21-268 Class Notes

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1 The Euclidean Space

Definition 1.1. $\mathbb{R}^n$ is the space of all "n-tuples" (or vectors) $(x_1, \cdots, x_n)$ of real numbers. i.e. $x_i \in \mathbb{R}$ for each $1 \leq i \leq n$.

Definition 1.2.

$$\vec{x} + \vec{y} = (\vec{x}_1 + \vec{y}_1, \cdots, \vec{x}_n + \vec{y}_n)$$
$$t\vec{x} = (t\vec{x}_1, \cdots, t\vec{x}_n)$$

Example 1.3. $\vec{0} = (0, \cdots, 0)$ with n zeros.

Theorem 1.4. Given $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, the following properties hold:

(i) (Positivity) $\vec{x} \cdot \vec{x} \geq 0$; also $\vec{x} \cdot \vec{x} = 0 \iff \vec{x} = \vec{0}$.

(ii) (Symmetry) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$

(iii) (Bilinearity) $(s\vec{x} + t\vec{y}) \cdot \vec{z} = s(\vec{x} \cdot \vec{z}) + t(\vec{y} \cdot \vec{z})$.

Proof. (i) First,

$$\vec{x} \cdot \vec{x} = x_1^2 + \cdots + x_n^2 \geq 0$$

since each $x_i^2 \geq 0$ and the sum of non-negative numbers i non-negative. Second, suppose $\vec{x} \cdot \vec{x} = 0$. Since the only way for n non-negative numberes to sum to 0 is if they're each 0, $x_i^2 = 0$ for each $1 \leq i \leq n$. Thus $x_i = 0$. Conversely, if $\vec{x} = \vec{0}$, then $\vec{x} \cdot \vec{x} = 0^2 + \cdots + 0^2 = 0$. □

(ii) By the definition of the dot product followed by the commutativity of multiplication,

$$\vec{x} \cdot \vec{y} = \sum_{i=0}^{n} x_i y_i = \sum_{i=0}^{n} y_i x_i = \vec{y} \cdot \vec{x}$$

$\therefore \vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$. □

(iii) We do some computation.

$$\begin{aligned}
(s\vec{x} + t\vec{y}) \cdot \vec{z} &= (sx_1 + ty_1, \cdots, sx_n + ty_n) \cdot (z_1, \cdots, z_n) && (1)\\
&= (sx_1 + ty_1)z_1 + \cdots (sx_n + ty_n)z_n && (2)\\
&= sx_1z_1 + \cdots + sx_nz_n + ty_1z_1 + \cdots ty_nz_n && (3)\\
&= s(x_1z_1 + \cdots x_nz_n) + t(y_1z_1 + \cdots y_nz_n) && (4)\\
&= s(\vec{x} \cdot \vec{z}) + t(\vec{y} \cdot \vec{z}) && (5)
\end{aligned}$$

□

Definition 1.5. A function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that satisfies i) - iii) of the above theorem is called an *inner product* in $\mathbb{R}^n$.

Remark. *Inner products can also be defined on general vector spaces.*

Example 1.6. Let $x_1, \cdots, x_n \in \mathbb{R}$. We have an inner product in the polynomial vector space by defining, for polynomials q and p with degree equal to or less than n,

$$f(p,q) = \sum_{i=1}^{n} p(x_i)q(x_i)$$

Example 1.7. For a continuous vector space $\mathbb{C}_{[a,b]}$, the following are inner products:

$$f(g,h) = \int_a^b g(x)h(x)dx$$

$$f(g,h) = \int_a^b g(x)h(x)w(x)dx$$

where $w(x)$ is bounded, piecewise continuous, and $w > 0$ everywhere on $[a,b]$.

Definition 1.8. The Euclidean norm of $\vec{x} \in \mathbb{R}^n$ is

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$$

Theorem 1.9 (Cauchy-Schwarz). For every $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \, \|\vec{y}\|$$

Proof. We case on whether $\|\vec{y}\| = 0$ or $\|\vec{y}\| \neq 0$. Suppose $\|y\| = 0$. Computing $\vec{x} \cdot \vec{y}$ gives you zero. Computing $\|\vec{x}\| \, \|\vec{y}\|$ also gives you zero, as desired. Otherwise, suppose that $\|\vec{y}\| \neq 0$. We introduce an auxiliary scalar t and choose a suitable value for t later. By the definition of dot product,

$$0 \leq (\vec{x} + t\vec{y}) \cdot (\vec{x} + t\vec{y}) \tag{6}$$

$$= \|\vec{x}\|^2 + 2t\vec{x} \cdot \vec{y} + t^2 \|\vec{y}\|^2 \tag{7}$$

Now, let

$$t = \frac{-\vec{x} \cdot \vec{y}}{\|\vec{y}\|^2}.$$

Note that this is well-defined since $\|\vec{y}\| \neq 0$. By substitution, we have

$$0 \leq \|\vec{x}\|^2 \frac{-2(\vec{x} \cdot \vec{y})^2}{\|\vec{y}\|^2} + \frac{(\vec{x} \cdot \vec{y})^2}{\|\vec{y}\|^4} \|\vec{y}\|^2 \tag{8}$$

$$= \|x\|^2 - \frac{(\vec{x} \cdot \vec{y})^2}{\|\vec{y}\|^2} \tag{9}$$

Rearranging this yields $(\vec{x} \cdot \vec{y})^2 \leq \|\vec{x}\|^2 \|\vec{y}\|^2$, which implies

