# Real Variables Test

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1.) Suppose $(x_n)$ is a bounded sequence of real numbers such that $x_k \leq x_{k+1}$ for each $k \in \mathbb{N}$. Prove that $(x_n)$ converges in $\mathbb{R}$

*Proof.* We shall define the nonempty set of all $x_n$'s as,

$$X = \{n \in \mathbb{N} | x_n\}.$$

$(x_n)$ is bounded so $X$ is bounded thus there exists a least upper bound $l$ on $X$. For any $\epsilon > 0$ there exists an element $x_N \in X$ such that $X_N > l - \epsilon$ by the definition of l.u.b.. Then for any $n > N$, $x_n \geq x_N$ because $x_n$ is an increasing sequence. This gives us the inequality,

$$l \geq x_n > l - \epsilon,$$

for all $n > N$. Then $|x_n - l| < \epsilon$ for all $n > N$. This proves that $(x_n)$ converges to $l$.

$\square$

2.) Suppose $(x_n)$ is a sequence of real numbers such that $(x_{2k}), (x_{2k+1})$ and $(x_{3k})$ are all convergent subsequences. Prove that $(x_n)$ converges.

*Proof.* Since $(x_{2k}), (x_{2k+1})$ and $(x_{3k})$ converge, they are all Cauchy sequences. Thus for $(x_k)$ there exists an $N_1 \in \mathbb{N}$ such that for all *even* $n, m > N_1$,

$$|x_n - x_m| < \frac{\epsilon}{3}. \tag{1}$$

Similarly for $(x_{2k+1})$, there exists an $N_2 \in \mathbb{N}$ such that for all *odd* $n, m > N_2$,

$$|x_n - x_m| < \frac{\epsilon}{3}. \tag{2}$$

As well as for $(x_{3k})$, there exists an $N_3 \in \mathbb{N}$ such that for all $n, m > N_3$ that are divisible by 3,

$$|x_n - x_m| < \frac{\epsilon}{3}. \tag{3}$$

Now we take $N = \max(N_1, N_2, N_3)$ for any $j, k > N$, we have 4 different cases for the $j, k$ and we shall prove in each that case the requirement for Cauchy sequence holds. The first two cases are when $j$ and $k$ have the same parity, the next two cases are when $j$ and $k$ have distinct parity.

**Case 1:** Let $j$ be even and $k$ be even, then by Equation 1 we know that

$$|x_k - x_j| < \frac{\epsilon}{3} < \epsilon.$$

Thus when $j$ and $k$ are even $(x_n)$ follows the Cauchy requirements.
**Case 2:** Let $j$ be odd and $k$ be odd, then by Equation 2 we know that

$$|x_k - x_j| < \frac{\epsilon}{3} < \epsilon.$$

Thus when $j$ and $k$ are odd $(x_n)$ follows the Cauchy requirements.
**Case 3:** Let $j$ be odd and $k$ be even. Then $3j$ is odd and $3k$ is even and both $3j$ and $3k$ are divisible by 3. We can rewrite the Cauchy condition as follows,

$$|x_k - x_j| = |x_k - x_{3k} + x_{3k} - x_{3j} + x_{3j} - x_j|.$$

Using the Triangle Inequality we get,

$$|x_k - x_j| \leq |x_k - x_{3k}| + |x_{3k} - x_{3j}| + |x_{3j} - x_j|.$$

Then by Equation 1, $|x_k - x_{3k}| < \frac{\epsilon}{3}$, by Equation 3, $|x_{3k} - x_{3j}| < \frac{\epsilon}{3}$ and by Equation 2, $|x_{3j} - x_j| < \frac{\epsilon}{3}$. Synthesizing this information we get,

$$|x_k - x_j| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus when $j$ is odd and $k$ is even, $(x_n)$ follows the Cauchy requirements.
**Case 4:** Let $j$ be even and $k$ be odd. Then $3j$ is even and $3k$ is odd and both $3j$ and $3k$ are divisible by 3. We can rewrite the Cauchy condition as follows,

$$|x_k - x_j| = |x_k - x_{3k} + x_{3k} - x_{3j} + x_{3j} - x_j|.$$

Using the Triangle Inequality we get,

$$|x_k - x_j| \leq |x_k - x_{3k}| + |x_{3k} - x_{3j}| + |x_{3j} - x_j|.$$

Then by Equation 2, $|x_k - x_{3k}| < \frac{\epsilon}{3}$, by Equation 3, $|x_{3k} - x_{3j}| < \frac{\epsilon}{3}$ and by Equation 1, $|x_{3j} - x_j| < \frac{\epsilon}{3}$. Synthesizing these results returns,

$$|x_k - x_j| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus when $j$ is even and $k$ is odd, $(x_n)$ follows the Cauchy requirements.
Therefore since $|x_j - x_k| < \epsilon$ in each case, $(x_n)$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, every Cauchy sequence converges thus $(x_n)$ also converges.

