Transfer Payments, Sacrifice Ratios, and Inflation in a Fiscal Theory HANK[1]

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Abstract

I numerically solve a calibrated Heterogeneous Agent New-Keynesian (HANK) model that features nominal rigidities, incomplete markets, hand-to-mouth households, nominal long-term government debt, and active fiscal policy with a passive monetary policy rule to analyze the implications of the fiscal theory of the price level (FTPL) in a setting with wealth and income inequality. In model simulations, the total cumulative inflation generated by a fiscal helicopter drop is largely determined by the size of the initial stimulus and is relatively insensitive to the initial distribution of the payments. In contrast, the total real GDP and employment response depends much more strongly on the balance sheets of the transfer recipients, such that payments to and from households with few assets and high marginal propensities to consume (MPCs) move aggregate output much more strongly than payments to or from households with low MPCs. Stimulus checks to poor households thus yield a bigger economic boom but are about as inflationary in the medium-to-long run as checks sent to wealthy households in the FTPL environment, while lump-sum taxes on wealthier households yield disinflation at a much lower cost to employment than taxes on poorer households. Despite the incomplete markets setting, tight monetary policy is still able to reduce inflation in the short run through a forward guidance wealth effect, while still driving up inflation in the long term.

Keywords: FTPL, HANK, sacrifice ratios, continuous time models
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1 Introduction

Recent theoretical work related to the fiscal theory of the price level (FTPL) has highlighted the ways in which higher government deficits financed with nominally-denominated bonds can determine the overall price level and the path of inflation, as summarized in Cochrane (2023). However, the vast majority of the work dealing with FTPL has thus far been constrained to representative agent models, with little discussion of how inequality and household heterogeneity might inform the theory and its predictions for monetary and fiscal policy at business cycle frequencies. Fiscal policy in particular can be (and often is) tailored to direct stimulus transfers to particular segments of the population based on their position in the joint distribution of household assets and incomes. For a policymaker contemplating what to do in a recession, several natural questions might arise, which the existing fiscal theory literature has not yet directly addressed. First, in an environment where the FTPL is active, do transfers to some households generate more or less inflation than transfers to other households? Second, do some policies lead to a bigger real economic expansions than others, relative to the amount of inflation that they produce? This could be conversely framed for policymakers attempting to fight inflation: are there some targeted fiscal actions that could reduce inflation at a lower cost to output and employment than others? Thirdly, is tight monetary policy still an effective tool for fighting inflation in a fiscal theory world with incomplete markets and household heterogeneity, and how does it compare to monetary policy?

In this paper, I demonstrate that fiscal theory has strong implications for each of these three questions when combined with recent models of inequality and the business cycle. To do so, I integrate the fiscal theory of the price level with a canonical heterogeneous agent New-Keynesian (HANK) model. More precisely, I modify a Bewley-Aiyagari incomplete markets environment with uninsurable income risk, borrowing constraints, and nominal rigidities (a setting similar to the one of McKay, Nakamura, and Steinsson (2016)) to include long-term nominal government debt. The resulting household inequality allows me to discuss what happens when stimulus checks are sent to some groups, but not to others; the inclusion of long-term government debt, as developed in the representative agent framework of Cochrane (2018), provides a parsimonious channel for interest rate hikes to lower the price level in the FTPL. To use the terminology first popularized by Leeper (1991), I further assume that monetary policy is "passive" in that the central bank does not systematically increase nominal interest rates by more than the inflation rate in response to an inflationary episode (in contrast to McKay, Nakamura, and Steinsson (2016), which assumes the opposite), while I allow fiscal policy in my model to be "active" such that the fiscal authority does not automatically adjust the path of its primary surpluses in response to changes in the real interest rate or the price level; this allows the FTPL to be active in my setting. At the time of writing, the FTPL and HANK ingredients themselves are standard, but their combination is new: I incorporate the minimum elements necessary to explore the three policy questions that I have posed.

Before running the simulations, one can plausibly intuit what the answers to the questions are, at least when the FTPL mechanism for determining inflation applies. First, while New Keynesian elements like the Phillips Curve might complicate the short-run dynamics of inflation, the dominant forces in fiscal theory suggest that transfers to low-income (on average, poorer) households will generate roughly the same amount of inflation as transfers to high-income (on average, wealthier) households, holding the total size of the stimulus spending equal: the cumulative inflationary impact of issuing stimulus checks "should" be determined largely by how much the government spends in total, not by who the recipients of the transfers are.

Why might one predict this? Fiscal theory's central equation is an asset pricing equation, which stipulates that if a government's liabilities (both bonds and actual currency notes) are a claim to its future primary fiscal surpluses, then the *real* market value of government debt outstanding must be equal to the present value of those primary surpluses, where the relevant discount rate is the real interest rate[3] or a more general (but related) stochastic discount factor. Furthermore, if the government's liabilities are nominally denominated, then the government can settle its debts using inflated dollars. A government can thereby run deficits and not raise future surpluses by either an immediate devaluation of the currency in the case of fully flexible prices, or by an increase in the rate of inflation that reduces its future real interest expense, in the more realistic setting where prices do not jump immediately.[4] Either way, the fiscal authority is able to effectively default on part of its real obligations to bondholders. Though this channel, the FTPL thereby establishes its primary equilibrium determination mechanism for the path of inflation: inflation must be whatever it has to be to keep the real value of government debt from exploding and the asset pricing equation an equality, such that whatever the government spends but does back by new taxes today or in the future, it must inflate away. Taking the mechanism seriously, it is then not surprising that if two stimulus packages deploy the same amount of spending, then the model predicts that they will generate the same amount of inflation, regardless of where the money went.

How about the second question, which asks which stimulus policies produce the largest economic expansions relative to the amount of inflation they produce? The recent HANK literature provides a natural explanation as to why the output response could be predictably stronger when checks are disbursed to low-income households. One of the literature's seminal (and representative) papers is Kaplan, Moll, and Violante (2018), which re-examines the transmission mechanism of monetary policy in a setting where a large fraction of agents lack liquid assets (and thus live largely hand-to-mouth, passing through changes in their income directly into their spending). They find that monetary policy in their simulated environment acts primarily through the automatic real fiscal adjustments that governments must make in standard (passive-fiscal) models to balance their inter-temporal budgets and through general equilibrium effects that change labor. Both of these effects move resources

[3] If the buyer of government debt is risk-neutral or in a certainty-equivalent setting

[4] Of course, it's worth re-iterating that the maneuver only works when monetary policy is "passive" and real rates do not rise in response to inflation.

toward low-asset agents living paycheck-to-paycheck with a high marginal propensity to consume (MPC) out of their current income, which jump-starts an expansionary process wherein spending begets income begets production which begets spending, in a dynamic loop reminiscent of textbook "old" Keynesian static models. Auclert, Rognlie, and Straub (2018) and Auclert, Rognlie, and Straub (2023) discuss how these feedbacks from a stimulus continue until excess savings eventually "trickle up" to wealthier low-MPC households, arresting the cycle. Of course, all of these models assume passive fiscal policy that reacts to real interest rates and balances the government's budget over time, and where active monetary policy ensures inflation determinacy (via a threat to induce hyperinflation, as recounted in Cochrane (2011)) – which is why this paper focuses on an active fiscal/passive monetary policy regime instead. Even so, one would think that MPC heterogeneity should remain a quantitatively important concern for the real output impulse responses to policy shocks.

One can then combine these two predictions. If inflation in fiscal theory is driven by the government's need to inflate away debt, and is therefore mostly invariant to who received the checks disbursed by the deficits, but bigger economic expansions follow when the checks are sent to agents who will pass more of that income into spending, then it follows that transfers to segments of the population that are more likely to be poor and hand-to-mouth will generate the more real GDP and employment relative to the amount of cumulative inflation produced by the stimulus, relative to payments to wealthier groups. Running this process in reverse, one can then intuit that the issuance of new lump-sum taxes on households with large amounts of liquid assets will mean that the government will have to inflate away less of its debt, reassuring bondholders that their investments are safe and generating less inflation while leading to a smaller contraction to the real economy compared to taxes on the poor (as high-liquid asset households will not respond to the taxes with as severe cuts to their consumption spending as low-asset households).

But what about the third question? What are the effects of a transitory rise of nominal interest rates on the price level and output, and how do they compare to a counter-inflationary tax regime? Sims (2011) and Cochrane (2018) provide valuable intuition here: if the government issues long-term nominal government debt, the expected path of nominal interest rates can change long-term nominal bond prices, generating a wealth effect that can temporarily lower inflation (before the neo-Fisherian, long-term money neutrality properties of the model eventually lead the higher nominal rates to pull inflation higher). Cochrane (2018) describes the mechanism in considerable detail, and notes that the wealth effect is not driven by *current* rates per se, but rather by expectations, making the effect much more like forward guidance. But does this work in an incomplete markets setting? After all, McKay, Nakamura, and Steinsson (2016) is all about how the (unrealistically strong) theoretical power of forward guidance is attenuated in a typical HANK environment, and Cochrane (2018) documents how the neo-Fisherian force is a powerful one for a model to overcome in generating to generate a negative inflation response to higher interest rates. The key here is that McKay, Nakamura, and Steinsson

(2016) and Cochrane (2018) are talking about two slightly *different* mechanisms for forward guidance: while the latter's mechanism is through a wealth effect (which exists even in a frictionless model where the real rate does not move), the former's mechanism is a story about the intertemporal substitution channel propagating backward in time through the Euler equation to generate a substantial output gap, which then produces a large inflationary impulse through the Phillips curve. It makes sense for this standard new-Keynesian type of forward guidance to be blunted by introducing factors like hand-to-mouth agents and precautionary savings motives, which reduce the importance of the Euler equation's intertemporal substitution channel. The FTPL monetary disinflation impulse, however, comes from how the nominal yield curve prices bonds. One might thus suppose that the Cochrane (2018) fiscal theory model's description of tight money's deflationary power is therefore likely to be significantly less affected by the introduction of incomplete markets, despite being a forward guidance effect. Contractionary monetary policy in an FTPL-HANK model should still resemble contractionary monetary policy in the FTPL-RANK model of Cochrane (2018).

How do these predicted answers to my three questions hold up, when compared to the actual numerical solution of my HANK-FTPL model? In a word, they hold up well; this does appear to be how fiscal theory's ramifications for policy in the presence of inequality and heterogeneity. There are of course strong caveats to the above reasoning, which make it necessary to actually solve a model. For instance, fiscal surpluses are not entirely exogenous, thanks to the presence of automatic stabilizers like proportional income taxes (included in my model) and unemployment benefits (which I do not include). If one policy leads to a big economic expansion, the government may be able to effectively "tax the boom," either by recouping some of its stimulus money by capturing some fraction of the surge in economic output, or by reducing the amount of benefits that it pays out (as the households become more economically prosperous in the expansion). Additionally, the precise timing of interest rates might matter for a program's total cumulative inflation: a big economic boom might generate more inflation earlier, through the Phillips Curve and increases in the marginal labor costs necessary to bring more hours worked into the production process. This inflation might erode more of a government's real liabilities earlier, necessitating less inflation in the long run and lowering the cumulative inflationary impact overall. Interestingly, both of these forces would suggest that policies that boost aggregate spending by more are actually less inflationary, holding the total number of federal dollars spent constant. However, I find that both of these complications are very minor in numerical simulations. In any case, the heterogeneous agent framework brings into stark relief how the FTPL has a very different equilibrium determination mechanism from popular notions of "too much spending chasing too few goods" theories of inflation; the amount of private spending in the private economy doesn't drive inflation, the amount of public-sector spending (relative to taxes) does. For all of its similarities, the FTPL really is distinct from competing monetarist and New Keynesian stories of inflation, and the implications for policy are not trivial. In contrast, adding HANK to the FTPL does not substantially change the FTPL's predictions for monetary policy.

The rest of this paper sketches out a HANK world in which the FTPL is in play, to show that MPC heterogeneity and hand-to-mouth households combined with the FTPL price determination mechanism yields stark predictions for the trade-offs between output and inflation. Section 2 describes the incomplete-markets dynamic stochastic general equilibrium model. Section 3 details the model's calibration and moments from its non-stochastic steady-state. Section 4 describes the results of the simulations, including the impulse response functions for i) expansionary and contractionary monetary policy, ii) the impact of deficit-financed spending, iii) stimulus payments to high and low income households in particular, and iv) the implications of balanced-budget redistribution policy. Section 5 concludes.

