

Mathematics Education Foundations:
Developing Math Materials for Students and a System for Peer-Educators
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Abstract

Learning mathematics is hard, and learning college mathematics is made even harder by textbooks which are primarily used as vessels for homework exercises. These two facts motivated my Senior Integrated Project (SIP) or senior thesis, which consisted of 21 L^AT_EX documents and 5 videos on introductory mathematics topics. The documents were made to be more personalized to the Kalamazoo College curriculum than the all-purpose textbooks often circulated, and with a much larger emphasis on reading comprehension, examples, and conceptual motivations. The thesis also included a system for the creation of more materials in the future, in the hopes that such materials will be made for other subjects and that they will grow as the college does. Following this, I developed a system for the materials' online and in-person accessibility through the Kalamazoo College Math and Physics Center (MPC).

Acknowledgements

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I would be remiss not to thank the mathematics department at Kalamazoo College, which is filled with only the most inspiring educators, mathematicians, and office coordinators possible. They served as my “north star” of pedagogical excellence during the entire thesis development process, and my thesis would not be the same without them. In this same vein, I would like to thank Linda Williams of Berea Community High School, who showed me just how incredible mathematics educators could be, and who was integral in my mathematical excitement. I would not be studying mathematics without her.

Finally, I would like to thank my parents, Ketaki Bhattacharyya and Joshua Prentice, and my brother Zachary Prentice. Much of this work was completed at my home, and they provided an enormous amount of day-to-day encouragement and positive energy during my time there, as they have always done.

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Introduction

It started with a textbook. In Fall of 2021, I took Dr. Eric Barth's course "Complex and Vector Variables" (MATH-310), whose title shares a name with the textbook that was used. On page 15 of the textbook, Example 2.2 began, which described how to use the ϵ, δ definition of a limit to prove that $\lim_{x \rightarrow 2} 3x - 2 = 4$. It gave this solution:

Solution:

Let $\epsilon > 0$ be given.

We want to find a $\delta > 0$ so that

$$\begin{aligned} 0 < |x - 2| < \delta \text{ implies } & |(3x - 2) - 4| < \epsilon \\ & \text{or } |3x - 6| < \epsilon \\ & \text{or } |3(x - 2)| < \epsilon \\ & \text{or } |(x - 2)| < \epsilon/3, \end{aligned}$$

so we can take $\delta = \epsilon/3$.

$0 < |x - 2| < \delta = \epsilon/3$ implies

$$|(3x - 2) - 4| = |3(x - 2)| < 3\delta = \epsilon. \blacksquare$$

All ϵ, δ proofs begin this way

Start by saying what you want to prove
Now do some simplification

It's now obvious what δ should be
For all ϵ, δ exists: $\epsilon/3$
A final summary statement

Figure 1: Solution to Example 2.2

I found this example extremely helpful. The combination of the step-by-step solution and the commentary worked incredibly well for my understanding of the material.

Part of the reason this experience of mine was so incredible is that, before taking Dr. Barth's classes, the only use which STEM textbooks had for me was in the exercises which my professors would assign as homework; they provided no assistance for my understanding of the content. Dr. Barth's textbooks were different. Why? Well, as Figure 1 above shows, they provided accessible approaches to the exercises and described the content in plain English. Also, they were made for one particular class at one particular school, and as such they hardly covered any material which wasn't taught in the course. General textbooks, on the other hand, often seem to cover every detail of every topic.

I had not yet realized the importance of this experience, though. It was only during an academic advising meeting with Dr. Barth in Fall term when I had the realization. He asked how my work at the Math and Physics Center (MPC), a peer tutoring center, was going. I explained that students were coming in from Calculus I, asking for help understanding the

problem-solving method of one of the most commonly struggled-with topics in that course: *implicit differentiation*. He asked if there was anything that would help them, and the first thing that came to my mind was Example 2.2 from the Complex Analysis textbook. We discussed what some kind of analogy for Calculus topics would look like, and how they might be used in the MPC or outside of it, and Dr. Barth mentioned that it sounded like it could be my Senior Integrated Project (SIP), or senior thesis. And thus this project was born.

A week or so later, Dr. Barth and I met with Dr. Rachel Love, Director of the Science, Math, Business and Economics Centers (which contains the MPC), and discussed more what the project might involve and how it would best be integrated into the MPC. We asked her if she would be willing to join the team, and she accepted.

Our Goals

As the idea of the project progressed, we developed several goals and hopes for the project. These may be roughly divided into two categories: *pedagogical goals* and *systemic goals*. The pedagogical goals are mainly focused around (1) the course material which we wanted to cover, and (2) the student's relationship to that material. The systemic goals, on the other hand, are mainly focused on the accessibility of the project, establishing a system of production, etc. Let's now take a closer look at what these goals are.

Pedagogical Goals

Through the two discussions we'd already had and the many discussions to follow, we set several pedagogical goals for the project. Perhaps the most important goal was this: that we are not trying to create or replace any textbook. Above all else, this project was intended to be aimed at the students, not the instructors or the classes. After all, the primary inspiration for the project was Dr. Barth's textbook's ability to reach the students.

Through this goal we also discovered a secondary (but still important) goal: make the

documents accessible to the MPC consultants. The genesis of the project, in Dr. Barth's office, involved a discussion of how to help the consultants teach the most commonly struggled-with material. So it was clear: if the project can help the consultants teach the material, then it has, by proxy, helped the students.

To help the students and the consultants, and to actually make progress on the project, we needed an idea of what the scope of the project would be. We decided that my math proficiencies lied mainly in Calculus, so we originally limited our scope to Calculus I-III. Then, as a guideline, we gave priority to the topics which students seemed to struggle with the most. These “usual suspects” would be our main targets.

In addition to helping the students with the content of the material, we realized we must also keep in mind what skills we would like the students to develop through their utilization of the project. A list of such skills was created in the early part of Summer 2022, and it showcases some other values we had in the project, such as accessibility and feedback. The list included (1) reading comprehension, (2) use/understanding of definitions and theorems, (3) problem solving, (4) conceptual understanding, (5) applying concepts to real-world problems, and (6) translating ideas from English to math and vice-versa. These skills would be explicitly communicated to the student in something of an “Intro to the Learning Commons Math Materials” document and video.

Systemic Goals

Then, there were our systemic goals. Motivating this side of the project were several core values such as accessibility, feedback from students and peer consultants, and clarity of our goals and recommendations. Let's discuss some of these now.

Accessibility

From the beginning, the intention would be that the materials would be accessible both online and in-person at the MPC. The in-person materials allow students to see the materials and

get clarification whenever they were in the area or attending the MPC during the in-person consultant hours. Furthermore, this greatly increases the accessibility to consultants during their in-person hours. For example, if a consultant wanted clarity on how to approach a topic they hadn't worked with in a while, or if they wanted a way to clearly express the steps to take to solve a given problem, they would be able to quickly access that information through the materials housed in the MPC.

To even further improve their use for the consultants and students, the in-person materials will be laminated and dry-erase markers will be kept in the MPC to allow for notes, markings, and more, right on the materials themselves.

Of course, online accessibility is also extremely important. Not only does online accessibility allow students or consultants to access the information at any time of day, but it also allows them to look back at information to review for quizzes or exams, which may require them to remember content from several weeks before. Additionally, it would allow anyone to print out the materials and have paper copies with them if they please and would allow instructors or professors to access the information if they should want to.

The benefits continue. But they all lead to the question of where the online materials will be housed. Several options are available for that purpose. One of the most appealing options and the ultimate goal is to house the materials in some way on the MPC website. Housing the materials there would allow anyone with an internet connection to access them, and they could be easily accessed through the Hornet Hive. However, housing the materials on the website brings difficulties for those creating the materials or managing the website. For example, there would be a significant time delay between when material is created or updated and when that change is reflected in the online version, and it would take significant time and effort for the person who has the correct administrative privileges to do the updating or uploading. As Dr. Barth put it in a meeting, “it would take a crash course in WordPress any time a change was made.”

The other prominent option that came to mind was housing the materials on Microsoft

Teams. Many of Kalamazoo College's courses are heavily integrated with Teams, and any student of the college has access to the MPC Team Site. Furthermore, anyone creating or updating the materials could easily upload them to the Team Site and have the changes instantly reflected. So, using Teams would effectively avoid the difficulty issues that using the website brings. Unfortunately, it lacks many of the benefits of using the website, too. For one, you would have to have access to the Team Site to be able to view the materials, effectively meaning you would have to find out what the Team Site code is. The code to access the Team Site would be easily added to the MPC website or on a flyer in the physical MPC space, but it would still add an extra step to accessing the materials and, together with needing access to Microsoft Teams itself, would certainly reduce the accessibility to the materials to an undesirable extent.

The solution we went with was to realize that we need not pick only one option. For the time being, especially while the materials and the system surrounding them are being developed, they will be housed on Teams, where updates can be seen instantly and changes can be made with ease. The goal for the system in the future is to have the materials accessible on the website, but, even if that goal is achieved, there would be nothing to prevent Teams being used as well. In fact, having the materials accessible from several different places would significantly increase their accessibility in general.

Feedback

Collecting feedback from students and consultants is also incredibly important for the project. Given the student- and consultant-oriented nature of the project, it is necessary to hear what students and consultants think about the project and what possible improvements might be helpful.

Feedback is primarily run through a Google Form, which contains questions such as “What part of the resources are you giving feedback on?” and “Would you like us to respond to you directly?” but none of the questions are required, and many contain the option to

select “Other,” letting people write in their own response. In between these questions is a “Long answer” text box, allowing anyone to write as much feedback as they would like to.

In order to collect feedback, though, the feedback form needs to be accessible. Online materials can easily have the link, but in-person or visual materials don’t have that luxury. To solve this issue and for ease-of-access, we created a QR code. The link to the feedback form is <https://forms.gle/JZkH5ZUTARsB62bD9>, and the QR code is:



Figure 2: Feedback form QR code

The Plan

That completes our discussion of the high-level goals and aspirations of the project, but that is only half of the story. The other half lies in the details of how we were going to cover the content. A plan for the project was developed at the beginning of Summer 2022, and it included these details. The plan was to (1) create documents describing how to solve and understand the most difficult problems from introductory math courses, (2) make a document describing the goals of the project and how to best utilize them, (3) laminate the documents and have a home for them in the MPC, (4) establish a way for documents to be accessed online, (5) establish a system with which future documents can be added for math or other departments along with a document on how to add documents to the system, and (6) create videos on topics where possible, including how to best utilize the materials and how to create more for the system.

I want to be clear what is meant by our use of the word “document,” as that is truly the crux of the project. The plan was to create written materials describing the concepts

of the topic and the steps to solving the associated problems. The latter would, of course, include the side-by-side examples which constituted the original inspiration of the project. To do this, I would use the typesetting software which Dr. Barth used to write the textbook and which I had been practicing throughout the 2021-2022 school year: L^AT_EX. L^AT_EX had the versatility and resources to allow me complete control over what the documents would look like, and it would allow for an online consolidation of the documents by way of using the website Overleaf (<https://www.overleaf.com>).

Development

The system, then, developed as follows: we created a Google account for the Learning Commons, which we used to sign up for a free account on Overleaf. This account hosts the documents in their .zip form, which allows for future changes to any documents if required. The Google account is also the owner of a YouTube channel, which would host any videos created as part of the project as well. Let's look at the specifics of the document and video development now.

The Documents

The first document was developed in Spring 2022. It was on implicit differentiation, the topic which started the project, and was a pretty bare-bones L^AT_EX document but was nevertheless very similar to the final document structure we would use. As can be seen in Figure 3 below, it contained the “Abstract,” “Introduction,” “Steps and Examples,” and “Extra Exercises” sections which would persist throughout the development process.

Implicit Differentiation

Abstract
Implicit differentiation is a technique used for *implicit equations*, equations which have not isolated the output variable (such as y). It primarily involves the same differentiation rules as before, but with a new use of the chain rule.

1 Introduction.
Consider the two following equations:

$$(1) \quad y = 3x^2 + 2 \quad (2) \quad \cos(y) = \sin(x)$$

While you've likely learned how to differentiate (1), differentiating (2) may seem more intimidating. Your first instinct might be to solve for y , turning (2) into something that you can solve like (1), but cases like these often run into difficult functions to differentiate (such as inverse trig functions) or are out-right impossible. Instead, there is a technique to find the derivative $\frac{dy}{dx}$ for these kind of equations, called *implicit equations*.

Implicit equations are equations which have not been solved for the output variable (such as y). Examples of these equations include:

$$x^2 + y^2 = 1 \quad x^2 \cdot \sin(y) = 7 \quad x - \ln(xy) = e^{2x-3y}$$

As mentioned before, solving for y in these examples is either very difficult or impossible. So, there is a different method of finding $\frac{dy}{dx}$ which does not require isolating y and still uses the same differentiation rules you've previously learned. This method is called *implicit differentiation*.

2 Steps and Examples.

This is the general process:

- Take the derivative of both sides with respect to the desired variable (the bottom of the derivative fraction, such as x in $\frac{dy}{dx}$).
- Apply derivative rules to both sides, treating functions of y the same as functions of x . Use chain rule for functions of y , multiplying by $\frac{dy}{dx}$.
- Solve for $\frac{dy}{dx}$.

Example 1
Find $\frac{dy}{dx}$ given $x^2 + y^2 = 1$.

Solution:

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$ $\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$ $2x + 2y \cdot \frac{dy}{dx} = 0$ $\frac{dy}{dx}(2y) = -2x$ $\frac{dy}{dx} = -\frac{x}{y}$	Begin with the equation you are given. Take the derivative of both sides with respect to the variable you want, in this case x . Use any derivative rules you can to simplify. Evaluate the derivatives. For all functions of y , remember that y is a function of x . So, use chain rule and multiply by $\frac{dy}{dx}$. Isolate all terms with a $\frac{dy}{dx}$ and factor. Finally, divide and simplify.
--	---

Note that our final answer is in terms of both x and y . Though this may seem odd or new, don't worry! It's perfectly normal when working with implicit equations.

The main thing to remember with implicit differentiation is how to differentiate functions of y , such as y^2 in the above example. The reason $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$ is because y is a function of x . In this sense, y^2 seems to have an *inside* (y) and an *outside* (something squared), just like examples utilizing the chain rule. So, following chain rule, you take the derivative of the outside and leave the inside the same (resulting in $2y$), and multiply by the derivative of the inside ($\frac{dy}{dx}$). Let's follow this again in another example.

(a) Page 1

(b) Page 2

Example 2
Find $\frac{dy}{dx}$ given $y^2 \sin(y) = x^5$.

Solution:

$y^2 \sin(y) = x^5$ $\frac{d}{dx}(y^2 \sin(y)) = \frac{d}{dx}(x^5)$ $2y(\frac{dy}{dx})\sin(y) + y^2 \cos(y)(\frac{dy}{dx}) = 5x^4$ $\frac{dy}{dx}(2y \sin(y) + y^2 \cos(y)) = 5x^4$ $\frac{dy}{dx} = \frac{5x^4}{2y \sin(y) + y^2 \cos(y)}$	Begin with the equation you are given. Take the derivative of both sides with respect to the variable you want, in this case x . Evaluate the derivatives, remembering to use chain rule for functions of y . Note that in this case we have to use product rule on the left side. Isolate all terms with a $\frac{dy}{dx}$ and factor. Finally, divide and simplify.
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This example shows how you may need to apply regular derivative rules (power rule, product rule, etc.) to functions involving y . This is a very common way to go wrong, and as such is a very good thing to remember. Here's a final example, where we show the variables combined in various ways with a small rename.

Example 3
Find $\frac{dz}{dt}$ given $e^{2t-3z} = t^2 - \ln(tz^3)$ with z and $t > 0$.

Solution:

$e^{2t-3z} = t^2 - \ln(tz^3)$ $\frac{d}{dt}(e^{2t-3z}) = \frac{d}{dt}(t^2 - \ln(tz^3))$ $e^{2t+3z}(2 + 3\frac{dz}{dt}) = \frac{1}{t^2}(t^2 + 3tz^2\frac{dz}{dt})$ $\frac{dz}{dt}(2e^{2t+3z} + \frac{3}{t}) = 2t - \frac{1}{t} - 2e^{2t+3z}$ $\frac{dz}{dt} = \frac{2t - \frac{1}{t} - 2e^{2t+3z}}{3e^{2t+3z} + 3z - 1}$	Begin with the equation you are given. Take the derivative of both sides with respect to the variable you want, in this case t . Evaluate the derivatives, remembering to use chain rule for functions of z . Isolate all terms with a $\frac{dz}{dt}$ and factor. Note that because z and $t > 0$, we can cancel $\frac{1}{t}$ and $\frac{1}{t}$ without dividing by 0. Finally, divide and simplify.
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(c) Page 3

(d) Page 4

Figure 3: The first L^AT_EX document

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One aspect of this that instantly catches the eye is the “Extra Exercises” section, which has its questions right-side-up and in blue, with answers immediately below them, upside-down, and in yellow. This was done using code found on the L^AT_EX StackExchange website (tex.stackexchange.com), and, though we tried different options such as having the answers on a separate page, this system was what we stuck with.

Another aspect is worth briefly mentioning: the introduction. One of the skills we hope to help improve with this project is reading comprehension, but we also want the documents to be as accessible as possible. An initial, unofficial test on students attending the MPC confirmed our fears that a large amount of text on the first page would make the document seem daunting or inaccessible. Now, the introduction is on the same page as the abstract and title, which makes this slightly better, but we also decided to keep the introduction as concise as possible.

Now onto some things which would change. At the time of writing the document, I was unfamiliar with the `displaystyle` environment in L^AT_EX, which allows for larger text-size in math mode. This made the exercises and step-by-step examples difficult to read, especially when they contained fractions. Furthermore, there were some details that made the document and reading it “not nice” or difficult, such as the large 1.5 inch book margins or 11pt font which L^AT_EX defaults to. Eventually, we would settle on using 12pt font and varied margins, using 1.5 inch book margins for the introduction and exercises, but shifting to 1 inch margins for the examples. This can all be seen in Figure 3 above.

Once the style was settled, Summer 2022 became all about making as many documents as possible. I began by focusing on Calculus I, as it had the simplest topics and many topics were similar enough that the workflow was very easy. I dabbled with Calculus II and making a template/guide for other peer supports, and as of Tuesday, July 12th (one month since my summer began), I had completed documents on implicit differentiation, product rule, quotient rule, chain rule, critical numbers, the closed interval method for maxima and minima, and the template and guide for peer supports, 8 documents in total.

I then moved on to topics in Calculus II, such as integration by u -substitution, integration by parts, trigonometric integration, and trigonometric substitution. Simultaneously, I finished documents on partial fraction decomposition of real, rational functions with only linear roots of the denominator and how to use the limit definition of the derivative. I did some work on tank problems from the “Applications to Physics and Engineering” section of the James Stewart calculus book used at Kalamazoo College until the end of my summer on August 19th, when I was back in Kalamazoo for Resident Assistant training. The training ran for 2.5 weeks from about 9am to about 5pm, and was immediately followed by the opening of the residence halls and the beginning of classes the following week. This meant that my production of documents was effectively halted for 3 weeks.

In the small amount of time I had, though, I began work on Calculus III documents, promising I would return to Calculus II tank problems another time. I began with a document on the equation of a plane and how to find it from different kinds of information, and then followed that with a document on line integrals of scalar functions. I began work on a document on line integrals over vector fields, and continued work on it throughout the beginning of the school term until its completion on September 25th.

Once this document was done, I resumed work on Calculus II tank problems, which I finally completed on October 24th. I then took a hiatus making documents, though I updated the look of them in the middle of the term as per consultant feedback. These changes were small but took some time to implement them in all of the documents made. We added the course name and document title to the top of the page, in italics, and added the total page count of each document to the page numbers at the bottom. Examples of these new headers and footers can be seen here:

Example 1

A tank is in the shape of a rectangular prism (a box). It is 4 meters deep, 3 meters wide, and 5 meters long. The tank is filled with water up to the 3 meter mark. How much work is required to empty the tank from a spout at the top of the tank?

As you might see, unlike the previous examples, the shape of the layer is cylindrical, with a circle at the top and bottom. This means the volume of the layer is $\pi R_t^2 \Delta x$, where R_t is the radius of the circle. Like the last example, though, this radius changes as x_t varies. To find this relationship, we'll take the same approach of the last example, and draw a side-on view of the tank.

(a) Updated header

(b) Updated footer

Figure 4: Document changes in Fall 2022

My document-making hiatus ended over Winter break, on December 17th, at which point I completed a document on a brief guide to L^AT_EX, covering some of the basics of using the software and Overleaf to create documents in general, with a particular aim towards peer supports at Kalamazoo College making documents for the Learning Commons system. This was followed in quick succession with a document aimed at students utilizing the materials. It was meant as an introduction to the materials, and it gives a brief word of encouragement, it describes what to expect from the materials and how to best use them, and then it gives a quick note on feedback, with a QR code and link to the feedback form. The document was completed on December 20th, and it was kept intentionally short so as to maximize the number of people who will choose to read it.

The final document was completed shortly thereafter, on December 21st, right before I took a break for the holidays. The document was on how to use the limit definition of the definite integral, a topic covered in Calculus II. That means we can now move onto the shorter story of how the videos were developed.

The Videos

The videos, in general, were not the primary focus of my thesis, for several reasons. First, they were hard to make and took a lot of time. This made them relatively inefficient despite the benefits that video brings to mathematics education. Second, they were additionally deprioritized due to (a) the original inspiration of the thesis, that being Dr. Barth's example in his Complex Analysis textbook, and (b) the higher importance of establishing the system through which more materials could be made by peer supports, if they wanted to. That

said, several videos were made before the thesis deadline.

The first video was made on a relatively simple topic: integration by parts from Calculus II. It was made in the Summer with Microsoft OneNote and was completed on July 20th. At this point, a YouTube channel was created to host all of the videos made in the Learning Commons System. The YouTube channel is available at the following QR code and link <https://www.youtube.com/@kzoolearningcommons>:

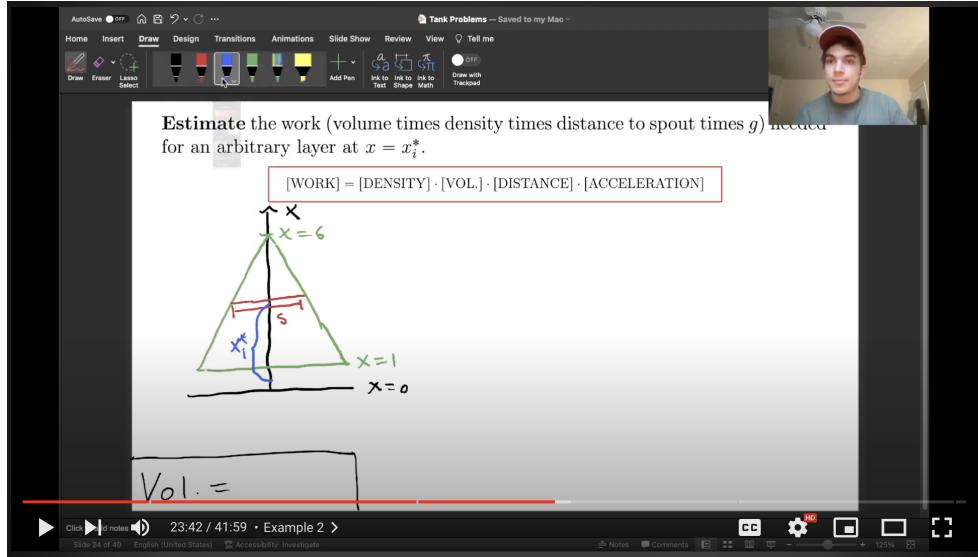


Figure 5: YouTube channel QR code

These videos would then be organized into playlists and separated by course. The state of moderation of the YouTube channel was still ultimately undecided, but I felt it important to allow comments so that feedback from one person could spark a similar “a-ha” moment in others, letting the most popular pieces of feedback rise to the top. However, for the sake of preventing harassment to anyone involved in the video-making process, these comments must be reviewed by a channel administrator before they could be seen publicly. Additionally, timestamps for important moments were added to the description of the videos so that people could skip to what they needed.

In the Fall, I then uploaded two other videos: one was on the concept of the derivative at a point, and the other was on the concept of the definite integral. They were quite prepared, and they used GeoGebra to give students an intuitive understanding of these fundamental parts of Calculus. These videos, as well as the video made on integration by parts, were all made by recording a Teams meeting with only myself, as this was the easiest way to get them made. This process changed upon the creation of the next video, though, which was on tank problems and which was uploaded on December 17th, over Winter break. A similar video was made on implicit differentiation and was uploaded on December

18th. These videos used the recording software OBS, which removed any potential internet problems, and they also utilized Microsoft PowerPoint instead of OneNote, which allowed for more prepared material at each step in the problem-solving process. These were also the last videos made on account of my break for the holidays. An example of the videos can be seen here:



(a) A screenshot of the video

11 views Dec 17, 2022
In this video we discuss some of the hardest Calc II problems: tank problems. They come from the "Applications to Physics and Engineering" section in the Stewart textbook, and we take a definitional approach to solving them.

Timestamps:
0:00 Introduction and steps
5:39 Example 1
17:36 Example 2
31:54 Example 3
41:34 Conclusion

Have feedback? Tell us about it here:
<https://forms.gle/z1eU8ESpX42ZgKYr7>

Chapters

0:00	5:39	17:36	31:54	41:34
Introduction and steps	Example 1	Example 2	Example 3	Conclusion

[View all](#)

Show less

(b) The video description, with chapters

Figure 6: The “Tank Problems” video

Conclusion

Overall, I am very proud of my work and very happy with the volume of materials I had created and the development of the system to allow for future development of the Learning Commons peer-support materials. I have completed a total of 21 documents and 5 videos on topics from Calculus, and I have no desire to stop production of these materials or stop development of the system. In particular, more materials on topics from Calculus III, Linear Algebra, and maybe introductory proof-writing are where my interests lie now that my thesis is done.

In addition to the materials, I have also worked to establish the system by which future documents can be created or edited by peer tutors and accessed by students. While the system will surely be changed in the future, the goals of the project have motivated our original choices and the system seems to be in the best place it could be for its debut to students and peer supports. In other words, our pedagogical goals and systemic goals which we set out at the beginning of the project have been satisfied, though my work with the project will continue.

Supplementary Materials

In the following pages are the documents that were completed at the thesis deadline, in their final forms. They are ordered by topic, beginning with the documents on the developed system itself, followed by Calculus I, II, and III in that order. Here is a list of those documents, followed by a list of the videos:

L^AT_EX Documents

Learning Commons System

- Intro to the Learning Commons Math Materials
- Documents Guide
- A Brief Guide to L^AT_EX
- Document Template

Math Review

- Partial Fraction Decomposition (Linear Terms)

A document on the basics of trigonometry was also in progress upon the project's completion.

