Real Variables Homework 2 Final

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September 2018

1.) Let $f : A \to B$ be a function. That is, f is some rule or device which, when presented with any element $a \in A$, produces an element $b = f(a)$ of B. The **graph** of f is the set S of all pairs $(a, b) \in A \times B$ such that $b = f(a)$.

(a) If you are given a subest $S \subset A \times B$, how can you tell if it is the graph of some function? (What are the set theoretic properties of a graph?).

Let $s \in S$, then $s = (s_1, s_2)$. It is known that $(a, b) \in A \times B$, then $b = f(a)$. Thus $s_2 = f(s_1)$. S is a graph of the function iff for each $(s_1, s_2) \in S$, s_1 is the only value for which the function f returns s_2.

(b) Let $g : B \to C$ be a second function and consider the composed function $g \circ f : A \to C$. Assume that $A = B = C = [0, 1]$, draw $A \times B \times C$ as the unit cube in 3-space, and try to relate the graphs of f, g, and $g \circ f$ in the cube.

2.) Prove: Every 'box' of the form $[a_1, b_1] \times \cdots \times [a_m, b_m]$ is a convex subset of $\mathbb{R}^m$.

We shall define the box as $\mathbb{B} : I_i \times \cdots \times I_m$, where $I_i = [a_i, b_i] \subset \mathbb{R}$ for $i = 1, 2, \cdots, m$. Let $x, y \in \mathbb{B}, x \neq y$, then $x_i \in [a_i, b_i]$ and $y_i \in [a_i, b_i]$, this can also be written as $a_i \leq x \leq b_i$ and $a_i \leq y \leq b_i$ for $i = 1, 2, \cdots, m$. We shall define a transformation $z = (1-t)x + ty$, where $t : [0, 1]$. This transformation is known as a convex combination. It includes all points between x and y so $z(t), t : [0, 1]$ is the equation for the line between x and y . Let $z_i = (1-t)x_i + ty_i$, where $t : [0, 1]$ is the ith component of z. If we can show that $z_i \in I_i$ for $i = 1, 2, \cdots$ then the line lies inside the box. We shall show this in two parts, first giving an upper bound:

$$z_i \leq (1-t)b_i + tb_i = b_i - tb_i + tb_i = b_i,$$

so $z_i \leq b_i$. Next we show the lower bound:

$$z_i \geq (1-t)a_i + ta_i = a_i - ta_i + ta_i = a_i,$$

so $z_i \geq a_i$. Therefore $a_i \leq z_i \leq b_i$ for all $i = 1, 2, \cdots$ thus every point between x and y is inside $\mathbb{B}$. This proves a box is convex.

3.) Prove: The unit sphere $S^{m-1} = \{x \in \mathbb{R}^m : |x| = 1\}$ is not convex.

Let $x \in S^{m-1}$ where $x = \{x_1, x_2, \cdots, x_m\}$ and $|x| = 1$. So

$$\sqrt{x_1^2 + x_2^2 + \cdots + x_m^2} = 1,$$

by definition of magnitude. Let $y = \{-x_1, -x_2, \cdots, -x_m\}$ then we can show that this point is on the unit sphere,

$$\begin{aligned} |y| =& \sqrt{(-x_1)^2 + (-x_2)^2 + \cdots + (-x_m)^2}, \\ |y| =& \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2}, \\ |y| =& 1. \end{aligned}$$

Thus $y \in S^{m-1}$. Define the convex combination as the transformation $z = (t-1)x + ty$ where $t \in [0, 1]$. So z includes all points between x and y. We shall define a point m between x and y when $t = 0.5$ and m_n as the nth component of m then,

$$m_n = (0.5 - 1)x_n - 0.5y_n,$$

$$m_n = -0.5x_n - 0.5(-x_n),$$

$$m_n = 0.$$

Thus $m = 0$, the zero vector and $|m| = 0$ so $m \notin S$ which implies the unit sphere is not convex.

