Math 104: Introduction to Real Analysis
Lecture Notes

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Contents

1 Lecture 1: January 20, 2026

Lecture Overview: We begin by proving $\sqrt{2}$ is irrational, motivating the need for a number system without "gaps." This leads us to define **ordered sets** and the crucial **Least Upper Bound Property (LUBP)**—the defining feature of $\mathbb{R}$ that $\mathbb{Q}$ lacks. We then introduce **fields** as algebraic structures with addition and multiplication, and combine these ideas into **ordered fields**. The real numbers are the unique complete ordered field.

1.1 Ordered sets and the least-upper-bound property

Section Overview: This section motivates the need for the real numbers by showing that $\mathbb{Q}$ has "gaps"—$\sqrt{2}$ is irrational, yet we can get arbitrarily close to it with rationals. We develop the machinery of **ordered sets**: partial orders, total orders, upper/lower bounds, and the supremum/infimum. The central concept is the **Least Upper Bound Property (LUBP)**: every non-empty bounded-above subset has a supremum. This property distinguishes $\mathbb{R}$ from $\mathbb{Q}$ and is the foundation for all of real analysis. We prove that LUBP implies GLBP.

Consider the ancient problem from Greek times: can we write $\sqrt{2}$ as a quotient of two natural numbers?

Theorem 1.1. *$\sqrt{2}$ is irrational; that is, there do not exist $p, q \in \mathbb{N}$ such that $\sqrt{2} = \frac{p}{q}$.*

Proof. Suppose, for contradiction, that $\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$ (i.e., the fraction is in lowest terms).

Then $2 = \frac{p^2}{q^2}$, so $p^2 = 2q^2$.

This means p^2 is even, so p is even. Write $p = 2k$ for some $k \in \mathbb{N}$.

Then $(2k)^2 = 2q^2$, so $4k^2 = 2q^2$, hence $q^2 = 2k^2$.

This means q^2 is even, so q is even.

But then both p and q are even, contradicting $\gcd(p, q) = 1$. □

Now consider two sets:

$$A = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 < 2\}, \quad B = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 > 2\}.$$

Proposition 1.2. *A contains no largest element and B contains no smallest element.*

Proof. Let $p_0 \in A$. Define

$$q = p_0 + \frac{2 - p_0^2}{p_0^2 + 2}.$$

Since $p_0 \in A$, we have $p_0^2 < 2$, so $2 - p_0^2 > 0$. Thus $q > p_0$.

We claim $q \in A$, i.e., $q^2 < 2$. One can verify that

$$q^2 - 2 = \frac{(p_0^2 - 2)^2 \cdot (\text{positive})}{(p_0^2 + 2)^2}$$

which shows $q^2 < 2$ when $p_0^2 < 2$.

Hence A has no largest element.

A similar argument shows B has no smallest element. □

Definition 1.3 (1.3)**.** If A is any set, we write $x \in A$ to say that x is a **member** of A. Otherwise, $x \notin A$. The set that contains no elements is called the **empty set**, denoted $\emptyset$. If $A \neq \emptyset$, we say that A is **non-empty**.

If A, B are sets and $\forall x \in A$ we have $x \in B$, we say that $A \subset B$, or A is a **subset** of B. If there exists an element $x \in B$ with $x \notin A$, then A is a **proper subset** of B, denoted $A \subsetneq B$.

Example. $3 \in \mathbb{N}$, but $-1 \notin \mathbb{N}$. We have $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{N} \subsetneq \mathbb{Z}$ (since $-1 \in \mathbb{Z}$ but $-1 \notin \mathbb{N}$).

Definition 1.4. A **binary relation** on a set S is a set of ordered pairs $\langle x, y \rangle$ with $x, y \in S$.

Example. On $\mathbb{Z}$, the relation $\leq$ is the set $\{\langle x, y \rangle : x, y \in \mathbb{Z}, x \leq y\}$, e.g., $\langle 2, 5 \rangle$ is in the relation.

Definition 1.5. A **partial order** is a binary relation $\leq$ on S such that:

1. **Reflexive:** $\forall x \in S$, $x \leq x$.
2. **Anti-symmetric:** $\forall x, y \in S$, if $x \leq y$ and $y \leq x$, then $x = y$.
3. **Transitive:** $\forall x, y, z \in S$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Example. On the power set $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, the subset relation $\subseteq$ is a partial order (but not a total order, since $\{1\} \not\subseteq \{2\}$ and $\{2\} \not\subseteq \{1\}$).

Definition 1.6. A **total order** is a partial order with the additional axiom that any two elements are comparable. That is, for any $x, y \in S$, either $x \leq y$ or $y \leq x$ (non-exclusive).

Example. The usual $\leq$ on $\mathbb{R}$ is a total order: for any $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$.

Definition 1.7. An **ordered set** is a set equipped with a total order.

Example. $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are ordered sets.

Definition 1.8. Suppose S is an ordered set and $E \subset S$. If there exists $\beta \in S$ such that $x \leq \beta$ for all $x \in E$, we say β is an **upper bound** of E. Similarly, if there exists $\alpha \in S$ such that $\alpha \leq x$ for all $x \in E$, we say α is a **lower bound** of E.

Example. Let $E = (0, 1) \subset \mathbb{R}$. Then $1, 2, 100$ are all upper bounds of E, and $0, -5$ are lower bounds of E.

Definition 1.9. Suppose S is an ordered set and $E \subset S$ is bounded above. If there exists $\alpha \in S$ such that:

1. α is an upper bound of E, and
2. if $\gamma < \alpha$, then γ is not an upper bound of E,

then α is called the **least upper bound** of E (or **supremum**), denoted $\sup E$.

Example. $\sup(0, 1) = 1$ and $\sup[0, 1] = 1$ in $\mathbb{R}$.

Definition 1.10. Suppose S is an ordered set and $E \subset S$ is bounded below. If there exists $\alpha \in S$ such that:

1. α is a lower bound of E, and
2. if $\gamma > \alpha$, then γ is not a lower bound of E,

then α is called the **greatest lower bound** of E (or **infimum**), denoted $\inf E$.

Example. $\inf(0,1) = 0$ and $\inf[0,1] = 0$ in $\mathbb{R}$.

