Notes on Dedekind

Dedekind is concerned with formalizing the intuitive "continuity" of the real line. As it develops, we see that what he calls continuity we call "completeness". Specifically, he is motivated to prove the following:

Thm: Let (a_n) be an increasing sequence that is bounded above. Then $\lim_{n\to\infty} a_n$ exists.

As we will see, this statement is equivalent to the:

Least Upper Bound Property: If $S \subseteq \mathbb{R}$ is bounded, then there is some $y \in \mathbb{R}$ such that $y \geq s$ for all $s \in S$ and such that y is the smallest such real number with this property.

In most courses and textbooks on Real Analysis, the L.U.B. property is an axiom, i.e. we assume that it is true and go from there. The beauty of Dedekind's work is that he devised an explicit construction of $\mathbb{R}$, using only $\mathbb{Q}$, for which the L.U.B. property can be proved. Practically speaking this won't affect us much, but philosophically it puts us on much stabler ground.

Here's the idea.

The Natural Numbers $\mathbb{N} = \{1, 2, \ldots\}$ are 'natural' enough that we can take them as given. The operations of addition and multiplication on $\mathbb{N}$ are also quite natural.

If we want to be able to subtract natural numbers, then we most define the meaning of $a-b$ if $b \geq a$. From this we get the integers $\mathbb{Z}$.

In order to have division, we must introduce rational numbers, $\mathbb{Q}$.

$\mathbb{Q}$ is a so-called "ordered field"; this means $+, -, \cdot,$ and $\div$ are all well-behaved binary operations on $\mathbb{Q}$ and further $\mathbb{Q}$ is ordered, that is $\forall a, b \in \mathbb{Q}$ either $a < b$, $a = b$, or $a > b$.

From this description, $\mathbb{Q}$ seems almost perfect. We created $\mathbb{Z}$, and then $\mathbb{Q}$, to address deficiencies in $\mathbb{N}$ and $\mathbb{Z}$, respectively. but what deficiency exists in $\mathbb{Q}$? The answer is completeness. For instance, the length of the diagonal of a square with side 1 is $\sqrt{2}$. But:

<u>Thm</u>: $\sqrt{2}$ is not rational

<u>Proof</u>: Suppose on the contrary that $\sqrt{2} \in \mathbb{Q}$. Then we have: $\sqrt{2} = \frac{a}{b}$, where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $\gcd(a,b) = 1$.

Then, squaring both sides, we have $2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, or $2b^2 = a^2$. $(*)$

Now since $a \in \mathbb{Z}$, it is either even or odd.

If a is even, then b must be odd; we have $a = 2m$ and $b = 2n+1$ for integers m & n.

$\Rightarrow a^2 = 4m^2$ is divisible by 4, while $b^2 = 4n^2 + 4n + 1$ is odd.

$\therefore 4 \mid a^2$ but $4 \nmid 2b^2$, a contradiction of $(*)$. Thus a cannot be even.

Suppose instead that a is odd; then b must be even. But again, $(*)$ is contradicted, so a cannot be odd.

There is no integer which is neither even nor odd, so no $\frac{a}{b}$ exists with $\sqrt{2} = \frac{a}{b}$. $\square$

So irrational numbers definitely exist. Dedekind likens them to "holes" in the line of rational numbers, which we must "fill" to attain the continuity of the real line.

His key insight is the following: If we let $A = \{q \in \mathbb{Q} \mid q^2 < \sqrt{2} \text{ or } q \leq 0\}$ and $B = A^c = \mathbb{Q} \setminus A$, then we can uniquely specify the irrational number $\sqrt{2}$ by referring only to A & B — i.e., using only rational numbers!

He then makes the leap:

Def. Given any two sets A, B of rationals, with $a < b$ $\forall a \in A, \forall b \in B$, and such that $A \cup B = \mathbb{Q}$, there is a <u>cut</u> $A|B$ corresponding to a real number.

In case A has a largest element or B has a smallest, then $A|B$ corresponds to that rational number. Otherwise, $A|B$ corresponds to an irrational number.

The irrational cuts fill the holes in $\mathbb{Q}$ in an intuitive sense. What Dedekind does <u>not</u> prove is that this actually gives us $\mathbb{R}$! For that we must prove: to every real number there is a cut that corresponds to it.