Chapter 1

ONE VARIABLE OPTIMIZATION

Problems in optimization are the most common applications of mathematics. Whatever the activity in which we are engaged, we want to maximize the good that we do and minimize the unfortunate consequences or costs. Business managers attempt to control variables in order to maximize profit or to achieve a desired goal for production and delivery at a minimum cost. Managers of renewable resources such as fisheries and forests try to control harvest rates in order to maximize long–term yield. Government agencies set standards to minimize the environmental costs of producing consumer goods. Computer system managers try to maximize throughput and minimize delays. Farmers space their plantings to maximize yield. Physicians regulate medications to minimize harmful side effects. What all of these applications and many more have in common is a particular mathematical structure. One or more variables can be controlled to produce the best outcome in some other variable, subject in most cases to a variety of practical constraints on the control variables. Optimization models are designed to determine the values of the control variables which lead to the optimal outcome, given the constraints of the problem.

We begin our discussion of optimization models at a place where most students will already have some practical experience. One–variable optimization problems, sometimes called maximum–minimum problems, are typically discussed in first–semester calculus. A wide variety of practical applications can be handled using just these techniques. The purpose of this chapter, aside from a review of these basic techniques, is to introduce the fundamentals of mathematical modeling in a familiar setting.

1.1 The five-step Method

In this section we outline a general procedure that can be used to solve problems using mathematical modeling. We will illustrate this procedure, called the *five-*

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step method, by using it to solve a one–variable maximum–minimum problem typical of those encountered by most students in the first semester of calculus.

Example 1.1. A pig weighing 200 pounds gains 5 pounds per day and costs 45 cents a day to keep. The market price for pigs is 65 cents per pound, but is falling 1 cent per day. When should the pig be sold?

The mathematical modeling approach to problem solving consists of five steps:

1. Ask the question.

2. Select the modeling approach.

3. Formulate the model.

4. Solve the model.

5. Answer the question.

The first step is to ask a question. The question must be phrased in mathematical terms, and it often requires a good deal of work to do this. In the process we are required to make a number of assumptions or suppositions about the way things really are. We should not be afraid to make a guess at this stage. We can always come back and make a better guess later on. Before we can ask a question in mathematical terms we need to define our terms. Go through the problem and make a list of variables. Include appropriate units. Next make a list of assumptions about these variables. Include any relations between variables (equations and inequalities) that are known or assumed. Having done all of this, we are ready to ask a question. Write down in explicit mathematical language the objective of this problem. Notice that the preliminary steps of listing variables, units, equations and inequalities, and other assumptions are really a part of the question. They frame the question.

In Example 1.1 the weight w of the pig (in lbs), the number of days t until we sell the pig, the cost C of keeping the pig t days (in dollars), the market price p for pigs ($/lb), the revenue R obtained when we sell the pig ($), and our resulting net profit P ($) are all variables. There are other numerical quantities involved in the problem, such as the initial weight of the pig (200 lbs). However, these are not variables. It is important at this stage to separate variables from those quantities that will remain constant.

Next we need to list our assumptions about the variables identified in the first stage of step 1. In the process we will take into account the effect of the constants in the problem. The weight of the pig starts at 200 lbs and goes up by 5 lbs/day so we have

$$(w \text{ lbs}) = (200\,\text{lbs}) + \left(\frac{5\,\text{lbs}}{\text{day}}\right)(t\,\text{days}).$$

Variables: t = time (days)
w = weight of pig (lbs)
p = price for pigs (\$/lb)
C = cost of keeping pig t days (\$)
R = revenue obtained by selling pig (\$)
P = profit from sale of pig (\$)

Assumptions: $w = 200 + 5t$
$p = 0.65 - 0.01t$
$C = 0.45t$
$R = p \cdot w$
$P = R - C$
$t \geq 0$

Objective: Maximize P

Figure 1.1: Results of step 1 of the pig problem.

