

## Chapter 3

# COMPUTATIONAL METHODS FOR OPTIMIZATION

The preceding chapters have discussed some of the analytic techniques for solving optimization problems. These techniques form the basis for most optimization models. In this chapter we will study some of the computational problems that arise in real applications and discuss a few of the most popular methods for dealing with them.

### 3.1 One Variable Optimization

Even for simple one-variable optimization problems, the task of locating global extreme points can be exceedingly difficult. Real problems are usually messy. Even when the functions involved are differentiable everywhere, the computation of the derivative is often complicated. The worst part, however, is solving the equation  $f'(x) = 0$ . The plain and simple fact is that most equations cannot be solved analytically. The best we can do in most instances is to find an approximate solution by graphical or numerical techniques.

**Example 3.1.** Reconsider the pig problem of Example 1.1, but now take into account the fact that the growth rate of the pig is not constant. Assume that the pig is young, so that the growth rate is increasing. When should we sell the pig for maximum profit?

We will use the five-step method. Step 1 will be to modify the work we did in Section 1.1, as summarized in Fig. 1.1. Now we cannot simply assume that  $w = 200 + 5t$ . What would be a reasonable assumption to represent an increasing rate of growth? There are, of course, many possible answers to this question. Let us suppose for now that the growth rate of the pig is proportional

to its weight. In other words, let us assume that

$$\frac{dw}{dt} = cw. \quad (3.1)$$

From the fact that  $dw/dt = 5$  lbs/day when  $w = 200$  lbs, we conclude that  $c = 0.025$ . This leaves us with a simple differential equation to solve for  $w$ , namely

$$\frac{dw}{dt} = 0.025w, \quad w(0) = 200. \quad (3.2)$$

We can solve Eq. (3.2) by separation of variables to obtain

$$w = 200e^{0.025t}. \quad (3.3)$$

Since all of our other assumptions are unchanged from what was presented in Fig. 1.1, this concludes step 1.

Step 2 is to select our modeling approach. We will model the problem as a one-variable optimization problem. The general solution procedure for one-variable optimization problems was outlined in Section 1.1. In this section we will explore some computational methods that can be used to implement this general solution procedure. Computational methods such as those we present here are often needed in real problems when calculations become either too hard or too tedious to perform by hand.

Step 3 is to formulate the model. The only difference between the present case and the problem formulation of Section 1.1 is that we have to replace the weight equation  $w = 200 + 5t$  by Eq. (3.3). This leads to the new objective function

$$\begin{aligned} y &= f(x) \\ &= (0.65 - 0.01x)(200e^{0.025x}) - 0.45x, \end{aligned} \quad (3.4)$$

and our problem is to maximize the function in Eq. (3.4) over the set  $S = \{x : x \geq 0\}$ .

Step 4 is to solve the model. We will use the graphical method. Good graphing utilities for personal computers, and graphing calculators, are widely available. We start our graphical analysis of this problem by producing a graph of the function in Eq. (3.4) on the same scale as Fig. 1.2, our graph of the original objective function. In this case we are left with the feeling that there is more to see on the graph of this function. We would say that Fig. 3.1 is not a *complete graph* of this function over the set  $S = [0, \infty)$ .

Figure 3.2 *is* a complete graph. It shows all of the important features we need for the solution of our problem.

How do we know when we have a complete graph? There is no simple answer to this question. Graphing is an exploratory technique. You need to experiment and to use good judgment. Of course, we do not need to look at negative values of  $x$ , but we also need not look beyond  $x = 65$ . After this point our formula says that the price for pigs is negative, which is clearly nonsense.

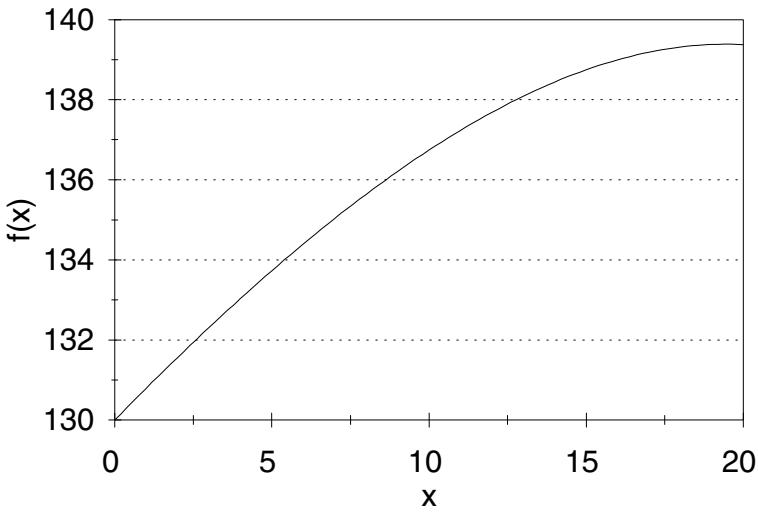


Figure 3.1: Graph of net profit  $f(x)$  from (3.4) versus time to sell  $x$  for the pig problem with nonlinear weight model.

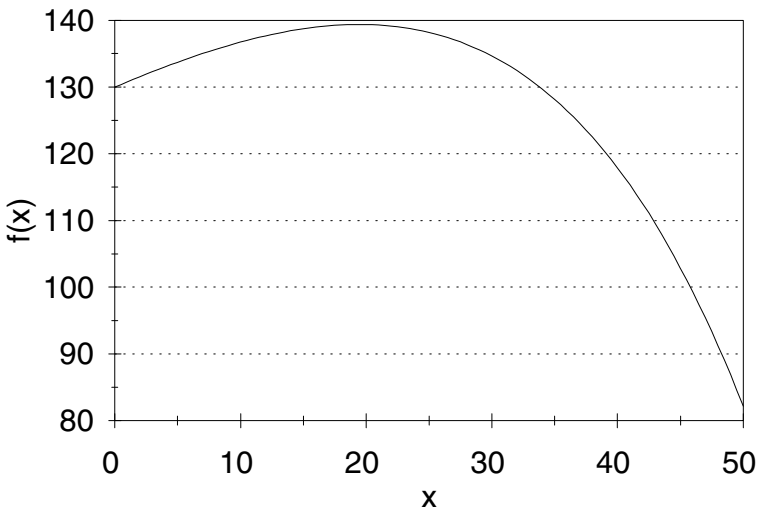


Figure 3.2: Complete graph of net profit  $f(x)$  from (3.4) versus time to sell  $x$  for the pig problem with nonlinear weight model.

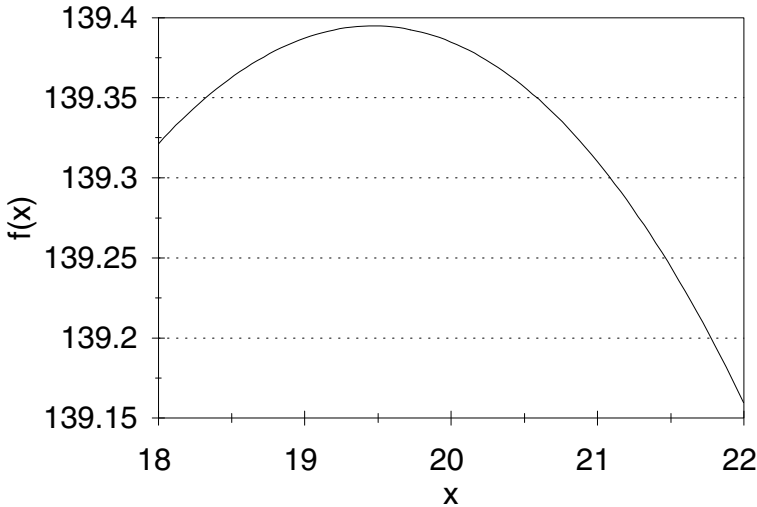


Figure 3.3: First zoom-in on the graph of net profit  $f(x)$  from (3.4) versus time to sell  $x$  for the pig problem with nonlinear weight model.

From the graph in Fig. 3.2 we can conclude that the maximum occurs around  $x = 20$  and  $y = f(x) = 140$ . To obtain a better estimate, we can zoom in on the maximum point of the graph. Figures 3.3 and 3.4 show the outcome of successive zooms.

On the basis of the graph in Fig. 3.4, we would estimate that the maximum occurs at

$$\begin{aligned} x &= 19.5 \\ y &= f(x) = 139.395. \end{aligned} \tag{3.5}$$

At this point we have found the location of the maximum to three significant digits, and the value of the maximum to six significant digits. Since  $f'(x) = 0$  at the maximum, the function  $f(x)$  is quite insensitive to changes in  $x$  near this point, so we are able to obtain more accuracy for  $f(x)$  than for  $x$ .

Step 5 is to answer the question. After taking into account the fact that the growth rate of the young pig is still increasing, we now recommend waiting 19 or 20 days to sell. This should result in a net profit of approximately \$140.

The graphical method used in step 4 to locate the optimum point (see Eq. (3.5)) did not produce a high degree of accuracy. This is acceptable for the present problem because we do not need a high degree of accuracy. While it is true that the graphical method can be made to produce higher accuracy (by zooming in on the optimum point over and over again), there are more efficient computational methods that should be used in such cases. We will look at some of these next, in the course of our sensitivity analysis.

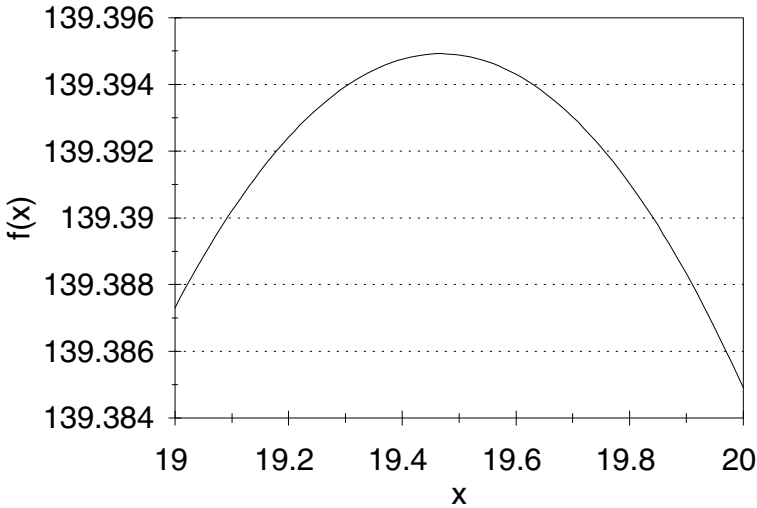


Figure 3.4: Second zoom-in on the graph of net profit  $f(x)$  from (3.4) versus time to sell  $x$  for the pig problem with nonlinear weight model.

Let us examine the sensitivity of the optimum coordinates in Eq. (3.5) to the growth rate  $c = 0.025$  for the young pig. One way to do this would be to repeat our graphical analysis for several different values of the parameter  $c$ . However, this would be tedious. We would prefer a more efficient method.

Let us begin by generalizing the model. Now we are assuming that

$$\frac{dw}{dt} = cw, \quad w(0) = 200, \quad (3.6)$$

so that

$$w = 200e^{ct}. \quad (3.7)$$

This leads to the objective function

$$f(x) = (0.65 - 0.01x)(200e^{cx}) - 0.45x. \quad (3.8)$$

From our graphical analysis, we know that for  $c = 0.025$ , the optimum occurs at an interior critical point, at which point  $f'(x) = 0$ . Since  $f$  is a continuous function of  $c$ , it seems reasonable to conclude that the same holds for values of  $c$  near 0.025. In order to locate this interior critical point, we need to compute the derivative  $f'(x)$  and solve the equation  $f'(x) = 0$ . The first part of this process (computing the derivative) is relatively easy. There is a standard method for computing derivatives that you learned in one-variable calculus. It can be applied to virtually any differentiable function. For complicated expressions the derivative can also be computed using a computer algebra system (Maple,

Mathematica, Derive, etc.) or a hand calculator (such as the HP-48). In our problem, it is not too difficult to compute by hand that

$$f'(x) = 200ce^{cx}(0.65 - 0.01x) - 2e^{cx} - 0.45. \quad (3.9)$$

The second part of the process is to solve the equation

$$200ce^{cx}(0.65 - 0.01x) - 2e^{cx} - 0.45 = 0. \quad (3.10)$$

You can try to solve this equation by hand, but you are unlikely to succeed in obtaining an algebraic solution. Some computer algebra systems can now solve (3.10) in terms of the  $W$  function, which was developed specifically to solve equations of this type. Ongoing research in mathematics and mathematical software will continue to expand our ability to solve equations, but the plain and simple fact is that most equations still cannot be solved by algebraic methods. While there does exist a general algebraic method for computing derivatives, there is no general algebraic method for solving equations. Even for polynomials, it is known that there can be no generally useful algebraic method of finding roots (i.e., there can be no analog of the quadratic formula) for degree five or above. This is why it is often necessary to resort to *numerical approximation* methods to solve algebraic equations.

We will use Newton's method to solve Eq. (3.10). You probably learned about Newton's method in one-variable calculus.

We are given a differentiable function  $F(x)$  and an approximation  $x_0$  to the root  $F(x) = 0$ . Newton's method works by linear approximation. Near  $x = x_0$  we have  $F(x) \approx F(x_0) + F'(x_0)(x - x_0)$  by the usual tangent line approximation. To obtain a better estimate  $x = x_1$  of the actual root to  $F(x) = 0$ , we set  $F(x_0) + F'(x_0)(x - x_0) = 0$  and solve for  $x = x_1$  to get  $x_1 = x_0 - F(x_0)/F'(x_0)$ . Geometrically, the tangent line to  $y = F(x)$  at the point  $x = x_0$  intersects the  $x$ -axis at the point  $x = x_1$ . As long as  $x_1$  is not too far away from  $x_0$ , the tangent line approximation should be reasonably good, and so  $x_1$  should be close to the actual root. Newton's method produces a sequence of increasingly accurate estimates  $x_1, x_2, x_3, \dots$ , to the actual root by repeating this tangent line approximation. Once we are sufficiently close to the root, each successive approximation produced by Newton's method is accurate to about twice as many decimal places as the preceding estimate.

Figure 3.5 presents Newton's method in a form called *pseudocode*. This is a standard method for describing a numerical algorithm.

It is a fairly simple matter to transform pseudocode into a working computer program in any high-level computer language (BASIC, FORTRAN, C, PASCAL, etc.) or to implement on a spreadsheet. It is also possible to program in most computer algebra systems. For the numerical methods in this book, any of these options can be used. It is not recommended that these algorithms be implemented by hand.

**Algorithm:** NEWTON'S METHOD

**Variables:**  $x(n)$  = approximate location of root after  $n$  iterations.  
 $N$  = number of iterations

**Input:**  $x(0), N$

**Process:** Begin  
for  $n = 1$  to  $N$  do  
    Begin  
         $x(n) \leftarrow x(n-1) - F(x(n-1))/F'(x(n-1))$   
    End  
End

**Output:**  $x(N)$

Figure 3.5: Pseudocode for Newton's Method in one variable.

$c$	$x$
0.022	11.626349
0.023	14.515929
0.024	17.116574
0.025	19.468159
0.026	21.603681
0.027	23.550685
0.028	25.332247

Table 3.1: Sensitivity of best time to sell  $x$  to growth rate parameter  $c$  for the pig problem with nonlinear weight model.

