Chapter 2

MULTIVARIABLE OPTIMIZATION

Many optimization problems require the simultaneous consideration of a number of independent variables. In this chapter we consider the simplest category of multivariable optimization problems. The techniques should be familiar to most students from multivariable calculus. In this chapter we also introduce the use of computer algebra systems to handle some of the more complicated algebraic computations.

2.1 Unconstrained Optimization

The simplest type of multivariable optimization problems involves finding the maximum or minimum of a differentiable function of several variables over a nice set. Further complications arise, as we will see later, when the set over which we optimize takes a more complex form.

Example 2.1. A manufacturer of color TV sets is planning the introduction of two new products, a 19–inch LCD flat panel set with a manufacturer's suggested retail price (MSRP) of \$339 and a 21–inch LCD flat panel set with an MSRP of \$399. The cost to the company is \$195 per 19–inch set and \$225 per 21–inch set, plus an additional \$400,000 in fixed costs. In the competitive market in which these sets will be sold, the number of sales per year will affect the average selling price. It is estimated that for each type of set, the average selling price drops by one cent for each additional unit sold. Furthermore, sales of the 19–inch set will affect sales of the 21–inch set, and vice–versa. It is estimated that the average selling price for the 19–inch set will be reduced by an additional 0.3 cents for each 21–inch set sold, and the price for the 21–inch set will decrease by 0.4 cents for each 19–inch set sold. How many units of each type of set should be manufactured?

Fourth edition
© 2013, Elsevier Inc. All rights reserved.

Variables: s = number of 19–inch sets sold (per year)
t = number of 21–inch sets sold (per year)
p = selling price for a 19–inch set ($)
q = selling price for a 21–inch set ($)
C = cost of manufacturing sets ($/year)
R = revenue from the sale of sets ($/year)
P = profit from the sale of sets ($/year)

Assumptions: $p = 339 - 0.01s - 0.003t$
$q = 399 - 0.004s - 0.01t$
$R = ps + qt$
$C = 400{,}000 + 195s + 225t$
$P = R - C$
$s \geq 0$
$t \geq 0$

Objective: Maximize P

Figure 2.1: Results of step 1 of the color TV problem.

The five-step approach to mathematical modeling, introduced in the preceding chapter, will be used to solve this problem. Step 1 is to ask a question. We begin by making a list of variables. Next we write down the relations between variables and any other assumptions, such as nonnegativity. Finally, we formulate a question in mathematical terms, using the established notation. The results of step 1 are summarized in Figure 2.1.

Step 2 is to select the modeling approach. We will solve this problem as a multivariable unconstrained optimization problem. This type of problem is typically treated in introductory courses in multivariable calculus. We will outline the model and the general solution procedure here. We refer the reader to any introductory calculus textbook for details and mathematical proofs.

> We are given a function $y = f(x_1, \ldots, x_n)$ on a subset S of the n–dimensional space $\mathbb{R}^n$. We wish to find the maximum and/or minimum values of f on the set S. There is a theorem that states that if f attains its maximum or minimum at an interior point $(x_1, \ldots, x_n)$ in S, then $\nabla f = 0$ at that point, assuming that f is differentiable at that point. In other words, at the extreme point
>
> $$\begin{aligned} \frac{\partial f}{\partial x_1}(x_1, \ldots, x_n) &= 0 \\ \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n) &= 0. \end{aligned} \tag{2.1}$$
>
> The theorem allows us to rule out as a candidate for max–min any point in the interior of S for which any of the partial derivatives of f

do not equal zero. Thus, to find the max–min points we must solve simultaneously the n equations in n unknowns defined by Equation (2.1). Then we must also check any points on the boundary of S, as well as any points where one or more of the partial derivatives is undefined.

Step 3 is to formulate the model, using the standard form chosen in step 2. Let

$$\begin{aligned} P &= R - C \\ &= ps + qt - (400,000 + 195s + 225t) \\ &= (339 - 0.01s - 0.003t)s \\ &+ (399 - 0.004s - 0.01t)t \\ &- (400,000 + 195s + 225t). \end{aligned}$$

Now let $y = P$ be the quantity we wish to maximize, and let $x_1 = s,\ x_2 = t$ be our decision variables. Our problem now is to maximize

$$\begin{aligned} y &= f(x_1, x_2) \\ &= (339 - 0.01x_1 - 0.003x_2)x_1 \\ &+ (399 - 0.004x_1 - 0.01x_2)x_2 \\ &- (400,000 + 195x_1 + 225x_2) \end{aligned} \tag{2.2}$$

over the set

$$S = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}. \tag{2.3}$$

Step 4 is to solve the problem, using the standard solution methods outlined in step 2. The problem is to maximize the function f given by Eq. (2.2) over the set S defined in Eq. (2.3). Figure 2.2 shows a 3–D graph of the function f. This plot indicates that f attains its maximum in the interior of S. Figure 2.3 shows a plot of the level sets of f.

From this plot we can estimate that the maximum value of the function f occurs around $x_1 = 5,000$ and $x_2 = 7,000$. The function f is a paraboloid, and the vertex of the paraboloid is the unique solution to Eq. (2.1) obtained by setting $\nabla f = 0$. We compute that

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 144 - 0.02x_1 - 0.007x_2 = 0 \\ \frac{\partial f}{\partial x_2} &= 174 - 0.007x_1 - 0.02x_2 = 0 \end{aligned} \tag{2.4}$$

at the point

$$\begin{aligned} x_1 &= \frac{554,000}{117} \approx 4,735 \\ x_2 &= \frac{824,000}{117} \approx 7,043. \end{aligned} \tag{2.5}$$

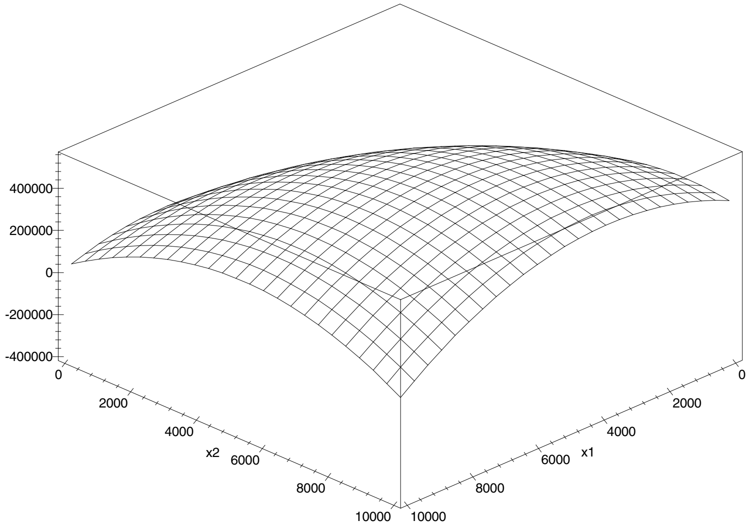

Figure 2.2: 3–D graph of profit $y = f(x_1, x_2)$ from (2.2) versus production levels x_1 of 19–inch sets and x_2 of 21–inch sets for the color TV problem.

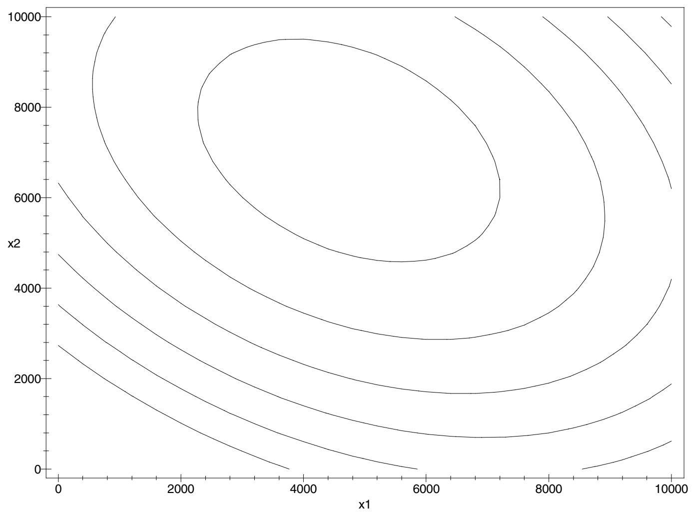

Figure 2.3: Contour plot showing level sets of profit $y = f(x_1, x_2)$ from (2.2) versus production levels x_1 of 19–inch sets and x_2 of 21–inch sets for the color TV problem.

The point (x_1, x_2) defined by Eq. (2.5) represents the global maximum of f over the entire real plane. It is therefore also the maximum of f over the set S defined in Eq. (2.3). The maximum value of f is obtained by substituting Eq. (2.5) back into Eq. (2.2), which yields

$$y = \frac{21,592,000}{39} \approx 553,641. \tag{2.6}$$

The calculations of step 4 in this problem are a bit cumbersome. In cases like this one, it is appropriate to use a *computer algebra system* to perform the necessary calculations. Computer algebra systems can differentiate, integrate, solve equations, and simplify algebraic expressions. Most packages can also perform matrix algebra, draw graphs, and solve some differential equations. Several good computer algebra systems (Maple, Mathematica, Derive, etc.) are available for both mainframe and personal computers, and many systems offer a student version at a substantially reduced price. The graphs in Figure 2.2 and Figure 2.3 were drawn using the computer algebra system Maple. Computer algebra systems are an example of the kind of "appropriate technology" we referred to in Fig. 1.3, in our summary of the five-step method. Figure 2.4 shows the results of using the computer algebra system Mathematica to solve the current model.

There are several advantages to using a computer algebra system for a problem like this. It is more efficient and more accurate. You will be more productive if you learn to use this technology, and it will give you the freedom to concentrate more on the larger issues of problem solving instead of getting bogged down in the calculations. We will illustrate the use of computer algebra systems again in our sensitivity analysis calculations below, where the algebra is even more exacting.