Notice that we have included units as a check that our equation makes sense. The other assumptions inherent in our problem are as follows:

$$\begin{aligned}
\left(\frac{p \text{ dollars}}{\text{lb}}\right) &= \left(\frac{0.65 \text{ dollars}}{\text{lb}}\right) - \left(\frac{0.01 \text{ dollars}}{\text{lb} \cdot \text{day}}\right)(t \text{ days}) \\
(C \text{ dollars}) &= \left(\frac{0.45 \text{ dollars}}{\text{day}}\right)(t \text{ days}) \\
(R \text{ dollars}) &= \left(\frac{p \text{ dollars}}{\text{lb}}\right)(w \text{ lb}s) \\
(P \text{ dollars}) &= (R \text{ dollars}) - (C \text{ dollars})
\end{aligned}$$

We are also assuming that $t \geq 0$. Our objective in this problem is to maximize our net profit, P dollars. Figure 1.1 summarizes the results of step 1, in a form convenient for later reference.

The three stages of step 1 (variables, assumptions, and objective) need not be completed in any particular order. For example, it is often useful to determine the objective early in step 1. In Example 1.1, it may not be readily apparent that R and C are variables until we have defined our objective, P, and we recall that $P = R - C$. One way to check that step 1 is complete is to see whether our objective P relates all the way back to the variable t. The best general advice about step 1 is to *do something.* Start by writing down whatever is immediately apparent (e.g., some of the variables can be found simply by reading over the problem and looking for nouns), and the rest of the pieces will probably fall into place.

Step 2 is to select the modeling approach. Now that we have a problem stated in mathematical language, we need to select a mathematical approach to use to get an answer. Many types of problems can be stated in a standard form for which an effective general solution procedure exists. Most research in applied mathematics consists of identifying these general categories of problems and inventing efficient ways to solve them. There is a considerable body of literature in this area, and many new advances continue to be made. Few, if any, students in this course will have the experience and familiarity with the literature to make a good selection for the modeling approach. In this book, with rare exceptions, problems will specify the modeling approach to be used. Our example problem will be modeled as a one–variable optimization problem, or maximum–minimum problem.

We outline the modeling approach we have selected. For complete details we refer the reader to any introductory calculus textbook.

> We are given a real–valued function $y = f(x)$ defined on a subset S of the real line. There is a theorem that states that if f attains its maximum or minimum at an interior point $x \in S$, then $f'(x) = 0$, assuming that f is differentiable at x. This allows us to rule out any interior point $x \in S$ at which $f'(x) \neq 0$ as a candidate for max–min. This procedure works well as long as there are not too many exceptional points.

Step 3 is to formulate the model. We need to take the question exhibited in step 1 and reformulate it in the standard form selected in step 2, so that we can apply the standard general solution procedure. It is often convenient to change variable names if we will refer to a modeling approach that has been described using specific variable names, as is the case here. We write

$$\begin{aligned} P &= R - C \\ &= p \cdot w - 0.45t \\ &= (0.65 - 0.01t)(200 + 5t) - 0.45t. \end{aligned}$$

Let $y = P$ be the quantity we wish to maximize and $x = t$ the independent variable. Our problem now is to maximize

$$\begin{aligned} y &= f(x) \\ &= (0.65 - 0.01x)(200 + 5x) - 0.45x \end{aligned} \tag{1.1}$$

over the set $S = \{x : x \geq 0\}$.

Step 4 is to solve the model, using the standard solution procedure identified in step 2. In our example we want to find the maximum of the function $y = f(x)$ defined by Eq. (1.1) over the set $x \geq 0$. Figure 1.2 shows a graph of the function $f(x)$. Since f is quadratic in x, the graph is a parabola. We compute that

$$f'(x) = \frac{(8 - x)}{10},$$

so that $f'(x) = 0$ at the point $x = 8$. Since f is increasing on the interval $(-\infty, 8)$ and decreasing on $(8, \infty)$, the point $x = 8$ is the global maximum. At

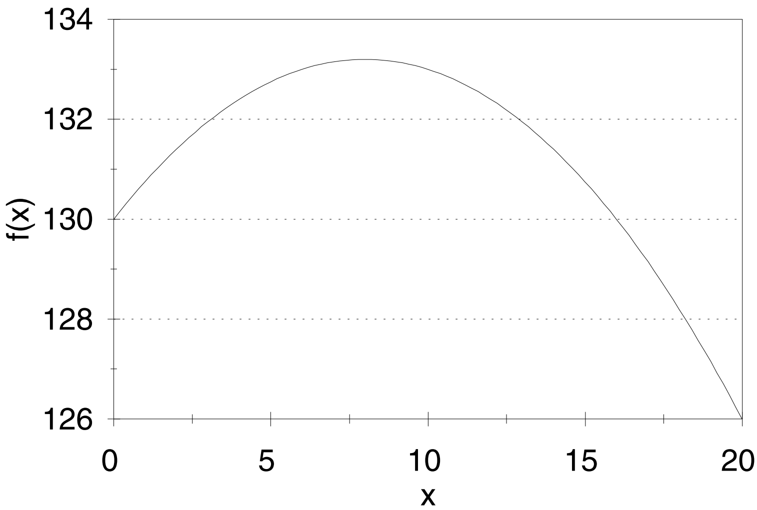

Figure 1.2: Graph of net profit $f(x) = (0.65 - 0.01x)(200 + 5x) - 0.45x$ versus time to sell x for the pig problem.

