Real Variables Homework 3 Final

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1.) A function defined on an interval $[a, b]$ or (a, b) is **uniformly continuous** if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - t| < \delta$ implies that $|f(x) - f(t)| < \epsilon$. (Note that this δ cannot depend on x, it can only depend on ϵ. With ordinary continuity, the δ can depend on both x and ϵ.)

(a) Show that a uniformly continuous function is continuous but continuity does not imply uniform continuity.

Let f be uniformly continuous then for any $\epsilon > 0$, there exists a $\delta_1 > 0$, such that $|x - y| < \delta_1$ implies that for $x, y \in (a, b)$, $|f(x) - f(y)| < \epsilon$. Then f is also continuous with the same δ since δ is not determined by x, y we know that for all ϵ and point $p \in (a, b)$ we know that $|x - p| < \delta_1$ and $|f(x) - f(p)| < \epsilon$.
This doesn't work in the reverse order. Examine the function $sin(\frac{1}{x})$ on the interval $(0, 1)$. $sin(\frac{1}{x})$ is continuous on the interval $(0, 1)$. However we can show it is not uniformly continuous. Set $\epsilon = 2$ and

$$x_n = \frac{1}{\pi/2 + 2n\pi},$$

$$y_n = \frac{1}{3\pi/2 + 2n\pi}.$$

We know that that $|x_n - y_n|$ converges to 0 since each $x_n \to 0$ and $y_n \to 0$ however

$$f(x_n) = sin(\frac{1}{x_n}) = sin(\pi/2 + 2n\pi) = 1,$$

$$f(y_n) = sin(\frac{1}{y_n}) = sin(3\pi/2 + 2n\pi) = -1.$$

So $|f(x_n) - f(y_n)| = 2 \geq 2 = \epsilon$, therefore $sin(1/x)$ is not uniformly continuous.

(b) Is the function $2x$ uniformly continuous on the unbounded interval $(-\infty, \infty)$?
Let $\delta = \frac{\epsilon}{2}$, then $\forall \epsilon > 0$ then for every $x, y \in (-\infty, \infty)$ we have $|x - y| < \delta = \frac{\epsilon}{2}$

$$|f(x) - f(y)| = |2x - 2y| = 2|x - y| < 2\frac{\epsilon}{2} = \epsilon$$

Thus $2x$ is uniformly continuous.

(c) What about x^2?

We can show that for $f(x) = x^2$,

$$|x^2 - y^2| = |x + y||x - y| < \delta|x + y.|$$

So $\delta \leq \frac{\epsilon}{|x+y|}$, however is is a problem. Since for any choice of $\delta = \frac{\epsilon}{c}$, where $c \in \mathbb{R}^+$, there exists an $|x + y| > c$, thus giving us a $|x^2 - y^2| > \epsilon$. Therefore there is no choice of δ that will work. Thus x^2 is not uniformly continuous.

2.) Let (a_n) be a sequence of real numbers. It is **bounded** if the set $A = \{a_1, a_2, \cdots\}$ is bounded. The **limit supremum**, or lim sup, of a bounded sequence (a_n) as $n \to \infty$ is

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} (\sup_{k\geq n} a_k)$$

(a) Why does the lim sup exist?

The limit supremum gives the bound on the sequence as n increases. The whole sequence of a_n wouldn't be bounded by the limit supremum, but it will give us the bound as $n \to \infty$ on a_n. If $a_n \to a$, then

$$\limsup_{n\to\infty} a_n = a.$$

(b) If $\sup\{a_n\} = \infty$, how should we define $\limsup_{n\to\infty} a_n$.

$\limsup_{n\to\infty} a_n = \infty$, because as a_n diverges the supremum of a_n also diverges.

(c) If $\lim_{n\to-\infty} a_n = -\infty$, how should we define $\limsup a_n$

$\limsup_{n\to\infty} a_n = -\infty$

(d) When is it true that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

$$\limsup_{n\to\infty} ca_n = c\limsup_{n\to\infty} a_n?$$

When is it true they are unequal? Draw pictures that illustrate these relations.

These equations hold when $|a_n| < K_1$ and $|b_n| < K_2$, when both a_n and b_n are bounded. When $a_n \to a$ and $b_n \to b$ we know that $\limsup a_n = a$ and $b_n = b$. Then $\limsup(a_n + b_n) = a + b$ as well as $\limsup ca_n = ca$. Looking at a oscillating case for example $a_n = sin(n\pi/3)$ and $b_n = -sin(n\pi/3)$ we see that $\limsup(a_n + b_n) = 0$ while $\limsup a_n + \limsup b_n = 2$.

