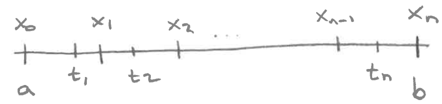
3.2 Riemann Integration

Def: Fix some closed interval $[a,b] \subseteq \mathbb{R}$. A <u>partition pair</u> for $[a,b]$ is given by (P,T), where $P = \{x_0, \dots, x_n\}$, $T = \{t_1, \dots, t_n\}$, and:

$$a = x_0 \leq t_1 \leq x_1 \leq t_2 \leq \dots \leq t_n \leq x_n = b.$$



Given $f : [a,b] \to \mathbb{R}$, the <u>Riemann Sum of f wrt P,T</u> is:

$$R(f,P,T) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(t_i)\Delta x_i.$$

The <u>mesh</u> of P is: $\operatorname{mesh} P = \max_{i=1,\dots,n} \{\Delta x_i\}$

We would like to define the integral of f on $[a,b]$ as the limit of Riemann Sums as mesh P goes to zero — but this is not well defined. R is not a function of mesh P! Here's how we <u>really</u> do it:

Def: $I \in \mathbb{R}$ is the <u>Riemann Integral of f on $[a,b]$</u> if:

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall P,T$, $\operatorname{mesh} P < \delta \implies |R - I| < \varepsilon$.

[This should feel a lot like continuity!]

If such an I exists, we write: $\boxed{I = \int_a^b f(x)\,dx.}$

If I exists, then f is <u>Riemann Integrable</u>

<u>Q</u>: Is I unique?

Def: $\mathcal{R} = \{f : [a,b] \to \mathbb{R} \mid f \text{ is integrable}\}.$

Thm: If f is in $\mathcal{R}$ then it is bounded.

Proof: Suppose for contradiction that $f \in \mathcal{R}$ is unbounded. Let $I = \int_a^b f(x)\,dx$.

$\exists\, \delta$ s.t. $\forall\, (P,T)$ with mesh $P < \delta$, $|R(f,P,T) - I| < 1$; choose some (P,T) with mesh $P < \delta$.

Since f is unbounded on $[a,b]$, it is unbounded on at least one of the subintervals $[x_{i_0-1}, x_{i_0}]$. Choose a $t'_{i_0} \in [x_{i_0-1}, x_{i_0}]$ such that $|f(t'_{i_0}) - f(t_{i_0})|\,(x_{i_0} - x_{i_0-1}) > 2$.

Define $T' = \{t_1, t_2, \dots, t'_{i_0}, \dots, t_n\}$, and let $R' = R(f,P,T')$.

Then
$$|R - R'| = \Big|\big(f(t_0) - f(t_0)\big)(x_1 - x_0) + \dots + \big(f(t_{i_0}) - f(t'_{i_0})\big)(x_{i_0} - x_{i_0-1}) + \dots + \big(f(t_n) - f(t_n)\big)(x_n - x_{n-1})\Big|$$
$$= \Big|\big(f(t'_{i_0}) - f(t_{i_0})\big)(x_{i_0} - x_{i_0-1})\Big| > 2$$

But also, $|R - R'| \le |R - I| + |I - R'| < 2$ $\lightning$ $\square$

Thm: $\mathcal{R}$ is a vector space over $\mathbb{R}$ and $f \longmapsto \int_a^b f(x)\,dx$ is linear.

The function $h(x) \equiv k$ is in $\mathcal{R}$ and $\int_a^b k\,dx = k(b-a)$.

For $f, g \in \mathcal{R}$, if $f(x) \le g(x)$ $\forall x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Proof: Riemann Sums. $\square$

The definition given above is fine, but practically speaking it is not always what we're looking for. We may prefer:

Def: Given a function $f:[a,b] \to \mathbb{R}$ and a partition P of $[a,b]$, the upper sum of f is: $U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$, where $M_i = \sup\{f(t) \mid x_{i-1} \le t \le x_i\}$. The lower sum is: $L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i$, where $m_i = \inf\{f(t) \mid x_{i-1} \le t \le x_i\}$.

