**MTH 654-2024 Homework C1.** **Noah Prentice**
Date: January 3, 2025
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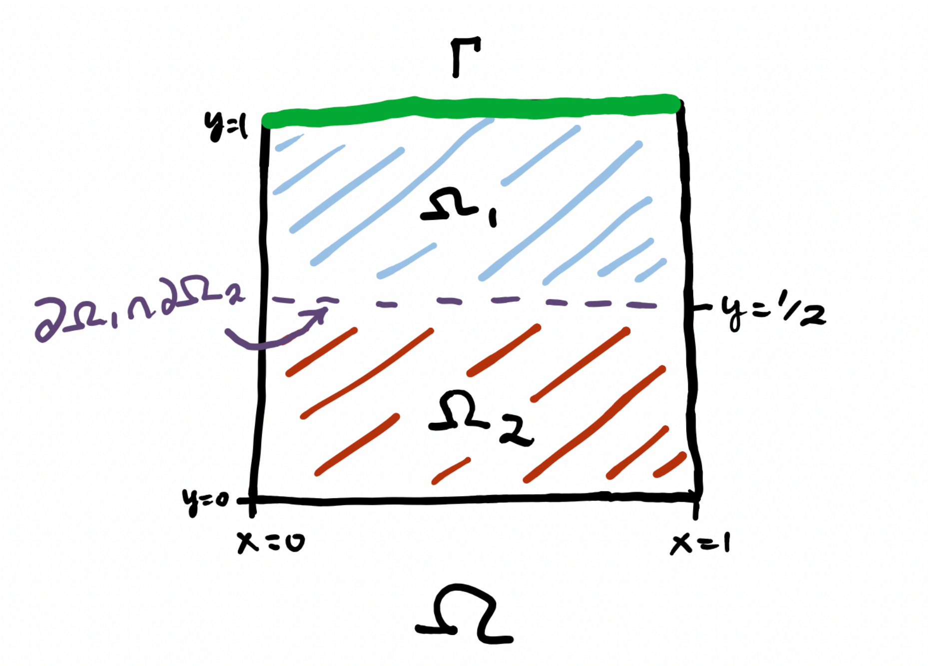
For the project in part C of the course, I plan on working on an optimal control problem. In particular, I plan on working on a 2D boundary control-problem governed by the heat equation.

Let $\Omega_1 = (0,1) \times (1/2, 1)$ be the top half of the unit square, $\Omega_2 = (0,1) \times (0, 1/2)$ the bottom half, $\Omega = (0,1) \times (0,1)$ the unit square, and $\Gamma = (0,1) \times \{1\}$ the top boundary.




We consider temperatures $u : \Omega_1 \times [0,T] \to \mathbb{R}$ and $v : \Omega_2 \times [0,T] \to \mathbb{R}$ governed by the coupled PDEs

$$\begin{cases} u_t - k_1 \Delta u = f & \text{on } \Omega_1, \\ -k_2 \Delta v = g & \text{on } \Omega_2, \\ u_y = -v_y & \text{on } \partial\Omega_1 \cap \partial\Omega_2, \end{cases}$$

with additional boundary conditions and initial conditions as necessary, for instance, Dirichlet and Neumann boundary conditions for $u$ and $v$ on $\partial\Omega \setminus \Gamma$, and initial conditions $u|_{t=0} = u_{init}, v|_{t=0} = v_{init}$.[1]

The control problem, then, involves controlling $u$ on the piece $\Gamma$ of the boundary $\partial\Omega_1$ so that $v$ is as close as possible (for instance, in the $L^2$ norm) to a desired temperature distribution $v^* : \Omega_2 \to \mathbb{R}$. Thus boundary conditions are specified for $u$ only on $\partial\Omega_1 \setminus \Gamma$.

The primary references used will be Ulbrich's *Semismooth Newton Methods for Variational Inequalities and Constrained Optimization Problems in Function Spaces* and the lecture notes "Optimal Control of Partial Differential Equations" by Dr. Peter Philip of Ludwig-Maximilians University (LMU) Munich, available online here.

[1]This particular problem was motivated by a discussion with the instructor, but could easily be modified if needed. These initial/boundary conditions, for instance, were guesses at what might create a well-posed problem, but may need to be revised.