Chapter 5

ANALYSIS OF DYNAMIC MODELS

In this chapter we consider some of the most broadly applicable techniques for the analysis of discrete and continuous time dynamical systems. Except for a few special cases, these methods do not yield exact analytical solutions. Such exact analytical methods are more properly treated in a course on differential equations. In any case, most dynamical system models which arise in practice are not amenable to exact solution by any known technique. In this chapter, we will present techniques which can be applied to the analysis of almost any dynamical system model. These methods can provide important qualitative information about the behavior of dynamical systems, even when exact analytic solutions are not obtainable.

5.1 Eigenvalue Methods

When the equations of a dynamic model are linear, it is possible to obtain an exact analytical solution. While linear dynamics are rare in real life, the majority of dynamic systems can be approximated by linear systems, at least locally. Such linear approximations, especially in the neighborhood of an isolated equilibrium point, provide the basis for many of the most important analytical techniques available for dynamic modeling.

Example 5.1. Reconsider the tree problem of Example 4.1. Assume that hardwoods grow at a rate of 10% per year and softwoods at a rate of 25% per year. An acre of forest land can support about 10,000 tons of hardwoods or 6,000 tons of softwoods. The extent of competition has not been numerically determined. Can both types of trees coexist in stable equilibrium?

Step 1 of the five-step method was laid out in Figure 4.1. In this particular

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case we have

$$\begin{aligned} r_1 &= 0.10 \\ r_2 &= 0.25 \\ a_1 &= \frac{0.10}{10,000} \\ a_2 &= \frac{0.25}{6,000}. \end{aligned}$$

Step 2 is to select the modeling approach, including a method of analysis. We will analyze this nonlinear dynamical system by the eigenvalue method.

Suppose we are given a dynamical system $x' = F(x)$ where $x = (x_1, \ldots, x_n)$ is an element of the state space $S \subseteq \mathbb{R}^n$ and $F = (f_1, \ldots, f_n)$. A point $x_0 \in S$ is an equilibrium or steady state if and only if $F(x_0) = 0$. There is a theorem that states that an equilibrium point x_0 is asymptotically stable if the matrix

$$A = \begin{pmatrix} \partial f_1/\partial x_1\,(x_0) & \cdots & \partial f_1/\partial x_n\,(x_0) \\ \vdots & & \vdots \\ \partial f_n/\partial x_1\,(x_0) & \cdots & \partial f_n/\partial x_n\,(x_0) \end{pmatrix} \tag{5.1}$$

has eigenvalues with all negative real parts. If any eigenvalue has a positive real part, then the equilibrium is unstable. In the remaining cases (pure imaginary eigenvalues) the test is inconclusive (Hirsch and Smale (1974) p. 187).

The eigenvalue method is based on a linear approximation. Even if $x' = F(x)$ is not linear, we will have

$$F(x) \approx A(x - x_0)$$

in the neighborhood of the equilibrium point. This is the same sort of linear approximation you saw in one–variable calculus, except that now the derivative of F is represented by a matrix. Some authors will call this matrix DF in analogy to the one–variable derivative. The linear approximation is good enough so that if the origin is a stable equilibrium of $x' = Ax$ (i.e., the point x_0 is a stable equilibrium of $x' = A(x - x_0)$), then x_0 is a stable equilibrium of $x' = F(x)$ as well. Therefore, it is enough to understand the eigenvalue test in the case of a linear system.

Undoubtedly, you solved some linear systems of differential equations in your introductory differential equations course, and you probably learned about the relation between solutions and eigenvalues. For example, if $Au = \lambda u$ (i.e., u is an eigenvector of A belonging to the eigenvalue λ), then $x(t) = ue^{\lambda t}$ is a solution to the initial value problem

$$x' = Ax, \; x(0) = u.$$

It is actually possible to write down the general solution to an $n \times n$ system of linear differential equations, although it is rather messy and requires a lot of linear algebra. One good thing that comes out of doing so, however, is a general description of solution behavior. This theorem says that for any solution $x(t)$ to the differential equation $x' = Ax$, where A is a matrix of constants, each coordinate is a linear combination of terms that look like one of

$$t^k e^{at}\cos(bt), t^k e^{at}\sin(bt)$$

where $a \pm ib$ is an eigenvalue of A (if the eigenvalue is real then $b = 0$), and k is a nonnegative integer less than n. From this general description it is easy to calculate that the origin is an asymptotically stable equilibrium of the system $x' = Ax$ if and only if every eigenvalue $a \pm ib$ has $a < 0$. (Hirsch and Smale (1974) p. 135)

Of course, a successful application of the eigenvalue method requires us to be able to compute the eigenvalues. In simple cases (e.g. on $\mathbb{R}^2$) it will be possible to compute eigenvalues by hand, or possibly with the aid of a computer algebra system. Otherwise, we will have to rely on approximate methods. Fortunately, there do exist numerical analysis software packages to compute the eigenvalues of an $n \times n$ matrix, and these are effective in most cases. (e.g., Press (1986))

Returning to Example 5.1, recall from Section 4.1 that there is an equilibrium at the point

$$\begin{aligned} x_1 &= \frac{r_1 a_2 - r_2 b_1}{D} \\ x_2 &= \frac{a_1 r_2 - b_2 r_1}{D} \end{aligned}$$

where $D = a_1 a_2 - b_1 b_2$. We have now specified values for $a_1, a_2, r_1,$ and r_2 but not for b_1 and b_2. We will, however, continue to assume that $b_i < a_i$. For the moment let us take $b_i = a_i/2$. Then the coordinates of the equilibrium point are $x_0 = (x_1^0, x_2^0)$, where

$$\begin{aligned} x_1^0 &= \frac{28000}{3} \approx 9333 \\ x_2^0 &= \frac{4000}{3} \approx 1333 \end{aligned} \tag{5.2}$$

The dynamical system equations are $x' = F(x)$ where $F = (f_1, f_2)$ and

$$\begin{aligned} f_1(x_1, x_2) &= 0.10x_1 - \frac{0.10}{10000}x_1^2 - \frac{0.05}{10000}x_1 x_2 \\ f_2(x_1, x_2) &= 0.25x_2 - \frac{0.25}{6000}x_2^2 - \frac{0.125}{6000}x_1 x_2 \end{aligned} \tag{5.3}$$

The partial derivatives are

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= \frac{20000 - x_2}{200000} - \frac{x_1}{50000} \\
\frac{\partial f_1}{\partial x_2} &= \frac{-x_1}{200000} \\
\frac{\partial f_2}{\partial x_1} &= \frac{-x_2}{48000} \\
\frac{\partial f_2}{\partial x_2} &= \frac{-x_1}{48000} - \frac{x_2}{12000} + \frac{1}{4}
\end{aligned} \tag{5.4}$$

Evaluating the partial derivatives (5.4) at the equilibrium point (5.2) and substituting back into (5.1), we obtain

$$A = \begin{pmatrix} -7/75 & -7/150 \\ -1/36 & -1/18 \end{pmatrix}. \tag{5.5}$$

The eigenvalues of this 2×2 matrix can be computed as the roots of the equation

$$\begin{vmatrix} \lambda + 7/75 & 7/150 \\ 1/36 & \lambda + 1/18 \end{vmatrix} = 0.$$

Evaluating the determinant, we obtain the equation

$$\frac{1800\lambda^2 + 268\lambda + 7}{1800} = 0,$$

and then we obtain

$$\lambda = \frac{-67 \pm \sqrt{1339}}{900}.$$

Since both eigenvalues have negative real parts, the equilibrium is stable.

The eigenvalue test for continuous time dynamical systems involves quite a bit of computation. This is an appropriate application for a computer algebra system. Figure 5.1 illustrates the use of the computer algebra system Mathematica to perform the computations in step 4 for the present problem.

Finally, we proceed to step 5. We have found that hardwoods and softwoods can coexist in stable equilibrium. There will be approximately 9,300 tons per acre of hardwoods and 1,300 tons per acre of softwoods in a mature, stable forest. These conclusions are based on certain plausible assumptions about the degree of competition between the two types of trees. A sensitivity analysis will be conducted to determine the effect of these assumptions on our broad conclusions.

