Interior Point Methods: The Short-Step Barrier Method

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Nov. 20, 2025

1 Introduction

NOTE TO PROFS: I worked this out as a draft, I expect there to be some mathematical errors and typos throughout. I tried to do the proofs covering up the materials I was citing, sometimes I couldn't do so, so some math may be incorrect or messy. Theorem 4 is left unproved so I will need to fix that, although I haven't figured out how to do so just yet. I look to tighten up my language and try to make things more concise and formal. Please comment if you see anything noteworthy, especially notational clarity or any major math errors, thanks!

Linear algebra is a powerful area of mathematics that is used throughout many disciplines such as computer science, engineering, economics, and statistics. One area where linear algebra plays a foundational role is in optimization, which is the study of selecting the best decision or input to a function according to some constraints or criterion. Optimization has applications ranging from resource allocation to machine learning.

An influential algorithmic framework for solving optimization problems is seen through interior point methods(IPM). IPMs are especially useful on what are called convex optimization problems. The method elegantly moves through the interior of our region we are optimizing over. Although IPMs are slower than more modern methods, they have strong theoretical guaranties and have been studied extensively. They also retain a great practical effectiveness and are still used in many solvers today. The analysis of IPMs bridges many disciplines, such as linear algebra, convex analysis, and calculus.

2 Mathematical Tools

2.1 Convex Analysis

Definition 1. *A set $Q \subseteq \mathbb{R}^n$ is convex iff for all $\lambda \in [0,1]$ and $x, y \in Q$*

$$\lambda x + (1-\lambda)y \in Q$$

That is the entire line segment between any two points in Q is also contained within Q.

Definition 2. *A function $f : Q \to \mathbb{R}$ defined on a convex set $Q \subseteq \mathbb{R}^n$ is convex if for all $x, y \in Q$ and $\lambda \in [0,1]$,*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

If Q is open and f is differentiable, this is equivalent to

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle$$

If Q is open and f is twice differentiable, this is equivalent to

$$\nabla^2 f(x) \succeq 0 \quad \text{for all} \quad x \in Q$$

2.2 Self Concordance

We refer to strong nondegenerate self concordance as self concordance in this paper. Note this is not a universal convention. Assume for this section that $Q \subseteq \mathbb{R}^n$ is open and convex.

Definition 3. *Let f be a twice differentiable convex function on Q with $\nabla^2 f(x) \succ 0$ for all $x \in Q$. The intrinsic norm at x is the norm on vectors $v \in \mathbb{R}^n$ defined by*

$$\|v\|_x := \sqrt{v^\top \nabla^2 f(x) v}$$

Definition 4. *A three-times differentiable convex function $f : Q \to \mathbb{R}$ is self-concordant if for all $x \in Q$ and all directions $h \in \mathbb{R}^n$,*

$$| D^3 f(x)[h,h,h] | \leq 2(D^2 f(x)[h,h])^{3/2}$$

Proposition 1. *The function $f(x) = -\sum_{i=1}^n \log x_i$ is self concordant on $\mathbb{R}^n_{++}$ where $\mathbb{R}^n_{++}$ is the positive orthant.*

Proof. For any direction $h \in \mathbb{R}^n$ we have

$$D^2 f(x)[h,h] = \sum_{i=1}^n \frac{h_i^2}{x_i^2}, \qquad D^3 f(x)[h,h,h] = -2 \sum_{i=1}^n \frac{h_i^3}{x_i^3}$$

Hence

$$\left| D^3 f(x)[h,h,h] \right| = 2 \sum_{i=1}^n \frac{|h_i|^3}{x_i^3}$$

Set $a_i := \dfrac{h_i}{x_i}$ for $i = 1, \ldots, n$. Then the previous expressions become

$$\left| D^3 f(x)[h,h,h] \right| = 2 \sum_{i=1}^n |a_i|^3 = 2\|a\|_3^3 \qquad D^2 f(x)[h,h] = \sum_{i=1}^n a_i^2 = \|a\|_2^2$$

