# Sticky Expectations, Fiscal Transfers, Inflation, and Unemployment in HANK

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**Abstract**

I develop a new technique to numerically solve sticky-expectation heterogeneous agent models in state space. I use this methodology to study the effects of the post-COVID fiscal stimulus in an estimated medium-scale heterogeneous agent New Keynesian (HANK) model. Sticky information frictions triple the one-year transfer multiplier of a stimulus check intervention from 0.10 to 0.30. In the absence of fiscal transfer stimulus in 2020 and 2021, the model predicts the COVID-19 recession would have induced a 33% larger average cumulative real GDP per capita loss relative to the pre-pandemic trend. Despite being highly stimulative in the incomplete markets setting, the model suggests that transitory fiscal transfers have relatively modest impacts on inflation and unemployment.

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# 1 Introduction

Deviations from full information rational expectations (often abbreviated as FIRE) are qualitatively and quantitatively important for understanding business cycles and are often necessary to reconcile heterogeneous agent New Keynesian (HANK) models with macroeconomic data. I develop a new technique to expediently convert the linearized state-space Jacobians of a full information HANK continuous time system into their sticky expectation counterparts, wherein only a fraction of agents update their beliefs about the macroeconomy to full information at any moment in time.

With this machinery in place, I develop a medium-scale HANK model to study output, inflation, and unemployment in a sticky expectation environment with incomplete markets and household inequality. The stickiness of household beliefs after a macroeconomic shock generates persistent slow-moving and hump-shaped impulse response functions after a monetary policy shock, a key feature from the empirical literature on the effect of interest rate change. Sticky expectations thereby allow me to estimate the model's structural parameters by matching the empirical response to an identified monetary policy shock while still matching realistic moments of households' marginal propensities to consume. Under the assumption that the structural parameters (including the learning rate, nominal rigidities, and so on) are independent of policy, I am then able to leverage data about a monetary policy shock to explore the model's implications for fiscal policy.

I find that in the sticky expectations environment with active monetary and passive fiscal policy, exogenous fiscal transfers that mean revert quickly are about a third more stimulative on impact than they would be in a similar full information rational expectations (FIRE) version of the model. After a year, the total transfer multiplier from the fiscal stimulus is 0.30, three times higher than in the FIRE setting and despite the fact that the one-time transfer I model is not targeted. In the model, households only gradually realize that aggregate income has risen thanks to the stimulus policy, but are also slow to understand the policy's implications for future debt stabilization. The second effect dominates, making the transfers more powerful.

Despite the highly stimulative nature of transfers, I also find that they do not substantially boost employment or inflation. This lack of an effect is primarily because the surge in demand and economic activity that they produce is sharp and transitory, while price increases and hiring take time and resources. As such, the firms in the model are too slow to capitalize significantly on the abrupt surge in demand. The empirical estimates from the identified monetary shock also imply a very flat Phillips curve, further dampening the responsiveness of prices to fiscal policy.

I then evaluate the relative effects of monetary and fiscal transfer policy in the historical time series of output and inflation. I focus on the extraordinary fiscal stimulus enacted to boost the economy during the COVID-19 pandemic: after the passage of the CARES Act and the American Rescue Plan, discretionary[3] federal social benefit transfers to households as a percentage of pre-crisis GDP jumped sharply twice, to over 8% in 2020q2 and to 11.5% in 2021q1 (see Figure 1). I estimate the persistence and variance parameters of a set of 9 shocks in the model via Sequential Monte Carlo (SMC), using postwar U.S. data and a measurement equation reminiscent of Smets and Wouters (2007) but with the

[3]Excluding social security, Medicare/Medicaid, veterans' benefits, and unemployment insurance.

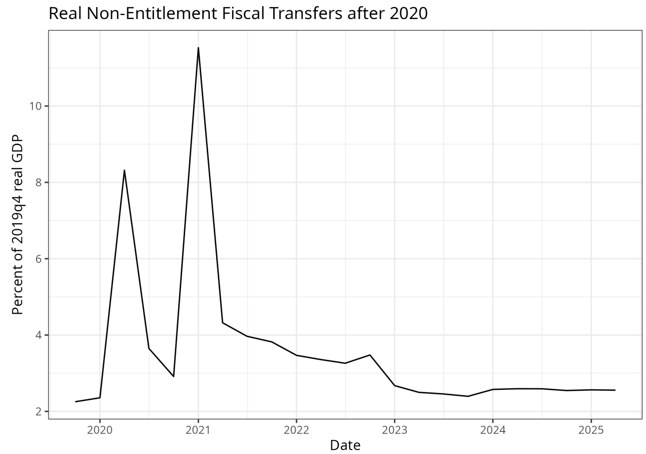


**Figure 1:** Discretionary fiscal transfers after 2020 as a percentage of 2019q4 GDP. Data includes all government social benefits to persons (personal current transfer receipts) excluding payments for Social Security, Medicare/Medicaid, veteran's benefits, and unemployment insurance. The series is deflated using the GDP deflator.

addition of fiscal transfers to GDP and unemployment rate data. The historical shock decomposition suggests that had such fiscal transfers not occurred, the recession's GDP losses relative to trend would have been 33% larger. The model finds little role for fiscal stimulus in lowering unemployment or driving up inflation.

I briefly survey related work on HANK models and departures from FIRE in the literature review, along with work related to fiscal stimulus during and after the pandemic. In Section 2, I describe the layout of a broad class of sticky expectation HANK models and the mathematical arguments that justify my solution technique. In Section 3, I detail the simple matrix manipulation that implements my methodology. In the Appendix, I show that the methodology agrees with closed form solutions for simple models and the sequence-space technique of Auclert, Rognlie, and Straub (2020) for numerical examples.

In Section 4, I introduce the medium-scale HANK model with sticky expectations and unemployment frictions that I use to study fiscal transfers. In section 5, I describe the model's steady-state calibration, the estimation of key structural parameters using impulse response function matching, and the Bayesian estimation of the remaining parameters using Sequential Monte Carlo. Section 6 presents the results for the effects of fiscal transfers with and without sticky expectations and the implications of the estimated model for the COVID era. Section 7 concludes.

## 1.1 Literature Review

*i. Models with Information frictions and HANK*

Numerous papers have explored how information frictions and rational inattention are key to explaining the slow movement of macroeconomic aggregates following changes in macroeconomic news relative to the speed of adjustment achieved in full information rational expectations models. In particular, Campbell and Deaton (1989) documents the "excess smoothness" of aggregate consumption's movement in response to news about permanent income – even as Campbell and Mankiw (1989) finds that aggregate consumption is excessively sensitive to *current* income relative to the permanent income hypothesis of Friedman (1957). Mankiw and Reis (2002) describe a model in which price-setters update their information about the macroeconomic only periodically, and in so doing generates realistic inertia in the movement of inflation. Reis (2006) suggests that such periodic updating may be the result of rational inattention if macroeconomic information is costly to obtain or process, and then shows that it can explain aggregate consumption's excess smoothness – along with its excess sensitivity to current income, if costly processing shortens some households' planning horizons. Finally, using survey data evidence, Carroll (2003) shows that households are slow to update their expectations about macroeconomic aggregates, and this phenomenon is well captured by a Calvo-style (effectively, Poisson) process in which households only update their forecasts of the future periodically – a framework now known as "sticky expectations."

In a seminal paper that merges HANK with the sticky expectation structure of Carroll (2003), Carroll et al. (2020) use a Krusell and Smith (1998) approach to solve a simple state-space HANK model by tracking the entire distribution of infrequently-updating household expectations. They then demonstrate that their simulated model replicates the empirically observed sluggish response of household consumption to macroeconomic events, providing a key friction that explains consumption's "excess smoothness" puzzle. Simultaneously, the high marginal propensities to consume of liquidity-constrained households in the incomplete markets environment (as documented earlier in Kaplan, Moll, and Giovanni L. Violante (2018), Auclert, Rognlie, and Straub (2024), and other HANK papers) explain the excess sensitivity of consumption to current income and aggregate demand. The authors then study the stimulus checks issued in the United States following the 2008 Global Financial Crisis using a small open economy model and find that households in the model react little to the checks upon announcement but strongly to the checks upon receiving them, consistent with empirical estimates from household microdata.

In a related branch of the literature, Auclert, Rognlie, and Straub (2020) also combine a HANK model with sticky expectations. However, the authors instead focus on reconciling the high marginal propensities to consume documented in Fagereng, Holm, and Natvik (2021) with empirical evidence pertaining to the effects of monetary policy related to the vector autoregression (VAR) and local projections (LP) literatures. They too show that information frictions are critical to reconcile the high MPCs of HANK models required to match microeconomic survey data with the empirical estimates of a slow-moving "hump" shaped response to a cut in interest rates. They find that forward guidance general equilibrium effects are muted with sticky expectations, but investment becomes an important driver of business cycles through its impact on aggregate demand and household income.

Like my paper, both Carroll et al. (2020) and Auclert, Rognlie, and Straub (2020) introduce both empirical findings and methodological innovations; I comment on the latter in the next subsection of the literature review. Empirically, Auclert, Rognlie, and Straub (2020) is the nearest neighbor to my paper – but the former is primarily concerned with monetary policy, and does not study the impacts of stimulus payments on output and inflation, nor does it model hiring and unemployment. In short, Auclert, Rognlie, and Straub (2020) studies monetary policy as a matter of direct interest, while my paper uses identified estimates of monetary policy to discipline the model's implications for *fiscal* transfer policy. Similarly, while Carroll et al. (2020) investigates a fiscal stimulus check response, it primarily does so as a laboratory to test the household-level empirical implications of HANK with information frictions. In contrast, I study the *macroeconomic* implications of stimulus check policy in the HANK environment in an estimated, medium-scale model.

*ii. Solving models with information frictions and HANK*

Methodologically, Auclert, Rognlie, and Straub (2020) represents a computational breakthrough by introducing sticky expectations to the sequence-space Jacobian (SSJ) approach of Auclert, Bardóczy, et al. (2021). Like my conversion, their technique also requires only a few small changes to the computation by recycling FIRE Jacobians in an efficient and computationally straightforward way. Indeed, in the appendix, I provide a numerical example of a canonical HANK model solved with sticky expectations in both my state-space form and in a continuous time variation of their sequence-space methodology to show that both methods produce very similar impulse response functions, up to an approximation error. However, while the SSJ framework is a powerful tool, some applications may still be more easily handled in state-space: higher-order perturbation approximations, regime-switching models shock filtering, missing data and measurement error handling during estimation, and more are for the time being still more straightforward in state space.[4] In contrast, my approach is able to bridge the gap between the information frictions literature and the growing full information numerical state-space HANK work of Bayer, Born, and Luetticke (2024), Bayer, Born, and Luetticke (2024), Acharya et al. (2023), and others.

Similarly, Carroll et al. (2020) solve their model using an adaptation of the Krusell and Smith (1998) methodology. While this is sufficient for the question they study, the model imposes strong dimensionality reduction by requiring households to only need a linear aggregate law of motion to forecast the macroeconomy. The computational burden also rises steeply with more complex models, and their solution technique requires function forms for utility and state transition dynamics that my methodology does not require. In contrast, I make full use of the fact that only average beliefs matter for economic dynamics when information updating is orthogonal to the idiosyncratic state space, a point established in Guerreiro (2023), to solve a sticky expectations HANK framework for a model that includes a search-and-matching labor market and the full set of Smets and Wouters (2007) frictions.

My approach naturally builds off of previous work on solving FIRE state-space HANK models. In my HANK numerical example, I repeatedly employ a continuous time analogue of the approach used in Bayer and Luetticke (2020) to solve for the systems' FIRE Jacobians. Their approach – based on Reiter (2009) – reduces the dimensionality of the heterogeneous agent problem using a discrete cosine transformation for the households' value function and a copula for the distribution of households. I also draw from the work of Achdou et al. (2021) and Ahn et al. (2018), which explains steady-state

[4] Additionally, determinacy and uniqueness of a stationary solution can be difficult to assess in sequence-space, but are more straightforward to assess with the Blanchard and Kahn (1980) methodology in state-space. Most sequence-space determinacy checks involve the use of Onatski (2006) winding criteria and the approximation of the solution with a state-space model. Auclert, Rognlie, and Straub (2023) use the quasi-Toeplitz structure as of the sequence-space Jacobians to approximate their model after a large number of time periods to assess determinacy. M. Hagedorn (2023) assumes households decisions only depend on aggregate states instead of the full distribution, making the dimension-reduced sequence-space model exactly Toeplitz. However, neither approach directly evaluates the stationarity of the model of interest – only a distant future or dimension reduced approximation.

perturbation solutions in continuous time HANK models more broadly. For converting the continuous time model to the discretely sampled data frequencies, I rely on the methods employed by Christensen, Neri, and Parra-Alvarez (2024), which provides a guide for properly integrating continuous time equations to discretized measurements of stocks and flows.

*iii. Fiscal transfers, output, and inflation*

When considering the interaction between information frictions and fiscal stimulus, Gabaix (2020) builds a simple behavioral New Keynesian model with cognitive discounting, in which agents attenuate their forecasts of the future toward the steady-state at a rate geometric in the forecasting horizon. The relative down-weighting of future macroeconomic forecasts leads the representative agent to violate Ricardian equivalence, leading deficit-financed fiscal transfers to become stimulative, as their expectations of the future taxation to stabilize the debt. In numerical simulations, I show that a similar phenomena occurs in my estimated HANK model, leading transfers stimulus to be significantly more stimulative with sticky expectations.

Recent fiscal policy has been analyzed in policy frameworks that involve fiscal dominance. Bianchi, Faccini, and Melosi (2023) estimate a rational expectations two-agent New Keynesian (TANK) model in a Fiscal Theory of the Price Level (FTPL) paradigm with active fiscal policy and passive monetary policy in the sense of Leeper (1991) and find that fiscal deficits could have accounted for the post-COVID inflation in their framework. Smets and Wouters (2024) estimate a similar RANK model and find transfer shocks under an active fiscal policy could have accounted for a significant fraction – but not a majority – of inflation in that same era while still boosting real GDP significantly. Kwicklis (2025) explores the tradeoff between output and inflation in a more stylized rational expectations HANK setting when fiscal policy is active and finds that the ratios of positive output gaps to inflation broadly match the experience of the post-COVID era. I do not model such fiscally dominant regimes in this paper, and instead employ a more conventional active monetary, passive fiscal framework. As a consequence, I show that fiscal transfers in an estimated HANK model with realistically high MPCs can be powerfully stimulative, but only slightly inflationary, if one is not in a fiscally-led policy regime.

My paper is also related to several recent, less model-based analyses. Baker et al. (2023) study the stimulus payments from the 2020 CARES Act using high-frequency banking data and estimate the average recipient spent about 25-30 cents on the dollar in the first quarter of receiving their transfer, a number broadly in line with the quarterly version of the MPCs estimated in Fagereng, Holm, and Natvik (2021) used to calibrate my HANK model. Consistent with HANK theory, they find that households with few liquid assets were the most responsive to the payments.

Orchard, V. A. Ramey, and Wieland (2025) argue that the high transfer multipliers associated with heterogeneous agent models produce implausible counterfactuals for no-stimulus scenarios following the 2008 Global Financial Crisis. The authors argue this is because the MPC estimates used to calibrate them are upwardly biased; many popular two-way fixed effect differences-in-differences MPC estimates inadvertently mix treatment and control groups when the control group receives a delayed treatment, while the social security numbers used to distribute stimulus checks are not entirely random. Notably, though, the lottery data MPC estimates of Fagereng, Holm, and Natvik (2021) does not have this econometric problem – although Orchard, V. A. Ramey, and Wieland (2025) note that the former must impute consumption for its estimates using wealth and income data. Even so, as a high-frequency event study that observes the transfers directly the aforementioned Baker et al. (2023) does not have "forbidden comparison" concerns and observes consumption expenditures directly.

## 2 General Framework

Several factors allow my numerical method to offer an expedient solution by recycling FIRE Jacobians. First, the machinery of continuous time naturally handles the interior and boundary of the state-space separately via partial differential equations (PDEs) and their accompanying boundary conditions, which ensure that agents do not violate borrowing constraints and similar restrictions. Second, as explained in Guerreiro (2023), only the average beliefs of the households in the standard sticky information setting matter for aggregate allocations.

As such, my first step is to solve the linearized problem for a household with *average* beliefs about the macroeconomy, given that the average household treats its beliefs (to first order) as the true future when calculating its value function and forming its plan for its control variables. Additionally, in continuous time, only a vanishing measure of households update their beliefs to full information in any given moment, so updates only lead the average belief (and the average behavior it induces) to drift, not jump. Information updates can therefore be incorporated entirely as additive drift terms. Lastly, the average belief value function can be parsimoniously updated using the value functions of full information households, as both solve the same partial equilibrium decision problem, just for different (incorrect and correct) sequences of forecasted prices.

Time is $t \geq 0$ is continuous. Households are ex-post heterogeneous and know the vector of their idiosyncratic state variables $x_t \in \mathcal{X}$ with full information. These state variables are assumed to evolve via a standard stochastic differential equation with the law of motion

$$dx_t = f(x_t, c_t, p_t)dt + \sigma_x(x)dW_t$$

where $c_t$ is the vector of the household's choice of controls, $p_t$ is a vector of macroeconomic variables outside of the individual household's control (like prices or inflation), $f$ is the law of motion governing the state variable's deterministic drift, and $\sigma_x(x)$ is a diagonal matrix through which an independent vector of Brownian motions $W_t$ feeds back into the state equations.[5] Note that $f$ is itself vector-valued; if $f_i$ depends on $c$, then the coordinate $x_i$ is an endogenous idiosyncratic state variable. If not, then $x_i$ is an exogenous idiosyncratic state variable.

In addition, I assume that idiosyncratic dynamics must satisfy a boundary condition along at least one of its dimensions:

$$x_{j,t} \geq \underline{x}.$$

For simplicity, I assume that $\sigma_{x,j,j}(x) = 0$ if $x_{j,t} = \underline{x}$, such that endogenous idiosyncratic states with

[5]Naturally, the logic in this text can accommodate other kinds of random processes for the state variables, like Poisson jump processes.

a boundary constraint do not evolve with a stochastic diffusion term on the boundary $\partial\mathcal{X}$.

Households plan to choose control variables to maximize their expected discounted utility. In contrast to full information rational expectations, however, the household uses its beliefs (indexed by $i \in \mathcal{I}$ with CDF $\Gamma(i)$) about the macroeconomy to forecast the macroeconomy's evolution and its impact on its decision problem, which may or may not be correct. The perceived problem is:

$$\begin{aligned} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = & \max_{(c_\tau^i(x_\tau, p_\tau, \mu_\tau))_{\tau \geq t}} \tilde{\mathbb{E}}_t^i \int_t^\infty e^{-(\tau - t)\rho} u(c_\tau) d\tau \\ \text{s.t.} \quad & \mathbb{E}_t^i[dx_t | dW_t] = \mathbb{E}_t^i[f(x_t, c_t; p_t)dt] + \sigma_x(x) dW_t, \\ & \mathbb{E}_t^i[\partial_t \mu_t] = \mathbb{E}_t^i[\mathcal{D}_t^*(V, p)[\mu_t](x)] \\ & \frac{\mathbb{E}_t^i[dp_t]}{dt} = \mathbb{E}_t^i[g(\mu_t, p_t)] \\ & x_{jt} \geq \underline{x} \ \forall t \geq 0 \end{aligned} \tag{1}$$

where $\mathcal{D}^*(V, p)$ is the true infinitessimal generator for the Kolmogorov Forward equation (KFE) of the distribution $\mu$, while $g$ is the true law of motion for aggregates $p$. $\tilde{\mathbb{E}}^i$ is the expectation taken with the subjective probability measure of a household with belief $i$ at time $t$. Here, $p_t^i \equiv \tilde{\mathbb{E}}_t^i[p_t]$ and $\mu_t^i \equiv \tilde{\mathbb{E}}_t^i[\mu_t]$

Although households may have incorrect beliefs about the trajectory of prices, they only use those incorrect beliefs for forecasting and constructing their value function $V_t^i$. For the first-order conditions that arise from their decision problem in the interior $\mathcal{X} \setminus \partial\mathcal{X}$, I assume households plug the actual $p_t$ into their choices at time $t$, such that

$$c_t^i(x) = h(x, V_t^i, p_t).$$

These consumption choices and contemporaneous prices are assumed to have no impact on the household's value function forecast. Similarly, I assume that households on a boundary $\partial\mathcal{X}$ with $x_{j,t} = \underline{x}$ choose consumption according to

$$f_j(x_t, h(V_t^i, p_t), p_t) = 0.$$

This is tantamount to a sequence of boundary constraints for the value function over time. In later sections, I show that this boundary condition is implied by a "mass-preserving" KFE infinitessimal generator and does not need to be imposed directly.

As in the sticky information framework of Caroll et al (2020) and Auclert et al (2023) (MJMH), households in each moment either update to full information about the aggregate system or not at all for the purpose of constructing $V_t^i$. They do so with a constant, independent Poisson intensity $\lambda$; in an infinitessimal increment of time, a random $\lambda dt$ mass from the cross section of households updates to full information.

While households are able to reason through the dynamics of $g$ given their beliefs about its inputs, I assume that the $p$ equations follow the general structural relation

$$Q dp_t = q(\mu_t, p_t, \{h(V_t^i, p_t)\}_{i \in \mathcal{I}}) dt. \tag{2}$$

If $Q$ is invertible, then $g = Q^{-1}q$. In other cases some rows of $Q$ are entirely zero, such that the equation denotes a static fixed point relationship for which $g$ is a solution. In this way, equation (2) encompasses the dynamics of macroeconomic jump variables (like inflation), macroeconomic state variables (like the capital stock), and static variables (like prices).

### 2.1 Households in the Interior

I consider the interior $\mathcal{X} \setminus \partial\mathcal{X}$ and $\partial\mathcal{X}$ separately, as in the former case the $i$-indexed beliefs affect households' decisions, while in the latter they do not. Working with the two cases separately is straightforward in continuous time, as the recursive Hamilton Jacobi Bellman (HJB) equation describes only the state-space's interior.

Discretizing the value function (1) in with infinitessimal time increments $dt$, the analogue to the discrete time value function is

$$\begin{aligned} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) &= \max_{(c_\tau^i)_{\tau \geq t}} u(c_t^i)dt + e^{-\rho dt}\tilde{\mathbb{E}}_t^i V_{t+dt}^i(x_{t+dt}, p_{t+dt}, \mu_{t+dt}, p_{t+dt}^i, \mu_{t+dt}^i) + \\ \text{s.t. } & \mathbb{E}_t^i[dx_t|dW_t] = \mathbb{E}_t^i[f(x_t, c_t^i; p_t)] + \sigma_x(x)dW_t,\ \dot{p}_t^i = \mathbb{E}_t^i[g(\mu_t, p_t)],\ \partial_t\mu_t^i = \mathbb{E}_t^i[\mathcal{D}_t^*(V,p)\mu_t] \end{aligned}$$

where conditioning on the subjective $p^i, \mu^i$, the evolution of $p, \mu$ is irrelevant for the household's decision problem (although the *level* is still relevant).

**Proposition 2.1.** *The Hamilton Jacobi Bellman (HJB) equation for $x \in \mathcal{X} \setminus \partial\mathcal{X}$ takes the form*

$$\begin{aligned} \rho V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{\tilde{c}_t^i} \Big\{ & u(\tilde{c}_t^i) + \nabla_x V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\mathbb{E}_t^i[f(x_t, c_t^i; p_t)] + \frac{1}{2}tr(\sigma_x(x)\sigma_x(x)'\nabla_x^2 V^i)dt \\ & + \nabla_p V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\tilde{\mathbb{E}}_t^i[g(p, \mu)] + \int_{\mathcal{X}} \delta_{\mu(x')} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)\tilde{\mathbb{E}}_t^i[\mathcal{D}_t^*(V, p_t)\mu_t(x')]dx' \Big\}. \end{aligned} \tag{3}$$

*Here, I write $\delta_{\mu(x')}F(x)$ as a shorthand for $\frac{\delta F(x)}{\delta\mu(x')}$, the functional (Frechét) derivative of $F(x)$ with respect to $\mu(x')$.*

*Proof.* See Appendix A.1. □

**Definition 2.2.** I define a *non-stochastic steady state* as one in which the value function no longer explicitly depends on time, and the macroeconomic variables $\mu, p$ are equal to their expected values across the entire economy and are no longer changing. All households have the correct belief, while $\dot{p} = 0$ and $\partial_t\mu_t = 0$.

