Chapter 6

SIMULATION OF DYNAMIC MODELS

The technique of simulation has become the most important and popular method of analysis for dynamic models. Exact solution methods such as those taught in introductory courses in differential equations are of limited scope. The fact is that we do not know how to solve very many differential equations. The qualitative methods introduced in the two preceding chapters are more widely applicable, but for some problems we need a quantitative answer and a high degree of accuracy. Simulation methods provide both. Almost any dynamic model that will occur in practical applications can be simulated to a reasonable degree of accuracy. Furthermore, simulation techniques are very flexible. It is not hard to introduce more complex features such as time delays and stochastic elements, which are difficult to treat analytically.

The principal drawback of simulation comes in the area of sensitivity analysis. Without recourse to an analytic formula, the only way to test sensitivity to a particular parameter is to repeat the entire simulation for several values and then interpolate. This can be very expensive and time–consuming if there are several parameters to test. Even so, simulation is the method of choice for many problems. If we cannot solve analytically, and if we need quantitative solutions, then we really have no alternative but to simulate.

6.1 Introduction to Simulation

There are essentially two ways to approach the analysis of a dynamic system model. The analytic approach attempts to predict what will happen according to the model in a variety of circumstances. In the simulation approach, we build the model, turn it on, and find out.

Example 6.1. Two forces, which we will call red (R) and blue (B), are engaged in battle. In this conventional battle, attrition is due to direct fire (infantry) and

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Variables:	R = number of red divisions (units) B = number of blue divisions (units) D_R = red attrition rate due to direct fire (units/hour) D_B = blue attrition rate due to direct fire (units/hour) I_R = red attrition rate due to indirect fire (units/hour) I_B = blue attrition rate due to indirect fire (units/hour)
Assumptions:	$D_R = a_1 B$ $D_B = a_2 R$ $I_R = b_1 R B$ $I_B = b_2 R B$ $R \geq 0,\ B \geq 0$ $R(0) = 5,\ B(0) = 2$ $a_1,\ a_2,\ b_1,\ b_2$ are positive reals $a_1 > a_2,\ b_1 > b_2$
Objective:	Determine the conditions under which $R \to 0$ before $B \to 0$

Figure 6.1: Results of step 1 of the war problem.

area fire (artillery). The attrition rate due to direct fire is assumed proportional to the number of enemy infantry. The attrition rate due to artillery depends on both the amount of enemy artillery and the density of friendly troops. Red has amassed five divisions to attack a blue force of two divisions. Blue has the advantage of defense, and superior weapon effectiveness besides. How much more effective does blue have to be in order to prevail in battle?

We will use the five-step method. The results of step 1 are summarized in Figure 6.1. We have assumed that the attrition rate due to area fire is directly proportional to the product of enemy force level and friendly force level. It seems reasonable to assume at this stage that force level is proportional to force density. And since we have no information about the number of artillery versus infantry units, for this analysis we simply assume that artillery and infantry units are attrited in proportion to their numbers. Hence, the number of remaining artillery or infantry units on each side is assumed to remain in proportion to the total number of units.

Next is step 2. We will use a discrete–time dynamical system model, which we will solve by simulation. Figure 6.2 gives an algorithm for simulating a discrete–time dynamical system in two variables:

$$\begin{aligned} \Delta x_1 &= f_1(x_1, x_2) \\ \Delta x_2 &= f_2(x_1, x_2). \end{aligned} \tag{6.1}$$

Step 3 is next. We will model the war problem as a discrete–time dynamical system with two state variables: $x_1 = R$, the number of red force units; and

Algorithm: DISCRETE–TIME SIMULATION

Variables: $x_1(n)$ = first state variable at time n
$x_2(n)$ = second state variable at time n
N = number of time steps

Input: $x_1(0)$, $x_2(0)$, N

Process:
```
Begin
for n = 1 to N do
   Begin
   x1(n) ← x1(n − 1) + f1(x1(n − 1), x2(n − 1))
   x2(n) ← x2(n − 1) + f2(x1(n − 1), x2(n − 1))
   End
End
```

Output: $x_1(1), \ldots, x_1(N)$
$x_2(1), \ldots, x_2(N)$

Figure 6.2: Pseudocode for discrete–time simulation.

$x_2 = B$, the number of blue force units. The difference equations are

$$\begin{aligned} \Delta x_1 &= -a_1 x_2 - b_1 x_1 x_2 \\ \Delta x_2 &= -a_2 x_1 - b_2 x_1 x_2. \end{aligned} \tag{6.2}$$

We will start with $x_1(0) = 5$ and $x_2(0) = 2$ divisions of troops. We will use a time step of $\Delta t = 1$ hour. We will also need numerical values for a_i and b_i in order to run the simulation program. Unfortunately, we have been given no idea what they are supposed to be. We will have to make an educated guess. Suppose that a typical conventional battle lasts about 5 days and that engagement takes place about 12 hours per day. That means that one force is depleted in about 60 hours of battle. If a force were to be depleted by 5% per hour for 60 hours, the fraction remaining would be $(0.95)^{60} = 0.05$, which looks about right. We will assume that $a_2 = 0.05$. Since area fire is not generally as effective as direct fire in terms of attrition, we will assume $b_2 = 0.005$. (Recall that b_i is multiplied by both x_1 and x_2, which is why we made it so small.) Now blue is supposed to have greater weapon effectiveness than red, so we should have $a_1 > a_2$ and $b_1 > b_2$. Let us assume that $a_1 = \lambda a_2$ and $b_1 = \lambda b_2$ for some $\lambda > 1$. The analysis objective is to determine the smallest λ that will make $x_1 \to 0$ before $x_2 \to 0$. Now the difference equations are

$$\begin{aligned} \Delta x_1 &= -\lambda(0.05)x_2 - \lambda(0.005)x_1 x_2 \\ \Delta x_2 &= -0.05 x_1 - 0.005 x_1 x_2. \end{aligned} \tag{6.3}$$

In step 4 we will solve the problem by running the simulation program for several values of λ. We begin by exercising the model for $\lambda = 1$, 1.5, 2, 3, and 5.

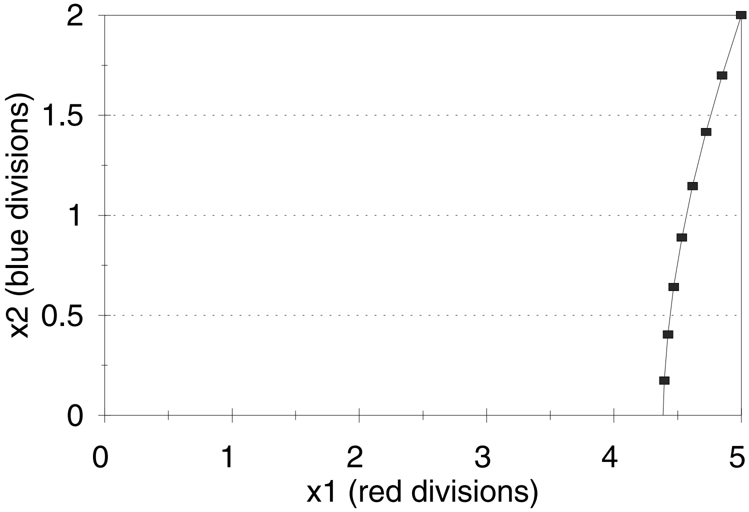

Figure 6.3: Graph of blue divisions x_2 versus red divisions x_1 for the war problem: case $\lambda = 1.0$.

This should give us a good idea of how large λ needs to be, and it allows us to check our simulation against our intuitive grasp of the situation. For example, we should check that the larger λ is, the better blue does.

The results of this first batch of model runs are shown in Figures 6.3 through 6.7. A summary of our findings is contained in Table 6.1.

For each run we have recorded the value of λ, the duration of the battle, the identity of the winner, and the number of units remaining on the winning side. We decided to run the simulation for up to 14 days of combat (or $N = 168$ hours). The duration of battle is defined to be the number of hours of actual combat (there are 12 hours of fighting per day) until one of the variables x_1 or x_2 becomes zero or negative. If both sides survive 168 hours of combat, we call it a draw.

Advantage (λ)	Hours of Combat	Winning Side	Remaining Forces
1.0	8	red	4.4
1.5	9	red	4.1
2.0	9	red	3.7
3.0	10	red	3.0
5.0	17	red	1.0

Table 6.1: Summary of simulation results for the war problem.

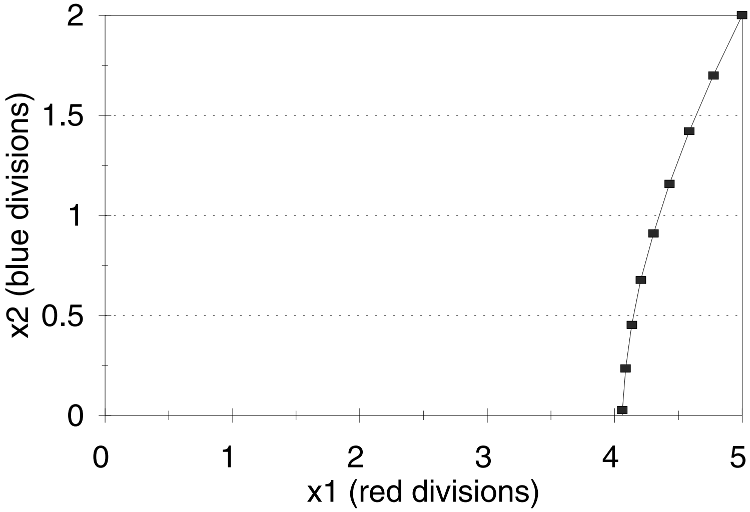

Figure 6.4: Graph of blue divisions x_2 versus red divisions x_1 for the war problem: case $\lambda = 1.5$.

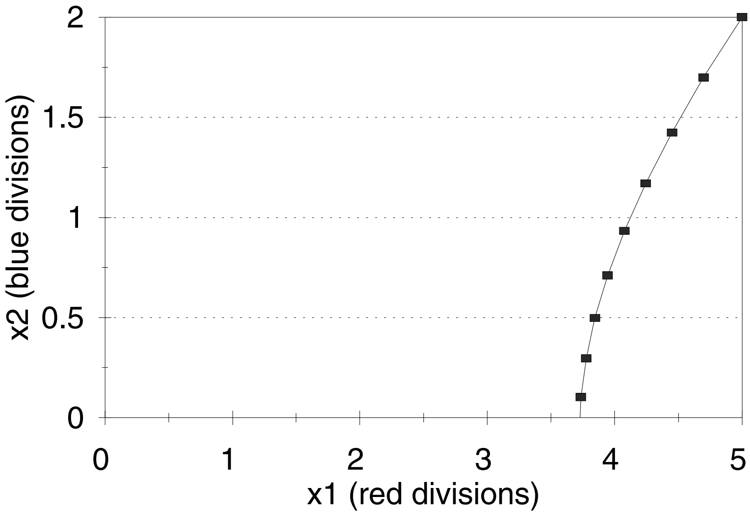

Figure 6.5: Graph of blue divisions x_2 versus red divisions x_1 for the war problem: case $\lambda = 2.0$.

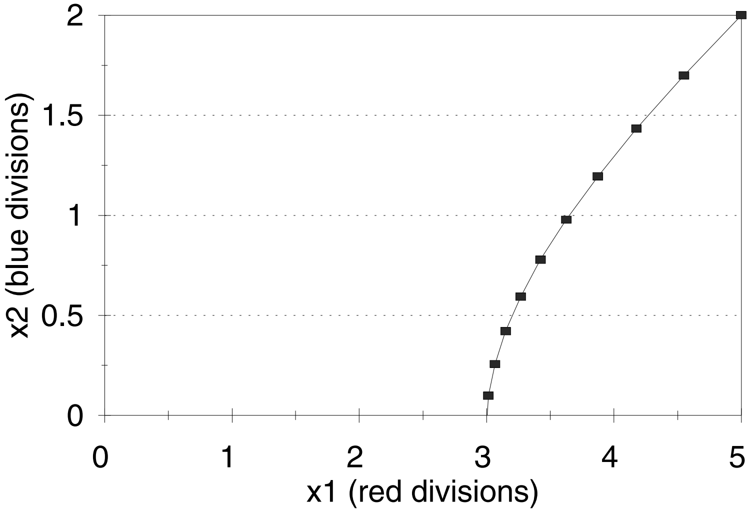

Figure 6.6: Graph of blue divisions x_2 versus red divisions x_1 for the war problem: case $\lambda = 3.0$.

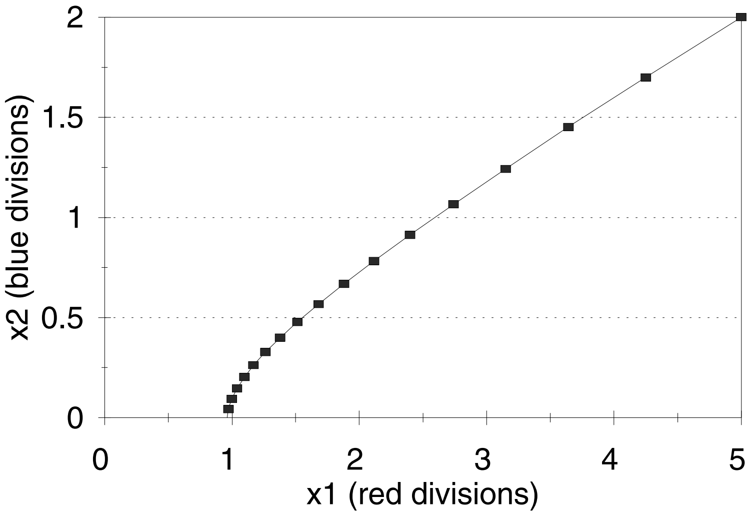

Figure 6.7: Graph of blue divisions x_2 versus red divisions x_1 for the war problem: case $\lambda = 5.0$.

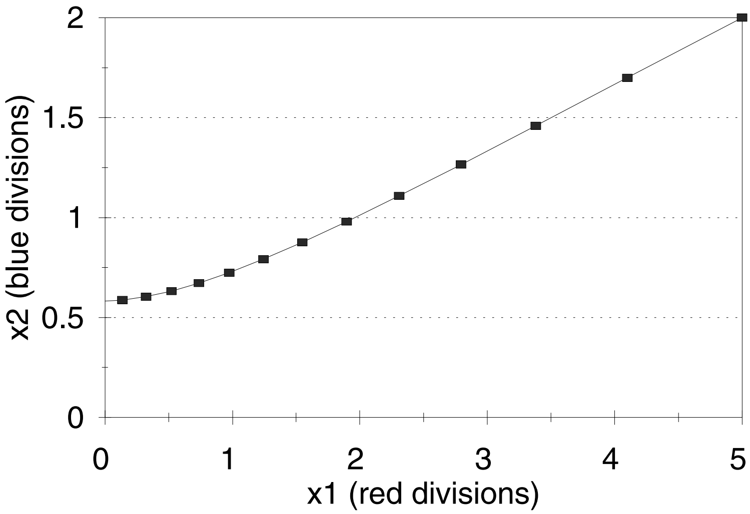

Figure 6.8: Graph of blue divisions x_2 versus red divisions x_1 for the war problem: case $\lambda = 6.0$.

