$\boxed{2.7}$ Coverings

Now we'll get to say 'compact' instead of 'sequentially compact.'

Let M be a metric space, and let $\mathcal{U}$ be a collection of open subsets in M. Suppose $A \subseteq M$.

If $\bigcup_{U \in \mathcal{U}} U \supseteq A$, then $\mathcal{U}$ is an <u>open cover</u> of A. [We can write $\bigcup \mathcal{U}$]

<u>Ex:</u> $\mathcal{U} = \{(-n, n) \mid n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$

$\mathcal{U} = \{(\frac{1}{n}, 1 - \frac{1}{n}) \mid n \in \mathbb{N}\}$ is an open cover of $(0,1)$

(but <u>not</u> of $[0,1]$)

$\mathcal{U} = \{\mathbb{R}\}$ is an open cover of $\mathbb{R}$

Suppose $\mathcal{V} \subseteq \mathcal{U}$, i.e. for all $V \in \mathcal{V}$, $V \in \mathcal{U}$. If $\mathcal{V}$ is an open cover of A, then it is a <u>subcover</u> of $\mathcal{U}$, or we can say <u>that $\mathcal{U}$ has a subcover $\mathcal{V}$</u>.

<u>Def:</u> Let M be a metric space. The set $A \subseteq M$ is <u>compact</u> if every open cover of A has a finite subcover [covering compact].

To get an intuitive feel for this definition, it helps to start with some examples of sets which are <u>not</u> compact.

Ex: • $\mathbb{R}$ is not compact, since the open cover $\{(-n,n)\}_{n\in\mathbb{N}}$ has no finite subcover.

• $(0,1)$ is not compact, since $\{(\frac{1}{n}, 1-\frac{1}{n})\}$ has no finite subcover.

• $\mathbb{N}_{disc.}$ is not compact, since $\{\{n\}\}_{n\in\mathbb{N}}$ has no finite subcover.

But why is $[0,1]$ compact? It's not immediately obvious; the following will help:

Thm: $\boxed{\text{In a metric space } M}$, $A\subseteq M$ is (covering) compact iff it is sequentially compact.

Proof: ($\Rightarrow$) Suppose A is covering compact but not sequentially compact. Then there is some sequence $(p_n)\subseteq A$ that has no convergent subsequences. Then for every $a\in A$, a is not the limit of any subsequence of (p_n). So we can find an $\varepsilon_a > 0$ such that $\{n\in\mathbb{N} \mid p_n \in B(a,\varepsilon_a)\}$ is finite.

Let $\mathcal{U} = \{B(a,\varepsilon_a) \mid a\in A\}$. Then $\mathcal{U}$ is an open cover of A, so it has a finite subcover $\mathcal{V} = \{B(a_1,\varepsilon_1), \dots, B(a_k,\varepsilon_k)\}$.

For each $i=1,\dots,k$, we have $\{n\in\mathbb{N} \mid p_n\in B(a_i,\varepsilon_i)\}$ is finite.

Then $\{n\in\mathbb{N} \mid p_n \in \bigcup_{i=1}^{k} B(a_i,\varepsilon_i)\}$ is finite. But $\bigcup_{i=1}^{k} B(a_i,\varepsilon_i) \supseteq A \supseteq (p_n)$, so this is a contradiction. The sequence must have a convergent subsequence, i.e. A must be sequentially compact.

The other direction will be the most involved proof we've done so far.

Lemma: (Lebesgue Numbers): Suppose M is sequentially compact, and let $\mathcal{U}$ be an open cover of M. Then there is some $\lambda > 0$ such that: $\forall x \in M$, $\exists U \in \mathcal{U}$ with $B(x, \lambda) \subseteq U$.

Proof: Suppose otherwise, that is, suppose $\mathcal{U}$ is an open cover without a positive Lebesgue number.

Then for each $n \in \mathbb{N}$ we can find an $a_n \in M$ such that $B(a_n, \frac{1}{n})$ is not contained in any $U \in \mathcal{U}$.

Since M is sequentially compact, the sequence (a_n) has a convergent subsequence, (a_{n_k}), with $a_{n_k} \to p \in M$

Since $\mathcal{U}$ is a cover, there is some $U \in \mathcal{U}$ with $p \in U$.

Since $\mathcal{U}$ is an open cover, U contains an open ball $B(p, \varepsilon)$ for some $\varepsilon > 0$.

There is some $K_1 \in \mathbb{N}$ s.t. $k \geq K_1 \Rightarrow d(a_{n_k}, p) < \varepsilon/2$. There is also some $K_2 \in \mathbb{N}$ s.t. $k \geq K_2 \Rightarrow \frac{1}{n_k} < \varepsilon/2$. Thus if $k \geq \max\{K_1, K_2\}$, $B(a_{n_k}, \frac{1}{n_k}) \subseteq B(p, \varepsilon) \subseteq U$.