Calculus I

- The Product Rule
- The Quotient Rule
- The Chain Rule
- Implicit Differentiation
- Critical Numbers
- The Closed Interval Method for Maxima and Minima

Documents on the concept of the derivative, how to use the limit definition of the derivative, and optimization problems were also in progress upon the project's completion.

Calculus II

- Using the Limit Definition of the Definite Integral
- Integration by U-Substitution
- Integration by Parts (IBP)
- Trigonometric Integration: sine and cosine
- Trigonometric Integration: secant and tangent, cosecant and cotangent
- Trigonometric Substitution
- Tank Problems

Documents on finding the volume of solids of rotation were also in progress upon the project's completion.

Calculus III

- The Equation of a Plane
- Line Integrals of Scalar Functions
- Line Integrals of Vector Fields

A document on the directional derivative was also in progress upon the project's completion.

Videos

- “Integration by Parts (IBP)” video: <https://youtu.be/pcmaZEsrY-4>
- “The Concept of the Derivative” video: <https://youtu.be/-LI2vmuXA58>
- “The Concept of the Definite Integral” video: <https://youtu.be/msb.zCCQscQ>
- “Tank Problems” video: https://youtu.be/bF0KDjpo_Pc
- “Implicit Differentiation” video: <https://youtu.be/FLRMkn2b6Zk>

Intro to the Learning Commons Math Materials

Abstract

In this document, we discuss the Learning Commons System for documents and videos such as the one you are reading right now.

1 A Brief Word of Encouragement.

Hi, and thank you for checking out these materials today. Before we begin on some of the more important stuff, we want to briefly say a couple of things:

As we've made these documents, we've always kept it in mind that **math is hard**. So, if you are struggling with a particular topic, be sure to keep in mind that you are attempting something difficult, and that being a mathematician involves struggling with difficult content. In other words, try not to be hard on yourself. The material is tough, and it's not your fault if you have tried and failed to understand something.

We also want to say that, though math is hard, we believe that anyone can do it. Let us say that again: **you can do it**. You might learn differently from other people, but you can absolutely understand whatever math material you are trying to learn.

It is also true that learning math takes effort, and this is where you are already excelling: that you are reading this document and utilizing these materials shows that you are already putting in above-average effort to learn the material that you are trying to learn. So, way to be a go-getter! The people who seek out help from places like these documents are often the most successful students in their courses.

2 How to Best Utilize These Materials.

As we said before, math is hard. We made these documents to try to reduce that very difficulty. In other words, these documents and the corresponding videos are made to help students with school. Specifically, the materials are

focused on the Kalamazoo College curriculum. If you are a Kalamazoo College student in a course with math topics covered by these documents, you have some skills which we hope to improve through these materials:

1. Mathematical reading comprehension
2. Use and understanding of definitions and important results/theorems
3. Problem solving
4. Understanding of concepts
5. Applying concepts to real-world problems
6. Translating ideas from math to plain English and vice-versa

We believe that the best thing you can do to grow as a mathematician is to try and improve these skills. We also recognize that it can be difficult to see, while in a math course, how to work on improving these skills. So, we've boiled the skills down into three general recommendations for using these materials and learning math in general:

I. Just keep reading.

First and foremost, these documents, including that mathematics ones, are written in text. We completely understand that reading about math and other technical topics is hard, and that it is easy to lose focus and motivation when you are given a large amount of text to read about math. However, **being able to read technical text is an important skill, no matter your career interests.** So, the biggest recommendation we can give is this: just keep reading. Even if you don't understand every word, attempting to read through these documents will help you develop the vital skills of reading technical language.

Furthermore, these documents are written in a somewhat narrative style. In other words, they are written to be read from front to back. So, we recommend you try to do just that. Of course, if you are a skilled math learner and you want to just jump into the step-by-step process to solving problems, you can do that. Confusion may arise, though, if you skip to the middle of the document and don't read the introduction.

II. Try to understand definitions and important results.

If you're going into a STEM field, there will likely be definitions and results which are important to understanding the content of that field. This is truest in mathematics. In essence, **the key to understanding mathematics is understanding definitions**. Much of mathematics is starting with a definition and seeing what results follow from that. Because of that, trying to grapple with the definitions given to you, as well as the most important results or theorems, will give you an enormous leg up compared to those who don't pay attention to these things at all.

III. Balance problem solving with concepts.

Chances are, if you are coming to these documents or videos, it's because you want to know how to solve a textbook problem which you don't know how to solve. However, we believe it is important to balance understanding the problem solving techniques with the concepts. Fundamentally, **understanding the concepts behind what problem you're trying to solve will make learning much easier**.

Furthermore, your understanding of concepts will help you translate between math language and plain English, so, if you're in a math class and are given a kind of real-world word problem, practice translating your intuitions in plain English into the mathematical language and concepts you're learning.

3 A Note on the LC System.

Finally, we have two short notes about the system for developing these materials.

First, do keep in mind that these are "living materials." So, more documents and videos will be created over time, and they will be updated as necessary. If we don't have a document or video on a particular topic yet, we may make one in the future!

Second, we really value and appreciate any respectful feedback, whether critical or otherwise. Getting feedback helps us make the documents better, and keeps open a line of communication so we can know how we're doing. So, if you have feedback for the documents, feel free to provide it at this link or using the following QR code:
<https://forms.gle/PpZsKyymnxyqLn1P7>



Documents Guide

Abstract

This is a guide for documents in the learning commons system. The hope is that this document (along with looking at examples from other, previously made documents) will help others make similar documents in a systematic way. In the abstract, give a brief description of the topic, not mentioning any specifics. Good things to mention include the goal of the document (are you explaining an idea or a problem-solving technique?), and, if the document is problem-solving oriented, it's helpful to mention the context in which the technique will be used.

1 Introduction.

1.1 Documents.

The introduction, much like the rest of the document, will change depending on the goal or topic being covered. There are many ways to introduce a topic, including but not limited to:

1. Introducing the topic through a concept that motivates it, such as, “How might we model this real-world event using math/physics?”
2. Introducing the topic through a problem-solving challenge, for instance, “We don’t know how to solve this kind of problem yet. Let’s see if we can manipulate our previous tools to find a way to solve these problems.”

For the sake of expositional clarity, it’s best to **pick one and only one way** to introduce the topic per document. This may mean providing several documents to introduce the same idea, or devoting *one* document to the concept and *one* separate document to the problem-solving techniques. One example of a concept that might require this is the derivative, which can be motivated either through the idea of *slope* (generalizing a math tool from high-school),

or through the idea of *velocity* and other every-day rates of change (motivated by real-world concepts).

When doing expository writing for educational purposes, your main worry, in my limited experience, should be *losing the reader's attention*. Ultimately, you want them to understand the concept or technique. This means writing about it. However, too much depth will be daunting or even scary if they're looking for a quick-and-easy explanation. This is another benefit of separating the topic into one document for the concept and one document for the problem-solving technique: if they want to read about the concept, they know where to go. If they want to just see how to solve the problems, there's an option for that, too.

Because of the difficulties surrounding expository writing and the many different ways to introduce the concept, I highly recommend getting as much feedback from peers, professors, or students using the documents as possible. This will help you gauge whether you've gone into too much depth, or whether you haven't explained the topic clearly enough. You can definitely do this actively by going to specific people with the document, but it's also a good idea to set up a passive system, such as a feedback form which people can fill out whenever they like. We'll be using Google Forms for this, and will be putting it in as many places as possible, such as on the website, the YouTube channel and descriptions, and maybe even a QR code in the MPC itself.

1.2 Videos.

Speaking of the YouTube channel, let's talk about videos. Video explanations are, in my experience, one of the most helpful ways to learn a concept. It allows people to see the visuals of the concept or problem-solving technique in real time, as well as hearing an audible explanation of the ideas. Many students are unfortunately not very good readers of difficult topics, and while one of our main goals is to increase the reading comprehension skills of those who use the documents, having videos available will be such a helpful thing to have, it's really a no-brainer to include them.

The issue is, of course, that videos take time. A lot of time. So, it's important to prioritise topics in a sensible manner and try to be as consistent with the documents as much as possible. If you introduce one way of solving a problem in the documents, solving the same problem a different way in the videos will simply make things far more confused. The best way to make sure that you're doing everything in a consistent way is to remember why this project is being started in the first place: the documents and videos are not to make something better than Khan Academy for everyone who needs math

help. Instead, these resources are supposed to be specifically guided towards the Kalamazoo College curriculum, and so consistency will arise naturally out of following what the professors are teaching and how they are solving the problems. So, keeping in touch with them will likely be a very good way to stay “on track.”

1.3 Tools.

There are many ways to help make the documents and videos look better and be made clearer. One of the best ways to start off is to familiarize yourself with LaTeX, if you haven’t already. LaTeX is very powerful, and is capable of using many packages and lots of customization. There are ways to graph things, create tables, and import images with lots of customization, though it takes some practice to get used to. The hope is that we will have some kind of introduction to LaTeX as a document and YouTube video so that peer supports can familiarize themselves with the basics or at least know what to Google. Until then, the best place to learn more about LaTeX is [the overleaf “learn” section](#), which has a TON of information on how to use LaTeX.

LaTeX will cover most of the type-setting and text, but sometimes additional visuals are needed or helpful. There are many ways to do visuals, of course, but here are a couple recommendations, in order of how steep the learning curve is:

1. [Desmos.](#) Desmos is easy to use, and many will be very familiar with it. This means that it should be very easy to pick up for simple visuals of graphs, and any kind of live build on a video should be easy to follow. That said, its power is somewhat limited by the software.
2. [Geogebra.](#) Many people know Geogebra only from Calculus III because of its 3D graphing capabilities, but Geogebra is actually far more powerful than meets the eye. If you go to “app downloads,” for instance, and download Geogebra Classic 5, you’ll see that it functions as a computer algebra system, 2D graphing calculator, 3D graphing calculator, spreadsheet, and much more. Because of all of the possibilities, though, learning to use it may be somewhat difficult. There are other ways to learn the basics, but there are many videos on [SparksMaths](#) on YouTube to get inspiration or an understanding of how the program functions.
3. [Python.](#) Python is full-on Turing-complete coding language, meaning anything that a computer can do, can be done in Python. It is also known for its relatively easy syntax and powerful math capabilities, including

parts of Calculus, Statistics, and Differential Equations. It also has some pretty in-depth graphing capabilities, mainly done through a package known as PyPlot. This all makes it, by far, the most powerful of all of the tools mentioned here. However, it also is the most difficult to learn, as it is an entire programming language which will likely take months to become familiar. There are many ways to get help with Python, though, including talking to math/computer science professors, YouTube, Stack Overflow, or [w3schools](#).

2 Steps and Examples.

This section is designated for working through examples of a problem-solving technique. So, if your document is mainly about a concept, you may not include this section.

That said, this section is the main thing that sparked the idea for this project: examples of *how* to solve a problem with commentary *directly next to the example* is something that I had not seen before reading the textbooks created by Dr. Eric Barth. In my experience, this was an incredibly helpful way to condense problem-solving techniques into bite-sized, easy-to-remember steps. Following through several examples (in increasing difficulty) using the same technique will naturally give the student a kind of instinctual, problem-solving “muscle-memory” that does not require them to memorize the particular steps. Here’s the general structure we’ve decided on:

1. Give a list of the general steps, not worrying about making complete sense. If you say something like “treat y as a function of x ,” the reader may not immediately know what you mean, but they will come to understand it through the examples. Still, they can always revisit the steps to make sure they have it down.
2. Come up with three (or more) examples and solve them, step-by-step.
3. Use the “tabular” environment to provide comments directly next to the step-by-step examples. Then, after each example, add any additional comments as required.

Here's an example from the first document made. It was done on implicit differentiation from Calculus I.

Example 1

Find $\frac{dy}{dx}$ given $x^2 + y^2 = 1$.

Solution:

$x^2 + y^2 = 1$ $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$ $\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$ $2x + 2y \cdot \frac{dy}{dx} = 0$ $\frac{dy}{dx}(2y) = -2x$ $\frac{dy}{dx} = -\frac{x}{y}$	<p>Begin with the equation you are given.</p> <p>Take the derivative of both sides with respect to the variable you want, in this case x.</p> <p>Use any derivative rules you can to simplify.</p> <p>Evaluate the derivatives. For all functions of y, remember that y is a function of x. So, use chain rule and multiply by $\frac{dy}{dx}$.</p> <p>Isolate all terms with a $\frac{dy}{dx}$ and factor.</p> <p>Finally, divide and simplify.</p>
---	--

And here's an example from the limit definition of a derivative document:

Example 2

Find the derivative of $f(x) = x^2 + 3$ at the point $(1, 4)$ using the limit definition.

Solution:

$$f(x) = x^2 + 3, a = 1$$

Begin by rewriting the function and noting the value of a (it will be the x value of any given point).

$$f(x + h) = (x + h)^2 + 3$$

Find $f(x + h)$ by plugging $(x + h)$ into f . To do so, anytime you see x in the original $f(x)$ equation, replace it with $(x + h)$ in parentheses. Be very careful with this!

$$f'(x) = \lim_{h \rightarrow 0} \frac{[(x + h)^2 + 3] - [x^2 + 3]}{h}$$

Plug your function into the definition.

Be careful with the minus sign!

Simplify.

Separate the fraction.

Cancel the hs because $h \neq 0$.

Finally, evaluate the limit and plug in a .

$$= 2x + h$$

$$= 2x = 2 \text{ at } x = 1.$$

And finally, another exercise from the limit definition document:

Example 3

Prove that, for a constant c , $\frac{d}{dx}[cf(x)] = c\left[\frac{d}{dx}f(x)\right]$.

Solution:

Our function is $cf(x)$

$$cf(x + h)$$

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h \rightarrow 0} \frac{cf(x + h) - cf(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c(f(x + h) - f(x))}{h} \\ &= \lim_{h \rightarrow 0} c \frac{f(x + h) - f(x)}{h} \\ &= c \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\ &= c\left[\frac{d}{dx}f(x)\right].\end{aligned}$$

Begin by rewriting the function.

Plug $(x + h)$ into f .

Plug your function into the definition.

Do some rearranging to get the derivative definitions on their own. Here, we have to use a limit property to do so.

Finally, use the definition.

3 Extra Exercises.

Finally, we leave the reader with some extra exercises, if they'd like them. The current system used is the one that follows, with numbered questions followed by answers, which are upside-down. An alternative way of doing this would be to list the questions right-side-up on a separate page. What is chosen is perhaps more aesthetic than anything else, but would be a great opportunity to get feedback from others.

One of the main things that motivates including this section is how helpful it can be for any peer-supports who are forced to come up with examples to run through on the spot. These extra exercises will give them some examples to work through, as well as the answers so they can check them. Here are the extra exercises from the implicit differentiation document.

Question 1 Find $\frac{dy}{dx}$ given $3x^2 + 4y^2 = 36$.

Answer 1
$$\frac{dy}{dx} = \frac{x}{4y}$$

Question 2 Find $\frac{dy}{dx}$ given $y^2 - \ln(x) = y\cos(y)$.

Answer 2
$$\frac{dy}{dx} = \frac{2yx - x\cos(y) + xy\sin(y)}{1}$$

Question 3 Find $\frac{dx}{dt}$ given $\cos(t)\sin(x) = \frac{t}{x}$.

Answer 3
$$\frac{dx}{dt} = \frac{x\cos(x)\cos(t) + x\sin(x)\sin(t)}{x^2 + x}$$

A Brief Guide to L^AT_EX

Abstract

In this document, we'll go over some introductory functionality in L^AT_EX. The content in this document will primarily be guided towards peer-supports at Kalamazoo College.

1 Introduction and Setup.

L^AT_EX (pronounced lah-tek) is a type-setting tool. In other words, if you want to type something and make it look really nice, L^AT_EX will have the tools for you to do that. Additionally, there is an online website that lets you use it for free: overleaf.com. We use Overleaf at the Kalamazoo College Learning Commons (LC) to keep every document in our system in one place. To get access to the LC Google account (which is required to access the Overleaf account), contact Dr. Rachel Love.

Once you have the login information for the LC Google account, go to overleaf.com and click “Log In” at the top right. Then, click “Log in with Google” and input the LC Google account information. This will bring you to the Projects home. In the Projects home, you can see the projects in the center, as well as folders or filters on the left side. In the LC account, these folders will be the courses offered. You can click on the folders to see the projects in that folder, or you can look at all projects by clicking on “All Projects” on the top left. This is also where you will find the big, green “New Project” button which allows you to make a new, blank project, or to copy one from a template. If you are planning on jumping right into creating documents, we recommend using the custom template we made for this system. The template will be in the “Learning Commons System” folder. To copy the template and make a new document, you can go under the “Actions” column on the right side and click the copy icon. When you make a copy of this document, it will appear in the “Uncategorized” folder at the bottom left.

One last word of note before we get to the main content: this will be a very simple introduction to the basics of L^AT_EX, but there is already plenty

of helpful information on Overleaf. There is an entire guide on how to use Overleaf and L^AT_EX in general on the web page overleaf.com/learn. So, we highly recommend going there for information whenever you get stuck, or even to introduce the software to yourself in a more user-driven way (in contrast to the linear narrative way set in this document).

2 L^AT_EX and Overleaf Basics.

In every L^AT_EX project, you type what you want to and do all formatting and commands in the “Editor” tab, and then you see the output in the “PDF” tab. The “Editor” tab is where you will do all your work; the “PDF” tab is where you will check to make sure it looks how you want it to. To change which tab you see and how you see them on Overleaf, open a project and click the “Layout” menu in the top right. Then, we recommend beginning with “Editor & PDF” so that you can see both tabs at the same time, side-by-side.

For changes made in the “Editor” section to be reflected in the “PDF” section, you have to *compile* the document. To do this on Overleaf, simply click the big, green “Recompile” button on the top left of the PDF tab. You can also click the arrow next to the button and turn on “Auto Compile” which will reflect changes to the PDF as soon as they are made in the editor.

Side note: If you want to add pictures or other files to your Overleaf project, you do this in the tab to the left of the “Editor” tab. At the top, just beneath the “menu” button, there is an “upload” button which lets you upload any files you need to use within the project. L^AT_EX doesn’t by default allow the insertion of these images into documents, though. To do that, you’ll need a package, which we will discuss later.

3 The “Editor” Tab.

Since the “Editor” tab is where you’ll be working, let’s focus on it now. Every project has two sections: the *preamble* and the *document*. The preamble is everything before the “\begin{document}” line in the editor, and the document is everything that comes after that line. Everything you want to explicitly appear in the PDF output needs to go in the document section. Anytime you want to set a parameter, such as changing the font of the document, or load a package, you’ll do that in the preamble.

Now, to see simple text in the output, you can simply type stuff in the document section of the editor. But, if you want to do anything more than have

normal text, you have to use *commands*. Commands in L^AT_EX are achieved by putting a backslash (\) before the command name. In Overleaf, when you type a backslash and then begin typing a command name, it will make suggestions as to what command you may be trying to use. You can auto-fill this suggestion by pressing “tab.”

3.1 Simple Commands.

Here are some simple L^AT_EX commands to get you started with simple stuff:

Command	What it does	Shortcut
\textit{text}	italicizes the bracketed text	command + i
\textbf{text}	bolds the bracketed text	command + b
\underline{text}	underlines the bracketed text	N/A
\usepackage{name}	loads a package (preamble only)	N/A
\maketitle	puts the title in the PDF	N/A
\section{text}	creates section header	N/A
\subsection{text}	creates subsection header	N/A
\textbackslash	adds a backslash in line	N/A
\\$	adds a \$ in line	N/A
\%	adds a % in line	N/A
\&	adds a & in line	N/A

A note on the last 4: you may ask “why can’t you just use a backslash, dollar sign, percent sign, or ampersand to add it to your text in line?” and that’s a great question. The reason is that these all have functionality within L^AT_EX. When you put a backslash in the editor, L^AT_EX is told to interpret whatever follows it as a command. So, you need a special command to print a backslash. Similarly, a dollar sign tells L^AT_EX to interpret what comes next as an equation, and a percent sign tells L^AT_EX to ignore whatever comes next (this is how you can comment in the editor tab, like in other programming languages). Finally, an ampersand is used to signify alignment. Its functionality lies in environments, so let’s talk about them now.

3.2 Environments and Equations.

There is a special use for commands called *environments*. Environments are signified by a command starting it and a command ending it: \begin{environment} and \end{environment}. You’ve already met one such environment: the document environment (remember \begin{document}?). These environments let you do special stuff in between them. Some examples are:

Environment	What it does
equation	creates a centered equation on a new line
split	splits and centers an equation around an &
tabular	creates a table on a new line
center	centers the text in the environment
align	allows for centered equation/text alignment on a new line

Let me give you some examples. An example of an equation environment without using the split environment is:

$$\sin^2(x) + \cos^2(x) = 1 \quad (1)$$

or, you can omit the equation number on the right side by inserting asterisks after the environment name (e.g. `\begin{equation*}`):

$$\sin^2(x) + \cos^2(x) = 1$$

Then, an equation with the split environment is best used for a series of equalities, such as:

$$\begin{aligned} \frac{\partial P}{\partial y} &= f_{yx} \\ &= \frac{\partial f_y}{\partial x} \\ &= \frac{\partial Q}{\partial x}. \end{aligned} \quad (2)$$

Then, the tabular environment has been used in this document so far to make tables, such as the one above. You can vary where the lines of the table are, though, by changing where you insert `\hline` and the vertical lines in the argument of the environment. Some examples are:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Name	Favorite fruit
Sam	Apple
Stacy	Pear

As you can see, tables are not, by default, centered. So, you have to use the center environment to center them.

Finally, for the align environment. Here's an example:

$$(a) y = 3x^2 + 2 \quad (b) \cos(y) = \sin(x) \quad (3)$$

or, without the equation number, using asterisks:

$$(a) y = 3x^2 + 2 \quad (b) \cos(y) = \sin(x)$$

3.2.1 Commands in Equations.

Now, within equations (such as the equation or align environment, or within single dollar signs for in-line equations), there are a variety of commands at your disposal which can be very helpful.

One such command is the `\frac{}{}` command, which adds a fraction to your equation, with the first brackets as the numerator and the second brackets as the denominator. An example of an in-line fraction is $\frac{dy}{dx}$. You'll notice, though, that when fractions (or other equations) are put in-line, they are shrunk to fit with the other text and not mess with the spacing. You can get around this by inserting `\displaystyle{}` inside of the equation, and putting all of the equation text inside the brackets. This will make the above fraction look like this: $\frac{dy}{dx}$. This is also useful for a variety of other math commands, including limits, integrals, and sums:

Command	in-line	displaystyle
<code>\lim_{x \rightarrow \infty}</code>	$\lim_{x \rightarrow \infty}$	$\lim_{x \rightarrow \infty}$
<code>\int_0^1</code>	\int_0^1	\int_0^1
<code>\sum_{n=0}^{\infty}</code>	$\sum_{n=0}^{\infty}$	$\sum_{n=0}^{\infty}$

In general, for readability, using `\displaystyle{}` is encouraged.

3.3 Packages.

There are a couple of helpful packages which every L^AT_EX user should know about. As a recap, you load these packages by using the `\usepackage{name}` command in the preamble. Here are these packages and what they do:

Package name	What it does
amsmath	Adds vital math functionality/new commands
amssymb	Adds math symbol commands
graphicx	Lets you add images to documents
geometry	Allows for varied margin sizes

As with all packages, the best way to learn what functionality they have is to Google “geometry LaTeX package” and read about what it does. Often, the documentation is very good and thorough. So, we won’t explain how to use every single feature of every package, but we will instead encourage you to research the packages as you need them.

4 Conclusion.

That concludes this brief guide to L^AT_EX. These are the absolute basics to let you begin creating, but you shouldn’t try to memorize any more than this before you begin to play around with the application. From here on out, overleaf.com/learn is your best friend, as is Google for questions like “how to make upside-down text in L^AT_EX.” Good luck, and happy creating!

Your Title Here

Abstract

Your abstract here. You can write some stuff here, but try not to keep it too long. Just do your best. I believe in you.

1 Introduction.

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

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2 Steps and Examples.

This is the general process:

1. Add a step one.
2. Then a step two.
3. Maybe a step three.

Example 1

Give the prompt for Example 1 here.

Solution:

Equation in math mode.	Commentary on step one. If your commentary is large, you may have to include a second line for text.
Equation in math mode.	Commentary on step two. If your commentary is large, you may have to include a second line for text.
Equation in math mode.	Commentary on step three. If your commentary is large, you may have to include a second line for text.
Equation in math mode.	Commentary on step four. If your commentary is large, you may have to include a second line for text.
Equation in math mode.	Commentary on step five. If your commentary is large, you may have to include a second line for text.
Equation in math mode.	Commentary on step six. If your commentary is large, you may have to include a second line for text.

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Example 2

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Solution:

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Example 3

Give the prompt for Example 3 here.

Solution:

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Equation in math mode.	Commentary on step three. If your commentary is large, you may have to include a second line for text.
Equation in math mode.	Commentary on step four. If your commentary is large, you may have to include a second line for text.
Equation in math mode.	Commentary on step five. If your commentary is large, you may have to include a second line for text.
Equation in math mode.	Commentary on step six. If your commentary is large, you may have to include a second line for text.

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3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 *Text for question 1.*

Answer 1 *Text for answer 1.*

Question 2 *Text for question 2.*

Answer 2 *Text for answer 2.*

Question 3 *Text for question 3.*

Answer 3 *Text for answer 3.*

Partial Fraction Decomposition (Linear Terms)

Abstract

In this document, we review an important part of algebra: decomposing rational functions into the sum of partial fractions. This process is called *partial fraction decomposition*, and it has many uses.

1 Introduction.

The ability to turn a rational function into the sum of partial fractions stems from one fact about polynomials over the real numbers: they can always be factored into a product of linear or irreducible quadratic factors (fun fact: polynomials over the complex numbers are even nicer, as there are no irreducible quadratics there). This means if you give me a rational function, I can always factor its denominator into linear terms and/or irreducible quadratic terms.

Now rational functions which factor into irreducible quadratic terms are harder and they are not often taught at Kalamazoo College. As such, this document will only cover linear terms. This gives us two cases, though: (1) one or more linear terms can be repeated (such as $(x-1)^2(x+5)$, where $(x-1)$ has a higher power than one and so is “repeated”), or (2) there are no repeated factors. We’ll handle both cases at once.

