Potential Flow around a Disc

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1 Introduction

This paper describes the mechanics of potential flow for incompressible fluids. It follows closely *A Mathematical Introduction to Fluid Mechanics by Alexandre Chorin and Jerrold Marsden* especially sections 1.1, 1.2 and 2.1. We will start by setting up the underlying equations of fluid dynamics known as Euler's Equations. These essential equations are partial differential equations whose solutions describe the velocity field, pressure and mass density of the fluid. We derive Euler's equation from physical laws, *conservation of mass* and *balance of momentum*, as well a property of fluid, *incompressible flow*. Euler's equations ignore the *viscosity*, or thickness, of the fluid. Therefore they are approximations of the real solutions. For certain problems in fluid dynamics they serve as a realistic model of flow. For the simulation of potential flow around a disc, they adequately model the flow. After defining Eulers' Equations, we prove an important theorem *Bernoulli's Theorem*. Bernoulli's describes how to model the *streamlines*, lines that show the general direction of particles in the flow. It is important to our next section on potential flow used to prove Blasius' Theorem as well.

The second portion of the paper is dedicated to the concept of *potential flow*. Potential flow is flow that admits a *velocity potential*. It has the property of *irrotational flow*, which implies the curl of the velocity field is zero. The *vorticity* field, or the curl of the velocity field, measures spinning motion so potential flow is suited to model smooth flow with little to no turbulence. We show that 2 dimensional potential flow can be modeled in the complex plane with a *complex velocity* and *complex potential* functions. Examining potential flow around an obstacle and the forces acting on the obstacle, we define and prove two important theorems: *Blasius' Theorem* and *Kutta-Joukowski Theorem*. Bernoulli's Theorem is used to prove the theorem. These two theorems show the advantage of modeling potential flow in the complex plane. Finally we bring up two simple examples of *stationary* potential flow around a disc. One with constant flow at infinity and another with a rotating disc. By taking the superposition of the two examples and Kutta-Joukowski theorem it can be proven the disc will have a lift force associated with it. Potential flow can model the lift of a rotating disc similar to the *Magnus Effect*.

2 Euler's Equations

Let D be a three dimensional region that is filled with a fluid. We will use the standard Euclidean coordinates (x, y, z) to describe location in D. We make the assumption that we are able to describe the fluid completely by functions for velocity, pressure and density. In detail, particles in D follow a well-defined path. Let $\mathbf{u}(x, y, z, t)$ be the velocity field of the fluid in D so that $\mathbf{u}(x, y, z, t)$ returns the velocity of a particle at location (x, y, z) at time t. Let $(x(t), y(t), z(t))$ be the well-defined *trajectory* followed by a particle thus $\mathbf{u}$ is given by

$$\mathbf{u}(x(t), y(t), z(t), t) = \Big(\frac{dx}{dt}(t), \frac{dy}{dy}(t), \frac{dz}{dt}(t)\Big).$$

We shall let

$$\mathbf{u}(x, y, z, t) = \big(u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)\big),$$

where (u, v, w) describe the velocity in each of the three dimensions, respectively (x, y, z). Assume that the fluid has a well-defined *mass density* $\rho(x, y, z, t)$ an assumption known as the continuum assumption [1]. The function ρ return the mass density of point (x, y, z) at time t. Thus for any subregion R of D the mass is given by

$$m(R, t) = \int_R \rho(x, y, z, t) dV. \tag{1}$$

Where dV is a volume element along the subregion R. Also assume that the fluid is an *ideal fluid*

> For any motion of the fluid there is a function $p(x, y, z, t)$ called the *pressure* such that if S is a surface in the fluid with a chosen unit normal n, the force of stress exerted across the surface S per unit area at $x \in S$ at time t is $p(x, y, z, t)n$. [1]

We will introduce the concept of pressure further below. Now we define Euler's Equations which are based on physic properties of *conservation of mass*, *balance of momentum* and *incompressibility*, but first we introduce the material derivative which describes physical quantities as they change location in time from the Eulerian frame.

2.1 Material Derivative

We will derive the *material derivative* by examining the acceleration of the fluid. We assume that $\mathbf{u}$ is smooth and well-defined. Therefore the derivative, acceleration, exists,

$$\mathbf{a}(x, y, z, t) = \frac{du}{dt}(x(t), y(t), z(t), t).$$

By the chain rule and the fact that x, y, and z depend on time,

$$\begin{aligned}\mathbf{a}(t) &= \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathbf{u}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \mathbf{u}}{\partial z}\frac{\partial z}{\partial t}, \\ &= \partial_t \mathbf{u} + u\frac{\partial \mathbf{u}}{\partial x} + v\frac{\partial \mathbf{u}}{\partial y} + w\frac{\partial \mathbf{u}}{\partial z}.\end{aligned}$$

For any function $f(x, y, z)$, $\nabla f(x, y, z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$. This simplifies the acceleration equation to

$$\mathbf{a}(t) = \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} \tag{2}$$

