Chapter 4

INTRODUCTION TO DYNAMIC MODELS

Many problems of practical interest involve processes that evolve over time. Dynamic models are used to represent the changing behavior of these systems. Space flight, electrical circuits, chemical reactions, population growth, investments and annuities, military battles, the spread of disease, and pollution control are just a few of the many areas in which extensive use is made of dynamic models.

The five-step method and the fundamental principles of sensitivity analysis and robustness are as relevant and useful for dynamic models as they are for optimization models. We will continue to rely on them as we explore some of the most popular and generally applicable dynamic modeling techniques. In the course of this study, we will also introduce the important modeling concepts of state space, equilibrium, and stability. All of this will also be very useful in the last part of this book, where we explore stochastic models.

As a general rule, dynamic models are easy to formulate and hard to solve. Exact analytic solutions are available only for a few special cases, such as linear systems. Numerical methods usually do not provide a good qualitative understanding of system behavior. Therefore, the application of graphical techniques is usually employed as at least one part of the analysis of dynamic models. Because of the inherent simplicity of graphical techniques, along with their geometrical nature, this chapter also provides us with an ideal opportunity to introduce some of the deepest and most fundamental modeling concepts used for dynamic systems.

4.1 Steady State Analysis

In this section we will consider the simplest type of dynamic model. The mathematics required are elementary indeed. Even so, the practical applications for this model are numerous, and the absence of too much sophisticated technique

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leaves us free to concentrate on some of the most fundamental ideas of dynamic modeling.

Example 4.1. In an unmanaged tract of forest area, hardwood and softwood trees compete for the available land and water. The more desirable hardwood trees grow more slowly, but are more durable and produce more valuable timber. Softwood trees compete with the hardwoods by growing rapidly and consuming the available water and soil nutrients. Hardwoods compete by growing taller than the softwoods can and shading new seedlings. They are also more resistant to disease. Can these two types of trees coexist on one tract of forest land indefinitely, or will one type of tree drive the other to extinction?

We will use the five-step method. Let H and S denote the populations of hardwood and softwood trees, respectively. A convenient unit often used by biologists is the biomass (tons per acre of living tree). We need to make some assumptions about the dynamics of these two populations. To begin with, we want to make assumptions that are as simple as possible without neglecting the most fundamental aspects of the problem. Later on, we can improve or enrich our model if necessary. It is reasonable to assume that in conditions of unrestricted growth (plenty of room, sunshine, water, and soil nutrients), the growth rate of a species is roughly proportional to the size of the species. Twice as many trees give rise to twice as many little trees. As population increases, members of the same species must compete for resources, and this inhibits growth. Thus, it is reasonable to assume that growth rate is roughly linear in population size for small populations and then falls off as population increases. The simplest growth rate function with these properties is

$$g(P) = rP - aP^2.$$

Here r is the intrinsic growth rate, and $a << r$ is a measure of the strength of resource limitations. If a is smaller, there is more room to grow.

The effect of competition is also due to resource limitations. The presence of hardwood trees limits the amount of sunlight, water, etc., available for the softwoods, and vice versa. The loss in growth rate due to competition depends on the size of both populations. A simple assumption is that this loss is proportional to the product of the two. Given these assumptions about growth and competition, we wish to know whether we can expect one species to die out over time. Figure 4.1 summarizes the results of step 1.

Step 2 is to select the modeling approach. We will model this problem as a dynamic model in steady state.

We are given functions

$$\begin{gathered} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{gathered}$$

Variables:
H = hardwood population (tons/acre)
S = softwood population (tons/acre)
g_H = growth rate for hardwoods (tons/acre/year)
g_S = growth rate for softwoods (tons/acre/year)
c_H = loss due to competition for hardwoods (tons/acre/year)
c_S = loss due to competition for softwoods (tons/acre/year)

Assumptions:
$g_H = r_1 H - a_1 H^2$
$g_S = r_2 S - a_2 S^2$
$c_H = b_1 SH$
$c_S = b_2 SH$
$H \geq 0, S \geq 0$
r_1, r_2, a_1, a_2, b_1, b_2 are positive reals

Objective: Determine whether $H \to 0$ or $S \to 0$

Figure 4.1: Results of step 1 for the tree problem.

defined on a subset S of $\mathbb{R}^n$. The functions $f_1, \ldots, f_n$ represent the rate of change of each variable $x_1, \ldots, x_n$ respectively. A point $(x_1, \ldots, x_n)$ in the set S is called an *equilibrium point* provided that

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, \ldots, x_n) &= 0 \end{aligned} \tag{4.1}$$

at this point. The rate of change of each of the variables $x_1, \ldots, x_n$ is then equal to zero, and so the system is at rest.

The variables $x_1, \ldots, x_n$ are called *state variables*, and S is called the *state space*. Since the functions $f_1, \ldots, f_n$ depend only on the current state $(x_1, \ldots, x_n)$ of the system, knowledge of the current state suffices to determine the entire future of the system. What happened in the past does not matter. We only need to know where we are now, not how we got here. When we are at an equilibrium point, defined by Eq. (4.1), we say that the system is in *steady state*. At this point all of the rates of change are equal to zero. All of the forces acting on the system are in balance. For this reason the equations in (4.1) are sometimes referred to as the *balance equations*. When a dynamic system is in steady state, it remains there forever. Since all of the rates of change are equal to zero, any future time will find us in exactly the same place we are right now.

In order to find the equilibrium states of a dynamic system, we need to solve the n equations in n unknowns given by Eq. (4.1). In very easy cases we can solve by hand. Sometimes we can solve using a computer algebra system. All of the problems in this chapter, including the exercises, can be solved using these techniques. Of course, many real problems give rise to systems of equations that cannot be solved analytically. We will treat such problems in Chapter 6, where we discuss computational methods for dynamical systems. (Alternatively, we could use the multivariable version of Newton's method introduced in Chapter 3.)

