Transfer Payments, Sacrifice Ratios, and Inflation in a Fiscal Theory HANK[1]

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Abstract

I numerically solve a calibrated heterogeneous agent new-Keynesian (HANK) model in which the fiscal theory of the price level (FTPL) is active, with incomplete markets and long-term debt. In model simulations, the total cumulative inflation generated by a fiscal helicopter drop is largely determined by the size of the initial stimulus and is relatively insensitive to the initial distribution of the payments, while in contrast the responses of GDP and employment are highly sensitive to the transfer payment distribution. Stimulus checks to poor households yield a bigger economic boom but are about as inflationary in the medium-to-long run as checks sent to wealthy households. Conversely, lump-sum taxes on wealthier households yield disinflation at a much lower cost to employment than taxes on poorer households. Despite the incomplete markets setting, tight monetary policy is able to reduce inflation in the short run through a forward guidance wealth effect when the government issues long-term nominal bonds – but still drives up inflation in the long run.

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1 Introduction

The fiscal theory of the price level (FTPL), summarized in Cochrane (2023), explains how government deficits financed with nominally-denominated bonds can determine the overall price level and the path of inflation. However, the FTPL literature has thus far been constrained to representative agent models, with little discussion of how inequality and household heterogeneity might impact the theory's predictions for monetary and fiscal policy at business cycle frequencies. Fiscal policy in particular can be (and often is) tailored to direct stimulus transfers to particular segments of the population based on their asset positions and incomes. For a policymaker contemplating what to do in a recession, several natural questions might arise which the existing fiscal theory literature has not yet directly addressed. First, in an environment where the FTPL is active, do transfers to some households generate more or less inflation than transfers to other households? Second, do some policies lead to a bigger real economic expansions than others, relative to the amount of inflation that they produce? Conversely, are there some targeted fiscal actions that could reduce inflation at a lower cost to output and employment than others? Thirdly, is tight monetary policy still an effective tool for fighting inflation in a fiscal theory world with incomplete markets and household heterogeneity, and how does it compare to fiscal policy?

In this paper, I demonstrate that fiscal theory has strong implications for each of these three questions. To do so, I integrate the fiscal theory of the price level and long-term nominal government debt with a canonical heterogeneous agent New-Keynesian (HANK) model. First, the model simulations suggest that the total amount of cumulative inflation generated by a fiscal policy is largely a function of how much deficit spending is committed by the program net of future tax revenues, *not* who the of recipients of the spending are: a deficit-financed stimulus amounting to 1% of steady-state GDP leads to an increase in the overall price level of about 0.6%, regardless of whether the payments went to the rich, the poor, or a mixture of both. Secondly, stimulus checks sent to the poor boost the real economy by more than those disbursed to the rich. Finally, monetary policy with long-term debt in an FTPL-HANK world works much the same as in a similar representative agent model; the forward guidance mechanism of Cochrane (2018b) is still able to bring down inflation temporarily when nominal rates rise, only for high nominal rates to drive up inflation further down the line.

For the first result, the irrelevance of who receives stimulus checks in determining inflation may seem surprising given the standard new-Keynesian model's intuition connecting the output gap and inflation through the Phillips curve – but the result is actually a natural consequence of the FTPL. Fiscal theory's central equation is an asset pricing equation, which stipulates that if a government's liabilities (both bonds and actual currency notes) are nominally denominated and are a claim to its future primary fiscal surpluses, then the *real* market value of government debt outstanding must be equal to the present value of those primary surpluses, where the relevant discount rate is the real interest rate or a more general (but related) stochastic discount factor. Inflation in the fiscal theory world thus occurs when buyers of nominal government liabilities (like money) sense that the pieces of paper that the government is exchanging with them are a claim to fewer real resources in the future; since the government's liabilities are nominally denominated, the Treasury can settle its debts using inflated dollars. A government can thereby run deficits and not raise future surpluses if the currency it transacts in devalues either immediately or over time. Inflation then comes from the market's willingness to hold government liabilities given the government's plans for repayment, where whatever the government spends but does back by current or future real surpluses must be inflated away. Taking the mechanism seriously, it is then not surprising that if two stimulus packages deploy the same amount of spending, then the model predicts that they will generate the same amount of inflation, regardless of where the money went: both policies have to "inflate away" the same amount of debt.

However, the model makes clear that the response of *real* variables like output and employment *are* sensitive to who receives the transfers; sending checks to poorer low-income households with higher marginal propensities to consume (MPCs) generates a larger economic expansion than sending checks to wealthier high-income households. It therefore follows that sending checks to low-income (and thus on average, lower-asset) groups stimulates the economy significantly more relative to the total cumulative inflation incurred by the policy than sending payments to other groups. Similarly, levying lump-sum taxes on wealthier households reduces inflation at a lower cost to the economy than taxes on other groups, as measured by the total amount of inflation abated per cumulative percentage points of real GDP lost, the policy's *sacrifice ratio*.

Why is monetary policy in the HANK-FTPL setting not that different from the *representative agent* new-Keynesian (RANK) fiscal theory model of Cochrane (2018b)? The latter paper suggests that persistently elevated nominal interest rates temporarily lower the price level through a forward-guidance wealth effect on long-term government debt prices. For constrained and poor households, this wealth effect may not be a considerable factor, but in the HANK setting the marginal bond's buyer and seller are much wealthier than the average household: with less of a precautionary savings motive and less MPC heterogeneity, the heterogeneous agent bond market still behaves similarly to the representative agent one. In consequence, long-term debt still means that high nominal rates translate to lower inflation in the short term in the FTPL-HANK environment (and still lead to higher inflation in the long-run, given the model's long-run monetary neutrality and neo-Fisherian properties).

1.1 Related Literature

The building blocks of this paper draw from both the FTPL and HANK literatures. The HANK component is a continuous-time version of the discrete-time model of McKay, Nakamura, and Steinsson (2016), which is itself a standard incomplete markets framework with nominal rigidities. For the FTPL side, I then include long-term nominal government debt as in Cochrane (2018b) (which in turn builds off of Sims (2011)). To use the terminology first popularized by Leeper (1991), I further assume that monetary policy is "passive" in that the central bank does not systematically increase nominal interest rates by more than the inflation rate in response to an inflationary episode (in contrast to McKay, Nakamura, and Steinsson (2016), which assumes the opposite), while I allow fiscal policy in my model to be "active" such that the fiscal authority does not automatically adjust the path of its primary surpluses in response to changes in the real interest rate or the price level; when combined with nominal debt, this allows the FTPL to be present in my setting. At the time of writing, the FTPL and HANK ingredients themselves are standard, but their combination is new: I incorporate the minimum elements necessary to explore the three policy questions that I have posed.

While I study both monetary and fiscal policy in a model that merges McKay, Nakamura, and Steinsson (2016) and Cochrane (2018b), both of the papers I cite are focused on monetary policy via *forward guidance* in particular. The aforementioned Cochrane (2018b) monetary policy mechanism is all about the market's expectations about the path of future rates, while McKay, Nakamura, and Steinsson (2016) shows that the power of more conventional new-Keynesian forward guidance is attenuated in incomplete markets settings where agents face pro-cyclical income risk (an important feature noted in Acharya and Dogra (2020)) and a precautionary savings motive. Combining these papers' two results, one could speculate that perhaps the Cochrane (2018b) monetary policy mechanism would be significantly less effective in a HANK setting. However, as noted in the above paragraphs, I show that this is *not* the case. In short, McKay, Nakamura, and Steinsson (2016) and Cochrane (2018b) assess two *different* mechanisms of forward guidance: while the latter's inflation mechanism is through a wealth effect (which exists even in a frictionless model where the real rate does not move), the former's mechanism is a story about the intertemporal substitution channel propagating backward in time through the Euler equation to generate a substantial output gap, which then produces a large inflationary impulse through the Phillips curve. It makes sense for the standard new-Keynesian type of forward guidance to be blunted by introducing factors like hand-to-mouth agents and precautionary savings motives, which reduce the importance of the Euler equation's intertemporal substitution channel. Indeed, totally apart from HANK, the FTPL itself has also been proposed as a way to resolve the forward guidance puzzle, as in Canzoneri et al. (2018) and Cochrane (2018a). The FTPL monetary disinflation impulse, in contrast, comes from how the nominal yield curve prices long-term bonds and how the price level must adjust to make the resultant real bond prices attractive enough to investors for the bond market to clear, which still proves to be a potent force for determining prices in the heterogeneous agent model.

This paper also shares the rest of the HANK literature's emphasis on the importance of household balance sheet heterogeneity – particularly with respect to fiscal policy's implications for real output. One of the HANK literature's seminal papers is Kaplan, Moll, and Violante (2018), which re-examines the transmission mechanism of monetary policy in a setting where a large fraction of agents lack liquid assets and thus pass changes in their current income directly into their spending. They find that monetary policy in their simulated environment acts primarily through the automatic real *fiscal* adjustments that governments must make in standard (passive-fiscal) models to balance their inter-temporal budgets and through general equilibrium effects that change labor income. Both of these effects move resources toward low-asset agents living paycheck-to-paycheck with a high marginal propensity to consume out of their current income, which jump-starts an expansionary process wherein spending begets income begets production which begets spending, in a dynamic loop reminiscent of textbook "old" Keynesian static models. Auclert, Rognlie, and Straub (2018) and Auclert, Rognlie, and Straub (2023) discuss how these feedbacks from expansionary policy continue until excess savings eventually "trickle up" to wealthier low-MPC households, arresting the cycle. My HANK-FTPL model shares these features. However, the two groups of models are less similar on the inflation side. Conventional HANK models typically assume a passive fiscal policy that reacts to real interest rates and balances the government's budget either in each period or over time, while active monetary policy ensures inflation determinacy via a threat to induce hyperinflation, as recounted in Cochrane (2011). When combined with the new-Keynesian Phillips curve and the central bank's threat of reacting to inflation with higher rates, the rest of the HANK literature leaves open the possibility that MPC heterogeneity may be important for the inflation side, too. In the active fiscal/passive monetary policy regime explored in my paper, this is largely not the case.

I have written my paper at a time of surging interest in fiscal theory, and while I believe it is the first to combine a HANK framework with the FTPL, a contemporaneous working paper, Kaplan, Nikolakoudis, and Violante (2023), has also made the important step of combining the FTPL with a setting that includes incomplete markets and household heterogeneity, but no nominal rigidities. Their paper does the important work of examining the determinacy of passive monetary/active fiscal Bewley-Aiyagari models and finds that although FTPL can deliver a unique saddle-path equilibrium if the government runs surpluses in the steady-state, multiple equilibria (a higher-welfare saddle-path stable one and a lower welfare locally stable one) can emerge when the government runs perpetual deficits. They estimate that the U.S. can plausibly run perpetual deficits of up to 4.8% of GDP without the bond market collapsing, but make the point that a long-term policy that is more redistributive from the wealthy to the poor better insures the public against idiosyncratic income risk, leading the public to be less willing to hold government liabilities as assets – potentially reducing the fiscal space that the government has available to take on new debt at a time when it is inflating debt away and offering a negative real rate of return. They also look at the effect of one-time targeted and un-targeted helicopter drops in a constant aggregate endowment setting without price frictions and find that while the jump in the price level is larger in the heterogeneous agent setting is larger than in the representative agent one, the price level jumps a similar amount regardless of whether or not the transfer was targeted (although this is not a point the coauthors emphasize). I confirm this latter property in my own model wherein output is endogenous and prices do not adjust immediately, and I discuss the business-cycle level implications for sacrifice ratios and GDP. However, I restrict my analysis to the slightly less realistic but more theoretically straightforward case where the government runs surpluses in the steady state.

Lastly, my paper also provides a small methodological contribution: to solve my model in the style of Sims (2002), I couple the continuous time tools of Achdou et al. (2021) and Ahn et al. (2018) with a continuous-time version of the discrete cosine transformation dimension reduction routine of Bayer and Luetticke (2020). I know of no other papers that have yet employed this useful combination.

The rest of this paper sketches out a HANK world in which the FTPL is in play, to show that MPC heterogeneity and hand-to-mouth households combined with the FTPL price determination mechanism yields stark predictions for the trade-offs between output and inflation. Section 2 describes the incomplete-markets dynamic stochastic general equilibrium model. Section 3 details the model's calibration and moments from its non-stochastic steady-state. Section 4 describes the results of the simulations, including the impulse response functions for i) expansionary and contractionary monetary policy, ii) the impact of deficit-financed spending, iii) stimulus payments to high and low income households in particular, and iv) the implications of a balanced-budget redistribution policy. Section 5 concludes.

2 The Model

In order to study the interactions between fiscal theory and inequality, I employ a model that exhibits both. This paper consists of a block of heterogeneous households who purchase consumption goods, have the option to save a perfectly liquid financial asset, and supply their labor to a monopolistically competitive intermediary production section. These intermediary firms then supply their output to final goods firms, which produce consumer goods via a constant elasticity of substitution (CES) aggregator. The main elements of these blocks are indeed highly reminiscent of McKay, Nakamura, and Steinsson (2016) (and other papers in the HANK literature) – although I do also add in nominal wage rigidity and union power to the labor market as in Auclert, Rognlie, and Straub (2018), as the presence of nominal wage rigidities and acyclic real wages appear to be better supported by empirical evidence than their absence.[3] The government consists of a monetary authority (central bank) that sets nominal interest rates and a fiscal authority that taxes households via a proportional income tax and lump-sum transfers. In my model, however, fiscal policy is active, and the government can run deficits by spending in excess of its tax revenue, which it does not necessarily have to balance later (thanks to inflation); it finances those deficits by issuing long-term nominal debt at the rate set by the central bank, which

[3]See the vector autoregressions in Christiano, Trabandt, and Walentin (2010) on the acyclicality of real wages in response to a monetary policy shock. For evidence that markups are not counter-cyclical in response to a demand shock, see Nekarda and Ramey (2020). For evidence against countercyclical markups during the 2021-2023 inflationary spell, see Glover, Mustre-del-Rio, and Ende-Becker (2023) and Glover, Mustre-del-Rio, and Nichols (2023).

an investment fund purchases using household savings.

The solution method I employ linearizes the model around a non-stochastic steady-state (NSS). As such, the dynamics of the model are certainty-equivalent with respect to macroeconomic (but not idiosyncratic) uncertainty. Unforeseen shocks to policy are thus like "MIT shocks" in perfect foresight models: the initial shock is a total surprise to the inhabitants of the model economy, but the ensuing transition dynamics are deterministic. An important consideration here is that agents do *not* price aggregate risk in this setting.

2.1 The Households' Problem

The heterogeneous household block of this model is built around a standard Bewley-Aiyagari incomplete markets setting, wherein a measure-one continuum of agents can save and borrow in a single liquid asset and insure themselves against fluctuations in their personal income.

