1 Lecture 9: February 19, 2026

Lecture Overview: [High-level summary of the entire lecture. What are the main goals? What key concepts are introduced? How do they connect to the bigger picture of the course?]

1.1 Series

Given a sequence $\{a_n\}$, we can consider the **partial sums** $\{s_n\}$ where

$$s_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k.$$

We say this series **converges** to s if

$$s = \sum_{n=1}^{\infty} a_n,$$

that is, if $s_n \to s$, i.e. the partial sums converge. This means that many of the same properties of sequences apply to series.

1.2 Review: Cauchy Sequences

Last time we discussed Cauchy sequences: a sequence $\{s_n\}$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$|s_m - s_n| < \varepsilon.$$

Reader's Note: The Cauchy criterion gives us a way to test for convergence *without knowing the limit*. Instead of checking that terms get close to some value s, we check that terms get close to *each other*. This is especially useful for series, where computing the exact sum is often difficult or impossible.

Without loss of generality we can assume $m > n$. Then

$$|s_m - s_n| = \left| \sum_{k=n+1}^{m} a_k \right|.$$

1.3 Monotonic Sequences

Definition 1.1. A monotonic sequence converges if and only if it is bounded.

$\{s_n\}$ is **monotonically increasing** if and only if $s_n \leq s_{n+1}$, i.e. all the terms a_n are non-negative.

Reader's Note: Why does $s_n \leq s_{n+1}$ mean all terms are non-negative? Since $s_{n+1} - s_n = a_{n+1}$, requiring $s_n \leq s_{n+1}$ is the same as requiring $a_{n+1} \geq 0$. If all terms were negative, the partial sums would be monotonically *decreasing*, not increasing.

Corollary 1.2. *If $m = n + 1$ and $\sum a_n$ converges, then* $\lim_{n\to\infty} a_n = 0$.

Example. The **harmonic series**

$$\sum \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

1.4 Comparison Tests for Convergence

Theorem 1.3. *Given two sequences $\{a_n\}$ and $\{c_n\}$:*

(a) If $|a_n| \leq c_n$ for all $n \geq N$ and $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for all $n \geq N$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof. (a) Since $\sum c_n$ converges, by the Cauchy criterion, for every $\varepsilon > 0$ there exists N_0 such that for all $m > n \geq N_0$,

$$\sum_{k=n+1}^{m} c_k < \varepsilon.$$

For $n \geq \max(N, N_0)$, we have

$$\left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k| \leq \sum_{k=n+1}^{m} c_k < \varepsilon.$$

So $\sum a_n$ converges by the Cauchy criterion.

(b) Since $a_n \geq d_n \geq 0$ for $n \geq N$, we have $s_m - s_n = \sum_{k=n+1}^{m} a_k \geq \sum_{k=n+1}^{m} d_k$. Since $\sum d_n$ diverges, the partial sums of d_n are not Cauchy, so the partial sums of a_n are not Cauchy either. Thus $\sum a_n$ diverges.

□

Theorem 1.4 (Cauchy Condensation Test)**.** *If $a_0 \geq a_1 \geq \cdots \geq 0$, then $\sum a_n$ converges if and only if $\sum 2^k a_{2^k}$ converges.*

Proof. Since a_n is decreasing and non-negative, we group terms in blocks of powers of 2. For the partial sums $s_n = \sum_{k=0}^{n} a_k$, consider s_{2^n}:

$$\begin{aligned} s_{2^n} &= a_0 + a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^{n-1}} + \cdots + a_{2^n - 1}) \\ &\leq a_0 + a_1 + 2a_2 + 4a_4 + \cdots + 2^{n-1} a_{2^{n-1}} \\ &= a_0 + \sum_{k=0}^{n-1} 2^k a_{2^k}. \end{aligned}$$

So if $\sum 2^k a_{2^k}$ converges, then the partial sums s_{2^n} are bounded. Since $\{s_n\}$ is monotonically increasing (all terms are non-negative), $\{s_n\}$ is bounded, so $\sum a_n$ converges.

Conversely,

$$\begin{aligned} s_{2^n} &= a_0 + a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^{n-1}} + \cdots + a_{2^n - 1}) \\ &\geq a_0 + a_1 + 2a_3 + 4a_7 + \cdots + 2^{n-1} a_{2^n - 1} \\ &\geq a_0 + \frac{1}{2}(2a_2 + 4a_4 + \cdots + 2^n a_{2^n}) \\ &= a_0 + \frac{1}{2} \sum_{k=1}^{n} 2^k a_{2^k}. \end{aligned}$$

So if $\sum a_n$ converges (i.e. the partial sums are bounded), then $\sum 2^k a_{2^k}$ is bounded and monotonically increasing, hence converges. □

1.5 Special Series

Definition 1.5. The **geometric series** is

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n.$$

Let $s_n = 1 + x + x^2 + \cdots + x^n$. Then

$$x \cdot s_n = x + x^2 + x^3 + \cdots + x^{n+1}.$$

Subtracting,

$$x \cdot s_n - s_n = x^{n+1} - 1,$$

so $s_n(x-1) = x^{n+1} - 1$, and thus for $x \neq 1$,

$$s_n = \frac{x^{n+1}-1}{x-1}.$$

The geometric series converges if and only if $|x| < 1$, in which case $s_n \to \dfrac{1}{1-x}$.

Proof. ($\Rightarrow$) If $|x| \geq 1$, then $|x^n| \geq 1$ for all n, so $a_n = x^n \not\to 0$. Since the terms do not tend to zero, the series diverges.

