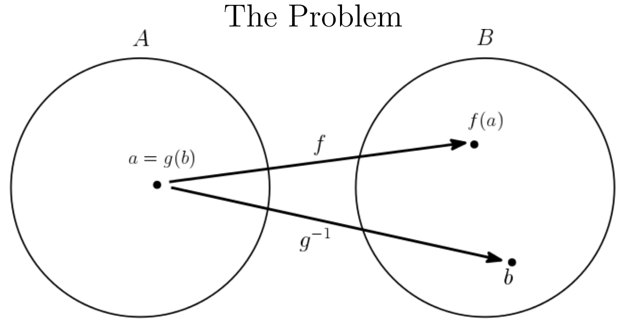
MATH 5201 **September 12, 2018**
Notes on Schroeder-Bernstein

Theorem: Suppose $f : A \to B$ and $g : B \to A$ are injective. Then there is a bijection $\phi : A \leftrightarrow B$.

Idea of the proof: There are two different routes to take from A to B: f and g^{-1}. f is defined on all of A, and g^{-1} is defined for all of B, so by cleverly combining them we will construct a ϕ that is defined *on* all of A and *for* all of B. Why do we have to be clever about it? There are probably places where f and g^{-1} don't agree on what a given $a \in A$ should map to. This is illustrated in the figure at right – using f, we would define $\phi(a) = f(a)$. But using g^{-1} it would instead be $\phi(a) = b$. This situation can only arise when $a \in g(B)$, and so the solution is to eliminate that possibility.

The Problem



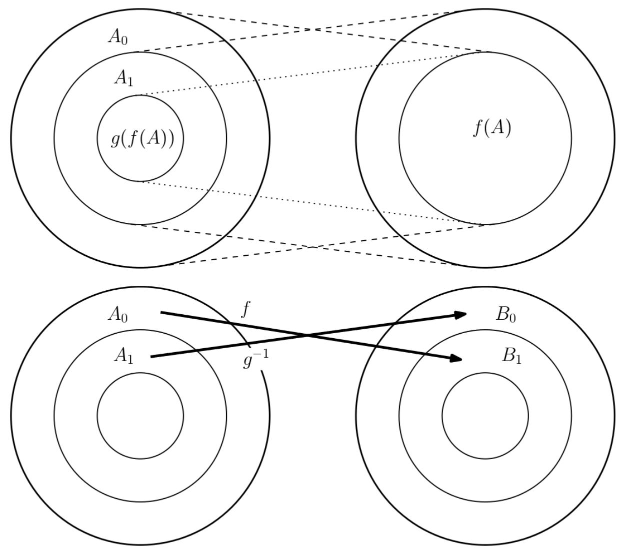
How should we define $\phi(a)$?

Start by defining $A_0 = A \setminus g(B)$ and $B_0 = B \setminus f(A)$. Then for every $a \in A_0$, we know that a is *not* equal to $g(b)$ for any $b \in B$, and a similar fact is true about B_0. Next, let $A_1 = g(B) \setminus g(f(A))$ and $B_1 = f(A) \setminus (f(g(B))$. Note that $A_0 \cap A_1 = \emptyset$ (and the same for B_0 and B_1).

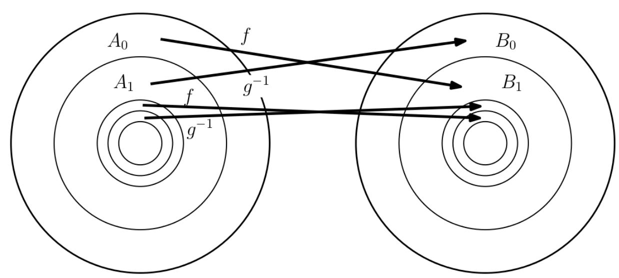
We know that f is injective, and we can see that f is a *bijection* between A_0 and B_1, since $B_1 \subseteq f(A)$ and f is trivially surjective onto its image. For exactly the same reasons, we also know that g is a bijection between B_0 and A_1. Thus we can define the bijection ϕ between $A_0 \cup A_1$ and $B_0 \cup B_1$ by setting $\phi(a) = f(a)$ for $a \in A_0$ and $\phi(a) = g^{-1}(a)$ for $a \in A_1$.

What's left? We still have to define ϕ on $A \setminus (A_0 \cup A_1)$, which is $g(f(A))$, and ϕ must be a bijection between this set and $f(g(B))$. We are back where we started! We know f is an injection from $g(f(A))$ into $f(g(B))$ and that g is an injection in the other direction.

The Solution



So we recurse: we set $A_2 = g(f(A)) \setminus g(f(g(B)))$, and we set $B_2 = f(g(B)) \setminus f(g(f(A)))$.
Then we define $A_3 = g(f(g(B))) \setminus g(f(g(f(A))))$ and $B_3 = f(g(f(A))) \setminus f(g(f(g(B))))$ . . . obviously it gets pretty messy from here. But the upshot is: we can define $\phi(a) = f(a)$ when $a \in A_2$ and $\phi(a) = g^{-1}(a)$ when $a \in A_3$, exactly analogously to the first step.