2 Steps and Examples.

This is the general process:

1. If the degree (highest power) of the numerator is larger than the degree of the denominator, do polynomial long division and only do steps 2-6 on the remaining fraction.
2. Factor the denominator.
3. If none of the factors are repeated, equate the function to $\frac{A}{(a_1x + b_1)} + \frac{B}{(a_2x + b_2)} + \dots$ where the $(ax + b)$ s are the factors of the denominator. Include one term for every factor.
4. For any repeated factors, such as $(ax + b)^n$, include a term for every power of the repeated factor less than or equal to n . This is called “building up” to n .
5. Multiply both sides by the denominator of the rational function.
6. Solve for A, B , etc. To do so, plug the zeroes associated with the factors in for x . If you have repeated factors, you’ll need to plug in other points (such as $x = 0$) also.
7. Plug A, B, \dots into the original equation.

Example 1

Decompose $\frac{1}{x^2 - 3x + 2}$ into partial fractions.

Solution: Before we get started, note that the degree of the numerator is less than the degree of the denominator, so we don’t have to do polynomial long division.

$x^2 - 3x + 2 = (x - 1)(x - 2)$ $\frac{1}{x^2 - 3x + 2} = \frac{1}{(x - 1)(x - 2)}$ $= \frac{A}{(x - 1)} + \frac{B}{(x - 2)}$ $1 = \frac{A(x - 1)(x - 2)}{x - 1} + \frac{B(x - 1)(x - 2)}{(x - 2)}$	<p>Factor the denominator. Here, we have no repeated factors.</p> <p>Equate the function to these partial fractions.</p> <p>Multiply both sides by the denominator.</p>
---	---

$$1 = A(x - 2) + B(x - 1)$$

At $x = 1$: $1 = A(1 - 2) + B(1 - 1)$

So $1 = -A$ or $A = -1$.

At $x = 2$: $1 = A(2 - 2) + B(2 - 1)$

So $1 = B$ or $B = 1$.

$$\begin{aligned} \text{Then } \frac{1}{x^2 - 3x + 2} &= \frac{A}{(x - 1)} + \frac{B}{(x - 2)} \\ &= \frac{-1}{(x - 1)} + \frac{1}{(x - 2)} \end{aligned}$$

Simplify. Tip: don't distribute A and B .

Plug in the zeroes of the denominator.

Solve for A and B .

Plug A and B back into the original equation.

Great, hopefully that isn't too hard to understand. The main thing to remember here is how to set up the partial fractions in the first place: putting A and B and such letters over the different factors of the denominator and then solving for them. This may take some practice, but just try to take it one step at a time. Now let's move on to a harder example where we can see what happens when we have a repeated factor.

Example 2

Decompose $\frac{2x+7}{2x^3 - 5x^2 - 4x + 12}$ into partial fractions. (Hint: the denominator has only rational zeroes).

Solution: Again, the degree of the numerator is less than the degree of the denominator so we don't have to do long division. Our first step, then, is factoring the denominator. The hint tells us that the denominator has only rational zeroes, meaning we can use the rational root test to find them all. Doing so gives us zeroes at $x = 2$ and $x = -\frac{3}{2}$. When we factor out these zeroes, we see that we can write the denominator as $(2x+3)(x-2)^2$. (Note here that we have a repeated factor of $x-2$.) We're done factoring the denominator, then, so let's move on to step 3.

$$\frac{2x+7}{2x^3 - 5x^2 - 4x + 12} = \frac{2x+7}{(2x+3)(x-2)^2}$$

$$= \frac{A}{(2x+3)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$2x+7 = A(x-2)^2 + B(2x+3)(x-2) + C(2x+3)$$

$$\text{At } x = 2: 2(2) + 7 = C(2(2) + 3)$$

$$\text{So } 11 = 7C \text{ or } C = \frac{11}{7}$$

$$\text{At } x = -\frac{3}{2}: 2\left(-\frac{3}{2}\right) + 7 = A\left(-\frac{3}{2} - 2\right)^2$$

$$\text{So } 4 = A\frac{49}{4} \text{ or } A = \frac{16}{49}$$

Here's our factorization.

Make your partial fractions.

Because $(x-2)$ is repeated, we need 2 terms, one for each power $\leq n$.

Multiply by the denominator.

Plug in the zeroes.

Solve for A, B, C .

Now you'll notice something here. We've plugged in all of our zeroes but still haven't found B . This is very common when the denominator has repeated factors. To find B , plug in another point that isn't a zero (pick one easy to plug in), and substitute in A and C . Let's do that now with $x = 0$.

$$\text{At } x = 0: 7 = 4A - 6B + 3C \quad \text{Plug in 0.}$$

$$= \frac{64}{49} - 6B + \frac{33}{7} \quad \text{Sub in } A, C.$$

$$\text{So } -6B = \frac{48}{49} \text{ or } B = -\frac{8}{49} \quad \text{Solve for } B.$$

Then $\frac{2x + 7}{2x^3 - 5x^2 - 4x + 12}$ $= \frac{A}{(2x + 3)} + \frac{B}{(x - 2)} + \frac{C}{(x - 2)^2}$ $= \frac{16/49}{(2x + 3)} + \frac{-8/49}{(x - 2)} + \frac{11/7}{(x - 2)^2}$	Plug A, B, C back into the original equation.
--	---

Those are some gross fractions, but the process shouldn't be too bad if you've practiced without repeated factors before. The main thing to remember is that you have to include several terms for the same factor because it is raised to a higher power. Then, it's all about plugging in x -values and solving for A , B , and C . Let's now take a look at what happens when the degree of the numerator is higher than the degree of the denominator.

Example 3

Decompose $\frac{x^4 - 3x^3 - 7x^2 + 16x + 14}{x^3 - 5x^2 + 3x + 9}$ into partial fractions.

Solution: Here the degree of the numerator is higher than the degree of the denominator, so we have to use polynomial long division first. Doing so looks like this:

$$\begin{array}{r}
 \begin{array}{c} X+2 \\ \hline X^3 - 5X^2 + 3X + 9 \end{array} \overline{) X^4 - 3X^3 - 7X^2 + 16X + 14} \\
 \begin{array}{c} -(X^4 - 5X^3 + 3X^2 + 9X) \\ \hline 2X^3 - 10X^2 + 7X + 14 \end{array} \\
 \begin{array}{c} -(2X^3 - 10X^2 + 6X + 18) \\ \hline R: X - 4 \end{array}
 \end{array}$$

And so we have our original fraction equal to $x + 2 + \frac{x - 4}{x^3 - 5x^2 + 3x + 9}$. This means we can now do partial fraction decomposition on the remaining fraction. Factoring the denominator as in Example 3, using the rational root test, gives us the factorization of $(x + 1)(x - 3)^2$, so we have one repeated factor. Let's move on from there.

$$\frac{x^4 - 3x^3 - 7x^2 + 16x + 14}{x^3 - 5x^2 + 3x + 9}$$

This was our given fraction.

$$= x + 2 + \frac{x - 4}{(x + 1)(x - 3)^2}$$

The result of division and factoring.

$$\frac{x - 4}{(x + 1)(x - 3)^2} = \frac{A}{(x + 1)} + \frac{B}{(x - 3)} + \frac{C}{(x - 3)^2}$$

Now decompose the remainder. We have a repeated factor, so we need 3 terms.

$$x - 4 = A(x - 3)^2 + B(x + 1)(x - 3) + C(x + 1)$$

Multiply by the denominator.

$$\text{At } x = 3: -1 = 4C \text{ so } C = -\frac{1}{4}$$

Plug in the zeroes.

$$\text{At } x = -1: -5 = A(-4)^2 \text{ so } A = -\frac{5}{16}$$

At $x = 0$: $-4 = 9A - 3B + C$

$$\implies -4 = 9 \left(\frac{-5}{16} \right) - 3B - \frac{1}{4}$$

$$\implies -3B = -\frac{15}{16} \text{ or } B = \frac{5}{16}$$

$$\text{So } \frac{x^4 - 3x^3 - 7x^2 + 16x + 14}{x^3 - 5x^2 + 3x + 9}$$

$$= x + 2 + \frac{-5/16}{(x+1)} + \frac{5/16}{(x-3)} + \frac{-1/4}{(x-3)^2}$$

Plug in another value.

Plug in A and C .

Solve for B .

Put everything together.

Hopefully the jump from Example 2 to 3 isn't too severe: the main thing is remembering to do polynomial long division (and knowing how to do so). It won't be the most common thing you see, but it is good to review it if you haven't already.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Decompose $\frac{1}{x^2 + 3x + 2}$ into partial fractions.

Answer 1 $\frac{1}{1 - x} + \frac{x + 1}{x + 2}$.

Question 2 Decompose $\frac{8}{x^3 - 7x^2 - 5x + 75}$ into partial fractions.

Answer 2 $\frac{1/8}{1 - x} + \frac{-1/8}{x + 3} + \frac{(x + 3)}{x - 5}$.

Question 3 Decompose $\frac{x^4 + x^3 - 34x^2 - 54x + 96}{x^3 + 2x^2 - 32x - 96}$ into partial fractions.

Answer 3 $x - 1 + \frac{-3/5}{x + 4} + \frac{4}{x - 3} + \frac{9 - x}{x + 5}$.

The Product Rule

Abstract

In this document, we discuss one of the first techniques of differentiation, called the *product rule*. The product rule gives us a formula with which we can take derivatives of two functions multiplied together.

1 Introduction.

Consider the function $y = x^2 \cdot e^x$. We know that the derivative of x^2 is $2x$ by the power rule, and the derivative of e^x is e^x . This might lead us to guess that the derivative y' is $2x \cdot e^x$, but this is actually not correct. In general, if we have two functions $f(x)$ and $g(x)$ and we consider their product $f(x) \cdot g(x)$, the derivative will **not** be $f'(x) \cdot g'(x)$. Instead, we have to use the *product rule*, which gives this formula:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Or, if we rename the functions to be more helpful,

$$\frac{d}{dx}[\text{LEFT} \cdot \text{RIGHT}] = \text{LEFT}' \cdot \text{RIGHT} + \text{LEFT} \cdot \text{RIGHT}'$$

Applied to our example of $y = x^2 \cdot e^x$, then, we have $\text{LEFT} = x^2$ and $\text{RIGHT} = e^x$. So $\text{LEFT}' = 2x$ and $\text{RIGHT}' = e^x$. Using the product rule then gives $y' = 2x \cdot e^x + x^2 \cdot e^x$. Let's take a closer look at how to apply the formula with a couple examples.

2 Steps and Examples.

This is the general process:

1. Identify the two functions.
2. Find their derivatives.
3. Use the product rule formula.

Example 1

Find the derivative of $5xe^x$.

Solution:

$\text{LEFT} = 5x \text{ and } \text{RIGHT} = e^x$ $\text{LEFT}' = 5 \text{ and } \text{RIGHT}' = e^x$ $\begin{aligned} \frac{d}{dx}[5xe^x] &= \frac{d}{dx}[(5x) \cdot (e^x)] \\ &= \frac{d}{dx}[\text{LEFT} \cdot \text{RIGHT}] \\ &= \text{LEFT}' \cdot \text{RIGHT} + \text{LEFT} \cdot \text{RIGHT}' \\ &= \frac{d}{dx}[5x] \cdot e^x + 5x \cdot \frac{d}{dx}[e^x] \\ &= 5 \cdot e^x + 5x \cdot e^x \\ &= e^x(5x + 5) \end{aligned}$	<p>Identify the functions.</p> <p>Find their derivatives.</p> <p>Use the product rule formula.</p>
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Great, that wasn't so bad! Let's look at another example.

Example 2

Find the derivative of $(5x^4 + 2x^3 - 7x^2 - 34x + 17) \cdot e^x$.

Solution:

$$\text{LEFT} = 5x^4 + 2x^3 - 7x^2 - 34x + 17 \text{ and } \text{RIGHT} = e^x$$

Identify your functions.

$$\text{LEFT}' = 20x^3 + 6x^2 - 14x - 34$$

Find their derivatives.

$$\text{RIGHT}' = e^x$$

$$\frac{d}{dx}[(5x^4 + 2x^3 - 7x^2 - 34x + 17) \cdot e^x]$$

Use the product rule formula.

$$= \frac{d}{dx}[\text{LEFT} \cdot \text{RIGHT}]$$

$$\begin{aligned} &= \text{LEFT}' \cdot \text{RIGHT} + \text{LEFT} \cdot \text{RIGHT}' \\ &= \frac{d}{dx}[5x^4 + 2x^3 - 7x^2 - 34x + 17] \cdot e^x \\ &\quad + (5x^4 + 2x^3 - 7x^2 - 34x + 17) \cdot \frac{d}{dx}[e^x] \end{aligned}$$

$$\begin{aligned} &= (20x^3 + 6x^2 - 14x - 34) \cdot e^x \\ &\quad + (5x^4 + 2x^3 - 7x^2 - 34x + 17) \cdot e^x \end{aligned}$$

$$= e^x(5x^4 + 22x^3 - x^2 - 48x - 17)$$

That's a long, large answer. This is pretty common, and it can make product rule problems look pretty daunting. The key is to break the problem up into the steps of finding the functions, finding the derivatives, and using the formula. If you follow those steps, every problem should be manageable. Let's look at a final, more complex example.

Example 3

Use product rule to find the derivative of $t^2 + t\sqrt{t}$.

Solution:

$$\begin{aligned}\frac{d}{dt}[t^2 + t\sqrt{t}] &= \frac{d}{dt}[t^2] + \frac{d}{dt}[t\sqrt{t}] \\ &= 2t + \frac{d}{dt}[t\sqrt{t}]\end{aligned}$$

$$\text{LEFT} = t \text{ and } \text{RIGHT} = \sqrt{t} = t^{\frac{1}{2}}$$

$$\text{LEFT}' = 1 \text{ and } \text{RIGHT}' = \frac{1}{2}t^{-\frac{1}{2}}$$

$$\begin{aligned}&\frac{d}{dt}[t\sqrt{t}] \\ &= \frac{d}{dt}[(t) \cdot (t^{\frac{1}{2}})] \\ &= \frac{d}{dt}[\text{LEFT} \cdot \text{RIGHT}]\end{aligned}$$

$$= \text{LEFT}' \cdot \text{RIGHT} + \text{LEFT} \cdot \text{RIGHT}'$$

$$\begin{aligned}&= \frac{d}{dt}[t] \cdot t^{\frac{1}{2}} + t \cdot \frac{d}{dt}[t^{\frac{1}{2}}] \\ &= 1 \cdot \sqrt{t} + t \cdot \frac{1}{2}t^{-\frac{1}{2}}\end{aligned}$$

$$\text{So } \frac{d}{dt}[t^2 + t\sqrt{t}] = 2t + \sqrt{t} + t \cdot \frac{1}{2}t^{-\frac{1}{2}}$$

Before we use product rule here, we have to use the sum rule to split up the derivative.

Now we can use product rule on the right term.

Find their derivatives.

Use the product rule formula.

Put it together for your final answer.

The last example is tricky. First, you have to realize that you can separate the derivative at the beginning using the sum rule, and then subsequently use the product rule on the right term. Then, at the end, once you've used product rule, you have to remember to put that back together with the part from the beginning for your final answer. This is very easy to forget, and so it's very important that you remember to put everything together at the end!

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find the derivative of $3e^x \cdot x^2$.

Answer 1 $3e^x \cdot x^2 + 3e^x \cdot 2x$

Question 2 Find the derivative of $(x^5 + 7x + 4) \cdot e^x$.

Answer 2 $e^x \cdot (5x^4 + 7) + (x^5 + 7x + 4) \cdot e^x$

Question 3 Find the derivative of $x^4 + x^3e^x$.

Answer 3 $4x^3 + x^2e^x + x^3e^x$

The Quotient Rule

Abstract

In this document, we discuss another technique of integration, called the *quotient rule*. The quotient rule gives us a formula with which we can take derivatives of a fraction with functions in the numerator and denominator.

1 Introduction.

Consider the function $y = \frac{x^2}{e^x}$. We know that the derivative of x^2 is $2x$ by the power rule, and the derivative of e^x is e^x . This might lead us to guess that the derivative y' is $\frac{2x}{e^x}$, but this is actually not correct. In general, if we have two functions $f(x)$ and $g(x)$ and we consider their quotient $\frac{f(x)}{g(x)}$, the derivative will **not** be $\frac{f'(x)}{g'(x)}$. Instead, we have to use the *quotient rule*, which gives this formula:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Now in comparison to the product rule, this is pretty hard to remember. Especially since, with the minus sign in the numerator, getting the order of the terms in the numerator is crucial to using it correctly. A phrase that might help to remember this formula is “low ‘d’ high minus high ‘d’ low over low squared,” where “low” refers to $g(x)$ (the bottom function), “high” refers to $f(x)$ (the top function), and “d” means “derivative.” However you choose to remember it, it’s crucial that you do. Here’s the formula with different naming:

$$\frac{d}{dx} \left[\frac{\text{LOW}}{\text{HI}} \right] = \frac{\text{LOW} \cdot \text{HI}' - \text{HI} \cdot \text{LOW}'}{\text{LOW}^2}$$

Let’s see how to apply it with a couple of examples.

2 Steps and Examples.

This is the general process:

1. Identify the functions.
2. Find their derivatives.
3. Use the quotient rule formula.

Example 1

Find the derivative of $\frac{x^2}{e^x}$ (our example from the introduction).

Solution:

$\text{HI} = x^2 \text{ and } \text{LOW} = e^x$ $\text{HI}' = 2x \text{ and } \text{LOW}' = e^x$ $\frac{d}{dx} \left[\frac{\text{HI}}{\text{LOW}} \right] = \frac{d}{dx} \left[\frac{\text{HI}}{\text{LOW}} \right]$ $= \frac{\text{LOW} \cdot \text{HI}' - \text{HI} \cdot \text{LOW}'}{\text{LOW}^2}$ $= \frac{(e^x)(2x) - (x^2)(e^x)}{(e^x)^2}$ $= \frac{2x - x^2}{e^x}$	Identify the functions. Find their derivatives. Use the quotient rule formula.
---	--

Note that the steps here are identical to those of the product rule. This means learning the quotient rule should not be that much more work; you need only know the new formula, the step-by-step process is identical. Let's look at another example

Example 2

Find the derivative of $\frac{e^x}{4x^3 - 2x^2 + x - 3}$.

Solution:

$$\text{HI} = e^x \text{ and } \text{LOW} = 4x^3 - 2x^2 + x - 3$$

$$\text{HI}' = e^x \text{ and } \text{LOW}' = 12x^2 - 4x + 1$$

$$\begin{aligned} \frac{d}{dx} \left[\frac{e^x}{4x^3 - 2x^2 + x - 3} \right] &= \frac{d}{dx} \left[\frac{\text{HI}}{\text{LOW}} \right] \\ &= \frac{\text{LOW} \cdot \text{HI}' - \text{HI} \cdot \text{LOW}'}{\text{LOW}^2} \end{aligned}$$

$$\begin{aligned} &\frac{(4x^3 - 2x^2 + x - 3)(e^x) - (e^x)(12x^2 - 4x + 1)}{(4x^3 - 2x^2 + x - 3)^2} \\ &= \frac{e^x(4x^3 - 14x^2 + 5x - 4)}{(4x^3 - 2x^2 + x - 3)^2} \end{aligned}$$

Identify the functions.

Find their derivatives.

Use the quotient rule formula.

You may ask why the denominator in the final answer was left “unsquared.” The truth is that this is probably the simplest way to write the answer, and keeping the denominator like that will likely (but not always) be the simplest way to write the answer in general. Let’s look at a final, more complex example.

Example 3

Find the derivative of $t^2 - \frac{\sqrt{t}}{t + e^t}$.

Solution:

$$\begin{aligned} \frac{d}{dt} \left[t^2 - \frac{\sqrt{t}}{1 + e^t} \right] &= \frac{d}{dt}[t^2] - \frac{d}{dt} \left[\frac{\sqrt{t}}{t + e^t} \right] \\ &= 2t - \frac{d}{dt} \left[\frac{\sqrt{t}}{t + e^t} \right] \end{aligned}$$

$$\text{HI} = \sqrt{t} = t^{\frac{1}{2}} \text{ and } \text{LOW} = t + e^t$$

$$\text{HI}' = \frac{1}{2}t^{-\frac{1}{2}} \text{ and } \text{LOW}' = 1 + e^t$$

$$\begin{aligned} \frac{d}{dt} \left[\frac{\sqrt{t}}{t + e^t} \right] &= \frac{d}{dt} \left[\frac{\text{HI}}{\text{LOW}} \right] \\ &= \frac{\text{LOW} \cdot \text{HI}' - \text{HI} \cdot \text{LOW}'}{\text{LOW}^2} \\ &= \frac{(t + e^t)(\frac{1}{2}t^{-\frac{1}{2}}) - (\sqrt{t})(1 + e^t)}{(t + e^t)^2} \\ \text{So } \frac{d}{dt} \left[t^2 - \frac{\sqrt{t}}{t + e^t} \right] &= \\ 2t - \frac{(t + e^t)(\frac{1}{2}t^{-\frac{1}{2}}) - (\sqrt{t})(1 + e^t)}{(t + e^t)^2} \end{aligned}$$

Before we use quotient rule here, we have to use the difference rule to split up the derivative.

Now we can use quotient rule on the right term.

Find the derivatives.

Use the quotient rule formula.

Put it together for your final answer.

The last example is tricky. First, you have to realize that you can separate the derivative at the beginning using the difference rule, and then subsequently use the quotient rule on the right term. Then, at the end, once you've used product rule, you have to remember to put that back together with the part from the beginning for your final answer. This is very easy to forget, and so it's very important that you remember to put everything together at the end!

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find the derivative of $\frac{3e^x}{x^2}$.

$$\text{ANSWER 1} \quad \cdot \frac{x}{(2x-x^2)(3e^x)} = \frac{x}{(x^2)(2e^x) - (x^2)(e^x)}$$

Question 2 Find the derivative of $\frac{x^5 + 7x + 4}{e^x}$.

$$\text{ANSWER 2} \quad \cdot \frac{x^2}{(x+xe^x-x^2e^x)(x+xe^x)} = \frac{x^2}{(x^2)(2e^x) - (x^2)(e^x)}$$

Question 3 Find the derivative of $x^4 - \frac{e^x}{x^3}$.

$$\text{ANSWER 3} \quad \cdot \frac{9x}{(x-x^2e^x)x^2} - e^x \cdot \frac{9x}{(x^2e^x)(x^2) - (x^2)(e^x)} = \frac{9x}{(x^2)(2e^x) - (x^2)(e^x)} - e^x$$

The Chain Rule

Abstract

In this document, we discuss another technique of differentiation, called the *chain rule*. The chain rule gives us a formula with which we can take derivatives of the composition of functions.

1 Introduction.

Consider the function $y = \sin(e^x)$. There seems to be one function (e^x) inside of another function ($\sin(x)$) or what you might call a *composition of functions*. If $\sin(x) = f(x)$ and $e^x = g(x)$, then $y = f(g(x)) = f \circ g(x)$, since e^x has been “plugged into” $\sin(x)$.

Now up to now, we have had no way of evaluating derivatives of such functions. We can handle two functions multiplied together (the product rule), or even one function divided by another (the quotient rule), but we have yet to handle the case in which one function is inside of another. This is exactly where the *chain rule* comes in. It gives us this formula:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

In other words, you take the derivative of the outside, keeping the inside the same, and then you multiply by the derivative of the inside. In terms of an inner and outer function, it looks like this:

$$\frac{d}{dx}[\text{OUTER}(\text{INNER})] = \text{OUTER}'(\text{INNER}) \cdot \text{INNER}'$$

Let's look at some examples of how to put this into practice.

2 Steps and Examples.

This is the general process, just like product and quotient rule:

1. Identify the outer and inner functions.
2. Find their derivatives.
3. Use the chain rule formula.

Example 1

Find the derivative of e^{5x} .

Solution:

$$\text{OUTER} = e^x \text{ and INNER} = 5x$$

Identify your outer and inner functions. Here, $5x$ is “inside” the exponential, so we take it to be $g(x)$ and the exponential to be $f(x)$.

$$\text{OUTER}' = e^x \text{ and INNER}' = 5x$$

Find their derivatives.

$$\frac{d}{dx}[e^{5x}] = \frac{d}{dx}[\text{OUTER}(\text{INNER})]$$

Use the chain rule formula.

$$= \text{OUTER}'(\text{INNER}) \cdot \text{INNER}'$$

$$= e^{5x} \cdot 5$$

$$= 5e^{5x}$$

One note here: the most common place people mess up in the chain rule is understanding how to handle $f'(g(x))$ or $\text{OUTER}'(\text{INNER})$. Many people have trouble knowing what to plug into it, or how to interpret it. Let’s have a little refresher.

If I told you that $f'(x)$ was $x^2 + 2x + 1$, and I wanted you to find $f'(5)$, what would you do? You would likely take the $f'(x)$ formula, and replace x with 5 every time it appears, and would give me something like $(5)^2 + 2(5) + 1$. That’s great, and is the perfect way to handle that. **Doing this process with functions is no different.** If I said, instead, to find $f'(\cos(x))$, you do basically the same thing: every time you see an x in the original $f'(x)$ formula, you replace it with $\cos(x)$. So, $f'(\cos(x))$ would be $(\cos(x))^2 + 2(\cos(x)) + 1$.

When you’re handling the chain rule formula, then, $f'(g(x))$ means “take $f'(x)$ and, every time you see an x , replace it with $g(x)$.” Similarly with INNER and OUTER, which are just renamed versions of that same thing. Let’s look at another example.

Example 2

Find the derivative of $x^2 \sin(5x)$.

Solution:

$$\frac{d}{dx}[x^2 \sin(5x)] = \frac{d}{dx}[x^2](\sin(5x)) + \frac{d}{dx}[\sin(5x)](x^2)$$

We have to use product rule here first.

$$= 2x \sin(5x) + \frac{d}{dx}[\sin(5x)](x^2)$$

$$\text{For } \frac{d}{dx}[\sin(5x)]:$$

Now use chain rule.

$$\text{OUTER} = \sin(x) \text{ and INNER} = 5x$$

$$\text{OUTER}' = \cos(x) \text{ and INNER}' = 5$$

$$\frac{d}{dx}[\sin(5x)] = \frac{d}{dx}[\text{OUTER}(\text{INNER})]$$

$$= \text{OUTER}'(\text{INNER}) \cdot \text{INNER}' = \cos(5x) \cdot 5$$

$$\text{So } \frac{d}{dx}[x^2 \sin(5x)] =$$

Put it together for your final answer.

$$2x \sin(5x) + 5 \cos(x)(x^2)$$

As you can see, when the differentiation rules are combined into one problem, things can get pretty complex. The key is to be quite familiar with product and quotient rule, so that you can easily use their formulas. Then “break off” the piece that needs chain rule and do it separately, before finally putting everything back together for your final answer. Let’s look at a final, more complex example.

