Munkres Topology Notes

Topological Spaces and Continuous Functions

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2 Abstract

Abstract

In a valiant effort to stop my Topology knowledge from dissipating, I have decided to self-study Munkres until my next Analysis course, which will be in about a years time. I know that if I don't keep my math skills sharp, they'll end up as flat as a non-differentiable function. Hopefully this self-study will help me retain my math skills like a homomorphism preserves topological properties ;)

Topological Spaces

3.1 What is a Topology?

Def: (Topology)

A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

i. $\emptyset$ and X are both in $\mathcal{T}$

ii. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$

iii. The intersection of the elements of any finite collection of $\mathcal{T}$ is in $\mathcal{T}$

If a set X satisfies the above three properties we call X a topological space.

3.2 Open Sets

Def: (Open Set)

If X is a topological space with a topology $\mathcal{T}$, we say that a subset U of X is an open set of X if U belongs to the collection of $\mathcal{T}$.

The whole idea here is to have this notion of openness be very general, the above definition is literally just saying that a set is open if it is a member of $\mathcal{T}$. YOU choose what is open by placing sets in $\mathcal{T}$, the only catch is that what you decide to declare open must satisfy the requirements for a topology; that is, $\emptyset$ and X are in $\mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary union and finite intersection. To tie together the notion of a topological space nicely Munkres has a very nice way of summarizing the concept of a topological space using the new vocabulary we have learned.

3.3 A more Concise Definition for Topological Space's

Def: (Concise Definition for Topological Space's)

A topological space is a set X together with a collection of subsets of X, called open sets, such that $\emptyset$ and X are both open, and such that the arbitrary union and finite intersection of open sets are open.

Example

- Choose X to be any set, the collection of all subsets of X, i.e., the power set of X denoted $\mathcal{P}(X)$ is a topology, it is called the discrete topology.

- The collection of only X and $\emptyset$ is a topology on X, it is called the indiscrete topology, or more commonly just the trivial topology.

We will now describe some terms for when we compare different topologies.

3.4 Fine and Course Topologies

Def: (Fine and Course Topologies)

Let $\mathscr{T}$ and $\mathscr{T}^{'}$ be two topologies on a set X. If $\mathscr{T}^{'} \supset \mathscr{T}$, we say that $\mathscr{T}^{'}$ is finer than $\mathscr{T}$. If $\mathscr{T}^{'}$ properly contains $\mathscr{T}$, we say that $\mathscr{T}^{'}$ is strictly finer than $\mathscr{T}$. One can also say that $\mathscr{T}$ is courser than $\mathscr{T}^{'}$, or strictly courser than $\mathscr{T}^{'}$,in the two respective cases.

Comparable Topologies

Def: (Comparable)

We say $\mathscr{T}$ is comparable to $\mathscr{T}^{'}$ if either $\mathscr{T}^{'} \supset \mathscr{T}$ or $\mathscr{T} \supset \mathscr{T}^{'}$.

Note, We can also use the terms

- "Weak topology" when $\mathscr{T}^{'} \supset \mathscr{T}$.
- "Strong topology" when $\mathscr{T} \supset \mathscr{T}^{'}$.

There tends to be some miscommunication as to the correct definitions of weak and strong topologies. I use the above associations because that is how I learned them, some people define weak/strong in the opposite way, so for future reading make sure you understand how one is defining there weak/strong topologies. Similarly, we can say that a topology is "Larger" or "Weaker" with the above definitions/caution being the same. To finish this portion of chapter 2 I'll provide a rather straightforward example that illustrates the above ideas.

Example

Let

$$X = \{a, b, c\}, \quad \mathscr{T}_1 = \{\emptyset,\ X\}, \quad \mathscr{T}_2 = \mathscr{P}(X)$$

- It's simple to see that $\mathscr{T}_1$ is just the trivial topology, we can say that $\mathscr{T}_1$ is the "Coarsest" topology.
- We call $\mathscr{T}_2$ the "Finest" topology.
- Lastly note that $\mathscr{T}_1 \subset \mathscr{T}_2$, thus $\mathscr{T}_1$ is coarser than $\mathscr{T}_2$ and $\mathscr{T}_2$ is finer than $\mathscr{T}_1$.

Basis for a Topology

4.1 Bases

Def: (bases)

Let X be a set, a basis for a topology on X is a collection $\mathscr{B}$ of subsets of X,(Called basis elements) such that,

- For each $x \in X$, there is at least one basis element $\mathscr{B}$ containing x.
- If x belongs to the intersection of two basis elements $\mathscr{B}_1$ and $\mathscr{B}_2$, then there is a basis element $\mathscr{B}_3$ containing x such that $\mathscr{B}_3 \subset \mathscr{B}_1 \cap \mathscr{B}_2$.

If $\mathscr{B}$ satisfies the above two properties, then we define the topology $\mathscr{T}$ generated by $\mathscr{B}$ as follows, A subset U of X is said to be open in X (that is, to be an element of $\mathscr{T}$) if for each $x \in U$, there is a basis element $B \in \mathscr{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself in $\mathscr{T}$.

5 The Order Topology

6 The Product Topology on $X \times Y$

7 The Subspace Topology

8 Closed Sets and Limit Points

9 Continuous Functions

10 The Product Topology

11 The Metric Topology

12 The Quotient Topology

13 Topological Groups