

21-268 CLASS NOTES

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1 The Euclidean Space

Definition 1.1. \mathbb{R}^n is the space of all “ n -tuples” (or vectors) (x_1, \dots, x_n) of real numbers. i.e. $x_i \in \mathbb{R}$ for each $1 \leq i \leq n$.

Definition 1.2.

$$\begin{aligned}\vec{x} + \vec{y} &= (\vec{x}_1 + \vec{y}_1, \dots, \vec{x}_n + \vec{y}_n) \\ t\vec{x} &= (t\vec{x}_1, \dots, t\vec{x}_n)\end{aligned}$$

Example 1.3. $\vec{0} = (0, \dots, 0)$ with n zeros.

Theorem 1.4. Given $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, the following properties hold:

- (i) (Positivity) $\vec{x} \cdot \vec{x} \geq 0$; also $\vec{x} \cdot \vec{x} = 0 \iff \vec{x} = \vec{0}$.
- (ii) (Symmetry) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
- (iii) (Bilinearity) $(s\vec{x} + t\vec{y}) \cdot \vec{z} = s(\vec{x} \cdot \vec{z}) + t(\vec{y} \cdot \vec{z})$.

Proof. (i) First,

$$\vec{x} \cdot \vec{x} = x_1^2 + \dots + x_n^2 \geq 0$$

since each $x_i^2 \geq 0$ and the sum of non-negative numbers is non-negative. Second, suppose $\vec{x} \cdot \vec{x} = 0$. Since the only way for n non-negative numbers to sum to 0 is if they're each 0, $x_i^2 = 0$ for each $1 \leq i \leq n$. Thus $x_i = 0$. Conversely, if $\vec{x} = \vec{0}$, then $\vec{x} \cdot \vec{x} = 0^2 + \dots + 0^2 = 0$. \square

(ii) By the definition of the dot product followed by the commutativity of multiplication,

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_i = \sum_{i=1}^n y_i x_i = \vec{y} \cdot \vec{x}$$

$$\therefore \vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}. \square$$

(iii) We do some computation.

$$(s\vec{x} + t\vec{y}) \cdot \vec{z} = (sx_1 + ty_1, \dots, sx_n + ty_n) \cdot (z_1, \dots, z_n) \quad (1)$$

$$= (sx_1 + ty_1)z_1 + \dots + (sx_n + ty_n)z_n \quad (2)$$

$$= sx_1z_1 + \dots + sx_nz_n + ty_1z_1 + \dots + ty_nz_n \quad (3)$$

$$= s(x_1z_1 + \dots + x_nz_n) + t(y_1z_1 + \dots + y_nz_n) \quad (4)$$

$$= s(\vec{x} \cdot \vec{z}) + t(\vec{y} \cdot \vec{z}) \quad (5)$$

\square

Definition 1.5. A function $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies i) - iii) of the above theorem is called an *inner product* in \mathbb{R}^n .

Remark. Inner products can also be defined on general vector spaces.

Example 1.6. Let $x_1, \dots, x_n \in \mathbb{R}$. We have an inner product in the polynomial vector space by defining, for polynomials q and p with degree equal to or less than n ,

$$f(p, q) = \sum_{i=1}^n p(x_i)q(x_i)$$

Example 1.7. For a continuous vector space $\mathbb{C}_{[a,b]}$, the following are inner products:

$$f(g, h) = \int_a^b g(x)h(x)dx$$

$$f(g, h) = \int_a^b g(x)h(x)w(x)dx$$

where $w(x)$ is bounded, piecewise continuous, and $w > 0$ everywhere on $[a, b]$.

Definition 1.8. The Euclidean norm of $\vec{x} \in \mathbb{R}^n$ is

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + \dots + x_n^2}$$

Theorem 1.9 (Cauchy-Schwarz). For every $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

Proof. We case on whether $\|\vec{y}\| = 0$ or $\|\vec{y}\| \neq 0$. Suppose $\|\vec{y}\| = 0$. Computing $\vec{x} \cdot \vec{y}$ gives you zero. Computing $\|\vec{x}\| \|\vec{y}\|$ also gives you zero, as desired. Otherwise, suppose that $\|\vec{y}\| \neq 0$. We introduce an auxiliary scalar t and choose a suitable value for t later. By the definition of dot product,

$$0 \leq (\vec{x} + t\vec{y}) \cdot (\vec{x} + t\vec{y}) \tag{6}$$

$$= \|\vec{x}\|^2 + 2t\vec{x} \cdot \vec{y} + t^2 \|\vec{y}\|^2 \tag{7}$$

Now, let

$$t = \frac{-\vec{x} \cdot \vec{y}}{\|\vec{y}\|^2}.$$

Note that this is well-defined since $\|\vec{y}\| \neq 0$. By substitution, we have

$$0 \leq \|\vec{x}\|^2 - \frac{2(\vec{x} \cdot \vec{y})^2}{\|\vec{y}\|^2} + \frac{(\vec{x} \cdot \vec{y})^2}{\|\vec{y}\|^4} \|\vec{y}\|^2 \tag{8}$$

$$= \|\vec{x}\|^2 - \frac{(\vec{x} \cdot \vec{y})^2}{\|\vec{y}\|^2} \tag{9}$$

Rearranging this yields $(\vec{x} \cdot \vec{y})^2 \leq \|\vec{x}\|^2 \|\vec{y}\|^2$, which implies