For the sensitivity analysis, we will still assume that $b_i = t\, a_i$ but we will relax the assumption that $t = 1/2$. The conditions

$$\begin{gathered} b_i < a_i \\ (r_i/a_i) < (r_j/b_j) \end{gathered}$$

```
In[1]:= f1 = x1 / 10 – (x1 ^ 2 / 10) / 10000 – (5 x1 x2 / 100) / 10000
```

Out[1]= $\frac{x1}{10} - \frac{x1^2}{100000} - \frac{x1\,x2}{200000}$

```
In[2]:= f2 = 25 x2 / 100 – (25 x2 ^ 2 / 100) / 6000 – (125 x1 x2 / 1000) / 6000
```

Out[2]= $\frac{x2}{4} - \frac{x1\,x2}{48000} - \frac{x2^2}{24000}$

```
In[3]:= s = Solve[{f1 / x1 == 0, f2 / x2 == 0}, {x1, x2}]
```

Out[3]= $\left\{\left\{x1 \to \frac{28000}{3},\ x2 \to \frac{4000}{3}\right\}\right\}$

```
In[4]:= df = {{D[f1, x1], D[f1, x2]}, {D[f2, x1], D[f2, x2]}};
```

```
In[6]:= MatrixForm[df]
```

Out[6]//MatrixForm=

$$\begin{pmatrix} \frac{1}{10} - \frac{x1}{50000} - \frac{x2}{200000} & -\frac{x1}{200000} \\ -\frac{x2}{48000} & \frac{1}{4} - \frac{x1}{48000} - \frac{x2}{12000} \end{pmatrix}$$

```
In[7]:= A = df /. s
```

Out[7]= $\left\{\left\{\left\{-\frac{7}{75},\ -\frac{7}{150}\right\},\ \left\{-\frac{1}{36},\ -\frac{1}{18}\right\}\right\}\right\}$

```
In[8]:= Eigenvalues[A]
```

Out[8]= $\left\{\frac{1}{900}\left(-67 - \sqrt{1339}\right),\ \frac{1}{900}\left(-67 + \sqrt{1339}\right)\right\}$

Figure 5.1: Calculations for step 4 of the tree problem using the computer algebra system Mathematica.

imply that $0 < t < 0.6$. The coordinates of the equilibrium point (x_1^0, x_2^0) are

$$\begin{aligned} x_1^0 &= \frac{10000 - 6000t}{1 - t^2} \\ x_2^0 &= \frac{6000 - 10000t}{1 - t^2}. \end{aligned} \tag{5.6}$$

The differential equations of this system are $x_i' = f_i(x_1, x_2)$ where

$$\begin{aligned} f_1(x_1, x_2) &= 0.10x_1 - \frac{0.10x_1^2}{10000} - \frac{0.10tx_1x_2}{10000} \\ f_2(x_1, x_2) &= 0.25x_2 - \frac{0.25x_2^2}{6000} - \frac{0.25tx_1x_2}{6000} \end{aligned} \tag{5.7}$$

and the partial derivatives are

$$
\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= \frac{10000 - tx_2}{100000} - \frac{x_1}{50000} \\
\frac{\partial f_1}{\partial x_2} &= \frac{-tx_1}{100000} \\
\frac{\partial f_2}{\partial x_1} &= \frac{-tx_2}{24000} \\
\frac{\partial f_2}{\partial x_2} &= \frac{-tx_1}{24000} - \frac{x_2}{12000} + \frac{1}{4}.
\end{aligned} \tag{5.8}
$$

Evaluating the partial derivatives (5.8) at the equilibrium point (5.6) and substituting back into (5.1) yields

$$
A = \begin{pmatrix} \dfrac{5-3t}{50(t^2-1)} & \dfrac{t(5-3t)}{50(t^2-1)} \\ \dfrac{t(3-5t)}{12(t^2-1)} & \dfrac{3-5t}{12(t^2-1)} \end{pmatrix}. \tag{5.9}
$$

The characteristic equation we must solve to find the eigenvalues is

$$
\left[\lambda - \frac{5-3t}{50(t^2-1)}\right]\left[\lambda - \frac{3-5t}{12(t^2-1)}\right] - \left[\frac{t(3-5t)}{12(t^2-1)}\right]\left[\frac{t(5-3t)}{50(t^2-1)}\right] = 0. \tag{5.10}
$$

Solving equation (5.10) for λ yields two roots:

$$
\begin{aligned}
\lambda_1 &= \frac{143t - 105 + \sqrt{9000t^4 - 20400t^3 + 20449t^2 - 9630t + 2025}}{600(1-t^2)} \\
\lambda_2 &= \frac{143t - 105 - \sqrt{9000t^4 - 20400t^3 + 20449t^2 - 9630t + 2025}}{600(1-t^2)}.
\end{aligned} \tag{5.11}
$$

Figure 5.2 illustrates the use of the computer algebra system Maple to compute the eigenvalues for this problem. Computer algebra systems are especially useful for problems like this one, where the calculations become complicated, and there is an increased risk of error when doing all of the algebra by hand. Most computer algebra systems also include a graphing utility. The combination of graphics and algebra is important in problems such as the present one. Drawing a graph is often the easiest way to solve an inequality.

Figure 5.3 shows a graph of λ_1 and λ_2 versus t over the interval $0 < t < 0.6$. From this graph we can see that λ_1, λ_2 are always negative, so that the equilibrium is stable regardless of the strength of competition. (If you did Exercise 1 of Chapter 4, you probably drew the same conclusion from a graphical analysis.)

5.2 Eigenvalue Methods for Discrete Systems

The methods of the previous section apply only to continuous time dynamical systems. In this section we will present analogous methods for the stability

```
> with(linalg):
> f1:=x1/10-(x1^2/10)/10000-(t*x1*x2/10)/10000;
```

$$f1 := \frac{1}{10}x1 - \frac{1}{100000}x1^2 - \frac{1}{100000}t\,x1\,x2$$

```
> f2:=25*x2/100-(25*x2^2/100)/6000-(25*t*x1*x2/100)/6000;
```

$$f2 := \frac{1}{4}x2 - \frac{1}{24000}x2^2 - \frac{1}{24000}t\,x1\,x2$$

```
> df1dx1:=diff(f1,x1);
```

$$df1dx1 := \frac{1}{10} - \frac{1}{50000}x1 - \frac{1}{100000}t\,x2$$

```
> df1dx2:=diff(f1,x2);
```

$$df1dx2 := -\frac{1}{100000}t\,x1$$

```
> df2dx1:=diff(f2,x1);
```

$$df2dx1 := -\frac{1}{24000}t\,x2$$

```
> df2dx2:=diff(f2,x2);
```

$$df2dx2 := \frac{1}{4} - \frac{1}{12000}x2 - \frac{1}{24000}t\,x1$$

```
> s:=solve({f1/x1=0,f2/x2=0},{x1,x2});
```

$$s := \{x2 = 2000\frac{-3+5\,t}{-1+t^2},\ x1 = 2000\frac{-5+3\,t}{-1+t^2}\}$$

```
> assign(s);
> A:=array([[df1dx1,df1dx2],[df2dx1,df2dx2]]);
```

$$A := \begin{bmatrix} \frac{1}{10} - \frac{1}{25}\frac{-5+3\,t}{-1+t^2} - \frac{1}{50}\frac{t\,(-3+5\,t)}{-1+t^2} & -\frac{1}{50}\frac{t\,(-5+3\,t)}{-1+t^2} \\ -\frac{1}{12}\frac{t\,(-3+5\,t)}{-1+t^2} & \frac{1}{4} - \frac{1}{6}\frac{-3+5\,t}{-1+t^2} - \frac{1}{12}\frac{t\,(-5+3\,t)}{-1+t^2} \end{bmatrix}$$

```
> eigenvals(A);
```

$$\frac{1}{2}\frac{-286\,t + 210 + 2\sqrt{20449\,t^2 - 9630\,t + 2025 - 20400\,t^3 + 9000\,t^4}}{600\,t^2 - 600},$$

$$\frac{1}{2}\frac{-286\,t + 210 - 2\sqrt{20449\,t^2 - 9630\,t + 2025 - 20400\,t^3 + 9000\,t^4}}{600\,t^2 - 600}$$

Figure 5.2: Calculations for sensitivity analysis in the tree problem of Example 5.1 using the computer algebra system Maple.

analysis of discrete time dynamical systems. Once again, the basis for our analysis is a linear approximation, together with a calculation of eigenvalues.

Example 5.2. Reconsider the docking problem of Example 4.3, and assume now that it takes 5 seconds to make the control adjustments, and another 10 seconds until we can return from other tasks to observe the velocity indicator once again. Under these conditions, will our strategy for matching velocities be successful?

Step 1 of the five-step method was summarized in Figure 4.7. Now we will assume $c_n = 5, w_n = 10$. For the moment we will set $k = 0.02$, and then later we will perform a sensitivity analysis on k.

Step 2 is to select the modeling approach, including the method of solution. We will use an eigenvalue method.