4.) A function $f : (a, b) \to \mathbb{R}$ is a **convex function** if for all $x, y \in (a, b)$ and all $s, t \in [0, 1]$ with $s + t = 1$ we have

$$f(sx + ty) \leq sf(x) + tf(y).$$

(a) Prove that f is convex iff the set S of points above its graph is convex in $\mathbb{R}^2$. The set S is $\{(x, y) : f(x) \leq y\}$.

The conjecture we need to prove is an if and only if statement so we will break this into 2 cases.

Case 1: Assume S is convex, prove f is convex. Let $a = (x_1, y_1)$ and $b = (x_2, y_2)$ such that $a \neq b$ be points on f so $f(x_1) = y_1$ and $f(x_2) = y_2$. Since a, b are also in S, the points between a and b are in S (since S is convex). Define the transformation $z = sx + ty$ where $s + t = 1$ as the convex combination so $z \in S$ when $t + s = 1$. Then

$$z = sa + tb,$$
$$z = s(x_1, y_1) + t(x_2, y_2),$$
$$z = (sx_1 + tx_2, sy_1 + ty_2).$$

Since $z \in S$,

$$f(sx_1 + tx_1) \leq sy_1 + ty_2,$$
$$f(sx_1 + tx_1) \leq sf(x_1) + tf(x_2),$$

from our assumptions above. Therefore f is a convex function.

Case 2: Assume f is convex prove S is convex. Start by defining two points in S, $x, y \in S$ such that $x \neq y$ so $x = (x_1, y_1)$ and $y = (x_2, y_2)$. Then by the definition of S, $f(x_1) \leq y_1$ and $f(x_2) \leq y_2$. Define the convex combination as the transformation $z = sx + sy$ where $s + t = 1$. Then

$$z = s(x_1, y_1) + t(x_2, y_2),$$
$$z = (sx_1 + tx_2, sy_1 + ty_2).$$

Then let z_x and z_y be the coordinates of z, then $f(z_x) = f(sx_1 + tx_2) \leq sy_1 + ty_2 = f(z_y)$. So $z \in S$, therefore S is convex.

(c) Suppose that f is convex and $a < x < u < b$. The slope σ of the line through $(x, f(x))$ and $(u, f(u)$ depends on x and u, say $\sigma = \sigma(x, u)$. Prove that σ increases when x increases, and σ increases when u increases.

We shall split this problem into two distinct parts. **Part 1:** Define $\hat{u} > u$

we can show that the slope between $\hat{u}$ and x is greater than or equal to the slope between u and x. Start by writing u as a convex combination of x and $\hat{u}$ as $u = sx + t\hat{u}$ where $s + t = 1$. Then we can rearrange $\hat{u} = \frac{u-sx}{t}$. Since f is a convex function,

$$f(u) \leq sf(x) + tf(\hat{u}),$$

$$\frac{f(u) - sf(x)}{t} \leq f(\hat{u}),$$

$$\frac{\frac{f(u)-sf(x)}{t} - f(x)}{\hat{u} - x} \leq \frac{f(\hat{u}) - f(x)}{\hat{u} - x},$$

$$\frac{\frac{f(u)-sf(x)}{t} - f(x)}{\frac{u-sx}{t} - x} \leq \frac{f(\hat{u}) - f(x)}{\hat{u} - x},$$

$$\frac{f(u) - sf(x) - tf(x)}{u - sx - tx} \leq \frac{f(\hat{u}) - f(x)}{\hat{u} - x},$$

$$\frac{f(u) - (s+t)f(x)}{u - (s+t)x} \leq \frac{f(\hat{u}) - f(x)}{\hat{u} - x},$$

since $t + s = 1$,

$$\frac{f(u) - f(x)}{u - x} \leq \frac{f(\hat{u}) - f(x)}{\hat{u} - x}.$$

Therefore the slope will increase or remain constant as u increases.