Remark 1.11. If $\sup E$ or $\inf E$ exists, it need not be an element of E. For example, the set $A = \{p \in \mathbb{Q} : p > 0, p^2 < 2\}$ has $\sup A = \sqrt{2}$ (in $\mathbb{R}$), but $\sqrt{2} \notin A$ since $\sqrt{2} \notin \mathbb{Q}$.

Definition 1.12. Let S be an ordered set.

1. S has the **least upper bound property** if for any non-empty $E \subset S$ that is bounded above, $\sup E$ exists in S.

2. S has the **greatest lower bound property** if for any non-empty $E \subset S$ that is bounded below, $\inf E$ exists in S.

Example. $\mathbb{R}$ has the LUBP (and hence GLBP). However, $\mathbb{Q}$ does not: the set $A = \{p \in \mathbb{Q} : p > 0, p^2 < 2\}$ is bounded above in $\mathbb{Q}$, but $\sup A = \sqrt{2} \notin \mathbb{Q}$.

Theorem 1.13 (LUBP implies GLBP)**.** *Suppose S is an ordered set with the least upper bound property. Let $B \subset S$, $B \neq \emptyset$, and suppose B is bounded below. Let L be the set of all lower bounds of B. Then $\alpha = \sup L$ exists in S, and $\alpha = \inf B$.*

Proof. First, $L \neq \emptyset$ since B is bounded below.

Second, L is bounded above: every $b \in B$ is an upper bound for L (since if $\ell \in L$, then $\ell \leq b$ by definition of lower bound).

By the LUBP, $\alpha = \sup L$ exists in S.

We claim $\alpha = \inf B$:

1. α is a lower bound of B: For any $b \in B$, b is an upper bound of L, so $\alpha \leq b$ (since α is the *least* upper bound of L).

2. α is the greatest lower bound: If $\gamma > \alpha$ and γ were a lower bound of B, then $\gamma \in L$, so $\gamma \leq \sup L = \alpha$, contradicting $\gamma > \alpha$. Thus γ is not a lower bound of B.

Thus $\alpha = \inf B$. $\square$

1.2 Fields

Section Overview: This section introduces the algebraic structure underlying $\mathbb{R}$. We define **groups** (sets with an operation having identity, inverses, and associativity) and **fields** (sets with addition and multiplication that behave like we expect from $\mathbb{Q}$ or $\mathbb{R}$). We sketch how to construct $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q}$ using equivalence relations. The key definition is an **ordered field**: a field that is also an ordered set, allowing us to combine algebraic operations with comparison. $\mathbb{R}$ is the unique complete ordered field.

Definition 1.14. A **binary operation** on S is a map $S \times S \to S$.

Definition 1.15. A **group** is a set G with a binary operation $+$ satisfying the following axioms:

1. **Identity:** There exists $0 \in G$ such that $a + 0 = 0 + a = a$ for all $a \in G$.

2. **Existence of inverse:** For every $a \in G$, there exists $-a \in G$ such that $a + (-a) = 0$.

3. **Associativity:** For all $a, b, c \in G$, $(a+b)+c = a+(b+c)$.

If we add a fourth axiom:

4. **Commutativity:** For all $a, b \in G$, $a+b = b+a$,

then G is called an **abelian group**.

Definition 1.16. A **field** is a set F with two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that:

1. $(F, +)$ is an abelian group with identity 0.
2. $(F \setminus \{0\}, \cdot)$ is an abelian group with identity 1.
3. **Distributivity:** For all $a, b, c \in F$, $a \cdot (b+c) = a \cdot b + a \cdot c$.

Example. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields. $\mathbb{Z}$ is not a field (e.g., 2 has no multiplicative inverse in $\mathbb{Z}$).

Zooming out, we can construct the number systems as follows:
The **natural numbers** $\mathbb{N}$ can be defined by the cardinality of iterated power sets of $\emptyset$:

$$0 = |\emptyset|, \quad 1 = |\mathcal{P}(\emptyset)|, \quad 2 = |\mathcal{P}(\mathcal{P}(\emptyset))|, \quad \ldots$$

The **integers** are defined as:

$$\mathbb{Z} = \{a - b : a, b \in \mathbb{N}\}.$$

Definition 1.17. An **equivalence relation** $\sim$ on a set S has the following properties:

1. **Reflexive:** $x \sim x$ for all $x \in S$.
2. **Symmetric:** If $x \sim y$, then $y \sim x$.
3. **Transitive:** If $x \sim y$ and $y \sim z$, then $x \sim z$.

The **rational numbers** are defined as:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{N},\ q \neq 0 \right\} / \sim$$

We can verify this is an equivalence relation: $\frac{p}{q} \sim \frac{r}{s}$ if and only if $ps = rq$.

Definition 1.18. An **ordered field** is a field F which is also an ordered set such that:

1. If $x, y, z \in F$ and $y < z$, then $x + y < x + z$.
2. If $x, y \in F$, $x > 0$, and $y > 0$, then $xy > 0$.

Proposition 1.19. *If $x > 0$ and $y < z$, then $xy < xz$.*

Proof. Since $y < z$, we have $z - y > 0$. Since $x > 0$ and $z - y > 0$, we have $x(z-y) > 0$. Thus $xz - xy > 0$, so $xy < xz$. □

2 Lecture 2: January 22, 2026

Lecture Overview: We construct the real numbers $\mathbb{R}$ as an ordered field with the Least Upper Bound Property (LUBP) containing $\mathbb{Q}$ as a subfield, using Dedekind cuts. We prove key properties of $\mathbb{R}$: the Archimedean property and density of $\mathbb{Q}$ in $\mathbb{R}$. Using the LUBP, we establish existence of nth roots of positive reals via a supremum argument. We discuss decimal/ternary representations and the Cantor set. We introduce the complex numbers $\mathbb{C}$ and prove $\mathbb{C}$ is not an ordered field. Finally, we define Euclidean spaces $\mathbb{R}^n$ with inner products and norms, and prove the Cauchy-Schwarz inequality.

2.1 Dedekind Cuts

Section Overview: We define Dedekind cuts as a way to construct the real numbers from the rationals.

Definition 2.1. A **cut** $\alpha \subset \mathbb{Q}$ is a nonempty, proper subset such that:

1. **Downward closed:** If $p \in \alpha$ and $q < p$, then $q \in \alpha$.
2. If $\sup \alpha$ exists, then $\sup \alpha \notin \alpha$.

