MATH221 Mathematics for Computer Science

Outline Solutions to Tutorial Sheet Week 5

Autumn 2017

1. **Step 0:** Let $\mathrm{CLAIM}(n)$ represent the statement $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Step 1: $\mathrm{CLAIM}(1)$ says that $1^2 = \dfrac{(1)(1+1)(2(1)+1)}{6}$, and this is true, completing Step 1.

Step 2: We assume $\mathrm{CLAIM}(k)$ is true. So $1^2 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$.
We need to prove $\mathrm{CLAIM}(k+1)$, which says

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Now we have

$$\begin{aligned}
\text{LHS of } \mathrm{CLAIM}(k+1) &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \text{ using } \mathrm{CLAIM}(k) \\
&= \frac{k+1}{6}(k(2k+1) + 6(k+1)) \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \text{ after simplification—check!} \\
&= \text{RHS of } \mathrm{CLAIM}(k+1).
\end{aligned}$$

So $\mathrm{CLAIM}(k)$ implies $\mathrm{CLAIM}(k+1)$, completing Step 3. Therefore, by mathematical induction, $\mathrm{CLAIM}(n)$ is true for all $n \in \mathbb{N}$.

2. **Step 0:** Let $\mathrm{CLAIM}(n)$ represent the statement $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.
Step 1: $\mathrm{CLAIM}(1)$ says that $1^3 = \frac{1}{4} \cdot 1^2 \cdot 2^2$, and this is true, completing Step 1.
Step 2: We assume $\mathrm{CLAIM}(k)$ is true. So $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2$.
We need to prove $\mathrm{CLAIM}(k+1)$, which says

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2.$$

Now we have

$$\begin{aligned}
\text{LHS of } \mathrm{CLAIM}(k+1) &= 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 \\
&= \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3 \text{ using } \mathrm{CLAIM}(k) \\
&= \tfrac{1}{4}(k+1)^2\big[k^2 + 4(k+1)\big] \\
&= \tfrac{1}{4}(k+1)^2(k+2)^2 \text{ after simplification} \\
&= \text{RHS of } \mathrm{CLAIM}(k+1),
\end{aligned}$$

so $\mathrm{CLAIM}(k)$ implies $\mathrm{CLAIM}(k+1)$, completing Step 3. Therefore, by mathematical induction, $\mathrm{CLAIM}(n)$ is true for all $n \in \mathbb{N}$.

3. (i) $\sum_{i=1}^{5}(2i-5) = -3 + (-1) + 1 + 3 + 5 = 5$
(ii) $\sum_{j=-2}^{2} 2^j = \frac{1}{4} + \frac{1}{2} + 1 + 2 + 4 = \frac{31}{4}$
(iii) $\sum_{k=0}^{3} \frac{k!}{2} = \frac{1}{2} + \frac{1}{2} + 1 + 3 = 5$
(iv) $\sum_{\ell=0}^{99} \frac{(-1)^\ell}{3} = \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{3} - \frac{1}{3}\right) = 0$

4. (i) $2 + 6 + 10 + \cdots + (4n-2) = \displaystyle\sum_{i=1}^{n}(4i-2)$.

(ii) We will only do the key argument in Step 2 here. (*Your* argument should be *full*, with all details!)

$$\begin{aligned}\text{LHS for } k+1 &= 2+6+10+\cdots+(4k-2)+(4(k+1)-2)\\ &= 2k^2+(4k+4-2) \text{ using CLAIM}(k)\\ &= 2(k^2+2k+1) = 2(k+1)^2 = \text{RHS for } k+1\end{aligned}$$

5. Let's write $\text{CLAIM}(n)$ to represent the statement $2^n \geq n^2$. We find that $\text{CLAIM}(n)$ is true for $n=1$ and $n=2$, then false for $n=3$, then true again for $n=4$, and it appears to remain true for larger values of n. Hence our conjecture is that $\text{CLAIM}(n)$ is true for all $n \geq 4$.
Once again, we just do key argument of Step 2.

$$\begin{aligned}&\text{LHS for } k+1 = 2^{k+1} = 2\cdot 2^k \geq 2k^2 \text{using CLAIM}(k)\text{, so}\\ &\text{LHS for } k+1 \geq k^2+k^2 \geq k^2+4k \text{ since } k\geq 4\text{, so}\\ &\text{LHS for } k+1 \geq k^2+2k+2k \geq k^2+2k+1 = (k+1)^2 = \text{RHS for } k+1\end{aligned}$$

6. We define CLAIM(n) to be "$n! > 2^n$."
Establishing CLAIM(4) is true. $4! = 24$ and $2^4 = 16$, establishing CLAIM(4).
Assume CLAIM(k) is true: $k! > 2^k$.
We wish to prove CLAIM(k+1): $(k+1)! > 2^{k+1}$.

$$\begin{aligned}(k+1)! = k!\times(k+1) &> 2^k\times(k+1) \text{ (by applying CLAIM(k))}\\ &> 2^k\times 2 = 2^{k+1} \text{ (as } k+1>2\text{)},\end{aligned}$$

demonstrating CLAIM(k+1), and establishing the result by Generalised Principle of Mathematical Induction.

7. CLAIM(1) says that *before the first loop iteration* `X+Y` *has value* $x+y$. This is true, since before the first loop iteration `X` has value x and `Y` has value y.
We assume CLAIM(k): *before the kth loop iteration* `X+Y` *has value* $x+y$, and wish to prove CLAIM(k+1): *before the (k+1)th loop iteration* `X+Y` *has value* $x+y$. Now, "before the $(k+1)$th loop iteration" means the same things as "after the kth loop iteration." But the kth iteration decremented `X` and incremented `Y`, meaning that the new value of `X+Y` is $x-1+y+1 = x+y$, proving the claim.
Therefore, the CLAIM is established by mathematical induction.

For the loop to terminate, `X` must have value 0, and if so, `X+Y` has the value $x+y$, and so `Y` has value $x+y$.
Note that there is no guarantee that the loop will terminate.

8. Assume CLAIM(k): $k+1 < k$. We want to establish CLAIM(k+1); that is, $k+2 < k+1$.
From CLAIM(k): $k+1 < k \Rightarrow (k+1)+1 < k+1$, establishing CLAIM(k+1).
But, as we have no basis step, we haven't shown that CLAIM(k) is true anywhere, and thankfully as there is no starting point for our inductive step, this CLAIM is in fact not true for any value.

9. The issue is with $n=1$. In this case the two sets $H_0 = \{h_1\}$ and $H_1 = \{h_2\}$ don't overlap, and there is no horse in common. We should have $n \geq 2$.