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

□

Theorem 1.10. If $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, then

1. $\|\vec{x}\| \geq 0$ and $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$.
2. $\|t\vec{x}\| = |t| \|\vec{x}\|$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Proof. Left as an exercise for the reader. □

Remark. *A function satisfying the above three properties is known as a norm and can be defined on a general vector space*

Example 1.11. Some other examples of norms in $\mathbb{R}^n$:

- $\|\vec{x}\|_{\ell\infty} := \max\{|x_1|, \cdots, |x_n|\}$
- $\|\vec{x}\|_{\ell^1} := |x_1| + \cdots + |x_n|$
- $\|\vec{x}\|_{\ell^p} := (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$

2 Topological Properties of Euclidean Space

Definition 2.1. Given $\vec{x}_0 \in \mathbb{R}^n$ and $r > 0$, then the *ball* centered at x_0 of radius r is

$$B(\vec{x}_0, r) = \{x \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| < r\}.$$

Remark. *Note that the ball doesn't include the "boundary" and it's sometimes called an "open" ball.*

Definition 2.2. Given a set $E \subseteq \mathbb{R}^n$, a point $\vec{x} \in E$ is called an *interior point* of E if there exists $r > 0$ such that $B(\vec{x}, r) \subseteq E$.

Definition 2.3. The *interior* of E - denoted by E° - is the set of all interior points of E.

Definition 2.4. A subset $U \subseteq \mathbb{R}^n$ is *open* if every $\vec{x} \in U$ is an interior point of U. In other words, for every $\vec{x} \in U$ there exists $r > 0$ such that $B(\vec{x}, r) \subseteq U$.

Example 2.5. If $\vec{x}_0 \in \mathbb{R}^n$ and $r > 0$ then $B(\vec{x}_0, r)$ is open.

Proof. Fix $\vec{y} \in B(\vec{x}_0, r)$. We need to find s such that

$$B(\vec{y}, s) \subseteq B(\vec{x}_0, r).$$

Set $s = r - \|\vec{x}_0 - \vec{y}\|$. Then, if $x \in B(\vec{y}, s)$, we have

$$\begin{aligned}
\|\vec{x} - \vec{x}_0\| &= \|\vec{x} - \vec{y} + \vec{y} - \vec{x}_0\| && (10)\\
&\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{x}_0\| && (11)\\
&< s + r - s && (12)\\
&= r. && (13)
\end{aligned}$$

Thus, $\|\vec{x} - \vec{x}_0\| < r$ and so $x \in B(\vec{x}_0, r)$. □

Remark. *To see where $s = r - \|\vec{x}_0 - \vec{y}\|$ comes from, draw a picture.*

Example 2.6. Here are some examples concerning open sets:

1. (a, ∞) is open.

   Proof. If $x \in (a, \infty)$, one can check that $B(x, x - a) \subseteq (a, \infty)$. □

2. (a, b) is open.

   Proof. Note that $(a, b) = B(\frac{a+b}{2}, \frac{b-a}{2})$. Using the fact that a ball is an open set, (a, b) is open. □

   Corollary 2.7. $[a, b]^\circ = (a, b)$

3. $(a, b]$ is not open.

   Proof. $B(b, r)$ is not a subset of $(a, b]$ for any $r > 0$ since it contains elements larger than b. □

4. Any union of open sets is open. That is, if U_i are all open $i \in I$ then $U = \bigcup_{i \in I} U_i$ is open.

   Proof. If $\vec{x} \in U$ then $x \in U_i$ for some i. So there exists $B(\vec{x}, r) \subseteq U_i$ for some $r > 0$. Since $U_i \subseteq U$ and $B(\vec{x}, r) \subseteq U_i$, then $B(\vec{x}, r) \subseteq U$. So $\vec{x}$ is an interior point of U. □

Definition 2.8. Given $E \subseteq \mathbb{R}^n$, a point $x \in \mathbb{R}^n$ is an *accumulation point* of E if for every $r > 0$, the ball $B(\vec{x}, r)$ contains at least one point of E different from $\vec{x}$. Note that $\vec{x}$ may or may not be in E, so the membership of $\vec{x}$ is irrelevant.

Example 2.9. 0 is an accumulation point of $\{\frac{1}{n}\}_{n \in \mathbb{N}}$.

Proof. It suffices to show that for every $r > 0$, the ball $B(0, r)$ contains a point in E. This is equivalent to showing that for all $r < 0$, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < r$. It should be clear that we can always find some n such that $n > \frac{1}{r}$, and therefore 0 is an accumulation point. ☐

Remark. *An interior point of $E \subseteq \mathbb{R}^n$ is an accumulation point of E.*

Example 2.10.

$$E^\circ \subseteq \text{acc } E$$

If $\vec{x} \in E^\circ$, then for some $r_0 > 0$, $B(\vec{x}, r_0) \subseteq E$. Now given $r > 0$, set $\vec{y} = \vec{x} + \frac{\min\{r, r_0\}}{2}(1, 0, \cdots, 0)$
Then $\vec{y} \neq \vec{x}$ and

$$\begin{aligned} \|\vec{y} - \vec{x}\| &= \left\| \vec{x} + \frac{\min\{r, r_0\}}{2}(1, 0, \cdots, 0) - \vec{x} \right\| \\ &= \left\| \frac{\min\{r, r_0\}}{2}(1, 0, \cdots, 0) \right\| \\ &= \frac{\min\{r, r_0\}}{2} \end{aligned}$$

So in particular, $\vec{y} \in B(\vec{x}, r) \cap B(\vec{x}, r_0)$. Therefore $\vec{y} \in E$. ☐

Example 2.11.

$$E = \mathbb{R}/(\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\})$$

Show that E is open.