Q: Why sup & inf rather than min & max?

Prop: $L(f,P) \le R(f,P,T) \le U(f,P)$ for any T.

Def: The upper integral of f on $[a,b]$ is: $\boxed{\overline{I} = \inf_P U(f,P)}$

The lower integral is $\boxed{\underline{I} = \sup_P L(f,P)}$

If $\underline{I} = \overline{I}$, then f is Darboux Integrable

Thm: f is Darboux Integrable iff it is Riemann Integrable; in this case $\underline{I} = I = \overline{I}$.

Proof: Looooong. ☹

Thm: $f \in \mathcal{R}$ iff it is bounded and $\forall \varepsilon > 0\ \exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Ex: Continuous functions are Riemann Integrable.

Proof: LTS. □

The remainder of this section is dedicated to coming up with practical ways to classify the set of Riemann Integrable functions. Because as it turns out, not every function is integrable...

Def: Given a set $S \subseteq \mathbb{R}$, the characteristic or indicator function of S is:

$$\chi_S(x) = \mathbb{1}_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}.$$

Ex: $\mathbb{1}_{[a,b]} \in \mathcal{R}$, $\mathbb{1}_{(a,b)} \in \mathcal{R}$, $\mathbb{1}_{\{0\}} \in \mathcal{R}$. [This last one is maybe surprising!]

$\mathbb{1}_{\mathbb{Q}} \notin \mathcal{R}$ is our first example of a non-integrable function.

Why? Because $L(\mathbb{1}_{\mathbb{Q}}, P) = 0 \ \forall P$ and $U(\mathbb{1}_{\mathbb{Q}}, P) = 1 \ \forall P$.

Prop: $\mathbb{1}_{\mathbb{Q}}$ is discontinuous at every real number.

Proof: LTS. □

This gives us some evidence that continuity is closely related to integrability — but also that there exist integrable functions which are not everywhere continuous.

The eventual conclusion of this mystery will be intimately tied up in the definition of the Riemann Sum.

Thm: If, for every $\varepsilon > 0$, there are g & h in $\mathcal{R}$ with $g \leq f \leq h$, and $\int_a^b (h(x) - g(x))\,dx \leq \varepsilon$, then $f \in \mathcal{R}$.

Proof: For a fixed P, we must have:

$$\boxed{L(g,P) \leq \boxed{L(f,P) \leq U(f,P)} \leq U(h,P).}$$

Let $\varepsilon > 0$. Then there is some $\delta > 0$ s.t. whenever mesh $P < \delta$,

$$\int_a^b g(x)\,dx - L(g,P) < \varepsilon/3, \qquad U(h,P) - \int_a^b h(x)\,dx < \varepsilon/3.$$

Then $$\int_a^b g(x)\,dx - \varepsilon/3 < L(g,P) \leq U(h,P) < \int_a^b h(x)\,dx + \varepsilon/3.$$

By hypothes we have $\int_a^b (h(x) - g(x))\,dx \leq \varepsilon/3$, so that:

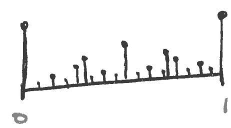
$$\begin{aligned} U(f,P) - L(f,P) &< \int_a^b h(x)\,dx + \varepsilon/3 - L(f,P) \\ &= \int_a^b (h(x) - g(x))\,dx + \int_a^b g(x)\,dx - L(f,P) + \varepsilon/3 \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

$\therefore$ f is Darboux Integrable, and thus Riemann Integrable. $\square$

The preceding theorem helps find some other interesting examples:

Ex: "Rational Ruler" function: $f : [0,1] \to \mathbb{Q}$

$$f(x) = \begin{cases} 1/q, & x = \frac{p}{q} \text{ in lowest terms} \\ 0, & x \notin \mathbb{Q} \end{cases}$$



[by convention, $0 = \frac{0}{1}$]

Prop: f is continuous at $a \in [0,1]$ iff $a \notin \mathbb{Q}$.

