Math 104: Introduction to Real Analysis

Lecture Notes

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Contents

1 Lecture 1: January 20, 2026

Lecture Overview: We begin by proving $\sqrt{2}$ is irrational, motivating the need for a number system without "gaps." This leads us to define **ordered sets** and the crucial **Least Upper Bound Property (LUBP)**—the defining feature of $\mathbb{R}$ that $\mathbb{Q}$ lacks. We then introduce **fields** as algebraic structures with addition and multiplication, and combine these ideas into **ordered fields**. The real numbers are the unique complete ordered field.

1.1 Ordered sets and the least-upper-bound property

Section Overview: This section motivates the need for the real numbers by showing that $\mathbb{Q}$ has "gaps"—$\sqrt{2}$ is irrational, yet we can get arbitrarily close to it with rationals. We develop the machinery of **ordered sets**: partial orders, total orders, upper/lower bounds, and the supremum/infimum. The central concept is the **Least Upper Bound Property (LUBP)**: every non-empty bounded-above subset has a supremum. This property distinguishes $\mathbb{R}$ from $\mathbb{Q}$ and is the foundation for all of real analysis. We prove that LUBP implies GLBP.

Consider the ancient problem from Greek times: can we write $\sqrt{2}$ as a quotient of two natural numbers?

Theorem 1.1. *$\sqrt{2}$ is irrational; that is, there do not exist $p, q \in \mathbb{N}$ such that $\sqrt{2} = \frac{p}{q}$.*

Proof. Suppose, for contradiction, that $\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$ (i.e., the fraction is in lowest terms).

Then $2 = \frac{p^2}{q^2}$, so $p^2 = 2q^2$.

This means p^2 is even, so p is even. Write $p = 2k$ for some $k \in \mathbb{N}$.

Then $(2k)^2 = 2q^2$, so $4k^2 = 2q^2$, hence $q^2 = 2k^2$.

This means q^2 is even, so q is even.

But then both p and q are even, contradicting $\gcd(p, q) = 1$. $\square$

Now consider two sets:

$$A = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 < 2\}, \quad B = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 > 2\}.$$

Proposition 1.2. *A contains no largest element and B contains no smallest element.*

Proof. Let $p_0 \in A$. Define

$$q = p_0 + \frac{2 - p_0^2}{p_0^2 + 2}.$$

Since $p_0 \in A$, we have $p_0^2 < 2$, so $2 - p_0^2 > 0$. Thus $q > p_0$.

We claim $q \in A$, i.e., $q^2 < 2$. One can verify that

$$q^2 - 2 = \frac{(p_0^2 - 2)^2 \cdot (\text{positive})}{(p_0^2 + 2)^2}$$

which shows $q^2 < 2$ when $p_0^2 < 2$.

Hence A has no largest element.

A similar argument shows B has no smallest element. $\square$

Definition 1.3 (1.3)**.** If A is any set, we write $x \in A$ to say that x is a **member** of A. Otherwise, $x \notin A$. The set that contains no elements is called the **empty set**, denoted $\emptyset$. If $A \neq \emptyset$, we say that A is **non-empty**.

If A, B are sets and $\forall x \in A$ we have $x \in B$, we say that $A \subset B$, or A is a **subset** of B. If there exists an element $x \in B$ with $x \notin A$, then A is a **proper subset** of B, denoted $A \subsetneq B$.

Example. $3 \in \mathbb{N}$, but $-1 \notin \mathbb{N}$. We have $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{N} \subsetneq \mathbb{Z}$ (since $-1 \in \mathbb{Z}$ but $-1 \notin \mathbb{N}$).

Definition 1.4. A **binary relation** on a set S is a set of ordered pairs $\langle x, y \rangle$ with $x, y \in S$.

Example. On $\mathbb{Z}$, the relation $\leq$ is the set $\{\langle x, y \rangle : x, y \in \mathbb{Z}, x \leq y\}$, e.g., $\langle 2, 5 \rangle$ is in the relation.