Since for any finite-dimensional vector a we have $\|a\|_3 \leq \|a\|_2$, it follows that

$$\|a\|_3^3 \leq \|a\|_2^3 = \left(\|a\|_2^2\right)^{3/2} = \left(D^2 f(x)[h,h]\right)^{3/2}$$

Combining the equalities and inequality above yields

$$\left| D^3 f(x)[h,h,h] \right| = 2\|a\|_3^3 \leq 2\left(D^2 f(x)[h,h]\right)^{3/2}$$

which is exactly the self-concordance inequality. Thus f is self-concordant on $\mathbb{R}^n_{++}$. □

We introduce a second definition of self concordant that is equivalent, except for the fact that this definition only requires twice differentiability. We won't prove equivalence of these here, but for the interested reader check [4].

Definition 5. *A twice differentiable function $f : Q \to \mathbb{R}$ is self concordant if*

1. $\nabla^2 f(x) \succ 0$ *for all* $x \in Q$
2. *Let $B_x(y, r)$ denote a radius r open ball with center at y where the radius is measured with respect to the intrinsic norm $\| \cdot \|_x$. We have that if $x \in Q$ then $B_x(x, 1) \subseteq Q$ and if $y \in B_x(x, 1)$ then*

$$(1 - \|y - x\|_x)\|v\|_x \leq \|v\|_y \leq \frac{\|v\|_x}{1 - \|y - x\|_x}$$

3 Newton's Method

3.1 Newton Method

Algorithm 1 Newton's Method for Unconstrained Convex Optimization

Input: Twice differentiable convex function $f : \mathbb{R}^n \to \mathbb{R}$, initial point x_0, error $\epsilon > 0$
$k \leftarrow 0$
while $\|\nabla f(x_k)\| > \epsilon$ **do**
  Compute the Newton direction
$$N(x_k) := -\nabla^2 f(x_k)^{-1} \nabla f(x_k)$$
  Choose a step size $\eta_k \in (0, 1]$ (e.g., via backtracking line search)
  Update
$$x_{k+1} = x_k + \eta_k N(x_k)$$
  $k \leftarrow k + 1$
end while
return x_k

Theorem 1. *Let $f : Q \to \mathbb{R}$ be self concordant. Denote $T^2_{f,x}(y)$ as the second order Taylor expansion of f around x then for all $x \in Q$ and $y \in B_x(x, 1)$ we have*

$$|f(y) - T^2_{f,x}(y)| \leq \frac{\|y-x\|_x^3}{3(1-\|y-x\|_x)}$$

Proof. Define the function $\phi(a) := f(x + a(y - x))$ for $a \in [0, 1]$. Note we have

$$\phi'(t) = \langle \nabla f(x + a(y-x)), y - x\rangle$$

and

$$\phi''(t) = \langle y - x, \nabla f^2(x + a(y-x))(y-x)\rangle$$

By the fundamental theorem of calculus we have

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2}\phi''(0) + \int_0^1 \int_0^b \phi''(a) - \phi''(0) da db$$

It follows that

$$|f(y) - T^2_{f,x}(y)| \leq \int_0^1 \int_0^b \langle y - x, (\nabla^2 f(x + a(y-x)) - \nabla^2 f(x))(y-x)\rangle dadb$$