The final step, step 5, is to answer the question in plain English. Simply stated, the company can maximize profits by manufacturing 4,735 of the 19–inch sets and 7,043 of the 21–inch sets, resulting in a net profit of \$553,641 for the year. The average selling price for a 19–inch set is \$270.52 and \$309.63 for a 21–inch set. The projected revenue is \$3,461,590, resulting in a profit margin (profit/revenue) of 16 percent. These figures indicate a profitable venture, so we would recommend that the company proceed with the introduction of these new products.

The conclusions of the preceding paragraph are based on the assumptions illustrated in Fig. 2.1. Before reporting our findings to the company, we should perform sensitivity analysis to insure that our conclusions are robust with respect to our assumptions about both the market for TV sets and the manufacturing process. Our main concern is the value of the decision variables x_1 and x_2, since the company must act on this information.

We illustrate the procedure for sensitivity analysis by examining the sensitivity to price elasticity for 19–inch sets, which we will denote by the variable a. Price elasticity is the term used by economists to describe the sensitivity of quantity sold to the asking price. In our model we assumed that $a = 0.01$

```
In[1]:= y = (339 - x1 / 100 - 3 x2 / 1000) x1 +
          (399 - 4 x1 / 1000 - x2 / 100) x2 -
          (400000 + 195 x1 + 225 x2)
```

$$\text{Out[1]=}\quad \left(-\frac{x1}{100}-\frac{3\,x2}{1000}+339\right)x1-195\,x1+\left(-\frac{x1}{250}-\frac{x2}{100}+399\right)x2-225\,x2-400000$$

```
In[2]:= dydx1 = D[y, x1]
```

$$\text{Out[2]=}\quad -\frac{x1}{50}-\frac{7\,x2}{1000}+144$$

```
In[3]:= dydx2 = D[y, x2]
```

$$\text{Out[3]=}\quad -\frac{7\,x1}{1000}-\frac{x2}{50}+174$$

```
In[4]:= s = Solve[{dydx1 == 0, dydx2 == 0}, {x1, x2}]
```

$$\text{Out[4]=}\quad \left\{\left\{x1\to\frac{554000}{117},\ x2\to\frac{824000}{117}\right\}\right\}$$

```
In[5]:= N[s]

Out[5]= {{x1 → 4735.04, x2 → 7042.74}}

In[6]:= y /. s
```

$$\text{Out[6]=}\quad \left\{\frac{21592000}{39}\right\}$$

```
In[7]:= N[%]

Out[7]= {553641.}
```

Figure 2.4: Optimal solution to the color TV problem using the computer algebra system Mathematica.

dollars per set. Substituting into our previous formula, we obtain

$$\begin{aligned} y &= f(x_1, x_2) \\ &= (339 - ax_1 - 0.003x_2)x_1 \\ &\quad + (399 - 0.004x_1 - 0.01x_2)x_2 \\ &\quad - (400,000 + 195x_1 + 225x_2). \end{aligned} \tag{2.7}$$

When we compute partial derivatives and set them equal to zero, we obtain

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 144 - 2ax_1 - 0.007x_2 = 0 \\ \frac{\partial f}{\partial x_2} &= 174 - 0.007x_1 - 0.02x_2 = 0. \end{aligned} \tag{2.8}$$

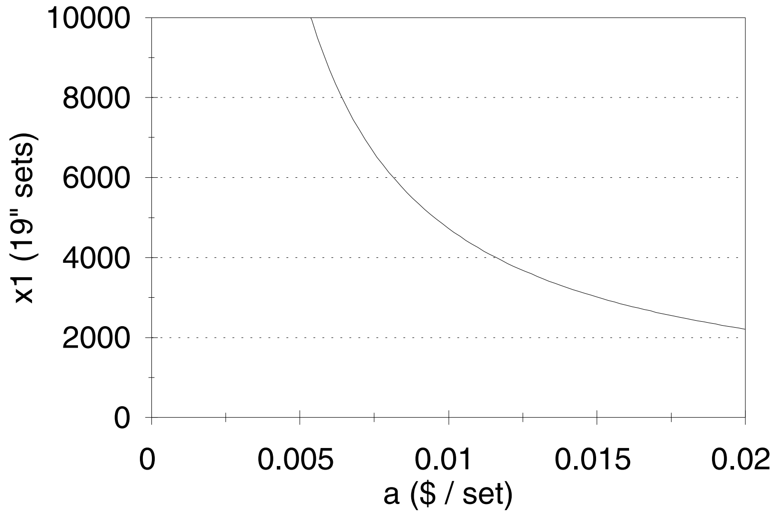

Figure 2.5: Graph of optimum production level x_1 of 19–inch sets versus price elasticity a for the color TV problem.

Solving for x_1 and x_2 as before yields

$$\begin{aligned} x_1 &= \frac{1,662,000}{40,000a - 49} \\ x_2 &= 8,700 - \frac{581,700}{40,000a - 49}. \end{aligned} \tag{2.9}$$

See Figs. 2.5 and 2.6 for the graphs of x_1 and x_2 versus a.

From these graphs it appears that a higher price elasticity a for 19–inch sets will reduce the optimal production level x_1 for 19–inch sets and increase the optimal production level x_2 for 21–inch sets. It also appears that x_1 is more sensitive to a than x_2, which seems to make sense. To get a numerical measure of these sensitivities we compute

$$\begin{aligned} \frac{dx_1}{da} &= \frac{-66,480,000,000}{(40,000a - 49)^2} \\ &= \frac{-22,160,000,000}{41,067} \end{aligned}$$

at $a = 0.01$, so that

$$\begin{aligned} S(x_1,\, a) &= \left(\frac{-22,160,000,000}{41,067}\right)\left(\frac{0.01}{554,000/117}\right) \\ &= -\frac{400}{351} \approx -1.1. \end{aligned}$$

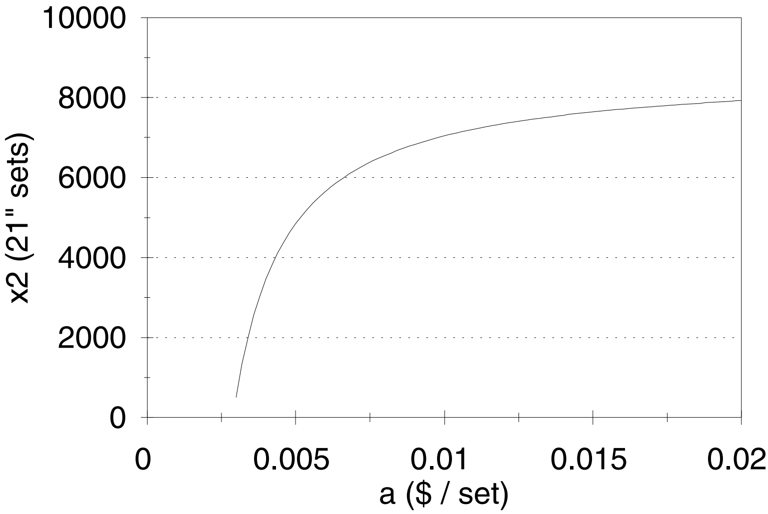

Figure 2.6: Graph of optimum production level x_2 of 21–inch sets versus price elasticity a for the color TV problem.

A similar calculation yields

$$S(x_2,\, a) = \frac{9,695}{36,153} \approx 0.27.$$

If the price elasticity for 19–inch sets were to increase by 10%, then we should make 11% fewer 19–inch sets and 2.7% more 21–inch sets.

Next, we consider the sensitivity of y to a. What effect will a change in the price elasticity for 19–inch sets have on our profits? To obtain a formula for y in terms of a, we substitute Eq. (2.9) back into Eq. (2.7) to get

$$\begin{aligned} y = &\left[339 - a\left(\frac{1,662,000}{40,000a-49}\right) - 0.003\left(8,700 - \frac{581,700}{40,000a-49}\right)\right] \\ &\times \left(\frac{1,662,000}{40,000a-49}\right) \\ &+ \left[399 - 0.004\left(\frac{1,662,000}{40,000a-49}\right) - 0.01\left(8,700 - \frac{581,700}{40,000a-49}\right)\right] \\ &\times \left(8,700 - \frac{581,700}{40,000a-49}\right) \\ &- \left[400,000 + 195\left(\frac{16,620,000}{40,000a-49}\right) + 225\left(8,700 - \frac{581,700}{40,000a-49}\right)\right]. \end{aligned} \tag{2.10}$$

See Fig. 2.7 for a graph of y versus a. It appears that an increase in price elasticity for 19–inch sets will result in lower profits.

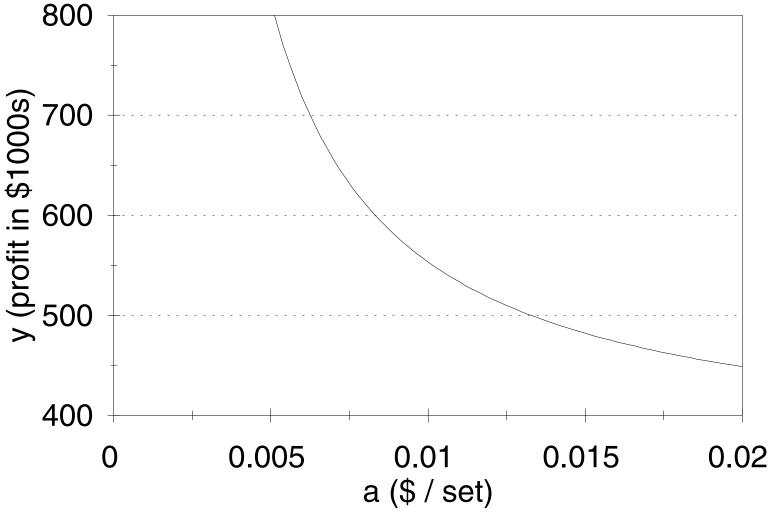

Figure 2.7: Graph of optimum profit y versus price elasticity a for the color TV problem.

