=$	EQUALITY	$\Leftrightarrow$

EX: LET $\mathcal{U} = \mathbb{Z}$. WRITE DOWN $\bar{A}$ FOR THE FOLLOWING.

a) $A = \{1,2,3\}$

b) $A = \{x \in \mathbb{Z} : x \text{ IS EVEN}\}$

c) $A = \{x \in \mathbb{Z} : x > 0 \vee x < 0\}$

EX: LET $\mathcal{U} \in \mathbb{R}$. WRITE DOWN $A \cup B$ FOR THE FOLLOWING, AND $A \cap B$.

a) $A = \{1\}, B = \{2\}$

b) A IS THE SET OF EVEN INTEGERS, B IS THE SET OF ODD INTEGERS.

c) $A = \{x \in \mathbb{R} : 0 \le x \le 2\}$ AND $B = \{x \in \mathbb{R} : 1 \le x \le 3\}$

EX: PROVE OR DISPROVE: $(A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cup B \subseteq C$.

A: $[(A \Rightarrow C) \wedge (B \Rightarrow C)] \Rightarrow [(A \vee B) \Rightarrow C]$. USE A TRUTH TABLE, OR SUPPOSE THE MAIN CONNECTIVE IS FALSE AND SHOW A CONTRADICTION.

$$[(A \Rightarrow C) \wedge (B \Rightarrow C)] \underset{F}{\Rightarrow} [(A \vee B) \Rightarrow C]$$

$$\underset{T}{(A \Rightarrow C) \wedge (B \Rightarrow C)} \text{ AND } \underset{F}{(A \vee B) \Rightarrow C}$$

$$\underset{T}{A \Rightarrow C} \text{ AND } \underset{T}{B \Rightarrow C} \text{ AND } \underset{T}{A \vee B} \text{ AND } \underset{F}{C}.$$

$$\underset{F}{A} \text{ AND } \underset{F}{B} \text{ AND } \underset{T}{A \vee B} \quad \nabla!$$

$\therefore (A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cup B \subseteq C$. $\square$

EXERCISE PROVE OR DISPROVE: $(A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cap B \subseteq C$.

EX: LET $U = \mathbb{R}$, $A = \{1, 2, 3\}$, $B = \{2\}$, $C = \{2, 3, 4\}$, $D = [0, 1]$.

a) $A - C =$

b) $B - C =$

c) $D - B =$

d) $D - A =$

e) $A - D =$

DEF: THE SETS A AND B ARE <u>DISJOINT</u> IF $A \cap B = \emptyset$.

EX: LET $U=\mathbb{R}$. WRITE DOWN SOME SETS THAT ARE DISJOINT TO THE FOLLOWING.

a) $A=\{x\in\mathbb{Z} : x \text{ IS EVEN}\}$

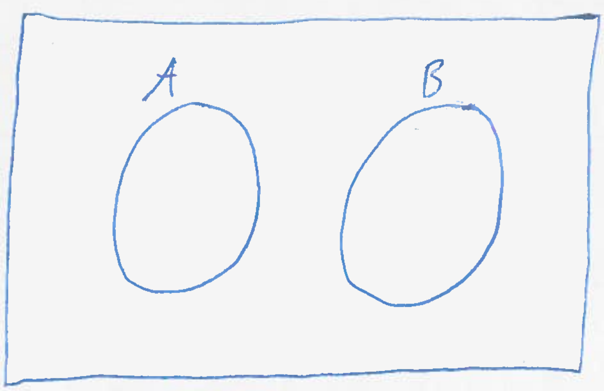
b) $A=\{x\in\mathbb{R} : x^2-5x+6\geq 0\}$

c) $A=\mathbb{Q}$

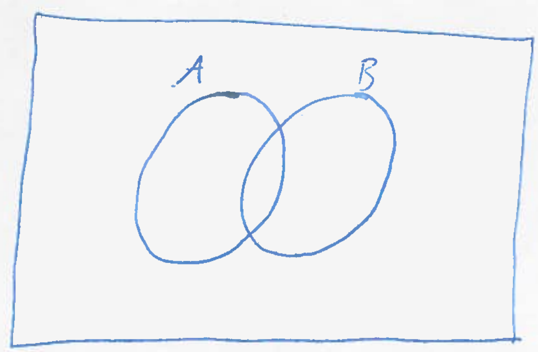
DEF (ADDITION RULE): LET A, B BE FINITE, DISJOINT SETS. THEN $A\cup B$ IS FINITE AND $|A\cup B| = |A|+|B|$.