Recall that a union of open sets is open. Also, we showed that intevals $(a, b), (-\infty, a), (a, \infty)$ are all open. Now,

$$E = (\infty, 0) \cup (1, \infty) \cup \bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n}).$$

So E is a union of open intervals and thus open sets, which implies that E is open. ☐

Definition 2.12. If $E \subseteq \mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$ is called a *boundary point* of E, $\vec{x} \in \partial E$, if, for every $r > 0$, $B(\vec{x}, r)$ contains at least one point in E and at least one point not in E.

Example 2.13.

$$\partial B(\vec{x}, r) = \{\vec{y} \in \mathbb{R}^n : \|\vec{y} - \vec{x}\| = r\}$$

Proposition 2.14. $\partial E \cup E^\circ = \emptyset$

Proof. It suffices to show that if $\vec{x} \in E^\circ$ then it cannot belong to the boundary. Now if $x \in E^\circ$, then there exists $r > 0$ such that $B(\vec{x}, r) \subseteq E$. Since the ball is fully countained by E, $\vec{x}$ fails the definition of boundary for this radius because this ball can not contain any points in the complement of E. ☐

Example 2.15.

$$E = (\infty, 0) \cup (1, \infty) \cup \bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n})$$

Find $E^\circ, \text{acc } E, \partial E$.

Solution. $E^\circ = E$. By definition, $E^\circ \subseteq E$. Now take $\vec{x} \in E$. Then $\vec{x}$ must belong to one of the open intervals. Note that open intervals are, well, open (lol) so there is some ball with $B(\vec{x}, r)$ and $r > 0$ contained in that interval. So $E \subseteq E^\circ$ and therefore $E^\circ = E$.

acc $E = E$. We must show that $\mathbb{R} \subseteq \text{acc } E$. Split $\mathbb{R} = (\mathbb{R}/E) \cup E$ and show that $\mathbb{R}/E \subset \text{acc } E$, $E \subseteq \text{acc } E$. First,

$$\mathbb{R}/E = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$$

In the case that $\vec{x} = 0$, take $\vec{y} = \frac{-r}{2} \in (-\infty, 0)$. $\vec{y} \in E$ and $\vec{y} \neq 0$, so it satisfies the definition of accumulation point for this r.

If $\vec{x} = \frac{1}{n}$, then we have the ball $B(\frac{1}{n}, r)$. If $r \geq \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1})$, take $\vec{y} = \frac{1}{n} - \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1}) \in (\frac{1}{n+1}, \frac{1}{n})$. Otherwise, take $\vec{x} - r$ and we're done.

$\partial E = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Take $\vec{x} \in \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and $r > 0$. Then by the definition of E, $\vec{x} \notin E$.But also, $\vec{x} \in \text{acc } E$ by the precious step, so I can find $\vec{y} \in B(\vec{x}, r)$ such that $\vec{y} \neq \vec{x}$ and $\vec{y} \in E$. So

$$\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq E.$$

To show the reverse inclusion, recall that $\partial E \cap E^\circ = \emptyset$, which implies that $\partial E \subseteq \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Therefore by double containment $\partial E = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

□

Definition 2.16 (Closed Sets). $C \subseteq \mathbb{R}^n$ is *closed* if R^n/C is open.

Proposition 2.17. E is closed if and only if acc $E \subseteq E$.

Proof. Suppose E is closed. Then $\mathbb{R}^n/E$ is open and so given $\vec{x} \in R^n/E$, there is $r > 0$ such that

$$B(\vec{x}, r) \subseteq \mathbb{R}^n/E.$$

So, $\vec{x} \notin \text{acc } E$ and thus

$$\mathbb{R}^n/E \subseteq \mathbb{R}^n/\text{acc } E$$

therefore acc $E \subseteq E$.

Now for the other direction, assume acc $E \subseteq E$. So given any $\vec{x} \in \mathbb{R}^n/E$, $\vec{x} \notin$ acc E. Therefore, for some $r > 0$, $B(\vec{x}, r)/\{\vec{x}\}$ does not contain any points of E. i.e.

$$B(\vec{x}, r)/\{\vec{x}\} \subseteq \mathbb{R}^n/E$$

since also $\vec{x} \in \mathbb{R}^N/E$. So therefore the whole ball doesn't belong to E meaning that it belongs to the complement.

$$B(\vec{x}, r) \subseteq \mathbb{R}^n/E$$

So $\mathbb{R}^n/E$ is open, which implies that E is closed. $\square$

3 Functions

$f : E \to \mathbb{R}^m$, where $E \subseteq R^n$. For every $x \in E$, $f(\vec{x}) \in R^m$.

Definition 3.1 (Domain). The domain of $f = E$ is the largest set on which f is well-defined. i.e. no logs of negatives, division by 0, etc.

Definition 3.2 (Bounded Sets). $A \subseteq \mathbb{R}^n$ is bounded if $A \subseteq B(\vec{x}, r)$ for some $x \in \mathbb{R}^n$, $r > 0$.

Definition 3.3 (Images). Given $F \subseteq E$, the image $f(F) = \{\vec{y} \in \mathbb{R}^m : \vec{y} = f(\vec{x})$ for some $x \in F\}$.