In our problem we want to use Newton's method to find a root of the equation

$$F(x) = 200ce^{cx}(0.65 - 0.01x) - 2e^{cx} - 0.45 = 0. \quad (3.11)$$

For values of  $c$  near 0.025, we expect to find a root near the point  $x = 19.5$ . We used a computer implementation of Newton's method to produce the results in Table 3.1. For each value of  $c$ , we performed  $N = 10$  iterations starting at the point  $x(0) = 19.5$ . An additional sensitivity run with  $N = 15$  was used to verify the accuracy of our results.

Notice that our method of solution for Eq. (3.11) has two steps. First, we apply a *global method* (graphing) to locate an approximate solution. Then, we apply a fast *local method* to determine the exact solution to the desired accuracy. These are the two stages of numerical solution, and they are common to most generally useful solution methods. For one-variable optimization, graphing is the simplest and most useful global method. Newton's method is easy to program, and built-in numerical equation solvers are also available on most graph-

ing calculators, spreadsheets, and computer algebra systems. While the details vary, most of these solvers are based on some variation of Newton's method. These solvers can be used safely and effectively in the same manner as Newton's method. First, use a global method to approximate the root. Most solvers require either a point estimate or an interval estimate of the root. Then, exercise the solver and verify the result, either by performing sensitivity analysis on the available tolerance parameter(s), or by substituting the numerical solution back into the original equation. One word of caution: Casual and uncritical faith in the results of numerical solvers is dangerous. For many real problems, including some of the exercises in this book, inappropriate use of numerical solvers has been known to produce significant errors. Initial application of an appropriate global method, and subsequent verification of the root, are important parts of the numerical solution procedure. Some calculators, computer algebra systems, and spreadsheets also have numerical optimizers. Usually, these routines apply a variant of Newton's method based on a numerical approximation of the derivative. The same advice applies to these routines. Use a global method to approximate the optimum, exercise the numerical optimizer, and then perform sensitivity analysis on the tolerance parameters to ensure accuracy.

In order to relate our sensitivity results back to the original data in the problem, in Figure 3.6 we have plotted the root  $x$ , which represents the optimal time to sell, against the growth rate

$$g = 200c, \quad (3.12)$$

which was originally given as  $g = 5$  lbs/day.

To obtain a numerical estimate of sensitivity, we solve Eq. (3.11) once more, setting  $c = 0.02525$  (a 1% increase). The solution found was

$$x = 20.021136,$$

which represents a 2.84% increase in  $x$ . Thus, we estimate that  $S(x, c) = 2.84$ . Since

$$g = 200c,$$

we can easily show that

$$S(x, g) = S(x, c) = 2.84.$$

Also, if we let  $h$  denote the initial weight of the pig (we assumed  $h = 200$  lbs), then since

$$h = 5/c,$$

we have

$$\begin{aligned} S(x, h) &= \frac{dx}{dh} \cdot \frac{h}{x} \\ &= \left( \frac{dx/dc}{dh/dc} \right) \left( \frac{5/c}{x} \right) \\ &= -S(x, c) = -2.84. \end{aligned}$$



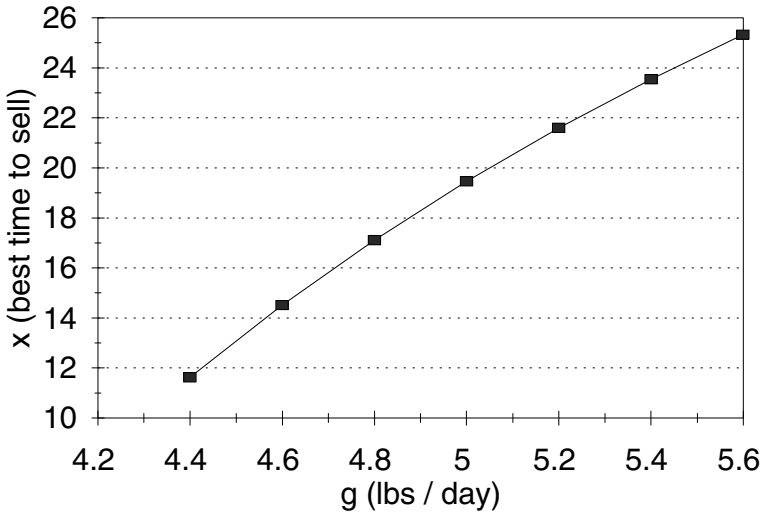


Figure 3.6: Graph of best time to sell  $x$  versus growth rate  $g$  for the pig problem with nonlinear weight model.

In fact, it is generally true that if  $y$  is proportional to  $z$ , then

$$S(x, y) = S(x, z), \quad (3.13)$$

and if  $y$  is inversely proportional to  $z$ , then

$$S(x, y) = -S(x, z). \quad (3.14)$$

Now we have computed the sensitivity of  $x$  to both the initial weight of the pig and the growth rate of the pig. The other sensitivities are unchanged from the original problem considered in Chapter 1, since the other parameters appear in the objective function in the same manner as before.

The fact that our optimal solution in the present case differs so markedly from what we found in Chapter 1 (19 or 20 days here versus 8 days there) raises some important questions about the robustness of our model. There is also serious concern about whether the assumption

$$p = 0.65 - 0.01t$$

would be valid over a three-week period. Alternative models for price could certainly be considered, along with more sophisticated models of the growth process for a representative pig. Some of these issues of robustness are addressed in the exercises at the end of this chapter. What we can say now is this: If the pig's rate of growth does not diminish, and if the price decline does not accelerate, then we should hold onto the pig for another week. At that time, we can reevaluate the situation on the basis of new data.

3	0	1	4	2	1
2	1	1	2	3	2
5	3	3	0	1	2
8	5	2	1	0	0
10	6	3	1	3	1
0	2	3	1	1	1

Figure 3.7: Map showing the number of emergency calls per year in each one square mile area of the city. Up is north and right is east.

## 3.2 Multivariable Optimization

The practical problems associated with locating the global optimum of a function of several variables are similar in many ways to those discussed in the preceding section. Additional complications arise because of the dimension of the problem. Graphical techniques are not available in dimensions  $n > 3$ , and solving  $\nabla f = 0$  becomes more complicated as the number of independent variables increases. Constrained optimization is also more difficult because the geometry of the feasible region can be more complicated.

**Example 3.2.** A suburban community intends to replace its old fire station with a new facility. The old station was located at the historical city center. City planners intend to locate the new facility more scientifically. A statistical analysis of response-time data yielded an estimate of  $3.2 + 1.7r^{0.91}$  minutes required to respond to a call  $r$  miles away from the station. (The derivation of this formula is the subject of Exercises 17 and 18 in Chapter 8.) Estimates of the frequency of calls from different areas of the city were obtained from the fire chief. They are presented in Figure 3.7. Each block represents one square mile, and the numbers inside each block represent the number of emergency calls per year for that block. Find the best location for the new facility.

We will represent locations on the city map by coordinates  $(x, y)$ , where  $x$  is the distance in miles to the west side of town, and  $y$  is the distance in miles to the south side. For example,  $(0, 0)$  represents the lower left-hand corner of the map, and  $(0, 6)$  represents the upper left-hand corner. For simplicity we will divide the city into nine  $2 \times 2$ -mile squares and assume that each emergency

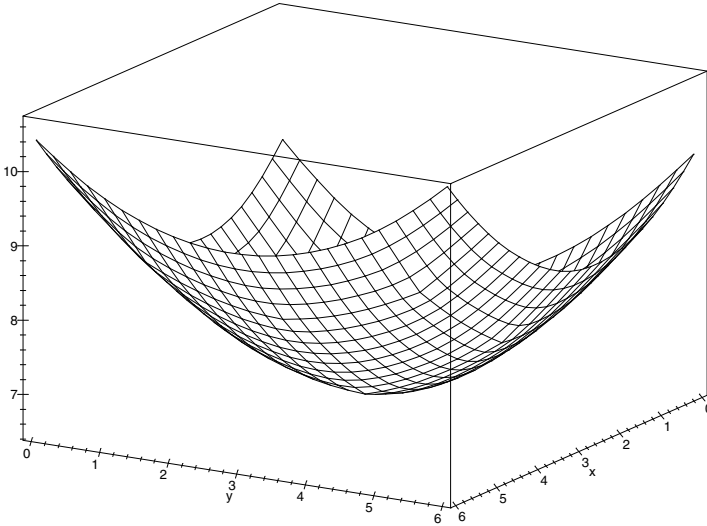


Figure 3.8: 3-D graph of average response time  $z = f(x, y)$  from (3.15) versus map location  $(x, y)$  for the facility location problem.

is located at the center of the square. If  $(x, y)$  is the location of the new fire station, the average response time to a call is  $z = f(x, y)$ , where

$$\begin{aligned}
 z = & 3.2 + 1.7[6\sqrt{(x-1)^2 + (y-5)^2}^{0.91} \\
 & + 8\sqrt{(x-3)^2 + (y-5)^2}^{0.91} + 8\sqrt{(x-5)^2 + (y-5)^2}^{0.91} \\
 & + 21\sqrt{(x-1)^2 + (y-3)^2}^{0.91} + 6\sqrt{(x-3)^2 + (y-3)^2}^{0.91} \\
 & + 3\sqrt{(x-5)^2 + (y-3)^2}^{0.91} + 18\sqrt{(x-1)^2 + (y-1)^2}^{0.91} \\
 & + 8\sqrt{(x-3)^2 + (y-1)^2}^{0.91} + 6\sqrt{(x-5)^2 + (y-1)^2}^{0.91}]/84.
 \end{aligned} \tag{3.15}$$

The problem is to minimize  $z = f(x, y)$  over the feasible region  $0 \leq x \leq 6$ ,  $0 \leq y \leq 6$ .

Figure 3.8 shows a 3-D graph of the objective function  $f$  over the feasible region. It appears as though  $f$  attains its minimum at the unique interior point at which  $\nabla f = 0$ . Figure 3.9 shows a contour plot of the level sets of  $f$ , indicating that  $\nabla f = 0$  near the point  $x = 2$  and  $y = 3$ .

Now, it is certainly possible to compute  $\nabla f$  in this problem, but it is not possible to solve  $\nabla f = 0$  algebraically. Further graphical analysis is possible, but this is especially cumbersome for functions of more than one variable. What is needed here is a simple global method for estimating the minimum.

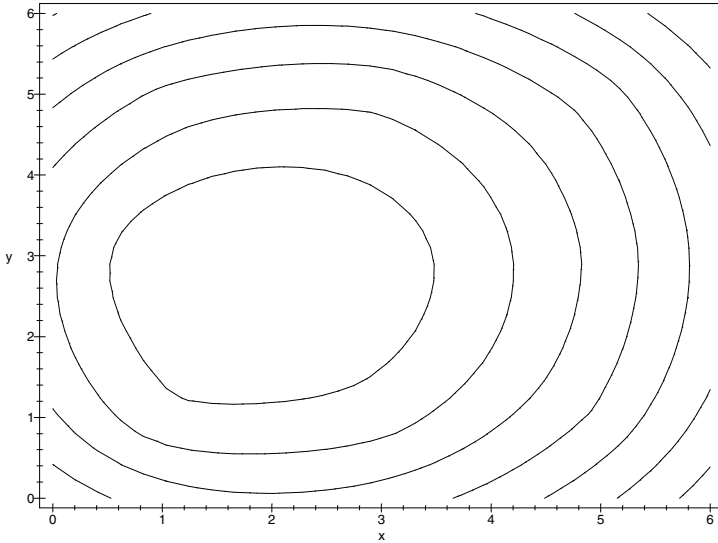


Figure 3.9: Contour plot of average response time  $z = f(x, y)$  from (3.15) versus map location  $(x, y)$  for the facility location problem.

Figure 3.10 presents an algorithm for random search. This optimization method simply picks  $N$  feasible points at random and selects the one that yields the smallest value of the objective function. The notation  $\text{Random } \{S\}$  denotes a randomly selected point from the set  $S$ . A computer implementation of random search was applied to the function in Eq. (3.15) with  $a = 0$ ,  $b = 6$ ,  $c = 0$ ,  $d = 6$ , and  $N = 1,000$ . The resulting estimate of the minimum occurred at

$$\begin{aligned} x \min &= 1.66 \\ y \min &= 2.73 \\ z \min &= 6.46. \end{aligned} \tag{3.16}$$

Since this algorithm involves random numbers, a repetition of the same computer implementation with the same inputs may produce slightly different outputs. The accuracy of random search is roughly the same as if the  $N$  points were to lie on an equally spaced grid over the entire feasible set. Such a grid would contain  $32 \times 32$  points ( $32^2 \approx 1,000$ ), so we would be accurate to within  $6/32 \approx 0.2$  in both  $x$  and  $y$ . Since  $\nabla f = 0$  at the minimum, we obtain much better accuracy in  $z$ . An alternative to random search would be a grid search (examine  $z = F(x, y)$  at  $N$  equally spaced points). The performance of grid search is essentially the same as random search, but random search is more flexible and easier to implement.

The estimates in Eq. (3.16) of the optimum location  $(1.7, 2.7)$  and the re-

**Algorithm:** METHOD OF RANDOM SEARCH

**Variables:**  $a$  = lower limit on  $x$   
 $b$  = upper limit on  $x$   
 $c$  = lower limit on  $y$   
 $d$  = upper limit on  $y$   
 $N$  = number of iterations  
 $x \min$  = approximate  $x$  coordinate of minimum  
 $y \min$  = approximate  $y$  coordinate of minimum  
 $z \min$  = approximate value of  $F(x, y)$  at minimum

**Input:**  $a, b, c, d, N$

**Process:** Begin  
 $x \leftarrow \text{Random} \{[a, b]\}$   
 $y \leftarrow \text{Random} \{[c, d]\}$   
 $z \min \leftarrow F(x, y)$   
 for  $n = 1$  to  $N$  do  
   Begin  
      $x \leftarrow \text{Random} \{[a, b]\}$   
      $y \leftarrow \text{Random} \{[c, d]\}$   
      $z \leftarrow F(x, y)$   
     if  $z < z \min$  then  
        $x \min \leftarrow x$   
        $y \min \leftarrow y$   
        $z \min \leftarrow z$   
     End  
 End

**Output:**  $x \min, y \min, z \min$

Figure 3.10: Pseudocode for the method of random search.

sulting average response time of 6.46 minutes were obtained by evaluating the objective function at  $N = 1,000$  random points in the feasible region. Better accuracy could be obtained by increasing  $N$ . However, the behavior of this simple, global method does not encourage such an approach. Each additional decimal place of accuracy requires increasing  $N$  by *a factor of one hundred*. Hence, this method is only suitable for obtaining a rough approximation of the optimum. For the present problem the answer we found is good enough. The simplifying assumptions we made earlier introduced errors on the order of one mile in emergency location, so there is no point in demanding more accuracy now. It is enough to state that the facility should be located around (1.7, 2.7) on the map, for an average response time of around 6.5 minutes. The exact location will, in any case, depend on several factors not incorporated into our

model, such as the location of roads and the availability of land in the area of the optimal site. It is also reasonable to consider different definitions of an “optimal” location; see Exercise 3.6.

It is important to provide an estimate of the sensitivity of response time to the eventual facility location. Since  $\nabla f = 0$  at the optimal location, we do not expect  $f$  to vary much near (1.7, 2.7). To obtain a more concrete understanding of the sensitivity of  $f$  to  $(x, y)$  near the optimum, we reran the random search program, replacing  $f$  by  $-f$  and bounding  $1.5 \leq x \leq 2$ ,  $2.5 \leq y \leq 3$ . After  $N = 100$  observations we found that the maximum of  $f$  over this region was approximately 6.49 minutes, or about 0.03 minutes longer than the observed optimum. It does not matter in any practical sense where in this half-mile square area we locate the station.

