$\boxed{4.1}$ Uniform Convergence and $C^0([a,b])$

Now we shift our perspective. Where before we were concerned with points and sequences of points in $\mathbb{R}$, now we will begin to ask about functions and sequences of functions.

Specifically, suppose we have, for each n, some $f_n : [a,b] \to \mathbb{R}$.

What would it mean to say $\lim_{n\to\infty} f_n = f$?

Def: If $\forall x \in [a,b]$, $\lim_{n\to\infty} f_n(x) = f(x)$, then $\lim_{n\to\infty} f_n = f$, or $f_n \to f$.

We say (f_n) converges pointwise to f.

Ex: Let $f_n(x) = \frac{1}{n} e^x$. Then $\lim f_n = 0$ on $[0,1]$.

Q: What about e^{-nx} ? [it converges to zero on any subset of $\mathbb{R}_{>0}$]

Q: What about $f_n = x^n$ on $[0,1]$? What is $\lim_{n\to\infty} x^n$?

What about on $(0,1)$?

So if $x^n \to 0$ on $(0,1)$, [and it does], it kind of does so "slowly". Which is to say, no matter how big n gets, we can find an $x \in (0,1)$ so that $x^n > \frac{1}{2}$ (e.g.).

We might even say the convergence is not "uniform".

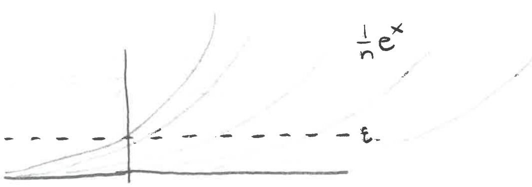
Def: The sequence (f_n) <u>converges to f uniformly on $[a,b]$</u> if

for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall x \in [a,b]$, $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq N$.

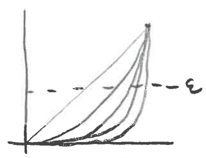
<u>Q</u>: How is this different from pointwise convergence? How would we write pointwise convergence in "math"?

If (f_n) converges uniformly to f, we write $f_n \rightrightarrows f$ or $\underset{n\to\infty}{\text{unif}\lim} f_n = f$.

<u>Intuitively</u>: the entire graph of f_n is eventually trapped inside a "tube" or "sausage" around f, for arbitrarily small tubes.



<u>not</u> uniform



not uniform

<u>Q</u>: Why do we need uniform convergence to be defined?

<u>A</u>: $f_n(x) = x^n$ is continuous for all n.

But $\lim f_n = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$ is <u>not</u> continuous.

pointwise limits of functions can change "type".

uniform limits cannot.

Thm: Suppose for each n that $f_n : [a,b] \to \mathbb{R}$ is continuous at $x_0 \in [a,b]$.
If $f_n \rightrightarrows f$, then f is also continuous at x_0.

Proof: Let $\varepsilon > 0$. By uniform convergence, there is some $N \in \mathbb{N}$ s.t.
whenever $n \geq N$, for any x, $|f_n(x) - f(x)| < \varepsilon/3$.

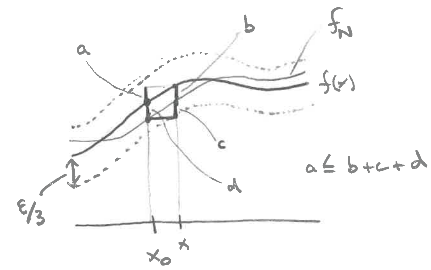
Since f_N is continuous at x_0, there is some $\delta > 0$ s.t.
whenever $|x - x_0| < \delta$, $|f_N(x) - f_N(x_0)| < \varepsilon/3$.

Then whenever $|x - x_0| < \delta$, we have:

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

$\therefore f$ is continuous at x_0. $\square$

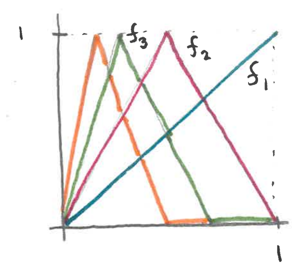


We have: the uniform limit of continuous functions is continuous.

Q: If $f_n \to f$ (pointwise), all f_n and f are continuous, must the convergence be uniform?

A: No.

Ex: Let $f_n(x) = \begin{cases} nx, & 0 \leq x \leq \frac{1}{n} \\ 2 - nx, & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \frac{2}{n} \leq x \leq 1 \end{cases}$



Prop: $f_n \to 0$ but $f_n \not\rightrightarrows 0$, even though all functions are uniformly continuous.

So, back to the question that opened this section: how should we think about functions and sequences of functions? The answer is: "uniformly."

Def: $C_b([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is bounded} \}$.

$C([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous} \}$.

We're going to think of these sets as metric spaces, so we can use all the machinery we've already developed. But will define them first as normed spaces.

Def: For $f \in C_b([a,b])$ or $C([a,b])$, Let $\|f\|_\infty = \sup\limits_{[a,b]} f(x)$.

$\|\cdot\|_\infty$ is the sup norm or the infinity norm, meaning:

- $\|f\|_\infty \geq 0 \quad \forall f$ and $\|f\|_\infty = 0$ iff $f = 0$. [f is the zero function]
- $\|cf\|_\infty = |c| \, \|f\|_\infty$
- $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Aside: The reason for the ∞ is: there are **lots of choices** that Pugh ignores.

$$\|f\|_2 = \left(\int_a^b (f(x))^2 \, dx \right)^{\frac{1}{2}} \qquad \|f\|_p = \left(\int_a^b (f(x))^p \, dx \right)^{\frac{1}{p}}$$

$$\|f\|_1 = \int_a^b f(x) \, dx$$

And the sup norm is basically $\lim\limits_{p \to \infty} \|\ \|_p$.

