Homework 1. Probs 1-4

1. Show by Induction
    a. Show $5^{2n} + 3n - 1$ is divisible by 9
        i. Hypothesis: $5^{2n} + 3n - 1$ is divisible by 9, n is integer
        ii. Base case: $n = 1 \Rightarrow 5^2 + 3 - 1 = 25 + 2 = 27$
            1. 27 is divisible by 9 ($27 / 9 = 3$)
        iii. Induction proof: Assume $5^{2n} + 3n - 1$ is divisible by 9 ($= 9m$), show this holds for $(n + 1)$ also
            1. $5^{2(n+1)} + 3(n+1) - 1 = 5^{(2n+2)} + 3n + 3 - 1$
            2. $= 5^2 * 5^{2n} + 3n + 2 = (9m - 3n + 1)*25 + 3n + 2$
            3. $= 9m*25 - 75n + 25 + 3n + 2 = 9*25*m - 72n + 27$
            4. 72 and 27 ar multiples of 9 so can pull 9 out the whole thing
            5. $= 9 * (25m - 8n + 3)$
            6. Since our final equation is a multipe of 9, the equation as a whole will always be divisible by 9 and therefore we have proved $5^{2(n+1)} + 3(n+1) - 1$ is divisible by 9
    b. Show $n! > 3^n$ for $n >= 7$ and n is integer
        i. Hypothesis: $n! > 3^n$ for $n >= 7$ and n is integer
        ii. Base case: $n = 7. \Rightarrow 7! > 3^7 = 5040 > 2187$
            1. This is true $5040 > 2187$
        iii. Induction proof: Assume $n! > 3^n$ for $n >= 7$ and n is integer, show this holds for n+1
            1. $(n+1)! > 3^{(n+1)} = (n+1)*n! > 3*3^n$

2. With n starting at 7, n+1 is a minimum of 8
3. With the assumption of n!>3^n, multiplying the already larger side by 8 and the smaller side by 3 will yield the same results therefore (n+1)! > 3^(n+1) is true

c. Show $\sum_{i=1}^{n} \frac{1}{i(i+1)}$ = n / (n+1) for positive integers n

   i. Hypothesis: $\sum_{i=1}^{n} \frac{1}{i(i+1)}$ = n / (n+1) for positive integers n is true

   ii. Base case: n = 1 => 1 / 1(2) = 1 / (1+1) => ½ = ½

      1. This is true ½ = ½

   iii. Induction proof: Assume $\sum_{i=1}^{n} \frac{1}{i(i+1)}$ = n / (n+1) for positive integers n, is true. Show it holds for n+1

      1. $\sum_{i=1}^{n+1} \frac{1}{i(i+1)}$ = (n+1) / (n+1+1)
      2. Left side
         a. $\sum_{i=1}^{n+1} \frac{1}{i(i+1)}$ = 1/(n+1)(n+2)+ $\sum_{i=1}^{n} \frac{1}{i(i+1)}$
         b. Based on assumption
         c. = 1/(n+1)(n+2) + n/(n+1)
         d. = (1 + n(n+2)) / (n+1)(n+2)
         e. = n^2 + 2n + 1 / (n+1)(n+2)
         f. (n+1)^2 / (n+1)(n+2) = (n+1)/(n+2)

   iv. Since the left side now equals the right, the assumed holds for n+1 and we have proved $\sum_{i=1}^{n} \frac{1}{i(i+1)}$ = n / (n+1)

d. Show $\sum_{i=1}^{n} i$ = (n*(n+1))/2

   i. Hypothesis: $\sum_{i=1}^{n} i$ = (n*(n+1))/2 is true

ii. Base case: n = 1 => 1 = 1(2)/2 => 1= 1

1. This is true

iii. Induction Proof: Assume $\sum_{i=1}^{n} i$ = (n*(n+1))/2 is true, show this holds for n+1

1. $\sum_{i=1}^{n+1} i$ = (n+1)*(n+2))/2
2. LHS
   a. $\sum_{i=1}^{n+1} i$ = n+1 + $\sum_{i=1}^{n} i$
   b. Using assumed
   c. (n+1) + (n*(n+1))/2 = ½ 2(n+1) +n^2 +n
   d. = ½ (2n + 2 + n^2 + n) = ½ (n^2 +3n + 2)
   e. = (n+1)(n+2) / 2
3. Since the left side now equals the right side, the case holds for n+1 and we have proved $\sum_{i=1}^{n} i$ = (n*(n+1))/2 is true for all positive n.