To compute $S(y,\, a)$ we will need to obtain a formula for dy/da. One way to do this would be to employ one–variable methods directly on Eq. (2.10), perhaps with the aid of a computer algebra system. Another method, which is computationally more efficient, is to use the multivariable chain rule, which implies that

$$\frac{dy}{da} = \frac{\partial y}{\partial x_1}\frac{dx_1}{da} + \frac{\partial y}{\partial x_2}\frac{dx_2}{da} + \frac{\partial y}{\partial a}. \tag{2.11}$$

Since both $\partial y/\partial x_1$ and $\partial y/\partial x_2$ are zero at the optimum, we have

$$\frac{dy}{da} = \frac{\partial y}{\partial a} = -x_1^2$$

directly from Eq. (2.7), so

$$\begin{aligned} S(y,a) &= -\left(\frac{554,000}{117}\right)^2 \frac{0.01}{(21,592,000/39)} \\ &= -\frac{383,645}{947,349} \approx -0.40. \end{aligned}$$

Thus, a 10% increase in price elasticity for 19–inch sets will result in a 4% drop in profit.

The fact that the term

$$\frac{\partial y}{\partial x_1}\frac{dx_1}{da} + \frac{\partial y}{\partial x_2}\frac{dx_2}{da} = 0$$

```
> y:=(339-a*x1-3*x2/1000)*x1
>   +(399-4*x1/1000-x2/100)*x2-(400000+195*x1+225*x2);
```

$$y := \left(339 - a\,x1 - \frac{3}{1000}\,x2\right)x1 + \left(399 - \frac{1}{250}\,x1 - \frac{1}{100}\,x2\right)x2 - 400000 - 195\,x1 - 225\,x2$$

```
> dydx1:=diff(y,x1);
```

$$dydx1 := -2\,a\,x1 + 144 - \frac{7}{1000}\,x2$$

```
> dydx2:=diff(y,x2);
```

$$dydx2 := -\frac{7}{1000}\,x1 - \frac{1}{50}\,x2 + 174$$

```
> s:=solve({dydx1=0,dydx2=0},{x1,x2});
```

$$s := \left\{x2 = \frac{48000\,(-21 + 7250\,a)}{-49 + 40000\,a},\; x1 = \frac{1662000}{-49 + 40000\,a}\right\}$$

```
> assign(s);dx1da:=diff(x1,a);
```

$$dx1da := -\frac{66480000000}{(-49 + 40000\,a)^2}$$

```
> assign(a=1/100);x1;
```

$$\frac{554000}{117}$$

```
> sx1a:=dx1da*(a/x1);
```

$$sx1a := \frac{-400}{351}$$

```
> evalf(sx1a);
```

$$-1.139601140$$

```
>
```

Figure 2.8: Calculation of the sensitivity $S(x_1, a)$ for the color TV problem using the computer algebra system Maple.

in Eq. (2.11) has its own real–world significance. This part of the derivative dy/da represents the effect on profits of the changing optimal production levels x_1 and x_2. The fact that it sums to zero means that small changes in production levels have (at least in the linear approximation) no effect on profits. Geometrically, since we are at the maximum point where the curve $y = f(x_1, x_2)$ is flat, small changes in x_1 and x_2 have little effect on y. Almost all of the drop in optimal profits caused by a 10% increase in price elasticity for 19–inch sets is due to the change in selling price. Therefore, the production levels given by our model will be very nearly optimal. For example, suppose that we have assumed $a = 0.01$, but that this price elasticity is in fact 10% higher. We will set our production levels using Eq. (2.5), which means that we will produce 11% too many 19–inch sets and around 3% too few 21–inch sets, compared to the optimal solution given by Eq. (2.9) with $a = 0.011$. Also, our profits will be 4% lower than expected. But what have we actually lost by applying the results of our model? Using Eq. (2.5) with $a = 0.011$, we will net a \$531,219 profit. The optimal profit would be \$533,514 (set $a = 0.011$ in Eq. (2.9) and substitute back into Eq. (2.7)). Hence, we have lost only 0.43 percent of the potential maximum profit by applying the results of our model, even though our actual production levels were quite a way off from the optimum values. Our model appears to be extremely robust in this regard. Furthermore, a similar conclusion should hold for many similar problems, since it is basically due to the fact that $\nabla f = 0$ at a critical point.

All of the previous sensitivity analysis calculations could also be performed using a computer algebra system. In fact, this is the preferred method, assuming that one is available. Figure 2.8 illustrates how the computer algebra system Maple can be used to compute the sensitivity $S(x_1, a)$. The calculations of the other sensitivities are similar.

Sensitivity analysis for the other elasticities could be carried out in the same manner. While the particulars will differ, the form of the function f suggests that each affects y in essentially the same manner. In particular, we have a high degree of confidence that our model will lead to a good (nearly optimal) decision about production levels even in the presence of small errors in the estimation of price elasticities.

We will say just a few words on the more general subject of robustness. Our model is based on a linear price structure. Certainly, this is only an approximation. However, in practical applications we are likely to proceed as follows. We begin with an educated guess about the size of the market for our new products and with a reasonable average sale price. Then we estimate elasticities either on the basis of past experience with similar situations or on the basis of limited marketing studies. We should be able to get reasonable estimates for these elasticities over a certain range of sales levels. This range presumably includes the optimal levels. So in effect we are simply making a linear approximation of a nonlinear function over a fairly small region. This sort of approximation is well known to exhibit robustness. After all, this is the whole idea behind calculus.

2.2 Lagrange Multipliers

In this section we begin to consider optimization problems with a more sophisticated structure. As we noted at the beginning of the previous section, complications arise in the solution of multivariable optimization models when the set over which we optimize becomes more complex. In real problems we are led to consider these more complicated models by the existence of constraints on the independent variables.

Example 2.2. We reconsider the color TV problem (Example 2.1) introduced in the previous section. There we assumed that the company has the potential to produce any number of TV sets per year. Now we will introduce constraints based on the available production capacity. Consideration of these two new products came about because the company plans to discontinue manufacture of some older models, thus providing excess capacity at its assembly plant. This excess capacity could be used to increase production of other existing product lines, but the company feels that the new products will be more profitable. It

Variables:	s = number of 19–inch sets sold (per year)
	t = number of 21–inch sets sold (per year)
	p = selling price for a 19–inch set (\$)
	q = selling price for a 21–inch set (\$)
	C = cost of manufacturing sets (\$/year)
	R = revenue from the sale of sets (\$/year)
	P = profit from the sale of sets (\$/year)
Assumptions:	$p = 339 - 0.01s - 0.003t$
	$q = 399 - 0.004s - 0.01t$
	$R = ps + qt$
	$C = 400,000 + 195s + 225t$
	$P = R - C$
	$s \leq 5000$
	$t \leq 8000$
	$s + t \leq 10,000$
	$s \geq 0$
	$t \geq 0$
Objective:	Maximize P

Figure 2.9: Results of step 1 for the color TV problem with constraints.

is estimated that the available production capacity will be sufficient to produce 10,000 sets per year ($\approx$ 200 per week). The company has an ample supply of 19–inch and 21–inch LCD panels and other standard components; however, the circuit boards necessary for constructing the sets are currently in short supply. Also, the 19–inch TV requires a different board than the 21–inch model because of the internal configuration, which cannot be changed without a major redesign, which the company is not prepared to undertake at this time. The supplier is able to supply 8,000 boards per year for the 21–inch model and 5,000 boards per year for the 19–inch model. Taking this information into account, how should the company set production levels?

Once again we will employ the five-step method. The results of step 1 are shown in Figure 2.9. The only change is the addition of several constraints on the decision variables s and t. Step 2 is to select the modeling approach.

This problem will be modeled as a multivariable constrained optimization problem and solved using the method of Lagrange multipliers.

We are given a function $y = f(x_1, \ldots, x_n)$ and a set of constraints. For the moment we will assume that these constraints can be expressed in the form of k functional equations

$$
\begin{aligned}
g_1(x_1, \ldots, x_n) &= c_1 \\
g_2(x_1, \ldots, x_n) &= c_2 \\
&\vdots \\
g_k(x_1, \ldots, x_n) &= c_k.
\end{aligned}
$$

Later on we will explain how to handle inequality constraints. Our job is to optimize

$$y = f(x_1, \ldots, x_n)$$

over the set

$$S = \{(x_1, \ldots, x_n) : \ g_i(x_1, \ldots, x_n) = c_i \text{ for all } i = 1, \ldots, k\}.$$

There is a theorem that states that at an extreme point $x \in S$, we must have

$$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k.$$

We call $\lambda_1, \ldots, \lambda_k$ the *Lagrange multipliers*. This theorem assumes that $\nabla g_1, \ldots, \nabla g_k$ are linearly independent vectors (see Edwards (1973), p. 113). Then in order to locate the max–min points of f on the set S, we must solve the n Lagrange multiplier equations

$$
\begin{aligned}
\frac{\partial f}{\partial x_1} &= \lambda_1 \frac{\partial g_1}{\partial x_1} + \cdots + \lambda_k \frac{\partial g_k}{\partial x_1} \\
&\vdots \\
\frac{\partial f}{\partial x_n} &= \lambda_1 \frac{\partial g_1}{\partial x_n} + \cdots + \lambda_k \frac{\partial g_k}{\partial x_n}
\end{aligned}
$$

together with the k constraint equations

$$
\begin{aligned}
g_1(x_1, \ldots, x_n) &= c_1 \\
&\vdots \\
g_k(x_1, \ldots, x_n) &= c_k
\end{aligned}
$$

for the variables $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_k$. We must also check any exceptional points at which the gradient vectors $\nabla g_1, \ldots, \nabla g_k$ are not linearly independent.

