More on continuity

We defined continuity in terms of convergent sequences, and we saw that closed-ness is related to convergence of sequence. It should not be surprising then that open-ness, closed-ness, and continuity are themselves closely related concepts.

Thm: If $f: M \to N$ is continuous, then for every closed $K \subseteq N$, its preimage $f^{-1}(K)$ is closed in M.

Proof: Let $K \subseteq N$ be closed. We want to show $f^{-1}(K)$ is closed, so let $(a_n) \subseteq K$ be a convergent sequence, i.e. $a_n \to a \in M$. Since f is continuous, $f(a_n) \to f(a)$ in N. But K is closed and $(f(a_n)) \subseteq K$, so we have that $f(a) \in K$. Then by definition $a \in f^{-1}(f(a)) \subseteq f^{-1}(K)$, so $f^{-1}(K)$ is closed. □

Thm: If $f: M \to N$ is continuous, then for every open $U \subseteq N$, its preimage $f^{-1}(U)$ is open in M.

Proof: Let $U \subseteq N$ be open. Then $N \setminus U$ is closed, so by the preceding theorem we have $f^{-1}(N \setminus U)$ is closed. Note that $\boxed{M \setminus f^{-1}(N \setminus U) = M \setminus f^{-1}(U)}$, so $f^{-1}(U)$ is open, and we are done. □

Student presentation.

Thm: If for every open $U \subseteq N$ $f^{-1}(U)$ is open, then f is ε-δ continuous.

Proof: Let $x \in M$, and let $\varepsilon > 0$. Let $U = B(f(x), \varepsilon)$; then U is open in N. Thus $f^{-1}(U)$, which contains x, is open. There must then be some $\delta > 0$ s.t. $B(x, \delta) \subseteq f^{-1}(U)$.
Now if $y \in M$ s.t. $d(x,y) < \delta$, then $y \in B(x,\delta) \subseteq f^{-1}(U)$, so that $f(y) \in U = B(f(x), \varepsilon)$. In other words, $d(f(x), f(y)) < \varepsilon$.
But this means f is ε-δ continuous at x. $\square$

Here is the situation:

$$\begin{tikzcd}
\boxed{\text{f is sequentially continuous}} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \boxed{\text{f satisfies the closed set property}} \arrow[d, Leftrightarrow] \\
\boxed{\text{f is } \varepsilon\text{-}\delta \text{ continuous}} & \boxed{\text{f satisfies the open set property.}} \arrow[l, Rightarrow]
\end{tikzcd}$$

We can get from any statement to any other by following implications; this means:

Theorem: The Following Are Equivalent:

1) f is (sequentially) continuous
2) f is ε-δ continuous at every point in its domain
3) f satisfies the open preimage condition
4) f satisfies the closed preimage condition.

Cor: A homeomorphism is a bijection between the collections of open sets in its domain and range.

Proof: LTS. $\square$

New Spaces from Old — Subspace Metrics

Let M be a metric space, and suppose $N \subseteq M$.

Q: Considering N as a metric subspace of M, which sets in N are open?

A: Thm: Suppose N is a metric subspace of M with inherited metric d. Then $S \subseteq N$ is open iff $S = U \cap N$ for some open $U \subseteq M$.

Proof: LTS. $\square$

Obviously the same is true for closed sets. The consequence is we need to be careful when we say "this set is open". We must be clear about which space it is open in.

Ex: Consider $\mathbb{N}$ and $\mathbb{Q}$ as subspaces of $\mathbb{R}$.

Then $\{1\}$ is open in $\mathbb{N}$ but closed in $\mathbb{Q}$ and $\mathbb{R}$.

$\{\pi - \frac{1}{n} \mid n \in \mathbb{N}\}$ is closed in $\mathbb{Q}$ but neither open nor closed in $\mathbb{R}$.

Every subset of $\mathbb{R}$ is open with respect to itself!

$[0,1)$ is closed in $(-1,1)$, but not in $\mathbb{R}$.

Thm: Let N be a metric subspace of M. If N is closed in M, then $K \subseteq N$ is closed in N iff it is closed in M.
If N is open, then $U \subseteq N$ is open in N iff it is open in M.