Note that we have the following $\frac{\|v\|_y}{\|v\|_x} = \frac{\nabla^2 f(y)}{\nabla^2 f(x)}$. Combining this with (2) of our second characteristic of self concordant functions we have

$$\nabla^2 f(x + a(y-x)) \preceq \frac{1}{(1 - a\|y-x\|_x)^2}\nabla^2 f(x)$$

Plugging this into our previous inequality and taking absolute values we can write

$$\begin{aligned}
|f(y) - T^2_{f,x}(y)| &\leq \int_0^1 \int_0^b \langle y - x, (\nabla^2 f(x + a(y-x)) - \nabla^2 f(x))(y-x)\rangle dadb \\
&\leq \int_0^1 \int_0^b (\frac{1}{(1 - a\|y-x\|_x)^2} - 1)|\langle y - x, \nabla^2 f(x)(y-x)\rangle| dadb \\
&= \|y-x\|_x^2 \int_0^1 \int_0^b (\frac{1}{(1 - a\|y-x\|_x)^2} - 1) dadb \\
&= \|y-x\|_x^2 \int_0^1 (\frac{b}{1 - b\|y-x\|_x} - b) db \\
&= \|y-x\|_x^3 \int_0^1 (\frac{b^2}{1 - b\|y-x\|_x}) db
\end{aligned}$$

Now since $0 \leq b \leq 1$ we have $1 - br \geq 1 - r$ and so

$$\int_0^1 \frac{b^2}{1-br} db \leq \frac{1}{1-r}\int_0^1 b^2 db = \frac{1}{3(1-r)}$$

Applying this with $r = \|y - x\|_x$ we have

$$|f(y) - T^2_{f,x}(y)| \leq \frac{\|y-x\|_x^3}{3(1-\|y-x\|_x)}$$

□

Theorem 2. *For a self-concordant function $f : Q \to \mathbb{R}$, if for an iterate $x_k \in Q$ we have $\|N(x_k)\|_{x_k} < 1$, then*

$$\|N(x_{k+1})\|_{x_{k+1}} \leq \left(\frac{\|N(x_k)\|_{x_k}}{1 - \|N(x_k)\|_{x_k}}\right)^2.$$

Proof. Let $x = x_k$ and $x_{k+1} = x_k + N(x_k)$ be the Newton update. Assuming $\|N(x_k)\|_{x_k} < 1$ we have

$$\begin{aligned}\|N(x_{k+1})\|^2_{x_{k+1}} &= \|\nabla^2 f(x_{k+1})^{-1}\nabla f(x_{k+1})\|^2_{x_{k+1}} \\ &= \langle \nabla f(x_{k+1}), \nabla^2 f(x_{k+1})^{-1}\nabla f(x_{k+1})\rangle \\ &\leq \|\nabla^2 f(x_{k+1})^{-1}\|_{x_k} \|\nabla f(x_{k+1})\|^2_{x_k}\end{aligned}$$

From the second definition of self-concordance, for $y \in B_x(x,1)$ and applying this to $y = x_{k+1}$, we get

$$\|\nabla^2 f(x_{k+1})^{-1}\|_{x_k} \leq \frac{1}{(1 - \|N(x_k)\|_{x_k})^2}$$

Thus,

$$\|N(x_{k+1})\|_{x_{k+1}} \leq \frac{\|\nabla f(x_{k+1})\|_{x_k}}{1 - \|N(x_k)\|_{x_K}}$$

Now we bound $\|\nabla f(x_{k+1})\|_{x_k}$. Using the fundamental theorem of calculus,

$$\nabla f(x_{k+1}) = \nabla f(x_k) + \int_0^1 \nabla^2 f(x_k + tN(x_k))N(x_k)dt$$

Since $\nabla f(x_k) = -\nabla^2 f(x_k)N(x_k)$, this becomes

$$\nabla f(x_{k+1}) = \int_0^1 (\nabla^2 f(x_k + tN(x_k)) - \nabla^2 f(x_k))N(x_k)dt$$

Now taking the norms

$$\begin{aligned}\|\nabla f(x_{k+1})\|_{x_k} &\leq \int_0^1 \|\nabla^2 f(x_k + tN(x_k)) - \nabla^2 f(x_k)\|_{x_k}\|N(x_k)\|_{x_k}dt \\ &\leq \|N(x_k)\|_{x_k}\int_0^1 (\|\nabla^2 f(x_k + tN(x_k))\|_{x_k} - 1)dt\end{aligned}$$