Definition 1.5. A **partial order** is a binary relation $\leq$ on S such that:

1. **Reflexive:** $\forall x \in S$, $x \leq x$.
2. **Anti-symmetric:** $\forall x, y \in S$, if $x \leq y$ and $y \leq x$, then $x = y$.
3. **Transitive:** $\forall x, y, z \in S$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Example. On the power set $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, the subset relation $\subseteq$ is a partial order (but not a total order, since $\{1\} \not\subseteq \{2\}$ and $\{2\} \not\subseteq \{1\}$).

Definition 1.6. A **total order** is a partial order with the additional axiom that any two elements are comparable. That is, for any $x, y \in S$, either $x \leq y$ or $y \leq x$ (non-exclusive).

Example. The usual $\leq$ on $\mathbb{R}$ is a total order: for any $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$.

Definition 1.7. An **ordered set** is a set equipped with a total order.

Example. $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are ordered sets.

Definition 1.8. Suppose S is an ordered set and $E \subset S$. If there exists $\beta \in S$ such that $x \leq \beta$ for all $x \in E$, we say β is an **upper bound** of E. Similarly, if there exists $\alpha \in S$ such that $\alpha \leq x$ for all $x \in E$, we say α is a **lower bound** of E.

Example. Let $E = (0, 1) \subset \mathbb{R}$. Then $1, 2, 100$ are all upper bounds of E, and $0, -5$ are lower bounds of E.

Definition 1.9. Suppose S is an ordered set and $E \subset S$ is bounded above. If there exists $\alpha \in S$ such that:

1. α is an upper bound of E, and
2. if $\gamma < \alpha$, then γ is not an upper bound of E,

then α is called the **least upper bound** of E (or **supremum**), denoted $\sup E$.

Example. $\sup(0, 1) = 1$ and $\sup[0, 1] = 1$ in $\mathbb{R}$.

Definition 1.10. Suppose S is an ordered set and $E \subset S$ is bounded below. If there exists $\alpha \in S$ such that:

1. α is a lower bound of E, and
2. if $\gamma > \alpha$, then γ is not a lower bound of E,

then α is called the **greatest lower bound** of E (or **infimum**), denoted $\inf E$.

Example. $\inf(0,1) = 0$ and $\inf[0,1] = 0$ in $\mathbb{R}$.

Remark 1.11. If $\sup E$ or $\inf E$ exists, it need not be an element of E. For example, the set $A = \{p \in \mathbb{Q} : p > 0, p^2 < 2\}$ has $\sup A = \sqrt{2}$ (in $\mathbb{R}$), but $\sqrt{2} \notin A$ since $\sqrt{2} \notin \mathbb{Q}$.

Definition 1.12. Let S be an ordered set.

1. S has the **least upper bound property** if for any non-empty $E \subset S$ that is bounded above, $\sup E$ exists in S.

2. S has the **greatest lower bound property** if for any non-empty $E \subset S$ that is bounded below, $\inf E$ exists in S.

Example. $\mathbb{R}$ has the LUBP (and hence GLBP). However, $\mathbb{Q}$ does not: the set $A = \{p \in \mathbb{Q} : p > 0, p^2 < 2\}$ is bounded above in $\mathbb{Q}$, but $\sup A = \sqrt{2} \notin \mathbb{Q}$.

Theorem 1.13 (LUBP implies GLBP)**.** *Suppose S is an ordered set with the least upper bound property. Let $B \subset S$, $B \neq \emptyset$, and suppose B is bounded below. Let L be the set of all lower bounds of B. Then $\alpha = \sup L$ exists in S, and $\alpha = \inf B$.*

Proof. First, $L \neq \emptyset$ since B is bounded below.

Second, L is bounded above: every $b \in B$ is an upper bound for L (since if $\ell \in L$, then $\ell \leq b$ by definition of lower bound).

By the LUBP, $\alpha = \sup L$ exists in S.

We claim $\alpha = \inf B$:

1. α is a lower bound of B: For any $b \in B$, b is an upper bound of L, so $\alpha \leq b$ (since α is the *least* upper bound of L).

2. α is the greatest lower bound: If $\gamma > \alpha$ and γ were a lower bound of B, then $\gamma \in L$, so $\gamma \leq \sup L = \alpha$, contradicting $\gamma > \alpha$. Thus γ is not a lower bound of B.