Step 3 is to formulate the model. Let $x_1 = H$ and $x_2 = S$ denote our two state variables, defined on the state space

$$\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}.$$

The steady–state equations are

$$\begin{aligned} r_1 x_1 - a_1 x_1^2 - b_1 x_1 x_2 &= 0 \\ r_2 x_2 - a_2 x_2^2 - b_2 x_1 x_2 &= 0. \end{aligned} \tag{4.2}$$

We are interested in solutions of this system of equations that lie in the state space. These solutions represent the equilibrium points of our dynamic model.

Step 4 is to solve the model. Factoring out x_1 from the first equation and x_2 from the second, we find four solutions, three at the following coordinates:

$$\begin{aligned} &(0, 0) \\ &(0, r_2/a_2) \\ &(r_1/a_1, 0) \end{aligned}$$

and the fourth at the intersection of these two lines:

$$\begin{aligned} a_1 x_1 + b_1 x_2 &= r_1 \\ b_2 x_1 + a_2 x_2 &= r_2. \end{aligned}$$

See Fig. 4.2 for an illustration.

Solving by Cramer's rule yields

$$\begin{aligned} x_1 &= \frac{r_1 a_2 - r_2 b_1}{a_1 a_2 - b_1 b_2} \\ x_2 &= \frac{a_1 r_2 - b_2 r_1}{a_1 a_2 - b_1 b_2}. \end{aligned}$$

If the two lines do not cross inside the state space, then there are only three equilibria. In this case the two species of trees cannot coexist in peaceful equilibrium.

We are interested to know the conditions under which $x_1 > 0$ and $x_2 > 0$. It is reasonable to assume that $a_i > b_i$, since the effect of competition between members of the same species should be stronger than the competition between species. The growth rate is

$$r_i x_i - a_i x_i x_i - b_i x_i x_j,$$

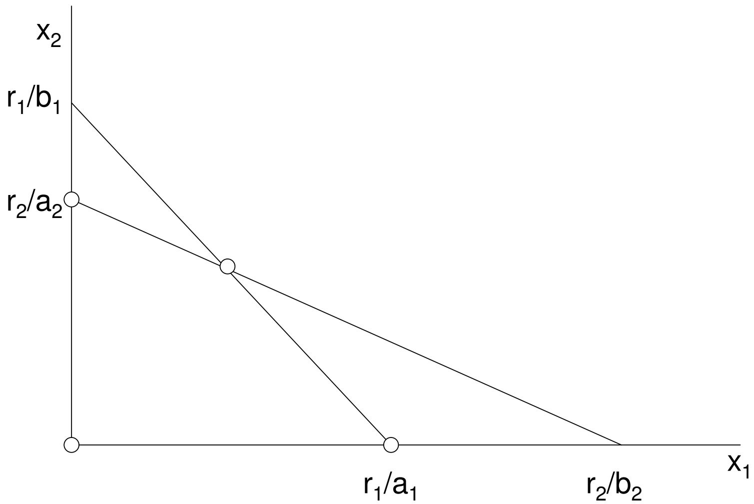

Figure 4.2: Graph of softwoods x_2 versus hardwoods x_1 showing equilibria for the tree problem.

where the first term represents unrestricted growth, the second represents the effect of competition within a population, and the third represents competition between populations. Since the two types of trees do not occupy exactly the same ecological niche, we would suppose that for $x_i = x_j$ the effect of competition within a population would be the stronger. Hence $a_i > b_i$, so

$$a_1 a_2 - b_1 b_2 > 0.$$

The condition for coexistence is, therefore, that

$$\begin{aligned} r_1 a_2 - r_2 b_1 &> 0 \\ a_1 r_2 - b_2 r_1 &> 0, \end{aligned}$$

or, in other words,

$$\frac{r_2}{a_2} < \frac{r_1}{b_1} \text{ and } \frac{r_1}{a_1} < \frac{r_2}{b_2},$$

as shown in Fig. 4.2.

Step 5 is to report in plain English the results obtained from our model analysis. It is difficult to do so in this case, because our answer is qualified, and the qualifications involve unknown parameters. In order to communicate our results clearly, we would like to find a more tangible interpretation of our conditions for coexistence. Let us reexamine our model formulation to see if we can interpret the meaning of the ratios r_i/a_i and r_i/b_i in some straightforward way.

The parameters r_i measure growth tendency, and the parameters a_i and b_i measure the strength of competition within and between populations, respectively. Thus, the ratios r_i/a_i and r_i/b_i must measure the relative strength of growth versus competition. Let us try to go further. In the absence of competition between species, the growth rate is

$$r_i x_i - a_i x_i^2 = x_i(r_i - a_i x).$$

The ratio r_i/a_i represents the equilibrium population level in the absence of competition between species, or the level at which the population will stop growing of its own accord. Similarly, if we neglect the factor of competition within a population, the net growth rate is

$$r_i x_i - b_i x_i x_j = x_i(r_i - b_i x_j).$$

The ratio r_i/b_i thus represents the level of population j necessary to put an end to growth of population i. In light of this, we can now give our analysis results the following concrete interpretation.

For each type of tree (hardwood and softwood), there are two kinds of limits to growth. The first comes from competition with the other type of tree, and the second comes from competition between trees of the same type under crowded conditions. Thus, for each type of tree there is one point where growth will halt itself due to crowding, and another point where the growth of one type of tree will halt the growth of the other type due to competition. The condition for coexistence of both types is that each type reaches the point where it limits its own growth before it reaches the point where it limits the other's growth.

The steady–state analysis of this section leaves one important question unanswered. Given that a dynamic model has an equilibrium solution, will we ever get there? The answer depends on the dynamics of the model. An equilibrium point

$$x_0 = (x_1^0, \ldots, x_n^0)$$

is said to be *asymptotically stable* (or just *stable*) if whenever the state variables

$$(x_1(t), \ldots, x_n(t))$$

pass sufficiently close to x_0, they are drawn into the equilibrium. In other words,

$$(x_1(t), \ldots, x_n(t)) \to x_0.$$

Steady–state analysis cannot answer the question of stability, so we will have to defer further discussion of this topic to the next section.