VENN DIAGRAMS

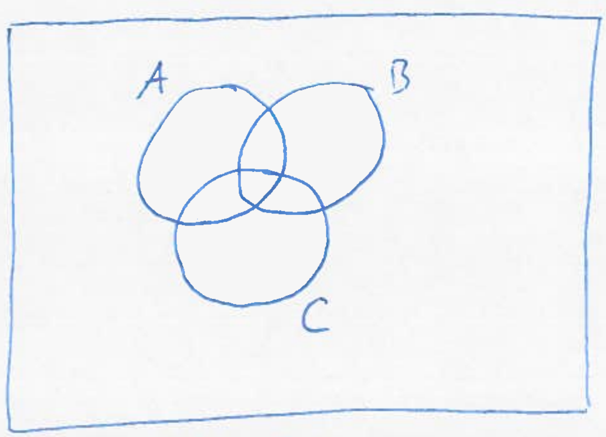
IF WE REPRESENT SETS AS REGIONS IN THE PLANE, THEN THE RELATIONSHIPS AMONG SETS CAN BE REPRESENTED BY DRAWINGS CALLED VENN DIAGRAMS.



2 DISJOINT SETS



2 INTERSECTING SETS



3 INTERSECTING SETS

ALGEBRA OF SETS

THERE ARE MANY RULES THAT GOVERN SET THEORY AND THE RELATIONSHIPS AMONG SETS. ALL OF THE FOLLOWING STATEMENTS CAN BE PROVED USING THE DEFINITIONS WE HAVE SEEN SO FAR.

<u>THEOREM</u>: LET U BE A SET, AND A, B, C BE ELEMENTS OF $P(U)$. THEN

1) $(A \subseteq B \wedge B \subseteq C) \Rightarrow A \subseteq C$

   $A \subseteq A \cup B$; $B \subseteq A \cup B$

   $A \cap B \subseteq A$; $A \cap B \subseteq B$

2) $A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A)$

   $A \subseteq B \Leftrightarrow A \cup B = B$

   $A \subseteq B \Leftrightarrow A \cap B = A$

3) $A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$

   $A \subseteq B \Rightarrow A \cap C \subseteq B \cap C$

4) COMMUTATIVE LAWS:

   $A \cup B = B \cup A$

   $A \cap B = B \cap A$

5) ASSOCIATIVE LAWS:

   $(A \cup B) \cup C = A \cup (B \cup C)$

   $(A \cap B) \cap C = A \cap (B \cap C)$

6) DISTRIBUTIVE LAWS:

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

7) DEMORGAN'S LAWS:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

8) $\overline{(\bar{A})} = A$

$$A \subseteq B \iff \bar{B} \subseteq \bar{A}$$

$$A - B = A \cap \bar{B}$$

$$\bar{U} = \phi$$

$$\bar{\phi} = U$$

9) $A \cap U = A$; $A \cup \phi = A$

$A \cap \phi = \phi$; $A \cup U = U$

$A \cap \bar{A} = \phi$; $A \cup \bar{A} = U$

10) $(A \subseteq C \wedge B \subseteq C) \iff (A \cup B) \subseteq C$

$(A \subseteq B \wedge A \subseteq C) \iff A \subseteq (B \cap C)$

<u>EXERCISE</u>: PROVE IT!

<u>EX</u>: PROVE (7) $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

($\subseteq$) LET $x \in \overline{A \cup B}$. THEN $x \notin A \cup B$.

$\Rightarrow x \notin A$ AND $x \notin B$.

$\Rightarrow x \in \bar{A}$ AND $x \in \bar{B}$

$\Rightarrow x \in \bar{A} \cap \bar{B}$.

(⊇) LET $x \in \bar{A} \cap \bar{B}$. THEN $x \in \bar{A}$ AND $x \in \bar{B}$

$\Rightarrow x \notin A$ AND $x \notin B$

$\Rightarrow x \in A \cup B$

$\Rightarrow x \in \overline{A \cup B}$

$\therefore \overline{A \cup B} = \bar{A} \cap \bar{B}$. $\square$

<u>Ex</u>: PROVE (2) $A \subseteq B \Leftrightarrow A \cup B = B$.