Part 2: Define a point $\hat{x}$, such that $u > \hat{x} > x$, we will show that the slope between $\hat{x}$ and u is greater or equal to than the slope at between u and x. Write $\hat{x}$ as the convex combination of x and u, $\hat{x} = sx + tu$ when $s + t = 0$. f is convex which results in

$$f(\hat{x}) \leq sf(x) + tf(u),$$

$$f(\hat{x}) - f(u) \leq sf(x) + tf(u) - f(u).$$

The next steps involves dividing both sides by $\hat{x} - u$, since $u > \hat{x}$, $\hat{x} - u < 0$, thus,

$$\frac{f(\hat{x}) - f(u)}{\hat{x} - u} \geq \frac{sf(x) + tf(u) - f(u)}{\hat{x} - u},$$

$$\frac{f(\hat{x}) - f(u)}{\hat{x} - u} \geq \frac{sf(x) + f(u)(t-1)}{sx + tu - u},$$

$$\frac{f(\hat{x}) - f(u)}{\hat{x} - u} \geq \frac{sf(x) + f(u)(t-1)}{sx + u(t-1)}.$$

From our assumptions on convex combination $s + t = 1$ implies $t - 1 = -s$, so

$$\frac{f(\hat{x}) - f(u)}{\hat{x} - u} \geq \frac{sf(x) - sf(u)}{sx - su},$$

$$\frac{f(\hat{x}) - f(u)}{\hat{x} - u} \geq \frac{f(x) - f(u)}{x - u}.$$

Thus the slope increases or remains constant as x increases.

5.) The proof of Corollary 11 (p.33) states without proof:

There are bijections from (a, b) onto $(-1, 1)$ onto the unit semicircle onto $\mathbb{R}$ shown in Figure 15.

Give these bijections explicitly and prove they are bijections.

The bijection between from (a, b) onto $(-1, 1)$ is given by the map f :

$$f(x) = \frac{x - \frac{a+b}{2}}{\frac{b-a}{2}}.$$

First we will prove this function is an injection. So let $a_1, a_2 \in (a, b)$ and $a_1 \neq a_2$. We must show that $f(a_1) \neq f(a_2)$, so assume they are equal and show a contradiction

$$\begin{aligned}
f(a_1) &= f(a_2), \\
\frac{a_1 - \frac{a+b}{2}}{\frac{b-a}{2}} &= \frac{a_2 - \frac{a+b}{2}}{\frac{b-a}{2}}, \\
a_1 - \frac{a+b}{2} &= a_2 - \frac{a+b}{2}, \\
a_1 &= a_2
\end{aligned}$$

. This is a contradiction since we stated earlier that $a_1 \neq a_2$, so $f(a_1) \neq f(a_2)$. Thus the function is injective. Next we must show that f is surjective. Take any $y \in [-1, 1]$ and we can show there exists an $x \in [a, b]$ such that $f(x) = y$,

$$\begin{aligned}
f(x) =& y, \\
\frac{x - \frac{a+b}{2}}{\frac{b-a}{2}} =& y, \\
x - \frac{a+b}{2} =& y(\frac{b-a}{2}), \\
x =& y(\frac{b-a}{2}) + \frac{a+b}{2}.
\end{aligned}$$

This is the inverse to f. It shows that there exist an $x \in [a, b]$ for every $y \in [-1, 1]$ such that $f(x) = y$. Thus f is surjective and injective so f is bijective. Next we define the map g from $(-1, 1)$ to the unit semicircle. Denote the unit semicircle as $\mathbb{U}$, so $g : [-1, 1] \to \mathbb{U}$ is defined as,

$$g(x) = (cos(\frac{\pi x - \pi}{2}), sin(\frac{\pi x - \pi}{2})).$$

First we will show that g is injection, so let $a_1, a_2 \in (-1, 1)$ be distinct elements, we must show that $g(a_1) \neq g(a_2)$, so assume they are equal and show a contradiction,

$$g(a_1) = g(a_2),$$
$$(cos(\frac{\pi a_1 - \pi}{2}), sin(\frac{\pi a_1 - \pi}{2})) = (cos(\frac{\pi a_2 - \pi}{2}), sin(\frac{\pi a_2 - \pi}{2})).$$