It does not look good for blue. Even with a 5 : 1 edge in weapon effectiveness, blue will lose the battle. We decided to make a few more model runs to find out just how big λ would have to be for blue to win. At $\lambda = 6.0$, the blue side won after 13 hours of battle, with 0.6 units remaining (see Figure 6.8). A few more model runs, bisecting the interval $5.0 \le \lambda \le 6.0$, yielded a lower bound of $\lambda = 5.4$ for blue to win. At $\lambda = 5.3$, red was the winner.

Finally, we need to summarize our results. We simulated an engagement between an attacking red force of five divisions and a defending blue force of two divisions. We assumed that the two forces would engage and continue to fight until there emerged a clear victor. We wanted to investigate the extent to which a greater weapon effectiveness (kill rate) could offset a 5 : 2 numerical disadvantage. We simulated a number of battles for different ratios of weapon effectiveness. We found that blue would need at least a 5.4 : 1 advantage in weapon effectiveness to defend successfully against a numerically superior red force of 5 divisions.

Having finished the five-step method and answered the question stated in step 1, we need to perform a sensitivity analysis. This is particularly important in a problem such as this one, where most of our data came from sheer guesswork. We will begin by investigating the relationship between the magnitude of the attrition coefficients and the outcome of the battle. We had assumed that $a_2 = 0.05$, $b_2 = a_2/10$, $a_1 = \lambda a_2$, and $b_1 = \lambda b_2$. We will now vary a_2, keeping the same relative relationship between it and the other variables.

We will investigate the dependence of $\lambda_{\min}$ on a_2, where $\lambda_{\min}$ is defined to

Force ratio (red : blue)	Advantage required ($\lambda_{\min}$)
8 : 2	11.8
7 : 2	9.5
6 : 2	7.3
5 : 2	5.4
4 : 2	3.6
3 : 2	2.2
2 : 2	1.1

Table 6.2: Summary of simulation results showing the effect of force ratio for the war problem.

be the smallest value of λ for which blue wins. This requires making a number of model runs for each value of a_2. It turns out that there is no need to tabulate these results, because in every case we checked (from $a_2 = 0.01$ to $a_2 = 0.10$), we found $\lambda_{\min} = 5.4$, the same as our baseline case ($a_2 = 0.05$). There is apparently no sensitivity to the magnitude of the attrition coefficients.

Several more kinds of sensitivity analysis are possible, and the process probably should go on as long as time permits, curiosity persists, and the pressures of other obligations do not intrude. We were curious about the relationship between $\lambda_{\min}$ and the numerical superiority ratio of red versus blue, currently assumed to be 5 : 2. To study this, we returned to the baseline case, $a_2 = 0.05$, and made several model runs to determine $\lambda_{\min}$ for several values of the initial red force strength x_1, keeping $x_2 = 2$ fixed. The results of this model excursion are tabulated in Table 6.2. The case $x_1 = 2$ was run as a check. We found $\lambda_{\min} = 1.1$ in this case, because the case $\lambda = 1$ produced a draw.

6.2 Continuous–Time Models

In this section we discuss the fundamentals of simulating continuous–time dynamical systems. The methods presented here are simple and usually effective. The basic idea is to use the approximation

$$\frac{dx}{dt} \approx \frac{\Delta x}{\Delta t}$$

to replace our continuous–time model (differential equations) by a discrete–time model (difference equations). Then we can use the simulation methods we introduced in the preceding section.

Example 6.2. Reconsider the whale problem of Example 4.2. We know now that starting at the current population levels of $B = 5,000$, $F = 70,000$, and assuming a competition coefficient of $\alpha < 1.25 \times 10^{-7}$, both populations of whales will eventually grow back to their natural levels in the absence of any further harvesting. How long will this take?

We will use the five-step method. Step 1 is the same as before (see Fig. 4.3), except that now the objective is to determine how long it takes to get to the equilibrium starting from $B = 5,000,\ F = 70,000$.

Step 2 is to select the modeling approach. We have an analysis question that seems to require a quantitative method. The graphical methods of Chapter 4 tell us what will happen, but not how long it will take. The analytical methods reviewed in Chapter 5 are local in nature. We need a global method here. The best thing would be to solve the differential equations, but we don't know how. We will use a simulation; this seems to be the only choice we have.

There is some question as to whether we want to adopt a discrete–time or a continuous–time model. Let us consider, more generally, the case of a dynamic model in n variables, $x = (x_1, \ldots, x_n)$, where we are given the rates of change $F = (f_1, \ldots, f_n)$ for each of the variables $x_1, \ldots, x_n$, but we have not yet decided whether to model the system in discrete–time or continuous–time. The discrete–time model looks like

$$\begin{aligned} \Delta x_1 &= f_1(x_1, \ldots, x_n) \\ &\ \vdots \\ \Delta x_n &= f_n(x_1, \ldots, x_n), \end{aligned} \tag{6.4}$$

where Δx_i represents the change in x_i over 1 unit of time ($\Delta t = 1$). The units of time are already specified. The method for simulating such a system was discussed in the previous section.

If we decided on a continuous–time model instead, we would have

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, \ldots, x_n) \\ &\ \vdots \\ \frac{dx_n}{dt} &= f_n(x_1, \ldots, x_n), \end{aligned} \tag{6.5}$$

which we would still need to figure out how to simulate. We certainly can't expect the computer to calculate $x(t)$ for every value of t. That would take an infinite amount of time to get nowhere. Instead we must calculate $x(t)$ at a finite number of points in time. In other words, we must replace the continuous–time model by a discrete–time model in order to simulate it. What would the discrete–time approximation to this continuous–time model look like? If we use a time step of $\Delta t = 1$ unit, it will be exactly the same as the discrete–time model we could have chosen in the first place. Hence, unless there is something wrong with choosing $\Delta t = 1$, we don't have to choose between discrete and continuous. Then we are done with step 2.

Step 3 is to formulate the model. As in Chapter 4, we let $x_1 = B$ and $x_2 = F$ represent the population levels of each species. The dynamical system equations are

$$\begin{aligned}\frac{dx_1}{dt} &= 0.05x_1\left(1-\frac{x_1}{150,000}\right)-\alpha x_1x_2\\ \frac{dx_2}{dt} &= 0.08x_2\left(1-\frac{x_2}{400,000}\right)-\alpha x_1x_2\end{aligned} \tag{6.6}$$

on the state space $x_1 \geq 0,\ x_2 \geq 0$. In order to simulate this model we will begin by transforming to a set of difference equations

$$\begin{aligned}\Delta x_1 &= 0.05x_1\left(1-\frac{x_1}{150,000}\right)-\alpha x_1x_2\\ \Delta x_2 &= 0.08x_2\left(1-\frac{x_2}{400,000}\right)-\alpha x_1x_2\end{aligned} \tag{6.7}$$

over the same state space. Here, Δx_i represents the change in population x_i over a period of $\Delta t = 1$ year. We will have to supply a value for α in order to run the program. We will assume that $\alpha = 10^{-7}$ to start with. Later on, we will do a sensitivity analysis on α.

Step 4 is to solve the problem by simulating the system in Eq. (6.7) using a computer implementation of the algorithm in Fig. 6.2. We began by simulating $N = 20$ years, starting with

$$\begin{aligned}x_1(0) &= 5,000\\ x_2(0) &= 70,000.\end{aligned}$$

Figures 6.9 and 6.10 show the results of our first model run. Both blue whale and fin whale populations grow steadily, but in 20 years they do not get close to the equilibrium values

$$\begin{aligned}x_1 &= 35,294\\ x_2 &= 382,352\end{aligned}$$

predicted by our analysis back in Chapter 4.

Figures 6.11 and 6.12 show our simulation results when we input a value of N large enough to allow this discrete–time dynamical system to approach equilibrium.

Step 5 is to put our conclusions into plain English. It takes a long time for the whale populations to grow back: about 100 years for the fin whale, and several centuries for the more severely depleted blue whale.

We will now discuss the sensitivity of our results to the parameter α, which measures the intensity of competition between the two species. Figures 6.13 through 6.18 show the results of our simulation runs for several values of α. Of course, the equilibrium levels of both species change along with α.

However, the time it takes our model to converge to equilibrium changes very little. Our general conclusion is valid whatever the extent of competition: It will take centuries for the whales to grow back.

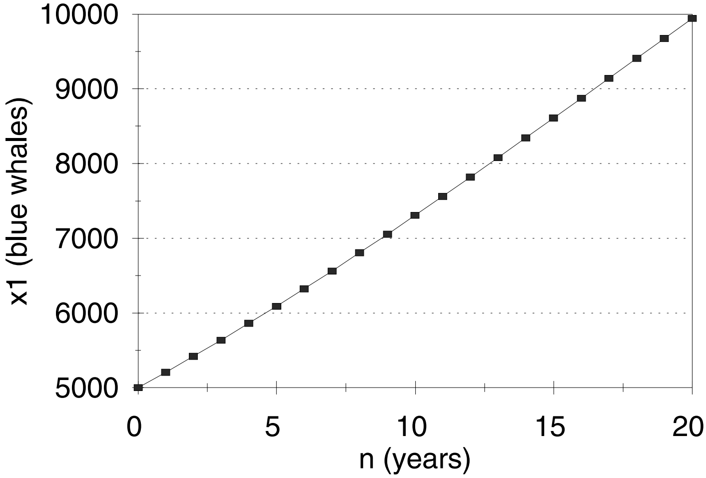

Figure 6.9: Graph of blue whales x_1 versus time n for the whale problem: case $\alpha = 10^{-7}, N = 20$.

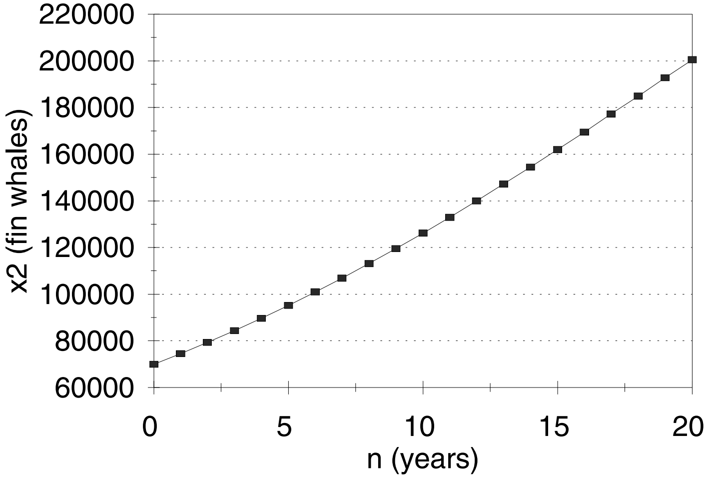

Figure 6.10: Graph of fin whales x_2 versus time n for the whale problem: case $\alpha = 10^{-7}, N = 20$.

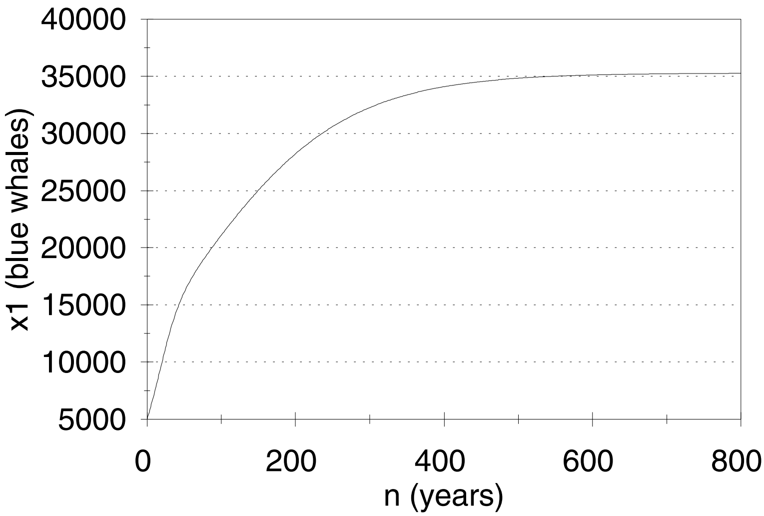

Figure 6.11: Graph of blue whales x_1 versus time n for the whale problem: case $\alpha = 10^{-7}, N = 800$.

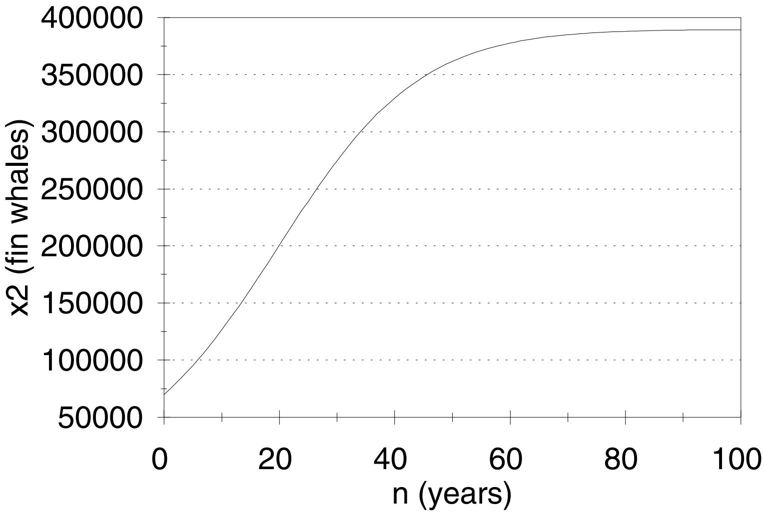

Figure 6.12: Graph of fin whales x_2 versus time n for the whale problem: case $\alpha = 10^{-7}, N = 100$.

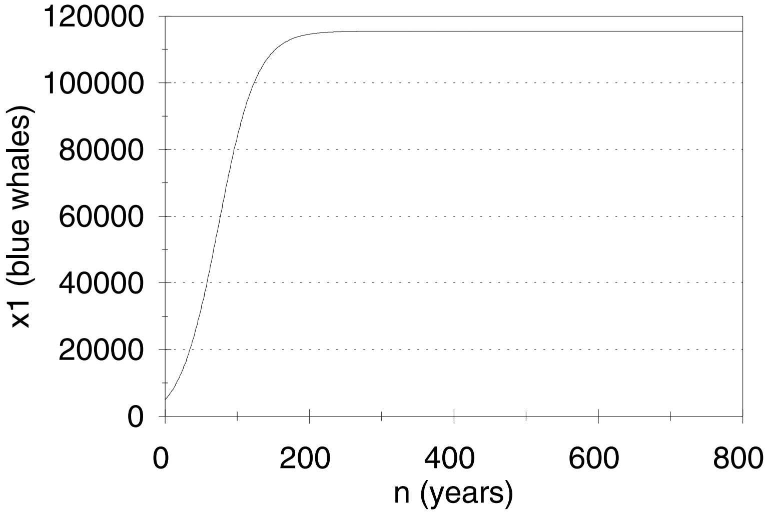

Figure 6.13: Graph of blue whales x_1 versus time n for the whale problem: case $\alpha = 3 \times 10^{-8}, N = 800$.