Proof: LTS (#2.34). $\square$

Product Metrics

Suppose X & Y are both metric spaces. We can put a metric structure on $M = X \times Y$. (Think $\mathbb{R}^2$, e.g.!)

The thing is, there is more than one way to do this! So we will look at three such, and then discuss why the choice doesn't matter very much, as well as when it does.

Def: Let $M = X \times Y$ and for $p = (x_1, y_1)$, $q = (x_2, y_2)$, define:

$$d_E(p,q) = \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2}$$

$$d_{max}(p,q) = \max(d_X(x_1,x_2), d_Y(y_1,y_2))$$

$$d_{sum}(p,q) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$

Thm: d_E, d_{max}, and d_{sum} are all metrics on M.

Proof: LTS (#2.38) $\square$

Def: Two metrics d_1 & d_2 on a space M are equivalent if

$id: (M, d_1) \longrightarrow (M, d_2)$ is a homeomorphism.

(It is automatically a bijection, so it's the continuity that matters)

Equivalent: sets open wrt d_1 are also open wrt d_2 and vice-versa.

Thm: d_E, d_{max}, and d_{sum} are all equivalent.

Proof: LTS. $\square$

Thm: Let $M = X \times Y$ and $(p_n) = ((x_n, y_n)) \subseteq M$. TFAE:

1) (p_n) converges wrt d_E
2) (p_n) converges wrt d_{max}
3) (p_n) converges wrt d_{sum}
4) (x_n) converges in X & (y_n) converges in Y

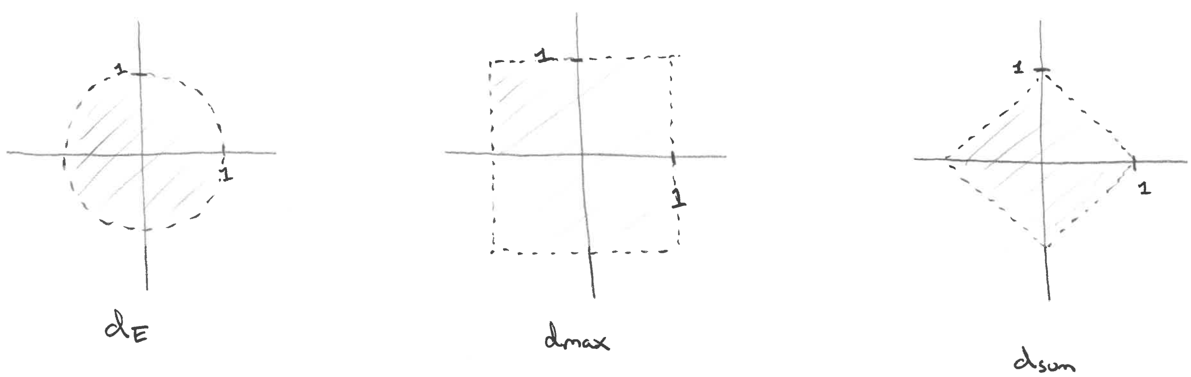
Proof: This can be proved directly from equivalence of the metrics. But instead, we note that $\forall p, q \in M$;

$$\boxed{d_{max}(p,q) \leq d_E(p,q) \leq d_{sum}(p,q) \leq 2\, d_{max}(p,q)}$$

which gives the results immediately. $\square$

In light of the inequalities above, it's worth pausing to ask: what do open balls look like in each of these metrics?

$B(0,1) \subseteq \mathbb{R}^2$:



Note that the largest open ball corresponds with the smallest metric!

These three metrics can be thought of as special cases of the "p-metric":

$$d_p(\underline{x},\underline{y}) = \left(d(x_1,y_1)^p + \cdots + d(x_m,y_m)^p \right)^{1/p}$$

Q: what is p for our choices? What do open balls look like for other values of p?

Thm: If $p \geq 1$, then d_p is a metric. $\boxed{\text{If } p<0, \text{ it is not.}}$

Proof: omitted. $\square$ student presentation

Thm: Let $(\underline{x}_n) \subseteq \mathbb{R}^m$ and write $\underline{x}_n = (x_{1n}, x_{2n}, \ldots, x_{mn})$. Then:

$(\underline{x}_n)$ converges iff each x_{kn} does, and if $\underline{x}_n \longrightarrow \underline{x}$,

then $x_{kn} \longrightarrow x_k$.