4.2 Dynamical Systems

Dynamical system models are the most commonly used type of dynamic model. In a dynamical system model the forces of change are represented by differential equations. In this section we will focus on the graphical method for obtaining qualitative information about a dynamical system. The emphasis will be on questions of stability.

Variables:	B = number of blue whales
	F = number of fin whales
	g_B = growth rate of blue whale population (per year)
	g_F = growth rate of fin whale population (per year)
	c_B = effect of competition on blue whales (whales per year)
	c_F = effect of competition on fin whales (whales per year)
Assumptions:	$g_B = 0.05B(1 - B/150,000)$
	$g_F = 0.08F(1 - F/400,000)$
	$c_B = c_F = \alpha BF$
	$B \geq 0, F \geq 0$
	α is a positive real constant
Objective:	Determine whether dynamic system can reach stable equilibrium starting from $B = 5,000$, $F = 70,000$

Figure 4.3: Results of step 1 for the whale problem.

Example 4.2. The blue whale and fin whale are two similar species that inhabit the same areas. Hence, they are thought to compete. The intrinsic growth rate of each species is estimated at 5% per year for the blue whale and 8% per year for the fin whale. The environmental carrying capacity (the maximum number of whales that the environment can support) is estimated at 150,000 blues and 400,000 fins. The extent to which the whales compete is unknown. In the last 100 years intense harvesting has reduced the whale population to around 5,000 blues and 70,000 fins. Will the blue whale become extinct?

We will use the five-step method. Notice that this problem is very similar to Example 4.1. Step 1 is to ask a question. We will use the number of blue and fin whales as state variables and make the simplest possible assumptions about growth and competition. The question we begin with is this: Can the two populations of whales grow to stable equilibrium starting from their current levels? The results of step 1 are summarized in Figure 4.3.

Step 2 is to select the modeling approach. We will model this problem as a dynamical system.

> A *dynamical system* consists of n state variables $(x_1, \ldots, x_n)$ and a system of differential equations
>
> $$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, \ldots, x_n) \\ \vdots \quad & \quad \vdots \\ \frac{dx_n}{dt} &= f_n(x_1, \ldots, x_n) \end{aligned} \tag{4.3}$$
>
> defined on the state space $(x_1, \ldots, x_n) \in S$, where S is a subset of $\mathbb{R}^n$. The existence and uniqueness theorem of differential equations

states that if $f_1, \ldots, f_n$ have continuous first partial derivatives in a neighborhood of a point

$$x_0 = (x_1^0, \ldots, x_n^0),$$

then there exists a unique solution to this system of differential equations through this initial condition. See any introductory text on differential equations for details (e.g., Hirsch, et al. (1974) p. 162). Many other differential equation models can be reduced to the form (4.3). If the dynamics depend on time, we can introduce time as another state variable. If second derivatives are involved, we can include first derivatives as state variables, and so forth.

It is best to think of a solution to a dynamical system as a path through the state space. As long as differentiability assumptions are satisfied, there is a path through each point, and paths cannot cross except at an equilibrium. Let

$$\begin{aligned} x &= (x_1, \ldots, x_n) \\ F(x) &= (f_1(x), \ldots, f_n(x)). \end{aligned}$$

Then the dynamical system equation is

$$\frac{dx}{dt} = F(x). \tag{4.4}$$

For a path $x(t)$, the derivative dx/dt represents the velocity vector. Hence, for every solution curve $x(t)$, we have that $F(x(t))$ is the velocity vector at each point. The vector field $F(x)$ tells us in what direction and how fast we are moving through the state space. Usually, a good idea of the qualitative behavior of a dynamical system in two variables can be obtained by drawing the vector field at selected points. The points where $F(x) = 0$ are the equilibria, and we will pay special attention to the vector field nearby these points.

Step 3 is to formulate the model. Let $x_1 = B$ and $x_2 = F$, and write

$$\begin{aligned} x_1' &= f_1(x_1, x_2) \\ x_2' &= f_2(x_1, x_2), \end{aligned}$$

where

$$\begin{aligned} f_1(x_1, x_2) &= 0.05x_1\left(1 - \frac{x_1}{150{,}000}\right) - \alpha x_1 x_2 \\ f_2(x_1, x_2) &= 0.08x_2\left(1 - \frac{x_2}{400{,}000}\right) - \alpha x_1 x_2. \end{aligned} \tag{4.5}$$

The state space is

$$S = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}.$$

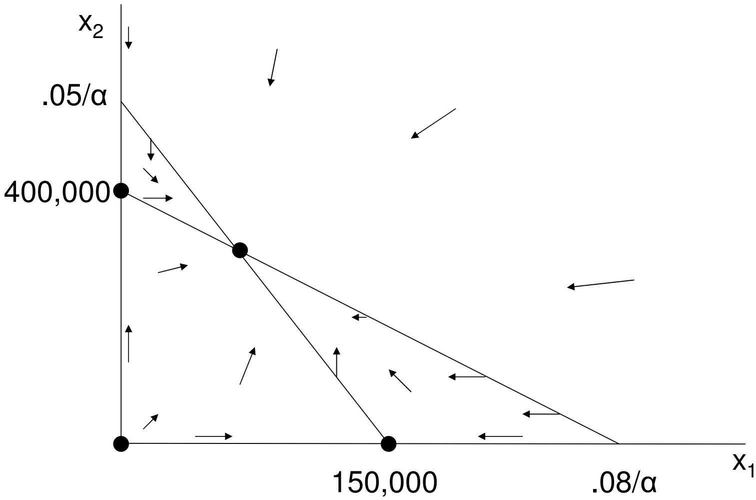

Figure 4.4: Graph of fin whales x_2 versus blue whales x_1 showing the vector field (4.5) for the whale problem.