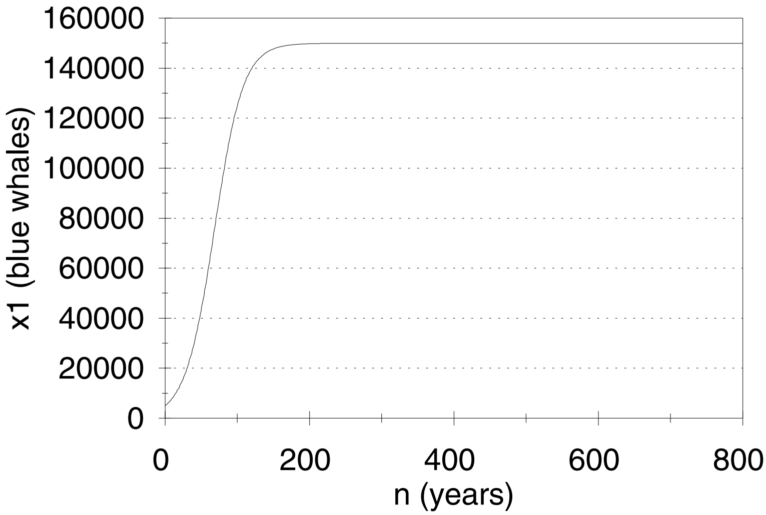

Figure 6.14: Graph of blue whales x_1 versus time n for the whale problem: case $\alpha = 10^{-8}, N = 800$.

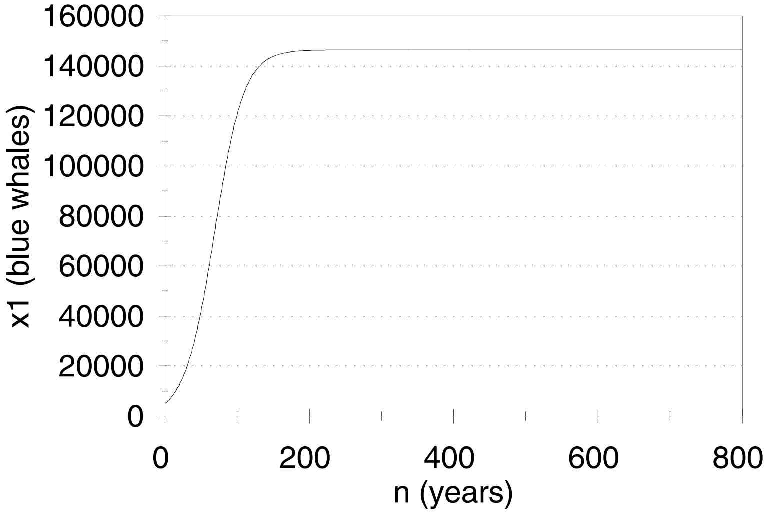

Figure 6.15: Graph of blue whales x_1 versus time n for the whale problem: case $\alpha = 10^{-9}, N = 800$.

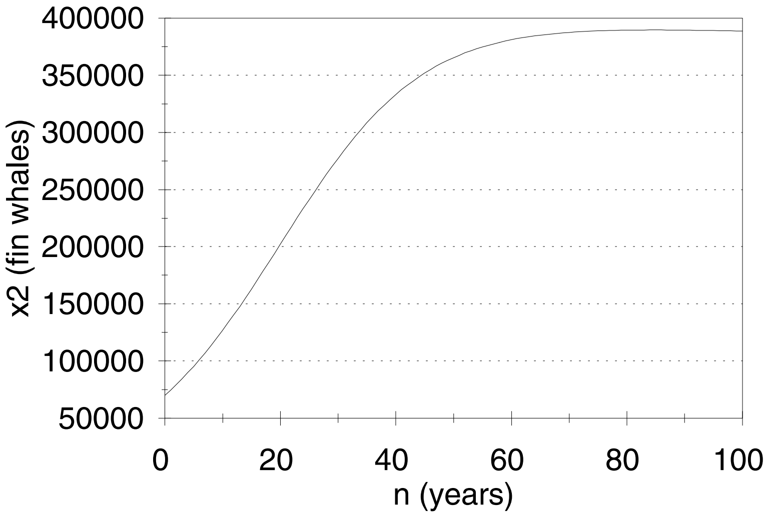

Figure 6.16: Graph of fin whales x_2 versus time n for the whale problem: case $\alpha = 3 \times 10^{-8}, N = 100$.

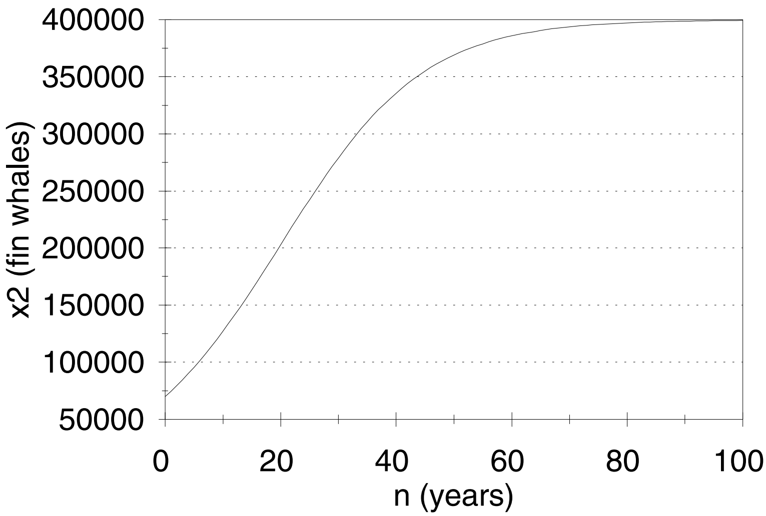

Figure 6.17: Graph of fin whales x_2 versus time n for the whale problem: case $\alpha = 10^{-8}, N = 100$.

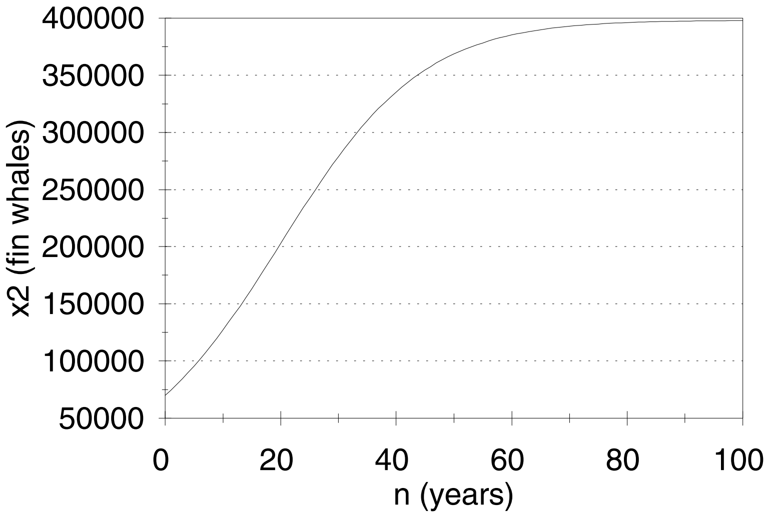

Figure 6.18: Graph of fin whales x_2 versus time n for the whale problem: case $\alpha = 10^{-9}, N = 100$.

6.3 The Euler Method

One of the reasons we simulate dynamic models is to obtain accurate quantitative information about system behavior. For some applications the simple simulation techniques of the previous section are too imprecise. More sophisticated numerical analysis techniques are available, however, that can be used to provide accurate solutions to initial value problems for almost any differential equation model. In this section we present the simplest generally useful method for solving systems of differential equations to any desired degree of accuracy.

Example 6.3. Reconsider the RLC circuit problem of Example 5.4 in the previous chapter. Describe the behavior of this circuit.

Our analysis in Section 5.3 was successful only in determining the local behavior of the dynamical system

$$\begin{aligned} x_1' &= x_1 - x_1^3 - x_2 \\ x_2' &= x_1 \end{aligned} \tag{6.8}$$

in the neighborhood of $(0, 0)$, which is the only equilibrium of this system. The equilibrium is unstable, with nearby solution curves spiraling counterclockwise and outward. A sketch of the vector field (see Fig. 5.11) reveals little new information. There is a general counterclockwise rotation to the flow, but it is hard to tell whether solution curves spiral inward, outward, or neither, in the absence of additional information.

We will use the Euler method to simulate the dynamical system in Eq. (6.8). Figure 6.19 gives an algorithm for the Euler method. Consider a continuous–time dynamical system model

$$x' = F(x)$$

with $x = (x_1, \ldots, x_n)$ and $F = (f_1, \ldots, f_n)$, along with the initial condition $x(t_0) = x_0$.

Starting from this initial condition, at each iteration the Euler method produces an estimate of $x(t + h)$ based on the current estimate of $x(t)$, using the fact that

$$x(t + h) - x(t) \approx h\, F(x(t)).$$

The accuracy of the Euler method increases as the step size h becomes smaller; i.e., as the number of steps N becomes larger. For small h the error in the estimate $x(N)$ of the final value of the state variable x is roughly proportional to h. In other words, using twice as many steps (i.e., reducing h by half) produces results twice as accurate.

Figures 6.20 and 6.21 illustrate the results obtained by applying a computer implementation of the Euler method to Eq. (6.8). Each graph in Figs. 6.20 and 6.21 is the result of several simulation runs. For each set of initial conditions, we need to perform a sensitivity analysis on the input parameters T and N.

First, we enlarged T until any further enlargements produced essentially the same picture (the solution just cycled around a few more times). Then

Algorithm: THE EULER METHOD

Variables:
$t(n)$ = time after n steps
$x_1(n)$ = first state variable at time $t(n)$
$x_2(n)$ = second state variable at time $t(n)$
N = number of steps
T = time to end simulation

Input: $t(0)$, $x_1(0)$, $x_2(0)$, N, T

Process:
Begin
$h \leftarrow (T - t(0))/N$
for $n = 0$ to $N - 1$ do
Begin
  $x_1(n+1) \leftarrow x_1(n) + hf_1(x_1(n), x_2(n))$
  $x_2(n+1) \leftarrow x_2(n) + hf_2(x_1(n), x_2(n))$
  $t(n+1) \leftarrow t(n) + h$
  End
End

Output:
$t(1), \ldots, t(n)$
$x_1(1), \ldots, x_1(N)$
$x_2(1), \ldots, x_2(N)$

Figure 6.19: Pseudocode for the Euler method.

we enlarged N (i.e., decreased the step size) to check accuracy. If doubling N produced a graph that was indistinguishable from the one before, we judged that N was large enough for our purposes.

In Fig. 6.20 we started at $x_1(0) = -1$, $x_2(0) = -1.5$. The resulting solution curve spirals in toward the origin, with a counterclockwise rotation. However, before it gets too close to the origin, the solution settles into a more–or–less periodic behavior, cycling around the origin. When we start nearer the origin in Fig. 6.21, the same behavior occurs, except now the solution curve spirals outward. In both cases the solution approaches the same closed loop around the origin. This closed loop is called a *limit cycle*.

Figure 6.22 shows the complete phase portrait for this dynamical system. For any initial condition except $(x_1, x_2) = (0, 0)$, the solution curve tends to the same limit cycle. If we begin inside the loop, the curve spirals outward; if we begin outside the loop, the curve moves inward. The kind of behavior we see in Fig. 6.22 is a phenomenon that cannot occur in a linear dynamical system. If a solution to a linear dynamical system spirals in toward the origin, it must spiral all the way into the origin. If it spirals outward, then it spirals all the way out to infinity. This observation has modeling implications, of course. Any dynamical system exhibiting the kind of behavior shown in Fig. 6.22 cannot be

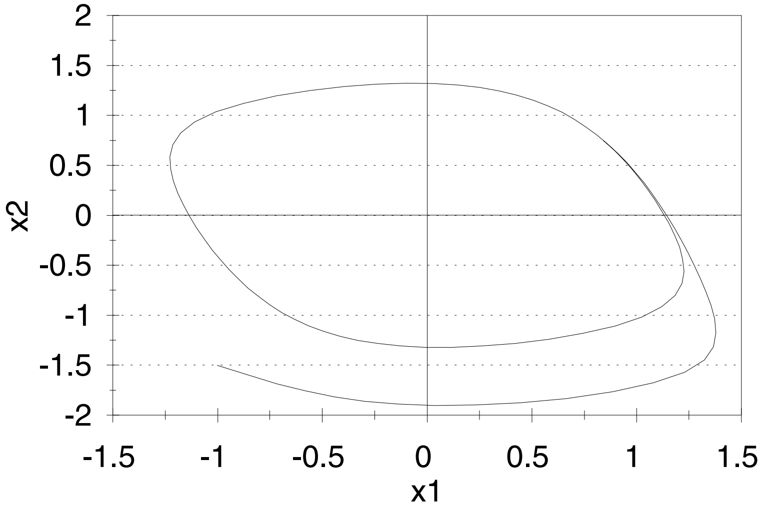

Figure 6.20: Graph of voltage x_2 versus current x_1 for the nonlinear RLC circuit problem: case $x_1(0) = -1.0$, $x_2(0) = -1.5$.

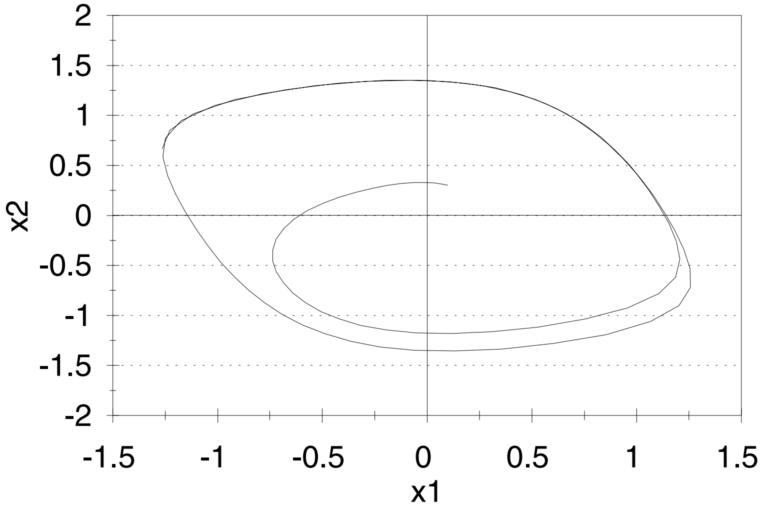

Figure 6.21: Graph of voltage x_2 versus current x_1 for the nonlinear RLC circuit problem: case $x_1(0) = 0.1$, $x_2(0) = 0.3$.

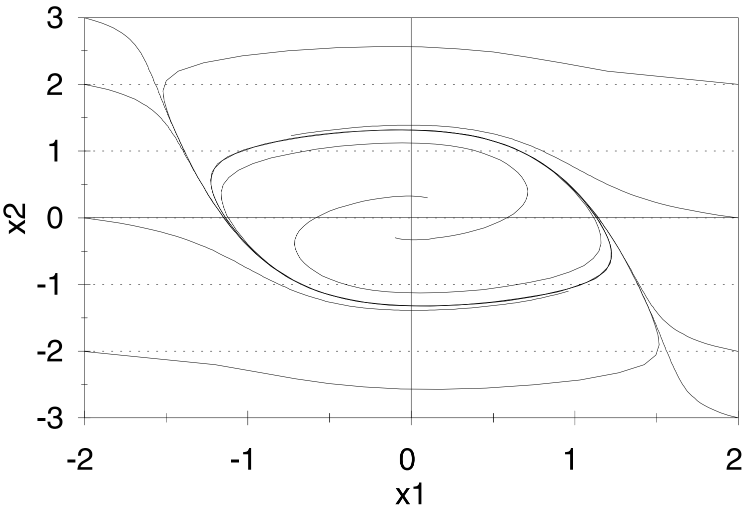

Figure 6.22: Graph of voltage x_2 versus current x_1 showing the complete phase portrait for the nonlinear RLC circuit problem of Example 6.3.

modeled adequately using linear differential equations.