Proof: LTS. $\square$

Note: The above theorem works because m is finite. It is not true in $\mathbb{R}^{\mathbb{N}}$; the details would be confusing at this point, but know that things get weird when infinite values are involved.

Thm: metrics are continuous.

– Before the proof: What does this even mean? Recall: a metric d on a space M is a function $d: M\times M \longrightarrow \mathbb{R}$. We assume that $\mathbb{R}$ will have its usual metric, $d_{\mathbb{R}}(x,y) = |x-y|$.

To show that d is continuous, we will show that it is ε-δ continuous at every point of its domain. But its domain is $M\times M$, so we need to define what distance means there, too. For this we will use d_{sum}, which is given in terms of d!

Proof: Let $x = (p,q)$ and $x' = (p',q')$ be points in $M\times M$.
Let $\varepsilon > 0$ be given, and we will let $\delta = \varepsilon$. We want to show that as long as $d_{sum}(x,x') < \delta$, then

$d_{\mathbb{R}}(d(p,q), d(p',q')) < \varepsilon$, i.e. that $|d(p,q) - d(p',q')| < \varepsilon$.

If $\boxed{d_{sum}(x,x')} < \delta$, then $\boxed{d(p,p') + d(q,q')} < \delta = \varepsilon$, so:

equal

$$d(p,q) \overset{\Delta}{\leq} d(p,p') + d(p',q') + d(q',q) < d(p',q') + \varepsilon$$

and

$$d(p',q') \overset{\Delta}{\leq} d(p',p) + d(p,q) + d(q,q') < d(p,q) + \varepsilon$$

Rewriting these, keeping only the first & last terms:

$$d(p,q) < d(p',q') + \varepsilon \qquad \text{AND} \qquad d(p',q') < d(p,q) + \varepsilon$$

Combining gives:

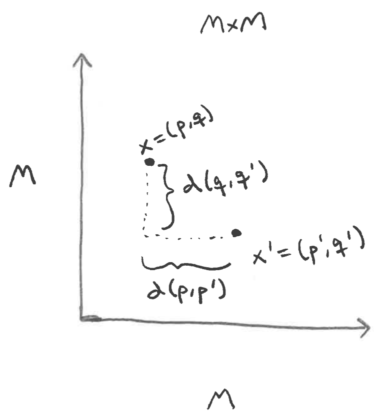
$$-\varepsilon < d(p,q) - d(p',q') < \varepsilon, \quad \text{or} \quad |d(p,q) - d(p',q')| < \varepsilon.$$

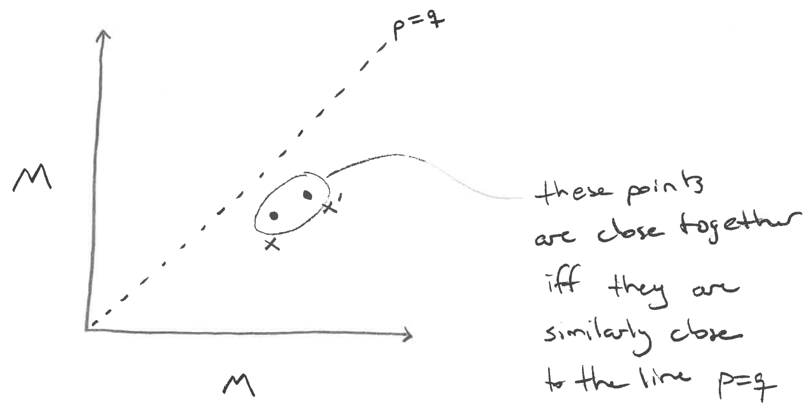
But this was exactly our goal! When x & x' are close in $M\times M$, then $d(x)$ & $d(x')$ are close in $\mathbb{R}$. □

The preceding requires at least two comments:

1) Pugh's proof in the text is at best a 3/4. He omits several key details which makes his proof, IMO, quite hard to read

2) There's a picture for this theorem:





We can even visualize the value of d as a surface above M×M:

3) Is it ok that we used d_{sum} rather than d_E or d_{max}?
Yes, it's fine, actually, since the metrics are all equivalent.