The method of Lagrange multipliers is based on a geometrical interpretation of the gradient vector. Suppose for the moment that there is only one constraint equation,

$$g(x_1, \ldots, x_n) = c,$$

so that the Lagrange multiplier equation becomes

$$\nabla f = \lambda \nabla g.$$

The set $g = c$ is a curved surface of dimension $n - 1$ in $\mathbb{R}^n$, and for any point $x \in S$ the gradient vector $\nabla g(x)$ is perpendicular to S at that point. The gradient vector ∇f always points in the direction in which f increases the fastest. At a local max or min, the direction in which f increases fastest must also be perpendicular to S, so at that point we must have ∇f and ∇g pointing along the same line; i.e., $\nabla f = \lambda \nabla g$.

In the case of several constraints, the geometrical argument is similar. Now the set S represents the intersection of the k level surfaces $g_1 = c_1, \ldots, g_k = c_k$. Each one of these is an $(n-1)$–dimensional subset of $\mathbb{R}^n$, so their intersection is an $(n-k)$–dimensional subset. At an extreme point, ∇f must be perpendicular to the set S. Therefore it must lie in the space spanned by the k vectors $\nabla g_1, \ldots, \nabla g_k$. The technical condition of linear independence ensures that the k vectors $\nabla g_1, \ldots, \nabla g_k$ actually do span a k–dimensional space. (In the case of a single constraint, linear independence simply means that $\nabla g \neq 0$.)

Example 2.3. Maximize $x + 2y + 3z$ over the set $x^2 + y^2 + z^2 = 3$.

This is a constrained multivariable optimization problem. Let

$$f(x, y, z) = x + 2y + 3z$$

denote the objective function, and let

$$g(x, y, z) = x^2 + y^2 + z^2$$

denote the constraint function. Compute

$$\begin{aligned} \nabla f &= (1, 2, 3) \\ \nabla g &= (2x, 2y, 2z). \end{aligned}$$

At the maximum, $\nabla f = \lambda \nabla g$; in other words,

$$\begin{aligned} 1 &= 2x\lambda \\ 2 &= 2y\lambda \\ 3 &= 2z\lambda. \end{aligned}$$

This gives three equations in four unknowns. Solving in terms of λ, we obtain

$$\begin{aligned} x &= 1/2\lambda \\ y &= 1/\lambda \\ z &= 3/2\lambda. \end{aligned}$$

Using the fact that

$$x^2 + y^2 + z^2 = 3,$$

we obtain a quadratic equation in λ, with two real roots. The root $\lambda = \sqrt{42}/6$ leads to

$$\begin{aligned} x &= \frac{1}{2\lambda} = \frac{\sqrt{42}}{14} \\ y &= \frac{1}{\lambda} = \frac{\sqrt{42}}{7} \\ z &= \frac{3}{2\lambda} = \frac{3\sqrt{42}}{14}, \end{aligned}$$

so that the point

$$a = \left(\frac{\sqrt{42}}{14}, \frac{\sqrt{42}}{7}, \frac{3\sqrt{42}}{14} \right)$$

is one candidate for the maximum. The other root, $\lambda = -\sqrt{42}/6$, leads to another candidate, $b = -a$. Since $\nabla g \neq 0$ everywhere on the constraint set $g = 3$, a and b are the only two candidates for the maximum. Since f is a continuous function on the closed and bounded set $g = 3$, f must attain its maximum and minimum on this set. Then, since

$$f(a) = \sqrt{42}, \qquad \text{and} \qquad f(b) = -\sqrt{42},$$

the point a is the maximum and b is the minimum. Consider the geometry of this example. The constraint set S defined by the equation

$$x^2 + y^2 + z^2 = 3$$

is a sphere of radius $\sqrt{3}$ centered at the origin in $\mathbb{R}^3$. Level sets of the objective function

$$f(x, y, z) = x + 2y + 3z$$

are planes in R^3. The points a and b are the only two points on the sphere S at which one of these planes is tangent to the sphere. At the maximum point a, the gradient vectors ∇f and ∇g point in the same direction. At the minimum point b, ∇f and ∇g point in opposite directions.

Example 2.4. Maximize $x + 2y + 3z$ over the set $x^2 + y^2 + z^2 = 3$ and $x =1$.

The objective function is

$$f(x, y, z) = x + 2y + 3z,$$

so

$$\nabla f = (1,\, 2,\, 3).$$

The constraint functions are

$$\begin{aligned} g_1(x,\, y,\, z) &= x^2 + y^2 + z^2 \\ g_2(x,\, y,\, z) &= x. \end{aligned}$$

Compute

$$\begin{aligned} \nabla g_1 &= (2x,\, 2y,\, 2z) \\ \nabla g_2 &= (1,\, 0,\, 0). \end{aligned}$$

Then the Lagrange multiplier formula $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ yields

$$\begin{aligned} 1 &= 2x\lambda_1 + \lambda_2 \\ 2 &= 2y\lambda_1 \\ 3 &= 2z\lambda_1. \end{aligned}$$

Solving for $x,\ y,$ and z in terms of λ_1 and λ_2 gives

$$\begin{aligned} x &= \frac{1 - \lambda_2}{2\lambda_1} \\ y &= \frac{2}{2\lambda_1} \\ z &= \frac{3}{2\lambda_1}. \end{aligned}$$

Substituting into the constraint equation $x = 1$ gives $\lambda_2 = 1 - 2\lambda_1$. Substituting all of this into the remaining equation

$$x^2 + y^2 + z^2 = 3$$

yields a quadratic equation in λ_1, which gives $\lambda_1 = \pm\sqrt{26}/4$. Substituting back into the equations for x, y, and z yields the two following solutions:

$$\begin{aligned} c &= \left(1,\, \frac{2\sqrt{26}}{13},\, \frac{3\sqrt{26}}{13}\right) \\ d &= \left(1,\, \frac{-2\sqrt{26}}{13},\, \frac{-3\sqrt{26}}{13}\right). \end{aligned}$$

Since the two gradient vectors ∇g_1 and ∇g_2 are linearly independent everywhere on the constraint set, the points c and d are the only candidates for a maximum. Since f must attain its maximum on this closed and bounded set, we need only evaluate f at each candidate point to find the maximum. The maximum is

$$f(c) = 1 + \sqrt{26},$$

and the point d is the location of the minimum. The constraint set S in this example is a circle in $\mathbb{R}^3$ formed by the intersection of the sphere

$$x^2 + y^2 + z^2 = 3$$

and the plane $x = 1$. As before, the level sets of the function f are planes in $\mathbb{R}^3$. At the points c and d these planes are tangent to the circle S.

Inequality constraints can be handled by a combination of the Lagrange multiplier technique and the techniques for unconstrained problems. Suppose that the problem in Example 2.4 is altered by replacing the $x = 1$ constraint with the inequality constraint $x \geq 1$. We can consider the set

$$S = \{(x,\, y,\, z) : x^2 + y^2 + z^2 = 3,\ x \geq 1\}$$

as the union of two components. The maximum over the first component

$$S_1 = \{(x,\, y,\, z) : x^2 + y^2 + z^2 = 3,\ x = 1\}$$

was found to occur at the point

$$c = \left(1,\, \sqrt{\frac{8}{13}},\, 1.5\sqrt{\frac{8}{13}}\right)$$

in our previous analysis, and we can calculate that

$$f(x,\, y,\, z) = 1 + 6.5\sqrt{\frac{8}{13}} = 6.01$$

at this point. To consider the remaining part

$$S_2 = \{(x,\, y,\, z) : x^2 + y^2 + z^2 = 3,\ x > 1\},$$

we return to our analysis from Example 2.3, noting that there is no local maximum of f anywhere on this set. Therefore, the maximum of f on S_1 must be the maximum of the function f on the set S. If we had considered the maximum of f over the set

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 3,\ x \leq 1\},$$

then the maximum would be at the point

$$a = \left(\frac{1}{2}, \frac{2}{2}, \frac{3}{2}\right) \cdot \sqrt{\frac{6}{7}}$$

found in our analysis of Example 2.3.

Returning now to the problem introduced at the beginning of this section, we are ready to continue the modeling process with step 3. We will formulate the revised color TV problem as a constrained multivariable optimization problem. We wish to maximize $y = P$ (profit) as a function of our two decision variables, $x_1 = s$ and $x_2 = t$. We have the same objective function

$$\begin{aligned} y &= f(x_1, x_2) \\ &= (339 - 0.01x_1 - 0.003x_2)x_1 + (399 - 0.004x_1 - 0.01x_2)x_2 \\ &\quad - (400,000 + 195x_1 + 225x_2). \end{aligned}$$

We wish to maximize f over the set S consisting of all x_1 and x_2 satisfying the constraints

$$\begin{aligned} x_1 &\le 5,000 \\ x_2 &\le 8,000 \\ x_1 + x_2 &\le 10,000 \\ x_1 &\ge 0 \\ x_2 &\ge 0. \end{aligned}$$

The set S is called the *feasible region* because it represents the set of all feasible production levels. Figure 2.10 shows a graph of the feasible region for this problem.