Example 3

Use product rule to find the derivative of $\cos(\sin(\tan(x)))$.

Solution:

Before we begin here, we have to note that we don't just have a simple composition of functions here, as was originally described with the chain rule formula. Instead of having 2 functions, we have 3. The key is to begin with the outermost "layer" of functions, and gradually move inward, using chain rule multiple times. So, in this case we'll first let OUTER be $\cos(x)$. Then INNER is **everything inside** of OUTER, so we let INNER be $\sin(\tan(x))$. Here's what that looks like:

$$\text{OUTER} = \cos(x) \text{ and } \text{INNER} = \sin(\tan(x)).$$

$$\text{OUTER}' = -\sin(x) \text{ and } \text{INNER}' = ???$$

$$\text{INNER}'(x) = \frac{d}{dx}[\sin(\tan(x))]$$

$$\text{OUTER}_2 = \sin(x) \text{ and } \text{INNER}_2 = \tan(x)$$

$$\text{OUTER}'_2 = \cos(x) \text{ and } \text{INNER}'_2 = \sec^2(x)$$

$$\text{INNER}' = \frac{d}{dx}[\sin(\tan(x))]$$

$$= \frac{d}{dx}[\text{OUTER}_2(\text{INNER}_2)]$$

$$= \text{OUTER}'_2(\text{INNER}_2) \cdot \text{INNER}'_2$$

$$= \cos(\tan(x)) \cdot \sec^2(x)$$

$$\text{So } \frac{d}{dx}[\cos(\sin(\tan(x)))] =$$

$$= \frac{d}{dx}[\text{OUTER}(\text{INNER})]$$

$$= \text{OUTER}'(\text{INNER}) \cdot \text{INNER}'(x)$$

$$= -\sin(\sin(\tan(x))) \cdot (\cos(\tan(x)) \cdot \sec^2(x))$$

$$= -\sin(\sin(\tan(x))) \cdot \cos(\tan(x)) \cdot \sec^2(x)$$

Identify OUTER and INNER, starting at the outmost "layer."

Find their derivatives... To find INNER' here, we need to use another chain rule. Let's do that separately.

Find the inner and outer functions. Here, I've called them OUTER_2 and INNER_2 .

Find their derivatives.

Use the chain rule formula.

Use INNER' for the first chain rule.

Use the chain rule formula again.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find the derivative of $\sin(x^4)$.

Answer 1 $\cos(x^4) \cdot 4x^3 = 4x^3 \cos(x^4)$.

Question 2 Find the derivative of $3x^2e^{7x}$.

Answer 2 $6x^2e^{7x} + 3x^2e^{7x}(21) = 6x^2e^{7x}(21 + 3)$.

Question 3 Find the derivative of $\sin(e^{3x})$.

Answer 3 $\cos(e^{3x}) \cdot e^{3x} \cdot 3e^{3x} = 3e^{3x} \cos(e^{3x})$.

Implicit Differentiation

Abstract

Implicit differentiation is a technique used for *implicit equations*, equations which have not isolated the output variable (such as y). It primarily involves the same differentiation rules as before, but with a new use of the chain rule.

1 Introduction.

Consider the two following equations:

$$(1) \quad y = 3x^2 + 2 \qquad (2) \quad \cos(y) = \sin(x)$$

While you've likely learned how to differentiate (1), differentiating (2) may seem more intimidating. Your first instinct might be to solve for y , turning (2) into something that you can solve like (1), but cases like these often run into functions that are hard to differentiate (such as inverse trig functions) or are out-right impossible. Instead, there is a technique to find the derivative $\frac{dy}{dx}$ for these kind of equations, called *implicit equations*.

Implicit equations are equations which have not been solved for the output variable (such as y). Examples of these equations include:

$$x^2 + y^2 = 1 \qquad x^3 \cdot \sin(y) = 7 \qquad x - \ln(xy) = e^{2x-3y}$$

As mentioned before, solving for y in these examples is either very difficult or impossible. So, there is a different method of finding $\frac{dy}{dx}$ which does not require isolating y and still uses the same differentiation rules you've previously learned. This method is called *implicit differentiation*.

One quick note: as mentioned before, implicit equations do not express y has some explicit function of x . Instead, they express a *relationship* between two variables, x and y , and it's not given that one is a function of the other. Because of this, some equations such as $z^2 + 5 \cos(t) = 7$ will not obviously have an independent or dependent variable, like x and $y = f(x)$. It's important to notice, but it doesn't affect the process of finding derivatives. Let's see how to do that.

2 Steps and Examples.

This is the general process:

1. Take the derivative of both sides with respect to the desired variable (the bottom of the derivative fraction, such as x in $\frac{dy}{dx}$).
2. Apply derivative rules to both sides, treating functions of y the same as functions of x . Use chain rule for functions of y , multiplying by $\frac{dy}{dx}$.
3. Solve for $\frac{dy}{dx}$.

Example 1

Find $\frac{dy}{dx}$ given $x^2 + y^2 = 1$.

Solution:

$x^2 + y^2 = 1$ $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$ $\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$ $2x + 2y \cdot \frac{dy}{dx} = 0$ $\frac{dy}{dx}(2y) = -2x$ $\frac{dy}{dx} = -\frac{x}{y}$	<p>Begin with the equation you are given.</p> <p>Take the derivative of both sides with respect to the variable you want, in this case x.</p> <p>Use any derivative rules you can to simplify.</p> <p>Evaluate the derivatives. For all functions of y, remember that y is a function of x. So, use chain rule and multiply by $\frac{dy}{dx}$.</p> <p>Isolate all terms with a $\frac{dy}{dx}$ and factor.</p> <p>Finally, divide and simplify.</p>
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Note that our final answer is in terms of both x and y . Though this may seem odd or new, don't worry! It's perfectly normal when working with implicit equations.

The main thing to remember with implicit differentiation is how to differentiate functions of y , such as y^2 in the above example. The reason $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$ is because y is a function of x . In this sense, y^2 seems to have an *inside* (y) and an *outside* (something squared), just like examples utilizing the chain rule. So, following chain rule, you take the derivative of the outside and leave the inside the same (resulting in $2y$), and multiply by the derivative of the inside ($\frac{dy}{dx}$). Let's follow this again in another example.

Example 2

Find $\frac{dy}{dx}$ given $y^2 \sin(y) = x^5$.

Solution:

$$y^2 \sin(y) = x^5$$

Begin with the equation you are given.

$$\frac{d}{dx}(y^2 \sin(y)) = \frac{d}{dx}(x^5)$$

Take the derivative of both sides with respect to the variable you want, in this case x .

$$2y \left(\frac{dy}{dx} \right) \sin(y) + y^2 \cos(y) \left(\frac{dy}{dx} \right) = 5x^4$$

Evaluate the derivatives, remembering to use chain rule for functions of y . Note that in this case we have to use product rule on the left side.

$$\frac{dy}{dx}(2y \sin(y) + y^2 \cos(y)) = 5x^4$$

Isolate all terms with a $\frac{dy}{dx}$ and factor.

$$\frac{dy}{dx} = \frac{5x^4}{2y \sin(y) + y^2 \cos(y)}$$

Finally, divide and simplify.

This example shows how you may need to apply regular derivative rules (power rule, product rule, etc.) to functions involving y . This is a very common way to go wrong, and as such is a very good thing to remember. Here's a final example, where we show the variables combined in various ways with a small rename.

Example 3

Find $\frac{dz}{dt}$ given $e^{2t-3z} = t^2 - \ln(tz^3)$ with z and $t > 0$.

Solution:

$e^{2t+3z} = t^2 - \ln(tz^3)$ $\frac{d}{dt}(e^{2t+3z}) = \frac{d}{dt}(t^2 - \ln(tz^3))$ $e^{2t+3z}(2 + 3\frac{dz}{dt}) = \frac{1}{tz^3}(z^3 + 3tz^2\frac{dz}{dt})$ $\frac{dz}{dt}(3e^{2t+3z} + \frac{3}{z}) = 2t - \frac{1}{t} - 2e^{2t+3z}$ $\frac{dz}{dt} = \frac{2t - t^{-1} - 2e^{2t+3z}}{3e^{2t+3z} + 3z^{-1}}$	<p>Begin with the equation you are given.</p> <p>Take the derivative of both sides with respect to the variable you want, in this case t.</p> <p>Evaluate the derivatives, remembering to use chain rule for functions of z.</p> <p>Isolate all terms with a $\frac{dz}{dt}$ and factor. Note that because z and $t > 0$, we can cancel $\frac{z}{z}$ and $\frac{t}{t}$ without dividing by 0.</p> <p>Finally, divide and simplify.</p>
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3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some examples with solutions, if you'd like them.

Question 1 Find $\frac{dy}{dx}$ given $3x^2 + 4y^2 = 36$.

Answer 1
$$\frac{dy}{dx} = \frac{xp}{yp}$$

Question 2 Find $\frac{dy}{dx}$ given $y^2 - \ln(x) = y\cos(y)$.

Answer 2
$$\frac{(y)\sin(yx) + (y)\cos(y) - x\cos(y)}{1} = \frac{xp}{yp}$$

Question 3 Find $\frac{dx}{dt}$ given $\cos(t)\sin(x) = \frac{t}{x}$.

Answer 3
$$\frac{t + (\cos(x)\cos(t))}{x + x^2\sin(x)\sin(t)} = \frac{dt}{dp}$$

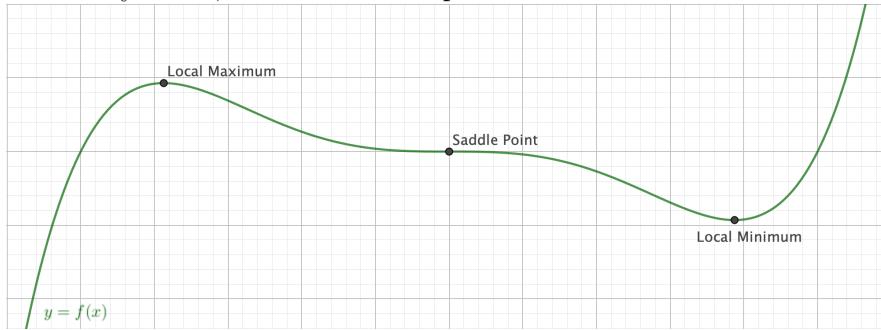
Critical Numbers

Abstract

Critical numbers are special points that a function may or may not have. In this document, we describe what critical numbers are and how to find them.

1 Introduction.

Now that we have some techniques for differentiation down, we can start to look at some applications. One of the most useful applications lies in *optimization problems*, which we'll visit soon. To be able to understand optimization problems, though, we have to take a look at special points that a function may have, called *critical points*. Take a look at this function:



You'll notice that, on the left, there's a point that's higher than all other points around it. That's called a *local maximum*. On the right, there's a point that's lower than all other points around it. That's called a *local minimum*. These points are special, as the function "levels off" there. The function also levels off in the middle, but it isn't a maximum or a minimum. Instead, it's what's called a *saddle point*. All of these points together are called the *critical points* of the function, and their x -values are called the *critical numbers*. Because these are all points where the function "levels off," they all represent points where the tangent line is horizontal or undefined. In other words, a function f has a critical number at a if either $f'(a) = 0$ or $f'(a)$ is undefined.

2 Steps and Examples.

To find the critical numbers of a function, here are the steps you take:

1. Take the derivative of the function using typical derivative rules.
2. Set the derivative equal to 0 and solve for x . All x -values found will be critical numbers.
3. Find any x -values that cause the derivative to be undefined. If the original function is defined at any of those x -values, they are also critical numbers.

Example 1

Find any and all critical numbers of $f(x) = x^2 + 2x + 1$.

Solution:

$$f(x) = x^2 + 2x + 1$$

Begin with the function you're given.

$$f'(x) = 2x + 2$$

Find the derivative of the function.
Here, we just have to use power rule.

$$f'(x) = 2x + 2 = 0$$

Set the derivative equal to 0.

$$\begin{aligned} 2x &= -2 \\ x &= -1 \end{aligned}$$

Solve for x .

$f'(x)$ is never undefined.

Find any x -values that make the derivative undefined. We have none here.

$x = -1$ is the only critical number.

Re-state all x -values found previously.

As you can see from the previous example, you won't always have a function that is undefined somewhere. In particular, polynomials, sin and cos, and exponential functions will never be undefined. Let's look at an example where the derivative is somewhere undefined.

Example 2

Find any and all critical numbers of $y = \frac{x^2 + 3x + 7}{x - 3}$.

Solution:

$$\begin{aligned}y &= \frac{x^2 + 3x + 7}{x - 3} \\y' &= \frac{(x - 3)(2x + 3) - (x^2 + 3x + 7)(1)}{(x - 3)^2} \\&\frac{(x - 3)(2x + 3) - (x^2 + 3x + 7)(1)}{(x - 3)^2} = 0 \\(x - 3)(2x + 3) - (x^2 + 3x + 7) &= 0 \\x^2 - 6x - 16 &= 0 \\(x - 8)(x + 2) &= 0 \\x &= 8 \text{ or } x = -2.\end{aligned}$$

y' is undefined if the denominator is zero.

This means $(x - 3)^2 = 0$

So $(x - 3) = 0$ and thus $x = 3$.

y is also undefined at $x = 3$,
so it isn't a critical number.

Our critical numbers are $x = 8$ and $x = -2$.

Begin with the function you're given.

Find the derivative of the function.

Here, we have to use quotient rule.

Set the derivative equal to 0.

Solve for x . Here, we can multiply
by the denominator and factor.

Find any x -values that make
the derivative undefined. Here, we
have one because of division by 0.

Check if the original function is undefined
at that value. It is, so $x = 3$ doesn't count.

Re-state your critical numbers.

This example shows how, if you find a point at which the derivative is undefined, **it is only a critical point if the function itself is still defined there**. Let's look at one final example.

Example 3

Find any and all critical numbers of $h(t) = 5te^{2-t^2}$.

Solution:

$$h(t) = 5te^{2-t^2}$$

$$\begin{aligned} h'(t) &= (5)(e^{2-t^2}) + (5t)(e^{2-t^2})(-2t) \\ &= (5e^{2-t^2})(1 - 2t^2) \end{aligned}$$

$$(5e^{2-t^2})(1 - 2t^2) = 0$$

Since $5e^{2-t^2} \neq 0$,

$$1 - 2t^2 = 0$$

$$\text{So } t^2 = \frac{1}{2}$$

$$t = \frac{1}{\sqrt{2}} \text{ or } t = -\frac{1}{\sqrt{2}}.$$

$h'(t)$ is never undefined, so our critical numbers are $t = \frac{1}{\sqrt{2}}$ and $t = -\frac{1}{\sqrt{2}}$.

Begin with the function you're given.

Find the derivative of the function. Here, we have to use product rule and chain rule.

Set the derivative equal to 0.

Solve for t . Here, we use the fact that exponentials are never 0.

Exponentials and polynomials are never undefined.

Re-state your critical numbers.

Once again, the most difficult part of this problem is knowing when the derivative function is undefined. This would be a great point to review when functions are undefined, but, in general, the only cases you'll come across will be (1) rational functions where you have to worry about dividing by 0, or (2) logarithmic functions, which are undefined for $x \leq 0$.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find any and all critical numbers of $f(x) = 6x^5 + 33x^4 - 30x^3 + 100$. Hint: you will want to factor the derivative as much as possible, including a greatest common factor.

Answer 1 $x = 0$, $x = -\frac{5}{3}$, $x = \frac{2}{3}$

Question 2 Find any and all critical numbers of $y = \frac{x^2 + 1}{x^2 - x - 6}$.

Answer 2 $x = -1$ and $x = 6$

Question 3 Find any and all critical numbers of $r(t) = te^{t^2}$.

Answer 3 This function has no critical numbers.

The Closed Interval Method for Maxima and Minima

Abstract

Now that we have looked at critical numbers, which give tell us where there may be local maxima or minima, we can use this idea to find *absolute maxima* and *absolute minima* (the highest and lowest points of a function) on a closed interval. This is called the *closed interval method*.

1 Introduction.

Suppose we have a function $f(x)$ and a closed interval $[a, b]$ such that $a \leq x \leq b$. In other words, think about restricting the domain of a function $f(x)$ to a smaller “piece” that has its endpoints included. It would be very useful if we could know whether it has an absolute maximum or absolute minimum on that interval. In other words, we’d like to know if there is a value of $f(x)$ that is the biggest it can be (or the smallest it can be) on the interval $[a, b]$. Then, if it does, it would be very helpful if we knew how to find it. We’re in luck, because we have tools for both of those things. Let’s first take a look at the **Extreme Value Theorem**, which guarantees that such points exist:

The Extreme Value Theorem. If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ for some numbers c and d in $[a, b]$.

Great, so we know that such points always exist for functions continuous on the interval, but how do we find them?

2 Steps and Examples.

This is the general process:

1. Write down your function and the endpoints of your interval.
2. Find the critical numbers of $f(x)$.
3. For any critical numbers on the interval $[a, b]$, evaluate the function at that number.
4. Evaluate the function at the two endpoints, a and b .
5. The largest value found between steps 3 and 4 is the absolute maximum, and the smallest value found between steps 3 and 4 is the absolute minimum.

Example 1

Find the absolute maximum and absolute minimum of $f(x) = x^2 + 7x + 10$ on the interval $[-4, 2]$.

Solution:

$$f(x) = x^2 + 7x + 10, \quad a = -4 \text{ and } b = 2$$

Write down the function and endpoints.

$$f'(x) = 2x + 7 = 0$$

Find the critical numbers of $f(x)$.

$$x = -\frac{7}{2} \text{ is the only critical number}$$

$$\begin{aligned} f\left(-\frac{7}{2}\right) &= \left(-\frac{7}{2}\right)^2 + 7\left(-\frac{7}{2}\right) + 10 \\ &= -\frac{9}{4} \end{aligned}$$

Evaluate f at any critical numbers in the interval.

$$\begin{aligned} f(a) &= f(-4) = -2 \\ f(b) &= f(2) = 28 \end{aligned}$$

Evaluate f at the endpoints.

$$\begin{aligned} \text{Absolute max at } (2, 28) \\ \text{Absolute min at } \left(-\frac{7}{4}, -\frac{9}{4}\right) \end{aligned}$$

Absolute max is the largest value of f found.

Absolute min is the smallest value of f found.

One comment on how you present your answer. In the previous section, we found that the absolute maximum value that $f(x)$ attains is $f(x) = 28$. So, we could simply say “Absolute maximum value of 28.” However, this doesn’t say what x -value gives us that maximum value, which is something that we really need to know. So, in your answer, it’s helpful to give your absolute maximum and absolute minimum as points, including both the x -values and y -values. Let’s look at another example.

Example 2

Find the absolute maximum and minimum of $y = (2x^2 - 2)^3$ on the interval $[-2, 1]$.

Solution:

$$y = (2x^2 - 2)^3, a = -2, \text{ and } b = 1.$$

Write down the function and endpoints.

$$\begin{aligned} y' &= 3(2x^2 - 2)^2(4x) = 0 \\ 4x &= 0 \text{ or } 3(2x^2 - 2)^2 = 0 \\ x &= 0 \text{ or } (2x^2 - 2)^2 = 0 \\ &\implies 2x^2 - 2 = 0 \\ &\implies x^2 = 1 \\ &\implies x = \pm 1 \end{aligned}$$

Find the critical numbers. To do so here, we have to use chain rule.

Critical numbers are $x = 0, 1, -1$

$$y(0) = -8$$

Evaluate y at the critical numbers in the interval.

$$y(1) = 0$$

$$y(-1) = 0$$

$$y(a) = y(-2) = 216$$

Evaluate y at the endpoints.

$$y(b) = y(1) = 0$$

Absolute max at $(-2, 216)$

Absolute max is the largest y -value found.

Absolute min at $(0, -8)$

Absolute min is the smallest y -value found.

Again, to find the absolute maximum and minimum values, we just look at the y -values of our critical numbers and endpoints, and pick the largest (or smallest) between them. In this example, you can see what our maximum and minimum values are by looking at the right hand side of all of the equations in the “Evaluate y at...” steps. Then, for our final answer, we simply put them into point form.

Now examples 1 and 2 have shown the process, but they both have something in common that will not be true in general: they only have one absolute maximum and one absolute minimum. This won’t be the case in our next example. Let’s take a look.

Example 3

Find all absolute maxima and minima of $h(t) = t^4 - 6t^3 + 9t^2 + 1$ on the interval $[0, 4]$.

Solution:

$$h(t) = t^4 - 6t^3 + 9t^2 + 1, \text{ } a = 0, \text{ and } b = 4.$$

Write down the function and endpoints.

$$\begin{aligned} h'(t) &= 4t^3 - 18t^2 + 18t = 0 \\ t(4t^2 - 18t + 18) &= 0 \\ t = 0 \text{ or } 4t^2 - 18t + 18 &= 0 \\ \implies t &= \frac{-18 \pm \sqrt{(-18)^2 - 4(4)(18)}}{2(4)} \\ \implies t &= 3 \text{ or } t = \frac{3}{2} \\ \text{Critical numbers are } t &= 0, 3, \frac{3}{2} \end{aligned}$$

Find the critical numbers. To do so here, we have to use the quadratic formula.

$$h(0) = 1$$

Evaluate h at the critical numbers.

$$h(3) = 1$$

$$h\left(\frac{3}{2}\right) = \frac{97}{16} = 6.0625$$

Evaluate h at the endpoints.

$$h(a) = h(0) = 1$$

$$h(b) = h(4) = 17$$

Absolute max at $(4, 17)$
Absolute min at $(0, 1)$ and $(3, 1)$.

Absolute max is the largest value of h found.
Absolute min is the smallest value of h found.

As you can see, in the previous example, we had an absolute minimum value of $h(t) = 1$, but it occurred at two different points: $t = 0$ and $t = 3$. This means we have two absolute minima on the same interval. This is common, and, if you encounter something like this, simply list both points in your final answer.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 As a decimal, find the absolute maximum and minimum of $f(x) = -5x^3 + 3x^2 + 2x - 3$ on the interval $[-2, 1]$.

Answer 1 Absolute max at $(-2, 45)$, absolute min at $(-0.2163, -2.4776)$.

Question 2 Find the absolute maximum and minimum of $y = te^{t^2}$ on the interval $[-1, 1]$.

Answer 2 Absolute max at $(1, e)$, absolute min at $(-1, -e)$.

Question 3 Find all absolute maxima and minima of $r(u) = \cos(u)$ on the interval $[0, 2\pi]$.

Answer 3 Absolute max at $(0, 1)$ and $(2\pi, 1)$, absolute min at $(\pi, -1)$.

Using the Limit Definition of the Definite Integral

Abstract

In this document, we discuss the main character of Calculus II: the *integral*. In particular, we review the exact, mathematical definition of the definite integral and its interpretation, and then look at examples of how to apply the definition to particular curves.

1 Introduction.

Let's recall what the definition of the definite integral is from first principles:

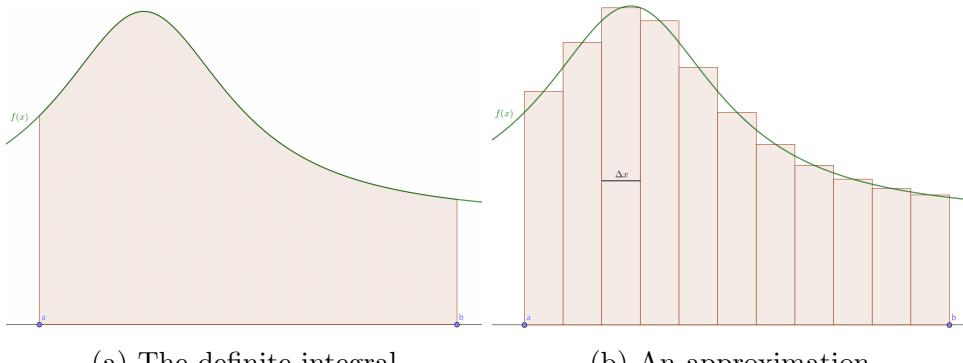


Figure 1: Building the definite integral

Say we have a function f and we want to find the area under f from $x = a$ to $x = b$. This is the definite integral $\int_a^b f(x) dx$, shown in Figure 1a above. The way we find this area is to start with an approximation, as seen in Figure 1b. We begin by splitting the interval $[a, b]$ into n sub-intervals with width $\Delta x = \frac{b - a}{n}$ and find a rectangle in that sub-interval with

height $f(x_i^*)$, where x_i^* is any point in the sub-interval of the corresponding rectangle. Then, we add the areas of all of the rectangles together. This can be seen in the picture above. Mathematically, the area of one of the rectangles is [HEIGHT] \times [WIDTH] = $f(x_i^*)\Delta x$. Then, we add the areas of all of the rectangles together, which, in the language of math, looks like $f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^n f(x_i^*)\Delta x$. This approximation is called a *Riemann sum*.

Now that we have an approximation, how do we make it better, or even exactly the area that we're trying to find? Well, we make the rectangles smaller. This means we want more rectangles and so, since n is the number of rectangles, we want to take n larger and larger, which is exactly what limits are for. This leads to the limit definition of the definite integral, which represents signed area under a curve:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

Now, there's a lot of pieces in this definition. But if you ever need help understanding it, it's always helpful to look back at the graphical idea of approximating an area under a curve with rectangles, adding the rectangles together, and making the rectangles smaller and smaller. Now that we have the definition under our belt, let's look at how to use it to evaluate some integrals.

Before we begin with the integrals, though, we have to talk briefly about sums. First off, we're gonna use some important equations:

$$\sum_{i=1}^n 1 = n \quad \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$$

Second, everything that doesn't have to do with the variable on the bottom of the sum notation (in this case, i) can be moved outside of the sum. This means that something like $\sum_{i=1}^n ni^3 = n \sum_{i=1}^n i^3$. Then, third, you can separate sums at addition and subtraction signs, so $\sum A \pm B = \sum A \pm \sum B$.

2 Steps and Examples.

This is the general process:

1. Find the function you're integrating, as well as a and b .
2. Find $\Delta x = \frac{b-a}{n}$ (n will just be a variable, whereas a and b will be numbers).
3. Find x_i^* , which for our purposes will be given by $x_i^* = a + i\Delta x$ (i will just be a variable).
4. Find $f(x_i^*)$ by plugging x_i^* into f .
5. Plug $f(x_i^*)$ and Δx into the definition formula. That is, multiply the two together and put the limit and sum in front.
6. Simplify as much as possible, using the sum properties to put the sums just in terms of 1 , i , i^2 , and i^3 .
7. Use the sum formulas above to evaluate the sums, and then evaluate the limit.

Example 1

Find $\int_1^5 3x - 2 dx$.