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

□

Theorem 1.10. If $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, then

1. $\|\vec{x}\| \geq 0$ and $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$.
2. $\|t\vec{x}\| = |t| \|\vec{x}\|$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Proof. Left as an exercise for the reader. □

Remark. A function satisfying the above three properties is known as a *norm* and can be defined on a general vector space

Example 1.11. Some other examples of norms in \mathbb{R}^n :

- $\|\vec{x}\|_{\ell_\infty} := \max\{|x_1|, \dots, |x_n|\}$
- $\|\vec{x}\|_{\ell_1} := |x_1| + \dots + |x_n|$
- $\|\vec{x}\|_{\ell_p} := (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$

2 Topological Properties of Euclidean Space

Definition 2.1. Given $\vec{x}_0 \in \mathbb{R}^n$ and $r > 0$, then the *ball* centered at x_0 of radius r is

$$B(\vec{x}_0, r) = \{x \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| < r\}.$$

Remark. Note that the ball doesn't include the "boundary" and it's sometimes called an "open" ball.

Definition 2.2. Given a set $E \subseteq \mathbb{R}^n$, a point $\vec{x} \in E$ is called an *interior point* of E if there exists $r > 0$ such that $B(\vec{x}, r) \subseteq E$.

Definition 2.3. The *interior* of E - denoted by E° - is the set of all interior points of E .

Definition 2.4. A subset $U \subseteq \mathbb{R}^n$ is *open* if every $\vec{x} \in U$ is an interior point of U . In other words, for every $\vec{x} \in U$ there exists $r > 0$ such that $B(\vec{x}, r) \subseteq U$.

Example 2.5. If $\vec{x}_0 \in \mathbb{R}^n$ and $r > 0$ then $B(\vec{x}_0, r)$ is open.

Proof. Fix $\vec{y} \in B(\vec{x}_0, r)$. We need to find s such that

$$B(\vec{y}, s) \subseteq B(\vec{x}_0, r).$$

Set $s = r - \|\vec{x}_0 - \vec{y}\|$. Then, if $x \in B(\vec{y}, s)$, we have

$$\|\vec{x} - \vec{x}_0\| = \|\vec{x} - \vec{y} + \vec{y} - \vec{x}_0\| \quad (10)$$

$$\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{x}_0\| \quad (11)$$

$$< s + r - s \quad (12)$$

$$= r. \quad (13)$$

Thus, $\|\vec{x} - \vec{x}_0\| < r$ and so $x \in B(\vec{x}_0, r)$. \square

Remark. To see where $s = r - \|\vec{x}_0 - \vec{y}\|$ comes from, draw a picture.

Example 2.6. Here are some examples concerning open sets:

1. (a, ∞) is open.

Proof. If $x \in (a, \infty)$, one can check that $B(x, x - a) \subseteq (a, \infty)$. \square

2. (a, b) is open.

Proof. Note that $(a, b) = B(\frac{a+b}{2}, \frac{b-a}{2})$. Using the fact that a ball is an open set, (a, b) is open. \square

Corollary 2.7. $[a, b]^\circ = (a, b)$

3. $(a, b]$ is not open.

Proof. $B(b, r)$ is not a subset of $(a, b]$ for any $r > 0$ since it contains elements larger than b . \square

4. Any union of open sets is open. That is, if U_i are all open $i \in I$ then $U = \bigcup_{i \in I} U_i$ is open.

Proof. If $\vec{x} \in U$ then $x \in U_i$ for some i . So there exists $B(\vec{x}, r) \subseteq U_i$ for some $r > 0$. Since $U_i \subseteq U$ and $B(\vec{x}, r) \subseteq U_i$, then $B(\vec{x}, r) \subseteq U$. So \vec{x} is an interior point of U . \square

Definition 2.8. Given $E \subseteq \mathbb{R}^n$, a point $x \in \mathbb{R}^n$ is an *accumulation point* of E if for every $r > 0$, the ball $B(\vec{x}, r)$ contains at least one point of E different from \vec{x} . Note that \vec{x} may or may not be in E , so the membership of \vec{x} is irrelevant.

Example 2.9. 0 is an accumulation point of $\{\frac{1}{n}\}_{n \in \mathbb{N}}$.

Proof. It suffices to show that for every $r > 0$, the ball $B(0, r)$ contains a point in E . This is equivalent to showing that for all $r > 0$, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < r$. It should be clear that we can always find some n such that $n > \frac{1}{r}$, and

therefore 0 is an accumulation point. \square

Remark. An interior point of $E \subseteq \mathbb{R}^n$ is an accumulation point of E .

