Lab 2

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Note.

Code for this assignment was done in Python. Only important outputs are shown here, but complete files can be found at this GitHub repository.[a] All code for Exercise N is in `ExN.py`.

[a] URL: https://github.com/NoahPrentice/Numerical-Linear-Algebra-MTH551-F24/tree/main/Lab2

Exercise 1. For each of the matrices, using $m = 20$, do the following:

(a) Compute the singular values.

(b) Plot the singular values.

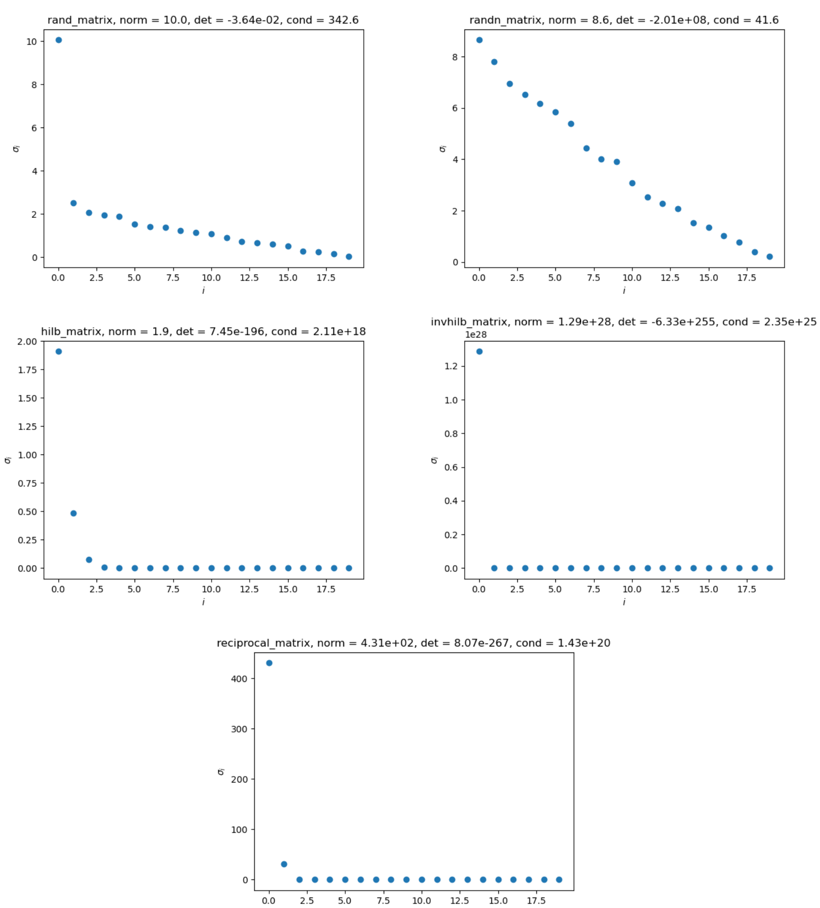
(c) Calculate various standard measures of matrices including `norm`, `det`, and `cond`.

Solution. The singular values of the matrices are:

`rand_matrix`	`randn_matrix`	`hilb_matrix`
10.047881867032626	8.643658689768936	1.907134720407253
2.5083099957036796	7.802343355897271	0.48703840657204867
2.0660149939920567	6.935430502726022	0.07559582130544103
1.9613059066082739	6.507148202380013	0.008961128614856484
1.904731964041219	6.155574880654011	0.0008676711091714912
1.5350170099774882	5.8414584157817036	7.033431473193452e-05
1.4000308859475652	5.387482926161078	4.830510048801921e-06
1.372130436668825	4.4388646279226736	2.827652055261804e-07
1.2310802844240314	4.010111203492225	1.4139547585415592e-08
1.131600217392952	3.901235651233231	6.036095329835451e-10
1.0841064378254488	3.0827102377388145	2.1928899740584974e-11
0.9079179947913204	2.532658601846031	6.740811876720582e-13
0.7145240830781547	2.274981291149573	1.7383178906896263e-14
0.6632399275045004	2.0716105818907273	3.735843783651779e-16
0.6016256373235976	1.531850935454308	1.404802420251727e-17
0.517582384624043	1.3449545779710121	1.1844225365325005e-17
0.2854889126953595	1.029373808929732	7.074389562107556e-18
0.24206395075179518	0.7725526432352972	3.840301693572978e-18
0.15468530682661466	0.3914068335621167	2.557327728003468e-18
0.029329929593921243	0.2076502176036207	9.053439614729207e-19