Now for self concordant functions the Hessian is bounded by

$$\|\nabla^2 f(x_k + tN(x_k))\|_{x_k} \leq \frac{1}{(1 - t\|N(x_k)\|_{x_k})^2}$$

It follows that we have

$$\begin{aligned}\|\nabla f(x_{k+1})\|_{x_k} &\leq \|N(x_k)\|_{x_k}\int_0^1 (\frac{1}{(1 - t\|N(x_k)\|_{x_k})^2} - 1)dt \\ &= \|N(x_k)\|_{x_k}[\frac{1}{1 - t\|N(x_k)\|_{x_k}} - t]_0^1 \\ &= \|N(x_k)\|_{x_k}(\frac{1}{1 - \|N(x_k)\|_{x_k}} - 1) \\ &= \frac{\|N(x_k)\|{x_k}^2}{1 - \|N(x_k)\|_{x_k}}\end{aligned}$$

Combining our bounds we have

$$\|N(x_{k+1})\|_{x_{k+1}} \leq \frac{1}{1 - \|N(x_k)\|_{x_k}} \cdot \frac{\|N(x_k)\|^2_{x_k}}{1 - \|N(x_k)\|_{x_k}} = \left(\frac{\|N(x-k)\|_{x_k}}{1 - \|N(x_k)\|_{x_k}}\right)^2$$

□

Theorem 3. *Let $f : Q \to \mathbb{R}$ be self concordant. If it follows that for a point $x \in Q$, $\|N(x)\|_x \leq \frac{1}{9}$, then there exists a minimizer of f, denoted by x^* within distance*

$$\|x^* - x\|_x \leq 3\|N(x)\|_x$$

Moreover, this can be strengthened to that if $\|N(x_k)\|_{x_k} \leq \frac{1}{4}$, then there exists a minimum within distance

$$\|x^* - x_{k+1}\|_{x_k} \leq \|N(x_k)\|_{x_k} + \frac{3\|N(x_k)\|^2_{x_k}}{(1 - \|N(x_k)\|_{x_k})^3}$$

Proof. From theorem 1 we have that

$$T^2_{f,x}(y) - \frac{\|y-x\|_x^3}{3(1-\|y-x\|_x)} \leq f(y)$$

Now expand out the second order Taylor approximation yielding

$$f(x) + \langle \nabla f(x), y-x\rangle + \frac{1}{2}\langle y-x, \nabla^2 f(x)(y-x)\rangle - \frac{\|y-x\|_x^3}{3(1-\|y-x\|_x)} \leq f(y)$$

We have that $\frac{1}{2}\langle y-x, \nabla^2 f(x)(y-x)\rangle = \frac{1}{2}\|y-x\|_x^2$ from the definition of the intrinsic norm. Now see that we can write

$$\langle \nabla f(x), y-x\rangle = \langle [\nabla^2 f(x)]^{-\frac{1}{2}}\nabla f(x), [\nabla^2 f(x)]^{\frac{1}{2}}(y-x)\rangle$$

Combining these observations gives us

$$f(x) + \langle [\nabla^2 f(x)]^{-\frac{1}{2}}\nabla f(x), [\nabla^2 f(x)]^{\frac{1}{2}}(y-x)\rangle + \frac{1}{2}\|y-x\|_x^2 - \frac{\|y-x\|_x^3}{3(1-\|y-x\|_x)} \leq f(y)$$

Applying the Cauchy-Schwartz inequality and using the definition of the Newton direction from earlier we have

$$|\langle [\nabla^2 f(x)]^{-\frac{1}{2}}\nabla f(x), [\nabla^2 f(x)]^{\frac{1}{2}}(y-x)\rangle| \leq \|N(x)\|_x \|y-x\|_x$$

Now returning to our inequality we see that

$$f(x) - \|N(x)\|_x\|y-x\|_x + \frac{1}{2}\|y-x\|_x^2 - \frac{\|y-x\|_x^3}{3(1-\|y-x\|_x)} \leq f(y)$$

From here it follows by inspection that if $\|y-x\|_x \geq \|N(x)\|_x$ then $f(y) \geq f(x)$. For $y \in \overline{B}_x(x, \frac{1}{3})$, note we used $\overline{B}$ to denote the closed ball, we have a minimum x^* here such that $\|x^* - x\|_x \leq 3\|N(x)\|$. Since $f(y) \geq f(x)$ if y is outside the ball it follows that x^* is the minimizer over Q.