2. Compute the sums
   a. $\sum_{i=3}^{n+1} 1$ = (n+1 – 3 +1)1 = n-1
      i. Second simple series from class notes
   b. $\sum_{i=3}^{n+1} i$ = n+1 + $\sum_{i=3}^{n} i$ = n+1 + $\sum_{i=1}^{n} i$ - $\sum_{i=1}^{3} i$

      = n+1 + $\sum_{i=1}^{n} i$ – 6 = n+1 + n(n+1)/2 – 6

      =1/2 (2n + 2 + n^2 + n – 12) = ½ (n^2 + 3n – 10)
      i. Simple series from 1d above is close so got there
   c. $\sum_{i=1}^{n} \quad \sum_{j=1}^{n} i\, j$ = $\sum_{i=1}^{n} i \sum_{j=1}^{n} j$ = $\sum_{i=1}^{n} i$ (n*(n+1)/2)

      = (n*(n+1)/2) $\sum_{i=1}^{n} i$ = (n*(n+1)/2) * (n*(n+1)/2)

      = n^2 (n+1)^2 / 4
      i. 2 of that same 1d simple series, pull out "constants"
3. Determine order of growth (prove)

a. (n^2 + 1)^10 = n^20 + … + 1

    i. O (n^20) by the polynomial theorem (#3)

b. (10n^2 + 7n + 3)^1/2

    i. Limit test with n

    ii. Lim n>inf of (10n^2 + 7n + 3)^1/2 / n >> l’h

    iii. Lim n>inf of 20n + 7 / (10n^2 + 7n + 3)^1/2 >> l’h

    iv. Lim n>inf of 20(20n + 7)/ (10n^2 + 7n + 3)^3/2 >>

    v. This is approaching 1/inf = 0

    vi. Therefore (10n^2 + 7n + 3)^1/2 is in O(n)

c. 2n log ((n+2)^2) + (n+2)^2 log (n/2)

    i. = 4n log(n+2)+ (n+2)^2(log(n) – log(2))

    ii. = 4n log(n+2) + (n+2)^2 log(n) – (n+2)^2

    iii. Limit test with n log(n)

    iv. Lim n>inf of (4n log(n+2) + (n+2)^2 log(n) – (n+2)^2) / (n log(n)) >> l’h

    v. After looking at the limit (it is very big and ugly) I am unsure I am doing this right.

d. 2^(n+1) + 3^(n-1)

    i. 2^n or 3^n is a complexity class given

    ii. As n approaches infinity the modifier to either will be negligible and so this will be big theta of 3^n

    iii. Lim n > inf of 2^n+1 + 3 ^ n-1 / 3^n

    iv. = Lim n > inf of (2^n+1 / 3^n) + 1/3

    v. = 1/3 ? therefore 2^(n+1) + 3^(n-1) is in big theta of 3^n

e. Log(n)

i. Log n is a complexity class given so this would simply be Big theta(log(n))?

1. This doesn't seem right (too easy) not sure what I am missing

4. Order of growth (justify)

a. $\sum_{i=1}^{n} \sum_{j=i}^{n} n$

i. N is not I or j so is constant and pulled out front

ii. n $\sum_{i=1}^{n} \sum_{j=i}^{n} 1$ = n $\sum_{i=1}^{n} n - i + 1$ =

iii. n ($\sum_{i=1}^{n} n$ -$\sum_{i=1}^{n} i$ + $\sum_{i=1}^{n} 1$) = n(n^2 – n(n+1)/2 +n)

iv. n^3 – n(n+1)/2 + n

v. This will have order of growth of n^3 by the polynomial theorem

b. $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (i + j)$

i. $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} i$ + $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} j$ =

ii. $\sum_{i=0}^{n-1} i \; \sum_{j=0}^{i-1} 1$ + $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} j$ =

iii. $\sum_{i=0}^{n-1} i^2$ + $\sum_{i=0}^{n-1} \frac{i(i+1)}{2}$ = n(n+1)(2n+1)/6 + 1/2 $\sum_{i=0}^{n-1} \mathrm{i}^2 + \mathrm{i}$

iv. n(n+1)(2n+1)/6 + ½ (n(n+1)(2n+1)/6 + n(n+1)/2

v. This will have order of growth of n^3 by the polynomial theorem

c. $\sum_{i=0}^{n-1} (i^2 + 1)^2$

i. $\sum_{i=0}^{n-1} i^4 + 2i^2 + 1$ =

ii. (n(n+1)(2n+1)/6)^2 + 2(n(n+1)(2n+1)/6) + n

iii. This will have order of growth of n^6 by the polynomial theorem

d. $\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l-1}^{n} l$

i. $\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{n(n+1)}{2}$

ii. Since no other variables all other summations result in n

iii. N^3 (n(n+1)/2)

iv. This will have order of growth of n^6 by the polynomial theorem

e. $\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-2} i$

i. $\sum_{i=0}^{n-1} i\ \sum_{j=i+1}^{n-2} 1$ = $\sum_{i=0}^{n-1} i\ \ (n-2-i-1+1)$

ii. $\sum_{i=0}^{n-1}\ ni-2i-i^2$ = n^2(n+1)/2 – n(n+1) – n(n+1)(2n+1)/6

iii. This will have order of growth of -n^2 by the polynomial theorem