Up to a first-order approximation in the macroeconomic variables around the non-stochastic steady state, the household will treat its forecast for prices and the distribution as if they are the true prices and distribution, as in Carroll et al. (2020). As such, I can write the HJB as a function of $p_t^i$ and $\mu_t^i$ alone:

$$\begin{aligned} \rho V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i) = \max_{\tilde{c}_t^i} \Big\{ & u(c_t^i) + \nabla_x V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i)' f(x_t, c_t^i; \tilde{p}_t^i) \\ & + \nabla_{p^i} V_t^i(x_t, \tilde{p}_t^i, \tilde{\mu}_t^i)' g(\tilde{p}_t^i, \tilde{\mu}_t^i) \\ & + \int_{\mathcal{X}} \delta_{\mu^i(x')} V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i(x'))\mathcal{D}_t^*(V_t^i(x'), \tilde{p}_t^i)\tilde{\mu}_t^i(x')]dx' \Big\} \end{aligned} \tag{4}$$

Note that the dependence of the value function on $i$ is entirely through the expected aggregates $p^i$ and $\mu^i$, while actual $p$ and $\mu$ do not affect the problem. The interior household will plan to choose consumption to maximize its expected utility only using its subjective beliefs about prices, the distribution, and other macro aggregates. Assuming that the optimization problem is concave, the household's planned control choice will thereby satisfy

$$\nabla u(\tilde{c}_t^i) = -\nabla_x V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i)' \partial_{c^i} f(x_t, \tilde{c}_t^i; \tilde{p}_t^i).$$

For many problems, the budget constraint can be rewritten so that the household chooses only a numeraire consumption good, as in Caroll et al (2020). In such cases, $c$ can be written entirely in terms of $\nabla_x V^i$. In more complicated settings, however, one could consider cases with variable control prices that the consumer is able to observe (but does *not* use to update their forecast). The consumer *actually* choose $c^i$ such that

$$\nabla u(c_t^i) = -\nabla_x V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i)' \partial_{c^i} f(x_t, c_t^i; p_t),$$

which equates the instantaneous value of the control with its perceived opportunity cost (e.g. the value of consumption with the *subjective* value of the savings given actual present prices). If $p_t$ does *not* change the consumption plan, however, it still does not enter into the forecast of $V_t^i$. Rather, actual $p_t$ only enters into how the distribution is updated.

I denote this control variable choice that satisfies the FOC

$$c_t^i(x_t; p_t, \mu_t, \tilde{p}_t^i, \tilde{\mu}_t^i) = h\Big(V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i), p_t\Big).$$

## 2.2 The Average-Belief Household

At this stage, it is useful to define an agent with *average* beliefs about the state of the macroeconomy. This agent does not actually exist; in the model, households either have full information or don't following a macroeconomic shock. Still, the construct is useful, as the average agent will behave as if the average beliefs about prices are the true dynamics, and their value function can be used to determine the average choices in the economy at every point in the idiosyncratic state-space $\mathcal{X}$. To see why this is useful and convenient, I show that in a first-order expansion, only the average belief matters for the households' aggregate control variables, as Guerreiro (2023) argues in a sequence-space setting.

From there, I derive the evolution of average beliefs under sticky expectations in continuous time. The result is a system of intuitive and tractable differential equations that are straightforward to add to the model.

### 2.2.1 The Average-Belief Value Function

Recall that the economy is populated with agents who have beliefs indexed by $i$ about macroeconomic states like the distribution $\mu$ and prices and aggregates $p$, and denote this subjective probability density $\tilde{\psi}_t^i$. Subjective expectations about a macroeconomic random variable $Y$ are calculated with the subjective measure:

$$\tilde{\mathbb{E}}_t^i[Y] \equiv \int_S y\tilde{\psi}_t^i(y)dy$$

Let the measure of housholds with belief $i$ be $\Gamma(i)$. Define the average belief about a macroeconomic variable $Y$ as

$$\overline{\mathbb{E}}_t[Y] \equiv \int_S y\overline{\psi}_t(y)dy$$

where $\overline{\psi}_t(y) \equiv \int_i \tilde{\psi}_i(y)d\Gamma(i)$ is the average agent's belief about the PDF of $Y$ at time $t$. Just like $\tilde{p}^i$ and $\tilde{\mu}^i$ were calculated using the $i$ probability measures, I similarly define $\overline{p}_t$ and $\overline{\mu}_t$ as the *average* beliefs about the macroeconomy:

$$\overline{p}_t \equiv \overline{\mathbb{E}}_t[p_t],\ \overline{\mu}_t \equiv \overline{\mathbb{E}}_t[\mu_t].$$

Define the "expectations-averaged" value function over the entire population *before* any agents update their beliefs as

$$\begin{aligned}\rho\overline{V}_t(x_t;\overline{p}_t,\overline{\mu}_t) = \max_{\overline{c}_t} \Bigg\{ & u(\overline{c}_t) + \nabla_x \overline{V}_t(x_t;\overline{p}_t,\overline{\mu}_t)' f(x_t,\overline{c};\overline{p}_t) \\ & + \nabla_{\overline{p}}\overline{V}_t(x_t;\overline{p}_t,\overline{\mu}_t)' g(\overline{p}_t,\tilde{\mu}_t) + \int_{\mathcal{X}} \delta_{\overline{\mu}(x')}\overline{V}_t(x_t;\overline{p}_t,\overline{\mu}_t)\mathcal{D}_t^*(\overline{V}_t,\overline{p}_t)[\overline{\mu}_t](x')dx' \Bigg\} \\ \text{s.t. } & x_{j,t} \geq \underline{x}\ \forall t\end{aligned} \tag{5}$$

The value function is simply the preceding agents' value function, but specifically using the *average* belief about macroeconomic variables.

**Proposition 2.3.** *As in Guerreiro (2023), the value function averaged over beliefs $\overline{V}_t$ and actual prices and aggregates $p_t$ characterize the average control variable choice given the households' idiosyncratic states to first order. In other words, in a neighborhood around the non-stochastic steady-state with deviations thereof denoted by $\Delta$,*

$$h(\overline{V}_t, p_t) = \int_i c_t^i(x;p,\mu,p^i,\mu^i)d\Gamma(i) + \mathcal{O}(\|\Delta p^2, \Delta\mu^2, (\Delta p^i)^2, (\Delta\mu^i)^2\|).$$

*Proof.* See Appendix A.3. □

### 2.2.2 Sticky Expectations in Continuous Time

For a random variable that is changing over time, the total change in the average forecast over time will be

$$\begin{aligned}\frac{d}{dt}\left(\overline{\mathbb{E}}_t[Y_{t+s}]\right) &= \frac{d}{dt}\left(\int_\Omega y_{t+s}(\omega)\overline{\psi}_t(\omega)d\omega\right) \\ &= \underbrace{\int_\Omega \dot{y}_{t+s}(\omega)\overline{\psi}_t(\omega)d\omega}_{\text{Subj. Forecast}} + \underbrace{\int_\Omega y_{t+s}(\omega)\partial_t\overline{\psi}_t(\omega)dy_t}_{\text{"Gain"}} \\ &= \overline{\mathbb{E}}_t[\dot{Y}_{t+s}] + \frac{d\overline{\mathbb{E}}}{dt}[Y_{t+s}].\end{aligned}$$

This structure has a Kalman Filter-like intuition: the agents' ex-post belief about a macroeconomic variable For a sticky-information environment like the one detailed in Mankiw and Reis (2002), Carroll et al. (2020), Auclert, Rognlie, and Straub (2020), and many others, the average belief is:

$$\overline{\mathbb{E}}_t[Y_{t+s}] = \int_0^\infty \lambda e^{-\lambda\tau}\mathbb{E}_{t-\tau}[Y_{t+s}]d\tau.$$

Differentiating with respect to $t$, I show in the Appendix A.2 that the average expectation then follows

$$\frac{d}{dt}\overline{\mathbb{E}}_t[Y_{t+s}] = \overline{\mathbb{E}}_t[\partial_t Y_{t+s}] + \lambda\Big(\mathbb{E}_t[Y_{t+s}] - \overline{\mathbb{E}}_t[Y_{t+s}]\Big).$$

Crucially, when a household updates from stale beliefs about the macroeconomy to full information, they do not just update their forecast for the variable at time $t$. Rather, they update their entire sequence of forecasts for the entire future, such that the update takes the form of an entire sequence of revisions
$\big\{\lambda(\mathbb{E}_t[Y_{t+s}] - \overline{\mathbb{E}}_t[Y_{t+s}]),\ s \geq 0\big\}$.

While the change in the entire forecast sequence is crucial for proper updating, I later show that it suffices to track just the zero-horizon forecasts for macroeconomic variables. For prices and aggregates $p_t$ in the economy and the distribution $\mu_t$, I define the zero-horizon expected values

$$\overline{p}_t \equiv \lim_{dt\to 0}\overline{\mathbb{E}}_t[p_{t+dt}]$$

$$\overline{\mu}_t \equiv \lim_{dt\to 0}\overline{\mathbb{E}}_t[\mu_{t+dt}].$$

The zero-horizon forecasts will then evolve according to

$$\frac{d\overline{p}_t}{dt} = \overline{\mathbb{E}}_t[\partial_t p_t] + \lambda(p_t - \overline{p}_t)$$

$$\frac{d\overline{\mu}_t}{dt} = \overline{\mathbb{E}}_t[\partial_t \mu_t] + \lambda(\mu_t - \overline{\mu}_t)$$

where the first term of each expression is the agent's perceived belief of how prices and the distribution evolve before new information updates agents' forecasts. In other words, to first order the average belief is updated as follows:

$$\frac{d\overline{p}_t}{dt} = g(\overline{p}_t, \overline{\mu}_t, \overline{V}) + \lambda(p_t - \overline{p}_t) \tag{6}$$

$$\partial_t\overline{\mu}_t(x) = \mathcal{D}_t^*(\overline{V}, \overline{p})[\overline{\mu}_t](x) + \lambda(\mu_t(x) - \overline{\mu}_t(x)) \tag{7}$$

## 2.3 The Distribution of Agents (and Households on the Boundary)

Up to this point, I have referenced the $\mathcal{D}^*$ infinitessimal generator; I now define it explicitly and discuss how it implicitly enforces the boundary constraints referenced in the Overview section. Consider the average value function $\overline{V}$ that induces consumption choices according to the first-order conditions $c_t(x; p_t, \overline{p}_t, \overline{\mu}_t) = h(\overline{V}_t(x; \overline{p}_t, \overline{\mu}_t), p_t)$. For an agent on the boundary $\partial\mathcal{X}$, e.g. at a borrowing constraint, the appropriate boundary condition on $\overline{V}$ to describe the agent's behavior is

$$f_i(x, h(\overline{V}_t, p_t); p_t, \mu_t) = 0 \quad \text{if } x_i = \underline{x} \tag{7.5}$$

such that $V_t(x)$, $x \in \partial\mathcal{X}$ satisfies the above implicit relationship. (Technically, this constraint should hold for every $V_t^i$, but only $\overline{V}$ is computationally important). In the particular context of a borrowing constraint, the above implies that the household consumes exactly its income when its assets are zero – such that the asset state variable does not drift past the constraint.

To enforce the appropriate sequence of boundary conditions on the HJB relationship, one need only ensure that the distribution $\mu$ whose mass starts within $\mathcal{X}$ stays within $\mathcal{X}$.

First, note that the evolution of the distribution of households with the value function $V$ may be expressed via a standard Kolmogorov Forward Equation (KFE)

$$\partial_t \mu_t(x) = -\nabla_x \cdot (f(x, h(\overline{V}_t, p_t); p_t)\mu_t(x)) + \nabla^2 \mathrm{tr}\left[(\sigma(x)\sigma(x)'\mu_t(x))\right] \tag{8}$$

given the $\sigma(x)$ diffusion matrix is diagonal. Equation (44) depends on the *actual* prices and the *actual* distribution. Expectations about prices only enter into the households' value function $V$, which may reflect some more complicated information or belief structure. This equation can be more compactly represented with the KFE infinitessimal generator operator $\mathcal{D}^*$, such that

$$\partial_t \mu_t = \mathcal{D}^*(\overline{V}_t, p_t)[\mu](x).$$

**Definition 2.4.** Define the KFE operator's *kernel* $D^*(V, p)(x, y) : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that

$$\mathcal{D}^*(V, p)[\mu](x) = \int_{\mathcal{X}} D^*(V, p)(x, x')\mu(x')dx'.$$

**Definition 2.5.** A KFE infinitessimal generator $\mathcal{D}^* : F[\mathcal{X}] \to F[\mathcal{X}]$ is *mass-preserving* if its kernel satisfies

$$\int_{\mathcal{X}} D^*(V, p)(x, x')dx = 0 \quad \forall x' \in \mathcal{X}.$$

By analogy, let $d^* = [d^*_{i,j}]$ be a matrix finite difference approximation the kernel of $\mathcal{D}^*(V, p)$, e.g. $D^*(V, p)(x, y)$. $d^*$ will be mass-preserving if all of its columns sum to zero, such that:

$$\sum_i d^*_{i,j} = 0.$$

**Proposition 2.6.** *Suppose the economy starts in its non-stochastic steady state when a macroeconomic shock occurs. If the KFE generator is mass-preserving, then the value function of households at the boundary will satisfy equation (7.5).*

*Proof.* See Appendix A.4. ☐

The proof goes roughly as follows: if the KFE operator is mass-preserving, then the net flux across the boundary defined by the state constraints must be zero for all time. If all the probability mass starts within or on the boundary of the space, then no mass crosses the boundary, and so probability mass exactly on the boundary must be traveling tangent to it. This tangent motion is equivalent to the value function satisfying equation (7.5).

Tracking the evolution of the distribution with a mass preserving KFE operator therefore naturally imposes a time-varying boundary condition for the value function that depends on the realization of actual aggregates. For example, if households are unable to borrow and are at a constraint of 0 assets, they will consume a maximum of their current income, regardless of their beliefs.

For most applications involving a first-order perturbation solution, the mass-preserving KFE generator will indeed be mass-preserving.

**Proposition 2.7.** *If the KFE infinitessimal generator $\mathcal{D}^*(V,p)$ is a first-order perturbation of the steady-state one with respect to macroeconomic variables, then it will be mass-preserving if the Jacobians evaluated at the steady-state are mass-preserving.*

*Proof.* See Appendix A.5. □

The result follows immediately from the linearity of the operators. In effect, if the perturbation solution enforces the correct idiosyncratic constraints in the FIRE case via the KFE operator, it will also enforce the correct constraints in the non-FIRE case along the boundary.

### 2.4 Aggregation

Because the average belief value function determines the value of average controls conditional on the idiosyncratic state-space point in the state-space, $\overline{V}$ will also be sufficient to characterize macroeconomic aggregates. Aggregate controls will then be

$$C_t = \int_{\mathcal{X}} h(\overline{V}_t, p_t)\mu_t(x)dx$$

such that the aggregate variables will depend on the *actual measure* of individuals given their choices derived from the average expectations. Similarly, aggregate states can be computed as

$$X_t = \int_{\mathcal{X}} x\mu_t(x)dx.$$

As such, the actual $p_t$ in the economy will evolve with the true distribution $\mu_t$, the true $p_t$, and the subjective belief-averaged $\overline{V}_t$:

$$Qdp_t = q(\mu_t, p_t, h(\overline{V}_t, p_t)dt. \tag{9}$$

### 2.5 Full Information Households

Every period, a mass $\lambda dt$ mass od new households becomes full information. It's necessary to track these households as well, as they become a greater and greater share of the population over time (and thus have a greater and greater effect on the average). Define the full information value function as

$$
\begin{aligned}
\rho\widehat{V}_t(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t) = \max_{c_t} \Big\{ & u(c_t) + \nabla_x \widehat{V}_t(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t)' f(x_t,c_t;p_t) \\
& + \nabla_p \widehat{V}(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t)' \frac{\mathbb{E}[dp_t]}{dt} + \int_{\mathcal{X}} \delta_\mu \widehat{V}_t(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t)\partial_t \mu_t dx \\
& + \nabla_{\overline{p}} \widehat{V}(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t)' \frac{d\overline{p}_t}{dt} + \int_{\mathcal{X}} \delta_{\overline{\mu}} \widehat{V}_t(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t)\partial_t \overline{\mu}_t dx \\
& + \int_{\mathcal{X}} \delta_{\overline{V}} \widehat{V}_t(x_t;p_t,\mu_t,\overline{p}_t,\overline{V}_t)\partial_t \overline{V}_t dx \Big\}.
\end{aligned}
\tag{10}
$$

$\widehat{V}$ uses the true law of motion for prices – given the actual prices and the mean beliefs across the economy, and the belief-averaged value function $\overline{V}$, which influences average choices. A rational full information household thus forecasts 1) their idiosyncratic state variables' evolution, given the true prices, 2) the evolution of those prices, given the true distribution of agents and their average choices (encapsulated by average belief $\overline{V}$), 3) the evolution of the total distribution, 4) the evolution of expected prices and 5) expected distributions for the average household, and 6) the average value function (and therefore decisions) of the average agent in the economy, which when combined with the true distribution is used to formulate a forecast for prices.

Altogether, equations (5-10) nearly describe how the system evolves for the purposes of calculating macroeconomic and microeconomic (but expectations-averaged) variables, but with a caveat: equation (5) is incomplete, and only models the average household dynamics if the composition of households did not change. With a probability $\lambda dt$, a household is uniformly selected (after choosing their consumption) to update their beliefs about the macroeconomic variables to full information. As such, $\overline{V}$ should evolve according equation (5) – but with an additional $\lambda(\widehat{V}_t - \overline{V})$ that the average household does not anticipate or plan for in their optimization problem. I discuss how to incorporate this adjustment into the dynamics in the next section.

### 2.6 A two-part problem

To solve the model, one can solve for two different stable manifolds (subspaces in the linearized model), consecutively. First, one can solve for the behavior of the fictitious average agent to determine the evolution of $\overline{V}$. Then, one can solve the full information households' problem, taking the average agent as a state variable (and where the full information agents internalize how they will update the average agent over time).

#### 2.6.1 The mean belief household solution

First, consider the perceived problem of the fictitious average household. For a given household, the value function may be concentrated to have an explicit time dependence, such that it represents the choices of the household for a given sequence of macroeconomic aggregates. Denoting the drift of the macroecomic variables $\partial_t \overline{V}(x_t)$, one can write

$$\partial_t \overline{V}(x_t) = \nabla_{\overline{p}} \overline{V}_t(x_t; \overline{p}_t, \overline{\mu}_t)' g(\overline{p}_t, \tilde{\mu}_t) + \int_{\mathcal{X}} \delta_{\overline{\mu}} \overline{V}_t(x_t; p_t, \mu_t, \overline{p}_t, \overline{\mu}_t) \mathcal{D}_t^*(\overline{V}_t, \overline{p}_t)[\overline{\mu}_t] dx.$$

By subsuming the macroeconomic variable dependence into the value function, the decision problem or "partial equilibrium" value function is then

$$\begin{aligned} \rho \overline{V}_t(x_t) = & \max_{\overline{c}_t} \left\{ u(\overline{c}_t) + \nabla_x \overline{V}_t(x_t)' f(x_t, \overline{c}; \overline{p}_t) \right\} + \partial_t \overline{V}(x_t) \\ & \text{s.t. } x_t \geq \underline{x} \ \forall t \end{aligned} \tag{11}$$

where

$$\partial_t \overline{\mu}_t = \mathcal{D}_t^*(\overline{V}_t, \overline{p}_t)\mu_t, \tag{12}$$

$$Q\frac{d\overline{p}_t}{dt} = q(\overline{\mu}_t, \overline{p}_t, \overline{V}_t). \tag{13}$$

This system exactly resembles the FIRE system – except with expected prices in lieu of the real ones. The reason for this is that the average expectation agent believes that their forecast of prices is correct (or at least, on average correct in a certainty equivalent setting).

The concentrated HJB can then be linearized around the non-stochastic steady-state with respect to the macroeconomic variables as

$$\partial_t \Delta \overline{V}(x_t) = \int_{\mathcal{X}} \mathcal{A}_{VV}(x, x') \Delta \overline{V}_t(x') dx' + \int_{\mathcal{X}} \mathcal{A}_{V\mu}(x, x') \Delta \overline{\mu}_t(x') dx' + \mathcal{A}_{Vp}(x) \Delta \overline{p}_t + \mathcal{O}([...]^2). \tag{14}$$

The $\mathcal{A}$ operators denote the partial equilibrium Jacobians of the household's concentrated HJB with respect to its own value function and the average beliefs about prices and the distribution evaluated in the non-stochastic steady state. In other words,

$$\mathcal{A}_{VV}(x, x') = \rho \delta(x - x') - \frac{\delta}{\delta v(x')} \Big[ h(v(x)) + \nabla_x v(x)' f(x, h(v(x)); \overline{p}) \Big],$$

$$\mathcal{A}_{V\mu}(x,x') = -\frac{\delta}{\delta\overline{\mu}(x')}\left[h(v(x)) + \nabla_x v(x)' f(x, h(\overline{V}); p)\right] (= 0).$$

$$\mathcal{A}_{Vp}(x) = -\nabla_x v(x)' \frac{\partial}{\partial p} f(x, h(v); p).$$

where $\delta(x - x')$ is a Dirac-delta function and $\frac{\delta f(x,g(x))}{\delta g(x')}$ refers to the functional (Frechét) derivative of $f$ with respect to $g$. Note that in the steady-state, actual and expected prices are equal and all households have the same value function $v(x)$; these Jacobians are exactly the same as their FIRE counterparts.

Suppose the average household's perceived problem can be solved for the value function's dynamics on the stable manifold, such that for a sequence of beliefs about prices and the distribution, the value function will satisfy (at least, under the household's average beliefs)

$$\overline{\mathbb{E}}_t[\partial_t \Delta\overline{V}(x_t)] = \int_{\mathcal{X}} \mathcal{B}_{VV}(x,x')\Delta\overline{V}_t(x'_t)dx' + \int_{\mathcal{X}} \mathcal{B}_{V\mu}(x,x')\Delta\overline{\mu}_t(x')dx' + \mathcal{B}_{Vp}(x)\Delta\overline{p}_t + \mathcal{O}([...]^2).$$

The forecasts of the expected household (prior to updating) will also be

$$\overline{\mathbb{E}}_t[\partial_t \Delta\mu_t(x)] = \int_{\mathcal{X}} \mathcal{B}_{V\mu}(x')\Delta\overline{V}_t(x'_t)dx' + \int_{\mathcal{X}} \mathcal{B}_{\mu\mu}(x,x')\Delta\overline{\mu}_t(x')dx' + \mathcal{B}_{\mu p}(x)\Delta\overline{p}_t + \mathcal{O}([...]^2)$$

$$\overline{\mathbb{E}}_t[\partial_t \Delta p_t] = \int_{\mathcal{X}} \mathcal{B}_{pV}(x,x')\Delta\overline{V}_t(x'_t)dx' + \int_{\mathcal{X}} \mathcal{B}_{p\mu}(x')\Delta\overline{\mu}_t(x')dx' + \mathcal{B}_{pp}\Delta\overline{p}_t + \mathcal{O}([...]^2)$$

In the actual economy, however, the average beliefs are updated with the realizations of the actual $p_t$ and $\mu_t$. Unfortunately, this is slightly complicated by the fact that learning at time $t$ updates the whole forecast sequence of $(\overline{p}_\tau, \overline{\mu}_\tau)_{\tau \geq t}$, not just their contemporaneous values. To see why this is important, consider integrating forward equation (14), with the assumption that $\lim_{t\to\infty} \Delta V_t(x) = 0$. Using the partial equilibrium Jacobians and treating the linear operators analogously to matrices, the value function becomes:

$$\Delta\overline{V}_t(x) = \int_t^\infty \int_{\mathcal{X}} \left[e^{-\mathcal{A}_{VV}(\tau - t)}\right](x,x'')\left[\int_{\mathcal{X}} \mathcal{A}_{V\mu}(x,x')\Delta\overline{\mu}_\tau(x')dx' + \mathcal{A}_{Vp}\Delta\overline{p}_\tau\right] dx'' d\tau.$$

where the exponential operator is $\left[e^{-\mathcal{A}_{VV}t}\right](x,x')$ is the kernel equivalent to a matrix exponential.[6] An update results in a change to the value function that takes the entire future path of the new forecast into the account – a complicated object. Fortunately, there's a simpler approach: use the present values already calculated in the value functions of the full information agents.

