# 2.4 Compactness

Q: What does "compact" mean?

(not quite small, not quite finite — more like "packed" together, but not quite that, either)

Def: A space $X$ is sequentially compact if every sequence $(a_n) \subseteq X$ has a subsequence that converges in $X$.

Ex: $\emptyset$ is sequentially compact

Any finite set is sequentially compact

$\mathbb{R}$ is not sequentially compact (which means this is different from completeness, too)

Thm: $[a,b] \subseteq \mathbb{R}$ is sequentially compact

Proof: Let $(a_n) \subseteq [a,b]$. Then $(a_n)$ is bounded above & below. We proved that $(a_n)$ has a monotone subsequence $(a_{n_k})$. Then $(a_{n_k})$ is bounded & monotone, so it converges in $\mathbb{R}$. Since $[a,b]$ is closed, $\lim a_{n_k} \in [a,b]$, and we are done. $\square$

Q: Is it the "closed-ness" of $[a,b]$?

A: Sort of...

Def: If $A$ is a subset of a metric space $M$, we say $A$ is bounded if there is some $x \in M$ and $r \in \mathbb{R}$ with $A \subseteq B(x,r)$.

<u>Thm:</u> If $A \subseteq M$ is sequentially compact, then it is closed & bounded.

<u>Proof:</u> (Closed): Let $p$ be a limit of $A$, i.e. there is some sequence $(a_n) \subseteq A$ with $a_n \longrightarrow p$ in $M$.

Since $A$ is sequentially compact, there is some $(a_{n_k}) \subseteq (a_n)$ that converges <u>in $A$</u>.

But we proved that subsequential limits equal sequential limits when the latter exist; this means $a_{n_k} \longrightarrow p$.

Combined with the above, we have $p \in A$, so $A$ is closed. $\square$

(Bounded): Choose any $p \in M$. If $A \nsubseteq B(p,1)$, there is some $a_1 \in A$ with $d(a_1,p) > 1$. If $A \nsubseteq B(p,2)$, there is some $a_2 \in A$ with $d(a_2,p) > 2$. In general, for each $n \in \mathbb{N}$, if $A \nsubseteq B(p,n)$, then there is some $a_n \in A$ with $d(a_n,p) > n$.

If $A \nsubseteq B(p,n)$ $\forall n \in \mathbb{N}$, we can build a sequence $(a_n) \subseteq A$.

Since $A$ is sequentially compact, $(a_n)$ would have a convergent subsequence $(a_{n_k})$ — but this is impossible because every subsequence of $(a_n)$ would be unbounded.

Thus there must be some $n \in \mathbb{N}$ with $A \subseteq B(p,n)$, and we are done. $\square$

<u>WARNING:</u> Closed & Bounded are <u>necessary</u> but <u>not sufficient</u> for sequential compactness.

<u>Challenge:</u> Find a metric space $M$ and a closed, bounded $A \subseteq M$, such that $A$ is not sequentially compact.

**Thm:** Let $A \subseteq M$ and $B \subseteq N$ be sequentially compact. Then $A \times B$ is sequentially compact.

**Proof:** LTS. (Proof in book is 3/4) $\square$

**Cor:** Let $A_k \subseteq M_k$ be sequentially compact for $k = 1, \ldots, m$. Then $A_1 \times A_2 \times \cdots \times A_m$ is sequentially compact.

**Proof:** Induction. $\square$

**Cor:** Boxes in $\mathbb{R}^m$ are sequentially compact.

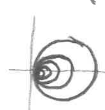
**Cor:** (Bolzano-Weierstrass): Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.