Definition 3.4 (Bounded Functions). A function f is bounded if $f(E)$ is a bounded subset of $\mathbb{R}^m$.

Definition 3.5 (Bounded Directions). If $m = 1$ - that is, if the dimension of the image is 1 - we can say that f is bounded from above if $f(E) \subseteq (-\infty, a)$ where $a \in \mathbb{R}$ and similarly, we can define when a function is bounded from below.

Definition 3.6 (Inverse Image). Given $G \subseteq \mathbb{R}^m$, $f^{-1}(G) = \{\vec{x} \in E : f(\vec{x}) \in G\}$. Note that this is not the inverse function, so be careful.

Definition 3.7 (Graphs). A graph of f is denoted as gr $f \subseteq \mathbb{R}^n \times \mathbb{R}^n = \{(\vec{x}, f(\vec{x})) : \vec{x} \in E\}$.

Definition 3.8 (-jectivity). When $f : E \to F, E \subseteq \mathbb{R}^n, R \subseteq \mathbb{R}^m$, f is said to be

1. Injective if $\vec{x} \neq \vec{z} \implies f(\vec{x}) \neq f(\vec{z})$.
2. Surjective if $f(E) = F$.
3. Bijective if it is both surjective and injective. If this is the case, then you can indeed define an inverse function $f^{-1} : F \to E$ which assigns to every $\vec{y} \in F$ the unique $\vec{x} \in E$ such that $f(\vec{x}) = \vec{y}$.

Definition 3.9 (-creasing). When $f : E \to \mathbb{R}$ and $E \subseteq \mathbb{R}$, f is

1. increasing if $f(x) \leq f(y)$ whenever $x < y$
2. decreasing if $f(x) \geq f(y)$ whenever $x > y$.
3. strictly increasing if $f(x) < f(y)$ whenever $x < y$
4. strictly decreasing if $f(x) > f(y)$ whenever $x > y$.
5. *monotone* if it is one of these 4.

4 Limits of Functions

Definition 4.1 (Limit of $f(x)$). Let $E \subseteq \mathbb{R}^n$, $f : E \to \mathbb{R}^m$, $\vec{x}_0 \in \operatorname{acc} E$. We say $\vec{y} \in \mathbb{R}^m$ is the *limit* of f as $\vec{x}$ approahes $\vec{x}_0$ if: given $\varepsilon > 0$, there exists $\delta > 0$ (depending on f and $\vec{x}_0, \varepsilon$) such that

$$\|f(\vec{x}) - \vec{y}\| < \varepsilon$$

for all $\vec{x} \in E$ with $0 < \|\vec{x} - \vec{x}_0\| < \delta$.

We write $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \vec{y}$ or $f(\vec{x}) \to \vec{y}$ as $\vec{x} \to \vec{x}_0$.

Remark. *Limits also sometimes do not exist. The proof strategy here is basically the following: you tell me how close you want $f(\vec{x})$ to be to $\vec{y}$ and I tell you how close $\vec{x}$ has to be to $\vec{x}_0$. Basically, formulate a δ such that whatever you're showing has to be true.*

There are some criteria for the nonexistence of $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x})$. If $F \subseteq E$, let $f \restriction F$ be the restriction of f to F. If $F, G \subseteq E$ with $F \cap G = \emptyset$, $\vec{x}_0 \in \operatorname{acc} E \cap \operatorname{acc} G$ and such that $\lim_{\vec{x} \to \vec{x}_0} f \restriction F(\vec{x}) \neq \lim_{\vec{x} \to \vec{x}_0} f \restriction G(\vec{x})$ then $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x})$ does not exist.

Remark. *The limit of a constant function is that constant. (Thanks Sherlock)*

Example 4.2.

$$f(x, y) = \begin{cases} 1 & y = x^2 \ \& \ x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Take $F = \{(x, y) : y = x^2\}$ and $G = \{(x, 0) : x \in \mathbb{R}\}$.

$$\lim_{(x,y) \to (0,0)} f(x, y) \restriction F = 1$$

$$\lim_{(x,y) \to (0,0)} f(x, y) \restriction G = 0$$

$$\therefore \lim_{(x,y) \to (0,0)} f(x, y) \text{ does not exist.}$$

Example 4.3.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$$

Solution. Fix arbitrary $\varepsilon > 0$. Let $\delta = \varepsilon$. If $0 < \|(x,y)-(0,0)\| < \delta = \varepsilon$, then

$$\left\| \frac{x^2 y}{x^2+y^2} - 0 \right\| = \frac{x^2|y|}{x^2+y^2}$$

$$\leq \frac{(x^2+y^2)\sqrt{x^2+y^2}}{x^2+y^2}$$

$$= \sqrt{x^2+y^2}$$

$$\|(x,y)\|$$

$$< \delta = \varepsilon$$

$\square$

At this point, you might be wondering how we made that choice of δ. The idea is that you deal with funny inequalities. Here, once you have your inequality in terms of $\|(x,y)\|$, you then choose your δ in terms of epsilon accordingly. More examples should illuminate these.