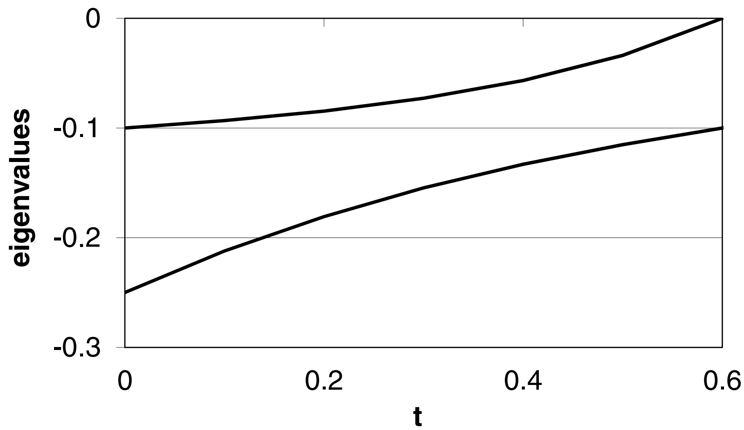

Figure 5.3: Graph of the eigenvalues λ_1 and λ_2 from (5.11) versus the parameter t in the tree problem.

Given a discrete time dynamical system

$$\Delta x = F(x)$$

where $x = (x_1, \ldots, x_n)$ and $F = (f_1, \ldots, f_n)$, let us define the iteration function

$$G(x) = x + F(x).$$

The sequence $x(0), x(1), x(2), \ldots$ is a solution to this system of difference equations if and only if

$$x(n+1) = G(x(n))$$

for all n. An equilibrium point x_0 is characterized by the fact that x_0 is a fixed point of the function $G(x)$; i.e., $G(x_0) = x_0$.

There is a theorem that says that an equilibrium point x_0 is (asymptotically) stable if every eigenvalue of the matrix of partial derivatives

$$A = \begin{pmatrix} \partial g_1/\partial x_1(x_0) & \cdots \partial g_1/\partial x_n(x_0) \\ \vdots & \vdots \\ \partial g_n/\partial x_1(x_0) & \cdots \partial g_n/\partial x_n(x_0) \end{pmatrix} \tag{5.12}$$

has absolute value less than one. If the eigenvalue is complex $a \pm ib$ then by "absolute value" we mean the complex absolute value $\sqrt{a^2 + b^2}$. This simple test for stability is analogous to the eigenvalue test for continuous time dynamical systems presented in the preceding section (Hirsch and Smale (1974) p. 280).

As in the continuous case, the eigenvalue method for discrete time dynamical systems is based on a linear approximation. Even if the iteration function $G(x)$ is not linear, we will have

$$G(x) \approx A(x - x_0)$$

in the neighborhood of the equilibrium point x_0. In other words the behavior of the iteration function G in the neighborhood of the equilibrium point x_0 is approximately the same as the behavior of the linear function Ax near the origin. Therefore, the behavior of our original nonlinear system in the neighborhood of x_0 is approximately the same as the behavior of the linear discrete time dynamical system defined by the iteration function

$$x(n+1) = Ax(n)$$

in the neighborhood of the origin. The linear approximation is good enough so that if the origin is a stable equilibrium of the linear system, then x_0 is a stable equilibrium of the original nonlinear system. So it only remains to discuss the conditions for stability of the linear system.

A matrix A is called a *linear contraction* if $A^n x \to 0$ for every x. There is a theorem that states that if every eigenvalue of a matrix A has absolute value less than one, then A is a linear contraction (Hirsch and Smale (1974) p. 279). It follows that the origin is a stable equilibrium of the discrete time dynamical system with iteration function A whenever the eigenvalues of A all have absolute value less than one. We illustrate the proof of this result in a simple case. Suppose that $Ax = \lambda x$ for *all* x. Then λ is an eigenvalue of A, and every nonzero vector x is an eigenvector belonging to λ. In this simple case we will always have

$$x(n+1) = Ax(n) = \lambda x(n)$$

so that the origin is a stable equilibrium if and only if $|\lambda| < 1$.

Now we return to the docking problem. Step 3 is to formulate the model as necessary for the application of the techniques identified in step 2. In this case we already have a linear system with equilibrium $x_0 = (0, 0)$. The iteration function is

$$G(x_1, x_2) = x + F(x_1, x_2) = (g_1, g_2)$$

where

$$\begin{aligned} g_1(x_1, x_2) &= 0.8x_1 - 0.1x_2 \\ g_2(x_1, x_2) &= x_1. \end{aligned}$$

Moving on to step 4, we calculate

$$\begin{vmatrix} \lambda - 0.8 & 0.1 \\ -1 & \lambda - 0 \end{vmatrix} = 0$$

or $\lambda^2 - 0.8\lambda + 0.1 = 0$, from which we obtain

$$\lambda = \frac{4 \pm \sqrt{6}}{10}.$$

There are $n = 2$ distinct eigenvalues, and both are real and lie between -1 and $+1$. Hence the equilibrium $x_0 = 0$ is stable, and so we will get $x(t) \to (0,0)$ for any initial condition.

Step 5 is to state our results in plain English. We assumed 15 seconds between control adjustments; 5 seconds to make the adjustment and 10 seconds slack time. Using a correction factor of 1 : 50, we can guarantee success for our proportional method of control. In practical terms, a correction factor of 1 : 50 means that if the velocity indicator reads 50 m/sec we will set the acceleration controls for -1 m/sec^2; if the reading is 25 m/sec, we set the controls at -0.5 m/sec^2, and so on.

What follows is a sensitivity analysis for the parameter k. For a general k the iteration function is given by $G = (g_1, g_2)$ where

$$\begin{aligned} g_1(x_1, x_2) &= (1 - 10k)x_1 - 5kx_2 \\ g_2(x_1, x_2) &= x_1 \end{aligned}$$

which leads to the characteristic equation

$$\lambda^2 - (1 - 10k)\lambda + 5k = 0.$$

The eigenvalues are

$$\begin{aligned} \lambda_1 &= \frac{(1 - 10k) + \sqrt{(1 - 10k)^2 - 20k}}{2} \\ \lambda_2 &= \frac{(1 - 10k) - \sqrt{(1 - 10k)^2 - 20k}}{2}. \end{aligned} \tag{5.13}$$

The quantity under the radical in (5.13) becomes negative between

$$k_1 = \frac{4 - \sqrt{12}}{20} \approx 0.027$$

and

$$k_2 = \frac{4 + \sqrt{12}}{20} \approx 0.373.$$

Figure 5.4 shows a graph of λ_1 and λ_2 over the interval $0 < k \leq k_1$. From the graph we can see that both eigenvalues have absolute value less than one, so that the equilibrium $(0,0)$ is stable over this entire range of k. For $k_1 < k < k_2$ both eigenvalues are complex, and the condition for stability is that

$$\left[\frac{(1 - 10k)}{2}\right]^2 + \left[\frac{\sqrt{20k - (1 - 10k)^2}}{2}\right]^2 < 1 \tag{5.14}$$

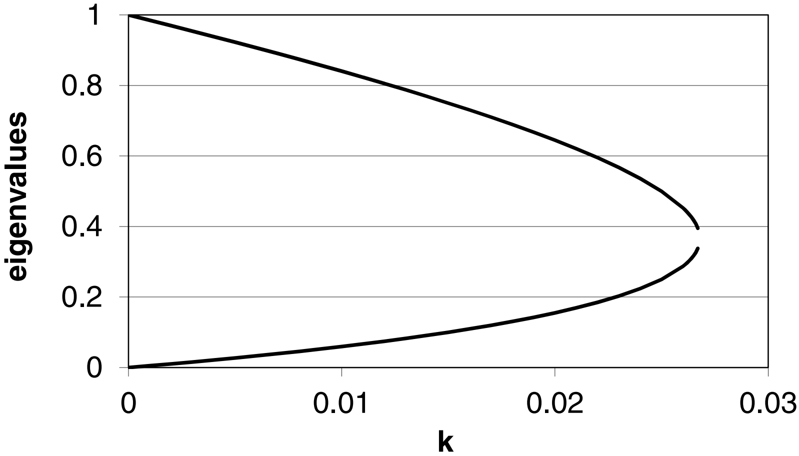

Figure 5.4: Graph of the eigenvalues λ_1 and λ_2 from (5.13) versus control parameter k in the docking problem: case $0 < k \leq k_1$.