This equation uses the material derivative,

$$\frac{Df}{Dt} = \partial_t f + \mathbf{u} \cdot \nabla f. \tag{3}$$

The material derivative describes a physical quantity as it changes position with time. It relates the Eulerian frame to the Lagrangian frame by taking into account the velocity field of the fluid. Equation (3) takes a function f, which describes a physical quantity and derives it along the trajectory of the fluid. It accounts for the change in location. Material derivative is useful for deriving many functions including mass density which we explain below. So we can rewrite Equation (2) as

$$\mathbf{a}(t) = \frac{D\mathbf{u}}{Dt}.$$

2.2 Conservation of Mass

Conservation of mass is the property that mass cannot be created or destroyed. In a closed system mass remains constant over time. Therefore for the domain D the total mass does not change since no mass enters or leaves the domain. Looking at a subregion R of D, the amount of fluid leaving the region is equal to the change of mass of the region. The *volume flow rate* of R is the amount of fluid passing across the surface of R, given by $S = \partial R$. Volume flow rate is the velocity field dot product with the unit outward normal of S. Integrating over the boundary of R yields the volume flow rate,

$$\int_S \mathbf{u} \cdot n \, dA,$$

dA is an infinitesimal area element on the boundary. Mass is volume times density. This implies the *outward mass flow rate* per unit areas will be

$$\int_S \rho \, \mathbf{u} \cdot n \; dA.$$

The principle of conservation of mass can be restated as the property: the inward mass flow rate of a region must be equal to rate of change of the mass. This statement ensures that no mass is created or destroyed in a region. This can be written as

$$\frac{d}{dt} m(R, t) = -\int_S \rho \, \mathbf{u} \cdot n \, dA.$$

Mass flow rate is negative since it is the inward flow required and n is defined as the outward normal force. From Equation (1) and by bringing the right hand side over to the left, we get an important result,

$$\frac{d}{dt} \int_R \rho \, dV + \int_S \rho \, \mathbf{u} \cdot n dA = 0. \tag{4}$$

Using divergence theorem on the mass flow rate term we will get,

$$\int_S \rho \, \mathbf{u} \cdot n dA = \int_R \nabla \cdot (\rho \mathbf{u}) \, dV.$$

The term $\rho\mathbf{u}$ is equivalent to the flux of the fluid through the surface S, so $(\nabla \cdot \rho\mathbf{u})$ is the divergence of the flux. Putting the left hand side of this equation back into Equation (4) results in,

$$\int_R \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \, dV = 0.$$

This equation is true for any subregion R in the domain. Therefore it is identical to the *differential form of the law of conversation of mass*, which can be written,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{5}$$

Expanding the term $\nabla \cdot (\rho \mathbf{u})$ we arrive at

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \left(\nabla \cdot \mathbf{u} \right) = 0$$

The first two terms on the left form the material derivative for mass density (Equation (3)) this finally yields

$$\frac{D\rho}{Dt} = -\rho \left(\nabla \cdot \mathbf{u} \right). \tag{6}$$

This is the first equation that makes up Eulers' Equations. It will be modified later when *incompressible flow* is introduced

2.3 Balance of Momentum

Next we derive Eulers' Equation from the *balance of momentum*. The balance of momentum is a consequence of Newton's second law that states, in a fluid the change in momentum over time will equal the force that is applied to the fluid and be in the direction of the force. This can be stated as $F = \frac{d}{dt} mv$ or the colloquial form of $F = ma$. We must examine the forces that act on a fluid.

There are two types of forces. The forces of stress, aforementioned in the ideal fluid definition, are local forces on the body of the region. They are caused by the pressure of the fluid in the continuum. We use the scalar *pressure* function, $p(x, y, z, t)$ to describe the stress forces. We will use $\mathbf{F}_S(\partial R, t)$ to represent all the stress forces on a surface ∂R at time t. From the ideal fluid definition, define the forces across a surface ∂R per unit area to be

$$\mathbf{F}_S(\partial R, t) = -\int_{\partial R} (pn) \, dA,$$

where p is the pressure and n is the outward unit normal vector. Applying the divergence theorem results in

$$\mathbf{F}_S(R, t) = -\int_R \mathrm{grad}(p) \, dV. \tag{7}$$

The other force acting on a region is called body force. Body force is external and can represent force either from gravity or a magnetic or electric field. We will define body forces on a region R as they change with time by $\mathbf{F}_B(R, t)$.