Example 4.4. Compute and prove:

$$\lim_{(x,y)\to(0,0)} \frac{x^m y}{x^2+y^2}. \quad (m \in \mathbb{N}, m \geq 2)$$

Solution. We immediately suspect that the limit is 0 because shoving in $y = 0$ yields $\frac{0}{x^2}$ where $x \neq 0$. Fix $\varepsilon > 0$. Now we do some scratch work to figure out what a good choice for δ should be.

$$|\frac{x^m y}{x^2+y^2}| = \frac{|x^m|\sqrt{y^2}}{x^2+y^2}$$

$$= \frac{(x^2)^{\frac{m}{2}}\sqrt{y^2}}{x^2+y^2}$$

$$\leq \frac{\sqrt{x^2+y^2}^{\frac{m}{2}}\sqrt{x^2+y^2}}{x^2+y^2}$$

$$= \sqrt{x^2+y^2}^{m-1}$$

$$< \|(x,y)-(0,0)\|$$

$$\leq \delta^{m-1}$$

$$< \delta \quad (\delta < 1)$$

Now we realize that it's a good idea to let $\delta = \min\{1, \epsilon\}$. So we then have that

$$\delta \leq \epsilon$$

which finishes the proof since then

$$\|(x, y) - (0, 0)\| < \epsilon$$

as desired.

$\square$

If given a limit, you have two options.

- Prove that the limit exists
- Prove it doesn't exist by finding F and G both subsets of the domain such that the limits along F and G disagree.

Example 4.5.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

We should suspect by inspection that if the limit exists, it should be zero. But surely we should try $y = kx$. Then let

$$F = \{(x, y) : x = y\}$$

$$G = \{(x, y) : x = -y\}$$

Note that using these subsets yields constant functions. If $(x, y) \in F$, the limit is $\frac{1}{2}$. If $(x, y) \in G$, the limit is $\frac{-1}{2}$. Since the limits of constant functions are constants,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \frac{1}{2} \quad ((x, y) \in F)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = -\frac{1}{2} \quad ((x, y) \in G)$$

So then the limit doesn't exist because they don't agree.

Example 4.6.

$$\lim_{(x,y)\to(0,0)} \frac{x^{100}y}{x - y}$$

Note that the domain is all $(x, y) \in \mathbb{R}^2$ such that $x \neq y$. The idea here is that we try to use the fact that the denominator blows up to get the non-existence of the limit. Suppose

$$F = \{(x, y) : y = 0\}.$$

If the limit exists, it must be the case that it must be equal to 0. Now produce a set

G such that the limit with the domain restricted to G doesn't approach 0. Say

$$G = \{(x, y) : y = x + x^m\}.$$

We haven't chosen m yet, but we're free to do that later. Then if $(x, y) \in G/\{(0, 0)\}$,

$$f(x, y) = \frac{x^{100}(x + x^m)}{-x^m}$$

Let $m = 101$

$$= \frac{x^{101}(1 + x^{100})}{-x^{101}}$$

$$= -1 \text{ as } x \to 0.$$

So $\lim\limits_{(x,y)\to(0,0)} f \restriction G = -1$. Then by our criteria for non-existence, the limit does not exist.

Theorem 4.7. Let $E \subseteq \mathbb{R}^n$ and $x_0 \in \text{acc } E$. Suppose $f : E \to \mathbb{R}^m$ and $g : E \to \mathbb{R}^m$ satisfy

$$\lim_{x\to x_0} f(\vec{x}) = \ell_1$$

$$\lim_{x\to x_0} g(\vec{x}) = \ell_2.$$

Then,

1. $\lim\limits_{\vec{x}\to\vec{x}_0} (f + g)(\vec{x}) = \ell_1 + \ell_2$
2. $\lim\limits_{\vec{x}\to\vec{x}_0} xf(\vec{x}) = c\ell_1$
3. $\lim\limits_{\vec{x}\to\vec{x}_0} (fg)(\vec{x}) = \ell_1\ell_2$
4. If $m = 1$ and $\ell_2 \neq 0$ and $g(\vec{x}) \neq 0$ for all $\vec{x} \in E$, then

$$\lim_{x\to x_0} \left(\frac{f}{g}\right)(\vec{x}) = \frac{\ell_1}{\ell_2}.$$

(4). Fix $\varepsilon > 0$,

$$\left|\frac{f(x)}{g(x)} - \frac{\ell_1}{\ell_2}\right| = \left|\frac{f(x)\ell_2 - g(x)\ell_1}{g(x)\ell_2}\right|$$

$$= \frac{|f(x)\ell_2 - \ell_1\ell_2 + \ell_1\ell_2 - g(x)\ell_1|}{|g(x)\ell_2|}$$

$$\leq \frac{|\ell_2||f(x) - \ell_1|}{|g(x)\ell_2|} + \frac{|\ell_1||\ell_2 - g(x)|}{|g(x)\ell_2|}$$

$$= \frac{|f(x) - \ell_1|}{|g(x)|} + \frac{|\ell_1||\ell_2 - g(x)|}{|g(x)\ell_2|}$$

Let δ_1 be small enough so that for $0 < \|x - x_0\| < \delta_1$,

$$|g(x) - \ell_2| < \frac{|\ell_2|}{2}$$

$$\implies |g(x)| < \frac{|\ell_2|}{2}$$
$$\implies \frac{1}{|g(x)|} < \frac{2}{|\ell_2|}$$

Plugging this estimate into the above, we obtain

$$\left|\frac{f(x)}{g(x)} - \frac{\ell_1}{\ell_2}\right| \leq \frac{|f(x)-\ell_1|2}{|\ell_2|} + \frac{2|\ell_1|}{|\ell_2|^2}|\ell_2 - g(x)|$$