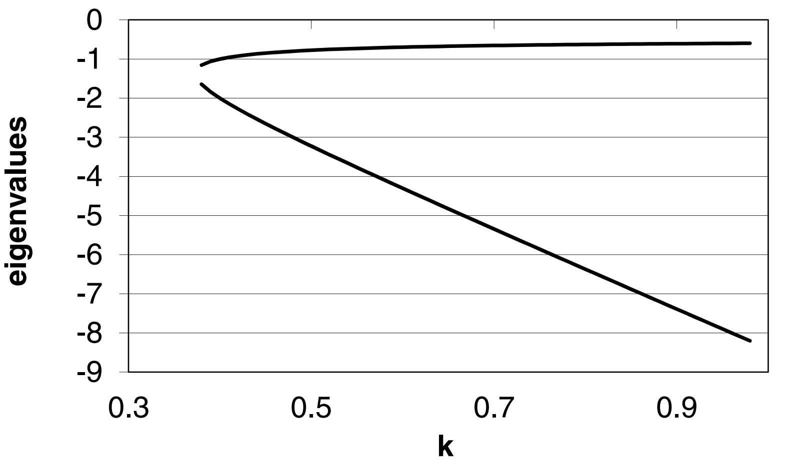

Figure 5.5: Graph of the eigenvalues λ_1 and λ_2 from (5.13) versus control parameter k in the docking problem: case $k \geq k_2$.

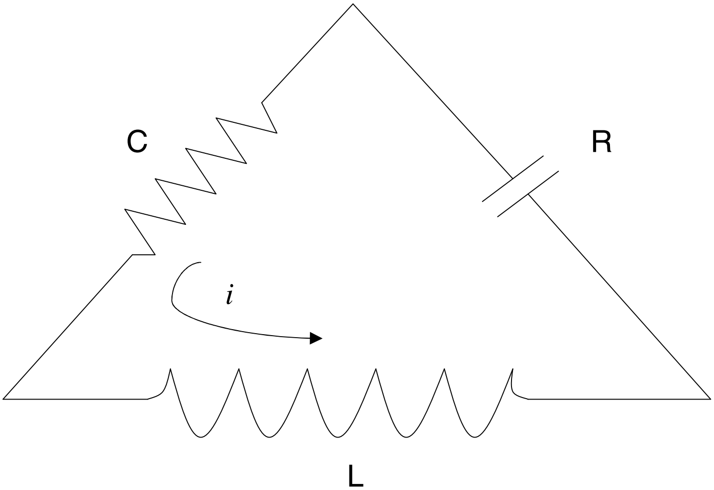

Figure 5.6: RLC circuit diagram for Example 5.3.

which reduces to $k < 1/5$. Figure 5.5 shows a graph of λ_1 and λ_2 for $k \geq k_2$. It is easy to see that the smaller eigenvalue λ_2 has absolute value greater than one for all such k. To summarize, the method will achieve matched velocities as long as $k < 0.2$, or at least a $1:5$ correction factor. Of course, it is of some interest to know which value of k is the most efficient. We will leave this problem for the exercises.

5.3 Phase Portraits

In Section 5.1 we introduced the eigenvalue test for stability in continuous time dynamical systems. This test is based on the idea of a linear approximation in the neighborhood of an isolated equilibrium point. In this section we will show how this simple idea can be used to obtain a graphical description of the behavior of a dynamical system near an equilibrium point. This information can then be used along with a sketch of the vector field to obtain a graphical description of the dynamics over the entire state space, called the *phase portrait.* Phase portraits are important in the analysis of nonlinear dynamical systems because, in most cases, it is not possible to obtain exact analytical solutions. At the end of this section we also include a brief discussion of some similar techniques for discrete time dynamical systems, again based on the idea of linear approximation.

Example 5.3. Consider the electrical circuit diagrammed in Figure 5.6. The circuit consists of a capacitor, a resistor, and an inductor in a simple closed

Variables:	v_C = voltage across capacitor i_C = current through capacitor v_R = voltage across resistor i_R = current through resistor v_L = voltage across inductor i_L = current through inductor
Assumptions:	$C\, dv_C/dt = i_C$ $v_R = f(i_R)$ $L\, di_L/dt = v_L$ $i_C = i_R = i_L$ $v_C + v_R + v_L = 0$ $L = 1$ $C = 1/3$ $f(x) = x^3 + 4x$
Objective:	Determine the behavior of all six variables over time

Figure 5.7: Step 1 of the RLC circuit problem.

loop. The effect of each component of the circuit is measured in terms of the relationship between current and voltage on that branch of the loop. An idealized physical model gives the relations

$$\begin{aligned} C\frac{dv_C}{dt} &= i_C \text{ (capacitor)} \\ v_R &= f(i_R) \text{ (resistor)} \\ L\frac{di_L}{dt} &= v_L \text{ (inductor)} \end{aligned}$$

where v_C represents the voltage across the capacitor, i_R represents the current through the resistor, and so on. The function $f(x)$ is called the *v-i characteristic* of the resistor. Usually $f(x)$ has the same sign as x. This is called a passive resistor. Some control circuits use an active resistor, where $f(x)$ and x have opposite sign for small x, see Example 5.4. In the classical linear model of the RLC circuit, we assume that $f(x) = Rx$ where $R > 0$ is the resistance. Kirchoff's current law states that the sum of the currents flowing into a node equals the sum of the currents flowing out. Kirchoff's voltage law states that the sum of the voltage drops along a closed loop must add up to zero. Determine the behavior of this circuit over time in the case where $L = 1$, $C = 1/3$, and $f(x) = x^3 + 4x$.

We will use the five-step method. The results of step 1 are summarized in Figure 5.7. Step 2 is to select a modeling approach. We will model this problem using a continuous time dynamical system, which we will analyze by sketching the complete phase portrait.

Suppose that we are given a dynamical system $x' = F(x)$ where $x = (x_1 \dots, x_n)$ and F has continuous first partial derivatives in the neighborhood of an equilibrium point x_0. Let A denote the matrix of first partial derivatives evaluated at the equilibrium point x_0 as defined by (5.1). We have stated previously that for x near x_0 the system $x' = F(x)$ behaves like the linear system $x' = A(x - x_0)$. Now we will be more specific.

The *phase portrait* of a continuous time dynamical system is simply a sketch of the state space showing a representative selection of solution curves. It is not hard to draw the phase portrait for a linear system (at least on $\mathbb{R}^2$) because we can always find an exact solution to a linear system of differential equations. Then we can just graph the solutions for a few initial conditions to get the phase portrait. We refer the reader to any textbook on differential equations for the details on how to solve linear systems of differential equations. For nonlinear systems, we can draw an approximate phase portrait in the neighborhood of each isolated equilibrium point by using the linear approximation.

A *homeomorphism* is a continuous function with a continuous inverse. The idea of a homeomorphism has to do with shapes and their generic properties. For example, consider a circle in the plane. The image of this circle under a homeomorphism

$$G : \mathbb{R}^2 \to \mathbb{R}^2$$

might be another circle, an ellipse, or even a square or a triangle. But it could not be a line segment. This would violate continuity. It also could not be a figure eight, because this would violate the property that G must have an inverse (so it must be one-to-one). There is a theorem that states that if the eigenvalues of A all have nonzero real parts, then there is a homeomorphism G that maps the phase portrait of the system $x' = Ax$ onto the phase portrait of $x' = F(x)$, with $G(0) = x_0$ (Hirsch and Smale (1974) p. 314). This theorem says that the phase portrait of $x' = F(x)$ around the point x_0 looks just like that of the linear system, except for some distortion. It would be as if we drew the phase portrait of the linear system on a sheet of rubber which we could stretch any way we like, but could not tear. This is a very powerful result. It means that we can get an actual picture (good enough for almost all practical purposes) of the behavior of a nonlinear dynamical system near each isolated equilibrium point just by analyzing its linear approximation. Then, to finish up the phase portrait on the rest of the state space, we combine what we have learned about the behavior of solutions near the equilibrium points with the information contained in a sketch of the vector field.

Step 3 is to formulate the model. We begin by considering the state space.

There are six state variables to begin with, but we can use Kirchoff's laws to reduce the number of degrees of freedom (the number of independent state variables) from six to two. Let $x_1 = i_R$ and notice that $x_1 = i_L = i_C$ as well. Let $x_2 = v_C$. Then we have

$$\begin{aligned}\frac{x_2'}{3} &= x_1\\ v_R &= x_1^3 + 4x_1\\ x_1' &= v_L\\ x_2 + v_R + v_L &= 0.\end{aligned}$$

Substitute to obtain

$$\begin{aligned}\frac{x_2'}{3} &= x_1\\ x_2 + x_1^3 + 4x_1 + x_1' &= 0,\end{aligned}$$

and then rearrange to get

$$\begin{aligned}x_1' &= -x_1^3 - 4x_1 - x_2\\ x_2' &= 3x_1.\end{aligned} \tag{5.15}$$

Now if we let $x = (x_1, x_2)$, then Eq. (5.15) can be written in the form $x' = F(x)$ where $F = (f_1, f_2)$ and

$$\begin{aligned}f_1(x_1, x_2) &= -x_1^3 - 4x_1 - x_2\\ f_2(x_1, x_2) &= 3x_1.\end{aligned} \tag{5.16}$$

This concludes step 3.

