2.2 Continuity

Why do we care about continuity?

- It is a topological property (but why do we care about topology?)
- Often when we want to prove something about a specific function, it will be much easier to prove that it holds for all continuous functions & then show the fn. is continuous.
- There are deep connections to complex analysis
- It is the basis on which we build Riemann Integration.

Def: Let M & N be metric spaces, and suppose $f: M \to N$. We say f is continuous if for every convergent sequence $(a_n) \subseteq M$, the sequence $(f(a_n))$ is convergent in N.

What, you prefer ε-δ definitions?

Def: $f: M \to N$ is ε-δ continuous at a point $x \in M$ if
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall t \in M$, $d(x,t) < \delta \Rightarrow d(f(x), f(t)) < \varepsilon$.

Thm: f is continuous iff it is ε-δ continuous at every $x \in M$.

Proof: LTS. $\square$

Thm: The composition of continuous functions is continuous.

Proof: Let $f: M \to N$, $g: N \to P$ be continuous.

Suppose $a_n \to a \in M$. Then $f(a_n) \to f(a)$ in N.

Then $g(f(a_n)) \longrightarrow g(f(a))$ in P.

i.e., $g \circ f(a_n)$ converges. $\square$

Q: When are two metric spaces "essentially the same"?

A: When they are homeomorphic.

Def: A function $f: M \to N$ is a homeomorphism if it is a continuous bijection and f^{-1} is also continuous.

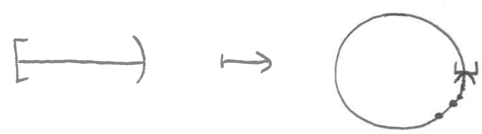
(This is actually a topological idea and not specific to metric spaces)

Ex: There exists a continuous bijection with an inverse that is not continuous:

Let $M = [0, 2\pi)$, $N = S^1 = \{(\cos\theta, \sin\theta),\ \theta \in M\}$.

Then $f: M \to N$ by $f(\theta) = (\cos\theta, \sin\theta)$ is continuous, and is a bijection (Q: how would we prove this?).

But f^{-1} is not continuous at $(1,0)$:



Let $(a_n) \subseteq S^1$ be defined by:

$$a_n = \left(\cos\left(2\pi - \tfrac{1}{n}\right), \sin\left(2\pi - \tfrac{1}{n}\right)\right).$$

Then $a_n \to (1,0)$ in S^1. But $f^{-1}(a_n)$ does not converge in $[0, 2\pi)$, since $f^{-1}(a_n) = 2\pi - \frac{1}{n}$.