Example 2.10.

$$E^\circ \subseteq \text{acc } E$$

If $\vec{x} \in E^\circ$, then for some $r_0 > 0$, $B(\vec{x}, r_0) \subseteq E$. Now given $r > 0$, set $\vec{y} = \vec{x} + \frac{\min\{r, r_0\}}{2}(1, 0, \dots, 0)$

Then $\vec{y} \neq \vec{x}$ and

$$\begin{aligned} \|\vec{y} - \vec{x}\| &= \left\| \vec{x} + \frac{\min\{r, r_0\}}{2}(1, 0, \dots, 0) - \vec{x} \right\| \\ &= \left\| \frac{\min\{r, r_0\}}{2}(1, 0, \dots, 0) \right\| \\ &= \frac{\min\{r, r_0\}}{2} \end{aligned}$$

So in particular, $\vec{y} \in B(\vec{x}, r) \cap B(\vec{x}, r_0)$. Therefore $\vec{y} \in E$. \square

Example 2.11.

$$E = \mathbb{R} / (\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\})$$

Show that E is open.

Recall that a union of open sets is open. Also, we showed that intervals (a, b) , $(-\infty, a)$, (a, ∞) are all open. Now,

$$E = (\infty, 0) \cup (1, \infty) \cup \bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n}).$$

So E is a union of open intervals and thus open sets, which implies that E is open. \square

Definition 2.12. If $E \subseteq \mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$ is called a *boundary point* of E , $\vec{x} \in \partial E$, if, for every $r > 0$, $B(\vec{x}, r)$ contains at least one point in E and at least one point not in E .

Example 2.13.

$$\partial B(\vec{x}, r) = \{\vec{y} \in \mathbb{R}^n : \|\vec{y} - \vec{x}\| = r\}$$

Proposition 2.14. $\partial E \cup E^\circ = \emptyset$

Proof. It suffices to show that if $\vec{x} \in E^\circ$ then it cannot belong to the boundary. Now if $x \in E^\circ$, then there exists $r > 0$ such that $B(\vec{x}, r) \subseteq E$. Since the ball is fully contained by E , \vec{x} fails the definition of boundary for this radius because this ball can not contain any points in the complement of E . \square

Example 2.15.

$$E = (\infty, 0) \cup (1, \infty) \cup \bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n})$$

Find $E^\circ, \text{acc } E, \partial E$.

Solution. $E^\circ = E$. By definition, $E^\circ \subseteq E$. Now take $\vec{x} \in E$. Then \vec{x} must belong to one of the open intervals. Note that open intervals are, well, open (lol) so there is some ball with $B(\vec{x}, r)$ and $r > 0$ contained in that interval. So $E \subseteq E^\circ$ and therefore $E^\circ = E$.

$\text{acc } E = E$. We must show that $\mathbb{R} \subseteq \text{acc } E$. Split $\mathbb{R} = (\mathbb{R}/E) \cup E$ and show that $\mathbb{R}/E \subset \text{acc } E, E \subseteq \text{acc } E$. First,

$$\mathbb{R}/E = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

In the case that $\vec{x} = 0$, take $\vec{y} = \frac{-r}{2} \in (-\infty, 0)$. $\vec{y} \in E$ and $\vec{y} \neq 0$, so it satisfies the definition of accumulation point for this r .

If $\vec{x} = \frac{1}{n}$, then we have the ball $B(\frac{1}{n}, r)$. If $r \geq \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1})$, take $\vec{y} = \frac{1}{n} - \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1}) \in (\frac{1}{n+1}, \frac{1}{n})$. Otherwise, take $\vec{x} - r$ and we're done.

$\partial E = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Take $\vec{x} \in \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and $r > 0$. Then by the definition of E , $\vec{x} \notin E$. But also, $\vec{x} \in \text{acc } E$ by the previous step, so I can find $\vec{y} \in B(\vec{x}, r)$ such that $\vec{y} \neq \vec{x}$ and $\vec{y} \in E$. So

$$\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subseteq E.$$

To show the reverse inclusion, recall that $\partial E \cap E^\circ = \emptyset$, which implies that $\partial E \subseteq \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Therefore by double containment $\partial E = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. □

Definition 2.16 (Closed Sets). $C \subseteq \mathbb{R}^n$ is *closed* if \mathbb{R}^n/C is open.

Proposition 2.17. E is closed if and only if $\text{acc } E \subseteq E$.

Proof. Suppose E is closed. Then \mathbb{R}^n/E is open and so given $\vec{x} \in \mathbb{R}^n/E$, there is $r > 0$ such that

$$B(\vec{x}, r) \subseteq \mathbb{R}^n/E.$$

So, $\vec{x} \notin \text{acc } E$ and thus

$$\mathbb{R}^n/E \subseteq \mathbb{R}^n/\text{acc } E$$

therefore $\text{acc } E \subseteq E$.

Now for the other direction, assume $\text{acc } E \subseteq E$. So given any $\vec{x} \in \mathbb{R}^n/E$, $\vec{x} \notin \text{acc } E$. Therefore, for some $r > 0$, $B(\vec{x}, r)/\{\vec{x}\}$ does not contain any points of E . i.e.

$$B(\vec{x}, r)/\{\vec{x}\} \subseteq \mathbb{R}^n/E$$

since also $\vec{x} \in \mathbb{R}^n/E$. So therefore the whole ball doesn't belong to E meaning that it belongs to the complement.

$$B(\vec{x}, r) \subseteq \mathbb{R}^n/E$$

So \mathbb{R}^n/E is open, which implies that E is closed. \square

3 Functions

$f : E \rightarrow \mathbb{R}^m$, where $E \subseteq \mathbb{R}^n$. For every $x \in E$, $f(\vec{x}) \in \mathbb{R}^m$.