($\Rightarrow$) LET $A \subseteq B$.

($\subseteq$) LET $x \in A \cup B$. THEN $x \in A$ OR $x \in B$.

SINCE $A \subseteq B$, IF $x \in A$, THEN $x \in B$.

$\Rightarrow x \in B$.

$\therefore x \in A \cup B \Rightarrow x \in B$, i.e. $A \cup B \subseteq B$.

(⊇) LET $x \in B$. THEN $x \in A \cup B$.

$\therefore B \subseteq A \cup B$.

$\therefore A \subseteq B \Rightarrow A \cup B = B$.

($\Leftarrow$) LET $A \cup B = B$.

LET $x \in A$. THEN $x \in A \cup B$. BUT $A \cup B = B$, SO $x \in B$.

$\Rightarrow A \subseteq B$.

$\therefore A \cup B = B \Rightarrow A \subseteq B$.

$\therefore A \subseteq B \Leftrightarrow A \cup B = B$. $\square$

EX: PROVE THAT THE DIFFERENCE OPERATOR IS NOT COMMUTATIVE.

A: SHOW THAT $A-(B-C) \neq (A-B)-C$.

① $A-(B-C) = A-(B\cap\bar{C}) = A\cap\overline{B\cap\bar{C}} = A\cap(\bar{B}\cup C) = (A\cap\bar{B})\cup(A\cap C)$

② $(A-B)-C = (A\cap\bar{B})-C = A\cap\bar{B}\cap\bar{C}$.

LET $x\in A\cap C$. THEN BY ①, $x\in A-(B-C)$, AND SINCE $x\in A$ AND $x\in C$, WE HAVE $x\notin\bar{C}$. SO BY ②, $x\notin(A-B)-C$.

$\therefore A-(B-C)\neq(A-B)-C$. □

DEF: THE SETS $A_1, A_2, \ldots, A_k$ ARE PAIRWISE DISJOINT IF $A_i\cap A_j=\phi$ FOR ALL $i\neq j$.

THM (EXTENSION OF ADDITION RULE): LET $A_1, A_2, \ldots, A_k$ BE FINITE, PAIRWISE DISJOINT SETS. THEN $A_1\cup A_2\cup\cdots\cup A_k$ IS FINITE AND

$$|A_1\cup A_2\cup\cdots\cup A_k| = |A_1|+|A_2|+\cdots+|A_k|.$$

COMBINATORICS

SEQUENCES AND WORDS

A SEQUENCE IS AN ORDERED LIST OF OBJECTS, WITH REPETITIONS OF THE SAME OBJECTS ALLOWED (AS OPPOSED TO A SET). THE OBJECTS OF A SEQUENCE ARE CALLED TERMS. A SEQUENCE MAY BE FINITE:

$(1,2,3,4)$; $(a,b,\ldots,z)$;

OR INFINITE:

$(2,4,6,\ldots)$; $(\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots)$

THE ORDER MATTERS; $(1,2,3)$ IS A DIFFERENT SEQUENCE THAN $(3,2,1)$. IF ALL TERMS OF A SEQUENCE ARE FROM A SET U, THE SEQUENCE IS A SEQUENCE IN U OR A U-SEQUENCE. FOR EXAMPLE, $(1,2,3)$ IS A SEQUENCE IN $\mathbb{N}$. IT'S ALSO A SEQUENCE IN $\{0,1,2,3,4\}$, IN $\mathbb{Q}$, IN $\mathbb{Z}$ AND IN $\mathbb{R}$. #

A SEQUENCE CAN ALSO BE CALLED A WORD IN THE ALPHABET U. THE SEQUENCE $(t_1, t_2, \dots, t_k)$ IS EQUIVALENT TO THE WORD $t_1 t_2 \cdots t_k$.

THM (MULTIPLICATION RULE): LET S DENOTE THE NUMBER OF DISTINCT SEQUENCES $(t_1, t_2, \dots, t_k)$ WITH n_i POSSIBLE VALUES FOR EACH t_i. THEN $S = n_1 n_2 \cdots n_k$.

COROLLARY: LET $|A| = n$. THEN THERE ARE n^k SEQUENCES OF LENGTH k IN A.

EX: HOW MANY 3-LETTER WORDS CAN BE FORMED WITH THE ENGLISH ALPHABET?