The set of all cuts is ordered by inclusion: $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.

2.2 Field Operations on Cuts

Section Overview: We define addition and multiplication on cuts to make them into an ordered field.

Addition:

$$\alpha + \beta = \{r + s \mid r \in \alpha, s \in \beta\}$$

Additive identity:

$$0^* = \{p \in \mathbb{Q} \mid p < 0\}$$

Multiplication: For $\alpha > 0^*$ and $\beta > 0^*$:

$$\alpha\beta = \{p \in \mathbb{Q} \mid p \leq rs \text{ for some } r \in \alpha, r > 0 \text{ and } s \in \beta, s > 0\}$$

2.3 Least Upper Bound Property

Section Overview: We show that the set of cuts has the LUBP.

For a nonempty set E of cuts that is bounded above:

$$\sup E = \bigcup_{\alpha \in E} \alpha$$

2.4 Embedding $\mathbb{Q}$ into $\mathbb{R}$

Section Overview: We embed the rationals into the reals as a subfield.

For $p \in \mathbb{Q}$, define the cut:

$$p^* := \{q \in \mathbb{Q} \mid q < p\}$$

This embedding $p \mapsto p^*$ identifies $\mathbb{Q}$ as a subfield of $\mathbb{R}$.

2.5 Properties of $\mathbb{R}$

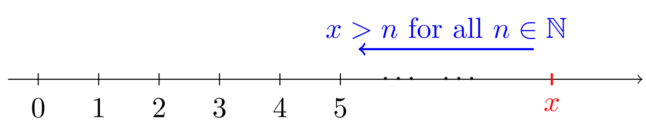
Section Overview: Having constructed $\mathbb{R}$, we now explore its key properties.

Theorem 2.2 (Archimedean Property)**.** *For any $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof. By contradiction. Suppose no such n exists, i.e., $nx \leq y$ for all $n \in \mathbb{N}$. Then the set $A = \{nx : n \in \mathbb{N}\}$ is bounded above by y. By the LUBP, $\sup A$ exists. Let $\alpha = \sup A$. Since $x > 0$, we have $\alpha - x < \alpha$, so $\alpha - x$ is not an upper bound for A. Thus there exists $m \in \mathbb{N}$ with $mx > \alpha - x$, which gives $(m+1)x > \alpha$. But $(m+1)x \in A$, contradicting that $\alpha = \sup A$. $\square$

Remark 2.3. The Archimedean property ensures there are no **infinitely large** elements (every element is bounded by some natural number) and no **infinitesimals** (positive elements smaller than $1/n$ for all n). This property is essential for proving that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example of a non-Archimedean field: Consider the field of rational functions $\mathbb{R}(x)$ with the ordering where x is declared to be larger than every real number (i.e., $x > r$ for all $r \in \mathbb{R}$). Then $x > n$ for all $n \in \mathbb{N}$, so the Archimedean property fails. In this field, $1/x$ is an infinitesimal: it is positive but smaller than $1/n$ for all $n \in \mathbb{N}$.



Theorem 2.4 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For any $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.*

Proof. Since $b - a > 0$, by the Archimedean property there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$, i.e., $nb - na > 1$. Thus there exists an integer m with $na < m < nb$. Then $a < \frac{m}{n} < b$, and $q = \frac{m}{n} \in \mathbb{Q}$. $\square$

2.6 The Roots of Reals

Section Overview: Having constructed $\mathbb{R}$ with the LUBP, we can now prove that nth roots of positive reals exist, resolving the gap in $\mathbb{Q}$ where $\sqrt{2}$ was missing.

Previously, we showed that $\sqrt{2} \notin \mathbb{Q}$. Now that we have constructed $\mathbb{R}$ with the LUBP, we can prove that nth roots exist.

Theorem 2.5 (Existence of nth Roots)**.** *For all $x \in \mathbb{R}_{>0}$ and for all $n \in \mathbb{Z}_{>0}$, there exists a unique $y \in \mathbb{R}_{>0}$ such that $y^n = x$.*

Proof. Let $E = \{t \in \mathbb{R}_{>0} : t^n < x\}$.

E **is non-empty:** We have

$$\left(\frac{x}{x+1}\right)^n < \frac{x}{x+1} < x,$$

so $\frac{x}{x+1} \in E$.

E **is bounded above:** (to be shown)

By the LUBP, $y = \sup E$ exists.

Claim: $y^n = x$.

Aside (Trichotomy): Since $\mathbb{R}$ is a totally ordered set, for any $a, b \in \mathbb{R}$, exactly one of the following holds: $a < b$, $a = b$, or $a > b$. Thus for y^n and x, exactly one of $y^n < x$, $y^n = x$, or $y^n > x$ holds. We show the first and third cases lead to contradictions.

Case 1: Suppose $y^n < x$. Then there exists $h > 0$ small enough such that $(y+h)^n < x$. But then $y + h \in E$, contradicting that $y = \sup E$.

Case 2: Suppose $y^n > x$. Then there exists $h > 0$ small enough such that $(y-h)^n > x$. But then $y - h$ is still an upper bound for E, contradicting that $y = \sup E$ (the *least* upper bound).

Therefore $y^n = x$. □

Note to the reader: This proof employs a fundamental technique in real analysis called a *supremum argument*. The strategy is:

1. **Define a set:** Construct a set E of elements that are "too small" (i.e., $t^n < x$).
2. **Apply LUBP:** Since E is nonempty and bounded above, $\sup E$ exists—this is where we crucially use that $\mathbb{R}$ has the Least Upper Bound Property.
3. **Use trichotomy:** By the trichotomy of total orders, the supremum y satisfies exactly one of $y^n < x$, $y^n = x$, or $y^n > x$.
4. **Eliminate by contradiction:** Show that $y^n < x$ contradicts y being an *upper* bound (we can go higher), and $y^n > x$ contradicts y being the *least* upper bound (we can find a smaller upper bound).

This technique appears repeatedly throughout analysis whenever we need to prove existence of a value with a specific property. A similar technique is employed in Exercise 7 (showing the existence of the logarithm).

2.7 Decimals, Binaries, Ternaries

Section Overview: We discuss representations of real numbers in different bases.