Definition 3.1 (Domain). The domain of $f = E$ is the largest set on which f is well-defined. i.e. no logs of negatives, division by 0, etc.

Definition 3.2 (Bounded Sets). $A \subseteq \mathbb{R}^n$ is bounded if $A \subseteq B(\vec{x}, r)$ for some $x \in \mathbb{R}^n$, $r > 0$.

Definition 3.3 (Images). Given $F \subseteq E$, the image $f(F) = \{\vec{y} \in \mathbb{R}^m : \vec{y} = f(\vec{x}) \text{ for some } x \in F\}$.

Definition 3.4 (Bounded Functions). A function f is bounded if $f(E)$ is a bounded subset of \mathbb{R}^m .

Definition 3.5 (Bounded Directions). If $m = 1$ - that is, if the dimension of the image is 1 - we can say that f is bounded from above if $f(E) \subseteq (-\infty, a)$ where $a \in \mathbb{R}$ and similarly, we can define when a function is bounded from below.

Definition 3.6 (Inverse Image). Given $G \subseteq \mathbb{R}^m$, $f^{-1}(G) = \{\vec{x} \in E : f(\vec{x}) \in G\}$. Note that this is not the inverse function, so be careful.

Definition 3.7 (Graphs). A graph of f is denoted as $\text{gr } f \subseteq \mathbb{R}^n \times \mathbb{R}^m = \{(\vec{x}, f(\vec{x})) : \vec{x} \in E\}$.

Definition 3.8 (-jectivity). When $f : E \rightarrow F$, $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^m$, f is said to be

1. Injective if $\vec{x} \neq \vec{z} \implies f(\vec{x}) \neq f(\vec{z})$.
2. Surjective if $f(E) = F$.
3. Bijective if it is both surjective and injective. If this is the case, then you can indeed define an inverse function $f^{-1} : F \rightarrow E$ which assigns to every $\vec{y} \in F$ the unique $\vec{x} \in E$ such that $f(\vec{x}) = \vec{y}$.

Definition 3.9 (-creasing). When $f : E \rightarrow \mathbb{R}$ and $E \subseteq \mathbb{R}$, f is

1. increasing if $f(x) \leq f(y)$ whenever $x < y$
2. decreasing if $f(x) \geq f(y)$ whenever $x > y$.
3. strictly increasing if $f(x) < f(y)$ whenever $x < y$
4. strictly decreasing if $f(x) > f(y)$ whenever $x > y$.
5. *monotone* if it is one of these 4.

4 Limits of Functions

Definition 4.1 (Limit of $f(x)$). Let $E \subseteq \mathbb{R}^n$, $f : E \rightarrow \mathbb{R}^m$, $\vec{x}_0 \in \text{acc } E$. We say $\vec{y} \in \mathbb{R}^m$ is the *limit* of f as \vec{x} approaches \vec{x}_0 if: given $\varepsilon > 0$, there exists $\delta > 0$ (depending on f and \vec{x}_0, ε) such that

$$\|f(\vec{x}) - \vec{y}\| < \varepsilon$$

for all $\vec{x} \in E$ with $0 < \|\vec{x} - \vec{x}_0\| < \delta$.

We write $\lim_{\vec{x} \rightarrow \vec{x}_0} f(\vec{x}) = \vec{y}$ or $f(\vec{x}) \rightarrow \vec{y}$ as $\vec{x} \rightarrow \vec{x}_0$.

Remark. *Limits also sometimes do not exist. The proof strategy here is basically the following: you tell me how close you want $f(\vec{x})$ to be to \vec{y} and I tell you how close \vec{x} has to be to \vec{x}_0 . Basically, formulate a δ such that whatever you're showing has to be true.*

There are some criteria for the nonexistence of $\lim_{\vec{x} \rightarrow \vec{x}_0} f(\vec{x})$. If $F \subseteq E$, let $f \upharpoonright F$ be the restriction of f to F . If $F, G \subseteq E$ with $F \cap G = \emptyset$, $\vec{x}_0 \in \text{acc } E \cap \text{acc } G$ and such that $\lim_{\vec{x} \rightarrow \vec{x}_0} f \upharpoonright F(\vec{x}) \neq \lim_{\vec{x} \rightarrow \vec{x}_0} f \upharpoonright G(\vec{x})$ then $\lim_{\vec{x} \rightarrow \vec{x}_0} f(\vec{x})$ does not exist.

Remark. *The limit of a constant function is that constant. (Thanks Sherlock)*

Example 4.2.

$$f(x, y) = \begin{cases} 1 & y = x^2 \text{ \& } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Take $F = \{(x, y) : y = x^2\}$ and $G = \{(x, 0) : x \in \mathbb{R}\}$.