Solution:

$$a = 1, b = 5, \text{ and } f(x) = 3x - 2$$

Find a , b , and the function.

$$\Delta x = \frac{b-a}{n} = \frac{4}{n}$$

Find Δx .

$$x_i^* = a + i\Delta x = 1 + i\frac{4}{n}$$

Find x_i^* .

$$f(x_i^*) = 3 \left(1 + i\frac{4}{n} \right) - 2$$

Find $f(x_i^*)$.

$$\int_1^5 3x - 2 dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Use the definition formula.

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[3 \left(1 + i\frac{4}{n} \right) - 2 \right] \cdot \frac{4}{n}$$

Plug in $f(x_i^*)$ and Δx .

$$= \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n 3 + i\frac{12}{n} - 2$$

Simplify. We can bring the $\frac{4}{n}$ out by sum properties.

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n 1 + i \frac{12}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[\sum_{i=1}^n 1 + \sum_{i=1}^n i \frac{12}{n} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{4}{n} \sum_{i=1}^n 1 + \frac{4}{n} \sum_{i=1}^n i \frac{12}{n} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{4}{n} \sum_{i=1}^n 1 + \frac{48}{n^2} \sum_{i=1}^n i \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{4}{n}(n) + \frac{48}{n^2} \left(\frac{n(n+1)}{2} \right) \right] \\
 &= \lim_{n \rightarrow \infty} \left[4 + 24 \frac{n+1}{n} \right] \\
 &= 4 + 24 = 28
 \end{aligned}$$

Evaluate the limits.

Phew, that was a lot of work, and it can be hard to follow. Our recommendation is, be careful, and take it slowly, step by step. Little mistakes here, such as sign errors or forgetting parentheses, will cascade into issues which make the problem seem much harder than it actually is.

That said, most of the work we actually did was simplifying, using algebra and sum properties to get to a point where we can use our formula. Then, we just evaluated the limit as per usual. Let's look at this process with another, more difficult example.

Example 2

Find $\int_3^4 3x^2 dx$.

Solution:

$$a = 3, b = 4, \text{ and } f(x) = 3x^2 \quad \text{Find } a, b, \text{ and } f.$$

$$\Delta x = \frac{b-a}{n} = \frac{1}{n} \quad \text{Find } \Delta x.$$

$$x_i^* = a + i\Delta x = 3 + i \frac{1}{n} \quad \text{Find } x_i^*.$$

$$f(x_i^*) = 3 \left(3 + i \frac{1}{n} \right)^2 \quad \text{Find } f(x_i^*).$$

$$\begin{aligned}
\int_3^4 3x^2 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x && \text{Use the definition.} \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n 3 \left(3 + i \frac{1}{n} \right)^2 \cdot \frac{1}{n} && \text{Plug in } f(x_i^*) \text{ and } \Delta x. \\
&= \lim_{n \rightarrow \infty} 3 \cdot \frac{1}{n} \sum_{i=1}^n \left(3 + i \frac{1}{n} \right)^2 && \text{Simplify.} \\
&= \lim_{n \rightarrow \infty} 3 \cdot \frac{1}{n} \sum_{i=1}^n 9 + i \frac{6}{n} + i^2 \frac{1}{n^2} \\
&= \lim_{n \rightarrow \infty} \frac{3}{n} \left[\sum_{i=1}^n 9 + \sum_{i=1}^n i \frac{6}{n} + \sum_{i=1}^n i^2 \frac{1}{n^2} \right] \\
&= \lim_{n \rightarrow \infty} \left[\frac{27}{n} \sum_{i=1}^n 1 + \frac{18}{n^2} \sum_{i=1}^n i + \frac{3}{n^3} \sum_{i=1}^n i^2 \right] && \text{Evaluate the sums.} \\
&= \lim_{n \rightarrow \infty} \left[\frac{27}{n}(n) + \frac{18}{n^2} \left(\frac{n(n+1)}{2} \right) + \frac{3}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) \right] \\
&= \lim_{n \rightarrow \infty} \left[27 + 9 + \frac{1}{n} + 1 + \frac{3}{2n} + \frac{1}{2n^2} \right] \\
&= 27 + 9 + 1 = 37 && \text{Evaluate the limit.}
\end{aligned}$$

Like before, the main work was simplifying the definition, remembering sum properties and formulas. Here, though, we also had to square x_i^* , which meant FOIL-ing out a binomial and getting a term with i^2 . You'll notice that the same process applies, though: use sum rules to make them look like our formulas from the introduction, and then evaluate the limit as normal. Let's look at a final example, where we need to use the i^3 formula.

Example 3

Find $\int_{-1}^1 x^3 + 2x + 3 dx$.

Solution:

$$\begin{aligned}
a &= -1, b = 1, && \text{Find } a, b, \text{ and } f. \\
\text{and } f(x) &= x^3 + 2x + 3 \\
\Delta x &= \frac{b-a}{n} = \frac{2}{n} && \text{Find } \Delta x.
\end{aligned}$$

$$x_i^* = a + i\Delta x = -1 + i\frac{2}{n}$$

Find x_i^* .

$$f(x_i^*) = \left(-1 + i\frac{2}{n}\right)^3 + 2\left(-1 + i\frac{2}{n}\right) + 3$$

Find $f(x_i^*)$.

$$\int_{-1}^1 x^3 + 2x + 3 dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Use the definition.

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(-1 + i\frac{2}{n}\right)^3 + 2\left(-1 + i\frac{2}{n}\right) + 3 \right] \cdot \frac{2}{n}$$

Plug in $f(x_i^*)$ and Δx .

$$= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left(i^3 \frac{8}{n^3} - i^2 \frac{12}{n^2} + i \frac{6}{n} - 1 \right) + \left(-2 + i \frac{4}{n} \right) + 3$$

Simplify.

$$= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n i^3 \frac{8}{n^3} - i^2 \frac{12}{n^2} + i \frac{10}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{2}{n} \left[\sum_{i=1}^n i^3 \frac{8}{n^3} - \sum_{i=1}^n i^2 \frac{12}{n^2} + \sum_{i=1}^n i \frac{10}{n} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{16}{n^4} \sum_{i=1}^n i^3 - \frac{24}{n^3} \sum_{i=1}^n i^2 + \frac{20}{n^2} \sum_{i=1}^n i \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{16}{n^4} \left(\frac{n(n+1)}{2} \right)^2 - \frac{24}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{20}{n^2} \left(\frac{n(n+1)}{2} \right) \right]$$

Evaluate the sums.

$$= \lim_{n \rightarrow \infty} \left[\left(4 + \frac{8}{n} + \frac{4}{n^2} \right) - \left(8 + \frac{12}{n} + \frac{4}{n^2} \right) + \left(10 + \frac{10}{n} \right) \right]$$

Simplify.

$$= 4 - 8 + 10 = 6$$

Evaluate the limit.

As you can see, this can end up being a lot of work. We've even hidden the step where we expanded the results of the sum formulas, but you should definitely practice expanding them carefully.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find $\int_3^6 2x + 7 \, dx$.

Answer 1 48.

Question 2 Find $\int_1^2 3x^2 + 3 \, dx$.

Answer 2 10.

Question 3 Find $\int_0^2 2x^3 + 3x^2 + 1 \, dx$.

Answer 3 18.

Integration by U-Substitution

Abstract

The first main technique of integration is known as *the substitution method*, or, more commonly, *u-substitution*. In this document, we discuss when to use a u-substitution, and how to do so.

1 Introduction.

In one of the previous documents, we looked at the Fundamental Theorem of Calculus. Part 2 of the Fundamental Theorem of Calculus gives us a simple yet powerful rule for evaluating definite integrals, as it reduces all integral problems into problems of finding an antiderivative or indefinite integral. Some antiderivatives are easy, as we can just use our knowledge from Calculus I to “reverse-engineer” the answer. But some integrals are harder, and we can’t immediately identify what the antiderivative is. Here are a couple examples:

$$(1) \int \cos(3x + 4) dx \quad (2) \int e^{\sin(x)} \cdot \cos(x) dx$$

In (1), we could solve the integral, if only the inside of the cosine function were a single variable. In (2), you may notice that the integrand (inside of the integral) has both a function ($\sin(x)$) and its derivative ($\cos(x)$) multiplied together somehow. The method for solving both of these problems is u-substitution. In sum: **a u-substitution is used in two cases:**

1. To reduce a linear term, or
2. When there is a function and a constant multiple of its derivative in the same integral.

Let’s cover both of these cases separately.

2 Steps and Examples.

Case I: Reducing a Linear Term

This is the general process:

1. Let u be the entire linear term.
2. Take the derivative of u with respect to x , that is, find $\frac{du}{dx}$.
3. Treat $\frac{du}{dx}$ as a fraction and isolate dx .
4. Replace your linear term and dx in the integral with what you found. Change your bounds if the integral is definite.
5. Evaluate the integral.
6. Re-substitute any remaining u 's and add a $+C$ if the integral is indefinite. If the integral is definite, use FTC.

Example 1

Find $\int \cos(3x + 4) dx$.

Solution:

<p>Let $u = 3x + 4$</p> $\frac{du}{dx} = 3$ $du = 3 \cdot dx$ $dx = \frac{1}{3} \cdot du$ $\begin{aligned} \int \cos(3x + 4) dx &= \int \cos(u) \cdot \frac{1}{3} du \\ &= \frac{1}{3} \int \cos(u) du \\ &= \frac{1}{3} \sin(u) \\ &= \frac{1}{3} \sin(3x + 4) + C \end{aligned}$	<p>Let u be the entire linear term.</p> <p>Find $\frac{du}{dx}$.</p> <p>Treat $\frac{du}{dx}$ as a fraction and isolate dx.</p> <p>Substitute in u and dx.</p> <p>Simplify.</p> <p>Evaluate the integral.</p> <p>Re-substitute the remaining u. Add a $+C$.</p>
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Hopefully that isn't too hard to follow. Let's do another one.

Example 2

Find $\int_2^4 e^{100x+13} dx$.

Solution:

$$\text{Let } u = 100x + 13$$

$$\frac{du}{dx} = 100$$

$$\begin{aligned} du &= 100 \cdot dx \\ dx &= \frac{1}{100} \cdot du \end{aligned}$$

$$\int_2^4 e^{100x+13} dx = \int_{u(2)}^{u(4)} e^u \cdot \frac{1}{100} du$$

$$= \frac{1}{100} \int_{u(2)}^{u(4)} e^u du$$

$$= \frac{1}{100} e^u \Big|_{u(2)}^{u(4)}$$

$$= \frac{1}{100} e^{100x+13} \Big|_2^4$$

$$= \frac{1}{100} e^{413} - \frac{1}{100} e^{213}$$

Let u be the entire linear term.

$$\text{Find } \frac{du}{dx}.$$

Treat $\frac{du}{dx}$ as a fraction and isolate dx .

Substitute in u and dx , and change the bounds. To do so, simply replace a with $u(a)$ and b with $u(b)$.

Simplify.

Evaluate the integral.

Re-substitute the remaining u 's.

Replace your bounds with a and b again.

Use FTC.

The main thing to remember about what changes when doing a definite integral is the change in bounds. Because your integral variable changes from x to u , you have to change what the bounds are. This is a pretty simple change (you just put the bounds in parentheses and add a u in the front), but it's important to remember. Let's look at a final example for Case I.

Example 3

Use a u -substitution to find $\int_0^3 (5x + 7)^4 dx$.

Solution:

$$\text{Let } u = 5x + 7$$

$$\frac{du}{dx} = 5$$

$$du = 5dx$$

$$dx = \frac{1}{5}du$$

$$\int_0^3 (5x + 7)^4 dx = \int_{u(0)}^{u(3)} u^4 \cdot \frac{1}{5} du$$

$$= \frac{1}{5} \int_{u(0)}^{u(3)} u^4 du$$

$$= \frac{1}{5} \cdot \frac{u^5}{5} \Big|_{u(0)}^{u(3)}$$

$$= \frac{(5x + 7)^5}{25} \Big|_0^3$$

$$= \frac{(5(3) + 7)^5}{25} - \frac{(5(0) + 7)^5}{25}$$

$$= 205473$$

Let u be the entire linear term.

$$\text{Find } \frac{du}{dx}.$$

Treat $\frac{du}{dx}$ as a fraction and isolate dx .

Sub in u and dx , and change the bounds. To do so, simply replace a with $u(a)$ and b with $u(b)$.

Simplify.

Evaluate the integral.

Re-substitute the remaining u 's.

Replace your bounds with a and b again.

Use FTC.

Great. Now that we've done a lot of work on Case I, we can move on to Case II, which is a little more complicated. Any practice you've done with Case I will benefit you greatly while you're trying to learn Case II, though.

Case II: A Function and a Constant Multiple of its Derivative

Here's the general process:

1. Make sure there's a function and a constant multiple of its derivative in the same integral.
2. Identify the function (call it $g(x)$). Note that the derivative itself may not be in the problem, but a constant multiple of it should be.
3. Let u be $g(x)$.
4. Take the derivative of u with respect to x , that is, find $\frac{du}{dx}$.
5. Treat $\frac{du}{dx}$ as a fraction and isolate du . Move any coefficients to the du side.
6. Substitute in u and du in the integral. Change your bounds if the integral is definite.
7. Evaluate the integral.
8. Re-substitute any remaining u 's and add a $+C$ if the integral is indefinite. If the integral is definite, use FTC.

Example 1

Find $\int \sin(x) \cdot \cos(x) dx$.

Solution:

$$g(x) = \sin(x)$$

Identify $g(x)$.

$$\text{Let } u = \sin(x)$$

Let u be $g(x)$.

$$\frac{du}{dx} = \cos(x)$$

Find $\frac{du}{dx}$.

$$du = \cos(x) dx$$

Treat $\frac{du}{dx}$ as a fraction and isolate du .
Move any coefficients to the du side.

$$\int \sin(x) \cos(x) dx = \int u du$$

Substitute in u and du .

$$= \frac{1}{2}u^2$$

Evaluate the integral.

$$= \frac{1}{2}\sin^2(x) + C$$

Re-substitute the remaining u 's.

Add a $+C$.

The main thing to note here is that we're isolating du instead of dx here, as we did in Case I. But we also want to keep any constants on the du side. Let's look at an example of that.

Example 2

Use a u -substitution to find $\int x(x^2 + 3)^3 dx$.

Solution: In this example, you may notice that we don't exactly have a function and its derivative in the problem, since the derivative of $x^2 + 3$ is $2x$, not x . This is totally normal and OK, and a u -substitution can be used because we have a constant multiple of $2x$ in the problem. Let's look at how that works.

$$g(x) = x^2 + 3$$

$$\text{Let } u = x^2 + 3$$

$$\frac{du}{dx} = 2x$$

$$du = 2x dx$$

$$\frac{1}{2}du = x dx$$

$$\int x(x^2 + 3)^2 dx = \int (x^2 + 3)^2 \cdot x dx$$

$$= \int u^2 \cdot \frac{1}{2} du$$

$$= \frac{1}{2} \int u^2 du$$

$$= \frac{1}{2} \cdot \frac{u^3}{3}$$

$$= \frac{1}{6}(x^2 + 3)^3 + C$$

Identify $g(x)$.

Let u be $g(x)$.

Find $\frac{du}{dx}$.

Treat $\frac{du}{dx}$ as a fraction and isolate du .

Move any coefficients to the du side.

Substitute in u and du .

Evaluate the integral.

Re-substitute the remaining u 's.

Add a $+C$.

These kinds of problems are tricky. Here's the key thing to remember: u -substitutions are great if there's a function and its derivative in the same integral, but the derivative need not appear exactly. A u -substitution for $u = \sin(x)$ will work if you have $\cos(x)$ in the problem, sure, but $5\cos(x)$ or $0.5\cos(x)$ will work just as well. Same thing if you're missing a coefficient of the derivative of the function. Let's look at a final example.

Example 3

Find $\int_1^2 \frac{\sin(2 \ln t)}{t} dt$.

Solution:

$g(t) = 2 \ln t$ <p>Let $u = 2 \ln t$</p> $\frac{du}{dt} = \frac{2}{t}$ $du = \frac{2}{t} dt$ $\frac{1}{2} du = \frac{1}{t} dt$ $\int_1^2 \frac{\sin(2 \ln t)}{t} dt = \int_1^2 \sin(2 \ln t) \cdot \frac{1}{t} dt$ $= \int_{u(1)}^{u(2)} \sin(u) \cdot \frac{1}{2} du$ $= \frac{1}{2} \int_{u(1)}^{u(2)} \sin(u) du$ $= \frac{1}{2} (-\cos(u)) \Big _{u(1)}^{u(2)}$ $= -\frac{1}{2} \cos(2 \ln t) \Big _1^2$ $= -\frac{1}{2} \cos(2 \ln 2) + \frac{1}{2} \cos(2 \ln 1)$ $= -\frac{1}{2} \cos(2 \ln 2) + \frac{1}{2}$	<p>Identify $g(t)$.</p> <p>Let u be $g(t)$.</p> <p>Find $\frac{du}{dt}$.</p> <p>Treat $\frac{du}{dt}$ as a fraction and isolate du.</p> <p>Move any coefficients to the du side.</p> <p>Substitute in u and du.</p> <p>Change your bounds like in Case I.</p> <p>Evaluate the integral.</p> <p>Re-substitute the remaining u's.</p> <p>Replace your bounds with a and b again.</p> <p>Use FTC.</p>
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3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find $\int (4x + 2)^2 dx$.

Answer 1 $\frac{12}{(4x+2)^3} + C$.

Question 2 Find $\int_0^\pi \sin(2x + 2) dx$

Answer 2 0.

Question 3 Find $\int_{-1}^1 e^{3t+37} dt$

Answer 3 $\frac{e^{37}-e^{-37}}{3}$.

Question 4 Find $\int 10x \cos(5x^2) dx$.

Answer 4 $\sin(5x^2) + C$.

Question 5 Find $\int x \sin(x^2) dx$

Answer 5 $-\frac{1}{2} \cos(x^2)$.

Question 6 Find $\int_0^1 \frac{e^{\sqrt{t}}}{\sqrt{t}} dt$

Answer 6 $2e - 2$.

Integration by Parts (IBP)

Abstract

In this document, we discuss another technique of integration, called *integration by parts* or *IBP* for short. Integration by parts gives us a formula which corresponds to the product rule, similar to how *u*-substitutions correspond to the chain rule.

1 Introduction.

Previously in the documents we looked at *u*-substitutions, which were very helpful when we had an integrand that looked like the chain rule, that is, when a function and its derivative were in the integrand. But what if that isn't the case? What if we had an integral that looked something like $\int xe^x dx$? Well, these kinds of problems are exactly the kinds of problems that *integration by parts* or *IBP* is meant to handle. Here is the formula it gives us, which is derived from the product rule formula from Calc I in the Stewart textbook:

$$\boxed{\int u \, dv = u \cdot v - \int v \, du}$$

This is the *integration by parts formula*, or *parts formula*. It's very important, and so you'll definitely want to memorize it. It's read as "the integral of $u \, dv$ is equal to u times v minus the integral of $v \, du$." Let's take a look at how to use it.

2 Steps and Examples.

This is the general process:

1. Select your u and dv .
2. Find du and v .
3. Plug your information from steps 1 and 2 into the integration by parts formula.
4. Repeat steps 1-3 if necessary.
5. If your integral is indefinite, add a $+C$. Otherwise, use FTC.

Some more information on step 1. When using the integration by parts formula, you'll likely have something that looks like $\int f(x) \cdot g(x) dx$, that is, two functions multiplied together. What step 1 says to do is to pick one of those functions to be u , and one of the functions (plus the dx) to be dv . That is, you can make $u = f(x)$ and $dv = g(x) dx$.

The choice for u and dv is not arbitrary, though. In general, u will be differentiated, and dv will be integrated. One general tip (that won't work all of the time), is to use the acronym "I LATE" to aid in your choice of u and dv . "I LATE" Stands for:

- Inverse trigonometric functions (\arctan , \sin^{-1} , etc.)
- Logarithmic functions (\log_{10} , \ln , etc.)
- Algebraic functions (x^5 , $3x^{17}$, etc.)
- Trigonometric functions (\sin , \cos , etc.)
- Exponential functions (e^x , e^{5x} , 2^x , etc.)

To use the acronym, the general tip is to pick u to be the function that appears **first** in the acronym. So, if your integrand has a logarithmic function and a trig function, the "I LATE" method says to make the logarithmic function your u .

One last quick note: you may hear variations on this method, but we find this to be the easiest to remember. Ok, let's get into the examples.

Example 1

Find $\int xe^x dx$.

Solution:

Let $u = x$ and $dv = e^x dx$

Select u and dv . Here, the “I LATE” method tells us to make this choice, which will work.

$$\frac{du}{dx} = 1, \text{ so } du = 1 \cdot dx$$

Find du and v . To find du , treat $\frac{du}{dx}$ as a fraction. To find v , integrate dv .

$$v = \int dv = \int e^x dx = e^x$$

Plug u , v , du , and dv into the parts formula.

$$\int xe^x dx = \int u dv.$$

$$= u \cdot v - \int v du$$

Integrate and simplify.

$$= xe^x - \int e^x \cdot 1 dx$$

Add a $+C$.

$$= xe^x - e^x$$

$$= e^x(x - 1) + C$$

Great. This problem wasn’t too complex: the “I LATE” method worked great, and the integral at the end, in the parts formula, was totally doable. This won’t always be the case, though. In particular, you’ll often get integrals at the end that need a u -sub or even IBP again. Let’s look at one such example.

Example 2

Find $\int_0^\pi x^2 \cos(x) dx$.

Solution: One note on definite integrals using parts: the best way to approach it is to simply treat it as an indefinite integral until the very end. Let's see how that goes.

Let $u = x^2$ and $dv = \cos(x) dx$

$$\frac{du}{dx} = 2x, \text{ so } du = 2x dx$$

$$v = \int dv = \int \cos(x) dx = \sin(x)$$

$$\int_0^\pi x^2 \cos(x) dx = \int u dv \Big|_0^\pi$$

$$= u \cdot v - \int v du \Big|_0^\pi$$

$$= x^2 \sin(x) - \int \sin(x) \cdot 2x dx \Big|_0^\pi$$

$$= x^2 \sin(x) - 2 \int x \sin(x) dx \Big|_0^\pi$$

Let $u = x$ and $dv = \sin(x) dx$

$$\frac{du}{dx} = 1, \text{ so } du = dx$$

$$v = \int dv = \int \sin(x) dx = -\cos(x)$$

$$\text{So } x^2 \sin(x) - 2 \int x \sin(x) dx \Big|_0^\pi =$$

$$x^2 \sin(x) - 2[x(-\cos(x)) - \int -\cos(x) dx] \Big|_0^\pi$$

$$= x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) \Big|_0^\pi$$

$$= -2\pi$$

Select u and dv . Here, the "I LATE" method tells us to make this choice, which will work.

Find du and v .

Use the parts formula. Because we have bounds, we put them at the end with an evaluation bar.

Simplify. Here we have to use parts again.

Select u and dv . It's important to make the same choice of u and dv as before.

Find du and v .

Use the parts formula.

Be careful with parentheses!

Simplify.

Use FTC.

Example 3

Find $\int e^x \cdot \sin(x) dx$.

Solution:

Let $u = \sin(x)$ and $dv = e^x dx$

$$\frac{du}{dx} = \cos(x), \text{ so } du = \cos(x) dx$$

$$v = \int dv = \int e^x dx = e^x$$

$$\int e^x \sin(x) dx = \int u dv$$

$$= u \cdot v - \int v du$$

$$= e^x \sin(x) - \int e^x \cos(x) dx$$

Let $u = \cos(x)$ and $dv = e^x dx$

$$\frac{du}{dx} = -\sin(x), \text{ so } du = -\sin(x) dx$$

$$v = \int dv = \int e^x dx = e^x$$

$$\text{So } e^x \sin(x) - \int e^x \cos(x) dx =$$

$$e^x \sin(x) - [e^x \cos(x) - \int e^x (-\sin(x)) dx]$$

$$= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x) dx$$

Select u and dv . Here, the “I LATE” method tells us to make this choice, which will work.

Find du and v .

Use the parts formula.

Here we have to use parts again.

Select u and dv . It’s important to make the same choice of u and dv as before.

Find du and v .

Use the parts formula.

Be careful with parentheses!

Simplify.

Now before we finish off this problem, you may notice an issue and be unsure how to continue. The integral we’re trying to find is $\int e^x \sin(x) dx$, but that very integral has appeared again when using the parts formula. To handle problems like these, we need a trick, and the trick is to add that same integral to both sides. Let’s see what that looks like.

$$\begin{aligned}
 \int e^x \sin(x) dx &= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x) dx && \text{This is where we left off.} \\
 \int e^x \sin(x) dx + \int e^x \sin(x) dx &= e^x \sin(x) - e^x \cos(x) && \text{Add the integral to both sides.} \\
 2 \int e^x \sin(x) dx &= e^x \sin(x) - e^x \cos(x) && \text{Combine the integrals.} \\
 \int e^x \sin(x) dx &= \frac{e^x \sin(x) - e^x \cos(x)}{2} && \text{Divide by 2.} \\
 &= \frac{1}{2} e^x \sin(x) - \frac{1}{2} e^x \cos(x) && \text{Simplify.} \\
 &= \frac{1}{2} e^x (\sin(x) - \cos(x)) + C && \text{Add a } +C.
 \end{aligned}$$

That finishes off that problem. It's a tricky one, and you need to know the trick of adding the integral to both sides and dividing by 2. This is a good and important thing to remember for almost every Calculus II course, as it often comes up on quizzes or homework.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find $\int x \ln(x) dx$.

Answer 1 $\frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C$.

Question 2 Find $\int_0^{2\pi} x^2 \sin(x) dx$.

Answer 2 $-4\pi^2$.

Question 3 Find $\int e^{2x} \cos(2x) dx$.

Answer 3 $\frac{1}{4}e^{2x}(\sin(2x) + \cos(2x)) + C$.

Trigonometric Integration: sine and cosine

Abstract

In this document, we discuss one of the additional techniques of integration, called *trigonometric integration*. We focus here on integrals which use sin and cos, as they are the most common.

1 Introduction.

In the past, when you had sin and cos in the same integral, it was usually the best to use integration by parts or a u -substitution. But this won't always work. So, we have another technique of integration for particular integrals of trig functions, called *trigonometric integration*. It helps us look at integrals like

$$\int \sin^m(x) \cdot \cos^n(x) dx$$

We can also handle integrals of **only** sin and cos by thinking of the other one as being raised to the power 0.