These equations aren't equal when a_n or b_n is not bounded. For example $a_n = n$ and $b_n = -n$ when $\limsup a_n + b_n = \limsup 0 = 0$, while $\limsup a_n + \limsup b_n = \infty + -\infty$, which is a undefined quantity.

(e) Define the **limit infimum**, or lim inf, of a sequence of real numbers, and find a formula relating it to the limit supremum.

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} (\inf_{k\ge n} a_n)$$

Both lim sup and lim inf are looking at a sequence at infinity, but lim sup is the upper bound and lim inf is the lower bound so

$$\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n.$$

They will only be equal when a_n converges. Since if $a_n \to a$, then the upper and lower bound at infinity will be a.

(f) Prove that $\lim_{n\to\infty} a_n$ exists if and only if the sequence (a_n) is bounded and $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$

$$\lim_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$$

$$\lim_{n\to\infty} a_n \ge \liminf_{n\to\infty} a_n$$

$$\liminf_{n\to\infty} a_n \le \lim_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$$

3.) Let $X = [0, 1)$ and define $d : X \times X \to R$ by:

$$d(a, b) = \min\{|a - b|, 1 - |a - b|\}.$$

Prove that d is a metric on X. Describe the metric space (X, d) geometrically.

First we prove positive definite or that $d(a, b) \geq 0$. Since $|a - b| \geq 0$, we are only concerned with $1 - |a - b|$. Let $a, b \in X$, then $|a - b| < 1$, so $1 - |a - b| > 1 - 1 = 0$, the minimum of two number greater than or equal to zero is also greater than or equal to zero so $d(a, b) \geq 0$.

Next we show that $d(a, b) = 0$ iff $a = b$, we will prove both directions. Let $d(a, b) = 0$, then $0 = \min\{|a - b|, 1 - |a - b|\}$. We know from above that $1 - |a - b| > 0$ so $0 = |a - b|$, then $a = b$. Now the other way, let $a = b$, then $d(a, b) = \min\{|a - b|, 1 - |a - b|\} = \min\{|a - a|, 1 - |a - a|\} = \min\{0, 1\} = 0$. Bingo!

We know $|a - b| = |b - a|$, so symmetry is simple. We start with $d(a, b) = \min\{|a - b|, 1 - |a - b|\} = \min\{|b - a|, 1 - |b - a|\} = d(b, a)$. Thus this metric is symmetric.

Finally the triangle inequality $d(a, c) = \min\{|a - c|, 1 - |a - c|\}$. We want the following to be true,

$$\min\{|a - c|, 1 - |a - c|\} \leq \min\{|a - b|, 1 - |a - b|\} + \min\{|b - c|, 1 - |b - c|\}.$$

The right side is the sum of two minimums so it would be equivalent to the minimum of all four possible sums, this can be written as

$$\min\{|a - c|, 1 - |a - c|\} \leq \min\{|a - b| + |b - c|,\ 1 - |a - b| + |b - c|, 1 - |b - c| + |a - b|, 2 - |a - b| - |b - c|\}.$$

If we can show that all four possibilities on the right side are greater than or equal to the value on the left then we can say this is true.

Case 1: We shall start with $|a - b| + |b - c|$, by the triangle inequality

$$|a - c| \leq |a - b| + |b - c|.$$

We only need to show $|a - b| + |b - c|$ is greater than $|a - c|$ or $1 - |a - c|$. Here we showed that $|a - c| \leq |a - b| + |b - c|$. Since either $|a - c| \geq 1 - |a - c|$ or $|a - c| < 1 - |a - c|$. If $|a - c| \leq 1 - |a - c|$ we have shown $|a - c| \geq 1 - |a - c|$ thus $1 - |a - c| \leq |a - b| + |b - c|$ so $\min\{|a - c|, 1 - |a - c|\} \leq |a - b| + |b - c|$. However if $|a - c| < 1 - |a - c|$ then $\min\{|a - c|, 1 - |a - c|\} = |a_c| \leq |a - b| + |b - c|$. This works in both directions, so in general given $x, y, z \in \mathbb{R}$ if you show that $x \leq z$, then you know that $\min\{x, y\} \leq z$. So in each subsequent case we only prove our value is greater than or equal to either $|a - c|$ or $1 - |a - c|$.

