Math 5201 Homework 1
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Due September 8th

1.) Prove: There is no smallest positive real number.

We can prove this conjecture by contradiction, let p be the smallest positive number. So for any $b \in \mathbb{R}^+, p < b$. This implies that for $t = \frac{p}{2}$, $p < t$ thus $p < \frac{p}{2}$, which leads to $1 < \frac{1}{2}$ given that p is positive. This is a contradiction so our assumption must be false, therefore there is no smallest positive real number.

2.) Prove: For any two positive real numbers $x, y \in \mathbb{R}^+$, there is some natural number $n \in \mathbb{N}$ with $nx > y$.

There are three cases $x > y$, $x = y$ and $y < x$
case 1 ($x > y$): $n = 1$ because $x(1) = x > y$.

case 2 ($x = y$): Assume that $x = y$, then choose $n = 2$ since $nx = 2y > y$ for any positive real x, y.

case 3 ($x < y$): we can find an n that will satisfy $nx > y$. Define the ceiling function as $\lceil x \rceil = \min\{m \in \mathbb{Z} | m \geq x\}$. If we let $n = \lceil \frac{2y}{x} \rceil$ so that $n \geq \frac{2y}{x}$. Then $nx \geq \frac{2yx}{x} = 2y$ thus $nx \geq 2y$ which implies that $nx > y$.

3.) Prove: If k is even, then k^2 is even, and if k is odd, then k^2 is odd.

We will separate this problem into two proofs.

Conjecture 1: If k is even, then k^2 is even.
Assume that $k \in \mathbb{Z}$ and $k \pmod 2 = 0$. This implies $\exists\, l \in \mathbb{Z}$ such that $k = 2l$. Therefore $k^2 = (2l)^2 = 4l^2 = 2(2l^2)$. Since $2l^2 \in \mathbb{Z}$, it is implies that $2(2l^2) \pmod 2 = 0$ or that $4l^2$ is even. So k^2 is even when k is even.

Conjecture 2: If k is odd, then k^2 is odd.
Assume that $k \in \mathbb{Z}$ and $k \pmod 2 = 1$. This implies $\exists\, l \in \mathbb{Z}$ such that $k = 2l+1$. Therefore $k^2 = (2l+1)^2 = 4l^2 + 4l + 1 = 2(2l^2 + 2l) + 1$. Since $(2l^2 + 2l) \in \mathbb{Z}$, we know that $2(2l^2 + 2l) + 1 \pmod 2 = 1$, or that $(2l + 1)^2$ is odd. Therefore k^2 is odd when k is odd.

4.) A real number is called *algebraic* if it is a root of some polynomial with integer coefficients. Prove: if $p \in \mathbb{R}$ is not algebraic, then it is not rational.

We will solve prove this conjecture by contrapositive. The contrapositive states that if $p \in \mathbb{R}$ is rational then it is algebraic. Begin by taking $u, t \in \mathbb{Z}, t \neq 0$ such that $p = \frac{u}{t}$. Then there exists a polynomial $f(x) = tx - u$, that has a root $\frac{u}{t}$, since

$$t(\frac{u}{t}) - u,$$
$$u - u = 0.$$

Therefore p is algebraic since there exists a polynomial $f(x)$ such that p is a root of $f(x)$. Thus the contrapositive of the conjecture is proven which implies the conjecture is also true.

5.) Prove that $\lim_{x\to 3} x^2 - 1 = 8$. Use an ϵ-δ argument.

For all $\epsilon > 0$, let $\delta = \min(1, \frac{\epsilon}{7})$ when $|x-3| \leq \delta$, we have $|x-3| < 1$ implies,

$$|x+3| = |x-3+6| \leq |x-3| + 6 < 1 + 6 = 7.$$

So we have a bound on $|x+3|$ of 7. So given $f(x) = x^2 - 1$,

$$|x^2 - 1 - 8| = |x^2 - 9| = |x-3||x+3| < |x-3| \cdot 7 < 7\delta.$$

Since $\delta = \min(1, \frac{\epsilon}{7})$ then $7\delta < \frac{\epsilon}{7} \cdot 7 = \epsilon$. Therefore $\lim_{x\to 3} x^2 - 1 = 8$.

6.) Prove that $\lim_{n\to\infty} \frac{1}{n^3-1} = 0$. Use and ϵ-N argument.