$\overline{26} \cdot \overline{26} \cdot \overline{26} = 17576$. THE WORDS ARE AAA, AAB, AAC, ..., ZZZ.

PERMUTATIONS

A SEQUENCE IN WHICH ALL TERMS ARE DISTINCT IS CALLED A PERMUTATION. IF $|S| = n$, A SEQUENCE OF LENGTH $k \leq n$ OF ALL DISTINCT OBJECTS IS CALLED A PERMUTATION OF n OBJECTS TAKEN k AT A TIME.

IF $k = n$, WE JUST SAY PERMUTATION OF n OBJECTS.

$\underline{\text{EX}}$: LET $S = \{1,2,3,4,5,6\}$. THE FOLLOWING WORDS IN S ARE PERMUTATIONS OF 6 OBJECTS TAKEN 3 AT A TIME.

$$s_1 = 146, \quad s_2 = 324, \quad s_3 = 531.$$

THE FOLLOWING WORDS IN S ARE PERMUTATIONS OF 6 OBJECTS.

$$s_4 = 123456, \quad s_5 = 132645, \quad s_6 = 651324.$$

THERE ARE $P_k^n = \dfrac{n!}{(n-k)!}$ PERMUTATIONS OF n OBJECTS TAKEN k AT A TIME. NOTICE THAT $\dfrac{n!}{(n-k)!} = \dfrac{1 \cdot 2 \cdot \ldots \cdot n}{1 \cdot 2 \cdot \ldots \cdot k} = n(n-1)\cdots(n-k+1)$.

THIS HAS A SHORTER NOTATION CALLED "FALLING FACTORIAL" $n^{\underline{k}}$, WHICH IS ALSO USED FOR $k \geq n$. WHEN $k \leq n$, WE HAVE $n^{\underline{k}} = \dfrac{n!}{(n-k)!}$, AND WHEN $k > n$, $n^{\underline{k}} = 0$.

$\underline{\text{EX}}$: LET $n = 7$, $k = 10$. THEN

$$7^{\underline{10}} = 7 \cdot 6 \cdot \ldots \cdot 1 \cdot 0 \cdot (-1) \cdot (-2) = 0.$$

$\underline{\text{THM}}$: FOR ALL $n, k \in \mathbb{N}$, THERE ARE $n^{\underline{k}}$ PERMUTATIONS OF n OBJECTS TAKEN k AT A TIME.

$\underline{\text{PROOF}}$: IF $k \geq n$, THERE IS NO WAY TO PERMUTE n OBJECTS k AT A TIME, SO THE ANSWER MUST BE ZERO.

$$n^{\underline{k}} = n(n-1) \cdot \ldots \cdot 2 \cdot 1 \cdot 0 \cdot (-1) \cdot \ldots \cdot (n-k+1) = 0.$$

IF $k \leq n$, THERE ARE n CHOICES FOR THE 1st ELEMENT, THEN $(n-1)$ CHOICES FOR THE 2nd, ETC., SO THE TOTAL POSSIBILITIES ARE

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = n^{\underline{k}} = \frac{n!}{(n-k)!} \qquad \square$$

COR: FOR ALL $n \in \mathbb{N}$, THERE ARE $n!$ PERMUTATIONS OF n OBJECTS.

COUNTING STRATEGIES

CONSIDER THE PROBLEM, "HOW MANY SEQUENCES SATISFY A CERTAIN SET OF PROPERTIES"? WE USE COUNTING STRATEGY TO ANSWER THIS QUESTION METHODICALLY. FOR A SEQUENCE OF LENGTH k, USE k EMPTY SLOTS:

$$\underset{1}{\underline{\quad}}\ \underset{2}{\underline{\quad}}\ \underset{3}{\underline{\quad}}\ \cdots\ \underset{k}{\underline{\quad}}.$$

FILL EACH SLOT ONE AT A TIME, WITH THE NUMBER OF POSSIBLE VALUES FOR EACH TERM, GIVEN THE RESTRICTIONS OF THE PROPERTIES.

$$\underset{1}{\underline{n_1}}\ \underset{2}{\underline{n_2}}\ \underset{3}{\underline{n_3}}\ \cdots\ \underset{k}{\underline{n_k}}.$$

BY THE MULTIPLICATION RULE, THERE ARE $n_1 n_2 n_3 \cdots n_k$ POSSIBLE SEQUENCES.