Step 4 is to solve the model. We will analyze the dynamical system (5.15) by sketching the complete phase portrait. Figure 5.8 shows a Maple graph of the vector field for this dynamical system. It is also a fairly simple matter to sketch the vector field by hand. Velocity vectors are horizontal on the curve $x_1 = 0$ where $x_2' = 0$, and vertical on the curve $x_2 = -x_1^3 - 4x_1$ where $x_1' = 0$. There is one equilibrium point $(0, 0)$ at the intersection of these two curves. From the vector field it is difficult to tell whether the equilibrium is stable or unstable. To obtain more information, we will analyze the linear system that approximates the behavior of (5.15) near the equilibrium $(0, 0)$.

Computing the partial derivatives from (5.16), we obtain

$$\begin{aligned}\frac{\partial f_1}{\partial x_1} &= -3x_1^2 - 4\\ \frac{\partial f_1}{\partial x_2} &= -1\\ \frac{\partial f_2}{\partial x_1} &= 3\\ \frac{\partial f_2}{\partial x_2} &= 0.\end{aligned} \tag{5.17}$$

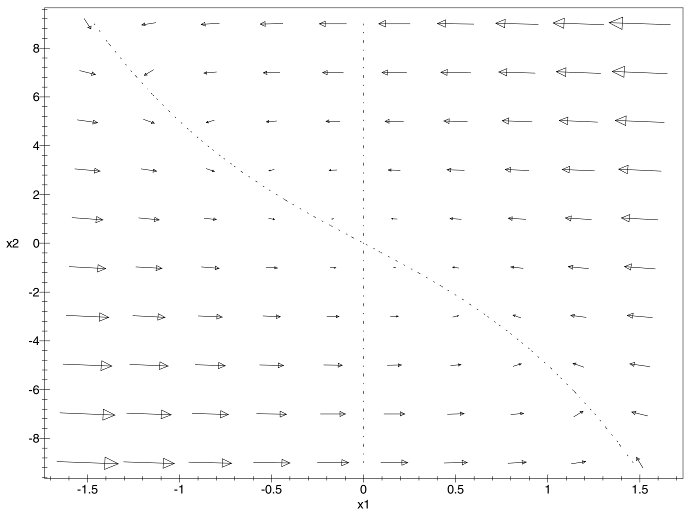

Figure 5.8: Graph of voltage x_2 versus current x_1 showing vector field (5.16) for the RLC circuit problem of Example 5.3.

Evaluating the partial derivatives (5.17) at the equilibrium point $(0, 0)$ and substituting back into Eq. (5.1), we obtain

$$A = \begin{pmatrix} -4 & -1 \\ 3 & 0 \end{pmatrix}.$$

The eigenvalues of this 2×2 matrix can be computed as the roots of the equation

$$\begin{vmatrix} \lambda + 4 & 1 \\ -3 & \lambda \end{vmatrix} = 0.$$

Evaluating the determinant, we obtain the equation

$$\lambda^2 + 4\lambda + 3 = 0,$$

and then we obtain

$$\lambda = -3, -1.$$

Since both eigenvalues are negative, the equilibrium is stable.

To obtain additional information, we will solve the linear system $x' = Ax$. In this case we have

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -4 & -1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{5.18}$$

We will solve the linear system (5.18) by the method of eigenvalues and eigenvectors. We have already calculated the eigenvalues $\lambda = -3, -1$. To compute the eigenvector corresponding to the eigenvalue λ, we must find a nonzero solution to the equation

$$\begin{pmatrix} \lambda+4 & 1 \\ -3 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $\lambda = -3$ we have

$$\begin{pmatrix} 1 & 1 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which we obtain

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

so that

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

is one solution to the linear system (5.18). For $\lambda = -1$ we have

$$\begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which we obtain

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix},$$

so that

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-t}$$

is another solution to the linear system (5.18). Then, the general solution to (5.18) can be written in the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-t} \tag{5.19}$$

where c_1, c_2 are arbitrary real constants.

Figure 5.9 shows the phase portrait for the linear system (5.18). This graph was obtained by plotting the solution curves (5.19) for a few select values of the constants c_1, c_2. For example, when $c_1 = 1$ and $c_2 = 1$, we plotted a parametric graph of

$$\begin{aligned} x_1(t) &= -e^{-3t} - e^{-t} \\ x_2(t) &= e^{-3t} + 3e^{-t}. \end{aligned}$$

We superimposed a graph of the linear vector field in order to indicate the orientation of the solution curves. Whenever you plot a phase portrait, be sure to add arrows to indicate the direction of the flow.

Figure 5.10 shows the complete phase portrait for the original nonlinear dynamical system (5.15). This picture was obtained by combining the information in Figures 5.8 and 5.9 and using the fact that the phase portrait of the nonlinear

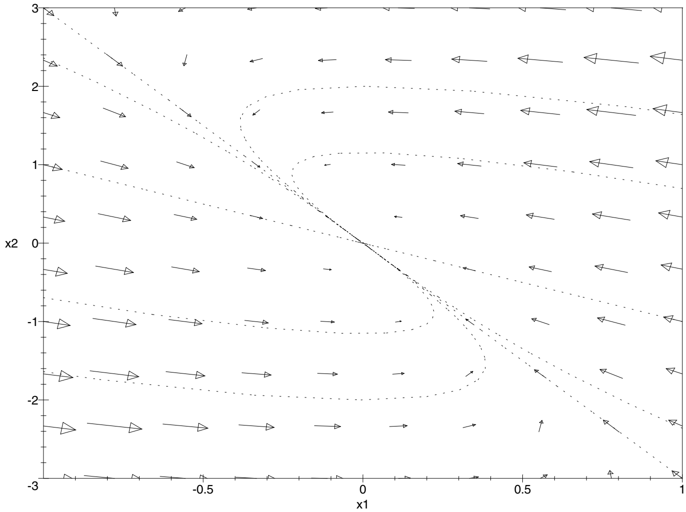

Figure 5.9: Graph of voltage x_2 versus current x_1 showing linear approximation to the phase portrait near (0, 0) for the RLC circuit problem of Example 5.3.

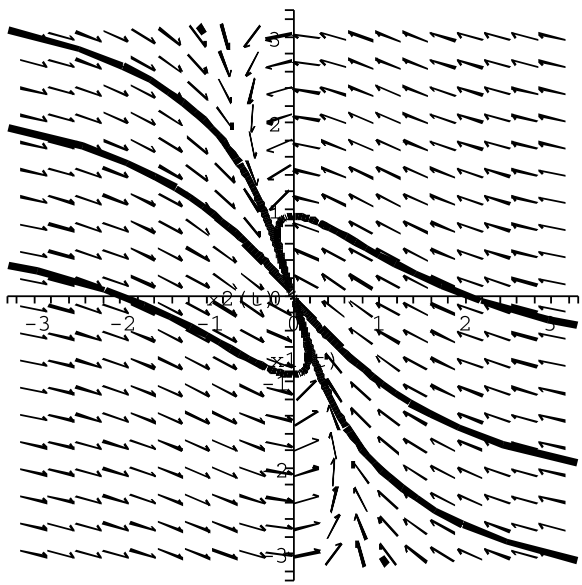

Figure 5.10: Graph of voltage x_2 versus current x_1 showing complete phase portrait for the RLC circuit problem of Example 5.3.

system (5.15) is homeomorphic to the phase portrait of the linear system (5.18). In this example there is not much qualitative difference between the behavior of the linear and the nonlinear systems.

Step 5 is to answer the question. The question was to describe the behavior of the RLC circuit. The overall behavior can be described in terms of two quantities, the current through the resistor and the voltage drop across the capacitor. Regardless of the initial state of the circuit, both quantities eventually tend to zero. Furthermore, it is eventually true that either voltage is positive and current is negative, or vice versa. For a complete graphical description of the way that current and voltage behave over time, see Figure 5.10, where x_1 represents current and x_2 represents voltage. The behavior of other quantities of interest can easily be described in terms of these two variables (see Figure 5.7 for details). For example, the variable x_1 actually represents the current through any branch of the circuit loop.

Next, we will perform a sensitivity analysis to determine the effect of small changes in our assumptions on our general conclusions. First let us consider the capacitance C. In our example we assumed that $C = 1/3$. Now we will generalize our model by letting C remain indeterminate. In this case we obtain the dynamical system

$$\begin{aligned} x_1' &= -x_1^3 - 4x_1 - x_2 \\ x_2' &= \frac{x_1}{C} \end{aligned} \tag{5.20}$$

in place of (5.15). Now we have

$$\begin{aligned} f_1(x_1, x_2) &= -x_1^3 - 4x_1 - x_2 \\ f_2(x_1, x_2) &= \frac{x_1}{C}. \end{aligned} \tag{5.21}$$

For values of C near 1/3, the vector field for (5.21) is essentially the same as in Figure 5.8. Velocity vectors are still horizontal on the curve $x_1 = 0$ and vertical on the curve $x_2 = -x_1^3 - 4x_1$. There is still one equilibrium point $(0, 0)$ at the intersection of these two curves.

