Graded Homework 1
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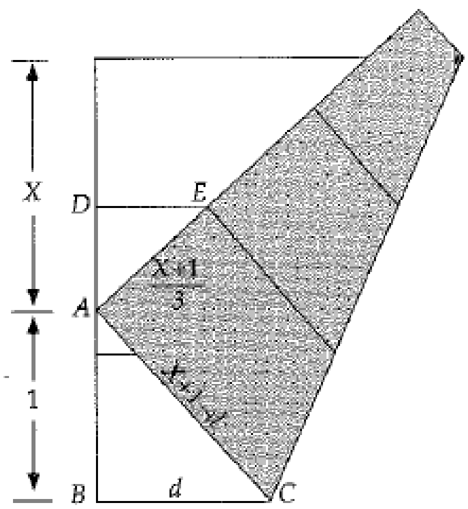
1.) (20 points) In the following diagram, a piece of paper is first folded into thirds. By performing the origami "move" of folding two points onto two lines, we obtain the picture below. Prove that $X = \sqrt[3]{2}$.



We start by finding an equation that gets d in terms of X. Inspect $\triangle ABC$, since $\angle ABC$ is a right angle applying the Pythagorean theorem to the triangle's sides yields,

$$1^2 + d^2 = (X + 1 - d)^2.$$

Rearrange the terms to get d alone as follows,

$$\begin{aligned} 1 + d^2 =& X^2 + 2X + 1 - 2Xd + d^2 - 2d, \\ 2Xd + 2d =& X^2 + 2X, \\ d =& \frac{X^2 + 2X}{2X + 2}. \end{aligned}$$

From here we need a clever way to relate the terms X and d again, examine the triangles formed by the fold. Notice the angle $\angle EAC$ is a right angle, since BAD all lie on the same line and both $\triangle ADE$ and $\triangle ABC$ are right triangles we can infer that $\angle DAE = \angle ACB$ and that $\angle DEA = \angle BAC$. This facts imply that $\triangle ABC$ is similar to $\triangle ADE$. There exists a ratio between sides of similar triangles so we can set up an equation relating the sides of these triangles such as,

$$\frac{BC}{AC} = \frac{AD}{AE}.$$

We have all the lengths in terms of X and d except for AD. Here we label the upper left hand corner of the sheet of paper L. Given that the segment $AL = X$ and $DL = \frac{X+1}{3}$, since it is 1/3 of the total length of a side (BL), and that A, D, and L are all on a straight line, $AD = X - \frac{X+1}{3}$. This can be simplified to $AD = \frac{2X-1}{3}$. Next we plug in all the side lengths of our similar triangles and simplify,

$$\begin{aligned}
\frac{d}{X+1-d} &= \frac{\frac{2X-1}{3}}{\frac{3}{X+1}}, \\
\frac{d}{X+1-d} &= \frac{2X-1}{X+1}.
\end{aligned}$$

We now substitute the equation found for d above into this equation and solve for X,

$$\begin{aligned}
\frac{\frac{X^2+2X}{2X+2}}{X+1-\frac{X^2+2X}{2X+2}} &= \frac{2X-1}{X+1}, \\
\frac{\frac{X^2+2X}{2X+2}}{\frac{(X+1)(2X+2)-X^2-2X}{2X+X}} &= \frac{2X-1}{X+1}, \\
\frac{X^2+2X}{(X+1)(2X+2)-X^2-2X} &= \frac{2X-1}{X+1}, \\
\frac{X^2+2X}{X^2+2X+2} &= \frac{2X-1}{X+1}, \\
(X^2+2X)(X+1) &= (X^2+2X+2)(2X-1), \\
X^3+3X^2+2X &= 2X^3+3X^2+2X-2, \\
2 &= X^3, \\
X &= \sqrt[3]{2}.
\end{aligned}$$

This proves the conjecture that $X = \sqrt[3]{2}$!

2.) (20 points) Write a proof to solve the equation $ax^2 + bx + c = 0$ for x. Explain each step.