Before we get started on the process, though, we'll need to remember 3 trigonometric identities: the Pythagorean Identity (PI), the half-angle formulas, and the sin double angle formula. Let's recall what those look like. The Pythagorean identity is

$$\sin^2(x) + \cos^2(x) = 1$$

while the half angle formulas are

$$\begin{aligned}\sin^2(x) &= \frac{1}{2}(1 - \cos(2x)) \\ \cos^2(x) &= \frac{1}{2}(1 + \cos(2x))\end{aligned}$$

and sin's double angle formula (rewritten) is:

$$\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$$

2 Steps and Examples.

This is the general process:

1. Determine your powers of sin and cos (call them m and n).
2. If both m and n are even and $m = n$, use sin's double angle formula. If $m \neq n$, use the half-angle formulas.
3. If the power of sin is odd, keep one factor of sin and use the Pythagorean Identity (PI) to turn all other sin's into cos's. Then use a u -sub with $u = \cos(x)$.
4. If the power of cos is odd, keep one factor of cos and use the PI to turn all other cos's into sin's. Then use a u -sub with $u = \sin(x)$.
5. If both are odd, keep the factor of the function with the smallest power.
6. If the integral is indefinite, add a $+C$. Otherwise, put your bounds at the end of the integral and use FTC as the last step.

Example 1

Find $\int \sin^3(x) \cos^5(x) dx$.

Solution:

$m = 3$ and $n = 5$.

Find your m and n .

(It doesn't matter which is which).

Both are odd, so keep a factor of the smaller power.

In this case, we keep a power of sin.

Use the PI to turn the other sin's into cos's.

Simplify.

Make a u -substitution.

$$\frac{du}{dx} = -\sin(x) \text{ so } -du = \sin(x) dx$$

Let $u = \cos(x)$.

$$\begin{aligned}
 & \text{So } \int (\cos^5(x) - \cos^7(x)) \sin(x) dx \\
 &= \int (u^5 - u^7) \cdot -du && \text{Sub in your } u \text{ and } du. \\
 &= \int u^7 - u^5 du \\
 &= \frac{1}{8}u^8 - \frac{1}{6}u^6 \\
 &= \frac{1}{8}\cos^8(x) - \frac{1}{6}\cos^6(x) + C && \text{Integrate.} \\
 &&& \text{Add a } +C.
 \end{aligned}$$

Great! These problems where one or both of the powers are odd are really nice, since you only need to use the Pythagorean Identity and then make a u -substitution. Let's look at one more of these.

Example 2

Find $\int_0^\pi \cos^3(2x) \sin^2(2x) dx$.

Solution: Before we start: when you have a coefficient in the argument of trig functions, you follow the same exact process. Just be careful when you make the u -substitution or use the half-angle formulas.

$$m = 2 \text{ and } n = 3.$$

Find your m and n .

$$\begin{aligned} & \int_0^\pi \cos^3(2x) \sin^2(2x) dx \\ &= \int \cos^2(2x) \sin^2(2x) \cos(2x) dx \Big|_0^\pi \\ &= \int (1 - \sin^2(2x)) \sin^2(2x) \cos(2x) dx \Big|_0^\pi \\ &= \int (\sin^2(2x) - \sin^4(2x)) \cos(2x) dx \Big|_0^\pi \end{aligned}$$

Put the bounds at the end

and keep a factor of the odd power.

Use the PI to turn the remaining cos's into sin's.

Simplify.

Make a u -sub.

Careful with the chain rule here!

$$\frac{du}{dx} = \cos(2x) \cdot 2 \text{ so } \frac{1}{2} du = \cos(2x) dx$$

$$\begin{aligned} \text{So } & \int (\sin^2(2x) - \sin^4(2x)) \cos(2x) dx \Big|_0^\pi \\ &= \int (u^2 - u^4) \frac{1}{2} du \Big|_{u(0)}^{u(\pi)} \\ &= \frac{1}{2} \int u^2 - u^4 du \Big|_{u(0)}^{u(\pi)} \\ &= \frac{1}{2} \left[\frac{1}{3}u^3 - \frac{1}{5}u^5 \right]_{u(0)}^{u(\pi)} \\ &= \left[\frac{1}{6}u^3 - \frac{1}{10}u^5 \right]_{u(0)}^{u(\pi)} \\ &= \left[\frac{1}{6}\sin^3(2x) - \frac{1}{10}\sin^5(2x) \right]_0^\pi \\ &= 0. \end{aligned}$$

Treat your bounds like a u -sub.

Simplify.

Integrate.

Substitute back in.

Use FTC.

Example 3

Find $\int \sin^4(t) \cos^4(t) dt$.

Solution:

$$m = 4 \text{ and } n = 4$$

Find m and n .

$$\begin{aligned} & \int \sin^4(t) \cos^4(t) dt \\ &= \int [\sin(t) \cos(t)]^4 dt \\ &= \int \left[\frac{1}{2} \sin(2t) \right]^4 dt \\ &= \frac{1}{16} \int \sin^4(2t) dt \\ &= \frac{1}{16} \int [\sin^2(2t)]^2 dt \\ &= \frac{1}{16} \int \left[\frac{1}{2}(1 - \cos(4t)) \right]^2 dt \\ &= \frac{1}{64} \int 1 dt - \frac{2}{64} \int \cos(4t) dt + \frac{1}{64} \int \cos^2(4t) dt \\ &= \frac{1}{64}t - \frac{1}{32} \int \cos(u) \cdot \frac{1}{4} du + \frac{1}{64} \int \frac{1}{2}(1 + \cos(8t)) dt \\ &= \frac{1}{64}t - \frac{1}{128}[\sin(4t)] + \frac{1}{128} \int 1 + \cos(8t) dt \\ &= \frac{1}{64}t - \frac{1}{128} \sin(4t) + \frac{1}{128}t + \frac{1}{128} \int \cos(u) \cdot \frac{1}{8} du \\ &= \frac{3}{128}t - \frac{1}{128} \sin(4t) + \frac{1}{1024} \sin(8t) + C \end{aligned}$$

Both are even and $m = n$, so we use the double angle formula first.

Simplify.

Now we have another integral with two even powers ($m = 0$). So, use the half-angle formulas.

Careful with the argument!

Doubling $2t$ gives you $4t$.

Simplify.

Integrate, make a u -sub ($u = 4t$), and apply the half-angle formula for cos.

Integrate and make another u -sub.

Here, $u = 8t$.

Integrate and add a $+C$.

That should give you all you need to know about even powers.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find $\int \cos^3(x) \sin^{\frac{1}{2}}(x) dx$. (Hint: don't be put off by the fractional power on sin! One of the powers is still odd.)

Answer 1 $\frac{3}{2} \sin^{\frac{3}{2}}(x) - \frac{7}{2} \sin^{\frac{7}{2}}(x) + C$.

Question 2 Find $\int_0^{\frac{\pi}{4}} \sin^3(2x) dx$.

Answer 2 $\frac{3}{2}$.

Question 3 Find $\int \sin^6(t) dt$.

Answer 3 $\frac{5}{16}t - \frac{4}{1}\sin(2t) + \frac{64}{3}\sin(4t) + \frac{48}{1}\sin^3(2t) + C$.

Trigonometric Integration: secant and tangent, cosecant and cotangent

Abstract

In this document, we continue and complete our discussion of trigonometric integration by discussing integrals containing either sec and tan or csc and cot. These frequently come up in the next technique of integration, called trigonometric substitution.

1 Introduction.

We previously looked at trigonometric integrals containing sin and cos. But what about the other four trig functions, sec, tan, csc, and cot? Well, we can handle two different cases which are very similar:

$$\int \sec^m(x) \tan^n(x) dx \quad \int \csc^m(x) \cot^n(x) dx$$

We can handle these cases primarily because the derivative of $\tan(x)$ is $\sec^2(x)$, and the derivative of sec is $\sec(x) \tan(x)$. Similarly, the derivative of $\cot(x)$ is $-\csc^2(x)$, and the derivative of csc is $-\csc(x) \cot(x)$. In a table, this yields:

Function	Derivative
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x) \tan(x)$
$\cot(x)$	$-\csc^2(x)$
$\csc(x)$	$-\csc(x) \cot(x)$

We also need two more trig identities, though, which both stem from the Pythagorean Identity ($\sin^2(x) + \cos^2(x) = 1$). If we divide the PI by $\cos^2(x)$, we get our first identity. If we divide by $\sin^2(x)$ we get our second identity. They are, in order:

$$\begin{aligned} \tan^2(x) + 1 &= \sec^2(x) \\ 1 + \cot^2(x) &= \csc^2(x) \end{aligned}$$

2 Steps and Examples.

This is the general process:

1. If the power of $\tan(x)$ [$\cot(x)$] is odd, keep a factor of $\tan(x) \sec(x)$ [$\cot(x) \csc(x)$] and use our identities to turn all other $\tan(x)$ into $\sec(x)$ [$\cot(x)$ into $\csc(x)$].
2. If the power of $\sec(x)$ [$\csc(x)$] is even, keep a factor of $\sec^2(x)$ [$\csc^2(x)$] and use our identities to turn all other $\sec(x)$ into $\tan(x)$ [$\csc(x)$ into $\cot(x)$].
3. For the cases not covered here, convert the problem into one of sin and cos.
4. Make a u -substitution, integrate, and re-substitute.
5. If the integral is indefinite, add a $+C$. Otherwise, use FTC.

Example 1

Find $\int \sqrt{\tan(x)} \sec^4(x) dx$.

Solution:

$\begin{aligned} & \int \sqrt{\tan(x)} \sec^4(x) dx \\ &= \int \tan^{\frac{1}{2}}(x) \sec^2(x) \sec^2(x) dx \\ &= \int \tan^{\frac{1}{2}}(x)(\tan^2(x) + 1) \sec^2(x) dx \\ &\quad \text{Let } u = \tan(x) \\ &\quad \frac{du}{dx} = \sec^2(x) \text{ so } du = \sec^2(x) dx \\ &\quad \text{So } \int \tan^{\frac{1}{2}}(x)(\tan^2(x) + 1) \sec^2(x) dx \\ &\quad = \int u^{\frac{1}{2}}(u^2 + 1) du \\ &\quad = \int u^{\frac{1}{2}} + u^{\frac{5}{2}} du \\ &\quad = \frac{2}{3}u^{\frac{3}{2}} + \frac{2}{7}u^{\frac{7}{2}} \\ &\quad = \frac{2}{3}\tan^{\frac{3}{2}}(x) + \frac{2}{7}\tan^{\frac{7}{2}}(x) + C \end{aligned}$	<p>We have an even sec power here.</p> <p>Keep a $\sec^2(x)$.</p> <p>Turn the remaining $\sec(x)$ into $\tan(x)$.</p> <p>Make a u-sub.</p> <p>Integrate.</p> <p>Sub back in and add a $+C$.</p>
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Example 2

Find $\int \cot^5(x) \csc^5(x) dx$.

Solution:

$$\begin{aligned} & \int \cot^5(x) \csc^5(x) dx \\ &= \int \cot^4(x) \csc^4(x) (\cot(x) \csc(x)) dx \\ &= \int [\csc^2(x) - 1]^2 \csc^4(x) (\cot(x) \csc(x)) dx \end{aligned}$$

Let $u = \csc(x)$

$$\frac{du}{dx} = -\cot(x) \csc(x) \text{ so } -du = \cot(x) \csc(x) dx$$

$$\text{So } \int [\csc^2(x) - 1]^2 \csc^4(x) (\cot(x) \csc(x)) dx$$

$$\begin{aligned} &= \int [u^2 - 1]^2 \cdot u^4 \cdot -du \\ &= - \int [u^4 - 2u^2 + 1] \cdot u^4 du \\ &= - \int u^8 - 2u^6 + u^4 du \end{aligned}$$

$$= - \left[\frac{1}{9}u^9 - \frac{2}{7}u^7 + \frac{1}{5}u^5 \right]$$

$$= - \frac{1}{9} \csc^9(x) + \frac{2}{7} \csc^7(x) - \frac{1}{5} \csc^5(x) + C$$

We have an odd power of cot here.

Keep a factor of $\cot(x) \csc(x)$.

Turn all cot's into csc's.

Be careful with the power of cot here.

Make a u -sub.

Sub in u and du .

Simplify.

Integrate.

Sub back in and add a $+C$.

Those two examples should show off basically how to do all indefinite integrals with csc and cot or sec and tan. For csc and cot, you just treat it identical to sec and tan. You just get a negative sign when you make the u -sub. Let's look at a final example, with a name change and bounds.

Example 3

Find $\int_0^{\frac{\pi}{3}} \tan^5(t) \sec^3(t) dt$.

Solution:

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} \tan^5(t) \sec^3(t) dt \\ &= \int \tan^4(t) \sec^2(t) (\tan(t) \sec(t)) dt \Big|_0^{\frac{\pi}{3}} \\ &= \int [\sec^2(t) - 1]^2 \sec^2(t) (\tan(t) \sec(t)) dt \Big|_0^{\frac{\pi}{3}} \end{aligned}$$

Let $u = \sec(t)$

$$\frac{du}{dt} = \sec(t) \tan(t) \text{ so } du = \sec(t) \tan(t) dt$$

$$\begin{aligned} \text{So } & \int [\sec^2(t) - 1]^2 \sec^2(t) (\tan(t) \sec(t)) dt \Big|_0^{\frac{\pi}{3}} \\ &= \int [u^2 - 1]^2 \cdot u^2 du \Big|_{u(0)}^{u(\frac{\pi}{3})} \\ &= \int [u^4 - 2u^2 + 1] \cdot u^2 du \Big|_{u(0)}^{u(\frac{\pi}{3})} \\ &= \int u^6 - 2u^4 + u^2 du \Big|_{u(0)}^{u(\frac{\pi}{3})} \\ &= \left[\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 \right]_{u(0)}^{u(\frac{\pi}{3})} \\ &= \left[\frac{1}{7}\sec^7(t) - \frac{2}{5}\sec^5(t) + \frac{1}{3}\sec^3(t) \right]_0^{\frac{\pi}{3}} \\ &= \frac{848}{105} \end{aligned}$$

We have an odd power of tan here.

Keep a factor of $\sec(t) \tan(t)$.

Put the bounds at the end, also.

Turn all other tan's into sec's.

Again, careful with the power of tan.

Make a u -sub.

Sub in u and du .

Simplify.

Integrate.

Sub back in.

Use FTC.

Just like with a u -sub and a trigonometric integral with sin and cos, we just keep the bounds at the end in an evaluation bar and use FTC as our last step.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find $\int \sec^6(x) dx$.

Answer 1 $\frac{5}{2} \tan^2(x) + \frac{3}{2} \tan^4(x) + C$.

Question 2 Find $\int \csc^3(x) \cot(x) dx$.

Answer 2 $-\frac{3}{4} \csc^2(x) + C$.

Question 3 Find $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc^4(t) \cot^{\frac{2}{3}}(t) dt$.

Answer 3 $\frac{26 \cdot 3^{\frac{5}{6}}}{11} \approx 3.54$.

Trigonometric Substitution

Abstract

Trigonometric substitution is yet another technique of integration that helps with integrals that have a piece which looks similar to the Pythagorean Theorem. It turns integrals into trigonometric integrals, so make sure you know trigonometric integration before beginning.

1 Introduction.

So far, our techniques of integration cover a lot of different cases, but certain integrals are still very difficult or even impossible with our current tool-belt. Some integrals, though, have a piece that looks a lot like the Pythagorean Theorem when solved for a side. When you solve the Pythagorean for the hypotenuse, you get $\sqrt{a^2 + b^2}$, and when you solve for a leg, you get $\sqrt{c^2 - b^2}$. This fact allows us to make a really nice substitution, called a *trigonometric substitution*, when we have integrals involving one of these 3 pieces:

$$(1) \sqrt{x^2 + a^2} \quad (2) \sqrt{x^2 - a^2} \quad (3) \sqrt{a^2 - x^2}$$

Now the Stewart textbook recommends just memorizing the substitution to make in each of the cases: for (1) you substitute $x = a \tan \theta$, for (2) you substitute $x = a \sec \theta$, and for (3) you substitute $x = a \sin \theta$.

The easiest way to remember this is the following: if you have a plus sign, use tangent. If x^2 is the only positive term, use secant. Otherwise, use sine. Once you do this, you're going to need to use the Pythagorean Identity (PI) or the "sec and tan version." Before we start on examples, let's revisit what these are:

$$\begin{aligned}\sin^2(\theta) + \cos^2(\theta) &= 1 \\ \tan^2(\theta) + 1 &= \sec^2(\theta)\end{aligned}$$

Remember also that you can find the second equation by dividing the first by $\cos^2(\theta)$, meaning you only truly need to memorize the first.

2 Steps and Examples.

This is the general process:

1. Determine what substitution to make.
2. Find dx in terms of $d\theta$.
3. Make your substitution.
4. Simplify the square root using the PI or the “sec and tan version.”
5. Integrate.
6. Substitute back in for all remaining θ ’s. For any remaining trig functions in terms of theta, draw a right triangle with an angle θ using your substitution and label the sides.
7. If your integral is indefinite, add a $+C$. Otherwise, use FTC.

Example 1

Use a trigonometric substitution to find $\int \frac{1}{\sqrt{1-x^2}} dx$.

Solution:

The first step is to determine what our substitution is. To do so, we look at the inside of the square root. Because we have a negative sign and it’s attached to x^2 , we use the substitution $x = a \sin(\theta)$, but what’s our a ? Well, if you look back at our 3 cases on page 1, we see its the square root of our constant, that is, $a^2 = 1$ and so $a = 1$. This means that our substitution is $x = 1 \cdot \sin(\theta)$ and so $\theta = \arcsin(x)$. Now for step 3.

$x = \sin(\theta)$ $\frac{dx}{d\theta} = \cos(\theta) \text{ so } dx = \cos(\theta) d\theta$ $\int \frac{1}{\sqrt{1-x^2}} dx$ $= \int \frac{1}{\sqrt{1-\sin^2(\theta)}} \cdot \cos(\theta) d\theta$ $= \int \frac{1}{\sqrt{\cos^2(\theta)}} \cos(\theta) d\theta$ $= \int \frac{\cos(\theta)}{\cos(\theta)} d\theta$ $= \int 1 d\theta$	<p>This was our substitution.</p> <p>Find dx like a u-sub.</p> <p>Make your substitution.</p> <p>Simplify with the PI.</p> <p>Simplify.</p>
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$$\begin{array}{l|l} = \theta & \text{Integrate.} \\ = \arcsin(x) + C & \begin{array}{l} \text{Substitute back in for } \theta. \\ \text{Add a } +C. \end{array} \end{array}$$

One note on our re-substituting for this problem. You'll notice we didn't actually draw or use a triangle at all. This is because there were no trig functions after we integrated, so we only needed the fact that $\theta = \arcsin(x)$. Making the triangle is good practice, but in examples like these, it's not necessary. Let's move onto our next example.

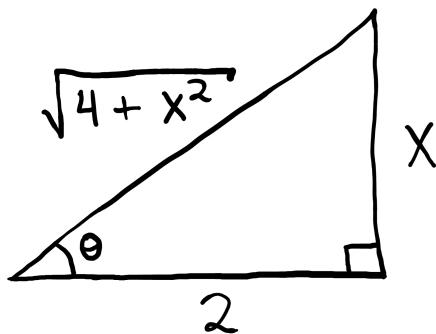
Example 2

Find $\int \frac{1}{(4+x^2)^{\frac{3}{2}}} dx$.

Solution: Now before we can choose what substitution to make, we need to get this into a form that looks like our typical 3 cases. To do so, we change the fractional exponent into the denominator, turning our integral into $\int \frac{1}{(\sqrt{4+x^2})^3} dx$. Now because we have a plus sign in the square root, we'll use the substitution $x = a \tan(\theta)$. What's a ? Well, you may be tempted to say that it's 4, but remember that this is of the form $\sqrt{a^2 + x^2}$, so 4 is actually our a^2 . This means $a = 2$ and so $x = 2 \tan(\theta)$ or $\tan(\theta) = \frac{x}{2}$.

$x = 2 \tan(\theta)$ $\frac{dx}{d\theta} = 2 \sec^2(\theta) \text{ so } dx = 2 \sec^2(\theta) d\theta$ $\int \frac{1}{(\sqrt{4+x^2})^3} dx$ $= \int \frac{1}{(\sqrt{4+4\tan^2(\theta)})^3} \cdot 2 \sec^2(\theta) d\theta$ $= 2 \int \frac{\sec^2(\theta)}{(\sqrt{4(1+\tan^2(\theta))})^3} d\theta$ $= 2 \int \frac{\sec^2(\theta)}{2^3 (\sqrt{\sec^2(\theta)})^3} d\theta$ $= \frac{2}{8} \int \frac{\sec^2(\theta)}{\sec^3(\theta)} d\theta$ $= \frac{1}{4} \int \frac{1}{\sec(\theta)} d\theta$ $= \frac{1}{4} \int \cos(\theta) d\theta$ $= \frac{1}{4} \sin(\theta)$	<p>This was our sub.</p> <p>Find dx.</p> <p>Sub in x and dx.</p> <p>Simplify. When $a \neq 1$, factor it out.</p> <p>Use the PI. Simplify.</p>
---	--

Now we have a trig function in terms of theta that we need in terms of x . This is where you need to draw the right triangle with an angle of θ , and then fill in its sides using the substitution we made before. This gives us the following triangle:



We know that x and 2 go where they do because $\tan(\theta) = \frac{\text{OPP.}}{\text{ADJ.}} = \frac{x}{2}$. Then, the hypotenuse is what it is by the Pythagorean Theorem. Using this triangle, we see that $\sin(\theta) = \frac{\text{OPP.}}{\text{HYP.}} = \frac{x}{\sqrt{4 + x^2}}$ which means our final answer is $\frac{x}{8\sqrt{4 + x^2}} + C$.

Good. One tip for the triangle: you're going to always want to draw the triangle the same way, with a right angle and one angle of θ . The hardest part is writing in the sides of the triangle. This takes practice, and it comes down to using the substitution you made and the Pythagorean Theorem. It may be good practice to go back to Example 1 and draw the triangle for that substitution. Otherwise, let's move on to Example 3.

Example 3

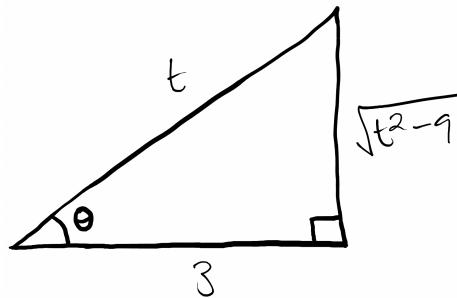
Find $\int_3^4 \frac{\sqrt{t^2 - 9}}{t^3} dt.$

Solution: From our typical 3 cases, this fits perfectly into our case where we substitute using sec. So, our substitution is going to be $t = a \sec(\theta)$. Like in the previous example, our a^2 is our constant term, that is, 9, and so $a = 3$. This means our substitution will be $t = 3 \sec(\theta)$.

$t = 3 \sec(\theta)$ $\frac{dt}{d\theta} = 3 \sec(\theta) \tan(\theta)$ $dt = 3 \sec(\theta) \tan(\theta) d\theta$ $\int_3^4 \frac{\sqrt{t^2 - 9}}{t^3} dt$ $= \int \frac{\sqrt{9 \sec^2(\theta) - 9}}{\sec^3(\theta)} \cdot 3 \sec(\theta) \tan(\theta) d\theta \Big _{\theta(3)}^{\theta(4)}$ $= 9 \int \frac{\sqrt{\sec^2(\theta) - 1}}{\sec^2(\theta)} \tan(\theta) d\theta \Big _{\theta(3)}^{\theta(4)}$ $= 9 \int \frac{\sqrt{\tan^2(\theta)}}{\sec^2(\theta)} \tan(\theta) d\theta \Big _{\theta(3)}^{\theta(4)}$ $= 9 \int \frac{\tan^2(\theta)}{\sec^2(\theta)} d\theta \Big _{\theta(3)}^{\theta(4)}$ $= 9 \int \frac{\tan^2(\theta)}{\sec^2(\theta)} d\theta \Big _{\theta(3)}^{\theta(4)}$ $= 9 \int \frac{\sin^2(\theta)}{\cos^2(\theta)} \cdot \cos^2(\theta) d\theta \Big _{\theta(3)}^{\theta(4)}$ $= 9 \int \sin^2(\theta) d\theta \Big _{\theta(3)}^{\theta(4)}$	<p>This was our sub.</p> <p>Find dt.</p> <p>Sub in t and dt.</p> <p>Treat the bounds like a trig integral.</p> <p>Simplify.</p> <p>Use the PI.</p> <p>This trig integral with tan and sec doesn't fit any of the cases, so we turn it into sin and cos.</p> <p>Here we use $\sec(\theta) = \frac{1}{\cos(\theta)}$.</p> <p>This is a standard trig integral in sin and cos. To do it, we use the half angle formula for sin.</p>
---	--

$$\begin{aligned}
 &= 9 \int \frac{1}{2} (1 - \cos(2\theta)) d\theta \Big|_{\theta(3)}^{\theta(4)} \\
 &= \left[\frac{9}{2} \int 1 d\theta - 9 \int \cos(2\theta) d\theta \right]_{\theta(3)}^{\theta(4)} \quad \text{Simplify.} \\
 &= \left[\frac{9}{2}\theta - 9 \int \cos(u) \cdot \frac{1}{2} du \right]_{\theta(3)}^{\theta(4)} \quad \text{Make a } u\text{-sub with } u = 2\theta. \\
 &= \left[\frac{9}{2}\theta - \frac{9}{2} \sin(2\theta) \right]_{\theta(3)}^{\theta(4)}
 \end{aligned}$$

Now we need to substitute back in in terms of t , which means we'll need a triangle. The one for this problem looks something like this:



Again, this comes from the fact that $\sec(\theta) = \frac{\text{HYP.}}{\text{ADJ.}} = \frac{t}{3}$ and the Pythagorean Theorem.

Using this, we can see that $\theta = \arccos\left(\frac{3}{t}\right)$ (you could use arcsecant, but it's more common to see arccos). But what about $\sin(2\theta)$? Well, to handle that, we have to use another trig identity. You won't normally be expected to have this memorized, but here it comes in handy: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Looking at our triangle, $\sin(\theta) = \frac{\sqrt{t^2 - 9}}{t}$ and $\cos(\theta) = \frac{3}{t}$. This means that $\sin(2\theta) = \frac{6\sqrt{t^2 - 9}}{t^2}$ which means our final antiderivative is $\frac{9}{2} \arccos\left(\frac{3}{t}\right) - 27 \frac{\sqrt{t^2 - 9}}{t^2}$, which we need to evaluate from $t = 3$ to $t = 4$. Doing so with FTC gives us a final answer of -1.212 .

