1.3 Euclidean Space

Product spaces are ubiquitous in Mathematics and its applications.

Def: The Cartesian Product of sets A & B is $A \times B$, defined as:

$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

We will write $\mathbb{R} \times \mathbb{R}$ as $\mathbb{R}^2$, $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ as $\mathbb{R}^3$, and so on

Note that $\mathbb{R}^m$ is a vector field over $\mathbb{R}$ — this means that addition is distributive, associative, and invertible, and scalar multiplication likewise.

Def: The Inner Product, or Dot Product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ is:

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_m y_m = \mathbf{x} \cdot \mathbf{y}.$$

Thm: The inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^m$ satisfies:

$\langle \mathbf{x}, \mathbf{y} + c\mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + c \langle \mathbf{x}, \mathbf{z} \rangle$, [Bilinearity]

$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$, [Symmetry] and

$\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, with $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ iff $\mathbf{x} = \mathbf{0}$. [non-negative]

Proof: L.T.S. □

Def: The magnitude of $\mathbf{x}$ is $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

(Note that this corresponds with the magnitude of a real #, and provides an alternate definition: For $x \in \mathbb{R}$, $|x| = \sqrt{x^2}$.)

Thm (Cauchy-Schwartz): For any $\underline{x}, \underline{y} \in \mathbb{R}^m$, $\langle \underline{x}, \underline{y} \rangle \leq |\underline{x}| \cdot |\underline{y}|$.

Proof: Define the function $Q: \mathbb{R} \to \mathbb{R}$ by $Q(t) = \langle \underline{x} + t\underline{y}, \underline{x} + t\underline{y} \rangle$.

Then by bilinearity, we have:

$$\begin{aligned} Q(t) = \langle \underline{x}+t\underline{y}, \underline{x}+t\underline{y} \rangle &= \langle \underline{x}+t\underline{y}, \underline{x} \rangle + t\langle \underline{x}+t\underline{y}, \underline{y} \rangle \\ &= \langle \underline{x}, \underline{x} \rangle + t\langle \underline{y}, \underline{x} \rangle + t(\langle \underline{x}, \underline{y} \rangle + t\langle \underline{y}, \underline{y} \rangle) \\ &= \langle \underline{y}, \underline{y} \rangle t^2 + 2t\langle \underline{x}, \underline{y} \rangle + \langle \underline{x}, \underline{x} \rangle \\ &= at^2 + bt + c \end{aligned}$$

Then the graph of $Q(t)$ is a parabola.
Note that since $Q(t)$ is the value of an inner product, we have $Q(t) \geq 0$ $\forall t$. In particular, there can be at most one real root of the equation $Q(t) = 0$.
This means the discriminant must be nonpositive, i.e.:

$$b^2 - 4ac = (2\langle \underline{x}, \underline{y} \rangle)^2 - 4\langle \underline{y}, \underline{y} \rangle \langle \underline{x}, \underline{x} \rangle \leq 0.$$

Rearranging, and cancelling a 4, gives:

$$\langle \underline{x}, \underline{y} \rangle^2 \leq \langle \underline{x}, \underline{x} \rangle \langle \underline{y}, \underline{y} \rangle, \quad \text{or}$$

$$\langle \underline{x}, \underline{y} \rangle \leq \sqrt{\langle \underline{x}, \underline{x} \rangle} \sqrt{\langle \underline{y}, \underline{y} \rangle} = |\underline{x}|\,|\underline{y}|. \quad \blacksquare$$

Pedantic Note: What if $\underline{y} = \underline{0}$?

<u>Cor:</u> The triangle inequality: $|\underline{x}+\underline{y}| \leq |\underline{x}| + |\underline{y}| \quad \forall \underline{x}, \underline{y} \in \mathbb{R}^m$.