The other forces acting on a region are called body forces. They are external and can represent force from gravity or magnetic or electric fields. Let $\mathbf{b}(x, y, z, t)$ represent the external force per unit mass so that the total body force is given by

$$\mathbf{F}_B(t) = m(R, t) \int_R \mathbf{b} dV.$$

Recall from Equation (1)

$$\mathbf{F}_B(t) = \int_R \rho dV * \int_R \mathbf{b} dV = \int_R \rho \mathbf{b} dV.$$

So our total force on a region is

$$\mathbf{F}_B(R,t) + \mathbf{F}_S(R,t) = \int_R \big(\rho \mathbf{b} - \text{grad}(p)\big)\, dV$$

The left hand side is equal to the mass times the acceleration of fluid in R by Newton's second law. Using Equation (2) and (1), this equation becomes

$$\frac{D\mathbf{u}}{Dt} * \int_R \rho\, dV = \int_R \big(\rho \mathbf{b} - \text{grad}(p)\big)\, dV.$$

Since this equation holds for every region R in D arrive at the balance of momentum equation,

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{b} - \text{grad}(p). \tag{8}$$

This is the second of the Eulers' Equations.

2.4 Incompressible Flows

We start by defining the fluid flow map which will lead us to the definition for incompressible flow. The fluid flow map is denoted by ϕ so that $\phi(x,y,z,t)$ is the trajectory of a particle that starts at point (x,y,z) at $t=0$. So that ϕ_t is map of the particle to the new location of the particle at time t. ϕ maps the trajectory of this particle as time passes. So if W is a region in the domain D, $\phi_t(W) = W_t$ where W_t is the region of particles that started as W at $t=0$, now at time t. W_t describes the fluid moving through the domain D.

Flow that is incompressible means the fluid will not change volume over time. As the fluid moves and distorts over time a fixed region will not change volume. Given a region W we know that volume of W_t will remain constant through all time, this can be expressed as

$$\frac{d}{dt} \int_{W_t} dV = 0.$$

Next we utilize the transport theorem to derive a more applicable equation from this. The transport theorem is given as

Theorem 1 (Transport Theorem) *For any function f of x,y,z and t, we have*

$$\frac{d}{dt} \int_{W_t} f dV = \int_{W_t} \left(\frac{\partial f}{\partial t} + \nabla \cdot (f \boldsymbol{u})\right) dV. \tag{9}$$

[1]

Applying this to our imcompressible equation with $f = 1$ we arrive at,

$$\frac{d}{dt} \int_{W_t} 1 dV = \int_{W_t} \left(\frac{\partial}{\partial t}(1) + \nabla \cdot \mathbf{u}\right) dV,$$
$$\frac{d}{dt} \int_{W_t} dV = \int_{W_t} \nabla \cdot \mathbf{u}.$$

The left hand side is equal to zero and this integral holds for any subregion W_t this implies,

$$\nabla \cdot \mathbf{u} = 0. \tag{10}$$

This equation holds for any t. We can use this to simplify the Euler Equation for *conservation of mass*. Equation (6) now becomes,

$$\frac{D\rho}{Dt} = 0$$

In summary, Euler's equations for incompressible fluids reads as follows

$$\begin{cases} \frac{D\rho}{Dt} &= 0, \\ \nabla \cdot \mathbf{u} &= 0, \\ \rho \frac{D\mathbf{u}}{Dt} &= \rho \mathbf{b} - \mathrm{grad}(p). \end{cases} \tag{11}$$

Equations for velocity, pressure and density that satisfy these equations describe an incompressible flow.

3 Bernoulli's Theorem

Now a brief digression from defining potential flow to define Bernoulli's Theorem. This theorem is used later for a proof on potential flow. A couple more ingredients before Bernoulli's Theorem is defined. First *trajectory* is the curve traced out by a particle as time progresses [1]. It is given by the solution of the differential equation,

$$\frac{dx}{dt} = \mathbf{u}(x(t), t), \tag{12}$$

where x is a location in space. Next we define *homogeneous flow* which is flow with constant density $\rho(x, y, z, t) = \rho_0$, where ρ_0 is a constant. This holds for the whole domain D and all time t. From Equation (11) it can be seen that *incompressible flow* is also *homogeneous*.

Stationary flow is flow that is independent of time, no matter the time the velocity field is the same. This can be written as $\frac{\partial \mathbf{u}}{\partial t} = 0$. *Streamlines* differ from the trajectory which is the curved traced out by a particle as time passes. The *streamline* has a fixed time and is tangent to the velocity field $\mathbf{u}$. It can be written with parametric variable s at fixed t as

$$\frac{dx}{ds} = \mathbf{u}(x(s), t), \tag{13}$$

where $x(s)$ is the streamline. *Stationary flow* has the property, $\frac{\partial \mathbf{u}}{\partial t} = 0$. This implies that the *streamlines* coincide with the trajectories. This is because the velocity field does not change so the trajectory will follow the tangents of the velocity vectors.

From Chorin and Marsden Bernoulli's theorem is as follows

Theorem 2 (Bernoulli's Theorem) *In stationary homogenous flows and in the absence of external forces, the quantity*

$$\frac{1}{2}||\boldsymbol{u}||^2 + \frac{p}{\rho_0}$$

is constant along streamlines.