2 The Model

In order to study the interactions between fiscal theory and inequality, one must employ a model that exhibits both. To this end, this paper's model is largely the merger of two workhorse ones, the McKay, Nakamura, and Steinsson (2016) HANK model and the Cochrane (2018) FTPL long-term debt framework. As such, it consists of a block of heterogeneous households who purchase consumption goods, have the option to save a perfectly liquid financial asset, and supply their labor to a monopolistically competitive intermediary production section, which in turn supplies its output to final goods firms, which produce consumer goods via a constant elasticity of substitution aggregator. The main elements of these blocks are indeed highly reminiscent of McKay, Nakamura, and Steinsson (2016) (and other papers in the HANK literature) − although I do also add in nominal wage rigidity and union power to the labor market as in Auclert, Rognlie, and Straub (2018), as the presence of nominal wage rigidities and acyclic real wages appear to be better supported by empirical evidence than their absence.[5] The government consists of a monetary authority (central bank) that sets nominal interest rates and a fiscal authority that determines taxes households via a proportional income tax and lump-sum transfers. In my model, however, fiscal policy is active, and the government can run deficits by spending in excess of its tax revenue that it does not necessarily have to balance later; it finances those deficits by issuing long-term nominal debt at the rate set by the central bank, which an investment fund purchases using household savings. Aggregate exogenous shock processes are denoted with the vector ζ; the model is first solved around its non-stochastic steady-state (NSS), where the aggregate shocks are $\zeta_t = 0$. It is then linearized around the NSS to approximate the dynamics of the economy in a business cycle when the shocks deviate from zero and then mean-revert.

[5]See the vector autoregressions in Christiano, Trabandt, and Walentin (2010) on the acyclicality of real wages in response to a monetary policy shock. For evidence that markups are not counter-cyclical in response to a demand shock, see Nekarda and Ramey (2020). For evidence against countercyclical markups during the 2021-2023 inflationary spell, see Glover, Mustre-del-Rio, and Ende-Becker (2023) and Glover, Mustre-del-Rio, and Nichols (2023).

2.1 The Households' Problem

The heterogeneous household block of this model is built around a standard Bewley-Aiyagari incomplete markets setting, wherein a measure-one continuum of agents can save and borrow in a single liquid asset and insure themselves against fluctuations in their personal income. The model does, however, include a new feedback from expectation errors regarding the price of nominal government debt into agents' wealth, which is necessitated by the fact that government debt is long-term and bond prices can jump.

Households discount the future at a rate ρ and maximize their time-separable expected lifetime utility V by choosing paths of consumption $(c_t)_{t\geq 0}$ given their liquid asset position a_t and their labor skills z_t (where a and z are *idiosyncratic* state variables specific to each household). The aggregate state of the economy is described by the distribution of households across idiosyncratic states $\mu_t(a, z)$ and a vector of *aggregate* shocks ζ to tax policy and the central bank's monetary policy rule. Since each household is infinitesimally small relative to the aggregate market, they take the distribution of the economy and the resultant prices (wages w_t and real interest rates r_t) as given. Inter-temporal discounting is set by the preference parameter ρ.

Households earn their income from the effective units of labor $(z_t h_t)$ that they supply to the marketplace, and their log skills evolve according to an Ornstein-Uhlenbeck AR(1) process with mean reversion governed by θ_z and shocks driven by a Brownian motion W_t with a variance of σ_z^2. Given the assumed market structure, households are not on their typical labor supply curves. Rather, they earn wages with a small markup over their marginal rate of substitution between labor and leisure, and supply their labor to small decentralized unions in the quantities determined by labor demand.

Households must pay a fraction τ of their wage income to the government in taxes. They must also pay lump-sum taxes $T(a_t, z_t, \zeta)$ to the fiscal authority (although $T(a_t, z_t, \zeta)$ can also be negative, in which case it is a lump-sum transfer, like a stimulus check). Households can earn a prevailing real market rate of return of r_t on their asset holdings, denoted a_t. These assets are perfectly liquid, and are claims to government liabilities (which could be interpreted as interest-bearing reserves or near-cash instruments, literal government bonds, or shares of an investment firm or bank that owns government bonds). Borrowing is subject to first a soft constraint, and then a hard constraint: households' lending rates are $r(a) = r_t + \Delta_r \mathbf{1}_{\{a\leq 0\}}$, such that interest rates go up by a wedge Δ_r for households with negative holdings. The hard borrowing constraint demands that $a \geq \underline{a}$ for every agent in the economy; both it and the higher borrowing rate cause a substantial measure of agents to amass at or near the zero-asset level, living largely hand-to-mouth for fear of taking on costly debt or maxing out their unsecured lines of credit.

The total ex-post realized return on assets dR_t in an infinitessimal time increment dt will thus include the asset's ex-ante expected real realized return $r_t(a)$, plus whatever surprise capital gains $d\delta_{a,t}$ the households make on their entire portfolio driven by the movement of asset prices. In a rational expectations equilibrium, these errors are mean-zero and endogenous to the model, as in Sims (2002).

In the absence of capital stock dynamics, all assets are purely financial in nature. Corporate profits are also remitted back to households, as in other HANK papers like Kaplan, Moll, and Violante (2018) and McKay, Nakamura, and Steinsson (2016), in payments $\Pi_t(a,z)$ contingent on the households' position in the state space.

All told, the household problem can be succinctly summarized mathematically as

$$V(a_0, z_0; \mu_0, \zeta_0) = \max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left[\frac{c_t^{1-\gamma}}{1-\gamma} - \frac{h_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] dt$$

$$\begin{aligned}
\text{s.t. } da_t &= [(1-\tau)w_t z_t h_t - T_t(a_t, z_t; \zeta_t) + \Pi_t(a,z) - c_t]dt + a_t dR_t \\
d\log(z_t) &= -\theta_z \log(z_t)dt + \sigma_z dW_{t,z} \\
dR_t &= r_t(a_t)dt + d\delta_{a,t} \\
r_t(a) &= r_t + \Delta_r \mathbf{1}_{a\leq 0} \\
a_t &\geq \underline{a}
\end{aligned}$$

Recursively, this value function can be reformulated as a Hamilton-Jacobi-Bellman (HJB) equation (using a first-order Taylor approximation around the point where the capital gain $d\delta_a$ is zero):

$$\begin{aligned}
\rho V(a,z;\mu,\zeta) = \max_c \Bigg\{ & \left[\frac{c^{1-\gamma}}{1-\gamma} - \frac{h^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] \\
&+ \frac{\partial V}{\partial a}(a,z;\mu,\zeta) \left[(1-\tau)wzh - T_t(a_t, z_t; \zeta_t) + \Pi_t(a,z) - c + r(a)a + \frac{d\delta_a}{dt} a \right] \\
&+ \frac{\partial V}{\partial z}(a,z;\mu,\zeta) z \left[\frac{1}{2}\sigma_z^2 - \theta_z \log(z) \right] + \frac{\partial^2 V}{\partial z^2}(a,z;\mu,\zeta) \frac{1}{2}\sigma_z^2 z^2 + \frac{\mathbb{E}_t^{\mu,\zeta}[dV]}{dt} \Bigg\}
\end{aligned} \tag{1}$$

Here, I write $\mathbb{E}_t^{\mu,\zeta}$ as the expectation operator taken over *only* the aggregate state variables, *not* the idiosyncratic ones. $\frac{\mathbb{E}_t^{\mu,\zeta}[dV]}{dt}$ is thus the aggregate variables' stochastic infinitessimal generator applied to the households' value function.

From the optimal choices of $c(a,z;\mu,\zeta)$ and the demand-determined dynamics of $h(a,z;\mu,\zeta)\}$, the distribution of households then evolves according to a Kolmogorov forward equation (KFE):

$$\frac{\partial \mu_t}{\partial t}(a,z) = \overbrace{-\frac{\partial}{\partial a}\left(\frac{da}{dt}\mu_t(a,z) \right) - \frac{\partial}{\partial z}\left(\frac{\mathbb{E}_t[dz_t]}{dt}\mu_t(a,z) \right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2 z^2 \mu_t(a,z) \right)}^{\text{Idiosyncratic state drifts}} \tag{2}$$

Combining the evolution of the distribution with the market clearing conditions and the policy actions of the government is then sufficient to characterize the evolution of macroeconomic aggregates in the model environment.

2.2 Organization of Labor (Unions)

In order to rationalize nominal wage rigidities (and to generate real wages and firm profits that do not respectively move strongly pro-cyclically and counter-cyclically, as per empirical evidence), I adapt the decentralized union approach of Auclert, Rognlie, and Straub (2018) to continuous time, which in turn is related to Schmitt-Grohé and Uribe (2005). Namely, I follow a stylized framework wherein small unions (or alternatively, workers collaborating informally in a workplace) indexed by k supply their work to labor-aggregating agencies. For simplicity, these unions each employ a slice of all workers in the economy, such that:

$$L_{kt} = \int\int zh_t(a,z)da\ dz$$

These employment agencies (or alternatively, hiring departments at production firms) transform the labor supplied by workers using a constant-elasticity-of-substitution aggregator into the mix of labor used by production firms by hiring at each union's nominal wage rate $\tilde{w}_{kt}$:

$$L_t = \left(\int_0^1 L_{kt}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk\right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1}}$$

Intermediate firms thus desire to hire according to

$$\max_{\{L_{kt}\}_{k\in[0,1]}} w_t \left(\int_0^1 L_{kt}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk\right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1}} - \int_0^1 \tilde{w}_{kt} L_{kt} dk$$

where $\tilde{w}_t$ is the prevailing nominal wage at time t. Employment is demand-determined in the economy, and labor unions internalize this demand when choosing the wages they demand, all the while maximizing the collective welfare of their members. Negotiating new wages, however, is subject to Rotemberg adjustment costs, making the objective of union k

$$\max_{\pi^w_{kt}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\left[\int\int \left\{\frac{c_t(a,z)^{1-\gamma}}{1-\gamma} - \frac{h_t(a,z)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right\}\mu_t(a,z)da\ dz - \frac{\theta_w}{2}(\pi^w_{k,t})^2\right]dt$$

$$\text{s.t. } \frac{d\tilde{w}_t}{dt} = \pi^w_{t,k}\tilde{w}_t, \quad L_{kt} = \int\int zh_t(a,z)\mu_t(a,z)da\ dz, \quad \frac{L_{kt}}{L_t} = \left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell}$$

Where $\pi^w_{t,k}$ is wage inflation. Further details and derivations are provided in the appendix; in a symmetric equilibrium, the resulting wage Phillips Curve is

$$\frac{\mathbb{E}_t[d\pi^w_t]}{dt} = \rho\pi^w_t - \frac{\varepsilon_\ell}{\theta_w}L_t\int\int\left(\frac{1}{Z}h_t(a,z)^\eta - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zw_tc_t(a,z)^{-\gamma}\right)da\ dz \tag{3}$$

and the real wage rate evolves according to

$$\frac{dw_t}{dt} = \pi_t^w - \pi_t \tag{4}$$

Unlike Auclert, Rognlie, and Straub (2018), I do not assume that households all supply the same amount of labor. Rather, I assume that in the steady-state households work hours according to

$$\frac{1}{Z}h(a,z)^\eta = \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zwc(a,z)^{-\gamma}$$

where $Z = \int\int z\mu(a,z)da\ dz$ (and in my calibration, $Z \approx 1$). In other words, households in the steady-state work approximately what their labor-leisure choice would be in absence of the union, but with a wage markup commensurate with the elasticity of demand for their hours. Outside of the steady-state, I assume all members of the workplace increase their hours worked commensurate with the change in labor demand following a shock:

$$h_t(a,z) = h_{nss}(a,z)\frac{d\mu_{nss}}{d\mu_t}(a,z) + \frac{1}{Z}(L_t - L_{nss}) \tag{5}$$

where $\frac{d\mu_{nss}}{d\mu_t}(a,z)$ is the Radon-Nikodym derivative of the measure of workers in the steady state relative to the measure of workers at time t.[6]

2.3 The Government

2.3.1 Government Debt

The model's fiscal authority collects aggregate taxes (net of transfers) equal to T_t; real government expenditures G_t are included in the following equations for generality, but are set to be zero in equilibrium. The government is able to borrow using long-term nominal bonds, as in Cochrane (2018). As such, it can pay off existing nominal debt $\tilde{B}$ maturing at time t by either running a primary surplus or by selling new bonds with a maturity of τ at a nominal price of $Q^B_{t,t+\tau}$, which pay off a unitary nominal return when they come due. The debt flow equation is thus

$$\underbrace{\tilde{B}_t dt}_{\substack{\text{Debt}\\ \text{maturing}\\ \text{at time } t}} = \underbrace{p_t(T_t - G_t)dt}_{\text{Surplus}} + \underbrace{\int_0^\infty Q^B_{t,t+\tau} d\tilde{B}_{t,t+\tau} d\tau}_{\substack{\text{Financing from}\\ \text{new bond sales}}}$$

I denote the real value of total government debt outstanding at time t as B_t, such that

$$B_t \equiv \frac{\int_0^\infty Q^B_{t,t+\tau}\tilde{B}_{t,t+\tau}d\tau}{p_t}$$

[6] The rule stipulates that if firms want more hours worked, everyone in the workplace increases their hours by a the same amount, even if they were working different amounts before, adjusting for how the distribution of workers is also different relative to the non-stochastic steady-state.

