$\boxed{4.3}$ Compactness & Equicontinuity

Consider the zero function in $C([0,1])$.

Let $B = \overline{B(0,1)} \subseteq C([0,1])$. Then B is closed & bounded, by definition. But!

Thm: B is not compact.

Proof: Let $f_n = x^n$. Then the uniform limit of (f_n) is not continuous; (f_n) does not converge in $C([0,1])$ much less B. □

This is a sucky, anticlimactic proof. We can do better!

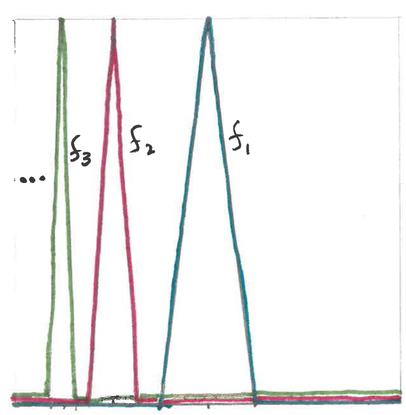
Thm: Let $B = \overline{B(0,1)} \subseteq C_b([0,1])$. Then B is not compact.

Proof: We will construct an open cover having no finite subcover.

Let $\mathcal{U} = \{B(g, \frac{1}{2}) \mid g \in B\}$. $\mathcal{U}$ is clearly an open cover of B.

For each $n \in \mathbb{N}$, let $I_n = \left[\frac{1}{2^n} - \frac{1}{2^{n+2}}, \frac{1}{2^n} + \frac{1}{2^{n+2}}\right]$, and define:

$$f_n = \begin{cases} -2^{n+2}\left|x - \frac{1}{2^n}\right| + 1, & x \in I_n \\ 0, & \text{o.w.} \end{cases}$$



Note that each f_n is in B, since $d(f_n, 0) = \|f_n\|_\infty = 1$.

Note also that if $m \neq n$, $I_m \cap I_n = \emptyset$, and so $\|f_m - f_n\|_\infty = d(f_m, f_n) = 1$.

But for any $f, g \in U \in \mathcal{U}$, $d(f,g) < 1$. $\therefore$ Infinitely many of the open sets in $\mathcal{U}$ are required to cover B. □

If closed balls aren't compact, a natural question is: which sets are? Leaving aside the obvious answer — closed & totally bounded — we can find a very natural condition on sequences of functions that guarantees compactness.

[The concept is defined for general families of functions; Pugh considers only sequences]

Def: Let (f_n) be a sequence in $C([a,b])$.

The sequence is <u>equicontinuous</u> if:

For every $\varepsilon > 0$ there is a $\delta > 0$ s.t. $|s-t| < \delta \Longrightarrow |f_n(s) - f_n(t)| < \varepsilon \ \forall n$.

This is a strong condition! It says all the functions are continuous, and the same ε-δ pair will work for any of them.

Ex: Any finite set of continuous functions on $[a,b]$ is equicontinuous.

Proof: LTS. □

Thm: (Arzelà-Ascoli): If (f_n) is bounded & equicontinuous in $C([a,b])$, then (f_n) has a uniformly convergent subsequence.

Proof: Let $D = \mathbb{Q} \cap [a,b]$. Then D is countable and dense in $[a,b]$. Write $D = \{d_1, d_2, \ldots\}$, and consider the sequence $(f_n(d_1))$. The sequence is bounded (since (f_n) is), which means it has a convergent subsequence $(f_{1,k}(d_1))$. Let $y_1 = \lim_{k\to\infty} f_{1,k}(d_1)$.

Now consider the sequence $(f_{1,k}(d_2))$. It is bounded, so it has a convergent subsequence $f_{2,k}(d_2) \longrightarrow y_2$.

Note that $(f_{1,k})$ is a subsequence of (f_n), and $(f_{2,k})$ is a subsequence of $(f_{1,k})$, and $f_{2,k}(d_1) \longrightarrow y_1$, $f_{2,k}(d_2) \longrightarrow y_2$.

Continuing in this way for every n we can find a sequence $(f_{n,k})$ s.t.:

- $(f_{n,k}) \subseteq (f_{n-1,k}) \subseteq \cdots \subseteq (f_{1,k}) \subseteq (f_n)$
- $\lim_{k\to\infty} f_{n,k}(d_i) = y_i \quad \forall\, i \leq n$

Let $(g_m) = (f_{m,m})$. Then $(g_m) \subseteq (f_n)$. It is our desired uniformly convergent subsequence.

Note that $g_m = f_{m,m} = f_{m-1,r}$ for some $r > m$, and this can be repeated inductively.

Given any $d_j \in D$, $\forall\ m > j$ we can thus write:

$$g_m(d_j) = f_{m,m}(d_j) = f_{m-1,r_1}(d_j) = f_{m-2,r_2}(d_j) = \cdots = f_{j,r}(d_j)$$

for some $r > m$. Thus:

$$\lim_{m\to\infty} g_m(d_j) = \lim_{r\to\infty} f_{j,r}(d_j) = y_j. \qquad [\,r \text{ is "accelerated" by } m\,]$$

So $g_m(d_j) \to y_j$ for every j. The sequence (g_m) converges pointwise on D. We must show it converges uniformly on $[a,b]$.