Households discount the future at a rate ρ and maximize their time-separable expected lifetime utility V by choosing paths of consumption $(c_t)_{t\geq 0}$ given their liquid asset position a_t and their labor skills z_t (where a and z are *idiosyncratic* state variables specific to each household). The aggregate state of the economy is described by the distribution of households across idiosyncratic states $\mu_t(a, z)$ and a vector of *aggregate* shocks ζ_t that characterize tax policy and the central bank's monetary policy fluctuations. Since each household is infinitesimally small relative to the aggregate market, they take the distribution of the economy and the resultant prices (wages w_t and real interest rates r_t) as given. Inter-temporal discounting is set by the preference parameter ρ.

Households earn their income from the effective units of labor $(z_t h_t(a, z))$ that they supply to the marketplace, and their log skills evolve according to an Ornstein-Uhlenbeck AR(1) process with mean reversion governed by θ_z and shocks driven by a Brownian motion W_t (incremented dW_t) with a variance of σ_z^2. Given the assumed labor market structure, households are not on their typical labor supply curves. Rather, they earn wages with a small markup over their marginal rate of substitution between labor and leisure, and supply their labor to small decentralized unions in the quantities determined by labor demand. As such, household hours worked $h_t(a, z)$ is *not* an explicit choice variable in the household's problem.

Households must pay a fraction τ of their wage income to the government in taxes, along with lump-sum taxes $T(a_t, z_t, \zeta)$ (although $T(a_t, z_t, \zeta)$ can also be negative, in which case it is a lump-sum transfer, like a stimulus check). Households can earn a prevailing real market rate of return of r_t on their asset holdings, denoted a_t. These assets are perfectly liquid, and are claims to government liabilities (which could be interpreted as interest-bearing reserves or near-cash instruments, literal government bonds, or shares of an investment firm or bank that owns government bonds). Borrowing is subject to first a soft constraint, and then a hard one: households' lending rates are $r(a) = r_t + \Delta_r \mathbf{1}_{\{a\leq 0\}}$, such that interest rates go up by a wedge Δ_r for households with negative holdings, with a hard limit that $a \geq \underline{a}$ for every agent in the economy. Both the borrowing constraint and the higher borrowing rate cause a substantial measure of agents to amass at or near the zero-asset level, living largely hand-to-mouth for fear of taking on costly debt or maxing out their unsecured lines of credit.

The total ex-post realized return on assets dR_t in an infinitessimal time increment dt will thus include the asset's ex-ante expected real realized return $r_t(a)$, plus whatever surprise capital gains $d\delta_{q,t}$ the households make on their entire portfolio driven by the movement of asset prices. In a rational expectations equilibrium, these errors are mean-zero and endogenous to the model, as in Sims (2002).

In the absence of capital stock dynamics, all assets are purely financial in nature. Corporate profits are also remitted back to households, as in other HANK papers like Kaplan, Moll, and Violante (2018) and McKay, Nakamura, and Steinsson (2016), in payments $\Pi_t(a, z)$ contingent on the households' position in the state space.

All told, the household problem is

$$
\begin{aligned}
V(a_0, z_0; \mu_0, \zeta_0) = \max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} \left[\frac{c_t^{1-\gamma}}{1-\gamma} - \frac{h_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] dt& \\
\text{s.t. } da_t = [(1-\tau)w_t z_t h_t - T_t(a_t, z_t; \zeta_t) + \Pi_t(a, z) - c_t]dt + a_t dR_t& \\
d\log(z_t) = -\theta_z \log(z_t)dt + \sigma_z dW_{t,z}& \\
dR_t = r_t(a_t)dt + d\delta_{a,t}& \\
r_t(a) = r_t + \Delta_r \mathbf{1}_{a\leq 0}& \\
a_t \geq \underline{a}.&
\end{aligned}
$$

This value function solves a Hamilton-Jacobi-Bellman (HJB) equation in the absence of aggregate shocks:

$$
\begin{aligned}
\rho V_t(a, z) = \max_c \Bigg\{ & \left[\frac{c^{1-\gamma}}{1-\gamma} - \frac{h^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] \\
& + \frac{\partial V_t}{\partial a}(a, z) \left[(1-\tau)wzh - T_t(a_t, z_t; \zeta_t) + \Pi_t(a, z) - c + r(a)a \right] \\
& + \frac{\partial V}{\partial z}(a, z) z \left[\frac{1}{2}\sigma_z^2 - \theta_z \log(z) \right] + \frac{\partial^2 V}{\partial z^2}(a, z; \mu, \zeta) \frac{1}{2}\sigma_z^2 z^2 + \frac{\partial V_t}{\partial t}(a, z) \Bigg\}.
\end{aligned} \tag{1}
$$

Here, $\frac{\partial V_t}{\partial t}(a, z)$ is the change in the value function over time due to changes not related to the idiosyncratic state variables – making it the aggregate variables' (μ_t, ζ_t) stochastic infinitessimal generator applied to the households' value function.

The household value function is a jump variable, and thus is perturbed by expectation errors when aggregate shocks strike the economy. In addition, the value function is also perturbed (to a first order) by a factor of $\frac{\partial V_t(a,z)}{\partial a} d\delta_{q,t}$; the asset price expectation error raises the value function of the household by the size of the shock times the household's marginal value of wealth. From there, the value function resumes the transition dynamics explicated in equation (1).

Given the optimal choices of $c_t(a,z)$ and the demand-determined dynamics of $h_t(a,z)$, the distribution of households then evolves according to a Kolmogorov forward equation (KFE):

$$\frac{\partial \mu_t}{\partial t}(a,z) = -\frac{\partial}{\partial a}\left(\frac{da_t}{dt}\mu_t(a,z)\right) - \frac{\partial}{\partial z}\left(\frac{\mathbb{E}_t[dz_t]}{dt}\mu_t(a,z)\right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2 z^2 \mu_t(a,z)\right) \tag{2}$$

Combining the evolution of the distribution with the market clearing conditions and the policy actions of the government is then sufficient to characterize the evolution of macroeconomic aggregates in the model environment.

2.2 Organization of Labor (Unions)

In order to rationalize nominal wage rigidities and to generate real wages and firm profits that do not respectively move strongly pro-cyclically and counter-cyclically, as per empirical evidence, I adapt the decentralized union approach of Auclert, Rognlie, and Straub (2018) to continuous time, which in turn is related to Schmitt-Grohé and Uribe (2005). Namely, I follow a stylized framework wherein small unions indexed by k supply their work to labor-aggregating agencies. For simplicity, these unions each employ a representative slice of all workers in the economy, such that:

$$L_{kt} = \int\int z h_t(a,z) da\ dz$$

These employment agencies transform the labor supplied by workers using a constant-elasticity-of-substitution aggregator into the mix of labor used by production firms by hiring at each union's nominal wage rate $\tilde{w}_{kt}$:

$$L_t = \left(\int_0^1 L_{kt}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk\right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1}}$$

Intermediate firms thus desire to hire according to

$$\max_{\{L_{kt}\}_{k\in[0,1]}} \tilde{w}_t \left(\int_0^1 L_{kt}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk\right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1}} - \int_0^1 \tilde{w}_{kt} L_{kt} dk$$

where $\tilde{w}_t$ is the prevailing nominal wage at time t. Employment is demand-determined in the economy, and labor unions internalize this demand when choosing the wages they demand, all the while maximizing the collective welfare of their members. Negotiating new wages, however, is subject to Rotemberg adjustment costs, making the objective of union k

$$\max_{\pi^w_{kt}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\left[\int\int \left\{\frac{c_t(a,z)^{1-\gamma}}{1-\gamma} - \frac{h_t(a,z)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right\}\mu_t(a,z) da\ dz - \frac{\theta_w}{2}(\pi^w_{k,t})^2\right] dt$$

$$\text{s.t. } \frac{d\tilde{w}_t}{dt} = \pi^w_{t,k}\tilde{w}_t, \quad L_{kt} = \int\int z h_t(a,z)\mu_t(a,z) da\ dz, \quad \frac{L_{kt}}{L_t} = \left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell},$$

where $\pi^w_{t,k}$ is wage inflation. Further details and derivations are provided in the appendix; in a symmetric equilibrium, the resulting wage Phillips Curve is

$$\frac{\mathbb{E}_t[d\pi^w_t]}{dt} = \rho\pi^w_t - \frac{\varepsilon_\ell}{\theta_w} L_t \int\int \left(\frac{1}{Z} h_t(a,z)^{\frac{1}{\eta}} - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau) z w_t c_t(a,z)^{-\gamma}\right) da\ dz \tag{3}$$

and the real wage rate evolves according to

$$\frac{dw_t}{dt} = \pi^w_t - \pi_t \tag{4}$$

Unlike Auclert, Rognlie, and Straub (2018), I do not assume that households all supply the same amount of labor. Rather, I assume that in the steady-state households work hours according to

$$\frac{1}{Z} h(a,z)^\eta = \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau) z w c(a,z)^{-\gamma}$$

where $Z = \int\int z\mu(a,z) da\ dz$ (and in my calibration, $Z \approx 1$). In other words, households in the steady-state work approximately what their labor-leisure choice would be in absence of the union, but with a wage markup commensurate with the elasticity of demand for their hours. Outside of the steady-state, I assume all members of the workplace increase their hours worked commensurate with the change in labor demand following a shock:

$$h_t(a,z) = h_{nss}(a,z)\frac{d\mu_{nss}}{d\mu_t}(a,z) + \frac{1}{Z}(L_t - L_{nss}) \tag{5}$$

where $\frac{d\mu_{nss}}{d\mu_t}(a,z)$ is the Radon-Nikodym derivative of the measure of workers in the steady state relative to the measure of workers at time t.[4]

2.3 The Government

2.3.1 Government Debt

The model's fiscal authority collects aggregate taxes (net of transfers) equal to T_t; real government expenditures G_t are included in the following equations for generality, but are set to be zero in equilibrium. The aggregate price level in the economy is p_t. The government is able to borrow using long-term nominal bonds as in Cochrane (2018b) by issuing nominal perpetuities $\tilde{B}_t$ at a nominal price of q^B_t that pay out exponentially declining coupon payments of $\omega e^{-\omega t}$ per unit of time, where ω effectively determines the overall maturity of the government's

[4]The rule stipulates that if firms want more hours worked, everyone in the workplace increases their hours by a the same amount, even if they were working different amounts before, adjusting for how the distribution of workers is also different relative to the non-stochastic steady-state.

debt portfolio.[5] Note that as $\omega \to \infty$, government debt becomes instantaneously short-term and must be rolled over immediately with new bonds (analogous to the continuous-time equivalent of a one-period bond in discrete time), while as $\omega \to 0$, each new bond issued becomes a perpetuity. If the government debt is priced by a risk-neutral fund, as it is in this setting since the linearized model is certainty-equivalent with respect to aggregate shocks, the central fiscal theory asset pricing equation takes the same form as the one presented in Cochrane (2018b):

$$\underbrace{\frac{q_t^B \tilde{B}_t}{p_t}}_{\substack{\text{Real debt} \\ \text{outstanding}}} = \mathbb{E}_t \left[\int_t^\infty e^{-\int_t^\tau r_s ds} [T_\tau - G_\tau] d\tau \right]$$

To simplify the notation, I write the market value of real debt outstanding as

$$B_t \equiv \frac{q_t^B \tilde{B}_t}{p_t}$$

Changes in the value of government debt outstanding can thus be influenced by both the price of government bonds and by the value of primary surpluses net of interest expense. In the appendix, I follow similar steps as in Cochrane (2018b) to show that the evolution of real government debt will be

$$dB_t = -(T_t - G_t)dt + B_t \left[i_t - \pi_t \right] dt + \frac{d\delta_{qB,t}}{q_t^B} B_t \tag{6}$$

Here, $d\delta_{qB,t}$ denotes the endogenous expectation error on the nominal price of government debt. Since households save by holding a share of a portfolio that is a representative slice of government liabilities, it further follows that $d\delta_{q,t} = d\delta_{qB,t}$. Expected nominal bond prices are in turn governed by

$$E_t[dq_t^B] = q_t^B \left(i_t + \omega - \frac{\omega}{q_t^B} \right) dt \tag{7}$$

and so bond prices evolve according to

$$dq_t^B = q_t^B \left(i_t + \omega - \frac{\omega}{q_t^B} \right) dt + d\delta_{qB,t}$$

as derived in the Appendix. Notably, since the bonds offer nominal payments, the path of nominal interest rates (and the bond portfolio's maturity structure) determines the evolution of nominal bond prices.

[5] While this may seem like an arbitrary structure, it can be rationalized by having the government issue debt to maintain an exponentially distributed maturity structure, as shown in Cochrane (2018b). From there, one can imagine that government debt is bought by a mutual fund, whose shares are in turn owned by households as assets, such that every household effectively owns a representative share of the government's debt portfolio. The details of this are relegated to the appendix.

2.3.2 Taxes

As a baseline, the fiscal authority in the model taxes labor income at a rate of τ, such that if total effective labor employment in the economy is L_t and real wages are w_t, total income taxes are $\tau w_t L_t$ per unit of time. Households are additionally subject to lump-sum transfers, which aggregate to total lump-sum taxes $T_t(\zeta_t)$. These taxes aggregate naturally from taxes on households:

$$T_t(\zeta_t) = \int_0^\infty \int_{\underline{a}}^\infty T_t(a, z; \zeta_t)\mu_t(a, z)da\ dz$$

In the steady-state, these transfers are assumed to balance the budget; after paying for the interest expense on the debt, the government rebates its tax revenue evenly to the rest of society, such that

$$T_{nss} + \tau w_{nss} L_{nss} - r_{nss} B_{nss} = 0$$

where the NSS subscript denotes variables in the non-stochastic steady-state. Since more tax revenue is collected from high-earners, but the rebate is evenly distributed when there are no aggregate perturbations, the result is a progressive net transfer scheme in the NSS. Outside of steady-state, however, the government does *not* necessarily balance the budget. Rather, it adjusts lump-sum transfers according to exogeneous aggregate processes:

$$T_t(a, z; \zeta_t) = T_{nss} + 4Y_{nss} \times \left(T_t^{\text{All}}(a, z; \zeta_t^{\text{All}}) + T_t^{\text{High}}(a, z; \zeta_t^{\text{High}}) + T_t^{\text{Low}}(a, z; \zeta_t^{\text{Low}}) + T_t^{\text{BB}}(a, z; \zeta_t^{\text{BB}})\right)$$

These tax policies (driven by aggregate "shocks" ζ_t) may all be viewed as programs that send out or demand transfers of varying kinds. All of them are shocks expressed relative to annual GDP in the non-stochastic steady-state (which is $4 \times Y_{nss}$, since the model is quarterly). The first, T_t^{All}, denotes a mean-reverting increase in lump-sum taxes (or cut in benefits) on all members of society:

$$T_t^{\text{All}}(a, z; \zeta_t^{\text{All}}) = \zeta_t^{\text{All}}$$

A negative shock to $T_t^{\text{All}}(a, z; \zeta_t^{\text{All}})$ is analogous to the government printing or borrowing stimulus checks and sending them out to everyone in society. A positive shock is its (somewhat less realistic) opposite, in which the government demands its citizens to pay it flat fees.

