Relations and Functions

Functions

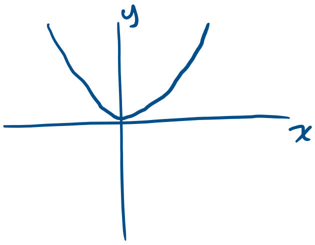
Definition:

- A relation F from A to B is a **function** from A to B IFF:
  1) $dom\ F = A$, and
  2) For each $x \in A$ there is at most **one** $y \in B$ such that $(x, y) \in F$.
     a. Then B is the **codomain** of F.
- A function from A to B is denoted by $f: A \to B$.
- The equation $y = f(x)$ means $(x, y) \in f$.
- In that case, y is the **image** of x under f.
- Relations on $\mathbb{R}$ can be plotted by drawing all the points.
- Such relations are functions if they satisfy the **vertical line test**
  - Every vertical line cuts the graph at most once.

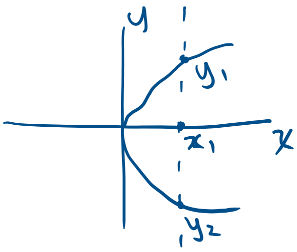
Exercise:

Sketch the relations and determine which are functions.

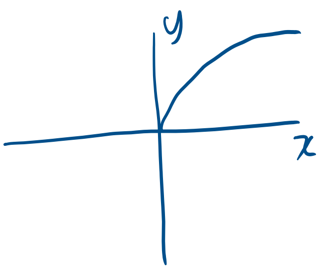
a) On $\mathbb{R}, R = \{(x, y): y = x^2\}$
b) On $R, R = \{(x, y): x = y^2\}$
c) On $\mathbb{R}_+ = \{x \in \mathbb{R} :\ x \geq 0\}, R = (x, y): x = y^2\}$
d) On $\mathbb{R}, R = \{(x, y): y = \sqrt{x}\}$

a)


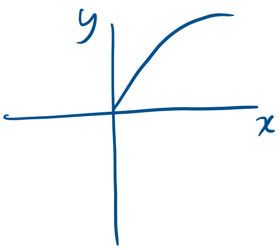
Yes function

b)


Not a function.
Fails vertical line test.
x_1 has two values y_1, y_2.

c)


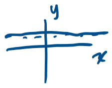
Yes function.

d)


Yes function.

Exercise:

Which are functions?

a) The identity relation on $A = \{1,5,10\}$. $\longrightarrow$ Yes
b) $A = \{2,4,6\}, B = \{1,3,5\}, R \text{ on } A \times B, R = \{(x,y): x+1 = y\}$ $\rightarrow$ No,
Let $x = 6 \Rightarrow y = 7$
$7 \notin$ codomain B
c) On $\mathbb{Z}, F = \{(x,y): x+1 = y\}$ $\rightarrow$ Yes
d) On $\mathbb{R}, R = \{(x,y): y = 1\}$ $\rightarrow$,  Yes

Definition (Injective):

- Let $f: A \to B$ be a function.
- We say that f is **one-to-one (injective)** IFF for all $x_1, x_2 \in A$.

$$f(x_1) = f(x_2) = x_1 = x_2$$

- That is, each element of the range is the image of only **one** element of the domain.

Exercise:

Let $A = \{0,1,2,3\}, f: f(A) \to \mathbb{N}, f(A_i)$ is the number of elements in A_i. Prove or disprove that f is one-to-one.

Let $A_1 = 1$, $A_2 = 2$, $A_1 \neq A_2$
$f(\{1\}) = 1$
$f(\{2\}) = 1$
$f(A_1) = f(A_2)$
$\therefore$ This function is not one-to-one.

Exercise:

Which are one-to-one?

a) On $A = \{1,2,3\}, F = \{(1,2),(2,3),(3,1)\}$ $\rightarrow$ Yes
b) On $A = \{1,2,3\}, F = \{(1,2),(2,1),(3,1)\}$ $\rightarrow$ No $\rightarrow$ $f(2) = f(3) = 1$
c) On $\mathbb{Z}, F = \{(x,y): y = 2x\}$ $\rightarrow$ Yes
d) On $\mathbb{Z}\backslash\{0\} \to \mathbb{R}, F = \{(x,y): y = \sqrt{x^2 - 1}\}$ $\rightarrow$ No $\rightarrow$ for $x_1 = 3$, $x_2 = -3$
$y = 2.8$

Definition (Surjective):

- A function $f: A \to B$ is **onto (surjective)** IFF $ran\ f = B$.
- That is, for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Exercise:

Let $A = \{1,2,3,4,5\}, B = \{a,b,c,d\}$. Which are onto?

a) $f: A \to B, f = \{(1,a),(2,c),(3,c),(4,d),(5,d)\}$ $\rightarrow$ No, no b element
b) $f: A \to B, f = \{(1,a),(2,b),(3,c),(4,d),(5,a)\}$ $\rightarrow$ Yes
c) $f: \mathbb{R} \to \mathbb{R}, f(x) = 4x - 1$
d) $f: \mathbb{Z} \to \mathbb{Z}, f(x) = 4x - 1$

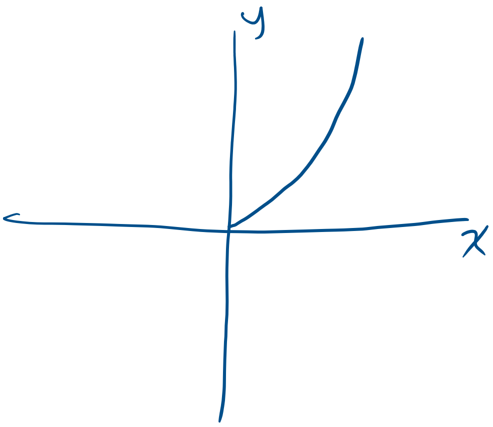
Theorem (Inverse):

- The **inverse** of a function f, written f^{-1}, is also a function IFF f is one-to-one and onto **(bijective)**.

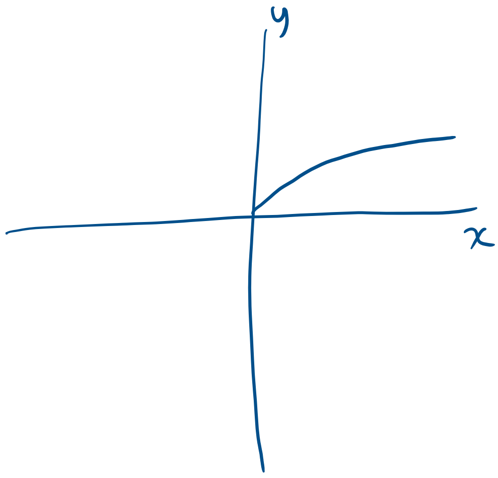
Exercise:

Sketch $f: \mathbb{R}_+ \to \mathbb{R}, f = \{(x, y): y = x^2\}$. Find and sketch f^{-1}. Is f^{-1} a function?

f:



f^{-1}:



$f^{-1}: y = \sqrt{x}$

f^{-1} is also a function