Step 4 is to solve the model. We want to sketch a graph of the vector field for this problem. Start out by sketching the level sets $f_1 = 0$ and $f_2 = 0$. The equilibria will be at the intersection of these two. Furthermore, the velocity vectors will be vertical along $f_1 = 0$ ($x_1' = 0$) and horizontal along $f_2 = 0$ ($x_2' = 0$). Draw the velocity vectors along these two curves. Then fill in some of the velocity vectors in between. It helps to remember that (as long as $F(x)$ is continuous, which it usually is) both the length and direction of the vectors change continuously. In fact, for this kind of analysis, the length of the velocity vectors is not very important. See Figure 4.4 for the finished graph.

There are four equilibrium solutions, three at

$$\begin{aligned} &(0,\ 0) \\ &(150{,}000,\ 0) \\ &(0,\ 400{,}000) \end{aligned} \tag{4.6}$$

and another at a point whose coordinates depend on α. Our graph assumes that

$$400{,}000 < (0.05/\alpha).$$

In this case it is easy to see that the equilibrium in the interior is the only stable one. In fact, any solution through a point in the interior of the state space will eventually converge upon this equilibrium. In particular, the solution with initial conditions $x_1(0) = 5{,}000$ and $x_2(0) = 70{,}000$ tends to this equilibrium as $t \to \infty$.

Step 5 is to summarize the results of our model analysis in nonmathematical terms. Based on our analysis, in the absence of further harvesting, the whale populations will grow back to their natural levels, and the ecological system will remain in stable equilibrium.

Of course, our conclusions are based on some rather broad assumptions. For example, we have assumed that the effect of competition is relatively small. If it were larger (i.e., if $(0.05/\alpha) < 400,000$), then the two species could not coexist. It does seem reasonable to make the assumption that α is small, since we know that the two species have coexisted for a long time before we began to harvest them. We have also made several simplifying assumptions about the growth process. The most critical of these is that for very small population levels, the population still tends to increase at the intrinsic growth rate. It is believed that some species have a minimum size (called the *minimum viable population level*) below which the growth rate is negative, ensuring the eventual extinction of the species. This assumption would, of course, change the behavior of our dynamical system. See Exercise 5 at the end of this chapter.

Finally, we address the questions of sensitivity analysis and robustness. First let us consider the sensitivity to the parameter α, for which we have very little information. For any value of $\alpha < 1.25 \times 10^{-7}$ there is a stable equilibrium $x_1 > 0,\ x_2 > 0$ at

$$\begin{aligned} x_1 &= \frac{150,000(8,000,000\alpha - 1)}{D} \\ x_2 &= \frac{400,000(1,875,000\alpha - 1)}{D}, \end{aligned} \tag{4.7}$$

where

$$D = 15,000,000,000,000\alpha^2 - 1,$$

which we found by Cramer's Rule. For example, if $\alpha = 10^{-7}$, then

$$\begin{aligned} x_1 &= \frac{600,000}{17} \approx 35,294 \\ x_2 &= \frac{6,500,000}{17} \approx 382,353. \end{aligned} \tag{4.8}$$

The sensitivities at this point are

$$S(x_1,\, \alpha) = -\frac{21,882,352,927}{6,000,000,000} \approx -3.6$$

and

$$S(x_2,\, \alpha) = \frac{27}{221} \approx 0.122.$$

The above calculations could be performed either by hand or by using a computer algebra system. Figure 4.5 illustrates the computation of the sensitivity $S(x_1,\, \alpha)$ using the computer algebra system Maple.

```
> e1:=(5/100)*(1-x1/150000)-alpha*x2;
```

$$e1 := \frac{1}{20} - \frac{1}{3000000}x1 - \alpha\,x2$$

```
> e2:=(8/100)*(1-x2/400000)-alpha*x1;
```

$$e2 := \frac{2}{25} - \frac{1}{5000000}x2 - \alpha\,x1$$

```
> s:=solve({e1=0,e2=0},{x1,x2});
```

$$s := \left\{ x2 = \frac{400000\,(-1 + 1875000\,\alpha)}{-1 + 15000000000000\,\alpha^2},\ x1 = \frac{150000\,(-1 + 8000000\,\alpha)}{-1 + 15000000000000\,\alpha^2} \right\}$$

```
> assign(s);
> dx1dalpha:=diff(x1,alpha);
```

$$dx1dalpha := \frac{1200000000000}{-1 + 15000000000000\,\alpha^2} - \frac{4500000000000000000\,(-1 + 8000000\,\alpha)\,\alpha}{(-1 + 15000000000000\,\alpha^2)^2}$$

```
> assign(alpha=10^(-7));
> sx1alpha:=dx1dalpha*(alpha/x1);
```

$$sx1alpha := \frac{-62}{17}$$

```
> evalf(sx1alpha);
```

$$-3.647058824$$

Figure 4.5: Calculation of the sensitivity $S(x_1, \alpha)$ for the whale problem using the computer algebra system Maple.

The blue whale population is most sensitive to α. If $\alpha = 10^{-8}$, then

$$x_1 = \frac{276,000,000}{1,997} \approx 138,207$$
$$x_2 = \frac{785,000,000}{1,997} \approx 393,090.$$

Of course we will always have $x_1 < 150,000$ and $x_2 < 400,000$, as is apparent from the graph in Fig. 4.4. But the most important features of this equilibrium are not its coordinates, but rather the fact that an equilibrium exists on $x_1 > 0$, $x_2 > 0$ and is stable. These conclusions remain valid over the entire range of $\alpha < 1.25 \times 10^{-7}$, which we believe to be plausible. Hence, we should say that our main conclusion is not at all sensitive to α. Similarly, our main conclusion is not at all sensitive to our data on intrinsic growth rate and carrying capacity, or even to the current whale populations.

