# 3 Inverse Problems Arising in a Mathematical Model of Homelessness

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# Introduction

In [1], O'Flaherty gives the following model for housing consumption:

$$\frac{dA}{dt} = (rA - c) + s(A)\frac{dw}{dt}, \tag{1}$$

| Variable | Interpretation |
|---|---|
| $t$ | Time (continuous) |
| $A = A(t)$ | Net assets of the household at time $t$ |
| $c = c(t)$ | Consumption of housing at time $t$ |
| $r$ | Rate of return on assets (constant) |
| $s(A)\frac{dw}{dt}$ | Random, unforseen changes in assets |

# Our model

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As in [1], this leads to the nonlinear differential equation

$$(c(A) - rA)c'(A) = \frac{\sigma^2}{2} c''(A) \tag{2}$$

where $r$ and $\sigma$ are the relevant parameters.

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# Noiseless data

Suppose that the true value of $q = \begin{bmatrix} r & \sigma \end{bmatrix}^T$ is $q^* = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^T$ and that $c(0) = c'(0) = 0.1$, yielding true solution $c^* = c(A; q^*)$.

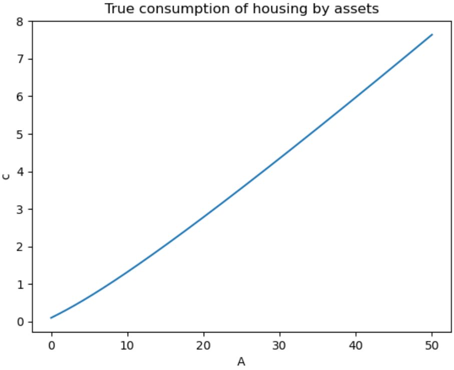


Figure: $c^*$ on $[0, 50]$

# Noiseless data

We observe $c^*$ on a uniform mesh $\{A_i\}_{i=1}^M \subseteq [0, 50]$, obtaining $d_i = c^*(A_i)$ for $i = 1, \ldots, M$. Given a candidate $q$, we define the residual $R(q) = \begin{bmatrix} R_1(q) & \cdots & R_M(q) \end{bmatrix}^T$ by

$$R_i(q) = |c(A_i; q) - d_i|.$$

Then, we define the least squares objective function by

$$T_M^h(q) = \frac{1}{2} R(q)^T R(q)$$

# Levenberg-Marquardt Implementation

Least squares optimization was done via a custom implementation of the Levenberg-Marquardt version of Newton's method:

github.com/NoahPrentice/Inverse-Problems-MTH655-W25

# Results of noiseless parameter ID

Obtained convergence to $q^*$ (residual less than $10^{-6}$) in fewer than 10 iterations for $M = 2^k$, $k = 5, \ldots, 12$ and an initial guess of $q_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

This is to be expected, since Levenberg-Marquardt is a robust method for nonlinear least squares problems whose solutions has $R = 0$.

# Results of noiseless parameter ID

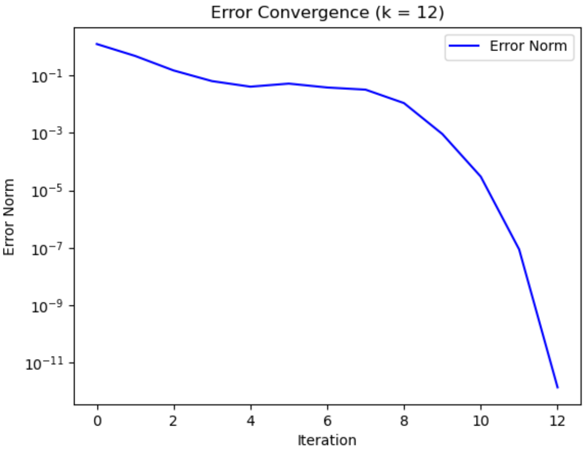


Figure: $\|q_n - q^*\|_2$ against the iteration count $i$

# Conclusions from noiseless parameter ID

The parameter identification methods discussed in the course were extremely successful for noiseless data and a nonlinear least squares objective.

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Because of fast convergence and unexpected short-term iteration behavior, estimates of convergence order are difficult to obtain.

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# Noisy data

Suppose $d_i = c^*(A_i) + \Sigma_i$, where $\Sigma_i \sim N(0, 10^{-4})$ for $i = 1, \ldots, M$.

