Homework 2

1 a-e , 2 a-e , 3, 4, 5, 6 a-b , 7a-b

a/c

1) Solve recurrence $T(0) = c$ $T(1) = c$ $c > 0$

a) $T(n) = T(n-1)+1$

$x - T(x) = T(x-1)+1$

$T(n) = T(n-2)+1 \quad +1$

$x = n-1 \Rightarrow T(n-1) = \underbrace{T(n-2)+1}$

$T(n) = T(n-3)+1 \quad +1+1$

$x = n-2 \quad T(n-2) = T(n-3)+1$

$T(n) = T(n-k)+k$

Anchor

$T(n-k) = T(1)$

$n-k = 1 \Rightarrow n = k+1$

$k = n-1$

$T(n) = T(k+1-k)+k$

$T(n) = T(1) + n-1$

$T(n) = c + n-1 \in \Theta(n)$

b) $T(n) = T(n-5)+2$ $=1$

$T(x) = T(x-5)+2$

$T(n) = T(n-5\cdot 2) + 2+2$ $=2$

$x = n-5 \quad T(n-5) = T(n-5-5)+2$

$T(n) = T(n-5\cdot 3) + 2+2+2$ $=3$

$x = n-10 \quad T(n-10) = T(n-15)+2$

$T(n) = T(n-5k)+2k$

Anchor

$T(n-5k) = T(0)$

$n-5k = 0$

$n = 5k$ or $k = \frac{n}{5}$

$\Rightarrow T(n) = T(5k-5k)+2\frac{n}{5}$

$T(n) = T(0) + \frac{2}{5}n$

$\in \Theta(n)$

c) $T(n) = 2T(\frac{n}{2})+2$

$T(x) = 2T(\frac{x}{2})+2$

$T(n) = 2(2T(\frac{n}{2^2})+2)+2$

$x = \frac{n}{2} \quad T(\frac{n}{2}) = 2T(\frac{n}{2^2})+2$

$T(n) = 2\cdot 2\cdot(2T(\frac{n}{2^3})+2)+2+2$

$x = \frac{n}{2^2} \quad T(\frac{n}{2^2}) = 2T(\frac{n}{2^3})+2$

$T(n) = 2^k T(\frac{n}{2^k})+2k$

Anchor

$T(\frac{n}{2^k}) = T(1)$

$n = 2^k$

$k = \log_2 n$

$\Rightarrow$

$T(n) = nT(\frac{n}{n}) + 2\log n$

$= nc + 2\log n$

$\in \Theta(\log n)$

1 d e, 2a-c, 3, 4, 5, 6a-b, 7a-b

1)

d) $T(n) = 3T\left(\frac{n}{3}\right) + n$

$T(x) = 3T\left(\frac{x}{3}\right) + x$

$T(n) = 3\left(3T\left(\frac{n}{9}\right) + \frac{n}{3}\right) + n$

$x = \frac{n}{3} \Rightarrow 3T\left(\frac{n}{9}\right) + \frac{n}{3}$

$= 9T\left(\frac{n}{9}\right) + n + n$

$T(n) = 9\left(3T\left(\frac{n}{27}\right) + \frac{n}{9}\right) + n + n$

$x = \frac{n}{9} \Rightarrow 3T\left(\frac{n}{27}\right) + \frac{n}{9}$

$= 27T\left(\frac{n}{27}\right) + n + n + n$

$\Rightarrow T(n) = 3^k T\left(\frac{n}{3^k}\right) + kn$

Anchor:

$T\left(\frac{n}{3^k}\right) = T(1) \Rightarrow n = 3^k$ and $\frac{\log n}{\log 3} = k \Rightarrow \log_3 n = k$ log rule (i think)