The random search methods employed in the preceding example are simple but slow. For some problems, where greater accuracy is required, such methods are unsuitable. If the functions involved are as complex as the objective in Example 3.2, then it will be very hard to obtain an accurate answer. More accurate and efficient methods for global optimization of functions of more than one variable are almost always based on the gradient. We now present an example in which these methods are more tractable, because the gradient is easier to compute.

**Example 3.3.** A manufacturer of lawn furniture makes two types of lawn chairs, one with a wood frame and one with a tubular aluminum frame. The wood-frame model costs \$18 per unit to manufacture, and the aluminum-frame model costs \$10 per unit. The company operates in a market where the number of units that can be sold depends on the price. It is estimated that in order to sell  $x$  units per day of the wood-frame model and  $y$  units per day of the aluminum-frame model, the selling price cannot exceed  $10 + 31x^{-0.5} + 1.3y^{-0.2}$  \$/unit for wood-frame chairs, and  $5 + 15y^{-0.4} + 0.8x^{-0.08}$  \$/unit for aluminum-frame chairs. Find the optimal production levels.

The objective is to maximize the profit function  $z = f(x, y)$  \$/day over the feasible set of production levels  $x \geq 0$ ,  $y \geq 0$ , where

$$\begin{aligned} z &= x(10 + 31x^{-0.5} + 1.3y^{-0.2}) - 18x \\ &\quad + y(5 + 15y^{-0.4} + 0.8x^{-0.08}) - 10y. \end{aligned} \tag{3.17}$$

Figure 3.11 shows a graph of  $f$ . The graph indicates that  $f$  has a unique interior maximum at the point where  $\nabla f = 0$ . Figure 3.12 shows the level sets of  $f$ . This graph suggests that the maximum occurs around  $x = 5$  and  $y = 6$ . We calculate the gradient  $\nabla f(x, y) = (\partial z / \partial x, \partial z / \partial y)$  to get

$$\begin{aligned} \frac{\partial z}{\partial x} &= 15.5x^{-0.5} - 8 + 1.3y^{-0.2} - 0.064yx^{-1.08} \\ \frac{\partial z}{\partial y} &= 9y^{-0.4} - 5 + 0.8x^{-0.08} - 0.26xy^{-1.2}. \end{aligned} \tag{3.18}$$

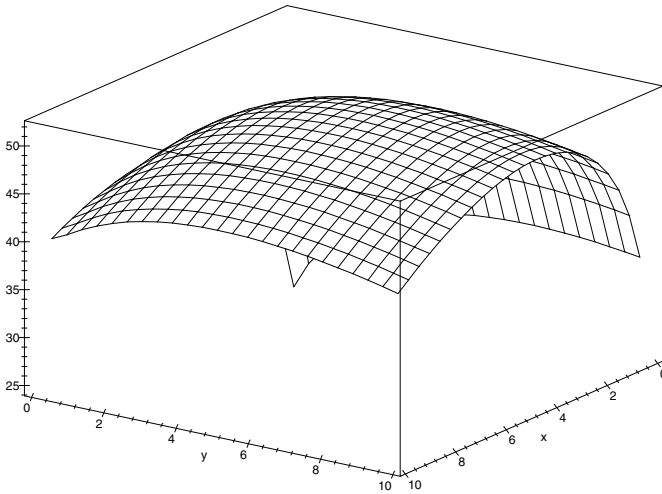


Figure 3.11: 3-D graph of profit  $z = f(x, y)$  from (3.17) versus the number  $x$  of wood-frame chairs and the number  $y$  of aluminum-frame chairs produced per day in the lawn chair problem.

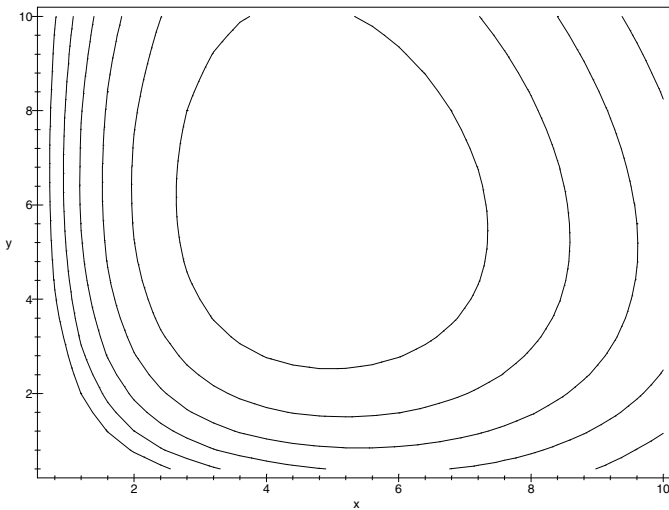


Figure 3.12: Contour graph of profit  $z = f(x, y)$  from (3.17) versus the number  $x$  of wood-frame chairs and the number  $y$  of aluminum-frame chairs produced per day in the lawn chair problem.

A random search of  $N = 1,000$  points over the region  $0 \leq x \leq 10$ ,  $0 \leq y \leq 10$  yields an observed maximum of \$52.06 per day at the production level  $x = 4.8$  and  $y = 5.9$  chairs per day. To obtain a more accurate numerical estimate of the optimum point, we will use a multivariable version of Newton's method to solve the gradient equation  $\nabla f = 0$ . Figure 3.13 gives the algorithm for Newton's method in two variables.

We are given a set of differentiable functions,  $f_1, \dots, f_n$ , and an approximate root  $(x_1(0), \dots, x_n(0))$  to the system of equations

$$\begin{aligned} f_1(x_1, \dots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, \dots, x_n) &= 0. \end{aligned} \tag{3.19}$$

In order to simplify the explanation, and to clarify the connection between the one-variable and multivariable Newton's method, we will use vector notation. Let  $x = (x_1, \dots, x_n)$  and  $F(x) = (f_1(x), \dots, f_n(x))$  so that we can rewrite Eq. (3.19) in the form  $F(x) = 0$  and  $x(0)$  is the initial estimate of the root. Newton's method produces a sequence of increasingly accurate estimates  $x(1)$ ,  $x(2)$ ,  $x(3)$ ,  $\dots$  to the actual root  $F(x) = 0$  using a linear approximation. Near  $x = x(0)$  we have  $F(x) \approx F(x(0)) + A(x - x(0))$ , where  $A$  is the matrix of partial derivatives

$$A = \begin{pmatrix} \partial f_1 / \partial x_1 & \cdots & \partial f_1 / \partial x_n \\ \vdots & & \vdots \\ \partial f_n / \partial x_1 & \cdots & \partial f_n / \partial x_n \end{pmatrix}$$

evaluated at  $x = x(0)$ . This is the multivariable version of the tangent line approximation. To obtain a better estimate  $x = x(1)$  to the actual root  $F(x) = 0$ , we set  $F(x(0)) + A(x - x(0)) = 0$  and solve for  $x = x(1)$  to get  $x(1) = x(0) - A^{-1}F(x(0))$ . This is exactly the same formula as in the one-variable case, except that we cannot divide by the derivative matrix, so we take the inverse. Once we are sufficiently close to the root, each successive approximation is accurate to about twice as many decimal places as the preceding estimate. Just as in the one-variable case, the sequence of estimates produced by Newton's method converges rapidly to the actual root.

The pseudocode in Figure 3.13 implements Newton's method in two variables using the formula for the inverse of a  $2 \times 2$  matrix

$$\begin{pmatrix} q & r \\ s & t \end{pmatrix}^{-1} = \frac{1}{qt - rs} \begin{pmatrix} t & -r \\ -s & q \end{pmatrix}.$$

In the general case, it is necessary to compute the inverse of the derivative matrix  $A$  by other methods. For more details on Newton's method in several variables, see Press, et al. (2002).



**Algorithm:** NEWTON'S METHOD IN TWO VARIABLES

**Variables:**  $x(n)$  = approximate  $x$  coordinate of the root after  $n$  iterations.  
 $y(n)$  = approximate  $y$  coordinate of the root after  $n$  iterations.  
 $N$  = number of iterations

**Input:**  $x(0), y(0), N$

**Process:** Begin  
 for  $n = 1$  to  $N$  do  
   Begin  
      $q \leftarrow \partial F / \partial x(x(n-1), y(n-1))$   
      $r \leftarrow \partial F / \partial y(x(n-1), y(n-1))$   
      $s \leftarrow \partial G / \partial x(x(n-1), y(n-1))$   
      $t \leftarrow \partial G / \partial y(x(n-1), y(n-1))$   
      $u \leftarrow -F(x(n-1), y(n-1))$   
      $v \leftarrow -G(x(n-1), y(n-1))$   
      $D \leftarrow qt - rs$   
      $x(n) \leftarrow x(n-1) + (ut - vr)/D$   
      $y(n) \leftarrow y(n-1) + (qv - su)/D$   
   End  
 End

**Output:**  $x(N), y(N)$

Figure 3.13: Pseudocode for Newton's method in two variables.

In our problem we have

$$\begin{aligned} F(x, y) &= 15.5x^{-0.5} - 8 + 1.3y^{-0.2} - 0.064yx^{-1.08} \\ G(x, y) &= 9y^{-0.4} - 5 + 0.8x^{-0.08} - 0.26xy^{-1.2}, \end{aligned} \quad (3.20)$$

and so we can compute that

$$\begin{aligned} \frac{\partial F}{\partial x} &= 0.06912yx^{-2.08} - 7.75x^{-1.5} \\ \frac{\partial F}{\partial y} &= -0.064x^{-1.08} - 0.26y^{-1.2} \\ \frac{\partial G}{\partial x} &= -0.064x^{-1.08} - 0.26y^{-1.2} \\ \frac{\partial G}{\partial y} &= 0.312xy^{-2.2} - 3.6y^{-1.4}. \end{aligned} \quad (3.21)$$

We used a computer implementation of Newton's method in two variables start-

ing at  $x(0) = 5$ ,  $y(0) = 5$ . After  $N = 10$  iterations we obtained

$$\begin{aligned}x &= 4.68959 \\ y &= 5.85199\end{aligned}\tag{3.22}$$

as our estimate of the root. A further sensitivity run with  $N = 15$  was made to confirm these results. Substituting back into Eq. (3.17) yields  $z = 52.0727$ . Hence, the optimal solution to the lawn chair problem is to produce 4.69 wood-frame chairs and 5.85 aluminum-frame chairs per day, which should result in a net profit of \$52.07 per day.

As in the one-variable case, the accurate numerical solution of systems of equations is a two-step process. First, we must use a global method to estimate the location of the root. Graphical methods are useful if there are only two variables, but for most problems, a numerical method such as random search is necessary. A variety of more sophisticated global methods is also available, most of which are tailored to special classes of problems. Next, we use a fast local method to obtain an accurate solution, which must be verified either by sensitivity analysis of the parameters which control accuracy, or by substituting the numerical solution back into the original system of equations. Newton's method is a very fast local method that requires computing the partial derivatives. Various modifications exist which employ estimates of the partial derivatives; e.g., see Press, et al. (2002). Most spreadsheets and computer algebra systems have multivariable equation solvers, and solvers are also available in most numerical analysis software libraries and packages. To use one of these, begin with a global method to estimate the location of the root, then exercise the solver, and finally verify the solution as usual. Do not simply plug in the equations, use the solver, and accept the reported solution, as this can lead to serious errors. Many spreadsheets, computer algebra systems, and numerical analysis packages also have multivariable numerical optimizers. Usually these routines apply a variation of Newton's method based on numerical approximation of the derivatives. The same advice applies to these routines. Use a global method to approximate the optimum, exercise the numerical optimizer, and then perform sensitivity analysis on the tolerance parameters to ensure accuracy.

Multivariable optimization problems with constraints may be harder to solve. In some cases the optimum occurs in the interior of the feasible region, so that we may treat the problem as if there were no constraints. When the optimum occurs on the boundary, the situation is more complex. There are no simple, generally effective computational algorithms for solving such problems. The methods that do exist are tailored to the particular features of some special class of problems. In the next section we will discuss the most important of these.

### 3.3 Linear Programming

Multivariable optimization problems with constraints are almost always difficult to solve. A variety of computational techniques has been developed to handle

Requirements per acre	Corn	Wheat	Oats
Irrigation (acre-ft)	3.0	1.0	1.5
Labor (person-hrs/week)	0.8	0.2	0.3
Yield (\$)	400	200	250

Table 3.2: Data for the farm problem of Example 3.4.

special types of multivariable optimization problems, but good general methods do not yet exist, even at the most sophisticated levels. The area of research that considers the development of new computational methods for such problems is called *nonlinear programming*, and it is very active.

The simplest type of multivariable constrained optimization problem is one where both the objective function and the constraint functions are linear. The study of computational methods for such problems is called *linear programming*. Software packages for linear programming are widely available and are in frequent use for problems in manufacturing, investment, farming, transportation, and government. Typical large-scale problems involve thousands of decision variables and thousands of constraints. There are many well-documented cases where an operations analysis based on a linear programming model has generated savings in the millions of dollars. Details can be found in the literature on operations research and management science.

**Example 3.4.** A family farm has 625 acres available for planting. The crops the family is considering are corn, wheat, and oats. It is anticipated that 1,000 acre-ft of water will be available for irrigation, and the farmers will be able to devote 300 hours of labor per week. Additional data are presented in Table 3.2. Find the amount of each crop that should be planted for maximum profit.

We will use the five-step method. The results of step 1 are shown in Figure 3.14. Step 2 is to select the modeling approach. We will model this problem as a linear programming problem.

The standard (inequality) form of a linear programming model is as follows: Maximize the objective function  $y = f(x_1, \dots, x_n) = c_1x_1 + \dots + c_nx_n$  over the feasible region defined by the constraints

$$\begin{array}{rcl}
 a_{11}x_1 + \dots + a_{1n}x_n & \leq & b_1 \\
 \vdots & & \vdots \\
 a_{m1}x_1 + \dots + a_{mn}x_n & \leq & b_m
 \end{array} \tag{3.23}$$

and  $x_1 \geq 0, \dots, x_n \geq 0$ . This is a special case of the multivariable

**Variables:**  $x_1$  = acres of corn planted  
 $x_2$  = acres of wheat planted  
 $x_3$  = acres of oats planted  
 $w$  = irrigation required (acre-ft)  
 $l$  = labor required (person-hrs/wk)  
 $t$  = total acreage planted  
 $y$  = total yield (\$)

**Assumptions:**  $w = 3.0x_1 + 1.0x_2 + 1.5x_3$   
 $l = 0.8x_1 + 0.2x_2 + 0.3x_3$   
 $t = x_1 + x_2 + x_3$   
 $y = 400x_1 + 200x_2 + 250x_3$   
 $w \leq 1,000$   
 $l \leq 300$   
 $t \leq 625$   
 $x_1 \geq 0; x_2 \geq 0; x_3 \geq 0$

**Objective:** Maximize  $y$

Figure 3.14: Results of step 1 of the farm problem.

constrained optimization problem discussed in Chapter 2. Let

$$\begin{aligned} g_1(x_1, \dots, x_n) &= a_{11}x_1 + \dots + a_{1n}x_n \\ &\vdots \\ g_m(x_1, \dots, x_n) &= a_{m1}x_1 + \dots + a_{mn}x_n \end{aligned} \quad (3.24)$$

and

$$\begin{aligned} g_{m+1}(x_1, \dots, x_n) &= x_1 \\ &\vdots \\ g_{m+n}(x_1, \dots, x_n) &= x_n. \end{aligned}$$

The constraints can be written in the form  $g_1 \leq b_1, \dots, g_m \leq b_m$  and  $g_{m+1} \geq 0, \dots, g_{m+n} \geq 0$ . The set of  $(x_1, \dots, x_n)$  that satisfies these constraints is called the *feasible region*. It represents all feasible values for the decision variables  $x_1, \dots, x_n$ . Since  $\nabla f = (c_1, \dots, c_n)$  can never be zero, the function cannot attain its maximum in the interior of the feasible region. At a maximum on the boundary we must have

$$\nabla f = \lambda_1 \nabla g_1 + \dots + \lambda_{m+n} \nabla g_{m+n} \quad (3.25)$$

with  $\lambda_i \neq 0$  only if the  $i$ th constraint is binding. For  $i = 1, \dots, m$ , the value of the Lagrange multiplier  $\lambda_i$  represents the potential increase in the maximum value of the objective function  $f(x_1, \dots, x_n)$

that would result from relaxing the  $i$ th constraint by one unit (changing it to  $g_i(x_1, \dots, x_n) \leq b_i + 1$ ). Computation of the optimal solution to a linear programming problem is usually obtained by computer, using a variation of the *simplex method*. This method is based on the fact (which you can easily verify) that the optimal solution must occur at one of the corner points of the feasible region. Rather than go into the details involved in the algebra of the simplex method, we will concentrate on what you need to know in order to use a linear programming package correctly.