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Use trigonometric substitution to find $\int \frac{1}{t\sqrt{t^2 - 1}} dt$.

Answer 1 $\arctan(\sqrt{t^2 - 1}) + C$ or $\arccos\left(\frac{1}{t}\right) + C$

Question 2 Use trigonometric substitution to find $\int x^3 \sqrt{x^2 + 1} dx$.

Answer 2 $\frac{(x^2 + 1)^{\frac{5}{2}}}{\frac{5}{3}} - \frac{(x^2 + 1)^{\frac{3}{2}}}{\frac{3}{2}} + C$.

Question 3 Find $\int_0^1 \sqrt{9 - x^2} dx$.

Answer 3 2.943.

Tank Problems

Abstract

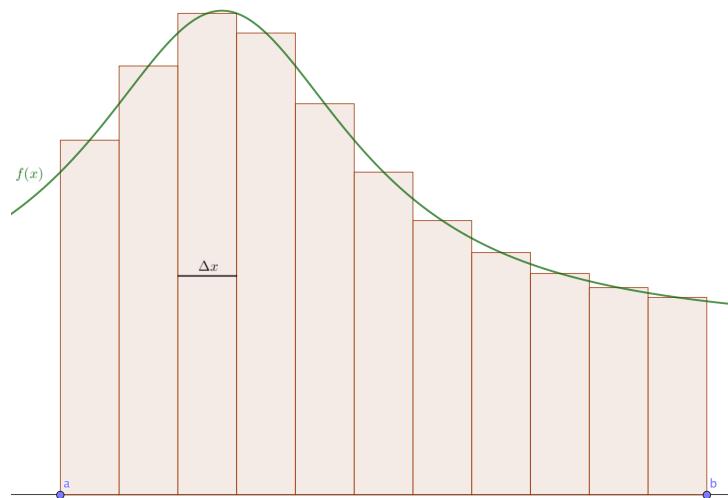
Here we discuss an application of Calculus II to physics and engineering: the work needed to empty a tank. Also called *tank problems*, these problems are some of the hardest a Calc II student will see.

1 Introduction.

Calc II is, as you've seen, mainly about the integral. But do you remember how the integral is actually defined? It's the area under a curve between two points, true, but what was the formal definition? Well, if your memory is good, you may remember this definition:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Graphically, this definition means breaking the interval $[a, b]$ into n equal sub-intervals, and considering the rectangle in that sub-interval with a height of the function at some point within the sub-interval. That looks something like this:



Then, we add more and more rectangles, making the sub-intervals smaller and smaller. When we add up the area of the rectangles, then, we get the area under the curve between a and b .

This all brings about a general process, helpfully shortened to the acronym **DEAL**:

- Divide what you're trying to find into equal sections.
- Estimate the quantity you're trying to find for one section.
- Add all of the sections together.
- Limit: take a limit to make the sections smaller and smaller.

This process needn't be restricted to area under a curve, though. In fact, it's incredibly helpful in the realm of applying Calculus to other contexts. Tank problems are one such context.

Now, for these problems you could memorize the formula for work in terms of distance, mass, volume, etc., but we think it will be most useful to learn the DEAL process and how to apply it to different situations.

In the context of tank problems, the quantity we're trying to find is “the work required to pump a liquid out of a tank” or, more simply, work. So, to be able to handle these work problems, you'll need to know the formula for work. In general, there are a couple different relationships to know, such as $W = Fd$, $F = ma$, or $m = \rho V$, but this all boils down to the main formula we care about, which is

$$\boxed{[\text{WORK}] = [\text{DENSITY}] \cdot [\text{VOL.}] \cdot [\text{DISTANCE}] \cdot [\text{ACCELERATION}]}$$

2 Steps and Examples.

This is the general process:

1. Draw a picture and give it a coordinates system.
2. DEAL:
 - **Divide** the tank into layers with thickness Δx .
 - **Estimate** the work (volume times density times distance to spout times g) needed for an arbitrary layer at $x = x_i^*$.
 - **Add** the work for all of the layers together.
 - Take the **limit** as the number of layers approaches infinity.
3. Use the definition of the definite integral, with bounds at the start and end of the liquid.
4. Integrate, use FTC, and add units.

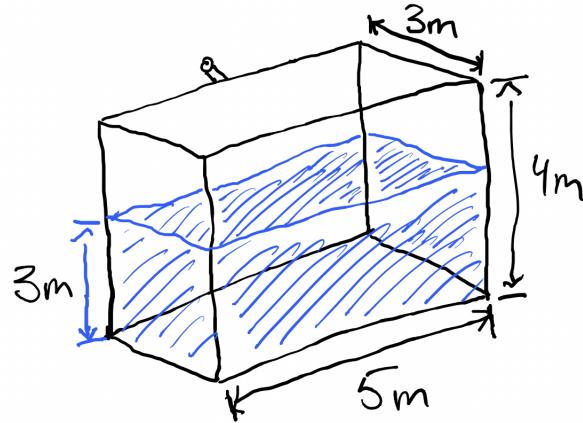
If you want specific, step-by-step instructions, though, they're here:

1. Draw a picture of the tank if you don't have one already.
2. Give the picture a coordinate system where down is the positive x direction. Call the surface of the liquid a and the bottom of the liquid b .
3. Divide the water in the tank into n "layers," each with a small thickness of Δx .
4. Find the volume of an arbitrary layer, that is, the i th layer. The layer's volume will likely depend on its height, x_i^*
5. Find the distance from the layer's height (x_i^*) to the spout's height. This will be in terms of the x -value of the layer.
6. Multiply the volume of the layer, the density of the liquid the distance from the layer to the spout, and the gravitational constant g (9.8 meters per second or 32 feet per second) all together. This will be the work for the i th layer.
7. Add the work of all n layers together and take the limit as n approaches infinity.
8. Use the definition of a definite integral to turn this into an integral, where the bounds reflect where the liquid starts and stops.
9. Integrate, use FTC, and add units (Joules (J) for meters, foot-pounds (ft-lbs) for feet).

Example 1

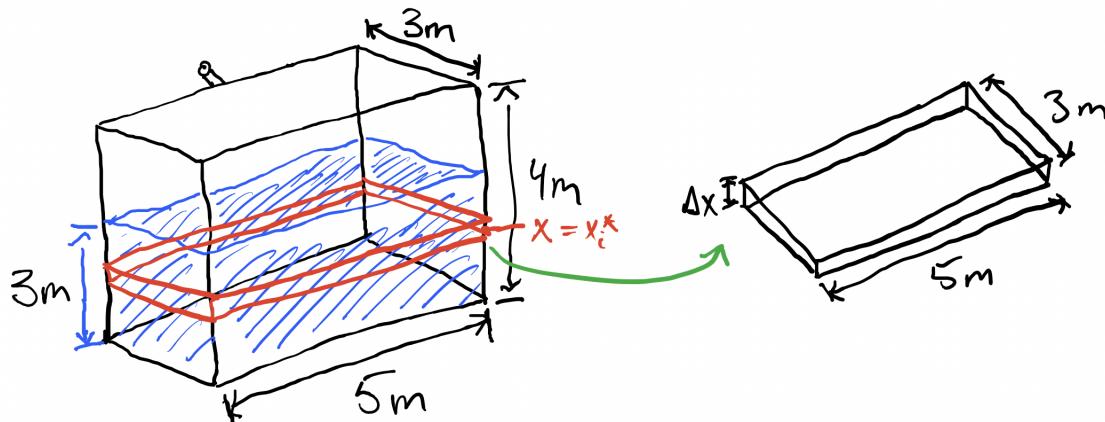
A tank is in the shape of a rectangular prism (a box). It is 4 meters deep, 3 meters wide, and 5 meters long. The tank is filled with water up to the 3 meter mark. How much work is required to empty the tank from a spout at the top of the tank?

Solution: Let's start with step 1 and draw a picture of the tank.



Now, we want to give it a coordinates system. Ultimately, this is an arbitrary choice that shouldn't affect your answer, but we will follow the convention of the Stewart book and make down the positive x direction. Then, with non-spherical tanks, it is easiest if you make the height of the spout $x = 0$. This will mean the bottom of the tank will be $x = 4$, and the surface of the water is at $x = 1$.

Now for steps 3 and 4. We consider an arbitrary layer of water in the tank with height Δx at $x = x_i^*$, that is, x_i^* meters from $x = 0$ where the spout is. A picture of this layer might look as follows:



Now we want to find a volume of this layer. What you'll notice is that, since the cross-sectional area of the tank is the same at each value of x , the volume of each layer will be $[\text{VOL.}] = [\text{LENGTH}] \cdot [\text{WIDTH}] \cdot [\text{HEIGHT}] = 15\Delta x$. Let's use this information, along with the fact that the density of water is 1000 in SI units, to move on to step 5.

$$[\text{VOL.}]_i = 15\Delta x$$

$$[\text{DIST.}]_i = x_i^*$$

$$\begin{aligned} [\text{WORK}]_i &= [\text{DENS.}] \cdot [\text{VOL.}]_i \cdot [\text{DIST.}]_i \cdot g \\ &= (1000)(15\Delta x)(x_i^*)(9.8) \end{aligned}$$

$$[\text{WORK}] = \lim_{n \rightarrow \infty} \sum_{i=1}^n [\text{WORK}]_i$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (1000)(15)(x_i^*)(9.8)\Delta x \\ &= \int_1^4 (1000)(15)(9.8)x \, dx \\ &= (1000)(15)(9.8) \left[\frac{1}{2}x^2 \right]_1^4 \\ &= (1000)(15)(9.8) \left[8 - \frac{1}{2} \right] \\ &= 1.1025 \cdot 10^6 \text{J or } 1102.5 \text{KJ} \end{aligned}$$

This is the volume of the i th layer.

Find the distance from the i th layer to the spout.

Find the work for the i th layer.

We use $g = 9.8$ since we're in meters.

Add the work of each layer together, and take a limit.

Use the integral definition. The bounds are where the water starts and ends.

Integrate.

Use FTC.

Simplify and add units.

Phew! That's a long, complicated problem, and it's very easy to go wrong in one or more steps. The key is to remember the process of DEAL, while keeping in mind the formula that we boxed in red on page 2.

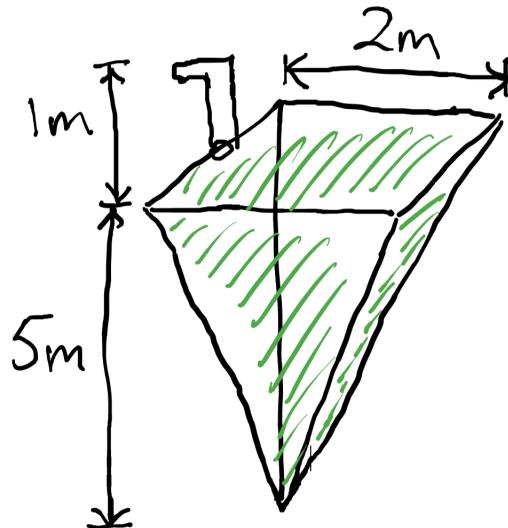
Now, the final step is to add units. We'll be doing everything in meters, where the units are Joules (J) or Kilo-Joules (KJ). Both will be accepted, but sometimes Kilo-Joules is easier for large numbers, as $1\text{KJ} = 1000\text{J}$. If your distances are in feet, then the final answer will have units of foot-pounds (ft-lbs).

Now that we have one example under our belt, let's look at a more complicated tank shape.

Example 2

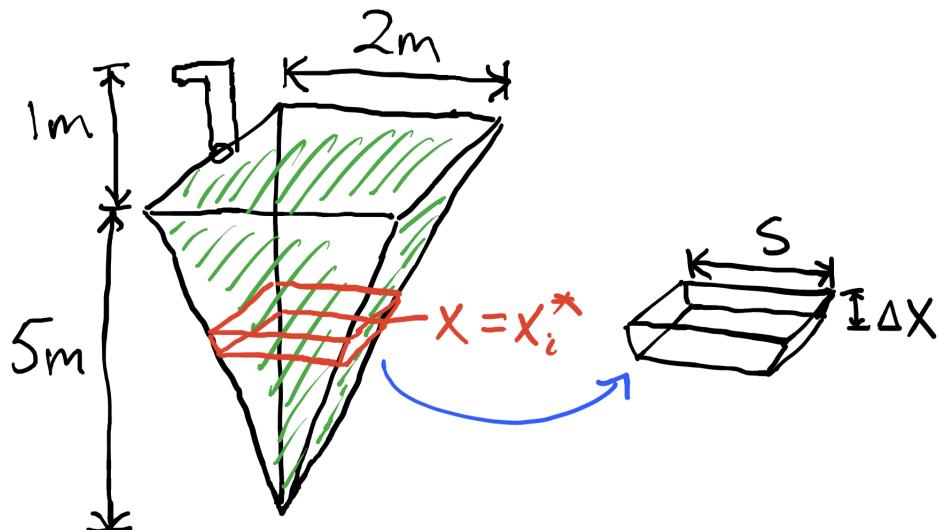
A tank is in the shape of an inverted (upside-down) square pyramid. The pyramid is 5 meters tall and the top of the pyramid is a square with a side length of 2 meters. The tank is full of olive oil, which has a density of about 900kg/m^3 . What is the work required to pump the oil out of a spout which is one meter above the top of the tank?

Solution: Again, we'll start with a picture. The tank looks something like this:



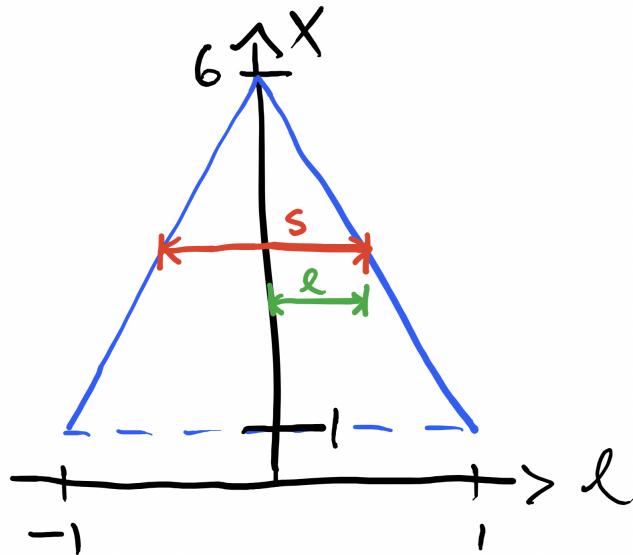
Then, like in Example 1, we'll pick a coordinates system in accordance to the Stewart book, making down the positive x -axis and the spout at a height of $x = 0$. This means the top of the tank (and surface of the oil) is at $x = 1$, and the bottom of the tank (and end of the oil) is at $x = 6$.

Now to look at an arbitrary layer at $x = x_i^*$. A picture of one layer might look like this:



As you might have seen, the cross section of the pyramid (the shape on the top of a layer) is a square, and we can call its side length s . This means that the volume of the layer is approximately $[\text{VOL.}] = [\text{LENGTH}] \cdot [\text{WIDTH}] \cdot [\text{HEIGHT}] = s^2 \Delta x$. Now the key is finding s . You might think that it's 2m, as that is the length that was given in the problem, but notice that the side length of the square is only 2m at the top of the pyramid; at the bottom of the pyramid, the square will be much smaller. So, we have to find s in terms of the height of the layer, x_i^* .

To do this, we'll draw a front-on view of our tank with the x -axis pointed upwards. Something like this:



You can see that I've called the horizontal axis l , but this is basically just a placeholder name of the horizontal coordinate. We can see, though, that $s = 2l$. This means that if we can find l in terms of x , we can find s in terms of x , which is our goal. Luckily, we can find l in terms of x simply using the equation of a line from high school algebra. Using slope-intercept form, we get that $x = -5l + 6$, and so $l = -\frac{1}{5}x + \frac{6}{5}$. Given $s = 2l$, then, we have $s = -\frac{2}{5}x + \frac{12}{5}$, and therefore our volume of the layer at $x = x_i^*$ will be $s^2 \Delta x = \left(-\frac{2}{5}x_i^* + \frac{12}{5}\right)^2 \Delta x = \left(\frac{4}{25}(x_i^*)^2 - \frac{48}{25}x_i^* + \frac{144}{25}\right) \Delta x$. We can finally move on to the next step now.

$$[\text{VOL.}]_i = \left(\frac{4}{25}(x_i^*)^2 - \frac{48}{25}x_i^* + \frac{144}{25}\right) \Delta x \quad \text{This was our volume of the } i\text{th layer.}$$

$$[\text{DIST.}]_i = x_i^*$$

Find the distance from
the i th layer to the spout.

$$\begin{aligned}
 [\text{WORK}]_i &= [\text{DENS.}] \cdot [\text{VOL.}]_i \cdot [\text{DIST.}]_i \cdot g \\
 &= (900) \left(\frac{4}{25}(x_i^*)^2 - \frac{48}{25}x_i^* + \frac{144}{25} \right) (x_i^*)(9.8)\Delta x \\
 [\text{WORK}] &= \lim_{n \rightarrow \infty} \sum_{i=1}^n [\text{WORK}]_i \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (900) \left(\frac{4}{25}(x_i^*)^2 - \frac{48}{25}x_i^* + \frac{144}{25} \right) (x_i^*)(9.8)\Delta x \\
 &= \int_1^6 (900) \left(\frac{4}{25}x^2 - \frac{48}{25}x + \frac{144}{25} \right) (x)(9.8) dx \\
 &= (900)(9.8) \left(\frac{4}{25} \right) \int_1^6 x^3 - 12x^2 + 36x dx \\
 &= (900)(9.8) \left(\frac{4}{25} \right) \left[\frac{1}{4}x^4 - 4x^3 + 18x^2 \right]_1^6 \\
 &= (900)(9.8) \left(\frac{4}{25} \right) \left[108 - \frac{57}{4} \right] \\
 &= 826675\text{J} \text{ or } 826.675\text{KJ}
 \end{aligned}$$

Find the work for the i th layer.

We use $g = 9.8$ since we're in meters.

Add the work of each layer together, and take a limit.

Use the integral definition. The bounds are where the water starts and ends.

Simplify.

Integrate.

Use FTC.

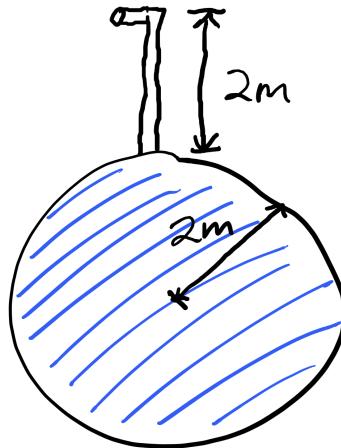
Simplify and add units.

Again, the main part of the problem is DEALing, and finding the volume of a layer is even more difficult when the tank is shaped this way. But, even though it takes time, it's still possible. Now we move to a spherical tank, which takes this method to its extreme.

Example 3

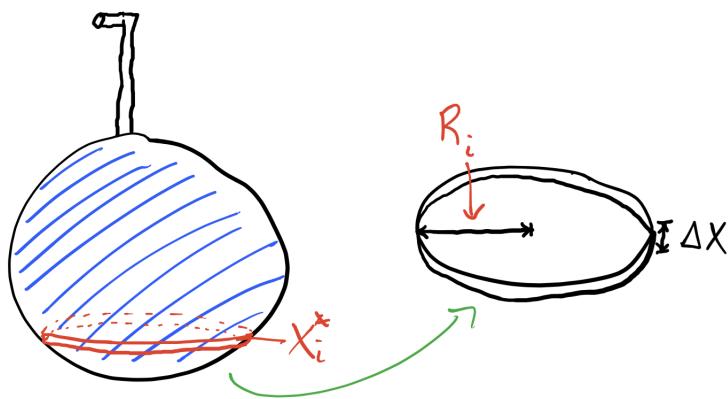
A tank is a sphere with a radius of 2 meters. There's a spout which is 2 meters above the height of the sphere. If the tank is filled with water, how much work is required to empty the tank?

Solution: Again, let's start by drawing a tank:

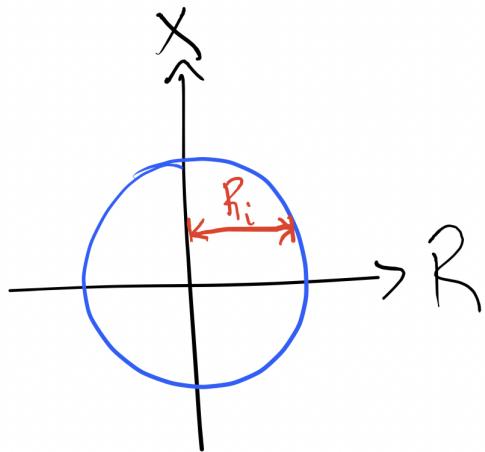


Now, to pick our coordinates system, we will do something different from the other examples. We will still make down the positive x direction, but instead of making $x = 0$ at the top of the spout, spherical tanks are easier if you make $x = 0$ at their center. This puts the spout at $x = -4$, the top of the tank at $x = -2$, and the bottom of the tank at $x = 2$.

Then, an arbitrary layer at $x = x_i^*$ with width Δx might look like this:



As you might see, unlike the previous examples, the shape of the layer is cylindrical, with a circle at the top and bottom. This means the volume of the layer is $\pi R_i^2 \Delta x$, where R_i is the radius of the circle. Like the last example, though, this radius changes as x_i^* varies. To find this relationship, we'll take the same approach of the last example, and draw a side-on view of the tank.



As you can see, I've oriented the axes so that the positive x -axis is now upwards, so as to best find the radius. Using the equation of a circle, we know that, for a point with height x_i^* , its distance from the vertical axis, which is the R_i we are trying to find, will be related through $(x_i^*)^2 + R_i^2 = 4$. Solving for R_i , we get $R_i = \sqrt{4 - (x_i^*)^2}$. Plugging this into the formula we had for volume, we get $V_i = \pi R_i^2 \Delta x = \pi(4 - (x_i^*)^2) \Delta x$.

The final thing before we move on is to find the formula for distance from the i th layer to the spout. Where in past examples the distance has been x_i^* , that won't be the case here because we have a different coordinates system. It's helpful to draw a picture here, but ultimately the distance comes out to be $4 + x_i^*$. This lets us move on to the next steps.

$$[\text{VOL.}]_i = \pi(4 - (x_i^*)^2)\Delta x$$

$$[\text{DIST.}]_i = 4 + x_i^*$$

$$[\text{WORK}]_i = [\text{DENS.}] \cdot [\text{VOL.}]_i \cdot [\text{DIST.}]_i \cdot g$$

$$= (1000)(\pi(4 - (x_i^*)^2))(4 + x_i^*)(9.8)\Delta x$$

$$[\text{WORK}] = \lim_{n \rightarrow \infty} \sum_{i=1}^n [\text{WORK}]_i$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n (1000)(\pi(4 - (x_i^*)^2))(4 + x_i^*)(9.8)\Delta x$$

$$= \int_{-2}^2 (1000)\pi(4 - x^2)(4 + x)(9.8) dx$$

$$= (1000)(9.8)(\pi) \int_{-2}^2 16 + 4x - 4x^2 - 4x^3 dx$$

$$= (1000)(9.8)(\pi) \left[16x + 2x^2 - \frac{4}{3}x^3 - x^4 \right]_{-2}^2$$

$$= (1000)(9.8)(\pi) \left[\frac{128}{3} \right]$$

$$= 1.313604 \cdot 10^6 \text{J or } 1313.604 \text{KJ}$$

This was our volume of the i th layer.

This was our distance.

Find the work for the i th layer.

We use $g = 9.8$ since we're in meters.

Add the work of each layer
together, and take a limit.

Use the integral definition.

Simplify.

Integrate.

Use FTC.

Simplify and add units.

The main difference here is the coordinates system and how it affects the distance of each layer to the spout. The way we did things is typical of tank problems, and it's good to know, but again, ultimately the coordinates choice shouldn't change your final answer.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 A cylindrical tank has a radius of 1m and is 5m tall. If it is full to the 3m mark with water, how much work is required to empty the tank from a spout at the top?

Answer 1 $102900\pi J = 3.23269 \cdot 10^5 J$ or $3232.69 KJ$.

Question 2 A tank in the shape of an upside-down cone is 6m tall and the circle on the top has a radius of 2m. If the tank is full of olive oil (the density of which is 900kg/m^3), how much work is required to empty the tank through a spout 1m above the top of the tank?

Answer 2 $\frac{1960000\pi}{3} J = 2.052507 \cdot 10^6 J$ or $2052.07 KJ$.

Question 3 A spherical tank with radius 4m is filled with water. How much work is required to empty the tank from a spout at the top?

Answer 3 $\frac{10035200\pi}{3} J = 1.050884 \cdot 10^7 J$ or $10508.84 KJ$.

The Equation of a Plane

Abstract

In this document, we look at our first example of a 3D *surface*: a *plane*. A plane is a flat surface that stretches on forever. Though it is flat, it may be at an angle, that is, it won't necessarily be horizontal or vertical. We discuss how to find the equation of a plane when given either a normal vector or 3 points.

1 Introduction.

From high school, you may remember that the easiest curve to think about was a straight line, which was of the form $y = mx + b$, where y is a function of x . You may also remember a different form to represent a line: the *general form* of the equation of a line was $ax + by + c = 0$. What made lines so nice to work with was twofold: (1) there were no fancy functions (no trig functions, no exponentials, etc.), and (2) x and y were all to the first power (no squares or cubes). As it is a property of lines, these two properties together form equations that are called *linear equations*.

Now that we're moving from 2 dimensions (x and y) to 3 dimensions (x , y , and z), we can take inspiration from lines and look at a linear 3D equation: $ax + by + cz + d = 0$. This equation, the simplest equation in 3D, is the equation of a *plane*. A plane is defined by two things: a starting point (x_0, y_0, z_0) on the plane, and a *normal vector* $\mathbf{n} = \langle a, b, c \rangle$ which is *orthogonal* or *perpendicular* to the plane. Given these two objects, we can construct the **scalar equation of a plane**:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where x_0 , y_0 and z_0 are the coordinates of any point on the plane and a , b , and c are the coordinates of the normal vector to the plane. Let's look at how to put this into practice.

2 Steps and Examples.

This is the general process:

1. If given 3 points on the plane, find the two vectors connecting the first point to the other two. Find the cross product of these two vectors to get the normal vector.
2. Find a , b , and c , which are the x , y , and z components of the normal vector, respectively.
3. Find x_0 , y_0 , and z_0 , the coordinates of any point on the plane.
4. Plug a , b , c , x_0 , y_0 , and z_0 into the scalar equation of a plane.

Example 1

Find the equation of the plane through the point $(1, 1, 1)$ and with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$.

