1.4 Cardinality

Pugh defines a function $f: A \to B$ as a 'rule' that turns any element $a \in A$ into some $b \in B$. This is not rigorous.

Def: Let A & B be sets. A function $f: A \to B$ is a binary relation between the sets A & B, s.t. such that each member of A is related to at most one element of B.

We say that $f(a)=b$ if $a \in A$ is related to $b \in B$.

A is the domain of f and B is the codomain.

The range is $f(A) \subseteq B$. [for $S \subseteq A$, we let $f(S) = \{f(a) : a \in S\}$]

For $S \subseteq A$, $f(S)$ is the image of S under f.

For $Y \subseteq B$, the preimage of Y is:

$$f^{-1}(Y) = f^{pre}(Y) = \{a \in A : f(a) \in Y\}.$$

We say f is injective if for every $b \in f(A)$, $f^{-1}(b)$ is a singleton. Equivalently, if whenever $a \neq a' \in A$ then $f(a) \neq f(a')$. (This is usually what you use in proofs)

We say f is surjective / onto B if $f^{-1}(b)$ is nonempty for every $b \in B$.

If f is injective and surjective, then it is bijective.

If f is a bijection, then f^{-1}, the inverse function, is bijective. (Note that f^{-1} exists whenever f is injective, and is a bijection between A and $f(A)$ in that case.)

→ This is an abuse of notation! ← $f^{-1}(b)$ might be a set of elements in A, and it might be an element.

The identity map $id : A \to A$ has $id(a) = a$, and is a bijection.

If $f(a) = b$ for every $a \in A$ (and a single $b \in B$), then f is the constant valued function, and we write $f(a) \equiv b$.

If $g : B \to C$, then $g \circ f : A \to C$ is given by $g(f(a))$.
$g \circ f$ is the composition of f with g.

Q: If f & g are injective, is $g \circ f$?
If f & g are surjective, is $g \circ f$?

Def: Define the relation $\sim$ on sets by: $A \sim B$ iff there exists a bijection $f : A \to B$.

Prop: $\sim$ is an equivalence relation, i.e.:

- $A \sim A$ $\forall$ sets A
- If $A \sim B$ then $B \sim A$
- If $A \sim B$ and $B \sim C$ then $A \sim C$

Proof: Clear. $\square$

We can use $\sim$ to get a handle on the "sizes" of sets.
The cardinality of a set captures this idea

Def: Let $S = \{1, \ldots, n\} \subseteq \mathbb{N}$. Then $card(S) = |S| = \# S = n$.

Any set A with $A \sim \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ is a finite set.

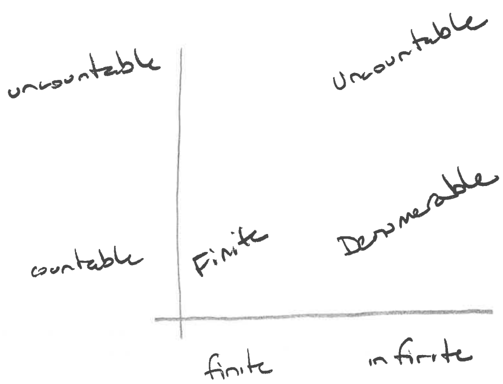
If A is not finite, then it is <u>infinite</u>

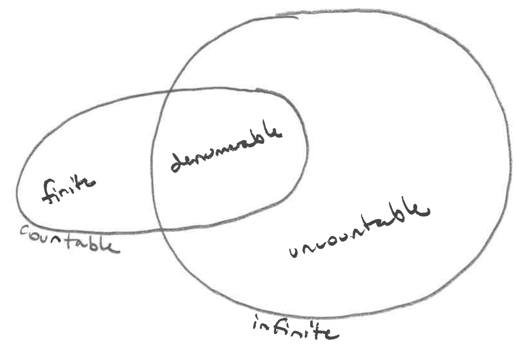
[Alternate definition: A is infinite iff there is some proper subset $S \subset A$ with $A \sim S$]

If $A \sim \mathbb{N}$, then A is <u>denumerable</u> or <u>countably infinite</u>.

If $A \sim S$ for <u>any</u> subset $S \subseteq \mathbb{N}$, then A is <u>countable</u>.