Deeper questions of robustness revolve around the assumed form of the functions f_1 and f_2. We assumed that x_1'/x_1 and x_2'/x_2 are linear functions of x_1 and x_2, respectively. These lines represent the point where one species or the other stops growing. Suppose that we relax this linearity assumption. Let

$$x_1' = x_1\, g_1(x_1, x_2)$$
$$x_2' = x_2\, g_2(x_1, x_2).$$

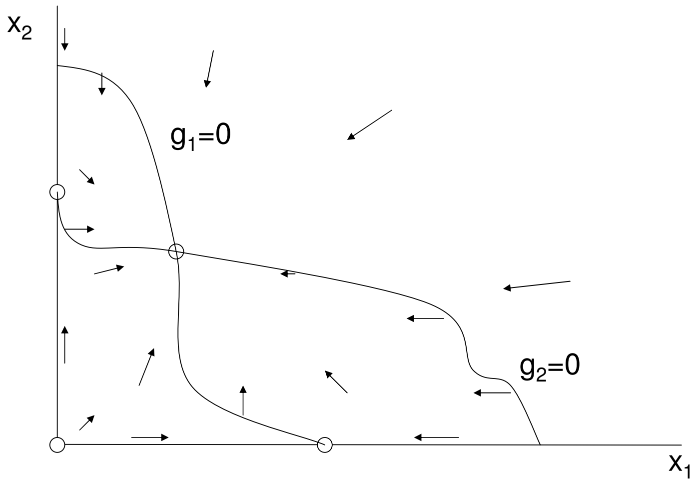

Figure 4.6: Graph of fin whales x_2 versus blue whales x_1 showing vector field for the generalized whale problem.

All of our analysis results would certainly remain true if g_1 and g_2 were not linear, so long as the vector field had the same general features. See Figure 4.6 for an illustration.

4.3 Discrete Time Dynamical Systems

In some problems it is most natural to model the time variable as being discrete. When this happens, the usual differential equations are replaced by their discrete–time analog: difference equations. The relationship between discrete and continuous dynamics is the relationship between $\Delta x/\Delta t$ and dx/dt, so it is often assumed that the behavior of a dynamical system will be roughly the same whether we assume that time is continuous or discrete. However, this kind of logic overlooks one important point. There is a kind of time delay built into every discrete–time dynamical system, which is the length of the time step Δt. For systems in which the dynamic forces are very strong, this time delay can lead to unexpected results.

Example 4.3. Astronauts in training are required to practice a docking maneuver under manual control. As a part of this maneuver, it is required to bring an orbiting spacecraft to rest relative to another orbiting craft. The hand controls provide for variable acceleration and deceleration, and there is a device on board that measures the rate of closing between the two vehicles. The following strategy has been proposed for bringing the craft to rest. First, look at the closing velocity. If it is zero, we are done. Otherwise, remember the closing velocity and look at the acceleration control. Move the acceleration control so that it is opposite to the closing velocity (i.e., if closing velocity is positive, we slow down, and we speed up if it is negative) and proportional in magnitude (i.e., we brake twice as hard if we find ourselves closing twice as fast). After a time, look at the closing velocity again and repeat the procedure. Under what circumstances will this strategy be effective?

We will use the five-step method. Let v_n denote the closing velocity observed at time t_n, the time of the nth observation. Let

$$\Delta v_n = v_{n+1} - v_n$$

denote the change in closing velocity as a result of our adjustments. We will denote the time between observations of the velocity indicator by

$$\Delta t_n = t_{n+1} - t_n.$$

This time interval naturally divides into two parts: the time it takes to adjust the velocity controls and the time between adjustment and the next observation of the velocity indicator. Write

$$\Delta t_n = c_n + w_n,$$

where c_n is the time to adjust the controls and w_n is the waiting time until the next observation. The parameter c_n is a function of astronaut response time, and we are free to choose w_n.

Let a_n denote the acceleration setting after the nth adjustment. Elementary physics yields

$$\Delta v_n = a_{n-1}c_n + a_n w_n.$$

The control law says to set acceleration proportional to $(-v_n)$; hence

$$a_n = -kv_n.$$

The results of step 1 are summarized in Figure 4.7.

Step 2 is to select the modeling approach. We will model this problem as a discrete–time dynamical system.

> A discrete–time dynamical system consists of a number of state variables $(x_1, \ldots, x_n)$ defined on the state space $S \subseteq \mathbb{R}^n$ and a system of difference equations
>
> $$\begin{aligned} \Delta x_1 &= f_1(x_1, \ldots, x_n) \\ \vdots & \qquad \vdots \\ \Delta x_n &= f_n(x_1, \ldots, x_n). \end{aligned} \tag{4.9}$$

Variables:

t_n = time of nth velocity observation (sec)
v_n = velocity at time t_n (m/sec)
c_n = time to make nth control adjustment (sec)
a_n = acceleration after nth adjustment (m/sec^2)
w_n = wait before $(n+1)$th observation (sec)

Assumptions:

$t_{n+1} = t_n + c_n + w_n$
$v_{n+1} = v_n + a_{n-1}c_n + a_n w_n$
$a_n = -kv_n$
$c_n > 0$
$w_n \geq 0$

Objective: Determine whether $v_n \to 0$

Figure 4.7: Results of step 1 of the docking problem.

Here Δx_n represents the change in x_n over one time step. It is common to take time steps of length 1, which just amounts to selecting appropriate units. If time steps are of variable length, or if the dynamics of the system vary over time, then we include time as a state variable. If we let

$$
\begin{aligned}
x &= (x_1, \ldots, x_n) \\
F &= (f_1, \ldots, f_n),
\end{aligned}
$$

then the equations of motion can be written in the form

$$\Delta x = F(x).$$

A solution to this difference equation model is a sequence of points

$$x(0), x(1), x(2), \ldots$$

in the state space with

$$
\begin{aligned}
\Delta x(n) &= x(n+1) - x(n) \\
&= F(x(n))
\end{aligned}
$$

for all n. An equilibrium point x_0 is characterized by

$$F(x_0) = 0,$$

and the equilibrium is stable if

$$x(n) \to x_0$$

whenever $x(0)$ is sufficiently close to x_0. As in the continuous time case, many other difference equation models can be reduced to the form (4.9) by introducing additional state variables.