In the simplex method, the coordinates of a corner point are computed using the alternative (equality) form of a linear programming problem: Maximize  $y = c_1x_1 + \dots + c_nx_n$  over the set

$$\begin{array}{rcl} a_{11}x_1 + \dots + a_{1n}x_n + x_{n+1} & & = b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n & + x_{n+2} & = b_2 \\ \vdots & & \\ a_{m1}x_1 + \dots + a_{mn}x_n & + x_{n+m} & = b_m \end{array} \quad (3.26)$$

and  $x_1 \geq 0, \dots, x_{n+m} \geq 0$ . The variable  $x_{n+i}$  is called a *slack variable* because it represents the amount of slack remaining in the  $i$ th constraint. The  $i$ th constraint is binding when the slack variable  $x_{n+i} = 0$ . The coordinates of a corner point can be obtained by setting  $n$  of the variables  $x_1, \dots, x_{n+m}$  equal to zero, and then solving the resulting  $m$  equations in  $m$  unknowns. The corner point is feasible if the remaining  $m$  variables (called *basic variables*) turn out to be nonnegative.

Suppose that we have a moderate size linear programming problem, with  $n = 50$  variables and  $m = 100$  constraints. The number of corner points is equal to the number of possible choices for which 50 of the 150 variables (50 decision variables plus 100 slack variables) we set equal to zero. The number of possible ways to choose 50 out of 150 is  $(150!)/(50!(100!))$ , or around  $2 \times 10^{40}$ . A computer program that could check one corner point every nanosecond would take around  $8 \times 10^{30}$  years to solve this problem, which is representative of a typical linear programming application. The simplex method typically calculates only a selected subset of the corner points (a very small fraction of the total). A linear program of size  $n = 50$ ,  $m = 100$  can be solved quickly on a mainframe computer. As of this writing, a problem of this size is at the upper limits of what can be solved in a reasonable amount of time on a personal computer, but technological advances will certainly push these limits further in the near future. As a general rule, the execution time for a linear program using the simplex method is proportional to  $m^3$ , so that an order-of-magnitude improvement in processor speed more than doubles the size of the linear program that can be handled by a

given machine. For the problems in this book, any good computer implementation of the simplex method will be adequate. It is not recommended that these problems be solved by hand.

Step 3 is to formulate the linear programming model in the standard form. In our problem the decision variables are the number of acres of each crop  $x_1$ ,  $x_2$ , and  $x_3$ . We want to maximize the total yield,  $y = 400x_1 + 200x_2 + 250x_3$ , over the set

$$\begin{aligned} 3.0x_1 + 1.0x_2 + 1.5x_3 &\leq 1,000 \\ 0.8x_1 + 0.2x_2 + 0.3x_3 &\leq 300 \\ x_1 + x_2 + x_3 &\leq 625 \end{aligned} \tag{3.27}$$

and  $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ .

Step 4 is to solve the problem. We used a computer implementation of the simplex method called LINDO, written by Linus Schrage. The results of step 4 are shown in Figure 3.15. The optimal solution is  $Z = 162,500$  at  $x_1 = 187.5$ ,  $x_2 = 437.5$ ,  $x_3 = 0$ . Since the slack variables for rows 2 and 4 are zero, the first and third constraints are binding. Since the slack variable for row 3 is equal to 62.5, the second constraint is not binding.

Step 5 is to answer the question. The question was how much of each crop to plant. The optimal solution is to plant 187.5 acres of corn, 437.5 acres of wheat, and no oats. This should yield \$162,500. The optimal crop mixture we found uses all 625 acres and all 1,000 acre-ft of irrigation water, but only 237.5 of the available 300 person-hours of labor per week. Thus, there will be 62.5 person-hours per week that may be devoted to other profitable activities, or to leisure.

We will begin our sensitivity analysis by considering the amount of water available for irrigation. This amount will vary as a result of rainfall and temperature, which determine the status of the farm's irrigation pond. It would also be possible to purchase additional irrigation water from a nearby farm. Figure 3.16 illustrates the effect of one additional acre-ft of irrigation water on our optimal solution. Now we can plant an additional half-acre of corn (a more profitable crop), and in fact we save a little bit of labor (0.3 person-hours per week). The net result is an additional \$100 in yield.

The \$100 is the shadow price for this resource (irrigation water). The farm should be willing to purchase additional irrigation water for up to \$100 per acre-ft. Alternatively, it should be unwilling to part with its own irrigation water for less than \$100 per acre-ft. In Fig. 3.15, the shadow prices for the three resources (water, labor, and land) are called *dual prices*. They appear next to the corresponding slack variables. An additional acre of land would also be worth \$100. Additional labor is worth \$0, since there is an excess.

The dollar amount per acre that each crop will yield varies with the weather and the market. Figure 3.17 shows the effect of a slightly higher yield for corn. This would not affect our decision variables  $x_1$ ,  $x_2$ , and  $x_3$  (the amount of each crop we should plant). Our yield increases, of course, by  $50x_1 = \$9,375$ . It is

```
MAX      400 X1 + 200 X2 + 250 X3
SUBJECT TO
    2)    3 X1 + X2 + 1.5 X3 <=    1000
    3)    0.8 X1 + 0.2 X2 + 0.3 X3 <=    300
    4)    X1 + X2 + X3 <=    625
END

LP OPTIMUM FOUND AT STEP      2

      OBJECTIVE FUNCTION VALUE

    1)    162500.000

      VARIABLE            VALUE            REDUCED COST
      X1                187.500000            .000000
      X2                437.500000            .000000
      X3                 .000000           - .000015

      ROW    SLACK OR SURPLUS    DUAL PRICES
      2)            .000000        100.000000
      3)            62.500000            .000000
      4)            .000000        99.999980

NO. ITERATIONS=      2
```

Figure 3.15: Optimal solution to the farm problem using the linear programming package LINDO.

also interesting to note the change in the shadow prices. Water is at more of a premium when corn is more valuable. (Although both the water and the land constraints are binding, it is the water constraint that keeps us from planting more corn in place of wheat.)

Figure 3.18 shows what happens if oats yield a bit more than expected. A very small change in this parameter has a very significant effect on our optimal decision. Now we plant oats instead of wheat. We also plant considerably less corn than before. Apparently, our model is quite sensitive to this parameter. This being the case, it seems appropriate to consider the sensitivity to this parameter in more depth.

Let  $c$  denote the yield (\$/acre) for oats, so that our objective function  $f(x) = 400x_1 + 200x_2 + cx_3$ . Note that the value of  $c$  does not affect the shape of the feasible region  $S$ . Several additional model runs were made varying  $c$ . For  $c \leq 250$  the optimal solution is at the corner point  $(187.5, 437.5, 0)$ , and for  $c > 250$  the optimal solution is at the adjacent corner point  $(41.66, 0, 583.33)$ .

```

MAX      400 X1 + 200 X2 + 250 X3
SUBJECT TO
      2)   3 X1 + X2 + 1.5 X3 <=   1001
      3)   0.8 X1 + 0.2 X2 + 0.3 X3 <=   300
      4)   X1 + X2 + X3 <=   625
END

```

LP OPTIMUM FOUND AT STEP        0

OBJECTIVE FUNCTION VALUE

1)        162600.000

VARIABLE	VALUE	REDUCED COST
X1	188.000000	.000000
X2	437.000000	.000000
X3	.000000	-.000015

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.000000	100.000000
3)	62.200000	.000000
4)	.000000	99.999980

NO. ITERATIONS=            0

Figure 3.16: Optimal solution to the farm problem with one extra acre-foot of water using the linear programming package LINDO.

Both points lie on the line formed by the intersection of the two planes

$$\begin{aligned}
 3.0x_1 + 1.0x_2 + 1.5x_3 &= 1,000 \\
 x_1 + x_2 + x_3 &= 625.
 \end{aligned}
 \tag{3.28}$$

Consider the gradient vector  $\nabla f = (400, 200, c)$  at any point along this line. For  $c < 250$  the gradient vector points toward the corner point with  $x_3 = 0$ . For  $c > 250$  the gradient vector points toward the corner point with  $x_2 = 0$ . As  $c$  increases, the gradient vector turns away from the former and toward the latter. At  $c = 250$  the gradient vector  $\nabla f$  is perpendicular to the line through these two points. For this value of  $c$ , any point along the line segment joining the two corner points will be optimal.

The practical ramification of our model's sensitivity to the parameter  $c$  is that we don't know whether we should plant oats or wheat. A small variation in yield would change our optimal decision. In light of the fact that the \$/acre yield varies considerably with the weather and the market, it might be best to



```
MAX      450 X1 + 200 X2 + 250 X3
SUBJECT TO
    2)    3 X1 + X2 + 1.5 X3 <=   1000
    3)    0.8 X1 + 0.2 X2 + 0.3 X3 <=   300
    4)    X1 + X2 + X3 <=   625
END

LP OPTIMUM FOUND AT STEP      0

      OBJECTIVE FUNCTION VALUE

    1)    171875.000

      VARIABLE            VALUE            REDUCED COST
      X1                187.500000            .000000
      X2                437.500000            .000000
      X3                 .000000            12.500000

      ROW    SLACK OR SURPLUS    DUAL PRICES
      2)            .000000        125.000000
      3)        62.500000            .000000
      4)            .000000        75.000000

NO. ITERATIONS=      0
```

Figure 3.17: Optimal solution to the farm problem with a higher yield for corn using the linear programming package LINDO.

present the farmer with more than one option. Any crop mixture of the form

$$\begin{aligned}x_1 &= 187.5t + 41.6\bar{6}(1 - t) \\x_2 &= 437.5t + 0(1 - t) \\x_3 &= 0t + 583.3\bar{3}(1 - t)\end{aligned}\tag{3.29}$$

for any  $0 \leq t \leq 1$  will use all the available land and irrigation water. There is too much uncertainty in the figures for the \$/acre yield to tell which option would produce the most profit.

Sometimes sensitivity analysis is performed in response to client feedback after the results of an initial study have been presented. Suppose that after the farmer has seen the results of our analysis, a new seed catalog arrives with an advertisement for a new variety of corn. This variety of corn is more expensive, but is supposed to require less irrigation. Figure 3.19 shows the results of a sensitivity run in which we assumed that the corn planted requires only 2.5 acre–feet of irrigation water per acre (instead of 3.0). The new seed corn yields

```

MAX      400 X1 + 200 X2 + 260 X3
SUBJECT TO
    2)    3 X1 + X2 + 1.5 X3 <=    1000
    3)    0.8 X1 + 0.2 X2 + 0.3 X3 <=    300
    4)    X1 + X2 + X3 <=    625
END

LP OPTIMUM FOUND AT STEP      1

      OBJECTIVE FUNCTION VALUE

    1)      168333.300

      VARIABLE                VALUE                REDUCED COST
      X1                   41.666670                .000000
      X2                      .000000               13.333340
      X3                   583.333300                .000000

      ROW    SLACK OR SURPLUS      DUAL PRICES
      2)           .000000           93.333340
      3)          91.666660           .000000
      4)           .000000          120.000000

NO. ITERATIONS=      1

```

Figure 3.18: Optimal solution to the farm problem with a higher yield for oats using the linear programming package LINDO.

an additional \$12,500, and in this case we will, of course, plant more corn than before. It is also interesting to note that in this case the shadow price for water has increased 33%.

Finally, suppose that the farmer wishes to investigate the addition of a new crop, barley. An acre of barley requires 1.5 acre-ft of water, 0.25 person-hours of labor, and is expected to yield \$200. We represent the new crop in our model by adding a decision variable  $x_4$  = acres of barley. Figure 3.20 shows our model run. The results are essentially unchanged from our base case. A mixture of corn and wheat remains our optimal solution, and it is easy to see why. Although barley and wheat yield the same, barley requires more water and more labor.

**Example 3.5.** A large construction firm is currently excavating at three sites. Meanwhile, they are also building at four additional sites, where they require fill dirt. The excavations at sites 1, 2, and 3 produce 150, 400, and 325 cubic yards of dirt per day. The building sites A, B, C, and D require 175, 125, 225, and 450 cubic yards of dirt per day. Additional fill dirt can also be obtained from

```
MAX      400 X1 + 200 X2 + 250 X3
SUBJECT TO
    2)    2.5 X1 + X2 + 1.5 X3 <=   1000
    3)    0.8 X1 + 0.2 X2 + 0.3 X3 <=   300
    4)    X1 + X2 + X3 <=   625
END

LP OPTIMUM FOUND AT STEP      1

      OBJECTIVE FUNCTION VALUE

    1)    175000.000

      VARIABLE            VALUE            REDUCED COST
      X1                250.000000            .000000
      X2                375.000000            .000000
      X3                  .000000           16.666680

      ROW    SLACK OR SURPLUS    DUAL PRICES
      2)            .000000           133.333300
      3)           25.000000            .000000
      4)            .000000           66.666660

NO. ITERATIONS=      1
```

Figure 3.19: Optimal solution to the farm problem with a lower water requirement for corn using the linear programming package LINDO.

site 4 at a cost of 5 dollars per cubic yard. The cost of shipping fill dirt is about 20 dollars per mile for one truckload, and a truck carries 10 cubic yards of dirt. Table 3.3 gives the distance between sites in miles. Find the transportation plan that minimizes the cost to the company.

We will use the five-step method. The results of step 1 are shown in Figure 3.21. For example, we can ship no more than 150 cubic yards of dirt per day from site 1, and we must ship at least 175 cubic yards of dirt to site A. Since it costs \$20 per mile to ship 10 cubic yards of dirt, the cost of shipping one cubic yard of dirt from site  $i = 1, 2$ , or 3 to site  $j = A, B, C$ , or D is \$2 per mile. There is an additional cost of \$5 per cubic yard if we ship from site  $i = 4$ . We assume that there is no limit to the amount of dirt we can obtain from site 4, if we are willing to pay the price.