The $T_t^{\text{High}}(a, z; \zeta_t^{\text{High}})$ is similar, but in this case, the tax is levied only on households whose wages are above the median in the population. If $\bar{z}$ is the median skill level z, it therefore follows that

$$T_t^{\text{High}}(a, z; \zeta_t^{\text{High}}) = \mathbf{1}_{\{z \geq \bar{z}\}} \zeta_t^{\text{High}}$$

The $T_t^{\text{Low}}(a, z; \zeta_t^{\text{Low}})$ is the same, but only levied on households with wages below the median:

$$T_t^{\text{Low}}(a, z; \zeta_t^{\text{Low}}) = \mathbf{1}_{\{z < \bar{z}\}} \zeta_t^{\text{Low}}$$

Naturally, if both $T_t^{\text{High}}(a, z; \zeta_t^{\text{High}})$ and $T_t^{\text{Low}}(a, z; \zeta_t^{\text{Low}})$ are active at the same time and are of the same magnitude in a linearized model, they are additively equivalent to a change in $T_t^{\text{All}}(a, z; \zeta_t^{\text{All}})$. Conversely, changes in the economy in reactions to the paths of $T_t^{\text{High}}(a, z; \zeta_t^{\text{High}})$ and $T_t^{\text{Low}}(a, z; \zeta_t^{\text{Low}})$ may be seen as a decomposition of the effects of $T_t^{\text{All}}(a, z; \zeta_t^{\text{All}})$ in the effect driven by taxes on the lower-income and the effect driven by taxes on the higher-income.

$T_t^{BB}(a, z; \zeta_t^{BB})$ is a slightly different policy than the preceding ones, in that it leaves net lump sum surpluses unchanged, and only acts through redistribution. It imagines a change in tax policy that taxes those above median income and remits those transfers to those below median income. The policy is thus set up such that

$$T_t^{BB}(a, z; \zeta_t^{BB}) = \frac{z}{Z_t} \zeta_t^{BB} \mathbf{1}_{\{z \geq \hat{z}\}} - \kappa_t^{BB} \mathbf{1}_{\{z < \bar{z}\}}$$

Where ζ_t^{BB} is the tax shock, here now scaled by $\frac{z}{Z_t}$ to make the tax yet more progressive on higher above-median earners, and where κ_t^{BB} is the flat remittance to lower-income households. If $T_t^{BB}(a, z)$ aggregates to zero to leave the federal budget unchanged (aside from the feedbacks from automatic taxes τ), it must then be that

$$\kappa_t^{BB} = \zeta_t^{BB} \frac{\int_{\bar{z}}^{\infty} \int_{\underline{a}}^{\infty} \frac{z}{Z_t} \mu_t(a, z) \; da \; dz}{\int_0^{\bar{z}} \int_{\underline{a}}^{\infty} \mu_t(a, z) \; da \; dz}$$

2.3.3 Monetary Block

The central bank directly sets nominal interest rates in the economy according to

$$i_t = r^* + \phi_\pi \pi_t + \zeta_{\text{MP},t} \tag{8}$$

where r^* is the interest rate that would prevail in equilibrium in the absence of any aggregate shocks. In theory, the model can be solved so long as the interest rate rule is "passive," such that $\phi_\pi < 1$. For clarity, I make the interest rate rule totally unreactive, such that $\phi_\pi = 0$.

2.4 Firms

Intermediate and final goods firms in the model behave as in Kaplan et al (2018), although in this version of the model labor is the only input factor used in production. Final output Y_t is produced by final goods firms at the end of the supply chain using a constant elasticity of substitiution (CES) production and the output of a continuum of monopolistically competitive intermediary firms indexed by j, denoted $y_t(j)$:

$$Y_t = \left(\int_0^1 y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

These monopolistically competitive intermediate goods firms hire labor and produce output via the linear production function

$$y_t(j) = h_t(j)$$

such that their marginal costs are simply their wage costs:

$$m_t = w_t$$

These intermediaries are subject to Rotemberg (1982) quadratic adjustment costs when changing their prices. As such, intermediate firms set their prices by solving the profit-maximization problem:

$$\begin{aligned} J_0(p_0(j)) = &\max_{\pi_t(j)} \mathbb{E}_0 \int_0^\infty e^{-\int_0^t r_s ds} \left[\frac{p_t(j)}{P_t} y_t(j) - w_t h_t(j) - \frac{\theta}{2} \pi_t(j)^2 Y_t \right] dt \\ \text{s.t. } & dp_t(j) = \pi_t(j) dt \\ & y_t(j) = h_t(j) \\ & y_t(j) = \left(\frac{p_t(i)}{P_t} \right)^{-\varepsilon} Y_t \end{aligned}$$

whose recursive formulation is

$$rJ_t(p_t(j)) = \max_{\pi_t(j)} \left\{ \left(\frac{p_t(j)}{P_t} - m_t \right) \left(\frac{p_t(j)}{P_t} \right)^{-\varepsilon} Y_t - \frac{\theta}{2} \pi_t(j)^2 Y_t + \frac{\partial J_t}{\partial p_t(j)} (p_t(j)) \pi_t(j) p_t(j) + \frac{\partial J_t}{\partial t} (p_t(j)) \right\}$$

As shown in Kaplan, Moll, and Violante (2018), the solution to this problem in a symmetric equilibrium yields the New-Keynesian Phillips Curve

$$\frac{\mathbb{E}[d\pi_t]}{dt} = \left(r_t - \frac{\mathbb{E}[dY_t]}{dt} \frac{1}{Y_t} \right) \pi_t - \frac{\varepsilon}{\theta} (m_t - m^*) \tag{9}$$

where $m^* = \frac{\varepsilon-1}{\varepsilon}$ is a firms' marginal cost in the absence of shocks.

2.5 The Distribution of Monopoly Profits

Firms in the economy collect monopoly profits, which are in total equal to

$$\Pi_t^F = (1 - m_t)Y_t - \frac{\theta_\pi}{2}\pi_t^2 Y_t$$

To abstract away from the price adjustment costs, I assume that these are rebated back to households as the income of a separate firm, such that profits from the entire corporate sector are $\Pi_t = \Pi_t^F + \frac{\theta_\pi}{2}\pi_t^2 Y_t = (1 - m_t)Y_t$.

Like in McKay, Nakamura, and Steinsson (2016), households are rebated these profits directly; this allows the model to avoid a discussion of the evolution of equity markets outside of the steady-state. However, as in Kaplan, Moll, and Violante (2018), households receive these profits as "bonuses," with higher-income households receiving a larger share, to keep the redistribution scheme from significantly changing the income inequality features of the economy. As such, the profit flows to households take the simple form

$$\Pi_t(a, z) = \frac{z}{Z_t}\Pi_t \tag{10}$$

2.6 Aggregate Exogenous Stochastic Processes

The aggregate exogenous stochastic process vector is $\zeta_t = (\zeta_t^{\text{MP}}, \zeta_t^{\text{All}}, \zeta_t^{\text{High}}, \zeta_t^{\text{Low}}, \zeta_t^{\text{BB}})'$. The first entry pertains to a monetary policy shock, while the latter four refer to perturbations to tax policy, as discussed in Section 2.3.2. Collectively, the shocks evolve according to the Ornstein-Uhlenbeck process

$$d\zeta_t = -\Theta_\zeta \zeta_t dt + d\epsilon_{\zeta,t} \tag{11}$$

where $d\epsilon_{\zeta,t}$ is a vector of mean-zero independent stochastic innovation terms. The model considers shocks to each of the vectors independently, one at a time, where the monetary policy shock reverts at a rate of θ_{MP} while the tax shocks revert at a common rate of θ_{Tax}.

2.7 Market Clearing and Equilibrium

Note that in a symmetric equilibrium, total output is

$$Y_t = L_t \tag{12}$$

And if the inflation adjustment costs $\frac{\theta}{2}\pi_t^2 Y_t$ are rebated back to households, then in the absence of government spending and physical investment

$$Y_t = C_t + \underbrace{\int_0^\infty \int_{\underline{a}}^\infty -\mathbf{1}_{\{a<0\}}\Delta_r a\mu_t(a,z)da\ dz}_{\text{Financial Fees}} \tag{13}$$

where aggregate consumption is

$$C_t = \int_0^\infty \int_{\underline{a}}^\infty c_t(a,z)\mu_t(a,z)da\ dz$$

and the aggregate labor demand is equal to the total effective hours worked by households:

$$L_t = \int_0^\infty \int_{\underline{a}}^\infty zh_t(a,z)\mu_t(a,z)da\ dz \tag{14}$$

The total amount of assets also further be equal to the total amount of government debt

$$\begin{aligned} A_t &= \int_0^\infty \int_{\underline{a}}^\infty a\mu_t(a,z)da\ dz \\ A_t &= B_t \end{aligned} \tag{15}$$

A equilibrium given a sequence of aggregate shocks $(\zeta_t)_{t\geq 0}$ is therefore

i. a sequence of value functions $(V_t(a,z))_{t\geq 0}$ and household decisions for consumption $(c_t(a,z))_{t\geq 0}$ across idiosyncratic states and over time that solves (1) given the path of prices prices, transfers, profits, and aggregate shocks

ii. a distribution of idiosyncratic household states μ_t that evolves over time according to (2)

iii. A sequence of aggregate effective hours $(L_t)_{t\geq 0}$ worked determined by firms' labor demand (which is in turn determined by output and the demand for consumer goods), and idiosyncratic hours worked $h_t(a,z)$ determined by unions' labor rules (5)

iii. A sequence of inflation consistent with firms' profit maximization problem given prices, i.e. the Phillips Curve (9) (where marginal cost is determined by wages)

iv. A sequence of nominal wage inflation consistent with the unions' maximization problem and resulting wage Phillips Curve (3)

iv. A sequence of nominal government bond prices $(q_t^B)_{t\geq 0}$ consistent with the dynamic equation (7)

v. Profit disbursal to households $(\Pi_t(a,z))_{t\geq 0}$ in accordance with (10)

vi. Sequences of macro aggregates $(Y_t, C_t, L_t, A_t, B_t)_{t\geq 0}$ consistent with their definitions (and therefore the household decision rules and distribution equations and the production functions)

vii. Sequences of real wages and real rates of return $(w_t, r_t)_{t\geq 0}$, where w_t evolves according to (4) and r_t obeys the Fisher equation $r_t = i_t - \pi_t$

viii. Government taxes and transfers across the population and over time $(T_t(a, z))_{t\geq 0}$

such that

1. Total output is produced as in (12)
2. the asset market clears, as in (15)
3. the labor market clears (and so by Walras' law, the goods market clears) (14 and 13)
4. interest rates $((i_t)_{t\geq 0})$ are set according to the central bank's policy rule (8)
5. tax policy is consistent with the government's fiscal rules (summarized in section 2.3.2).

Computationally, the structure of the numerical solution is similar to the approach proposed by Reiter (2009). The model is first solved around its non-stochastic steady-state (NSS) using a finite difference scheme similar to the kind put forward by Achdou et al. (2021). To generate the impulse response functions of the economy in response to aggregate shocks, the system is subjected to a dimension reduction routine demonstrated in Bayer and Luetticke (2020) and linearized around its NSS and perturbed in continuous time as in Ahn et al. (2018) et al (2018). In doing so, the value function is projected down onto a hierarchical Chebychev polynomial basis via a discrete cosine transform (DCT), where only the Chebychev polynomials that explain the largest amount of variation in the value function are perturbed from their steady-state values. Additionally, the distribution function is projected onto a fixed copula, where the idiosyncratic variables' joint distribution is assumed to be characterized by the evolution of the idiosyncratic marginal distributions.[6] The entire process treats the differential equations in the model (like the Hamilton-Jacobi-Bellman equations and the Kolmogorov Forward Equation) as a large system of inter-related stochastic ordinary differential equations. Once this discretization and dimension reduction has been completed, the model is then solved using methodologies standard to the solution of linear rational expectation models, namely a QZ (Schur) decomposition, as in Klein (2000) and Sims (2002). For the model to have a uniquely determined stochastic solution, its Jacobian must have exactly as many explosive (positive) generalized eigenvalues as the system has jump variables (which here include the discretized value function, inflation, and the bond and equity prices); I verify that this is indeed the case for my system. Further details are provided in the appendix.

[6] Bayer and Luetticke (2020) note that this will be a good approximation if the rank correlations of the distributions are not strongly affected by the shocks, which they observe to be the case in models like the one in Krusell and Smith (1998), to which my model's household sector is highly similar.

3 Calibration

3.1 Model Parameters

I calibrate ρ to achieve a an annual interest rate of 2% in the non-stochastic steady-state. I retain the same relative risk aversion coefficient of $\gamma = 2$ as McKay, Nakamura, and Steinsson (2016) – although coefficients ranging from between 1 and 2 are common in the HANK literature. The parameters, along with their targets or sources, are displayed in Table 1.

I additionally target the same income process moments used in McKay, Nakamura, and Steinsson (2016), except using the continuous-time analogue of their discrete-time process. As such, for a given parameterization, I simulate a large number of independent Ornstein-Uhlenbeck process to generate a panel dataset of $\log z$. I then integrate the exponential of the process to the annual level to attain the model's predictions for the panel distribution of annual wages in the non-stochastic steady-state. From there, I choose the income process persistence parameter θ_z and the Brownian motion variance σ_z^2 to match the Floden and Lindé (2001) estimates of the permanent component of annual wage autocorrelation and autoregression variance, residualized for age, occupation, education, and other covariates. As such, I choose (θ_z, σ_z^2) and fit the regression

$$\text{wage}_{it}^{\text{Annual}} = \beta_0 + \beta_1 \text{wage}_{it-1}^{\text{Annual}} + \epsilon_{it}$$

on the simulated data in order to match the Floden and Lindé (2001) estimates of $\beta_1 = 0.9136$ and $\text{var}(\epsilon_{it}) = 0.0426$. I fit both moments up to machine precision; the parameters are also reported in Table 1.

