$\boxed{1.2}$ Cuts

Def: Suppose A and B are subsets of $\mathbb{Q}$ satisfying:

1) $A \cup B = \mathbb{Q}$, $A \neq \emptyset$, $B \neq \emptyset$

In the text, Pugh adds: $A \cap B = \emptyset$. I omit it here. Why?

2) $\forall a \in A$, $\forall b \in B$, $a < b$

3) A contains no largest element

Then A & B define a cut, denoted $A|B$

Def: Let $\mathcal{R}$ be the set of all cuts.

Pugh goes ahead and calls this $\mathbb{R}$, because it is. But we will prove this.

Now we will go about establishing the ordering properties of $\mathcal{R}$:

Def: Given two cuts $x = A|B$ and $y = C|D$, we say $x < y$ iff $A \subset C$.

Pugh uses $\subset$ to mean subset or equal to; I do not. This definition requires strict subset.

Thm: Given any two cuts x & y, either $x < y$, $x = y$, or $x > y$.

Proof: LTS. $\square$

Def: Given a set of cuts $S \subseteq \mathcal{R}$, The cut $y \in \mathcal{R}$ is an upper bound for S if: $\forall x \in S$, $y \geq x$. S is bounded above by y.

<u>Thm:</u> If $S \subseteq \mathcal{R}$ is bounded above, then there is a <u>least</u> upper bound for S in $\mathcal{R}$. (<u>$\mathcal{R}$ has the L.U.B. property</u>)

<u>Proof:</u> Let $x = X|Y$ be an upper bound for S, and define:

$C = \{a \in \mathbb{Q} \mid a \in A \text{ for some } A|B \in S\}$. $\boxed{C = \bigcup_{A|B \in S} A}$

$D = \mathbb{Q} \setminus C$

Then $C|D$ is a cut.

Pugh thinks this is obvious; I think one should prove that C has no largest element

In fact, $C|D$ is an upper bound for S, since for every $A|B \in S$ we have $A \subseteq C$.

Suppose $C'|D'$ is any other upper bound for S. This means: for every $A|B \in S$, $A \subseteq C'$. But since $C = \bigcup_{A|B \in S} A$, we also have $C \subseteq C'$, i.e. $C|D \leq C'|D'$.

$\therefore$ $C|D$ is the <u>least</u> upper bound for S. $\square$

By defining arithmetic on cuts carefully, one can see that $\mathcal{R}$ is a field; see the text for details. Thus:

<u>Fact:</u> $\mathcal{R}$ is an ordered field, with the L.U.B. property, which contains $\mathbb{Q}$ as an ordered subfield.

Of course we know that this fact is <u>also</u> true of $\mathbb{R}$.

But might there be multiple such objects? No!

<u>Thm:</u> There is exactly one ordered field with the L.U.B. property that contains $\mathbb{Q}$ as an ordered subfield.

<u>Proof:</u> Postponed until §2.10. $\square$

So we have it — $\mathcal{R} = \mathbb{R}$. This is deeply weird, and will only get weirder in section 2.10. Nonetheless, we will soldier on, thinking only of $\mathbb{R}$ as an ordered field with the L.U.B. property, containing $\mathbb{Q}$ as an ordered subfield.

<u>Def:</u> Given $x \in \mathbb{R}$, its <u>magnitude</u> / <u>absolute value</u> is:

$$|x| = \begin{cases} x ; & x \geq 0 \\ -x ; & x < 0 \end{cases}$$

<u>Thm:</u> (Triangle Inequality) For all $x, y \in \mathbb{R}$, we have:

$$|x+y| \leq |x| + |y|$$

<u>Proof:</u> Text. $\square$

It's not so obvious why we call it the triangle inequality when we write it here. But it will be in a general metric space.

<u>Thm:</u> (Reverse Triangle Inequality): For all $x, y \in \mathbb{R}$, we have:

$$\big||x| - |y|\big| \leq |x-y|$$

<u>Proof:</u> LTS. $\square$

Def: Let $S \subseteq \mathbb{R}$.

- If $S \neq \emptyset$,
  - If S is bdd above, $\sup S$ is its L.U.B.
  - If S is not bdd above, $\sup S = \infty$
  - If S is bdd below, $\inf S$ is its G.L.B.
  - If S is not bdd below, $\inf S = -\infty$
- $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

(The supremum and infimum of S).

These two are usually surprising. Take the time to understand why they are defined this way!