EX: THERE ARE 2 HIGWAYS FROM BRISBANE TO SYDNEY, AND 3 HIGHWAYS FROM SYDNEY TO ADELAIDE. HOW MANY DIFFERENT ROUND TRIPS FROM BRISBANE TO ADELAIDE VIA SYDNEY ARE THERE? HOW MANY ARE THERE WITHOUT TAKING THE SAME HIGHWAY TWICE?

A: 1st PROBLEM: $\underset{B-S}{\underline{\quad}}\ \underset{S-A}{\underline{\quad}}\ \underset{A-S}{\underline{\quad}}\ \underset{S-B}{\underline{\quad}}$

2nd PROBLEM: $\underset{B-S}{\underline{\quad}}\ \underset{S-A}{\underline{\quad}}\ \underset{A-S}{\underline{\quad}}\ \underset{S-B}{\underline{\quad}}$

YOU DON'T NECESSARILY HAVE TO START WITH THE 1st POSITION. START WHERE IT'S MOST CONVENIENT.

EX: HOW MANY 5-DIGIT ODD NUMBERS WITH NO REPEATED DIGITS ARE THERE?

_ _ _ _ _

THERE'S A BIG RESTRICTION ON DIGIT 5, AND A SMALLER ONE ON DIGIT 1, SO START WITH THOSE.

SOMETIMES, WE NEED TO BREAK A PROBLEM UP INTO SUBPROBLEMS.

EX: HOW MANY 5-DIGIT EVEN NUMBERS WITH NO REPEATED DIGITS ARE THERE?

_ _ _ _ _

THERE'S THE SAME RESTRICTION ON DIGIT 5, BUT THE RESTRICTION ON DIGIT 1 IS DIFFERENT IF DIGIT 5 IS ZERO.

FOR A REQUIRED ADJACENCY, TREAT THE ADJACENCY AS A SINGLE OBJECT, THEN MULTIPLY BY THE NUMBER OF ARRANGEMENTS OF THE ADJACENCY.

EX: THREE SINGLE PEOPLE AND A MARRIED COUPLE ARE TO BE SEATED IN A ROW OF CHAIRS. IN HOW MANY WAYS CAN IT BE DONE SUCH THAT THE SPOUSES SIT TOGETHER?

FOR A FORBIDDEN ADJACENCY, CALCULATE IT AS A REQUIRED ADJACENCY, THEN SUBTRACT FROM THE TOTAL POSSIBLE ARRANGEMENTS.

EX: IN HOW MANY WAYS CAN YOU ALIGN A COW, A GOAT, A FOX AND A CHICKEN SUCH THAT THE FOX AND THE CHICKEN ARE NOT NEXT TO EACH OTHER?

BINOMIAL COEFFICIENTS

RECALL THE POWER SET OF X: $P(X) = \{A : A \subseteq X\}$.

$P(\{1,2,3\}) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. ANOTHER NOTATION FOR $P(X)$ IS 2^X. THIS IS BECAUSE OF THE FOLLOWING.

THM: LET $|X| = n \in \mathbb{N} \cup \{0\}$. THEN X HAS 2^n SUBSETS, i.e. $|P(X)| = 2^{|X|}$.

PROOF: INDUCTION.

a) LET $n=0$. THEN $X = \phi$, AND $P(x) = \phi$, SO $|P(x)| = 1 = 2^0$. ✓

b) LET $k \in \mathbb{N}$, SUPPOSE $|X| = k$ AND $|P(x)| = 2^k$. DEFINE

$$Y = X \cup \{y\} = \{x_1, x_2, \dots, x_k, y\}.$$

THE SUBSETS OF Y ARE THOSE THAT CONTAIN y, AND THOSE THAT DO NOT. THOSE THAT DO NOT ARE EXACTLY THE SUBSETS OF X, OF WHICH THERE ARE 2^k. THOSE THAT DO CONTAIN y ARE OF THE FORM $Z \cup \{y\}$, WHERE $Z \in P(x)$, SO THERE ARE EXACTLY 2^k OF THOSE TOO. THEREFORE, $|Y| = 2^k + 2^k = 2^{k+1}$. ✓

$\therefore$ $|P(x)| = 2^{|X|}$. $\forall X$ FINITE. □

LET $|X| = n \in \mathbb{N} \cup \{0\}$. FOR EVERY $k \in \mathbb{N} \cup \{0\}$, WE DENOTE BY $\binom{n}{k}$ THE NUMBER OF SUBSETS OF X WITH k ELEMENTS.

$$\binom{n}{k} = |\{A : A \subseteq X \text{ AND } |A| = k\}|$$

THE SYMBOL $\binom{n}{k}$ IS READ "n CHOOSE k", OR "THE k^{th} BINOMIAL COEFFICIENT OF ORDER n." SOME $\binom{n}{k}$ VALUES ARE OBVIOUS:

$\binom{n}{0} = 1$, SINCE THE ONLY SUBSET OF CARDINALITY 0 IS $\emptyset$.