<u>Proof:</u> Consider instead $|\underline{x}+\underline{y}|^2$:

$$|\underline{x}+\underline{y}|^2 = \langle \underline{x}+\underline{y}, \underline{x}+\underline{y} \rangle = \langle \underline{x}, \underline{x} \rangle + 2\langle \underline{x}, \underline{y} \rangle + \langle \underline{y}, \underline{y} \rangle$$

$$\leq |\underline{x}|^2 + 2|\underline{x}||\underline{y}| + |\underline{y}|^2 = (|\underline{x}| + |\underline{y}|)^2$$

Taking the square root of both sides gives:

$$|\underline{x}+\underline{y}| \leq |\underline{x}| + |\underline{y}|,$$ as required. ∎

<u>Def:</u> The <u>Euclidean Distance</u> between $\underline{x}$ and $\underline{y}$ in $\mathbb{R}^m$ is:

$$|\underline{x}-\underline{y}| = \sqrt{\langle \underline{x}-\underline{y}, \underline{x}-\underline{y} \rangle} = \sqrt{(x_1-y_1)^2 + \cdots + (x_m-y_m)^2}$$

<u>Note:</u> We will soon be considering other distances, so don't drop the 'Euclidean'!

<u>Prop:</u> For all $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^m$, $|\underline{x}-\underline{z}| \leq |\underline{x}-\underline{y}| + |\underline{y}-\underline{z}|$.

(Actual triangles are involved this time!)

<u>Proof:</u> LTS. □

Some common sets in $\mathbb{R}^m$:

- For each $i=1,\dots,m$, let $I_i = [a_i, b_i] \subseteq \mathbb{R}$. Then:

  $I_1 \times I_2 \times \cdots \times I_m$ is a box in $\mathbb{R}^m$.

- The unit cube is $[0,1]^m$.

- The unit ball in $\mathbb{R}^m$ is:

  $B^m = \{ x \in \mathbb{R}^m \mid |x| \leq 1 \}$

- The unit sphere is the boundary of B^m, $S^{m-1} = \{ x \in \mathbb{R}^m \mid |x| = 1 \}$.

Aside: What do you think happens when we consider $\mathbb{R}^\infty$? The answer may suprise you!

In optimization problems, it is often important that the set of potential solutions (the feasible set) is convex:

Def: $E \subseteq \mathbb{R}^m$ is convex if for all $x, y \in \mathbb{R}^m$, the line segment $\overline{xy}$ is contained in E. In other words, if:

$$\underbrace{(1-t)x + ty}_{\text{This is called a convex combination of } x \text{ \& } y.} \in E \quad \text{for all } t \in [0,1].$$

Prop: Every box in $\mathbb{R}^m$ is convex.

B^m is convex.

S^{m-1} is not convex.

Proof: We show the second; the first and third are L.T.S..

Let $x, y \in B^m$. Then $|x| \leq 1$ and $|y| \leq 1$.

Let $z = (1-t)x + ty$. We have:

$$|\mathbf{z}|^2 = \langle \mathbf{z}, \mathbf{z} \rangle = (1-t)^2 \langle \mathbf{x}, \mathbf{x} \rangle + 2(1-t)t \langle \mathbf{x}, \mathbf{y} \rangle + t^2 \langle \mathbf{y}, \mathbf{y} \rangle$$
$$\leq (1-t)^2 |\mathbf{x}|^2 + 2(1-t)t\, |\mathbf{x}|\,|\mathbf{y}| + t^2 |\mathbf{y}|^2$$
$$\leq (1-t)^2 + 2(1-t)t + t^2 = (1-t+t)^2 = 1$$

But $|\mathbf{z}|^2 \leq 1 \implies |\mathbf{z}| \leq 1$. $\square$

Other inner product spaces exist; consider: $C([a,b])$, the space of continuous, real-valued functions on $[a,b]$.
For $f, g \in C([a,b])$, define: $\langle f, g \rangle = \int_a^b f(x)\,g(x)\,dx$.

Given an inner product $\langle \cdot, \cdot \rangle$, one can always define a <u>norm</u> on the space by: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

(A norm is a real-valued function satisfying:
- $\|\mathbf{x}\| \geq 0$; $\|\mathbf{x}\| = 0 \implies \mathbf{x} = \mathbf{0}$
- $\|a\mathbf{x}\| = |a|\,\|\mathbf{x}\|$
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$)

But not all norms come from inner products!

We will explore this idea again much later.