$T(n) = nT(\cancel{\tfrac{n}{n}})^{c} + (\log_3 n)n \Rightarrow cn + n(\log_3 n) \in \Theta(\log_3 n)$

e) $T(n) = 2T\left(\frac{n}{2}\right) + n^3$ $k=1$

$T(x) = 2T\left(\frac{x}{2}\right) + x^3$

$\Sigma n^i = \frac{a(r^n - 1)}{r - 1}$

r a = base

n = num terms

$T(n) = 2\left(2T\left(\frac{n}{4}\right) + \frac{n^3}{8}\right) + n^3$ $k=2$

$x = \frac{n}{2} \Rightarrow T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \frac{n^3}{8}$

$= 4T\left(\frac{n}{4}\right) + \frac{n^3}{4} + n^3$

$T(n) = 4\left(2T\left(\frac{n}{8}\right) + \frac{n^3}{4^3}\right) + \frac{n^3}{4} + n^3$ $k=3$

$x = \frac{n}{4} \Rightarrow T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \frac{n^3}{4^3}$

This seems overly complex so I think I messed up.

$= 8T\left(\frac{n}{8}\right) + \frac{n^3}{16} + \frac{n^3}{4} + n^3$

$T(n) = 2^k T\left(\frac{n}{2^k}\right) + n^3 \sum_{i=0}^{k-1} \frac{1}{4^i}$

$= 2^k T\left(\frac{n}{2^k}\right) + n^3 \left(\frac{\left(\frac{1}{4}\right)^k - 1}{-\frac{3}{4}}\right) = 2^k T\left(\frac{n}{2^k}\right) - \frac{4n^3\left(\left(\frac{1}{4}\right)^k - 1\right)}{3}$

Anchor

$T\left(\frac{n}{2^k}\right) = T(1)$ $n = 2^k \Rightarrow k = \log_2 n$

$T(n) = nT(\cancel{\tfrac{n}{n}})^{c} - \frac{4n^3\left(\left(\frac{1}{4}\right)^{\log_2 n} - 1\right)}{3} = cn - \frac{4n^3\left(\frac{1}{n^2} - 1\right)}{3} = cn - \frac{4}{3}n + \frac{n^3}{3}$

$\in \Theta(n^3)$

2 a-e, 3, 4, 5, 6a-b, 7 a-5

2) Consider algorithm

	0	1	2	3
0	M_0	M_1	M_1	M_1
1	M_1	M_2	M_3	M_3
2	M_1	M_3	M_4	M_5
3	M_1	M_3	M_5	X — not checked

Mystery (A[0 ... n-1, 0 ... n-1])

```
1    for (i=0 to n-2)
2        for (j=i+1 to n-1)
3            if A(i,j) ≠ A(j,i)
4                return false
5    return true
```

a) What does algo do?

It appears it verifies rows and colums have matching values (excluding central diagonal)

b) basic op? = Comparison of values on line 5

c) Worst case and algo? Worst case is Matrix appears as drawn above (no different values)

$$T(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} n \cancel{-1} - (i+1) \cancel{+1} = \sum_{i=0}^{n-2} n-i-1 = -(n-1) \sum_{i=0}^{n-2} i$$

$$= \frac{-(n-1)(n-1)n}{2} \in \Theta(n^3)$$

d) Best Case: $A(0,1) \neq A(1,0)$

Worst Case: Matrix Values align with matrix drawn above

e) improvement for algorithm?

In architecture we discussed how unrolling loops may help efficiency though not sure how that would work for re-writing / computing complexity.