Now we assume $\|N(x)\|_x \leq \frac{1}{4}$. Theorem (2) implies

$$\|N(x_{k+1})\|_{x_{k+1}} \leq (\frac{\|N(x_k)\|_{x_k}}{1-\|N(x_k)\|_{x_k}})^2 \leq \frac{1}{9}$$

We use our first result of this theorem to see that f has a minimizer x^* and $\|x^* - x_{k+1}\|_{x_{k+1}} \leq 3\|n(x_{k+1})\|_{x_{k+1}}$. It follows that

$$\begin{aligned} \|x^* - x_{k+1}\|_{x_k} &\leq \frac{\|x^* - x_{k+1}\|_{x_{k+1}}}{1-\|N(x_k)\|_{x_k}} \\ &\leq \frac{3\|N(x_{k+1})\|_{x_{k+1}}}{1-\|N(x_k)\|_{x_k}} \\ &\leq \frac{3\|N(x_k)\|_{x_k}^2}{(1-\|N(x_k)\|_{x_k})^3} \end{aligned}$$

□

Definition 6. *The complexity parameter of a self concordant function f is given as*

$$\theta_f := \sup_x \|N(x)\|_x^2$$

4 Interior Point Methods

First we write a short note on barrier functions

4.1 Barrier Functions

Definition 7. *A barrier function $\phi : Q^\circ \to \mathbb{R}$ must satisfy the following:*

1. *ϕ is self concordant over Q°*

2. *For a sequence $\{x_k\}$ that approaches ∂Q we have that $\phi(x_k)$ approaches $+\infty$*

Proposition 2. *The function $f(x) = -\sum_{i=1}^n \log x_i$ is a barrier function on $\mathbb{R}^n_{++}$. Moreover f is self-concordant and its complexity parameter satisfies $\theta_f = n$.*

Proof. From proposition (1) we have the function $f(x) = -\sum_{i=1}^n \log x_i$ is self concordant on $\mathbb{R}^n$.

If a sequence $x^{(k)} \in \mathbb{R}^n_{++}$ approaches the boundary $\partial\mathbb{R}^n_+$, then at least one coordinate $x_i^{(k)} \to 0^+$. Because $-\log x_i \to +\infty$ as $x_i \to 0^+$, it follows that $f(x^{(k)}) = -\sum_{i=1}^n \log x_i^{(k)} \to +\infty$.

Recall the Newton direction for a function f is given by $N(x) = -\nabla^2 f(x)^{-1} \nabla f(x)$. It follows that

$$\nabla f(x) = -\begin{bmatrix} 1/x_1 \\ \vdots \\ 1/x_n \end{bmatrix} \qquad \nabla^2 f(x)^{-1} = \operatorname{diag}(x_1^2, \ldots, x_n^2)$$

Hence

$$N(x) = -\nabla^2 f(x)^{-1} \nabla f(x) = -\operatorname{diag}(x_i^2)(-[1/x_i]) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x$$

We have that

$$\|N(x)\|_x^2 = N(x)^\top \nabla^2 f(x)\, N(x) = x^\top \operatorname{diag}(1/x_i^2)\, x = \sum_{i=1}^n 1 = n.$$

Taking the supremum over x yields the complexity parameter $\theta_f = n$. $\square$