$\square$

3.) Find an example of a continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence $(x_n) \subseteq (0, 1)$ such that $(f(x_n))$ is not a convergent sequence.

*Proof.* First we will show that the sequence $(x_n) = \frac{1}{n}$, starting at $n = 2$, is a Cauchy sequence. For any $\epsilon > 0$, let $N = \frac{2}{\epsilon}$ then for any $n, m > N$, we have

$$|x_n - x_m| = |\frac{1}{n} - \frac{1}{m}| \leq |\frac{1}{n}| + |\frac{1}{m}| = \frac{1}{n} + \frac{1}{m} < \frac{2}{N} = \epsilon.$$

Thus $x_n$ is a Cauchy sequence. However in $(0, 1)$ $x_n$ does not converge, since $0 \notin (0, 1)$. Next we define a function $f : (0, 1) \to \mathbb{R}$ as $f(x) = 1/x$. It is clear that $f$ is continuous. Then the sequence

$$f(x_n) = \frac{1}{x_n} = n,$$

does not converge in $\mathbb{R}$. It diverges to infinity. Therefore the continuous function $f$ and Cauchy sequence $x_n$, starting at $n = 2$, is such an example. $\square$

4.) A rational point in $\mathbb{R}^2$ is an ordered pair $(p, q)$ with both $p$ and $q$ in $\mathbb{Q}$. Prove that there exists a circle in the plane $\mathbb{R}^2$ that contains no rational points.

*Proof.* A circle has the equation $(x-a)^2 + (y-b)^2 = r^2$. Where $r$ is the radius of the circle, $(a, b)$ is the center of the circle and $x$ and $y$ are all the points that lie on the circle. We shall set the center of the circle to the origin or $(0, 0)$, this gives us the formula,

$$x^2 + y^2 = r^2.$$

Now let $r$ be an irrational number whose square is also irrational $(\pi, \sqrt[3]{2}, \cdots)$. Then every point on the circle is not a rational point. We can prove this by contradiction, assume there exists a rational point $(x_1, y_1)$ with $x_1, y_1 \in \mathbb{Q}$ on the circle. Thus $x_1 = \frac{p_1}{q_1}$ and $y_1 = \frac{p_2}{q_2}$ for some $p_1, p_2, q_1, q_2 \in \mathbb{Z}$ with $q_1, q_2 \neq 0$. Our circle equation becomes

$$\left(\frac{p_1}{q_1}\right)^2 + \left(\frac{p_2}{q_2}\right)^2 = r^2,$$

$$\frac{p_1^2}{q_1^2} + \frac{p_1^2}{q_1^2} = r^2,$$

$$\frac{p_1^2 q_2^2}{q_1^2 q_2^2} + \frac{p_2^2 q_1^2}{q_1^2 q_2^2} = r^2,$$

$$\frac{p_1^2 q_2^2 + p_2^2 q_1^2}{q_1^2 q_2^2} = r^2.$$

Since $p_1^2 q_2^2 + p_2^2 q_1^2, q_1^2 q_2^2 \in \mathbb{Z}$ and $q_1^2 q_2^2 \neq 0$, the left hand side is rational, however we have chosen $r$ so that, $r^2$, the right side is irrational this is a contradiction. Thus there does not exist a rational point on this circle. $\square$

5.) Suppose $A$ and $B$ are sets and $f : A \to B$ and $g : B \to A$ are both surjective. Prove that there exists a bijection $\phi : A \to B$.

*Proof.* Since $f$ is onto, for each $b_1 \in B$ there's at least one $a_1 \in A$ such that $f(a_1) = b_1$. Then we can define a nonempty set for each $b \in B$ called $K_b$ as

$$K_b = \{a \in A | f(a) = b\}.$$

From this we shall define a function $\phi_f : B \to A$ that maps $b$ to one and only one element from every $K_b$. Then if $x, y \in B$ and $x \neq y$ we know $\phi_f(x) \neq \phi_f(y)$ since $\phi_f(x)$ and $\phi_f(y)$ can't belong to the same $K_b$. Thus $\phi_f$ is injective. We can define a similar function with $g$, first since $g$ is onto, for each $a_2 \in A$, there's at least one $b_2 \in B$ such that $g(b_2) = a_2$. Then we can define a nonempty set for $a \in A$ called $J_a$ as,

$$J_a = \{b \in B | g(b) = a\}.$$

From this set we can define a function $\phi_g : A \to B$ that maps $a$ to one and only one element from its corresponding $J_a$. Then if $x, y \in A$ and $x \neq y$, we know $\phi_g(x) \neq \phi_g(y)$ since $\phi_g(x)$ and $\phi_g(y)$ can't belong to the same $J_a$. Thus $\phi_g$ is injective. Then by the Schroeder-Bernstein Theorem we have shown an injection from $AB$ and from $B \to A$ thus there exists a bijection $\phi : A \to B$. $\square$