Thus $\alpha = \inf B$. □

1.2 Fields

Section Overview: This section introduces the algebraic structure underlying $\mathbb{R}$. We define **groups** (sets with an operation having identity, inverses, and associativity) and **fields** (sets with addition and multiplication that behave like we expect from $\mathbb{Q}$ or $\mathbb{R}$). We sketch how to construct $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q}$ using equivalence relations. The key definition is an **ordered field**: a field that is also an ordered set, allowing us to combine algebraic operations with comparison. $\mathbb{R}$ is the unique complete ordered field.

Definition 1.14. A **binary operation** on S is a map $S \times S \to S$.

Definition 1.15. A **group** is a set G with a binary operation $+$ satisfying the following axioms:

1. **Identity:** There exists $0 \in G$ such that $a + 0 = 0 + a = a$ for all $a \in G$.

2. **Existence of inverse:** For every $a \in G$, there exists $-a \in G$ such that $a + (-a) = 0$.

3. **Associativity:** For all $a, b, c \in G$, $(a + b) + c = a + (b + c)$.

If we add a fourth axiom:

4. **Commutativity:** For all $a, b \in G$, $a + b = b + a$,

then G is called an **abelian group**.

Definition 1.16. A **field** is a set F with two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that:

1. $(F, +)$ is an abelian group with identity 0.

2. $(F \setminus \{0\}, \cdot)$ is an abelian group with identity 1.

3. **Distributivity:** For all $a, b, c \in F$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Example. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields. $\mathbb{Z}$ is not a field (e.g., 2 has no multiplicative inverse in $\mathbb{Z}$).

Zooming out, we can construct the number systems as follows:
The **natural numbers** $\mathbb{N}$ can be defined by the cardinality of iterated power sets of $\emptyset$:

$$0 = |\emptyset|, \quad 1 = |\mathcal{P}(\emptyset)|, \quad 2 = |\mathcal{P}(\mathcal{P}(\emptyset))|, \quad \dots$$

The **integers** are defined as:

$$\mathbb{Z} = \{a - b : a, b \in \mathbb{N}\}.$$

Definition 1.17. An **equivalence relation** $\sim$ on a set S has the following properties:

1. **Reflexive:** $x \sim x$ for all $x \in S$.

2. **Symmetric:** If $x \sim y$, then $y \sim x$.

3. **Transitive:** If $x \sim y$ and $y \sim z$, then $x \sim z$.

The **rational numbers** are defined as:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z},\, q \in \mathbb{N},\, q \neq 0 \right\} / \sim$$

We can verify this is an equivalence relation: $\frac{p}{q} \sim \frac{r}{s}$ if and only if $ps = rq$.

Definition 1.18. An **ordered field** is a field which is also an ordered set.

Proposition 1.19. *If $x > 0$ and $y < z$, then $xy < xz$.*

Proof. Since $y < z$, we have $z - y > 0$. Since $x > 0$ and $z - y > 0$, we have $x(z - y) > 0$. Thus $xz - xy > 0$, so $xy < xz$. $\square$

2 Lecture 2: January 22, 2026

Lecture Overview: We construct the real numbers $\mathbb{R}$ as an ordered field with the Least Upper Bound Property (LUBP) containing $\mathbb{Q}$ as a subfield, using Dedekind cuts. We prove key properties of $\mathbb{R}$: the Archimedean property and density of $\mathbb{Q}$ in $\mathbb{R}$. Using the LUBP, we establish existence of nth roots of positive reals via a supremum argument. We discuss decimal/ternary representations and the Cantor set. We introduce the complex numbers $\mathbb{C}$ and prove $\mathbb{C}$ is not an ordered field. Finally, we define Euclidean spaces $\mathbb{R}^n$ with inner products and norms, and prove the Cauchy-Schwarz inequality.

2.1 Dedekind Cuts

Section Overview: We define Dedekind cuts as a way to construct the real numbers from the rationals.

Definition 2.1. A **cut** $\alpha \subset \mathbb{Q}$ is a nonempty, proper subset such that:

1. **Downward closed:** If $p \in \alpha$ and $q < p$, then $q \in \alpha$.
2. If $\sup \alpha$ exists, then $\sup \alpha \notin \alpha$.

The set of all cuts is ordered by inclusion: $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.