Case 2: Next we examine $1 - |b - c| + |a - b|$. By the triangle inequality $|b - c| \leq |b - a| + |a - c|$, multiplying each side by -1 gives us $-|b - c| \geq -|b - a| - |a - b|$. Using that with $1 - |b - c| + |a - b|$ gives us.

$$1 - |b - c| + |a - b| \geq 1 - |b - a| - |a - c| + |a - b|,$$

$$1 - |b - c| + |a - b| \geq 1 - |a - b| - |a - c| + |a - b|,$$

$$1 - |b - c| + |a - b| \geq 1 - |a - c|.$$

Thus case 2 stands as well.

Case 3: $1 - |a - b| + |b - c|$ works similar to case 2. By the triangle inequality $|a - b| \leq |a - c| + |b - c|$ as well as $-|a - b| \geq -|a - c| - |b - c|$ so we can rewrite it as,

$$1 - |a - b| + |b - c| \geq 1 - |a - c| - |b - c| + |b - c|,$$

$$1 - |a - b| + |b - c| \geq 1 - |a - c|.$$

So case 3 stands as well.

Case 4: The last one is $2 - |a - b| - |b - c|$. Without loss of generality we assume that $a \geq b \geq c$, then we know that

$$a - c < 1,$$

$$2(a - c) < 2$$

$$a + a - c - c < 2,$$
$$a - b + a - c + b - c < 2.$$

Since $a \geq b \geq c$, we know that $a - b$, $a - c$ and $b - c$ are all positive so we can write them with absolute values giving us,

$$|a - b| + |a - c| + |b - c| < 2,$$
$$|a - c| < 2 - |a - b| - |b - c|.$$

Thus the last case stands.
Therefore we have a metric!
Geometrically we can imagine X as a unit circle with 0 and 1 connecting at the same point called this point O (for origin). For two points a and b on the circle the metric $d(a, b)$ selects the shortest route from a to b. If the shortest route doesn't cross the point O then $|a - b|$ is used by the metric if the shortest route crosses O then $1 - |a - b|$ is used.

4.) Assume that every bounded increasing sequence in $\mathbb{R}$ converges. Prove that this implies the Least Upper Bound property of $\mathbb{R}$.

Let X be a bounded nonempty set. We will show there is a least upper bound given an increasing bounded sequence converges. Let l be a bound on X and let $x \in X$ be some point in X. We shall define two sequences a_n and b_n recursively with $a_1 = x$ and $b_1 = l$,

$$a_n = \begin{cases} \frac{a_{n-1}+b_{n-1}}{2} & \text{if } \frac{a_{n-1}+b_{n-1}}{2} \text{ is not an upper bound on X} \\ a_{n-1} & \text{otherwise,} \end{cases}$$

$$b_n = \begin{cases} \frac{a_{n-1}+b_{n-1}}{2} & \text{if } \frac{a_{n-1}+b_{n-1}}{2} \text{ is an upper bound on X} \\ b_{n-1} & \text{otherwise.} \end{cases}$$

At each step either a_n or b_n will move half the distance between the two. The initial distance between a_1 and b_1 is $|l - x|$. This implies that $|a_n - b_n| = \frac{l-x}{2^{n-1}}$, thus $|a_n - b_n|$ converges to 0. So if $a_n \to k$, then $b_n \to k$. Since a_n is an increasing bounded sequence ($a_n < l$) we know it converges. Hence $b_n \to k$, which by our definition of k is an upper bound. k is the least upper bound, because if k is not the least upper bound then we could find some N so that for $n > N$, $b_n < k$ which implies that $b_n \not\to k$. Therefore the Least Upper Bound Property holds for $\mathbb{R}$.

5.) Let $(x_n) \subseteq \mathbb{R}$. Prove that (x_n) contains a monotone subsequence (that is, a subsequence which is either increasing or decreasing).

We shall define an important definition to solve this problem. Let a_k be a **peak** in a_n if for all $m > k$ we have $a_k \geq a_m$. So all points of a_n after a_k are less than a_k. Now inspecting two distinct cases we can define a monotone subsequence for each.

Case 1: There are a finite number of peaks, we shall list them as $a_{k_1}, a_{k_2}, a_{k_3}, \cdots a_{k_i}$, with a_{k_i} as the last peak. Then all points after a_{k_i} can't be peaks. So let $s_1 = k_i + 1$, then a_{s_1} is not a peak so there exists a point s_2, such that $s_2 > s_1$ and $a_{s_2} > a_{s_1}$. Similarly since a_{s_2} is not a peak there exists a point s_3 such that $s_3 > s_2$ and $a_{s_3} > a_{s_2}$. Since each subsequent point is not a peak we can continue this process to form an infinite subsequence $s_1, s_2, s_3, \cdots$ such that $a_{s_1} < a_{s_2} < a_{s_3} < \cdots$ thus there exists a monotone subsequence.