I next assume that bonds are purchased and priced not directly by households, but rather by a risk-neutral profit-maximizing investment fund that buys debt from the government and sells shares to the public. The central fiscal theory equation alluded to in the introduction of this paper therefore takes the form presented in Cochrane (2018):

$$\underbrace{\frac{\int_0^\infty Q^B_{t,t+\tau}\tilde{B}_{t,t+\tau}d\tau}{p_t}}_{\substack{\text{Real debt}\\\text{outstanding}}} = \mathbb{E}_t\left[\int_t^\infty e^{-\int_t^\tau r_s ds}[T_\tau - G_\tau]d\tau\right]$$

Each household that holds liquid assets by holding shares in the fund thus effectively owns a cross-sectional slice of the entire government portfolio, and receives whatever interest payments are distributed and absorbs whatever capital gains and losses the government debt accrues.

Again as in Cochrane (2018), I make the simplifying assumption that the government issues and rolls over debt such that the density of of government liabilities by maturity is always exponentially distributed with a rate of ω, such that the cumulative distribution of outstanding government treasury maturities τ is $CDF(\tau) = 1 - e^{-\omega\tau}$ and the density function is $PDF(\tau) = \omega e^{-\omega\tau}$. Additionally, I make the simplifying assumption that in the non-stochastic steady-state of the model, all households hold the same slice of government debt, just in varying amounts. For an individual holding a unitary slice of the total government portfolio, their assets entitle them to a payment of ωdt almost immediately (this is the shortest-term debt being repaid), plus payments of $\omega e^{-\omega\tau}dt$ for all periods thereafter. The entire bond portfolio is then effectively a perpetuity which pays out a geometrically declining coupon $\omega e^{-\omega\tau}dt$ at each time $t+\tau$ for the rest of time. Note that as $\omega \to \infty$, government debt becomes instantaneously short-term and must be rolled over immediately with new bonds (analogous to the continuous-time equivalent of a one-period bond in discrete time), while as $\omega \to 0$, each new bond issued becomes a perpetuity.

Changes in the value of government debt outstanding can thus be influenced by both the price of government bonds. In the appendix, I follow similar steps as in Cochrane (2018) to show that the evolution of real government debt will be

$$dB_t = -(T_t - G_t)dt + B_t\left[i_t - \pi_t\right]dt + \frac{d\delta_{qB,t}}{q^B_t}B_t \tag{6}$$

Here, $d\delta_{qB,t}$ denotes the endogenous expectation error on the nominal price of government debt. Since households save by holding a share of a portfolio that is a representative slice of government liabilities, it further follows that $d\delta_{a,t} = d\delta_{qB,t}$. Expected nominal bond prices are in turn governed by

$$E_t[dq^B_t] = q^B_t\left(i_t + \omega - \frac{\omega}{q^B_t}\right)dt \tag{7}$$

and so bond prices evolve according to

$$dq_t^B = q_t^B \left(i_t + \omega - \frac{\omega}{q_t^B} \right) dt + d\delta_{qB,t}$$

as derived in the Appendix. Notably, since the bonds offer nominal payments, the path of nominal interest rates (and the bond portfolio's maturity structure) determines the evolution of nominal bond prices.

2.3.2 Taxes

As a baseline, the fiscal authority in the model taxes labor income at a rate of τ, such that if total effective labor employment in the economy is L_t and real wages are w_t, total income taxes are $\tau w_t L_t$ within the time increment. Households are additionally subject to lump-sum transfers, which aggregate to total lump-sum taxes $T_t(\zeta_t)$ (where if this tax is negative, it is instead a lump-sum transfer). These taxes aggregate naturally from taxes on households:

$$T_t(\zeta_t) = \int_0^\infty \int_{\underline{a}}^\infty T_t(a, z; \zeta_t) \mu_t(a, z) da\ dz$$

In the steady-state, these transfers are assumed to balance the budget; after paying for the interest expense on the debt, the government rebates its tax revenue evenly to the rest of society, such that

$$T_{nss} + \tau w_{nss} L_{nss} - r_{nss} B_{nss} = 0$$

Since more tax revenue is collected from high-earners, but the rebate is evenly distributed, the result is a progressive net transfer scheme. Outside of steady-state, however, the government does *not* necessarily balance the budget. Rather, it adjusts lump-sum transfers according to exogeneous aggregate processes:

$$T_t(a, z; \zeta_t) = T_{nss} + 4Y_{nss} \times \left(T_t^{\text{All}}(a, z; \zeta_t^{\text{All}}) + T_t^{\text{High}}(a, z; \zeta_t^{\text{High}}) + T_t^{\text{Low}}(a, z; \zeta_t^{\text{Low}}) + T_t^{\text{BB}}(a, z; \zeta_t^{\text{BB}}) \right)$$

These tax policies (driven by aggregate "shocks" ζ_t) may all be viewed as programs that send out or demand transfers of varying kinds. All of them are shocks expressed relative to annual GDP in the non-stochastic steady-state (which is $4 \times Y_{nss}$, since the model is quarterly). The first, T_t^{All}, denotes a mean-reverting increase in lump-sum taxes (or cut in benefits) on all members of society:

$$T_t^{\text{All}}(a, z; \zeta_t^{\text{All}}) = \zeta_t^{\text{All}}$$

A negative shock to $T_t^{\text{All}}(a, z; \zeta_t^{\text{All}})$ is analogous to the government printing or borrowing stimulus checks and sending them out to everyone in society. A positive shock is its (somewhat less realistic) opposite, in which the government demands its citizens to pay it flat fees.

The $T_t^{\text{High}}(a, z; \zeta_t^{\text{High}})$ is similar, but in this case, the tax is levied only on households whose wages are above the median in the population. If $\bar{z}$ is the median skill level z, it therefore follows that

$$T_t^{\text{High}}(a, z; \zeta_t^{\text{High}}) = \mathbf{1}_{\{z \geq \bar{z}\}} \zeta_t^{\text{High}}$$

The $T_t^{\text{Low}}(a, z; \zeta_t^{\text{Low}})$ is the same, but only levied on households with wages below the median:

$$T_t^{\text{Low}}(a, z; \zeta_t^{\text{Low}}) = \mathbf{1}_{\{z < \bar{z}\}} \zeta_t^{\text{Low}}$$

Naturally, if both $T_t^{\text{High}}(a, z; \zeta_t^{\text{High}})$ and $T_t^{\text{Low}}(a, z; \zeta_t^{\text{Low}})$ are active at the same time and are of the same magnitude in a linearized model, they are additively equivalent to a change in $T_t^{\text{All}}(a, z; \zeta_t^{\text{All}})$. Conversely, changes in the economy in reactions to the paths of $T_t^{\text{High}}(a, z; \zeta_t^{\text{High}})$ and $T_t^{\text{Low}}(a, z; \zeta_t^{\text{Low}})$ may be seen as a decomposition of the effects of $T_t^{\text{All}}(a, z; \zeta_t^{\text{All}})$ in the effect driven by taxes on the lower-income and the effect driven by taxes on the higher-income.

$T_t^{BB}(a, z; \zeta_t^{BB})$ is a slightly different policy than the preceding ones, in that it leaves net lump sum surpluses unchanged, and only acts through redistribution. It imagines a change in tax policy that taxes those above median income and remits those transfers to those below median income. The policy is thus set up such that

$$T_t^{BB}(a, z; \zeta_t^{BB}) = \frac{z}{Z_t} \zeta_t^{BB} \mathbf{1}_{\{z \geq \hat{z}\}} - \kappa_t^{BB} \mathbf{1}_{\{z < \bar{z}\}}$$

Where ζ_t^{BB} is the tax shock, here now scaled by $\frac{z}{Z_t}$ to make the tax yet more progressive on higher above-median earners, and where κ_t^{BB} is the flat remittance to lower-income households. If $T_t^{BB}(a, z)$ aggregates to zero to leave the federal budget unchanged (aside from the feedbacks from automatic taxes τ), it must then be that

$$\kappa_t^{BB} = \zeta_t^{BB} \frac{\int_{\bar{z}}^{\infty} \int_{\underline{a}}^{\infty} \frac{z}{Z_t} \mu_t(a, z) \, da \, dz}{\int_0^{\bar{z}} \int_{\underline{a}}^{\infty} \mu_t(a, z) \, da \, dz}$$

2.3.3 Monetary Block

The central bank directly sets nominal interest rates in the economy according to

$$i_t = r^* + \phi_\pi \pi_t + \zeta_{\text{MP},t} \tag{8}$$

where r^* is the interest rate that would prevail in equilibrium in the absence of any aggregate shocks. In theory, the model can be solved so long as the interest rate rule is "passive," such that $\phi_\pi < 1$. For clarity, I make the interest rate rule totally unreactive, such that $\phi_\pi = 0$.

2.4 Firms

Intermediate and final goods firms in the model behave as in Kaplan et al (2018), although in this version of the model labor is the only input factor used in production. Final output Y_t is produced by final goods firms at the end of the supply chain using a constant elasticity of substitiution (CES) production and the output of a continuum of monopolistically competitive intermediary firms indexed by i, denoted $y_t(i)$:

$$Y_t = \left(\int_0^1 y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

These monopolistically competitive intermediate goods firms hire labor and produce output via the linear production function

$$y_t(i) = h_t(i)$$

such that their marginal costs are simply their wage costs divided by their total factor productivity (TFP).

$$m_t = w_t$$

These intermediaries are subject to Rotemberg (1982) quadratic adjustment costs when changing their prices. As such, intermediate firms set their prices by solving the profit-maximization problem:

$$\begin{aligned} J(p_0; \zeta_0) = & \max_{\pi_t(i)} \mathbb{E}_0 \int_0^\infty e^{-\int_0^t r_s ds} \left[\frac{p_t(i)}{P_t} y_t(i) - w_t h_t(i) - \frac{\theta}{2} \pi_t(i)^2 Y_t \right] dt \\ \text{s.t. } & dp_t(i) = \pi_t(i) dt \\ & y_t(i) = e^{\zeta_t^{\text{tfp}}} h_t(i) \\ & y_t(i) = \left(\frac{p_t(i)}{P_t} \right)^{-\varepsilon} Y_t \end{aligned}$$

whose recursive formulation is

$$rJ(p; \zeta) = \max_\pi \left\{ \left(\frac{p_t(j)}{P_t} - m_t \right) \left(\frac{p_t(j)}{P_t} \right)^{-\varepsilon} Y_t - \frac{\theta}{2} \pi^2 Y + \partial_p J(p; \zeta) \pi p + \underbrace{\frac{\mathbb{E}^\zeta [dJ(p, \zeta)]}{dt}}_{\mathcal{D}_\zeta J(p;\zeta)} \right\}$$

As shown in Kaplan, Moll, and Violante (2018), the solution to this problem yields the New-Keynesian Phillips Curve

$$\frac{\mathbb{E}[d\pi_t]}{dt} = \left(r_t - \frac{\mathbb{E}[dY_t]}{dt} \frac{1}{Y_t} \right) \pi_t - \frac{\varepsilon}{\theta} (m_t - m^*) \tag{9}$$

where $m^* = \frac{\varepsilon - 1}{\varepsilon}$ is a firms' marginal cost in the absence of shocks.

2.5 The Distribution of Monopoly Profits

Firms in the economy collect monopoly profits, which are in total equal to

$$\Pi_t^F = (1 - m_t)Y_t - \frac{\theta_\pi}{2}\pi_t^2 Y_t$$

To abstract away from the price adjustment costs, I assume that these are rebated back to households as the income of a separate firm, such that profits from the entire corporate sector are $\Pi_t = \Pi_t^F + \frac{\theta_\pi}{2}\pi_t^2 Y_t = (1 - m_t)Y_t$.