Like Kaplan, Moll, and Violante (2018), I target a slope of the Phillips Curve of 0.10. However, Auclert, Bardóczy, and Rognlie (2023) suggest that nominal wage rigidities capture several important features of the micro data, like low marginal propensities to earn in the micro data, than conventional final goods price rigidities do. As such, I set final goods price nominal rigidities to be very low (1% of the wage rigidities), and make the nominal wage Phillips Curve to have a slope of 0.10. This allows the model to capture similar overall inflation dynamics as other calibrations in the literature, but with arguable more realistic real wage dynamics: with more flexible prices, monopolistically competitive firms are better able to completely pass changes to their marginal costs along to households, largely keeping their ideal markups intact. Workers, in contrast, face a compressed surplus due to nominal wage rigidities during an economic expansion, ascribable to the time and effort that it takes to change their wages. Since they are not on their competitive labor supply curves, workers still increase their hours to meet their employers' demands, but their surplus per hour worked falls as they wait for their real wages to be negotiated to higher levels.

For the mean reversion of the shocks, a monetary policy shock is assumed to have a half-life of 4 quarters. In contrast, the mean reversion of fiscal shocks is made to be much stronger with $\theta_{\text{Tax}} = 1.0$. This is intended

Table 1: Baseline Parameters

Parameter	Symbol	Value	Source or Target
Relative Risk Aversion	γ	2.0	McKay et al (2016)
Quarterly Time Discounting	ρ	0.0163	$r = 2\%$ annually
Borrowing Limit	$\underline{a}$	-1.0	≈30% of avg income
Frisch Elasticity of Labor	η	0.5	Chetty (2012)
Borrowing Wedge Rate	Δ_r	0.04	Credit APR of 18%
Idiosyncratic Shock Variance	σ_z^2	0.017	Calibrated
Idiosyncratic Shock Mean Reversion	θ_z	0.034	Calibrated
Intermediary Elasticity of Substitution	ε	10	10% profit share of GDP
Rotemberg price adjustment cost	θ_π	1.0	(Roughly) acyclic real wages
Labor Elasticity of Substitution	ε	10	Philips Curve slope of 0.10
Rotemberg wage adjustment cost	θ_w	100	Phillips curve slope of 0.10
Steady-state government debt	B_{nss}	5.26	Debt/GDP of 1.37
Geometric maturity structure of debt	ω	0.043	Avg. maturity of 70 months
Income Tax Rate	τ	0.25	
Mean reversion of monetary shock	θ_{MP}	0.175	4-quarter shock half-life
Mean reversion of fiscal shocks	θ_{Tax}	1.0	

to better reproduce the speed with which stimulus checks may be sent out; after 4 quarters, the fiscal shocks almost entirely dissipate. Since the path of the shock in the absence of further perturbations may be described with

$$\zeta_t^{\text{All}} = e^{-\theta_{\text{Tax}} t}$$

this also means that the cumulative effect of an initial shock of $\zeta_0^{\text{All}} = -0.01$ has the interpretation of a 1%-of-annual-GDP disbursal of lump-sum stimulus checks.[7]

3.2 Non-Stochastic Steady-State

The non-stochastic steady state is solved by setting all aggregate shocks $\zeta_t = 0$. The model is not able to match the United States income Gini coefficient, which the Census Bureau reported as around 0.48 (before taxes) in 2018. The model also slightly undershoots measures of liquid wealth inequality, which Kaplan, Moll, and Violante (2018) estimates to be roughly 0.98. Being unable to match the observed wealth inequality in the data with Gaussian uncertainty is a common feature in Bewley-Aiyagari models of the type I study, as documented in Aiyagari (1994). Even so, the model generates a large mass of households with little-to-no savings; 24% of the households have a net worth of less than 0, and median liquid wealth is only about 25% of quarterly GDP per capita. This is lower than measures of total household wealth, but roughly congruous with a 2018 Federal Reserve survey that find roughly 40% of U.S. adults would be unable to cover a $400 emergency payment with cash, savings, or easily repaid credit, as reported in Chen et al. (2019). Both the distribution of assets a and

[7]For example, if the United States economy in 2019 was taken to be the non-stochastic steady-state, this would be a spending program of $210 billion.

Table 2: Non-Stochastic Steady-State Moments

Moment	Model Value
Debt/GDP Ratio	1.37
Earnings Gini	0.315
Wage Gini	0.317
Wealth Gini	0.824
Wage and Asset Correlation	0.518

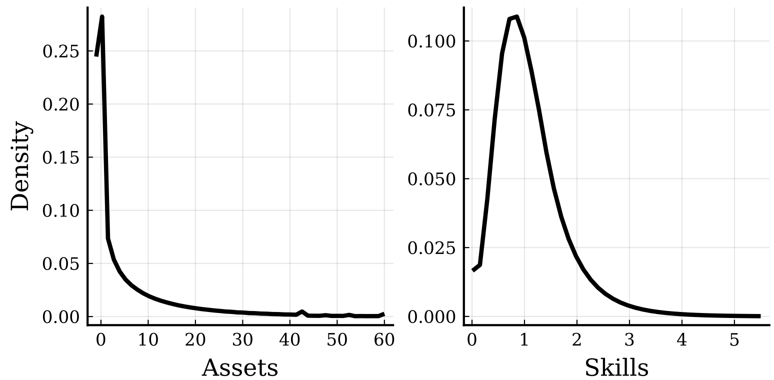

Figure 1: Distribution of assets and income (skills) in the non-stochastic steady-state.

skills z (which mechanically matches the model's distribution of pre-tax wages) are depicted in Figure 1.

4 Results

In simulating the economy's response to various shocks, I consider the contemporaneous and cumulative effects of the policy perturbations. To calculate the cumulative inflation rate, I simply calculate the price level over time as

$$\Delta p_t = e^{\int_0^t \pi_\tau d\tau}$$

and normalize the pre-shock price level to 1. For the growth of GDP, I similarly accumulate

$$\Delta Y_t = \frac{\int_0^t (Y_\tau - Y_{nss}) d\tau}{Y_{nss}}$$

to gauge the percentage growth of the economy following the shock relative to the non-stochastic steady-state. I define my measure of the sacrifice ratio at horizon t as the average decline in GDP required to bring prices down by a cumulative 1% by the time t quarters have elapsed:

$$SR_t = \frac{\Delta Y_t}{\Delta p_t}$$

As $t \to \infty$, the model returns to its non-stochastic steady-state, such that over long horizons SR_t approaches its non-truncated version.

The first simulation I conduct is a 1% *increase* to the nominal interest rate by the country's central bank, as documented in Figure 2. For comparison, I also plot a RANK version of the heterogeneous agent model and its response to the same monetary policy shock, where the RANK model simply omits the HJB system of equation (1) and the KFE equation (2) and instead uses the standard Euler equation $\frac{\mathbb{E}[dc]}{dt}\frac{1}{c} = \frac{1}{\gamma}(i_t - \pi_t - \rho)$ to characterize the solution of the representative agent's problem.

Both models exhibit remarkably similar dynamics, with the primary mechanism first employed by Sims (2011) and later elaborated by Cochrane (2018b) at work. In both environments, the monetary tightening precipitates an immediate fall in nominal bond prices to prevent arbitrage with the policy rate, where since since the debt portfolio is long-term, its price responds to the entire expected future path of nominal interest rates. These lower nominal bond prices mean that bonds become cheaper in real terms than the present value of their coupon payments unless the price level adjusts – provided the maturity of the bonds is long enough for nominal bond prices to fall more than the real present value of primary surpluses if inflation does not react.[8] Of course, this would result in the households wanting to buy more bonds than there are for sale on the market – so the goods market begins to experience a deflation from the lack of demand, further raising real rates until markets clear. At this point, the real market value of the bonds will be equal to the real present value of their cash flows, while the price level will gradually begin to fall (from a lack of "aggregate demand," in the words of Cochrane (2018b)).

Eventually, inflation begins to rise, and actually reverses the deflation of both simulations. The HANK model adds some realism to the dynamics; the fall in asset prices pushes bondholders closer to the borrowing constraint, prompting households with moderate asset holdings to cut back on their consumption following the loss of value of their precautionary savings. In both the HANK and RANK settings, though, the path of inflation depicted in Figure 2 is largely the same. The wealth effect forward guidance mechanism of Cochrane (2018b), (which Cochrane (2018b) notes does not rely on a new-Keynesian IS curve or sticky prices to produce the temporary deflation) has a largely unchanged affect on prices. The only significant difference in the HANK-FTPL model compared to the RANK version is in the real GDP response: while both models yield recessions following the monetary tightening, the recession is actually more severe in the RANK setting. In HANK, there are a large number of hand-to-mouth and near-hand-to-mouth agents; it stands to reason that these agents without wealth would not contract their demand or actual expenditures in response to a wealth effect. Although they are still affected by the fall in hours worked, they are less affected by financial markets directly, and do not propagate the mechanism in the first round.

[8] Note that since the model exhibits long-run money neutrality, real rates in the distant future are largely unchanged, making the model similar to the frictionless version if the maturity of debt is high relative to the pricing frictions in the economy.

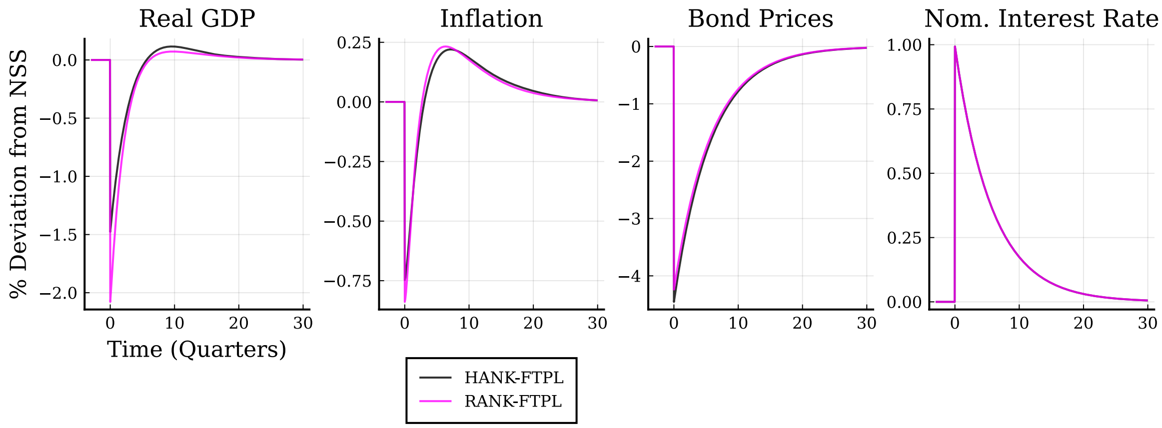

Figure 2: Impulse responses to a 1% increase in nominal interest rates in both a HANK model (in black) and a RANK model (in magenta), where the FTPL is active in both.

At what point does the "stepping-on-a-rake" dynamic win out? A year after the 1% increase in rates, the price level in the economy is 1.0% lower (as policymakers may have hoped, if they raised rates to lower inflation), but after just 9.5 quarters the price level has already returned to and exceeded its pre-shock value. After that, the high rates become inflationary, to the point that after 30 quarters (long after the shock has played out), prices are 1.34% *higher* in response to the persistent monetary tightening. As such, the sacrifice ratio appears to be 1.93 to the agents living in the economy for the first quarter following the shock. However, it appears over 40% larger after a year has elapsed, and becomes unbounded before flipping its sign entirely after 10 quarters (when on net, the price level begins to inflate).

To examine the effect of *lowering* interest rates 1% in model, one can simply flip the signs of the impulse responses, as the model is linear with respect to aggregate shocks. I plot the result in Figure 3; with the similarity to the RANK model established, I plot the dynamics of the FTPL-HANK model alone. With a lower policy rate, the mechanism runs in reverse, raising nominal and real asset prices and stimulating a boom to the economy.

Returning to the case plotted in the impulse response functions, wherein interest rates were lowered by the central bank, this means that the low rates eventually drive the price level *lower*, as documented in the first four columns of Panel A of Table 3.

How much does the federal government's ability to "tax the boom" matter for the amount of inflation or deflation generated by the shock? To assess this, I repeat the same experiment, but instead have the government spend its automatic stabilizer revenue instead of using it to back and pay down the debt. The government thus purchases

$$G_t = \tau(w_t L_t - w_{nss} L_{nss}) \tag{16}$$

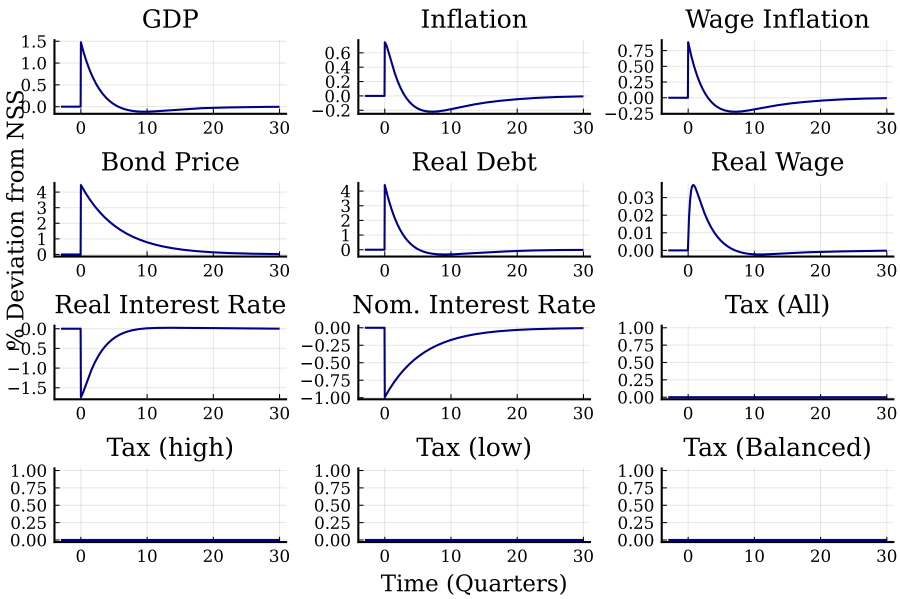

Figure 3: Impulse response functions to a 1% decrease in nominal interest rates.

Table 3: Monetary Policy Shock: Cumulative Inflation and Real GDP

	Quarters After Shock				Government Spends New Revenue
Cumulative Variable	1	4	8	30	30 Quarters
Monetary Expansion					
Real GDP (%)	1.26	2.75	2.72	1.73	2.45
Inflation (%)	0.65	1.00	0.21	-1.34	-1.35
Sacrifice Ratio	1.93	2.75	-	-	-

Note: Since the the sacrifice ratio for monetary policy blows up and flips sign at around 10 quarters due to neo-Fisherian effects, I do not calculate it for monetary policy's 8 and 30 quarter horizons.

from firms directly, modifying the goods market clearing equation to

$$Y_t = C_t + G_t + \int_{\underline{a}}^{\infty} -\mathbf{1}_{\{a<0\}} \Delta_r a \mu_t(a,z) da \; dz \tag{17}$$

The resulting cumulative changes in GDP and inflation after 30 quarters are reported in the fifth column of Table 3. The governments' new purchases act to yet further stimulate the economy during the monetary policy-driven expansion, causing output to grow by even more than it did in the baseline experiment. The government forsaking the use of the new revenue for debt repayment does produce a slight inflationary force, as more debt must instead be inflated away in equilibrium. However, the effect is very small, on the order of adding 0.07 percentage points to the overall amount of inflation generated by monetary policy.