Observe that decimal representations come in the form

$$n_0 + \frac{n_1}{10} + \frac{n_2}{100} + \cdots = n_0 + \sum_{k=1}^{\infty} \frac{n_k}{10^k}$$

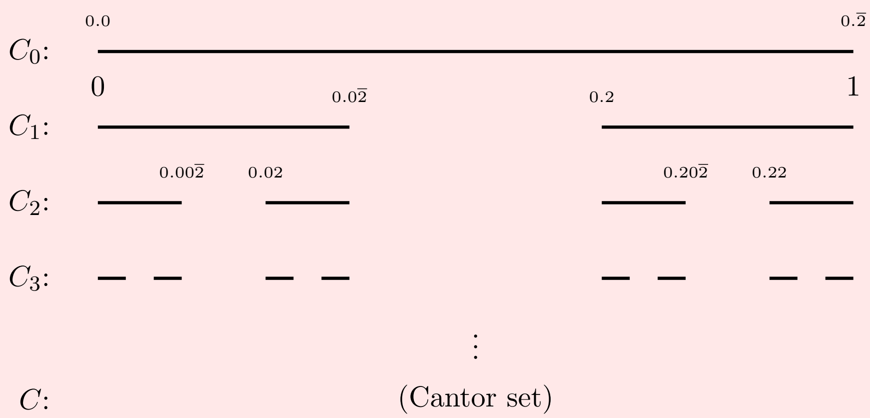
where $n_0 \in \mathbb{Z}$ and $n_k \in \{0, 1, 2, \ldots, 9\}$ for $k \geq 1$.

If we consider the set of partial sums

$$E = \left\{ n_0,\ n_0 + \frac{n_1}{10},\ n_0 + \frac{n_1}{10} + \frac{n_2}{100},\ \ldots \right\}$$

then $x = \sup E$.

Note: This construction is used to build the **Cantor set**. Starting with the interval $[0, 1]$, we iteratively remove the open middle third of each remaining interval:



Here the labels are *ternary* (base-3) expansions: e.g., $0.2_3 = \frac{2}{3}$, $0.02_3 = \frac{2}{9}$, $0.22_3 = \frac{8}{9}$, and $0.\overline{2}_3 = 0.222\ldots_3 = 1$.
The Cantor set $C = \bigcap_{n=0}^{\infty} C_n$ consists of all points in $[0, 1]$ whose ternary (base-3) expansion contains only the digits 0 and 2.

2.8 The Complex Field

Section Overview: We introduce the complex numbers $\mathbb{C}$ as an extension of $\mathbb{R}$.

Definition 2.6. The **complex numbers** are defined as

$$\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$$

where each element is of the form $z = a + bi$ with $a, b \in \mathbb{R}$ and $i^2 = -1$.

Theorem 2.7. $\mathbb{C}$ *is not an ordered field.*

Proof. By contradiction. Suppose $\mathbb{C}$ is an ordered field. By trichotomy, either $i > 0$ or $i < 0$ (since $i \neq 0$).

Case 1: If $i > 0$, then $i^2 > 0$ (since squares of nonzero elements are positive in an ordered field). But $i^2 = -1 < 0$, a contradiction.

Case 2: If $i < 0$, then $-i > 0$, so $(-i)^2 > 0$. But $(-i)^2 = i^2 = -1 < 0$, a contradiction.

Therefore $\mathbb{C}$ cannot be an ordered field. □

2.9 The Euclidean Spaces

Section Overview: We introduce Euclidean spaces $\mathbb{R}^n$ as spaces of ordered n-tuples.

Definition 2.8. The **Euclidean space** $\mathbb{R}^n$ is the set of all ordered n-tuples

$$\vec{x} = (x_1, x_2, x_3, \ldots, x_n)$$

where $x_i \in \mathbb{R}$ for each $i = 1, \ldots, n$.

For $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

Addition:

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

Scalar multiplication:

$$c\vec{x} = (cx_1, cx_2, \ldots, cx_n)$$

Inner product:

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Definition 2.9. An **inner product** over $\mathbb{R}$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that is:

- Symmetric bilinear and positive definite (for real vector spaces), or
- Hermitian sesquilinear and positive definite (for complex vector spaces).

Properties:

- **Symmetric:** $\langle x, y \rangle = \langle y, x \rangle$
- **Hermitian:** $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)
- **Bilinear:** Linear in both arguments:

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$$
$$\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$$

- **Sesquilinear:** Linear in one argument, conjugate-linear in the other:

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$$
$$\langle x, ay + bz \rangle = \bar{a}\langle x, y \rangle + \bar{b}\langle x, z \rangle$$

- **Positive definite:** $\langle x, x \rangle \geq 0$, with equality if and only if $x = 0$

Definition 2.10. The **norm** of a vector $\vec{x}$ is defined as

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Properties of a norm:

- **Triangle inequality:** $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
- **Absolute homogeneity:** $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$ for all $c \in \mathbb{R}$
- **Positive definite:** $\|\vec{x}\| \geq 0$, with equality if and only if $\vec{x} = \vec{0}$

Theorem 2.11 (Schwarz Inequality)**.** *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then*

$$\left| \sum_{j=1}^{n} a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

Proof. Put $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \bar{b}_j$ (all sums run over $j = 1, \ldots, n$). If $B = 0$, then $b_1 = \cdots = b_n = 0$ and the conclusion is trivial. Assume therefore that $B > 0$. We have

$$\begin{aligned}
\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)\overline{(Ba_j - Cb_j)} \\
&= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2 \\
&= B^2 A - B|C|^2 \\
&= B(AB - |C|^2).
\end{aligned}$$

Since the left side is a sum of non-negative terms, $B(AB - |C|^2) \geq 0$. Since $B > 0$, we conclude $AB - |C|^2 \geq 0$, i.e.,

$$\left| \sum a_j \bar{b}_j \right|^2 \leq \sum |a_j|^2 \sum |b_j|^2. \qquad \square$$

3 Lecture 3: January 27, 2026

Lecture Overview: We develop the set-theoretic foundations needed for analysis. We define **functions** as special relations and introduce key properties: surjectivity, injectivity, and bijectivity. Using bijections, we define when two sets have the same **cardinality**, leading to the notions of **finite**, **infinite**, and **countable** sets.

3.1 Functions

Section Overview: We define functions as relations with a uniqueness property, introduce images and preimages, and classify functions as injective (one-to-one), surjective (onto), or bijective (both).