**Thm:** If $A$ is sequentially compact and $K \subseteq A$ is closed, then $K$ is sequentially compact.

**Proof:** Suppose $(a_n) \subseteq K$. Then also, $(a_n) \subseteq A$, so it has a convergent subsequence $(a_{n_k})$ <u>in $A$</u>.
But $K$ is closed, so $(a_{n_k})$ converges in $K$, and we are done. $\square$

**Thm:** (Heine-Borel) If $A \subseteq \mathbb{R}^m$ is closed and bounded, then it is sequentially compact.

**Proof:** Since $A$ is bounded, there is some box $B$ with $A \subseteq B$. We already have that $B$ is sequentially compact, and since $A$ is closed this means $A$ is sequentially compact as well. $\square$

**Ex:**
- Any finite subset of any space is sequentially compact
- Any closed subset of any compact metric space is sequentially compact
- Any finite union of sequentially compact sets
- Any product of sequentially compact spaces
- Any intersection of sequentially compact sets
- Closed balls in $\mathbb{R}^m$ (but not in other spaces...)
- The boundary of any sequentially compact set
  (Def: $bd(S) = \bar{S} \setminus S^\circ$)
- $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$ (B/c it's closed & bdd, natch)
- The Hawaiian Earring  $= \bigcup_{n \in \mathbb{N}} \{(x,y) \in \mathbb{R}^2 \mid (x-\frac{1}{n})^2 + y^2 = \frac{1}{n}\}$
- The Cantor Middle 1/3 set (To come...)

## Nested sequences of sets

Consider: $A_n = (-\frac{1}{n}, \frac{1}{n})$. Then $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$.

The sets are called "nested".

Note that for this example, $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$, i.e. their intersection is nonempty.

Now consider: $A_n = (1-\frac{1}{n}, 1)$. Then $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$.

So nested intervals don't always have nonempty intersection.

Thm: Suppose $A_n$ is a nested sequence of nonempty sequentially compact sets. Then $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$, and it is compact.

Proof. Since each $A_n$ is closed, $\bigcap A_n$ is closed.

Since $\bigcap A_n \subseteq A_1$, then $\bigcap A_n$ is sequentially compact

Since each $A_n$ is nonempty, we can choose an $a_n$ for each $n$; consider the sequence $(a_n)$.

Since $A_1$ is sequentially compact, $(a_n)$ has a subsequence $(a_{n_k})$ that converges to some $p \in A_1$.

But $(a_{n_k})_{k=2}^{\infty} \subseteq A_2$, so $p \in A_2$ as well.

In fact, $(a_{n_k})_{k=m}^{\infty} \subseteq A_m$ for every $m \in \mathbb{N}$, so $p \in A_m \ \forall m$.

But this means $p \in \bigcap A_n$, so $\bigcap A_n \neq \emptyset$. □

Def: The diameter of a set $A$ is: $\operatorname{diam} A = \sup\{d(a,b) \mid a,b \in A\}$

Thm: If $A_n$ is a nested sequence of nonempty sequentially compact sets and $\operatorname{diam}(A_n) \longrightarrow 0$, then $\bigcap A_n = \{a\}$.

Proof: LTS. □

Recall: A continuous function on the interval $[a,b]$ attains its maximum & minimum values on that interval. But why?

- It's not because $[a,b]$ is closed; the statement is not true if $f$ is defined on all of $\mathbb{R}$
- It's not because $[a,b]$ is bounded; the statement is not true for $(a,b)$
- It's true because $[a,b]$ is compact, which means it behaves as though it were finite, and the statement is certainly true for any finite set as well

(You may recall that we used the LUB property of $\mathbb{R}$ — that is, completeness — when we proved this theorem in Section 1.6. It turns out that every compact metric space must be complete, so this is entirely consistent.)

Thm: Let $f: M \to N$ be continuous, and suppose $A \subseteq M$ is sequentially compact. Then $f(A) \subseteq N$ is sequentially compact.

Proof: Suppose $(b_n) \subseteq f(A)$. We must show it has a convergent subsequence. By definition, $f^{-1}(b_n) \neq \emptyset$ for all $n$, so let $a_n \in f^{-1}(b_n) \subseteq A$, and consider the sequence $(a_n) \subseteq A$.

Since $A$ is compact, $(a_n)$ has a convergent subsequence $(a_{n_k})$ which converges in $A$, say $a_{n_k} \to a \in A$.

And since $f$ is continuous, $(f(a_{n_k}))$ converges to $f(a)$.