Proceeding to the next simulation, I generate the impulse response functions reported in Figure 4. A 1% of steady-state GDP lump-sum payment to all households leads to a sharp increase in GDP as relatively hand-to-mouth households pass the increase in income into lower spending, which in turn causes a jump in hiring and further increases in income and spending. Inflation ensues to diminish the real amount of government

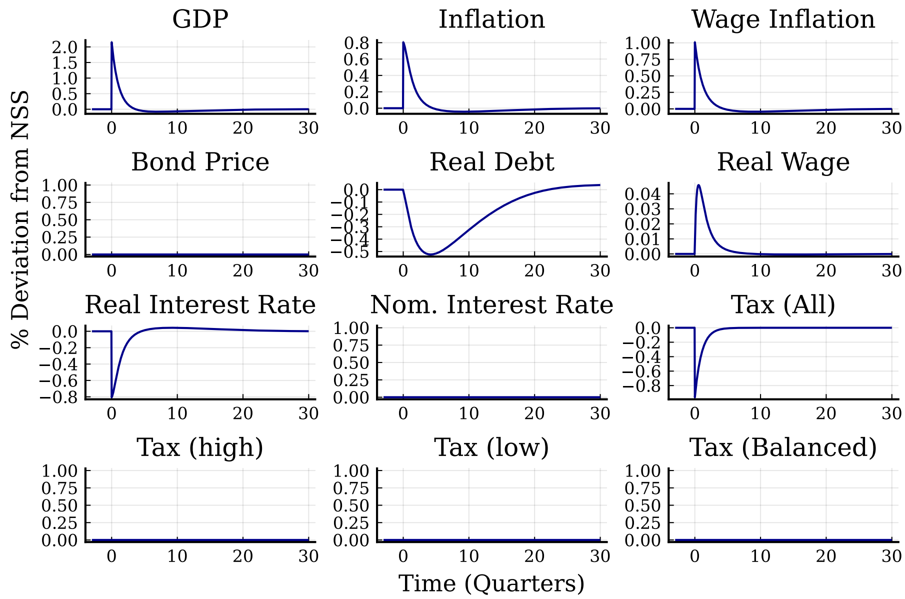

Figure 4: Impulse response functions to a mean-reverting decrease in lump-sum taxes on all households.

debt outstanding, as the expenditure was not backed by new taxes, although automatic stabilizers are able to partially offset the effect by providing more net income to the government's coffers. Surprisingly, despite the tax increase, the real value of the debt temporarily *falls*.[9] This is because the surge in inflation dramatically lowers real rates, to the point that the real value of the nominal debt actually declines.

All told, sending stimulus checks to all agents in the economy in an expenditure equal to 1% of GDP ends up growing the economy by 1.27% over time at the cost of an additional 0.61 percentage points of inflation (where in the very short-term, the rate of additional inflation experienced by agents in the first year is over 1.15%). If the government had gone the other way, and demanded fees from the public instead of distributing checks, the sacrifice ratio would thus have amounted to 2.08 by the end of the simulation window.

[9]Note that in the previous experiment with monetary policy, this dynamic was masked by the surge in bond prices. Since the central bank keeps the interest rate constant in the tax policy experiment, nominal bond prices do not move.

Table 4: Fiscal Policy Shocks: Cumulative Inflation and Real GDP

Cumulative Variable	Quarters After Shock 1	4	8	30	Government Spends New Revenue 30 Quarters
Panel A: Tax Shock (All)					
Real GDP (%)	1.43	2.08	1.81	1.27	1.89
Inflation (%)	0.67	1.15	1.05	0.61	0.67
Sacrifice Ratio	2.13	1.81	1.72	2.08	2.84
Panel B: Tax Shock (High Income)					
Real GDP (%)	0.94	1.57	1.48	1.03	1.48
Inflation (%)	0.48	1.01	1.04	0.64	0.68
Sacrifice Ratio	1.94	1.56	1.43	1.62	2.19
Panel C: Tax Shock (Low Income)					
Real GDP (%)	2.00	2.68	2.19	1.55	2.37
Inflation (%)	0.89	1.31	1.07	0.58	0.65
Sacrifice Ratio	2.25	2.04	2.05	2.66	3.64

Note: All tax shocks are equal to stimulus transfers amounting to 1.0% of GDP, while the monetary shock is a 1.0% nominal interest rate decrease. The cumulative effects in Panel A may be recovered by averaging the cumulative effects in Panels B and C with the weights 0.54 and 0.46, respectively. Since the the sacrifice ratio for monetary policy blows up and flips sign at around 10 quarters due to neo-Fisherian effects, I do not calculate it for monetary policy's 8 and 30 quarter horizons.

I next decompose the overall fiscal shock into two components: the stimulus check rebate to high-wage agents (who are on average wealthier), and the stimulus check rebate to low-wage agents (who are on average poorer). Note that now, since each of the shocks fall on only half of the population, I adjust the size of the tax shocks to keep the implied expenditure plans equal to 1.0% of annual steady-state GDP (and thus comparable to the earlier simulations); averaging the results of the two shocks yields the effects of Figure 4 composite shock.[10] I plot both sets of impulse response functions in Figure 5; the policy where checks are sent out to low-income households is plotted in bright red, while the policy that sends checks out to high-income households is depicted in dark blue.

Looking first at the effects of sending checks to the high-income agents, the economy still expands upon impact, but only by 1.3% instead of delivering the 3.1% output jump experienced immediately in the world where the equivalent value of checks are sent out to lower-income households. Automatic stabilizer revenue from proportional income taxes (not depicted) surge by similar amounts, as real wages are highly stable (companies defend their markups almost completely, passing along the increase in labor costs directly to consumers, resulting in little change to wages' overall purchasing power) and hours worked go up. Debt, too, falls even faster in the simulation where the low-income receive the stimulus checks, as it is devalued even more by the stronger initial inflation response (set in motion by the Phillips curve). In the first quarter, the differences in inflation are large; the scenario in which the low-income receive checks experiences 0.89% inflation, while in the case where the higher-income households received checks, the cumulative inflation in the first quarter reached only 0.48%.

[10]Due to the coarseness of the income space grid, however, the upper-income group is actually divided at those making more than the 46th percentile, as opposed to the 50th. As such, the weights for averaging the high-income tax simulation with the low income tax to obtain the Figure 4 are 0.56 and 0.44, respectively. The initial tax shocks are similarly scaled accordingly to keep the total expenditure shock the same across simulations.

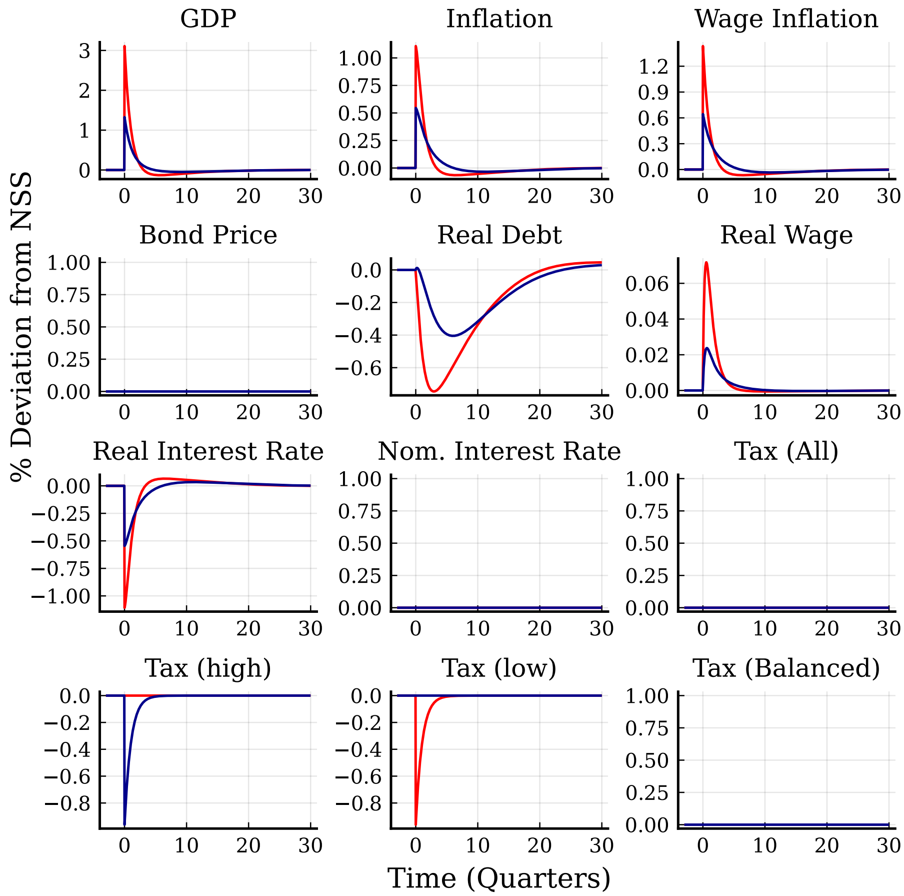

Figure 5: Impulse response functions to mean-reverting stimulus check payments to high-income and low-income households. The policy that sends checks to high-income households is plotted in dark blue, while the policy that sends checks to low-income households is plotted in bright red.

By the end of the year, however, the difference in the cumulative amount of inflation experienced in the two different scenarios narrows considerably, before almost vanishing by the time the second year concludes. Over a longer time horizon (30 quarters), it is actually the case that total inflation endured in the setting where the low-income received checks is actually slightly *lower* than the amount endured when the high-income were the recipients, as shown the fourth column of panels B and C in Table 4. The gap is about 0.06 percentage points of inflation, a tiny amount. In the last column of Table 4, panels B and C indicate that the tax-the-boom effect to balance the budget contributed about 0.03 percentage points to this difference; when the government simply spends its new labor income tax revenue instead of saving it, inflation (and output) rise in both scenarios, but the difference in inflation between the low-income stimulus check and high-income stimulus check scenarios falls to 0.03. Either way, the restoration of the government's balance sheet and the exact timing of the fall in real interest rates does not appear to significantly affect the total amount of inflation generated by the policies over time. By contrast, the difference in the implied ratios of additional GDP growth to cumulative inflation in the scenarios where the recipients of the checks are different is striking. When the high-income receive the checks, the economy goes on to grow by roughly 1.62% for every percentage point of inflation after all of the shocks have played out at the end of the 30th quarter. When the low-income get the checks, in contrast, the real economy grows by 2.66% for every additional percentage point in inflation, a ratio that is nearly 65% higher. As recounted in the first 3 columns of Table 4, the pattern of higher cumulative GDP relative to inflation induced when checks are distributed to the low-income is present at all of the intermediate time horizons as well.

Going in the other direction, toward fiscal contraction, taxing the low-income carries a dramatically higher sacrifice ratio. The new tax revenue instills confidence that the government will not partially default by inflating away its obligations, restoring faith in the currency and the government's debt, but at a grievous cost to real output and the welfare of the economy's poorest inhabitants. The same dis-inflationary effect is roughly accomplished by taxes higher-income households, but with a far less severe economic downturn.

All of the cumulative responses to the different policies are depicted in Figure 6.

What happens when the government pursues a policy wherein it taxes the rich and rebates the proceeds back to the poor? The aftermath of such a policy is presented in Figure 7. The movement of resources from low-MPC to high-MPC households generates an immediate economic boom, slightly lower than that experienced when checks were distributed to low-income households alone, but still larger than when they were distributed to high-income households. The increase in economic activity drives wage inflation as households negotiate higher wages to work more hours, which companies pass on to end consumers to protect their profits. However, since no deficits are actually incurred by the policy redistribution policy, FTPL implies that the cumulative amount of inflation experienced should be zero; there is no new unbacked debt to inflate away. Figure 8 shows that this is the case as the time horizon under consideration grows longer. After the first year following the policy, the

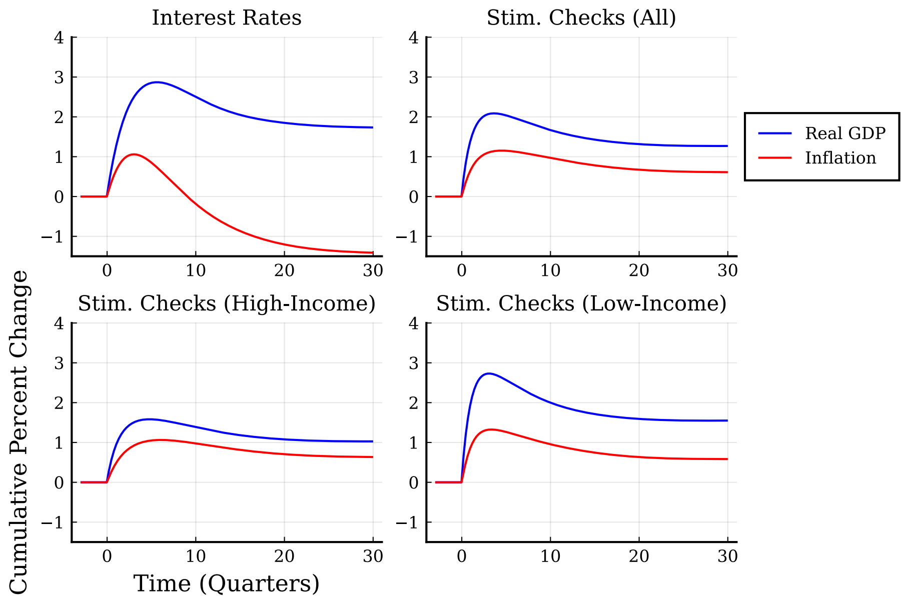

Figure 6: Cumulative responses of real GDP and inflation to expansionary shocks.