Definition 3.1. A **function** $f : A \to B$ is a relation $R \subseteq A \times B$ such that for all $x \in A$, there exists a unique $y \in B$ with $(x, y) \in R$. This y is denoted $f(x)$. The set A is called the **domain** of f, and B is the **codomain** (or **range**).

Remark 3.2. Not all relations are functions. A relation fails to be a function if some element of A maps to multiple elements of B, or to no element at all.

Definition 3.3. Let $f : A \to B$ be a function.

- If $E \subseteq A$, the **image** of E under f is $f(E) = \{f(x) : x \in E\}$.
- If $F \subseteq B$, the **preimage** of F under f is $f^{-1}(F) = \{x \in A : f(x) \in F\}$.

Definition 3.4. Let $f : A \to B$ be a function.

- f is **surjective** (or **onto**) if $f(A) = B$, i.e., for every $y \in B$, there exists $x \in A$ with $f(x) = y$.
- f is **injective** (or **one-to-one**) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, i.e., distinct inputs map to distinct outputs.
- f is **bijective** if it is both injective and surjective.

3.2 Equivalence Relations

Section Overview: We review equivalence relations and observe how function properties relate to composition and inverses.

Remark 3.5. Recall from Lecture 1 that an equivalence relation satisfies:

- **Reflexivity:** $x \sim x$ (like the identity function)
- **Symmetry:** $x \sim y \Rightarrow y \sim x$ (like invertible functions: if $f : A \to B$, then $f^{-1} : B \to A$)
- **Transitivity:** $x \sim y$ and $y \sim z \Rightarrow x \sim z$ (like composition: $f : A \to B$ and $g : B \to C$ give $g \circ f : A \to C$)

3.3 Cardinality

Section Overview: We define cardinality using bijections, then classify sets as finite, infinite, or countable.

Definition 3.6. Two sets A and B have the same **cardinality**, written $A \sim B$, if there exists a bijection $f : A \to B$.

Definition 3.7. Let $[n] = \{1, 2, 3, \dots, n\}$. We say that a set A is:

1. **finite** if there exists $n \in \mathbb{N}$ such that $A \sim [n]$,
2. **infinite** if A is not finite,
3. **countable** if $A \sim \mathbb{N}$.

Example. $\mathbb{Z}$ is countable. We can list the integers as:

$$0, 1, -1, 2, -2, 3, -3, \dots$$

This defines a bijection $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

(where we start $\mathbb{N}$ at 0). Thus $\mathbb{Z} \sim \mathbb{N}$, so $\mathbb{Z}$ is countable.

Definition 3.8. A **sequence** is a function $f : \mathbb{N} \to X$ where X is some set. If we denote $f(n)$ by x_n, then we write the sequence as $(x_n)_{n=1}^{\infty}$.

Remark 3.9. If A is a countable set, then there exists a surjection $f : \mathbb{N} \to A$. We say that A can be **arranged in a sequence**: the elements of A can be listed as $f(1), f(2), f(3), \dots$

Note: The **set difference** $A \setminus B$ (read "A minus B") is defined as

$$A \setminus B = \{x \in A : x \notin B\}.$$

Theorem 3.10. *Every infinite subset of a countable set is countable.*

Proof. Let A be a countable set and let $B \subseteq A$ be infinite. Since A is countable, we can arrange its elements as a sequence $(x_n)_{n=1}^{\infty}$.

We construct a bijection $f : \mathbb{N} \to B$ inductively. Let $B_0 = \emptyset$. For each $k \geq 1$:

- Let $n_k = \min\{n \in \mathbb{N} : x_n \in B \setminus B_{k-1}\}$
- Define $f(k) = x_{n_k}$
- Set $B_k = B_{k-1} \cup \{x_{n_k}\}$

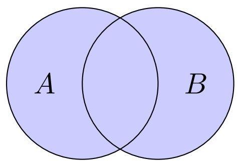
At each step, x_{n_k} is assigned position $k = |B_{k-1}| + 1$ in our enumeration of B. Since B is infinite, we never exhaust it, so each n_k exists.

This f is a bijection: it is injective since each element is added exactly once, and surjective since every element of B appears at some position n in the sequence (x_n) and will eventually be enumerated.

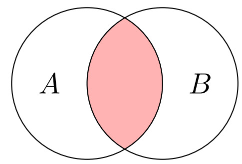
Thus $B \sim \mathbb{N}$, so B is countable. $\square$

3.4 Unions and Intersections of Sets

Section Overview: We define unions and intersections of collections of sets, including arbitrary (possibly infinite) unions and intersections.



$A \cup B$ (union)



$A \cap B$ (intersection)

Definition 3.11. Let A and B be sets, and let $f : A \to \mathcal{P}(B)$ be a function assigning to each $\alpha \in A$ a subset $f(\alpha) \subseteq B$. We define:

- The **union** of the family is

$$\bigcup_{\alpha \in A} f(\alpha) = \{x \in B : x \in f(\alpha) \text{ for some } \alpha \in A\}.$$

- The **intersection** of the family is

$$\bigcap_{\alpha \in A} f(\alpha) = \{x \in B : x \in f(\alpha) \text{ for all } \alpha \in A\}.$$

Theorem 3.12 (De Morgan's Laws)**.** *Let A and B be subsets of a universal set U. Then:*

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

Proof. We prove (1); the proof of (2) is similar.

($\subseteq$) Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$, so $x \notin A$ and $x \notin B$. Thus $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.

($\supseteq$) Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$. Thus $x \notin A \cup B$, so $x \in (A \cup B)^c$. $\square$

Note: To prove two sets are equal, $X = Y$, a standard technique is the **mutual subset argument**: show $X \subseteq Y$ and $Y \subseteq X$. For each direction, take an arbitrary element of one set and show it belongs to the other.

Theorem 3.13 (Distributive Law)**.** *Let A, B, and C be sets. Then*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof. ($\subseteq$) Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, either $x \in B$ or $x \in C$.

- If $x \in B$, then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.

- If $x \in C$, then $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$.

($\supseteq$) Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$.

- If $x \in A \cap B$, then $x \in A$ and $x \in B \subseteq B \cup C$, so $x \in A \cap (B \cup C)$.
- If $x \in A \cap C$, then $x \in A$ and $x \in C \subseteq B \cup C$, so $x \in A \cap (B \cup C)$.