[1]

Proof of the Bernoulli's Theorem is as follows:

External forces are ignored. This implies $\mathbf{b}$ that from Equation (8) is equal to zero, $\mathbf{b} = 0$. Also since the flow is *stationary* ($\partial_t \mathbf{u} = 0$), the density is constant, $\rho = \rho_0$ and Equation (8) becomes

$$\rho_0((\mathbf{u} \cdot \nabla)\mathbf{u}) = -\operatorname{grad} p.$$

Next divide both steps by ρ_0 which results in

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\operatorname{grad} p.$$

Now turning towards the vector identity

$$\frac{1}{2}\nabla(||\mathbf{u}||^2) = (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{14}$$

Bring $(\mathbf{u} \cdot \nabla)\mathbf{u}$ onto the left hand side of this equation and substitute $-\frac{1}{\rho_0} grad(p)$ into the identity to get

$$\nabla(\frac{1}{2}||\mathbf{u}||^2 + \frac{p}{\rho_0}) = \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{15}$$

Picking an arbitrary streamline $x(s)$ that fulfills Equation (13) and two values of s: s_1 and s_2 such that $x(s_1), x(s_2)$ are along the streamline. It must be proven that for any two points along the streamline, the scalar given by $\frac{1}{2}||\mathbf{u}||^2 + \frac{p}{\rho_0}$ at those two points is constant. Therefore the difference between those points should be equal to zero. This can be converted into an integral shown below.

$$\begin{aligned} &(\frac{1}{2}||\mathbf{u}||^2 + \frac{p}{\rho_0})\Big|_{x(s_1)}^{x(s_2)}, \\ \iff \quad &\int_{x(s_1)}^{x(s_2)} \nabla(\frac{1}{2}||\mathbf{u}||^2 + \frac{p}{\rho_0}) \cdot x'(s)ds. \end{aligned}$$

From Equation (13) on the definition of a streamline we know that $x'(s) = \mathbf{u}(x(s), t)$ also from Equation (15) the other term in the integral can be changed to $\mathbf{u} \times (\nabla \times \mathbf{u})$ which give

$$\int_{x(s_1)}^{x(s_2)} \mathbf{u} \times (\nabla \times \mathbf{u}) \cdot \mathbf{u}(x(s), t)ds = 0$$

This equation is equal to zero since $\mathbf{u}$ is orthogonal to $\mathbf{u} \times (\nabla \times \mathbf{u})$ for all values of s_1, s_2. This proves that the

$$\frac{1}{2}||\mathbf{u}||^2 + \frac{p}{\rho_0}$$

is constant along streamlines.

4 Potential Flow

Here we dive into the world of *potential flow*. *Potential flow* is useful in certain modeling problem with little to no spin in the flow. This will become apparent from Equation (17). After defining properties and important theorems for *potential flow*, we shall introduce a modeling example. Formally emphpotential flow is flow that admits a local potential, also known as a *velocity potential*. The velocity field is defined as the gradient of the *velocity potential* φ, this is given as

$$\mathbf{u} = \nabla\varphi. \tag{16}$$

This implies that components of two-dimensional $\mathbf{u}$ give,

$$u = \partial_x\varphi \text{ and } v = \partial_y\varphi.$$

The *vorticity field* of a flow is defined as $\xi = \nabla \times \mathbf{u}$, given Equation (16) and the vector identity that the curl of the gradient of gradient of a scalar field is zero results in

$$\xi = \nabla \times \mathbf{u} = \nabla \times \nabla\varphi = 0. \tag{17}$$

A flow with vorticity field equal to zero is called *irrotational*. Properties of irrotational flow and incompressible flow lead to modeling *potential flow* on the complex plane. The velocity field $\mathbf{u}$ holds satisfies the Cauchy-Riemann equations. Take $\mathbf{u}$ to be the two-dimensional velocity field. We first use the irrotational property from Equation (17) to get

$$\xi = \nabla \times \mathbf{u} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \tag{18}$$

$$\Leftrightarrow \qquad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0. \tag{19}$$

The flow is also incompressible, rewriting Equation (10) in two-dimensional form as

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{20}$$

Equation (18) and (20) are the Cauchy-Riemann equations. Define a complex-valued function as

$$\mathbf{F} = u - iv.$$

Thus $\mathbf{F}$, the *complex velocity*, is complex differentiable. Next we will define the *complex potential*, $\mathbf{W}$ as satisfying

$$\mathbf{F} = \frac{d\mathbf{W}}{dz}.$$

The complex potential can also be written as

$$\mathbf{W} = \varphi + i\psi. \tag{21}$$

Where φ is the velocity potential and ψ is the *stream function*. The stream function ψ has streamlines that lie on the level curves of ψ. It has the properties that

$$u = \partial_y\psi \text{ and } v = \partial_x\psi.$$

With the potential flow we can adjust Bernoulli's Theorem for a stronger result. From Equation (17) ($\nabla \times \mathbf{u} = 0$), revisiting the proof the vector identity, Equation (15) becomes

$$\nabla\left(\frac{1}{2}||\mathbf{u}||^2 + \frac{p}{\rho_0}\right) = 0,$$

$$\Longrightarrow \qquad \frac{1}{2}||\mathbf{u}||^2 + \frac{p}{\rho_0} = \text{constant}.$$