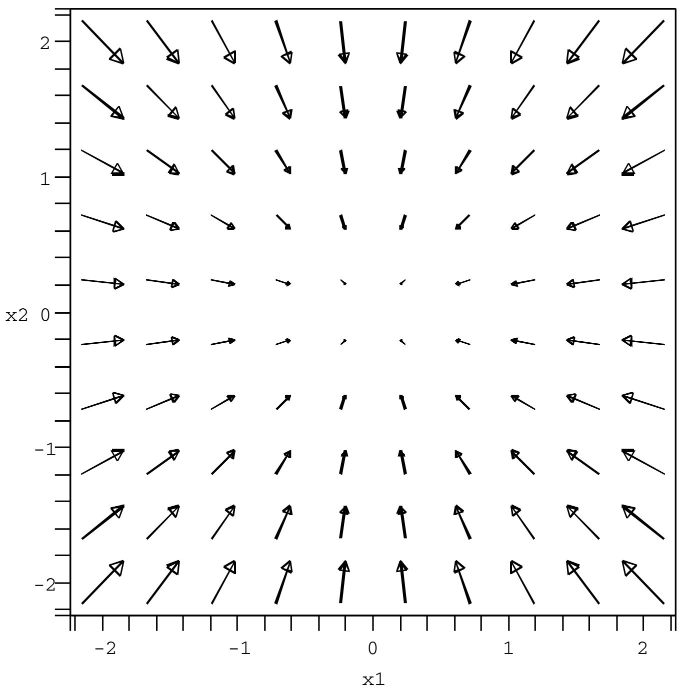

Figure 4.8: Vector field for Example 4.4.

Think of a solution as a sequence of points in the state space. The vector $F(x(n))$ connects the point $x(n)$ to the point $x(n+1)$. A graph of the vector field $F(x)$ can reveal much about the behavior of a discrete–time dynamical system.

Example 4.4. Let $x = (x_1, x_2)$, and consider the difference equation

$$\Delta x = -\lambda x, \tag{4.10}$$

where $\lambda > 0$. What is the behavior of solutions near the equilibrium point $x_0 = (0, 0)$?

Figure 4.8 shows a graph of the vector field $F(x) = -\lambda x$ in the case where $0 < \lambda < 1$. It is clear that $x_0 = (0, 0)$ is a stable equilibrium. Each step moves closer to x_0. Now let us consider what happens when λ becomes larger. Each of the vectors in Fig. 4.8 will stretch as λ increases. For $\lambda > 1$ the vectors are so long that they overshoot the equilibrium. For $\lambda > 2$ they are so long that the terminal point $x(n+1)$ is actually farther away from $(0, 0)$ than the starting point $x(n)$. In this case x_0 is an unstable equilibrium.

This simple example clearly illustrates the fact that discrete–time dynamical systems do not always behave like their continuous–time analogs. Solutions to the differential equation

$$\frac{dx}{dt} = -\lambda x \tag{4.11}$$

are all of the form

$$x(t) = x(0)e^{-\lambda t},$$

and the origin is a stable equilibrium regardless of $\lambda > 0$. The difference in behavior for the analogous difference equation in Eq. (4.10) is due to the inherent time delay. The approximation

$$\frac{dx}{dt} \approx \frac{\Delta x}{\Delta t}$$

is only valid for small Δt, where the term *small* depends on the sensitivity of x to t. It should only be relied upon in cases where Δx represents a small relative change in x. When this is not the case, the difference in behavior between discrete and continuous systems can be dramatic.

We return now to the docking problem of Example 4.3. Step 3 of the five-step method is to formulate the model. We are modeling the docking problem as a discrete–time dynamical system. From Fig. 4.7 we obtain

$$(v_{n+1} - v_n) = -k\, v_{n-1}\, c_n - k\, v_n w_n.$$

Hence, the change in velocity over the nth time step depends on both v_n and v_{n-1}. To simplify the analysis, let us assume that $c_n = c$ and $w_n = w$ for all n. Then the length of each time step is

$$\Delta t = c + w$$

seconds, and we do not need to include time as a state variable. We do, however, need to include both v_n and v_{n-1}. Let

$$\begin{aligned} x_1(n) &= v_n \\ x_2(n) &= v_{n-1}. \end{aligned}$$

Compute

$$\begin{aligned} \Delta x_1 &= -kwx_1 - kcx_2 \\ \Delta x_2 &= x_1 - x_2. \end{aligned} \tag{4.12}$$

The state space is $(x_1, x_2) \in \mathbb{R}^2$.

Step 4 is to solve the model. There is one equilibrium point (0, 0) found at the intersection of the two lines

$$\begin{aligned} kwx_1 + kcx_2 &= 0 \\ x_1 - x_2 &= 0. \end{aligned}$$

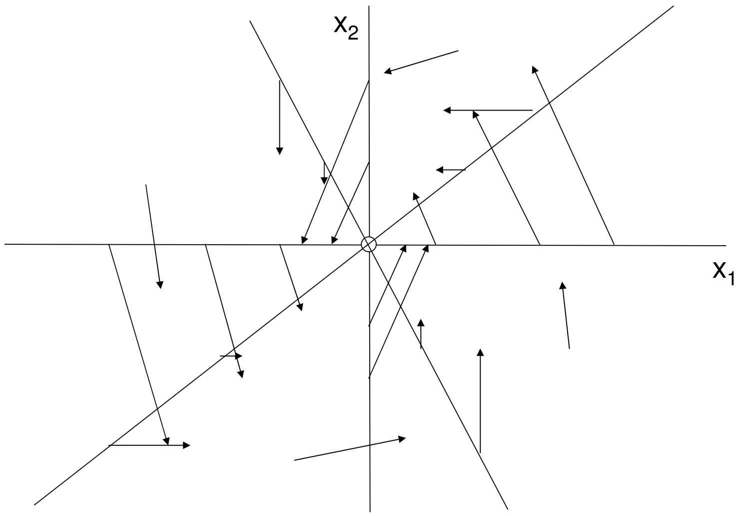

Figure 4.9: Graph of previous velocity x_2 versus current velocity x_1 showing vector field for the docking problem.