### 2.6.2 Full information households and updating

Consider the full information agent's decision problem, given a sequence of macro aggregates and the behavior and beliefs of other agents in the economy (e.g. $\overline{V}, \overline{\mu}, \overline{p}$). By concentrating equation (10),

[6] More explicitly, $\left[e^{\mathcal{A}_{VV}t}\right](x,x') \equiv \delta(x - x') + \sum_{n=1}^{\infty} \frac{t^n}{n!}\mathcal{A}^{(n)}_{VV}(x,x')$,
where $\mathcal{A}^{(n)}_{VV}(x,x') \equiv \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} \mathcal{A}_{VV}(x,x_1)\mathcal{A}_{VV}(x_1,x_2)(\ldots)\mathcal{A}_{\mathcal{VV}}(x_{n-1},x')dx_1 \ldots dx_{n-1}$.

the full information value function is

$$\begin{aligned}\rho \widehat{V}_t(x_t) &= \max_{c_t} \left\{ u(c_t) + \nabla_x \widehat{V}_t(x_t)' f(x_t, c_t; p_t) \right\} + \partial_t \widehat{V}(x_t) \\ \text{s.t. } & x_t \geq \underline{x} \ \forall t.\end{aligned} \tag{15}$$

Now, however, the sequence of actual prices evolve using the *average* value function (which characterizes average household actions) and the actual distribution, along with actual prices and the actual distribution.

$$\partial_t \mu_t = \mathcal{D}^*(\overline{V}_t, p_t)\mu_t$$

$$Q\dot{p}_t = q(\mu_t, p_t, \overline{V})$$

The full information rational agent knows that the other agents will learn over time. The law of motion for the average macroeconomic beliefs is the solution to the non-updating household's problem, but modified for the $\lambda dt$ measure of agents that update using the true values. As such,

$$\frac{d\Delta \overline{p}_t}{dt} = \overline{\mathbb{E}}_t[\Delta \partial_t p_t] + \lambda(\Delta p_t - \Delta \overline{p}_t) \tag{16}$$

$$\frac{\partial \Delta \overline{\mu}_t}{\partial t} = \overline{\mathbb{E}}_t[\partial_t \Delta \mu_t] + \lambda(\Delta \mu_t - \Delta \overline{\mu}_t) \tag{17}$$

where $\overline{\mathbb{E}}_t[\partial_t \Delta p_t]$ and $\overline{\mathbb{E}}_t[\partial_t \Delta \mu_t]$ are the solutions from the average expectation block. By analogy, one could reasonably guess:

$$\frac{\partial \Delta \overline{V}_t}{\partial t} = \overline{\mathbb{E}}_t[\partial_t \Delta V_t] + \lambda(\Delta \widehat{V}_t - \Delta \overline{V}_t)$$

where $\overline{\mathbb{E}}_t[\partial_t V_t]$ is again the solution from the average belief households' problem. This turns out to be correct, as per the following proposition:

**Proposition 2.8.** *To a first-order approximation, the average belief household updates its value function with a constant factor of* $\lambda(\Delta \widehat{V} - \Delta V)$.

*Proof.* See Appendix A.6. □

The intuition behind the result is straightforward: although rational expectations households and non-updating households have very different information sets, both solve essentially the same partial equilibrium decision problem when planning their consumption, just with different beliefs. The value functions themselves are linearized with respect to those beliefs, so the effect of a change in a sequence of beliefs is equal to a difference between value functions.

One could also think about the intuition in a slightly different, but equivalent, way: as time progresses following a time-zero shock, the mass of households who have updated grows at a rate of $\lambda$ per unit of time, while the mass who think they are still in the steady state shrinks at the same rate. This pulls the overall average belief households toward the full information ones at a rate of $\lambda$, as more and more FIRE households become averaged into the entire population.

## 3 Linearized Solution

The preceding section described the linearized solution in a more abstract functional form. To actually calculate the solution on the computer, one discretizes the functions onto grids as described in Achdou et al (2020). Functions become vectors, while integrals becomes sums.

Altogether, the process can be summarized in three steps:

1. Solve the full information rational expectations model:

$$\begin{aligned} \rho V_t(x_t) &= \max_{c_t} \left\{ u(c_t) + \nabla_x V_t(x_t)' f(x_t, c_t; p_t) \right\} + \partial_t V_t(x_t) \\ &\text{s.t. } x_t \geq \underline{x} \ \forall t \\ \partial_t \mu_t &= \mathcal{D}_t^*(p_t, V_t)\mu_t \\ Q\dot{p}_t &= q(\mu_t, p_t, V_t) \end{aligned} \tag{18}$$

2. Construct the solution to the average belief households' problem in the absence of updating. This is simply the FIRE solution, but with the subjectively expected variables instead of the true ones. The new system describes $\overline{\mathbb{E}}_t[\partial_t V_t]$, $\overline{\mathbb{E}}_t[\partial_t \mu_t]$, and $\overline{\mathbb{E}}_t[\partial_t p_t]$.

3. Solve the full information households' rational expectations problem given the average behavior of the other agents, accounting for how the average information agent updates.

In what follows, I assume knowledge Bayer and Luetticke (2020) and Ahn et al. (2018), which are in turn based on the methodology of Reiter (2009). After discretizing the value functions and distributions over a grid, one may solve for the non-stochastic steady-state. Thereafter, one constructs a first-order perturbation of the economy from that steady-state due to aggregate shocks. The $A$ and $B$ block matrices are essentially the discretized matrix representations of the $\mathcal{A}$ and $\mathcal{B}$ terms introduced earlier in the text. I also dispense with the $\Delta$ notation; $V$, $\mu$, and $p$ in this section are all discretized vectors that represent deviations from the non-stochastic steady-state.

First, one starts with the FIRE Jacobians for equation (18):

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} \mathbb{E}[dV] \\ d\mu \\ \mathbb{E}[dp] \end{bmatrix} = \begin{bmatrix} A_{VV} & A_{V\mu} & A_{Vp} \\ A_{\mu V} & A_{\mu\mu} & A_{\mu p} \\ A_{pV} & A_{p\mu} & A_{pp} \end{bmatrix} \begin{bmatrix} V \\ \mu \\ p \end{bmatrix} dt, \tag{19}$$

If the Blanchard and Kahn (1980) conditions are satisfied, one can solve the system as in Sims (2002) using a generalized Schur decomposition to determine its dynamics on its stable manifold – the stable subspace in the linearized, discretized model. The solved rational expectations model is then

$$\begin{bmatrix} dV \\ d\mu \\ dp \end{bmatrix} = \begin{bmatrix} B_{VV} & B_{V\mu} & B_{Vp} \\ B_{\mu V} & B_{\mu\mu} & B_{\mu p} \\ B_{pV} & B_{p\mu} & B_{pp} \end{bmatrix} \begin{bmatrix} V \\ \mu \\ p \end{bmatrix} dt. \tag{20}$$

Once again, the $B_{ij}$ matrices represent the Jacobians of the equilibrium system, restricted to the stable subspace.

Before any updating occurs, agents behave with the belief that the feedbacks of the system are in the stable subspace spanned by the $B$ system in the absence of shocks. Over time, however, households are awakened with a Calvo Poisson rate of $\lambda$ to the fact that a shock has perturbed the economy from its non-stochastic steady-state. The linearized average beliefs about prices and the distribution then evolve according to

$$\frac{d\overline{\mu}_t}{dt} = B_{\mu V} V_t + B_{\mu V}\overline{\mu}_t + B_{\mu p}\overline{p}_t + \lambda(\mu_t - \overline{\mu}_t)$$

$$\frac{d\overline{p}_t}{dt} = B_{pV} V_t + B_{pV}\overline{\mu}_t + B_{pp}\overline{p}_t + \lambda(p_t - \overline{p}_t).$$

$$\frac{d\overline{V}_t}{dt} = B_{VV}\overline{V}_t + B_{V\mu}\overline{\mu}_t + B_{Vp}\overline{p}_t + \lambda(\widehat{V}_t - \overline{V}_t).$$

with the initial conditions $\overline{p}_0 = 0$, $\overline{\mu}_0 = 0$, and $\overline{V}_0 = 0$ if the agents start with the belief that no shocks have occured such that they are in the non-stochastic steady-state. As discussed in the preceding sections, the distribution evolves according to the average control choices induced by the average belief. These affect the prices in the economy, which are determined via linearized market clearing conditions. Given the actual distribution (and the actual value of other macroeconomic variables), prices thus solve the same fixed point problem that they do in rational expectations – except that now, they must be consistent with market clearing under the evolution of control variables chosen with the non-FIRE belief-averaged value function:

$$A_{pp}p_t + A_{p\mu}\mu_t + A_{pV}\overline{V}_t = Qdp/dt$$

Actual prices in turn determine the actual decision problem for the full information value function $\widehat{V}$, which is sufficient for updating the average belief value function $\overline{V}$.

Altogether, the new system for the sticky expectation economy is:

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbb{E}[d\widehat{V}] \\ d\mu \\ \mathbb{E}[dp] \\ d\overline{V} \\ d\overline{\mu} \\ d\overline{p} \end{bmatrix} = \begin{bmatrix} A_{VV} & A_{V\mu} & A_{Vp} & 0 & 0 & 0 \\ 0 & A_{\mu\mu} & A_{\mu p} & A_{\mu V} & 0 & 0 \\ 0 & A_{p\mu} & A_{pp} & A_{pV} & 0 & 0 \\ \lambda I & 0 & 0 & B_{VV} - \lambda I & B_{V\mu} & B_{Vp} \\ 0 & \lambda I & 0 & B_{\mu V} & B_{\mu\mu} - \lambda I & B_{\mu p} \\ 0 & 0 & \lambda I & B_{pV} & B_{p\mu} & B_{pp} - \lambda I \end{bmatrix} \begin{bmatrix} \widehat{V} \\ \mu \\ p \\ \overline{V} \\ \overline{\mu} \\ \overline{p} \end{bmatrix} dt. \tag{21}$$

This modified system can then be solved with standard methods to determine the dynamics of an economy under a sticky information structure with a constant learning rate of $\lambda$.

With just a few additional lines of code, it is possible to recast a FIRE model into a sticky-expectation environment. Similarly to the FIRE system, the model's jump variables are $\widehat{V}_t$ and $p_t$, minus whatever predetermined variables are present in $p_t$. The system therefore satisfies the Blanchard and Kahn (1980) conditions when the number of explosive eigenvalues matches the cardinality of $\widehat{V}_t$ and the non-predetermined variables in $p_t$.

To demonstrate how the solution method works – and that it does work – I provide two worked examples in the appendix. In the first example in Appendix B.1, I demonstrate the solution technique for a simple representative agent New Keynesian model for which a closed form solution is known, and show that my matrix arithmetic generates the correct closed-form answer. In the second example in Appendix B.2, I numerically solve a canonical sticky information HANK model with both my state-space methodology and the sequence-space methodology described in Auclert, Rognlie, and Straub (2020). I simulate a monetary policy shock and a fiscal transfer shock in both versions of the model and show that the two solutions agree up to a small approximation error that declines as the sequence-space time discretization shrinks.

# 4 The Medium-Scale HANK Model

In this section, I outline the medium scale model that I estimate using the original 7 Smets and Wouters (2007) series, plus data on unemployment and transfers as a share of GDP. Most households participate in an incomplete markets setting, but the profits from managing the physical capital stock are rebated to a representative financier agent that follows a standard Euler equation. My framework is therefore conceptually similar to two agent New Keynesian (TANK) models – except instead of a hand-to-mouth mass of households, the economy has an entire heterogeneous agent block whose decisions dominate the consumption dynamics, as the financiers receive only 6% of aggregate income and have strong smoothing motives.

Time $t \geq 0$ is continuous. Markets are incomplete for an ex-post heterogeneous continuum of households, who pay and receive taxes and transfers to and from the government and trade a single non-contingent liquid asset (government bonds) to smooth consumption and insure themselves from idiosyncratic shocks to productivity and employment. Employed households earn wages in the labor market, while unemployed households receive unemployment benefits that are a fraction of their steady-state wage rate. All households work the same number of hours. Each household's expectations about macroeconomic variables are sticky, such that they only update their forecasts of macroeconomic variables periodically.

A frictional labor market connects unemployed job seekers to vacancies posted by labor agencies using an aggregate matching function. Wages are set via a flexible rule that slowly adjusts to labor market tightness, hours worked per worker, past wages, and inflation. Labor agencies then rent labor to monopolistically competitive intermediate firms with nominal rigidities, who produce a differentiated output good by combining effective hours worked and rented capital. These intermediate outputs are combined by final goods sector, which sell the resulting composites to households for consumption.

The capital stock is owned and managed by a competitive mutual fund sector, which pays adjustment costs for changes to the path of investment and utilization costs for the intensity of capital usage. These mutual funds are in turn owned by a representative block of financier households, who have access to complete markets and smooth consumption according to an Euler equation. Since financiers can participate in all markets, a single rate of return prevails in the economy. In addition, I assume that the financiers are attentive and operate with full information and rational expectations.

The government consists of a treasury and a central bank. The treasury pursues passive fiscal policy in the sense of Leeper (1991), but pays down its debt only very slowly. Similarly, the central bank pursues active monetary policy by adjusting nominal interest rates more than one-for-one with inflation according to a smoothed Taylor rule.

In what follows, I detail the model as it would be solved under a rational expectations solution, with the understanding that the linearized FIRE solution can be converted into a sticky expectations one using the methodology of Sections 2 and 3.

### 4.1 Households

A continuum of households is ex-post heterogeneous in terms of their liquid (risk-free) liquid real assets $a$, their idiosyncratic labor productivity $z$, and their employment state $\ell \in \{E, U\}$, where $E$ stands for an employed labor market state and $U$ stands for an unemployed labor market state. Idiosyncratic labor market productivity evolves according to a log Ornstein-Uhlenbeck process driven by a Brownian motion $W_t$:

$$d \log z_t = \theta_z \log z_t + dW_t.$$

Post-tax labor market income is denoted $y_t$, which depends on the household's labor market state and labor-augmenting productivity:

$$y_t(a, z, l) = \begin{cases} (1-\tau) z w_t h_t & \text{if } l = E, \\ \kappa_u (1-\tau) z w_{NSS} & \text{if } l = U \end{cases}$$

such that unemployment benefits are set via $\kappa_u$ to replace 40% of household labor income in steady-state. Lastly, households earn a rate of return of $\tilde{r}_t$ on their bonds, which is equal to

$$\tilde{r}_t = r_t + \zeta_{r,t}$$

where $r_t$ is the real interest rate related to the central bank's policy rate $i_t$ via the Fisher relation $r_t = i_t - \pi_t$. $\zeta_{r,t}$ is analogous to the bond premium shock of Smets and Wouters (2007).

In this framework, households maximize their expected constant relative risk averse (CRRA) utility with a risk aversion coefficient of $\gamma$. A household with beliefs indexed by $i$ will have a value function that takes the form

$$\begin{aligned} V_t(a, z, \ell) = & \max_{(c_s^i)s \geq t} \mathbb{E}_t^i \left[ \int_t^\infty e^{-\rho(s-t)} \frac{(c_s^i)^{1-\gamma} - 1}{1-\gamma} ds \right] \\ \text{s.t. } & \frac{\mathbb{E}_t^i [da_t]}{dt} = y_t(z, l) + \tilde{r}_t a_t - c_t^i + T(a, z) \\ & d \log z_t = \theta_z \log z_t + dW_t. \end{aligned}$$

Using the methodology of Sections 2 and 3, it suffices to write the households' full information rational expectations (FIRE) problem, with the understanding that it can be later converted into the sticky expectation framework. Households *believe* that they are solving the following Hamilton Jacobi Bellman (HJB) equation:

$$\begin{aligned} \rho V_t(a, z, l) = & \max_c \left\{ \frac{c^{1-\gamma} - 1}{1-\gamma} + \partial_a V_t(a, z, l)(\tilde{r}_t a + y_t(a, z, l) - c) + T_t(a, z) \right\} \\ & + \partial_z V_t(a, z, l) z \left[ \frac{1}{2} \sigma_z^2 - \theta_z \log(z) \right] + \frac{1}{2} \sigma_z^2 z^2 \partial_z^2 V_t(a, z, l) \\ & + \lambda_{ll',t} [V_t(a, z, l') - V_t(a, z, l)] + \partial_t V_t(a, z, l) \\ \text{s.t. } & a \geq 0. \end{aligned} \tag{22}$$

In the FIRE setting, the distribution of households over the idiosyncratic state space will evolve according to the Kolmogorov Forward Equation

$$\partial_t \mu_t(a,z,l) = -\frac{\partial}{\partial a}\left[\frac{da_t}{dt}\mu_t(a,z)\right] - \frac{\partial}{\partial z}\left[\frac{\mathbb{E}_t[dz_t]}{dt}\mu_t(a,z)\right] - (\lambda_{l,l',t}\mu_t(a,z,l) - \lambda_{l',l,t}\mu(a,z,l')). \tag{23}$$

### 4.2 Final and Intermediate Goods Firms

As is standard in the New Keynesian literature, a competitive final goods sector produces output with a constant elasticity of substitution $\varepsilon$:

$$Y_t = \left[\int y_t(j)^{\frac{1-\varepsilon}{\varepsilon}} dj\right]^{\frac{\varepsilon}{1-\varepsilon}}$$

where monopolistically competitive intermediate goods firms hire capital and labor to produce output $y_t(j)$, such that firm $j$ produces:

$$y_t(j) = e^{\zeta_t^{TFP}} K_t(j)^{\alpha} L_t(j)^{(1-\alpha)}.$$

Intermediate good firms choose their prices while internalizing the demand for their output and price adjustment costs as in Rotemberg (1982). To generate something like price indexing in continuous time, however, I assume that firms only pay to adjust costs that generate a deviation of their price growth from the previous backward-looking trend of aggregate inflation. That is, I assume they pay costs

$$\Theta\left(\frac{\dot{p}_t(j)}{p_t(j)} - \bar{\pi}_t\right) = \frac{\theta_\pi}{2}(\pi_t(j) - \bar{\pi}_t)^2 Y_t$$

where $\bar{\pi}_t$ is a moving exponential average of past aggregate inflation with decay rate $\kappa_\pi$:

$$\frac{d\bar{\pi}_t}{dt} = -\kappa_\pi(\bar{\pi}_t - \pi_t). \tag{24}$$

The recursive problem for profit maximizing firms is then the HJB

$$\begin{aligned} r_t J_t^F(p,\bar{\pi}_t) = \max_{\pi_t(j)} \Bigg\{ &\left(\frac{p_t(j)}{P_t} - m_t\right)\left(\frac{p_t(j)}{p_t}\right)^{-\varepsilon} Y_t - \frac{\theta_\pi}{2}(\pi_t - \bar{\pi}_t)^2 Y_t \\ &+ \partial_p J_t^F(p_t(j),\bar{\pi})\pi_t(j)p_t(j) + \partial_{\bar{\pi}} J_t^F(p_t(j),\bar{\pi}_t)\kappa_\pi(\pi_t - \bar{\pi}_t) + \partial_t J_t(p_t(j),\bar{\pi}_t)\Bigg\}. \end{aligned}$$

In the model appendix, I show that the symmetric equilibrium yields a backward-indexed Phillips Curve:

$$\frac{\mathbb{E}_t[d\pi_t]}{dt} = r(\pi_t - \bar{\pi}_t) + \frac{\varepsilon}{\theta_\pi}\left[\frac{\varepsilon - 1}{\varepsilon} - \mathrm{mc}_t\right] - \kappa_\pi(\bar{\pi} - \pi) + \zeta_{\pi,t} \tag{25}$$

where $\mathrm{mc}_t$ is the marginal cost of production $\mathrm{mc}_t(j) = \exp(-\zeta_t^{\mathrm{tfp}})\left(\frac{r_t^k}{\alpha}\right)^{\alpha}\left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}$. Lastly, $\zeta_{\pi,t}$ is a price markup shock, again in the style of Smets and Wouters (2007).

### 4.3 The labor market

Search in the labor market is facilitated with a standard Cobb-Douglas constant returns to scale matching function, as in much of the literature based on Diamond (1982) and Mortensen and Pissarides (1994). Writing $\iota < 1$ as the elasticity of matches with respect to the unemployment rate, the job finding rate $\lambda_{UE,t}$ and the vacancy filling rate $\lambda_{v,t}$ are respectively

$$\lambda_{UE,t} = \Psi\theta_t^{1-\iota}, \quad \lambda_{v,t} = \Psi\theta_t^{-\iota} \tag{26}$$

where $\Psi$ is the efficiency of the matching function and $\theta_t \equiv \frac{v_t}{u_t}$ is labor market tightness, the ratio of the mass of vacancies $v_t$ to the unemployment rate $u_t$.

Hiring in the labor market is managed identical labor agencies indexed by $j$. Each one hires such that their number of employees $N_t(j)$ evolves according to

$$\dot{N}_t(j) = \lambda_{v,t}v_t(j) - \lambda_{EU}N_t(j)$$

Worker hours are differentiated with a CES demand system by intermediate firms, in which $\varepsilon_\ell$ is the elasticity of substitution between workers. If hiring firm $j$ internalizes a downward-sloping demand curve for its laborers' hours $h_t(j)$, then a CES demand system implies

$$h_t(j) = \left(\frac{r_t^l(j)}{r_t^l}\right)^{-\varepsilon_\ell} h_t.$$

From there, I assume a labor market hiring structure that is similar to Bardóczy and Guerreiro (2024), which in turn is based off of Christiano, M. S. Eichenbaum, and Trabandt (2016): a representative labor agency firm pays costs $\eta_v$ for creating a vacancy, and the cost $\eta_l$ for actually filling the vacancy (essentially, the on-boarding costs of hiring). Besides the imperfect substitutability of workers' hours, I make one notable addition to dampen firms' ability to regulate the hours of existing workers instead of adding new hires: I add a quadratic adjustment cost to deviations in hours worked from the non-stochastic steady state (normalized to 1). These costs stand in for the negotiating costs that keep firms from effortlessly changing the number of labor hours a job demands, forcing it to hire or fire rather than instantly restructure employment agreements. The recursive problem for profit-maximizing labor agencies is therefore

$$\begin{aligned} r_t J_t(N_t(j)) = \max_{v_t(j), r_t^l(j)} \Bigg\{ & (r_t^l(j) - w_t)\left(\frac{r_t^l(j)}{r_t^l}\right)^{-\varepsilon_l} N_t(j) Z_t h_t - \frac{\phi_h}{2}(h_t(j) - 1)^2 N_t(j) Z_t \\ & - \eta_v v_t(j) - \eta_l \lambda_{v,t} v_t(j) + J_t'(N_t(j))[\lambda_{v,t}v_t(j) - \lambda_{EU}N_t(j)] + \partial_t J_t(N_t(j)) \Bigg\}. \end{aligned}$$

In the Appendix C.1, I show that the vacancy supply problem in a symmetric equilibrium yields the dynamic equation

$$\frac{\mathbb{E}_t[d\theta_t]}{dt}\frac{1}{\theta_t} = \frac{1}{\iota}(r_t + \lambda_{EU})\left[1 + \frac{\eta_l}{\eta_v}\lambda_{v,t}\right] - \frac{1}{\iota}(r_t^l - w_t)h_t\left(\frac{\lambda_{v,t}}{\eta_v}\right) \tag{27}$$

while the rental rate of labor will satisfy

$$r_t^l = \frac{\varepsilon_l}{\varepsilon_l - 1}(w_t + \phi_h(h_t - h^*)). \tag{28}$$

Similar to Bardóczy and Guerreiro (2024), I assume real wage growth follows an ad-hoc specification:

$$\frac{dw_t}{dt}\frac{1}{w_t} = (1-\rho_w)\left(\phi_{w,\theta}\Delta\theta_t + \phi_{w,h}\Delta h_t - \Delta w_t\right) - \rho_w \pi_t + \zeta_{w,t} \tag{29}$$

where $\Delta$ variables represent deviations from the non-stochastic steady-state, $\rho_w$ governs the smoothness of wage adjustment and the degree of erosion from inflation, and the $\phi_{w,h}, \phi_{w,\theta}$ coefficients regulate the passthrough of hours worked and labor market tightness into real wage growth. Lastly, $\zeta_{w,t}$ is an additive wage markup shock.

### 4.4 Capital Mutual Fund with Adjustment Costs

A representative competitive mutual fund sector manages the aggregate capital stock of the economy. To keep the setting similar to Smets and Wouters (2007) and previous papers in the literature, this firm pays adjustment costs for altering the rate of aggregate investment $I_t$ relative to a backward-looking moving average $\overline{I}_t$, with the costs denoted $\Phi(I_t, \overline{I}_t)$. It additionally pays costs $\psi_t(u_t^k)$ for utilizing capital, where $u_t^k$ is the capital utilization rate. $\zeta_t^I$ denotes a shock to the rate of transformation of investment into capital. As such, the representative mutual fund solves

$$\begin{aligned}
\Pi_0(I_0, K_0) = \max_{(I_t, u_t)_{t\geq 0}} \; & \mathbb{E}_0 \int_0^\infty e^{-\int_0^t r_\tau d\tau}\left[(r_t^k u_t K_t - I_t - \psi(u_t)K_t\right]dt \\
\text{s.t.} \; & \frac{dK_t}{dt} = e^{\zeta_t^I}[1 - \Phi(I_t, \bar{I}_t)]I_t - \delta K_t \\
& \frac{d\overline{I}_t}{dt} = (I_t - \overline{I}_t)
\end{aligned}$$

For tractability, I assume that the adjustment costs are quadratic and that the elasticity of utilization costs relative to steady-state utilization (normalized to 1) is $(1-\nu)/\nu$. As shown in Appendix C.2, the resulting dynamic equations are then

$$\begin{aligned}
r_t^k &= \psi'(u_t^k), \\
q_t^k e^{\zeta_t^I}\left[1 - \frac{\phi_I}{2}\left(\frac{x_t}{\overline{x}_t} - 1\right)^2 - \phi_I\left(\frac{I_t}{\overline{I}_t} - 1\right)\frac{I_t}{\overline{I}_t}\right] &= 1, \\
\frac{\mathbb{E}[dq_t^k]}{dt} &= (r_t + \delta)q_t^k - r_t^k u_t^k + \psi(u_t^k).
\end{aligned} \tag{30}$$

Here, $q_t^k$ denotes Tobin's marginal Q.