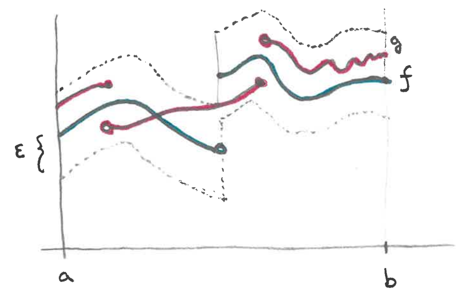
Def: For $f, g \in C_b([a,b])$ or $C([a,b])$, define $d(f,g) = \|f-g\|_\infty$.

Prop: d is a metric.

Proof: LTS. $\square$

Q: What do open balls look like in these metric spaces?

$g \in B(f,\varepsilon)$ and vice-versa.



Thm: If $(f_n) \subseteq C([a,b])$ or $C_b([a,b])$, and (f_n) converges to $f \in C([a,b])$, (i.e., with respect to d), then $f_n \rightrightarrows f$.

Proof: LTS. $\square$

Thm: $C_b([a,b])$ and $C([a,b])$ are complete.

Proof: First, C_b. Let $(f_n) \subseteq C_b([0,1])$ be Cauchy.

For any $x_0 \in [a,b]$, $|f_n(x_0) - f_m(x_0)| \leq \|f_n - f_m\|_\infty = d(f_n, f_m)$.

Thus $(f_n(x_0))$ is a Cauchy sequence of real numbers.

Since $\mathbb{R}$ is complete, $\lim f_n(x_0)$ exists for each $x_0 \in [a,b]$.

Define $f(x) = \lim f_n(x)$ for each x.

We will show that $f_n \rightrightarrows f$. Let $\varepsilon > 0$.

There is some $N \in \mathbb{N}$ s.t. whenever $m, n \geq N$, $\|f_n - f_m\|_\infty = d(f_n, f_m) < \frac{\varepsilon}{2}$.

But also, for each x there is an $m_x \geq N$ s.t. $|f_{m_x}(x) - f(x)| < \frac{\varepsilon}{2}$.

So for any x, and for any $n \geq N$,

$$|f_n(x) - f(x)| \leq |f_n(x) - f_{m_x}(x)| + |f_{m_x}(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore f_n \rightrightarrows f$. Now we must show $f \in C_b([a,b])$.

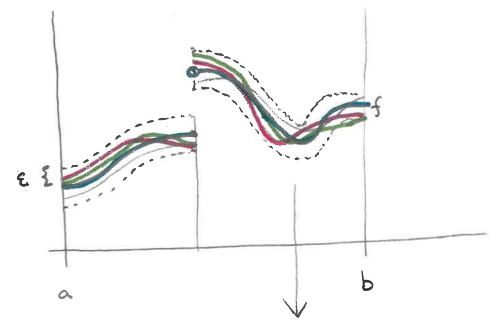
But f_N is bounded, and $\|f_N - f\| < \varepsilon$,

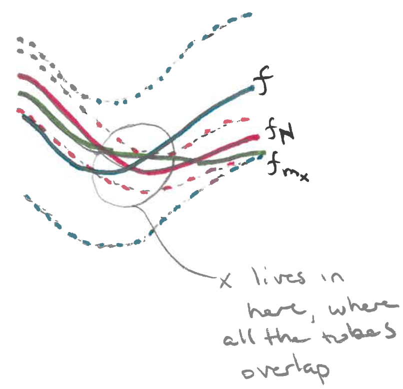
i.e. $|f_N(x) - f(x)| < \varepsilon \;\forall x$, so

f is bounded.

$\therefore (f_n)$ converges in $C_b([a,b])$,

so the space is complete. $\square$





Next, note that $C([a,b]) \subseteq C_b([a,b])$.

For a sequence $(f_n) \subseteq C([a,b])$ that converges in $C_b([a,b])$, say to f, we know that convergence must be uniform.

We have shown that the uniform limit of continuous functions is continuous.

$\therefore f \in C([a,b])$, which means $C([a,b])$ is closed.

It is a closed subset of a complete metric space, so it is complete. $\square$

Now let's take a bit of a rapid-fire tour of some consequences of uniform convergence.

Thm: If f_n is Riemann Integrable for each n, and $f_n \rightrightarrows f$, then f is Riemann Integrable and $\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$.

Cor: $\mathcal{R}$ is a closed subspace of $C_b([a,b])$.

Thm: If $\sum_{n=0}^{\infty} f_n$ converges uniformly [i.e., its partial sums do], then:

$$\int_a^b \sum_{n=0}^{\infty} f_n(x)\,dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)\,dx.$$

Thm: If f_n is differentiable for each n, $f_n \rightrightarrows f$, and $f_n' \rightrightarrows g$, Then f is differentiable and $f' = g$.

Cor: $\left(\sum_{n=0}^{\infty} f_n(x)\right)' = \sum_{n=0}^{\infty} f_n'(x)$ when the RHS converges uniformly.

Q: Why do we require the uniform convergence of f'?

A: Let $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$. Then $f_n \rightrightarrows |x|$.