Step 2 is to select the modeling approach. We will model this problem as a linear programming problem, and we will solve using a spreadsheet. Most spreadsheets include an equation solver or optimizer that implements the sim-

```

MAX      400 X1 + 200 X2 + 250 X3 + 200 X4
SUBJECT TO
    2)    3 X1 + X2 + 1.5 X3 + 1.5 X4 <=    1000
    3)    0.8 X1 + 0.2 X2 + 0.3 X3 + 0.25 X4 <=    300
    4)    X1 + X2 + X3 + X4 <=    625
END

LP OPTIMUM FOUND AT STEP      1

      OBJECTIVE FUNCTION VALUE

    1)    162500.000

      VARIABLE            VALUE            REDUCED COST
      X1                187.500000            .000000
      X2                437.500000            .000000
      X3                 .000000           - .000015
      X4                 .000000           49.999980

      ROW    SLACK OR SURPLUS    DUAL PRICES
      2)            .000000         100.000000
      3)           62.500000            .000000
      4)            .000000           99.999980

NO. ITERATIONS=      1

```

Figure 3.20: Optimal solution to the farm problem with the addition of a new crop, barley, using the linear programming package LINDO.

plex method. If the linear programming problem is not in the standard form defined by Eq. (3.23) or (3.26), a simple transformation usually suffices. For example, to minimize the objective function  $y = f(x_1, \dots, x_n)$ , we can maximize  $-y$ . Most implementations perform these transformations automatically, allowing for a more natural problem formulation. There is also a special streamlined version of the simplex method specially formulated for transportation problems. This transportation simplex method is much more efficient for large-scale problems. For the present problem, the ordinary simplex method will suffice.

Step 3 is to formulate the model. In this problem the decision variables are the number of cubic feet of dirt  $x_{ij}$  we want to ship from site  $i$  to site  $j$ , and

Excavation site	Site to receive fill dirt			
	A	B	C	D
1	5	2	6	10
2	4	5	7	5
3	7	6	4	4
4	9	10	6	2

Table 3.3: Mileage data for the dirt problem of Example 3.5: Miles between building sites.

**Variables:**  $x_{ij}$  = dirt shipped from site  $i$  to site  $j$  (cubic yards)  
 $s_i$  = dirt shipped from site  $i$  (cubic yards)  
 $r_j$  = dirt shipped to site  $j$  (cubic yards)  
 $c_{ij}$  = cost of shipping dirt from site  $i$  to site  $j$  (\$/cubic yard)  
 $d_{ij}$  = distance from site  $i$  to site  $j$  (miles)  
 $C$  = total shipping cost (\$)

**Assumptions:**  $s_1 = x_{1A} + x_{1B} + x_{1C} + x_{1D}$   
 $s_2 = x_{2A} + x_{2B} + x_{2C} + x_{2D}$   
 $s_3 = x_{3A} + x_{3B} + x_{3C} + x_{3D}$   
 $s_4 = x_{4A} + x_{4B} + x_{4C} + x_{4D}$   
 $r_A = x_{1A} + x_{2A} + x_{3A} + x_{4A}$   
 $r_B = x_{1B} + x_{2B} + x_{3B} + x_{4B}$   
 $r_C = x_{1C} + x_{2C} + x_{3C} + x_{4C}$   
 $r_D = x_{1D} + x_{2D} + x_{3D} + x_{4D}$   
 $s_1 \leq 150, s_2 \leq 400, s_3 \leq 325$   
 $r_A \geq 175, r_B \geq 125, r_C \geq 225, r_D \geq 450$   
 $c_{ij} = 2d_{ij}$  if  $i = 1, 2, 3$  or  $(2d_{ij} + 5)$  if  $i = 4$   
where  $d_{ij}$  is given in Table 3.3  
 $C = c_{1A}x_{1A} + c_{1B}x_{1B} + c_{1C}x_{1C} + c_{1D}x_{1D}$   
 $+ c_{2A}x_{2A} + c_{2B}x_{2B} + c_{2C}x_{2C} + c_{2D}x_{2D}$   
 $+ c_{3A}x_{3A} + c_{3B}x_{3B} + c_{3C}x_{3C} + c_{3D}x_{3D}$   
 $+ c_{4A}x_{4A} + c_{4B}x_{4B} + c_{4C}x_{4C} + c_{4D}x_{4D}$   
 $x_{ij} \geq 0$  for  $i = 1, 2, 3, 4$  and  $j = A, B, C, D$

**Objective:** Minimize  $C$

Figure 3.21: Results of step 1 of the dirt problem.

A	A	B	C	D	E	F	G
1	costs						
2	site	A	B	C	D		
3	1	10	4	12	20		
4	2	8	10	14	10		
5	3	14	12	8	8		
6	4	23	25	17	9		
7	solution						
8	site	A	B	C	D	shipped	available
9	1	0	0	0	0	0	150
10	2	0	0	0	0	0	400
11	3	0	0	0	0	0	325
12	4	0	0	0	0	0	
13	received	0	0	0	0		
14	needed	175	125	225	450		
15	total cost	0					

Figure 3.22: Spreadsheet formulation for the dirt problem.

our objective is to minimize the total cost  $y = C$  where

$$\begin{aligned}
 y = & 10x_{1A} + 4x_{1B} + 12x_{1C} + 20x_{1D} \\
 & + 8x_{2A} + 10x_{2B} + 14x_{2C} + 10x_{2D} \\
 & + 14x_{3A} + 12x_{3B} + 8x_{3C} + 8x_{3D} \\
 & + 23x_{4A} + 25x_{4B} + 17x_{4C} + 9x_{4D}
 \end{aligned} \tag{3.30}$$

subject to the constraints

$$\begin{aligned}
 x_{1A} + x_{1B} + x_{1C} + x_{1D} & \leq 150 \\
 x_{2A} + x_{2B} + x_{2C} + x_{2D} & \leq 400 \\
 x_{3A} + x_{3B} + x_{3C} + x_{3D} & \leq 325 \\
 x_{1A} + x_{2A} + x_{3A} + x_{4A} & \geq 175 \\
 x_{1B} + x_{2B} + x_{3B} + x_{4B} & \geq 125 \\
 x_{1C} + x_{2C} + x_{3C} + x_{4C} & \geq 225 \\
 x_{1D} + x_{2D} + x_{3D} + x_{4D} & \geq 450
 \end{aligned} \tag{3.31}$$

and  $x_{ij} \geq 0$  for  $i = 1, 2, 3, 4$  and  $j = A, B, C, D$ .

Figure 3.22 shows a spreadsheet set up for this problem. Most of the cells contain data. Cells F9 to F12, B13 to E13, and B15 contain formulas. For example, we input  $F9 = B9 + C9 + D9 + E9$  and  $B13 = B9 + B10 + B11 + B12$  while  $B15 = B3 * B9 + C3 * C9 + \dots + D6 * D12 + E6 * E12$ . Then we must tell the spreadsheet which cell contains the objective, which cells contain decision variables, and we must specify the constraints. The details vary from one spreadsheet to another; consult the manual or on-line help facility for your spreadsheet if you are not sure how to proceed.

A	A	B	C	D	E	F	G
1	costs						
2	site	A	B	C	D		
3	1	10	4	12	20		
4	2	8	10	14	10		
5	3	14	12	8	8		
6	4	23	25	17	9		
7	solution						
8	site	A	B	C	D	shipped	available
9	1	0	125	0	0	125	150
10	2	175	0	0	0	175	400
11	3	0	0	225	100	325	325
12	4	0	0	0	350	350	
13	received	175	125	225	450		
14	needed	175	125	225	450		
15	total cost	7650					

Figure 3.23: Spreadsheet solution for the dirt problem showing the optimal shipping plan.

Step 4 is to solve the problem. Figure 3.23 shows the spreadsheet solution for this problem obtained using the Quattro Pro optimizer. Notice that all of the receiving constraints are binding, but not all the shipping constraints are binding. Step 5 is to answer the question. The optimal solution is to ship 125 cubic yards of dirt per day from site 1 to site B, and 175 cubic yards from site 2 to site A. Site 3 sends 225 cubic yards to site C and another 100 cubic yards to site D. The remaining 350 cubic yards of dirt needed at site D come from site 4. The added cost of purchasing dirt is offset by the fact that site 4 is close to site D. The total cost of this transportation plan is \$7,650 per day. According to this shipping plan, we will not use all the dirt excavated from sites 1 and 2, so we will need to make other arrangements to dispose of the extra dirt.

We begin our sensitivity analysis by examining some of the reports that can be automatically generated by the spreadsheet optimizer. Figure 3.24 shows a sensitivity report on the constraints in our problem. Notice that the constraints on site 1 and site 2 are nonbinding, with a slack of 25 and 225 respectively. This means that there are 25 cubic yards of dirt being excavated from site 1, and 225 cubic yards from site 2, that are not being shipped under the optimal transportation plan. The dual values listed in Figure 3.24 are the shadow prices. The shadow prices for sites 1 and 2 are zero, since these constraints are nonbinding. The shadow price for site 3 is  $-1$  dollar, meaning that the total cost would increase by  $-1$  dollar if this constraint were increased by 1 cubic yard of dirt. One of the principal advantages of the spreadsheet implementation is the ease with which we can perform sensitivity analysis. We simply change the value in cell F12 from 325 to 326 and reoptimize. The resulting optimal solution (not shown) changes E11 to 101, E12 to 349 and B15 to 7649. All the other decision

Cell	Value	Constraint	Binding?	Slack	Dual Value	Increment	Decrement
site 1	125	$\leq 150$	No	25	0	Infinite	25
site 2	175	$\leq 400$	No	225	0	Infinite	225
site 3	325	$\leq 325$	Yes	0	-1	350	100
site A	175	$\geq 175$	Yes	0	8	225	175
site B	125	$\geq 125$	Yes	0	4	25	125
site C	225	$\geq 225$	Yes	0	9	100	225
site D	450	$\geq 450$	Yes	0	9	Infinite	350

Figure 3.24: Spreadsheet sensitivity report for the dirt problem showing sensitivity to changes in the amount of dirt available or the amount of dirt needed.

variables are unchanged. In other words, we ship one more cubic yard of dirt from site 3 to site D and we save \$1 per day.

The shadow prices for site A, B, C, D are all positive; e.g., the shadow price for site C is 9 dollars. If we require an additional 10 cubic yards at site C, then this will cost an additional \$90. To check this, we change cell D14 from 225 to 235 and reoptimize. The resulting optimal solution is shown in Figure 3.25. Note that the total cost increases by \$90 from \$7,650 to \$7,740. Now site C received 235 cubic yards from site 3, while site D receives 90 cubic yards from site 3 and 360 cubic yards from site 4. The other shipments remain the same.

The columns labelled “increment” and “decrement” show the amount of increase or decrease in the individual constraint amounts for which the shadow prices remain valid. If we increase the amount required at site C from 225 to 325 or less, then the total cost will increase by \$9 per cubic yard. If we go beyond this amount, then the nature of the solution will change. In this case it is easy to see why the nature of the solution changes if we need more than 325 cubic yards of dirt per day at site C. Right now we are getting all of the dirt for site C from site 3, but if we need more than 325 cubic yards at site C, then we must haul dirt from another site at a higher cost.

Figure 3.26 shows the spreadsheet report on sensitivity to objective function coefficients. The increment and decrement amounts indicate the amount of increase or decrease we can make in one of these cost-per-mile numbers without changing the optimal solution. For example, the current cost of shipping one cubic yard from site 1 to site B is 4 dollars. Lowering the cost by less than 4 dollars or raising the price by less than 6 dollars will not change the optimal shipping plan. The total cost will change, since we are shipping 125 cubic yards per day from site 1 to site B. Consider the geometry of the simplex method. When we change an objective function coefficient, we are not changing the feasible region at all. The optimal solution will only change if we alter the objective function enough so that the current corner point solution is no longer optimal. Then the optimal solution jumps to another corner point. We can



A	A	B	C	D	E	F	G
1	costs						
2	site	A	B	C	D		
3	1	10	4	12	20		
4	2	8	10	14	10		
5	3	14	12	8	8		
6	4	23	25	17	9		
7	solution						
8	site	A	B	C	D	shipped	available
9	1	0	125	0	0	125	150
10	2	175	0	0	0	175	400
11	3	0	0	235	90	325	325
12	4	0	0	0	360	360	
13	received	175	125	235	450		
14	needed	175	125	235	450		
15	total cost	7740					

Figure 3.25: Spreadsheet solution for the dirt problem with 10 additional cubic yards of dirt needed at site C.

Variable Cells	Starting	Final	Gradient	Increment	Decrement
x1A	0	0	10	Infinite	2
x2A	0	175	8	2	8
x3A	0	0	14	Infinite	7
x4A	0	0	23	Infinite	15
x1B	0	125	4	6	4
x2B	0	0	10	Infinite	6
x3B	0	0	12	Infinite	9
x4B	0	0	25	Infinite	21
x1C	0	0	12	Infinite	3
x2C	0	0	14	Infinite	5
x3C	0	225	8	3	9
x4C	0	0	17	Infinite	8
x1D	0	0	20	Infinite	11
x2D	0	0	10	Infinite	1
x3D	0	100	8	1	3
x4D	0	350	9	1	1

Figure 3.26: Spreadsheet sensitivity report for the dirt problem showing sensitivity to changes in the shipping costs.

A	A	B	C	D	E	F	G
1	costs						
2	site	A	B	C	D		
3	1	10	4	12	20		
4	2	8	10	14	10		
5	3	14	12	8	8		
6	4	23	25	17	9		
7	solution						
8	site	A	B	C	D	shipped	available
9	1	25	125	0	0	150	150
10	2	150	0	0	250	400	400
11	3	0	0	225	100	325	325
12	4	0	0	0	100	100	
13	received	175	125	225	450		
14	needed	175	125	225	450		
15	total cost	7950					

Figure 3.27: Spreadsheet solution for the dirt problem if we ship all of the excavated dirt.

verify this by changing the value in cell C3 and reoptimizing. If we insert any value between 0 and 10 into C3, we get the same shipping plan as in Figure 3.23. If we insert 11 into C3, then the shipping plan changes so that site B receives 125 cubic yards of dirt per day from site 2 and none from site 1.

Finally, we consider the robustness of our model. Our optimal solution in Figure 3.23 indicates that we should not ship all of the excavated dirt to another construction site. This leaves the company with a problem. Where do we put all the extra dirt? The company will need to put the extra dirt somewhere, and this will result in some additional cost. We do not have information about these costs, but we can explore some options. Suppose that we ship all of the excavated dirt to another construction site to be used as fill dirt. We know this is not the optimal solution, but how much more would it cost? We can find out by making a small change in our model. In step 1 we assumed that  $s_1 \leq 150$ ,  $s_2 \leq 400$ ,  $s_3 \leq 325$ . Assume now that  $s_1 = 150$ ,  $s_2 = 400$ ,  $s_3 = 325$ . The first three inequality constraints in Eq. (3.31) are replaced by

$$\begin{aligned}
 x_{1A} + x_{1B} + x_{1C} + x_{1D} &= 150 \\
 x_{2A} + x_{2B} + x_{2C} + x_{2D} &= 400 \\
 x_{3A} + x_{3B} + x_{3C} + x_{3D} &= 325
 \end{aligned} \tag{3.32}$$

and the rest of our linear programming problem remains the same. After changing these three constraints on our spreadsheet, we reoptimize to obtain the results shown in Figure 3.27. Sites B and C receive the same shipments as before, but now site A gets 25 cubic yards of fill dirt per day from site 1 and site D gets 250 cubic yards of fill dirt per day from site 2. According to this transportation plan, we only need to purchase 100 cubic yards from site 4. The new plan gets

rid of the extra 250 cubic yards of dirt and the additional cost is only \$300 per day, a little more than one dollar per cubic yard. We do not really know which of these two plans is best. If the company needs to haul the excavated dirt away from the site, then they may favor the alternate plan. If they can use the dirt at or near the excavation site, they may prefer the original plan. We should present both plans to the company management and let them choose.