Given any $\varepsilon > 0$, there is some δ s.t. $|s-t| < \delta \Rightarrow |g_m(s) - g_m(t)| < \frac{\varepsilon}{3}\ \forall m$.

The cover $\{B(d_j,\delta) \mid d_j \in D\}$ of $[a,b]$ has a finite subcover. Thus there is some $J \in \mathbb{N}$ s.t. $\{B(d_j,\delta) \mid j \leq J\}$ covers $[a,b]$.

Each <u>real</u> sequence $(g_m(d_j))$ converges to d_j and is Cauchy, so there is some $N \in \mathbb{N}$ s.t. $\forall\, \ell, m \geq N$ and $\forall j \leq J$,

$$|g_m(d_j) - g_\ell(d_j)| < \varepsilon/3.$$

Now given any $x \in [a,b]$, choose $j \le J$ s.t. $d(x, d_j) < \delta$. Then:

$$\ell, m \geq N \implies |g_m(x) - g_\ell(x)| \leq |g_m(x) - g_m(d_j)| + |g_m(d_j) - g_\ell(d_j)| + \cdots + |g_\ell(d_j) - g_\ell(x)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\therefore$ (g_m) is Cauchy with respect to the sup norm.

Since $C([a,b])$ is complete, (g_m) is convergent in $C([a,b])$.

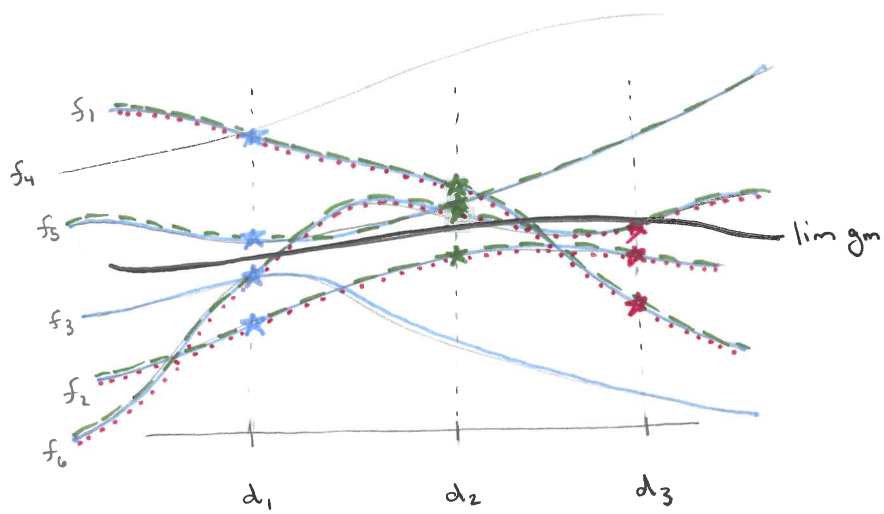
We have demonstrated that (f_n) has a convergent subsequence. $\square$

$(f_{1,k}) = (f_1, f_2, f_3, f_5, f_6, \ldots)$

$(f_{2,k}) = (f_1, f_2, f_5, f_6, \ldots)$

$(f_{3,k}) = (f_1, f_2, f_6, \ldots)$

$(g_m) = (f_{m,m}) = (f_1, f_2, f_6, \ldots)$



Outline of the preceding proof:

- Construct subsequences that each converge for points in a dense subset
- Construct "diagonal subsequence" that converges pointwise for every point in the dense subset
- Use compactness of $[a,b]$ to extend the convergence to the whole interval by showing the diagonal sequence is Cauchy.

Cor: A subset of $C([a,b])$ is compact iff it is closed, bounded, and equicontinuous.

Proof: Let $\mathcal{E} \subseteq C([a,b])$ be closed, bounded, and equicontinuous.

Let (f_n) be any sequence in $\mathcal{E}$. Then Arzelà-Ascoli gives a convergent subsequence.

$\therefore$ $\mathcal{E}$ is compact.

Now suppose $\mathcal{E}$ is compact. Then it is closed and totally bounded.

So the open cover $\{B(f, \varepsilon/3) \mid f \in \mathcal{E}\}$ has a finite subcover for any $\varepsilon > 0$;

call the subcover $\{B(f_1, \varepsilon/3), \ldots, B(f_n, \varepsilon/3)\}$.

Since each f_k is uniformly continuous, there is a $\delta > 0$ s.t.

whenever $|s-t| < \delta$, $|f_k(s) - f_k(t)| < \varepsilon/3$ for each $k = 1, \ldots, n$.

So for any $f \in \mathcal{E}$, we have $\|f - f_k\| < \varepsilon/3$ for some k, and:

$$|f(s) - f(t)| \leq |f(s) - f_k(s)| + |f_k(s) - f_k(t)| + |f_k(t) - f(t)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

which shows that $\mathcal{E}$ is equicontinuous. $\square$