The graphs in Figures 6.20–6.22 were produced using a spreadsheet implementation of the Euler method. The advantage of a spreadsheet implementation is that the computations and graphics are both performed on the same platform, and the results of changing initial conditions can be observed instantly. A simple computer program to implement this algorithm is effective, but the output is harder to interpret without graphics. Many graphing calculators and computer algebra systems also have built–in differential equation solvers, most of which are based on some variation of the Euler method. The Runge–Kutta method is one variation that uses a more sophisticated interpolation between $x(t)$ and $x(t + h)$; see Exercise 21 at the end of this chapter. No matter what kind of numerical method you use to solve differential equations, be sure to check your results by performing a sensitivity analysis on the parameters that control accuracy. Even the most sophisticated algorithms can produce serious errors unless they are used with care.

Next, we will perform a sensitivity analysis to determine the effect of small changes in our assumptions on our general conclusions. Here we will discuss the sensitivity to the capacitance C. Some additional questions of sensitivity and robustness are relegated to the exercises at the end of this chapter. In our example we assumed that $C = 1$. In the more general case we obtain the dynamical system

$$\begin{aligned} x_1' &= x_1 - x_1^3 - x_2 \\ x_2' &= \frac{x_1}{C}. \end{aligned} \tag{6.9}$$

For any value of $C > 0$, the vector field is essentially the same as in Fig. 5.11. The velocity vectors are vertical on the curve $x_2 = x_1 - x_1^3$ and horizontal on the x_2 axis. The only equilibrium is the origin, $(0, 0)$.

The matrix of partial derivatives is

$$A = \begin{pmatrix} 1 - 3x_1^2 & -1 \\ 1/C & 0 \end{pmatrix}.$$

Evaluate at $x_1 = 0$, $x_2 = 0$ to obtain the linear system

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1/C & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{6.10}$$

which approximates the behavior of our nonlinear system near the origin. To obtain the eigenvalues, we must solve

$$\begin{vmatrix} \lambda - 1 & 1 \\ -1/C & \lambda - 0 \end{vmatrix} = 0,$$

or $\lambda^2 - \lambda + 1/C = 0$. The eigenvalues are

$$\lambda = \frac{1 \pm \sqrt{1 - \dfrac{4}{C}}}{2}. \tag{6.11}$$

As long as $0 < C < 4$, the quantity under the radical is negative, so we have two complex conjugate eigenvalues with positive real parts, making the origin an unstable equilibrium.

Next, we need to consider the phase portrait for the linear system. It is possible to solve the system in Eq. (6.10) in general by using the method of eigenvalues and eigenvectors, although it would be rather messy. Fortunately, in the present case it is not really necessary to determine a formula for the exact analytical solution to Eq. (6.10) in order to draw the phase portrait. We already know that the eigenvalues of this system are of the form $\lambda = a \pm ib$, where a is positive. As we mentioned previously (in Section 5.1, during the discussion of step 2 for Example 5.1), this implies that the coordinates of any solution curve must be linear combinations of the two terms $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$. In other words, every solution curve spirals outward. A cursory examination of the vector field for Eq. (6.10) tells us that the spirals must rotate counterclockwise. We thus see that for any $0 < C < 4$, the phase portrait of the linear system in Eq. (6.10) looks much like the one in Fig. 5.10.

Our examination of the linear system in Eq. (6.10) shows that the behavior of the nonlinear system in the neighborhood of the origin must be essentially

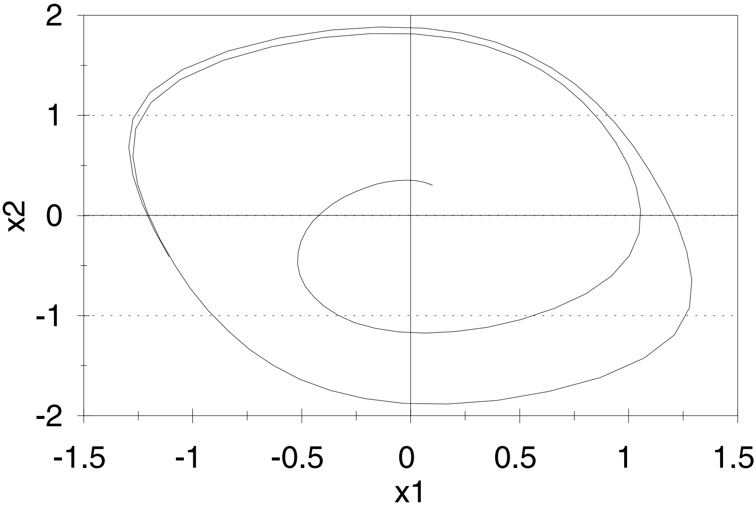

Figure 6.23: Graph of voltage x_2 versus current x_1 for the nonlinear RLC circuit problem: case $x_1(0) = 0.1$, $x_2(0) = 0.3$, $C = 0.5$.

the same as in Fig. 6.22 for any value of C near the baseline case $C = 1$. To see what happens farther away from the origin, we need to simulate. Figures 6.23 through 6.26 show the results of simulating the dynamical system in Eq. (6.9) using the Euler method for several different values of C near 1. In each simulation run we started at the same initial condition as in Fig. 6.21.

In each case the solution curve spirals outward and is gradually attracted to a limit cycle. The limit cycle gets smaller as C increases. Several different initial conditions were used for each value of C tested (additional simulation runs are not shown). In each case, apparently, a single limit cycle attracts every solution curve away from the origin. We conclude that the RLC circuit of Example 6.3 has the behavior shown in Fig. 6.22 regardless of the exact value of the capacitance C, assuming that C is close to 1.

6.4 Chaos and Fractals

One of the most exciting mathematical discoveries of the twentieth century is the chaotic behavior of some dynamic models. *Chaos* is characterized by the apparently random behavior of solutions, with extreme sensitivity to initial conditions. Chaotic dynamical systems can give rise to exotic limit sets called *fractals*. Chaotic dynamical system models have been used to handle problems in turbulent fluid flow, ecosystems with aperiodic population fluctuations, cardiac arrythmia, occasional reversal of the earth's magnetic poles, complex

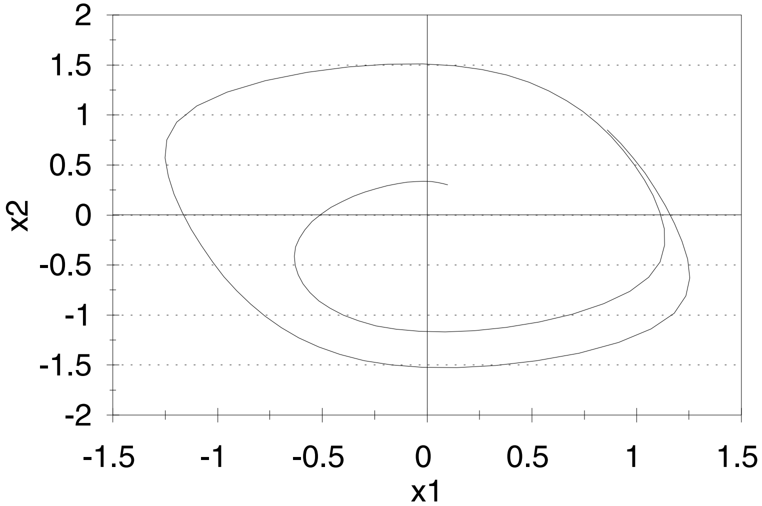

Figure 6.24: Graph of voltage x_2 versus current x_1 for the nonlinear RLC circuit problem: case $x_1(0) = 0.1$, $x_2(0) = 0.3$, $C = 0.75$.

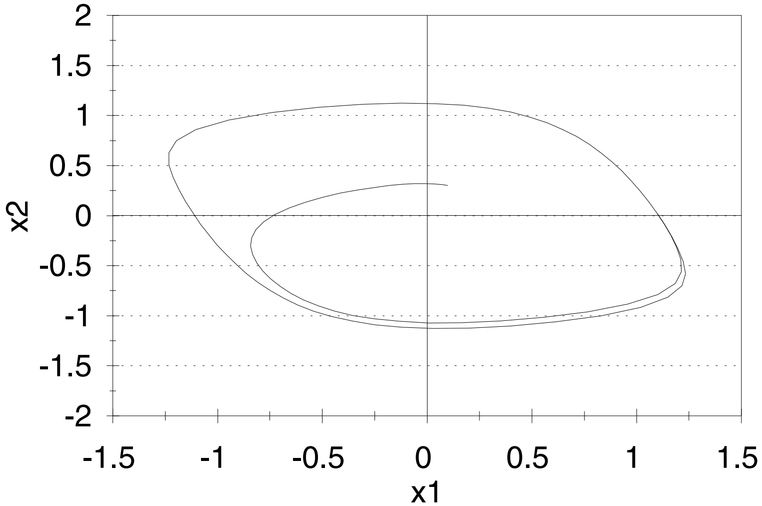

Figure 6.25: Graph of voltage x_2 versus current x_1 for the nonlinear RLC circuit problem: case $x_1(0) = 0.1$, $x_2(0) = 0.3$, $C = 1.5$.

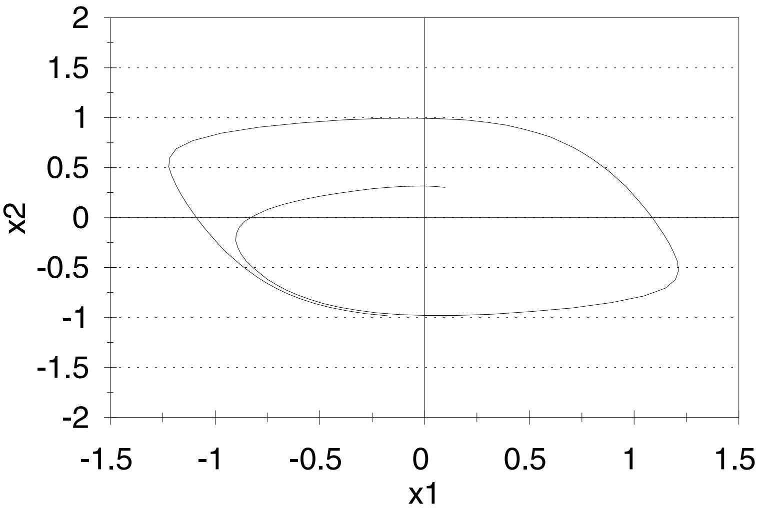

Figure 6.26: Graph of voltage x_2 versus current x_1 for the nonlinear RLC circuit problem: case $x_1(0) = 0.1$, $x_2(0) = 0.3$, $C = 2.0$.

chemical reactions, lasers, and the stock market. Many of these applications are controversial, and their implications are still being explored. One of the most surprising things about chaos is the way it can emerge from simple nonlinear dynamic models.

Example 6.4. Reconsider the whale problem of Example 4.2, but now suppose that we use a discrete model to project population growth with a time step of several years. We know that for a time step of one year, the discrete and continuous time models behave in essentially the same manner. How large a time step can we use and still retain the same qualitative behavior as the continuous time model? What happens to the model if we use too large a time step?

We will use the five-step method. The results of step 1 are the same as in Figure 4.3. In step 2 we specify a continuous–time dynamical system model, which we will solve by simulation using the Euler method.

Consider a continuous–time dynamical system model

$$\frac{dx}{dt} = F(x) \tag{6.12}$$

with $x = (x_1, \ldots, x_n)$ and $F = (f_1, \ldots, f_n)$, along with the initial condition $x(t_0) = x_0$. The Euler method uses a discrete time dynamical system

$$\frac{\Delta x}{\Delta t} = F(x) \tag{6.13}$$

to approximate the behavior of the continuous time system. One reason to use a large step size is to make long-range predictions. For example, if time t is measured in years, then $\Delta x = F(x)\Delta t$ is a simple projection of the change in the state variable x over the next Δt years, based on the current state information. If the step size Δt is chosen so that the relative change $\Delta x/x$ remains small, then the discrete time system in Eq. (6.13) will behave much like the original continuous time system in Eq. (6.12). If the step size is too big, then the discrete time system can exhibit very different behavior.

Example 6.5. Consider the simple linear differential equation

$$\frac{dx}{dt} = -x. \tag{6.14}$$

Compare the behavior of solutions to Eq. (6.14) to those of its discrete time analogue

$$\frac{\Delta x}{\Delta t} = -x. \tag{6.15}$$

Solutions to Eq. (6.14) are all of the form

$$x(t) = x(0)e^{-t}. \tag{6.16}$$

The origin is a stable equilibrium, and every solution curve decays exponentially fast to zero. For Eq. (6.15) the iteration function is

$$\begin{aligned} G(x) &= x + \Delta x \\ &= x - x\Delta t \\ &= (1 - \Delta t)x. \end{aligned}$$

Solutions to Eq. (6.15) are all of the form

$$x(n) = (1 - \Delta t)^n x(0).$$

If $0 < \Delta t < 1$, then $x(n) \to 0$ exponentially fast, and the behavior is much like that of the continuous time differential equation. If $1 < \Delta t < 2$, then we still have $x(n) \to 0$, but the sign of $x(n)$ oscillates between positive and negative. Finally, if $\Delta t > 2$, then $x(n)$ diverges to infinity as it oscillates in sign. In summary, the solutions of Eq. (6.15) behave much like those of Eq. (6.14) as long as the time step Δt is chosen so that the relative change $\Delta x/x$ remains small. If the time step Δt is too large, then Eq. (6.15) exhibits behavior entirely different from that of its continuous time analogue Eq. (6.14).

For linear dynamical systems, the time delays inherent in discrete approximations can lead to unexpected behavior. A stable equilibrium can become unstable, and new oscillations can occur.

For linear systems, this is about the only way in which the behavior of the discrete approximation can differ from that of the original continuous system. For nonlinear continuous time dynamical systems, however, discrete approximations can also exhibit chaotic behavior. In a chaotic dynamical system, there is an extreme sensitivity to initial conditions, along with apparently random behavior of individual solutions. Chaos is usually associated with systems in which nearby solutions tend to diverge, but overall remain bounded. This combination of factors can only occur in a nonlinear system.

The study of chaos in discrete time dynamical systems is an active area of research. Some iteration functions produce extremely complex sample paths, including *fractals.* A typical fractal is a set of points in the state space that is self–similar and whose *dimension* is not an integer. *Self–similar* means that the object contains smaller pieces that are exact scaled–down replicas of the whole. One simple way to measure dimension is to count the number of boxes needed to cover the object. For a one–dimensional object, it takes n times as many boxes if they are $1/n$ times as large; for a two–dimensional object, it takes n^2 times as many, and so on. For an object with fractal dimension d, the number of boxes it takes to cover the object increases like n^d as the box size $1/n$ tends to zero.