$\square$

3.5 The (Un)countability of Number Systems

Section Overview: We apply our results to the classical number systems. We show $\mathbb{Q}$ is countable (as a countable union of countable sets), but $\mathbb{R}$ is **uncountable** using Cantor's diagonal argument on binary sequences. This reveals a hierarchy of infinities: $|\mathbb{N}| = |\mathbb{Q}| < |\mathbb{R}|$.

Theorem 3.14. *Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of countable sets. Then*

$$\bigcup_{n=1}^{\infty} E_n$$

is countable.

Proof. Since each E_n is countable, we can enumerate its elements as

$$E_n = \{x_{n,1}, x_{n,2}, x_{n,3}, \ldots\}.$$

Arrange all elements in an infinite grid:

$$\begin{array}{cccccc}
E_1: & \overset{1}{x_{1,1}} & \overset{2}{x_{1,2}} & \overset{4}{x_{1,3}} & \overset{7}{x_{1,4}} & \cdots \\
E_2: & \overset{3}{x_{2,1}} & \overset{5}{x_{2,2}} & \overset{8}{x_{2,3}} & x_{2,4} & \cdots \\
E_3: & \overset{6}{x_{3,1}} & \overset{9}{x_{3,2}} & x_{3,3} & x_{3,4} & \cdots \\
E_4: & \overset{10}{x_{4,1}} & x_{4,2} & x_{4,3} & x_{4,4} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

We enumerate the union by traversing diagonals: first $x_{1,1}$, then $x_{1,2}, x_{2,1}$, then $x_{1,3}, x_{2,2}, x_{3,1}$, and so on. The k-th diagonal contains all $x_{n,m}$ with $n + m = k + 1$.

This gives a surjection from $\mathbb{N}$ onto $\bigcup_{n=1}^{\infty} E_n$ (skipping repeats if sets overlap). Thus the union is countable. $\square$

Note: The **diagonal argument** is a powerful technique for enumerating countable unions. By arranging elements in a grid and traversing along diagonals, we reduce a "two-dimensional" infinite collection to a "one-dimensional" sequence.

Theorem 3.15. *If A is countable, then A^n is countable for all $n \in \mathbb{N}$.*

Proof. By induction on n.

Base case ($n = 1$): $A^1 = A$ is countable by assumption.

Inductive hypothesis: Assume A^n is countable for some $n \geq 1$.

Inductive step: We show A^{n+1} is countable. Observe that

$$A^{n+1} = A^n \times A.$$

By the inductive hypothesis, A^n is countable, so we can enumerate it as $A^n = \{b_1, b_2, b_3, \ldots\}$. Since A is countable by assumption, we can write $A = \{a_1, a_2, a_3, \ldots\}$. The Cartesian product $A^n \times A$ can then be arranged in a grid:

$$\begin{array}{cccc}
(b_1, a_1) & (b_1, a_2) & (b_1, a_3) & \cdots \\
(b_2, a_1) & (b_2, a_2) & (b_2, a_3) & \cdots \\
(b_3, a_1) & (b_3, a_2) & (b_3, a_3) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

By the diagonal argument, $A^n \times A$ is countable.

Conclusion: By the principle of mathematical induction, A^n is countable for all $n \in \mathbb{N}$. $\square$

Corollary 3.16. $\mathbb{Q}$ *is countable.*

Proof. For each $n \in \mathbb{N}$, let $E_n = \left\{\frac{m}{n} : m \in \mathbb{Z}\right\}$ be the set of rationals with denominator n. Each E_n is countable (since $E_n \sim \mathbb{Z}$ and $\mathbb{Z}$ is countable). Then

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} E_n$$

is a countable union of countable sets, hence countable. $\square$

Theorem 3.17 ($\mathbb{R}$ is uncountable)**.** *The set of binary sequences $\{0, 1\}^{\mathbb{N}}$ is uncountable.*

Proof. By contradiction. Suppose $\{0, 1\}^{\mathbb{N}}$ is countable. Then we can list all binary sequences as $s_1, s_2, s_3, \ldots$ where each $s_n = (s_{n,1}, s_{n,2}, s_{n,3}, \ldots)$:

$$\begin{array}{lcccccc}
 & \text{pos 1} & \text{pos 2} & \text{pos 3} & \text{pos 4} & \text{pos 5} & \\
s_1: & 0 & 1 & 0 & 1 & 1 & \cdots \\
s_2: & 1 & 1 & 0 & 0 & 1 & \cdots \\
s_3: & 0 & 0 & 1 & 1 & 0 & \cdots \\
s_4: & 1 & 0 & 1 & 0 & 0 & \cdots \\
s_5: & 0 & 1 & 1 & 1 & 1 & \cdots \quad \text{flip diagonal} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
d: & 1 & 0 & 0 & 1 & 0 & \cdots
\end{array}$$

Construct a new sequence $d = (d_1, d_2, d_3, \ldots)$ by flipping the diagonal entries:

$$d_n = \begin{cases} 1 & \text{if } s_{n,n} = 0 \\ 0 & \text{if } s_{n,n} = 1 \end{cases}$$

Then d differs from s_n in the n-th position for every $n \in \mathbb{N}$. Thus $d \neq s_n$ for all n, so d is not in our list. But $d \in \{0,1\}^{\mathbb{N}}$, contradicting that our list contains all binary sequences.

Therefore $\{0,1\}^{\mathbb{N}}$ is uncountable. □

Note: This is **Cantor's diagonal argument**. Since there is a bijection between $\{0,1\}^{\mathbb{N}}$ and $\mathbb{R}$ (via binary expansions), this proves $\mathbb{R}$ is uncountable. The key insight is that any proposed enumeration can be "diagonalized" to produce a missing element.

Exercise: An **algebraic number** is a solution to a polynomial equation with coefficients in $\mathbb{Q}$:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad a_i \in \mathbb{Q}.$$

Is the set of algebraic numbers countable?

Proof: Yes. We proceed by induction on the degree of the polynomial.

Base case ($n = 1$): A degree-1 polynomial $mx + b = 0$ has solution $x = -b/m \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable, the set of algebraic numbers of degree 1 is countable.

