$\boxed{2.1}$ Metric Spaces

As Pugh notes, some problems in $\mathbb{R}$ are more easily solved in a more abstract setting. That setting is Metric Spaces:

Def: Let X be a set and let $d: X \times X \to \mathbb{R}$. Then (X, d) is a metric space, and d is a metric on X, provided:

- $\forall x, y \in X$, $d(x,y) \geq 0$ and $d(x,y) = 0 \implies x = y$
- $\forall x, y \in X$, $d(x,y) = d(y,x)$
- $\forall x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$

Ex:
- $\mathbb{R}^m$ is a metric space with metric:

$$d(\bar{x}, \bar{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_m - y_m)^2}$$

- Any subset of $\mathbb{R}^m$ is a metric subspace with inherited metric d.
- $\{0,1\}^{\mathbb{N}} = \{(a_n) \mid a_n = 0 \text{ or } 1 \ \forall n\}$ is a metric space with metric:

$$d((a_n), (b_n)) = \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}$$

Q: Proof that this is a metric?

- Any set X is a metric space with the discrete metric:

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

REMEMBER THIS ONE!

Q: Proof that this is a metric?

Note that $\mathbb{R}$ with $d(x,y) = |x-y|$ is also a metric space, so anything that's true for metric spaces in general will be true for $\mathbb{R}$, too!

Maybe the most important mathematical structure in a metric space is a sequence. Let's be rigorous:

Def: A sequence $(a_n) \subseteq X$ is an element of $X^{\mathbb{N}}$, i.e. it is a map $\sigma: \mathbb{N} \to X$ s.t. $\sigma(n) = a_n$.

Def: A sequence $(a_n) \subseteq X$ converges to a in X if:
$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. whenever $n \geq N$ we have $d(a_n, a) < \varepsilon$.

In almost every case, your intuition from $\mathbb{R}^2$ will suffice.
What does the above mean? Draw a picture!

If (a_n) converges to a, we will write $a_n \longrightarrow a$ or $\lim_{n\to\infty} a_n = a$.

Note: The "in X" part of the definition is important!
Let $X = (0,1) \subseteq \mathbb{R}$. Then $(a_n) = (1/n)$ does not converge in X!

Def: A subsequence $(b_k) \subseteq (a_n)$ is a sequence such that
$b_k = a_{n_k}$ for some strictly increasing sequence $(n_k) \subseteq \mathbb{N}$.

I.e., (b_k) consists of some of the elements of (a_n), in the same order, and infinitely many of them.

<u>Thm:</u> Suppose $(a_n) \subseteq X$ converges to $a \in X$. Then for any subsequence $(a_{n_k}) \subseteq (a_n)$, $\lim_{k \to \infty} a_{n_k} = a$.

<u>Proof:</u> Let $\varepsilon > 0$. Since $a_n \to a$, there is some $N \in \mathbb{N}$ s.t. whenever $n \geq N$, $d(a_n, a) < \varepsilon$.

Note that $n_k \geq k$ by the definition of a subsequence, so: whenever $k \geq N$, $n_k \geq N$, so that $d(a_{n_k}, a) < \varepsilon$.

$\therefore$ $a_{n_k} \to a$. $\square$

We will be using this result a lot!