Computing the partial derivatives from Eq. (5.21), we obtain

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= -3x_1^2 - 4 \\ \frac{\partial f_1}{\partial x_2} &= -1 \\ \frac{\partial f_2}{\partial x_1} &= \frac{1}{C} \\ \frac{\partial f_2}{\partial x_2} &= 0. \end{aligned} \tag{5.22}$$

Evaluating the partial derivatives (5.22) at the equilibrium point $(0, 0)$ and substituting back into Eq. (5.1), we obtain

$$A = \begin{pmatrix} -4 & -1 \\ 1/C & 0 \end{pmatrix}.$$

The eigenvalues of this matrix can be computed as the roots of the equation

$$\begin{vmatrix} \lambda + 4 & 1 \\ -1/C & \lambda \end{vmatrix} = 0.$$

Evaluating the determinant, we obtain the equation

$$\lambda^2 + 4\lambda + \frac{1}{C} = 0.$$

The eigenvalues are

$$\lambda = -2 \pm \sqrt{4 - \frac{1}{C}}.$$

If $C > 1/4$, then we have two distinct real negative eigenvalues, and so the equilibrium is stable. In this case, the general solution to the linear system is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 2+\alpha \end{pmatrix} e^{(-2+\alpha)t} + c_2 \begin{pmatrix} -1 \\ 2-\alpha \end{pmatrix} e^{(-2-\alpha)t} \tag{5.23}$$

where $\alpha^2 = 4 - 1/C$. The phase portrait of the linear system is about the same as Figure 5.9, except that the slope of the straight line solutions varies with C. Then for values of C greater than 1/4, the phase portrait for the original nonlinear system is a lot like the one shown in Figure 5.10. We conclude that our general conclusions about this RLC circuit are not sensitive to the exact value of C as long as $C > 1/4$. A similar result may be expected for the inductance L. Generally speaking, the important characteristics of our solution (e.g., eigenvectors) depend continuously on these parameters.

Next, we consider the question of robustness. We assumed that the RLC circuit had v-i characteristic $f(x) = x^3 + 4x$. Suppose more generally that $f(0) = 0$ and that f is strictly increasing. Now the dynamical system equations are

$$\begin{aligned} x_1' &= -f(x_1) - x_2 \\ x_2' &= 3x_1. \end{aligned} \tag{5.24}$$

Now we have

$$\begin{aligned} f_1(x_1, x_2) &= -f(x_1) - x_2 \\ f_2(x_1, x_2) &= 3x_1. \end{aligned} \tag{5.25}$$

Let $R = f'(0)$. The linear approximation uses

$$A = \begin{pmatrix} -R & -1 \\ 3 & 0 \end{pmatrix}$$

and so the eigenvalues are the roots to the equation

$$\begin{vmatrix} \lambda + R & 1 \\ 3 & \lambda \end{vmatrix} = 0.$$

We compute that

$$\lambda = \frac{-R \pm \sqrt{R^2 - 12}}{2}.$$

As long as $R > \sqrt{12}$, we have two distinct real negative eigenvalues, and the behavior of the linear system is as depicted in Figure 5.9. Furthermore the behavior of the original nonlinear system cannot be too different from Figure 5.10. We conclude that our model of this RLC circuit is robust with regard to our assumptions about the form of the v-i characteristic.

Example 5.4. Consider the nonlinear *RLC* circuit with $L = 1, C = 1$, and v-i characteristic $f(x) = x^3 - x$. Determine the behavior of this circuit over time.

The modeling process is, of course, the same as for the previous example. Letting $x_1 = i_R$ and $x_2 = v_C$, we obtain the dynamical system

$$\begin{aligned} x_1' &= x_1 - x_1^3 - x_2 \\ x_2' &= x_1. \end{aligned} \tag{5.26}$$

See Figure 5.11 for a plot of the vector field. The velocity vectors are vertical on the curve $x_2 = x_1 - x_1^3$ and horizontal on the x_2 axis. The only equilibrium is the origin $(0, 0)$. It is hard to tell from the vector field whether or not the origin is a stable equilibrium.

The matrix of partial derivatives is

$$A = \begin{pmatrix} 1 - 3x_1^2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Evaluate at $x_1 = 0, x_2 = 0$ to obtain the linear system

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

which approximates the behavior of our nonlinear system near the origin. To obtain the eigenvalues we must solve

$$\begin{vmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 0 \end{vmatrix} = 0$$

or $\lambda^2 - \lambda + 1 = 0$. The eigenvalues are

$$\lambda = 1/2 \pm i\sqrt{3}/2.$$

Since the real part of every eigenvalue is positive, the origin is an unstable equilibrium.

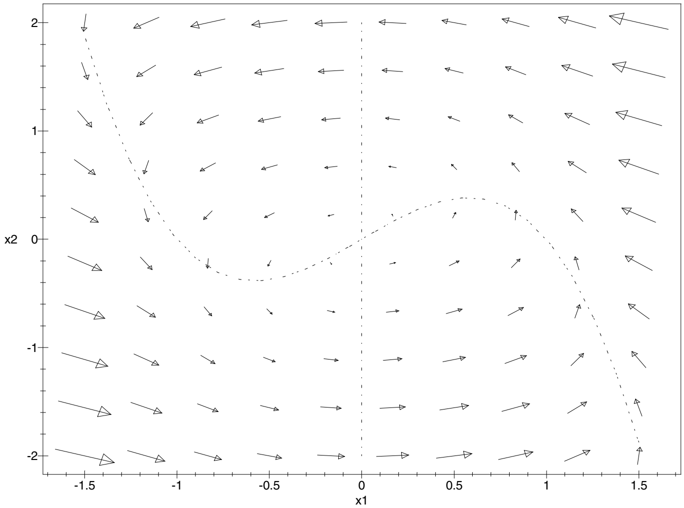

Figure 5.11: Graph of voltage x_2 versus current x_1 showing vector field from (5.26) for the RLC circuit problem of Example 5.4.

To obtain more information, we will solve the linear system. To find an eigenvector belonging to

$$\lambda = 1/2 + i\sqrt{3}/2,$$

we solve

$$\begin{pmatrix} -1/2 + i\sqrt{3}/2 & 1 \\ -1 & 1/2 + i\sqrt{3}/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we obtain

$$x_1 = 2,\ x_2 = 1 - i\sqrt{3}.$$

Then we have the complex solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 - i\sqrt{3} \end{pmatrix} e^{(\frac{1}{2} + i\frac{\sqrt{3}}{2})t}.$$

Taking real and imaginary parts yields two linearly independent real solutions $u = (x_1,\, x_2)$, where

$$\begin{aligned} x_1(t) &= 2e^{t/2}\cos(t\sqrt{3}/2) \\ x_2(t) &= e^{t/2}\cos(t\sqrt{3}/2) + \sqrt{3}\,e^{t/2}\sin(t\sqrt{3}/2) \end{aligned}$$

and $v = (x_1,\, x_2)$, where

$$\begin{aligned} x_1(t) &= 2e^{t/2}\sin(t\sqrt{3}/2) \\ x_2(t) &= e^{t/2}\sin(t\sqrt{3}/2) - \sqrt{3}\,e^{t/2}\cos(t\sqrt{3}/2). \end{aligned}$$

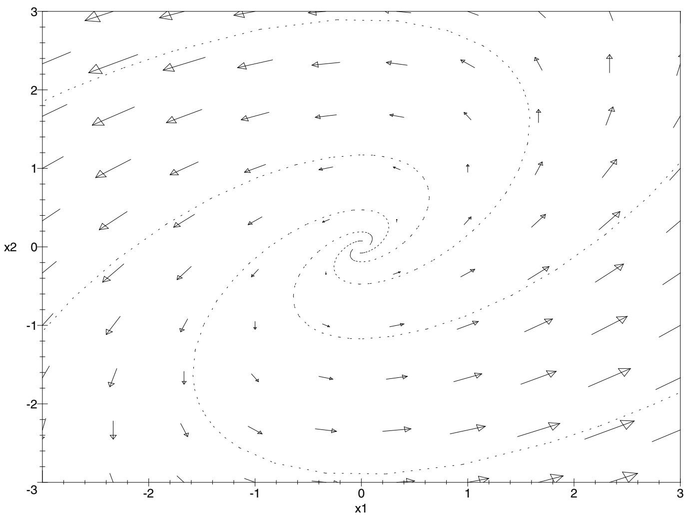

Figure 5.12: Graph of voltage x_2 versus current x_1 showing linear approximation to the phase portrait near (0, 0) for the RLC circuit problem of Example 5.4.