this point we have $y = f(8) = 133.20$. Since the point $(x, y) = (8, 133.20)$ is the global maximum of f over the entire real line, it is also the maximum over the set $x \geq 0$.

Step 5 is to answer the question posed originally in step 1; i.e., when to sell the pig in order to maximize profit. The answer obtained by our mathematical model is to sell the pig after eight days, thus obtaining a net profit of $133.20. This answer is valid as long as the assumptions made in step 1 remain valid. Related questions and alternative assumptions can be addressed by changing what we did in step 1. Since we are dealing with a real problem (A farmer owns pigs. When should they be sold?), there is an element of risk involved in step 1. For that reason it is usually necessary to investigate several alternatives. This process, called *sensitivity analysis*, will be discussed in the next section.

The main purpose of this section was to introduce the five-step method for mathematical modeling. Figure 1.3 summarizes the method in a form convenient for later reference. In this book we will apply the five-step method to solve a wide variety of problems in mathematical modeling. Our discussion of step 2 will generally include a description of the modeling approach selected, along with an example or two. The reader who is already familiar with the modeling approach may choose to skip this part, or just skim to pick up the notation. Some of the other points summarized in Fig. 1.3, such as the use of "appropriate technology," will be expanded upon later in this book.

The exercises at the end of each chapter also require the application of the

Step 1. Ask the question.

- Make a list of all the variables in the problem, including appropriate units.
- Be careful not to confuse variables and constants.
- State any assumptions you are making about these variables, including equations and inequalities.
- Check units to make sure that your assumptions make sense.
- State the objective of the problem in precise mathematical terms.

Step 2. Select the modeling approach.

- Choose a general solution procedure to be followed in solving this problem.
- Generally speaking, success in this step requires experience, skill, and familiarity with the relevant literature.
- In this book we will usually specify the modeling approach to be used.

Step 3. Formulate the model.

- Restate the question posed in step 1 in the terms of the modeling approach specified in step 2.
- You may need to relabel some of the variables specified in step 1 in order to agree with the notation used in step 2.
- Note any additional assumptions made in order to fit the problem described in step 1 into the mathematical structure specified in step 2.

Step 4. Solve the model.

- Apply the general solution procedure specified in step 2 to the specific problem formulated in step 3.
- Be careful in your mathematics. Check your work for math errors. Does your answer make sense?
- Use appropriate technology. Computer algebra systems, graphics, and numerical software will increase the range of problems within your grasp, and they also help reduce math errors.

Step 5. Answer the question.

- Rephrase the results of step 4 in nontechnical terms.
- Avoid mathematical symbols and jargon.
- Anyone who can understand the statement of the question as it was presented to you should be able to understand your answer.

Figure 1.3: The five-step method.

five-step method. Getting in the habit of using the five-step method now will make it easier to succeed on the more difficult modeling problems to come. Be sure to pay particular attention to step 5. In the real world, it is not enough to be right. You also need the ability to communicate your findings to others, some of whom may not be as mathematically knowledgeable as you.

1.2 Sensitivity Analysis

The preceding section outlines the five-step approach to mathematical modeling. The process begins by making some assumptions about the problem. We are rarely certain enough about things to be able to expect all of these assumptions to be exactly valid. Therefore, we need to consider how sensitive our conclusions are to each of the assumptions we have made. This kind of sensitivity analysis is an important aspect of mathematical modeling. The details vary according to the modeling approach used, and so our discussion of sensitivity analysis will continue throughout this book. We will focus here on sensitivity analysis for simple one–variable optimization problems.

In the preceding section we used the pig problem (Example 1.1) to illustrate the five-step approach to mathematical modeling. Figure 1.1 summarizes the assumptions we made in solving that problem. In this instance the data and assumptions were mostly spelled out for us. Even so, we need to be critical. Data are obtained by measurement, observation, and sometimes sheer guess. We need to consider the possibility that the data are not precise.