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \upharpoonright F = 1$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \upharpoonright G = 0$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ does not exist.}$$

Example 4.3.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2}$$

Solution. Fix arbitrary $\varepsilon > 0$. Let $\delta = \varepsilon$. If $0 < \|(x, y) - (0, 0)\| < \delta = \varepsilon$, then

$$\begin{aligned} \left\| \frac{x^2 y}{x^2 + y^2} - 0 \right\| &= \frac{x^2 |y|}{x^2 + y^2} \\ &\leq \frac{(x^2 + y^2) \sqrt{x^2 + y^2}}{x^2 + y^2} \\ &= \sqrt{x^2 + y^2} \\ &= \|(x, y)\| \\ &< \delta = \varepsilon \end{aligned}$$

□

At this point, you might be wondering how we made that choice of δ . The idea is that you deal with funny inequalities. Here, once you have your inequality in terms of $\|(x, y)\|$, you then choose your δ in terms of epsilon accordingly. More examples should illuminate these.

Example 4.4. Compute and prove:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^m y}{x^2 + y^2} \cdot (m \in \mathbb{N}, m \geq 2)$$

Solution. We immediately suspect that the limit is 0 because shoving in $y = 0$ yields $\frac{0}{x^2}$ where $x \neq 0$. Fix $\varepsilon > 0$. Now we do some scratch work to figure out what a good choice for δ should be.

$$\begin{aligned} \left| \frac{x^m y}{x^2 + y^2} \right| &= \frac{|x^m| \sqrt{y^2}}{x^2 + y^2} \\ &= \frac{(x^2)^{\frac{m}{2}} \sqrt{y^2}}{x^2 + y^2} \\ &\leq \frac{\sqrt{x^2 + y^2}^{\frac{m}{2}} \sqrt{x^2 + y^2}}{x^2 + y^2} \\ &= \sqrt{x^2 + y^2}^{m-1} \\ &< \|(x, y) - (0, 0)\| \\ &\leq \delta^{m-1} \\ &< \delta \quad (\delta < 1) \end{aligned}$$

Now we realize that it's a good idea to let $\delta = \min\{1, \epsilon\}$. So we then have that

$$\delta \leq \epsilon$$

which finishes the proof since then

$$\|(x, y) - (0, 0)\| < \epsilon$$

as desired. □

If given a limit, you have two options.

- Prove that the limit exists
- Prove it doesn't exist by finding F and G both subsets of the domain such that the limits along F and G disagree.

Example 4.5.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$$

We should suspect by inspection that if the limit exists, it should be zero. But surely we should try $y = kx$. Then let

$$F = \{(x, y) : x = y\}$$

$$G = \{(x, y) : x = -y\}$$

Note that using these subsets yields constant functions. If $(x, y) \in F$, the limit is $\frac{1}{2}$. If $(x, y) \in G$, the limit is $-\frac{1}{2}$. Since the limits of constant functions are constants,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \frac{1}{2} \quad ((x, y) \in F)$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = -\frac{1}{2} \quad ((x, y) \in G)$$

So then the limit doesn't exist because they don't agree.

Example 4.6.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^{100}y}{x - y}$$

Note that the domain is all $(x, y) \in \mathbb{R}^2$ such that $x \neq y$. The idea here is that we try to use the fact that the denominator blows up to get the non-existence of the limit. Suppose

$$F = \{(x, y) : y = 0\}.$$

If the limit exists, it must be the case that it must be equal to 0. Now produce a set

G such that the limit with the domain restricted to G doesn't approach 0. Say

$$G = \{(x, y) : y = x + x^m\}.$$

We haven't chosen m yet, but we're free to do that later. Then if $(x, y) \in G \setminus \{(0, 0)\}$,

$$f(x, y) = \frac{x^{100}(x + x^m)}{-x^m}$$

Let $m = 101$

$$\begin{aligned} &= \frac{x^{101}(1 + x^{100})}{-x^{101}} \\ &= -1 \text{ as } x \rightarrow 0. \end{aligned}$$

So $\lim_{(x,y) \rightarrow (0,0)} f \upharpoonright G = -1$. Then by our criteria for non-existence, the limit does not exist.

Theorem 4.7. Let $E \subseteq \mathbb{R}^n$ and $x_0 \in \text{acc } E$. Suppose $f : E \rightarrow \mathbb{R}^m$ and $g : E \rightarrow \mathbb{R}^m$ satisfy

$$\begin{aligned} \lim_{x \rightarrow x_0} f(\vec{x}) &= \ell_1 \\ \lim_{x \rightarrow x_0} g(\vec{x}) &= \ell_2. \end{aligned}$$

Then,

1. $\lim_{\vec{x} \rightarrow \vec{x}_0} (f + g)(\vec{x}) = \ell_1 + \ell_2$
2. $\lim_{\vec{x} \rightarrow \vec{x}_0} x f(\vec{x}) = c \ell_1$
3. $\lim_{\vec{x} \rightarrow \vec{x}_0} (fg)(\vec{x}) = \ell_1 \ell_2$
4. If $m = 1$ and $\ell_2 \neq 0$ and $g(\vec{x}) \neq 0$ for all $\vec{x} \in E$, then

$$\lim_{x \rightarrow x_0} \left(\frac{f}{g} \right) (\vec{x}) = \frac{\ell_1}{\ell_2}.$$