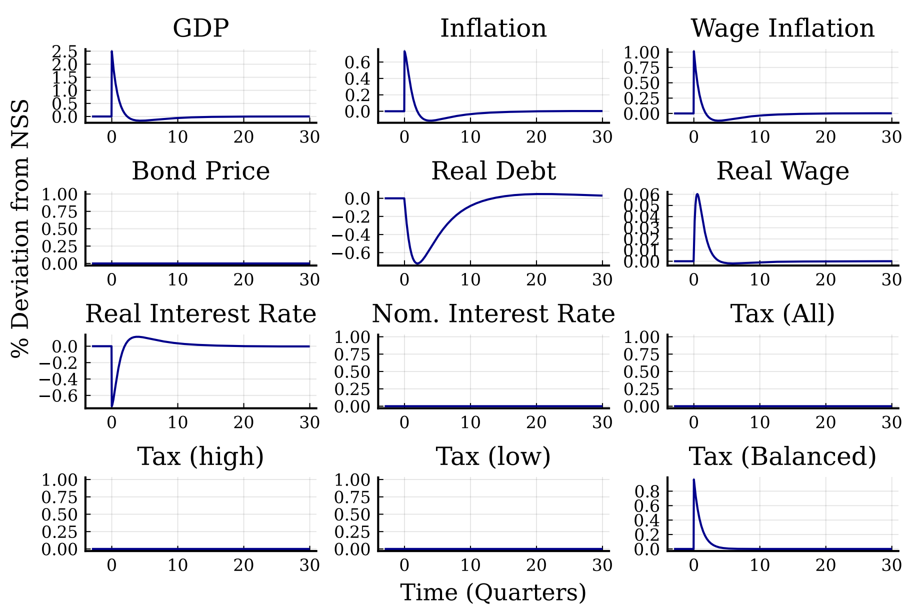

Figure 7: Impulse response functions to a mean-reverting increase in lump-sum taxes on high-income households with the proceeds remitted to low-income households.

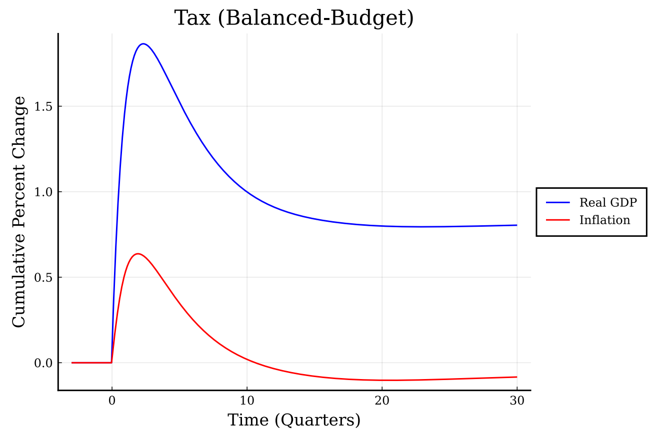

Figure 8: Cumulative effects on inflation and output following a balanced-budget transfer from high-to-low income households.

economy experiences a mild deflation, raising interest rates as households buy back bonds that were cheaper during the boom. Over time, the price level (not just the rate of inflation) converges back to where it started, before the fiscal transfers were made (the surge in automatic revenues also makes the government debt more valuable, making the transfer scheme slightly *deflationary* in the long-run, but the magnitude of the effect is small).

5 Discussion

The model presented in this paper drives to a few simple conclusions. First, household heterogeneity and incomplete markets alone are not enough to nullify the Cochrane (2018b) forward guidance mechanism. Second, the accounting of fiscal theory makes a fiscal policy's overall net inflationary force mainly determined by the overall size of the fiscal spending shock to the economy. Automatic stabilizers and the timing of inflation and real rates can potentially complicate this picture – but ultimately the simulations suggest that these complications are small. Thus, if a government sends out stimulus checks unbacked by future taxes, the model predicts that when it comes to the total cumulative amount of inflation sparked by a policy, the amount denominated by the checks matters much more than who gets them.[11]

The intuition of the fiscal policy result might be somewhat surprising in a framing outside of fiscal theory, in that it is contrary to the intuition that one might have about giving money to "high-velocity" versus "low-velocity" consumers having different consequences for inflation, wherein the average speed of subsequent spending might determine how inflationary a transfer ultimately is. However, within the framework of fiscal theory, the result is straightforward: if inflation is about markets participants' confidence in the government's ability to pay its debts, or the currency's usefulness as a means of settling taxes with the fiscal authority, then

[11] Of course, if acute inflation is more painful to consumers than a protracted but more moderate inflation, the exact path of inflation may matter more. However, in all of the simulations, the majority of inflation is over by the end of the first year following the policy shock.

the size of unfunded deficits is what is of first-order importance.

The next implication in this paper is similarly straightforward. The HANK literature suggests that transfers to people with different liquid asset positions generates heterogeneous effects on aggregate expenditure, income, and output. Namely, transfers to poor households who must spend their income almost as soon as they receive it start feedback loops that cause larger economic expansions than transfers to wealthier people do. If wealth is correlated to income, which governments can observe more easily than net worth, then transfers to low-income people are a more powerful stimulant to real GDP. Taking resources away from poor people, by contrast, provides a steeper contraction in economic activity, all else equal.

The upshot of these two points is that if a government is trying to stimulate the economy with the least amount of inflation possible, it should set the overall size of the program first and then tilt its stimulus efforts toward those it believes have a high marginal propensity to consume. If fiscal theory is correct, then the amount of inflation is relatively insensitive to the payments' recipients, but the amount of economic activity generated is larger when the recipients go out and spend their income relatively quickly. An insensitive denominator and a sensitive numerator combine to suggest that stimulus checks to those without liquid savings represent a superior GDP-to-inflation trade-off than alternative policies.

Alternatively, if a government is trying to *lower* inflation, fiscal theory stipulates that inflation from a one-time fiscal shock is eventually transitory, even without strong intervention from monetary policy; once the excess nominal debt is inflated away, the inflationary pressure will subside. Active monetary policy is also not necessary for determinacy, as Cochrane (2023) surveys at length, and higher interest rates may assist in the fight against inflation in the short-term, only to become counter-productive in the longer-term. If a country wants to tamp down on fiscally-caused inflation and end it earlier than later, however, FTPL suggests that fiscal policy and reform can play a greater role in restoring faith in a country's currency and a stable price level. However, from the logic of the previous paragraph, a fiscal reform accomplished by transferring resources away from those with low assets and low incomes likely carries with it a much higher sacrifice ratio. Markets can still get the message that the debt and currency will be honored from a fiscal reform that raises taxes on agents whose spending is less sensitive to their earnings, with less of a hit to employment and consumption.

But for all of its mathematical coherence, is fiscal theory right, in the sense that it is the the correct model of inflation out in the real world? In another sense, this paper broaches an interesting empirical test for those interested in the FTPL. For countries in inflationary episodes, particularly following deficit spending or new currency issuance with a signal of little willingness or ability to raise future surpluses, is the amount of ensuing inflation at all related to who gets the spending, or is it just a matter of how much spending or currency is issued? Finding clear examples of such cases in the real world is likely difficult, but if the relative size of the spending program appears to matter for inflation in the data, then that may be interpreted as evidence in support of fiscal theory. If this is not the case, however, then the FTPL may have to be reconsidered, amended, or ultimately rejected. Until then, the FTPL's interaction with heterogeneity and demand-determined economies with nominal rigidities opens interesting opportunities for new theory and its implications for policy.

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6 Appendix A: Derivations

6.1 Bond Math

6.1.1 General Maturities and Formula Derivations

To elaborate more upon the structure of government debt in my model, I more generally assume that the government is able to borrow using long-term nominal bonds of any maturity τ, as in Cochrane (2018b). As such, it can pay off existing nominal debt $\tilde{B}$ maturing at time t by either running a primary surplus or by selling new bonds with a maturity of τ at a price of $Q^B_{t,t+\tau}$. The debt flow equation is thus

$$\underbrace{\tilde{B}_{t,t}dt}_{\substack{\text{Debt}\\\text{maturing}\\\text{at time } t}} = \underbrace{p_t(T_t - G_t)dt}_{\text{Surplus}} + \underbrace{\int_0^\infty Q^B_{t,t+\tau} d\tilde{B}_{t,t+\tau} d\tau}_{\substack{\text{Financing from}\\\text{new bond sales}}}$$

I denote the real value of total government debt outstanding at time t as B_t, such that

$$B_t \equiv \frac{\int_0^\infty Q^B_{t,t+\tau}\tilde{B}_{t,t+\tau}d\tau}{p_t}$$

I next assume that bonds are purchased and priced not directly by households, but rather by a risk-neutral profit-maximizing investment fund that buys debt from the government and sells shares to the public. The central fiscal theory equation alluded to in the introduction of this paper therefore takes the form presented in Cochrane (2018b):

$$\underbrace{\frac{\int_0^\infty Q^B_{t,t+\tau}\tilde{B}_{t,t+\tau}d\tau}{p_t}}_{\substack{\text{Real debt}\\\text{outstanding}}} = \mathbb{E}_t\left[\int_t^\infty e^{-\int_t^\tau r_s ds}[T_\tau - G_\tau]d\tau\right]$$

Each household that holds liquid assets by holding shares in the fund thus effectively owns a cross-sectional slice of the entire government portfolio, and receives whatever interest payments are distributed and absorbs whatever capital gains and losses the government debt accrues.

For the bond portfolio, the total *real* return is the real capital gain on each bond type, weighted by the value of the bonds held, divided by the real value of the entire portfolio:

$$dR_t = \frac{\int_0^\infty \left[d\left(\frac{Q_{t,t+\tau}}{p_t}\right)/\frac{Q_{t+\tau}}{p_t}\right]\frac{Q_{t,t+\tau}}{p_t}\tilde{B}_{t,t+\tau}d\tau}{B_t}$$

$$\Rightarrow \quad B_t dR_t = \int_0^\infty d\left(\frac{Q_{t,t+\tau}}{p_t}\right)\tilde{B}_{t,t+\tau}d\tau$$

Such that

$$dB_t = d\left[\frac{\int_0^\infty Q_{t,t+\tau}\tilde{B}_{t,t+\tau}d\tau}{p_t}\right] = \underbrace{\frac{\int_0^\infty Q_{t,t+\tau}d\tilde{B}_{t,t+\tau}d\tau}{p_t} - \frac{\tilde{B}_t}{p_t}dt}_{-(T_t-G_t)dt} + \underbrace{\int_0^\infty \tilde{B}_{t,t+\tau}d\left(\frac{Q_{t,t+\tau}}{p_t}\right)d\tau}_{B_t dR_t}$$

It thus follows that

$$dB_t = -(T_t - G_t)dt + B_t dR_t$$

The first term is the primary deficit, while the second is the ex-post real rate of return on the bond portfolio. This ex-ante return will then be the expected return on the nominally riskless bonds, plus whatever capital gain has been unexpectedly accrued over the time increment.

Again as in Cochrane (2018b), I make the simplifying assumption that the government issues and rolls over debt such that the density of of government liabilities by maturity is always exponentially distributed with a rate of ω, such that the cumulative distribution of outstanding government treasury maturities τ is $CDF(\tau) = 1 - e^{-\omega\tau}$ and the density function is $PDF(\tau) = \omega e^{-\omega\tau}$. Additionally, I make the simplifying assumption that in the non-stochastic steady-state of the model, all households effectively hold the same representative slice of government debt by owning shares of a competitive profit-maximizing mutual fund, just in varying amounts. For an individual holding a unitary share of the total government portfolio, their assets entitle them to a payment of ωdt almost immediately (this is the shortest-term debt being repaid), plus payments of $\omega e^{-\omega\tau}dt$ for all periods thereafter. The entire bond portfolio is then effectively a perpetuity which pays out a geometrically declining coupon $\omega e^{-\omega\tau}dt$ at each time $t+\tau$ for the rest of time.

The nominal bond price of the entire portfolio will then be

$$q_t^B = \int_0^\infty e^{-\tau y_t}\omega e^{-\omega\tau}d\tau = \int_0^\infty \omega e^{-\tau(\omega+y_t)}d\tau = -\frac{\omega}{\omega+y_t}e^{-u}|_0^\infty = \frac{\omega}{\omega+y_t}$$

The nominal rate of return on the bond will be the the dividend yield, plus the capital gain.

$$dR_t^{nom} = \frac{(\omega - \omega q_t^B)dt + dq_t^B}{q_t^B} = y_t dt + \frac{dq_t^B}{q_t^B}$$

It then follows that if the ex-ante nominal rate of return is dR_t^{nom} is $i_t dt$ in expectation

$$i_t dt = \mathbb{E}_t[dR_t^{nom}] = y_t dt + \frac{\mathbb{E}_t[dq_t^B]}{q_t^B}$$

I define $\delta_{qB,t} = dq_t^B - \mathbb{E}_t[dq_t^B]$ as the unexpected gain in bond prices, which must in turn be equal to the ex-post nominal rate of return minus the expected (ex-ante) one:

$$\frac{\delta_{qB,t}}{q_t^B} \equiv dR_t^{nom} - i_t dt = \frac{dq_t^B - \mathbb{E}_t[dq_t^B]}{q_t^B}$$

Since the nominal rate will be the real one, plus inflation:

$$dR_t^{nom} = dR_t + \pi_t dt$$

$$\Rightarrow \quad \frac{\delta_{qB,t}}{q_t^B} - \pi_t dt = dR_t - i_t dt$$

$$\Rightarrow \quad dR_t = \frac{\delta_{qB,t}}{q_t^B} + (i_t - \pi_t)dt$$

The valuation equation becomes

$$dB_t = -(T_t - G_t)dt + B_t \left[i_t - \pi_t\right] dt + \frac{\delta_{qB,t}}{q_t^B} B_t \tag{18}$$

To derive the equation governing nominal bond prices, it also follows that if

$$dR_t^n = \frac{\omega dt + dq_t^B}{q_t^B} - \omega dt$$

such that

$$q_t^B dR_t^n = \omega dt + dq_t^B - \omega q_t^B dt$$

then in expectation

$$E_t[dq_t^B] = q_t^B \left(\mathbb{E}_t[dR_t^n] + \omega dt - \frac{dt}{q_t^B} \right)$$

$$\Rightarrow \quad E_t[dq_t^B] = q_t^B \left(i_t + \omega - \frac{\omega}{q_t^B} \right) dt \tag{19}$$

and so bond prices evolve according to

$$dq_t^B = q_t^B \left(i_t + \omega - \frac{\omega}{q_t^B} \right) dt + \delta_{qB,t}$$

6.1.2 Investment Fund

Suppose a risk-neutral competitive investment fund sector purchases bonds for the households, who hold shares of the fund as assets. The fund solves a profit maximization problem wherein it collects interest from the bonds in its portfolio, and can choose the size of its portfolio by buying more bonds at a cost of d_t^B. The interest payments that it receives in excess of its acquisition costs are rebated back to the fund's shareholders. As such, the present-value maximizing fund solves

$$A_t(B_t;\mu_t,\zeta_t) = \max_{(d_t^B)_{s\geq t}} \ \mathbb{E}_t \int_t^\infty e^{-\int_t^\tau r_s ds}\Big[r_\tau B_\tau - d_\tau^B\Big] d\tau$$
$$\text{s.t. } \frac{dB_t}{dt} = d_t^B$$