Solution: Note: because we've been given a normal vector, we can skip step 1 and move onto step 2.

$$\mathbf{n} = \langle 2, 3, 4 \rangle, \text{ so}$$

$$a = 2, b = 3, \text{ and } c = 4.$$

$(1, 1, 1)$ is on the plane, so we can take $x_0 = 1$, $y_0 = 1$, and $z_0 = 1$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\text{So } 2(x - 1) + 3(y - 1) + 4(z - 1) = 0$$

$$\text{or } 2x + 3y + 4z - 9 = 0$$

Find a , b , and c . They will be the coordinates of the normal vector.

Find x_0 , y_0 , and z_0 . They will be the coordinates of a point on the plane.

Plug into the scalar equation.

Simplify if desired.

Great, hopefully you found that relatively straightforward. The key to this is remembering the scalar equation for a plane, and remembering what a , b , c , and x_0 , y_0 , and z_0 stand for. Let's look at another example, this time including step 1.

Example 2

Find the plane through $P = (0, 0, 0)$, $Q = (1, 2, 3)$, and $R = (-2, -2, -2)$.

Solution: We weren't directly given a normal vector here, so we're going to have to find one ourselves. To do so, we find two vectors which start at the same point. So, we can use vectors \mathbf{PQ} (the vector from P to Q) and \mathbf{PR} (the vector from P to R). $\mathbf{PQ} = \langle 1, 2, 3 \rangle$, and $\mathbf{PR} = \langle -2, -2, -2 \rangle$. Then, to find our normal vector, we take the cross product of \mathbf{PQ} and \mathbf{QR} . Doing so gives us:

$$\begin{aligned}\mathbf{PR} \times \mathbf{PQ} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & -2 & -2 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 2 & 3 \\ -2 & -2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ -2 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} \\ &= 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}\end{aligned}$$

Now that we have our normal vector, $\langle 2, -4, 2 \rangle$, we can move onto step 2.

Normal vector: $\langle 2, -4, 2 \rangle$

Find a , b , and c .

so $a = 2$, $b = -4$, and $c = 2$.

$(0, 0, 0)$ is on the plane, so we can take $x_0 = 0$, $y_0 = 0$, and $z_0 = 0$.

Find x_0 , y_0 , and z_0 . You can pick any point on the plane for this.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Plug into the scalar equation.

$$\text{so } 2(x - 0) - 4(y - 0) + 2(z - 0) = 0$$

$$\text{or } 2x - 4y + 2z = 0$$

$$\text{or } x - 2y + z = 0$$

Simplify if desired.

This example, as you can see, doesn't add too much to the process that we used for Example 1. The only difference is we needed to find the normal vector using the cross product. This means you should be sure you know (1) how to find the vector from one point to another, and (2) how to find the cross product of two vectors. Let's put these skills into practice with one final example.

Example 3

Find the plane through the points $(2, 4, 6)$, $(3, 5, 9)$, and $(-1, -2, -4)$.

Solution: Once again, we need to find the vectors from one point to the other two. Let's pick $(2, 4, 6)$ to be our starting point, for convenience (you could pick any point, though). The vector from $(2, 4, 6)$ to $(3, 5, 9)$ is $\langle(3 - 2), (5 - 4), (9 - 6)\rangle = \langle1, 1, 3\rangle$. The vector from $(2, 4, 6)$ to $(-1, -2, -4)$ is $\langle(-1 - 2), (-2 - 4), (-4 - 6)\rangle = \langle-3, -6, -10\rangle$. If we call these vectors \mathbf{u} and \mathbf{v} respectively, our normal vector is

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ -3 & -6 & -10 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 1 & 3 \\ -6 & -10 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ -3 & -10 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ -3 & -6 \end{vmatrix} \\ &= 8\mathbf{i} + \mathbf{j} - 3\mathbf{k} \\ &= \langle 8, 1, -3 \rangle.\end{aligned}$$

Great. Let's move on to step 2 now.

Normal vector: $\langle 8, 1, -3 \rangle$

so $a = 8$, $b = 1$, and $c = -3$.

$(-1, -2, -4)$ is on the plane, so we can take $x_0 = -1$, $y_0 = -2$, and $z_0 = -4$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\text{so } 8(x - (-1)) + 1(y - (-2)) - 3(z - (-4)) = 0$$

$$\text{or } 8x + y - 3z - 2 = 0$$

Find a , b , and c .

Find x_0 , y_0 , and z_0 . You can pick any point on the plane for this.

Plug into the scalar equation.

Simplify if desired.

One thing that this example shows to be careful of: watch your parentheses when the point you're plugging in has any negative components! You could very easily end up with a sign error if you're not careful, which will result in the wrong equation.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find the equation of a plane through the point $P_0 = (1, 2, -3)$ with normal vector $\mathbf{n} = \langle 2, -5, 3 \rangle$.

Answer 1 $2(x - 1) - 5(y - 2) + 3(z + 3) = 0$ or $2x - 5y + 3z + 17 = 0$.

Question 2 Find the equation of a plane through points $P = (2, -10, 15)$, $Q = (0, 5, -2)$, and $R = (-1, 1, 4)$.

Answer 2 $22x + 29y + 23z - 99 = 0$.

Question 3 Find the equation of a plane through points $(-3, 3, 3)$, $(1, -1, 7)$ and $(3, 7, -4)$.

Answer 3 $3x + 13y + 10z - 60 = 0$.

Line Integrals of Scalar Functions

Abstract

In this document, we begin looking at one of the final (and most important) topics in Calculus: *line integrals*. Line integrals are integrals over a path. Here, we look specifically at what to do when the integrand is a scalar function (not a vector field).

1 Introduction.

Have you ever wanted to know the area of one side of a fence which has uneven height? Probably not. But, just in case, Calculus provides a way to help you if you know a parametric representation of the path C of the fence (as seen from above) and a scalar function $f(x, y)$ which describes the fence's height at each point. The area of one side of the fence will be precisely

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

where a and b are the t -values for the start and end of the fence. It's not such a nice formula, and it means more of these kinds of integrals will be hard (if not impossible) to symbolically compute. But we don't have to stop there. Why not let f be a function of three variables? There's not as good of a geometric interpretation for this, but the definition remains consistent:

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

The right side of these two expressions are often shortened to $\int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$, but this isn't so helpful to memorize. Do keep it in mind, though.

2 Steps and Examples.

This is the general process:

1. If you haven't been given an equation for the curve C , find one. Put it into parametric form. (Note: there will be a z -component of the curve if and only if there's a z -component of the scalar function you're given).
2. Find the starting and ending t -values of the curve if you don't already have them.
3. Find $x'(t)$ and $y'(t)$. Also find $z'(t)$ if there's a z -component.
4. Note what f is. It will be the integrand of the given integral.
5. Plug the information from steps 1-4 into the line integral equations from the last page.
6. Integrate as normal. Use FTC.

Example 1

Find $\int_C \sqrt{x} ds$ where C is the curve given by $\mathbf{r}(t) = \langle t^2, t \rangle$ from $(0, 0)$ to $(1, 1)$.

Solution:

$x(t) = t^2, y(t) = t$ $x'(t) = 2t, y'(t) = 1$ <p>At $(0, 0)$, $t = 0$. At $(1, 1)$, $t = 1$.</p> $f(x, y) = \sqrt{x}$ $\text{So } \int_C \sqrt{x} ds = \int_C f(x, y) ds$ $= \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2} dt$ $= \int_0^1 \sqrt{t^2} \sqrt{(2t)^2 + (1)^2} dt$ $= \int_0^1 t \sqrt{4t^2 + 1} dt$ <p>Let $u = 4t^2 + 1 \implies du = 8t dt$</p>	<p>Find the parametric form of C.</p> <p>Find $x'(t)$ and $y'(t)$.</p> <p>Find the starting and ending t-values.</p> <p>Note what f is.</p> <p>Use the line integral equations.</p> <p>Plug x and y into f.</p> <p>Also plug in x', y', a, and b.</p> <p>Simplify.</p> <p>Integrate as normal. Here, we need a u-sub.</p>
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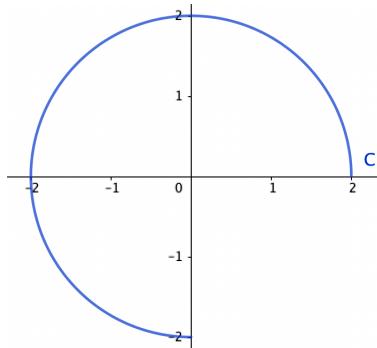
$$\begin{aligned}
 \text{Then } \int_0^1 t\sqrt{4t^2 + 1} dt &= \int_{u(0)}^{u(1)} \frac{1}{8}\sqrt{u} du && \text{Plug in } u \text{ and } du. \\
 &= \frac{1}{8} \left(\frac{2}{3}u^{\frac{3}{2}} \right) \Big|_{u(0)}^{u(1)} && \text{Re-substitute in for } u. \\
 &= \frac{(4t^2 + 1)^{\frac{3}{2}}}{12} \Big|_0^1 && \text{Use FTC.} \\
 &= \frac{5^{\frac{3}{2}} - 1}{12}
 \end{aligned}$$

Great. Hopefully that's decently easy to follow. It all comes down to following the steps and putting it all into the formula, so memorizing the formula is really important. Let's look at another example, one where we aren't explicitly given the equation of the curve.

Example 2

Find $\int_C x^3 y \, ds$ where C is the arc of the circle of radius 4 centered at the origin from 0 to $\frac{3\pi}{2}$ radians.

Solution: Let's start by getting a better picture of what our curve C is. C looks like this:



Where the curve is traveled counter-clockwise. To parameterize this, we'll use the classic parameterization of a circle, which you should definitely know if you don't already: $x(t) = r \cos(t)$, $y(t) = r \sin(t)$. Since $r = 4$, we get:

$$x(t) = 4 \cos(t), y(t) = 4 \sin(t)$$

Here's our parameterization.

$$x'(t) = -4 \sin(t), y'(t) = 4 \cos(t)$$

Find $x'(t)$ and $y'(t)$.

$$\text{At } 0 \text{ rad, } t = 0. \text{ At } \frac{3\pi}{2} \text{ rad, } t = \frac{3\pi}{2}.$$

Find the start and end t values.

$$f(x, y) = x^3 y$$

Note what f is.

$$\text{So } \int_C x^3 y \, ds = \int_C f(x, y) \, ds$$

Use the line integral equations.

$$= \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, dt$$

Plug in your info.

$$= \int_0^{\frac{3\pi}{2}} 64 \cos^3(t) \cdot 4 \sin(t) \sqrt{4 \sin^2(t) + 4 \cos^2(t)} \, dt$$

Simplify.

$$= 256 \int_0^{\frac{3\pi}{2}} \cos^3(t) \sin(t) \sqrt{4(\sin^2(t) + \cos^2(t))} \, dt$$

Now we have an integral to solve as usual.

$$= 512 \int_0^{\frac{3\pi}{2}} \cos^3(t) \sin(t) \, dt$$

Let $u = \cos(t) \implies du = -\sin(t) dt$

$$\begin{aligned} \text{So } &= 512 \int_0^{\frac{3\pi}{2}} \cos^3(t) \sin(t) dt = 512 \int_{u(0)}^{u(\frac{3\pi}{2})} u^3 \cdot -1 du \\ &= -512 \int_{u(0)}^{u(\frac{3\pi}{2})} u^3 du \\ &= -512 \frac{u^4}{4} \Big|_{u(0)}^{u(\frac{3\pi}{2})} \\ &= -128 \cos^4(t) \Big|_0^{\frac{3\pi}{2}} \\ &= 128 \end{aligned}$$

We need a u -sub here.

Sub in u and du .

Simplify.

Integrate.

Use FTC.

Great. As you can see, finding the equation of the curve usually isn't too bad, especially since really difficult curves are rarely encountered. The most common ones are circles and straight lines, so those are important to memorize. Speaking of which, let's look at an example where the curve is a straight line, and let's tackle a line integral in 3D space.

Example 3

Find $\int_C yz \, ds$ where C is the line segment from $(2, 0, 1)$ to $(3, 2, 5)$ followed by the line segment from $(3, 2, 5)$ to $(-1, 1, 2)$.

Solution: When you have two (or more) different curves you're integrating over, one after the other, the way to tackle the problem is to divide and conquer: find the parametric equation for each section and use the line integral equations separately. Then, combine them. Let's call the first line segment L_1 and the second line segment L_2 , and handle just L_1 for now.

To handle these line segments, we need to know how to parameterize line segments in general. Luckily, there's a pretty easy way to do so, given by $\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$ where \mathbf{r}_0 and \mathbf{r}_1 point to the starting and ending points, respectively. Then we take t from 0 to 1. Let's use this on L_1 :

$$\begin{aligned} L_1: x(t) &= 2(1 - t) + 3t = 2 + t \\ y(t) &= 2t \\ z(t) &= 1(1 - t) + 5t = 1 + 4t \end{aligned}$$

Find $x(t)$, $y(t)$, and $z(t)$.

$$x'(t) = 1, y'(t) = 2, z'(t) = 4$$

Find $x'(t)$, $y'(t)$, and $z'(t)$.

$$0 \leq t \leq 1$$

Find your start and end t -values.

$$f(x, y, z) = yz$$

Note what f is.

$$\text{So } \int_{L_1} yz \, ds = \int_{L_1} f(x, y, z) \, ds$$

Use the line integral equations.

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt$$

$$= \int_0^1 (2t)(1 + 4t) \sqrt{1^2 + 2^2 + 4^2} \, dt$$

Plug in your info.

$$= \sqrt{21} \int_0^1 2t + 8t^2 \, dt$$

Simplify.

$$= \sqrt{21} \left[t^2 + \frac{8}{3}t^3 \right]_0^1$$

Integrate.

$$\frac{11\sqrt{21}}{3}$$

Use FTC.

That takes care of L_1 . Now let's handle L_2 ; the process is identical.

$$\begin{aligned}L_2: \quad &x(t) = 3(1-t) - t = 3 - 4t \\&y(t) = 2(1-t) + t = 2 - t \\&z(t) = 5(1-t) + 2t = 5 - 3t\end{aligned}$$

$$x'(t) = -4, \quad y'(t) = -1, \quad z'(t) = -3$$

$$0 \leq t \leq 1$$

$$\begin{aligned}\text{So } \int_{L_2} yz \, ds &= \int_{L_2} f(x, y, z) \, ds \\&= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt \\&= \int_0^1 (2-t)(5-3t) \sqrt{(-4)^2 + (-1)^2 + (-3)^2} \, dt \\&= \sqrt{26} \int_0^1 10 - 11t + 3t^2 \, dt \\&= \sqrt{26} \left[10t - \frac{11}{2}t^2 + t^3 \right]_0^1 \\&= \frac{11\sqrt{26}}{2}\end{aligned}$$

Find $x(t)$, $y(t)$, and $z(t)$.

Find $x'(t)$, $y'(t)$, and $z'(t)$.

Find your start and end t -values.

Note what f is.

Use the line integral equations.

Plug in your info.

Simplify.

Integrate.

Use FTC.

That ends L_2 . Now, our original curve C was the combination of L_1 and L_2 , so we get

$$\begin{aligned}\int_C yz \, ds &= \int_{L_1} yz \, ds + \int_{L_2} yz \, ds \\&= \frac{11\sqrt{21}}{3} + \frac{11\sqrt{26}}{2} \\&= \frac{22\sqrt{21} + 33\sqrt{26}}{6}\end{aligned}$$

This is what we got for L_1 and L_2 .

Simplify if desired.

That's a lot to handle. The step up from 2D to 3D space shouldn't be too difficult: you handle it completely analogously to the 2D case. Having 2 different curves can be a lot, though. The key is to handle each curve separately and then put it together at the end. With enough work, that should get you to the answer.

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find $\int_C 1 \, ds$ where C is the curve given by $\mathbf{r}(t) = \langle 3t^2, t^3 \rangle$ from $(0, 0)$ to $(12, 8)$.

Answer 1 $2\frac{2}{9} - 8$.

Question 2 Find $\int_C x^2y \, ds$ where C is the line segment from $(1, 2)$ to $(3, -4)$.

Answer 2 $5 \cdot 10\frac{2}{3}$.

Question 3 Find $\int_C 3x + 4z \, ds$ where C is the line segment from $(1, 3, 5)$ to $(3, 5, 1)$ followed by the line segment from $(3, 5, 1)$ to $(2, 2, 2)$.

Answer 3 $\frac{8 \cdot 6\frac{2}{3} + 12\sqrt{11} + 27}{2}$.

Line Integrals of Vector Fields

Abstract

In this document, we continue our look at line integrals by looking at what a line integral of a vector field might mean and how to handle them.

1 Introduction.

Let's say there's a vector field associated with a force on an object. This could be the magnetic field, electric field, etc. And let's say we know what path an object follows through space, but we want to know what kind of work will be done on the object. This is precisely the kind of questions which integrals of vector fields are trying to solve.

Integrals over vector fields boil down to one equation to memorize: if you have a smooth curve C given by a vector function $\mathbf{r}(t)$ where t ranges from a to b , if the vector field \mathbf{F} is continuous and defined along C , then the line integral of \mathbf{F} along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

Keep in mind that since \mathbf{F} and \mathbf{r} are vectors, the dot in the equation above represents a dot product.

And how does this integral relate to work and forces? Well, if \mathbf{F} is the force field and $\mathbf{r}(t)$ describes the path taken by an object from $a \leq t \leq b$, then the line integral given by the equation above is precisely the work done on the object by the force. Let's now take a look at how to use this formula.

2 Steps and Examples.

This is the general process:

1. If you don't already have a parametrization $\mathbf{r}(t)$ of the given curve, find one. Put it in vector form.
2. Note what a and b are (your starting and ending values of t , respectively).
3. Find $\mathbf{r}'(t)$ in vector form.
4. Plug the components of \mathbf{r} in for their corresponding variables in \mathbf{F} . That is, find $\mathbf{F}(\mathbf{r}(t))$.
5. Plug that information into the equation on the last page. Integrate as usual. Use FTC.

Example 1

Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the upper half of the unit circle and \mathbf{F} is $\langle x^2, y^2 \rangle$.

Solution:

$x(t) = \cos(t), y(t) = \sin(t)$ $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ $a = 0$ and $b = \pi$ $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$ $\mathbf{F}(\mathbf{r}(t)) = \langle \cos^2(t), \sin^2(t) \rangle$ $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$ $= \int_0^\pi \langle \cos^2(t), \sin^2(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt$ $= \int_0^\pi \cos^2(t) \cdot (-\sin(t)) + \sin^2(t) \cos(t) dt$ $= \int_0^\pi \cos^2(t) \cdot (-\sin(t)) dt + \int_0^\pi \sin^2(t) \cos(t) dt$	This is the classic parametrization of a circle. Here it is in vector form. These are our t values for the top half. Find \mathbf{r}' . Plug the components of \mathbf{r} into \mathbf{F} . Use the line integral equation. Simplify.
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Let $u = \cos(t) \implies du = -\sin(t) dt$ | Make two separate u -substitutions.

Let $v = \sin(t) \implies dv = \cos(t) dt$ | Let's call the second one v .

$$\text{So } \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{u(0)}^{u(\pi)} u^2 du + \int_{v(0)}^{v(\pi)} v^2 dv$$

$$= \left[\frac{1}{3}u^3 \right]_{u(0)}^{u(\pi)} + \left[\frac{1}{3}v^3 \right]_{v(0)}^{v(\pi)}$$

$$= \left[\frac{1}{3}\cos^3(t) \right]_0^\pi + \left[\frac{1}{3}\sin^3(t) \right]_0^\pi$$

$$= 0$$

Integrate as usual.

Re-substitute u and v .

Use FTC.

Wow, that was quite a bit of work! Hopefully, though, it wasn't too hard to follow. Let's look at another example in 3D.

Example 2

Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the line segment from $(1, 2, 3)$ to $(6, 5, 4)$ and $\mathbf{F} = \langle z^2, x, y \rangle$.

Solution: First, we have to parameterize the curve C . Using the classic parametrization of a line segment, we get $x(t) = 1(1 - t) + 6t = 1 + 5t$, $y(t) = 2(1 - t) + 5t = 2 + 3t$, and $z(t) = 3(1 - t) + 4t = 3 + t$, where t ranges from 0 to 1. In vector form, this gives us $\mathbf{r}(t) = \langle 1 + 5t, 2 + 3t, 3 + t \rangle$. Now we can move on to step 3.

$\mathbf{r}(t) = \langle 1 + 5t, 2 + 3t, 3 + t \rangle$ $\mathbf{r}'(t) = \langle 5, 3, 1 \rangle$ $\mathbf{F}(\mathbf{r}(t)) = \langle (3 + t)^2, 1 + 5t, 2 + 3t \rangle$ $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$ $= \int_0^1 \langle (3 + t)^2, 1 + 5t, 2 + 3t \rangle \cdot \langle 5, 3, 1 \rangle dt$ $= \int_0^1 5(3 + t)^2 + 3(1 + 5t) + 2 + 3t dt$ $= \int_0^1 5(9 + 6t + t^2) + 3 + 15t + 2 + 3t dt$ $= \int_0^1 50 + 45t + 5t^2 dt$ $= \left[50t + \frac{45}{2}t^2 + \frac{5}{3}t^3 \right]_0^1$ $= \frac{445}{6}$	<p>This was our parametrization.</p> <p>Find $\mathbf{r}'(t)$.</p> <p>Plug the components of $\mathbf{r}(t)$ into \mathbf{F}.</p> <p>Use the line integral equation.</p> <p>Plug in your information from above.</p> <p>Simplify.</p> <p>Integrate as usual.</p> <p>Use FTC.</p>
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The step from 2D to 3D wasn't so bad, was it? The process is identical, it just gives you one extra term in your integral because of the dot product. Let's look at one final example, also in 3D, but phrased in terms of work and force.

Example 3

A supernatural electric field applies the following force to a proton with position (x, y, z) : $\mathbf{F} = \langle e^x, y, z^2 \rangle$. If a proton moves through the field in a straight line from $(1, 5, 2)$ to $(2, 1, 0)$, and from there in a straight line to $(3, 1, 4)$, what work will be done on the proton by the electric field?

Solution: Given the force field \mathbf{F} and the two line segments C_1 and C_2 parameterized by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, respectively, the work done on the proton will be $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

$\mathbf{r}_1(t)$, the parametrization of the first line segment C_1 , will be $\langle 1(1-t) + 2t, 5(1-t) + t, 2(1-t) \rangle$ or $\langle 1+t, 5-4t, 2-2t \rangle$ with $0 \leq t \leq 1$ as usual. Now let's continue with this first line segment C_1 .

$\mathbf{r}'(t) = \langle 1, -4, -2 \rangle$ $\mathbf{F}(\mathbf{r}(t)) = \langle e^{1+t}, 5-4t, (2-2t)^2 \rangle$ $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$ $= \int_0^1 \langle e^{1+t}, 5-4t, (2-2t)^2 \rangle \cdot \langle 1, -4, -2 \rangle dt$ $= \int_0^1 e \cdot e^t - 4(5-4t) - 2(2-2t)^2 dt$ $= \int_0^1 e \cdot e^t - 20 + 16t - 8 - 8t^2 + 16t dt$ $= \int_0^1 e^{t+1} - 8t^2 + 32t - 28 dt$ $= \left[e^{t+1} - \frac{8}{3}t^3 + 16t^2 - 28t \right]_0^1$ $= e^2 - \frac{8}{3} + 16 - 28 - e$ $= \frac{3e^2 - 3e - 44}{3}$ ≈ -10	<p>Find $\mathbf{r}'(t)$.</p> <p>Find $\mathbf{F}(\mathbf{r}(t))$.</p> <p>Use the line integral equation.</p> <p>Plug in the info from above.</p> <p>Simplify.</p> <p>Integrate.</p> <p>Use FTC.</p> <p>Simplify.</p>
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Now let's handle C_2 . The parametrization of C_2 will be $\mathbf{r}_2 = \langle 2(1-t) + 3t, 1, 4t \rangle$ or $\langle 2+t, 1, 4t \rangle$. Let's move on with this now:

$$\mathbf{r}'(t) = \langle 1, 0, 4 \rangle$$

$$\mathbf{F}(\mathbf{r}(t)) = \langle e^{2+t}, 1, (4t)^2 \rangle$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

$$= \int_0^1 \langle e^{2+t}, 1, 16t^2 \rangle \cdot \langle 1, 0, 4 \rangle dt$$

$$= \int_0^1 e^{2+t} + 4 \cdot 16t^2 dt$$

$$= \int_0^1 e^{2+t} dt + \int_0^1 64t^2 dt$$

$$= \left[\int e^{2+t} dt + \int 64t^2 dt \right]_0^1$$

Let $u = 2 + t \implies du = dt$

$$= \left[\int e^u du + \int 64t^2 dt \right]_{t=0}^{t=1}$$

$$= \left[e^u + \frac{64}{3}t^3 dt \right]_{t=0}^{t=1}$$

$$= \left[e^{2+t} + \frac{64}{3}t^3 \right]_0^1$$

$$= e^3 + \frac{64}{3} - e^2$$

$$= \frac{3e^3 - 3e^2 + 64}{3}$$

$$\approx 34.03$$

Find $\mathbf{r}'(t)$.

Find $\mathbf{F}(\mathbf{r}(t))$.

Use the line integral equation.

Plug in the info from above.

Simplify.

Make a u -sub in the first integral.

Sub in u and du .

Integrate.

Re-sub for u .

Use FTC.

Simplify if desired.

Then, for the final answer, the work done on the proton by the electric field will be the sum of the two different line integrals over C_1 and C_2 . So, our final answer will be

$$\frac{3e^2 - 3e + 20}{3} \approx 24.03.$$

3 Extra Exercises.

Now that you've hopefully gotten used to this method, here are some exercises with solutions, if you'd like them.

Question 1 Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is a segment of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ and $\mathbf{F} = \langle e^x, x^3 \rangle$.

Answer 1 $e - \frac{4}{3}$.

Question 2 Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle 1, y, x \rangle$ and C is the curve given by $\mathbf{r}(t) = \langle \sin(t), \cos(t), t^2 \rangle$ from $(0, 1, 0)$ to $(0, -1, \pi^2)$.

Answer 2 2π .

Question 3 An iron particle with position (x, y, z) travels on a straight line from $(0, 0, 0)$ to $(4, 2, 0)$ and from there to $(0, 6, 9)$. Find the work done on the particle at position traveling through a magnetic field which applies the following force to the particle: $\left\langle \frac{1}{1+x^2}, y, z \right\rangle$. (Hint: you may want to review integration by trigonometric substitution, or the anti-derivatives of inverse trig functions.)

Answer 3 $\frac{2}{117}$.