### 4.5 Financier Agents

Although the capital stock is managed by a mutual fund, that mutual fund receives a small share of GDP in profit, which must be received as income by another actor in the economy. In order to side-step questions of how those profits are distributed amongst households, and to neutralize how such profits could arbitrarily distort the cyclical nature of the economy, I assume that they are received by another kind of agent. In a departure from other HANK papers in the literature, I assume that a representative household owns the mutual fund consumes the entirety of their dividend income in the non-stochastic steady-state. Outside of the steady-state, they have access to complete financial markets (including the government debt market) and thereby choose consumption $c_t^{\mathrm{FIN}}$ in accordance with an Euler equation with habit formation:

$$\frac{\mathbb{E}_t[dc_t^{\mathrm{FIN}}]}{dt} = \frac{1}{\gamma}(r_t - \rho^{\mathrm{FIN}})(c_t^{FIN} - \beta^{\mathrm{FIN}}\bar{c}_t^{\mathrm{FIN}}) - \beta^{\mathrm{FIN}}(\bar{c}_t^{\mathrm{FIN}} - c_t^{\mathrm{FIN}}) \tag{31}$$

where the external habit $\bar{c}_t^{\mathrm{FIN}}$ accumulates according to

$$\frac{d\bar{c}_t^{\mathrm{FIN}}}{dt} = (c_t^{\mathrm{FIN}} - \bar{c}_t^{\mathrm{FIN}}). \tag{32}$$

The derivation of the system is in Appendix C.3. Here, $\rho^{\mathrm{FIN}}$ is equal to the steady-state interest rate by assumption, making the financier agents more patient than the rest of the heterogeneous agent block, while $\beta^{\mathrm{FIN}}$ moderates how financiers value present consumption relative to the retrospective moving average of other financier agents' consumption.

Because the value of these dividends is small (less than 5% of GDP in the steady-state) and are received by what are essentially very wealthy, insured households with very low marginal propensities to consume, their fluctuations have very little impact on current consumption and income.

### 4.6 Government Policy

#### 4.6.1 Monetary Policy

Monetary policy is managed by a central bank according to a standard delayed Taylor rule. Nominal interest rates $i$ evolve according to

$$di_t = \rho_i(r_{nss} + \phi_\pi \pi_t + \zeta_{MP,t} - i_t)dt + d\delta_{MP,t}. \tag{33}$$

For flexibility, nominal interest rates are affected both gradually through $\zeta_{MP,t}$ and immediately through $d\delta_{MP,t}$, the initial condition of $\zeta_{MP,t}$; the latter allows the interest rate to jump on impact. Monetary policy's responsiveness to inflation is moderated by the Taylor rule parameter $\phi_\pi$. The smoothing parameter $\rho_i$ governs the baseline speed at which interest rates catch up to what a static Taylor rule would proscribe.

As a baseline, I consider active monetary policy in the sense of Leeper (1991) by setting $\phi_\pi = 1.5$, a standard value in the literature. To consider the model economy under an alternative "passive" monetary policy configuration, I consider an interest rate peg that sets $\phi_\pi = 0$.

### 4.6.2 Fiscal Policy

Fiscal policy is managed by the Treasury, which is also assumed to follow a rule to stabilize the real value of government debt – or not, depending on the policy configuration. The government can also exogenously induce transfer "shocks" $\zeta_T$, effectively unforeseen stimulus payments sent to the heterogeneous household block.

The total government debt position $B_t$ evolves according to

$$\frac{dB_t}{dt} = [-(\text{Tax}_t - G_t) + r_t B_t]dt \tag{34}$$

where $\text{Tax}_t$ represents net tax revenue to the government and $G_t$ represents real government expenditures. The former term consists of labor income taxes, minus unemployment insurance claims, minus total transfers to the public $T_t$:

$$\text{Tax}_t = \underbrace{\tau w_t L_t}_{\substack{\text{Income}\\ \text{taxes}}} - \underbrace{\kappa_u w_{nss} L_{nss}}_{\text{UI claims}} - \underbrace{\sum_\ell \int\int T_t(a,z)\mu_t(a,z,\ell)da\ dz}_{\text{Total transfers}} \tag{35}$$

Non-UI transfers are rebated to households in the joint wealth and income space according to

$$T_t(a,z) = T_{nss} - \kappa_{Fiscal}\frac{z}{Z_t}(B_t - B_{nss}) + 4Y_{nss} \times \zeta_T. \tag{36}$$

Equation (36) implies that the government automatically adjusts the net lump-sum transfers that it pays out to households according to a rule that stabilizes the government's debt asymptotically so long as $\kappa_{fiscal}$ sufficiently exceeds the government's interest rate $r_t$. To reduce the effect of the transfers on the agents, however, the reduction in transfers (effectively, the debt stabilization tax) is proportional to the household's steady-state labor income. Steady-state transfers are set to balance the budget in the absence of macroeconomic shocks.

Transfer shocks $\zeta_T$ to all (non-financier) households are scaled by four times steady-state output. This gives the interpretation that a shock of $\zeta_t = 0.01$ is equivalent to transfers worth 1% of steady-state real GDP.

## 4.7 Market Clearing

For the labor market to clear, labor demand must equal effective aggregate labor supply:

$$L_t = \int\int z_t h_t \mu_t(a,z,\ell = E)da\ dz \tag{37}$$

Note that $h_t$ is not indexed by the idiosyncratic variables, as all employed households work the same number of hours to satisfy labor demand. Similarly, the unemployment rate $u_t$ must aggregate from the heterogeneous agents' distribution:

$$u_t = \int \int \mu(a, z, \ell = U) da\ dz$$

For the goods market to clear, total aggregate expenditures must equal aggregate income:

$$Y_t = (C_t + c_t^{\text{FIN}}) + I_t + G_t.$$

As in Marcus Hagedorn, Manovskii, and Mitman (2019), I assume that the frictional capital utilization and Rotemberg adjustment costs are "virtual," and do not contribute to the resource constraint or aggregate expenditures. Equivalently, one could assume that adjustment the costs for some firms are pure profit for others and therefore amount to a transfer that is net zero out when rebated back to households or financiers.

By Walras' law, the asset market will also clear, completing the mathematical description of the model.

# 5 Calibration and Estimation

In this section, estimation comes in three parts. First, I calibrate the model to be in line with the existing literature in the non-stochastic steady-state. A key feature is that the model matches the profile of marginal propensities to consume of Fagereng, Holm, and Natvik (2021). In the second step, I estimate the structural parameters of the model by matching the IRFs of the model to local projections estimates of a narratively identified monetary policy shock. This ensures that the model roughly matches the empirical dynamics of output, consumption, investment, inflation, unemployment, hours worked, and wage growth, for at least the effects of monetary policy. In the third step, I estimate the remaining parameters related to the stochastic shocks using a Bayesian method known as Sequential Monte Carlo (SMC).

## 5.1 Calibration and the Non-Stochastic Steady-State

Table 1 details the parameters of the model that are directly calibrated, along with the source for the parameters or the targeted values of the calibration. Most of these pertain to terms that directly influence the non-stochastic steady-state. The model is calibrated for a quarterly frequency.

In the heterogeneous household block, $\rho$ is set so that the asset market clears when $r = 2\%$ annually. The idiosyncratic income process mean reversion term $\theta_z$ and variance term $\sigma_z^2$ are calibrated to match micro data on the autocovariance of wages detailed in Floden and Lindé (2001), as is done in McKay, Nakamura, and Steinsson (2016) and Kwicklis (2025).

The labor market parameters are targeted so that the steady-state labor market tightness is $\theta_{nss} = 1$. Under this assumption, the parameter governing the efficiency of the Cobb-Douglas matching function in the labor market may be read as exactly the inverse of the average unemployment duration. By virtue of matches occurring according to a Poisson process, unemployment spells in the model are exponentially distributed. I pick an average duration of 1.15 quarters (14.95 weeks), roughly the average duration of the 1990s according to data from the Current Population Survey. For the match function's elasticity with respect to the unemployment rate, I use the empirical estimates of Pissarides (1986). I pick the separation rate to target a 4% level of unemployment in the absence of aggregate shocks; the average worker stays at a job for 27.6 quarters.

I assume that total vacancy posting costs are equal to 11% of the average worker's quarterly wage bill. These vacancy costs are split 51%-49% between the regular posting cost $\eta_v$ and the vacancy filling cost $\eta_l$.

For the quadratic hours adjustment costs, I set $\phi_h$ to be 0.5. This is roughly the value that generates a transitory 5 basis point reduction in the trough of the unemployment rate following a 25 basis point reduction in the policy rate, a value roughly consistent with the empirical impulse response functions of the next section.

**Table 1:** Calibrated HANK Model Parameters

| Parameter | Symbol | Value | Source or Target |
|---|---|---|---|
| *Households* | | | |
| **Internally Calibrated:** | | | |
| Quarterly Time Discounting | $\rho$ | 0.049 | $r = 2\%$ Annually |
| Idiosyncratic Income Shock Variance | $\sigma_z^2$ | 0.017 | Floden and Lindé (2001) |
| Idiosyncratic Shock Mean Reversion | $\theta_z$ | 0.034 | Floden and Lindé (2001) |
| Elasticity of Intermediate Subst. | $\varepsilon$ | 10 | 10% profit share |
| **Assumed from Literature:** | | | |
| Relative Risk Aversion | $\gamma$ | 2.0 | McKay et al (2016) |
| *Labor Market* | | | |
| Matching function efficiency | $\Psi$ | 0.86 | Avg. unemp. duration of $\approx 15$ weeks |
| Elasticity of matching w.r.t. $u$ | $\iota$ | 0.70 | Pissarides (1986) |
| UI replacement rate | $\kappa_u$ | 0.40 | UI benefits pay 40% of wages |
| Flat vacancy posting cost | $\eta_v$ | 0.10 | 5.7% of quarterly wage bill |
| Intensive vacancy posting cost | $\eta_l$ | 0.1218 | 5.4% of quarterly wage bill |
| Exogenous job loss rate | $\lambda_{EU}$ | 0.0362 | Target steady-state $u$ of 4% |
| Labor adjustment cost parameter | $\phi_h$ | 0.5 | Rate hike to peak unemp. passthrough of 20% |
| *Capital Market and Production* | | | |
| Capital depreciation rate | $\delta$ | 0.02075 | 8.65% annual depreciation rate |
| Elasticity of $Y$ w.r.t. $K$ | $\alpha$ | 0.33 | 60% labor share of income |
| Capital utilization cost | $\iota$ | 0.45 | Elasticity of $u_t^k$ to $r_t^k$ of 0.81 |
| *Financier Agents* | | | |
| Financier time discounting | $\rho^{\text{FIN}}$ | 0.005 | $r = 2\%$ annually |
| Financier habit weight | $\beta^{\text{FIN}}$ | 0.70 | Smets and Wouters (2007) prior mean |
| *Government* | | | |
| Steady state government debt | $B_{NSS}$ | 2.37 | HANK $iMPC_0 \approx 0.45$ |
| Income Tax Rate | $\tau$ | 0.30 | Approx. US tax wedge (OECD 2025) |
| Government share of expenditure | 0.15 | $G/Y$ | |
| Taylor rule coefficient | $\phi_\pi$ | 1.5 | Active monetary policy |
| Fiscal adjustment coefficient | $\kappa_{fiscal}$ | 0.05 | Passive fiscal policy |

The marginal distribution of liquid assets is depicted in the first panel of Figure 2. The unemployed households' asset distribution (in red) is shifted significantly closer to the borrowing constraint compared to the employed population (in black), where 19% of unemployed households in the model have completely exhausted their savings. The total bond position of the heterogeneous agent block is chosen to bring the average inter-temporal MPC (iMPC) out of a one-time increase in liquid wealth to 0.44 after one year, close to the estimates of Fagereng, Holm, and Natvik (2021) reproduced in Auclert, Rognlie, and Straub (2024). As shown in the second panel of Figure 2, the iMPCs decline thereafter, also broadly in line with the empirical estimates. In the last panel of Figure 2, I document how the households' one-year MPCs out of liquid wealth in the cross section decline in liquid wealth position and labor market income.

Outside of the non-stochastic steady-state, I study the conventional active monetary/passive fiscal framework, in the terminology of Leeper (1991). As such, I set the Taylor rule coefficient to $\phi_\pi = 1.5$, a standard value set in the literature. For a stationary linearized solution to exist, the government must then passively adjust its tax schedule to keep real government debt from exploding. As such, I set the fiscal adjustment parameter to $\kappa_{fiscal} = 0.05$, such that government debt above steady-state levels is paid back by raising net taxes only very gradually over time.

## 5.2 IRF Matching using a Monetary Policy Shock

I integrate the model to a discrete-time quarterly level using the methodology of Christensen, Neri, and Parra-Alvarez (2024). As in Christiano, M. Eichenbaum, and Evans (2005) and many other papers, I fit many of the model's remaining structural parameters by matching the model's impulse response functions (IRFs) to empirical estimates of the effect of a monetary policy shock. I follow a similar strategy as in Auclert, Rognlie, and Straub (2020): I use an identified C. D. Romer and D. H. Romer (2004) narrative shock to calculate the monthly impulse response function using the local projections method of Jordá (2005). As in V. Ramey (2016), I control for two lags of the narrative shock, the unemployment rate, the CPI, the GDP deflator, the commodity Producer Price Index, and an index of total U.S. industrial production, and the federal funds rate (FFR). I also include contemporaneous levels of the macroeconomic variables, except for the FFR (and, of course, the Romer and Romer (2004) shock). All real variables are deflated using the GDP deflator.

For the dependent variables in the projection, I use the logs of real GDP as total output, log total real personal consumption expenditures (PCE) as consumption, the log of real investment as $I$, the log of the PCE price index as $p$, and the log of real nonfarm hourly compensation and total nonfarm hours worked as $w$ and $h$. I use the headline unemployment rate to measure $u$. I measure nominal

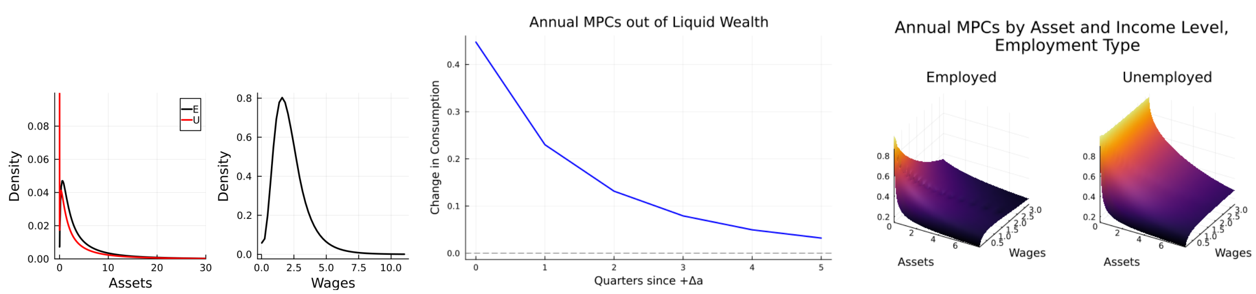


**Figure 2:** Leftmost panel: the distribution of assets and wages in the non-stochastic steady-state for employed workers (E) and unemployed households (U). Middle panel: the intertemporal MPCs (iMPCs) of households in the model out of liquid wealth after a one-time transfer at time 0 in the non-stochastic steady-state. Rightmost panel: the 1-year iMPCs out of liquid wealth in the non-stochastic steady-state, by liquid asset position, income, and employment type.

**Table 2:** HANK non-stochastic steady state quantities

| Description | Symbol | Value |
|---|---|---|
| First year iMPC out of liquid assets | | 0.443 |
| Debt to Annual Income | $B_{NSS}/(4Y_{NSS})$ | 0.170 |
| Correlation btw. Income and Assets | $\text{Corr}(a, z)$ | 0.688 |
| Share of households with $a = 0$ | $\int \mu_{NSS}(0, z)dz$ | 0.020 |
| Asset Gini Coefficient | | 0.648 |
| Income Gini Coefficient | | 0.334 |
| Unemployment Rate | $u_{nss}$ | 0.040 |
| Capital-to-Income Ratio | $K_{nss}/(4Y_{nss})$ | 2.883 |

interest rates using the end-of-month FFR. Like Auclert, Rognlie, and Straub (2020), I log-linearly interpolate variables reported at the quarterly frequency to obtain monthly series when necessary.

Gathering the controls in $X_t$, labeling the Romer and Romer (2004) shock as $RR_t$, and denoting outcome $j$ as $Y_t^j$, the local projection for outcome variable $j$ at forecast $h$ thus takes the form

$$Y_{t+h}^j = RR_t\beta_h^j + X_t'\eta_h^j + \varepsilon_{t,h}^j.$$

Like Auclert, Rognlie, and Straub (2020), I use the 1969-1996 time period – the time period covered by the original Romer and Romer (2004) paper – to generate my estimates. $\widehat{\beta}_h^j$ is then my estimate of the IRF of outcome $j$ at horizon $h$; I collect these estimates into a stacked empirical IRF vector $\widehat{\text{IRF}}$.

After obtaining the empirical estimates of the effect of a monetary shock, I estimate the model's parameters by minimizing a generalized method of moments-like objective function. Collecting the parameters to be estimated in $\Theta$ and writing the model impulse response function as $\text{IRF}(\Theta)$, the estimated parameters solve the problem

$$\widehat{\Theta} = \arg\min_{\Theta} \; (\text{IRF}(\Theta) - \widehat{\text{IRF}})'\widehat{\Sigma}^{-1}(\text{IRF}(\Theta) - \widehat{\text{IRF}}).$$

Here, the $\widehat{\Sigma}$ matrix contains the squared Newey and West (1987) standard errors of the empirical local projections estimates on its main diagonal and is zero elsewhere.

The GMM estimates of the model parameters are reported in Table 3. The local projection impulse response functions are presented in Figure 3; the shaded region represents a 95% confidence interval while the model impulse response functions are depicted in dark orange. All models are scaled to a 25 initial basis point drop in the federal funds rate.

**Table 3:** Estimated Model Parameters to Match IRFs

| Parameter | Symbol | Estimate | S.E. |
|---|---|---|---|
| Household update rate | $\lambda$ | 0.266 | (0.115) |
| Monetary policy smoother | $\rho_i$ | 0.375 | (0.028) |
| Monetary policy shock persistence | $\eta_{MP}$ | 0.0421 | (0.0088) |
| Capital Adjustment Cost | $\phi_k$ | 8.563 | (0.284) |
| Rotemberg Inflation Cost | $\theta_\pi/100$ | 37.829 | (0.145) |
| Real wage smoother | $\rho_w$ | 0.0912 | (0.0363) |
| Tightness to real wage feedback | $\phi_{w\theta}$ | 0.00350 | (0.00203) |
| Hours to real wage feedback | $\phi_{wh}$ | 0.813 | (0.120) |

Notably, the learning rate suggests that roughly a quarter of households have updated to a macroeconomic shock after one quarter, 40% have updated after two quarters, and nearly 65% have updated by the end of the year, a faster pace of learning than that estimated in Auclert, Rognlie, and Straub (2020).

The square root of the average mean square deviation between the orange and blue lines of 3 is 1.30 standard errors. In particular, the model struggles to fit the slow fall in unemployment captured by the local projection – hence why I calibrate $\phi_h$ to roughly match the magnitude of the drop, if not its timing. The model also generates an enormous amount of nominal rigidity, in order to acquiesce with the slow rise in the price level documented in the empirical estimates (a feature also depicted in the estimates of Auclert, Rognlie, and Straub (2020)). This implies that the effect of future output gaps on inflation movements is slow and muted in general.

### 5.3 Sequential Monte Carlo Estimation

Having estimated a subset of the structural model parameters using GMM and the empirical response to an identified monetary policy shock, I estimate the persistences and variances of the remaining shocks using a Bayesian technique known as Sequential Monte Carlo (SMC). SMC has recently gained prominence as a technique to estimate macroeconomic models; Herbst and Schorfheide (2014) demonstrates that SMC can better sample from the potentially irregularly shaped and multi-modal posteriors that arise in DSGE settings, while Cai et al. (2020) details how SMC can be used for efficient real-time estimation and forecasting. Acharya et al. (2023) uses the algorithm to estimate a full information medium-scale HANK model.

The reader can consult these papers for an in-depth technical discussion, but as a broad overview: SMC combines importance sampling with Metropolis-Hasting Markov Chain Monte Carlo methods and particle swarm techniques to sample from the model's posterior distribution. The user draws a large number of initial values ("particles") from the parameter space, often distributed according to the models' prior distribution, and constructs importance weights to sample from a series of bridge distributions. These bridge distributions are typically a log-linear weighting of the prior and likelihood

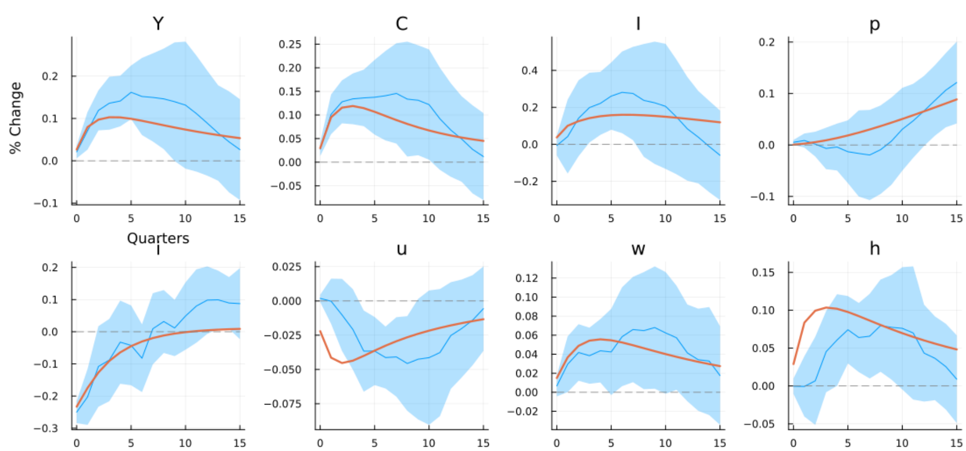


**Figure 3:** Model and local projection impulse response functions to a 25 basis point reduction in the nominal interest rate

that strongly weight the prior at initialization and gradually converge upon the model's true posterior. At each iteration, the algorithm conducts a Metropolis-Hastings step to change ("mutate") each of the particles stochastically – but weighted in the direction of increasing probability mass. The user periodically re-samples the particles from their distribution (constructed using their importance weights) in order to prevent any one particle from becoming too dominant. By the time the target distribution converges to the posterior, the particle swarm and its weights should be distributed according to the posterior, allowing the researcher to evaluate moments of the posterior by appealing to the law of large numbers.

I use the variant of the adaptive SMC algortithm presented in Cai et al. (2020) and employed in Acharya et al. (2023), which tunes the rate of bridge distribution convergence according to the effective sample size of the particles. I use standard state space Gaussian methods (e.g., the Kalman filter) to evaluate my likelihood function given the observable data. An advantage of SMC, compared to other Markov Chain Monte Carlo (MCMC) methods is that in addition to being robust to local but suboptimal posterior modes, the algorithm is highly parallelizable. For medium-scale HANK models where each likelihood evaluation can take a second or more, this is a highly useful property.

My measurement equation is similar to the one used in Acharya et al. (2023) and the classic Smets and Wouters (2007): I include i) real GDP per capita growth, ii) real personal consumption expenditures per capita growth, iii) real fixed private investment per capita growth (investment), iv) the change in the GDP deflator for inflation, v) the growth rate of real hourly compensation per worker, vi) the level of average weekly nonfarm hours per worker, and the level of the Federal Funds rate. The details of the construction of each series, taken from the Federal Reserve Economic Data (FRED) repository, are provided in the appendix. Each series in the measurement equation enters the model with its mean removed, as in Acharya et al. (2023) and Bayer, Born, and Luetticke (2024), to represent deviations from a log-linear balanced growth path and to avoid questions of balanced growth preferences in the HANK structure. In the training stage of the model, all data are from the United States over the period of 1965 to 2007, before the Great Financial Crisis, to assess the fit of the model in times of relative normalcy. I include post-crisis and post-pandemic data only to assess the model's fit over the latter periods.

In addition to the seven standard series, I introduce two new ones, which my model is able to describe: viii) the unemployment rate and ix) the level of transfers from the government to households as a share of GDP. I construct this last series as the current dollar value of government social benefits minus spending on Social Security, Medicare, and Medicaid, divided by nominal GDP.