Inductive step: Let A_n denote the set of algebraic numbers that are roots of some polynomial of degree at most n. Assume A_n is countable. Consider numbers of the form

$$\{a + b \cdot \sqrt[n+1]{z} : a, b, z \in A_n\}.$$

This set is countable since A_n^3 is countable (as a finite Cartesian product of a countable set). More generally, the set of polynomials of degree $n+1$ with coefficients in $\mathbb{Q}$ is $\mathbb{Q}^{n+2}$, which is countable. Each such polynomial has at most $n+1$ roots, so the roots form a countable union of finite sets, hence A_{n+1} is countable.

Conclusion: The set of all algebraic numbers is $\bigcup_{n=1}^{\infty} A_n$, a countable union of countable sets, hence countable.

Note: A real number that is *not* algebraic is called **transcendental**. Since the algebraic numbers are countable but $\mathbb{R}$ is uncountable, transcendental numbers must exist—in fact, "most" real numbers are transcendental! Examples include π, e, and $\tau = 2\pi$. Proving that a specific number is transcendental is typically very difficult: Lindemann proved π is transcendental in 1882, and Hermite proved e is transcendental in 1873.

Remark 3.18. Recall the **Cantor set** from Lecture 2: the set of all points in $[0,1]$ whose ternary expansion uses only the digits 0 and 2. The Cantor set is in bijection with $\{0,2\}^{\mathbb{N}} \sim \{0,1\}^{\mathbb{N}}$ (just map $0 \mapsto 0$ and $2 \mapsto 1$). By the same diagonal argument, the Cantor set is uncountable—despite being "sparse" (it contains no intervals and has measure zero).

4 Lecture 4: January 29, 2026

Lecture Overview: We introduce the abstract notion of a topological space and its key building blocks: open sets, bases, and the topology they generate. We then specialize to metric spaces, where open balls form a natural basis, and develop the metric topology with epsilon-delta techniques. Finally, we study closed sets, closures, and limit points — the dual perspective to open sets — along with boundedness, convexity, convergence, and the equivalence of the product and metric topologies on $\mathbb{R}^n$.

Symbol	Meaning
$\mathcal{T}$	A topology (collection of open sets) on X
$\mathcal{B}$	A basis for a topology
(X, d)	A metric space (set X with metric d)
$B(x, \varepsilon)$	Open ball of radius ε centered at x
E	A subset of X
$\overline{E}$	Closure of E (smallest closed set containing E)
E'	Derived set (set of all limit points of E)
$E \setminus E'$	Isolated points of E
Open set	A set $U \in \mathcal{T}$; every point has a neighborhood inside U
Closed set	Complement of an open set; contains all its limit points ($E' \subseteq E$)

4.1 Topological Spaces

Section Overview: We define a topology on a set X via three axioms governing open sets, then explore examples (discrete and cofinite topologies). We introduce the notion of a basis for a topology and show that every open set is a union of basis elements.

Definition 4.1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. For all $\mathcal{S} \subseteq \mathcal{T}$, $\bigcup_{S \in \mathcal{S}} S \in \mathcal{T}$ (arbitrary unions).
3. If $\mathcal{S} \subseteq \mathcal{T}$ is finite, then $\bigcap_{S \in \mathcal{S}} S \in \mathcal{T}$ (finite intersections).

Remark 4.2. The elements of $\mathcal{T}$ are called **open sets**.

Example. The **discrete topology** on a set X is $\mathcal{T} = \mathcal{P}(X)$, the power set of X. In this topology, every subset of X is open. For instance, $\mathbb{Z} \subseteq \mathbb{R}$ inherits the discrete topology from the standard topology on $\mathbb{R}$: every subset of $\mathbb{Z}$ is open.

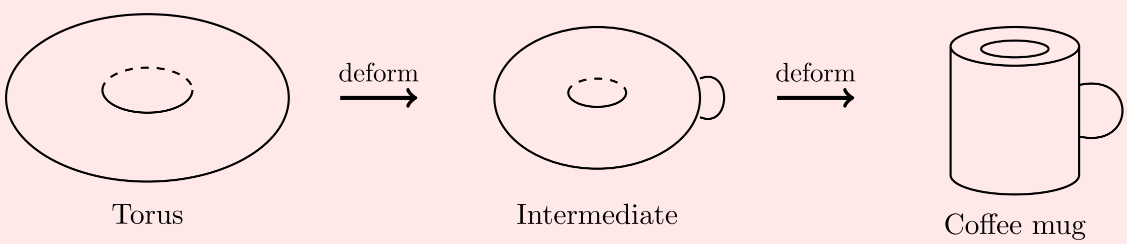
Why is $\mathbb{Z}$ discrete in $\mathbb{R}$? For any $n \in \mathbb{Z}$, the interval $(n - \frac{1}{2}, n + \frac{1}{2}) \cap \mathbb{Z} = \{n\}$, so every singleton $\{n\}$ is open in the subspace topology on $\mathbb{Z}$. Since arbitrary unions of open sets are open, every subset of $\mathbb{Z}$ is open. Intuitively, the integers are "isolated" — there is space around each one with no other integers nearby. The discrete topology is the topology where every set is open, which is the finest (most open sets) possible topology on a set.

Example. The **finite-complement topology** (or cofinite topology) on a set X: a subset $U \subseteq X$ is open if and only if $X \setminus U$ is finite (or $U = \emptyset$).

We verify the three topology axioms:

1. $\emptyset$ is open by convention. X is open since $X \setminus X = \emptyset$ is finite.
2. Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets. Then
$$X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} (X \setminus U_\alpha).$$
Each $X \setminus U_\alpha$ is finite, so the intersection is a subset of any one of them, hence finite. Thus $\bigcup U_\alpha$ is open.
3. Let $U_1, \ldots, U_n$ be finitely many open sets. Then
$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i).$$
Each $X \setminus U_i$ is finite, and a finite union of finite sets is finite. Thus $\bigcap_{i=1}^{n} U_i$ is open.

How does this relate to shapes like a torus? The axioms above define topology in full generality — they tell us what it means for sets to be "open" without any reference to distance or geometry. A shape like a torus is a *topological space*: a set of points (the surface) equipped with a topology (which subsets count as open). Two shapes are "the same" topologically if there is a continuous bijection with a continuous inverse (a homeomorphism) between them. The famous "coffee mug = donut" equivalence means there is such a map between them. The abstract axioms here are the foundation: they capture exactly the structure needed to define continuity, connectedness, and compactness, which are the properties that let us distinguish a torus from a sphere without ever measuring distances or angles.