Like in McKay, Nakamura, and Steinsson (2016), households are rebated these profits directly; this allows the model to avoid a discussion of the evolution of equity markets outside of the steady-state. However, as in Kaplan, Moll, and Violante (2018), households receive these profits as "bonuses," with higher-income households receiving a larger share, to keep the redistribution scheme from significantly changing the income inequality features of the economy. As such, the profit flows to households take the simple form

$$\Pi_t(a, z) = \frac{z}{Z_t}\Pi_t \tag{10}$$

2.6 Aggregate Exogenous Stochastic Processes

The aggregate exogenous stochastic process vector is $\zeta_t = (\zeta_t^{\text{MP}}, \zeta_t^{\text{All}}, \zeta_t^{\text{High}}, \zeta_t^{\text{Low}}, \zeta_t^{\text{BB}})'$. The first entry pertains to a monetary policy shock, while the latter four refer to perturbations to tax policy, as discussed in Section 2.3.2. Collectively, the shocks evolve according to the Ornstein-Uhlenbeck process

$$d\zeta_t = -\Theta_\zeta \zeta_t dt + dW_{\zeta,t} \tag{11}$$

where $dW_{\zeta,t}$ is a vector of mean-zero independent stochastic innovation terms. The model considers shocks to each of the vectors independently, one at a time, where the monetary policy shock reverts at a rate of θ_{MP} while the tax shocks revert at a common rate of θ_{Tax}.

2.7 Market Clearing and Equilibrium

Note that in a symmetric equilibrium, total output is

$$Y_t = \exp(\zeta_t^{\text{tfp}})L_t \tag{12}$$

And if the inflation adjustment costs $\frac{\theta}{2}\pi_t^2 Y_t$ is rebated back to households, then in the absence of government spending and physical investment

$$Y_t = C_t + \underbrace{\int_0^\infty \int_{\underline{a}}^\infty -\mathbf{1}_{\{a<0\}}\Delta_r a\mu_t(a,z)da\ dz}_{\text{Financial Fees}} \tag{13}$$

where aggregate consumption is

$$C_t = \int_0^\infty \int_{\underline{a}}^\infty c_t(a,z)\mu_t(a,z)da\ dz$$

and the aggregate labor demand is equal to the total effective hours worked by households:

$$L_t = \int_0^\infty \int_{\underline{a}}^\infty zh_t(a,z)\mu_t(a,z)da\ dz \tag{14}$$

The total amount of assets also further be equal to the total amount of government debt

$$\begin{aligned} A_t &= \int_0^\infty \int_{\underline{a}}^\infty a\mu_t(a,z)da\ dz \\ A_t &= B_t \end{aligned} \tag{15}$$

A equilibrium in the mean field game given a sequence of aggregate shocks $(\zeta_t)_{t\geq 0}$ is therefore

i. a set of household decisions for consumption across idiosyncratic states and over time $(c_t(a,z))_{t\geq 0}$ that solves (1) given the path of prices prices, transfers, profits, and aggregate shocks

ii. a distribution of idiosyncratic household states μ_t that evolves over time according to (2)

iii. A sequence of aggregate effective hours $(L_t)_{t\geq 0}$ worked determined by firms' labor demand (which is in turn determined by output and the demand for consumer goods), and idiosyncratic hours worked $h_t(a,z)$ determined by unions' labor rules (5)

iii. A sequence of inflation consistent with firms' profit maximization problem given prices, i.e. the Phillips Curve (9) (where marginal cost is determined by wages)

iv. A sequence of nominal wage inflation consistent with the unions' maximization problem and resulting wage Phillips Curve (3)

iv. A sequence of nominal government bond prices $(q_t^B)_{t\geq 0}$ consistent with the dynamic equation (7)

v. Profit disbursal to households $(\Pi_t(a,z))_{t\geq 0}$ in accordance with (10)

vi. Sequences of macro aggregates $(Y_t, C_t, L_t, A_t, B_t)_{t\geq 0}$ consistent with their definitions (and therefore the household decision rules and distribution equations and the production functions)

vii. Sequences of real wages and real rates of return $(w_t, r_t)_{t\geq 0}$, where w_t evolves according to (4) and r_t obeys the Fisher equation $r_t = i_t - \pi_t$

viii. Government taxes and transfers across the population and over time $(T_t(a, z))_{t\geq 0}$

such that

1. Total output is produced as in (12)
2. the asset market clears, as in (15)
3. the labor market clears (and so by Walras' law, the goods market clears) (14 and 13)
4. interest rates $((i_t)_{t\geq 0})$ are set according to the central bank's policy rule (8)
5. tax policy is consistent with the government's fiscal rules (summarized in section 2.3.2).

Computationally, the structure of the numerical solution is similar to the approach proposed by Reiter (2009). The model is first solved around its non-stochastic steady-state (NSS) using a finite difference scheme similar to the kind put forward by Achdou et al. (2021). To generate the impulse response functions of the economy in response to aggregate shocks, the system is subjected to a dimension reduction routine demonstrated in Bayer and Luetticke (2020) and linearized around its NSS and perturbed in continuous time as in Ahn et al. (2018) et al (2018). In doing so, the value function is projected down onto a hierarchical Chebychev polynomial basis via a discrete cosine transform (DCT), where only the Chebychev polynomials that explain the largest amount of variation in the value function are perturbed from their steady-state values. Additionally, the distribution function is projected onto a fixed copula, where the idiosyncratic variables' joint distribution is assumed to be characterized by the evolution of the idiosyncratic marginal distributions.[7] The entire process treats the differential equations in the model (like the Hamilton-Jacobi-Bellman equations and the Kolmogorov Forward Equation) as a large system of inter-related stochastic ordinary differential equations. Once this discretization and dimension reduction has been completed, the model is then solved using methodologies standard to the solution of linear rational expectation models, namely a QZ (Schur) decomposition, as in Klein (2000) and Sims (2002). For the model to have a uniquely determined stochastic solution, its Jacobian must have exactly as many explosive (positive) generalized eigenvalues as the system has jump variables (which here include the discretized value function, inflation, and the bond and equity prices); I verify that this is indeed the case for my system. Further details are provided in the appendix.

One may note that among the equations that I have listed, one of them is actually redundant: the aggregate law of motion (6) can be used to track the evolution of the market value of government debt, but since households

[7] Bayer and Luetticke (2020) note that this will be a good approximation if the rank correlations of the distributions are not strongly affected by the shocks, which they observe to be the case in models like the one in Krusell and Smith (1998), to which my model's household sector is highly similar.

hold the government's bonds as assets, the private sector's total bond position may be calculated by using the Kolmogorov forward equations (2) and aggregating using (15). In all of my numerical simulations, I calculate the evolution of the stock of government debt both ways, and then observe the percentage difference as a test of my model's numerical accuracy. Overall, the errors in the simulated time series are on the order of 5×10^{-6}.

3 Calibration

3.1 Model Parameters

I calibrate ρ to achieve a an annual interest rate of 2% (the selected value, $\rho = 0.0163$, is equivalent to an annual discount rate of $e^{-4\rho} = 0.937$). I retain the same relative risk aversion coefficient of $\gamma = 2$ – although coefficients ranging from between 1 and 2 are common in the HANK literature (for example, log utility is used in Kaplan, Moll, and Violante (2018)). The parameters, along with their rationals, targets, or sources, are displayed in Table 1.

I additionally target the same income process moments used in McKay, Nakamura, and Steinsson (2016), except using the continuous-time analogue of their discrete-time process. As such, for a given parameterization, I simulate a large number of independent Ornstein-Uhlenbeck process to generate a panel dataset of $\log z$. I then integrate the exponential of the process to the annual level to attain the model's predictions for the panel distribution of annual wages in the non-stochastic steady-state. From there, I choose the income process persistence parameter θ_z and the Brownian motion variance σ_z^2 to match the Floden and Lindé (2001) estimates of the permanent component of annual wage autocorrelation and autoregression variance, residualized for age, occupation, education, and other covariates. As such, I choose (θ_z, σ_z^2) and fit the regression

$$\text{wage}_{it}^{\text{Annual}} = \beta_0 + \beta_1 \text{wage}_{it-1}^{\text{Annual}} + \epsilon_{it}$$

on the simulated data in order to match the Floden and Lindé (2001) estimates of $\beta_1 = 0.9136$ and $\text{var}(\epsilon_{it}) = 0.0426$. I fit both moments up to machine precision; the parameters are also reported in Table 1.

Like Kaplan, Moll, and Violante (2018), I target a slope of the Phillips Curve of 0.10. However, Auclert, Bardóczy, and Rognlie (2023) suggest that nominal wage rigidities capture several important features of the micro data, like low marginal propensities to earn in the micro data, than conventional final goods price rigidities do. As such, I set final goods price nominal rigidities to be very low (1% of the wage rigidities), and make the nominal wage Phillips Curve to have a slope of 0.10. This allows the model to capture similar overall inflation dynamics as other calibrations in the literature, but with arguable more realistic real wage dynamics: with more flexible prices, monopolistically competitive firms are better able to completely pass changes to their marginal costs along to households, largely keeping their ideal markups intact. Workers, in contrast, face a compressed surplus due to nominal wage rigidities during an economic expansion, due to the time and effort that it takes to change their wages. Since they are not on their competitive labor supply curves, workers still increase their hours to meet their employers' demands, but their surplus per hour worked falls as they wait for their real wages to be negotiated to higher levels.

Table 1: Baseline Parameters

Parameter	Symbol	Value	Source or Target
Relative Risk Aversion	γ	2.0	McKay et al (2016)
Quarterly Time Discounting	ρ	0.0163	$r = 2\%$ annually
Borrowing Limit	$\underline{a}$	-1.0	$\approx$30% of avg income
Frisch Elasticity of Labor	η	0.5	Chetty (2012)
Borrowing Wedge Rate	Δ_r	0.04	Credit APR of 18%
Idiosyncratic Shock Variance	σ_z^2	0.017	Calibrated
Idiosyncratic Shock Mean Reversion	θ_z	0.034	Calibrated
Intermediary Elasticity of Substitution	ε	10	10% profit share of GDP
Rotemberg price adjustment cost	θ_π	1.0	(Roughly) acyclic real wages
Labor Elasticity of Substitution	ε	10	Philips Curve slope of 0.10
Rotemberg wage adjustment cost	θ_w	100	Phillips curve slope of 0.10
Steady-state government debt	B_{nss}	5.26	Debt/GDP of 1.37
Geometric maturity structure of debt	ω	0.043	Avg. maturity of 70 months
Income Tax Rate	τ	0.25	
Mean reversion of monetary shock	θ_{MP}	0.175	4-quarter shock half-life
Mean reversion of fiscal shocks	θ_{Tax}	1.0	

For the mean reversion of the shocks, a monetary policy shock is assumed to have a half-life of 4 quarters. In contrast, the mean reversion of fiscal shocks is made to be much stronger with $\theta_{\text{Tax}} = 1.0$. This is intended to better reproduce the speed with which stimulus checks may be sent out; after 4 quarters, the fiscal shocks almost entirely dissipate. Since the path of the shock in the absence of further perturbations may be described with

$$\zeta_t^{\text{All}} = e^{-\theta_{\text{Tax}} t}$$

this also means that the cumulative effect of an initial shock of $\zeta_0^{\text{All}} = -0.01$ has the interpretation of a 1%-of-annual-GDP disbursal of lump-sum stimulus checks, which in the United States would have implied a spending program of roughly \$210 billion dollars if 2019's GDP and prices represented the steady-state.

3.2 Non-Stochastic Steady-State

The non-stochastic steady state is solved by setting all aggregate shocks $\zeta_t = 0$. The model is not able to match the United States income Gini coefficient, which the Census Bureau reported as around 0.48 (before taxes) in 2018. The model also slightly undershoots measures of liquid wealth inequality, which Kaplan, Moll, and

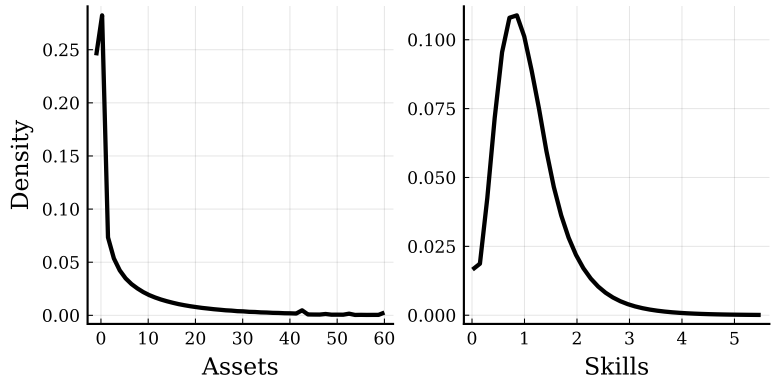

Figure 1: Distribution of assets and income (skills) in the non-stochastic steady-state.