If A is not countable, it is <u>uncountable</u>





Here's a famous theorem:

<u>Thm:</u> (Cantor): $\mathbb{R}$ is uncountable

<u>Proof:</u> Suppose otherwise. Then there is a bijection $\varphi: \mathbb{N} \to \mathbb{R}$.

Let $x_n \in \mathbb{R}$ be $\varphi(n)$ for $n \in \mathbb{N}$. We will consider the decimal expansion of x_n, namely $x_n = N_n . a_{n1} a_{n2} a_{n3} \cdots$

WOLOG we may assume the expansion doesn't end with an infinite string of 9s; then the expansion is uniquely defined.

For each $i \in \mathbb{N}$, let $b_i \in \{0, \ldots, 8\} \setminus \{a_{ii}\}$.

Let $y = 0.b_1 b_2 b_3 \cdots$. Then for each $n \in \mathbb{N}$, $y \neq x_n$!

Then $y \notin \varphi(\mathbb{N})$, which was supposed to be a bijection. This is a contradiction, so $\mathbb{R}$ must be uncountable. $\boxtimes$

The theorem has an elegant illustration:

n	$\varphi(n)$
1	$N_1 . a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15}\ \cdots$
2	$N_2 . a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25}\ \cdots$
3	$N_3 . a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{35}\ \cdots$
$\vdots$	$\vdots \quad \ddots$

And so the type of argument is called a "diagonal argument". We will see these again.

<u>Cor:</u> (a,b) & $[a,b]$ are uncountable.

<u>Proof:</u> HW. $\square$

<u>Thm:</u> If S is infinite, then it contains a countable subset.

<u>Proof:</u> (uses the Axiom of Choice): Since S is infinite, it is certainly nonempty. Choose any $s_1 \in S$.

The remaining set $S \setminus \{s_1\}$ must still be infinite (See #36b)

So choose $s_2 \in S \setminus \{s_1\}$. Similarly, choose $s_n \in S \setminus \{s_1, \ldots, s_{n-1}\}$.

Then the set $\{ s_i : i \in \mathbb{N} \}$ is countable. $\boxtimes$

<u>Thm:</u> If A is countable and $S \subseteq A$ is infinite, then S is countable.

<u>Proof:</u> LTS. $\boxtimes$

Thm: $\mathbb{N}\times\mathbb{N}$ is countable.

Proof: $\mathbb{N}\times\mathbb{N} = \{(1,1), (2,1), (1,2), (3,1), (2,2), (1,3), \dots\}$.

This list will reach every element of $\mathbb{N}\times\mathbb{N}$. $\square$

Q: Given $(m,n)\in\mathbb{N}\times\mathbb{N}$, can we explicitly give its position in the list?

Cor: $A\times B$ is countable whenever A & B are.

Proof: LTS. $\square$

Thm: If $f:\mathbb{N}\to A$ is surjective then A is countable.

Proof: Since f is a surjection, $f^{-1}(a)$ is nonempty for each $a\in A$.

Since $\mathbb{N}$ is well-ordered, $f^{-1}(a)$ has a smallest element K.

Define $h: A\to\mathbb{N}$ by $h(a) = \min\{f^{-1}(a)\}$.

Then $\boxed{h \text{ is a bijection}}$ between A and some subset of $\mathbb{N}$. $\square$

LTS. $\boxtimes$

Thm: Suppose A_n is countable for every $n\in\mathbb{N}$.

Then $A_1\cup A_2\cup\cdots = \bigcup_{n\in\mathbb{N}} A_n$ is countable.

Proof: For each $i\in\mathbb{N}$, let $\varphi_i:\mathbb{N}\to A_i$ be a bijection.

For $j\in\mathbb{N}$, call $a_{ij} = \varphi_i(j)$. Then $a_{ij}\in A_i\subseteq \bigcup A_n$.

Define $f:\mathbb{N}\times\mathbb{N}\to\bigcup A_n$ by $f(i,j) = \varphi_i(j)$. Then

f is surjective, since each φ_i is.

But $\mathbb{N}\times\mathbb{N}$ is countable, so there is some surjection

$g:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$. The composition $g\circ f:\mathbb{N}\to\bigcup A_n$

is thus surjective, so $\bigcup A_n$ is countable. $\square$