4.2 Short-Step Interior Point Method

We formulate a standard problem where $c \in \mathbb{R}^n$ and we maximize over the closure of Q

$$\begin{aligned} \text{minimize} \quad & \langle c, x \rangle \\ \text{subject to} \quad & x \in \overline{Q} \end{aligned}$$

Consider $\pi(y) = \operatorname{argmin}_x f_y(x)$ where $f_y(x) = yf(x) + \phi(x)$ where we have that ϕ is a barrier function on our set Q. Note that for a fixed t as we approach the boundary of our set Q that the barrier function value dominates. The idea behind the algorithm is to solve this problem for increasing values of t such that our algorithm draws out what is called a central path and only approaches a boundary when t approaches $+\infty$. For our problem this simplifies to $\pi(y) = \operatorname{argmin}_x yf(x) + \phi(x)$.

We let the solution of this problem be. Call $\pi(0)$ the analytic center of our feasible region. In the short step barrier IPM which is one of the simplest variants we start with a point that is close to our analytic center. Then we geometrically increasing steps in terms of our input to the central path function. We use the Newton method which will quadratically converge given the assumptions above, but this in this algorithm we can even shorten this to just one iteration of the Newton method since this is a good approximation.

Note that if apply this method on our problem above we have the setup that results in updates given by

$$y_{k+1} = qy_k \qquad x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1}(y_{k+1}c + \nabla f(x_k))$$

Algorithm 2 Short-Step Barrier IPM

Input: Geometric growth factor $q = 1 + \frac{1}{8\sqrt{\theta_\phi}}$, initial point $y_1 \approx \pi(0)$, number of iterations T
Initialize: Choose sequence $y_1, \ldots, y_T$ with $y_{t+1} = (1+q)y_i$
for $i = 1, \ldots, T$ **do**
    Compute

$$x_{k+1} = x_k - [\nabla^2 \phi(x_k)]^{-1}[y_{k+1}c + \nabla \phi(x_k)]$$

    using one step of Newtons method,
end for
Return: x_{k+1}

Proposition 3. *If we have a point close enough to our central path such that $\|N_{y_k}(x_k)\|_{x_k} \leq \frac{1}{9}$ then it follows that the next iterate maintains that same bound on proximity. That is $\|N_{y_{k+1}}(x_{k+1})\|_{x_{k+1}} \leq \frac{1}{9}$*

Proof. Examine the Newton distance and use the triangle inequality which yields

$$\begin{aligned}
\|N_{y_{k+1}}(x_k)\|_{x_k} &= \| - [\nabla^2 f(x_k)]^{-1}(y_{k+1}c + \nabla f(x_k))\|_{x_k} \\
&= \|\frac{y_{k+1}}{y_k} N_{y_k}(x_k) + (\frac{y_{k+1} - y_k}{y_k})[\nabla^2 f(x_k)]^{-1} \nabla f(x_k)\|_{x_k} \\
&\leq \frac{y_{k+1}}{y_k}\|N_{y_k}(x_k)\|_{x_k} + |\frac{y_{k+1} - y_k}{y_k}|\|[\nabla^2 f(x_k)]^{-1} \nabla f(x_k)\|_{x_k} \\
&\leq \frac{y_{k+1}}{y_k}\|N_{y_k}(x_k)\|_{x_k} + |\frac{y_{k+1} - y_k}{y_k}|\|[\nabla^2 f(x_k)]^{-1} \nabla f(x_k)\|_{x_k} \\
&\leq \frac{y_{k+1}}{y_k}\|N_{y_k}(x_k)\|_{x_k} + |\frac{y_{k+1} - y_k}{y_k}|\sqrt{\theta_\phi} \\
&\leq \frac{1}{9}(1 + \frac{1}{8\sqrt{\theta_\phi}}) + \frac{1}{8\sqrt{\theta_\phi}}\sqrt{\theta_\phi} \\
&\leq \frac{1}{4}
\end{aligned}$$