Step 3 is to formulate the model. We begin with the continuous time dynamical system model

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2) = 0.05x_1\left(1 - \frac{x_1}{150,000}\right) - \alpha x_1 x_2 \\ \frac{dx_2}{dt} &= f_2(x_1, x_2) = 0.08x_2\left(1 - \frac{x_2}{400,000}\right) - \alpha x_1 x_2 \end{aligned} \tag{6.17}$$

on the state space $x_1 \geq 0,\ x_2 \geq 0$, where x_1 denotes the population of blue whales and x_2 the population of fin whales. In order to simulate this model, we will transform to a set of difference equations

$$\begin{aligned} \Delta x_1 &= f_1(x_1, x_2)\Delta t \\ \Delta x_2 &= f_2(x_1, x_2)\Delta t \end{aligned} \tag{6.18}$$

over the same state space. Then, for example, Δx_1 represents the change in the population of Blue whales over the next Δt years. We will assume that $\alpha = 10^{-8}$ to start with, and later on we will do a sensitivity analysis on α. Our objective is to determine the behavior of solutions to the discrete time dynamical system in Eq. (6.18) and compare to what we know about solutions to the continuous time model in Eq. (6.17).

In step 4 we solve the problem by simulating the system in Eq. (6.17) using a computer implementation of the Euler method for several different values of $h = \Delta t$. We assume that $x_1(0) = 5,000$ and $x_2(0) = 70,000$, as in Example 6.2. Figure 6.27 illustrates the results of our simulation with $N = 50$ iterations and

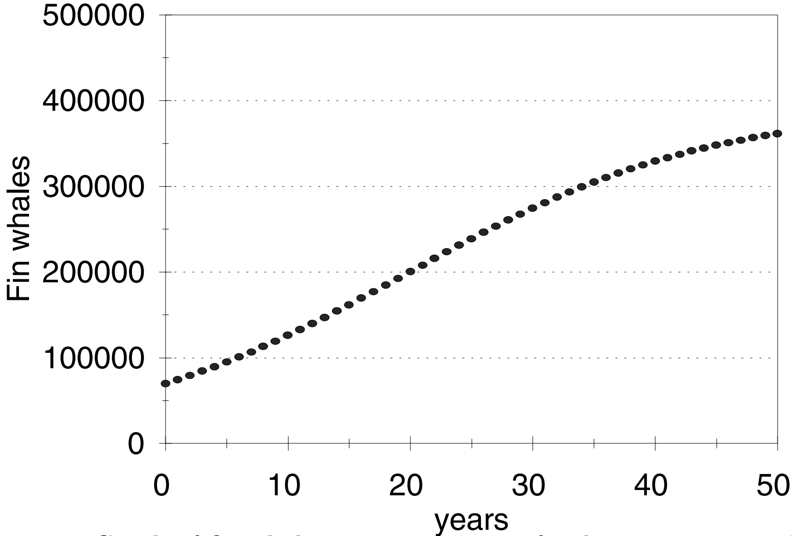

Figure 6.27: Graph of fin whales x_2 versus time t for discrete time simulation of the whale problem with time step $h = 1$.

a time step of $h = 1$ years. In 50 years the fin whales grow back steadily but do not quite reach their eventual equilibrium level. In Figure 6.28 we increase the step size to $h = 2$ years. Now our simulation with $N = 50$ iterations shows the fin whale population approaching its equilibrium value. Using a larger time step h is an efficient way to project further into the future, but then something interesting happens.

Figure 6.29 shows the result of using a time step of $h = 24$ years. The solution still approaches its equilibrium, but now there is an oscillation. Figure 6.30 shows what happens when we use a time step of $h = 27$ years. Now the population actually diverges from the equilibrium and eventually settles into a discrete limit cycle of period two.

When $h = 32$, the solution settles down into a limit cycle of period four; see Figure 6.31. Figure 6.32 shows that when $h = 37$, the solution exhibits chaotic behavior. The effect is similar to that of a random number generator. When $h = 40$ (not shown), the solution quickly diverges to infinity. The behavior of the blue whale population is similar. Different initial conditions and different values of α produce similar results. In every case there is a transition from stability to instability as the step size h increases. As the equilibrium becomes unstable, first oscillations appear, then discrete limit cycles, and then chaos. Finally, if h is too large the solutions simply diverge.

Step 5 is to answer the question. The discrete approximation to the continuous time model is useful as long as the time step is small enough to keep the relative change in the state variables small at each time step. Using a larger

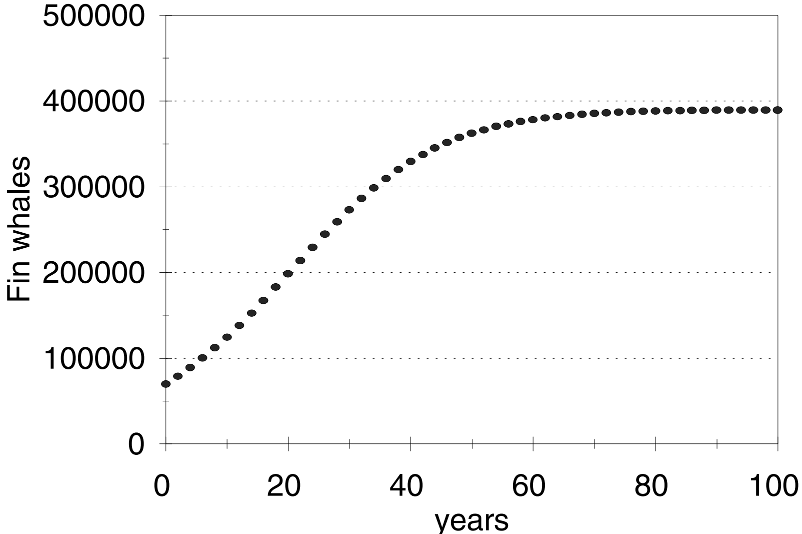

Figure 6.28: Graph of fin whales x_2 versus time t for discrete time simulation of the whale problem with time step $h = 2$.

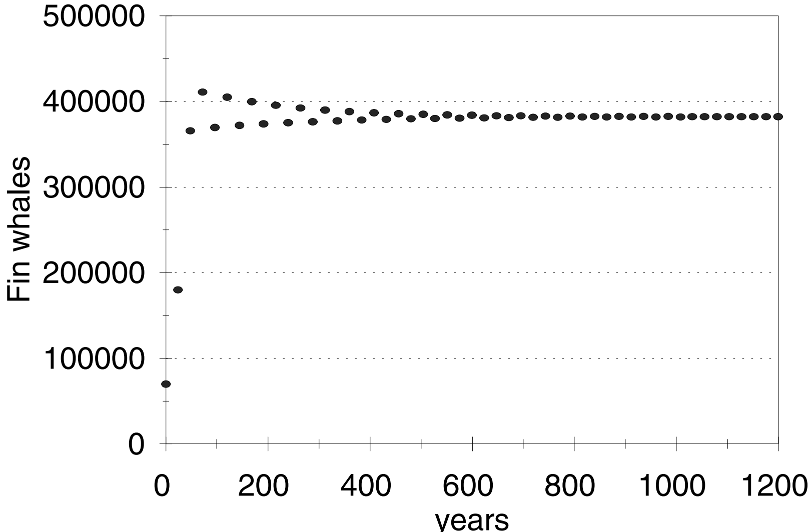

Figure 6.29: Graph of fin whales x_2 versus time t for discrete time simulation of the whale problem with time step $h = 24$.

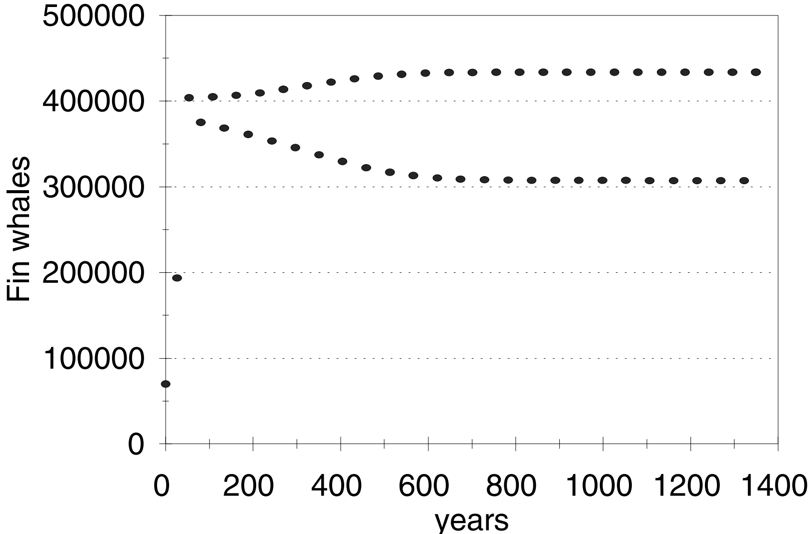

Figure 6.30: Graph of fin whales x_2 versus time t for discrete time simulation of the whale problem with time step $h = 27$.

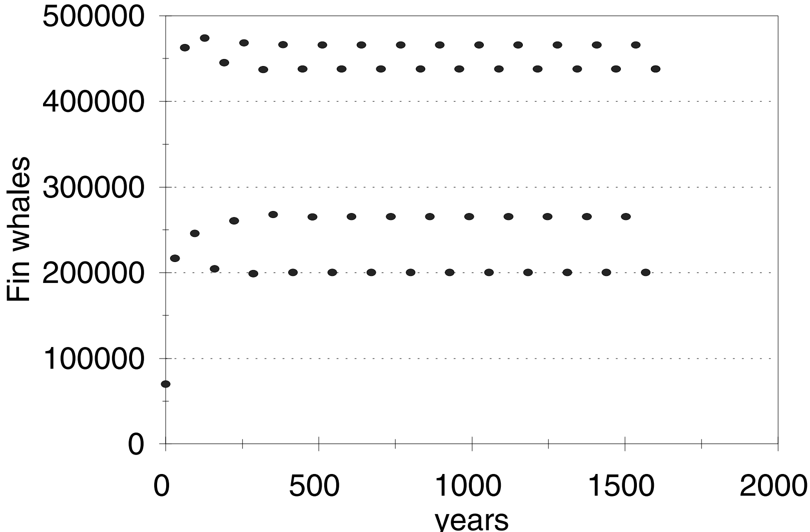

Figure 6.31: Graph of fin whales x_2 versus time t for discrete time simulation of the whale problem with time step $h = 32$.

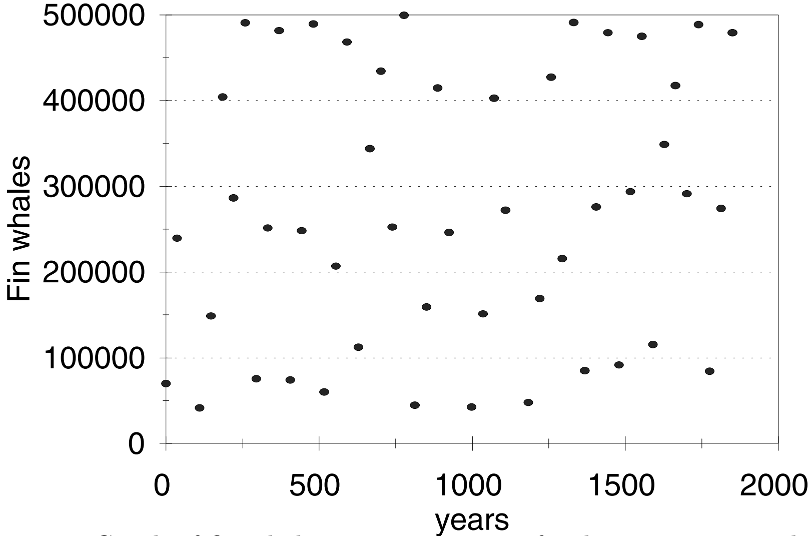

Figure 6.32: Graph of fin whales x_2 versus time t for discrete time simulation of the whale problem with time step $h = 37$.

time step allows us to project further into the future, but when the time step becomes too large, the behavior of the discrete time system no longer resembles that of the original continuous time model. It is interesting to observe the strange behavior of the discrete time system when a larger time step is employed; however, this behavior has no apparent connection to the real–world situation we are trying to model.

Many population models are based on some variation of the logistic model $x' = rx(1-x/K)$. Most of these models exhibit chaos in the discrete approximation. It is typical for the dynamics to transition from stable equilibrium to limit cycles to chaos (and then to unstable divergence) as the time step is increased. At the transition from limit cycles to chaos, the limit set typically becomes a fractal. See Exercise 25 at the end of this chapter for an illustration. There are many interesting books and articles on chaos and fractals. Strogatz (1994) is one good reference at an advanced undergraduate to beginning graduate level.

The emergence of chaos from the discrete approximation in the whale problem is interesting, but seems to have nothing to do with the real world. At best it is a mathematical curiosity, at worst a numerical headache. The next example, however, shows that chaos and fractals can also emerge naturally from realistic models of physical situations.

Example 6.6. Consider a layer of air that is heated from the bottom. In certain situations the warmer air rising up interacts with the colder air sinking down to form turbulent convection rolls. The complete derivation of the dynamics of motion involves a system of partial differential equations, which can be solved by the method of Fourier transforms; see Lorentz (1963). A simplified representation involves three state variables. The variable x_1 represents the rate at which the convection rolls rotate, x_2 represents the temperature difference between the ascending and descending air currents, and x_3 represents the deviation from linearity of the vertical temperature profile, a positive value indicating that the temperature varies faster near the boundary. The equations of motion for this system are

$$\begin{aligned} x_1' &= f_1(x_1, x_2, x_3) = -\sigma x_1 + \sigma x_2 \\ x_2' &= f_2(x_1, x_2, x_3) = -x_2 + r x_1 - x_1 x_3 \\ x_3' &= f_3(x_1, x_2, x_3) = -b x_3 + x_1 x_2, \end{aligned} \tag{6.19}$$

and we will consider the realistic case where $\sigma = 10$ and $b = 8/3$. The remaining parameter r represents the temperature difference between the top and bottom of the air layer. Increasing r pumps more energy into the system, creating more vigorous dynamics. The dynamical system of Eq. (6.19) is called the *Lorentz equations*, after the meteorologist E. Lorentz who analyzed them.

To find the equilibrium points of Eq. (6.19), we solve the system of equations

$$\begin{aligned} -\sigma x_1 + \sigma x_2 &= 0 \\ -x_2 + r x_1 - x_1 x_3 &= 0 \\ -b x_3 + x_1 x_2 &= 0 \end{aligned}$$

for the three state variables. Obviously, $(0, 0, 0)$ is one solution. The first equation implies $x_1 = x_2$. Substituting into the second equation, we obtain

$$\begin{aligned} -x_1 + r x_1 - x_1 x_3 &= 0 \\ x_1(-1 + r - x_3) &= 0 \end{aligned}$$

so that if $x_1 \neq 0$, then $x_3 = r - 1$. Then, from the third equation we obtain $x_1^2 = b x_3 = b(r - 1)$. If $0 < r < 1$, there are no real roots to this equation, and so the origin is the only equilibrium point. If $r = 1$, then $x_3 = 0$, and again the origin is the only equilibrium point. If $r > 1$, then there are three equilibrium points

$$\begin{aligned} E_0 &= (0, 0, 0) \\ E^+ &= (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r - 1) \\ E^- &= (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r - 1). \end{aligned}$$

Graphical analysis of the vector field is difficult, since we are now in three dimensions. Instead, we will perform an eigenvalue analysis to test the stability of these three equilibrium points. The matrix of partial derivatives is

$$DF = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - x_3 & -1 & -x_1 \\ x_2 & x_1 & -b \end{pmatrix}.$$

At the equilibrium $E_0 = (0, 0, 0)$ with the parameter values $\sigma = 10$ and $b = 8/3$, this matrix becomes

$$A = \begin{pmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix},$$

which has three real eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{-11 - \sqrt{81 + 40r}}{2} \\ \lambda_2 &= \frac{-11 + \sqrt{81 + 40r}}{2} \\ \lambda_3 &= \frac{-8}{3} \end{aligned}$$

for any value of $r > 0$. If $0 < r < 1$, then all of these eigenvalues are negative, so that the origin is a stable equilibrium. If $r > 1$, then $\lambda_2 > 0$, so the origin is an unstable equilibrium.