Violante (2018) estimates to be roughly 0.98. Even so, the model generates a large mass of households with little-to-no savings; 38% of the households have a net worth of zero or less. Both the distribution of assets a and skills z (which mechanically matches the model's distribution of pre-tax wages) are depicted in Figure 1.

Table 2: Non-Stochastic Steady-State Moments

Moment	Model Value
Debt/GDP Ratio	1.33
Earnings Gini	0.315
Wage Gini	0.317
Wealth Gini	0.824
Wage and Asset Correlation	0.518

4 Results

In simulating the economy's response to various shocks, I consider the contemporaneous and cumulative effects of the policy perturbations. To calculate the cumulative inflation rate, I simply calculate the price level over time as

$$p_t = e^{\int_0^t \pi_\tau d\tau}$$

and normalize the pre-shock price level to 1. For the growth of GDP, I similarly accumulate

$$\Delta Y_t = \frac{\int_0^t (Y_\tau - Y_{nss}) d\tau}{Y_{nss}}$$

to gauge the percentage growth of the economy following the shock relative to the non-stochastic steady-state. I define my measure of the sacrifice ratio as the average decline in GDP required to bring prices down by a cumulative 1%. If I truncate the cumulative changes to time horizon t, then this means the sacrifice ratio observed in the economy as of time t is

$$SR_t = \frac{\Delta Y_t}{p_t}$$

The first simulation I conduct is a 1% decrease to the nominal interest rate by the country's central bank, as documented in Figure 2 (note that since the model is linearized with respect to aggregate shocks, so the effects of instead *raising* interest rates can be deduced by flipping the sign of the impulse response functions). As expected from the introduction of this paper, the neo-Fisherian response still emerges after the initial jump in inflation, just like in the RANK model of Sims (2011), again providing a theoretical caution regarding the ability of high interest rates to tame inflation – the "stepping-on-a-rake" dynamic is still present. Additionally, low nominal rates in fiscal theory are a slightly deflationary force, since the government is able to roll over its debts without also having to roll over high levels of interest expense that it might have to inflate away in the future. Despite these influences, an economy with long-term nominally-denominated government debt still means that low interest rates deliver a transitory jump in inflation, or (flipping the signs portrayed in Figure 2) high interest rates deliver a transitory fall in inflation, much like the RANK models put forward in Sims (2011) and Cochrane (2018). Just as in those models, a fall in nominal interest rates precipitates a jump in nominal bond prices to prevent arbitrage with the central bank's nominal policy rate, where since the debt portfolio is long-term, its price responds to the entire expected future path of interest rates, which as Cochrane (2018) notes, is a kind of forward guidance mechanism. Since investors know the monetary policy shock will persist for a while, savers will be willing to pay even more to use Treasury bonds (and their relatively unaltered flow of coupon payments) as a saving vehicle. In a frictionless model, it's particularly clear that the price level must rise to counteract the higher nominal bond prices to make bonds cheap in real terms again; households will choose to purchase more of the output good unless the rise in the goods price level is enough to make bonds relatively cheap enough to be worth holding. With nominal rigidities, inflation instead must rise, lowering the path of real policy rates until the present value of the bond coupon payments are once again tempting enough get investors to hold the entire portfolio. McKay, Nakamura, and Steinsson (2016) show that forward guidance is dampened in this HANK setting, but monetary policy is still able to deliver a negative inflation impulse through the Cochrane (2018) channel.

At what point does the "stepping-on-a-rake" dynamic win out? A year after the *contractionary* version of the shock (a 1% *increase* in rates), the price level in the economy is 0.88% lower (as policymakers may hope, if they raised rates to lower inflation), but after just 8 quarters the price level has already returned to and exceeded its pre-shock value. After that, the high rates become inflationary, to the point that after 30 quarters (long after the shock has played out), prices are 1.42% *higher* in response to the persistent monetary tightening. As such, the sacrifice ratio appears to be 2.47 to the agents living in the economy for the first quarter following the shock. However, it appears almost 40% larger after a year has elapsed, and becomes unbounded before flipping its sign entirely after 10 quarters (when on net, the price level begins to inflate). As such, in the model economy, fiscal theory stipulates that interest rates can achieve disinflation in the short-term, only to have inverse effects

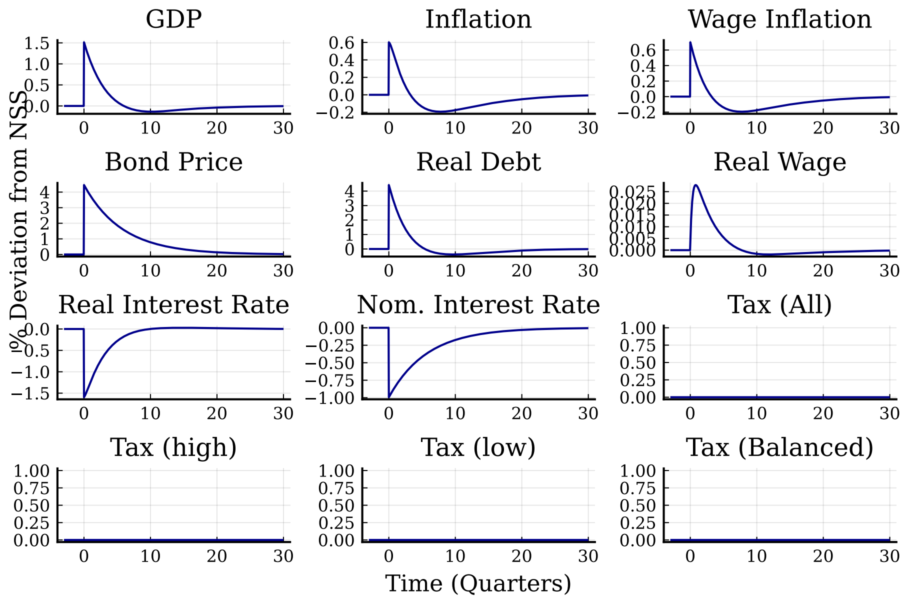

Figure 2: Impulse response functions to a 1% decrease in nominal interest rates.

in the long-term.

Returning to the case plotted in the impulse response functions, wherein interest rates were lowered by the central bank, this means that the low rates eventually drive the price level *lower*, as documented in the first four columns of Panel A of Table 3.

How much does the federal government's ability to "tax the boom" matter for the amount of inflation or deflation generated by the shock? To assess this, I repeat the same experiment, but instead have the government spend its automatic stabilizer revenue instead of using it to back and pay down the debt. The government thus purchases

$$G_t = \tau(w_t L_t - w_{nss} L_{nss}) \tag{16}$$

from firms directly, modifying the goods market clearing equation to

$$Y_t = C_t + G_t + \int_{\underline{a}}^{\infty} \mathbf{1}_{\{a<0\}} \Delta_r a \mu_t(a, z) da\ dz \tag{17}$$

The resulting cumulative changes in GDP and inflation after 30 quarters are reported in the fifth column of Table 3. The governments' new purchases act to yet further stimulate the economy during the monetary policy-driven expansion, causing output to grow by even more than it did in the baseline experiment. The government forsaking the use of the new revenue for debt repayment does produce a slight inflationary force, as more debt must instead be inflated away in equilibrium. However, the effect is very small, on the order of adding 0.07 percentage points to the overall amount of inflation generated by monetary policy.

Proceeding to the next simulation, I generate the impulse response functions reported in Figure 3. A 1%

Table 3: Monetary Policy Shock: Cumulative Inflation and Real GDP

Cumulative Variable	Quarters After Shock 1	4	8	30	Government Spends New Revenue 30 Quarters
Monetary Expansion					
Real GDP (%)	1.32	3.04	3.11	1.80	2.65
Inflation (%)	0.53	0.88	-0.23	-1.42	-1.35
Sacrifice Ratio	2.47	3.45	-	-	-

Note: Since the the sacrifice ratio for monetary policy blows up and flips sign at around 10 quarters due to neo-Fisherian effects, I do not calculate it for monetary policy's 8 and 30 quarter horizons.

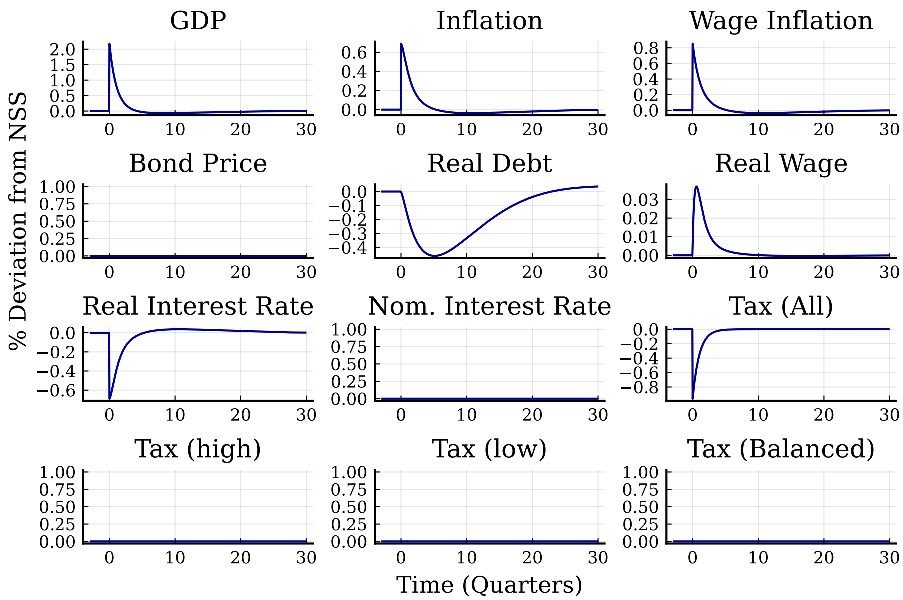

Figure 3: Impulse response functions to a mean-reverting decrease in lump-sum taxes on all households.

of steady-state GDP lump-sum payment to all households leads to a sharp increase in GDP as relatively hand-to-mouth households pass the increase in income into lower spending, which in turn causes a jump in hiring and further increases in income and spending. Inflation ensues to diminish the real amount of government debt outstanding, as the expenditure was not backed by new taxes, although automatic stabilizers are able to partially offset the effect by providing more net income to the government's coffers. Surprisingly, despite the tax increase, the real value of the debt temporarily *falls*.[8] This is largely because the inflation response is strong enough to move real rates into negative territory; this occurs as the supply side of the economy struggles to mobilize productive factors to support the boom and marginal costs rise.

All told, sending stimulus checks to all agents in the economy in an expenditure equal to 1% of GDP ends up growing the economy by 1.49% over time at the cost of an additional 0.61 percentage points of inflation (where in the very short-term, the rate of additional inflation experienced by agents in the first year is over 1.07%). If the government had gone the other way, and demanded fees from the public instead of distributing checks, the sacrifice ratio would thus have amounted to 2.47 by the end of the simulation window.

[8] Note that in the previous experiment with monetary policy, this dynamic was masked by the surge in bond prices. Since the central bank keeps the interest rate constant in the tax policy experiment, nominal bond prices do not move.

Table 4: Fiscal Policy Shocks: Cumulative Inflation and Real GDP

Cumulative Variable	Quarters After Shock 1	4	8	30	Government Spends New Revenue 30 Quarters
Panel A: Tax Shock (All)					
Real GDP (%)	1.47	2.27	2.09	1.49	2.23
Inflation (%)	0.58	1.07	1.04	0.61	0.67
Sacrifice Ratio	2.53	2.13	2.02	2.47	3.34
Panel B: Tax Shock (High Income)					
Real GDP (%)	0.97	1.74	1.76	1.27	1.84
Inflation (%)	0.41	0.92	1.01	0.63	0.68
Sacrifice Ratio	2.37	1.89	1.74	2.03	2.70
Panel C: Tax Shock (Low Income)					
Real GDP (%)	2.05	2.89	2.47	1.75	2.68
Inflation (%)	0.78	1.24	1.07	0.58	0.65
Sacrifice Ratio	2.62	2.34	2.32	3.04	4.12

Note: All tax shocks are equal to stimulus transfers amounting to 1.0% of GDP, while the monetary shock is a 1.0% nominal interest rate decrease. The cumulative effects in Panel A may be recovered by averaging the cumulative effects in Panels B and C with the weights 0.54 and 0.46, respectively. Since the the sacrifice ratio for monetary policy blows up and flips sign at around 10 quarters due to neo-Fisherian effects, I do not calculate it for monetary policy's 8 and 30 quarter horizons.

