MTH 655 Final Project

Noah Prentice

20 March 2025

Introduction

In [1], O'Flaherty models a household's assets and housing consumption through a time-invariant Ito process:

$$\frac{dA}{dt} = (rA - c) + s(A)\frac{dw}{dt}, \tag{1}$$

where we interpret the variables as follows:

Variable	**Interpretation**
t	Time (continuous)
$A = A(t)$	Net assets of the household at time t
$c = c(t)$	Consumption of housing at time t
r	Rate of return on assets (constant)
$s(A)\frac{dw}{dt}$	Random, unforseen changes in assets

In particular, w represents Brownian motion and $s(A)$ is a function that varies depending on the particular assumptions of the model.

For the particular model here, we assume that $s(A) = \sigma \in \mathbb{R}$. We treat c as a function of A by assuming the household consumes housing so as to maximize expected lifetime value according to the utility function $U(c) = c - bc^2$, where $b \in \mathbb{R}$. As discussed in [1], this leads to solving the nonlinear differential equation

$$(c(A) - rA)c'(A) = \frac{\sigma^2}{2}c''(A) \tag{2}$$

for $c(A)$. Thus r and σ are the relevant parameters for determining $A(t)$ and $c(t)$. In particular we want to know if $A(t) = 0$ or $c(t) = 0$ for some t: either case is interpreted as the household falling into homelessness at time t.

We will first perform basic parameter identification for the parameters r and σ using simulated data, which will require an implementation of least-squares minimization. Then, we will consider the effect of noise on the parameter identification problem, as well as on model validation.

Noiseless data

Suppose that the true value of $q = \begin{bmatrix} r & \sigma \end{bmatrix}^T$ is $q^* = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^T$ and that $c(0) = c'(0) = 0.1$, yielding true solution $c^* = c(A; q^*)$ as given in Figure 1.

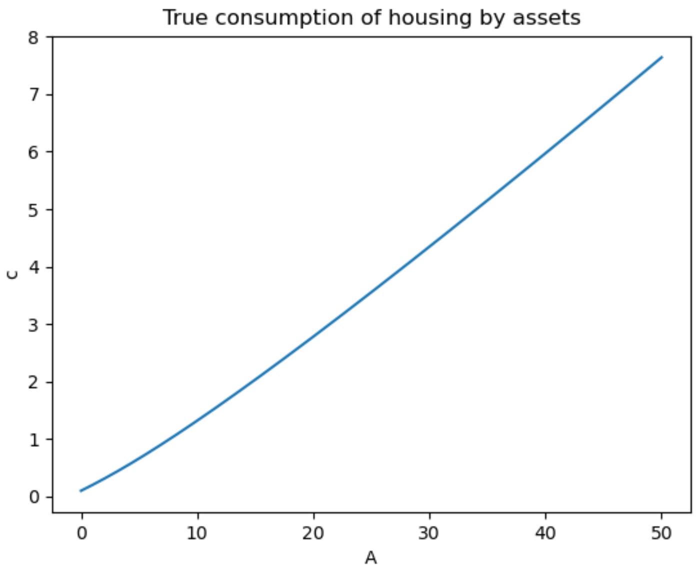

Figure 1: c^* on $[0, 50]$

We perform parameter identification to find q^*: we observe c^* on a uniform mesh $\{A_i\}_{i=1}^M \subseteq [0, 50]$, obtaining $d_i = c^*(A_i)$ for $i = 1, \dots, M$. Given a candidate q, we define the residual $R(q) = \begin{bmatrix} R_1(q) & \cdots & R_M(q) \end{bmatrix}^T$ by

$$R_i(q) = |c(A_i; q) - d_i|.$$

Then, we define the objective function[1] by

$$\begin{aligned} T_M^h(q) &= \frac{1}{2} R(q)^T R(q) \\ &= \frac{1}{2} \sum_{i=1}^{M} |c(A_i; q) - d_i|^2. \end{aligned}$$

