Real Variables Exam 2 Revision

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1 Introduction

1.) Prove or disprove: There exists a continuous function $f : \mathbb{R} \to \mathbb{Q}$ such that $f(q) = q$ for all $q \in \mathbb{Q}$.

Proof. Assume there exists a continuous function $f : \mathbb{R} \to \mathbb{Q}$ such that $f(q) = q$ for all $q \in \mathbb{Q}$. We can show a contradiction with this function. By sequential continuity of f the image of convergent sequence will converge. So if $(a_n) \to a$, then $f(a_n) \to f(a)$. Defining a convergent sequence (a_n) to be $(a_n) = \{1, 1.4, 1.41, 1.41, 1.4142, \dots\}$, this sequence converges to $\sqrt{2}$. Since all $(a_n) \in \mathbb{Q}$ for all n, $f(a_n) = \{1, 1.4, 1.41, 1.414, 1.4142, \dots\}$, however $f(\sqrt{2}) \neq \sqrt{2}$ since $\sqrt{2} \notin \mathbb{Q}$ thus $f(a_n) \not\to f(a)$. This is a contradiction of sequential continuity, thus it is not possible to have such a function. $\square$

8.) Suppose M is a covering compact space and $f : M \to N$ is continous. Prove that $f(M) \subseteq N$ is covering compact.

Proof. Let C be an open cover of $f(M)$,

$$C = \bigcup_{\alpha \in \lambda} U_\alpha.$$

Then by the open preimage condition the preimages of the open set in C, $f^{-1}(U_\alpha)$ are open and contained in M. Then

$$f^{-1}(C) = \bigcup_{\alpha \in \lambda} f^{-1}(U_\alpha)$$

covers M since every $m \in M$ is in some U_α. Since M is compact there exists a finite subcover of $f^{-1}(C)$ called D (denoted as $U_1, U_2, \ldots U_n$).

$$D = \bigcup_{i}^{n} f^{-1}(U_i)$$

Then $f(D)$ is a finite subcover of $f(M)$ since every $m \in M$ is covered in D and thus every $p \in f(M)$. Therefore $f(M)$ is covering compact. $\square$

6.) Let $M = \{(a_n) | a_n = 0 \text{ or } a_n = 1 \ \ \forall n\}$ be the metric space consisting of binary sequences, with metric:

$$d((a_n),(b_n)) = \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}$$

Prove: M is compact.

Proof. Given a sequence in M we can construct a subsequence that converges. We set up a series of steps that will construct a subsequence that converges. Let (A_n) denote the sequence in M so that $A_i = (a_i) = \{a_{i,1},\ a_{i,2},\ a_{i,3} \dots\}$ with each $a_{i,j} \in \{\{0\},\{1\}\}$. The first term in each sequence, $a_{1,1}, a_{2,1}, a_{3,1} \dots$, we will have 0 repeats infinite times and/or 1 repeats infinite times since only $0, 1$ are options and there are infinite sequences. Without loss of generality suppose $z_1 \in \{\{0\},\{1\}\}$ repeats infinitely many times, then define a subsequence,

$$B_n = \{a_i \in A_n, i > 1 | a_{i,1} = z_1\}.$$

Since z_1 repeats infinite number of times we know B_n is never ending. Let the sequences in B_n be defined as $b_1, b_2, \dots$, we know that every sequence in B_n it is true that $b_{i,1} = z_1$. Similarily we will define a subsequence of B_n. Again for the second term of $(b_1),(b_2),(b_3),\dots$ we assume without loss of generality for $z_2 \in \{\{0\},\{1\}\}$ that $b_{i,2} = z_2$ for $i\mathbb{N}$ happens an infinite number of times. Then define the subsequence as

$$C_n = \{b_i \in B_n, i > 1 | b_{i,2} = z_2\}.$$

We exclude (b_1) from our list to ensure the first sequence in $A_n, B_n, \dots$ is distinct. We shall call these sequences $c_1, c_2, \dots,$. We continue this process creating new subsequences that Remember that for each $A_n, B_n, C_n, \dots,$ it is a subsequence of the original A_n. Now define A_{n_k} as the first sequence in A_n followed by the first sequence in B_n, followed by the first sequence in C_n, and on forever. This is a subsequence of A_n and it converges to $Z_n = \{z_1, z_2, z_3, \dots\}$. We will now show that (A_{n_k}) converges to (Z_n).

For $A_{n_1} \in A_n$ so A_{n_1} can be any sequence, thus A_{n_1} could disagree with Z_n at every term, thus

$$d(A_{n_1}, Z_n) \leq 1,$$

since $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. The second term $A_{n_2} \in B_n$ so $A_{n_2,1} = z_1$ so the max distance of A_{n_2} is $1 - \frac{1}{2}$ since $A_{n_2,1}$ and Z_n agree at the first term, thus

$$d(A_{n_2}, Z_n) \leq \frac{1}{2}.$$

Continuing this trend we note that

$$d(A_{n_3}, Z_n) \leq \frac{1}{4},$$

since A_{n_3} agree on the first and second terms. We can generalize this to

$$d(A_{n_k}, Z_n) \leq \frac{1}{2^k}.$$

Thus for any $\epsilon > 0$, set $N = \log_2 \frac{1}{\epsilon}$ thus for all $n > N$, get we,

$$d(A_{n_k}, Z_n) \leq \frac{1}{2^k} < \frac{1}{2^{\log_2 \frac{1}{\epsilon}}} = \epsilon.$$

Thus (A_{n_k}) is a convergent subsequence of A_n, therefore M is compact. $\square$

7.) Suppose $A \subseteq \mathbb{R}$ is compact, and let $x \in A$. Prove: If (x_n) is a sequence in A such that every convergent subsequence of (x_n) converges to x.

Proof. Suppose for sake of contradiction that (x_n) does not converge x, then for any $\delta > 0$ for all $k \in \mathbb{N}$ there exists $n_k > k$ such that $x_{n_k} \notin B(x, \delta)$. We have the infinite set of n_k form a subsequence (x_{n_k}) of (x_n). By Bolanzo Weierstrass Property, x_n is bounded so since (x_{n_k}) is a subsequence of (x_n) it is also bounded. Also $(x_{n_k}) \in A \backslash B(x, \delta)$ which is closed since $B(x, \delta)$ is open, thus there exists a subsubsequence of (x_{n_k}) that converges. Let $(x_{n_{k_j}}) \to x'$. Since $B(x_{n_{k_j}}, x) > \delta$ for all n_{k_j}. Thus $(x_{n_{k_j}}) \to x' \neq x$, but this contradicts our assumption. Thus (x_n) must converge to x.

$\square$