(4). Fix $\varepsilon > 0$,

$$\begin{aligned} &\left| \frac{f(x)}{g(x)} - \frac{\ell_1}{\ell_2} \right| = \left| \frac{f(x)\ell_2 - g(x)\ell_1}{g(x)\ell_2} \right| \\ &= \frac{|f(x)\ell_2 - \ell_1\ell_2 + \ell_1\ell_2 - g(x)\ell_1|}{|g(x)\ell_2|} \\ &\leq \frac{|\ell_2||f(x) - \ell_1|}{|g(x)\ell_2|} + \frac{|\ell_1||\ell_2 - g(x)|}{|g(x)\ell_2|} \\ &= \frac{|f(x) - \ell_1|}{|g(x)|} + \frac{|\ell_1||\ell_2 - g(x)|}{|g(x)\ell_2|} \end{aligned}$$

Let δ_1 be small enough so that for $0 < \|x - x_0\| < \delta_1$,

$$|g(x) - \ell_2| < \frac{|\ell_2|}{2}$$

$$\begin{aligned} \implies |g(x)| &< \frac{|\ell_2|}{2} \\ \implies \frac{1}{|g(x)|} &< \frac{2}{|\ell_2|} \end{aligned}$$

Plugging this estimate into the above, we obtain

$$\left| \frac{f(x)}{g(x)} - \frac{\ell_1}{\ell_2} \right| \leq \frac{|f(x) - \ell_1|}{|\ell_2|} + \frac{2|\ell_1|}{|\ell_2|^2} |\ell_2 - g(x)|$$

Then choose δ_2 such that if $0 < \|x - x_0\| < \delta_2$

$$|f(x) - \ell_1| < \frac{\varepsilon |\ell_2|}{4}$$

and δ_3 such that if $0 < \|x - x_0\| < \delta_3$

$$|\ell_2 - g(x)| < \frac{|\ell_2|^2}{4|\ell_1| + 1} \varepsilon$$

Then if we choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, plugging in the above inequalities gives us

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{\ell_1}{\ell_2} \right| &< \varepsilon \frac{|\ell_2|}{4} \cdot \frac{2}{|\ell_2|} + \frac{2|\ell_1|}{|\ell_2|^2} \cdot \frac{|\ell_2|^2 \varepsilon}{4|\ell_1| + 1} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

□

Theorem 4.8 (Squeeze Theorem). Let $E \subseteq \mathbb{R}^n$ and $x_0 \in \text{acc } E$. Let f, g, h be functions from E to \mathbb{R} such that

$$\lim_{x \rightarrow x_0} g(x) = \ell = \lim_{x \rightarrow x_0} h(x)$$

and

$$g(x) \leq f(x) \leq h(x)$$

for all $x \in E$. Then

$$\lim_{x \rightarrow x_0} f(x) = \ell.$$

Proof. Fix $\varepsilon > 0$. Let δ_1 and δ_2 be small enough so that for all $|x_0 - x| < \delta_1$, or $|x_0 - x| < \delta_2$, $|g(x) - \ell| < \varepsilon$, or $|h(x) - \ell| < \varepsilon$ respectively. These inequalities imply that $g(x) > \ell - \varepsilon$ if $0 < |x - x_0| < \delta_1$ and $h(x) < \ell + \varepsilon$ if $0 < |x - x_0| < \delta_2$. Then,

$$\ell - \varepsilon \leq g(x) \leq f(x) \leq h(x) \leq \ell + \varepsilon$$

if $0 < \|x - x_0\| < \min\{\delta_1, \delta_2\}$. Thus

$$|f(x) - \ell| < \varepsilon$$

if $0 < \|x - x_0\| < \delta$.

□

Theorem 4.9. Let $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^m$, $x \in \text{acc } E$, and $f : E \rightarrow F$, $g : F \rightarrow \mathbb{R}^p$ be functions such that

$$\lim_{x \rightarrow x_0} f(x) = \ell \in \mathbb{R}^m$$

$$\ell \in \text{acc } F$$

$$\lim_{y \rightarrow \ell} g(y) = L \in \mathbb{R}^p$$

Assume that either

- (i) $\exists \delta_1 > 0$ such that $f(x) \neq \ell$ for all $x \in E$ with $0 < \|x - x_0\| < \delta_1$. or
- (ii) $\ell \in F$ and $g(\ell) = L$.

Then

$$\lim_{x \rightarrow x_0} g(f(x)) = L.$$

Proof left as an exercise for the reader (lol)

Remark. This is nice because we can use this to change variables and perform funny substitutions in limits.

Example 4.10. Compute:

$$\lim_{x \rightarrow 0} \frac{\log(1 + \sin x)}{x}.$$

Solution. Recall that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$. We have that

$$\lim_{x \rightarrow 0} \frac{\log(1 + \sin x)}{x} = \lim_{x \rightarrow 0} \frac{\log(1 + \sin x)}{\sin x} \cdot \frac{\sin x}{x}$$

Let $f(x) = \sin x$ and $g(y) = \frac{\log(1+y)}{y}$. Let's check the assumptions from the above theorem.