We will apply Lagrange multiplier methods to find the maximum of $y = f(x_1, x_2)$ over the set S. Compute

$$\nabla f = (144 - 0.02x_1 - 0.007x_2,\ 174 - 0.007x_1 - 0.02x_2).$$

Since $\nabla f \neq 0$ in the interior of S, the maximum must occur on the boundary. Consider first the segment of the boundary on the constraint line

$$g(x_1, x_2) = x_1 + x_2 = 10,000.$$

Here $\nabla g = (1, 1)$, so the Lagrange multiplier equations are

$$\begin{aligned} 144 - 0.02x_1 - 0.007x_2 &= \lambda \\ 174 - 0.007x_1 - 0.02x_2 &= \lambda. \end{aligned} \tag{2.12}$$

Solving these two equations together with the constraint equation

$$x_1 + x_2 = 10,000$$

yields

$$\begin{aligned} x_1 &= \frac{50,000}{13} \approx 3,846 \\ x_2 &= \frac{80,000}{13} \approx 6,154 \\ \lambda &= 24. \end{aligned}$$

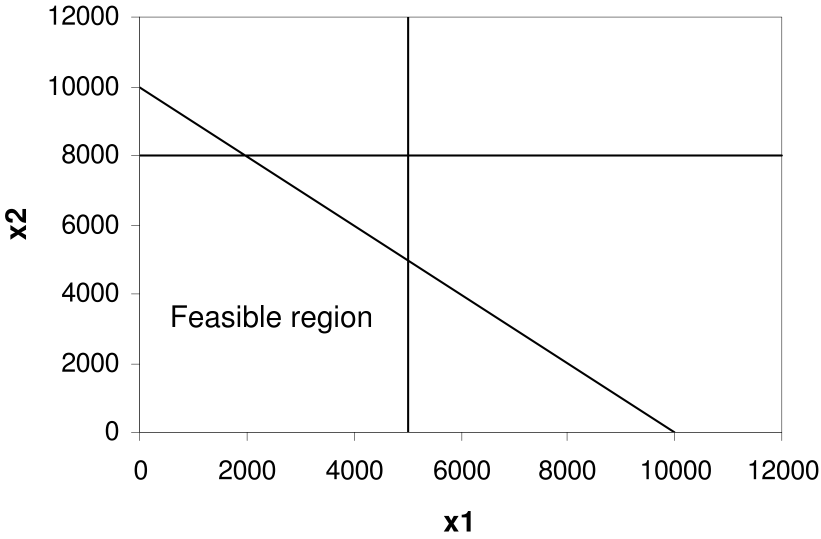

Figure 2.10: Graph showing the set of all feasible production levels x_1 of 19–inch sets and x_2 of 21–inch sets for the color TV problem with constraints.

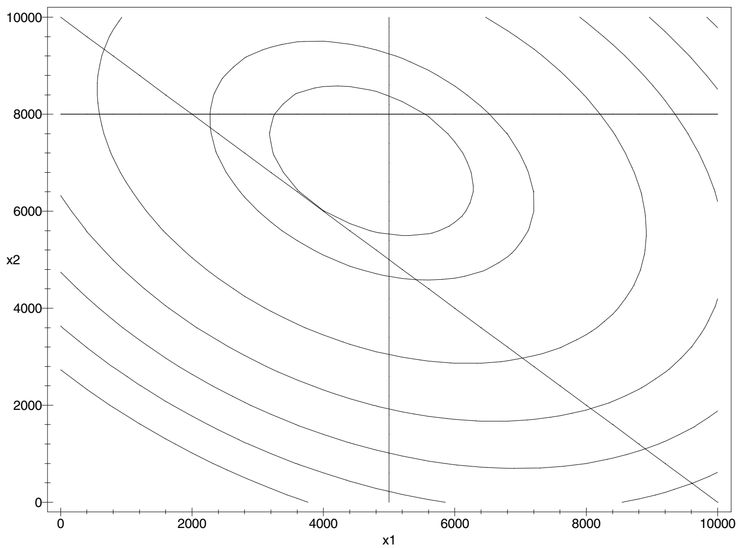

Figure 2.11: Graph showing level sets of profit $y = f(x_1, x_2)$ versus production levels x_1 of 19–inch sets and x_2 of 21–inch sets together with the set of all feasible production levels for the color TV problem with constraints.

Substituting back into Eq. (2.2), we obtain $y = 532,308$ at the maximum.

Figure 2.11 shows a Maple graph of the level sets of f together with the feasible region.

The level sets $f = C$ for $C = 0, 100,000, \ldots, 500,000$ form smaller and smaller concentric rings, all of which intersect the feasible region. The level set $f = 532,308$ forms the smallest ring. This set barely touches the feasible region S, and is tangent to the line $x_1 + x_2 = 10,000$ at the optimum point. This graphical evidence indicates that the critical point found by using Lagrange multipliers along the constraint line $x_1 + x_2 = 10,000$ is actually the maximum of the function f over the feasible region S.

In[1]:= **y = (339 − x1 / 100 − 3 x2 / 1000) x1 +**
(399 − 4 x1 / 1000 − x2 / 100) x2 −
(400000 + 195 x1 + 225 x2)

Out[1]= $\left(-\frac{x1}{100} - \frac{3\,x2}{1000} + 339\right) x1 - 195\,x1 + \left(-\frac{x1}{250} - \frac{x2}{100} + 399\right) x2 - 225\,x2 - 400000$

In[2]:= **dydx1 = D[y, x1]**

Out[2]= $-\frac{x1}{50} - \frac{7\,x2}{1000} + 144$

In[3]:= **dydx2 = D[y, x2]**

Out[3]= $-\frac{7\,x1}{1000} - \frac{x2}{50} + 174$

In[4]:= **s = Solve[{dydx1 == lambda, dydx2 == lambda, x1 + x2 == 10000}, {x1, x2, lambda}]**

Out[4]= $\left\{\left\{x1 \to \frac{50000}{13},\ x2 \to \frac{80000}{13},\ \text{lambda} \to 24\right\}\right\}$

In[5]:= **N[%]**

Out[5]= {{x1 → 3846.15, x2 → 6153.85, lambda → 24.}}

In[6]:= **y /. %**

Out[6]= {532308.}

Figure 2.12: Optimal solution to the color TV problem with constraints using the computer algebra system Mathematica.

An algebraic proof that this point is actually the maximum is a bit more complicated. By comparing values of f at this critical point with values at the endpoints (5,000, 5,000) and (2,000, 8,000), we can show that this critical point is the maximum over this line segment. Then we can optimize f over the remaining line segments and compare results. For example, the maximum of f over the line segment along the x_1 axis occurs at $x_1 = 5,000$. To see this, apply Lagrange multipliers with $g(x_1, x_2) = x_2 = 0$. Here $\nabla g = (0, 1)$, so the Lagrange multiplier equations are

$$\begin{aligned} 144 - 0.02x_1 - 0.007x_2 &= 0 \\ 174 - 0.007x_1 - 0.02x_2 &= \lambda. \end{aligned}$$

Solving these two equations together with the constraint equation $x_2 = 0$ yields $x_1 = 7,200$, $x_2 = 0$, and $\lambda = 123.6$. This is outside the feasible region, so the max–min along this segment must occur at the endpoints $(0, 0)$ and $(5,000, 0)$. The first is the minimum and the second is the maximum, since the value of f at the second is greater. It is also possible to optimize along this line segment by substituting $x_2 = 0$ and using one variable methods. Since the largest value of f occurs on the slanted line segment, we have found the maximum over S. Some of the calculations in step 4 are rather involved. In such cases it is appropriate to use a computer algebra system to simplify the process of computing derivatives and solving equations. Figure 2.12 shows the results of using the computer algebra system Mathematica to perform the calculations of step 4 for the constraint line $x_1 + x_2 = 10,000$.

In plain English, the company can maximize profits by producing 3,846 of the 19–inch sets and 6,154 of the 21–inch sets for a total of 10,000 sets per year. This level of production uses all of the available excess production capacity. The resource constraints on the availability of TV circuit boards are not binding. This venture will produce an estimated profit of $532,308 annually.

2.3 Sensitivity Analysis and Shadow Prices

In this section we discuss some of the specialized techniques for sensitivity analysis in Lagrange multiplier models. It turns out that the multipliers themselves have a real–world significance.

Before we report the results of our model analysis in Example 2.2, it is important to perform sensitivity analysis. At the end of Section 2.1 we investigated the sensitivity to price elasticity for a model without constraints. The procedure for our new model is not much different. We examine the sensitivity to a particular parameter value by generalizing the model slightly, replacing the assumed value with a variable. Suppose we want to look again at the price elasticity, a, for 19–inch sets. We rewrite the objective function as in Eq. (2.7) so that

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right),$$

where $\partial f/\partial x_1$ and $\partial f/\partial x_2$ are given by Eq. (2.8). Now the Lagrange multiplier equations are

$$\begin{aligned} 144 - 2ax_1 - 0.007x_2 &= \lambda \\ 174 - 0.007x_1 - 0.02x_2 &= \lambda. \end{aligned} \tag{2.13}$$

Solving together with the constraint equation

$$g(x_1, x_2) = x_1 + x_2 = 10{,}000$$

yields

$$\begin{aligned} x_1 &= \frac{50{,}000}{1{,}000a + 3} \\ x_2 &= 10{,}000 - \frac{50{,}000}{1{,}000a + 3} \\ \lambda &= \frac{650}{1{,}000a + 3} - 26. \end{aligned} \tag{2.14}$$

Then we have

$$\begin{aligned} \frac{dx_1}{da} &= \frac{-50{,}000{,}000}{(1{,}000a + 3)^2} \\ \frac{dx_2}{da} &= \frac{-dx_1}{da}, \end{aligned} \tag{2.15}$$

so that at the point $x_1 = 3{,}846$, $x_2 = 6{,}154$, $a = 0.01$, we obtain

$$\begin{aligned} S(x_1, a) &= \frac{dx_1}{da} \cdot \frac{a}{x_1} = -0.77 \\ S(x_2, a) &= \frac{dx_2}{da} \cdot \frac{a}{x_2} = 0.48. \end{aligned}$$

See Figures 2.13 and 2.14 for the graphs of x_1 and x_2 versus a in this case. If the price elasticity of 19–inch sets increases, we will shift production from 19–inch to 21–inch sets. If it decreases, then we will produce more 19–inch sets and fewer 21–inch sets.