Let's inspect the first coordinate of these two ordered pairs,

$$cos(\frac{\pi a_1 - \pi}{2}) = cos(\frac{\pi a_2 - \pi}{2}),$$
$$\frac{\pi a_1 - \pi}{2} = \frac{\pi a_2 - \pi}{2},$$
$$a_1 = a_2.$$

This contradicts the fact that a_1 and a_2 are distinct so g is injective. To prove that g is surjective we must show for any value $y = (y_1, y_2) \in \mathbb{U}$, there exists a $x \in [-1, 1]$ such that $g(x) = y$. So

$$\begin{aligned}
g(x) =& y, \\
(cos(\frac{\pi x - \pi}{2}), sin(\frac{\pi x - \pi}{2})) =& (y_1, y_2), \\
(\frac{\pi x - \pi}{2}, \frac{\pi x - \pi}{2}) =& (sin^{-1}(y_1), cos^{-1}(y_2)), \\
(\pi x, \pi x) =& (2sin^{-1}(y_1) + \pi, 2cos^{-1}(y_2) + \pi), \\
(x, x) =& (\frac{2sin^{-1}(y_1) + \pi}{\pi}, \frac{2cos^{-1}(y_1) + \pi}{\pi}).
\end{aligned}$$

This is the inverse of g, it shows that for every $y \in \mathbb{U}$, there exists an $x \in [-1, 1]$ such that $g(x) = y$. So g is surjective and injective which implies its bijective.

6.) The Well-Ordering Principle states that every nonempty subset of $\mathbb{N}$ has a least element. Any set equipped with an order $\prec$ having this property is said to be *well-ordered*. Use the fact that $\mathbb{Q}$ is countable to prove that there is some order $\prec$ on $\mathbb{Q}$ that makes it well-ordered. Note that $\prec$ will be different than our usual ordering on $\mathbb{Q}$.

$\mathbb{Q}$ is countable so $\mathbb{Q} \sim \mathbb{N}$, thus there exists $\phi : \mathbb{Q} \to \mathbb{N}$ which is a bijection. The ordering in $\mathbb{Q}$ works as follows: let $a, b \in \mathbb{Q}$, then $a \prec b$ iff $\phi(a) < \phi(b)$. Then for any nonempty subset of $\mathbb{Q}$, known as S. Let be S' be the set,

$$S' = \{s \in S | \phi(s)\}.$$

Then $S' \subseteq \mathbb{N}$ so S' by the well-ordering principle has a least element l. There exists the inverse mapping of $\phi : \phi^{-1} : \mathbb{N} \to \mathbb{Q}$ because ϕ is a bijection. Then $\phi^{-1}(l)$ is the least element in S. Therefore $\mathbb{Q}$ is well-ordered.

7.) Given a set X, the *power set* of X, denoted $\mathbb{P}(X)$, is the set containing all subsets of X. For example, if $X = \{a, b, c\}$, then

$$\mathbb{P} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

The power set has a natural ordering $\prec$ on it: for $A, B \in \mathbb{P}(X), A \prec B$ if and only if $A \subset B$. Note that $\subseteq$ corresponds to $\preceq$.

a.) Explain why, for a finite set X, $\#(\mathbb{P}(X)) = 2^{\#(X)}$.

We will prove this conjecture with induction, any finite set X will have finite cardinality or $n = \#(X), n \in \mathbb{N}$.

Base Case: If $n = 0$, or X is the empty set than the power set $\mathbb{P} = \{\emptyset\}$ thus $\#(\mathbb{P}(X)) = 1 = 2^0$. Note that the power set in this case is not the empty set, but the set containing the empty set. So the conjecture holds for $n = 0$.

