Type theory in Lean - 5

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Proof irrelevance

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This property (called proof irrelevance) is a feature of Lean's type theory, and it is built-in in the kernel.

Other proof assistants (for example Coq) use a *proof relevant* type theory, where proofs are not always definitionally equal.

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The idea is that a proof $(p : P)$ does not carry any information about P besides the fact that P holds. In practice, P is empty (meaning that we are not able to construct a term of type P) or it is a singleton. In the former case P is unprovable, in the latter case P holds.

Let's consider the following inductive constructions.

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inductive Inhabited (A : Type) : Type
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inductive Inhabited (A : Type) : Type
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and

```
inductive Nonempty (A : Type) : Prop
| intro (val : A) : Nonempty A
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- Formation rule: if A is a type, we have a well defined type `Inhabited A`.
- Constructors: there is only one constructor. If $(a : A)$, then

$$(\langle a \rangle : \mathrm{Inhabited}\ A)$$

- Eliminator: if $(B : \mathrm{Sort}\ u)$ and $(f : A \to B)$, then we have a function

$$\mathrm{rec}\ f : \mathrm{Inhabited}\ A \to B$$

- Computation rule: with the above notations we have

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In particular we can take $B = A$ and $f = \mathrm{id}$, getting a function

$$\mathrm{default} : \mathrm{Inhabited}\ A \to A$$

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Mathematically, giving a term of type $\mathrm{Inhabited}\ A$ is the same as giving a term of type A.

This is not the same as proving that A is not empty! Knowing that A is not empty should not give a well defined term of type A.

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  This is exactly the same as for `Inhabited A`.
- Eliminator and computation rule: we now show that *it is not possible* for `Nonempty A` to have the same eliminator and computation rule as `Inhabited A`.

If this were the case, following the same reasoning as above, we would be able to construct a function

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Now, to build terms of type $\text{Nonempty } \mathbb{N}$ one can use any term of type $\mathbb{N}$. In particular we have

$$(\langle 0 \rangle : \text{Nonempty } \mathbb{N}) \text{ and } (\langle 1 \rangle : \text{Nonempty } \mathbb{N})$$

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In particular, 0 and 1 would be definitionally equal, and this is not the case (we even know that they are not propositionally equal).

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It is called a *small eliminator*.

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- *If* $(x : \mathrm{Nonempty}\ A)$ *and* f *is as above, then*

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so $\mathrm{rec}\ f\ x$ *is a proof of* P *and* P *holds.*

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- *There is no need for a computation rule, as any two terms of type* P *are always definitionally equal.*
- *Similarly, the above argument to show that the eliminator does not exist does not work, since it would prove that any two terms of type* P *are definitionally equal.*

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$$g \ \langle p, q \rangle = f \ p \ q$$

and this is indeed the computation rule.

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The precise definition is technical, but essentially it is the following.

Definition

A syntactic subsingleton is an inductive proposition with at most one constructor whose arguments are either Prop or appear as immediate arguments in the output type.

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- $P \vee Q$ is *not* a syntactic subsingleton because it has two constructors. The idea is that if we have $f : P \to A$ and $g : Q \to A$, we cannot build a function $P \vee Q \to A$, because, given $(x : P \vee Q)$ we don't know whether x has been proved by proving P (and then we want to use f), or has been proved by proving Q (and then we want to use g).

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- $P \vee Q$ is *not* a syntactic subsingleton because it has two constructors. The idea is that if we have $f : P \to A$ and $g : Q \to A$, we cannot build a function $P \vee Q \to A$, because, given $(x : P \vee Q)$ we don't know whether x has been proved by proving P (and then we want to use f), or has been proved by proving Q (and then we want to use g). On the other hand, if A is a proposition, this does not matter because of proof irrelevance.

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Let's review the rules to build an inductive type, taking into account all the various examples and trying to be as general as possible.

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  - Arguments given by the inductive type itself. For example the constructor succ of $\mathbb{N}$ allows to construct a new natural number $\mathrm{succ}\ n$ given $(n : \mathbb{N})$. These are the constructors that really make the type *inductive*.

There are precise rules that say precisely how can an inductive type T appear in the constructors of T itself.

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  - Finally, for the constructors that use terms t of type T itself, one is allowed to use t to specify the image of the constructor. Again, the precise formulation is complicated, but think about what happens for $\mathbb{N}$.

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Inductive families

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The Curry–Howard correspondence

One can ask whether our definition of, say,

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In particular, there is a correspondence between classical proofs and Lean's proofs.