$\boxed{2.8}$ The Cantor Set

So we're interested in sets that have measure zero. So far, all the examples we've seen have been countable. We return to Chapter 2 for an uncountable example.

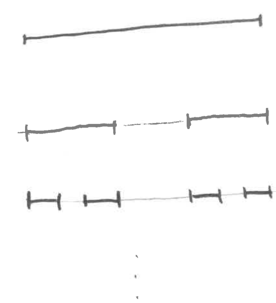
Def. Let $C_0 = [0,1]$

$$C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$

$$C_2 = \left([0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \right) \cup \left([\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1] \right)$$

$$\vdots$$

$$C_n = \left(\tfrac{1}{3} C_{n-1} \right) \cup \left(\tfrac{1}{3} (2 + C_{n-1}) \right)$$



Or:

$$C_2 = C_1 \setminus (\tfrac{1}{3}, \tfrac{2}{3})$$

$$C_3 = C_2 \setminus \left((\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9}) \right)$$

$$\vdots$$

$$C_n = C_{n-1} \setminus \left(\left(\tfrac{1}{3^{n-1}}, \tfrac{2}{3^{n-1}} \right) \cup \cdots \cup \left(\tfrac{3^{n-1}-2}{3^{n-1}}, \tfrac{3^{n-1}-1}{3^{n-1}} \right) \right)$$

This is why it's often called the "Cantor Middle-Thirds" set.

The <u>Cantor Middle-Thirds Set</u> is: $\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$

<u>Q</u>: Which points are in $\mathcal{C}$?

<u>A</u>: At least some points like $\frac{k}{3^n}$ ... but (uncountably) many more!

Prop: $\mathcal{C}$ is compact.

Proof: LTS. $\square$

Prop: For every $x \in \mathcal{C}$, x is the limit of a sequence in $\mathcal{C}$ that is not eventually constant (x is a cluster point of $\mathcal{C}$)

Proof: Given any $\varepsilon > 0$, consider $(x-\varepsilon, x+\varepsilon) \cap \mathcal{C}$.

There is an n large enough so that $\frac{1}{3^n} < \varepsilon$, and since $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$

we know that $x \in C_n$.

C_n is the disjoint union of intervals of width $\frac{1}{3^n}$; one of these intervals is thus contained in $(x-\varepsilon, x+\varepsilon)$.

$\Rightarrow (x-\varepsilon, x+\varepsilon) \cap C_n$ is infinite. But in fact this is true

for every n, so $(x-\varepsilon, x+\varepsilon) \cap \mathcal{C}$ is infinite, for every ε.

$\therefore$ Using $\varepsilon = \frac{1}{n}$, we can build a sequence in $\mathcal{C} \setminus \{x\}$ that converges to x. $\square$

Prop: For every $x \in \mathcal{C}$, and for every $\varepsilon > 0$, there is a clopen subset $U \in \mathcal{C}$ with $x \in U \subseteq (x-\varepsilon, x+\varepsilon)$. ($\mathcal{C}$ is totally disconnected)

Proof: Each subinterval of each C_n is clopen in C_n.

By Inheritance, then for each subinterval $I \subseteq C_n$, $I \cap \mathcal{C}$ is clopen in $\mathcal{C}$. But we can find arbitrarily small I by taking large n, and every $x \in \mathcal{C}$ is in one of these I. $\square$

Prop: $m(\mathcal{C}) = 0$

Proof: $m(C_n) = 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$, and C_n is a cover of $\mathcal{C}$ by closed intervals.

$\therefore$ $\forall \varepsilon > 0$, choose n so that $\left(\frac{2}{3}\right)^n < \varepsilon$. Then C_n is a cover of $\mathcal{C}$ by closed intervals with total width $\left(\frac{2}{3}\right)^n < \varepsilon$. $\square$

Prop: $\mathcal{C}$ is uncountable.

Proof: Let $W = \{(w_1, w_2, \dots) \mid w_i \in \{0, 2\}\}$.

Define $\varphi : W \to \mathcal{C}$ by: $\varphi(\varpi) = \sum_{n=1}^{\infty} \frac{w_n}{3^n}$.

1) for each $\varpi \in W$, $\varphi(\varpi) \in \mathcal{C}$:

Look at the partial sums $s_K = \sum_{n=1}^{K} \frac{w_n}{3^n}$. We see that $s_K \in C_K$ for each K, so $s_K \in \bigcap_{n=1}^{K} C_K$.

$\therefore$ $\varphi(\varpi) \in \mathcal{C}$.

2) For each $x \in \mathcal{C}$, $x = \varphi(\varpi)$ for exactly one $\varpi \in W$:

First consider $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] = I_0 + I_2$.

Note that the left endpoint of I_0 is $\frac{0}{3}$ and the left endpoint of I_2 is $\frac{2}{3}$.

Thus $I_0 = \{\varphi(\varpi) \mid w_1 = 0\}$ and $I_2 = \{\varphi(\varpi) \mid w_1 = 2\}$.

Next consider $C_2 = \left([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}]\right) \cup \left([\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]\right)$

$= (I_{00} \cup I_{02}) \cup (I_{20} \cup I_{22})$

Now $x \in I_{00}$ iff $x \in I_0$ and $x \leq \frac{1}{9}$; this forces $w_2 = 0$.

In fact, $I_{00} = \{\varphi(\varpi) \mid w_1 = 0, w_2 = 0\}$, $I_{02} = \{\varphi(\varpi) \mid w_1 = 0, w_2 = 2\}$,

and so on!

In general, there are 2^n intervals in C_n, and we can label them $I_{\alpha_1\alpha_2\cdots\alpha_n}$ where each $\alpha_i \in \{0,2\}$, and:

$$I_{\alpha_1\alpha_2\cdots\alpha_n} = \{\varphi(\varpi) \mid \varpi = (\alpha_1, \alpha_2, \dots, \alpha_n, \omega_{n+1}, \dots)\}.$$

In other words, each finite string $(\alpha_1, \dots, \alpha_n)$ uniquely determines an interval $I_{\alpha_1\cdots\alpha_n} \subseteq C_n$, and furthermore the left endpoint of $I_{\alpha_1\cdots\alpha_n}$ is $\sum_{k=1}^{n} \frac{\alpha_k}{3^k}$

Now if $x \neq y \in \mathcal{C}$, then there must be some C_n such that x and y are in different subintervals of C_n (Choose $\frac{1}{n} < |x-y|$).

So if $\varpi_1 \neq \varpi_2$ then $\varphi(\varpi_1) \neq \varphi(\varpi_2)$.

And since every x is in some $I_{\alpha_1\cdots\alpha_n}$ for every n, there is some $\varpi \in W$ with $\varphi(\varpi) = x$. $\square$

So there is an uncountable set with measure zero.

<u>Cor:</u> $\mathbb{1}_{[0,1]\setminus\mathcal{C}}$ is Riemann Integrable.

This function is one almost everywhere, so its derivative is zero a.e.

<u>Def:</u> <u>The Devil's Staircase</u>: For $x = \varphi(\varpi) \in \mathcal{C}$, let $H(x) = \sum_{i=1}^{\infty} \frac{w_i}{2^{i+1}}$

For $x \notin \mathcal{C}$, let $H(x) = \max\{H(y) \mid y \leq x\}$.

Then H is Riemann integrable, it is differentiable a.e. and $H'(x) = 0$ a.e.

But H & $\mathbb{1}_{[0,1]\setminus\mathcal{C}}$ do <u>not</u> differ by a constant!

So the FTC requires that antiderivatives differ by a constant <u>everywhere</u>, not just a.e.