We now from experience that the solution to $ax^2 + bx + c$ is known as the quadratic equation. We shall go step by step as to how you arrive at the quadratic equation. We start with $ax^2 + bx + c = 0$ and bring the c over to the right side resulting in

$$ax^2 + bx = -c.$$

Then we multiply both sides by $4a$,

$$4a^2x^2 + 4abx = -4ac.$$

The goal is to factor the left side so we add b^2 to both sides giving us,

$$4a^2 + 4abx + b^2 = b^2 - 4ac.$$

Then we can factor the left side into

$$(2ax + b)^2 = b^2 - 4ac.$$

Taking the squareroot of both sides results in,

$$2ax + b = \pm\sqrt{b^2 - 4ac}.$$

The last steps are getting x alone on the left side which will give us,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which as we have seen before is the quadratic equation!

3.) (20 points) Show that the following complex numbers are algebraic over $\mathbb{Q}$

(a) $\sqrt{2}$

The polynomial $f(x) = x^2 - 2$ proves that $\sqrt{2}$ is algebraic since $f(\sqrt{2}) = 0$.

(b) $\sqrt{n}$ for $n \in \mathbb{Z}^+$

For any $n \in \mathbb{Z}^+$, we define the function $g(x) = x^2 - n$, so that $\sqrt{n}$ is the root of $g(x)$ since $g(\sqrt{n}) = \sqrt{n}^2 - n = 0$. Therefore $\sqrt{n}$ is algebraic for any $n \in \mathbb{Z}^+$.

(c) $\sqrt{3} + \sqrt{5}$

The polynomial $f(x) = (x^2 - 8)^2 - 60$ proves that $\sqrt{3} + \sqrt{5}$ is algebraic since,

$$\begin{aligned} f(\sqrt{3} + \sqrt{5}) &= ((\sqrt{3} + \sqrt{5})^2 - 8)^2 - 60, \\ &= ((\sqrt{60} + 8) - 8)^2 - 60, \\ &= (\sqrt{60})^2 - 60, \\ &= 0. \end{aligned}$$

The method for finding the polynomial was simple. The term $\sqrt{3}+\sqrt{5}$ was squared and then whole number component of the resulting value was removed by subtraction, in this case it is 8. Since the value was still inside a radical it was squared again with the resulting value being a whole number, this in turn was also removed by subtraction (60). This worked because each time the value is squared it removed a radical from the term. This iterative method however fails to work for part (d) so we must invent a new way to approach the problem.

(d) $\sqrt[3]{2}+\sqrt{2}$

We start this problem by setting $X=\sqrt{2}+\sqrt[3]{2}$ if we follow algebraically sound steps to derive a polynomial with integer coefficients from this equation, then the root of the polynomial will be $\sqrt{2}+\sqrt[3]{2}$. This polynomial we can derive has the sixth degree, the steps to find it are as follows,

$$\begin{aligned}
X &= \sqrt{2}+\sqrt[3]{2},\\
X-\sqrt{2} &= \sqrt[3]{2},\\
(X-\sqrt{2})^3 &= (\sqrt[3]{2})^3,\\
X^3-3\sqrt{2}X^2+6X-2\sqrt{2} &= 2,\\
X^3+6X-2 &= 3\sqrt{2}X^2+2\sqrt{2},\\
X^3+6X-2 &= \sqrt{2}(3X^2+2),\\
(X^3+6X-2)^2 &= (\sqrt{2}(3X^2+2))^2,\\
X^6-6x^4-4X^3+36X^2-24X+4 &= 18X^4+24X^2+8,\\
X^6-6X^4-4X^3+12X^2-24X-4 &= 0.
\end{aligned}$$

Setting $X=\sqrt{2}+\sqrt[3]{2}$ into the resulting polynomial returns the value of 0 therefore $\sqrt{2}+\sqrt[3]{2}$ is an algebraic number since its the root of the polynomial $X^6-6X^4-4X^3+12X^2-24X-4$.