We want to find

$$\bar{q}_M^h = \operatorname*{argmin}_{q \in \mathbb{R}^2} T(q).$$

This is an unconstrained optimization problem with a non-linear least squares objective, hence we implement Newton's method for minimization via Levenberg-Marquardt:

[1]Note that this objective function has been discretized as a result of discretizing the domain.

```python
def build_residual(q):
    test_solution = simulate_ode(q) # ODE simulation with scipy.integrate
    return test_solution.sol(A_values)[0] - data

def build_jacobian(q, epsilon=1e-8):
    jacobian = np.zeros((len(A_values), len(q)))
    for i in range(len(q)):
        q_perturbed = q.copy()
        q_perturbed[i] += epsilon
        R_perturbed = build_residual(q_perturbed)
        R_original = build_residual(q)
        jacobian[:, i] = (R_perturbed - R_original) / epsilon
    return jacobian

def find_q_with_lm(initial_q, initial_nu=1, tolerance=1e-6,
    max_iterations=100):
    q = initial_q
    nu = initial_nu
    for _ in range(max_iterations):
        # Compute residual, test for convergence
        residual = build_residual(q)
        residual_norm = np.linalg.norm(residual)
        if residual_norm < tolerance:
            break

        # Otherwise, take a Newton step and update nu
        jacobian = build_jacobian(q)
        hessian = jacobian.T @ jacobian
        hessian += nu * np.eye(hessian.shape[0])
        gradient = jacobian.T @ residual
        step = np.linalg.solve(hessian, -gradient)
        q += step
        nu = initial_nu * np.linalg.norm(residual)
    return q
```

This is a basic Levenberg-Marquardt implementation using the algorithm given in class. We have implemented it so that the regularization parameter ν_k is $O(\|R\|)$; this should enforce quadratic convergence to the minimum.

We run the code for $M = 2^k$, $k = 5, \ldots, 12$ with an initial guess of $q_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. In all cases, the code converged to q^* (residual less than 10^{-6}) in fewer than 10 iterations. This is to be expected, since Levenberg-Marquardt is a robust method for nonlinear least squares problems whose solutions has $R = 0$. We check if the convergence is quadratic by fixing k and plotting the error in the Levenberg-Marquardt iterates q_n against the iteration count n, as in Figure 2.

Since the plot uses a logarithmically-scaled vertical axis, quadratic convergence would yield a straight line, but this is not what is shown. Instead, we seem to converge sub-quadratically

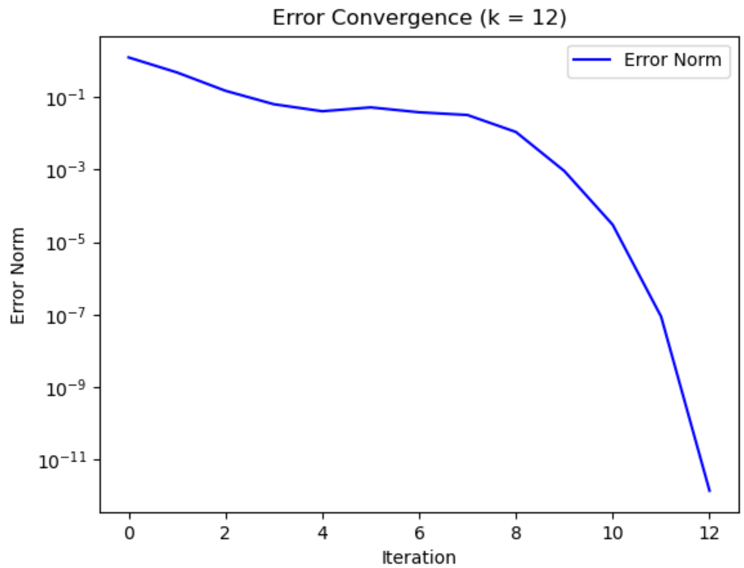

Figure 2: $\|q_n - q^*\|_2$ against the iteration count i

towards the beginning of the iteration, until eventually the convergence becomes super-quadratic.