Then choose δ_2 such that if $0 < \|x - x_0\| < \delta_2$

$$|f(x) - \ell_1| < \frac{\varepsilon|\ell_2|}{4}$$

and δ_3 such that if $0 < \|x - x_0\| < \delta_3$

$$|\ell_2 - g(x)| < \frac{|\ell_2|^2}{4||\ell_1| + 1|}\varepsilon$$

Then if we choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, plugging in the above inequalities gives us

$$\begin{aligned}\left|\frac{f(x)}{g(x)} - \frac{\ell_1}{\ell_2}\right| &< \epsilon\frac{|\ell_2}{4} \cdot \frac{2}{|\ell_2} + \frac{2|\ell_1|}{|\ell_2|^2} \cdot \frac{|\ell_2|^2\varepsilon}{4||\ell_1| + 1|} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

□

Theorem 4.8 (Squeeze Theorem). Let $E \subseteq \mathbb{R}^n$ and $x_0 \in \operatorname{acc} E$. Let f, g, h be functions from E to $\mathbb{R}$ such that

$$\lim_{x \to x_0} g(x) = \ell = \lim_{x \to x_0} h(x)$$

and

$$g(x) \leq f(x) \leq h(x)$$

for all $x \in E$. Then

$$\lim_{x \to x_0} f(x) = \ell.$$

Proof. Fix $\varepsilon > 0$. Let δ_1 and δ_2 be small enough so that for all $|x_0 - x| < \delta_1$, or $|x_0 - x| < \delta_2$, $|g(x) - \ell| < \varepsilon$, or $|h(x) - \ell| < \varepsilon$ respectively. These inequalities imply that $g(x) > \ell - \varepsilon$ if $0 < |x - x_0| < \delta_1$ and $h(x) < \ell + \varepsilon$ if $0 < |x - x_0| < \delta_2$. Then,

$$\ell - \epsilon \leq g(x) \leq f(x) \leq h(x) \leq \ell + \varepsilon$$

if $0 < \|x - x_0\| < \min\{\delta_1, \delta_2\}$. Thus

$$|f(x) - \ell| < \varepsilon$$

if $0 < \|x - x_0\| < \delta$. □

Theorem 4.9. Let $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^m$, $x \in \text{acc } E$, and $f : E \to F$, $g : F \to \mathbb{R}^p$ be functions such that

$$\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}^m$$

$$\ell \in \text{acc } E$$

$$\lim_{y \to \ell} g(y) = L \in \mathbb{R}^p$$

Assume that either

(i) $\exists \delta_1 > 0$ such that $f(x) \neq \ell$ for all $x \in E$ with $0 < \|x - x_0\| < \delta_1$. or

(ii) $\ell \in F$ and $g(\ell) = L$.

Then

$$\lim_{x \to x_0} g(f(x)) = L.$$

Proof left as an exercise for the reader (lol)

Remark. *This is nice because we can use this to change variables and perform funny substitutions in limits.*

Example 4.10. Compute:

$$\lim_{x \to 0} \frac{\log(1 + \sin x)}{x}.$$

Solution. Recall that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$. We have that

$$\lim_{x \to 0} \frac{\log(1 + \sin x)}{x} = \lim_{x \to 0} \frac{\log(1 + \sin x)}{\sin x} \cdot \frac{\sin x}{x}$$

Let $f(x) = \sin x$ and $g(y) = \frac{\log(1+y)}{y}$. Let's check the assumptions from the above theorem.

(i) $\lim_{x \to 0} f(x) = 0 = \ell$

(ii) $0 \in \text{acc}(F) = \text{acc}((-1, 0) \cup (0, 1))$.

(iii) $\lim_{y \to 0} g(y) = 1 = L$

(iv) For $|x - 0| < \frac{\pi}{2}$, $f(x) \neq 0$.

By the previous theorem, $\lim_{x \to 0} g(f(x)) = 1$. Then you can use the limit product rule to show that

$$\lim_{x \to 0} \frac{\log(1 + \sin x)}{\sin x} \cdot \frac{\sin x}{x} = 1 \cdot 1 = \boxed{1}$$

□

5 Continuity

Definition 5.1 (Isolated Points). Let $E \subseteq \mathbb{R}^n$. A point $x_0 \in E$ is called an isolated point of E if there exists $\delta > 0$ such that

$$B(x_0, \delta) \cap E = \{x_0\}.$$

In other words, a point is isolated if you can create a ball that has x_0 as the only element of E and elements in the complement of E.

Definition 5.2 (Continuous Functions). Let $E \subseteq \mathbb{R}^n$ and $x_0 \in \operatorname{acc} E$. Given a function $f : E \to \mathbb{R}^m$, we say f is continuous at x_0 if for any $\varepsilon > 0$, there exists δ such that for all $x \in E$ with $\|x_0 - x\| < \delta$,

$$\|f(x) - f(x_0)\| < \varepsilon.$$

If f is continuous at every point $x \in E$, then we say f is continuous on E and $f \in C(E)$ (or $f \in C^\circ(E)$).

