Type theory in Lean - 2

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October 21st 2023

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Moreover

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In general

$$\mathrm{Type}\ n = \mathrm{Sort}\ n+1 \text{ and } \mathrm{Prop} = \mathrm{Sort}\ 0$$

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In practice, if $x \equiv y$ then one can replace x by y everywhere.

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We will study in details two constructions:

- Dependent functions.
- Dependent pairs.

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- An optional *uniqueness principle*: roughly speaking it is the fact that the only terms of the new type are those obtained using the constructors. It is optional, and often it can be proved using the previous rules. In this case it is a design choice to make it a definitional equality (one cannot prove definitional equalities!).

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- Formation rule: if A and B are two types, we have another type $A \to B$, whose terms are called *functions* from A to B.
- Constructors: there is only one constructor, called *lambda abstraction*. If E is any expression containing a variable x such that $(E : B)$ if $(x : A)$, then

$$(\text{fun } x \mapsto E : A \to B)$$

  is of type $A \to B$.

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Here $E[x := a]$ is the expression E with each x replaced by a (syntactically).

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Not 100% precise: consider the case where E is the expression $x + (\mathrm{fun}\ x \mapsto \sin(x))\ 0$.

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At the moment there is no *functional extensionality principle*: if $f\ x = g\ x$ for all x, we cannot prove that $f = g$. The uniqueness principle implies that if $f\ x \equiv g\ x$ for all x, then $f \equiv g$, but this is difficult to state in Lean.

Functions of several variables

If A, B and C are types, the type

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If $(a : A)$, $(b : B)$ and $(f : A \to B \to C)$, then $(f\ a\ b : C)$.

This process is called *currying*.

Dependent functions

Consider the assignment

$$n \mapsto 0 \in \mathbb{R}^n,$$

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The real primitive construction is the Π-type, with functions as a special case.

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$$\prod_{(a:A)} B(a) = (a : A) \to B(a)$$
called Π-type. Its terms are called *dependent functions*.

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Example

The identity function can be considered as a dependent function.

$$\mathrm{id} : (A : \mathrm{Type}\ u) \to (A \to A)$$

Example

If $f : A \times B \to C$ is a function, let's define

$$\begin{aligned} \mathrm{swap}\ f &: B \times A \to C \\ (b, a) &\mapsto f(a, b) \end{aligned}$$

If we think to f as a term ($f : A \to B \to C$), then ($\mathrm{swap}\ f : B \to A \to C$) and

$$\mathrm{swap} : (A : \mathrm{Type}\ u) \to (B : \mathrm{Type}\ u) \to (A \to B \to C) \to (B \to A \to C)$$

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$$((a, b) : A \times B)$$

- Eliminators (non-dependent version): there is only one eliminator. Given $(x : A \times B)$ and a function $f : A \to B \to C$, we have a well defined term $(\mathrm{rec}_{A \times B}\ f\ x : C)$.

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The name $\mathrm{rec}_{A \times B}$ comes from the theory of inductive types.

Definition

We let $\pi_1 : A \times B \to A$ be the function given by the eliminator via

$$\pi_1 = \mathrm{rec}_{A \times B}\ ((\mathrm{fun}\ a\ b \mapsto a) : A \to B \to A)$$

The computation rule says that

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In Lean we write $x.1$ for $\pi_1\ x$. The function π_1 is also called fst. We similarly have a function $\pi_2 \colon A \times B \to B$ also called snd.

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$$\mathrm{rec}_{A \times B}\ f \equiv \mathrm{fun}\ (x : A \times B) \mapsto f\ x.1\ x.2$$

for all $(f : A \to B \to C)$.

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The Cartesian product is not a primitive notion in Lean's type theory: it is a special case of an inductive type.

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The eliminator and the computation rule we gave are for non-dependent functions from $A \times B$: we also need a dependent version.

- Eliminators (dependent version). Let C be a function $(C : A \times B \to \mathrm{Type}\ u)$. Given a dependent function $(f : \prod_{(a:A)} \prod_{(b:B)} C\ (a, b))$, we have a well defined term $(\mathrm{rec}_{A \times B}\ f\ x : C\ x)$ for all $(x : A \times B)$.

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We have the same remarks as before.

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$$\mathrm{rec}_{A \times B}\ C\ f\ (a, b) \equiv f\ a\ b$$

Note that in Lean the variable C is implicit.