- (i) $\lim_{x \rightarrow 0} f(x) = 0 = \ell$
- (ii) $0 \in \text{acc}(F) = \text{acc}((-1, 0) \cup (0, 1))$.
- (iii) $\lim_{y \rightarrow 0} g(y) = 1 = L$
- (iv) For $|x - 0| < \frac{\pi}{2}$, $f(x) \neq 0$.

By the previous theorem, $\lim_{x \rightarrow 0} g(f(x)) = 1$. Then you can use the limit product rule to show that

$$\lim_{x \rightarrow 0} \frac{\log(1 + \sin x)}{\sin x} \cdot \frac{\sin x}{x} = 1 \cdot 1 = \boxed{1}$$

□

5 Continuity

Definition 5.1 (Isolated Points). Let $E \subseteq \mathbb{R}^n$. A point $x_0 \in E$ is called an isolated point of E if there exists $\delta > 0$ such that

$$B(x_0, \delta) \cap E = \{x_0\}.$$

In other words, a point is isolated if you can create a ball that has x_0 as the only element of E and elements in the complement of E .

Definition 5.2 (Continuous Functions). Let $E \subseteq \mathbb{R}^n$ and $x_0 \in \text{acc } E$. Given a function $f : E \rightarrow \mathbb{R}^m$, we say f is continuous at x_0 if for any $\varepsilon > 0$, there exists δ such that for all $x \in E$ with $\|x_0 - x\| < \delta$,

$$\|f(x) - f(x_0)\| < \varepsilon.$$

If f is continuous at every point $x \in E$, then we say f is continuous on E and $f \in C(E)$ (or $f \in C^0(E)$).

Theorem 5.3. Let $E \subseteq \mathbb{R}^n$, $x_0 \in E$, $f : E \rightarrow \mathbb{R}^m$.

- i.) If x_0 is an isolated point of E , then f is continuous at x_0 .
- ii.) If x_0 is an accumulation point, then f is continuous at x_0 iff $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

Proof. i.) If $x_0 \in \text{iso } E$, then there exists $\delta > 0$ such that $B(x_0, \delta) \cap E = \{x_0\}$. In particular, if $\|x - x_0\| < \delta$ & $x \in E$, then $x = x_0$. To check continuity at x_0 , fix $\varepsilon > 0$. Take our δ from above. Then

$$\begin{aligned} \|x_0 - x\| < \delta \text{ \& } x \in E \\ \implies x = x_0 \\ \implies \|f(x) - f(x_0)\| = 0 < \varepsilon \end{aligned}$$

□

Theorem 5.4. Let $E \subseteq \mathbb{R}^n$, $x_0 \in E$, $f : E \rightarrow \mathbb{R}^m$, and $g : E \rightarrow \mathbb{R}^m$ such that f and g are continuous at x_0 . Then,

- i.) $f + g$, cf , cg , $f \cdot g$ are continuous at x_0 .
- ii.) If $m = 1$, then $\frac{f}{g}$ (restricted to the set where $g(x) \neq 0$) is also continuous at x_0 if $g(x_0) \neq 0$.

Proof. Trivial and left as an exercise for the reader.

□

Remark. If $f : E \rightarrow \mathbb{R}^m$ is given by

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

then f is continuous at $x_0 \in E \iff f_i$ is continuous at x_0 for each $1 \leq i \leq m$.

Theorem 5.5. Let $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^m$, $x_0 \in E$, $f : E \rightarrow F$, and $g : F \rightarrow \mathbb{R}^p$ be such that f is continuous at $f(x_0)$. Then $g \circ f : E \rightarrow \mathbb{R}^p$ is continuous at x_0 .

You have to be careful with continuity when taking inverses.

Example 5.6. This is continuous on the whole domain.

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ x - 1 & 2 < x \leq 3 \end{cases}$$

This isn't:

$$f^{-1}(x) = \begin{cases} x & 0 \leq x \leq 1 \\ x + 1 & 2 < x \leq 3 \end{cases}$$

So if a function is continuous, it's not necessarily the case that its inverse is also continuous. See Leoni's notes for example where continuity of f^{-1} is guaranteed.

6 Directional Derivatives and Differentiability

Definition 6.1 (Directional Derivative). Let $E \subseteq \mathbb{R}^n$, $f : E \rightarrow \mathbb{R}$, and $x_0 \in E$. Given a direction $v \in \mathbb{R}^n / \{0\}$, let L be the line through x_0 in direction v .

$$L = \{x_0 + tv : t \in \mathbb{R}\}$$

Assume x_0 is an accumulation point of $(L \cap E)$. Then the directional derivative of f at x_0 in direction v is

$$\frac{\partial f}{\partial v}(x_0) = \lim_{t \rightarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

Definition 6.2 (Partial derivatives in the coordinate directions). With the same setup as the previous definition, $v = e_i$ (Google standard basis)

$$\frac{\partial f}{\partial e_i}(x_0) = \frac{\partial f}{\partial x_i}(x_0) = D_i f(x)$$

is the partial derivative of f with respect to x_i at x_0 .