Theorem 5.3. Let $E \subseteq \mathbb{R}^n$, $x_0 \in E$, $f : E \to \mathbb{R}^m$.

i.) If x_0 is an isolated point of E, then f is continuous at x_0.

ii.) If x_0 is an accumulation point, then f is continuous at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$

Proof. i.) If $x_0 \in \operatorname{iso} E$, then there exists $\delta > 0$ such that $B(x_0, \delta) \cap E = \{x_0\}$. In particular, if $\|x - x_0\| < \delta$ & $x \in E$, then $x = x_0$. To check continuity at x_0, fix $\varepsilon > 0$. Take our δ from above. Then

$$\|x_0 - x\| < \delta \ \&\ x \in E$$

$$\implies x = x_0$$

$$\implies \|f(x) - f(x_0)\| = 0 < \varepsilon$$

□

Theorem 5.4. Let $E \subseteq \mathbb{R}^n$, $x_0 \in E$, $f : E \to \mathbb{R}^m$, and $g : E \to \mathbb{R}^m$ such that f and g are continuous at x_0. Then,

i.) $f + g$, cf, cg, $f \cdot g$ are continuous at x_0.

ii.) If $m = 1$, then $\frac{f}{g}$ (restricted to the set where $g(x) \neq 0$) is also continuous at x_0 if $g(x_0) \neq 0$.

Proof. Trivial and left as an exercise for the reader. □

Remark. *If $f : E \to \mathbb{R}^m$ is given by*

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

then f is continuous at $x_0 \in E \iff f_i$ is continuous at x_0 for each $1 \leq i \leq m$.

Theorem 5.5. Let $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^m$, $x_0 \in E$, $f : E \to F$, and $g : F \to \mathbb{R}^p$ be such that f is continuous at $f(x_0)$. Then $g \circ f : E \to \mathbb{R}^p$ is continuous at x_0.

You have to be careful with continuity when taking inverses.

Example 5.6. This is continuous on the whole domain.

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ x - 1 & 2 < x \leq 3 \end{cases}$$

This isn't:

$$f^{-1}(x) = \begin{cases} x & 0 \leq x \leq 1 \\ x + 1 & 2 < x \leq 3 \end{cases}$$

So if a function is continuous, it's not necessarily the case that its inverse is also continuous. See Leoni's notes for example where continuity of f^{-1} is guaranteed.

6 Directional Derivatives and Differentiability

Definition 6.1 (Directional Derivative). Let $E \subseteq \mathbb{R}^n, f : E \to \mathbb{R}$, and $x_0 \in E$. Given a direction $v \in \mathbb{R}^n/\{0\}$, let L be the line through x_0 in direction v.

$$L = \{x_0 + tv : t \in \mathbb{R}\}$$

Assume x_0 is an accumulation point of $(L \cap E)$. Then the directional derivative of f at x_0 in direction v is

$$\frac{\partial f}{\partial v}(x_0) = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

Definition 6.2 (Partial derivatives in the coordinate directions). With the same setup as the previous definition, $v = e_i$ (Google standard basis)

$$\frac{\partial f}{\partial e_i}(x_0) = \frac{\partial f}{\partial x_i}(x_0) = D_i f(x)$$

is the partial derivative of f with respect to x_i at x_0.

Example 6.3. $E = \mathbb{R}$, $v = 1$

$$\frac{\partial f}{\partial 1}(x_0) = \lim_{t \to 0} \frac{f(x_0 + 1 \cdot t) - f(x_0)}{t} = f'(x_0)$$

Recall on that on $\mathbb{R}$, if f is differentiable at x_0, then f is continuous at x_0.

Example 6.4. The existence of directional derivatives does not imply continuity.

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Take a vector $v \in \mathbb{R}^2$, with $v_1^2 + v_2^2 = 1$.

$$I_t := \frac{f(0+tv) - f(0)}{t} = \frac{f(t(v_1, v_2))}{t} = \frac{\frac{(tv_1)^2 tv_2}{(tv_1)^4 + (tv_2)^2}}{t}$$

$$I_t = \frac{1}{t}\left(\frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2}\right)$$

If $v_2 \neq 0$,

$$\lim_{t \to 0} I_t = \frac{v_1^2 v_2}{v_2^2} = \frac{\partial f}{\partial v}(0)$$

If $v_2 = 0$, then $I_t = 0$, which also implies that the directional derivative should be 0. In sum:

$$\frac{\partial f}{\partial v}(0,0) \begin{cases} \frac{v_1^2}{v_2} & \text{if } v_2 = 0 \\ 0 & \text{if } v_2 = 0 \end{cases}$$

All this tells us is that the directional derivative exists. So, is f continuous at $(0,0)$? Differentiability is stronger than continuity in functions from $\mathbb{R} \to \mathbb{R}$, but this isn't generally the case in higher-dimensional functions. Claim: f is not continuous at $(0,0)$.

$$f(x, x^2) = \frac{1}{2}$$

$$f(x, 0) = 0$$

The limit at these paths are not equal, which implies discontinuity at $(0,0)$.

Remark. *If $f : \mathbb{R} \to \mathbb{R}$, then directional differentiability implies continuity. Given the example above, the same doesn't hold for higher-dimensional functions. (Note the distinction between directional derivatives and the stronger notion of differentiability below.)*

Definition 6.5 (Differentiability). Let $E \subseteq \mathbb{R}^n$, $f : E \to \mathbb{R}$, and $x_0 \in E$ an accumulation point of E. Then f is differentiable at x_0 if there exists $\vec{b} \in \mathbb{R}^n$ depending on x_0 such that

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)-\vec{b}\cdot(x-x_0)}{\|x-x_0\|}=0$$

Remark. $T(x)=\vec{b}\cdot x$ *is linear.*

Definition 6.6 (Differentials). Say that f as before is continuous at $x_0\in E$, we call $df(x_0)\cdot v=\vec{b}\cdot v$ the differential of f at x_0.