2.2 Field Operations on Cuts

Section Overview: We define addition and multiplication on cuts to make them into an ordered field.

Addition:

$$\alpha + \beta = \{r + s \mid r \in \alpha, s \in \beta\}$$

Additive identity:

$$0^* = \{p \in \mathbb{Q} \mid p < 0\}$$

Multiplication: For $\alpha > 0^*$ and $\beta > 0^*$:

$$\alpha\beta = \{p \in \mathbb{Q} \mid p \leq rs \text{ for some } r \in \alpha, r > 0 \text{ and } s \in \beta, s > 0\}$$

2.3 Least Upper Bound Property

Section Overview: We show that the set of cuts has the LUBP.

For a nonempty set E of cuts that is bounded above:

$$\sup E = \bigcup_{\alpha \in E} \alpha$$

2.4 Embedding $\mathbb{Q}$ into $\mathbb{R}$

Section Overview: We embed the rationals into the reals as a subfield.

For $p \in \mathbb{Q}$, define the cut:

$$p^* := \{q \in \mathbb{Q} \mid q < p\}$$

This embedding $p \mapsto p^*$ identifies $\mathbb{Q}$ as a subfield of $\mathbb{R}$.

2.5 Properties of $\mathbb{R}$

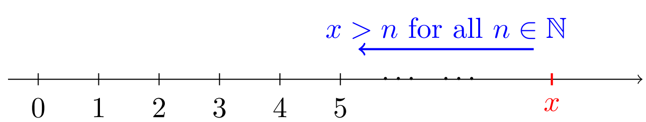
Section Overview: Having constructed $\mathbb{R}$, we now explore its key properties.

Theorem 2.2 (Archimedean Property)**.** *For any $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof. By contradiction. Suppose no such n exists, i.e., $nx \leq y$ for all $n \in \mathbb{N}$. Then the set $A = \{nx : n \in \mathbb{N}\}$ is bounded above by y. By the LUBP, $\sup A$ exists. Let $\alpha = \sup A$. Since $x > 0$, we have $\alpha - x < \alpha$, so $\alpha - x$ is not an upper bound for A. Thus there exists $m \in \mathbb{N}$ with $mx > \alpha - x$, which gives $(m+1)x > \alpha$. But $(m+1)x \in A$, contradicting that $\alpha = \sup A$. $\square$

Remark 2.3. The Archimedean property ensures there are no **infinitely large** elements (every element is bounded by some natural number) and no **infinitesimals** (positive elements smaller than $1/n$ for all n). This property is essential for proving that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example of a non-Archimedean field: Consider the field of rational functions $\mathbb{R}(x)$ with the ordering where x is declared to be larger than every real number (i.e., $x > r$ for all $r \in \mathbb{R}$). Then $x > n$ for all $n \in \mathbb{N}$, so the Archimedean property fails. In this field, $1/x$ is an infinitesimal: it is positive but smaller than $1/n$ for all $n \in \mathbb{N}$.



Theorem 2.4 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For any $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.*

Proof. Since $b - a > 0$, by the Archimedean property there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$, i.e., $nb - na > 1$. Thus there exists an integer m with $na < m < nb$. Then $a < \frac{m}{n} < b$, and $q = \frac{m}{n} \in \mathbb{Q}$. $\square$

2.6 The Roots of Reals

Section Overview: Having constructed $\mathbb{R}$ with the LUBP, we can now prove that nth roots of positive reals exist, resolving the gap in $\mathbb{Q}$ where $\sqrt{2}$ was missing.

Previously, we showed that $\sqrt{2} \notin \mathbb{Q}$. Now that we have constructed $\mathbb{R}$ with the LUBP, we can prove that nth roots exist.

Theorem 2.5 (Existence of nth Roots)**.** *For all $x \in \mathbb{R}_{>0}$ and for all $n \in \mathbb{Z}_{>0}$, there exists a unique $y \in \mathbb{R}_{>0}$ such that $y^n = x$.*

Proof. Let $E = \{t \in \mathbb{R}_{>0} : t^n < x\}$.

E **is non-empty:** We have

$$\left(\frac{x}{x+1}\right)^n < \frac{x}{x+1} < x,$$

so $\frac{x}{x+1} \in E$.