We see that $(f(a_{n_k}))$ is a subsequence of $(b_n)$, and $f(a) \in f(A)$.

Thus $f(A)$ is sequentially compact. $\square$

Thm: Suppose $f: M \to \mathbb{R}$ is continuous, and $A \subseteq M$ is sequentially compact. Then $f(A)$ is bounded, and $f$ attains its minimum & maximum values on $A$.

Proof: Since $f(A) \subseteq \mathbb{R}$ is sequentially compact, it must be bounded (and closed, as well).

Let $\alpha = \inf(f(A))$ and $\Omega = \sup(f(A))$. Then there are sequences $(a_n)$ and $(w_n) \in f(A)$ with $a_n \to \alpha$, $w_n \to \Omega$.

Since $f(A)$ is closed, this means $\alpha \in f(A)$ & $\Omega \in f(A)$.

But this is what we wanted to show, so we are done. □

So working with sequentially compact spaces is nice. But just knowing about sequential compactness can be useful!

Prop: $[0,1]$ is not homeomorphic to $\mathbb{R}$.
$[0, 2\pi)$ is not homeomorphic to $S^1$. etc.

Thm: Suppose $M$ is sequentially compact and $f: M \to N$ is a continuous bijection. Then $f$ must be a homeomorphism!

Proof: We only need to show that $f^{-1}$ is continuous. So suppose $(b_n) \subseteq N$ is a convergent sequence, say $b_n \to b$.

Consider the sequence $(f^{-1}(b_n)) \subseteq M$; if $f^{-1}(b_n) \longrightarrow f^{-1}(b)$ we are done, so suppose it does not. Then there must be some subsequence $(f^{-1}(b_{n_k}))$ that stays $\delta$-far away from $f^{-1}(b)$ for some $\delta > 0$. And there is some sub-sequence of that, $(f^{-1}(b_{n_{k_\ell}}))$, that converges in $M$, say to $f^{-1}(b^*) \neq f^{-1}(b)$.

But this is a contradiction, since $(b_{n_{k_\ell}}) \subseteq (b_n)$ and $b_n \to b$, and $f$ is continuous.

$\therefore$ $f^{-1}(b_n) \to f^{-1}(b)$, and we are done. □

Def: $f: M \to N$ is Uniformly continuous if for all $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $d_M(x,y) < \delta$, then $d_N(f(x), f(y)) < \varepsilon$.

Thm: Suppose $f: M \to N$ is continuous and $K \subseteq M$ is sequentially compact. Then $f$ is uniformly continuous on $K$.

Proof: Suppose for contradiction that it is not. Then there is some $\varepsilon$ such that for every $\delta > 0$ we can find $x$ & $y \in K$ with:

$d_M(x,y) < \delta$ but $d_N(f(x), f(y)) \geq \varepsilon$.

In particular this is true for $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$; then there are sequences $(x_n)$ & $(y_n)$ in $K$ with:

$d_M(x_n, y_n) < \frac{1}{n}$ but $d_N(f(x_n), f(y_n)) \geq \varepsilon$ $\forall n$.

Since $K$ is sequentially compact, there is a subsequence $(x_{n_k}) \subseteq (x_n)$ that converges in $K$, say $x_{n_k} \to x$.

Now $\lim_{k \to \infty} d(x_{n_k}, y_{n_k}) = 0$, so $y_{n_k} \to x$ as well.

Since $f$ is continuous, we then have $\lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(y_{n_k}) = f(x)$.

In other words, $\exists K \in \mathbb{N}$ s.t. $k \geq K \Rightarrow d(f(x_{n_k}), f(x)) < \varepsilon/2$ and $d_N(f(y_{n_k}), f(x)) < \varepsilon/2$. But then:

$$d_N(f(x_{n_k}), f(y_{n_k})) \leq d_N(f(x_{n_k}), f(x)) + d_N(f(x), f(y_{n_k})) < \varepsilon$$

which contradicts our assumption.

$\therefore$ $f$ must be uniformly continuous on $K$. $\square$