The eigenvalue analysis for the remaining two equilibria is rather messy. Fortunately, the eigenvalues are the same at both E^+ and E^-. For $1 < r < r_1 \approx 1.35$, all three eigenvalues are real and negative. For any value of $r > r_1$, there is one eigenvalue $\lambda_1 < 0$ and one complex conjugate pair $\lambda_2 = \alpha + i\beta$, $\lambda_3 = \alpha - i\beta$. The real part α is negative for $r_1 < r < r_0$ and positive for $r > r_0$, where $r_0 \approx 24.8$. Thus, for $1 < r < r_0$, there are two stable equilibria at E^+ and E^-, while for $r > r_0$, every equilibrium is unstable. Solutions nearby these two equilibria will behave much like those of the linear system $x' = Ax$, where A is the matrix of partial derivatives DF evaluated at the equilibrium point. Every component of every linear solution can be written as a linear combination of terms of the form $e^{\lambda_1 t}$, $e^{\alpha t}\cos(\beta t)$, and $e^{\alpha t}\sin(\beta t)$. When $r_1 < r < r_0$, nearby solution curves spiral into the nonzero equilibrium points, and when $r > r_0$, they spiral outwards. It also turns out that solutions do not diverge to infinity when $r > r_0$. We have seen this kind of behavior before, in the nonlinear RLC circuit of Example 5.4. In that case a computer simulation showed that the solution curves settled into a limit cycle. We will now simulate the dynamical system of Eq. (6.19) to determine the long–term behavior of solutions.

The Euler method for three state variables uses exactly the same algorithm as in Figure 6.19 except for the addition of another state variable. A computer implementation of that algorithm was used to simulate the solution of Eq. (6.19) in the case $\sigma = 10$ and $b = 8/3$. Figure 6.33 shows the results of a simulation using $r = 8$ and initial conditions $(x_1, x_2, x_3) = (1, 1, 1)$. We graphed x_2 versus x_1 since these variables are the easiest to interpret. We used $N = 500$ and $T = 5$, so that the step size was $h = 0.01$. Additional sensitivity runs were made to ensure that increasing the simulation time T or decreasing the step size h led to essentially the same graph. As we expected from our earlier analysis, the solution curve spirals into the equilibrium point at $x_1 = x_2 = 4.32$ (and $x_3 = 7$). Recall that x_1 represents the rate at which the convection rolls rotate, and x_2 represents the temperature difference between the ascending and descending air

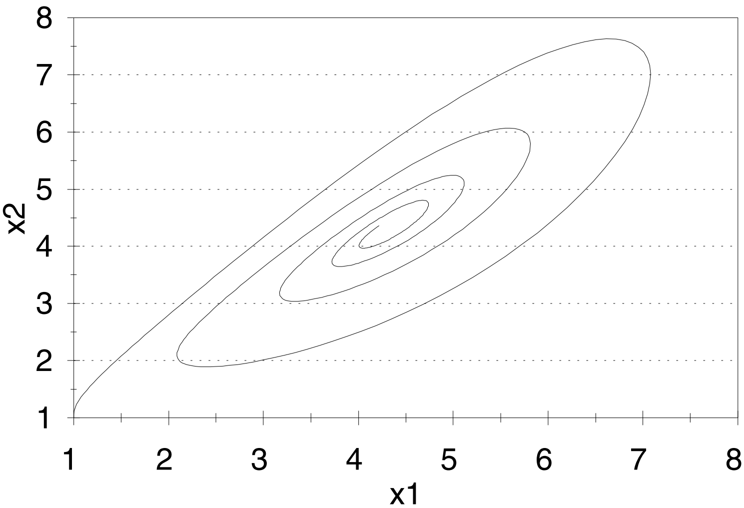

Figure 6.33: Graph of temperature differential x_2 versus convection rate x_1 for the weather problem with $r = 8$ and initial condition $(x_1, x_2, x_3) = (1, 1, 1)$.

currents. When $r = 8$, both of these quantities eventually settle down into a stable equilibrium. Figure 6.34 shows the simulation for the case $r = 8$ and initial condition $(x_1, x_2, x_3) = (7, 1, 2)$. Keep in mind that this graph is actually a projection of a three–dimensional picture. Of course, the real solution curve does not cross itself. That would violate the uniqueness of solutions. Once again, the solution spirals into the equilibrium.

As we increase the value of r, the simulation becomes extremely sensitive to discretization. Figure 6.35 shows the results of a simulation using $r = 18$ and initial conditions $(x_1, x_2, x_3) = (6.7, 6.7, 17)$. We used $N = 500$ and $T = 2.5$ for a step size of $h = 0.005$. The solution curve rotates rapidly about the equilibrium $E^+ = (6.733, 6.733, 17)$ while it spirals in very slowly. Figure 6.36 shows the results of the same simulation using $N = 500$ and $T = 5$ for a slightly larger step size of $h = 0.01$. In this case the solution spirals outward, away from the equilibrium. Of course, this is not really what is happening in the continuous time model, it is just an artifact of our simulation method. Because the system is very near the point between stability and instability, we must be careful to perform sensitivity analysis on the parameters N and T to ensure that the behavior of the discrete time system really reflects what is going on with the continuous time model.

Finally, we consider the case $r > r_0$, where we know that the equilibrium points are all unstable. Figure 6.37 shows the results of a simulation using $r = 28$ and initial conditions $(x_1, x_2, x_3) = (9, 8, 27)$. We used $N = 500$ and $T = 10$ for a step size of $h = 0.02$. A careful sensitivity analysis on N and T was performed

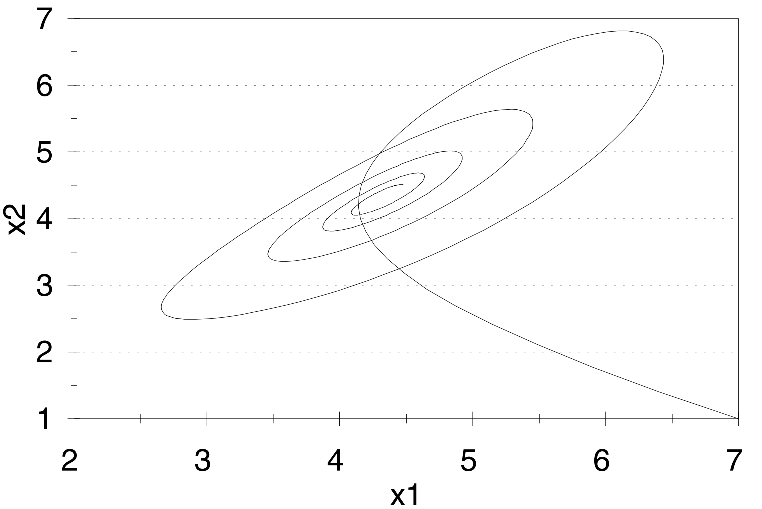

Figure 6.34: Graph of temperature differential x_2 versus convection rate x_1 for the weather problem with $r = 8$ and initial condition $(x_1, x_2, x_3) = (7, 1, 2)$.

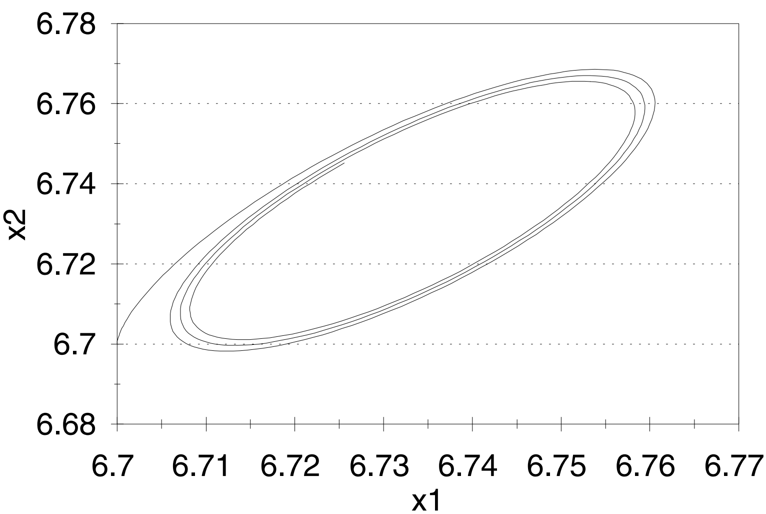

Figure 6.35: Graph of temperature differential x_2 versus convection rate x_1 for the weather problem with $r = 18$ and initial condition $(x_1, x_2, x_3) = (6.7, 6.7, 17)$ using step size $h = 0.005$.

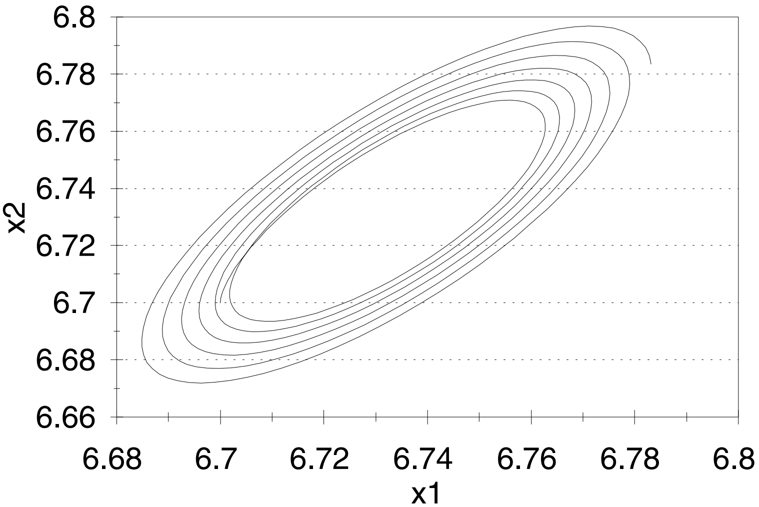

Figure 6.36: Graph of temperature differential x_2 versus convection rate x_1 for the weather problem with $r = 18$ and initial condition $(x_1, x_2, x_3) = (6.7, 6.7, 17)$ using step size $h = 0.01$.

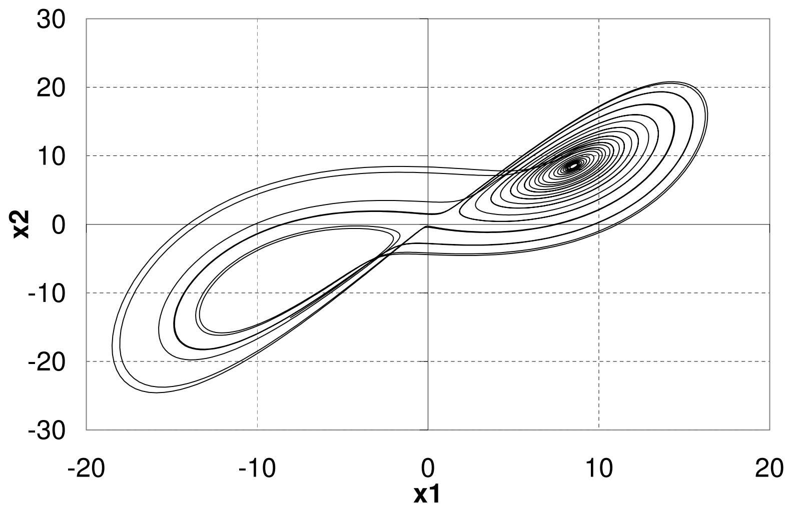

Figure 6.37: Graph of temperature differential x_2 versus convection rate x_1 for the weather problem with $r = 28$ and initial condition $(x_1, x_2, x_3) = (9, 8, 27)$.

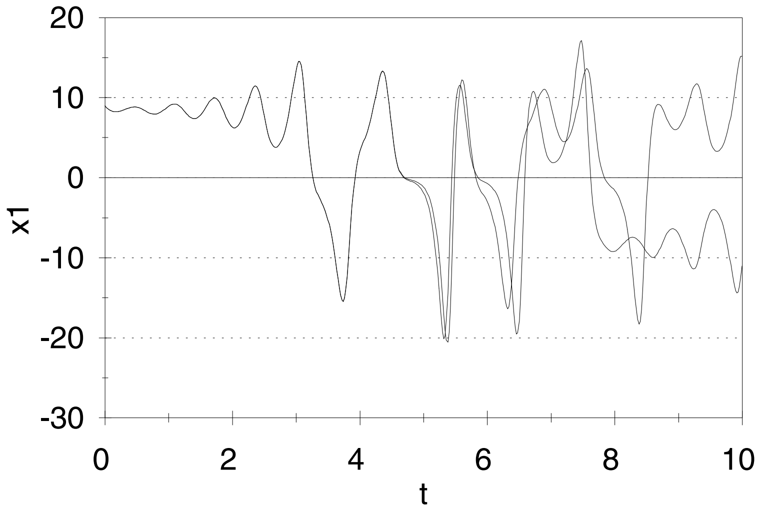

Figure 6.38: Graph of convection rate x_1 versus time t for the weather problem with $r = 28$ comparing two initial conditions $(x_1, x_2, x_3) = (9, 8, 27)$ and $(x_1, x_2, x_3) = (9.01, 8, 27)$.

to verify that this solution curve really represents the behavior of our continuous time model. At first the solution curve rotates rapidly about the equilibrium $E^+ = (8.485, 8.485, 27)$ while it spirals outward very slowly. Eventually, the solution curve heads out toward the equilibrium $E^- = (-8.485, -8.485, 27)$, where it spirals around for a while, and then eventually heads back towards E^+. Simulations using larger values of N and T indicate that the solution never repeats itself, yet remains bounded. It continues to cross over between the region around E^+ and the region around E^-.

Other solutions with nearby initial conditions show essentially the same behavior. There is also an extreme sensitivity to initial conditions. Solutions that begin very close together eventually move farther apart. The solution curve represented in Figure 6.37 started at the point $(x_1, x_2, x_3) = (9, 8, 27)$ at time $t = 0$, and by time $t = 10$, it was spiraling around E^-. A nearby solution curve, starting at the point $(x_1, x_2, x_3) = (9.01, 8, 27)$ at time $t = 0$, ends up spiraling around E^+ by time $t = 10$. Figure 6.38 compares the path of the x_1 coordinate for these two solution curves. By time $t = 10$ it would be impossible to guess that these two solutions began at almost the same initial condition. Yet if we let the simulation time T get larger and larger, we see that both solution curves trace out almost exactly the same shape in the state space. This limiting set is called a *strange attractor*. Although it is bounded, its length is infinite, and cross sections (for example, the collection of all points at which the curve intersects the plane $x_3 = 0$) are typically fractals. In fact, one reasonable analytic approach for dealing with the strange attractor is to consider the mapping that sends a given intersection point to the next. This iteration function acts much like the one we analyzed in Example 6.4 for the logistic model of population growth. In this rather unexpected way, the analysis of discrete and continuous time dynamical systems finds a new connection.