($\Leftarrow$) If $|x| < 1$, we have

$$\left| s_n - \frac{1}{1-x} \right| = \left| \frac{1-x^{n+1}}{1-x} - \frac{1}{1-x} \right| = \frac{|x|^{n+1}}{|1-x|}.$$

Since $|x| < 1$, we have $|x|^{n+1} \to 0$ as $n \to \infty$, so $s_n \to \dfrac{1}{1-x}$. □

Definition 1.6. The ***p*-series** is

$$\sum \frac{1}{n^p}.$$

If $p > 1$ it converges; if $p \leq 1$ it diverges.

Proof. By the Cauchy condensation test, $\sum \frac{1}{n^p}$ converges if and only if

$$\sum 2^k \cdot \frac{1}{(2^k)^p} = \sum 2^k \cdot 2^{-kp} = \sum 2^{k(1-p)}$$

converges. This is a geometric series with ratio $r = 2^{1-p}$. It converges if and only if $|r| < 1$, i.e. $2^{1-p} < 1$, which holds if and only if $1 - p < 0$, i.e. $p > 1$. □

Theorem 1.7. *If $p > 1$, then $\sum \dfrac{1}{n(\log n)^p}$ converges.*

Proof. By the Cauchy condensation test, $\sum \frac{1}{n(\log n)^p}$ converges if and only if

$$\sum 2^k \cdot \frac{1}{2^k (\log 2^k)^p} = \sum \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum \frac{1}{k^p}$$

converges. Since $p > 1$, the p-series $\sum \frac{1}{k^p}$ converges, so the original series converges. □

1.6 The Number e

Definition 1.8. We define the number e by

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

Theorem 1.9. *The series $\sum \frac{1}{n!}$ converges.*

Proof. For $n \geq 1$, we have $n! \geq 2^{n-1}$, so

$$\frac{1}{n!} \leq \frac{1}{2^{n-1}} = \frac{2}{2^n}.$$

Since $\sum \frac{1}{2^n}$ is a convergent geometric series (with $|x| = \frac{1}{2} < 1$), the series $\sum \frac{1}{n!}$ converges by the comparison test. $\square$

Theorem 1.10. *e is irrational.*

Proof. Suppose for contradiction that $e = p/q$ for some $p, q \in \mathbb{N}$ with $q \geq 1$. Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

Then $q!\, s_q$ is an integer (since $q!/k!$ is an integer for $0 \leq k \leq q$), and $q!\, e = q!\, p/q$ is also an integer. Therefore

$$q!\,(e - s_q) = q! \sum_{k=q+1}^{\infty} \frac{1}{k!}$$

is a positive integer. But we can bound this:

$$\begin{aligned} q! \sum_{k=q+1}^{\infty} \frac{1}{k!} &= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots \\ &\leq \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots \\ &= \frac{1}{q+1} \cdot \frac{1}{1 - \frac{1}{q+1}} = \frac{1}{q} \leq 1. \end{aligned}$$

So $q!(e - s_q)$ is a positive integer that is at most $\frac{1}{q} \leq 1$. Since it is strictly less than 1 when $q \geq 2$ (and one checks e is not an integer directly), this is a contradiction. $\square$

1.7 Root and Ratio Tests

Theorem 1.11 (Root Test)**.** *Given $\sum a_n$, let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then:*

(a) If $\alpha < 1$, the series converges.

(b) If $\alpha > 1$, the series diverges.

(c) If $\alpha = 1$, the test is inconclusive.

Proof. **(a)** If $\alpha < 1$, choose β with $\alpha < \beta < 1$. Since $\alpha = \limsup \sqrt[n]{|a_n|}$, there exists N such that $\sqrt[n]{|a_n|} < \beta$ for all $n \geq N$, i.e. $|a_n| < \beta^n$. Since $\sum \beta^n$ is a convergent geometric series ($\beta < 1$), $\sum a_n$ converges by the comparison test.

(b) If $\alpha > 1$, then $\sqrt[n]{|a_n|} > 1$ for infinitely many n, so $|a_n| > 1$ for infinitely many n. Thus $a_n \not\to 0$, so the series diverges.

(c) Both $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ have $\alpha = 1$, but one diverges and the other converges. □

Theorem 1.12 (Ratio Test)**.** *Given $\sum a_n$ with $a_n \neq 0$:*

(a) If $\limsup_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1$, *the series converges.*

(b) If $\left|\dfrac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq N$, *the series diverges.*

(c) If $\limsup_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$, *the test is inconclusive.*

Proof. **(a)** If $\limsup \left|\frac{a_{n+1}}{a_n}\right| = r < 1$, choose β with $r < \beta < 1$. There exists N such that $\left|\frac{a_{n+1}}{a_n}\right| < \beta$ for all $n \geq N$. Then by induction,

$$|a_{N+k}| < \beta^k |a_N|$$

for all $k \geq 0$. Since $\sum \beta^k |a_N|$ is a convergent geometric series, $\sum a_n$ converges by comparison.

(b) If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq N$, then $|a_n| \geq |a_N| > 0$ for all $n \geq N$, so $a_n \not\to 0$ and the series diverges.

(c) Same examples as the root test: $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. □

Remark 1.13. The root test is at least as powerful as the ratio test: whenever the ratio test gives a conclusion, so does the root test. This is because

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| \geq \limsup \sqrt[n]{|a_n|}.$$

However, there are series where the root test succeeds but the ratio test is inconclusive.