These are the steady–state equations obtained by setting $\Delta x_1 = 0$ and $\Delta x_2 = 0$.

Figure 4.9 shows a graph of the vector field

$$F(x) = (-kwx_1 - kcx_2,\, x_1 - x_2).$$

It appears as though solutions will tend toward equilibrium, but it is hard to be sure. If $k, c,$ and w are large, then the equilibrium is probably unstable, but once again it is difficult to tell.

In mathematics we often come across problems we cannot solve. Usually the best thing to do in such cases is to review our assumptions and consider whether we can reduce the problem to one that we can solve by making a further simplifying assumption. Of course this would be a meaningless and trivial exercise unless the simplified problem had some real significance.

In our docking problem we have expressed the change in velocity Δv_n as the sum of two components. One represents the change in velocity occurring between the time we read the velocity indicator and the time we adjust the acceleration controls. Suppose that we can do this very quickly. In particular, suppose that c is very much smaller than w. If v_n and v_{n-1} are not too different, the approximation

$$\Delta v_n \approx -kwv_n$$

should be reasonably accurate. The difference equation

$$\Delta x_1 = -kwx_1$$

is a familiar one from Example 4.4, and we know that we will get a stable equilibrium for any $kw < 2$. If $kw < 1$, we will approach the equilibrium asymptotically without overshooting.

Step 5 is to answer the analysis question in plain English. Maybe we should just say in plain English that we don't know the answer. However, we probably can do better than that. Let us report that a completely satisfactory solution is not obtainable by elementary graphical methods. In other words, it will take more work using more sophisticated methods to determine exactly the conditions under which the proposed control strategy will work. It does seem that the strategy will be effective in most cases as long as the time interval between control adjustments is not too long and the magnitude of those adjustments is not too large. The problem is complicated by the fact that there is a time delay between reading the velocity indicator and adjusting the controls. Since the actual closing velocity may change during this interval, we are acting on dated and inaccurate information. This adds an element of uncertainty to our calculations. If we ignore the effects of this time delay (which may be permissible if the delay is small), we can then draw some general conclusions, which are as follows.

The control strategy will work so long as the control adjustments are not too violent. Furthermore, the longer the interval between adjustments, the lighter those adjustments must be. In addition, the relationship is one of proportion. If we go twice as long between adjustments, we can only use half as much control. To be specific, if we adjust the controls once every 10 seconds, then we can only set the acceleration controls at 1/10 of the velocity setting to avoid overshooting the target velocity of zero. In order to allow for human and equipment error, we should actually set the controls somewhat lower, say 1/15 or 1/20 of velocity. More frequent adjustments require more frequent observations of the closing velocity indicator and more concentration on the part of the operator, but they do allow for the successful administration of more thrusting power under control. Presumably, this would be advantageous.

Normally, we would conclude our discussion of this problem with a fairly comprehensive sensitivity analysis. In view of the fact that we have not yet found a way to solve this problem, we will defer that discussion to a later chapter.

4.4 Exercises

1. Reconsider the tree problem of Example 4.1. Assume that

$$\frac{r_2}{a_2} < \frac{r_1}{b_1} \text{ and } \frac{r_1}{a_1} < \frac{r_2}{b_2}$$

so that the situation is as pictured in Fig. 4.2.

   (a) Draw the vector field for this model.

   (b) Classify each of the four equilibrium points as stable or unstable.

(c) Can the two species of trees coexist in stable equilibrium?

(d) Suppose that a logging operation removes all but a few of the valuable hardwood trees in this stand of forest. What does this model predict about the future of the two species of trees?

2. Reconsider the tree problem of Example 4.1, but now assume that

$$\frac{r_2}{a_2} < \frac{r_1}{b_1} \text{ and } \frac{r_1}{a_1} \geq \frac{r_2}{b_2}.$$

(a) Locate each of the equilibrium points (x_1, x_2) in the state space $x_1 \geq 0$, $x_2 \geq 0$.

(b) Draw the vector field for this case.

(c) Classify each equilibrium as stable or unstable.

(d) Suppose that we start out with an equal amount of hardwood and softwood trees. What does this model predict about the future of the two species?

3. Repeat Exercise 2, but now assume that

$$\frac{r_2}{a_2} \geq \frac{r_1}{b_1} \text{ and } \frac{r_1}{a_1} < \frac{r_2}{b_2}.$$

4. In the whale problem of Example 4.2 we used a logistic model of population growth, where the growth rate of population P in the absence of interspecies competition is

$$g(P) = rP\left(1 - \frac{P}{K}\right).$$

In this problem we will be using the simpler growth model

$$g(P) = rP.$$

(a) Can both species of whales coexist? Use the five-step method, and model as a dynamical system in steady state.

(b) Draw the vector field for this model. Indicate the location of each equilibrium point.

(c) Classify each equilibrium point in the state space as stable or unstable.

(d) Suppose that there are currently 5,000 blue whales and 70,000 fin whales. What does this model predict about the future of the two species?

5. In the whale problem of Example 4.2 we used a logistic model of population growth, where the growth rate of population P in the absence of interspecies competition is

$$g(P) = rP\left(1 - \frac{P}{K}\right).$$

In this problem we will be using a more complex model,

$$g(P) = rP\left(\frac{P-c}{P+c}\right)\left(1-\frac{P}{K}\right),$$

in which the parameter c represents a minimum viable population level below which the growth rate is negative. Assume that $\alpha = 10^{-8}$ and that the minimum viable population level is 3,000 for blue whales and 15,000 for fin whales.

(a) Can the two species of whales coexist? Use the five-step method, and model as a dynamical system in steady state.

(b) Sketch the vector field for this model. Classify each equilibrium point as stable or unstable.