Noisy data

Parameter identification

Now suppose that our observed data is noisy, so that $d_i = c^*(A_i) + \Sigma_i$, where $\Sigma_i \sim N(0, 10^{-4})$ for $i = 1, \ldots, M$. We can define the residual R just as for noiseless data and perform 1000 iterations of Levenberg-Marquardt. Doing so for the same discretizations yields the following results:

$k = \log_2(M)$	$\bar{r}_M^h$	$\bar{\sigma}_M^h$	$\|q_M^h - q^*\|_2$
5	0.8325	0.0374	7.352e-1
6	0.6519	0.0690	5.528e-1
7	0.0991	0.1006	1.039e-3
8	0.0977	0.1013	2.621e-3
9	0.0991	0.1005	9.648e-4
10	0.0992	0.1005	8.829e-4
11	0.0993	0.1004	8.324e-4
12	0.0999	0.1000	7.679e-5

The plot of the error against $h = 50/M$ is shown in Figure 3.

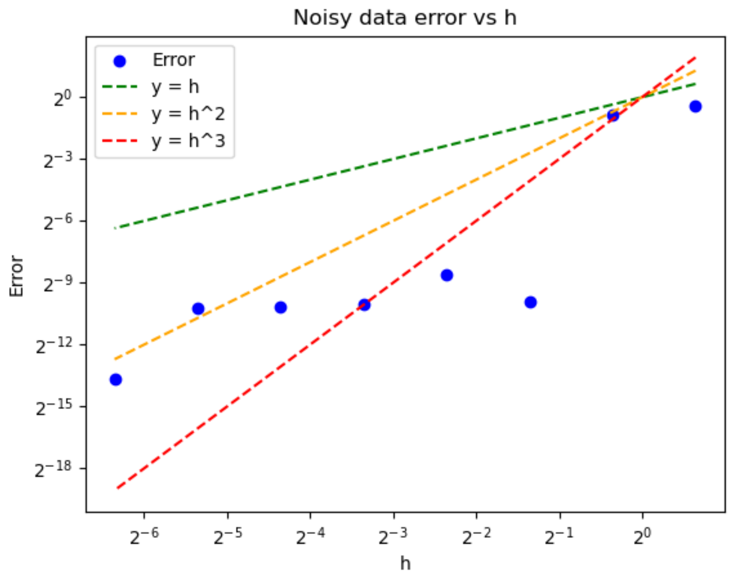

Figure 3: $\|q_M^h - q^*\|_2$ when $\Sigma_i \sim N(0, 10^{-4})$

It appears that the error is discontinuous with respect to the discretization parameter h as there is a large difference in error between $k = 6$ and $k = 7$. The error appears to be decreasing roughly linearly for $k = 7, \ldots, 12$, but the trend is unclear.

To test the sensitivity of these results to noise, we now suppose that $\Sigma_i \sim N(0, 10^{-2})$, so that the effect of noise on the observation is 10 times larger. This yields the following results:

$k = \log_2(M)$	$\bar{r}_M^h$	$\bar{\sigma}_M^h$	$\|q_M^h - q^*\|_2$
5	0.1195	0.0631	4.180e-2
6	0.1115	0.0828	2.073e-2
7	0.0998	0.1007	6.834e-4
8	0.0871	0.1149	1.970e-2
9	0.0886	0.1132	1.742e-2
10	0.0924	0.1091	1.188e-2
11	0.0955	0.1056	7.122e-3
12	0.0970	0.1037	4.728e-3

The error is plotted against $h = 50/M$ in Figure 4. It is clear that the increased presence of noise amplifies the overall error: 10 times the noise yielded 10 times the error, on average. Also, while a discretization of $k = 7$ yielded the best error for the noisier data, it is likely that this is due to random chance since other simulations did not yield the same result.

