Real Variables Homework 4

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1.) For a set $S \subseteq M$, the *interior* of S, usually denoted S°, is defined as:

$$\text{int } S = S^\circ = \{p \in S | B(p, \epsilon) \subseteq S \text{ for some } \epsilon > 0\}.$$

(a) Prove: For any $S \subseteq M, S^\circ$ is open.

Proof. By the proof in (c) we have shown that S° is the union over all open sets $U \subseteq S$. Every union of open sets is an open set, thus S° is open.

□

(b) Prove: For any open $U \subseteq S, U \subseteq S^\circ$. Thus, S° *is the largest open set contained in* S.

Proof. U is open in S so for every $p \in U$, there exists an $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U \subseteq S$. By definition of S°, $p \in S^\circ$ as well. Thus $U \subseteq S^\circ$ for any open set U. □

(c) Prove: For any $S \subseteq M$, $S^\circ = \bigcup U$, where the union is taken over all open sets $U \subseteq S$.

Proof. Let V be the union taken over all open sets $U \subseteq S$. From (b) we have proven that any open $U \subseteq S, U \subseteq S^\circ$. Since the union of open sets is an open set, then V is open set and $V \subseteq S$ thus $V \subseteq S^\circ$. For every $p \in S^\circ$ there exists an $\epsilon > 0$ such that an open ball $B(p, \epsilon) \in S$, $B(p, \epsilon)$ is an open set of S so $B(p, \epsilon) \in V$ which implies $p \in V$. Thus $S^\circ \subseteq V$. Then $S^\circ = V$.

□

(d) Prove: If $p \in S^\circ$ and $q \in (\overline{S})^c$, then $d(p, q) > 0$.

Proof. □

(e) Does $(S^\circ)^\circ = S^\circ$? If so, prove it; if not give a counterexample.

Proof. By (a) S° is an open set and $(S^\circ)^\circ$ is the largest open set contained in S° by (b). Well S° is the largest open set contained in S° so $(S^\circ)^\circ = S^\circ$. □

(f) Does $\overline{S^\circ} = \overline{S}$? If so, prove it; if not, give a counterexample.

Proof. Let $S = \{1\}$, then $S^\circ = \emptyset$ then $\overline{S^\circ} = \overline{\emptyset} = \emptyset$, since the closure is the largest closed set that contains the original set. However $\overline{S} = \{1\}$. Thus $\overline{S^\circ} = \overline{S}$ is not always true. □

(g) Does $(\overline{S})^\circ = S^\circ$? If so, prove it; if not, give a counterexample.

Proof. Let $S = (0, 1) \cup (1, 2)$, then $\overline{S} = [0, 2]$ and $(\overline{S})^\circ = (0, 2)$. While the $S^\circ = (0, 1) \cup (1, 2)$. □

2.) For a set $S \subseteq M$, the *boundary* of S, usually denoted ∂S, is defined as:

$$\text{bdry} S = \partial S = \overline{S} \backslash S^\circ$$

(a) For each of the following subsets of $\mathbb{R}$, find their boundary:

- $(0, 1)$

  $\overline{S} = [0, 1]$ since you can construct sequences in $(0, 1)$ that converges to $\{1\}$ and $\{0\}$ they are include in the $\overline{S}$ set. $S^\circ = (0, 1)$ since S° is the largest open set contained in $(0, 1)$. Thus $\partial S = \{\{0\}, \{1\}\}$.

- $[0, 1]$

  $\overline{S} = [0, 1]$ since the closure of a closed set is itself. $S^\circ = (0, 1)$, thus $\partial S = \{0\} \cup \{1\}$.

- $\mathbb{Q}$

  We proved in class that the reals can be created by sequences of rational numbers this implies that the closure of the rationals are the reals. The interior of the rationals is the empty set, since between two rational numbers there is a irrational number any open ball will include irrational numbers so there are no open balls contained in the rationals. Thus $\partial S = \mathbb{R}$.

- $\mathbb{R} \backslash \mathbb{Q}$

  The closure of this set is the real numbers. Every real number can be defined as a sequence of irrational numbers. Thus the closure is all real numbers. The interior of this set is the empty set, because between every two irrational numbers is a rational number so every open ball will include a rational number and thus not be contained in the set. So the boundary is the real numbers.

(b) Prove: A point p is in ∂S if and only if: for every $\epsilon > 0, B(p, \epsilon) \cap S \neq \emptyset$ and $B(p, \epsilon) \cap S^c \neq \emptyset$.

Proof. ($\implies$): Assume t Since p is a limit of S because $p \in \partial S$, there exists a sequence $(a_n)p$ with $(a_n) \in S$ such that for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ for $n > N$, $d(p, a_n) < \epsilon$. Then for a_n, with $n > N$ we know $a_n \in S$ and $a_n \in B(p, \epsilon)$. Thus for every $B(p, \epsilon)$ there exists a $q \in S$ such that $q \in B(p, \epsilon)$ and $B(p, \epsilon) \cap S \neq \emptyset$. We know that $S^\circ = \{p \in S | B(p, \epsilon) \subseteq S\}$, but $\partial S \not S^\circ$ by definition we know $\partial S \cap S = \emptyset$. Thus for $p \in \partial S$ it is true that for $\epsilon > 0$, $B(p, \epsilon) \not\subseteq S$ thus $B(p, \epsilon) \cap S^c \neq \emptyset$.