Let $N = \max\{1, \sqrt[3]{\frac{1}{\epsilon} + 1}\}$ then for all $\epsilon > 0$ we can show that our series $a_n = \frac{1}{n^3-1}$ converges to 0. We need to show that

$$|\frac{1}{n^3-1} - 0| < \epsilon,$$

holds for all $\epsilon > 0$. Since for $n > 1$, $\frac{1}{n^3-1}$ is positive, we can remove the absolute value term resulting in,

$$\frac{1}{n^3-1} < \epsilon,$$
$$n^3 - 1 > \frac{1}{\epsilon}$$
$$n > \sqrt[3]{\frac{1}{\epsilon} + 1},$$
$$n > N.$$

So for any ϵ choose $N = \max\{1, \sqrt[3]{\frac{1}{\epsilon} + 1}\}$, then $|\frac{1}{n^3-1} - 0| < \epsilon$, therefore a_n converges to 0.

8.) An example of a set X and subset S such that the least upper bound for S is not in X is:

$$X = \mathbb{Q}\, S = \{(1+1/n)^n | n \in \mathbb{N}\},$$

with l.u.b. $e \notin \mathbb{Q}$. Given two other examples of a set of real numbers $X \subseteq \mathbb{R}$ and a subset $S \subseteq X$ such that sup $S \notin X$.

If X is the set of rational numbers, and S is the set defined by

$$S = \left\{ 4 \sum_{i+1}^{n} \frac{(-1)^{i+1}}{2i-1} | n \in \mathbb{N} \right\}$$

This sum converges to π so the l.u.b on S is π which is not a rational number number. All elements in S however are rational numbers. So this set works.

For our second example, set X equal to $\mathbb{Q}$ and the set S is defined by

$$S = \{q \in \mathbb{Q} | q < \sqrt{2}\}.$$

Then the least upper bound on S is $\sqrt{2}$, which $\notin X$. Therefore our choice of X and S satisfies the properties listed above.

9.) Recast the following English sentence in mathematics, using correct mathematical grammar. Preserve the meaning.

2 is the smallest prime number

Let $\mathbb{P}$ denote the set of prime numbers.

$$\forall p \in \mathbb{P} : 2 \leq p, 2 \in \mathbb{P}$$

10.) Let $x = A|B, x' = A'|B'$ be cuts in $\mathbb{Q}$. We defined

$$x + x' = (A + A')|\, rest\, of\, \mathbb{Q}.$$

(a) Show that although $B + B'$ is disjoint from $A + A'$, it may happen in degenerate cases that $\mathbb{Q}$ is not the union of $A + A'$ and $B + B'$.

We will look at the case in which $p_1 = \sqrt{2} = A|B$ and $p_2 = -\sqrt{2} = A'|B'$. So the addition of two cuts results in $0 = A + A'|B + B$. It is known that $\sqrt{2} \notin \mathbb{Q}$, $p_1 \notin A$, likewise $-\sqrt{2} \notin \mathbb{Q}, p_2 \notin A'$, therefore $p_1 + p_2 = 0 \notin A + A'$. The same logic implies that $0 \notin B + B$. This implies $0 \notin (A + A') \cup (B + B')$, so $\mathbb{Q}$ is not the union of $A + A'$ and $B + B'$.

(b) Infer that the definition of $x + x'$ as $(A + A')|(B + B')$ would be incorrect.

From the definition of a cut part (a) specifies that $A \cup B = \mathbb{Q}$. In the problem above we showed a case in which this has been shown to not be true so addition with cuts would not always result in a cut therefore that definition of a cut is incorrect.

(c)Why did we not define $x \cdot x' = (A \cdot A')|$ rest of $\mathbb{Q}$?

We will show a contradiction of this definition, let $x < 0$ and $x' < 0$. Since every $a \in A$, $a \leq x < 0$ and $a' \in A'$, $a' \leq x' < 0$ so $a \cdot a' > 0$. Therefore $A \cdot A'$ includes only positive numbers. Thus the set of the rest of $\mathbb{Q}$ includes negative numbers. This contradicts the definition of a cut part (b) that says "If $a \in A$ and $b \in B$ then $a < b$.". So it would be unwise to use this definition for multiplication.