E **is bounded above:** (to be shown)

By the LUBP, $y = \sup E$ exists.

Claim: $y^n = x$.

Aside (Trichotomy): Since $\mathbb{R}$ is a totally ordered set, for any $a, b \in \mathbb{R}$, exactly one of the following holds: $a < b$, $a = b$, or $a > b$. Thus for y^n and x, exactly one of $y^n < x$, $y^n = x$, or $y^n > x$ holds. We show the first and third cases lead to contradictions.

Case 1: Suppose $y^n < x$. Then there exists $h > 0$ small enough such that $(y+h)^n < x$. But then $y + h \in E$, contradicting that $y = \sup E$.

Case 2: Suppose $y^n > x$. Then there exists $h > 0$ small enough such that $(y-h)^n > x$. But then $y - h$ is still an upper bound for E, contradicting that $y = \sup E$ (the *least* upper bound).

Therefore $y^n = x$. □

Note to the reader: This proof employs a fundamental technique in real analysis called a *supremum argument*. The strategy is:

1. **Define a set:** Construct a set E of elements that are "too small" (i.e., $t^n < x$).
2. **Apply LUBP:** Since E is nonempty and bounded above, $\sup E$ exists—this is where we crucially use that $\mathbb{R}$ has the Least Upper Bound Property.
3. **Use trichotomy:** By the trichotomy of total orders, the supremum y satisfies exactly one of $y^n < x$, $y^n = x$, or $y^n > x$.
4. **Eliminate by contradiction:** Show that $y^n < x$ contradicts y being an *upper* bound (we can go higher), and $y^n > x$ contradicts y being the *least* upper bound (we can find a smaller upper bound).

This technique appears repeatedly throughout analysis whenever we need to prove existence of a value with a specific property. A similar technique is employed in Exercise 7 (showing the existence of the logarithm).

2.7 Decimals, Binaries, Ternaries

Section Overview: We discuss representations of real numbers in different bases.

Observe that decimal representations come in the form

$$n_0 + \frac{n_1}{10} + \frac{n_2}{100} + \cdots = n_0 + \sum_{k=1}^{\infty} \frac{n_k}{10^k}$$

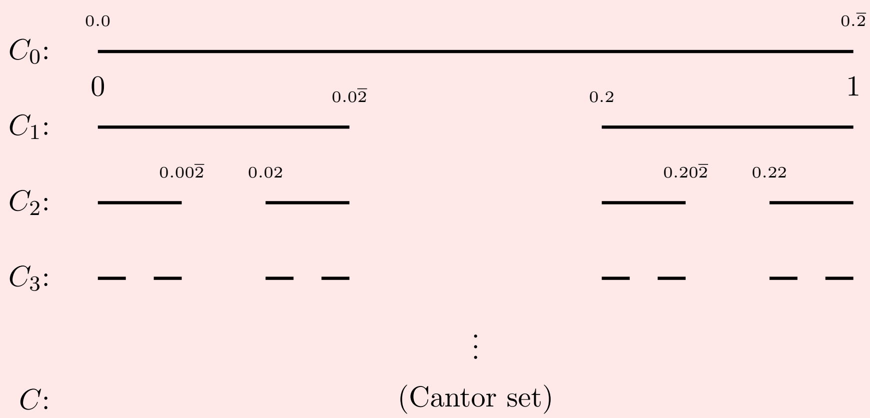
where $n_0 \in \mathbb{Z}$ and $n_k \in \{0, 1, 2, \ldots, 9\}$ for $k \geq 1$.

If we consider the set of partial sums

$$E = \left\{ n_0,\ n_0 + \frac{n_1}{10},\ n_0 + \frac{n_1}{10} + \frac{n_2}{100},\ \ldots \right\}$$

then $x = \sup E$.

Note: This construction is used to build the **Cantor set**. Starting with the interval $[0, 1]$, we iteratively remove the open middle third of each remaining interval:



Here the labels are *ternary* (base-3) expansions: e.g., $0.2_3 = \frac{2}{3}$, $0.02_3 = \frac{2}{9}$, $0.22_3 = \frac{8}{9}$, and $0.\overline{2}_3 = 0.222\ldots_3 = 1$.
The Cantor set $C = \bigcap_{n=0}^{\infty} C_n$ consists of all points in $[0, 1]$ whose ternary (base-3) expansion contains only the digits 0 and 2.