To summarize the nine macroeconomic time series and their connection to theoretical quantities in the model, I write the measurement equation with all variables in terms of deviations from their steady-state:

$$\text{Meas. Equation}_t = \begin{bmatrix} \text{Real GDP growth}_t \\ \text{Real consumption Growth}_t \\ \text{Real investment Growth}_t \\ \text{Real wage growth}_t \\ \text{Hours worked}_t \\ \text{GDP Deflator Growth}_t \\ \text{FFR}_t \\ \text{Unemployment Rate}_t \\ \text{Transfers to GDP}_t \end{bmatrix} = \begin{bmatrix} \log(Y_t) - \log(Y_{t-1}) \\ \log(C_t) - \log(C_{t-1}) \\ \log(I_t) - \log(I_{t-1}) \\ \log(w_t) - \log(w_{t-1}) \\ \log(h_t) - \log(h_{t-1}) \\ \pi_t \\ i_t \\ u_t \\ (T_t/Y_t) \end{bmatrix} \tag{38}$$

where here I denote $T_t$ (without arguments) as $T_t = \sum_\ell \int \int T_t(a,z)\mu_t(a,z,\ell)da\ dz$. Note that for flow variables, I integrate the quantities in the continuous time model again to match the data, in the style of Christensen, Neri, and Parra-Alvarez (2024), and then conduct the log differencing.

The result is a system with nine shocks (to monetary policy, TFP, the household bond premium, the productivity of investment, the price markup, the wage rate, the productivity of investment, the labor market matching function efficiency, and fiscal transfers) and nine observables. I assume most of the shocks follow an Ornstein-Uhlenbeck (AR(1)) process, except for the price and wage shocks. For these, I assume that they follow a continuous time autoregressive moving average process (CARIMA) with a (2,1) lag order that sets the second autoregressive component to be negligible, in order to be roughly consistent with the discrete time structure of Smets and Wouters (2007). I map the discretely sampled CARIMA(2,1) processes to their discrete time analogues using the methodology of Thornton and Chambers (2011).

Table 4 summarizes the priors for the shock and variance parameters in the leftmost columns.

# 6 Results

## 6.1 The Power of Persistence

To understand the implications of the timing of output gaps for unemployment and inflation, I linearize equations (25) and (27), which describe the New Keynesian Phillips curve and the evolution of labor market tightness. Both equations are largely standard elements in DSGE models. I then use them to solve for the dynamics of inflation and unemployment in response to changes in intermediate firms' marginal costs and labor agencies' profits per worker. Appendixes D.2 and D.1 contain the details the analytical derivations. In a partial equilibrium exercise, I then feed exogenous jumps in marginal costs and profits into these solutions, where the driving impulses are composed of simple exponentially decaying functions of varying persistences.

In the first plot of Figure 4, I display the exogenous jumps in per worker profitability that I feed into the labor agency's problem. The net present value of the increases are adjusted to be equal across the experiments, but the auto regressive (AR) persistences of the impulses are varied and range from

**Table 4:** Priors and Posterior estimates from SMC

| Parameter | Symbol | Prior | | | Posterior | |
|---|---|---|---|---|---|---|
| | | Dist. | Mode | S. Dev | Mean | S. Dev |
| AR of $\zeta_{TFP}$ | $\rho_{TFP}$ | Beta | 0.5 | 0.11 | 0.40 | 0.033 |
| AR of $\zeta_r$ | $\rho_r$ | Beta | 0.5 | 0.11 | 0.91 | 0.010 |
| AR of $\zeta_I$ | $\rho_I$ | Beta | 0.5 | 0.11 | 0.31 | 0.0013 |
| AR of $\zeta_\pi$ | $\rho_\pi$ | Beta | 0.5 | 0.11 | 0.61 | 0.089 |
| AR of $\zeta_w$ | $\rho_w$ | Beta | 0.5 | 0.11 | 0.92 | 0.024 |
| AR of $\zeta_G$ | $\rho_G$ | Beta | 0.5 | 0.11 | 0.26 | 0.038 |
| AR of $\zeta_u$ | $\rho_u$ | Beta | 0.5 | 0.11 | 0.81 | 0.028 |
| AR of $\zeta_T$ | $\rho_T$ | Beta | 0.5 | 0.11 | 0.85 | 0.022 |
| MA of $\zeta_p$ | $\mu_p$ | Beta | 0.5 | 0.21 | 0.61 | 0.093 |
| MA of $\zeta_w$ | $\mu_w$ | Beta | 0.5 | 0.21 | 0.45 | 0.063 |
| S. Dev. of $\zeta_{MP}$ | $100\sigma_{MP}$ | Inv. Gamma | 0.033 | 2.0 | 3.8 | 0.28 |
| S. Dev. of $\zeta_{TFP}$ | $100\sigma_{TFP}$ | Inv. Gamma | 0.033 | 2.0 | 0.45 | 0.033 |
| S. Dev. of $\zeta_r$ | $100\sigma_r$ | Inv. Gamma | 0.033 | 2.0 | 4.8 | 0.49 |
| S. Dev. of $\zeta_I$ | $100\sigma_I$ | Inv. Gamma | 0.033 | 2.0 | 4.2 | 0.37 |
| S. Dev. of $\zeta_\pi$ | $100\sigma_\pi$ | Inv. Gamma | 0.033 | 2.0 | 3.0 | 0.31 |
| S. Dev. of $\zeta_w$ | $100\sigma_w$ | Inv. Gamma | 0.033 | 2.0 | 3.5 | 0.36 |
| S. Dev. of $\zeta_g$ | $100\sigma_g$ | Inv. Gamma | 0.033 | 2.0 | 5.1 | 1.22 |
| S. Dev. of $\zeta_u$ | $100\sigma_u$ | Inv. Gamma | 0.033 | 2.0 | 0.10 | 0.0091 |
| S. Dev. of $\zeta_T$ | $100\sigma_T$ | Inv. Gamma | 0.033 | 2.0 | 5.1 | 0.25 |

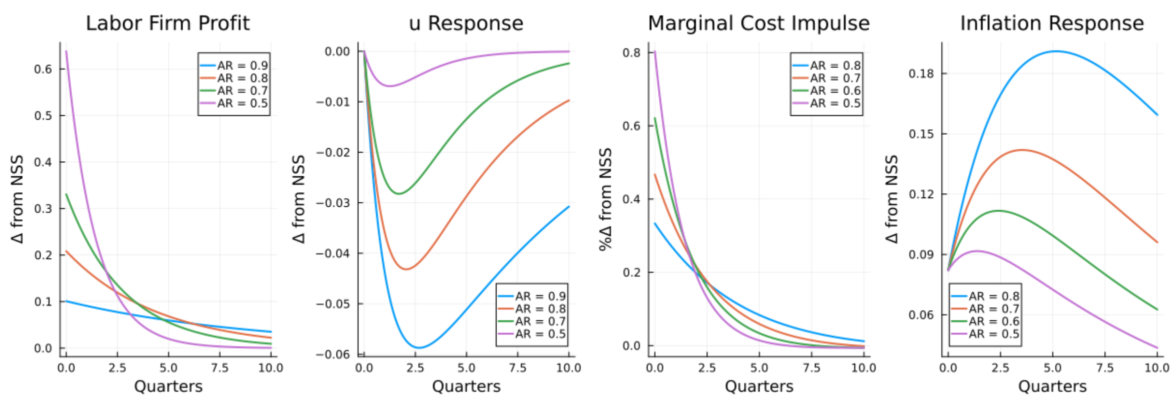


**Figure 4:** Impulse responses of unemployment and inflation in response to exogenous changes in worker profitability and total marginal cost.

AR(1) coefficients of 0.9 (highly persistent) to 0.5 (highly transitory). In the second plot, I display the resulting changes in the unemployment rate. Smaller but more persistent increases in per worker profit drive substantially more hiring and larger reductions in unemployment compared to large-but-fleeting profitability jumps.

I conduct a similar analysis of the equations that describe inflation. I generate jumps in marginal costs of varying persistence and normalize their net present value; I display these impulses in the third panel of Figure (4). The final plot in the panel depicts the effect of these marginal cost spikes on inflation.[7] Just as in the unemployment case, the response of inflation to increases in marginal cost is many times stronger when the persistence of the cost increase is larger, all else equal.

Both vacancy creation and inflation change according to strong forward-looking components in the model, even if the levels of unemployment and inflation have backward-looking components that induce a hump shape. Hiring and sticky price changes are an investment for firms that both take time and have persistent consequences. As such, even if stimulus checks drive large positive output gaps in the short term, forward-looking firms largely treat past output gaps as sunk. If output gaps fall rapidly or even become negative in the future after an initial peak, then this strongly attenuates the ability of stimulus checks to spur large movements in inflation and output compared to other macroeconomic events (like changes to monetary policy) that take years to fully play out.

### 6.2 Fiscal transfers with inattention

As documented in Auclert, Rognlie, and Straub (2020), greater degrees of inattention dampen the forward-looking general equilibrium effects that drive monetary policy, making it less stimulative. I show that the exact *opposite* is the case for deficit-financed fiscal transfers to households under an active monetary, passive fiscal policy mix. In Figure 5, I show the impulse responses in the model economy to a one-time increase in transfers equal to 1% of annual GDP on impact, which mean reverts with an AR coefficient of 0.16 to roughly simulate the speed at which transfers were disbursed during the COVID-19 pandemic. I plot a range of $\lambda$ values; the extreme case of $\lambda = 10$ essentially corresponds to the full information model, while $\lambda = 0.01$ gives a half-life for the non-updating share of the population of over 17 years. The estimates calculated using the learning parameter in 3 is plotted in a solid line.

When stimulus checks go out, hand-to-mouth households spend them immediately. Nominal rigidities are high in the estimated version of the model, so inflation is muted and the Federal Reserve makes only miniscule changes to the nominal interest rate. However, for forward-looking households on the margin of spending or saving, the departure from rational expectations dampens two important channels. First, households could internalize how future aggregate demand may be high from the surge in spending today, which further increases current aggregate demand as households attempt to dis-save or borrow against their future high income. Conversely, households may anticipate that future tax revenue may have to be raised in order to pay off the future debt incurred from the government deficit, the force behind Ricardian equivalence. As illustrated using a tractable RANK model in Gabaix (2020), stronger information frictions should therefore further break Ricardian equivalence, amplifying fiscal policy. As such, dampening the first channel weakens the stimulus effect of stimulus checks, while dampening the second strengthens it.

As shown in Figure 5, dampening the second effect is clearly far more important quantitatively – even despite the fact that debt stabilization taxes in my model are carried out in a progressive manner. After stimulus checks are sent out, aggregate spending and income rise at virtually every

[7] For this exercise, I increase the slope of the Phillips Curve to a standard 0.10, although the IRF matching exercise demands a baseline calibration with a far flatter curve. In the next sections, I explore robustness of the results to various NKPC slopes.

horizon. Households see the income in their bank accounts, but when they are less attentive to the implications of fiscal stabilization, the more powerful fiscal transfer policy becomes. Quantitatively, moving from the full information to the estimated sticky expectation model, the peak of the response of real GDP rises by a factor of one third. After the first year, the total cumulative sum of output gaps is three times higher with the estimated information frictions than without, for a fiscal transfer multiplier of 0.35 compared to 0.11.

Despite the stimulative potential of fiscal policy, however, it tends to have a highly muted effect on unemployment in the model (although the effect is still greater as information frictions increase). Hiring in the search-and-matching block of the model involves a costly investment that takes time, and the average employer-employee relationship lasts several years. The stimulative effect of stimulus checks, in contrast, largely play out over just a few quarters. As a result, while stimulus checks are effective at increasing consumption and output (and therefore welfare) in the model economy, most of the increase in labor demand under the estimated parameters is on the intensive margin, with firms increasing hours per worker (or alternatively, worker effort intensity). Compared to monetary policy, transitory fiscal transfers boost the extensive margin of the labor market, employment, only relative modestly. Indeed, because the government's debt is repaid gradually over a long period of time, the persistent fall in aggregate demand after the initial boost in the stimulus checks actually drives unemployment slightly higher – although the effects are quantitatively very small.

Similarly, fiscal transfers also contribute only slightly to inflation in the first few quarters, and actually become very weakly deflationary as time goes on and debt repayment begins. Part of the reason the magnitude is small is due to the very low Phillips Curve slope implied by the model estimates. However, even reducing the nominal rigidities 50-fold leads to only a transitory inflationary peak of roughly 0.05% when all of the other parameters are kept at their estimated values. Rather, much like the unemployment response, the lack of inflationary pressure from the transfers is mostly due to the transitory nature of their timing – a theme explored in Kwicklis (2025). Even with a backward-looking term, the Phillips Curve is forward-looking, and by the time firms are able to significantly adjust their prices, the output gaps of the fiscal stimulus are past and sunk, while the gradual (but prolonged) debt repayment lies in the future. Firms with nominal rigidities thus react less to the sharp stimulus than they do to the austerity period that follows.

### 6.3 Fiscal transfers and the time series

In Figure 6, I detail historical decompositions of the federal funds rate, real GDP growth, and inflation inflation for the United States using the nine exogenous shock processes detailed in the model. The HANK model is evaluated at the minimum-distance coefficients obtained from the monetary policy IRF matching and, for the parameters related to shock persistences and variances, at the posterior modes obtained from the SMC estimation. As in Smets and Wouters (2007), and to emphasize the policy-related shocks, I bin the bond premium, investment, and government expenditure shocks as "demand" shocks, and similarly bin the wage and price markup shocks together. For comparison, I also solve and integrate a continuous time RANK model similar to the model solved in Smets and Wouters (2007). The decompositions through the lens of the RANK model are also displayed in Figure

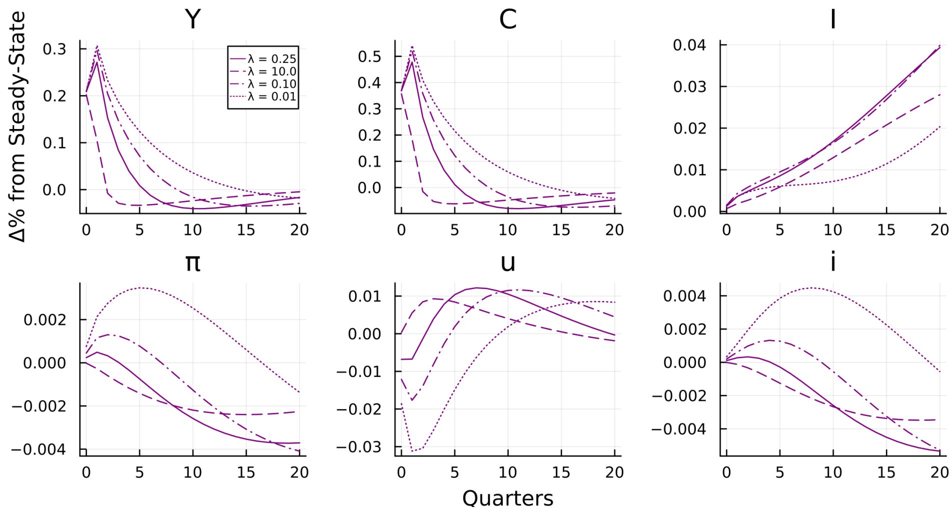


**Figure 5:** Impulse response functions to fiscal transfers to households with varying degrees of inattention. The solid line depicts the impulse response of the estimated model, while dashed and dotted lines represent alternative calibrations. Lower values of $\lambda$ correspond to slower rates of household updating.

6, on the right.

Despite the differences in the structure of the models, both the HANK and RANK frameworks tell a similar qualitative story regarding the U.S. macroeconomic policy following 1965. Both diagnose monetary policy as too "loose" in the 1970s and contributing to elevated levels of inflation, both identify contractionary interest rate hikes and subsequent disinflation following Paul Volker's appointment as Federal Reserve Chair in 1980, and both describe the 2010s as experiencing a large demand-side recession. For reasons previously discussed, the HANK model loads very little inflation and unemployment variation (the latter is not shown) onto stimulus transfers. Indeed, the HANK model does not attribute much GDP growth to fiscal transfers until the massive pandemic-era interventions.

Focusing on the period after 2020, the model suggests that stimulus checks significantly boosted the recovery. Figure 7 depicts the trajectory of real GDP, inflation, and unemployment following the 2020 recession, along with the relative contributions of fiscal and monetary policy to each line, according to the shocks after 2019 estimated via a Durbin and Koopman (2002) smoother. In keeping with the previous impulse response functions, the model suggests that the stimulus checks significantly boosted the recovery from the recession. In particular, the model predicts that by 2020q2 quarterly real GDP would have declined by 0.84 percentage points relative to 2019 had fiscal transfers checks not increased following the CARES act, roughly 9% of the realized decline. After the passage of the American Rescue Plan in the first quarter of 2021, real GDP per capita essentially returned to 2019 levels – but would have been roughly 2.5 percentage points lower in the model, had fiscal stimulus not taken place. Had this been this case, the economy would have taken an additional three months to return to its pre-2020 real GDP per capita levels.

To broadly assess how much worse the recession would have been without the stimulus checks, I integrate the difference between the realized GDP per capita line and then trend and compare that value to the blue area, the accumulated additional losses in the absence of the stimulus checks. Between the start of the recession and the start of 2022, the total sum of output losses is 33% larger in the

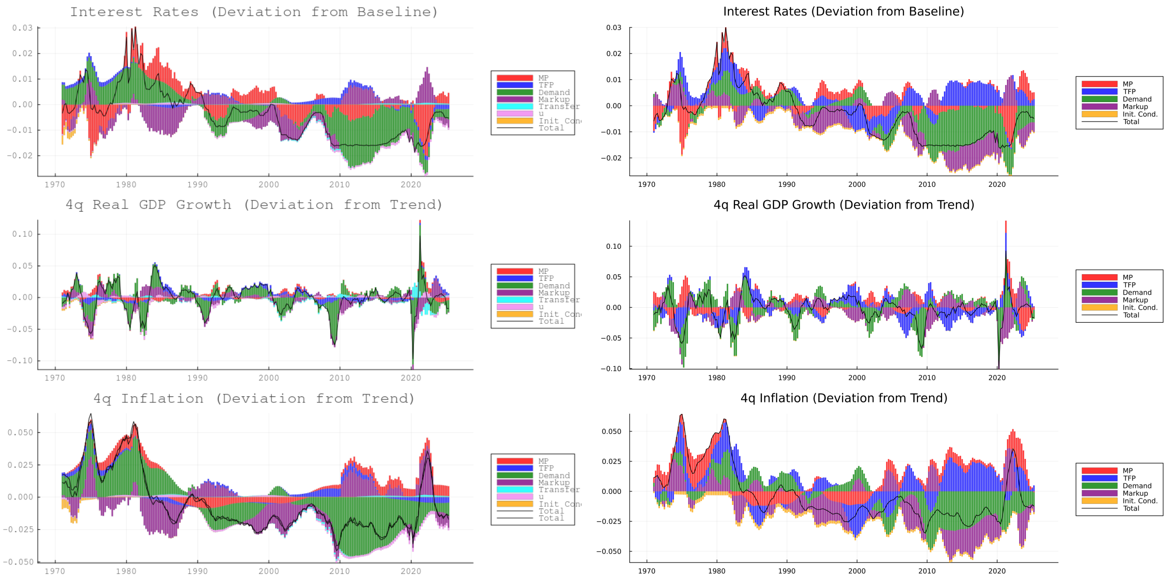


**Figure 6:** HANK (left) and RANK (right) historical shock decompositions of interest rates, output, and inflation.

simulated no-transfer scenario.

The second and third panels of 7 indicate that the model ascribes very little movement in the 4-quarter inflation rate and the unemployment rate to fiscal transfers. Instead, the first panel of Figure 6 reveals that the model prefers to load much of the variation during that period in prices onto markup shocks – i.e., shocks that are largely unrelated to aggregate demand and output gaps. This is unsurprising, as the local projections estimates force the model to adopt a very flat Phillips Curve. Even so, if one dramatically increases the slope of the Phillips Curve to 0.10, then fiscal transfers still only account for less than 15% of inflation's 6% above-trend peak in the second quarter of 2022.

In contrast to the blue shaded area of the effect of fiscal transfers, the red shaded areas in 7 denote the contribution of monetary policy to the macroeconomic time series. The monetary policy shocks are identified as deviations from the active slow-moving Taylor rule of equation (33); according to the estimated model, the Federal Reserve kept interest rates below the Taylor rule's proscription in the face of rising inflation and only reversed course in 2022, as shown in the graph on the left in 6. During this time, and taking it as given that the agents in the model reacted to each instance that rates were below the Taylor target with surprise, the monetary policy innovations drove an increasing share of the post-COVID expansion. The effects were too lagged to significantly change the dynamics of the macroeconomy until the United States was well into its recovery, however, and only contribute to a modest increase in inflation and decrease in the unemployment rate long after both had respectively peaked. Following the decline in inflation after 2024, however, the after-effects of loose monetary policy explains most of the remaining persistence of inflation above trend.

## 7 Conclusion

In this article, I demonstrate how to solve a linearized sticky-expectation HANK model in state-space by recycling the Jacobians obtained from the full information, rational expectations (FIRE) version

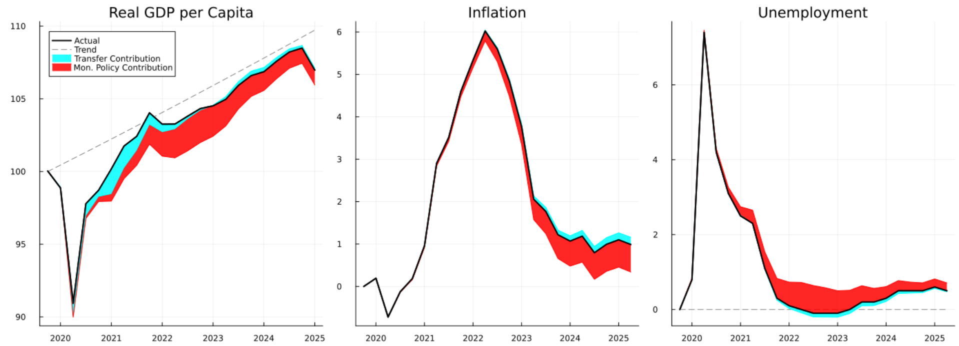


**Figure 7:** Decomposition of real GDP per capital, 4-quarter inflation, and the unemployment rate following 2020.

of the system. The approach is simple to implement and conveniently bridges the sticky expectation environment to the state space literature. I then use this technique to develop a medium-scale heterogeneous agent New Keynesian (HANK) model with search-and-matching employment frictions and sticky expectations that is both consistent with microdata on household marginal propensities to consume and with the slow moving empirical estimates of the effects of an identified monetary policy shock on output, consumption, wage growth, hours worked, the unemployment rate, and inflation.

I then use parameters that I estimate using an identified monetary shock to study the effects of fiscal policy in the estimated HANK model. I show that sharply transitory transfers like stimulus checks are substantially more effective in the sticky expectation environment than in the complete information one, as households in the former take time to realize the debt stabilization implications of the transfer policy. Despite this, the transitory nature of stimulus checks lead their inflationary impact to be limited, as the large output gaps that they produce are similarly transitory, while debt repayment is long. For this same reason, unemployment is little moved by stimulus policy, with capital intensity and worker hours and effort moving more sharply in response to the stimulus. Hiring and on-boarding new employees requires firms to invest substantial time and money, leading the unemployment rate to respond less strongly to shocks that come and go quickly.

Using the model to process macroeconomic time series, I find that movements in fiscal transfers played only a very small role in the post war variation in real economic aggregates. However, the linearized model suggests that the fiscal stimulus efforts during the COVID-19 pandemic substantially reduced the severity of the recession while contributing little to the ensuing surge in inflation or the fall in unemployment.

While my model provides a stylized mathematical framework to quantitatively analyze historical events, there are many directions of further investigation. First, my model is completely linear with respect to macroeconomic aggregates, and while it matches interest rates in the zero lower bound (ZLB) eras of the 21st century, it does so only by engineering unexpected interest rate movements as fresh shocks. It may be possible, however, to adapt my state space framework to study the ZLB in otherwise active monetary/passive fiscal policy regimes explicitly, as in Alves and Giovanni L Violante (2025), or to study policy regime switching models like Bianchi and Ilut (2017).

Further innovations to the general methodological technique outlined in this paper could also be easily made by introducing higher order terms in the style of Schmitt-Grohé and Uribe (2004) to the model to capture macroeconomic uncertainty shocks, similar to what Ilut, Luetticke, and Schneider (2025) does for a full information HANK framework.[8] With the computationally straightforward conversion from full information rational expectations to sticky expectations, it should be feasible to combine such models with other state-space tools and innovations as the researcher sees fit.

[8]For instance, higher-order terms due to the learning rate may not be tractable to include, but higher-order terms for other macroeconomic variables should be feasible.