($\impliedby$): Let for every $\epsilon > 0$ we have $B(p, \epsilon) \cap S \neq \emptyset$ and $B(p, \epsilon) \cap S^c \neq \emptyset$. We need to show $p \in \partial S$. Since $B(p, \epsilon) \cap S \neq \emptyset$ we know there exists a sequence in S that converges to p so $p \in \overline{S}$, similarly there exists a sequence in S^c that converges to p, so $p \in \overline{S^c}$. From (c) we know that $\partial S = \overline{S} \cap \overline{(S^c)}$ so $p \in \partial S$. ☐

(c) Prove: $\partial S = \overline{S} \cap \overline{(S^c)}$

Proof. ($\implies$): Let $p \in \partial S$, then $p \in \overline{S}$ and $p \notin S^c$. We need to show $p \in \overline{(S^c)}$. Which is the same as showing p is a limit of S^c. From (b) we know that for all $\epsilon > 0, B(p, \epsilon) \cap S^c \neq \emptyset$. So there exists a points *epsilon* away from p for any *epsilon*. Thus p is a limit point of S^c and $p \in \overline{S} \cap \overline{(S^c)}$ giving us $\partial S \subseteq \overline{S} \cap \overline{(S^c)}$.

($\impliedby$): Let $p \in \overline{S} \cap \overline{(S^c)}$ then $p \in \overline{S}$ and $p \in \overline{(S^c}$. Thus p is a limit point of S and of S^c. We know that $p \in \overline{S^c}$ so $p \notin S^\circ$. Then we know $p \in \partial S$. Thus $\overline{S} \cap \overline{(S^c)} \subseteq \partial S$. Thus $\partial S = \overline{S} \cap \overline{(S^c)}$. ☐

(d) Find a metric space M, a point $p \in M$, and a radius r, such that $\partial B(p, r) \neq \{q \in M | d(p, q) = r\}$.

Proof. Let $M = \mathbb{R}$ with the discrete metric, $q = 3$ and $r = 1$, then the open ball $B(3, 1) = \{x \in \mathbb{R} | d(x, 3) < 1\}$. Thus $B(3, 1) = \{3\}$. We know $\partial B(3, 1) = \overline{B(3, 1)} \backslash B(3, 1)^\circ$. Computing each component gives $\overline{B(3, 1)} = \{3\}$ and $B(3, 1)^\circ = \{3\}$. Thus $\partial B(3, 1) = \emptyset$. On the right hand side we have $\{q \in M | d(3, q) = 1\} = \mathbb{R} \backslash \{3\}$. Thus $\partial B(3, 1) \neq \{q \in M | d(3, q) = 1\}$. ☐

3.) A subset S of a metric space M is said to be *dense in* M if $\overline{M}$. (a) Find a dense subset S of $\mathbb{R}$ such that $S^\circ = \emptyset$.

Proof. The rational numbers are a dense subset of the real numbers. We proved in class that sequences of rational numbers construct the real numbers. This means for every real number p there exists a sequence of rational numbers that converges to p. Thus the closure of the rationals in the reals. However since there are no open balls in rationals, since each open ball will contain an irrational, the interior of the rationals is the empty set. $\square$

(b) Prove: If $S \subseteq M$ is dense, then for any open $U \subseteq M, U \cap S \neq \emptyset$.

Proof. Let $q \in U$, then $q \in M$ which means $q \in \overline{S}$. Since U is open there exists an open ball around point q with radius r, $B(q,r) = \{p \in M | d(q,p) < r\} \subseteq U$. Next we look at the closure of S, since $q \in \overline{S}$ there exists a sequence $(a_n) \in S$ such that $(a_n) \to q$. Thus for $r > 0$, there exists a $N \in \mathbb{N}$ such that for $n > N$, $d(a_n, q) < r$. This implies that $(a_n) \in B(q,r)$ for $n > N$. So $(a_n) \in U$ for $n > N$ and $(a_n) \in S$ for $n > N$. Therefore $U \cap S \neq \emptyset$. $\square$

(c) Does every infinite metric space have a proper dense subset? If so, prove it; if not, give a counterexample.

Proof. This conjecture does not always hold. Let's take the real numbers with the discrete metric. Then for a proper subset $C \subset \mathbb{R}$ there exists a $q \in \mathbb{R}$ and $q \notin C$. Then we will show that $q \notin \overline{C}$, because $q \in \overline{C}$ if there exists a sequence $(a_n) \to q$, for $a_n \in C$. We have gone over in class that the only convergent sequences with the discrete metric are the eventually constant ones. So if $a_n \to q$ then there exists $N \in \mathbb{N}$ such that for $n > N$, $a_n = q$, but $q \notin C$ so $a_n \notin C$. Thus this sequence is not possible, no sequence exists that converges to q. Thus $\overline{C} \neq \mathbb{R}$ since $q \in \mathbb{R}$ and $q \notin \overline{C}$. Therefore for $(\mathbb{R}, \text{discrete})$ does not have a proper dense subset.

$\square$

(d) If $S \subseteq M$ is dense, does $\partial S = M$? If so, prove it: if not give a counterexample.

Proof. This conjecture is not always true. Let's examine the subset of the reals $S = \mathbb{R} \backslash \mathbb{Z}$, this can also be written as $S = \cdots \cup (-2,-1) \cup (-1,0) \cup (0,1) \cup \ldots$. This set is dense because there exists a sequence that converges to every real number, for example to converge to $z \in \mathbb{Z}, (a_n) = z + 1/n$ starting at $n = 2$ converges to z. However the boundary is not the real numbers. $\overline{S} = \mathbb{R}$ and $S^\circ = S$ thus $\partial S = \mathbb{R} \backslash S = \mathbb{Z}$. Thus the conjecture above is not always true. $\square$