This result holds for any values, which implies that Bernoulli's Theorem is constant in space rather than only on streamlines. Also we can rearrange this formula to put in terms of the pressure,

$$p = \frac{-\rho_0||\mathbf{u}||^2}{2} + \text{ constant.} \tag{22}$$

4.1 Forces on an obstacle

We now look at the flow around an obstacle $\mathcal{B}$. The force, $\mathcal{F}$ on the obstacle will be given by the pressure on the surface $\partial\mathcal{B}$ of the obstacle. It can be expressed as

$$\mathcal{F} = -\int_{\partial\mathcal{B}} p\,\mathbf{n}\,ds, \tag{23}$$

where $\mathbf{n}$ is the outward normal. Our next theorem gives a more concise formula for $\mathcal{F}$.

Theorem 3 (Blasius' Theorem) *For incompressible potential flow exterior to a body $\mathcal{B}$ (with rigid boundary) and complex velocity $\boldsymbol{F}$, the force $\mathcal{F}$ on the body is given by*

$$\mathcal{F} = -\frac{i\rho}{2}\overline{\int_{\partial\mathcal{B}} \boldsymbol{F}^2 dz}, \tag{24}$$

where the $\bar{\cdot}$ denotes complex conjugation and where the vector $\mathcal{F}$ is identified with a complex number. [1]

Proof of Blasius' Theorem is as follows: Given the body $\mathcal{B}$ let $\partial\mathcal{B}$ be the boundary of the body. Let $dz = dx + idy$ represent an infinitesimal displacement along the curve $\partial\mathcal{B}$. So by Equation (23) we get that

$$\mathcal{F} = -\int_{\partial\mathcal{B}} p\,\mathbf{n}\,dl,$$

where dl is an infinitesimal displacement along the surface $\partial\mathcal{B}$. The normal displacement of $\mathbf{n}dl$ is given by $dz/i = -idx + dy$ since dividing by i is analogous to rotating the vector dz by $\pi/2$ radians clockwise. Replacing $n\,dl$ results in

$$\mathcal{F} = i\int_{\partial\mathcal{B}} pdx - \int_{\partial\mathcal{B}} pdy = i\int_{\partial\mathcal{B}} p(dx + idy).$$

Next substituting Equation (22) into this equation

$$\mathcal{F} = \frac{-i\rho}{2}\int_{\partial\mathcal{B}} (u^2 + v^2)dz \tag{25}$$

The constant in Equation (22) can be disregarded since integrating over the close surface $\partial\mathcal{B}$ shall eliminate any constant. The next step requires an equation we pull from the boundary condition on the obstacle $\mathcal{B}$. Called the *slip boundary-condition.* On the boundary of the surface of $\mathcal{B}$ we have $\mathbf{u}\cdot n = 0$. Breaking this equation into 2 dimensions we get $un_1 + vn_2 = 0$. Since dz is the displacement along the curve $n_1 = dy$ and $n_2 = -dx$, this gives

$$udy = vdx. \tag{26}$$

We will use this equation to prove that $(u^2 + v^2)dz = \overline{\mathbf{F}^2 dz}$. To show this we take the difference between the two and show it equals zero. We will use Equation (26) to prove this.

$$\begin{aligned}u^2+v^2-\overline{\mathbf{F}^2} =&(u^2+v^2)dz-(u-iv)^2\overline{dz},\\ =&(u^2+v^2)(dx-idy)-(u^2-2iuv-v^2)(dx+idy),\\ =&u^2dx-iu^2dy+v^2dx-iv^2dy-u^2dx-iu^2dy+2iuvdx-2uvdy+v^2dx+v^2idy,\\ =&-2iu^2dy+2v^2dx+2iuvdx-2uvdy.\end{aligned}$$

Substituting $udy=vdx$ for the terms $-2iuvdx$ and $2uvdy$ gives us

$$-2iu^2dy+2v^2dx+2iu^2dy-2v^2dx=0$$

This implies that $(u^2+v^2)dz=\overline{F^2dz}$, putting this back into Equation (25) is equivalent to Blasius' theorem.

We use this theorem to proof a stronger result.

Theorem 4 (Kutta-Joukowski Theorem) *Consider incompressible potential flow exterior to a region $\mathcal{B}$. Let the velocity field approach the constant value $(U_\infty, V_\infty)=\boldsymbol{U}_\infty$ at infinity. Then the force exerted on $\mathcal{B}$ is given by*

$$\mathcal{F}=-\rho\Gamma_{\partial\mathcal{B}}||\boldsymbol{U}_\infty||n \tag{27}$$

where $\Gamma_{\partial\mathcal{B}}$ is the circulation around $\mathcal{B}$ and n is a unit vector orthogonal to $\boldsymbol{U}$.