The general solution is $c_1u(t) + c_2v(t)$. The phase portrait for this linear system is shown in Figure 5.12. This graph shows parametric plots of the solution for a few select values of c_1 and c_2. We superimpose a plot of the vector field in order to show the direction of the flow.

Note that, if we zoom in or zoom out on the origin in this linear phase portrait, it will look essentially the same. One of the defining characteristics of linear vector fields and linear phase portraits is that they look the same on every scale. The phase portrait for the nonlinear system in the neighborhood of the origin looks about the same, with some distortions. The solution curves near $(0, 0)$ spiral outward, moving counterclockwise. If we continue to zoom in to the origin on a vector field or phase portrait for the nonlinear system, it will look more and more like the linear system. Further away from the origin, the behavior of the nonlinear system may vary significantly from that of the linear system.

In order to obtain the complete phase portrait of the nonlinear system, we need to combine the information from Figures 5.11 and 5.12. It is apparent from the vector field in Figure 5.11 that the behavior of solution curves changes dramatically farther away from the origin. There is still a general counterclockwise flow, but the solution curves do not spiral out to infinity as in the linear phase portrait. Solution curves that begin far away from the origin look like they are moving toward the origin as they continue their counterclockwise flow. Since the solution curves near the origin are spiraling outwards, and the solution curves far away from the origin tend inward, and we know that solution curves do not cross, something interesting must be happening in the phase portrait. Whatever is happening, it is something that can never happen in a linear system. If a solution curve spirals outward in a linear phase portrait, then it must continue to spiral all the way out to infinity. In Section 6.3 we will explore the behavior of the dynamical system (5.26) using computational methods. We will wait until then to draw the complete phase portrait.

Before we leave the subject of linear approximation techniques, we should point out a few facts about discrete time dynamical systems. Suppose we have a discrete time dynamical system

$$\Delta x = F(x)$$

where $x = (x_1, \ldots, x_n)$, and let

$$G(x) = x + F(x)$$

denote the iteration function. At an equilibrium point x_0 we have $G(x_0) = x_0$. In Section 5.2 we used the approximation

$$G(x) \approx A(x - x_0)$$

for values of x near x_0, where A is the matrix of partial derivatives evaluated at $x = x_0$ as defined by (5.12).

One way to obtain a graphical picture of the iteration function $G(x)$ is to draw the image sets

$$G(S) = \{G(x) : x \in S\}$$

for various sets

$$S = \{x : |x - x_0| = r\}.$$

In dimension $n = 2$ the set S is a circle, and in dimension $n = 3$ it is a sphere. It is possible to show that, as long as the matrix A is nonsingular, there is a diffeomorphism $H(x)$ that maps the image sets $A(S)$ onto $G(S)$ in a neighborhood of the point x_0. If a point x lies inside of S, then $G(x)$ will be inside of $G(S)$. This allows a graphical interpretation of the dynamics. Figures 5.13 through 5.15 illustrate the dynamics of the docking problem from Example 5.2. In this case, $G(S) = A(S)$, since G is linear. Starting at a state on (or inside) the set S, shown in Figure 5.13, the next state will be on (or inside, respectively) the set $A(S)$, shown in Figure 5.14, and then the next state will be on (or inside, respectively) the set $A^2(S) = A(A(S))$, shown in Figure 5.15. As $n \to \infty$, the set $A^n(S)$ gradually shrinks in toward the origin.

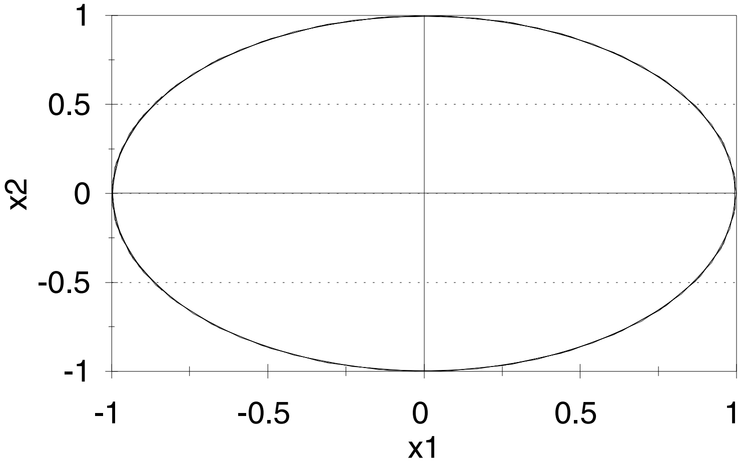

Figure 5.13: Dynamics of the docking problem showing the initial condition $S = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$.

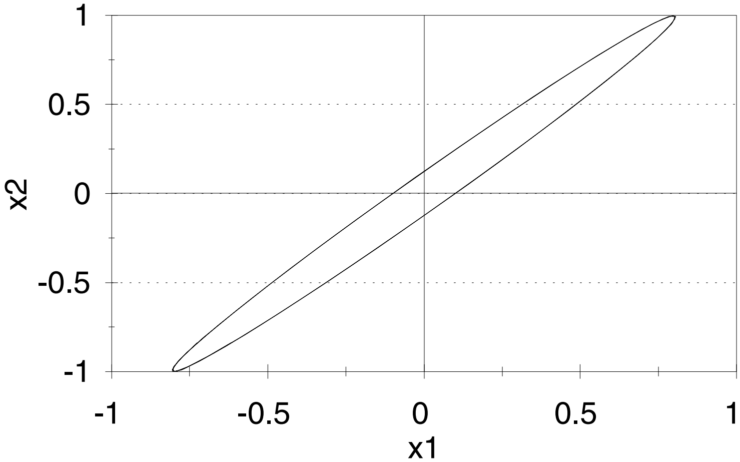

Figure 5.14: Dynamics of the docking problem showing $A(S)$ after one iteration.

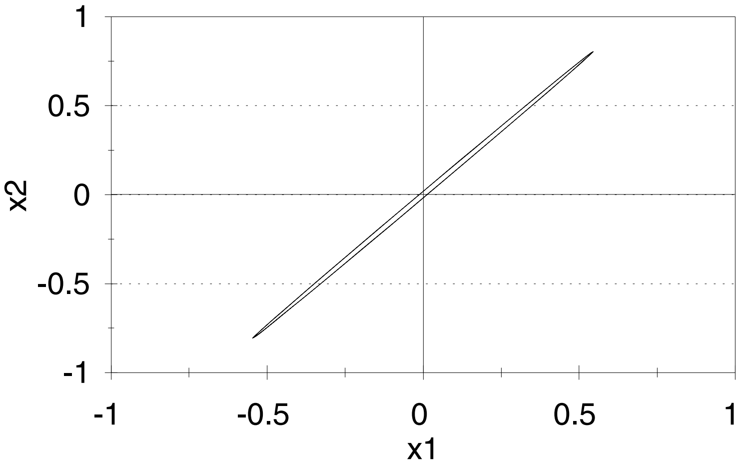

Figure 5.15: Dynamics of the docking problem showing $A^2(S)$ after two iterations.

5.4 Exercises

1. Reconsider Exercise 4 of Chapter 4.

   (a) Sketch the vector field for this model. Determine the location of each equilibrium in the state space. Can you tell from the vector field which of the equilibria are stable?

   (b) Use the eigenvalue method to test the stability of each equilibrium in the state space.

   (c) For each equilibrium point, determine the linear system that approximates the behavior of the original dynamical system in the neighborhood of the equilibrium point. Write the general solution to this linear system, and sketch the linear phase portrait.

   (d) Sketch the complete phase portrait for this model, using the results of parts (a) and (c).

   (e) Given the current estimates of 5000 blue whales and 70,000 fin whales, what does this model predict about the future of the two species?

2. Reconsider Exercise 5 of Chapter 4.

   (a) Sketch the vector field for this model. Determine the location of each equilibrium in the state space.

(b) Use the eigenvalue method to test the stability of each equilibrium in the state space.

(c) For each equilibrium point, determine the linear system that approximates the behavior of the original dynamical system in the neighborhood of the equilibrium point. Write the general solution to this linear system, and sketch the linear phase portrait.

(d) Sketch the complete phase portrait for this model, using the results of parts (a) and (c).

(e) Given the current estimates of 5000 blue whales and 70,000 fin whales, what does this model predict about the future of the two species?

3. Reconsider Exercise 6 of Chapter 4. Assume that the catchability coefficient $q = 10^{-5}$ and that the level of effort is $E = 3000$ boat-days per year.

   (a) Sketch the vector field for this model. Determine the location of each equilibrium in the state space.

   (b) Use the eigenvalue method to test the stability of each equilibrium in the state space.