## 3.4 Discrete Optimization

All of the models used so far in this book have been based on continuous variables. In many real-world problems we must deal with variables that are discrete, like the integers. Discrete mathematics was once considered an arcane field with little or no practical applications. With the invention of digital computers, discrete mathematics has become extremely important. Discrete optimization is useful for problems in scheduling, inventory, investment, transportation, manufacturing, ecology, and computer science. Discrete models play an important role in the remainder of the text, and the relationship between continuous and discrete variables is one of the major issues in modeling.

In some cases a discrete optimization problem can be solved by simply enumerating the possibilities. In other cases we can employ continuous methods and then round off to the nearest integer. Nonlinear programming problems usually become much harder to solve when continuous decision variables are replaced by discrete variables. The feasible region becomes more complex when continuity is lost, and is often represented by a graph or tree structure. Effective solution algorithms have been developed for some classes of problems, and the improvement of these algorithms is a very active area of research. However, as in the continuous case, there are still no universally effective methods for solving discrete optimization problems.

In this section we will concentrate on one type of discrete optimization problem called *integer programming*. Integer programming is the discrete analog of the linear programming model considered in the preceding section. Aside from being the most commonly applied discrete optimization algorithm, the parallel with linear programming will facilitate comparison between discrete and continuous models. An added bonus is that most linear programming implementations also solve integer programming problems, so that we can concentrate our attention on the model rather than learning a new software package.

**Example 3.6.** Reconsider the family farm problem of Example 3.4. The family has 625 acres available for planting. There are 5 plots of 120 acres each and another plot of 25 acres. The family wants to plant each plot with only one crop: corn, wheat, or oats. As before, 1,000 acre-ft of water will be available for irrigation, and the farmers will be able to devote 300 hours of labor per week. Additional data are presented in Table 3.2. Find the crop that should be planted in each plot for maximum profit.

We will use the five-step method. The results of step 1 are shown in Figure

3.28. Step 2 is to select the modeling approach. We will model this problem as an integer programming problem.

**Variables:**

- $x_1$  = number of 120 acre plots of corn planted
- $x_2$  = number of 120-acre plots of wheat planted
- $x_3$  = number of 120-acre plots of oats planted
- $x_4$  = number of 25-acre plots of corn planted
- $x_5$  = number of 25-acre plots of wheat planted
- $x_6$  = number of 25-acre plots of oats planted
- $w$  = irrigation required (acre-ft)
- $l$  = labor required (person-hrs/wk)
- $t$  = total acreage planted
- $y$  = total yield (\$)

**Assumptions:**

- $w = 120(3.0x_1 + 1.0x_2 + 1.5x_3)$   
 $\quad + 25(3.0x_4 + 1.0x_5 + 1.5x_6)$
- $l = 120(0.8x_1 + 0.2x_2 + 0.3x_3)$   
 $\quad + 25(0.8x_4 + 0.2x_5 + 0.3x_6)$
- $t = 120(x_1 + x_2 + x_3) + 25(x_4 + x_5 + x_6)$
- $y = 120(400x_1 + 200x_2 + 250x_3)$   
 $\quad + 25(400x_4 + 200x_5 + 250x_6)$
- $w \leq 1,000$
- $l \leq 300$
- $t \leq 625$
- $x_1 + x_2 + x_3 \leq 5$
- $x_4 + x_5 + x_6 \leq 1$
- $x_1, \dots, x_6$  are nonnegative integers

**Objective:** Maximize  $y$

Figure 3.28: Results of step 1 of the modified farm problem.

An integer programming (IP) problem is a linear programming (LP) problem in which the decision variables are further constrained to take integer values. Both the objective function and the constraints must be linear. The most commonly used method for solving an IP is the method of branch-and-bound. This method involves branching through repeated solutions of LPs chosen to bound the solution of the IP. If we remove the constraint that the decision variables in a given IP take only integer values, we obtain what is known as the LP relaxation. Since the feasible region of the LP relaxation is larger than the feasible region of the corresponding IP, any optimal solution of the LP relaxation for which all of the decision variables take integer values is also an optimal solution of the IP. If some decision variable is not an integer, we branch to create

two additional LP relaxations. For example, if the optimal solution to the LP relaxation yields  $x_1 = 11/3$ , then we consider the original problem together with the additional constraint  $x \leq 3$ , and we also consider the original problem together with the additional constraint  $x \geq 4$ . Any integer solution must satisfy one of these additional constraints. By continuing to branch any time we find an optimal solution with a fractional decision variable, we obtain a binary tree of LP relaxations. If one of the new LPs has an optimal solution with all integer decision variables, this becomes a candidate for the solution to the original IP. Since the new LP has a smaller feasible region than the original, this integer solution also gives a useful lower bound on the optimal solution of the original IP. By systematically branching through the tree, and using these solution bounds, we will eventually solve the original IP. Because the branch-and-bound method requires many iterations of the simplex method, integer programming problems generally take much longer to solve than linear programming problems of the same size.

Step 3 is to formulate the problem. In our problem the decision variables are the number of 120-acre plots and the number of 25-acre plots to plant with corn, wheat, or oats. Notice that the variables  $x_4$ ,  $x_5$ , and  $x_6$  are binary decision variables. They can only take the values 0 or 1. Our integer programming problem in standard form is to maximize the total yield  $y = 48000x_1 + 24000x_2 + 30000x_3 + 10000x_4 + 5000x_5 + 6250x_6$  over the set

$$\begin{array}{rcccccccl} 375x_1 + 125x_2 + 187.5x_3 + 75x_4 + 25x_5 + 37.5x_6 & \leq & 1000 & & & & & \\ 100x_1 + 25x_2 + 37.5x_3 + 20x_4 + 5x_5 + 7.5x_6 & \leq & 300 & & & & & \\ x_1 + x_2 + x_3 & \leq & 5 & & & & & (3.33) \\ & & & x_4 + x_5 + x_6 & \leq & 1 & & \end{array}$$

where  $x_1, \dots, x_6$  are nonnegative integers.

Step 4 is to solve the problem. Figure 3.29 shows the solution of our integer programming problem using the popular linear programming package LINDO. The GIN 6 command specifies that the first six decision variable are nonnegative integers. Everything else in the problem specification is the same for the IP and the LP. The optimal solution is  $y = 156,250$  which occurs when  $x_3 = 5$ ,  $x_6 = 1$ , and the other decision variables are all zero. At the optimal solution the first two constraints are nonbinding, and the last two are binding.

Step 5 is to answer the question. If the family does not wish to split up individual plots (plan B), then the best plan is to plant oats in every plot. This results in an expected total yield of \$156,250 for the season. This is about 4% less than the projected total yield of \$162,500 if we allow more than one crop per plot (plan A, the optimal solution found in Example 3.4). Plan A uses all of the acreage available, all of the irrigation water available, and all but 62.5 of the 300 person-hours of labor available each week. Plan B uses all of the acreage available, but only 975 of the 1000 available acre-feet of irrigation water, and

```

MAX  48000 X1 + 24000 X2 + 30000 X3 + 10000 X4 + 5000 X5 + 6250 X6
SUBJECT TO
    2)   375 X1 + 125 X2 + 187.5 X3 + 75 X4 + 25 X5 + 37.5 X6 <= 1000
    3)   100 X1 + 25 X2 + 37.5 X3 + 20 X4 + 5 X5 + 7.5 X6 <= 300
    4)    X1 + X2 + X3 <= 5
    5)    X4 + X5 + X6 <= 1
END
GIN      6

```

## OBJECTIVE FUNCTION VALUE

```

1)      156250.0

```

VARIABLE	VALUE	REDUCED COST
X1	0.000000	-48000.000000
X2	0.000000	-24000.000000
X3	5.000000	-30000.000000
X4	0.000000	-10000.000000
X5	0.000000	-5000.000000
X6	1.000000	-6250.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	25.000000	0.000000
3)	105.000000	0.000000
4)	0.000000	0.000000
5)	0.000000	0.000000

```

NO. ITERATIONS=      101

```

Figure 3.29: Optimal solution to the modified farm problem using the linear programming package LINDO.

only 195 of the available 300 person-hours of labor available each week. We leave it to the family to decide which plan is best.

Sensitivity analysis can be very time-consuming for integer programming problems, because IPs take so much longer to solve than LPs. There are no shadow prices to guide us, because the value of the objective function at the optimal solution does not change smoothly as the constraints vary. Integer solutions will not usually lie exactly on the constraint boundaries, so the optimum may also be sensitive to small changes in nonbinding constraints. We begin by considering the amount of irrigation water available. Suppose that an additional 100 acre-feet of water is available. This changes our IP in Eq. (3.33) by substituting 1100 for 1000 in the first constraint. Figure 3.30 shows the LINDO solution to this IP. Now we plant one 120-acre plot of corn, one 120-acre plot

of wheat, and we plant oats everywhere else. Our optimal solution is quite sensitive to the amount of irrigation water available, even though this constraint was not binding in our original IP solution. The new plan yields an additional \$12,000 in expected revenue.

```
MAX  48000 X1 + 24000 X2 + 30000 X3 + 10000 X4 + 5000 X5 + 6250 X6
SUBJECT TO
    2)   375 X1 + 125 X2 + 187.5 X3 + 75 X4 + 25 X5 + 37.5 X6 <= 1100
    3)   100 X1 + 25 X2 + 37.5 X3 + 20 X4 + 5 X5 + 7.5 X6 <= 300
    4)    X1 + X2 + X3 <= 5
    5)    X4 + X5 + X6 <= 1
END
GIN      6
```

OBJECTIVE FUNCTION VALUE

1)        168250.0

VARIABLE	VALUE	REDUCED COST
X1	1.000000	-48000.000000
X2	1.000000	-24000.000000
X3	3.000000	-30000.000000
X4	0.000000	-10000.000000
X5	0.000000	-5000.000000
X6	1.000000	-6250.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	0.000000
3)	55.000000	0.000000
4)	0.000000	0.000000
5)	0.000000	0.000000

NO. ITERATIONS=        62

Figure 3.30: Optimal solution to the modified farm problem with an additional 100 acre–feet of water available.

The optimal solution in Figure 3.29 shows that there are 25 acre–feet of unused irrigation water. If we lower the amount of water available, we will not change the optimal solution unless we go below 975 acre–feet. This is because the optimal solution will remain feasible, and since we are shrinking the feasible region, it must remain optimal as well. Figure 3.31 shows what happens if we have only 950 acre–feet of water available. The optimal IP solution is to plant one 120–acre plot of corn, four 120–acre plots of wheat, and to plant corn instead of oats in the smaller 25 acre plot. We use all the water and all the land, but we have 80 person–hours of labor per week to spare. The expected

```
MAX  48000 X1 + 24000 X2 + 30000 X3 + 10000 X4 + 5000 X5 + 6250 X6
SUBJECT TO
  2)   375 X1 + 125 X2 + 187.5 X3 + 75 X4 + 25 X5 + 37.5 X6 <= 950
  3)   100 X1 + 25 X2 + 37.5 X3 + 20 X4 + 5 X5 + 7.5 X6 <= 300
  4)    X1 + X2 + X3 <= 5
  5)    X4 + X5 + X6 <= 1
END GIN      6
```

OBJECTIVE FUNCTION VALUE

1)        154000.0

VARIABLE	VALUE	REDUCED COST
X1	1.000000	-48000.000000
X2	4.000000	-24000.000000
X3	0.000000	-30000.000000
X4	1.000000	-10000.000000
X5	0.000000	-5000.000000
X6	0.000000	-6250.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	0.000000
3)	80.000000	0.000000
4)	0.000000	0.000000
5)	0.000000	0.000000

NO. ITERATIONS=        4

Figure 3.31: Optimal solution to the modified farm problem with 50 acre–feet less water available.

total yield is \$154,000, which is only \$2,250 less than before. This illustrates the unpredictable nature of IP solutions. For a 5% decrease in the amount of available irrigation water, instead of planting oats everywhere, we should not plant oats anywhere.

The main robustness issue is the relation between discrete and continuous optimization. We have already seen that devoting large plots to a single crop dramatically changes the optimal planting scheme. Now we will return to the original farm problem and examine the effect of changing the minimum plot size. Table 3.4 shows the results of several LINDO runs with different minimum plot sizes. For example, when the minimum plot size is 2 acres, we maximize



Minimum Plot Size (acres)	Corn (acres)	Wheat (acres)	Oats (acres)	Yield (dollars)
0	187.5	437.5	0	162,500
1	42	1	582	162,500
2	188	436	0	162,400
5	45	10	570	162,500
10	190	430	0	162,000
20	60	40	520	162,000
50	200	400	0	160,000
100	200	400	0	160,000
125	125	250	250	162,500
150	150	300	150	157,000
200	200	400	0	160,000
250	250	250	0	150,000
300	0	0	600	150,000
500	0	0	500	125,000

Table 3.4: Comparison of optimal planting schemes for the farm problem for different minimum plot sizes.

total yield  $y = 800x_1 + 400x_2 + 500x_3$  over the set

$$\begin{aligned}
 6.0x_1 + 2.0x_2 + 3.0x_3 &\leq 1000 \\
 1.6x_1 + 0.4x_2 + 0.6x_3 &\leq 300 \\
 x_1 + x_2 + x_3 &\leq 312
 \end{aligned} \tag{3.34}$$

and  $x_1, x_2, x_3$  are nonnegative integers representing the number of 2-acre plots of corn, wheat, and oats to plant. Figure 3.32 shows the optimal solution for a minimum plot size of 2 acres using LINDO. The optimal solution is to plant 94 plots of corn (for a total of  $2 \times 94 = 188$  acres), 218 plots of wheat (436 acres), and no oats. For this model excursion we do not plant in any leftover acreage, so that when the minimum plot size is 2 acres, we will leave 1 acre unplanted. The optimal solution in Table 3.4 changes significantly as the minimum plot size increases. Recall that, in the continuous optimization problem (LP), there are two optimal corner point solutions  $(x_1, x_2, x_3) = (187.5, 437.5, 0)$  and  $(x_1, x_2, x_3) = (41.6\bar{6}, 0, 583.\bar{33})$  at which we achieve the optimal yield of  $y = 162,500$ . For small minimum plot sizes, the best planting scheme jumps between discrete approximations to these two solutions. For minimum plot sizes of 2, 10, and 50 acres, the solution is similar to the original corn-and-wheat planting scheme. For minimum plot sizes of 1, 5, and 20 acres, the solution is similar to the alternate corn-and-oats planting scheme. When the minimum plot size gets larger, both the optimal planting scheme and the expected total yield vary considerably. A larger minimum plot size may result in either a larger or smaller yield. For example, when we use a plot size of 125 acres, we obtain a scheme which matches the original \$162,500 expected total yield. Note that

```

MAX      800 X1 + 400 X2 + 500 X3
SUBJECT TO
      2)   6 X1 + 2 X2 + 3 X3 <=   1000
      3)   1.6 X1 + 0.4 X2 + 0.6 X3 <=   300
      4)   X1 + X2 + X3 <=   312

END
GIN      3

      OBJECTIVE FUNCTION VALUE

      1)      162400.0

VARIABLE      VALUE      REDUCED COST
X1             94.000000      -800.000000
X2            218.000000      -400.000000
X3              0.000000      -500.000000

      ROW    SLACK OR SURPLUS      DUAL PRICES
      2)           0.000000           0.000000
      3)        62.399998           0.000000
      4)           0.000000           0.000000

NO. ITERATIONS=      2

```

Figure 3.32: Optimal solution to the farm problem with a minimum plot size of 2 acres.

the 625 acres divide evenly into 125-acre plots, and that the optimal solution  $(x_1, x_2, x_3) = (125, 250, 250)$  lies on the line segment of Eq. (3.29) giving the optimal solutions for the original LP. Most larger plot sizes give a considerably smaller expected yield, however, and the planting schemes vary widely.