Proof of Kutta-Joukowski Theorem is as follows:

We proved above that $\mathbf{F}(z)$ is an analytic function since it satisfies the Cauchy-Riemann equations. Therefore we can expand $\mathbf{F}(z)$ can be expanded into a Laurent series. This implies

$$\mathbf{F}(z)=\cdots+\frac{a_{-2}}{z^2}+\frac{a_{-1}}{z}+a_0+a_1z+a_2z^2+\cdots \tag{28}$$

With constants $\cdots a_{-2}, a_{-1}, a_0, a_1\cdots\in\mathbb{C}$. We can simplify this equation by examining $\mathbf{F}(z)$ as z approaches infinity. $\mathbf{U}_\infty$, the flow velocity at infinity, is constant. So $\mathbf{F}(z)=\mathbf{U}_\infty$ as $z\to\infty$. This implies that the constants $a_1, a_2,\cdots$ are equal to zero, otherwise $\mathbf{F}(z)$ will diverge as z approaches infinity. Laurent expansion of $\mathbf{F}(z)$ can be rewritten as

$$\begin{aligned}\mathbf{F}(z)&=a_0=\mathbf{U}_\infty \text{ as } z\to\infty\\ \mathbf{F}(z)&=\cdots+\frac{a_{-2}}{z^2}+\frac{a_{-1}}{z}+\mathbf{U}_\infty\end{aligned}$$

Every analytical function outside the boundary of the obstacle $\mathcal{B}$ has a valid Laurent expansion. The coefficients of this expansion can be found by a generalized Cauchy's integral formula [1], that shows

$$a_{-1}=\frac{1}{2\pi i}\int_{\mathcal{P}}\mathbf{F}(z)dz,$$

Where $\mathcal{P}$ is any path around the body $\mathcal{B}$. Set $\mathcal{P}$ to be the boundary of the body, $\partial\mathcal{B}$ since $\mathbf{F}(z)$ around the boundary is the *circulation.* The *circulation* around $\partial\mathcal{B}$, $\Gamma_{\partial\mathcal{B}}$ is given by

$$\Gamma_{\partial\mathcal{B}}=\int_{\partial\mathcal{B}}\mathbf{F}(z)dz.\ \ [1]$$

This implies that the Laurent series coeffecient a_{-1} will be

$$a_{-1} = \frac{\Gamma_{\partial\mathcal{B}}}{2\pi i}.$$

The Laurent expansion of $\mathbf{F}(z)$ is now

$$\mathbf{F}(z) = \cdots + \frac{a_{-2}}{z^2} + \frac{\Gamma_{\partial\mathcal{B}}}{2\pi i z} + \mathbf{U}_\infty.$$

We now will use Blasius Theorem and substitute the Laurent expansion for $\mathbf{F}(z)$ we get

$$\mathcal{F} = -\frac{i\rho}{2}\overline{\int_{\partial\mathcal{B}} (\cdots + \frac{a_{-2}}{z^2} + \frac{\Gamma_{\partial\mathcal{B}}}{2\pi i z} + \mathbf{U}_\infty)^2 dz}.$$

We will eventually use Cauchy Residue Theorem again to evaluate the integral. So we will just look at the term inside the integral for now. We need to expand $\mathbf{F}(z)^2$ to find the a_{-1} term in the ensuing Laurent series since this is the only constant used in the Cauchy Residue Theorem. The constant a_{-1} is any terms with $\frac{1}{z}$. Expanding $\mathbf{F}(z)^2$ gives us

$$\begin{aligned}
\mathbf{F}(z)^2 =& (\cdots + \frac{a_{-2}}{z^2} + \frac{\Gamma_{\partial\mathcal{B}}}{2\pi i z} + \mathbf{U}_\infty)(\cdots + \frac{a_{-2}}{z^2} + \frac{\Gamma_{\partial\mathcal{B}}}{2\pi i z} + \mathbf{U}_\infty), \\
\mathbf{F}(z)^2 =& \cdots + 2\frac{\Gamma_{\partial\mathcal{B}}}{2\pi i}\mathbf{U}_\infty\frac{1}{z} + \mathbf{U}_\infty^2.
\end{aligned}$$

So $a_{-1} = \frac{\Gamma_{\partial\mathcal{B}}}{\pi i}\mathbf{U}_\infty$. Now using Cauchy Residue Theorem we get

$$\begin{aligned}
\int_{\partial\mathcal{B}} \mathbf{F}(z)^2 dz &= 2\pi i \frac{\Gamma_{\partial\mathcal{B}}}{\pi i}\mathbf{U}_\infty, \\
\int_{\partial\mathcal{B}} \mathbf{F}(z)^2 dz &= 2\Gamma_{\partial\mathcal{B}}\mathbf{U}_\infty.
\end{aligned}$$

