$\boxed{2.3}$ The Topology of a Metric Space

There are some familiar concepts of real numbers — like "open" and "closed" intervals — which we will now study in detail. But we will actually define them for general metric spaces!

Note: The following can be done in several different, but equivalent ways. We will take the path of least resistance, making our definitions as simple & intuitive as possible and leaving the interesting, equivalent formulations as theorems and exercises. It would be just as valid to do the opposite!

Def: Let M be a metric space. An open ball of radius r around the point $x \in M$ is: $B(x,\varepsilon) = B_\varepsilon(x) = \{y \in M \mid d(x,y) < \varepsilon\}$.

Q: What should ε be? Can it be 0? <0?

Def: A subset S of a metric space M is open if:

$\forall s \in S \;\exists \varepsilon > 0$ s.t. $B(s,\varepsilon) \subseteq S$. In words:

"S is open if it contains an open ball around each of its points."

Def: S is closed if $S = M - U$ for some open $U \subseteq M$.

Q: What are the open balls in $\mathbb{R}$? Why is (a,b) open and $[a,b]$ closed? What about $[a,b)$?

Q: What about $\emptyset$? $\mathbb{R}$?

Def: A set which is both open and closed is clopen

Q: Which sets in $\mathbb{R}$ are clopen? in $\mathbb{Q}$? $\mathbb{N}$?

Thm: $K \subseteq M$ is closed iff for every convergent sequence $(a_n) \subseteq K$ that converges to a in M, then $a \in K$.

Proof: ($\Rightarrow$) Suppose K is closed. Then $U = M \setminus K$ is open.

Suppose $(a_n) \subseteq K$ with $a_n \longrightarrow a \in M$. Assume for contradiction that $a \notin K$. Then $a \in U$, and since U is open there is some $\varepsilon > 0$ with $B(a,\varepsilon) \subseteq U$.

But then for each $a_n \in K$ we must have $d(a, a_n) \geq \varepsilon$.
This contradicts the fact that $a_n \longrightarrow a$ $\lightning$

$\therefore$ $a \in K$, as required. $\square$

($\Leftarrow$) Suppose K contains all its limits. Let $U = M \setminus K$; we must show that U is open, so let $a \in U$.

Consider the ball $B(a,1)$. If $B(a,1) \subseteq U$ we are done. If not, then $B(a,1)$ contains some element of K; call it a_1.

Similarly; If $B(a,\frac{1}{2}) \subseteq U$ we are done. If not, there is some $a_2 \in B(a,\frac{1}{2}) \cap K$.

In general, if $B(a,\frac{1}{n}) \nsubseteq U$, there is some $a_n \in B(a,\frac{1}{n}) \cap K$.

If for <u>every</u> $n \in \mathbb{N}$, $B(a,\frac{1}{n}) \nsubseteq U$, then we have constructed a sequence $(a_n) \subseteq K$, with $a_n \longrightarrow a$! But this can't be, since K is closed and $a \in M \setminus K$.

So our process must fail for some n, so that $B(a,\frac{1}{n}) \subseteq M$.
In other words, U is open, so $K = M \setminus U$ is closed. $\square$

<u>Theorem:</u> 1) Every union of open sets is open.
2) Every <u>finite</u> intersection of open sets is open.
3) ϕ & M are open.

<u>Proof:</u> 1) Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a collection of open sets, and let $U = \bigcup_{\lambda\in\Lambda} U_\lambda$.

Given any $x \in U$, then $x \in U_\lambda$ for some λ.
Since U_λ is open, there is some $\delta > 0$ with $B(x,\delta) \subseteq U_\lambda \subseteq U$.
$\therefore U$ is open. $\square$

2) Let $\{U_1, \dots, U_n\}$ be open sets, and let $U = \bigcap_{k=1}^{n} U_k$.

Given any $x \in U$, we have $x \in U_k$ for each $k = 1, \dots, n$.
Thus there are $\delta_1, \dots, \delta_n$ with $B(x, \delta_k) \subseteq U_k$ for each k.
Let $\delta = \min_{k=1,\dots,n} \{\delta_k\}$. Then $B(x,\delta) \subseteq U_k$ for each k, so in fact
$B(x,\delta) \subseteq U$. $\square$

3) For all $x \in M$, $B(x,\varepsilon) \subseteq M$ for any ε we like, so M is open.
Since ϕ contains no elements, it vacuously satisfies the definition of an open set. $\square$

The above theorem says that a metric space is a kind of <u>topological space</u>. Metric spaces provide the motivation for the development of topology so many of the concepts we will be considering have counterparts there.