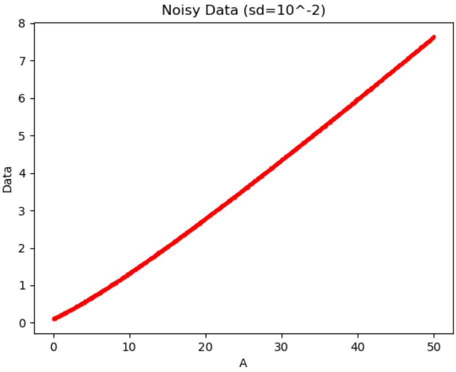


Figure: $d_i = c^*(A_i) + \Sigma_i$

# Isolated noise

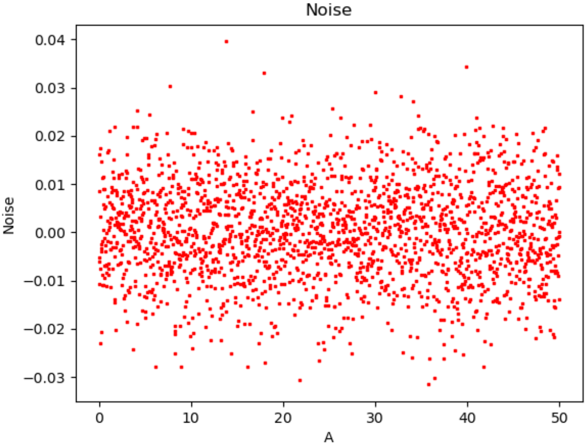


Figure: $\Sigma_i \sim N(0, 10^{-4})$

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# Noisy data parameter ID

As before, we define the residual $R(q) = \begin{bmatrix} R_1(q) & \cdots & R_M(q) \end{bmatrix}^T$ by

$$R_i(q) = |c(A_i; q) - d_i|,$$

and we use the same least squares objective as before.

$$T_M^h(q) = \frac{1}{2} R(q)^T R(q)$$

# Noisy data parameter ID

After 1000 Levenberg- Marquardt iterations on the same grids, we obtain:

| $k = \log_2(M)$ | $\bar{r}_M^h$ | $\bar{\sigma}_M^h$ | $\|q_M^h - q^*\|_2$ |
|---|---|---|---|
| 5 | 0.8325 | 0.0374 | 7.352e-1 |
| 6 | 0.6519 | 0.0690 | 5.528e-1 |
| 7 | 0.0991 | 0.1006 | 1.039e-3 |
| 8 | 0.0977 | 0.1013 | 2.621e-3 |
| 9 | 0.0991 | 0.1005 | 9.648e-4 |
| 10 | 0.0992 | 0.1005 | 8.829e-4 |
| 11 | 0.0993 | 0.1004 | 8.324e-4 |
| 12 | 0.0999 | 0.1000 | 7.679e-5 |

# Error in noisy parameter ID

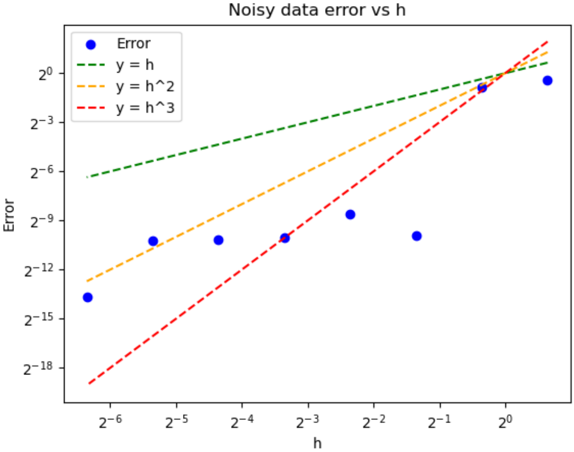


Figure: $\|q_M^h - q^*\|_2$ when $\Sigma_i \sim N(0, 10^{-4})$

# Effect of noise

Parameter identification was still successful with when the data contains a small amount of noise, though grid refinement yields unpredictable changes in error.

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How do the methods cope with an increased noise level, say $10\times$ the noise?

# 10× the noise

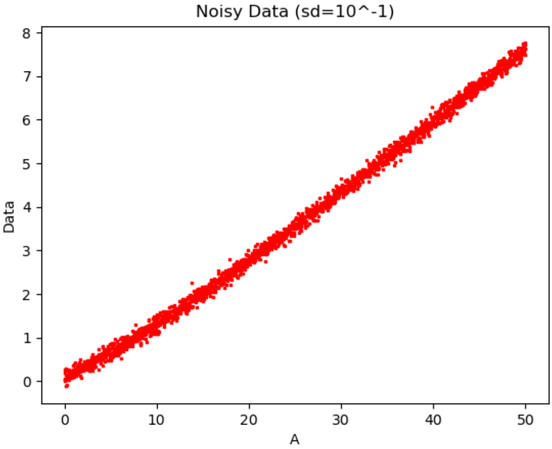


Figure: $d_i$ when $\Sigma_i \sim N(0, 10^{-2})$

# Effect of noise

If $\Sigma_i \sim N(0, 10^{-2})$:

| $k = \log_2(M)$ | $\bar{r}^h_M$ | $\bar{\sigma}^h_M$ | $\lVert q^h_M - q^* \rVert_2$ |
|---|---|---|---|
| 5 | 0.1195 | 0.0631 | 4.180e-2 |
| 6 | 0.1115 | 0.0828 | 2.073e-2 |
| 7 | 0.0998 | 0.1007 | 6.834e-4 |
| 8 | 0.0871 | 0.1149 | 1.970e-2 |
| 9 | 0.0886 | 0.1132 | 1.742e-2 |
| 10 | 0.0924 | 0.1091 | 1.188e-2 |
| 11 | 0.0955 | 0.1056 | 7.122e-3 |
| 12 | 0.0970 | 0.1037 | 4.728e-3 |

# Effect of noise

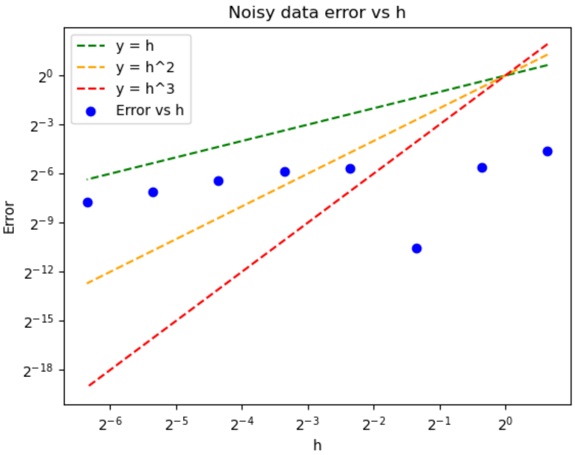


Figure: $\|q_M^h - q^*\|_2$ when $\Sigma_i \sim N(0, 10^{-2})$

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# Model validation

Suppose we want to know if $\sigma = 0$, i.e., we want to test if randomness is in fact present in the consumption of housing as modeled by [1]. Take

$$Q_{ad} = \mathbb{R}^2, \quad Q_0 = \mathbb{R} \times \{0\}$$

and formulate the null and alternative hypotheses

$$H_0 : \; q^* \in Q_0 \subseteq Q_{ad},$$

$$H_a : \; q^* \in Q_{ad} \setminus Q_0.$$

We test at an $\alpha = 0.05$ significance level.

# Hypothesis testing

To perform the hypothesis test, we take the $Q_{ad}$ minimizer $\bar{q}_M^h = \operatorname{argmin}_{q \in Q_{ad}} T_M^h(q)$ from the parameter identification section and compare against the $Q_0$-minimizer $\hat{q}_M^h = \operatorname{argmin}_{q \in Q_0} T_M^h(q)$ by constructing the test statistic

$$U_M^h = \frac{T_M^h(\hat{q}_M^h) - T_M^h(\bar{q}_M^h)}{T_M^h(\bar{q}_M^h)}.$$

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$$U_M^h = \frac{T_M^h(\hat{q}_M^h) - T_M^h(\bar{q}_M^h)}{T_M^h(\bar{q}_M^h)}.$$

We know that $U_M^h$ converges in distribution to $\chi^2(1)$ as $M \to \infty$ and $h \to 0$, so we compute the test statistic for the noisy data and compare to the 95th percentile.

# Result of model validation

$\Sigma_i \sim N(0, 10^{-4})$ yields $U_M^h > 150$ for all $M$ values tested above, allowing us to reject the null hypothesis in favor of the alternative; in this case, we can conclude with 95% confidence that $\sigma \neq 0$.

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When the effect of noise is greater, such as when $\Sigma_i \sim N(0, 10^{-2})$, we fail to reject the null hypothesis. This is because $\bar{q}_M^h$ is very sensitive to noise as demonstrated in the parameter identification section.

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# Conclusion

Despite the negative impacts of noise on the implemented numerical methods, the methods are robust enough to achieve relatively good estimates of the model parameters. This changes in the context of model validation, where the effect of noise is more detrimental.

# Thank you!

Thank you for listening.

I can now take any additional questions.

# References

[1] B. O'Flaherty, "Individual homelessness: Entries, exits, and policy," Journal of Housing Economics, vol. 21, no. 2, pp. 77–100, 2012. [Online]. Available: https://www.sciencedirect.com/-science/article/pii/S1051137712000277