Some data are naturally known with much more certainty than others. The current weight of the pig, the current price for pigs, and the cost per day of keeping the pig are easy to measure and are known to a great degree of certainty. The rate of growth of the pig is a bit less certain, and the rate at which the price is falling is even less certain. Let r denote the rate at which the price is falling. We assumed that $r = 0.01$ dollars per day, but let us now suppose that the actual value of r is different. By repeating the solution procedure for several different values of r, we can get an idea of the sensitivity of our answer to the value of r. Table 1.1 shows the results of solving our problem for a few selected values of r. Figure 1.4 contains the same sensitivity data in graphical form. We can see that the optimal time to sell is quite sensitive to the parameter r.

A more systematic method for measuring this sensitivity would be to treat r as an unknown parameter, following the same steps as before. Writing

$$p = 0.65 - rt,$$

we can proceed as before to obtain

$$\begin{aligned} y &= f(x) \\ &= (0.65 - rx)(200 + 5x) - 0.45x. \end{aligned}$$

Then we can compute

$$f'(x) = \frac{-2(25rx + 500r - 7)}{5}$$

r ($/day)	x (days)
0.008	15.0
0.009	11.1
0.010	8.0
0.011	5.5
0.012	3.3

Table 1.1: Sensitivity of best time to sell x to rate r at which price is falling for the pig problem.

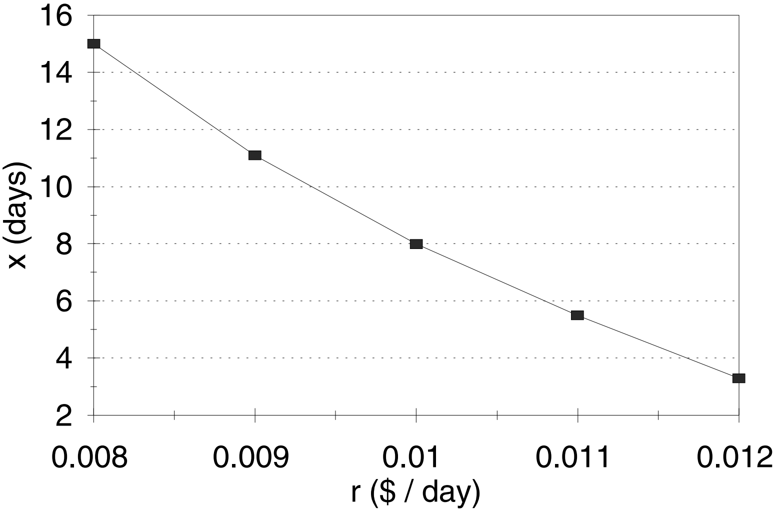

Figure 1.4: Graph of best time to sell x versus rate r at which price is falling for the pig problem.

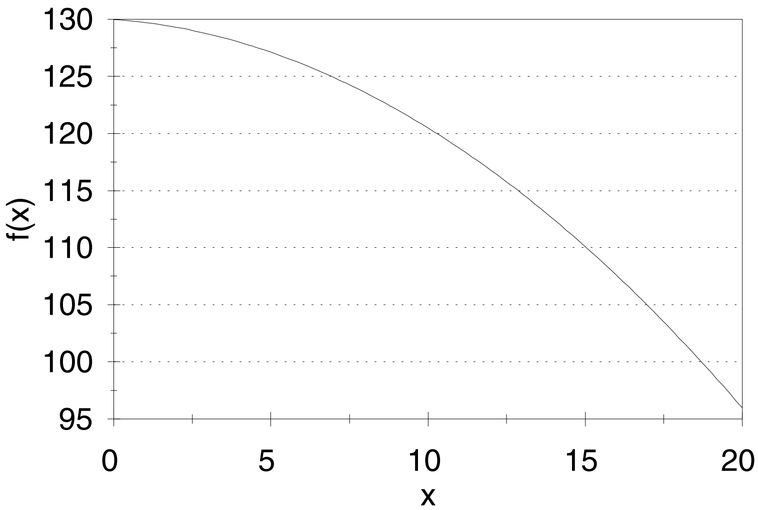

Figure 1.5: Graph of net profit $f(x) = (0.65 - 0.015x)(200 + 5x) - 0.45x$ versus time to sell x for the pig problem in the case $r = 0.015$.