$\binom{n}{n} = 1$, SINCE X IS THE ONLY SUBSET OF X WITH n ELEMENTS.

IF $k > n$, THEN $\binom{n}{k} = 0$, AS IT'S IMPOSSIBLE TO HAVE A SUBSET OF X WITH CARDINALITY LARGER THAN THAT OF X.

THM: FOR ALL $n, k \in \mathbb{N} \cup \{0\}$, $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!}$.

PROOF: FOR $k > n$, WE'VE SEEN THAT $n^{\underline{k}} = 0$, AND $\binom{n}{k} = 0$.

LET $k \leq n$. RECALL THAT THE NUMBER OF PERMUTATIONS OF n OBJECTS TAKEN k AT A TIME IS $P_k^n = \frac{n!}{(n-k)!}$. THIS NUMBER CAN BE OBTAINED BY TAKING ALL $\binom{n}{k}$ COMBINATIONS OF k ELEMENTS AND ORDERING THE ELEMENTS IN EACH COMBINATION, WHICH CAN BE DONE IN P_k^k WAYS. THUS,

$$P_k^n = \binom{n}{k} P_k^k \Rightarrow \binom{n}{k} = \frac{P_k^n}{P_k^k} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!}. \;\llcorner$$

THE SYMBOL $\binom{n}{k}$ IS ALSO DENOTED BY C_k^n, THE NUMBER OF COMBINATIONS OF n OBJECTS TAKEN k AT A TIME.

EX: HOW MANY DIFFERENT POKER HANDS ARE THERE?

A: THERE ARE 5 CARDS IN A POKER HAND, ORDER IS NOT IMPORTANT, AND THEY ARE TAKEN FROM A DECK OF 52 CARDS. SO THERE ARE

$$\binom{52}{5} = \frac{52!}{5!\,47!} = 2,598,960 \text{ POKER HANDS.}$$

THM: FOR ALL $n, k \in \mathbb{N} \cup \{0\} \ni 0 \le k \le n$, $\binom{n}{k} = \binom{n}{n-k}$.

PROOF: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)![n-(n-k)]!} = \binom{n}{n-k}$. $\square$

THM: FOR ALL $n, k \in \mathbb{N} \cup \{0\} \ni 0 \le k \le n$,

a) $\binom{n}{0} = 1$

b) $\binom{0}{k} = 0$

c) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

PROOF: EXERCISE.

THE BINOMIAL THEOREM

MOTIVATION: IN HOW MANY WAYS CAN 3 RED MARBLES AND 4 BLUE MARBLES BE ARRANGED IN A ROW? (OR A MORE PRACTICAL EXAMPLE: HOW MANY BINARY WORDS ARE THERE WITH 3 ZEROS AND 4 ONES?)

THE MULTIPLICATION RULE ISN'T VERY HELPFUL HERE; THERE ARE TOO MANY CASES. HOWEVER, CONSIDERING THE 7 SLOTS

_ _ _ _ _ _ _
1 2 3 4 5 6 7

NOTICE THAT ONCE YOU CHOOSE SLOTS FOR THE RED MARBLES, THE PLACEMENT OF THE BLUE ONES IS AUTOMATIC. SO THE QUESTION IS, HOW MANY WAYS ARE THERE TO CHOOSE 3 OF THE 7 SLOTS? WE KNOW THE ANSWER IS $\binom{7}{3} = 35$.