Now using theorem (2) we have

$$\|N_{y_{k+1}}(x_{k+1})\|_{x_{k+1}} \leq (\frac{\|N_{y_{k+1}}(x_k)\|_{x_k}}{1 - \|N_{y_{k+1}}(x_k)\|_{x_k}})^2 \leq \frac{1}{9}$$

□

Theorem 4. *For a barrier function ϕ and for any $t > 0$ we have that*

$$\langle c, \pi(y)\rangle \leq \min_{z \in \overline{Q}} \langle c, z\rangle + \frac{1}{y}\theta_\phi$$

Moreover, if we have a point $x \in Q$ such that $\|x - \pi(y)\|_x \leq \frac{1}{6}$, then

$$\langle c, x\rangle \leq \min_{z \in \overline{Q}} \langle c, z\rangle + \frac{6}{5y}\theta_\phi$$

Proof. □

Theorem 5. *If we begin the short step barrier IPM method an initial point x_1 such that $\|N_{y_1}(x_1)\|_{x_1} \leq \frac{1}{9}$ then it follows that for any relative error $\epsilon > 0$ we attain an ϵ-error $\langle c, x_T\rangle \leq \min_{x \in \overline{Q}}\langle c, x\rangle + \epsilon$ in a number of iterations*

$$T = \lceil 10\sqrt{\theta_\phi}\log(\frac{6\theta_\phi}{5\epsilon y_1})\rceil$$

Proof. For y_1 to reach a fixed $y \in \mathbb{R}$ it takes a number of iterations greater or equal to

$$T = \lceil \frac{\log \frac{y}{y_1}}{\log(1 + \frac{1}{8\sqrt{\theta_\phi}})}\rceil$$

Using the face $\frac{1}{\log(1+x)} \leq \frac{5}{4x}$ for all $x \in [0, \frac{1}{2}]$ we have

$$T \leq \lceil 10 \log(\frac{y}{y_1}) \sqrt{\theta_\phi} \rceil$$

By theorem (4) we know that the ϵ-error is given by $\frac{6\theta_\phi}{5y}$ if $\|x_T - \pi(y_T)\|_{x_T} \leq \frac{1}{6}$. We plug in $y = \frac{6\theta_\phi}{5\epsilon}$ which yields

$$T \leq \lceil 10 \sqrt{\theta_\phi} \log(\frac{6\theta_\phi}{5\epsilon y_1}) \rceil$$

In worst case we must have equality. □

Note on the case where our barrier function is given as in proposition (2) we have a complexity to attain an ϵ-suboptimal solution in

$$O(\sqrt{n} \log(\frac{n}{\epsilon}))$$

5 Conclusion

The interior point method is a class of algorithms that are important to the field of optimization. Here, we just scratched the surface of the most basic path following method known as the short step barrier IPM, but many methods used in commercial solvers and in research are much more involved. In future work I would look to expand to other common IPMs such as long step and predictor corrector methods. Although many modern methods have moved on from interior point methods, it remains an incredibly practical and rich framework for optimization problems.

References

[1] Farina, G. (2025). *Gabriele Farina - Central path and interior-point methods.* MIT.edu.
`https://www.mit.edu/~gfarina/2025/67220s25_L20_barrier/`

[2] Musco, C. (n.d.). *Lecture 17: Interior Point Methods.*
`https://www.cs.princeton.edu/courses/archive/fall18/cos521/Lectures/lec17.pdf`

[3] Nemirovski, A. S., & Todd, M. J. (2008). *Interior-point methods for optimization. Acta Numerica, 17*, 191–234.
`https://doi.org/10.1017/s0962492906370018`

[4] Renegar, J. (2001). *A Mathematical View of Interior-Point Methods in Convex Optimization.* SIAM.
`https://doi.org/10.1137/1.9780898718812`

[5] *Lecture 14: Barrier Method.* (n.d.). Retrieved November 19, 2025, from
`http://www.seas.ucla.edu/~vandenbe/ee236a/lectures/barrier.pdf`