The HJB equation is

$$r_t A_t(B_t;\mu_s,\zeta_s) = \max_{d_t^B} \ r_t B_t - d_t^B + \frac{\partial A_t}{dB_t} d_t^B + \frac{\partial A_t}{\partial t}$$

Taking FOCs,

$$\frac{\partial A_t}{dB_t} = 1$$

which is consistent with the solution

$$A_t(B_t;\mu_t,\zeta) = B_t$$

(and this solution is in turn consistent with market-clearing). As such, the setup becomes (using $dB_t = ([G_t - T_t] + r_t B_t)\,dt + \frac{d\delta_{qB,t}}{q_t^B} B_t$ and $\mathbb{E}_s[\delta_{qB,t}] = 0$ if $s < t$)

$$B_t = \mathbb{E}_t \int_t^\infty e^{-\int_t^\tau r_s ds}\Big[T_\tau - G_\tau\Big] d\tau$$

Recalling the definition of B_t, I recover the central fiscal theory asset pricing equation:

$$\frac{\int_0^\infty Q^B_{t,t+\tau}\tilde{B}_{t,t+\tau} d\tau}{p_t} = \mathbb{E}_t \int_t^\infty e^{-\int_t^\tau r_s ds}\Big[T_\tau - G_\tau\Big] d\tau$$

6.2 Wage Phillips Curve

This is a continuous-time version of Auclert, Rognlie, and Straub (2018), *The Intertemporal Keynesian Cross*. Say a labor-aggregator hires labor from households to create an aggregate unit of input labor.

$$L_{kt} = \int_0^1 (z_i h_{ikt}) di$$

$$L_t = \left(\int_0^1 L_{kt}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk\right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1}}$$

Let W_t be the nominal wage paid by employers to labor-aggregators, and let labor-aggregator pay its workers a nominal wage of W_{kt}. Labor-aggregating firms thus hire according to

$$\max_{\{L_{kt}\}_{k\in[0,1]}} \ W_t \left(\int_0^1 L_{kt}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk\right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1}} - \int_0^1 W_{kt} L_{kt} dk$$

such that from the FOCs, the demand for labor from union k is

$$W_t \left(\int_0^1 L_{kt}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk \right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1} - 1} L_{kt}^{-\frac{1}{\varepsilon_\ell}} - W_{kt} = 0$$

$$W_t L_t^{\frac{1}{\varepsilon_\ell}} L_{kt}^{-\frac{1}{\varepsilon_\ell}} = W_{kt}$$

$$W_t L_t^{\frac{1}{\varepsilon_\ell}} = W_{kt} L_{kt}^{\frac{1}{\varepsilon_\ell}}$$

$$\Rightarrow \quad \frac{L_{kt}}{L_t} = \left(\frac{W_t}{W_{kt}} \right)^{\varepsilon_\ell}$$

Unions face nominal wage adjustment costs:

$$\frac{\theta_w}{2} \int_0^1 \pi_{w,k}^2 dk, \quad \text{where} \quad \pi_{w,k} = \frac{dW_{kt}}{dt} \frac{1}{W_{kt}}$$

The labor union k sets wages to maximize its members' lifetime utilities:

$$\max_{\pi_{kt}^w} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left[\int \int \left\{ \frac{c(a,z)^{1-\gamma}}{1-\gamma} - \frac{h(a,z)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\} \mu_t(a,z) da\ dz - \frac{\theta_w}{2} (\pi_{k,t}^w)^2 \right] dt$$

$$\text{s.t. } \frac{dW_t}{dt} = \pi_t^w W_t$$

$$L_{kt} = \int_0^1 z_i h_{ikt} di$$

$$\frac{L_{kt}}{L_t} = \left(\frac{W_t}{W_{kt}} \right)^{\varepsilon_\ell}$$

Where the third equation follows from the first-order conditions from the households.

The HJB is then

$$\rho J^w = \left[\int \int \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{kt})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\} \mu_t(a,z) da\ dz - \frac{\theta_w}{2} (\pi_{k,t}^w)^2 \right] + \frac{\partial J^w(W_{kt})}{\partial W_{kt}} \pi_t^w W_{kt} + \frac{\mathbb{E}_t^\zeta dJ^w}{dt}$$

The FOC for wage inflation is then

$$-\theta_w \pi_{kt}^w + \frac{\partial J^w(W_{kt})}{\partial W_{kt}} W_{kt} = 0$$

$$\Rightarrow \quad \frac{\partial J^w(W_{kt})}{\partial W_{kt}} = \theta_w \frac{\pi_{kt}^w}{W_{kt}}$$

With Itô's lemma, we know that

$$d \left(\frac{\partial J^w(W_{kt}, \zeta)}{\partial W_{kt}} \right) = \partial_{W_{kt}}^2 J^w dW_{kt} + \partial_\zeta \partial_{W_{kt}} J^w d\zeta + \partial_\zeta^2 \partial_{W_{kt}} J^w d\langle \zeta \rangle_t$$

and applying Itô again to the LHS of the wage inflation FOC,

$$d\left(\theta_w \frac{\pi_t^w}{W_{kt}}\right) = \frac{\theta_w}{W_{kt}} d\pi_t^w - \frac{\theta_w \pi_t^w}{W_{kt}^2} dW_{kt}$$

such that by equating the two,

$$\frac{\theta_w}{W_{kt}} d\pi_t^w - \frac{\theta_w \pi_t^w}{W_{kt}^2} dW_{kt} = \partial_{W_{kt}}^2 J^w dW_{kt} + \partial_\zeta \partial_{W_{kt}} J^w d\zeta + \partial_\zeta^2 \partial_{W_{kt}} J^w d\langle \zeta \rangle_t$$

Taking expectations and dividing by dt yields

$$\frac{\theta_w}{W_{kt}} \frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{kt}} \underbrace{\frac{dW_{kt}}{dt} \frac{1}{W_{kt}}}_{\pi_{kt}^w} = \partial_{W_{kt}}^2 J^w \frac{dW_{kt}}{dt} + \underbrace{\frac{\partial_\zeta \partial_{W_{kt}} J^w \mathbb{E}_t[d\zeta] + \partial_\zeta^2 \partial_{W_{kt}} J^w d\langle \zeta \rangle_t}{dt}}_{\mathcal{D}_\zeta \partial_{W_{kt}} J^w}$$

such that

$$\frac{\theta_w}{W_{kt}} \frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{kt}} \pi_t^w = \partial_{W_{kt}}^2 J^w \pi_t^w W_{tk} + \partial_{W_{kt}} \mathcal{D}_\zeta J^w \tag{20}$$

Next, the Envelope condition stipulates that

$$\rho \partial_{W_{kt}} J^w =$$

$$\left[\int\int \partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{kt})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\} \mu_t(a,z) da\ dz \right] + \partial_{W_{kt}}^2 J^w \pi_t^w W_{kt} + \partial_{W_{kt}} J^w (W_{kt}) \pi_t^w + \partial_{W_{kt}} \mathcal{D}_\zeta J^w$$

Substituting in (20),

$$\rho \partial_{W_{kt}} J^w =$$

$$\left[\int\int \partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{kt})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\} \mu_t(a,z) da\ dz \right] + \partial_{W_{kt}} J^w (W_{kt}) \pi_t^w + \frac{\theta_w}{W_{kt}} \frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{kt}} \pi_t^w$$

and then the FOC,

$$\rho \theta_w \frac{\pi_{kt}^w}{W_{kt}} =$$

$$\left[\int\int \partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{kt})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\} \mu_t(a,z) da\ dz \right] + \theta_w \frac{\pi_{kt}^w}{W_{kt}} \pi_t^w + \frac{\theta_w}{W_{kt}} \frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{kt}} \pi_t^w$$

it follows that

$$\rho \pi_{kt}^w = \frac{W_{kt}}{\theta_w} \left[\int\int \partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{kt})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\} \mu_t(a,z) da\ dz \right] + \frac{\mathbb{E}_t[d\pi_t^w]}{dt} \tag{21}$$

From the households' envelope condition, the change in utility from wages will be equal to the marginal utility, times the change in earnings:

$$\partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} \right\} = c(a,z)^{-\gamma}(1-\tau)\partial_{W_{kt}} \left(\frac{W_{kt}}{P_t} h(a,z) \right)$$

Where if households uniformly supply their labor to union k and internalize their labor's demand:

$$h_{ikt}(a,z) = h_{nss}(a,z)\frac{d\mu_{nss}}{d\mu_t}(a,z) + \frac{1}{Z_t}(L_{kt} - L_{nss}) = h_{nss}(a,z)\frac{d\mu_{nss}}{d\mu_t}(a,z) + \frac{1}{Z_t}\left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell} L_t - \frac{1}{Z_t}L_{nss}$$

$$\Rightarrow \quad \partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} \right\} = c(a,z)^{-\gamma}(1-\tau)\partial_{W_{kt}} \left(z\frac{W_{kt}}{P_t}\left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell} L_t \right)$$

$$= c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{W_{kt}}\left(\frac{W_{kt}}{P_t}\left(\frac{W_t}{W_{kt}}\right)^{-\varepsilon_\ell} L_t\right)$$

$$= c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{W_{kt}}\frac{W_{kt}}{P_t}L_{kt}$$

$$= c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{P_t}L_{kt}$$

For the effect of wages on labor disutility, I can directly evaluate

$$\partial_{W_{kt}} h(a,z) = \frac{1}{Z_t}\partial_{W_{kt}}\left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell} L_t = -\varepsilon_\ell \frac{1}{Z_t}\frac{1}{W_{kt}}\left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell} L_t = -\varepsilon_\ell \frac{1}{Z_t}\frac{L_{kt}}{W_{kt}}$$

Plugging in the results into (21),

$$\rho\pi_{kt}^w = \frac{W_{kt}}{\theta_w}\left[\int\int \left\{ c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{P_t}L_{kt} + \frac{1}{Z_t}h(a,z)^{\frac{1}{\eta}}\varepsilon_\ell\frac{L_{kt}}{W_{kt}} \right\}\mu_t(a,z)da\ dz\right] + \frac{\mathbb{E}_t[d\pi_t^w]}{dt}$$

$$\rho\pi_{kt}^w = \frac{\varepsilon_\ell}{\theta_w}L_{kt}\int\int\left\{\frac{1}{Z_t}h(a,z)^{\frac{1}{\eta}} - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)z\frac{W_{kt}}{P_t}c(a,z)^{-\gamma}\right\}\mu_t(a,z)da\ dz + \frac{\mathbb{E}_t[d\pi_t^w]}{dt}$$

Leading to the wage Phillips Curve

$$\frac{\mathbb{E}_t[d\pi_t^w]}{dt} = \rho\pi_t^w - \frac{\varepsilon_\ell}{\theta_w}L_t\int\int\left(v'(h(a,z)) - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zw_t u'(c(a,z))\right)da\ dz \tag{22}$$

where w_t is the real wage.

Note that in the NSS, households supply labor while internalizing how supplying more labor reduces their wage, such that

$$v'(h(a,z)) = \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zw_t u'(c(a,z))$$

Marginal disutility from labor is slightly lower than in the competitive equilibrium, such that hours worked is slightly lower as well. The real wage is defined as

$$w_t = \frac{W_t}{P_t}$$

Taking the total time derivative:

$$dw_t = \frac{dW_t}{P_t} - \frac{W_t}{P_t}\frac{dP_t}{P_t}$$

$$\Rightarrow \quad \frac{dw_t}{dt} = \frac{dW_t}{dt}\frac{1}{W_t}\frac{W_t}{P_t} - \frac{W_t}{P_t}\pi_t$$

$$\Rightarrow \quad \frac{dw_t}{dt} = (\pi_t^w - \pi_t)w_t$$

6.2.1 Perturbations from NSS

Suppose labor supply after a shock can be characterized by

$$h_t(a,z) = h_{nss}(a,z)\frac{d\mu_{nss}}{d\mu_t}(a,z) + \frac{1}{Z_t}(L_t - L_{nss})$$

Aggregating,

$$\int\int zh_t(a,z)d\mu_t(a,z) = \int\int z[h_{nss}(a,z) + \frac{1}{Z_t}(L_t - L_{nss})]d\mu_t(a,z)$$

$$= \int\int zh_{nss}(a,z)d\mu_{nss}(a,z) + L_t - L_{nss} = L_t$$

so indeed

$$\int\int zh_t(a,z)d\mu_t(a,z) = L_t$$

7 Appendix B: Computations

This section is best read after having already read Achdou et al. (2021), Ahn et al. (2018), and particularly Bayer and Luetticke (2020) as background; the below section largely amounts to a brief sketch of adapting Bayer and Luetticke (2020) to continuous time. For notational brevity, I write the infinitessimal generator operator of the concentrated Hamilton Jacobi Bellman equation as

$$
\begin{aligned}
\mathcal{D}[V] =& \lim_{t \downarrow 0} \frac{\mathbb{E}_t^{a,z}[V_t(a_{t+dt}, z_{t+dt})] - V_t(a_t, z_t)}{dt} \\
=& \frac{\partial V_t}{\partial a}(a, z; \mu, \zeta)\left[(1-\tau)w_t z h_t(a,z) + T_t(a,z) - c_t(a,z) + r_t(a)a\right] \\
&+ \frac{\partial V_t}{\partial z}(a, z; \mu, \zeta) z\left[\frac{1}{2}\sigma_z^2 - \theta_z \log(z)\right]
\end{aligned}
$$

where the expectation operator is taken with respect to only the idiosyncratic variables. As in Achdou et al. (2021), I write the adjoint operator (which describes the Kolmogorov forward equation of the idiosyncratic state distribution) as $\mathcal{D}^*$, where the KFE operator is the adjoint of the maximized HJB operator in L^2 space. Additionally, I write expectation errors for a jump variable "J" as $d\delta_{J,t}$, such that $d\delta_{J,t} = dJ_t - \mathbb{E}_t[dJ_t]$.