(c) Assuming that there are currently 5,000 blue whales and 70,000 fin whales, what does this model predict about the future of the two populations?

(d) Suppose that we have underestimated the minimum viable population for the blue whale, and that it is actually closer to 10,000. Now what happens to the two species?

6. Reconsider the whale problem of Example 4.2, and assume that $\alpha = 10^{-8}$. In this problem we will investigate the effects of harvesting on the two whale populations. Assume that a level of effort E boat–days will result in the annual harvest of qEx_1 blue whales and qEx_2 fin whales, where the parameter q (catchability) is assumed to equal approximately 10^{-5}.

   (a) Under what conditions can both species continue to coexist in the presence of harvesting? Use the five-step method, and model as a dynamical system in steady state.

   (b) Draw the vector field for this problem, assuming that the conditions identified in part (a) are satisfied.

   (c) Find the minimum level of effort required to reduce the fin whale population to its current level of around 70,000 whales. Assume that we started out with 150,000 blue whales and 400,000 fin whales before mankind began to harvest them.

   (d) Describe what would happen to the two populations if harvesting were allowed to continue at the level of effort identified in part (c). Draw the vector field in this case. This is the situation which led the IWC to call for an international ban on whaling.

7. One of the favorite foods of the blue whale is called krill. These tiny shrimp–like creatures are devoured in massive amounts to provide the principal food source for the huge whales. The maximum sustainable population for krill is 500 tons/acre. In the absence of predators, in uncrowded

conditions, the krill population grows at a rate of 25% per year. The presence of 500 tons/acre of krill increases blue whale population growth rate by 2% per year, and the presence of 150,000 blue whales decreases krill growth rate by 10% per year.

(a) Determine whether the whales and the krill can coexist in equilibrium. Use the five-step method, and model as a dynamical system in steady state.

(b) Draw the vector field for this problem. Classify each equilibrium point in the state space as stable or unstable.

(c) Describe what happens to the two populations over time. Assume that we start off with 5,000 blue whales and 750 tons/acre of krill.

(d) How sensitive are your conclusions in part (c) to the assumption of a 25% growth rate per year for krill?

8. Two armies are to engage in battle. The red army enjoys a three–to–one numerical superiority, but the blue army is better trained and better equipped. Let R and B denote the force levels of red and blue forces. The Lanchester model of combat states that

$$\begin{aligned} R' &= -aB - bRB \\ B' &= -cR - dRB, \end{aligned}$$

where the first term accounts for direct fire (aimed at a specific target) and the second term accounts for attrition due to area fire (e.g., artillery). We are assuming that weapon effectiveness is higher for blue than for red; i.e., $a > c$ and $b > d$. But what kind of edge in weapon effectiveness would be necessary to counteract a $3:1$ numerical superiority?

(a) Assuming that $a = \lambda c$ and $b = \lambda d$ for some $\lambda > 1$, determine the approximate lower bound on λ necessary for blue to win the war. Use the five-step method, and model as a dynamical system.

(b) In part (a) you assumed red had an $n:1$ numerical superiority. Discuss the sensitivity of your results in part (a) to the parameter $n \in (2, 5)$.

9. The following simple model is intended to represent the dynamics of supply and demand. Let P denote the selling price of a certain product and Q the quantity of this product being produced. The supply curve $Q = f(P)$ tells how much should be produced at a given price to maximize profit. The demand curve $P = g(Q)$ tells what price buyers should pay given a certain level of production in order to maximize their utility.

(a) Select a specific product and make an educated guess as to the form of the supply curve $Q = f(P)$ and the demand curve $P = g(Q)$.

(b) Use the results of part (a) to determine the equilibrium levels of P and Q.

(c) Formulate a dynamic model based on the assumption that P will tend toward the level dictated by the demand curve, while Q will tend to the level given by the supply curve.

(d) According to your model, is the (P, Q) equilibrium stable? Does it matter whether you assume a discrete–time or continuous–time model? (Economists usually assume a discrete–time model in order to represent the effect of a time delay.)

(e) Perform a sensitivity analysis for the assumptions you made in part (a). Consider the question of stability.

10. A population of 100,000 members is subject to a disease that is seldom fatal and leaves the victim immune to future infections by this disease. Infection can only occur when a susceptible person comes in direct contact with an infectious person. The infectious period lasts approximately three weeks. Last week, there were 18 new cases of the disease reported. This week, there were 40 new cases. It is estimated that 30% of the population is immune due to previous exposure.

    (a) What is the eventual number of people who will become infected? Use the five-step method, and model as a discrete–time dynamical system.

    (b) Estimate the maximum number of new cases in any one week.

    (c) Conduct a sensitivity analysis to investigate the effect of any assumptions you made in part (a) that were not supported by hard data.

    (d) Perform a sensitivity analysis for the number of cases (18) reported last week. It is thought by some that in early weeks the epidemic might be underreported.

11. Reconsider the docking problem of Example 4.3, and now assume that $c = 5$ sec, $w = 10$ sec, and $k = 0.02$.

    (a) Assuming an initial closing velocity of 50 m/sec, calculate the sequence of velocity observations v_0, v_1, v_2, ..., predicted by the model. Is the docking procedure successful?

    (b) An easier way to compute the solution in part (a) is to use the iteration function $G(x) = x + F(x)$, with the property that $x(n+1) = G(x(n))$. Compute the iteration function for this problem, and use it to repeat the calculation in part (a).

    (c) Calculate the solution $x(1)$, $x(2)$, $x(3)$, ..., starting at $x(0) = (1, 0)$. Repeat, starting at $x(0) = (0, 1)$. What happens as $n \to \infty$? What does this imply about the stability of the equilibrium $(0, 0)$? [Hint: Every possible initial condition $x(0) = (a, b)$ can be written as a linear combination of the vectors $(1, 0)$ and $(0, 1)$, and $G(x)$ is a linear function of x.]

    (d) Are there any states x for which $G(x) = \lambda x$ for some real λ? If so, what happens to the system if we start with this initial condition?

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