Putting this equation back into Blasius' Theorem we will arrive at Kutta-Joukowski Theorem

$$\begin{aligned}
\mathcal{F} &= -\frac{i\rho}{2}\overline{(2\Gamma_{\partial\mathcal{B}}\mathbf{U}_\infty)}, \\
\mathcal{F} &= -i\rho\Gamma_{\partial\mathcal{B}}\overline{\mathbf{U}_\infty}.
\end{aligned}$$

Complex velocity at infinity $\mathbf{U}_\infty = U_\infty - iV_\infty$. Therefore $\overline{\mathbf{U}_\infty} = U_\infty + iV_\infty$, which is equivalent to $||\mathbf{U}_\infty||$. Also multiplying a vector by i is equivalent to the inward normal vector n. Culminating in

$$\mathcal{F} = -\rho\Gamma_{\partial\mathcal{B}}||\mathbf{U}_\infty||n.$$

This proves the Kutta-Joukowski Theorem.

5 Flow around a disc

This section explores the use of potential flow in modeling examples. It's usefulness in modeling physical phenomenon like the *Magnus effect*.

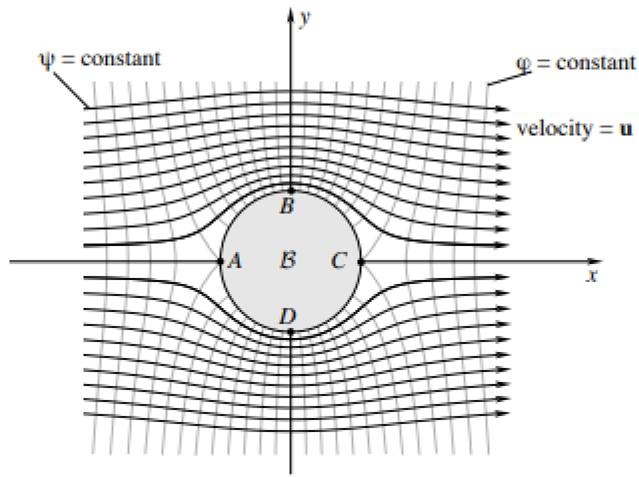

Figure 1: Potential Flow around a disc [1]

5.1 Potential Flow around a disc

Using the ideas presented in this paper we will show potential flow around a simple obstacle, in this case a disc. The disc $\mathcal{B}$ is a circle centered at the origin of the complex plane with radius a. The complex potential of the flow is

$$\mathbf{W}(z) = \mathbf{U}_\infty\left(z + \frac{a^2}{z}\right),$$

and the complex velocity is

$$\mathbf{F}(z) = \mathbf{U}_\infty\left(1 + \frac{a^2}{z^2}\right).$$

We can see that the complex velocity tends to $\mathbf{U}_\infty$ as z approaches infinity. We start with the boundary condition on obstacle $\mathcal{B}$. The boundary of $\mathcal{B}$ must follow a slip-boundary condition that the flow is tangent to the surface. This property can be written as $F(z) \cdot n = 0$ on $\partial\mathcal{B}$ where n is the inward normal vector on the surface. This boundary condition holds if $F(z)$ is tangent to the boundary at all points.

Equation (21) gives the complex potential as $\mathbf{W} = \varphi + i\psi$. To verify that the flow is tangent at the boundary we must show that ψ, the stream function, is equal to zero. When $|z| = a$ its on the boundary on obstacle $\mathcal{B}$. Squaring this we get

$$|z|^2 = z\bar{z} = a^2,$$

where $\bar{z}$ is the complex conjugate of z. Substituting this into the complex potential yields

$$\mathbf{W}(z) = \mathbf{U}_\infty(z + \bar{z}).$$

$z + \bar{z}$ will cancel out the imaginary part of z. Therefore $\psi = 0$ at the boundary which implies the flow is tangent and the slip-boundary condition holds.

Applying the Kutta-Joukowski Theorem it is found that the force on the obstacle $\mathcal{F} = 0$. This is due to the circulation, $\Gamma_{\partial\mathcal{B}}$ around the boundary can be found to be zero. Figure 1 shows a

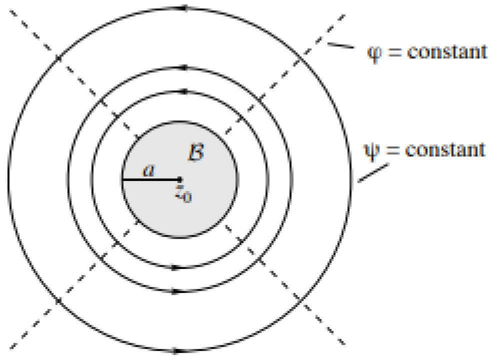

Figure 2: Potential flow centered at the origin z_0 [1]

symmetrical flow along the x-axis. This implies that the circulation along points A to B to C is the same as the circulation along A to D to C. The circulation around the boundary goes from A to B to C to D and thus these two circulation cancel each other out and $\Gamma_{\partial\mathcal{B}} = 0$. Thus Kutta Joukowski Theorem yields

$$\mathcal{F} = -\rho\Gamma_{\partial\mathcal{B}}||\mathbf{U}_\infty||n = 0.$$

There is no force on the body of the obstacle.