Suppose aggregate shocks in the economy evolve according to

$$
d\zeta_t = -\Theta_\zeta \zeta_t dt + d\epsilon_{\zeta,t}. \tag{23}
$$

A sequential equilibrium following a perturbation from the steady-state $W_{\zeta,0}$ is a resulting path of aggregate shocks $\{\zeta_t\}_{t\geq 0}$, a series of value functions $\{V_t(a,z)\}_{t\geq 0}$, consumption decisions and labor allocations $\{c_t(a,z), h_t(a,z)\}_{t\geq 0}$, distributions $\{\mu_t(a,z)\}_{t\geq 0}$, outstanding government debt $\{B_t\}_{t\geq 0}$, wages $\{w_t\}_{t\geq 0}$, nominal and real interest rates $\{i_t, r_t\}_{t\geq 0}$, bond prices $\{q_t^B\}_{t\geq 0}$, and inflation rates $\{\pi_t\}_{t\geq 0}$ where

$$
dV_t(a,z) = \left\{\rho V_t(a,z) - \left[u(c_t(a,z)) - v(h_t(a,z)) + \mathcal{D}[V]\right]\right\}dt - \frac{\partial V_t(a,z)}{\partial a} d\delta_{qB,t} + d\delta_{V(a,z),t} \tag{24}
$$

where if $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, it follows that the FOC for consumption is

$$
c_t(a,z)^{-\gamma} = \frac{\partial V_t}{\partial a}(a,z) \tag{25}
$$

The distribution evolves according to

$$
d\mu_t(a,z) = \mathcal{D}^*[\mu]dt - \frac{\partial}{\partial a}\big(\mu_t(a,z) d\delta_{qB,t}\big) \tag{26}
$$

while labor is supplied to meet market demand:

$$\begin{aligned} \frac{1}{Z}h_{nss}(a,z)^{\eta} &= \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zwc_{nss}(a,z)^{-\gamma} \\ h_t(a,z) &= h_{nss}(a,z)\frac{d\mu_{nss}}{d\mu_t}(a,z) + \frac{1}{Z}(L_t - L_{nss}) \end{aligned} \tag{27}$$

Goods inflation must be consistent with the goods market Phillips Curve derived from the firms' profit maximization problem:

$$d\pi_t = \left(r_t\pi_t - \frac{\varepsilon}{\theta_\pi}[m_t - m^*] \right) dt + d\delta_{\pi,t} \tag{28}$$

while wage inflation is dictated by the labor market Phillips Curve

$$d\pi_t^w = \left\{ \rho\pi_t^w - \frac{\varepsilon_\ell}{\theta_w}L_t \int\int \left(v'(h(a,z)) - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zw_t u'(c(a,z)) \right) da\ dz \right\} dt + d\delta_{\pi^w,t} \tag{29}$$

Real wages then follow

$$dw_t = (\pi_t^w - \pi_t)w_t dt \tag{30}$$

The government's budget constraint must satisfy

$$dB_t = -(T_t - G_t)dt + r_t B_t dt + \frac{d\delta_{qB,t}}{q_t^B}B_t \tag{31}$$

where nominal bond prices and equity prices satisfy

$$dq_t^B = q_t^B \left(i_t + \omega - \frac{\omega}{qB,t} \right) dt + d\delta_{qB,t} \tag{32}$$

Equilibrium must also be consistent with the Fisher equation, the marginal cost equation, and the profit equation:

$$r_t = i_t - \pi_t \tag{33}$$

$$m_t = w_t \tag{34}$$

$$\Pi_t = [1 - m_t]Y_t \tag{35}$$

All goods consumed must be produced:

$$Y_t = L_t \tag{36}$$

and the idiosyncratic variables must aggregate:

$$C_t = \int_0^\infty \int_{\underline{a}}^\infty c_t(a,z)\mu_t(a,z)da\ dz \tag{37}$$

$$L_t = \int_0^\infty \int_{\underline{a}}^\infty zh_t(a,z)\mu_t(a,z)da\ dz \tag{38}$$

Finally, goods and financial markets must clear:

$$Y_t = C_t + \int_0^\infty \int_{\underline{a}}^\infty -\mathbf{1}_{\{a<0\}}\Delta_r a\mu_t(a,z)da \tag{39}$$

$$B_t = \int_0^\infty \int_{\underline{a}}^\infty a\mu_t(a,z)da\ dz \tag{40}$$

It's then possible to write a vector of variables as

$$X_{C,t} = (V_t(a,z),\ \pi_t,\ \pi_t^w,\ q_t^B)',$$

the set of state variables as

$$X_{S,t} = (\mu_t(a,z),\ B_t,\ w_t, \zeta_t)',$$

and the vector of static constraints as

$$X_{L,t} = (Y_t,\ L_t)',$$

(where many of the static constraints like the Fisher equation and the employment rules can be re-written to solve out the other static variables from the model). Stacking the controls, states, and static variables, I write

$$X_t = (X_{C,t}, X_{S,t}, X_{L,t})'$$

where dX_t represents the differentials of X_t. Using this succinct notation, the entire system (23-40) can be written as

$$\Gamma_0 dX_t = \Omega(X_t, d\delta_{X,t}, d\epsilon_{\zeta,t}) \tag{41}$$

where the rows of Γ_0 corresponding to static constraints are equal to zero.

I discretize the partial differential equations on the computer in the non-stochastic steady-state where $X_t = X_{nss}$, $dX_t = 0$, $d\delta_{X,t} = 0$, and $d\epsilon_{\zeta,t} = 0$, using the finite-differences methodology described in Achdou et al. (2021). This entails discretizing (41) via an upwind finite difference approximation for the partial derivatives along an asset grid (which I index by $i \in I \equiv \{1, \ldots, N_a\}$) and an income grid (which I index by $j \in J \equiv$

$\{1\ldots,N_z\}$). The tensor $V_{i,j,nss}$ then approximates the value function $V_{nss}(a_i,z_j)$ in the discretized state space, while the tensor $\mu_{i,j,nss}$ approximates the distribution $\mu_{nss}(a_i,z_j)$.

Before proceeding, I find it useful to define $\widehat{X}_t \equiv X_t - X_{nss}$ as either the level deviations or the log deviations of the variables from their values in the non-stochastic steady-state. As such, the complete system can be rewritten to become

$$\Gamma_0 d\widehat{X}_t = \widehat{\Omega}(\widehat{X}_t, d\delta_{X,t}, d\epsilon_{\zeta,t}) \tag{42}$$

where the arguments are the deviation terms. The steady-state thus satisfies $\widehat{\Omega}(\mathbf{0})=\mathbf{0}$. I then proceed to solve for the dynamics of the economy following aggregate shocks. Practically, the dimensionality of the discretized value functions and distributions necessitate dimension reduction. However, for clarity, I first describe the process *without* dimension reduction.

7.1 Without Dimension Reduction

With the non-stochastic steady-state (NSS) in hand, I then calculate the numerical Jacobian of the system at the NSS using automatic differentiation. Differentiating the entire system with respect to just the arguments in X_t alone, I can write the Jacobian of the system with respect to its X_t variables at the non-stochastic steady-state as

$$\Gamma_{X,X} \equiv \nabla_X \widehat{\Omega}(\mathbf{0})$$

While the derivatives of the system with respect to the expectation errors and the perturbations are

$$\Gamma_{X,\delta} \equiv \nabla_{d\delta} \Omega(\mathbf{0})$$

$$\Gamma_{X,W} \equiv \nabla_{dW_\zeta} \Omega(\mathbf{0})$$

A first-order Taylor expansion of the system around the steady-state without any shocks (and where $d\widehat{X}_t = 0$) is then

$$\Gamma_0 d\widehat{X}_t = \Gamma_{X,X}\widehat{X}_t dt + \Gamma_{X,\delta} d\delta_{X,t} + \Gamma_{X,W} d\epsilon_{\zeta,t} + \mathcal{O}(\|\widehat{X}_t, d\delta_{X,t}, d\epsilon_{\zeta,t}\|^2)$$

I then solve

$$\Gamma_0 d\widehat{X}_t = \Gamma_{X,X}\widehat{X}_t dt + \Gamma_{X,\delta} d\delta_{X,t} + \Gamma_{X,W} d\epsilon_{\zeta,t} \tag{43}$$

using the generalized eigenvalue methodology described in Sims (2002). If the system has more stable generalized eigenvalues than it has control variables, the dimensionality of the linear subspace being used to approximate the system's stable manifold is too large to ensure that the dynamics are unique, such that multiple equilibria are possible (sunspots). If the system has fewer stable eigenvalues than state variables, then the equilibrium cannot exist. I verify that the number of stable eigenvalues in my system matches the number of state variables, such that the solution exists and is unique.

While straightforward, this approach is too computationally costly to be feasible with the number of grid-points that I employ to solve my full model. As such, I use the dimension reduction strategy of Bayer and Luetticke (2020) before calculating the Jacobian of (42).

7.2 With Dimension Reduction

I write the 2-dimensional discrete cosine transform (DCT) of a 2-dimensional array A as $\theta^A = \text{DCT}(A)$, where its inverse $\text{DCT}^{-1}(\theta^A) = A$. I can write the transformation of the value function in the non-stochastic steady-state as

$$\{\theta^V_{(i,j),nss}\}_{(i,j)\in I\times J} = \text{DCT}(\{V_{(i,j),nss}\}_{(i,j)\in I\times J})$$

I then compute the "energy" (to use the terminology of Bayer and Luetticke (2020)) of the $\theta^V_{i,j,nss}$ coefficients as

$$E_{ij} = \frac{[\theta^V_{(i,j),nss}]^2}{\sum_{(i,j)\in I\times J}[\theta^V_{(i,j),nss}]^2}$$

Sorting the coefficients by their energy from greatest to least, I then identify those coefficients that contain a cumulative $1-\kappa$ share of the coefficients' energy, where κ is a small number. I label the set of these coefficients (which are effectively the ones with the largest absolute value) as Θ_E; these coefficients explain most of the variation of the value function in the steady-state.

As in Bayer and Luetticke (2020), I then move toward constructing a perturbation solution of the equilibrium system, but perturbing only high-energy coefficients in Φ_E. Otherwise, I keep the lower-energy coefficients constant, at their steady-state values:

$$\tilde{\theta}^V_{i,j,t} = \theta^V_{(i,j),t} + \mathbf{1}_{\{(i,j)\in\Theta_E\}}\widehat{\theta}^V_{(i,j),t}$$

where $\widehat{\theta}^V_{i,j,t}$ is the coefficient's deviation at time t from its NSS value.

The DCT is a linear operator. As such, I can write the differentials of the coefficients as

$$\{d\theta^V_{(i,j),t}\}_{(i,j)\in I\times J} = d\left[\text{DCT}(\{V_{(i,j),t}\}_{(i,j)\in I\times J})\right] = \{d\theta^V_{(i,j),nss}\}_{(i,j)\in I\times J} = \left[\text{DCT}(\{dV_{(i,j),nss}\}_{(i,j)\in I\times J})\right]$$

and similarly I write

$$d\tilde{\theta}^V_{(i,j),t} = \mathbf{1}_{\{(i,j)\in\Theta_E\}}d\theta^V_{(i,j),t}$$

By perturbing only the $|\Theta_E|$ largest-magnitude coefficients instead of the full $N_a \times N_z$ elements of the discretized value function, I can greatly reduce the dimensionality of the problem. Of course, this only reduces the number of control variables. To reduce the number of state variables in the distribution, I also employ the fixed copula transformation of Bayer and Luetticke (2020).

I write the discretized joint cumulative distribution function $F_{\mu(a_i,z_j)}$, and the marginal CDFs as $F_{\mu(a_i)}$ and $F_{\mu(z_j)}$. The copula is then the joint distribution interpolated onto the marginal ones:

$$\text{Cop} = \text{Interp}(\{F_{\mu(a_i,z_j),nss}\}_{ij}, \{F_{\mu(a_i),nss}\}_i, F_{\mu(z_j),nss}\}_j)$$

where the nss subscript denotes the steady-state values. It then follows that $\text{Cop} : [0,1] \times [0,1] \rightarrow [0,1]$ maps cumulative marginal distributions to a joint distribution, as predicted by the rank correlations of the steady-state. Outside of the steady-state, I then approximate the joint cumulative distribution $F_{\mu(a_i,z_j),t}$ at time t as

$$F_{\mu(a_i,z_j),t} \approx \text{Cop}(F_{\mu(a_i),t}, F_{\mu(z_j),t}),$$

from which the marginal joint *density* function μ_{ij} may be derived. Using this object, I can then iterate the Kolmogorov Forward Equation to obtain $d\mu_{ij}$, which can be integrated (or summed, since the functions are discretized) to obtain the evolution of the differentials

$$\{(dF_{\mu(a_i),t}, dF_{\mu(z_j),t})\}_{ij}.$$

As Bayer and Luetticke (2020) note, this approximation allows me to track only the N_a and N_z dimensional marginal CDFs instead of their joint one to describe the economy, so long as the rank correlations outside of the steady-state are similar to those represented in the steady-state (which Bayer and Luetticke (2020) show is generally the case in Bewley-Aiyagari models).

I then define the dimension-reduced set of controls as

$$\tilde{X}_{C,t} = (\{\tilde{\theta}^V_{i,j,t}\}_{(i,j)\in\Theta_E},\ \pi_t,\ \pi^w_t,\ q^B_t)'$$

and the dimension-reduced set of states as

$$\tilde{X}_{S,t} = (\{F_{\mu(a_i),t}\}_i, \{F_{\mu(z_j),t}\}_j,\ B_t,\ w_t, \zeta_t)',$$

Once again stacking the reduced controls, states, and static variables, I write

$$\tilde{X}_t = (\tilde{X}_{C,t}, \tilde{X}_{S,t}, X_{L,t})'$$

and the system (41) is approximated by a smaller one:

$$\tilde{\Gamma}_0 d\tilde{X}_t = \tilde{\Omega}(\tilde{X}_t, d\delta_{X,t}, d\epsilon_{\zeta,t})$$

where $\tilde{\Omega}$ calculates the value function and joint distribution given the DCT coefficients and the marginal distribution, feeds them back into the original Ω function, and then from there recovers the resulting truncated DCT coefficients and marginal CDFs' time differentials. Just like before, this system can also be written in terms of just the differences (or log differences) of the variables from their non-stochastic steady-state values. The rest of the linearization steps and solution methods then proceed exactly in the same manner as they do in the version without dimension reduction, as reviewed in the prior subsection of this appendix.

7.3 Numerical Accuracy

I solve the model over a unifrom grid of $N_a = 50$ points from -1 to 60 and $N_z = 40$ grid points from 0.01 to 5.5.

The aggregate law of motion (6) can be used to track the evolution of the market value of government debt, but since households hold the government's bonds as assets, the private sector's total bond position may be calculated by using the Kolmogorov forward equations (2) and aggregating using (15). To assess the accuracy of the model, I calculate the evolution of the stock of government debt both ways, and then observe the percentage difference as a test of my model's numerical accuracy. Overall, the errors in the simulated time series are on the order of 5×10^{-6} in the fiscal policy experiments, where the nominal price of government bonds does not jump.

In the monetary policy experiments, however, the use of a first-order Taylor expansion around where $d\delta_{q,t} = 0$ for the value function and the KFE equation introduces additional numerical errors. When interest rates fall by 1%, the increase in the value of the government's debt should match the jump in bond prices on impact, as neither the price level nor the number of bonds outstanding jump on impact. However, while the bond price jumps 4.45%, the value of those real bonds B_0 aggregated from the KFE equation jumps by only 4.23%, a roughly 5% (0.22 percentage points) difference between the two responses. To reduce this error, I re-scale the debt impulse response function after linearization so that the initial bond and price movements match in the moment that the shock is realized.