Example 6.3. $E = \mathbb{R}$, $v = 1$

$$\frac{\partial f}{\partial 1}(x_0) = \lim_{t \rightarrow 0} \frac{f(x_0 + 1 \cdot t) - f(x_0)}{t} = f'(x_0)$$

Recall on that on \mathbb{R} , if f is differentiable at x_0 , then f is continuous at x_0 .

Example 6.4. The existence of directional derivatives does not imply continuity.

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Take a vector $v \in \mathbb{R}^2$, with $v_1^2 + v_2^2 = 1$.

$$I_t := \frac{f(0 + tv) - f(0)}{t} = \frac{f(t(v_1, v_2))}{t} = \frac{(tv_1)^2 tv_2}{(tv_1)^4 + (tv_2)^2}$$

$$I_t = \frac{1}{t} \left(\frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2} \right)$$

If $v_2 \neq 0$,

$$\lim_{t \rightarrow 0} I_t = \frac{v_1^2 v_2}{v_2^2} = \frac{\partial f}{\partial v}(0)$$

If $v_2 = 0$, then $I_t = 0$, which also implies that the directional derivative should be 0.
In sum:

$$\frac{\partial f}{\partial v}(0, 0) = \begin{cases} \frac{v_1^2}{v_2} & \text{if } v_2 \neq 0 \\ 0 & \text{if } v_2 = 0 \end{cases}$$

All this tells us is that the directional derivative exists. So, is f continuous at $(0, 0)$? Differentiability is stronger than continuity in functions from $\mathbb{R} \rightarrow \mathbb{R}$, but this isn't generally the case in higher-dimensional functions. Claim: f is not continuous at $(0, 0)$.

$$f(x, x^2) = \frac{1}{2}$$

$$f(x, 0) = 0$$

The limit at these paths are not equal, which implies discontinuity at $(0, 0)$.

Remark. If $f : \mathbb{R} \rightarrow \mathbb{R}$, then directional differentiability implies continuity. Given the example above, the same doesn't hold for higher-dimensional functions. (Note the distinction between directional derivatives and the stronger notion of differentiability below.)

Definition 6.5 (Differentiability). Let $E \subseteq \mathbb{R}^n$, $f : E \rightarrow \mathbb{R}$, and $x_0 \in E$ an accumulation point of E . Then f is differentiable at x_0 if there exists $\vec{b} \in \mathbb{R}^n$ depending

on x_0 such that

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - \vec{b} \cdot (x - x_0)}{\|x - x_0\|} = 0$$

Remark. $T(x) = \vec{b} \cdot x$ is linear.

Definition 6.6 (Differentials). Say that f as before is continuous at $x_0 \in E$, we call $df(x_0) \cdot v = \vec{b} \cdot v$ the differential of f at x_0 .

Theorem 6.7. Let $E \subseteq \mathbb{R}^n$, $f : E \rightarrow \mathbb{R}$, and $x_0 \in E$ an accumulation point of E . If f is differentiable at x_0 , then f is continuous at x_0 .

Proof.

$$\begin{aligned} f(x) - f(x_0) &= f(x) - f(x_0) - \vec{b} \cdot (x - x_0) + \vec{b} \cdot (x - x_0) \\ &= \frac{f(x) - f(x_0) - \vec{b} \cdot (x - x_0)}{\|x - x_0\|} \|x - x_0\| + \vec{b} \cdot (x - x_0) \\ &\leq \left\| \frac{f(x) - f(x_0) - \vec{b} \cdot (x - x_0)}{\|x - x_0\|} \right\| \|x - x_0\| + \|\vec{b}\| \|x - x_0\| \\ &\implies \lim_{x \rightarrow x_0} |f(x) - f(x_0)| \leq 0 = 0 \end{aligned}$$

□

Theorem 6.8. $E \subset \mathbb{R}^n$, $f : E \rightarrow \mathbb{R}$, $x_0 \in E^\circ$. If f is differentiable at x_0 , then

- i) There exists $\frac{\partial f}{\partial v}(x_0)$ for all $v \in \mathbb{R}^n$, $\|v\| = 1$ and $\frac{\partial f}{\partial v}(x_0) = df(x_0)v$.
- ii) For all $v \in \mathbb{R}^n$, $\|v\| = 1$,

$$\frac{\partial f}{\partial v}(x_0) = \sum_i^n \frac{\partial f}{\partial x_i}(x_0) v_i$$

In other words,

$$\frac{\partial f}{\partial v}(x_0) = \nabla f(x_0) \cdot v.$$

Proof. $x_0 \in E^\circ \implies \exists r > 0, B(x_0, r) \subset E$. Take $v \in \mathbb{R}^n$, $\|v\| = 1$, $x = x_0 + tv$. Then

$$\|x - x_0\| = \|x_0 + tv - x_0\| = t\|v\| = t$$

If we choose $t < r$, then $x \in B(x_0, r) \subset E$. (To be continued...)

□