MTH 654-2024 Homework A1. **FirstName LastName/Group**
Date: October 2, 2024 (Group member names)
Prof. M. Peszynska (Group member names)

Please try all (1-4), but turn in at least 2 pbms including (1,4).

Instructions: Your answers should be concise but also provide sufficient insight into the algorithms and theory rather than be "*a bunch of plots* stapled together. Avoid showing `obvious code snippets` and ELEMENTARY calculations. Code in any language, but using MATLAB might save you time and allow Instructor's support.

Problem 1 (+10). Consider the ODE

$$u' = u - u^3,\ u(u) = u_{init}. \tag{1}$$

Explore the stability of FE, BE and CS schemes for (1) allowing variable time step τ.
i) Write your code that allows variable τ (show me snippets for time stepping). For nonlinear solver, use `fzero` or see Pbm 2. Test your code on some ODE with an easy analytical solution. Also, you can compare to `ode45` (do not turn these tests in).
(ii) Experiment with $u_{init} = 2$, 0.1 or 0.9, with $\tau = 0.1, \tau = 0.9$ or even $\tau = 2$. (For each case run until $T = 5$.)
Report if the numerical solution for FE, BE, and CS is monotone and energy stable (should it be? Illustrate with plots or tables and theory if possible.

Solution:

Problem 2 (+10). Consider $F(u) = u^3 - u$. Use the Newton iteration to find its roots, i.e, the points where $F(u) = 0$. These roots can (perhaps) be also found with a fixed point set-up $u = G(u)$ (choose $G(u)$ wisely). Apply the theory and check what happens in practice.

A region of attraction is a region (set of initial guesses $u^{(0)}$) for which the algorithm for Newton converges quadratically. For fixed point iteration, this is a region in which the fixed point iteration converges. The attraction regions can be empty.

(i) Is the problem $F(u) = 0$ or $u = G(u)$ well-posed?
(ii) Find the attraction region for Newton iteration around the root $u^* = 1$ with initial guess $u^{(0)}$ by experimentation; compare with the theoretical estimates.
(iii) Do the same for fixed point iteration.
Extra: (iv) Attempt the "pseudo-transient continuation", e.g., apply Forward Euler method to the ODE $u' + F(u) = 0, u(0) = 2$ to advance the initial guess u_0 towards the root. Compare to what happens if you try instead the ODE $u' - F(u) = 0, u(0) = 2$? Explain what you see in practice, and what theory can explain what you see. How would you choose the time step in the Forward Euler method to succeed.

Solution:

Problem 3 (+10). Let $F(x) = (x_1^2 + x_2^2 - 2; \exp(x_1 - 1) + x_2^3 - 2)$. which has a root $x_* = (1, 1)$. Plot the region of attraction for Newton. (Grid the region around the root and mark the initial guesses, for which the code converges quadratically, say in 5 iterations; e.g, use `scatter` in MATLAB). **Hint:** $x^{(0)} = (1.088, 1.257)$ is a good initial guess. **Extra:** With $x^{(0)} = (2, 0.5)$, you would need backtracking. Implement backtracking and show that it can be successful (report on the iterations).

Solution:

Problem 4 (+10). Use the provided code `ELLIPTIC1d.m` for FV in 1D for

$$-(ku_x)_x = f, \; x \in (0, 1) \tag{2}$$

which allows Dirichlet or Neumann boundary conditions.

(Get acquainted with the code, try some easy examples when $k = 1$, and $u(x)$ is known: do not turn this in).

(i) Let $u(x) = e^{10x^2}/e^{10}$. Test using Dirichlet and Neumann boundary conditions with an appropriate "manufactured" f. Test convergence in $\| \cdot \|_{\Delta,2}$ on a sequence of uniform grids $h = 5, 10, 50, 100$. Determine an optimal non-uniform grid. (Be brief. I do not need to see the code unless you have difficulties).

(ii) Describe how to extend the algorithm (give the discrete equations) from (2) to

$$-(ku_x)_x + cu = f, x \in (0, 1) \tag{3}$$

Implement and test as in (i).

(iii) Calculate an analytical solution $u^{exact}(x)$ to the flow problem (2) when $u(0) = 1, u(1) = 0, f \equiv 0$ and when $k(x) = k_{left} = 1, x < a$ and $k(x) = k_{right} = 10, x > a$ with $a = 1/3$. Implement the numerical solution using well chosen grid with $M = 5$ and $M = 50$, and compare with u^{exact}. What do you see? What does the theory say?

Extra: calculate the flux out of the domain at $x = 1$ (you need to code this). This flux gives *an upscaled* value $\hat{k}$, a weighted harmonic average of k_{left}, k_{right}. Check!

Solution: