Def: A set X is infinite if it is not finite.

Theorem: A set X is infinite if and only if it has cardinality equal to some proper subset of itself.

Proof: ($\Rightarrow$) (*Due to Peter*) Assume X is infinite. By a previous result, we know that X must contain some countably infinite subset A. Write $A = \{a_1, a_2, \ldots\}$.

Define $A_1 = \{a_1, a_3, \ldots, a_{2k+1}, \ldots\}$ and $A_2 = \{a_2, a_4, \ldots, a_{2k} \ldots\}$, and define $\phi : A \to A_2$ by $\phi(a_k) = a_{2k}$.

We see that ϕ is a bijection, so that $|A| = |A_2|$. Now define $\psi : X \to X \setminus A_1$ by:

$$\psi(a) = \begin{cases} a, & a \notin A \\ \phi(a), & a \in A \end{cases}$$

Then ψ is the identity map on $X \setminus A$ and it is bijective on A, so that it is a bijection between X and $X \setminus A_1$.

But $X \setminus A_1$ is a proper subset of X, and we have just established that $|X| = |X \setminus A_1|$. $\square$

($\Leftarrow$) Assume there is some proper subset $Y \subset X$ with $|X| = |Y|$.

Suppose that in fact X is finite; then Y is too, and for some $\mathcal{N} = \{1, 2, \ldots, n\}$ there is a bijection $f : Y \to \mathcal{N}$.

Let $x \in X \setminus Y$ and define a new function $g : Y \cup \{x\} \to \{1, \ldots, n+1\}$ as follows:

$$g(y) = \begin{cases} f(y), & y \neq x \\ n+1, & y = x \end{cases}$$

Since f is a bijection on Y, then g must be a bijection. Thus $|Y \cup \{x\}| = n + 1$.

But $Y \cup \{x\}$ is a subset of X, so we must have $|Y \cup \{x\}| \leq |X|$, or $n + 1 \leq n$. $\lightning$

Therefore X cannot be finite, so it must be infinite. $\square$