6.5 Exercises

1. Reconsider the war problem of Example 6.1. In this problem we explore the effects of weather on combat. Bad weather and poor visibility decrease the effectiveness of direct fire weapons for both sides. The effectiveness of indirect fire weapons is relatively unaffected by the weather. We can represent the effects of bad weather in our model as follows. Let w denote the decrease in weapon effectiveness caused by bad weather conditions, and replace the dynamical system in Eq. (6.3) by

$$\begin{aligned}\Delta x_1 &= -w\lambda(0.05)x_2 - \lambda(0.005)x_1x_2 \\ \Delta x_2 &= -w0.05x_1 - 0.005x_1x_2.\end{aligned} \tag{6.20}$$

   Here, the parameter $0 \leq w \leq 1$ represents a range of weather conditions, with $w = 1$ indicating the best weather and $w = 0$ indicating the worst weather.

   (a) Use a computer implementation of the algorithm in Fig. 6.2 to simulate the discrete–time dynamical system in Eq. (6.20) in the case $\lambda = 3$. Assume that adverse weather conditions cause a 75% decrease in weapon effectiveness for both sides ($w = 0.25$). Who wins the battle, and how long does it take? How many divisions of troops remain on the winning side?

   (b) Repeat your analysis for each of the cases $w = 0.1$, 0.2, 0.5, 0.75, and 0.9, and tabulate your results. Answer the same questions as in part (a).

   (c) Which side benefits from fighting in adverse weather conditions? If you were the blue commander, would you expect red to attack on a sunny day or a rainy day?

   (d) Examine the sensitivity of your results in parts (a), (b), and (c) to the degree of weapon superiority of blue over red. Repeat the simulations in parts (a) and (b) for $\lambda = 1.5$, 2.0, 4.0, and 5.0, and tabulate your results as before. Reconsider your conclusions in part (c). Are they still valid?

2. Reconsider the war problem of Example 6.1. In this problem we will consider the effect of tactics on the outcome of the battle. The red commander is considering the option of holding two of his five divisions in reserve until the second or third day of combat. You can simulate each possibility as

a deviation from the baseline case by running two separate simulations. First, simulate the first one or two days of battle, matching two blue divisions against three red divisions. Then, use the final outcome of that simulation as the initial conditions for the rest of the battle, except add two more divisions to the red side.

(a) Use a computer implementation of the algorithm in Fig. 6.2 to simulate the first phase of battle, in which two blue divisions fight three red divisions. Assume $\lambda = 2$, and tabulate the final force levels for the two cases (12 or 24 hours of battle).

(b) Use the results of part (a) to simulate the next phase of battle. Add two divisions to the final force levels for red, and continue the simulation. In each case, tell which side wins the battle, how many units remain on the winning side, and how long the battle lasts (total time for both phases of battle).

(c) The red commander may choose to commit all of his forces on the first day, or he may keep two divisions in reserve for one or two days. Which of the three strategies is better? Optimize on the basis of achieving victory at the minimum cost in terms of lost manpower.

(d) Perform a sensitivity analysis on the parameter λ, which describes the extent to which blue has weapon superiority. Repeat parts (a) and (b) for $\lambda = 1.0, 1.5, 3.0, 5.0$, and 6.0, and identify the optimal strategy for each value of λ. State your general conclusions concerning the optimal strategy for red.

3. Reconsider the war problem of Example 6.1. In this problem we will investigate the effect of tactical nuclear weapons on the battle. As a desperation move, the blue commander considers calling for a tactical nuclear strike. It is estimated that such a strike will kill or incapacitate 70% of the red force, and 35% of the blue force as well.

(a) Use a computer implementation of the algorithm in Fig. 6.2 to simulate the discrete–time dynamical system in Eq. (6.3), assuming that the blue commander calls for an immediate nuclear strike. Start with initial conditions $x_1 = (0.30)5.0, x_2 = (0.65)2.0$, and assume $\lambda = 3$. Who wins the battle, and how long does it take? How many divisions survive on the winning side? How does blue benefit from calling for a nuclear attack in this case?

(b) Simulate the case where the blue commander waits for six hours and then calls for a nuclear attack. Simulate six hours of battle, starting with $x_1 = 5$ divisions and $x_2 = 2$ divisions. Reduce the remaining number of troops for both sides to represent the results of a nuclear strike, and then continue the simulation. Answer the same questions as in part (a).

(c) Compare the results of parts (a) and (b) to the case of conventional combat summarized in the chapter. Discuss the benefits of a tactical nuclear strike by blue. Can such a move be effective and, if so, when should the commander request the strike?

(d) Examine the sensitivity of your conclusions in part (c) to the extent λ of blue weapon superiority. Repeat the simulations of parts (a) and (b) for each of the cases $\lambda = 1.0, 1.5, 2.0, 5.0$, and 6.0, and answer the same questions as before.

4. Reconsider the space docking problem of Example 5.2. Suppose that our initial closing velocity is 50 m/sec under zero acceleration.

   (a) Determine the time required for docking, assuming that the control factor $k = 0.02$. Use a computer implementation of the algorithm for discrete–time dynamical systems described in Fig. 6.2. Assume docking is complete when the closing velocity has been reduced to less than 0.1 m/sec in absolute value for all future time.

   (b) Repeat the simulation of part (a) for each of the cases $k = 0.01, 0.02, 0.03, \ldots, 0.20$, and determine the time required for docking in each case. Which of these values of k results in the quickest time to dock?

   (c) Repeat part (b), assuming an initial closing velocity of 25 m/sec.

   (d) Repeat part (b), assuming an initial closing velocity of 100 m/sec. What conclusions can you draw about the optimal value of k for this docking procedure?

5. Reconsider the infectious disease problem introduced in Exercise 10 of Chapter 4. Use a computer implementation of the algorithm in Fig. 6.2 to simulate this discrete–time dynamical system model. Answer the questions in parts (a) through (d) from the original exercise.

6. In Exercise 4 of Chapter 4, we introduced a simplified model of population growth in the whale problem.

   (a) Simulate this model, assuming that there are currently 5,000 blue whales and 70,000 fin whales. Use the simple simulation technique of Section 6.2, and assume $\alpha = 10^{-7}$. What happens to the two species of whales over the long term, according to this model?

   (b) Examine the sensitivity of your conclusions in part (a) to the assumption that there are currently 5,000 blue whales. Repeat the simulation of part (a), assuming that there are originally 3,000, 4,000, 6,000, or 7,000 blue whales. How sensitive are your conclusions to the exact number of blue whales in the ocean at the present time?

   (c) Examine the sensitivity of your conclusions in part (a) to the assumption that the intrinsic growth rate of the blue whale is 5% per year. Repeat the simulation of part (a), assuming that the actual rate is 3, 4, 6, or 7% per year. How sensitive are your conclusions to the actual intrinsic growth rate for the blue whales?

(d) Examine the sensitivity of your conclusions in part (a) to the competition coefficient α. Repeat the simulation of part (a) for each of the cases $\alpha = 10^{-9}$, 10^{-8}, 10^{-6}, and 10^{-5}, and tabulate your results. How sensitive are your general conclusions to the extent of competition between the two species?

7. In Exercise 5 of Chapter 4, we introduced a more sophisticated model of population growth in the whale problem.

   (a) Simulate this model, assuming that there are currently 5,000 blue whales and 70,000 fin whales. Use the simple simulation technique of Section 6.2, and assume that $\alpha = 10^{-8}$. What happens to the two species of whales over the long term, according to this model? Do both species of whales grow back, or will one or both species become extinct? How long does this take?

   (b) Examine the sensitivity of your conclusions in part (a) to the assumption that there are currently 5,000 blue whales. Repeat the simulation of part (a), assuming that there are originally 2,000, 3,000, 4,000, 6,000, or 8,000 blue whales. How sensitive are your conclusions to the exact number of blue whales in the ocean at the present time?

   (c) Examine the sensitivity of your conclusions in part (a) to the assumption that the intrinsic growth rate of the blue whale is 5% per year. Repeat the simulation of part (a), assuming that the actual rate is 2, 3, 4, 6, or 7% per year. How sensitive are your conclusions to the actual intrinsic growth rate for the blue whales?

   (d) Examine the sensitivity of your conclusions in part (a) to the assumption that the minimum viable population level of the blue whale is 3,000 whales. Repeat the simulation of part (a), assuming that the actual level is 1,000, 2,000, 4,000, 5,000, or 6,000 whales. How sensitive are your conclusions to the actual minimum viable population level for the blue whales?

8. Reconsider Exercise 6 in Chapter 4, and assume $\alpha = 10^{-8}$. Assume that the current population levels are $B = 5{,}000$ and $F = 70{,}000$.

   (a) Use a computer implementation of the simple algorithm used in Section 6.2 to determine the effect of harvesting. Assume $E = 3{,}000$ boat–days per year. What happens to the two species of whales over the long term, according to this model? Do both species of whales grow back, or will one or both species become extinct? How long does this take?

   (b) Repeat part (a), assuming that $E = 6{,}000$ boat–days per year.

   (c) For what range of E does the number of whales of both species approach a nonzero equilibrium?

(d) Repeat part (c) for each of the cases $\alpha = 10^{-9}$, 10^{-8}, 10^{-6}, and 10^{-5}, and tabulate your results. Discuss the sensitivity of your conclusions to the extent of interspecies competition.

9. Reconsider the whale harvesting problem of Exercise 6 in Chapter 4. In this problem we will explore the economic incentives for whalers to drive one species of whale to extinction. Assume that there are currently 5,000 blue whales and 70,000 fin whales.

   (a) Simulate this model, assuming $E = 3{,}000$ boat–days. Use the simple simulation technique of Section 6.2, and assume that $\alpha = 10^{-7}$. Determine the long–term harvest rate in blue whale units per year (2 fin whales = 1 blue whale unit).

   (b) Determine the level of effort that maximizes the long–term harvest rate in blue whale units. Simulate each of the cases $E = 500$, 1,000, 1,500, ..., 7,500, boat–days per year. Which case results in the highest sustainable yield?

   (c) Assume that whalers harvest at the rate that maximizes their long–term sustainable yield. What happens to the two species of whales over the long term, according to this model? Do both species of whales grow back, or will one or both species become extinct? How long does this take?

   (d) Some economists argue that whalers will act to maximize the long–term sustainable yield for the entire industry. If so, would continued harvesting cause one or both species of whales to become extinct?

10. (Continuation of Exercise 9) Some economists argue that whalers will act in such a way as to maximize the total discounted revenue obtained by the entire whaling industry. Assume that harvesting produces revenue of \$10,000 per blue whale unit, and assume a discount rate of 10%. If revenue R_i is obtained in year i, the total discounted revenue is defined as

$$R_0 + \lambda R_1 + \lambda^2 R_2 + \lambda^3 R_3 + \cdots ,$$

where $1 - \lambda$ represents the discount rate ($\lambda = 0.9$ for this problem).

   (a) Simulate this model, assuming $E = 3{,}000$ boat–days. Use the simple simulation technique of Section 6.2, and assume that $\alpha = 10^{-7}$. Determine the total discounted revenue for this case.

   (b) Determine the level of effort that maximizes the total discounted revenue. Simulate each of the cases $E = 500$, 1,000, 1,500, ..., 7,500, boat–days per year. Which case results in the highest yield?

   (c) Assume that whalers harvest at the rate that maximizes their total discounted revenue. What happens to the two species of whales over the long term, according to this model? Do both species of whales grow back, or will one or both species become extinct? How long does this take?

(d) Perform a sensitivity analysis on the parameter α, which measures the extent of interspecies competition. Consider each of the cases $\alpha = 10^{-9}$, 10^{-7}, 10^{-6}, and 10^{-5}, and tabulate your results. Discuss the sensitivity of your conclusions to the extent of interspecies competition.

11. Reconsider the predator–prey model of Exercise 7 in Chapter 4.

    (a) Determine by simulation the equilibrium levels for whales and krill. Use the simple simulation technique discussed in Section 6.2. Begin at several different initial conditions and run the simulation until both population levels have settled down into steady state.

    (b) Suppose that after both population levels have settled down into steady state, an ecological disaster kills off 20% of the whales and 80% of the krill. Describe what happens to the two species, and how long it takes.

    (c) Suppose that harvesting has depleted the whales to 5% of their equilibrium population level, while krill remain at about the same level. Describe what happens once harvesting is stopped. How long does it take for the whales to grow back? What happens to the krill population?

    (d) Examine the sensitivity of your results in part (c) to the assumption that 5% of the whales remain. Simulate each of the cases where 1, 3, 7, or 10% remain, and tabulate your results. How sensitive is the time it takes for the whales to grow back to the extent to which the population is depleted?

12. Reconsider the tree problem of Example 5.1.

    (a) Determine how long it will take for both hardwoods and softwoods to grow to 90% of their stable equilibrium levels. Assume an initial population of 1,500 tons/acre of softwood trees and 100 tons/acre of hardwoods. This is the situation in which we are trying to introduce a new type of more valuable tree into an existing ecosystem. Assume $b_i = a_i/2$, and use the simple simulation technique introduced in Section 6.2.

    (b) Determine the point at which the biomass of hardwood trees is increasing at the fastest rate.

    (c) Assuming that hardwoods are worth four times as much as softwoods in $/ton, determine the point at which the value of the forest stand ($/acre) is increasing at the fastest rate.

13. (Continuation of Exercise 12) Clear–cutting is a technique in which all of the trees in the forest are harvested at one time and then replanted.

(a) Determine the optimal harvest policy for this forest; i.e., determine the number of years we should wait before cutting and replanting. Assume that replanting involves 100 tons/acre of hardwoods and 100 tons/acre of softwoods. Base your answer on the number of $/acre per year generated.

(b) Determine the optimal harvest policy, assuming that only hardwoods are replanted (200 tons/acre).

(c) Repeat part (b), but now assume that only softwoods are replanted (200 tons/acre).

(d) State the optimal clear–cutting policy for management of this tract of forest land. At what point would you consider selling the land rather than reforesting?

14. Reconsider the more sophisticated competing species model of Exercise 5 in Chapter 4. Assume $\alpha = 10^{-8}$.

    (a) Use a computer implementation of the Euler method to simulate the behavior of this model, starting with the initial conditions $x_1 =$ 5,000 blue whales and $x_2 =$ 70,000 fin whales. Perform a sensitivity analysis on both T and N to ensure the validity of your results, as in the text. What happens to the two species of whales over the long term, according to this model? Do both species of whales grow back, or will one or both species become extinct? How long does this take?

    (b) Repeat part (a) for a range of initial conditions for both blue and fin whales. Tabulate the results of your simulations, and answer the same questions as in part (a) for each case.

    (c) Use the results of parts (a) and (b) in order to draw the complete phase portrait for this system.

    (d) Indicate the region on the phase portrait where one or both species of whale are destined to become extinct.

15. Reconsider the RLC circuit problem of Example 6.3, and perform a sensitivity analysis on the parameter L, which represents inductance.

    (a) Generalize the dynamical system model in Eq. (6.8) to represent the case $L > 0$. How does the vector field for this model vary with L?

    (b) Determine the form of the linear system that approximates the behavior of the nonlinear RLC circuit model in the neighborhood of the origin. Calculate the eigenvalues for the linear system as a function of L. Determine the range of L over which both eigenvalues are complex with positive real part, as in the baseline case $L = 1$.

    (c) Use a computer implementation of the Euler method to simulate the behavior of the RLC circuit for each of the cases $L = 0.5, 0.75, 1.5$, and 2.0. Use the same initial condition $x_1 = 0.1$, $x_2 = 0.3$ as in Fig. 6.21. For each case, perform a sensitivity analysis on both T and N to ensure the validity of your results, as in the text.