so that $f'(x) = 0$ at the point

$$x = \frac{(7 - 500r)}{25r}. \tag{1.2}$$

The optimal time to sell is given by Eq. (1.2) as long as this expression is positive, i.e., as long as $0 < r \leq 0.014$. For $r > 0.014$, the vertex of the parabola $y = f(x)$ lies outside of the set $x \geq 0$ over which we are maximizing. In this case the optimal time to sell is at $x = 0$ since we have $f' < 0$ on the entire interval $[0, \infty)$. See Figure 1.5 for an illustration in the case $r = 0.015$.

We are also uncertain about the growth rate g of the pig. We have assumed that $g = 5$ lbs/day. More generally, we have that

$$w = 200 + gt,$$

which leads to the equation

$$f(x) = (0.65 - 0.01x)(200 + gx) - 0.45x, \tag{1.3}$$

so that

$$f'(x) = \frac{-(2gx + 5(49 - 13g))}{100}.$$

Now $f'(x) = 0$ at the point

$$x = \frac{5(13g - 49)}{2g}. \tag{1.4}$$

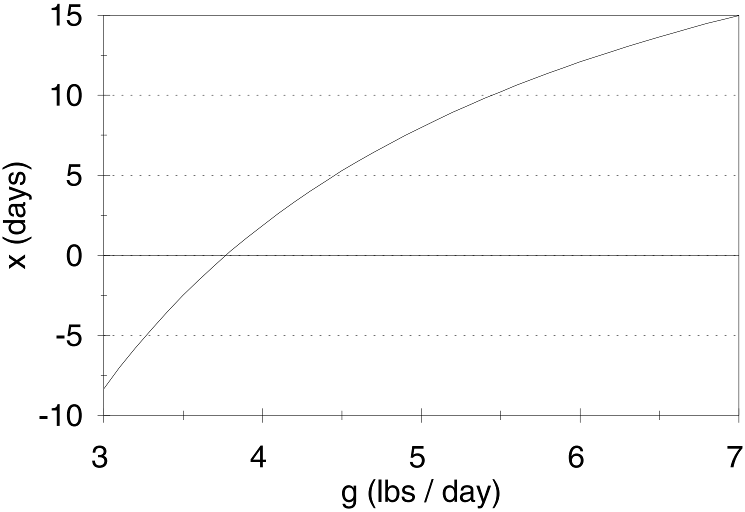

Figure 1.6: Graph of best time to sell x versus growth rate g for the pig problem.

The optimal time to sell is given by Eq. (1.4) so long as it represents a nonnegative value of x. Figure 1.6 shows the relationship between the growth rate g and the optimal time to sell.

It is most natural and most useful to interpret sensitivity data in terms of relative change or percent change, rather than in absolute terms. For example, a 10% decrease in r leads to a 39% increase in x, while a 10% decrease in g leads to a 34% decrease in x. If x changes by an amount Δx, the relative change in x is given by $\Delta x/x$, and the percent change in x is $100\,\Delta x/x$. If r changes by Δr, resulting in the change Δx in x, then the ratio between the relative changes is $\Delta x/x$ divided by $\Delta r/r$. Letting $\Delta r \to 0$ and using the definition of the derivative, we obtain

$$\frac{\Delta x/x}{\Delta r/r} \to \frac{dx}{dr} \cdot \frac{r}{x}.$$

We call this limiting quantity the *sensitivity* of x to r, and we will denote it by $S(x, r)$. In the pig problem we have

$$\begin{aligned}\frac{dx}{dr} &= \frac{-7}{25r^2}\\ &= -2,800\end{aligned}$$

at the point $r = 0.01$ and $x = 8$; thus

$$\begin{aligned} S(x, r) &= \frac{dx}{dr} \cdot \frac{r}{x} \\ &= (-2,800) \left(\frac{.01}{8} \right) \\ &= \frac{-7}{2}. \end{aligned}$$

If r goes up by 2%, then x goes down by 7%. Since

$$\begin{aligned} \frac{dx}{dg} &= \frac{245}{2g^2} \\ &= 4.9, \end{aligned}$$

we have

$$\begin{aligned} S(x, g) &= \frac{dx}{dg} \cdot \frac{g}{x} \\ &= (4.9) \left(\frac{5}{8} \right) \\ &= 3.0625, \end{aligned}$$

so that a 1% increase in the growth rate of the pig would cause us to wait about 3% longer to sell the pig.