prob ~~3~~, 4, 5, 6 a-5, 7 a-5 remain

3) Design tower of hanoi algorithm and find its # of moves

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```
TOH(n disks)
    if n == 1
        move to destination
    else
        TOH(n-1) // disks on top
        move bottom
        TOH(n-1) // put discs back
```

$T(n) = T(n-1) + 1 + T(n-1)$
$= 2T(n-1) + 1$

$T(n) = 2T(n-1) + 1$ $\quad k=1$

$T(n) = 2(2T(n-2) + 1) + 1$ $\quad k=2$

$= 4T(n-2) + 2 + 1$

$T(n) = 4(2T(n-3) + 1) + 2 + 1$ $\quad k=3$

$= 8T(n-3) + 4 + 2 + 1$

$T(n) = 2^k T(n-k) + \sum_{i=0}^{k-1} 2^i = 2^k T(n-k) + \frac{2^k - 1}{2-1}$

$T(x) = 2T(x-1) + 1$

$x = n-1 \Rightarrow T(n-1) = 2T(n-2) + 1$

$x = n-2 \Rightarrow T(n-2) = 2T(n-3) + 1$

Anchor: $T(n-k) = T(1)$ $\quad n = 1+k$ or $k = n-1$

$T(n) = 2^{n-1} T(\cancel{n-n+1}) + 2^{n-1} - 1 = 2(2^{n-1}) - 1 = \cancel{2}\frac{2^n}{\cancel{2}} - 1 = 2^n - 1$

$\boxed{\in \Theta(2^n)}$

probs 4, 5 a-b, 6, 7 a-b

4) $T(n) = T(\lfloor \frac{n}{2} \rfloor) + 1$

if assuming even, $n = 2^k$
and using the smoothness rule

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$\Rightarrow T(2^k) = T(\frac{2^k}{2}) + 1 = T(2^{k-1}) + 1$ $\quad T(x) = T(2^{x-1}) + 1$

$T(2^k) = T(2^{k-2}) + 2$ $\quad x = k-1 \Rightarrow T(2^{k-1}) + 1$

$T(2^k) = T(2^{k-3}) + 3$ $\quad x = k-2 \Rightarrow T(2^{k-2}) + 1$

$T(2^k) = T(2^{\cancel{k-k}}) + k = k$ or $T(n) = \log_2(n)$

$\in \Theta(\log_2 n)$

5) Do as follows

a) find algo to do 2^n such that $2^n = 2^{n-1} + 2^{n-1}$

FindSecond(n)
  if n == 1
    return 2
  else
    return FindSecond(n-1) + FindSecond(n-1)

b) recurrence relation

$T(n) = \underbrace{2T(n-1)}_{\text{recursions}} + \overbrace{1}^{\text{addition}}$ $\quad k=1$ $\quad T(x) = 2T(x-1) + 1$

$T(n) = 2(2T(n-2)+1) + 1 = 4T(n-2) + 2 + 1$ $\quad k=2$ $\quad x = n-1 \Rightarrow T(n-1) = 2T(n-2) + 1$

$T(n) = 4(2T(n-3)+1) + 2 + 1 = 8T(n-3) + 4 + 2 + 1$ $\quad k=3$ $\quad x = n-2 \Rightarrow T(n-2) = 2T(n-3) + 1$

$T(n) = 2^k T(n-k) + \sum_{i=0}^{k-1} 2^i = 2^k T(n-k) + 2^k - 1$

Anchor: Let $T(1) = c$ $\quad c > 0$

$T(n-k) = T(1)$

$n = k+1$ or $k = n-1$

$2^{n-1} T(\cancel{n-n+1}) + \frac{2^n}{2} - 1 = \frac{c\,2^n}{2} + \frac{2^n}{2} - 1$

$= c\,2^n - 1 \in \Theta(2^n)$ is not good probs 6, 7a-b

6) $Fib(n) < 2^{31}-1$

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$$\frac{1}{\sqrt{5}}\phi^{n} < 2^{31}-1$$

$$\lg\left(\frac{1}{\sqrt{5}}\right) + n\lg\phi < 31 - 0$$

$$-1.161 + n(0.6942) < 31$$

$$n(0.6942) < 32.161$$

$$n < 46.32 \quad \text{or} \quad \boxed{n \leq 46}$$

7) Fibonacchi rabit

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a) Fibo(n)

```
Fibo(n)
    if n == 1
        return 0
    if n == 2
        return 1
    else
        return Fibo(n-1) + Fibo(n-2)
```

not sure what is wanted here

Relation

from book and slides

$$T(n) = T(n-1) + T(n-2) + 1 \;=\; \frac{1}{\sqrt{5}}\phi^{n} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} \;???$$

b)

1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	5	8	13	21	34	55	89	144