I next decompose the overall fiscal shock into two components: the stimulus check rebate to high-wage agents (who are on average wealthier), and the stimulus check rebate to low-wage agents (who are on average poorer). Note that now, since each of the shocks fall on only half of the population, I adjust the size of the tax shocks to keep the implied expenditure plans equal to 1.0% of annual steady-state GDP (and thus comparable to the earlier simulations); averaging the results of the two shocks yields the effects of Figure 3 composite shock.[9] I plot both sets of impulse response functions in Figure 4; the policy where checks are sent out to low-income households is plotted in bright red, while the policy that sends checks out to high-income households is depicted in dark blue.

Looking first at the effects of sending checks to the high-wage agents, the economy still expands upon impact, but only by 1.3% instead of delivering the 3.1% output boom experienced immediately in the world where the equivalent value of checks are sent out to lower-income households. Automatic stabilizer revenue from proportional income taxes (not depicted) surge by similar amounts, as real wages are highly stable (companies defend their markups almost completely, passing along the increase in labor costs directly to consumers, resulting in little change to wages' overall purchasing power) and hours worked go up. Debt, too, falls even faster in the simulation where the low-income receive the stimulus checks, as it is devalued even more by the stronger initial inflation response. In the first quarter, the differences in inflation are large; the scenario in which the low-income receive checks experiences 0.78% inflation, while in the case where the higher-income households received checks, the cumulative inflation in the first quarter reached only 0.41%. By the end of the year, however, the difference

[9]Due to the coarseness of the income space grid, however, the upper-income group is actually divided at those making more than the 46th percentile, as opposed to the 50th. As such, the weights for averaging the high-income tax simulation with the low income tax to obtain the Figure 3 are 0.56 and 0.44, respectively. The initial tax shocks are similarly scaled accordingly to keep the total expenditure shock the same across simulations.

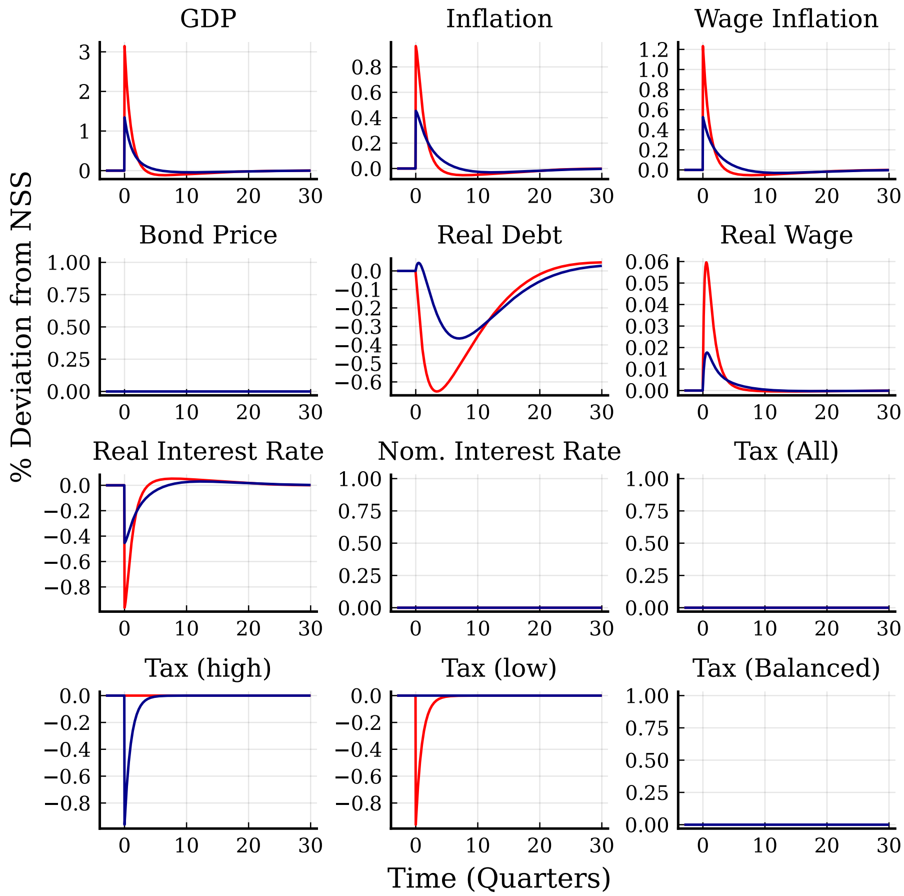

Figure 4: Impulse response functions to mean-reverting stimulus check payments to high-income and low-income households. The policy that sends checks to high-income households is plotted in dark blue, while the policy that sends checks to low-income households is plotted in bright red.

in the cumulative amount of inflation experienced in the two different scenarios narrows considerably, before almost vanishing by the time the second year concludes. Over a longer time horizon (30 quarters), it is actually the case that total inflation endured in the setting where the low-income received checks is actually slightly *lower* than the amount endured when the high-income were the recipients, as shown the fourth column of panels B and C in Table 4. The gap is about 0.05 percentage points of inflation, a tiny amount. The last column of Table 4, panels B and C indicates that the tax-the-boom effect to balance the budget contributed about 0.03 percentage points to this difference; when the government simply spends its new labor income tax revenue instead of saving it, inflation (and output) rise in both scenarios, but the difference in inflation between the low-income stimulus checks and high-income stimulus check scenarios falls to 0.03. Either way, the restoration of the government's balance sheet and the exact timing of the fall in real interest rates does not appear to significantly bear on the total amount of inflation generated by the policies over time. By contrast, the difference in the implied ratios of additional GDP growth to additional inflation in the scenarios where the recipients of the checks are different is striking. When the high-income receive the checks, the economy goes on to grow by roughly 2.03% for every percentage point of inflation. When the low-income get the checks, the ratio is about 50% higher (3.04) after all of the shocks have played out by the end of the 30th quarter. As recounted in the first 3 columns of Table 4, the pattern of higher cumulative GDP relative to inflation induced when checks are distributed to the low-income is present at all of the intermediate time horizons as well.

Going in the other direction, toward fiscal contraction, taxing the low-income carries a dramatically higher sacrifice ratio. The new revenue restores confidence that the government will not partially default by inflating away its obligations, restoring faith in the currency and the government's debt, but at a grievous cost to real output. The same dis-inflationary effect is roughly accomplished by taxes higher-income households, but with a far less severe economic downturn.

All of the cumulative responses to the different policies are depicted in Figure 5.

What happens when the government pursues a policy wherein it taxes the rich and rebates the proceeds back to the poor? The aftermath of such a policy is presented in Figure 6. The movement of resources from low-MPC to high-MPC households generates an immediate economic boom, slightly lower than that experienced when checks were distributed to low-income households alone, but still larger than when they were distributed to high-income households. The increase in economic activity drives wage inflation as households negotiate higher wages to work more hours, which companies pass on to end consumers to protect their profits. However, since no deficits are actually incurred by the policy redistribution policy, FTPL implies that the cumulative amount of inflation experienced should be zero; there is no new unbacked debt to inflate away. Figure 7 shows that this is the case as the time horizon under consideration grows longer. After the first year following the policy, the economy experiences a mild deflation, raising interest rates as households buy back bonds that were cheaper

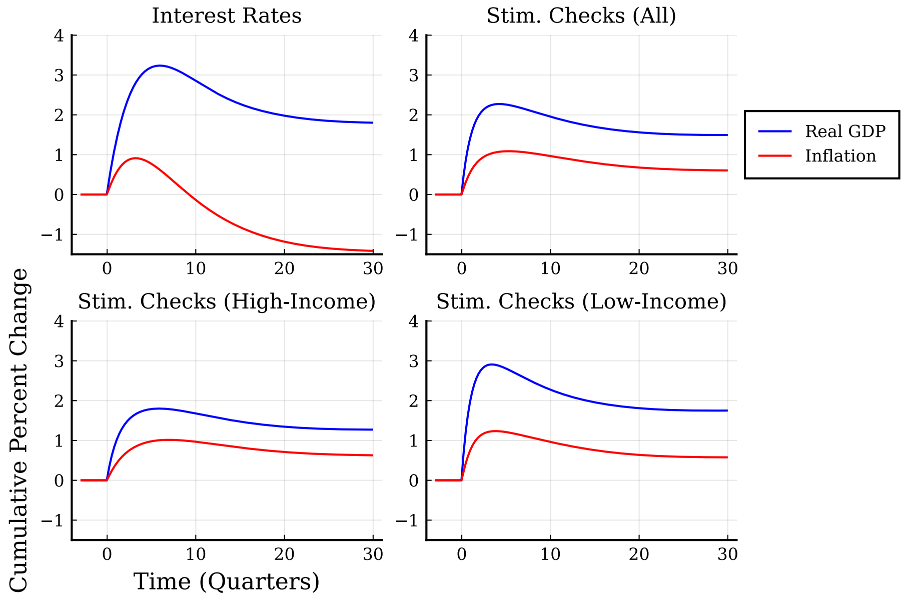

Figure 5: Cumulative responses of real GDP and inflation to expansionary shocks.

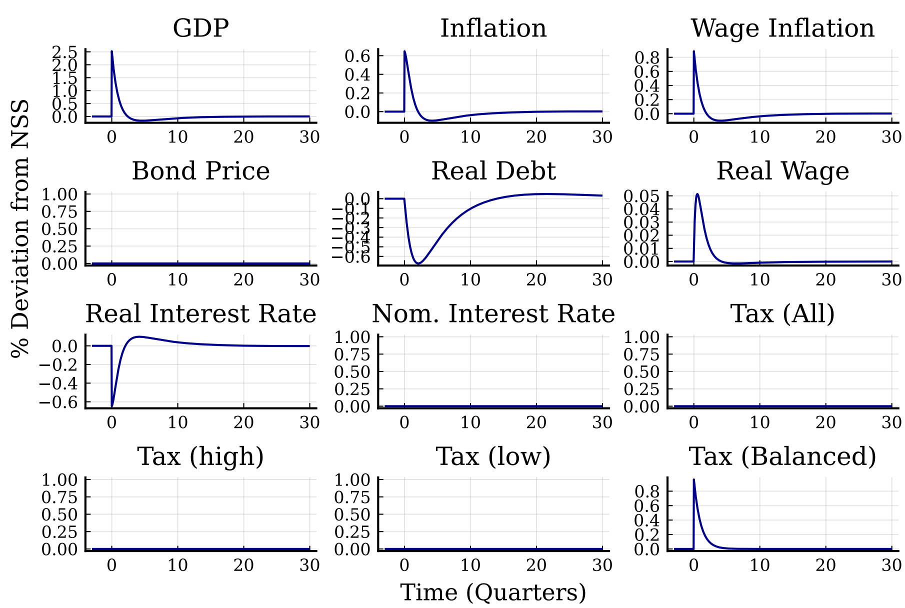

Figure 6: Impulse response functions to a mean-reverting increase in lump-sum taxes on high-income households with the proceeds remitted to low-income households.

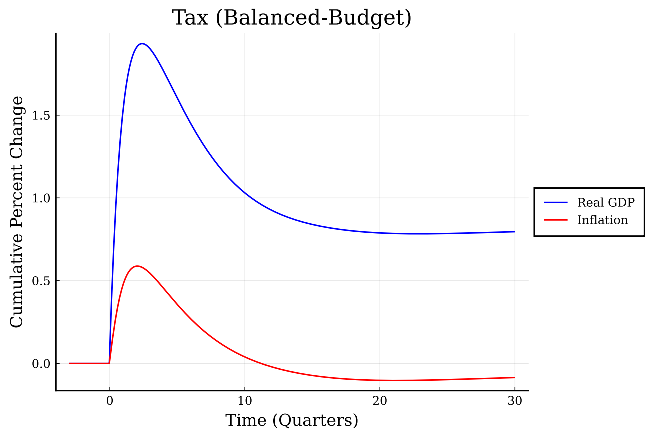

Figure 7: Cumulative effects on inflation and output following a balanced-budget transfer from high-to-low income households.

during the boom. Over time, the price level (not just the rate of inflation) converges back to where it started, before the fiscal transfers were made (the surge in automatic revenues also makes the government debt more valuable, making the transfer scheme slightly *deflationary* in the long-run, but the magnitude of the effect is small).