Case 2: There are infinite number of peaks. We shall list them as $a_{k_1}, a_{k_2}, a_{k_3}, \cdots$ such that $k_1 < k_2 < k_3 < \cdots$. Since each is a peak we also know that $a_{k_1} \geq a_{k_2} \geq a_{k_3} \geq \cdots$, thus the subsequence of a_n, $k_1, k_2, k_3, \cdots$ is a monotone subsequence.

Therefore there always exists a monotone subsequence of a_n.

6.) $(0,1)$ is an open subset of $\mathbb{R}$ but not of $\mathbb{R}^2$, when we think of $\mathbb{R}$ as the x-axis in $\mathbb{R}^2$. Prove this.

We are working in $I = (0,1) \times \{0\}$. So let $x \in I$, then our unit ball is a circle instead of a line segment (we are using Euclidean Distance as our metric). So $B(x, \epsilon)$ will include some points with a positive y value no matter how small ϵ, so B cannot be contained in I since the y values in I are only $\{0\}$. So $(0,1)$ is not open in $\mathbb{R}^2$.

7.) A map $f : M \to N$ is **open** if for each open set $U \subset M$, the image set $f(U)$ is open in N.

(a) If f is open, is it continuous?

From the lecture we define a function $f : [0, 2\pi) \to S^1$ as $f(x) = (cos(x), sin(x))$. We have already shown that $f(x)$ is a continuous bijection. By the open preimage condition f^{-1} is open. However f^{-1} is not continuous. Examine the sequence $z_n \to p$, where $p = (0,1) \in S^1$. Then $f^{-1}1(p) = 0$, but $f^{-1}(z_n) \not\to p$ so f^{-1} is not continuous.

(b) If f is a homeomorpism, is it open?

Let $f : M \to N$ be a homeomorphism so then f and f^{-1} are continuous and bijections. Since f is continuous by the open preimage condition f^{-1} is open. Also since f^{-1} is continuous then under the open preimage condition $f^{-1}(f^{-1})) = f$ is open.

(c) If f is an open, continuous bijection, is it a homeomorphism?

Since f is a bijection, f^{-1} is as well. So we need to show that f^{-1} continuous. Since f is open let $U \subseteq M$ then $f(U)$ is open in N. So by the open preimage condition f^{-1} is continuous. Therefore f is homeonmorphism.
(d) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous surjection, must it be open?

Define the function f as

$$f(x) = \begin{cases} (x+2)^2 & x \le 0 \\ -x+4 & x > 0 \end{cases}$$

This function is surjective since each $a \in \mathbb{R}$ has a corresponding $b \in \mathbb{R}$ such that $f(b) = a$. f is continuous as well. However f is not open. Take the open set $I = (-4, 0)$ then $f(I) = [0, 4)$, which is not open. Thus a continuous surjection is not necessarily open.

(e) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous, open surjection, must it be a homeomorphism?

We shall prove that f is an bijection, so we must show its injective. Assume f is not injective then there exists distinct $a, b \in M$ such that $f(a) = f(b)$ and $a < b$. We shall inspect the open interval in M (a, b). Since f is continuous it has a maximum K and a minimum M in $[a, b]$. If $f(a) = f(b) = K = M$, then (a, b) maps to $\{K\}$ a singleton set which is closed. If $f(a) = f(b) = M$, then (a, b) maps to $(M, K]$ which is not open. If $f(a) = f(b) = K$ then (a, b) maps to $[M, K)$ which is not open. Finally if $f(a) = f(b) \neq K \neq M$, then (a, b) maps to $[M, K]$ which is closed. This is contradiction since f is open. Thus f is injective and a bijection. Therefore f must be a homeomorphism.

(f) What happens in (e) if $\mathbb{R}$ is replaced by the unit S^1?

The function $f(x) : (cos(x), sin(x)) \to (cos(2x), sin(2x))$ is a surjective, open and continous, but it is not one-to-one, thus it is not a homeomorphism.
8.) Consider a two-point set $M = \{a, b\}$ who topology consists of the two sets, M and the empty set. Why does this toplogy not arise from a metric on M?

The topology would have to include $\{a\}$ and $\{b\}$ as well if there was a metric on M. If there exists a metric $d(a, b)$, then $\{a\}$ is open since $B(a, \epsilon) \in \{a\}$ when $epsilon = d(a, b)/2$. So a metric would require $\{a\}$ to be in the topology.