But this is a contradiction to our assumption; such an open cover cannot exist. □

($\Leftarrow$): Suppose A is sequentially compact, and let $\mathcal{U}$ be any open cover of A. Let λ be a Lebesgue number for $\mathcal{U}$.

Let $a_1 \in A$ and $U_1 \in \mathcal{U}$ s.t. $B(a_1, \lambda) \subseteq U_1$.

If $A \subseteq U_1$ we are done; if not we choose $a_2 \in A - U_1$ and find some $U_2 \in \mathcal{U}$ with $B(a_2, \lambda) \subseteq U_2$.

Now if $A \subseteq (U_1 \cup U_2)$ we are done; if not we can continue in this way with $a_3 \in A \setminus (U_1 \cup U_2)$ and so on.

In fact, if this process ever terminates, then we have proved the result. So suppose it doesn't.

Thus we can construct a sequence (a_n); this sequence must have a convergent subsequence (a_{n_k}) with $a_{n_k} \to p \in A$. There is some $K \in \mathbb{N}$ s.t. $k \geq K \Rightarrow d(a_{n_k}, p) < \lambda$ so that $p \in U_{n_K}$. If $l > K$, we know that $a_{n_l} \in A \setminus \bigcup_{i=1}^{K} U_{n_K}$; in particular, this would mean that $a_{n_l} \notin U_{n_K}$.

But this is a contradiction; if this were true then (a_{n_k}) would not converge to p. We cannot build such a sequence, and so eventually the process must terminate. Thus there is a finite subcover $\{U_1, \dots, U_m\}$ as required. $\square$

<u>Recall:</u> $A \subseteq \mathbb{R}^m$ is compact iff it is closed & bounded.

If $A \subseteq M$ is compact it is closed & bounded.

This leaves the door ajar; we can sneak in something that is closed & bounded but not compact [e.g. $\mathbb{N}_{\text{disc}}$]. How to shut that door?

<u>Def:</u> $A \subseteq M$ is <u>totally bounded</u> if, for every $\varepsilon > 0$, the open cover of A by balls of radius ε has a finite subcover.

<u>Q:</u> Is totally bounded <u>weaker</u> or <u>stronger</u> than being bounded?

<u>A:</u> LTS (presentation)

The following result seems almost obvious; total boundedness is basically "compactness for ε-balls." It's not surprising that it extends to "just plain compactness."

Thm: Let M be a complete metric space. Then $A \subseteq M$ is compact iff it is closed & totally bounded.

Proof: ($\Rightarrow$) If A is compact, then it is closed by a previous result.

Now $\{B(x,\varepsilon) \mid x \in A\}$ is an open cover of A for any $\varepsilon > 0$, and by compactness it has a finite subcover.

$\therefore$ A is totally bounded.

($\Leftarrow$): If A is closed & totally bounded, then for each $\varepsilon_k = \frac{1}{k}$ there is a finite open cover $\mathcal{U}_k$ of balls of radius $\frac{1}{k}$, e.g.:

$$\mathcal{U}_1 = \{B(q_{11}, 1), \dots, B(q_{1m_1}, 1)\} \text{ covers } A \text{ for some } q_1, \dots, q_m.$$

Let $(a_n) \subseteq A$ be any sequence. Then it must visit one of the open balls infinitely often; WOLOG say $B(q_1, 1)$.

Set $A_1 = B(q_{11}, 1)$, and choose $n_1 \in \mathbb{N}$ s.t. $a_{n_1} \in A_1$.

Now, A_1 is itself totally bounded. Thus it has a finite cover: $\{B(q_{21}, \frac{1}{2}), \dots, B(q_{2m_2}, \frac{1}{2})\}$. The sequence (a_n) must visit one of these balls, WOLOG $A_2 = B(q_{21}, \frac{1}{2})$, infinitely many times. Thus we can choose $n_2 > n_1$ s.t. $a_{n_2} \in A_2$.

Continuing in this way, we can construct a subsequence $(a_{n_k}) \subseteq (a_n)$ such that $a_{n_k} \in A_k = A_{k-1} \cap B(p_k, \frac{1}{k})$ for a specific sequence of points (p_k).

Claim: (a_{n_k}) is Cauchy. Proof: Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $\frac{2}{N} < \varepsilon$. Then for $k, \ell \geq N$ we have:

$$d(a_{n_k}, a_{n_\ell}) \leq \operatorname{diam}(A_N) = \frac{2}{N} < \varepsilon$$

Since M is complete, (a_{n_k}) converges in M. Since A is closed, it converges in A. Thus A is sequentially compact. And now we know that this just means compact. $\square$