   (c) For each equilibrium point, determine the linear system that approximates the behavior of the original dynamical system in the neighborhood of the equilibrium point. Write the general solution to this linear system, and sketch the linear phase portrait.

   (d) Sketch the complete phase portrait for this model, using the results of parts (a) and (c).

   (e) Given the current estimates of 5000 blue whales and 70,000 fin whales, what does this model predict about the future of the two species?

4. Repeat Exercise 3, but now assume a level of effort of $E = 6000$ boat-days per year.

5. Reconsider Exercise 7 of Chapter 4.

   (a) Sketch the vector field for this model. Determine the location of each equilibrium in the state space.

   (b) Use the eigenvalue method to test the stability of each equilibrium in the state space.

   (c) For each equilibrium point, determine the linear system that approximates the behavior of the original dynamical system in the neighborhood of the equilibrium point. Write the general solution to this linear system, and sketch the linear phase portrait.

   (d) Sketch the complete phase portrait for this model, using the results of parts (a) and (c).

(e) Suppose that an ecological disaster suddenly kills off 80% of the krill in the area, leaving 150,000 blue whales and only 100 tons per acre of krill. What does our model predict about the future of the whales and the krill?

6. Reconsider Exercise 9 of Chapter 4.

   (a) Use an eigenvalue method to determine the stability of the (P, Q) equilibrium, assuming a continuous time model.

   (b) Repeat part (a) assuming a discrete time model. To what do you attribute the difference in your results?

7. Reconsider the tree problem of Example 5.1. Assume $t = 1/2$.

   (a) Sketch the vector field for this model. Indicate the location of each equilibrium in the state space.

   (b) For each equilibrium point, determine the linear system that approximates the behavior of the original dynamical system in the neighborhood of the equilibrium point. Write the general solution to this linear system, and sketch the linear phase portrait.

   (c) Sketch the complete phase portrait for this model, using the results of parts (a) and (b).

   (d) Suppose that a small number of hardwood trees is introduced into a mature stand of softwood trees. What does our model predict about the future of this forest?

8. Reconsider the tree problem of Example 5.1, but now suppose that the strength of the competition factor is too great to allow the coexistence of both hardwood and softwood trees. Assume $t = 3/4$.

   (a) Sketch the vector field for this model. Indicate the location of each equilibrium in the state space.

   (b) For each equilibrium point, determine the linear system that approximates the behavior of the original dynamical system in the neighborhood of the equilibrium point. Write the general solution to this linear system, and sketch the linear phase portrait.

   (c) Sketch the complete phase portrait for this model, using the results of parts (a) and (b).

   (d) Suppose that a small number of hardwood trees is introduced into a mature stand of softwood trees. What does our model predict about the future of this forest?

9. Reconsider the RLC circuit problem of Example 5.3 and perform a sensitivity analysis on the parameter L, which gives the inductance of the capacitor. Assume that $L > 0$.

   (a) Describe the vector field for the general case $L > 0$.

(b) Determine the range of L for which the equilibrium $(0, 0)$ remains stable.

(c) Draw the phase portrait for the linear system in the case where there are two real eigenvalues.

(d) Use the results of parts (a) and (c) to draw the complete phase portrait. Comment on the sensitivity of our conclusions in Section 5.3 to the actual value of the parameter L.

10. Reconsider the RLC circuit problem of Example 5.3, but now suppose that the capacitance $C = 1/5$.

(a) Sketch the vector field for this model.

(b) Use the eigenvalue method to test the stability of the equilibrium at the origin.

(c) Determine the linear system that approximates the behavior of the original dynamical system in the neighborhood of the origin. Write the general solution to this linear system, and sketch the linear phase portrait.

(d) Sketch the complete phase portrait for this model, using the results of parts (a) and (c). How does the behavior of the RLC circuit change when the capacitance C is lowered?

11. (Continuation of Exercise 10) Reconsider the RLC circuit problem of Example 5.3, and now consider the effect of varying the capacitance C over the entire range $0 < C < \infty$.

(a) Sketch the phase portrait of the linear system that approximates the behavior of the RLC circuit in the neighborhood of the origin in the case $0 < C < 1/4$. Compare with the case $C > 1/4$, which was done in the text.

(b) Draw the complete phase portrait for the RLC circuit for the case $0 < C < 1/4$. Describe the changes that occur in the phase portrait as we transition between the two cases $0 < C < 1/4$ and $C > 1/4$.

(c) Draw the phase portrait of the linear system in the case $C = 1/4$ where there is only one eigenvalue. Sketch the phase portrait of the nonlinear system in this case. Explain how the phase portrait in this case represents an intermediate step between the case of two real distinct eigenvalues ($C > 1/4$) and one pair of complex conjugate eigenvalues ($0 < C < 1/4$).

(d) Reconsider the description of circuit behavior given in step 5 of Example 5.3 in the text. Describe in plain English the behavior of the RLC circuit in the more general case $C > 0$.

12. Reconsider the space docking problem of Example 5.2.

(a) Draw the vector field for this problem.

(b) Find the eigenvectors associated with the eigenvalues

$$\lambda = \frac{4 \pm \sqrt{6}}{10}$$

that were calculated in the text. Draw these eigenvectors on the picture from part (a). What do you notice about the vector field at these points?

(c) Calculate the rate at which closing velocity decreases (% per minute) if we start at an eigenvector.

(d) Generally speaking, what can you say about the rate of decrease in closing velocity for an arbitrary initial condition [Hint: Any (x_1, x_2) initial condition is a linear combination of the two eigenvectors found in part (b).]

13. (Continuation of Exercise 12) Reconsider the space docking problem of Example 5.2.

(a) As in Exercise 12, calculate the rate at which closing velocity decreases (% per minute) as a function of the control parameter k over the range of values $0 < k \leq 0.0268$.

(b) Find the value of k in part (a) that maximizes the rate of decrease in closing velocity.

(c) Explain the significance of your results in part (b) in terms of the efficiency of the control parameter k.

(d) Explain the problem with extending the approach used in this problem to find the most efficient value of k over the entire interval $0 < k < 0.2$ over which we have a stable control procedure.

14. Reconsider the space docking problem of Example 5.2, but now approximate the discrete time dynamical system

$$\begin{aligned}\Delta x_1 &= -kwx_1 - kcx_2\\ \Delta x_2 &= x_1 - x_2\end{aligned}$$

by its continuous time analogue

$$\begin{aligned}\frac{dx_1}{dt} &= -kwx_1 - kcx_2\\ \frac{dx_2}{dt} &= x_1 - x_2\end{aligned}$$

(a) Show that the continuous time model has a stable equilibrium at $(0, 0)$. Assume $w = 10, c = 5$, and $k = 0.02$.

(b) Solve the continuous model, using the method of eigenvalues and eigenvectors.

(c) Draw the complete phase portrait for this model.

(d) Comment on any differences between the behavior of the discrete and continuous models.

15. (Continuation of Exercise 14) Reconsider the docking problem of Example 5.2. As in Exercise 14, replace the discrete time model by its continuous time analogue.

    (a) Assume $w = 10$ and $c = 5$. For what values of k does the continuous model have a stable equilibrium at $(0, 0)$?

    (b) Solve the continuous model using the method of eigenvectors and eigenvalues.

    (c) Draw the complete phase portrait for this model. How does the phase portrait depend on k?

    (d) Comment on any differences between the continuous and discrete models.

16. Reconsider the tree problem of Example 5.1.

    (a) Can both types of trees coexist in equilibrium? Assume $b_i = a_i/2$. Use the five-step method, and model as a *discrete time* dynamical system with a time step of one year.

    (b) Use the eigenvalue test for discrete time dynamical systems to check the stability of the equilibrium you found in part (a).

    (c) Perform a sensitivity analysis on the parameter t, where $b_i = ta_i$. Determine the range of $0 < t < 0.6$ for which the equilibrium found in part (a) is stable.

    (d) Comment on any differences in results between the discrete time and continuous time models. As a practical matter, does it make any difference which we choose?

Further Reading

1. Beltrami, E. (1987) *Mathematics for Dynamic Modeling.* Academic Press, Orlando, Florida.

2. Frauenthal, J. (1979) *Introduction to Population Modeling.* UMAP Monograph.

3. Hirsch, M. and S. Smale (1974) *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press, New York.

4. Keller, M., *Electrical Circuits and Applications of Matrix Methods: Analysis of Linear Circuits.* UMAP modules 108 and 112.

5. Rescigno, A. and I. Richardson, The Struggle for Life I, Two Species. *Bulletin of Mathematical Biophysics*, vol. 29, pp. 377-388.

6. Smale, S. (1972) On the Mathematical Foundations of Circuit Theory. *Journal of Differential Geometry*, vol. 7, pp. 193-210.

7. Wilde, C., *The Contraction Mapping Principle.* UMAP module 326.