In order to compute the sensitivity $S(y, g)$, first substitute (1.4) into the objective function $y = f(x)$ from (1.3) to obtain

$$\begin{aligned} y = &\left(0.65 - 0.01 \left[\frac{5(13g - 49)}{2g} \right] \right) \left(200 + g \left[\frac{5(13g - 49)}{2g} \right] \right) \\ &- 0.45 \left[\frac{5(13g - 49)}{2g} \right] \\ = &\frac{150.0625}{g} + 50.375 + 10.5625g. \end{aligned}$$

Then compute the derivative

$$\frac{dy}{dg} = -\frac{150.0625}{g^2} + 10.5625,$$

and substitute $g = 5$ to get $dy/dg = 4.56$, which leads to

$$\begin{aligned} S(y, g) &= \frac{dy}{dg} \cdot \frac{g}{y} \\ &= (4.56) \left(\frac{5}{133.20} \right) \\ &= 0.17. \end{aligned}$$

If the pig grows 10% faster than expected, the expected net profit will be 1.7% larger. The computation of the derivative dy/dg in this case involves quite a bit of algebra. In Chapter 2, we will discuss how a *computer algebra system* can be used to perform the necessary algebraic computations.

The successful application of sensitivity analysis procedures requires good judgment. It is usually not possible to compute sensitivity coefficients for each parameter in the model, nor is this particularly desirable. We need to select those parameters about which there is the most uncertainty and perform sensitivity analysis on them. The interpretation of sensitivity coefficients also depends on the degree of uncertainty, the fundamental question being the extent to which our uncertainty about the data affects our confidence in the answer. In the pig problem, we are probably considerably more certain of the growth rate g than of the rate r at which prices fall. A 25% error in g would be quite surprising if we have observed the past history of growth in this pig or in similar animals. A 25% error in our estimate of r would not be at all surprising.

1.3 Sensitivity and Robustness

A mathematical model is robust if the conclusions it leads to remain true even though the model is not completely accurate. In real problems we will never have perfect information, and even if it were possible to construct a perfectly accurate model, we might be better off with a simpler and more tractable approximation. For this reason a consideration of robustness is a necessary ingredient in any mathematical modeling project.

In the preceding section we introduced the process of sensitivity analysis, which is a way to gauge the robustness of a model with respect to assumptions about the data. There are other assumptions made in step 1 of the mathematical modeling process which should also be examined. While it is often necessary to make assumptions for purposes of mathematical convenience and simplicity, it is the responsibility of the modeler to see to it that these assumptions are not so specialized as to invalidate the results of the modeling process.

Figure 1.1 contains a summary of the assumptions made in solving the pig problem. Aside from data values, the main assumptions are that both the weight of the pig and the selling price per pound are linear functions of time. These are obviously simplifying assumptions and cannot be expected to hold exactly. After all, according to these assumptions, a year from now the pig would weigh

$$\begin{aligned} w &= 200 + 5(365) \\ &= 2,025\,\text{lbs} \end{aligned}$$

and sell for

$$\begin{aligned} p &= 0.65 - 0.01(365) \\ &= -3.00\,\text{dollars/lb.} \end{aligned}$$

A more realistic model would take into account both the nonlinearity of these functions and the increasing uncertainty as time goes on.

How can a model give the right answer if the assumptions are wrong? While mathematical modeling strives for perfection, perfection can never be achieved. It would be more descriptive to say that mathematical modeling strives *toward* perfection. A well–constructed mathematical model will be robust, which is to say that while the answers it gives may not be perfectly correct, they will be close enough to be useful in a real–world context.