(d) Simulate several additional initial conditions for each value of L specified in part (c). Draw the complete phase portrait for each case. Describe how the phase portrait changes in response to changes in the inductance L.

16. Reconsider the RLC circuit problem of Example 6.3, and now consider what happens in the case of a large capacitance, $C > 4$.

    (a) Solve the linear system in Eq. (6.10) by the method of eigenvalues and eigenvectors in the case $C > 4$.

    (b) Draw the phase portrait for this linear system. How does the phase portrait change as a function of C?

    (c) Use a computer implementation of the Euler method to simulate the RLC circuit for each of the cases $C = 5, 6, 8$, and 10. Use the same initial condition $x_1 = 0.1$, $x_2 = 0.3$ as in Fig. 6.21. For each case, perform a sensitivity analysis on both T and N to ensure the validity of your results, as in the text.

    (d) Simulate several additional initial conditions for each value of C specified in part (c). Draw the complete phase portrait for each case. Contrast with the case $0 < C < 4$ discussed in the text. What changes occur in the phase portrait as we transition between the two cases?

17. Reconsider the RLC circuit problem of Example 6.3, and now consider the robustness of our general conclusions with respect to the assumption that the resistor in this RLC circuit has v–i characteristic $f(x) = x^3 - x$. In this problem we will assume that $f(x) = x^3 - ax$, where the parameter a may represent any positive real number. (The case $a = -4$ was the subject of Example 5.3.)

    (a) Generalize the dynamical system model in Eq. (6.8) to represent the general case $a > 0$. How does the vector field for this model vary with a?

    (b) Determine the form of the linear system that approximates the behavior of the nonlinear RLC circuit model in the neighborhood of the origin. Calculate the eigenvalues for the linear system as a function of a.

    (c) Draw the phase portrait for this linear system. How does the phase portrait change as a function of a?

    (d) Use a computer implementation of the Euler method to simulate the RLC circuit for each of the cases $a = 0.5, 0.75, 1.5$, and 2.0, and draw the complete phase portrait for each case. What changes occur in the phase portrait as we change a? What do you conclude about the robustness of this model?

18. Reconsider the RLC circuit problem of Example 6.3, and now consider the robustness of our general conclusions with respect to the assumption that the resistor in this RLC circuit has v–i characteristic $f(x) = x^3 - x$. In this problem we will assume that $f(x) = x|x|^{1+b} - x$, where $b > 0$.

    (a) Generalize the dynamical system model in Eq. (6.8) to represent the general case $b > 0$. How does the vector field for this model vary with b?

    (b) Determine the form of the linear system that approximates the behavior of the nonlinear RLC circuit model in the neighborhood of the origin. Calculate the eigenvalues for the linear system as a function of b.

    (c) Draw the phase portrait for this linear system. How does the phase portrait change as a function of b?

    (d) Use a computer implementation of the Euler method to simulate the RLC circuit for each of the cases $b = 0.5$, 0.75, 1.25, and 1.5. Draw the complete phase portrait for each case. What changes occur in the phase portrait as we change b? What do you conclude about the robustness of this model?

19. A pendulum consists of a 100 g weight at the end of a lightweight rod 120 cm in length. The other end of the rod is fixed, but can rotate freely. The frictional forces acting on the moving pendulum are thought to be roughly proportional to its angular velocity.

    (a) The pendulum is lifted manually until the rod makes a 45^0 angle with the vertical. Then the pendulum is released. Determine the subsequent motion of the pendulum. Use the five-step method, and model as a continuous–time dynamical system. Simulate using the Euler method. Assume that the force due to friction is of magnitude $k\theta'$, where θ' is the angular velocity in radians per second and the coefficient of friction is $k = 0.05$ g/sec.

    (b) Use a linear approximation to determine the approximate behavior of the system near equilibrium. Assume that the magnitude of the frictional force is $k\theta'$. How does the local behavior depend on k?

    (c) Determine the period of the pendulum. How does period vary with k?

    (d) This size pendulum will be used as part of the mechanism for a grandfather clock. In order to maintain a certain amplitude of oscillation, a force is to be applied periodically. How much force should be applied, and how often, to produce an amplitude of $\pm 30^0$? How does the answer depend on the amplitude desired? [Hint: Simulate one period of the pendulum oscillation. Vary the initial angular velocity $\theta'(0)$ to obtain periodic behavior.]

20. (Chaos) This problem illustrates the striking difference between the behavior of continuous–time and discrete–time dynamical systems that can occur even in simple models.

    (a) Show that the continuous–time dynamical system

    $$\begin{aligned} x_1' &= (a-1)x_1 - ax_1^2 \\ x_2' &= x_1 - x_2 \end{aligned}$$

    has a stable equilibrium at $x_1 = x_2 = (a-1)/a$ for any $a > 1$.

    (b) Show that the analogous discrete–time dynamical system

    $$\begin{aligned} \Delta x_1 &= (a-1)x_1 - ax_1^2 \\ \Delta x_2 &= x_1 - x_2 \end{aligned}$$

    also has an equilibrium at $x_1 = x_2 = (a-1)/a$ for any $a > 1$.

    (c) Use a simulation to explore the stability of the equilibrium $x_1 = x_2 = (a-1)/a$ and the behavior of nearby solutions for the discrete–time dynamical system. For each of the cases $a = 1.5$, 2.0, 2.5, 3.0, 3.5, and 4.0, try several different initial conditions near the equilibrium point and report what you see. (The case $a = 4.0$ represents a simple model of chaos, the apparently random behavior of a deterministic dynamical system.)

21. (Programming exercise) An alternative method that can be used to simulate dynamical systems is the Runge–Kutta method.

    Figure 6.39 gives an algorithm for the Runge–Kutta method to simulate a dynamical system in two variables,

    $$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2) \\ \frac{dx_2}{dt} &= f_1(x_1, x_2). \end{aligned}$$

    For a fairly small step size h, Runge–Kutta has the property that doubling the number of steps (halving h) produces results approximately 16 times more accurate.

    (a) Implement the Runge–Kutta method on a computer.

    (b) Verify your computer implementation by using it to solve the linear system given by Eq. (5.18) in Chapter 5. Compare your results to the analytic solution in Eq. (5.19) for the case $c_1 = 1$, $c_2 = 0$.

    (c) Verify the results obtained in Figs. 6.20 and 6.21 for the RLC circuit problem of Example 6.3.

22. Reconsider the whale problem of Example 6.4. In this problem we will explore the behavior of the blue whale population. Assume that $\alpha = 10^{-8}$ and begin with $x_1 = 5,000$ blue whales and $x_2 = 70,000$ fin whales.

Algorithm: RUNGE–KUTTA METHOD

Variables:
$t(n)$ = time after n steps
$x_1(n)$ = first state variable at time $t(n)$
$x_2(n)$ = second state variable at time $t(n)$
N = number of steps
T = time to end simulation

Input: $t(0)$, $x_1(0)$, $x_2(0)$, N, T

Process:
```
Begin
h ← (T − t(0))/N
for n = 0 to N − 1 do
    Begin
    r1 ← f1(x1(n), x2(n))
    s1 ← f2(x1(n), x2(n))
    r2 ← f1(x1(n) + (h/2)r1, x2(n) + (h/2)s1)
    s2 ← f2(x1(n) + (h/2)r1, x2(n) + (h/2)s1)
    r3 ← f1(x1(n) + (h/2)r2, x2(n) + (h/2)s2)
    s3 ← f2(x1(n) + (h/2)r2, x2(n) + (h/2)s2)
    r4 ← f1(x1(n) + hr3, x2(n) + hs3)
    s4 ← f2(x1(n) + hr3, x2(n) + hs3)
    x1(n + 1) ← x1(n) + (h/6)(r1 + 2r2 + 2r3 + r4)
    x2(n + 1) ← x2(n) + (h/6)(s1 + 2s2 + 2s3 + s4)
    t(n + 1) ← t(n) + h
    End
End
```

Output: $t(1), \ldots, t(N);\ x_1(1), \ldots, x_1(N);\ x_2(1), \ldots, x_2(N)$

Figure 6.39: Pseudocode for the Runge–Kutta method.

(a) Use the Euler method with a time step of $h = \Delta t = 1$ year. Simulate $N = 50$ time steps and describe the behavior of the blue whale population over time.

(b) Repeat the simulation of part (a) with $N = 50$ time steps for each of the cases $h = 5, 10, 20, 30, 35,$ and 40. How does the behavior of the blue whale population change as the time step h increases?

(c) Repeat part (b) for $\alpha = 10^{-7}$ and $\alpha = 10^{-9}$. How sensitive are your conclusions in part (b) to the assumption that $\alpha = 10^{-8}$?

(d) Repeat part (b) starting from the initial condition $x_1 = 150,000$ blue whales and $x_2 = 400,000$ fin whales, and assume that $\alpha = 10^{-8}$. How sensitive are your conclusions in part (b) to the assumption that we start with $x_1 = 5,000$ blue whales and $x_2 = 70,000$ fin whales?

23. Reconsider the whale problem of Example 6.4. In this problem we will explore the sensitivity to initial conditions for a chaotic dynamical system. Assume that $\alpha = 10^{-8}$ and that we begin with $x_2(0) = 70,000$ fin whales.

    (a) Use the Euler method with a time step of $h = \Delta t = 35$ years. Simulate to determine the number of blue whales $x_1(T)$ that remain after $T = 1750$ years ($N = 50$ time steps), using the initial condition $x_1(0) = 5,000$ blue whales.

    (b) Repeat the simulation of part (a) using the initial condition $x_1(0) = 5050$ and determine the resulting blue whale population $x_1(T)$ after $T = 1750$ years. Compare to the results of part (a) and compute the sensitivity of the eventual population level to the initial condition. Note that the relative change in the initial condition is $\Delta x_1(0)/x_1(0) = 0.01$ and the relative change in the eventual population level is $\Delta x_1(T)/x_1(T)$.

    (c) Repeat part (b) for each of the initial conditions $x_1 = 5005$, 5000.5, 5000.05, 5000.005 and comment on the relation between the sensitivity and the difference $\Delta x_1(0)$ in the initial condition.

    (d) How sensitive is this chaotic dynamical system to small changes in the initial condition? If we estimate the current state of such a system, can we reliably predict its future?

24. Reconsider the whale problem of Example 6.4. In this problem we will explore the transition from stability to instability in the discrete approximation of Eq. (6.18) as the step size $h = \Delta t$ increases. Assume that $\alpha = 10^{-8}$.

    (a) Compute the coordinates of the equilibrium in the positive first quadrant for the continuous time dynamical system of Eq. (6.17). Use the eigenvalue test for continuous time dynamical systems to show that this equilibrium is stable.

    (b) Explain why the iteration function for the discrete approximation is given by $G(x) = x + hF(x)$ where $h = \Delta t$. Write down the iteration function for the discrete time dynamical system of Eq. (6.18).

    (c) Write down the matrix of partial derivatives $A = DG$ evaluated at the equilibrium point found in part (a), and compute the eigenvalues of this matrix as a function of the step size h.

    (d) Use the eigenvalue test for discrete time dynamical systems to determine the largest step size h for which the equilibrium found in part (a) remains stable in the discrete approximation. Compare with the results in the text.

25. Reconsider the whale problem of Example 6.4. In this problem we will use simulation to explore the fractal limit sets in the discrete approximation of Eq. (6.18) for different step sizes $h = \Delta t$.

(a) Use a computer implementation of the Euler method to reproduce the results shown in Figure 6.31 in the text. Assume that $\alpha = 10^{-8}$ and use a step size of $h = \Delta t = 32$ years with initial conditions $x_1(0) = 5,000$ blue whales and $x_2(0) = 70,000$ fin whales.

(b) Plot fin whales $x_2(n)$ versus blue whales $x_1(n)$ for $n = 100, \ldots, 1000$. Your graph should show the limit set consisting of four points.

(c) Repeat the simulation of part (a) for step size $h = 33, 34, \ldots, 37$. For each case, plot the limit set as in part (b). How does the limit set change as the step size increases?

(d) Repeat part (c) for initial condition $x_1(0) = 150,000$ blue whales and $x_2(0) = 400,000$ fin whales. Does the limit set depend on the initial condition?

(e) Repeat part (c) for $\alpha = 3 \times 10^{-8}$. Does the limit set depend on the competition parameter α?

26. Reconsider the weather problem of Example 6.6.

(a) Use a computer implementation of the Euler method to reproduce the results of Figure 6.33 in the text. Assume $\sigma = 10$, $b = 8/3$, $r = 8$, and use the initial condition $(x_1, x_2, x_3) = (1, 1, 1)$.

(b) Use the results of part (a) to plot the deviation x_3 from linearity of the temperature profile versus the rate x_1 at which the convection rolls rotate. Perform a sensitivity analysis on the step size h to ensure that your plot represents the true behavior of the continuous time dynamical system.

(c) Repeat part (b) for the initial condition $(x_1, x_2, x_3) = (7, 1, 2)$. Does the solution curve approach the equilibrium found in the text?

(d) Repeat part (b) for $r = 18$ and $r = 28$. How does the solution behavior change as r increases?

27. Reconsider the weather problem of Example 6.6.

(a) Use a computer implementation of the Euler method with $N = 500$ and $T = 2.5$ (step size $h = 0.005$) to reproduce the results of Figure 6.35 in the text. Assume $\sigma = 10$, $b = 8/3$, $r = 18$, and use the initial condition $(x_1, x_2, x_3) = (6.7, 6.7, 17)$.

(b) Repeat part (a) for larger step sizes $h = 0.01, 0.015, \ldots, 0.03$. You can keep $N = 500$ and increase $T = 5, 7.5, 10, 12.5$, and 15. How does the simulated solution curve change as the step size increases?

(c) Find the coordinates of the equilibrium point E^+ when $\sigma = 10$, $b = 8/3$, and $r = 24$. Verify by simulation. Begin the simulation at the initial condition E^+ and check that the solution remains at this point.

(d) Is the equilibrium point E^+ found in part (c) stable? Verify by simulation. Begin the simulation at the initial condition $E^+ + (0.1, 0.1, 0)$ and determine whether the solution tends toward the equilibrium point E^+. How small a step size h must be used to ensure that the simulation results represent the true behavior of the continuous time dynamical system?

Further Reading

1. Acton, F. (1970) *Numerical Methods That Work.* Harper and Row, New York.

2. Brams, S., Davis, M. and Straffin, P. *The Geometry of the Arms Race.* UMAP module 311.

3. Dahlquist, G. and Bjorck, A. *Numerical Methods.* Prentice-Hall, Englewood Cliffs, New Jersey.

4. Gearhart, W. and Martelli, M. *A Blood Cell Population Model, Dynamical Diseases, and Chaos.* UMAP module 709.

5. Gleick, J. (1987) *Chaos: Making a New Science.* R. R. Donnelley, Harrisonburg, Virginia.

6. Press, W., Flannery, B., Teukolsky, S. and Vetterling, W. (1987). *Numerical Recipies.* Cambridge University Press, New York.

7. Smith, H. *Nuclear Deterrence.* UMAP module 327.

8. Strogatz, S. (1994) *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering.* Addison Wesley, Reading, Massachusetts.

9. Zinnes, D., Gillespie, J. and Tahim, G. *The Richardson Arms Race Model.* UMAP module 308.