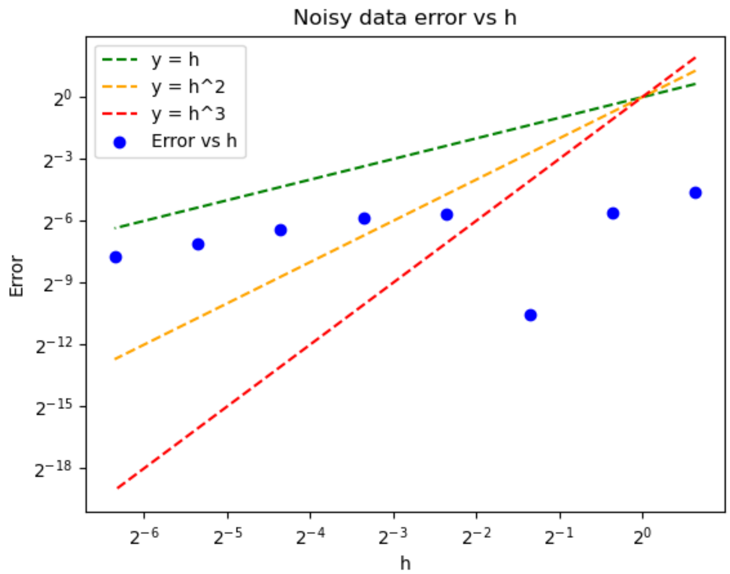

Figure 4: $\|q_M^h - q^*\|_2$ when $\Sigma_i \sim N(0, 10^{-2})$

Model validation

Suppose we want to know if $\sigma = 0$, i.e., we want to test if randomness is in fact present in the consumption of housing as modeled by the Ito process in (1). By default, we have that the set of admissible values for q is $Q_{ad} = \mathbb{R}^2$. We let $Q_0 = \mathbb{R} \times \{0\}$ be the set in which $\sigma = 0$ and formulate the null hypothesis

$$H_0: \; q^* \in Q_0 \subseteq Q_{ad},$$

which we test against the alternative hypothesis

$$H_a: \; q^* \in Q_{ad} \setminus Q_0$$

at an $\alpha = 0.05$ significance level. To perform the hypothesis test, we take $\bar{q}_M^h = \operatorname{argmin}_{q \in Q_{ad}} T_M^h(q)$ from the parameter identification section and compare against the Q_0-minimizer $\hat{q}_M^h = \operatorname{argmin}_{q \in Q_0} T_M^h(q)$ by constructing the test statistic

$$U_M^h = \frac{T_M^h(\hat{q}_M^h) - T_M^h(\bar{q}_M^h)}{T_M^h(\bar{q}_M^h)}.$$

We know that U_M^h converges in distribution to $\chi^2(1)$ as $M \to \infty$ and $h \to 0$, so a 95th percentile for U_M^h is ≈ 3.84. Computing the test statistic with $\Sigma_i \sim N(0, 10^{-4})$ yields $U_M^h > 150$ for all M values tested above, allowing us to reject the null hypothesis in favor of the alternative; in this case, we can conclude with 95% confidence that $\sigma \neq 0$.

When the effect of noise is greater, such as when $\Sigma_i \sim N(0, 10^{-2})$, we fail to reject the null hypothesis. This is because $\bar{q}_M^h$ is very sensitive to noise as demonstrated in the parameter identification section.

Conclusion

Despite the negative impacts of noise on the implemented numerical methods, the methods are robust enough to achieve relatively good estimates of the model parameters. This changes in the context of model validation, where the effect of noise is more detrimental.

References

[1] B. OFlaherty, "Individual homelessness: Entries, exits, and policy," *Journal of Housing Economics*, vol. 21, no. 2, pp. 77–100, 2012. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S1051137712000277