2.8 The Complex Field

Section Overview: We introduce the complex numbers $\mathbb{C}$ as an extension of $\mathbb{R}$.

Definition 2.6. The **complex numbers** are defined as

$$\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$$

where each element is of the form $z = a + bi$ with $a, b \in \mathbb{R}$ and $i^2 = -1$.

Theorem 2.7. $\mathbb{C}$ *is not an ordered field.*

Proof. By contradiction. Suppose $\mathbb{C}$ is an ordered field. By trichotomy, either $i > 0$ or $i < 0$ (since $i \neq 0$).

Case 1: If $i > 0$, then $i^2 > 0$ (since squares of nonzero elements are positive in an ordered field). But $i^2 = -1 < 0$, a contradiction.

Case 2: If $i < 0$, then $-i > 0$, so $(-i)^2 > 0$. But $(-i)^2 = i^2 = -1 < 0$, a contradiction.

Therefore $\mathbb{C}$ cannot be an ordered field. ☐

2.9 The Euclidean Spaces

Section Overview: We introduce Euclidean spaces $\mathbb{R}^n$ as spaces of ordered n-tuples.

Definition 2.8. The **Euclidean space** $\mathbb{R}^n$ is the set of all ordered n-tuples

$$\vec{x} = (x_1, x_2, x_3, \ldots, x_n)$$

where $x_i \in \mathbb{R}$ for each $i = 1, \ldots, n$.

For $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

Addition:

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

Scalar multiplication:

$$c\vec{x} = (cx_1, cx_2, \ldots, cx_n)$$

Inner product:

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Definition 2.9. An **inner product** over $\mathbb{R}$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that is:

- Symmetric bilinear and positive definite (for real vector spaces), or
- Hermitian sesquilinear and positive definite (for complex vector spaces).

Properties:

- **Symmetric:** $\langle x, y \rangle = \langle y, x \rangle$
- **Hermitian:** $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)
- **Bilinear:** Linear in both arguments:

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$$
$$\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$$

- **Sesquilinear:** Linear in one argument, conjugate-linear in the other:

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$$
$$\langle x, ay + bz \rangle = \bar{a}\langle x, y \rangle + \bar{b}\langle x, z \rangle$$

- **Positive definite:** $\langle x, x \rangle \geq 0$, with equality if and only if $x = 0$

Definition 2.10. The **norm** of a vector $\vec{x}$ is defined as

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Properties of a norm:

- **Triangle inequality:** $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
- **Absolute homogeneity:** $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$ for all $c \in \mathbb{R}$
- **Positive definite:** $\|\vec{x}\| \geq 0$, with equality if and only if $\vec{x} = \vec{0}$

Theorem 2.11 (Cauchy-Schwarz Inequality)**.** *For all $\vec{x}, \vec{y} \in \mathbb{R}^n$:*

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|$$

Proof. Consider the function $f(t) = \|\vec{x} + t\vec{y}\|^2$ for $t \in \mathbb{R}$. By positive definiteness, $f(t) \geq 0$ for all t.

Expanding:

$$f(t) = \langle \vec{x} + t\vec{y}, \vec{x} + t\vec{y} \rangle = \|\vec{x}\|^2 + 2t\langle \vec{x}, \vec{y} \rangle + t^2\|\vec{y}\|^2$$

This is a quadratic in t that is always non-negative. For a quadratic $at^2 + bt + c \geq 0$ for all t, the discriminant must satisfy $b^2 - 4ac \leq 0$.

Here $a = \|\vec{y}\|^2$, $b = 2\langle \vec{x}, \vec{y} \rangle$, $c = \|\vec{x}\|^2$, so:

$$4\langle \vec{x}, \vec{y} \rangle^2 - 4\|\vec{x}\|^2\|\vec{y}\|^2 \leq 0$$

Therefore $\langle \vec{x}, \vec{y} \rangle^2 \leq \|\vec{x}\|^2\|\vec{y}\|^2$, and taking square roots gives the result. □