5 Discussion

The model presented in this paper drives to a few simple conclusions. First, household heterogeneity and incomplete markets alone are not enough to nullify the Cochrane (2018) forward guidance mechanism. Second, the accounting of fiscal theory makes in the calibrated model makes a fiscal policy's overall net inflationary force mainly determined by the overall size of the fiscal spending shock to the economy. Automatic stabilizers and the timing of inflation and real rates can potentially complicate this picture – but ultimately the simulations suggest that these complications are small. Thus, if a government sends out stimulus checks unbacked by future taxes, the model predicts that when it comes to the total cumulative amount of inflation sparked by a policy, the amount denominated by the checks matters much more than who gets them.[10]

The intuition of the fiscal policy result might be somewhat surprising in a framing outside of fiscal theory, in that it is contrary to the intuition that one might have about giving money to "high-velocity" versus "low-velocity" consumers having different consequences for inflation, wherein the average speed of subsequent spending might determine how inflationary a transfer ultimately is. However, within the framework of fiscal theory, the result is straightforward: if inflation is about markets participants' confidence in the government's ability to pay its debts, or the currency's usefulness as a means of settling taxes with the fiscal authority, then the size of unfunded deficits is what is of first-order importance.

[10] Of course, if acute inflation is more painful to consumers than a protracted but more moderate inflation, the exact path of inflation may matter more. However, in all of the simulations, the majority of inflation is over by the end of the first year following the policy shock.

The next implication in this paper is similarly straightforward. The HANK literature suggests that transfers to people who of varying liquid asset positions generates heterogeneous effects on aggregate expenditure, income, and output. Namely, transfers to poor households who must spend their income almost as soon as they receive it start feedback loops that cause larger economic expansions than transfers to wealthier people do. If wealth is correlated to income, which governments can observe more easily than net worth, then transfers to low-income people are a powerful stimulant to real GDP. Taking resources away from poor people, by contrast, provides a steeper contraction in economic activity, all else equal.

The upshot of these two points is that if a government is trying to stimulate the economy with the least amount of inflation possible, it should set the overall size of the program first and then tilt its stimulus efforts toward those it believes have a high marginal propensity to consume. If fiscal theory is correct, then the amount of inflation is relatively insensitive to the payments' recipients, but the amount of economic activity generated is larger when the recipients go out and spend their income relatively quickly. An insensitive denominator and a sensitive numerator combine to suggest that stimulus checks to those without liquid savings represent a superior GDP-to-inflation trade-off than alternative policies.

Alternatively, if a government is trying to *lower* inflation, fiscal theory stipulates that inflation from a one-time fiscal shock is eventually transitive, even without strong intervention from monetary policy; once the excess nominal debt is inflated away, the inflationary pressure will subside. Active monetary policy is also not necessary for determinacy, as Cochrane (2023) surveys at length, and higher interest rates may assist in the fight against inflation in the short-term, only to become counter-productive in the longer-term. If a country wants to tamp down on fiscally-caused inflation and end it earlier than later, however, FTPL suggests that fiscal policy and reform can play a greater role in restoring faith in a country's currency and a stable price level. However, from the logic of the previous paragraph, a fiscal reform accomplished by transferring resources away from those with low assets and low incomes likely carries with it a much higher sacrifice ratio. Markets can still get the message that the debt and currency will be honored from a fiscal reform that raises taxes on agents whose spending is less sensitive to their earnings, with less of a hit to employment and consumption.

But for all of its mathematical coherence, is fiscal theory right, in the sense that it is the the correct model of inflation out in the real world? In another sense, this paper broaches an interesting empirical test for those interested in fiscal theory. For countries in inflationary episodes, particularly following deficit spending or new currency issuance with a signal of little willingness or ability to raise future surpluses, is the amount of ensuing inflation at all related to who gets the spending, or is it just a matter of how much spending or currency is issued? Finding clear examples of such cases in the real world is likely difficult (much less finding enough of them to conduct statistical inference), but if the relative size of the spending program appears to matter for inflation in the data, then that may be interpreted as evidence in support of fiscal theory. If this is not the case, however, then FTPL may have to be reconsidered, amended, or ultimately rejected. Until then, FTPL's interaction with heterogeneity and demand-determined economies with nominal rigidities opens interesting opportunities for new theory and its implications for policy.

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6 Appendix

6.1 Bond Math

6.1.1 Fiscal Block

In this model, I also assume that the government is able to borrow using long-term nominal bonds, as in Cochrane (2018). As such, it can pay off existing nominal debt $\tilde{B}$ maturing at time t by either running a primary surplus or by selling new bonds with a maturity of τ at a price of $Q^B_{t,t+\tau}$. The debt flow equation is thus

$$\underbrace{\tilde{B}_t dt}_{\substack{\text{Debt}\\ \text{maturing}\\ \text{at time } t}} = \underbrace{p_t(T_t - G_t)dt}_{\text{Surplus}} + \underbrace{\int_0^\infty Q^B_{t,t+\tau} d\tilde{B}_{t,t+\tau} d\tau}_{\substack{\text{Financing from}\\ \text{new bond sales}}}$$

The real value of total government debt outstanding at time t will thus be

$$B_t = \frac{\int_0^\infty Q^B_{t,t+\tau} \tilde{B}_{t,t+\tau} d\tau}{p_t}$$

For the bond portfolio, the cumulative real return is the real capital gain on each bond type, weighted by the value of the bonds held, divided by the real value of the entire portfolio:

$$dR_t = \frac{\int_0^\infty \left[d\left(\frac{Q_{t,t+\tau}}{p_t}\right) / \frac{Q_{t+\tau}}{p_t} \right] \frac{Q_{t,t+\tau}}{p_t} \tilde{B}_{t,t+\tau} d\tau}{B_t}$$

$$\Rightarrow \quad B_t dR_t = \int_0^\infty d\left(\frac{Q_{t,t+\tau}}{p_t}\right) \tilde{B}_{t,t+\tau} d\tau$$

Such that

$$dB_t = d\left[\frac{\int_0^\infty Q_{t,t+\tau} \tilde{B}_{t,t+\tau} d\tau}{p_t}\right] = \underbrace{\frac{\int_0^\infty Q_{t,t+\tau} d\tilde{B}_{t,t+\tau} d\tau}{p_t} - \frac{\tilde{B}_t}{p_t} dt}_{-(T_t - G_t)dt} + \underbrace{\int_0^\infty \tilde{B}_{t,t+\tau} d\left(\frac{Q_{t,t+\tau}}{p_t}\right) d\tau}_{B_t dR_t}$$

It thus follows that

$$dB_t = -(T_t - G_t)dt + B_t dR^B_t$$

The first term is the primary deficit, while the second is the ex-post real rate of return on the bond portfolio. This ex-ante return will then be the expected return on the nominally riskless bonds, plus whatever capital gain has been unexpectedly accrued over the time increment.

Suppose the government bond portfolio is effectively a perpetuity which pays out a geometrically declining coupon $\omega e^{-\omega\tau}$ at each time $t+\tau$. The nominal bond price will then be

$$Q_t^B = \int_0^\infty e^{-\tau y_t}\omega e^{-\omega\tau}d\tau = \int_0^\infty \omega e^{-\tau(\omega+y_t)}d\tau = -\frac{\omega}{\omega+y_t}e^{-u}|_0^\infty = \frac{\omega}{\omega+y_t}$$

The nominal rate of return on the bond will be the the dividend yield, plus the capital gain.

$$dR_t^{B,nom} = \frac{(\omega-\omega Q_t^B)dt + dQ_t^B}{Q_t^B} = y_t dt + \frac{dQ_t^B}{Q_t^B}$$

It then follows that

$$i_t dt = \mathbb{E}_t[dR_t^{B,nom}] = y_t dt + \frac{\mathbb{E}_t[dQ_t]}{Q_t}$$

Such that the unexpected capital gain on the bond portfolio is the ex-post nominal rate of return minus the expected (ex-ante) one:

$$\delta_{q,t} \equiv dR_t^{B,nom} - i_t dt = \frac{dQ_t^B - \mathbb{E}_t[dQ_t^B]}{Q_t^B}$$

Since the nominal rate will be the real one, plus inflation:

$$dR_t^n = dR_t + \pi_t dt$$

$$\delta_{q,t} - \pi_t dt = dR_t - i_t dt$$

$$\Rightarrow \quad dR_t = \delta_{q,t} + (i_t - \pi_t)dt$$

The valuation equation becomes

$$dB_t = -(T_t - G_t)dt + B_t\left[i_t - \pi_t\right]dt + \delta_{q,t}B_t$$

Equivalently, if I define

$$\delta_{Q,t} \equiv dQ_t - \mathbb{E}_t[dQ_t]$$

then the government debt equation becomes

$$dB_t = -(T_t - G_t)dt + B_t\left[i_t - \pi_t\right]dt + \frac{\delta_{Q,t}}{Q_t}B_t \tag{18}$$

It also follows that if

$$dR_t^n = \frac{\omega dt + dQ_t}{Q_t} - \omega dt$$

such that

$$Q_t dR_t^n = \omega dt + dQ_t - \omega Q_t dt$$

such that in expectation

$$E_t[dQ_t] = Q_t \left(\mathbb{E}_t[dR_t^n] + \omega dt - \frac{dt}{Q_t} \right)$$

$$\Rightarrow \quad E_t[dQ_t] = Q_t \left(i_t + \omega - \frac{\omega}{Q_t} \right) dt \tag{19}$$

and so bond prices evolve according to

$$dQ_t = Q_t \left(i_t + \omega - \frac{\omega}{Q_t} \right) dt + \delta_{Q,t}$$

6.2 Wage Phillips Curve

This is a continuous-time version of Auclert, Rognlie, and Straub (2018), *The Intertemporal Keynesian Cross.* Say a labor-aggregator hires labor from households to create an aggregate unit of input labor.

$$L_{kt} = \int_0^1 (z_i h_{ikt}) di$$

$$L_t = \left(\int_0^1 L_{kt}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk \right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1}}$$

Intermediate firms thus desire to hire according to

$$\max_{\{L_{kt}\}_{k\in[0,1]}} W_t \left(\int_0^1 L_{kt}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk \right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1}} - \int_0^1 W_{kt} L_{kt} dk$$

such that from the FOCs, the demand for labor from union k is

$$W_t \left(\int_0^1 L_{kt}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk \right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1} - 1} L_{kt}^{-\frac{1}{\varepsilon_\ell}} - W_{kt} = 0$$

$$W_t L_t^{\frac{1}{\varepsilon_\ell}} L_{kt}^{-\frac{1}{\varepsilon_\ell}} = W_{kt}$$

$$W_t L_t^{\frac{1}{\varepsilon_\ell}} = W_{kt} L_{kt}^{\frac{1}{\varepsilon_\ell}}$$

$$\Rightarrow \quad \frac{L_{kt}}{L_t} = \left(\frac{W_t}{W_{kt}} \right)^{\varepsilon_\ell}$$

Unions face nominal wage adjustment costs:

$$\frac{\theta_w}{2}\int_0^1 \pi_{w,k}^2 dk, \quad \text{where} \quad \pi_{w,k} = \frac{dW_{kt}}{dt}\frac{1}{W_{kt}}$$

The labor union k sets wages to maximize its members' lifetime utilities:

$$\max_{\pi_{kt}^w} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\left[\int\int\left\{\frac{c(a,z)^{1-\gamma}}{1-\gamma} - \frac{h(a,z)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right\}\mu_t(a,z)da\ dz - \frac{\theta_w}{2}(\pi_{k,t}^w)^2\right]dt$$

$$\text{s.t. } \frac{dW_t}{dt} = \pi_t^w W_t$$

$$L_{kt} = \int_0^1 z_i h_{ikt} di$$

$$\frac{L_{kt}}{L_t} = \left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell}$$

Where the third equation follows from the first-order conditions from the households.