Proof: LTS. $\square$

(Hint: count the number of points in $[0,1]$ with $f(x) = \frac{1}{q}$ for a given q)

Prop: f is Riemann Integrable

Proof: Let $g(x) \equiv 0$ and $h(x) = \varepsilon \mathbb{1}_{[0,1]}(x) + s(x)$, where:

$$s(x) = \begin{cases} 1/q, & x = p/q \text{ \& } \frac{1}{q} \geq \varepsilon \\ 0, & \text{o.w.} \end{cases}$$

(finitely many discontinuities — see hint above!)

Then $g \leq f \leq s$, and $g, s \in \mathcal{R}$.

$\therefore f \in \mathcal{R}$. $\square$

The mystery continues to clear — this function with countably many discontinuities is Riemann Integrable. Is this the final word?

Zeno's staircase seems to confirm... But,

Prop: There exists a function which is Riemann Integrable that is discontinuous at uncountably many points.

Proof: Must be postponed while we build some machinery.

So the thing that can 'ruin' integrability is not the number of discontinuities. The counterexample mentioned above happens to be closed, and to have no open subset, as well. What we really need is a new idea:

Def: Let $S \subseteq \mathbb{R}$. The measure of S is:

$$m(S) = \inf\left\{ \sum \text{width}(I_\alpha) \mid \{I_\alpha\} \text{ is a cover of } S \text{ by closed intervals} \right\}$$

Ex: $m([a,b]) = b-a$

$m(\{0\}) = 0$

$m(\mathbb{R}) = \infty$

$m(\mathbb{Q}) = \dots \quad 0.$

Thm: If $S \subseteq \mathbb{R}$ is countable, then $m(S)=0$.

Proof: Write $S = \{a_1, a_2, \dots\}$. Given $\varepsilon > 0$, we will find a cover of S by closed intervals whose total width is less than ε.

Let $I_1 = \left[a_1 - \frac{\varepsilon}{2^2}, a_1 + \frac{\varepsilon}{2^2}\right]$, $I_2 = \left[a_2 - \frac{\varepsilon}{2^3}, a_2 + \frac{\varepsilon}{2^3}\right], \dots$

$\dots \quad I_k = \left[a_k - \frac{\varepsilon}{2^{k+1}}, a_k + \frac{\varepsilon}{2^{k+1}}\right], \quad \dots$

Then $a_k \in I_k \ \forall k$, so $\{I_k\}$ covers S. And:

$\sum \text{width}(I_k) = \sum \frac{\varepsilon}{2^{k+1}} = \varepsilon \sum \frac{1}{2^{k+1}} < \varepsilon$, so $m(S) < \varepsilon \quad \forall \varepsilon > 0$

$\therefore m(S) = 0. \ \square$

When a property holds everywhere except on a set of measure zero, that property holds "almost everywhere". [In stats, they say "Almost Surely"].

Thm: A function $f:[a,b] \to \mathbb{R}$ is Riemann Integrable iff the set of points where f is discontinuous has measure zero. In other words, f is Riemann integrable iff it is continuous a.e. on $[a,b]$.

Lemma: Let $osc_x = \limsup_{t \to x} f(t) - \liminf_{t \to x} f(t) = \lim_{r \to 0} \{ diam (f([x-r, x+r])) \}$

Then f is continuous at x iff $osc_x f = 0$.

Proof: LTS. □

Proof of Theorem: Let $D = \{ x \in [a,b] \mid f \text{ is discontinuous at } x \}$, and for each n define $D_n = \{ x \in [a,b] \mid osc_x f \geq 1/n \}$. Then $D = \bigcup D_n$.