invhilb_matrix	reciprocal_matrix
1.2857804745391373e+28	430.8679276123039
2.9618178783065674e+25	30.86792761230391
1.4231954011036262e+23	3.8746695163071604e-14
1.0716742028442323e+21	7.957782698557319e-15
1.126116490939403e+19	3.507797096299257e-15
1.5511657937536413e+17	2.3128133898516145e-15
2694709822590430.0	1.5848462582619237e-15
57574457107308.23	1.227174201984047e-15
1469301827468.033	9.045859333282394e-16
509208449193.51447	8.324021266974405e-16
199573521094.30762	7.382491019511882e-16
53952305628.26082	5.808580026008901e-16
24306079637.82468	4.874371480428449e-16
5952739801.353444	3.18524143392855e-16
3617552348.6599727	2.608475881655981e-16
391745400.3549331	2.3315986435997126e-16
72146663.28400095	1.444566622337276e-16
3704632.125282324	5.0274764005188504e-17
274557.81057984795	2.8172463219286964e-17
547.75727440046	3.022245466304265e-18

Plotting these, with `norm`, `det`, and `cond` yields:



□

Exercise 2. Considering the plots you just made, which type of matrix is likely the best candidate for a *low-rank* approximation (with respect to the relative difference in the 2-norm, i.e., $\|A - A_\nu\|_2/\|A\|_2$)? Which is the worst candidate? Justify your answers.

Solution. The best candidate for a low-rank approximation is likely the the inverse Hilbert matrix. This is because the relative 2-norm error of a rank k approximation to A is $\frac{\sigma_{k+1}}{\sigma_1}$, and because $\sigma_1 \gg \sigma_k + 1$ for all k, as seen in the plot of its singular values. For similar reasons, the matrix with normally distributed entries will likely be the worst candidate for low rank approximation, as its singular values appear to be roughly uniformly distributed their range, and thus $\sigma_{k+1} - \sigma_1$ is not particularly large for small k. □

Exercise 3. Considering the measures you computed, including norm, determinant, and condition number, is there one measure that consistently determines a good candidate for low-rank approximation? If so, which one? If not, why not? Same question for determining a bad candidate for low-rank approximation.

Solution. By observing the plots of the matrices' singular values, it seems to be that condition number is a good measure of low-rank approximability. In general, the better candidates for low-rank approximation—that is, the matrices whose singular values fall off steeply—have a large condition number, while the worse candidates for low-rank approximation have a small condition number.[1] □

Exercise 5. The singular values of `randn(m)` seem to (roughly) follow a specific pattern. Explain how this might be exploited to compute an estimate to the Frobenius norm given m and the largest singular value.

Solution. Since the singular values are roughly uniformly distributed over their range, we can estimate

$$\sigma_i \approx \sigma_1 \cdot \left(1 - \frac{i-1}{m-1}\right).$$

Then the Frobenius norm becomes

$$\begin{aligned}
\|A\|_F &= \left(\sum_{i=1}^{m} \sigma_i^2\right)^{1/2} \\
&\approx \left(\sum_{i=1}^{m} \left(\sigma_1 \left(1 - \frac{i-1}{m-1}\right)\right)^2\right)^{1/2} \\
&= \sigma_1 \left(\sum_{i=1}^{m} \left(1 - \frac{i-1}{m-1}\right)^2\right)^{1/2} \\
&\approx \sigma_1 \left(\int_0^m \left(1 - \frac{x}{m}\right)^2 dx\right)^{1/2} \\
&= \sigma_1 \sqrt{\frac{m}{3}}.
\end{aligned}$$

□

[1] We may also measure the "steepness" of the plot of singular values by measuring $\max_k |\sigma_{k+1} - \sigma_k|$.