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# A Appendix to Section 2: Derivation Proofs

## A.1 Proposition 2.1: Derivation of the HJB Equation for $i$-Belief Households

Statement: *The Hamilton Jacobi Bellman (HJB) equation for $x \in \mathcal{X} \setminus \partial\mathcal{X}$ takes the form*

$$\rho V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{\tilde{c}_t^i} \Big\{ u(\tilde{c}_t^i) + \nabla_x V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)' \mathbb{E}_t^i[f(x_t, c_t^i; p_t)] + \frac{1}{2} tr(\sigma_x(x)\sigma_x(x)' \nabla_x^2 V^i) dt$$
$$+ \nabla_p V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)' \tilde{\mathbb{E}}_t^i[g(p, \mu)] + \int_{\mathcal{X}} \delta_\mu V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \tilde{\mathbb{E}}_t^i[\mathcal{D}_t^*(V, p_t)\mu_t] dx \Big\}.$$

*Proof.* The value function may be additively separated to write

$$\begin{aligned}
\tilde{\mathbb{E}}_t^i \int_t^\infty e^{-(\tau - t)\rho} u(c_\tau) d\tau &= \tilde{\mathbb{E}}_t^i \int_t^{t+dt} e^{-(\tau-t)\rho} u(c_\tau^i) d\tau + \tilde{\mathbb{E}}_t^i \int_{t+dt}^\infty e^{-(\tau-t)\rho} u(c_\tau^i) d\tau \\
&= u(c_t^i) dt + e^{-\rho dt} \tilde{\mathbb{E}}_t^i \underbrace{\tilde{\mathbb{E}}_{t+dt}^i \int_{t+dt}^\infty e^{-(\tau - [t+dt])\rho} u(c_\tau^i) d\tau}_{V_{t+dt}(x_{t+dt}; p_{t+dt}, \mu_{t+dt})} \\
&= u(c_t^i) dt + e^{-\rho dt} \tilde{\mathbb{E}}_t^i V_{t+dt}(x_{t+dt}; p_{t+dt}, \mu_{t+dt})
\end{aligned}$$

So if the subjective measure still obeys the Law of Iterated Expectations (LIE), the discretized HJB is indeed still

$$\begin{aligned}
V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) &= \max_{(c_\tau)_{\tau \geq t}} u(c_t^i) dt + e^{-\rho dt} \tilde{\mathbb{E}}_t^i V_{t+dt}^i(x_{t+dt}, p_{t+dt}, \mu_{t+dt}, p_{t+dt}^i, \mu_{t+dt}^i) \\
\text{s.t. } \mathbb{E}_t^i[x_t | dW_t] &= \mathbb{E}_t^i f(x_t, c_t^i; p_t) + \sigma_x(x) dW_t
\end{aligned}$$

Approximating $e^{-\rho dt} \approx 1 - \rho dt$,

$$V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{(c_\tau^i)_{\tau \geq t}} u(c_t^i) dt + (1 - \rho dt) \tilde{\mathbb{E}}_t^i V_{t+dt}^i(x_{t+dt}; p_{t+dt}, \mu_{t+dt}, p_{t+dt}^i, \mu_{t+dt}^i)$$

Using a Taylor expansion about $dt = 0$,

$$\begin{aligned}
V_{t+dt}^i(x_{t+dt}; p_{t+dt}, \mu_{t+dt}, p_{t+dt}^i, \mu_{t+dt}^i) = & V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \\
& + \nabla_x V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \mathbb{E}_t^i[dx_t] + \frac{1}{2} \mathrm{tr}(\sigma_x(x)\sigma_x(x)' \nabla_x^2 V^i) dt \\
& + \nabla_p V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)' \dot{p}_t dt + dt \int_{\mathcal{X}} \delta_\mu V^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \partial_t \mu_t dx \\
& + \nabla_{p^i} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)' \dot{p}_t^i dt + dt \int_{\mathcal{X}} \delta_{\mu^i} V^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \partial_t \mu_t^i dx \\
& + \mathcal{O}(dt^2)
\end{aligned}$$

where the Hessian $\nabla^2 V^i$ appears because the differential of the Brownian covariation process $d\langle x \rangle_t$ is proportional to $dW_t^2 = dt$. Taking subjective expectations,

$$\begin{aligned}\tilde{\mathbb{E}}_t^i V_{t+dt}^i(x_{t+dt}; p_{t+dt}, \mu_{t+dt}, p_{t+dt}^i, \mu_{t+dt}^i) =& \tilde{\mathbb{E}}_t^i V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \\ &+ \nabla_x V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)' \mathbb{E}_t^i[dx_t] + \frac{1}{2}\mathrm{tr}(\sigma_x(x)\sigma_x(x)'\nabla_x^2 V^i)dt \\ &+ \tilde{\mathbb{E}}_t^i[\nabla_p V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\dot{p}dt] \\ &+ \tilde{\mathbb{E}}_t^i\left[dt\int_{\mathcal{X}} \delta_\mu V^i(x_t; p_t, \mu_t)\partial_t \mu_t dx\right] + \mathcal{O}(dt^2).\end{aligned}$$

where I further assume the orthogonality conditions

$$\tilde{\mathbb{E}}_t^i[\nabla_p V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\dot{p}] = 0,$$

$$\tilde{\mathbb{E}}_t^i V^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)\partial_t \mu_t = 0$$

since holding subjective beliefs $p_t^i, \mu_t^i$ constant, households in the state-space interior do not forecast their value function to change based on the actual $p_t, \mu_t$.

I can substitute the subjectively expected Taylor expansion back into the HJB to write:

$$\begin{aligned}V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{(c_\tau)_{\tau \geq t}} \; & u(c_t^i)dt + \\ & (1-\rho dt)\Big[V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) + \nabla_x V_t^i(x_t, p_t)'\tilde{\mathbb{E}}_t^i[f(x_t, c_t^i; p_t)dt] \\ & + \frac{1}{2}\mathrm{tr}(\sigma_x(x)\sigma_x(x)'\nabla_x^2 V^i)dt \\ & + \nabla_{p^i} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\tilde{\mathbb{E}}_t^i[\dot{p}]dt + dt\int_{\mathcal{X}} \delta_{\mu^i} V^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)\tilde{\mathbb{E}}_t^i\left[\partial_t \mu_t\right]dx + \mathcal{O}(dt^2)\Big]\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad \rho V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)dt =& \max_{\tilde{c}_t^i} u(c_t^i)dt + \nabla_x V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\mathbb{E}_t^i f(x_t, c_t^i; p_t)dt \\ &+ \frac{1}{2}\mathrm{tr}(\sigma_x(x)\sigma_x(x)'\nabla_x^2 V^i)dt \\ &+ \partial_{p^i} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)\tilde{\mathbb{E}}_t^i[\dot{p}]dt \\ &+ \int_{\mathcal{X}} \delta_{\mu^i} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)\tilde{\mathbb{E}}_t^i\left[\partial_t \mu_t\right]dx + \tilde{\mathbb{E}}_t^i[\partial_t V_t(x_t; p_t, \mu_t)]dt + \mathcal{O}(dt^2)\end{aligned}$$

Dividing by $dt$ and taking $dt \to 0$:

$$\begin{aligned}\rho V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{\tilde{c}_t^i}\Bigg\{& u(c_t^i) + \nabla_x V_t^i(x_t; , p_t, \mu_t, p_t^i, \mu_t^i)'\mathbb{E}_t^i[f(x_t, c_t^i; p_t)] + \frac{1}{2}\mathrm{tr}(\sigma_x(x)\sigma_x(x)'\nabla_x^2 V^i) \\ &+ \nabla_{p^i} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\tilde{\mathbb{E}}_t^i[\dot{p}] + \int_{\mathcal{X}} \frac{\delta V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)}{\delta \mu_t^i(x')}\tilde{\mathbb{E}}_t^i\left[\partial_t \mu_t(x')\right]dx'\Bigg\},\end{aligned}$$

as proposed in equation (3).

$\square$

## A.2 Sticky mean expectation evolution

Start with the expression for the mean forecast:

$$\overline{\mathbb{E}}_t[Y_{t+s}] = \int_0^\infty \lambda e^{-\lambda\tau}\mathbb{E}_{t-\tau}[Y_{t+s}]d\tau.$$

Differentiating with respect to $t$,

$$\begin{aligned}
\frac{d}{dt}\overline{\mathbb{E}}_t[Y_{t+s}] &= \frac{d}{dt}\int_0^\infty \lambda e^{-\lambda\tau}\int_\Omega y_{t+s}(\omega)\psi_{t-\tau}(\omega)d\omega d\tau \\
&= \int_0^\infty \lambda e^{-\lambda\tau}\int_\Omega \left[\frac{dy_{t+s}}{dt}(\omega)\psi_{t-\tau}(\omega) + y_{t+s}(\omega)\frac{d\psi_{t-\tau}(\omega)}{dt}\right]d\omega d\tau \\
&= \int_0^\infty \lambda e^{-\lambda\tau}\int_\Omega \frac{dy_{t+s}}{dt}(\omega)\psi_{t-\tau}(\omega)d\omega d\tau + \int_0^\infty \lambda e^{-\lambda\tau}\int_\Omega y_{t+s}(\omega)\frac{d\psi_{t-\tau}(\omega)}{dt}d\omega d\tau \\
&= \underbrace{\int_0^\infty \lambda e^{-\lambda\tau}\mathbb{E}_{t-\tau}\left[\frac{dy_{t+s}(\omega)}{dt}\right]d\tau}_{\overline{\mathbb{E}}_t[\partial_t Y_{t+s}]} + \int_0^\infty \lambda e^{-\lambda\tau}\int_\Omega y_{t+s}(\omega)\frac{d\psi_{t-\tau}(\omega)}{dt}d\omega d\tau
\end{aligned}$$

For the second term, I can interchange the order of integration and integrate by parts with $u = \lambda e^{-\lambda\tau}y_{t+s}(\omega)$ and $v = -\psi_{t-\tau}(\omega)$:

$$\begin{aligned}
\int_\Omega\int_0^\infty \lambda e^{-\lambda\tau}y_{t+s}(\omega)\frac{d\psi_{t-\tau}(\omega)}{dt}d\tau d\omega &= \int_\Omega\int_0^\infty \lambda e^{-\lambda\tau}y_{t+s}(\omega)\left(-\frac{d\psi_{t-\tau}(\omega)}{d\tau}\right)d\tau d\omega \\
&= -\int_\Omega \lambda e^{-\lambda\tau}y_{t+s}(\omega)\psi_{t-\tau}(\omega)\Big|_{\tau=0}^{\infty}d\omega + \int_\Omega\int_0^\infty \psi_{t-\tau}(\omega)\partial_\tau[\lambda e^{-\lambda\tau}y_{t+s}(\omega)]d\tau \\
&= \lambda\int_\Omega y_{t+s}(\omega)\psi_t(\omega)d\omega - \lambda\int_\Omega\int_0^\infty \psi_{t-\tau}(\omega)\lambda e^{-\lambda\tau}y_{t+s}(\omega)d\tau d\omega \\
&= \lambda\Big(\mathbb{E}_t[Y_{t+s}] - \overline{\mathbb{E}}_t[Y_{t+s}]\Big).
\end{aligned}$$

As such, the complete evolution of the forecast is

$$\frac{d}{dt}\overline{\mathbb{E}}_t[Y_{t+s}] = \overline{\mathbb{E}}_t[\partial_t Y_{t+s}] + \lambda\Big(\mathbb{E}_t[Y_{t+s}] - \overline{\mathbb{E}}_t[Y_{t+s}]\Big).$$

## A.3 Proof of 2.3: Characterizing the household choices averaged over beliefs

*Proof.* Let $\Delta$ denote the difference of a variable from its non-stochastic steady-state (when there are no macroeconomic shocks). The consumption choice averaged over all idiosyncratic beliefs will be:

$$\int_i c_t^i(x;p,\mu,p^i,\mu^i)di = \int_i h\big(V_t^i(x;p,\mu,p^i,\mu^i),p_t\big)d\Lambda(i)$$

Let $G_{Vp^i}(x)$ and $G_{V\mu^i}(x)$ denote the Jacobians of $V^i$ with respect to $p^i$ and $\mu^i$ at the steady-state, and let $G_{Vp}(x)$ and $G_{V\mu}(x)$ be the Jacobians with respect to the actual $p$ and $\mu$. Additionally, let $\overline{G}_{Vp}(x)$ and $\overline{G}^i_{V\mu}(x)$ denote the Jacobians of the value function calculated with the average beliefs.

Note that since $G_{V\mu}(x)$ is itself a functional derivative, so the operator is in fact shorthand for

$$G_{V\mu}(x)\Delta\mu \equiv D_\mu V[\Delta\mu](x) = \int_{\mathcal{X}} \frac{\delta V(x)}{\delta\mu(x')}\Delta\mu(x')dx.$$

With a first-order Taylor expansion around the steady-state:

$$\begin{aligned}\int_i \Delta c_t^i(x;p,\mu,p^i,\mu^i)di = &\int_i \partial_V h(x)\left[G_{Vp}(x)\Delta p + G_{V\mu}(x)\Delta\mu + G_{Vp^i}(x)\Delta p^i + G_{V\mu^i}(x)\Delta\mu^i\right]d\Gamma(i) + \partial_p h(x)\Delta p \\ &+ \mathcal{O}(\|\Delta p^2, \Delta\mu^2, (\Delta p^i)^2, (\Delta\mu^i)^2\|) \\ =&\partial_V h(x)\left[G_{Vp}(x)\Delta p + G_{V\mu}(x)\Delta\mu + G_{Vp^i}(x)\Delta\overline{p} + G_{V\mu^i}(x)\Delta\overline{\mu}\right] + \partial_p h(x)\Delta p \\ &+ \mathcal{O}(\|\Delta p^2, \Delta\mu^2, (\Delta p^i)^2, (\Delta\mu^i)^2\|) \\ =&\Delta h(\overline{V}(p,\mu,\overline{p},\overline{\mu}),p) + \mathcal{O}(\|\Delta p^2, \Delta\mu^2, (\Delta p^i)^2, (\Delta\mu^i)^2\|)\end{aligned}$$

As such,

$$h(\overline{V}_t, p_t) = c_t(x;p,\mu,\overline{p},\overline{\mu}) = \int_i c_t^i(x;p,\mu,p^i,\mu^i)d\Gamma(i) + \mathcal{O}(\|\Delta p^2, \Delta\mu^2, (\Delta p^i)^2, (\Delta\mu^i)^2\|)$$

in a neighborhood around the non-stochastic steady-state. □

## A.4 Proof of Proposition 2.6

Statement: *Suppose the economy starts in its non-stochastic steady state when a macroeconomic shock occurs. If the KFE generator is mass-preserving, then the value function of households at the boundary will satisfy equation (7.5).*

*Proof.* Let $J(x, c(x), p, \mu)$ denote the probability *flux* vector field for all $x \in \mathcal{X}$, i.e. the instantaneous rate at which probability mass changes in a given direction at a point in the idiosyncratic state-space $x \in \mathcal{X}$ per increment of time. Note that probability mass enters or leaves a region of space in one of two ways: mass flows into the region advectively by being pushed directly by the flows of the mean state equations, or it diffuses out at a rate related to the directional gradient of the mass already in the region relative to surrounding regions. For my problem,

$$J(x, c_t(V_t(x)), p_t, \mu_t) = \underbrace{f(x, c_t(V_t(x)), p_t)\mu_t(x)}_{\text{Advective}} - \underbrace{\frac{1}{2}\nabla_x \cdot (\sigma_x(x)\sigma_x(x)'\mu_t(x))}_{\text{Diffusive}}$$

(Note that $\nabla \cdot A(x)$ is here defined as the gradient operator dotted with each row of the matrix, transforming the matrix into a vector.) The Kolmogorov Forward Equation is equivalent to the statement

$$\partial_t\mu_t(x) + \nabla_x \cdot J(x_t; c(V_t(x)), p_t, \mu_t) = 0.$$

such that the total change in probability density at a point $x \in \mathcal{X}$ is equal to the spatial divergence of the probability flux field. Writing this in terms of the KFE infinitessimal generator,

$$\mathcal{D}^*(V,p)[\mu](x) = -\nabla_x \cdot J(x_t; c_t(V_t(x)), p_t, \mu_t), \text{ such that } \partial_t \mu_t = \mathcal{D}^*(V,p)[\mu_t](x).$$

Note that as the name implies, if the operator is mass-preserving, the total flux of probability through the space $\mathcal{X}$ will always be zero:

$$\begin{aligned}\int_{\mathcal{X}} \partial_t \mu_t(x) &= \int_{\mathcal{X}} \mathcal{D}^*(p,V)[\mu](x)dx = \int_{\mathcal{X}}\int_{\mathcal{X}} D^*(p,V)(x,x')\mu(x')dx'dx \\ &= \int_{\mathcal{X}}\int_{\mathcal{X}} D^*(p,V)(x,x')\mu(x')dx\ dx' = \int_{\mathcal{X}} \underbrace{\left[\int_{\mathcal{X}} D^*(p,V)(x,x')dx\right]}_{0} \mu(x')dx' = 0.\end{aligned}$$

The first equality of the second line above follows from Fubini's theorem, since the boundary is assumed to be rectangular, allowing for the order of integration to be interchanged.

Using the relation between the probability flux field and the infinitessimal generator, it then follows that

$$0 = \int_{\mathcal{X}} \mathcal{D}^*(V,p)[\mu](x)dx = -\int_{\mathcal{X}} \nabla_x \cdot J\ dx^n = -\oint_{\partial\mathcal{X}} J(x)\cdot \vec{n}(x)\ dS$$

where the last inequality follows from Gauss' Divergence Theorem, and $S$ is the boundary of the idiosyncratic state-space (note that the surface integral on the right is one dimension lower than the volume integral on the left). Here, $\vec{n}(x)$ is a unit vector normal to the boundary $\partial\mathcal{X}$, at a point evaluated somewhere along said boundary.

As such, the total net flux of probability mass across the boundary of the state-space must be equal to zero. However, if all of the initial distribution is inside the idiosyncratic state-space (as it is in the non-stochastic steady-state), then this means there can be no flux *anywhere* along the boundary.

Note that the above equality must hold for any distribution, even those that that start with a Dirac delta mass on the boundary. As such, the final integral must hold point-wise. Intuitively, if no mass can cross the $\{x_i = \underline{x}_i\}$ hyperplane, then probability must flow along (tangent to) it. As such, a vector orthogonal to the hyperplane boundary must also be orthogonal to the probability flux:

$$J(x_t)\cdot \vec{n}(x_t) = 0.$$

For a boundary of the form $x_i > \underline{x}$, the orthogonal vector $\vec{n}$ is simply the $i$th standard basis vector.

Suppose there is no diffusive term for the constrained variable $x_i$. This then implies if $x \in \partial\mathcal{X}$,

$$J(x; h(V_t(x)), p_t, \mu_t)\cdot \vec{n}(x) = 0$$

$$\iff f_i(x, h(V_t(x)), p_t \mu_t)\mu_t(x) = 0$$

And if $\mu_t(x) > 0$, then

$$\iff f_i(x, h(V_t(x)), p_t) = 0.$$

A mass-preserving KFE infinitesimal generator at each point in time is thus tantamount to a *boundary* *condition*

$$f_i(x, h(V_t(x)), p_t) = 0$$

where $x \in \partial\mathcal{X}$ such that $x_i = \underline{x}_i$.

For example, suppose the law of motion for assets is $f(x_t, c_t, p_t) = r_t x_t + w_t + T_t - c_t$, where $p_t = [r_t, w_t, T_t]$ in this case is the real rate of return, the aggregate wage, and government transfers to households (all macroeconomic objects). The household at the boundary with assets $x = 0$ will then satisfy

$$c_t = h(V_t) = w_t + T_t$$

as an equilibrium condition. This condition is inherited from the fact that the HJB infinitessimal generator is mass preserving – even if the household does not correctly perceive macroeconomic wages and prices. □

### A.5 Proof of Proposition 2.7

Statement: *If the KFE infinitessimal generator $\mathcal{D}^*(V, p)$ is a first-order perturbation of the steady-state one with respect to macroeconomic variables, then it will be mass-preserving if the Jacobians evaluated at the steady-state are mass-preserving.*

*Proof.* This statement is nearly a tautology. To see this, take the kernel and approximate to first order:

$$\partial_t \mu_t(x) = \int_{\mathcal{X}} D^*(V_t, p_t)(x, x')\mu_t(x')dx' \approx \int_{\mathcal{X}} D^*_V(x, x')\Delta V_t(x')dx' + D^*_p(x)\Delta p_t + \int_{\mathcal{X}} D^*_\mu(x, x')\Delta\mu_t(x')dx'$$

Here, $D^*_V, D^*_\mu, D^*_p$ are the Jacobians (Frechét in the for $\mu$ and $V$) evaluated in the non-stochastic steady-state. Integrating over the entire distribution, if the Jacobians are all mass-preserving:

$$\begin{aligned}
\int_{\mathcal{X}} \partial_t \mu_t(x)dx &= \int_{\mathcal{X}}\int_{\mathcal{X}} D^*(V_t, p_t)(x, x')\mu_t(x')dx'\ dx \\
&\approx \int_{\mathcal{X}}\int_{\mathcal{X}} D^*_V(x, x')\Delta V_t(x')dx'\ dx + \int_{\mathcal{X}} D^*_p(x)\Delta p_t dx + \int_{\mathcal{X}}\int_{\mathcal{X}} D^*_\mu(x, x')\Delta\mu_t(x')dx'\ dx \\
&= \int_{\mathcal{X}} \left[\int_{\mathcal{X}} D^*_V(x, x')dx\right] \Delta V_t(x')\ dx' + \int_{\mathcal{X}} D^*_p(x)dx\Delta p_t + \int_{\mathcal{X}} \left[\int_{\mathcal{X}} D^*_\mu(x, x')dx\right] \Delta\mu_t(x')dx' \\
&= 0.
\end{aligned}$$

If for any distribution $\mu_t$ in the approximation,

$$\int_{\mathcal{X}}\int_{\mathcal{X}} D^*(V_t, p_t)(x, x')\mu_t(x')dx'dx = 0$$

then it must also be that up to a first order approximation

$$\int_{\mathcal{X}} D^*(V_t, p_t)(x, x')dx = 0$$

such that the linearized KFE operator will also be mass-preserving in the perturbation if the Jacobians integrate to zero over $x$. $\square$

### A.6 Proof of Proposition 2.8: Average belief value function updating

Statement: *To a first-order approximation, the average belief household updates its value function with a constant factor of* $\lambda(\Delta\widehat{V} - \Delta V)$.

*Proof.* First, note when an update occurs, the learning occurs to the entire sequence of macro aggregates. At time $t$, the sequences of beliefs (both about the current state and the future) are updated at rates of $\{\lambda(\mu_{t+s} - \overline{\mu}_{t+s})\}_{s\geq 0}$ and $\{\lambda(p_{t+s} - \overline{p}_{t+s})\}_{t\geq 0}$ multiplied by the time increment $dt$.