Cor: Any intersection of closed sets is closed; any finite union of closed sets is closed.

Proof: LTS. □

Ex: Let $K_n = [-\frac{1}{n}, \frac{1}{n}]$ for $n \in \mathbb{N}$. Then $\bigcup K_n = (-1,1)$ which is not closed.

Let $U_n = (-\frac{1}{n}, \frac{1}{n})$. Then $\bigcap_{n=1}^{\infty} U_n = \{0\}$ which is not open.

(Proofs?)

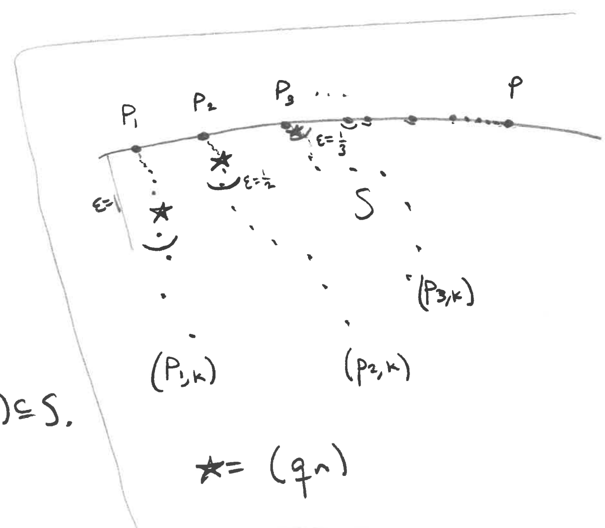
Def: The limit set of $S \subseteq M$ is: $\lim S = \{p \in M \mid \exists (a_n) \subseteq S \text{ with } a_n \to p\}$.

Thm: For any $S \subseteq M$, $\lim S$ is closed.

Proof: We must show that for any convergent sequence in $\lim S$, its limit is also in $\lim S$. This is not trivial since S' is different from S.

Suppose $(p_n) \subseteq \lim S$ with $p_n \to p$.

Then each p_n is a limit point of S, so there is some sequence $(p_{n,k})_{k=1}^{\infty}$ with $p_{n,k} \xrightarrow{k \to \infty} p_n$.



This means there is some $k_n \in \mathbb{N}$ so that $d(p_n, p_{n,k_n}) < \frac{1}{n}$

Let $q_n = p_{n,k_n}$ and consider the sequence (q_n).

Note that $(q_n) \subseteq S$, since each $(p_{n,k}) \subseteq S$.

And for a given n, we have:

$$d(q_n, p) \leq d(q_n, p_n) + d(p_n, p)$$

So let $\varepsilon > 0$. There is an N_q so that $n \geq N_q \Rightarrow d(q_n, p_n) < \varepsilon/2$.
There is also an N_p with $n \geq N_p \Rightarrow d(p_n, p) < \varepsilon/2$.
Thus if $n \geq \max\{N_p, N_q\}$, then $d(q_n, p) < \varepsilon$. $\therefore$ $p \in \lim S$. □

Cor: $\lim(\lim S) = \lim S$.

Thm: $\lim S$ is the "smallest" closed set containing S, in the following sense: If K is closed and $S \subseteq K$, then $\lim S \subseteq K$

Proof: If $x \in \lim S$, then there is a sequence $(x_n) \subseteq S$ that converges to x. But since K is closed, then $x \in K$.

Def: We usually will write $\lim S = \overline{S}$, the closure of S.

Thm: $\overline{S}$ is the intersection of all closed sets containing S.

Proof: Let $\mathcal{K} = \bigcap\limits_{\substack{K \text{ closed} \\ S \subseteq K}} K$. We have $\overline{S} \subseteq \mathcal{K}$ since $\overline{S} \subseteq K$ for each K in the intersection.

We have shown that $\overline{S}$ is closed, so it is one of the K, which gives $\mathcal{K} \subseteq \overline{S}$. Thus $\mathcal{K} = \overline{S}$. $\square$