SIMILARLY, IF YOU CHOOSE 4 SLOTS FOR THE BLUE MARBLES FIRST, THERE ARE $\binom{7}{4} = 35$ WAYS TO DO IT. THE ANSWER IS THE SAME, BECAUSE

$\binom{7}{3} = \binom{7}{4} = \frac{7!}{3!4!}$ $\quad \frac{7!}{3!4!}$

THM: THE NUMBER OF WORDS OF LENGTH n CONSISTING OF n_1 LETTERS OF ONE SORT, AND $n_2 = n - n_1$ LETTERS OF A SECOND SORT, IS

$$\binom{n}{n_1} = \binom{n}{n_2} = \frac{(n_1+n_2)!}{n_1!\,n_2!}.$$

CONSIDER THE BINOMIAL EXPANSION

$$(x+y)^2 = xx + xy + yx + yy,$$

WHICH IS THE SUM OF ALL WORDS OF LENGTH 2 IN THE ALPHABET $\{x, y\}$.

SIMILARLY,

$$(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

IS THE SUM OF ALL WORDS OF LENGTH 3 IN THE ALPHABET $\{x, y\}$.

SIMPLIFYING, WE GET THE FAMILIAR FORMULAE:

$$(x+y)^2 = x^2 + 2xy + y^2,$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

THE BINOMIAL THEOREM BELOW IS A FORMULA FOR THE COEFFICIENTS OF BINOMIAL EXPANSION TO ANY POWER IN $\mathbb{N}$.

THM (BINOMIAL THEOREM): FOR ALL $n \in \mathbb{N} \cup \{0\}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

PROOF: THE CASE $n=0$ IS EASILY VERIFIED BY HAND. FOR $n \in \mathbb{N}$, THE EXPANSION OF $(x+y)^n$ IS (BEFORE SIMPLIFICATION) THE SUM OF ALL 2^n WORDS OF LENGTH n IN THE ALPHABET $\{x,y\}$. THE NUMBER OF SUCH WORDS THAT CONSIST OF k x'S AND $(n-k)$ y'S IS $\binom{n}{k}$ BY THE PREVIOUS THEOREM. $\square$

THE BINOMIAL THEOREM AS WRITTEN GIVES THE EXPANSION IN ASCENDING POWERS OF x:

$$(x+y)^n = y^n + nxy^{n-1} + \binom{n}{2}x^2y^{n-2} + \binom{n}{3}x^3y^{n-3} + \dots + nx^{n-1}y + x^n.$$

EQUIVALENTLY, IT CAN BE WRITTEN IN REVERSE:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + nxy^{n-1} + y^n.$$

WE CAN SUBSTITUTE VALUES FOR x AND y TO OBTAIN IDENTITIES.

EX: LET $x=y=1$. THEN THE BINOMIAL THEOREM GIVES

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

EX: LET $x=-1$, $y=1$. THEN THE BINOMIAL THEOREM GIVES

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n\binom{n}{n} = 0;$$

$$\binom{n}{0} + \binom{n}{2} + \dots = \binom{n}{1} + \binom{n}{3} + \dots;$$

$$\sum_{k \text{ EVEN}}^{n} \binom{n}{k} = \sum_{k \text{ ODD}}^{n} \binom{n}{k}.$$

SOMETIMES, A USEFUL TRICK IS TO USE THE FACT THAT $x = x \cdot 1$.

EX: SIMPLIFY $\sum_{k=0}^{n} \binom{n}{k} a^k$.

A: $\sum_{k=0}^{n} \binom{n}{k} a^k = \sum_{k=0}^{n} \binom{n}{k} a^k \cdot 1^{n-k} = (a+1)^n$.

EX: SIMPLIFY $\sum_{k=1}^{17} (-1)^k \binom{17}{k} 13^{17-k}$.

$$\sum_{k=1}^{17} \binom{17}{k} 13^{17-k} (-1)^k = \sum_{k=0}^{17} \binom{17}{k} 13^{17-k} (-1)^k - \binom{17}{0} 13^{17-0} (-1)^0$$

$$= (13-1)^{17} - 1 \cdot 13^{17} \cdot 1$$

$$= 12^{17} - 13^{17}.$$

RELATIONS AND FUNCTIONS

CARTESIAN PRODUCT

DEF: LET A, B BE SETS, $a \in A$, $b \in B$. AN ORDERED PAIR (a,b) IS A PAIR OF ELEMENTS WITH THE PROPERTY

$$(a,b) = (c,d) \iff a = c \wedge b = d.$$

NOTE: THE OPEN INTERVAL $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ USES THE SAME NOTATION, BUT CONTEXT MAKES IT CLEAR.