

2.1 Metric Spaces

As Pugh notes, some problems in \mathbb{R} are more easily solved in a more abstract setting. That setting is Metric Spaces:

Def. Let X be a set and let $d: X \times X \rightarrow \mathbb{R}$. Then (X, d) is a metric space, and d is a metric on X , provided:

- $\forall x, y \in X, d(x, y) \geq 0$ and $d(x, y) = 0 \Rightarrow x = y$
- $\forall x, y \in X, d(x, y) = d(y, x)$
- $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$

Ex. • \mathbb{R}^m is a metric space with metric:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_m - y_m)^2}$$

• Any subset of \mathbb{R}^m is a metric subspace with inherited metric d .

• $\{0, 1\}^{\mathbb{N}} = \{(a_n) \mid a_n = 0 \text{ or } 1 \forall n\}$ is a metric space with metric:

$$d((a_n), (b_n)) = \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n} \quad \text{Q: Proof that this is a metric?}$$

• Any set X is a metric space with the discrete metric:

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

REMEMBER THIS ONE!

Q: Proof that this is a metric?

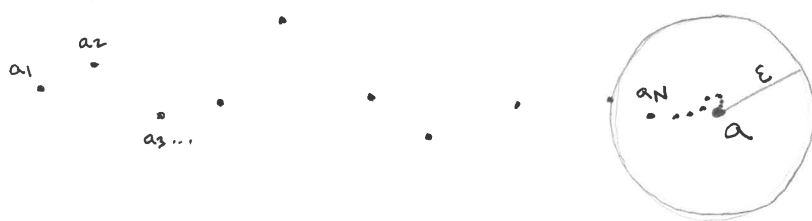
Note that \mathbb{R} with $d(x, y) = |x - y|$ is also a metric space, so anything that's true for metric spaces in general will be true for \mathbb{R} , too!

Maybe the most important mathematical structure in a metric space is a sequence. Let's be rigorous:

Def. A sequence $(a_n) \in X$ is an element of $X^{\mathbb{N}}$, i.e. it is a map $\sigma: \mathbb{N} \rightarrow X$ s.t. $\sigma(n) = a_n$.

Def. A sequence $(a_n) \in X$ converges to a in X if:
 $\forall \epsilon > 0 \exists N \in \mathbb{N}$ s.t. whenever $n \geq N$ we have $d(a_n, a) < \epsilon$.

In almost every case, your intuition from \mathbb{R}^2 will suffice.
 What does the above mean? Draw a picture!



If (a_n) converges to a , we will write $a_n \rightarrow a$ or $\lim_{n \rightarrow \infty} a_n = a$.

Note. The "in X " part of the definition is important!

Let $X = (0, 1) \subseteq \mathbb{R}$. Then $(a_n) = (1/n)$ does not converge in X !

Def. A subsequence $(b_k) \in (a_n)$ is a sequence such that
 $b_k = a_{n_k}$ for some ^{strictly} increasing sequence $(n_k) \in \mathbb{N}$.

I.e., (b_k) consists of some of the elements of (a_n) ,
 in the same order, and infinitely many of them.

Thm: Suppose $(a_n) \in X$ converges to $a \in X$. Then for any subsequence $(a_{n_k}) \in (a_n)$, $\lim_{k \rightarrow \infty} a_{n_k} = a$.

Proof: Let $\varepsilon > 0$. Since $a_n \rightarrow a$, there is some $N \in \mathbb{N}$ s.t. whenever $n \geq N$, $d(a_n, a) < \varepsilon$.

Note that $n_k \geq k$ by the definition of a subsequence, so: whenever $k \geq N$, $n_k \geq N$, so that $d(a_{n_k}, a) < \varepsilon$.

$\therefore a_{n_k} \rightarrow a$. \square

We will be using this result a lot!