In any case, as long as the point (x_1, x_2) given by Eq. (2.14) lies between the other constraint lines $(.007 \leq a \leq .022)$, we will always produce a total of 10,000 sets.

Now let us consider the sensitivity of our optimal profit y to the price elasticity a for 19–inch sets. To obtain a formula for y in terms of a, we substitute Eq. (2.14) back into Eq. (2.2) to get

$$\begin{aligned} y = {} & \left[339 - a\left(\frac{50{,}000}{1{,}000a + 3}\right) - 0.003\left(10{,}000 - \frac{50{,}000}{1{,}000a + 3}\right)\right]\left(\frac{50{,}000}{1{,}000a + 3}\right) \\ & + \left[399 - 0.004\left(\frac{50{,}000}{1{,}000a + 3}\right) - 0.01\left(10{,}000 - \frac{50{,}000}{1{,}000a + 3}\right)\right] \\ & \times \left(1{,}000 - \frac{50{,}000}{1{,}000a + 3}\right) \\ & - \left[400{,}000 + 195\left(\frac{50{,}000}{1{,}000a + 3}\right) + 225\left(10{,}000 - \frac{50{,}000}{1{,}000a + 3}\right)\right]. \end{aligned}$$

See Figure 2.15 for the graph of y versus a.

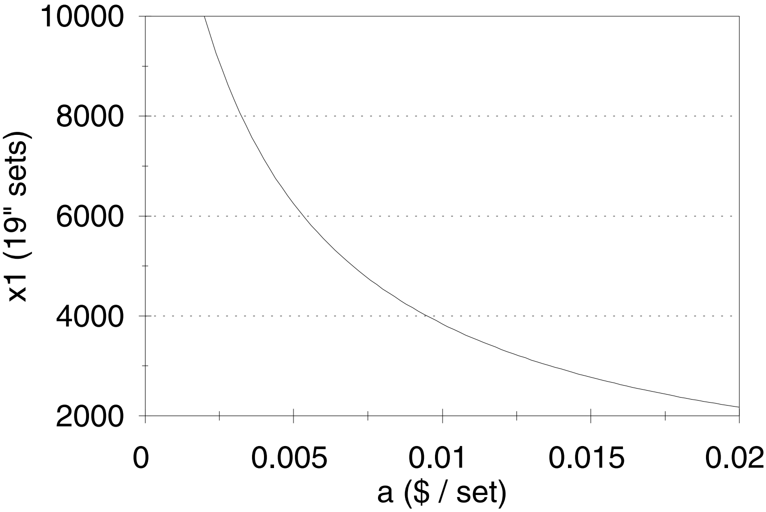

Figure 2.13: Graph of optimal production level x_1 of 19–inch sets versus price elasticity a for the color TV problem with constraints.

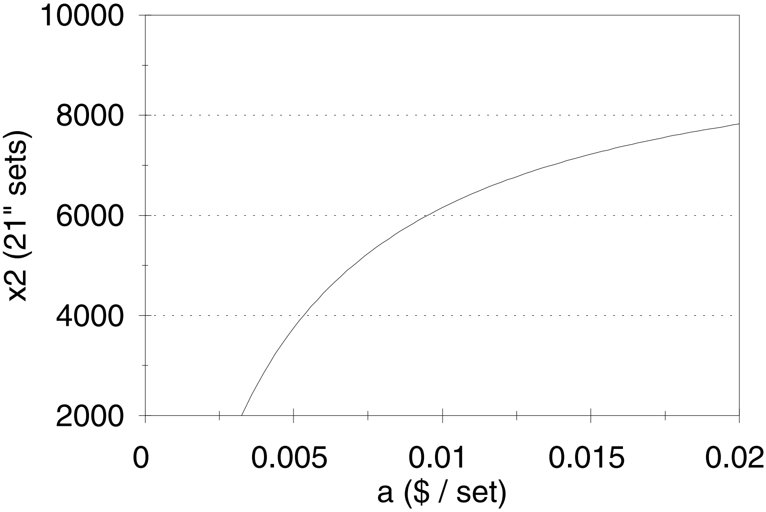

Figure 2.14: Graph of optimal production level x_2 of 21–inch sets versus price elasticity a for the color TV problem with constraints.

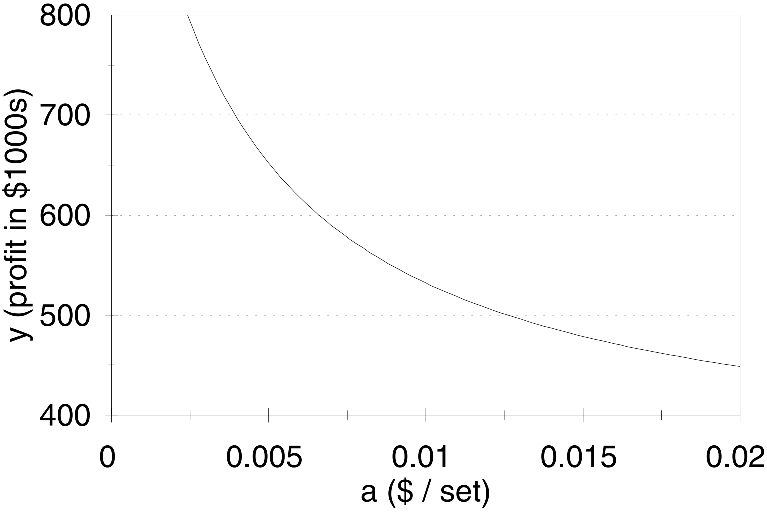

Figure 2.15: Graph of optimum profit y versus price elasticity a for the color TV problem with constraints.

In order to get a numerical measure of the sensitivity of y to a, we could apply one–variable techniques to the above expression, perhaps with the aid of a computer algebra system. Another method, which is much more efficient computationally, is to use the multivariable chain rule in Eq. (2.11). For any a, the gradient vector ∇f is perpendicular to the constraint line $g = 10,000$. Since

$$x(a) = (x_1(a), x_2(a))$$

is a point on the curve $g = 10,000$, the velocity vector

$$\frac{dx}{da} = \left(\frac{dx_1}{da}, \frac{dx_2}{da}\right)$$

is tangent to the curve. But then ∇f is perpendicular to dx/da; i.e., the dot product

$$\begin{aligned}\nabla f \cdot \frac{dx}{da} &= \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}\right) \cdot \left(\frac{dx_1}{da}, \frac{dx_2}{da}\right) \\ &= \frac{\partial y}{\partial x_1}\frac{dx_1}{da} + \frac{\partial y}{\partial x_2}\frac{dx_2}{da} = 0\end{aligned}$$

in general. Therefore, we again obtain

$$\frac{dy}{da} = \frac{\partial y}{\partial a} = -x^2$$

as in Section 2.1. Now we can easily compute that

$$\begin{aligned} S(y,\, a) &= \frac{dy}{da} \cdot \frac{a}{y} \\ &= -(3,846)^2 \frac{0.01}{532,308} \\ &= -0.28. \end{aligned}$$

As in the unconstrained problem, an increase in price elasticity leads to lost profits. It is also true here that, as in the unconstrained problem, almost all of the lost profit is due to the fact that the selling price for 19–inch sets has decreased. If $a = 0.011$ and we use $x_1 = 3,846$, $x_2 = 6,154$ instead of the new optimum given by Eq. (2.13), we will not lose much potential profit. The gradient vector ∇f points in the direction of fastest increase of the objective function f, which represents our profits. We are not at the optimum, but the path between the optimum and the point (3,846, 6,154) is perpendicular to ∇f. Therefore, we may expect that the value of f at this point does not vary much from the optimum value. Hence our model leads to a nearly optimal decision even in the presence of small variations in a.

We remark that for this problem it would also make sense to use a computer algebra system to perform the necessary calculations. Figure 2.16 illustrates the use of the computer algebra system Maple to compute the sensitivity $S(x_2,\, a)$. The other sensitivities can be computed in a similar manner.

We will now consider the sensitivity of the optimal production levels x_1 and x_2 and the resulting profit y to the available manufacturing capacity of $c = 10,000$ sets per year. In order to do this we will return to our original problem, replacing the constraint $g = 10,000$ by the more general form $g = c$. The feasible region is similar to that pictured in Fig. 2.10, but now the slanted constraint line is moved a bit (remaining parallel to the line $x_1 + x_2 = 10,000$). For values of c near $10,000$, the maximum still occurs at the point on the constraint line

$$g(x_1,\, x_2) = x_1 + x_2 = c \tag{2.16}$$

where $\nabla f = \lambda \nabla g$. Since both ∇f and ∇g are unchanged from our original problem, we have the same Lagrange multiplier equations from Eq. (2.12), which are to be solved along with the new constraint equation, Eq. (2.16). Solving, we obtain

$$\begin{aligned} x_1 &= \frac{13c - 30,000}{26} \\ x_2 &= \frac{13c + 30,000}{26} \\ \lambda &= \frac{3(106,000 - 9c)}{2,000}. \end{aligned} \tag{2.17}$$

```
> y:=(339-a*x1-3*x2/1000)*x1
>    +(399-4*x1/1000-x2/100)*x2-(400000+195*x1+225*x2);
```

$$y := \left(339 - a\,x1 - \frac{3}{1000}\,x2\right)x1 + \left(399 - \frac{1}{250}\,x1 - \frac{1}{100}\,x2\right)x2 - 400000 - 195\,x1 - 225\,x2$$

```
> dydx1:=diff(y,x1);
```

$$dydx1 := -2\,a\,x1 + 144 - \frac{7}{1000}\,x2$$

```
> dydx2:=diff(y,x2);
```

$$dydx2 := -\frac{7}{1000}\,x1 - \frac{1}{50}\,x2 + 174$$

```
> s:=solve({dydx1=lambda,dydx2=lambda,x1+x2=10000},{x1,x2,lambda});
```

$$s := \{\,x1 = \frac{50000}{1000\,a+3},\ x2 = 20000\,\frac{500\,a-1}{1000\,a+3},\ \lambda = -52\,\frac{500\,a-11}{1000\,a+3}\}$$

```
> assign(s);
> dx2da:=diff(x2,a);
```

$$dx2da := \frac{10000000}{1000\,a+3} - 20000000\,\frac{500\,a-1}{(1000\,a+3)^2}$$

```
> assign(a=1/100);
> sx2a:=dx2da*(a/x2);
```

$$sx2a := \frac{25}{52}$$

```
> evalf(sx2a);
```

$$.4807692308$$

Figure 2.16: Calculation of the sensitivity $S(x_2,\,a)$ for the color TV problem with constraints using the computer algebra system Maple.