Next, note that the average expectation household and the full information household both essentially solve the same HJB equation:

$$\begin{aligned}\rho v_t(x_t) =& \max_{c_t}\Big\{u(h(v_t(x_t),p_t)) + \nabla_x v_t(x_t)' f(x_t, h(v_t(x), p_t); p_t)\Big\} + \partial_t v(x_t)\\ &\text{s.t. } x_t \geq \underline{x}\ \forall t.\end{aligned}$$

which can be linearized such that

$$\partial_t \Delta v_t(x) = \int_{\mathcal{X}} \mathcal{A}_{VV}(x,x')\Delta v_t(x')dx' + \int_{\mathcal{X}} \mathcal{A}_{V\mu}(x,x')\Delta\mu_t(x')dx' + \mathcal{A}_{Vp}(x)\Delta p_t + \mathcal{O}([...]^2).$$

and then – assuming that $\lim_{T\to\infty}\Delta V_T(x) = 0$ – solved forward to write

$$\mathcal{V}(x, \{p_\tau, \mu_\tau\}_{\tau\geq t}) \equiv \int_t^\infty \left[e^{-\mathcal{A}_{VV}(\tau-t)}\right](x,x'')\left[\int_{\mathcal{X}} A_{V\mu}(x,x')\Delta\overline{\mu}_\tau(x')dx' + A_{Vp}\Delta\overline{p}_\tau\right]dx''d\tau.$$

Here, $\mathcal{V}(x, \{p_\tau, \mu_\tau\}_{\tau\geq t})$ represents the solution of the HJB given *sequences* of macro agreggates and distributions from time period $t$ onwards. In equibrium,

$$\begin{aligned}\Delta\overline{V}_t(x_t) =& \mathcal{V}(x, \{\overline{p}_\tau, \overline{\mu}_\tau\}_{\tau\geq t}),\\ \Delta\widehat{V}_t(p_t) =& \mathcal{V}(x, \{p_\tau, \mu_\tau\}_{\tau\geq t}).\end{aligned}$$

Because $\mathcal{V}$ is linear in the macro aggregates, the effect of an update can be expressed as

$$\begin{aligned}&\mathcal{V}(x, \{\lambda(p_\tau - \overline{p})dt, \lambda(\mu_\tau - \overline{\mu}_\tau)dt\}_{\tau\geq t}) =\\ \lambda dt \int_t^\infty \left[e^{-\mathcal{A}_{VV}(\tau-t)}\right](x,x'')&\left[\int_{\mathcal{X}} A_{V\mu}(x,x')[\Delta\mu_\tau(x') - \Delta\overline{\mu}_\tau(x')]dx' + A_{Vp}[\Delta p_\tau - \Delta\overline{p}_\tau]\right]dx''d\tau\\ &= [\mathcal{V}(x, \{p_\tau, \mu_\tau\}_{\tau\geq t}) - \mathcal{V}(x, \{\overline{p}_\tau, \overline{\mu}_\tau\}_{\tau\geq t})]dt\\ &= \lambda[\Delta V_t(x) - \Delta\overline{V}_t(x)]dt.\end{aligned}$$

$\square$

# B Appendix to Section 3: Analytical RANK and Canonical HANK Examples

## B.1 A toy representative agent example

My computational approach can be demonstrated using a simple representative agent macroeconomic model that can be exactly solved analytically. Consider the simple FIRE representative agent model with the log-linearized Euler equation:

$$\frac{\mathbb{E}_t[d\widehat{c}_t]}{dt} = \gamma^{-1}\widehat{r}_t$$

where the real interest rate follows $r_t = e^{-\kappa t}r_0$ with $r_0$ given. Using the households budget constraints and assuming that the household consumption path returns to steady-state, the consumption choice can be written as:

$$c_t = \rho \int_t^\infty e^{-(\tau-t)\rho}\mathbb{E}_t[\widehat{y}_\tau]d\tau - \gamma^{-1}\int_t^\infty e^{-(\tau-t)\rho}\mathbb{E}_t[\widehat{r}_\tau]d\tau. \tag{39}$$

With some calculus and a goods market clearing condition that $y_t = c_t$, the output response to the sequence of real interest rate deviations is

$$y_t = -\gamma^{-1}\frac{1}{\kappa}r_0 e^{-\kappa t}$$

Suppose instead households update their information about the macroeconomic environment at a rate of $\lambda$. Aggregate consumption is then chosen in a way that depends on the aggregate expectation $\overline{\mathbb{E}}_t$:

$$c_t = \rho \int_t^\infty e^{-(\tau-t)\rho}\overline{\mathbb{E}}_t[\widehat{y}_\tau]d\tau - \gamma^{-1}\int_t^\infty e^{-(\tau-t)\rho}\overline{\mathbb{E}}_t[\widehat{r}_\tau]d\tau. \tag{40}$$

In the appendix, I show using sequence-space solution techniques that the sticky-expectation law of motion will be

$$\frac{dy_t}{dt} = \left(\frac{d\mu_t}{dt}\frac{1}{\mu_t} + (1-\mu_t)\rho\right)y_t + \gamma^{-1}\mu_t r_t,$$

where $\mu_t = 1 - e^{-\lambda t}$ is the fraction of households who have updated to full information. In the limit as $\rho \to 0$, the exact closed form solution is:

$$y_t = -\gamma^{-1}\frac{1}{\kappa}\left(e^{-\kappa t} - e^{-(\lambda+\kappa)t}\right)r_0. \tag{41}$$

Using the machinery from the previous section, the $A$ matrix Jacobian for the FIRE system is thus

$$\begin{bmatrix}\mathbb{E}_t[d\widehat{c}] \\ d\widehat{r}\end{bmatrix} = \begin{bmatrix}0 & \gamma^{-1} \\ 0 & -\kappa\end{bmatrix}\begin{bmatrix}\widehat{c} \\ \widehat{r}\end{bmatrix}dt$$

while the $B$ matrix is here identical to the $A$ matrix, as there are no static variables. The new augmented system will be

$$\begin{bmatrix} \mathbb{E}_t[d\widehat{c}] \\ d\widehat{r} \\ d\overline{y} \\ d\overline{r} \end{bmatrix} = \begin{bmatrix} 0 & \gamma^{-1} & 0 & 0 \\ 0 & -\kappa & 0 & 0 \\ \lambda & 0 & -\lambda & \gamma^{-1} \\ 0 & \lambda & 0 & -\lambda-\kappa \end{bmatrix} \begin{bmatrix} \widehat{c} \\ \widehat{r} \\ \overline{y} \\ \overline{r} \end{bmatrix} dt \tag{42}$$

Clearly, the eigenvalues of the system are 0, $-\kappa$, $-\lambda$, and $-(\lambda+\kappa)$. Technically, the system is borderline indeterminate – as the rational expectations model I started with is borderline indeterminate. However, if we require that $\widehat{c}_t$ return to steady-state (and not just remain bounded), then this is a constraint on the zero eigenvector (the nullspace of the matrix). As I show in Appendix B.4, solving for the stable subspace of equation (42) recovers equation (41) for aggregate GDP exactly.

## B.2 A Canonical HANK model

In this section, I solve a canonical HANK model with sticky expectations using both my state-space approach and the sequence-space approach of Auclert, Rognlie, and Straub (2020). The model is essentially the one solved in Kwicklis (2025); the reader should refer to that paper for the model's derivation and details. There are only two important changes. First, while the original model was solved with full information and rational expectations, the model in this section is of course solved with sticky expectations. Second, the calibration in this section uses a more conventional active monetary/passive fiscal form, as opposed to the "active fiscal" experiments considered in Kwicklis (2025). The central bank raises nominal interest rates more than one-to-one with inflation with a Taylor rule coefficient of 1.5, while the government adjusts taxes over time to slowly stabilize its debt. All other parameters are unchanged and are listed in the appendix.

For illustration, I consider two different shocks: a monetary policy shock ($\zeta_{\text{mp}}$) that lowers the interest rate by 1% on impact, and a fiscal transfer shock ($\zeta_{\text{tax}}$) that sends flat transfers valued at 1% of annualized steady-state GDP to all households simultaneously. The monetary policy shock demonstrates how the methodology properly leads impulse responses generated by general equilibrium feedbacks to become hump-shaped, while the fiscal transfer shock demonstrates how the model handles instantaneous feedbacks to households' individual budget constraints.

### B.2.1 Abridged setup

Households choose consumption $c_t$ and take hours worked $L_t$ as given (chosen by their union to meet aggregate labor demand). They save via an non-contingent bonds $a_t$. subject to idiosyncratic risk about their labor productivity $z_t$, which follows a Gaussian log Ornstein-Uhlenbeck process with a mean reversion parameter of $\theta_z$ and a variance parameter of $\sigma_z^2$:

$$d\log(z_t) = -\theta_z \log(z_t)dt + \sigma_z dW_{z,t}$$

where $W_{z,t}$ is a standard normal Weiner process. The FIRE Hamilton Jacobi Bellman (HJB) equation is

$$\begin{aligned}\rho V_t(a_t, z_t) = &\max_{c_t} \left\{ \frac{c_t^{1-\gamma} - 1}{1-\gamma} - \frac{L_t^{\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \partial_a V_t(a_t, z_t)(r_t a_t + w_t L_t + T_t(z_t, \zeta_t) - c_t) \right\} \\ &+ \partial_z V_t(a_t, z_t) \left( \frac{1}{2}\sigma_z^2 - \theta_z \log z_t \right) + \frac{1}{2}\sigma_z^2 z_t^2 \partial_z^2 V_t(a_t, z_t) + \partial_t V_t(a_t, z_t) \\ &\text{s.t. } a_t \geq 0.\end{aligned} \tag{43}$$

The first-order conditions imply the household chooses $c_t = (\partial_a V_t)^{-1/\gamma}$, such that $h(V) = (\partial_a V)^{-1/\gamma}$. The distribution of households evolves according to

$$\frac{\partial \mu_t}{\partial t}(a, z) = -\frac{\partial}{\partial a}\left(\frac{da_t}{dt}(\overline{V}_t, p_t, a, z)\mu_t(a, z)\right) - \frac{\partial}{\partial z}\left(\frac{\mathbb{E}_t[dz_t]}{dt}\mu_t(a, z)\right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2 z^2 \mu_t(a, z)\right). \tag{44}$$

$$\frac{da}{dt}(\overline{V}_t, p_t, a, z) = r_t a_t + w_t L_t + T_t(z, \zeta) - h(\overline{V}_t)$$

Decentalized unions negotiate wages such that wage inflation (and overall inflation, if the passthrough from firms to consumers is complete) abides by a New Keynesian Phillips curve similar to the one in Auclert, Rognlie, and Straub (2024):

$$\frac{\mathbb{E}_t[d\pi_t]}{dt} = r_t \pi_t - \frac{\varepsilon_\ell}{\theta_w}\frac{L_t}{Z}\int\int\left(h_t(a, z)^{\frac{1}{\eta}} - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)z w_t c_t(a, z)^{-\gamma}\right)\mu_t(a, z)da\ dz. \tag{45}$$

The Fisher equation connects $r = i_t - \pi_t$. Tax policy is set via a slow-moving passive rule

$$T_t = \tau w_t L_t + \phi_B(B - B^*) + \zeta_{\text{tax},t}, \tag{46}$$

and government bonds evolve according to

$$\frac{dB_t}{dt} = -(T_t - G_t) + (i_t - \pi_t)\, B_t. \tag{47}$$

Monetary policy is set with a Taylor rule, plus a monetary policy shock:

$$i_t = r^* + \phi_\pi \pi_t + \zeta_{\text{mp},t}. \tag{48}$$

The aggregate shocks $\zeta$ follow a mean-reverting process $d\zeta_{i,t} = -\theta_i \zeta_{i,t} dt$, such that

$$\zeta_{i,t} = e^{-\theta_i t}\zeta_{i,0}. \tag{49}$$

I linearize equations (43-48) around the non-stochastic steady-state wherein the aggregate shocks are disabled: $\zeta_{i,0} = 0$. From there, I solve the linearized FIRE version of the model using the methodology of Bayer and Luetticke (2020) in state-space and using the sequence-space Jacobian (SSJ) algorithm of Auclert et al (2021) in sequence-space. I then solve the sticky expectation variation of the model using my methodology in state-space and the Auclert et al (2020) methodology in sequence-space.

#### B.2.2 Simulation results

In Figure 8, I depict the impulse response functions of output and inflation to a 1% reduction in the interest rate and a 1% of GDP increase in lump-sum government transfers using the two different

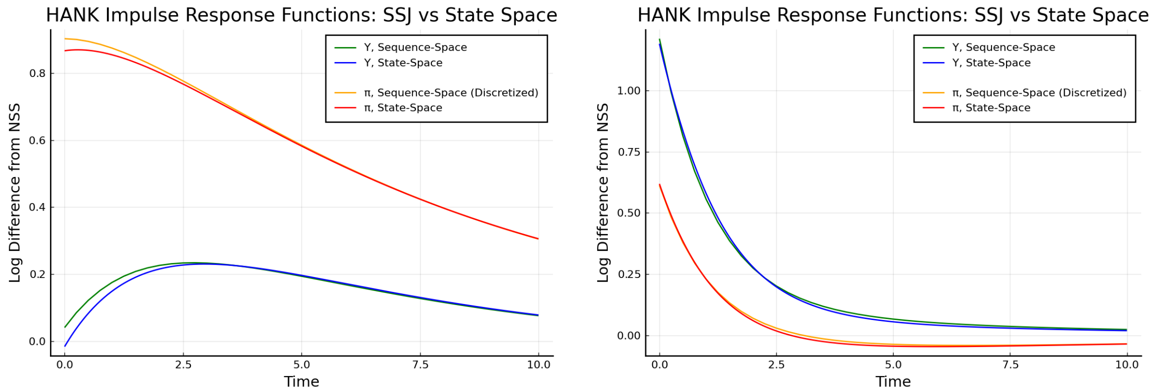


**Figure 8:** Response of a canonical HANK model to a 1% monetary policy shock and a 1% government transfer shock, as a percentage deviation from the non-stochastic steady-state. Orange and red denote inflation (using the Auclert et al SSJ framework and my state-space approach, respectively). Blue and green denote GDP.

**Table 5:** Numerical Solution: HANK Model Parameters

| Parameter | Symbol | Value | Source or Target |
|---|---|---|---|
| *Households* | | | |
| **Internally Calibrated:** | | | |
| Quarterly Time Discounting | $\rho$ | 0.021 | $r = 2\%$ Annually |
| Idiosyncratic Income Shock Variance | $\sigma_z^2$ | 0.017 | Floden and Lindé (2001) |
| Idiosyncratic Shock Mean Reversion | $\theta_z$ | 0.034 | Floden and Lindé (2001) |
| **Assumed from Literature:** | | | |
| Relative Risk Aversion | $\gamma$ | 2.0 | McKay et al (2016) |
| Frisch Elasticity of Labor | $\eta$ | 0.5 | Chetty (2012) |
| *Labor Market* | | | |
| Labor Elasticity of Substitution | $\varepsilon_L$ | 10 | Philips Curve slope of 0.07 |
| Rotemberg wage adjustment cost | $\theta_w$ | 100 | Philips Curve slope of 0.07 |
| *Government* | | | |
| steady state government debt | $B_{NSS}$ | 2.63 | HANK $iMPC_0 \approx 0.40$ |
| Geometric maturity structure of debt | $\omega$ | 0.043 | Avg. maturity of 70 months |
| Income Tax Rate | $\tau$ | 0.25 | |
| Taylor Rule Coefficient | $\phi_\pi$ | 1.5 | Active monetary policy |
| Fiscal Debt Coefficient | $\kappa$ | 0.10 | Passive fiscal policy |
| *Shocks* | | | |
| Mean reversion of fiscal shocks | $\theta_{\text{Tax}}$ | 1.0 | |

solution methods. Here, I set $\lambda = 0.30$, such that roughly half of the households have fully updated for the presence of the macroeconomic shock 2.5 quarters after the shock's impact.

On impact, a small gap appears on impact between the two impulse response functions. This is because the sequence-space solution becomes higher and higher dimensional in continuous time as the time grid becomes finer, which in turn limits the resolution of the continuous time SSJ solution.[9] Approximation error notwithstanding, the state-space methodology broadly coincides with the sequence-space one, even despite the fact that the state-space approach undergoes dimension reduction with a fixed copula. Note that even though stimulus checks are macroeoconomic variables that only a

[9]Naturally, Auclert, Bardóczy, et al. (2021) originally formulated the SSJ approach for discrete time. Greater numerical accuracy could be obtained by using a non-uniform time step mesh – but this further complicates the approach. As the size of the discretized $dt$ time steps falls, the solution methods align more closely.

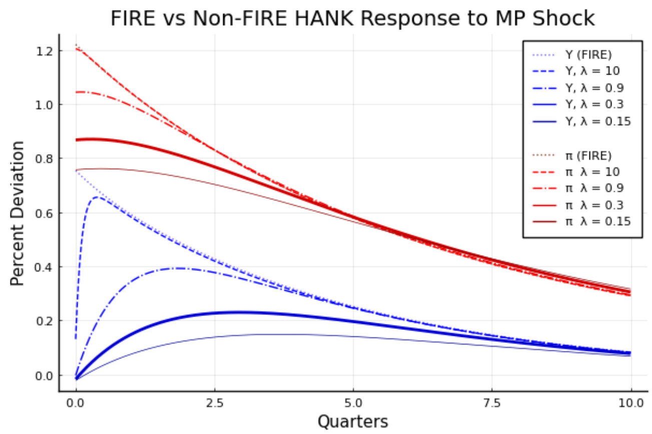


**Figure 9:** A canonical HANK model's response to a 1% monetary policy shock, for differing degrees of expectation stickyness $\lambda$.

zero measure of households observe upon impact, the households do immediately observe an influx of resources into their individual idiosyncratic accounts. As such, output jumps on impact. The model is linear with respect to macroeconomic shocks, so if government transfers are reduced aggregate demand also falls on impact, exactly inverting the pattern of a stimulus check disbursal.

As one might expect, increasing the learning rate by increasing $\lambda$ leads the impulse response functions to a monetary policy shock to more closely resemble those of the FIRE model. This property is displayed in Figure 9. In the FIRE setting, output and inflation jump as soon as interest rates are lowered. In the sticky expectations setting, however, the output response takes more time to build and peaks lower as $\lambda$ decreases.

### B.3 Sequence space derivation of a simple RANK model

$$c_t = \rho \int_t^\infty e^{-(\tau-t)\rho} \overline{\mathbb{E}}_t[y_\tau] d\tau - \gamma^{-1} \int_t^\infty e^{-(\tau-t)\rho} \overline{\mathbb{E}}_t[r_\tau] d\tau \tag{50}$$

where for the market to clear, $y_t = c_t$ for the representative agent. Suppose monetary policy sets $r_t = e^{-\kappa t} r_0$.

1. Rational expectations: Suppose $\overline{\mathbb{E}}_t = \mathbb{E}_t$. Then if there are no further shocks to the economy,

$$y_t = \rho \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau - \gamma^{-1} \int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau$$

where

$$\int_t^\infty e^{-(\tau-t)} r_\tau d\tau = \int_t^\infty e^{-(\tau-t)} r_0 e^{-\kappa\tau} d\tau = r_0 e^{\rho t} \int_t^\infty e^{-(\rho+\kappa)\tau} d\tau = r_0 e^{\rho t} \frac{-1}{\rho+\kappa} e^{-(\rho+\kappa)\tau}\Big|_t^\infty = \frac{r_0 e^{-\kappa t}}{\rho+\kappa}.$$

Meanwhile, setting $G(t) = \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau$,

$$G'(t) = -y_t + \rho G(t)$$

such that

$$0 = -y_t + \underbrace{\rho \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau}_{G(t)} - \gamma^{-1} \int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau$$

$$G'(t) - \gamma^{-1} \frac{r_0 e^{-\kappa t}}{\rho+\kappa} = 0.$$

Integrating both sides forward

$$\int_t^\infty G'(s)ds - \gamma^{-1} \frac{r_0}{\rho+\kappa} \int_t^\infty e^{-\kappa s} ds = 0$$

$$\Rightarrow \quad \lim_{s\to\infty} G(s) - G(t) = \gamma^{-1} \frac{r_0}{\rho+\kappa} \frac{1}{\kappa} e^{-\kappa t}$$

$$G(t) = -\gamma^{-1} \frac{r_0}{\rho+\kappa} \frac{1}{\kappa} e^{-\kappa t}.$$

Substituting this into the original expression,

$$y_t = -\rho \overbrace{\gamma^{-1} \frac{r_0}{\rho+\kappa} \frac{1}{\kappa} e^{-\kappa t}}^{G(t)} - \gamma^{-1} \frac{r_0}{\rho+\kappa} e^{-\kappa t} = -\gamma^{-1} \frac{r_0}{\rho+\kappa} \left(\frac{\rho}{\kappa} + 1\right) e^{-\kappa t}$$

$$y_t = -\gamma^{-1} \frac{1}{\kappa} r_0 e^{-\kappa t}$$

2. Sticky info. Suppose a fraction $\lambda$ updates their beliefs about the macro environment per increment $dt$, such that $d\mu_t = \lambda(1-\mu_t)dt$ where the fraction of households who have updated at time $t$ is $\mu_t = 1 - e^{-\lambda t}$. It further follows that $\dot{\mu}_t = \lambda e^{-\lambda t}$. Thus the average expectation is

$$\overline{\mathbb{E}}_t[x_\tau] = (1-\mu_t) \underbrace{(0)}_{\substack{\text{No}\\\text{update}}} + \mu_t \underbrace{x_\tau}_{\text{Actual}}.$$

Thus

$$\overline{\mathbb{E}}_t[x_\tau] = \mu_t x_\tau$$

Note that $\mu_0 = 0$, $\lim_{t\to\infty} \mu_t = 1$. As $\lambda \to \infty$, $\mu_t \to 1$, while $\lambda \to 0$ causes $\mu_t = 0$.

3. Substituting this into (50),

$$y_t = \mu_t \left[\rho \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau - \gamma^{-1} \int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau\right].$$

Differentiating with respect to time,

$$\begin{aligned}
\frac{dy_t}{dt} =& \frac{d\mu_t}{dt}\frac{y_t}{\mu_t} + \mu_t \frac{d}{dt}\left[\rho \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau - \gamma^{-1}\int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau\right] \\
=& \frac{d\mu_t}{dt}\frac{y_t}{\mu_t} + \mu_t\left[\rho\left(-y_t + \rho\int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau\right) - \gamma^{-1}\left(-r_t + \rho\int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau\right)\right]. \\
=& \frac{d\mu_t}{dt}\frac{y_t}{\mu_t} + \mu_t(-\rho y_t + \gamma^{-1} r_t) + \rho \underbrace{\mu_t\left[\rho\int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau - \gamma^{-1}\int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau\right]}_{y_t}
\end{aligned}$$

Thus

$$\frac{dy_t}{dt} = \left(\frac{d\mu_t}{dt}\frac{1}{\mu_t} + (1-\mu_t)\rho\right) y_t + \gamma^{-1}\mu_t r_t$$

Note that $\frac{d\mu_t}{dt}\frac{1}{\mu_t} = \lambda\frac{(1-\mu_t)}{\mu_t} = \lambda(\mu_t^{-1} - 1)$:

$$\frac{dy_t}{dt} = (1-\mu_t)\left(\frac{\lambda}{\mu_t} + \rho\right) y_t + \gamma^{-1}\mu_t r_t$$

4. In the limit as $\rho \to 0$,

$$\frac{dy_t}{dt} - \frac{d\mu_t}{dt}\frac{1}{\mu_t} y_t = \gamma^{-1}\mu_t r_t.$$

Write an integrating factor as $e^{\int_t^\infty \frac{d\mu_s}{ds}\frac{1}{\mu_s}ds}$. Multiplying both sides,

$$\underbrace{e^{\int_t^\infty \frac{d\mu_s}{ds}\frac{1}{\mu_s}ds}\left(\frac{dy_t}{dt} - \frac{d\mu_t}{dt}\frac{1}{\mu_t} y_t\right)}_{\frac{d}{dt}\left[y_t e^{\int_t^\infty \frac{d\mu_s}{ds}\frac{1}{\mu_s}ds}\right]} = e^{\int_t^\infty \frac{d\mu_s}{ds}\frac{1}{\mu_s}ds}\gamma^{-1}\mu_t r_t.$$

Solving out the integral, $u = \mu_s$, $du = \frac{d\mu_s}{ds}$,

$$\int_t^\infty \frac{d\mu_s}{ds}\frac{1}{\mu_s}ds = \int_{\mu_t}^1 \frac{1}{\mu_s}d\mu_s = \lim_{s\to\infty}\log(\mu_s) - \log(\mu_t) = -\log(\mu_t)$$

such that

$$\frac{d}{dt}\left[\frac{1}{\mu_t}y_t\right] = \frac{1}{\mu_t}\gamma^{-1}\mu_t r_t.$$

Integrating from $t$ to $\infty$, assuming $\lim_{\tau\to\infty} y_\tau = 0$:

$$\int_t^\infty \frac{d}{ds}\left(\frac{1}{\mu_\tau}y_s\right)d\tau = \gamma^{-1}\int_t^\infty r_\tau d\tau$$

$$\Rightarrow \quad 0 - \frac{1}{\mu_t}y_t = \gamma^{-1}\int_t^\infty r_\tau d\tau$$

$$y_t = -\gamma^{-1}\mu_t\int_t^\infty r_\tau d\tau$$

If $r_\tau = r_0 e^{-\kappa\tau}$, then

$$y_t = -\gamma^{-1}\mu_t \int_t^\infty r_0 e^{-\kappa\tau} d\tau$$

$$y_t = -\gamma^{-1}\mu_t \frac{1}{\kappa} r_0 e^{-\kappa t}$$

Substituting the definition of $\mu_t$ into the expression concludes the derivation.