5.2 Potential vortex flow

We know look at another example around the same obstacle $\mathcal{B}$. This example represents a rotating disc represents by Γ so the fluid flows in a circular motion. Let the complex potential be

$$\mathbf{W}(z) = \frac{\Gamma}{2\pi i}\log z.$$

Therefore the complex velocity is

$$\mathbf{F}(z) = \frac{\Gamma}{2\pi i z}.$$

Here we see that as z approaches infinity, the complex velocity goes to zero, $\mathbf{U}_\infty = 0$. We let the circulation be Γ. Again we must show that the boundary condition is met by showing ψ is constant along the boundary of $\mathcal{B}$. It is known that the logarithm of a complex number can be written as

$$\log z = \log|z| + i\arg z.$$

Substituting this in for log(z) in the complex potential gives us

$$\begin{aligned}\mathbf{W}(z) &= \frac{\Gamma}{2\pi i}(\log|z| + i\arg z),\\ &= \frac{\Gamma}{2\pi}(\arg z - i\log|z|).\end{aligned}$$

Since the function arg z and log$|z|$ both return real values, the stream function is defined by

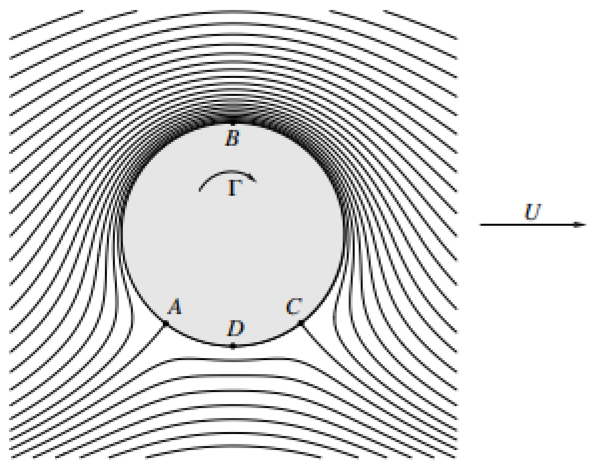

Figure 3: Flow around a disc with circulation [1]

$$\psi = -\frac{\Gamma}{2\pi} log|z|.$$

We can see that $|z|$ is constant along circles with center at the origin. Therefore φ is constant along any circle centered at the origin. This proves that slip-boundary condition is held on the obstacle $\mathcal{B}$. Therefore Kutta-Joukowski Theorem can be applied to find the force on the obstacle $\mathcal{B}$. Since $\mathbf{U}_\infty = 0$, $||\mathbf{U}_\infty|| = 0$ which implies that

$$\mathcal{F} = -\rho\Gamma||\mathbf{U}_\infty||n = 0.$$

So there is also no body force on the obstacle.

5.3 Flow around a disc with circulation

Now we take complex potential from the previous two sections and combine them. As shown below, this example arrives at an interesting result: the two flows that give no force on the obstacle combined will give a force on the obstacle $\mathcal{B}$. The complex potential is now a sum of the previous two given by

$$\mathbf{W}(z) = \mathbf{U}_\infty\left(z + \frac{a^2}{z}\right) + \frac{\Gamma}{2\pi i} log\, z.$$

With complex velocity given by

$$\mathbf{F}(z) = \mathbf{U}_\infty\left(1 + \frac{a^2}{z^2}\right) + \frac{\Gamma}{2\pi i\, z}.$$

We proved that ψ is constant for each complex potential independently therefore ψ is constant for the sum of the two. The flow outside the obstacle is incompressible potential flow. We also know that as z approaches infinity the complex velocity will converge to $\mathbf{U}_\infty$. Therefore applying the Kutta-Joukowski theorem results in,

$$\mathcal{F} = -\rho\Gamma||\mathbf{U}_\infty||n$$

Kutta-Joukowski demonstrates the force on the obstacle caused by flow around an obstacle as it rotates. The direction and intensity of the force is determined by the flow around the disc and the circulation from the body's rotation. We briefly touch on the physical phenomenon this example shows below.

6 Conclusion

We presented clearly some of the basic equations of fluid dynamics and introduce the concept of potential flow. This involved defining physical properties of flow, velocity, pressure and density and converting them into Euler's Equations for incompressible flow. Next potential flow's properties are discussed and an example is provide of the advantage of modeling potential flow.

Section 5.3 is an example of the *Magnus Effect* when a spinning cylinder or sphere experiences a force perpendicular to the flow. A good example is how professional soccer players can curve balls by kicking the ball with spin. The spin of the ball along with the flow of air from its velocity cause the ball to curve.

References

[1] Jerrold E. Marsden Alexandre J. Chorin. *A Mathematical Introduction to Fluid Mechanics*, volume 3rd edition. Springer-Verlag, 1993.