MTH 451/551 – Lab 2

The MATLAB command `svd` computes the SVD of a matrix, e.g., `[U,S,V]=svd(A)`. Type `help svd` to see the syntax and `doc svd` for an example. Consider the following types of matrices pre-defined in MATLAB:

(i.) `rand(m)` is a random $m \times m$ matrix with entries $\sim U(0,1)$

(ii.) `randn(m)` is a random $m \times m$ matrix with entries $\sim \mathcal{N}(0,1)$

(iii.) `hilb(m)` is the $m \times m$ Hilbert matrix

(iv.) `invhilb(m)` is the inverse of the $m \times m$ Hilbert matrix

(v.) `1./hilb(m)` is the $m \times m$ matrix defined by $A_{i,j} = i + j - 1$

Display these matrices for $m = 5$ in order to get an idea of what they look like. (Do not turn in.)

1. For each of the matrices above, using $m = 20$, do the following in MATLAB:
   (a) (5 points) Compute the singular values (e.g., `s=svd(hilb(20))`) Note: use `format long` to display many digits.
   (b) (5 points) Plot the singular values (e.g., `plot(s,'-o')`).
   (c) (15 points) Calculate various standard measures of matrices including `norm, det, cond` and possibly others (e.g., consider using `title(sprintf('Hilbert(20):  norm=%g, det=%g, cond=%g',norm(A),det(A),cond(A)))` to display this output on your plot)
2. (10 points) Considering the plots you just made, which type of matrix is likely the best candidate for a *low-rank* approximation (with respect to the relative difference in the 2−norm, i.e., $\|A - A_\nu\|_2 / \|A\|_2$? Which is the worst candidate? Justify your answers.
3. (5 points) Considering the measures you computed, including norm, determinant and condition number, or something else, is there one measure that consistently determines a good candidate for low-rank approximation? If so, which one? If not, why not? Same question for determining a bad candidate for low-rank approximation.
4. **451:** (5 points) The singular values of the Hilbert matrix are best displayed on a log scale. Try `figure; s=svd(hilb(20)); semilogy(s,'-o')` . The curve seems to be smooth until about the 15^{th} value. What may be happening here?

5. **551:** (5 points) The singular values of `randn(m)` seem to (roughly) follow a specific pattern. Explain how this might be exploited to compute an estimate to the Frobenius norm given m and the largest singular value. (Hint:

$$\sum_{i=1}^{m} \sigma_i^2 \approx \sum_{i=1}^{m} \left[\sigma_1 \left(1 - \frac{i-1}{m-1}\right)\right]^2$$
$$\approx \int_0^m \sigma_1^2 \left(1 - \frac{x}{m}\right)^2 dx$$

6. **Optional** The 2−norm of `randn(m)` for various m seem to follow a specific pattern:

```
figure; x=0; for i=1:100, x(i)=norm(randn(i))^2; end; plot(x,'-o')
```

Using the statement of the previous problem, explain how this might be exploited to compute an estimate to the Frobenius norm of `randn(m)` without even generating the random matrix!

> Note: the Frobenius norm of `randn(m)` for various m actually seems to follow a *very* simple pattern, (try `figure; x=0; for i=1:100, x(i)=norm(randn(i),'fro'); end; plot(x,'-o')`) although not exactly the same one as the above *approximation*. (The proof of `norm(randn(m),'fro')` $\approx m$ involves the expected value of the chi-square distribution!)
>
> FYI: The true *singular value function* for `randn(m)` is very nearly
>
> $$f(x) \approx 2\sqrt{m - \sqrt{m^2 - (x-m)^2}}.$$
>
> Try
>
> `figure; s=svd(randn(1000)); plot(s,'-o')`
>
> and
>
> `hold on; ezplot('2*sqrt(1000-sqrt(1000^2-(x-1000).^2))', [0 1000])`
>
> and
>
> `hold on; ezplot('2*sqrt(1000)(1-x/1000)', [0 1000])`
>
> to see the upper and lower bounds. Thus,
>
> $$\|A\|_F^2 \approx \int_0^m f(x)^2 dx \approx m^2(4-\pi),$$
>
> which is closer than the above approximation, but still not quite m^2 exactly.

7. **Optional** Could one also predict absolute value of determinant for `randn(m)`? What about condition number?

8. **Optional** Just for fun, compare the plot of singular values of random matrices that have entries $\sim \mathcal{N}(.5, .3)$ to the $U(0,1)$ plot from above (e.g., try `figure;s=svd(.5 + .3*randn(20));plot(s,'-o')`). Does this mean that the distributions are similar (plot the PDF's to find out!).