### B.4 Solving the Stable Subspace of Equation (42)

Start with equation (42):

$$\begin{bmatrix} \mathbb{E}_t[d\widehat{c}] \\ d\widehat{r} \\ d\overline{y} \\ d\overline{r} \end{bmatrix} = \begin{bmatrix} 0 & \gamma^{-1} & 0 & 0 \\ 0 & -\kappa & 0 & 0 \\ \lambda & 0 & -\lambda & \gamma^{-1} \\ 0 & \lambda & 0 & -\lambda-\kappa \end{bmatrix} \begin{bmatrix} \widehat{c} \\ \widehat{r} \\ \overline{y} \\ \overline{r} \end{bmatrix} dt$$

The eigenvectors of the system matrix can be collected into the change of bases matrices from ($P$) and to ($P^{-1}$) eigen coordinates, where the eigenvector columns correspond to the eigenvalues listed in descending order:

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -\kappa\gamma & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & -\kappa\gamma & 0 & -\kappa\gamma \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & \frac{1}{\gamma\kappa} & 0 & 0 \\ 0 & -\frac{1}{\gamma\kappa} & 0 & 0 \\ -1 & -\frac{1}{\gamma\kappa} & 1 & \frac{1}{\gamma\kappa} \\ 0 & \frac{1}{\gamma\kappa} & 0 & -\frac{1}{\gamma\kappa} \end{bmatrix}$$

Under the restriction that system dynamics are orthogonal to the zero eigenvector (such that the system strictly returns to steady-state), the first row of $P^{-1}$ dotted with the state system must be zero, such that

$$\widehat{c}_t = -\gamma^{-1}\frac{1}{\kappa}\widehat{r}_t.$$

The updating households will switch to the full information IRF once they become aware of the shock. The choice of $y_t$ will then correspond with the $\overline{y}$ expectations (after being updated for learning). Eliminating the solved consumption choice of the updating households, I arrive at the stable system

$$\begin{bmatrix} d\widehat{r} \\ d\overline{y} \\ d\overline{r} \end{bmatrix} = \begin{bmatrix} -\kappa & 0 & 0 \\ -\lambda\gamma^{-1}\kappa^{-1} & -\lambda & \gamma^{-1} \\ \lambda & 0 & -\lambda-\kappa \end{bmatrix} \begin{bmatrix} \widehat{r} \\ \overline{c} \\ \overline{r} \end{bmatrix} dt$$

With the control variable associated with updating agents substituted out, the system is in its stable subspace; it is simply a system of linear ODEs with a known set of initial conditions. Integrating this system forward (starting with interest rates, then expected interest rates, and then expected output),

$$\begin{bmatrix} \widehat{r}_t \\ \overline{y}_t \\ \overline{r}_t \end{bmatrix} = \begin{bmatrix} e^{-\kappa t} & 0 & 0 \\ -\gamma^{-1}\frac{1}{\kappa}\big(e^{-\kappa t} - e^{-(\lambda+\kappa)t}\big) & e^{-\lambda t} & -\gamma^{-1}\frac{1}{\kappa}\big(e^{-\lambda t} - e^{-(\lambda+\kappa)t}\big) \\ e^{-\kappa t} - e^{-(\lambda+\kappa)t} & 0 & e^{-(\lambda+\kappa)t} \end{bmatrix} \begin{bmatrix} \widehat{r}_0 \\ \overline{y}_0 \\ \overline{r}_0 \end{bmatrix}.$$

Using the initial conditions that $\overline{y}_0 = 0$ and $\overline{r}_0 = 0$,

$$y_t = -\gamma^{-1}\frac{1}{\kappa}\big(e^{-\lambda t} - e^{-(\lambda+\kappa)t}\big)r_0.$$

The linearized solution matches the closed form solution obtained with $\rho = 0$ exactly. Note that $\overline{y}_t$ (post updating) is equal to the actual realized $y_t$; aggregate output is equal to the consumption decisions chosen by all of the agents in the population, averaged over their beliefs.

# C Appendix to Section 4: Medium-Scale HANK Model Derivation

## C.1 Labor Agency Sector

Hiring in the labor market is managed identical labor agencies indexed by $j$. Each one hires such that their number of employees $N_t(j)$ evolves according to

$$\dot{N}_t(j) = \lambda_{v,t} v_t(j) - \lambda_{EU} N_t(j)$$

If hiring firm $j$ internalizes a downward-sloping demand curve for its laborers' hours $h_t(j)$, then a CES demand system implies

$$h_t(j) = \left(\frac{r_t^l(j)}{r_t^l}\right)^{-\varepsilon_\ell} h_t$$

such that

$$\begin{aligned} r_{a,t} J_t(N_t(j)) = \max_{v_t(j), r_t^l(j)} & \left\{ (r_t^l(j) - w_t) \left(\frac{r_t^l(j)}{r_t^l}\right)^{-\varepsilon_l} N_t(j) Z_t h_t - \frac{\phi_h}{2}(h_t(j) - h^*)^2 N_t(j) Z_t \right. \\ & \left. - \eta_v v_t(j) - \eta_l \lambda_{v,t} v_t(j) + J_t'(N_t(j))[\lambda_{v,t} v_t(j) - \lambda_{EU} N_t(j)] + \partial_t J_t(N_t(j)) \right\} \end{aligned}$$

$$\begin{aligned} r_{a,t} J_t(N_t(j)) = \max_{v_t(j), r_t^l(j)} & \left\{ (r_t^l(j) - w_t) \left(\frac{r_t^l(j)}{r_t^l}\right)^{-\varepsilon_l} N_t(j) Z_t h_t - \frac{\phi_h}{2} \left( \left(\frac{r_t^l(j)}{r_t^l}\right)^{-\varepsilon_\ell} h_t - h^* \right)^2 N_t(j) Z_t h_t \right. \\ & \left. - \eta_v v_t(j) - \eta_l \lambda_{v,t} v_t(j) + J_t'(N_t(j))[\lambda_{v,t} v_t(j) - \lambda_{EU} N_t(j)] + \partial_t J_t(N_t(j)) \right\} \end{aligned}$$

FOC:

$$\left(\frac{r_t^l(j)}{r_t^l}\right)^{-\varepsilon_l} + (r_t^l(j) - w_t) \left(\frac{r_t^l(j)}{r_t^l}\right)^{-\varepsilon_l} (-\varepsilon_l) \frac{1}{r_t^l} - \phi_h \left( \left(\frac{r_t^l(j)}{r_t^l}\right)^{-\varepsilon_\ell} h_t - h^* \right) \left(\frac{r_t^l(j)}{r_t^l}\right)^{-\varepsilon_\ell} (-\varepsilon_l) \frac{1}{r_t^l(j)} = 0$$

In a symmetric equilibrium,

$$\begin{aligned} 1 &= \varepsilon_l (r_t^l - w_t) \frac{1}{r_t^l} - \varepsilon_l \phi_h (h_t - h^*) \frac{1}{r_t^l} \\ r_t^l(\varepsilon_l - 1) &= \varepsilon_l w_t + \varepsilon_l \phi_h (h_t - h^*) \\ r_t^l &= \frac{\varepsilon_l}{\varepsilon_l - 1} [w_t + \phi_h (h_t - h^*)] \end{aligned}$$

The FOC with respect to $v_t(j)$ is (in a symmetric equilibrium, dropping the $j$ subscripts)

$$-(\eta_v + \eta_l \lambda_{v,t}) + J_t'(N_t) \lambda_{v,t} = 0$$

such that the marginal benefit of posting a vacancy is equated to its marginal cost:

$$J_t'(N_t) \lambda_{v,t} = (\eta_v + \eta_l \lambda_{v,t})$$

$$\Rightarrow \quad J_t'(N_t) = \frac{\eta_v}{\lambda_{v,t}} + \eta_l$$

Totally differentiating:

$$d(J_t'(N_t)) = d\left(\frac{\eta_v}{\lambda_{v,t}}\right) = -\left(\frac{\eta_v}{\lambda_{v,t}^2}\right) d\lambda_{v,t}$$

The envelope condition indicates

$$r_{a,t}J_t'(N_t) = (r_t^l - w_t)Z_t h_t - \lambda_{EU} J_t'(N_t) + \underbrace{J_t''(N_t)[\lambda_{v,t}v_t - \lambda_{EU} N_t] + \partial_t J_t'(N_t)}_{\mathbb{E}[dJ_t'(N_t)]/dt}$$

Thus

$$(r_{a,t} + \lambda_{EU})\left[\frac{\eta_v}{\lambda_{v,t}} + \eta_l\right] = (r_t^l - w_t)Z_t h_t - \left(\frac{\eta_v}{\lambda_{v,t}^2}\right)\frac{\mathbb{E}_t[d\lambda_{v,t}]}{dt}$$

Rearranging:

$$(r_{a,t} + \lambda_{EU})\left[1 + \frac{\eta_l}{\eta_v}\lambda_{v,t}\right]\lambda_{v,t} = (r_t^l - w_t)Z_t h_t\left(\frac{\lambda_{v,t}}{\eta_v}\right)\lambda_{v,t} - \frac{\mathbb{E}_t[d\lambda_{v,t}]}{dt}$$

such that

$$\frac{\mathbb{E}_t[d\lambda_{v,t}]}{dt}\frac{1}{\lambda_{v,t}} = (r_t^l - w_t)Z_t h_t\left(\frac{\lambda_{v,t}}{\eta_v}\right) - (r_{a,t} + \lambda_{EU})\left[1 + \frac{\eta_l}{\eta_v}\lambda_{v,t}\right]$$

Note that this means that in the NSS,

$$\frac{1}{\varepsilon_l} wZh\left(\frac{\lambda_v}{\eta_v}\right) = (r_a + \lambda_{EU})\left[1 + \frac{\eta_l}{\eta_v}\lambda_v\right]$$

To simplify the calibration, I set $\varepsilon_l$ such that steady-state hours worked are equal to 0.5:

$$\varepsilon_l = \frac{wZh\left(\frac{\lambda_v}{\eta_v}\right)}{(r_a + \lambda_{EU})\left[1 + \frac{\eta_l}{\eta_v}\lambda_v\right]}$$

Note that because $r_a$ and $\lambda_{EU}$ are relatively small, this implies $\varepsilon_l$ is large – such that labor hours are very substitutable across firms.

## C.2 Mutual Fund Problem

If there are adjustment costs to altering capital, each mutual fund solves

$$\begin{aligned}
\Pi_0(I_0, K_0) = \max_{(I_t,u_t)_{t\geq 0}} \ \mathbb{E}_0 \int_0^\infty e^{-\int_0^t r_\tau d\tau} \Big[ (r_t^k u_t K_t - I_t - \psi(u_t) K_t \Big] dt \\
\text{s.t. } \frac{dK_t}{dt} = e^{\zeta_t^I}[1 - \Phi(I_t, \bar{I}_t)] I_t - \delta K_t \\
\frac{d\bar{I}_t}{dt} = \kappa_I (I_t - \overline{I}_t)
\end{aligned}$$

Let $k_t \equiv \frac{K_t}{A_t}$ and $x_t \equiv \frac{I_t}{A_t}$, where $A_t = A_0 e^{g_Y t}$. Note that this means $\dot{x}_t = \dot{I}_t / A_t - x_t g_Y$, such that $\dot{I}_t / I_t = \dot{x}_t / x_t + g_Y$. Similarly, $\dot{k}_t = x_t - (\delta + g_Y) k_t$. The stationary problem is

$$\begin{aligned}
\Pi_0(x_0, k_0) = \max_{(x_t,u_t)_{t\geq 0}} \ \mathbb{E}_0 \int_0^\infty e^{-\int_0^t (r_\tau - g_Y) d\tau} \Big[ (r_t^k u_t k_t - x_t - \psi(u_t) k_t \Big] dt \\
\text{s.t. } \frac{dk_t}{dt} = e^{\zeta_t^I}[1 - \Phi(x_t, \overline{x}_t)] x_t - (\delta + g_Y) k_t \\
\frac{d\overline{x}_t}{dt} = \kappa_I (x_t - \overline{x}_t)
\end{aligned}$$

Writing the problem in recursive form,

$$(r_t - g_Y)\Pi_t = \max_{x_t, u_t} r_t^k u_t k_t - x_t - \psi(u_t) k_t + \overbrace{\frac{\partial \Pi_t}{\partial \overline{x}_t} \kappa_I (\tilde{x}_t - \overline{x}_t) + \frac{\partial \Pi_t}{\partial k_t} (e^{\zeta_t^I}[1 - \Phi(x_t, \overline{x}_t)] x_t - (\delta + g_Y) k_t) + \frac{\partial \Pi_t}{\partial t}}^{\mathcal{D}\Pi_t}$$

Taking FOCs,

$$-1 + \frac{\partial \Pi_t}{\partial k_t} e^{\zeta_t^I} [1 - \Phi(x_t, \overline{x}_t) - \partial_x \Phi(x_t, \overline{x}_t) x_t] = 0$$

$$r_t^k k_t - \psi'(u_t) k_t = 0$$

The last FOC equates the marginal value of renting more capital on the intensive margin with the cost of renting more intensive capital:

$$\psi'(u_t) = r_t^k$$

Rearranging the first FOC, and writing $\frac{\partial \Pi}{\partial k} \equiv q_t^K$ and $\frac{\partial \Pi}{\partial \overline{x}} \equiv \overline{q}_t^I$:

$$q_t^k e^{\zeta_t^I} [1 - \Phi(x_t, \overline{x}_t) - \partial_x \Phi(x_t, \overline{x}_t) x_t] = 1$$

Using the envelope theorem,

$$(r_t - g_Y)\partial_k \Pi_t = r_t^k u_t - \psi(u_t) + \partial_k \left[ \frac{\partial \Pi_t}{\partial \overline{x}_t} \kappa_I (\tilde{x}_t - \overline{x}_t) + \frac{\partial \Pi_t}{\partial k_t} \left( e^{\zeta_t^I}[1 - \Phi(x_t, \overline{x}_t)] x_t - (\delta + g_Y) k_t \right) + \frac{\partial \Pi_t}{\partial t} \right]$$

$$= r_t^k u_t - \psi(u_t) + \underbrace{\left[ \frac{\partial \partial_k \Pi_t}{\partial \overline{x}_t} \kappa_I (\tilde{x}_t - \overline{x}_t) + \frac{\partial \partial_k \Pi_t}{\partial k_t} (e^{\zeta_t^I}[1 - \Phi(x_t, \dot{x}_t)] x_t - (\delta + g_Y) k_t) + \frac{\partial \partial_k \Pi_t}{\partial t} \right]}_{\frac{\mathbb{E}[d \partial_k \Pi_t]}{dt}} - (\delta + g_Y) \frac{\partial \Pi_t}{\partial k_t}$$

such that

$$(r_t - g_Y)q_t^k = r_t^k u_t - (\delta + g_Y)q_t^k - \psi(u_t) + \frac{\mathbb{E}[dq_t^k]}{dt}.$$

$$\Rightarrow \frac{\mathbb{E}[dq_t^k]}{dt} = (r_t + \delta)q_t^k - r_t^k u_t + \psi(u_t).$$

### C.2.1 Quadratic Costs

Suppose the adjustment costs take the form

$$\Phi(I_t, \overline{I}_t) = \frac{\phi_I}{2}\left(\frac{I_t}{\overline{I}_t} - 1\right)^2 = \frac{\phi_I}{2}\left(\frac{x_t}{\overline{x}_t} - 1\right)^2$$

such that

$$\partial_x \Phi(x_t, \overline{x}_t) = \phi_I \left(\frac{x_t}{\overline{x}_t} - 1\right)\frac{1}{\overline{x}_t}$$

Plugging these functional forms in, the investment choice is

$$q_t^k e^{\zeta_t^I}\left[1 - \frac{\phi_I}{2}\left(\frac{x_t}{\overline{x}_t} - 1\right)^2 - \phi_I\left(\frac{x_t}{\overline{x}_t} - 1\right)\frac{x_t}{\overline{x}_t}\right] = 1$$

Solving this quadratic expression,

$$x_t = \left\{\frac{2}{3} + \frac{1}{2}\left(\frac{16}{9} + \frac{8}{3\phi_I}\left[1 - \frac{\phi_I}{2} - \frac{1}{q_t^k e^{\zeta_t^I}}\right]\right)^{1/2}\right\}\overline{x}_t$$

## C.3 Financier Households (external habit formation)

"Financier" households recieve dividend and interest income and maximize utility:

$$\max_{(c_t)_{t\geq 0}} \mathbb{E}_0 \left[ \int_0^\infty e^{-\rho^* t} \left[ e^{\zeta_t^c} \frac{(c_t^{FIN} - \beta_c \bar{c}_t)^{1-\gamma}}{1-\gamma} \right] \right]$$
$$\text{s.t. } \frac{da_t}{dt} = (i_t - \pi_t - g_Y) a_t + s_t - c_t^{FIN}$$
$$\frac{d\bar{c}_t}{dt} = \theta_c C_t^{FIN} - \theta_c \bar{c}_t$$

Where $i$ is the nominal interest rate, $w$ is the real wage, $p$ is the price of the consumption good, and $s$ is the dividend paid out by firms to shareholders. In equilibrium, $c_t^{FIN} = C_t^{FIN}$ – but this is not internalized by the individual household.

The household HJB is then

$$\begin{aligned} \rho^* V(a, \bar{c}^{FIN}; \zeta) = \max_{c,h} \Bigg\{ & \left[ e^{\zeta_t^c} \frac{(c_t^{FIN} - \beta_c \bar{c}_t^{FIN})^{1-\gamma}}{1-\gamma} \right] \\ & + \partial_a V(a, \bar{c}^{FIN}; \zeta)[(i - \pi - g_Y)a + s - c^{FIN}] \\ & + \partial_{\bar{c}^{FIN}} V(a, \bar{c}^{FIN}; \zeta)[\theta_c C^{FIN} - \theta_c \bar{c}^{FIN}] \\ & + \underbrace{\frac{\mathbb{E}_t^\zeta dV(a, \bar{C}; \zeta)}{dt}}_{\mathcal{D}_\zeta V(a, \bar{C}; \zeta)} \Bigg\} \end{aligned}$$

Taking FOCs,

$$e^{\zeta_c}(c^{FIN} - \beta_c \bar{c}^{FIN})^{-\gamma} - \partial_a V(a, \bar{c}^{FIN}; \zeta) = 0$$

Using the Envelope Theorem,

$$\rho^* \partial_a V = \partial_a V(i - \pi) + \partial_a^2 V([i - \pi]a + s - c) + \theta \partial_a \partial_{\bar{c}} V(\theta_c c - \theta_c \bar{c}) + \partial_a \partial_t V$$

Thus, using $\mathcal{D}$ as the macro-variable infinitessimal generator,

$$\partial_a V(\rho^* - [i - \pi]) = \mathcal{D} \partial_a V = \frac{\mathbb{E}_t[d\partial_a V]}{dt}$$

Thus the Euler equation becomes

$$\frac{\mathbb{E}_t[d(c_t^{FIN} - \beta_c \bar{c}_t^{FIN})^{-\gamma}]}{dt} = (\rho^* - [i - \pi])(c_t^{FIN} - \beta_c \bar{c}_t^{FIN})^{-\gamma}$$

With the chain rule,

$$-\gamma(c_t^{FIN} - \beta_c \bar{c}_t^{FIN})^{-\gamma-1} \frac{\mathbb{E}_t[d(c_t^{FIN} - \beta \bar{c}_t^{FIN})]}{dt} = (\rho^* - [i - \pi])(c_t^{FIN} - \beta_c \bar{c}_t)^{-\gamma}$$

$$\Rightarrow \quad \frac{\mathbb{E}_t[dc_t^{FIN}]}{dt} - \beta_c \frac{d\bar{c}_t^{FIN}}{dt} = \frac{1}{\gamma}([i - \pi] - \rho)(c_t^{FIN} - \beta_c \bar{c}_t)$$

Since $\frac{d\bar{c}_t}{dt} = -\theta(\bar{c}_t - c_t^{FIN})$, the financier agents' consumption thus evolves according to

$$\frac{\mathbb{E}_t[dc_t^{FIN}]}{dt} = \gamma^{-1}([i - \pi] - \rho^*)(c_t^{FIN} - \beta_c\bar{c}_t) - \beta_c\theta_c(\bar{c}_t^{FIN} - c_t^{FIN}) \tag{51}$$

# D Appendix to Section 6: Persistence Intuition

## D.1 Persistence and Unemployment

## D.2 Persistence and Inflation

Consider the NKPC

$$\frac{\mathbb{E}[d\pi_t]}{dt} = r_t(\pi_t - \overline{\pi}_t) + \frac{\varepsilon}{\theta_\pi}\left[\frac{\varepsilon-1}{\varepsilon} - m_t\right] + \kappa(\pi_t - \overline{\pi}_t),$$

where

$$\frac{d\overline{\pi}_t}{dt} = \kappa(\pi_t - \overline{\pi}_t)$$

and $\overline{\pi}_0 = 0$, $\mathbb{E}_t[\lim_{T\to\infty} e^{-\int_t^T r_s ds}\pi_T] = 0$. Rewriting the expression using $x_t \equiv \pi_t - \overline{\pi}_t$, such that

$$\frac{\mathbb{E}[dx_t]}{dt} = r_t x + \frac{\varepsilon}{\theta_\pi}\left[\frac{\varepsilon-1}{\varepsilon} - m_t\right],$$

such that the ODE can be integrated forward to write

$$x_t = \frac{\varepsilon}{\theta_\pi}\int_t^\infty e^{-\int_t^s r_\tau d\tau}\left[m_s - \frac{\varepsilon-1}{\varepsilon}\right] ds$$

The increase of inflation from its lagged values is proportional to the present value of the future marginal cost gaps.

From there, note that

$$\dot{\pi}_t = \dot{x}_t + \kappa x_t$$

while log-linearizing (with $x_t = 0$ in the NSS),

$$\dot{x}_t = r x_t - \frac{\varepsilon-1}{\theta_\pi}\widehat{m}_t$$

From the first equation, assuming $\lim_{T\to\infty}\pi_T = 0$,

$$x_t = \frac{\varepsilon-1}{\theta_\pi}\int_t^\infty e^{-r(s-t)}\widehat{m}_s ds$$

The backward-looking inflation average term is then

$$\overline{\pi}_t = \int_0^t \dot{\overline{\pi}}_s ds = \int_0^t \kappa x_s ds = \frac{\varepsilon-1}{\theta_\pi}\kappa\int_0^t\int_s^\infty e^{-r(\tau-s)}\widehat{m}_\tau d\tau \ ds$$

and changing the bounds of the integral,

$$= \frac{\varepsilon-1}{\theta_\pi}\frac{\kappa}{r}\left[\int_0^t (1-e^{-rs})\widehat{m}_s ds + \int_t^\infty (e^{-r(s-t)} - e^{-rs})\widehat{m}_s ds\right]$$

Using the fact that

$$\pi_t = x_t + \overline{\pi}_t,$$

it follows that inflation is

$$\pi_t = \frac{\varepsilon - 1}{\theta_\pi} \int_t^\infty e^{-r(s-t)} \widehat{m}_s ds + \frac{\varepsilon - 1}{\theta_\pi} \frac{\kappa}{r} \left[ \int_0^t (1 - e^{-rs}) \widehat{m}_s ds + \int_t^\infty (e^{-r(s-t)} - e^{-rs}) \widehat{m}_s ds \right].$$

Simplifying the algebra, inflation is then the sum of both a backward-looking and a forward-looking component.

$$\pi_t = \frac{\varepsilon - 1}{\theta_\pi} \frac{\kappa}{r} \left[ \int_0^t (1 - e^{-rs}) \widehat{m}_s ds + \int_t^\infty \left[ \left(1 + \frac{r}{\kappa}\right) e^{-r(s-t)} - e^{-rs} \right] \widehat{m}_s ds \right].$$

If the marginal costs jump and follow an exponential decay according to $\widehat{m}_t = m_0 e^{-\rho_2 s}$, then inflation at each point in time will be

$$\pi_t = \frac{\varepsilon - 1}{\theta_\pi} \frac{\kappa}{r} \left[ \int_0^t (1 - e^{-rs}) m_0 e^{-\rho_2 s} ds + \int_t^\infty \left[ \left(1 + \frac{r}{\kappa}\right) e^{-r(s-t)} - e^{-rs} \right] m_0 e^{-\rho_2 s} ds \right]$$

Three equation system:

$$\frac{\mathbb{E}[d\pi_t]}{dt} = (r + \kappa)(\pi_t - \overline{\pi}_t) + \frac{\varepsilon - 1}{\theta_\pi} \widehat{m}_t$$

$$\frac{d\overline{\pi}_t}{dt} = \kappa(\pi_t - \overline{\pi}_t)$$

$$\frac{d\widehat{m}_t}{dt} = -\rho_2 \widehat{m}_t$$

Write the system as a matrix:

$$\begin{bmatrix} \mathbb{E}[d\pi_t] \\ d\overline{\pi}_t \\ d\widehat{m}_t \end{bmatrix} = \begin{bmatrix} r + \kappa & -r - \kappa & \frac{\varepsilon - 1}{\theta_\pi} \\ \kappa & -\kappa & 0 \\ 0 & 0 & -\rho_2 \end{bmatrix} \begin{bmatrix} \pi_t \\ \overline{\pi}_t \\ \widehat{m}_t \end{bmatrix} dt$$

The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} r + \kappa - \lambda & -r - \kappa & \frac{\varepsilon - 1}{\theta_\pi} \\ \kappa & -\kappa - \lambda & 0 \\ 0 & 0 & -\rho_2 - \lambda \end{bmatrix}$$

$$= [(r + \kappa - \lambda)(-\kappa - \lambda) + \kappa(r + \kappa)](-\rho_2 - \lambda)$$

$$= \lambda(\lambda - r)(-\lambda - \rho_2)$$

Clearly, the roots are $-\rho_2 < 0$ (stable), $r > 0$ (explosive) and 0 (borderline).