Theorem 6.7. Let $E\subseteq\mathbb{R}^n$, $f:E\to\mathbb{R}$, and $x_0\in E$ an accumulation point of E. If f is differentiable at x_0, then f is continuous at x_0.

Proof.

$$\begin{aligned}
f(x)-f(x_0)&=f(x)-f(x_0)-\vec{b}\cdot(x-x_0)+\vec{b}\cdot(x-x_0)\\
&=\frac{f(x)-f(x_0)-\vec{b}\cdot(x-x_0)}{\|x-x_0\|}\|x-x_0\|+\vec{b}\cdot(x-x_0)\\
&\le\left\|\frac{f(x)-f(x_0)-\vec{b}\cdot(x-x_0)}{\|x-x_0\|}\right\|\|x-x_0\|+\|b\|\,\|x-x_0\|\\
&\implies\lim_{x\to x_0}|f(x)-f(x_0)|\le 0=0
\end{aligned}$$

□

Theorem 6.8. $E\subset\mathbb{R}^n$, $f:E\to\mathbb{R}$, $x_0\in E^\circ$. If f is differentiable at x_0, then

i) There exists $\frac{\partial f}{\partial v}(x_0)$ for all $v\in\mathbb{R}^n$, $\|v\|=1$ and $\frac{\partial f}{\partial v}(x_0)=df(x_0)v$.

ii) For all $v\in\mathbb{R}^n$, $\|v\|=1$,

$$\frac{\partial f}{\partial v}(x_0)=\sum_i^n\frac{\partial f}{\partial x_i}(x_0)v_i$$

In other words,

$$\frac{\partial f}{\partial v}(x_0)=\nabla f(x_0)\cdot v.$$

A consequence of this theorem is that $\vec{b}=(\frac{\partial f}{\partial x_1}(x_0),...,\frac{\partial f}{\partial x_n}(x_0))$. This is because $b_i=\vec{b}\cdot\vec{e_i}=\frac{\partial f}{\partial e_i}(x_0)=\frac{\partial f}{\partial x_i}(x_0)$.

Proof. $x_0\in E^\circ \implies \exists r>0$ such that $B(x_0,r)\subset E$. Take $v\in\mathbb{R}^n$, $\|v\|=1, x=x_0+tv$. Then

$$\|x-x_0\|=\|x_0+tv-x_0\|=t\,\|v\|=t$$

If we choose $t<r$, then $x\in B(x_0,r)\subset E$. This implies that you can compute $f(x_0+tv)$

for such t and v. Now we test the definition of differentiability with $x = x_0 + tv$. Thus

$$0 = \lim_{x \to x_0} \lim_{(x = x_0 + tv)} \frac{f(x) - f(x_0) - b \cdot (x - x_0)}{\|x - x_0\|}$$

$$= \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0) - b \cdot tv}{|t| \, \|v\|}$$

Multiply by $\|v\| > 0$.

$$0 = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0) - b \cdot tv}{|t|}$$

Then,

$$0 = \lim_{t \to 0} \left| \frac{f(x_0 + tv) - f(x_0) - b \cdot tv}{|t|} \right|$$

$$= \lim_{t \to 0} \left| \frac{f(x_0 + tv) - f(x_0) - b \cdot tv}{|t|} - 0 \right|$$

So,

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0) - b \cdot tv}{t} = 0.$$

Rearranging on the LHS,

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} - b \cdot v = 0.$$

And thus,

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = b \cdot v$$

which is precisely the definition of the directional derivative in direction v at x_0. So,

$$\frac{\partial f}{\partial v}(x_0) = b \cdot v.$$

$\square$

Definition 6.9. If $\frac{\partial f}{\partial x_i}(x_0)$ exists for each $1 \leq i \leq n$, we call

$$\left(\frac{\partial f}{\partial x_1}(x_0), ..., \frac{\partial f}{\partial x_n}(x_0)\right) = \nabla f(x_0) = \text{grad } f(x_0)$$

the gradient of f at x_0.

Remark (Proving or disproving differentiability at x_0)**.** *If we want to prove that something is differentiable and f is indeed differentiable at x_0, then $b = \nabla f(x_0)$ by the above theorem. In particular, to check differentiability of f at x_0, it suffices to verify that*

$$= \lim_{x_0 \to x} \frac{f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)}{\|x - x_0\|}.$$

We can disprove differentiability in a few ways.

i) Show f is not continuous at x_0 (Since differentiability implies continuity).

ii) if $x_0 \in E^\circ$ and $\frac{\partial f}{\partial v}(x_0)$ doesn't exist for any v, then f is not differentiable at x_0.

iii) if $x_0 \in E^\circ$ and

$$\frac{\partial f}{\partial v}(x_0) \neq \nabla f(x_0) \cdot v$$

for some $v \neq 0$, then f is not differentiable at x_0.

Example 6.10.

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$\frac{\partial f}{\partial v}(0, 0) = \frac{v_1^2}{v_2}$$

for $v_2 \neq 0$. This expression, if f were differentiable at $(0, 0)$, should be of the form

$$\frac{\partial f}{\partial v}(0, 0) = \frac{\partial f}{\partial x_1}(0, 0) v_1 + \frac{\partial f}{\partial x_2}(0, 0) v_2$$

so f is not differentiable at the origin. The key here is that $\frac{v_1^2}{v_2}$ is not linear.