Consider the geometry of the LP and its various IP approximations. When we choose a minimum plot size, we are restricting the feasible region to a lattice of integer points. If the plot size is small, then there are many such points, and they cover most of the feasible region, in the sense that there is a lattice point near every feasible point. Since the objective function is continuous, we can find a lattice point solution for the IP that is nearly optimal for the original LP. But when the lattice points are widely spaced, discretization will often change the optimal solution significantly, since there may be no lattice point near the LP optimum. Generally speaking, discretization makes little difference as long as the jumps between lattice points (in our example, the minimum plot size) represent a small percent change in the decision variables. Otherwise the IP solution may be very different than the solution to the corresponding LP relaxation.

Route	From	To	Miles	Cubic Yards
1	1	B	2	125
2	2	A	4	175
3	3	C	4	225
4	3	D	4	100
5	4	D	4	350

Table 3.5: Route data for the truck problem of Example 3.7.

**Example 3.7.** Reconsider the dirt problem of Example 3.5. The company has already determined the optimal shipping plan, assuming that dump trucks with a 10 cubic yard capacity are being used. The company also has 3 larger trucks available that can carry 20-cubic-yard loads. It may be possible to use these trucks to save some money on shipping. The 10-cubic-yard trucks take an average of 25 minutes to load, 5 minutes to unload, and travel at an average of 20 miles per hour. The cost of operating these trucks is \$20 per mile per load. The larger 20-cubic-yard trucks take 35 minutes to load, 5 minutes to unload, and travel at an average of 20 miles per hour. The cost of operating these trucks is \$30 per mile per load. How should these trucks be assigned in order to maximize our savings on shipping costs?

We will use the five-step method. Step 1 is to ask a question. The question is which trucks to assign to which routes. The optimal shipping routes found in Example 3.5 are shown in Table 3.5. There are 5 routes of varying length, and the amount of dirt shipped also varies. We will assume that either 10-cubic-yard trucks or 20-cubic-yard trucks, but not both, are used on each route. Since the larger trucks carry twice as much dirt at less than twice the cost, we want to assign these trucks to the routes where they will save us the most money. We need to compute the number of trucks needed and the total cost of shipping for each route and each type of truck, and then compute the potential savings for using larger trucks on each route. The optimal shipping plan calls for 125 cubic yards of dirt to be shipped from site 1 to site B, a distance of 2 miles. It takes the smaller trucks 20 minutes to load, 5 minutes to unload, and 6 minutes of travel time at 20 miles per hour, for a total of 31 minutes per load. We assume an 8-hour workday, so that each truck can be assigned no more than 480 minutes of work per day. It will take 13 loads to move the 125 cubic yards of dirt, for a total of  $13 \times 31 = 403$  minutes of work, so one small truck is sufficient to service route 1. The cost of servicing route 1 with one small truck is

$$13 \text{ loads} \times \frac{2 \text{ miles}}{\text{load}} \times \frac{20 \text{ dollars}}{\text{mile}} = 520 \text{ dollars.}$$

It takes the larger trucks 30 minutes to load, 5 minutes to unload, and 6 minutes of travel time on route 1, for a total of 41 minutes per load. It will take 7 loads to move the 125 cubic yards of dirt, for a total of  $7 \times 41 = 287$  minutes of work, so one large truck is enough to service route 1. The cost of servicing route 1

**Variables:**  $x_i = 1$  if we use large trucks on route  $i$   
 $x_i = 0$  if we use small trucks on route  $i$   
 $T$  = total number of large trucks used  
 $y$  = total cost savings (\$)

**Assumptions:**  $T = 1x_1 + 1x_2 + 2x_3 + 1x_4 + 2x_5$   
 $y = 100x_1 + 360x_2 + 400x_3 + 200x_4 + 640x_5$   
 $T \leq 3$

**Objective:** Maximize  $y$

Figure 3.33: Results of step 1 of the truck problem.

A	A	B	C	D	E	F
1		possible			actual	
2	route	trucks	savings	decision	trucks	savings
3	1	1	100	0	0	0
4	2	1	360	0	0	0
5	3	2	400	0	0	0
6	4	1	200	0	0	0
7	5	2	640	0	0	0
8	available	3		totals	0	0

Figure 3.34: Spreadsheet formulation for the truck problem.

with one large truck is

$$7 \text{ loads} \times \frac{2 \text{ miles}}{\text{load}} \times \frac{30 \text{ dollars}}{\text{mile}} = 420 \text{ dollars.}$$

The potential savings on route 1 is \$100. The savings and number of trucks required on the remaining routes is calculated similarly. The results of step 1 are summarized in Figure 3.33.

Step 2 is to select a modeling approach. We will model this problem as a binary integer programming (BIP) problem. A BIP is an IP with binary decision variables that can only take the integer values 0 or 1. It is common to use BIP to represent YES/NO decisions. Typical applications include assignment problems, scheduling, facility location, and investment portfolios. Specialized algorithms are available for BIPs that are much faster than the usual IP algorithms. For the small BIP problems in this book, any IP or BIP solver will suffice.

Step 3 is to formulate the problem. Figure 3.34 shows a spreadsheet formulation for this problem. Column D contains the binary decision variables  $x_i$ , 0 if small trucks are used and 1 if large trucks are used. Column E counts the number of large trucks used, and column F tallies our cost savings. For example,

A	A	B	C	D	E	F
1		possible			actual	
2	route	trucks	savings	decision	trucks	savings
3	1	1	100	0	0	0
4	2	1	360	1	1	360
5	3	2	400	0	0	0
6	4	1	200	0	0	0
7	5	2	640	1	2	640
8	available	3		totals	3	1000

Figure 3.35: Spreadsheet solution for the truck problem.

Trucks	Routes	Savings (\$)	Marginal Savings (\$)
1	2	360	360
2	5	640	280
3	2,5	1000	360
4	2,4,5	1200	200
5	2,3,5	1400	200
6	2,3,4,5	1600	200
7	all	1700	100

Table 3.6: Sensitivity to number of large trucks available for the truck problem.

$E3=B3*D3$  and  $F3=C3*D3$ , while  $E8=E3+\dots+E7$ . We use the spreadsheet solver to maximize the total  $F8=F3+\dots+F7$  with constraints  $E8\leq B8$ , and  $D3$  through  $D7$  binary integers.

Step 4 is to solve the problem. Figure 3.35 shows the optimal assignment of trucks to routes, obtained from the spreadsheet optimization utility. The optimum value of  $y = 1000$  is obtained at  $x_2 = 1$ ,  $x_5 = 1$ , and  $x_i = 0$  for  $i = 1, 3, 4$ . Step 5 is to answer the question. We wanted to know how to use the larger 20-cubic-yard trucks to save some money on our transportation costs. By using one of the large trucks on route 2 and the other two large trucks on route 5, we can save around \$1,000 per day.

The first question for sensitivity analysis is how the potential cost savings depends on the number of large trucks available. The company may have other large trucks that can be reassigned, or they may be able to lease additional trucks. They may also have other projects in mind for their own trucks, so we will consider the effect of increasing or decreasing the number of large trucks available. Table 3.6 shows the results of our sensitivity runs. In each case we alter the constraint cell B8 and reoptimize.

We record the optimal decision (which routes to service with large trucks) as well as the expected cost savings. We also tabulate the marginal savings.

For example, the company can save \$1,200 by using 3 large trucks, or \$1,400 by using 4 large trucks, so the 4th truck saves the company \$200 per day. If the company can obtain the use of 1, 2, or 3 additional trucks for less than \$200 per day, it will save money. On the other hand, if the company has another activity where a large truck would save more than \$360 per day, then they would be better off using one less large truck for the transportation plan under consideration. The marginal savings are like shadow prices, since they tell us the potential impact of altering the constraint by one unit.

Next, we consider the impact of changes in the potential cost savings per route. Currently, we estimate that assigning 2 large trucks to route 5 will save the company \$640 per day. Another way to view this is that each truck on route 5 saves the company \$320 per day. Large trucks on routes 3 and 4 save \$200 per day, and a large truck on route 1 saves \$100 per day. It seems reasonable that small changes in the cost savings on route 5 will not alter our conclusion that assigning large trucks to routes 2 and 5 is the best plan. To verify this, we alter the value in cell C7 and reoptimize. Any value greater than 400 yields the same optimal decision. A value less than 400 leads us to assign large trucks to routes 2 and 3.

In the course of our robustness analysis, we will illustrate the way in which binary constraints can be used to restrict the possible decisions. Suppose management decides that large trucks should not be used on route 2, because the larger trucks will cause a public relations problem in that neighborhood. We could reformulate the problem to exclude this decision, but it is far easier to add the constraint  $x_2 = 0$  to our current formulation. Now the optimal solution is to use large trucks on routes 4 and 5, with a total projected savings of \$840. This political concession costs the company \$160 per day. Now suppose that, for similar political reasons, management decides that if large trucks are used on route 4, then large trucks must be used on route 3 as well. This policy can be incorporated into our formulation by adding the linear constraint  $x_3 \geq x_4$ , or  $x_3 - x_4 \geq 0$ . If we add only this constraint, the optimal solution does not change, since the original optimal decision already satisfies this additional constraint (we don't use large trucks on either route). If we impose both constraints, then the optimal policy is to use large trucks on routes 3 and 4, with a total projected savings of \$600 per day.

### 3.5 Exercises

1. Reconsider the pig problem of Example 1.1, but now suppose that the price for pigs after  $t$  days is  $p = 0.65e^{-(.01/.65)t}$  dollars/pound.
  - (a) Show that the price for pigs is falling by one cent/day at time  $t = 0$ . What happens as  $t$  increases?
  - (b) Find the optimal time to sell the pig. Use the five-step method, and model as a one-variable optimization problem.

- (c) The parameter 0.01 represents the rate at which price is falling at time  $t = 0$ . Perform a sensitivity analysis on this parameter. Consider both the best time to sell and the resulting net profit.
- (d) Compare the results of (b) to what was done in Section 1.1, and comment on the robustness of our model.
2. Reconsider the pig problem of Example 1.1, but now suppose that the weight of the pig after  $t$  days is  $w = 800/(1 + 3e^{-t/30})$  lbs.
- (a) Show that the pig is gaining about 5 lbs/day at  $t = 0$ . What happens as  $t$  increases?
- (b) Find the optimal time to sell the pig. Use the five-step approach, and model as a one-variable optimization problem.
- (c) The parameter 800 represents the eventual mature weight of the pig. Perform a sensitivity analysis for this parameter. Consider both the best time to sell and the profit obtained.
- (d) Compare the results obtained in (b) to what was done in Sections 1.1 and 3.1. Comment on the robustness of our model. What general conclusions can we draw?
3. A statistical algorithm to determine the difference in effectiveness between two alternative treatments requires maximizing the quantity

$$\sum_{(k_1, k_2) \in E} \binom{n_1}{k_1} p_1^{k_1} (1 - p_1)^{n_1 - k_1} \binom{n_2}{k_2} p_2^{k_2} (1 - p_2)^{n_2 - k_2}$$

over the set

$$S = \{(p_1, p_2) : p_1 - p_2 = \Delta; p_1, p_2 \in [0, 1]\}.$$

Here  $E$  is a subset of the set

$$E_0 = \{(k_1, k_2) : k_1 = 0, 1, 2, \dots, n_1; k_2 = 0, 1, 2, \dots, n_2\}$$

and  $\Delta \in [-1, 1]$ . Find the maximum in the case  $n_1 = n_2 = 4$ ,  $\Delta = -0.1$ ,

$$E = \{(0, 4), (0, 3), (0, 2), (0, 1), (1, 4), (1, 3), (2, 4)\}$$

(Santner and Snell (1980)).

4. One method of evaluating the effectiveness of institutional trauma and burn medicine involves maximizing the function

$$f(p_1, \dots, p_n) = \frac{\left(A - \sum_{i=1}^n p_i\right)}{\sqrt{B + \sum_{i=1}^n p_i(1 - p_i)}}$$

over the set

$$\{(p_1, \dots, p_n) : a_i \leq p_i \leq b_i \text{ for all } i = 1, \dots, n\}.$$

Maximize  $f$  in the case  $n = 2$ ,  $A = -5.92$ ,  $B = 1.58$ ,  $a_1 = 0.01$ ,  $b_1 = 0.33$ ,  $a_2 = 0.75$ , and  $b_2 = 0.85$  (Falk, J. et al., (1992)).

5. Reconsider the competing species model of Exercise 3 in Chapter 2. Assume that a level of effort  $E$  boat-days will result in the annual harvest of  $qEx$  blue whales and  $qEy$  fin whales, where the parameter  $q$  (catchability) is assumed to equal approximately  $10^{-5}$ . Given a constant level of effort, assume that population levels will stabilize at the point where growth rate equals harvest rate.
  - (a) Assuming that the cost of a whaling expedition is \$250 per boat-day, find the level of effort that will maximize profit for the industry in the long run. Use the five-step method, and model as a one-variable optimization problem.
  - (b) Examine the sensitivity to catchability  $q$ . Consider profit, level of effort, and the eventual stable population levels of the whales.
  - (c) Increasing technology will certainly raise whale catchability. What will be the long-term effects on the whale populations and on the whaling industry?
6. Reconsider the facility location problem of Example 3.2, but now assume that the response time from point  $(x_0, y_0)$  to point  $(x_1, y_1)$  is proportional to the road travel distance  $|x_1 - x_0| + |y_1 - y_0|$ .
  - (a) Find the location that minimizes average response time. Use the five-step method, and model as a multivariable unconstrained optimization problem.
  - (b) Examine the sensitivity of the optimal location to the estimated number of emergencies in each  $2 \times 2$ -mile sector. Can you draw any general conclusions?
  - (c) Comment on the robustness of this model. Compare the optimal location to that obtained in the analysis of Section 3.2. What do you think would happen if we assumed that response time was proportional to the straight-line distance  $r = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$ ?
7. Reconsider the color TV problem of Example 2.1, but now use numerical methods instead of the analytic methods we employed in Chapter 2.
  - (a) Determine the production levels  $x_1$  and  $x_2$  that maximize the objective function  $y = f(x_1, x_2)$  in Eq. (2.2) of Chapter 2. Use the two-variable version of Newton's method.