Now

$$\begin{aligned}\frac{dx_1}{dc} &= \frac{1}{2}\\ \frac{dx_2}{dc} &= \frac{1}{2}.\end{aligned} \tag{2.18}$$

There is a simple geometric explanation for Eq. (2.18). Since ∇f points in the direction of the fastest increase of f, when we move the constraint line in Eq. (2.16), the new optimum $(x_1,\,x_2)$ should be located at approximately the point where ∇f intersects the line in Eq. (2.16). Then

$$\begin{aligned}S(x_1,\,c) &= \frac{1}{2}\cdot\frac{10,000}{3,846} \approx 1.3\\ S(x_2,\,c) &= \frac{1}{2}\cdot\frac{10,000}{6,154} \approx 0.8.\end{aligned}$$

To obtain the sensitivity of y to c, we compute

$$\begin{aligned}\frac{dy}{dc} &= \frac{\partial y}{\partial x_1}\frac{dx_1}{dc} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dc}\\ &= (24)\left(\frac{1}{2}\right) + (24)\left(\frac{1}{2}\right)\\ &= 24,\end{aligned}$$

which is the value of the Lagrange multiplier λ. Now

$$S(y,\, c) = (24)\left(\frac{10,000}{532,308}\right) \approx 0.45.$$

The geometric explanation for $dy/dc = \lambda$ is as follows. We have $\nabla f = \lambda \nabla g$, and as we increase c, we move out in the direction ∇f. As we move in this direction, f increases λ times as fast as g.

The derivative $dy/dc = 24$ has an important real–world interpretation. The addition of 1 unit of production capacity $\Delta c = 1$ results in an increased profit $\Delta y = 24$ dollars. This is called a *shadow price.* It represents the value to the company of a certain resource (production capacity). If the company is interested in the possibility of increasing production capacity, which is, after all, the binding constraint, it should be willing to pay up to $24 per unit of added capacity. Alternatively, it would be worthwhile to transfer production capacity from the manufacture of 19–inch and 21–inch LCD flat panel TV sets to an alternative product if and only if the new product would yield a profit greater than $24 per unit.

The calculation of sensitivities in this problem can be simplified by using a computer algebra system. Figure 2.17 illustrates the use of the computer algebra system Maple to compute the sensitivity $S(y,\, c)$.

The other sensitivities can be computed in a similar manner. If you have the good fortune to have access to a computer algebra system, you should use it for your own work. Real–world problems often involve lengthy computations. Some facility in the use of a computer algebra system will make you more productive. It is also a lot more fun than calculation by hand.

Of course, the optimal level of profit y and the production levels x_1 and x_2 are totally insensitive to the values of the other constraint coefficients, since the other constraints, $x_1 \le 5,000$ and $x_2 \le 8,000$, are not binding. A small change in the upper bounds on x_1 or x_2 would change the feasible region, but the optimal solution would remain at (3,846, 6,154). Thus, the shadow prices for these resources are zero. The company would be unwilling to pay a premium to increase the available number of TV circuit boards, since they do not need them. This situation would not change unless the number of boards available was reduced to 3,846 or less for 19–inch sets, or to 6,154 or less for 21–inch sets. In the next example, we will consider what happens in this case.

Example 2.5. Suppose that in the constrained color TV problem of Example 2.2, the number of circuit boards available for 19–inch TVs is only 3,000 per year. What is the optimum production schedule?

```
> y:=(339-x1/100-3*x2/1000)*x1
>    +(399-4*x1/1000-x2/100)*x2-(400000+195*x1+225*x2);
```

$$y := \left(339 - \frac{1}{100}x1 - \frac{3}{1000}x2\right)x1 + \left(399 - \frac{1}{250}x1 - \frac{1}{100}x2\right)x2 - 400000 - 195\,x1 - 225\,x2$$

```
> dydx1:=diff(y,x1);
```

$$dydx1 := -\frac{1}{50}x1 + 144 - \frac{7}{1000}x2$$

```
> dydx2:=diff(y,x2);
```

$$dydx2 := -\frac{7}{1000}x1 - \frac{1}{50}x2 + 174$$

```
> s:=solve({dydx1=lambda,dydx2=lambda,x1+x2=c},{x1,x2,lambda});
```

$$s := \{\lambda = -\frac{27}{2000}c + 159, x1 = \frac{1}{2}c - \frac{15000}{13}, x2 = \frac{1}{2}c + \frac{15000}{13}\}$$

```
> assign(s);
> dydc:=diff(y,c);
```

$$dydc := -\frac{27}{2000}c + 159$$

```
> assign(c=10000);
> dydc;
```

$$24$$

```
> syc:=dydc*(c/y);
```

$$syc := \frac{78}{173}$$

```
> evalf(syc);
```

$$.4508670520$$

Figure 2.17: Calculation of the sensitivity $S(y,\, c)$ for the color TV problem with constraints using the computer algebra system Maple.

In this case the point on the level curve $x_1 + x_2 = 10,000$ at which $f(x_1,\, x_2)$ is maximized occurs outside of the feasible region. The maximum of f on the feasible region occurs at the point (3,000, 7,000). This is the intersection of the constraint curves

$$g_1(x_1,\, x_2) = x_1 + x_2 = 10,000$$
$$g_2(x_1,\, x_2) = x_1 = 3,000.$$

At this point we have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.$$

In fact, we can easily compute

$$\nabla f = (35,\, 13)$$
$$\nabla g_1 = (1,\, 1)$$
$$\nabla g_2 = (1,\, 0)$$

at the point (3,000, 7,000); thus we have $\lambda_1 = 13$ and $\lambda_2 = 22$. Of course any vector in $\mathbb{R}^2$ can be written as a linear combination of (1, 1) and (1, 0). The point, however, of computing the Lagrange multipliers, is that even in the case of multiple constraints, they still represent the shadow prices for the binding constraints (production capacity and 19–inch boards). In other words, an additional unit of production capacity is worth \$13, and an additional circuit board is worth \$22.

For the convenience of the reader, we include here a proof of the fact that the Lagrange multipliers represent shadow prices. We are given a function $y = f(x_1, \ldots, x_n)$, which is to be optimized over the set defined by one or more constraint equations of the form

$$\begin{aligned} g_1(x_1, \ldots, x_n) &= c_1 \\ g_2(x_1, \ldots, x_n) &= c_2 \\ &\vdots \\ g_k(x_1, \ldots, x_n) &= c_k. \end{aligned}$$

Suppose that the optimum occurs at a point x_0 at which the hypotheses of the Lagrange multiplier theorem are satisfied, so that at this point we have

$$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k. \tag{2.19}$$

Since the constraint equations can be written in any order, it will suffice to show that λ_1 is the shadow price corresponding to the first constraint. Let $x(t)$ denote the location of the optimum point over the set $g_1 = t$, $g_2 = c_2, \ldots, g_k = c_k$. Since $g_1(x(t)) = t$, we have $\nabla g(x(t)) \cdot x'(t) = 1$, and in particular $\nabla g(x_0) \cdot x'(c_1) = 1$. Since for $i = 2, \ldots, k$ we have $g_i(x(t)) = c_i$ constant for all t, we have $\nabla g_i(x(t)) \cdot x'(t) = 0$, and in particular $\nabla g_i(x_0) \cdot x'(c_1) = 0$. The shadow price is

$$\frac{d(f(x(t)))}{dt} = \nabla f(x(t)) \cdot x'(t)$$

evaluated at the point $t = c_1$. Then, from the fact that Eq. (2.19) holds at the point x_0, we obtain $\nabla f(x_0) \cdot x'(c_1) = \lambda_1 \nabla g_1(x_0) \cdot x'(c_1) = \lambda_1$ as desired.

2.4 Exercises

1. Ecologists use the following model to represent the growth process of two competing species, x and y:

$$\frac{dx}{dt} = r_1 x \left(1 - \frac{x}{K_1}\right) - \alpha_1 xy$$
$$\frac{dy}{dt} = r_2 y \left(1 - \frac{y}{K_2}\right) - \alpha_2 xy.$$

The variables x and y represent the number in each population; the parameters r_i represent the intrinsic growth rates of each species; K_i represents the maximum sustainable population in the absence of competition; and α_i represents the effects of competition. Studies of the blue whale and fin whale populations have determined the following parameter values (t in years):

	Blue	Fin
r	0.05	0.08
K	150,000	400,000
α	10^{-8}	10^{-8}

   (a) Determine the population levels x and y that maximize the number of new whales born each year. Use the five-step method, and model as an unconstrained optimization problem.

   (b) Examine the sensitivity of the optimal population levels to the intrinsic growth rates r_1 and r_2.

   (c) Examine the sensitivity of the optimal population levels to the environmental carrying capacities K_1 and K_2.

   (d) Assuming that $\alpha_1 = \alpha_2 = \alpha$, is it ever optimal for one species to become extinct?

2. Reconsider the whale problem of Exercise 1, but now look at the total number of whales. We will say that the whale population levels x and y are feasible provided that both x and y are nonnegative. We will say that the population levels x and y are *sustainable* provided that both of the growth rates dx/dt and dy/dt are nonnegative.