Let us examine the linearity assumptions made in the pig problem. Our basic equation is

$$P = pw - 0.45t,$$

where p is the selling price of the pig in dollars per pound, and w is the weight of the pig in pounds. If the original data and assumptions of the model are not too far off, then the best time to sell the pig is obtained by setting $P' = 0$. Calculate to find

$$p'w + pw' = 0.45$$

dollars per day. The term $p'w + pw'$ represents the rate of increase in the value of the pig. Our model tells us to keep the pig as long as the value of the pig is increasing faster than the cost of feeding it. Furthermore, the change in the pig's value has two components, $p'w$ and pw'. The first term, $p'w$, represents the loss in value due to a drop in price. The second term, pw', represents the gain in value due to the pig gaining weight. Consider the practical problems involved in the application of this more general model. The data required include a complete specification of both the future growth of the pig and the future changes in price as differentiable functions of time. There is no way to know these functions exactly. There is even some question as to whether they make sense. Can the pig be sold at 3 A.M. Sunday morning? Can price be an irrational number? Let us construct a realistic scenario. The farmer has a pig weighing approximately 200 lbs. The pig has been gaining about five lbs/day during the last week. Five days ago the pig could have been sold for 70 cents/lb but by now the price has dropped to 65 cents/lb. What should we do? The obvious approach is to project on the basis of this data ($w = 200,\ w' = 5,\ p = 0.65,\ p' = -0.01$) and decide when to sell. This is exactly what we did. We understand that p' and w' will not remain constant over the next few weeks, and that therefore p and w will not be linear functions of time. However, as long as p' and w' do not change too much over this period, the error involved in assuming they remain constant will not be too great.

We are now prepared to give a somewhat broader interpretation to the results of our sensitivity analysis from the preceding section. Recall that the sensitivity of the best time to sell (x) to changes in the growth rate w' was calculated to be 3. Suppose that in fact the growth rate over the next few weeks is somewhere between 4.5 and 5.5 lbs/day. This is within 10% of the assumed value. Then the best time to sell the pig will be within 30% of 8 days, or between 5 and 11 days. The amount of lost profit by selling at 8 days is less than 1 dollar.

With regard to price, suppose that we feel the value $p' = -.01$, or a 1 cent/day drop in price over the next few weeks, is a worst–case scenario. Prices are likely to drop more slowly in the future and may even level off ($p' = 0$).

All we can really say now is that we should wait at least 8 days to sell. For small values of p' (near zero), our model suggests waiting a very long time to sell. However, our model is not valid over long time intervals. The best course of action in this case is probably to keep the pig for a week, reestimate the parameter values p, p', w', and w, and start over.

1.4 Exercises

1. An automobile manufacturer makes a profit of \$1,500 on the sale of a certain model. It is estimated that for every \$100 of rebate, sales increase by 15%.

   (a) What amount of rebate will maximize profit? Use the five-step method, and model as a one–variable optimization problem.

   (b) Compute the sensitivity of your answer to the 15% assumption. Consider both the amount of rebate and the resulting profit.

   (c) Suppose that rebates actually generate only a 10% increase in sales per \$100. What is the effect? What if the response is somewhere between 10 and 15% per \$100 of rebate?

   (d) Under what circumstances would a rebate offer cause a reduction in profit?

2. In the pig problem, perform a sensitivity analysis based on the cost per day of keeping the pig. Consider both the effect on the best time to sell and on the resulting profit. If a new feed costing 60 cents/day would let the pig grow at a rate of 7 lbs/day, would it be worth switching feed? What is the minimum improvement in growth rate that would make this new feed worthwhile?

3. Reconsider the pig problem of Example 1.1, but now assume that the price for pigs is starting to level off. Let

$$p = 0.65 - 0.01t + 0.00004t^2 \tag{1.5}$$

   represent the price for pigs (cents/lb) after t days.

   (a) Graph Eq. (1.5) along with our original price equation. Explain why our original price equation could be considered as an approximation to Eq. (1.5) for values of t near zero.

   (b) Find the best time to sell the pig. Use the five-step method, and model as a one–variable optimization problem.

   (c) The parameter 0.00004 represents the rate at which price is leveling off. Conduct a sensitivity analysis on this parameter. Consider both the optimal time to sell and the resulting profit.

(d) Compare the results of part (b) to the optimal solution contained in the text. Comment on the robustness of our assumptions about price.

4. An oil spill has fouled 200 miles of Pacific shoreline. The oil company responsible has been given 14 days to clean up the shoreline, after which a fine will be levied in the amount of \$10,000/day. The local cleanup crew can scrub five miles of beach per week at a cost of \$500/day. Additional crews can be brought in at a cost of \$18,000 plus \$800/day for each crew.

   (a) How many additional crews should be brought in to minimize the total cost to the company? Use the five-step method. How much will the clean–up cost?

   (b) Examine the sensitivity to the rate at which a crew can clean up the shoreline. Consider both the optimal number of crews and the total cost to the company.