The HJB is then

$$\rho J^w = \left[\int\int\left\{\frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{kt})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right\}\mu_t(a,z)da\ dz - \frac{\theta_w}{2}(\pi_{k,t}^w)^2\right] + \frac{\partial J^w(W_{kt})}{\partial W_{kt}}\pi_t^w W_{kt} + \frac{\mathbb{E}_t^\zeta dJ^w}{dt}$$

The FOC for wage inflation is then

$$-\theta_w \pi_{kt}^w + \frac{\partial J^w(W_{kt})}{\partial W_{kt}} W_{kt} = 0$$

$$\Rightarrow \quad \frac{\partial J^w(W_{kt})}{\partial W_{kt}} = \theta_w \frac{\pi_{kt}^w}{W_{kt}}$$

With Itô's lemma, we know that

$$d\left(\frac{\partial J^w(W_{kt},\zeta)}{\partial W_{kt}}\right) = \partial_{W_{kt}}^2 J^w dW_{kt} + \partial_\zeta \partial_{W_{kt}} J^w d\zeta + \partial_\zeta^2 \partial_{W_{kt}} J^w d\langle\zeta\rangle_t$$

and applying Itô again to the LHS of the wage inflation FOC,

$$d\left(\theta_w \frac{\pi_t^w}{W_{kt}}\right) = \frac{\theta_w}{W_{kt}} d\pi_t^w - \frac{\theta_w \pi_t^w}{W_{kt}^2} dW_{kt}$$

such that by equating the two,

$$\frac{\theta_w}{W_{kt}} d\pi_t^w - \frac{\theta_w \pi_t^w}{W_{kt}^2} dW_{kt} = \partial_{W_{kt}}^2 J^w dW_{kt} + \partial_\zeta \partial_{W_{kt}} J^w d\zeta + \partial_\zeta^2 \partial_{W_{kt}} J^w d\langle\zeta\rangle_t$$

Taking expectations and dividing by dt yields

$$\frac{\theta_w}{W_{kt}}\frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{kt}}\underbrace{\frac{dW_{kt}}{dt}\frac{1}{W_{kt}}}_{\pi_{kt}^w} = \partial^2_{W_{kt}} J^w \frac{dW_{kt}}{dt} + \underbrace{\frac{\partial_\zeta \partial_{W_{kt}} J^w \mathbb{E}_t[d\zeta] + \partial^2_\zeta \partial_{W_{kt}} J^w d\langle\zeta\rangle_t}{dt}}_{\mathcal{D}_\zeta \partial_{W_{kt}} J^w}$$

such that

$$\frac{\theta_w}{W_{kt}}\frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{kt}}\pi_t^w = \partial^2_{W_{kt}} J^w \pi_t^w W_{tk} + \partial_{W_{kt}} \mathcal{D}_\zeta J^w \tag{20}$$

Next, the Envelope condition stipulates that

$$\rho \partial_{W_{kt}} J^w =$$

$$\left[\int\int \partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{kt})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\} \mu_t(a,z) da\ dz \right] + \partial^2_{W_{kt}} J^w \pi_t^w W_{kt} + \partial_{W_{kt}} J^w (W_{kt}) \pi_t^w + \partial_{W_{kt}} \mathcal{D}_\zeta J^w$$

Substituting in (20),

$$\rho \partial_{W_{kt}} J^w =$$

$$\left[\int\int \partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{kt})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\} \mu_t(a,z) da\ dz \right] + \partial_{W_{kt}} J^w (W_{kt}) \pi_t^w + \frac{\theta_w}{W_{kt}}\frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{kt}}\pi_t^w$$

and then the FOC,

$$\rho \theta_w \frac{\pi_{kt}^w}{W_{kt}} =$$

$$\left[\int\int \partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{kt})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\} \mu_t(a,z) da\ dz \right] + \theta_w \frac{\pi_{kt}^w}{W_{kt}}\pi_t^w + \frac{\theta_w}{W_{kt}}\frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{kt}}\pi_t^w$$

it follows that

$$\rho \pi_{kt}^w = \frac{W_{kt}}{\theta_w}\left[\int\int \partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{kt})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\} \mu_t(a,z) da\ dz \right] + \frac{\mathbb{E}_t[d\pi_t^w]}{dt} \tag{21}$$

From the households' envelope condition, the change in utility from wages will be equal to the marginal utility, times the change in earnings:

$$\partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} \right\} = c(a,z)^{-\gamma}(1-\tau)\partial_{W_{kt}} \left(\frac{W_{kt}}{P_t} h(a,z) \right)$$

Where if households uniformly supply their labor to union k and internalize their labor's demand:

$$h_{ikt}(a,z) = h_{nss}(a,z)\frac{d\mu_{nss}}{d\mu_t}(a,z) + \frac{1}{Z_t}(L_{kt} - L_{nss}) = h_{nss}(a,z)\frac{d\mu_{nss}}{d\mu_t}(a,z) + \frac{1}{Z_t}\left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell} L_t - \frac{1}{Z_t} L_{nss}$$

$$\Rightarrow \quad \partial_{W_{kt}} \left\{ \frac{c(a,z;W_{kt})^{1-\gamma}}{1-\gamma} \right\} = c(a,z)^{-\gamma}(1-\tau)\partial_{W_{kt}} \left(z\frac{W_{kt}}{P_t} \left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell} L_t \right)$$

$$= c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{W_{kt}}\left(\frac{W_{kt}}{P_t}\left(\frac{W_t}{W_{kt}}\right)^{-\varepsilon_\ell} L_t\right)$$

$$= c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{W_{kt}}\frac{W_{kt}}{P_t}L_{kt}$$

$$= c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{P_t}L_{kt}$$

For the effect of wages on labor disutility, I can directly evaluate

$$\partial_{W_{kt}} h(a,z) = \frac{1}{Z_t}\partial_{W_{kt}}\left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell} L_t = -\varepsilon_\ell \frac{1}{Z_t}\frac{1}{W_{kt}}\left(\frac{W_t}{W_{kt}}\right)^{\varepsilon_\ell} L_t = -\varepsilon_\ell \frac{1}{Z_t}\frac{L_{kt}}{W_{kt}}$$

Plugging in the results into (21),

$$\rho\pi^w_{kt} = \frac{W_{kt}}{\theta_w}\left[\int\int\left\{c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{P_t}L_{kt} + \frac{1}{Z_t}h(a,z)^\eta \varepsilon_\ell \frac{L_{kt}}{W_{kt}}\right\}\mu_t(a,z)da\ dz\right] + \frac{\mathbb{E}_t[d\pi^w_t]}{dt}$$

$$\rho\pi^w_{kt} = \frac{\varepsilon_\ell}{\theta_w}L_{kt}\int\int\left\{\frac{1}{Z_t}h(a,z)^\eta - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)z\frac{W_{kt}}{P_t}c(a,z)^{-\gamma}\right\}\mu_t(a,z)da\ dz + \frac{\mathbb{E}_t[d\pi^w_t]}{dt}$$

Leading to the wage Phillips Curve

$$\frac{\mathbb{E}_t[d\pi^w_t]}{dt} = \rho\pi^w_t - \frac{\varepsilon_\ell}{\theta_w}L_t\int\int\left(v'(h(a,z)) - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zw_t u'(c(a,z))\right)da\ dz \tag{22}$$

where w_t is the real wage.

Note that in the NSS, households supply labor while internalizing how supplying more labor reduces their wage, such that

$$v'(h(a,z)) = \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zw_t u'(c(a,z))$$

Marginal disutility from labor is slightly lower than in the competitive equilibrium, such that hours worked is slightly lower as well. The real wage is defined as

$$w_t = \frac{W_t}{P_t}$$

Taking the total time derivative:

$$dw_t = \frac{dW_t}{P_t} - \frac{W_t}{P_t}\frac{dP_t}{P_t}$$

$$\Rightarrow \quad \frac{dw_t}{dt} = \frac{dW_t}{dt}\frac{1}{W_t}\frac{W_t}{P_t} - \frac{W_t}{P_t}\pi_t$$

$$\Rightarrow \quad \frac{dw_t}{dt} = (\pi^w_t - \pi_t)w_t$$

6.3 Perturbations from NSS

Suppose labor supply after a shock can be characterized by

$$h_t(a,z) = h_{nss}(a,z)\frac{d\mu_{nss}}{d\mu_t}(a,z) + \frac{1}{Z_t}(L_t - L_{nss})$$

Aggregating,

$$\begin{aligned}\int\int zh_t(a,z)d\mu_t(a,z) &= \int\int z[h_{nss}(a,z) + \frac{1}{Z_t}(L_t - L_{nss})]d\mu_t(a,z)\\ &= \int\int zh_{nss}(a,z)d\mu_{nss}(a,z) + L_t - L_{nss} = L_t\end{aligned}$$

so indeed

$$\int\int zh_t(a,z)d\mu_t(a,z) = L_t$$

6.4 Equilibrium in the Single-Asset Model

For notational brevity, I write the infinitessimal generator operator of the concentrated Hamilton Jacobi Bellman equation as

$$\begin{aligned}\mathcal{D}[V] =& \lim_{t\downarrow 0}\frac{\mathbb{E}_t^{a',z'}[V_{t+dt}(a',z';\mu,\zeta)] - V_t(a,z;\mu,\zeta)}{dt}\\ =& \frac{\partial V}{\partial a}(a,z;\mu,\zeta)\left[(1-\tau)wzh(a,z) + T(a,z) - c(a,z) + \left(r(a) + \frac{\delta_q}{q}\right)a\right]\\ &+ \frac{\partial V}{\partial z}(a,z;\mu,\zeta)z\left[\frac{1}{2}\sigma_z^2 - \theta_z\log(z)\right] + \int_{z'}[V(a,z';\mu,\zeta) - V(a,z;\mu,\zeta)]dF(z'|z)\end{aligned}$$

where the expectation operator is taken with respect to only the idiosyncratic variables. As in Achdou et al (2022), I write the adjoint operator (which describes the Kolmogorov forward equation of the idiosyncratic state distribution) as $\mathcal{D}^*$, where the KFE operator is the adjoint of the maximized HJB operator in L^2 space.

A recursive competitive equilibrium will thus be a set of consumption choices $c(a,z;\mu_t,\zeta_t)$, distributions μ_t, outstanding government debt $B_t(\mu_t,\zeta_t)$, wages $w_t(\mu_t,\zeta_t)$, nominal and real interest rates $i_t, r_t(\mu_t,\zeta_t)$), bond and equity prices (q_t^B, q_t^S), and inflation rates π_t such that household choices satisfy

$$\frac{\mathbb{E}_t^{\mu,\zeta}[dV(a,z;\mu,\zeta)]}{dt} = \rho V(a,z;\mu,\zeta) - \left[u(c(a,z;\mu,\zeta)) - v(h(a,z;\mu,\zeta)) + \mathcal{D}[V]\right]$$

and the distribution evolves according to

$$\frac{\partial_t\mu_t(a,z)}{\partial t} = \mathcal{D}^*[\mu]$$

Inflation must be consistent with the Phillips Curve derived from the firms' profit maximization problem:

$$\mathbb{E}_t[d\pi_t] = \left(r_t\pi_t - \frac{\varepsilon}{\theta_\pi}[m_t - m^*] \right) dt$$

The government's budget constraint must satisfy

$$dB_t = -(T_t - G_t)dt + r_t B_t dt + \frac{\delta_{q_t^B}}{q_t^B} B_t$$

where nominal bond prices and equity prices satisfy

$$\mathbb{E}_t[dq_t^B] = q_t^B \left(i_t + \omega - \frac{\omega}{q_t^B} \right) dt$$

$$\mathbb{E}_t[dq_t^S] = (r_t q_t^S - \Pi_t)dt$$

subject to the static equations

$$r_t = i_t - \pi_t$$

$$m_t = -\zeta_{\text{TFP},t} w_t$$

$$\Pi_t = [1 - m_t]Y_t$$

and market clearing:

$$Y_t = C_t = \zeta_{\text{TFP}} L_t$$

$$C_t = \int_0^\infty \int_{\underline{a}}^\infty c_t(a,z)\mu_t(a,z) da \ dz$$

$$L_t = \int_0^\infty \int_{\underline{a}}^\infty z h_t(a,z)\mu_t(a,z) da \ dz$$

Note that if $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ and $v(h) = \frac{h^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}$, it follows that the FOCs become

$$c(a,z)^{-\gamma} = \frac{\partial V}{\partial a}(a,z)$$

$$h(a,z)^{\frac{1}{\eta}} = (1-\tau)wz\frac{\partial V}{\partial a}(a,z) = (1-\tau)wzc(a,z)^{-\gamma}$$

The budget constraint in the absence of expectations errors is

$$\frac{da}{dz} = (1-\tau)wzh(a,z) + \Pi + T(a,z) + r(a)a - c(a,z)$$

When the asset state is no longer drifting, it follows that

$$c(a, z) = (1 - \tau)wzh(a, z) + \Pi(a, z) + T(a, z) + r(a)a$$

Such that

$$h(a, z)^{\frac{1}{\eta}} = [(1 - \tau)wzh(a, z) + \Pi(a, z) + T(a, z) + r(a)a]^{-\gamma}$$

thus

$$0 = h(a, z) - [(1 - \tau)wzh(a, z) + \Pi(a, z) + T(a, z) + r(a)a]^{-\gamma\eta}$$