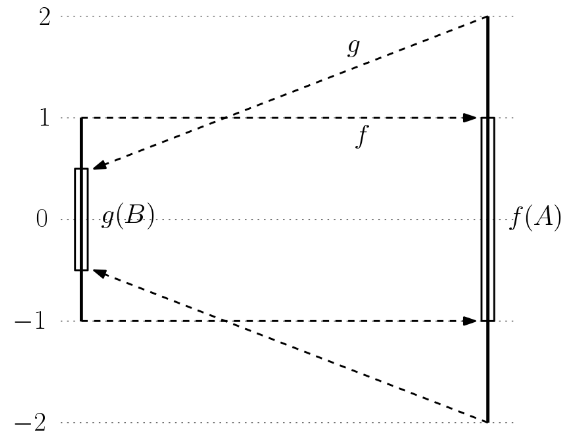


What's left? Well, $g(f(g(f(A))))$ and $f(g(f(g(B))))$ of course! We can continue to subdivide these sets recursively, obtaining sequences of disjoint sets A_i and B_i, such that $\bigcup_{i=1}^{\infty} A_i = A$ and $\bigcup_{i=1}^{\infty} B_i = B$. We define:

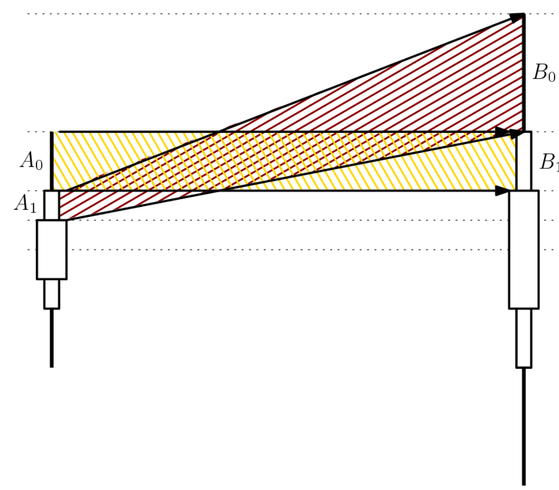
$$\phi(a) = \begin{cases} f(a) & a \in A_{2k} \\ g^{-1}(a) & a \in A_{2k+1} \end{cases}.$$

$\square$

This may be easier to follow if we specify particular sets and functions. Let's take $A = [-1, 1]$ and $B = [-2, 2]$, and let's take $f(a) = a$ and $g(b) = a/4$. The situation is pictured at right. You can imagine that the circles in the preceding figures have been rotated so that we are now viewing them edge-on (and one is smaller than the other, of course). I've chosen then intervals A and B to make the illustrations easier to follow, but the *notation* will be easier if we pretend that the negative numbers don't exist. So in what follows, I will describe the portion of A and B that is greater than or equal to zero only, so we pretend $A = [0, 1]$ and $B = [0, 2]$.

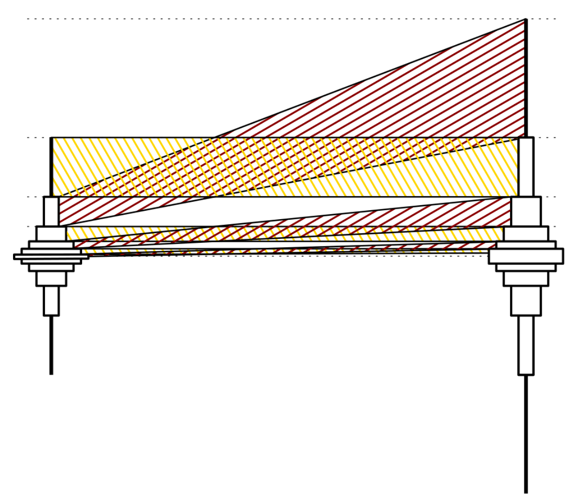


Now we will calculate the set A_0, A_1, B_0, and B_1.
We have $A_0 = (1/2, 1]$ and $B_0 = (1, 2]$, which means $A_1 = (0, 1/2] \setminus (0, 1/4] = (1/4, 1/2]$ and $B_1 = (1/2, 1]$. Then $f(a) = a$ is a bijection between A_0 and B_1, and $g(b) = b/4$ is a bijection between B_0 and A_1.



Now repeat *ad infinitum*. The next iteration gives $A_2 = (1/8, 1/4]$ and $A_3 = (1/16, 1/8]$ on the left, and $B_2 = (1/4, 1/2]$ and $B_3 = (1/8, 1/4]$ on the right. Repeating in this way gives a kind of interleaving pattern, which I've tried to illustrate at right. Meanwhile, there is a clear pattern for the sets:

$$A_n = (1/2^{n+1}, 1/2^n] \text{ and } B_n = (1/2^n, 1/2^{n-1}].$$



Furthermore, $\phi(a)$ will be equal to $f(a)$ for *even*-indexed A sets, and given by $G^{-1}(a)$ for *odd*-indexed A sets. Thus we can give the full definition of our Schroeder-Bernstein bijection:

$$\phi(a) = \begin{cases} a, & 1/2^{2k+1} < a \leq 1/2^{2k} \\ 4a, & 1/2^{2k} < a \leq 1/2^{2k+1} \end{cases}.$$

Piece of (layer) cake!