- (b) As in Section 2.1, let  $a$  denote the price elasticity for 19-inch sets. In part (a) we assumed  $a = 0.01$ . Now assume that  $a$  increases by 10% to  $a = 0.011$  and repeat the optimization problem in part (a). Use your results to obtain a numerical estimate of the sensitivities  $S(x_1, a)$ ,  $S(x_2, a)$ , and  $S(y, a)$ . Compare to the results obtained analytically in Section 2.1.
  - (c) Let  $b$  denote the price elasticity for 21-inch sets. Currently,  $b = 0.01$ . As in part (b), use numerical methods to estimate the sensitivities of  $x_1$ ,  $x_2$ , and  $y$  to the parameter  $b$ .
  - (d) Compare the analytic methods of Section 2.1 to the numerical methods employed in this problem. Which do you prefer? Explain.
8. Reconsider Exercise 6 in Chapter 2, but now suppose that management has been persuaded to lift the ceiling on advertising expenditures. The assumption that sales vary as a linear function of the advertising budget is probably not reasonable over the wider range of advertising budget figures we now wish to consider. Suppose instead that sales are increased by 1,000 units each time the advertising budget is doubled.
- (a) Find the price and the advertising budget that will maximize profit. Use the five-step method, and model as an unconstrained optimization problem.
  - (b) Determine the sensitivity of the decision variables (price and advertising budget) to price elasticity (the 50% number).
  - (c) Determine the sensitivity of the decision variables to the advertising agency's estimate of 1,000 new sales each time the advertising budget is doubled.
  - (d) What goes wrong in part (a) if we assume a linear relationship between advertising budget and sales? Why wasn't this an issue in Exercise 6 of Chapter 2?
9. (Continuation of Exercise 8) Repeat Exercise 8, but now assume an alternative model of the relationship between advertising expenditures and sales. Suppose that doubling the advertising budget results in 1,000 additional sales, but doubling it again results in only 500 additional sales, and so forth. Repeat parts (a) through (c) of Exercise 8. In part (c), determine the sensitivity to the assumption of 1,000 additional sales the first time the advertising budget is doubled. Compare your results to those obtained in Exercise 8, and comment on the robustness of the model.
10. Reconsider the newspaper problem of Exercise 7 in Chapter 2, but now suppose that we choose to maximize our profit margin (profit as a percentage of revenue). Assume that our business expenses remain fixed at \$200,000 per week.

- (a) Find the subscription price and the advertising price that maximize profit margin. Use the five-step method, and model as an unconstrained optimization problem. Find an approximate solution by the method of random search.
  - (b) Let  $z = f(x, y)$  denote the objective function you obtained in part (a). Use a computer algebra system to determine  $F = \partial f / \partial x$  and  $G = \partial f / \partial y$ . Then determine  $\partial F / \partial x$ ,  $\partial F / \partial y$ ,  $\partial G / \partial x$ , and  $\partial G / \partial y$ .
  - (c) Use Newton's method in two variables to obtain a precise answer to the question in part (a). Use the approximate solution from part (a) as your initial estimate. The required derivatives were calculated in part (b).
  - (d) If you have not previously solved part (a) of Exercise 7 in Chapter 2, then do so now. Use any method. Compare to the results of part (c). Does it matter whether we choose to maximize profit or profit margin? Explain.
11. Reconsider the lawn chair problem of Example 3.3. Notice that the objective function  $f(x, y)$  tends to infinity as  $x$  or  $y$  approaches zero, and that  $f(x, y)$  is undefined on the lines  $x = 0$  and  $y = 0$  that form the boundary of the feasible region. Presumably, the estimates of price elasticity are not accurate when extrapolated all the way to  $x = 0$  or  $y = 0$ .
  - (a) Correct this model deficiency by altering the feasible region.
  - (b) Comment on the robustness of the decisions you made in part (a).
  - (c) Show that, for your corrected model, the optimal solution lies in the interior of the feasible region. Locate any local maximum of  $f(x, y)$  on the boundary, and show that at every such point  $\nabla f$  points into the interior.
12. Reconsider the newspaper problem of Exercise 9 in Chapter 2. Solve as a linear programming problem using a computer. Answer questions (a), (b), and (c) from the original problem.
13. Reconsider the cargo problem (Exercise 10 of Chapter 2). Solve as a linear programming problem using a computer. Answer questions (a), (b), and (c) from the original problem.
14. Reconsider the color TV problem of Example 2.2, but make the simplifying assumption that the company makes a profit of \$80 per 19-inch set and \$100 per 21-inch set.
  - (a) Find the optimal production levels. Use the five-step method, and solve as a linear programming problem using a computer.
  - (b) Determine the shadow prices for each constraint and explain what they mean. Which constraints are binding on the optimal solution?

		MI	NY	CA	GA	Output
Textile Mill	1	430	550	680	700	105
	2	510	590	890	685	160
	3	395	425	910	450	85
Allocation		70	100	105	75	

Table 3.7: Data for the transportation problem of Exercise 15. Shipping cost per truckload in dollars.

- (c) Determine the sensitivity to the objective function coefficients (profit per set). Consider both profit and optimal production levels.
  - (d) Draw a graph of the feasible region (see Fig. 2.10) and include a picture of  $\nabla f$  at the optimum. Describe geometrically what happens to the vector  $\nabla f$  as we change one of the objective function coefficients. Use this geometric idea to determine how much each objective function coefficient can change before the current optimal solution is no longer optimal.
15. The Burningham textiles company has three textile mills located in the southern U.S. and four distribution centers located in Michigan, New York, California, and Georgia. The estimated yearly output from each mill, the allocation to each warehouse, and the shipping costs are tabulated in Table 3.7.
- (a) Find the transportation plan that minimizes shipping costs. Use the five-step method, and solve as a linear programming problem using a computer.
  - (b) Determine the shadow prices for each of the output constraints. Would it be beneficial to shift production capacity from one mill to another? How much should the company be willing to spend to facilitate the shift?
16. A manufacturer of personal computers sells three desktop models. Model A costs \$850 to manufacture and sells for \$1,250; model B costs \$950 to manufacture and sells for \$1,400; and model C costs \$1,500 to manufacture and sells for \$2,500. The company purchases 10,000 desktop cases each month, and each computer requires one case. Models A and B use a 15-inch monitor, and the company can obtain 5,000 of these monitors per month. Model C uses a 17-inch monitor, and 7,500 of these are available per month. The remaining components are freely available. The company has 20,000 hours of production capacity available per month, and the production of each unit of model A, B, and C requires 1, 1.25, and 1.75 hours respectively.
- (a) How many of each type of computer should the company produce? Use the five-step method and solve as a linear programming problem.

- (b) Determine the shadow prices for each constraint. Explain what each shadow price means in the context of this problem.
  - (c) Next month, the company plans to put model C computers on sale for \$2,199. How does this change the results of parts (a) and (b)?
  - (d) The company is considering a new desktop model D. This model will cost \$1,250 to manufacture and will sell for \$1,895. It requires a desktop case, a 17-inch monitor, and 1.5 hours of production. How will this change the results of parts (a) and (b)? Would you advise the company to go ahead with their plans to introduce this new model?
17. A retired engineer has \$250,000 to invest and is willing to spend five hours per week managing her investments. Municipal bonds earn 6% per year and require no management. Real estate investments are expected to appreciate at 8% per year and require one hour of management per \$100,000 invested. Blue chip stocks earn 10% per year and require 1.5 hours of management. Junk bonds earn 12% and require 2.5 hours, while grain futures earn 15% and require five hours per \$100,000 invested.
- (a) How should the retiree invest her money in order to maximize her expected earnings? Use the five-step method and solve as a linear programming problem.
  - (b) Determine the shadow prices for each constraint. Interpret each shadow price in the context of this problem.
  - (c) The retiree downloads software from the internet that allows her to effectively manage her grain futures in three hours per week per \$100,000. How does this change the results in parts (a) and (b)?
  - (d) After a few disasters in the futures market, the engineer decides that risk is a significant factor in her investment strategy. An investment self-help book ranks municipal bonds, real estate, blue chip stocks, junk bonds, and grain futures as risk level 1, 4, 3, 6, and 10 respectively. The engineer decides that her investment portfolio should have an average risk level of no more than 4. How does this change the results in parts (a) and (b)?
18. The Green Supply Company manufactures plastic grocery bags and milk jugs. The company can obtain 5,000 lbs of used plastic bags, 18,000 lbs of used plastic milk jugs, and 40,000 of industrial plastic scraps per week at a cost of \$18, \$12, and \$10 per 100 lbs respectively. The company has orders for 4,000 boxes of plastic bags and 80,000 milk jugs per week. One box of plastic bags requires 6 lbs of plastic, costs \$5 to manufacture, and sells for \$14. It costs \$9 and requires 14 lbs of plastic to make 100 milk jugs, and the jugs retail for \$20. The plastic bags must be at least 25% post-consumer recycled plastic (used milk jugs or bags) because of consumer preference, and the milk jugs must be at most 50% post-consumer recycled plastic for strength.

- (a) Determine the optimal mix of plastics for each product. Use the five-step method and model as a linear programming problem.
  - (b) Determine the shadow prices for the each constraint, and interpret each shadow price in the context of this problem.
  - (c) A new supplier can provide industrial plastic scraps at \$8 per 100 lbs. How does this change the results in parts (a) and (b)?
  - (d) A new customer has offered to buy 40,000 environmentally friendly milk jugs per week for \$30 per 100 jugs. The jugs must contain at least 35% post-consumer recycled plastic. How does this change the results in parts (a) and (b)? Should the company accept this new customer?
19. Reconsider the dirt problem of Example 3.5, and assume that the company will only ship full truckloads of dirt.
- (a) Suppose that the company uses dump trucks with a capacity of ten cubic yards of dirt, and find the optimal shipping plan. Use the five-step method and solve as an integer programming problem.
  - (b) Repeat part (a) assuming that the trucks have a capacity of five cubic yards.
  - (c) Repeat part (a) assuming that the trucks have a capacity of 20 cubic yards.
  - (d) Compare the results of parts (a), (b), and (c) to determine the robustness of the original linear programming model. Is the original shipping plan in Example 3.5 nearly optimal for any size truck?
20. (Variation of Exercise 14) Reconsider the color TV problem of Example 2.2, but make the simplifying assumption that the company makes a profit of \$80 per 19-inch set and \$100 per 21-inch set. In this problem we will explore the effect of discretization on the optimal solution.
- (a) Find the optimal production levels. Use the five-step method, and solve as a linear programming problem using a computer.
  - (b) Because of mass production, the TV sets are actually made in batches of 30. Find the optimal number of batches to maximize profit. Solve as an integer programming problem using a computer.
  - (c) Repeat part (b) assuming batch sizes of 10, 20, 50, 100, 200, and 300. For each case, use integer programming to compute the optimal number of batches for each type of TV set.
  - (d) Compare the results in parts (a), (b), and (c) and comment on the robustness of the original linear programming solution found in part (a). How does the discretization of the feasible region affect the optimal solution? Consider both the optimal levels of production and the optimal profit.

21. (Variation of Exercise 15) The Burningham textiles company has three textile mills located in the southern U.S. and four distribution centers located in Michigan, New York, California, and Georgia. The estimated yearly output from each mill, the allocation to each warehouse, and the shipping costs are tabulated in Table 3.7.
- Find the transportation plan that minimizes shipping costs. Use the five-step method, and solve as a linear programming problem using a computer.
  - The company has purchased three new trucks for shipping. The new trucks are more fuel-efficient, and are expected to reduce shipping costs by 50%. Assume that one truck can make one trip per week (52 per year), and that each route will use either all old or all new trucks. How should the company use the new trucks in order to maximize their cost savings? Use the five-step method, and solve as an integer programming problem using a computer.
  - Perform a sensitivity analysis on the number of new trucks. Determine the optimal shipping plan and the expected cost savings for  $n = 4, 5, 6$ , and 7 new trucks. If the cost of a new truck is \$12,000 per year amortized over the expected life of the vehicle, how many additional trucks should the company purchase?
  - Because the State of California has instituted new pollution laws for trucks, the company must use the new trucks to ship to CA. How does this change the results of parts (b) and (c)? What do the new CA pollution laws cost the company?
22. Reconsider the textile shipping problem of Exercise 21, but now assume that both old and new trucks can be used on the same route. Answer the same questions as before. [Hint: Use one decision variable to indicate whether or not one new truck is used on route  $i$ , another decision variable to indicate whether a second truck is used, and so forth.]
23. A computer operating system stores files on a hard disk. Five large files of sizes 18, 23, 12, 125, and 45 MB are to be stored. Contiguous blocks of storage are available with size 25, 73, 38, and 156 MB, and each file must be stored in one contiguous block. In this problem we will explore an integer programming algorithm to assign files to storage blocks.
- In order to reserve large contiguous blocks of storage for future use, we want to store each file in the smallest available block large enough to hold the file. Define the cost of storing file  $i$  in block  $j$  to be the size of block  $j$ , and determine the assignment of files to blocks that minimizes the total cost. Use the five-step method, and model as an integer programming problem.
  - Suppose that the 12 MB file expands to 19 MB. How does this effect the optimal solution found in part (a)? How much can this 12 MB file expand before the optimal solution changes?

- (c) Suppose that the 18 MB file and the 23 MB file are to be stored in the same block, since they are used by the same program. How does this affect the optimal solution found in part (a)?
  - (d) One “greedy” algorithm for allocating blocks to files is to place each file in the first available block that is large enough to hold it. Apply this algorithm (by hand) and compare to the results of part (a). Is the IP solution found in part (a) significantly better than the results of the greedy algorithm?
  - (e) Why not just maximize the size of the largest remaining contiguous block of storage? Can this optimization problem be solved as an IP?
24. A technical manager is scheduling engineers to work on several upcoming projects. Projects A, B, and C will require 18, 12, and 30 person-months to complete. Engineers 1, 2, 3, and 4 are available to work on these projects. Their monthly salaries are \$3000, \$3500, \$3200, and \$3900 respectively.
- (a) Determine the schedule (allocation of engineers to projects) that minimizes the total cost of finishing all the projects. Assume that engineers are assigned to only one project every six months, and that all projects are to be completed within 18 months. [Hint: Let the decision variable  $x_{ijk}$  indicate whether engineer  $i$  works on project  $j$  during period  $k$ .]
  - (b) Repeat part (a) assuming that engineer 1 is unavailable during period 2 because of a prior assignment. How does this affect the optimal solution? How much would it be worth to the technical manager to have engineer 1 reassigned to his team during period 2?
  - (c) Repeat part (a) assuming that engineers 2 and 3 cannot work together because of a personality conflict. How much does their personal animosity cost the company?
  - (d) The company is offered a \$10,000 performance bonus if they can complete project A within 6 months. How does this change the optimal schedule?

## Further Reading

1. Beltrami, E. (1977) *Models for Public Systems Analysis*. Academic Press, New York.
2. Dantzig, G. (1963) *Linear Programming and Extensions*. Princeton University Press, Princeton, New Jersey.
3. Falk, J., Palocsay, S., Sacco, W., Copes, W. and Champion, H. (1992) Bounds on the Trauma Outcome Function via Optimization. *Operations Research* 40, Supp. No. 1, S86–S95.

4. Gearhart, W. and Pierce, J. *Fire Control and Land Management in the Chaparral*. UMAP module 687.
5. Hillier, F. and Lieberman, J. (1990) *Introduction to Operations Research*. McGraw-Hill, New York.
6. Maynard, J. *A Linear Programming Model for Scheduling Prison Guards*. UMAP module 272.
7. Press, W., Flannery, B., Teukolsky, S. and Vetterling, W. (2002) *Numerical Recipes in C++: The Art of Scientific Computing*. 2nd Ed., Cambridge University Press, New York. See also [www.numerical-recipes.com](http://www.numerical-recipes.com)
8. Polack, E. (1971) *Computational Methods in Optimization*. Academic Press, New York.
9. Santner, T. and Snell, M. (1980) Small-sample confidence intervals for  $\rho_1 - \rho_2$  and  $\rho_1/\rho_2$  in  $2 \times 2$  contingency tables. *Journal of the American Statistical Association* 75, 386–394.
10. Straffin, P. *Newton's Method and Fractal Patterns*. UMAP module 716.