Induction Step: Assume that for $\#(X_k) = k$, the conjecture holds true, $\#(\mathbb{P}(X_k)) = 2^{\#(X)} = 2^k$. We must prove this is also true for $\#(X_{k+1}) = k + 1$. Another way to phrase this is we must prove that the power set of X_k will have twice as many elements as the power set of X_{k+1}, when $\#(X_{k+1}) = k + 1$.

Let a be the new element in X_{k+1} that is not in X_k then elements of the power set $p \in \mathbb{P}(X_{k+1})$ will fall into two categories. Either $p \in \mathbb{P}(X_k)$ or $p \in S$ where $S = \{\{a, b\} | b \in \mathbb{P}(X_k)\}$. Since $\{a\} \notin X_k$, the elements in $\mathbb{P}(X_k)$ and S are distinct. We already know that $\#(\mathbb{P}(X_k)) = 2^k$, next we that S and $\mathbb{P}(X_k)$ have equal order. Since each element in $\mathbb{P}(X_k)$ has a corresponding element in S just with an $\{a\}$ element as well. They have the same size. This implies that $\#(\mathbb{P}(X_k)) = \#(S) = 2^k$. Since these sets are distinct

$$\#(\mathbb{P}(X_{k+1})) = \#(\mathbb{P}(X_k)) + \#(S),$$
$$\#(\mathbb{P}(X_{k+1})) = 2^k + 2^k = 2(2^k) = 2^{k+1}.$$

This explain our conjecture.

b.) Prove that $\prec$ is *not* a total order on $\mathbb{P}(X)$. This makes $\mathbb{P}(X)$ a partially ordered set (poset).

Let $\{a\}, \{b\} \in X$ then $\{a\}, \{b\}$ and $\{a, b\} \in \mathbb{P}(x)$ so $\{a\} \prec \{a, b\}$ and $\{b\} \prec \{a, b\}$, but there is no ordering between $\{a\}$ and $\{b\}$ since $\{a\} \not\subset \{b\}$ and

$\{b\} \not\subset \{a\}$. Therefore this is not a total ordering.

c.) For which sets X is $\mathbb{P}(X)$ a totally ordered set under $\prec$?

The only sets X that have a total ordering under $\prec$ are $X = \emptyset$ and $X = \{a\}$ where X is a singleton set. All other sets will run into the issue described in part (b).

d.) Is $\mathbb{P}(X)$ well-ordered under $\prec$?

Yes $\mathbb{P}(X)$ is well-ordered because the least element in every $\mathbb{P}(X)$ is the empty set. Since for $s \in \mathbb{P}(X), s \neq \emptyset$, then $\emptyset \subset s$ thus $\emptyset \prec s$ for every element in $\mathbb{P}(X)$. So $\emptyset$ will always be the least element.

e.) Use a diagonal argument to show that $\mathbb{P}(\mathbb{N})$ is uncountable.

List all element of $\mathbb{P}(\mathbb{N})$ in a row

$$\begin{gathered}
N_1 = \{a_{1,1},\, a_{1,2},\, a_{1,3},\, a_{1,4},\, \cdots\} \\
N_2 = \{a_{2,1},\, a_{2,2},\, a_{2,3},\, a_{2,4},\, \cdots\} \\
N_3 = \{a_{3,1},\, a_{3,2},\, a_{3,3},\, a_{3,4},\, \cdots\} \\
N_4 = \{a_{4,1},\, a_{4,2},\, a_{4,3},\, a_{4,4}, \cdots\} \\
\cdots
\end{gathered}$$

Where $a_{i,j} \in \mathbb{N}$ and the $a_{i,j}$ are ordered and distinct in each row, so $a_{i,j} < a_{i,k}$ if $j < k$.Choose a new number Y with elements $y_i, i = 1, 2, \cdots$ and $y_i \in \mathbb{N}$. Elements of y_i are distinct and ordered. Choose each y_i so that it is not equal to $a_{i,i}$. Then Y is not in this list, because every value $N_1, N_2 \cdots$ differs from Y at least at one value, so y_i is not in the list $N_1, N_2, \cdots$