$\boxed{1.6}$ The "Skeleton" of Calculus

I think Pugh means "skeleton" in the sense of "supporting structure" here.

If the question is: "Why does Calculus work?", the answer is, amazingly, because of the LUB property of $\mathbb{R}$!

Let $[a,b] \subseteq \mathbb{R}$. Recall: $f:[a,b] \to \mathbb{R}$ is continuous if, for each $x \in [a,b]$, For all $\varepsilon > 0$ there is some $\delta > 0$ s.t. $\forall y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$.

Thm: If f is continuous on $[a,b]$, then $f([a,b])$ is bounded.

Proof: For $x \in [a,b]$, define $V_x = \{ f(t) \mid a \leq t \leq x \}$.

Let $X = \{ x \in [a,b] \mid V_x \text{ is bounded} \}$. If we can show that $b \in X$, we will be done.

First note that $a \in X$, since $V_a = \{ f(a) \}$, which is trivially bounded.

Next, we see that b is an upper bound for X since $X \subseteq [a,b]$.

Thus, the LUB property guarantees that $c = \text{lub } X$ exists.

Since f is continuous, there is some $\delta > 0$ s.t. when $|x-c| < \delta$, then $|f(x)-f(c)| < 1$.

We know there is some $x \in X$ with $c - \delta < x \leq c$. Consider the intervals: $[a,x] \cup [x,c]$ separately:

$f([a,x]) = V_x$ which is bdd since $x \in X$, and

$f([x,c])$ is also bounded; we have $f([x,c]) \subseteq (f(c)-1, f(c)+1)$.

Let $V_c = V_x \cup (f(c)-1, f(c)+1)$. V_c is bounded, so $c \in X$.

Now if $c < b$, consider the interval $[c, c+\delta)$. There would be some element y here which is less than b, and it would also be in $(f(c)-1, f(c)+1)$ ~ thus V_y would be bounded, contradicting that $c = \text{l.u.b. } X$.

$\therefore c = b$, so $b \in X$, and we are done. $\square$

<u>Thm:</u> If f is continuous on $[a,b]$, then there exist x_0 & $x_1 \in [a,b]$ s.t.:

for all $x \in [a,b]$, $f(x_0) \leq f(x) \leq f(x_1)$

(f achieves its minimum and maximum values)

<u>Proof:</u> let $M = \text{lub}\{f(t) \mid t \in [a,b]\}$ (<u>Q</u>: how do we know M exists?)

Define $X = \{x \in [a,b] \mid \text{lub } V_x < M\}$, and consider $f(a)$.

If $f(a) = M$, we are done, (<u>Q</u>: what about the minimum?)

so suppose $f(a) < M$. Then $a \in X$, so $c = \text{lub } X$ exists.

Assume that $f(c) < M$ as well. Then let $\varepsilon = \frac{M - f(c)}{2}$. There is some $\delta > 0$ s.t. whenever $|t - c| < \delta$, we have $|f(t) - f(c)| < \varepsilon$.

Now $c = \text{l.u.b. } X$, so this means $\text{lub } V_c < M$ so that $c \in X$.

As in the last proof, we cannot have $c < b$, so $c = b$.

But if $c = b$, then $b = \text{l.u.b. } X$ (and $b \in X$!), so in fact

$\text{lub } V_x < M$ for <u>all</u> $x \in [a,b]$. This gives:

$$M = \text{lub}\{f(t) \mid t \in [a,b]\} \leq \text{lub}\{V_t \mid t \in [a,b]\} < M \quad ↯$$

Our assumption must be wrong, and $f(c) = M$.

Thus f achieves its maximum on $[a,b]$ (at c). □

(<u>Q</u>: What about the minimum???)

Thm: <u>Intermediate Value Theorem</u>: Suppose f is continuous on $[a,b]$.
Then for every $\gamma \in [f(a), f(b)]$ there is some $c \in [a,b]$ with $f(c) = \gamma$.
(WOLOG, $f(a) < f(b)$)

Proof: Let $X = \{ x \in [a,b] \mid \text{lub}_x \leq \gamma \}$, and let $c = \text{lub}\, X$ (Q: why does c exist?)

- Suppose $f(c) < \gamma$. Then there is some $\delta > 0$ s.t. whenever $|t-c| < \delta$ then $|f(t) - f(c)| < \gamma - f(c)$. In other words, for all $t \in (c-\delta, c+\delta)$, $-(\gamma - f(c)) < f(t) - f(c) < \gamma - f(c)$

$$\Rightarrow -\gamma + 2f(c) < \boxed{f(t) < \gamma}$$

which means $c + \delta/2 \in X$, a contradiction.

- Suppose then that $f(c) > \gamma$. Then there is some $\delta > 0$ s.t. whenever $|t-c| < \delta$ then $|f(t) - f(c)| < f(c) - \gamma$, so that $f(t) > \gamma$ for all $t \in (c-\delta, c+\delta)$.

Thus $c - \delta/2$ is an upper bound for X, another contradiction.

- We must therefore have $f(c) = \gamma$, and we are done. $\square$

Thm: (#43): If f is continuous on $[a,b]$, then it is <u>uniformly</u> continuous;
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x, y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Proof: Let $\varepsilon > 0$. Define $A_\delta = \{ u \in [a,b] \mid \forall x, t \in [a,u]$ with $|x-t| < \delta$, then $|f(x) - f(t)| < \varepsilon \}$.
and $A = \bigcup_{\delta > 0} A_\delta$.

Note that $a \in A$, since if x & t are in $[a,a]$ then $x = t$.
Thus $c = \text{lub}\, A$ exists, and we must show that $c = b$.

Suppose that $c<b$ (it's clear that $c \leq b$ by the definition of A).

Since f is continuous, there is some $\delta_1 > 0$ s.t. $|t-c| < \delta \Rightarrow |f(t)-f(c)| < \varepsilon/2$.

And since $c = \text{lub} A$, there is some $\delta_2 > 0$ s.t. $\forall x,t \in [a,c)$,

$$|t-x| < \delta_2 \Rightarrow |f(x)-f(t)| < \varepsilon/2.$$

Let $\delta = \min(\delta_1, \delta_2)$, and let x & $t \in [a, c+\delta)$ s.t. $|t-x| < \delta$.

If x & t are both in $[a,c)$ then $|f(x)-f(t)| < \varepsilon$, since $c = \text{lub} A$.

If x & t are both in $(c-\delta, c+\delta)$, then $|f(x)-f(t)| < \varepsilon$ as well.

Suppose $x \in [a, c-\delta]$ and $t \in (c-\delta, c)$.

Then $|f(x)-f(t)| \leq \underbrace{|f(x)-f(c)|}_{\text{both in } [a,c-\delta]} + \underbrace{|f(t)-f(c)|}_{\text{both in } (c-\delta,\, c+\delta)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\therefore \forall x,t \in [a, c+\delta)$, $|x-t| < \delta \Rightarrow |f(x)-f(t)| < \varepsilon$, so that $c+\delta \in A_\delta \subseteq A$.

Thus c was not an upper bound for A, a contradiction.

$\therefore$ In fact $c = b$, so that f is uniformly continuous on $[a,b]$. $\square$