Universal constructions (non-dependent version)

The (non-dependent) eliminator allows to construct a function $A \times B \to C$ given a function $A \to B \to C$.
In practice it is given by a (dependent!) function $\mathrm{rec}_{A \times B}$ of type

$$\left(\mathrm{rec}_{A\times B} : \prod_{(C:\mathrm{Type}\ u)} (A \to B \to C) \to A \times B \to C\right)$$

The computation rule says

$$\mathrm{rec}_{A\times B}\ C\ f\ (a,b) \equiv f\ a\ b$$

Note that in Lean the variable C is implicit.
We have

$$\pi_1 \equiv \mathrm{rec}_{A\times B}\ A\ (\mathrm{fun}\ a\ b \mapsto a)$$

Universal constructions (dependent version)

Similarly, the (dependent) eliminator is given by a function $\mathrm{rec}_{A\times B}$ of type

$$\left(\mathrm{rec}_{A\times B} : \prod_{(C:A\times B\to\mathrm{Type}\ u)} \left(\prod_{(a:A)} \prod_{(b:B)} C\ (a, b)\right) \to \prod_{(x:A\times B)} C\ x\right)$$

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We will see that $\mathrm{rec}_{A\times B}$ is a special case of the *recursor* of an inductive type.

Dependent pair types

Dependent pair types (also called $\sum$-types) generalize Cartesian product in the same way as dependent functions generalize function types: we allow the type of the second component to depend on the first one.

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- Formation rule: if A is a type and $(B : A \to \mathrm{Type}\ u)$ is a function, we have another type, denoted

$$\sum_{(a:A)} B\ a = (a : A) \times B\ a$$

whose terms are called *dependent pairs*.

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Definition

A *magma* is a term of type

$$\sum_{(M : \mathrm{Type}\ u)} (M \times M \to M)$$

- Constructors: there is only one constructor. If we have two terms $(a : A)$ and $(b : B\ a)$, then we have a term, denoted $\langle a, b \rangle$, of type $\sum_{(a:A)} B\ a$.

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- Eliminators (non-dependent version): there is only one eliminator. Given $(x : \sum_{(a:A)} B\ a)$ and a (dependent) function $(f : \prod_{(a:A)}(B\ a \to C))$, we have a well defined term $(\mathrm{rec}_{\sum_{(a:A)} B\ a}\ f\ x : C)$.

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- Computation rules (non-dependent version): there is only one computation rule. If we have terms $(a : A)$ and $(b : B\ a)$, where notation is as above, then

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We can now define a function $(\pi_1 \colon \sum_{(a:A)} B\ a \to A)$ as above. It satisfies

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Note that for π_2 we need the dependent version of the eliminator and of the computation rule.

- Eliminators (dependent version). Let C be a function $(C : \sum_{(a:A)} B\ a \to \mathrm{Type}\ u)$. Given a dependent function $(f : \prod_{(a:A)} \prod_{(b:B\ a)} C\ \langle a, b \rangle)$, we have a well defined term $(\mathrm{rec}_{\sum_{(a:A)} B\ a}\ f\ x : C\ x)$ for all $(x : \sum_{(a:A)} B\ a)$.

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$$\left(\mathrm{rec}_{\sum_{(a:A)} B\ a}\ f : \prod_{(x:\sum_{(a:A)} B\ a)} C\ x\right)$$

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- Computation rules (dependent version). If we have terms $(a : A)$ and $(b : B\ a)$ and a dependent function f as above, then

$$\mathrm{rec}_{\sum_{(a:A)} B\ a}\ f\ \langle a, b\rangle \equiv f\ a\ b.$$

Using the dependent version of the eliminator, we can now define π_2 as a dependent function

$$\left(\mathrm{snd} : \prod_{(x : \sum_{(a:A)} B\ a)} B\ x.1\right)$$

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- Uniqueness principle: if $(x : \sum_{(a:A)} B\ a)$, then

$$x \equiv \langle \pi_1\ x, \pi_2\ x \rangle,$$

so all terms of $\sum_{(a:A)} B\ a$ are given by pair of elements.

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  $$x \equiv \langle \pi_1\ x, \pi_2\ x\rangle,$$

  so all terms of $\sum_{(a:A)} B\ a$ are given by pair of elements. This implies that

  $$\mathrm{rec}_{\sum_{(a:A)} B\ a}\ f \equiv \mathrm{fun}\ \left(x : \sum_{(a:A)} B\ a\right) \mapsto f\ x.1\ x.2$$

  for all $(f : \prod_{(a:A)}(B\ a \to C))$.

All the remarks above are still true. For example, the uniqueness principle implies that all functions of type $\prod_{(x : \sum_{(a:A)} B\ a)} C\ x$ can be defined via π_1 and π_2, and this is what we do in practice.

Also the dependent pair type is a special case of an inductive type.

We also the analogues universal constructions. The (dependent) eliminator is given by a function $\mathrm{rec}_{\sum_{(a:A)} B\ a}$ of type

$$\left(\mathrm{rec}_{\sum_{(a:A)} B\ a} : \prod_{(C:\sum_{(a:A)} B\ a \to \mathrm{Type}\ u)} \left(\prod_{(a:A)} \prod_{(b:B\ a)} C\ \langle a, b\rangle\right) \to \prod_{(x:\sum_{(a:A)} B\ a)} C\ x\right)$$

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