I haven't yet said that $\mathbb{R}$ is "complete" — only that it satisfies the L.U.B. property. That's because complete has a different — but equivalent — definition, in terms of sequences.

Def: A sequence $(a_n) \subseteq \mathbb{R}$ converges to b / has the limit b if:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |a_n - b| < \varepsilon$.

We write $\lim\limits_{n \to \infty} a_n = b$ or just $a_n \longrightarrow b$.

Thm: If $a_n \to b$ and $a_n \to c$, then $b = c$.

Proof: LTS. $\square$

Def. A sequence (a_n) is Cauchy / satisfies the Cauchy Condition if:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $m, n \geq N \implies |a_m - a_n| < \varepsilon$.

Thm: If $a_n \to b$ then (a_n) is Cauchy.

Proof: LTS.

Thm: If $(a_n) \subseteq \mathbb{R}$ is Cauchy, then it converges in $\mathbb{R}$. ($\mathbb{R}$ is complete)

Proof: Since (a_n) is Cauchy, there is some $N \in \mathbb{N}$ s.t. whenever $m, n \geq N$ we have $|a_m - a_n| < 1$.

There is a $K \in \mathbb{R}$ large enough so that $|a_i| \leq K$ for each $i = 1, \ldots, N$. Let $M = K + 1$.

Proof LTS.

Let $n \in \mathbb{N}$. Then $|a_n| < M$, so (a_n) is bounded.

Now let $X = \{x \in \mathbb{R} \mid a_n \geq x \text{ for infinitely many } n\}$.

Then $-M \in X$, but $M \notin X$. Thus X is nonempty and bounded above, so let $b = \sup X$. Note: $|b| \leq M$.

For any $\varepsilon > 0$, there is an N s.t. $m, n \geq N \implies |a_m - a_n| < \varepsilon/2$.

Note that $b - \varepsilon/2$ is not an upper bound for X ($b = \text{L.U.B.}(X)$), so there is some $x \in X$ with $b - \varepsilon/2 \leq x \leq b$, and there are infinitely many n with $a_n \geq x$.

But also, $b + \varepsilon/2$ is not in X, so only finitely many a_n satisfy $a_n \geq b + \varepsilon/2$.

Together, these imply that for infinitely many n, we have:

$$b - \varepsilon/2 \le x \le a_n \le b + \frac{\varepsilon}{2}$$

and in particular there are infinitely many such n with $n \ge N$. Choose one such n, called n_0. Then:

$$\forall n \ge N, \quad |a_n - b| = |a_n - a_{n_0} + a_{n_0} - b| \le |a_n - a_{n_0}| + |a_{n_0} - b|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

i.e., $a_n \longrightarrow b$, so (a_n) is convergent. □

Some fun $\mathbb{R}$-facts

If $x \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$, then $x + y \in \mathbb{R} - \mathbb{Q}$.

If $x \neq 0$, then $xy \in \mathbb{R} - \mathbb{Q}$.

Thm: Let $x < y \in \mathbb{Q}$. Then there is some $z \in \mathbb{R} \setminus \mathbb{Q}$ with $x < z < y$.

Define the transformation: $T: t \longmapsto (1-t)x + ty = x + (y-x)t$

Then $T(0) = x$ and $T(1) = y$, so $T: [0,1] \longrightarrow [x,y]$.

Note that $\sqrt{2} - 1 \in [0,1]$, and $\sqrt{2} - 1 \in \mathbb{R} - \mathbb{Q}$. Thus, $T(\sqrt{2}-1)$ is also irrational with $x < T(\sqrt{2}-1) < y$. □

In fact, between any two different real numbers there are infinitely many rationals and irrationals. In particular:

> Between any two irrational numbers is a rational
> Between any two rationals there is an irrational
> There are strictly more irrationals than rationals !

> To be proved a little later...

Thm: Let $x, y \in \mathbb{R}$.

a) If $\forall \varepsilon > 0$, $x \leq y + \varepsilon$, then $x \leq y$.

b) If $\forall \varepsilon > 0$, $|x - y| \leq \varepsilon$, then $x = y$.

Proof:

a) Either $x \leq y$ or $x > y$. If the latter were true, we would have $x > y + \frac{(x-y)}{2}$, which contradicts our assumption. Thus $x \leq y$ as required.

b) If $x \neq y$, then $|x - y| > \frac{|x-y|}{2}$, another contradiction. Thus $x = y$ as required. $\square$