   (a) Determine the population levels that are feasible, sustainable, and that maximize the total whale population $x + y$. Use the five-step method, and model as a constrained optimization problem.

   (b) Examine the sensitivity of the optimal population levels x and y to the intrinsic growth rates r_1 and r_2.

   (c) Examine the sensitivity of the optimal population levels x and y to the environmental carrying capacities K_1 and K_2.

(d) Assuming that $\alpha_1 = \alpha_2 = \alpha$, examine the sensitivity of the optimal population levels x and y to the strength of competition α. Is it ever optimal to drive one species to extinction?

3. Reconsider the whale problem of Exercise 1, but now consider the economic ramifications of harvesting.

   (a) A blue whale carcass is worth \$12,000, and a fin whale carcass is worth about half as much. Assuming that controlled harvesting can be used to maintain x and y at any desired level, what population levels will produce the maximum revenue? (Once population reaches the desired levels, the population levels will be kept constant by harvesting at a rate equal to the growth rate.) Use the five-step method. Model as an unconstrained optimization problem.

   (b) Examine the sensitivity of the optimal population levels x and y to the parameters r_1 and r_2.

   (c) Examine the sensitivity of revenue in \$/year to the parameters r_1 and r_2.

   (d) Assuming $\alpha_1 = \alpha_2 = \alpha$, study the sensitivity of x and y to α. At what point does it become economically optimal to drive a species to extinction?

4. In Exercise 1, suppose that the International Whaling Commission (IWC) has decreed that no population of whales may be sustained at a level less than half of the environmental carrying capacity K.

   (a) Find the population levels that maximize the sustained profit subject to these constraints. Use Lagrange multipliers.

   (b) Examine the sensitivity of the optimal population levels x and y and the sustained profit to the constraint coefficients.

   (c) The IWC feels that enforcement of the minimum population rules is most easily carried out in terms of quotas. Determine the quotas (maximum number of blue whales and fin whales harvested per year) that will have an equivalent effect to the $K/2$ rule.

   (d) The whalers, complaining that IWC quotas cost them a considerable amount of money, have petitioned for them to be relaxed. Analyze the potential effects of increased quotas on both the yearly revenue of the whalers and the population levels of the whales.

5. Consider the color TV problem without constraints (Example 2.1). Because the company's assembly plant is located overseas, the U.S. government has imposed a tariff of \$25 per unit.

   (a) Find the optimal production levels, taking the tariff into consideration. What does the tariff cost the company? How much of this cost is paid directly to the government, and how much represents lost sales?

(b) Would it be worthwhile for the company to relocate production facilities to the U.S. in order to avoid the tariff? Assume that the overseas facility can be leased to another manufacturer for $200,000 per year and that the cost of constructing and operating a new facility in the U.S. would amount to $550,000 annually. The construction costs have been amortized over the expected life of the new facility.

(c) The purpose of the tariff is to motivate manufacturing companies to operate plants in the U.S. What is the minimum tariff that would make it worthwhile for the company to relocate its facility?

(d) Given that the tariff is large enough to motivate the company to move its facility, how important is the actual tariff amount? Consider the sensitivity of both production levels and profit to the amount of the tariff.

6. A manufacturer of personal computers currently sells 10,000 units per month of a basic model. The cost of manufacture is $700/unit, and the wholesale price is $950. During the last quarter the manufacturer lowered the price $100 in a few test markets, and the result was a 50% increase in sales. The company has been advertising its product nationwide at a cost of $50,000 per month. The advertising agency claims that increasing the advertising budget by $10,000/month would result in a sales increase of 200 units/month. Management has agreed to consider an increase in the advertising budget to no more than $100,000/month.

   (a) Determine the price and the advertising budget that will maximize profit. Use the five-step method. Model as a constrained optimization problem, and solve using the method of Lagrange multipliers.

   (b) Determine the sensitivity of the decision variables (price and advertising) to price elasticity (the 50% number).

   (c) Determine the sensitivity of the decision variables to the advertising agency's estimate of 200 new sales each time the advertising budget is increased by $10,000 per month.

   (d) What is the value of the multiplier found in part (a)? What is the real–world significance of the multiplier? How could you use this information to convince top management to lift the ceiling on advertising expenditures?

7. A local daily newspaper has recently been acquired by a large media conglomerate. The paper currently sells for $1.50/week and has a circulation of 80,000 subscribers. Advertising sells for $250/page, and the paper currently sells 350 pages/week (50 pages/day). The new management is looking for ways to increase profits. It is estimated that an increase of ten cents/week in the subscription price will cause a drop in circulation of 5,000 subscribers. Increasing the price of advertising by $100/page will cause the paper to lose approximately 50 pages of advertising per week.

The loss of advertising will also affect circulation, since one of the reasons people buy the paper is for the advertisements. It is estimated that a loss of 50 pages of advertisements per week will reduce circulation by 1,000 subscriptions.

(a) Find the weekly subscription price and the advertising price that will maximize profit. Use the five-step method, and model as an unconstrained optimization problem.

(b) Examine the sensitivity of your conclusions in part (a) to the assumption of 5,000 lost sales when the price of the paper increases by ten cents.

(c) Examine the sensitivity of your conclusions in part (a) to the assumption of 50 pages/week of lost advertising sales when the price of advertising is increased by \$100/page.

(d) Advertisers who currently place advertisements in the newspaper have the option of using direct mail to reach their customers. Direct mail would cost the equivalent of \$500/page of newspaper advertising. How does this information alter your conclusions in part (a)?

8. Reconsider the newspaper problem of Exercise 7, but now suppose that advertisers have the option of using direct mail to reach their customers. Because of this, management has decided not to increase the price of advertising beyond \$400/page.

(a) Find the weekly subscription price and the advertising price that will maximize profit. Use the five-step method, and model as a constrained optimization problem. Solve by the method of Lagrange multipliers.

(b) Determine the sensitivity of your decision variables (subscription price and advertising price) to the assumption of 5,000 lost sales when the price of the paper increases by ten cents.

(c) Determine the sensitivity of the two decision variables to the assumption of 50 pages of advertisements lost per week when the advertising price increases by \$100/page.

(d) What is the value of the Lagrange multiplier found in part (a)? Interpret this number in terms of the sensitivity of profit to the \$400/page assumption.

9. Reconsider the newspaper problem of Exercise 7, but now look at the newspaper's business expenses. The current weekly business expenses for the paper are as follows: \$80,000 for the editorial department (news, features, editorials); \$30,000 for the sales department (advertising); \$30,000 for the circulation department; and \$60,000 in fixed costs (mortgage, utilities, maintenance). The new management is considering cuts in the editorial

department. It is estimated that the paper can operate with a minimum of a \$40,000/week editorial budget. Reducing the editorial budget will save money, but it will also affect the quality of the paper. Experience in other markets suggests that the paper will lose 2% of its subscribers and 1% of its advertisers for every 10% cut in the editorial budget. Management is also considering an increase in the sales budget. Recently the management of another paper in a similar market expanded its advertising sales budget by 20%. The result was a 15% increase in advertisements. The sales budget may be increased to as much as \$50,000/week, but the overall budget for business expenses will not be increased beyond the current level of \$200,000/week.

(a) Find the editorial and sales budget figures that maximize profit. Assume that the subscription price remains at \$1.50/week, and the advertising price stays at \$250/page. Use the five-step method, and model as a constrained optimization problem. Solve using the method of Lagrange multipliers.

(b) Calculate the shadow price for each constraint, and interpret their meaning.

(c) Draw a graph of the feasible region for this problem. Indicate the location of the optimal solution on this graph. Which of the constraints are binding at the optimal solution? How is this related to the shadow prices?

(d) Suppose that cuts in the editorial budget produce an unusually strong negative response in this market. Assume that a 10% cut in the editorial budget causes the paper to lose p% of its advertising and $2p$% of its subscribers. Determine the smallest value of p for which the paper would be better off not to cut the editorial budget.

10. A shipping company has the capacity to move 100 tons/day by air. The company charges \$250/ton for air freight. Besides the weight constraint, the company can only move 50,000 ft^3 of cargo per day because of limited volume of aircraft storage compartments. The following amounts of cargo are available for shipping each day:

Cargo	Weight (tons)	Volume (ft^3/ton)
1	30	550
2	40	800
3	50	400

(a) Determine how many tons of each cargo should be shipped by air each day in order to maximize revenue. Use the five-step method, and model as a constrained optimization problem. Solve using Lagrange multipliers.

(b) Calculate the shadow prices for each constraint, and interpret their meaning.

(c) The company has the capability to reconfigure some of its older planes to increase the size of the cargo areas. The alterations would cost \$200,000 per plane and would add 2,000 ft^3 per plane. The weight limits would stay the same. Assuming that the planes fly 250 days per year and that the remaining lifetime of the older planes is around five years, would it be worthwhile to make the alterations? To how many planes?

Further Reading

1. Beightler, C., Phillips, D. and Wilde, D. (1979) *Foundations of Optimization.* 2nd ed., Prentice–Hall, Englewood Cliffs, New Jersey.

2. Courant, R. (1937) *Differential and Integral Calculus.* vol. II, Wiley, New York.

3. Edwards, C. (1973) *Advanced Calculus of Several Variables.* Academic Press, New York.

4. Hundhausen, J., and Walsh, R. *The Gradient and Some of Its Applications.* UMAP module 431.

5. Hundhausen, J., and Walsh, R. *Unconstrained Optimization.* UMAP module 522.

6. Nievergelt, Y. *Price Elasticity of Demand: Gambling, Heroin, Marijuana, Whiskey, Prostitution, and Fish.* UMAP module 674.

7. Nevison, C. *Lagrange Multipliers: Applications to Economics.* UMAP module 270.

8. Peressini, A. *Lagrange Multipliers and the Design of Multistage Rockets.* UMAP module 517.