   (c) Examine the sensitivity to the amount of the fine. Consider the number of days the company will take to clean up the spill and the total cost to the company.

   (d) The company has filed an appeal on the grounds that the amount of the fine is excessive. Assuming that the only purpose of the fine is to motivate the company to clean up the oil spill in a timely manner, is the fine excessive?

5. It is estimated that the growth rate of the fin whale population (per year) is $rx(1 - x/K)$, where $r = 0.08$ is the intrinsic growth rate, $K = 400,000$ is the maximum sustainable population, and x is the current population, now around 70,000. It is further estimated that the number of whales harvested per year is about 0.00001 Ex, where E is the level of fishing effort in boat–days. Given a fixed level of effort, population will eventually stabilize at the level where growth rate equals harvest rate.

   (a) What level of effort will maximize the sustained harvest rate? Model as a one–variable optimization problem using the five-step method.

   (b) Examine the sensitivity to the intrinsic growth rate. Consider both the optimum level of effort and the resulting population level.

   (c) Examine the sensitivity to the maximum sustainable population. Consider both the optimum level of effort and the resulting population level.

6. In Exercise 5, suppose that the cost of whaling is \$500 per boat–day, and the price of a fin whale carcass is \$6,000.

   (a) Find the level of effort that will maximize profit over the long term. Model as a one–variable optimization problem using the five-step method.

(b) Examine the sensitivity to the cost of whaling. Consider both the eventual profit in \$/year and the level of effort.

(c) Examine the sensitivity to the price of a fin whale carcass. Consider both profit and level of effort.

(d) Over the past 30 years there have been several unsuccessful attempts to ban whaling worldwide. Examine the economic incentives for whalers to continue harvesting. In particular, determine the conditions (values of the two parameters: cost per boat–day and price per fin whale carcass) under which harvesting the fin whale produces a sustained profit over the long term.

7. Reconsider the pig problem of Example 1.1, but now suppose that our objective is to maximize our profit rate (\$/day). Assume that we have already owned the pig for 90 days and have invested \$100 in this pig to date.

   (a) Find the best time to sell the pig. Use the five-step method, and model as a one–variable optimization problem.

   (b) Examine the sensitivity to the growth rate of the pig. Consider both the best time to sell and the resulting profit rate.

   (c) Examine the sensitivity to the rate at which the price for pigs is dropping. Consider both the best time to sell and the resulting profit rate.

8. Reconsider the pig problem of Example 1.1, but now take into account the fact that the growth rate of the pig decreases as the pig gets older. Assume that the pig will be fully grown in another five months.

   (a) Find the best time to sell the pig in order to maximize profit. Use the five-step method, and model as a one–variable optimization problem.

   (b) Examine the sensitivity to the time it will take until the pig is fully grown. Consider both the best time to sell and the resulting profit.

9. A local daily newspaper with a circulation of 80,000 subscribers is thinking of raising its subscription price. Currently the price is \$1.50 per week, and it is estimated that the paper would lose 5,000 subscribers if the rate were to be raised by ten cents/week.

   (a) Find the subscription price that maximizes profit. Use the five-step method, and model as a one–variable optimization problem.

   (b) Examine the sensitivity of your answer in part (a) to the assumption of 5,000 lost subscribers. Calculate the optimal subscription rate assuming that this parameter is 3,000, 4,000, 5,000, 6,000, or 7,000.

   (c) Let $n = 5,000$ denote the number of subscribers lost when the subscription price increases by ten cents. Calculate the optimal subscription price p as a function of n, and use this formula to determine the sensitivity $S(p, n)$.

(d) Should the paper change its subscription price? Justify your conclusions in plain English.

Further Reading

1. Cameron, D., Giordano, F. and Weir, M. *Modeling Using the Derivative: Numerical and Analytic Solutions.* UMAP module 625.

2. Cooper, L. and Sternberg, D. (1970) *Introduction to Methods of Optimization.* W. B. Saunders, Philadelphia.

3. Gill, P., Murray, W. and Wright, M. (1981) *Practical Optimization.* Academic Press, New York.

4. Meyer, W. (1984) *Concepts of Mathematical Modeling.* McGraw–Hill, New York.

5. Rudin W. (1976) *Principles of Mathematical Analysis.* 3rd Ed., McGraw–Hill, New York.

6. Whitley, W. *Five Applications of Max–Min Theory from Calculus.* UMAP module 341.