Claim: If $m(D_n) = 0$ for each n, then $m(D) = 0$.
Proof: Let $\varepsilon > 0$. For each n, there is a cover of D_n by closed intervals with total width no more than $\frac{\varepsilon}{2^{n+1}}$.
The total width of the union of those covers is then a cover of D with total width less than ε. □

($\Rightarrow$) Suppose f is Riemann Integrable. Let $\varepsilon > 0$ and fix $k \in \mathbb{N}$. Then there is some partition P of $[a,b]$ with:

$$U(f,P) - L(f,P) = \sum (M_i - m_i) \Delta x_i < \varepsilon/k.$$

For each i, let $I_i = [x_{i-1}, x_i]$. If there is some $d \in D_k \cap I_i$, we will call I_i "bad". Then:

$$\frac{1}{k} \sum_{\text{bad } i} \Delta x_i \leq \sum_{\text{bad } i} (M_i - m_i) \Delta x_i \leq \sum_{\text{all } i} (M_i - m_i) \Delta x_i = U(f,P) - L(f,P) < \varepsilon/k$$

because $osc_d f \geq \frac{1}{k} \Rightarrow M_i - m_i \geq \frac{1}{k}$

$\Rightarrow \sum_{\text{bad } i} \Delta x_i < \varepsilon$. Since $\{\Delta x_i\}_{\text{bad } i}$ is a cover of D_k by closed intervals, $m(D_k) < \varepsilon \quad \forall \varepsilon > 0$.

$\therefore m(D_k) = 0 \ \forall k \implies m(D) = 0.$ □

($\Leftarrow$) Assume $m(D)=0$, and let $\varepsilon>0$. We will find a partition P such that $U(f,P)-L(f,P)<\varepsilon$.

Choose $k\in\mathbb{N}$ s.t. $\frac{1}{k}<\frac{\varepsilon}{2(b-a)}$. There is a countable covering of D_k by open intervals with total width no more than $\frac{\varepsilon}{4M}$, where $M=\sup_{[a,b]} f$. Call it $\mathcal{J}$.

Every interval $J_i\in\mathcal{J}$ is "bad", so $\sup_{t\in J_i} f(t)-\inf_{t\in J_i} f(t)\geq\frac{1}{k}$.

For each x not in D_k, there is an open interval I_x such that:

$\sup_{t\in I_x} f(t)-\inf_{t\in I_x} f(t)<\frac{1}{k}$. The collection $\mathcal{I}=\{I_x \mid x\notin D_k\}$

is a cover of $[a,b]\setminus D_k$.

Let $\mathcal{V}=\mathcal{I}\cup\mathcal{J}$. Then $\mathcal{V}$ is an open cover of $[a,b]$, which is compact, so it has a positive Lebesgue number $\lambda>0$.

Let $P=\{x_0,\dots,x_n\}$ be any partition with mesh $P<\lambda$.

For any interval $A_i=[x_{i-1},x_i]$, either $A_i\subseteq I_x$ for some $I_x\in\mathcal{I}$ or $A_i\subseteq J_j$ for some $J_j\in\mathcal{J}$.

Let $\mathbb{J}=\{i \mid A_i\subseteq J_j \text{ for some } J_j\in\mathcal{J}\}$, and consider $\bigcup_{i\in\mathbb{J}} A_i$.

There must be some finite $J_1\cup\dots\cup J_m\supseteq\bigcup_{i\in\mathbb{J}} A_i$, since $\mathbb{J}$ is finite, since P is finite.

Now:

$$U(f,P)-L(f,P)=\sum_{i=1}^{n}(M_i-m_i)\Delta x_i=\left(\sum_{i\in\mathbb{J}}+\sum_{i\notin\mathbb{J}}\right)(M_i-m_i)\Delta x_i$$

$$\leq\sum_{i\in\mathbb{J}}2M\Delta x_i+\sum_{i\notin\mathbb{J}}\frac{1}{k}\Delta x_i\leq 2M\sum_{j=1}^{m}(b_j-a_j)+\frac{(b-a)}{k}$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

$\therefore$ f is Riemann Integrable. $\square$