Both have exactly one hole — they are homeomorphic (genus 1 surfaces).

Definition 4.3. A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X such that:

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and for all $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$ and $x \in B_3$.

The **topology generated by $\mathcal{B}$** is defined as follows: $U \in \mathcal{T}$ if and only if for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

Example. The **order topology** on $\mathbb{R}$ is generated by the basis of open intervals (a, b) for $a < b$.

Lemma 4.4. *Let $\mathcal{B}$ be a basis for the topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Let $\mathcal{U}$ denote the collection of all unions of elements of $\mathcal{B}$. We show $\mathcal{T} = \mathcal{U}$.

($\supseteq$) Each $B \in \mathcal{B}$ is open (since for any $x \in B$, the element B itself witnesses $x \in B \subseteq B$). Since $\mathcal{T}$ is closed under arbitrary unions, every union of elements of $\mathcal{B}$ is in $\mathcal{T}$. Thus $\mathcal{U} \subseteq \mathcal{T}$.

($\subseteq$) Let $U \in \mathcal{T}$. For each $x \in U$, by definition of the topology generated by $\mathcal{B}$, there exists $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$. Then

$$U = \bigcup_{x \in U} B_x,$$

which is a union of elements of $\mathcal{B}$. Thus $U \in \mathcal{U}$, so $\mathcal{T} \subseteq \mathcal{U}$. □

4.2 Metric Topology

Section Overview: We define metrics, open balls, and the metric topology they generate. Key results include the characterization of open sets via epsilon-balls, the product topology equaling the metric topology on $\mathbb{R}^n$, and the epsilon-delta proof technique that recurs throughout analysis.

Definition 4.5. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V \mid U \text{ open in } X,\ V \text{ open in } Y\}.$$

Definition 4.6. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

1. $d(x, y) \geq 0$, with equality if and only if $x = y$ (positive definiteness).
2. $d(x, y) = d(y, x)$ (symmetry).
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Example. The **Euclidean metric** on $\mathbb{R}$ (or $\mathbb{R}^n$) is defined by $d(x, y) = |x - y|$. The topology induced by this metric is called the **Euclidean topology**.

Definition 4.7. Let d be a metric on X. The **ε-open ball** centered at $x \in X$ is

$$B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Theorem 4.8. *The collection $\{B(x, \varepsilon) \mid x \in X,\ \varepsilon > 0\}$ forms a basis for a topology on X. The resulting topology is called the* ***metric topology*** *induced by d.*

Proof. We verify the two basis axioms for $\mathcal{B} = \{B(x, \varepsilon) \mid x \in X,\ \varepsilon > 0\}$.

1. For any $x \in X$, we have $x \in B(x, 1)$, so every point of X is contained in some basis element.
2. Let $B(x_1, \varepsilon_1), B(x_2, \varepsilon_2) \in \mathcal{B}$ and let $y \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$. By the lemma, there exist $\delta_1, \delta_2 > 0$ such that $B(y, \delta_1) \subseteq B(x_1, \varepsilon_1)$ and $B(y, \delta_2) \subseteq B(x_2, \varepsilon_2)$. Setting $\delta = \min(\delta_1, \delta_2)$, we have

   $$y \in B(y, \delta) \subseteq B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2).$$

   Since $B(y, \delta) \in \mathcal{B}$, the second basis axiom is satisfied.

□

Epsilon-delta proof technique. This is a recurring pattern in analysis. The goal is to show some property holds "locally" — that is, within a small neighborhood of a point. The approach:

1. **Identify what you need:** You want to find a $\delta > 0$ such that something (a ball, a set, a bound) holds within distance δ of your point.

2. **Use what you're given:** You typically start with some $\varepsilon > 0$ that gives you room to work with (e.g., your point lies inside a ball of radius ε).

3. **Compute the gap:** Figure out how much room you have — often $\delta = \varepsilon - d(\text{point}, \text{center})$ or $\delta = \min(\delta_1, \delta_2)$ when intersecting constraints.

4. **Verify with the triangle inequality:** Chain distances together to show your choice of δ works.

When you see "for all . . . there exists $\delta > 0$," think: *How much room do I have, and how do I stay within it?*

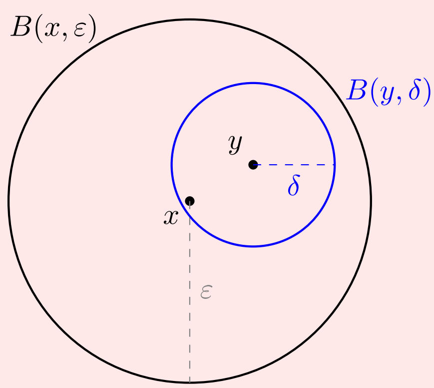
Lemma 4.9. *For every $y \in B(x, \varepsilon)$, there exists $\delta > 0$ such that $B(y, \delta) \subseteq B(x, \varepsilon)$.*

Proof. Let $y \in B(x, \varepsilon)$, so $d(x, y) < \varepsilon$. Set $\delta = \varepsilon - d(x, y) > 0$. For any $z \in B(y, \delta)$, we have $d(y, z) < \delta$, and by the triangle inequality:

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = d(x, y) + (\varepsilon - d(x, y)) = \varepsilon.$$

Thus $z \in B(x, \varepsilon)$, so $B(y, \delta) \subseteq B(x, \varepsilon)$. □

Intuition: If y is inside the open ball $B(x, \varepsilon)$, then y is strictly closer than ε to x, so there is some "room left over." We can fit a smaller ball around y that still stays inside the original ball. Concretely, $\delta = \varepsilon - d(x, y) > 0$ works.



Theorem 4.10. *A set U is open in the metric topology on X induced by d if and only if for all $y \in U$, there exists $\delta > 0$ such that $B_d(y, \delta) \subseteq U$.*

Proof. ($\Rightarrow$) This follows directly from the definition of the topology generated by a basis.

($\Leftarrow$) For each $y \in U$, choose $\delta_y > 0$ such that $B_d(y, \delta_y) \subseteq U$. Then

$$U = \bigcup_{y \in U} B_d(y, \delta_y).$$

Each $B_d(y, \delta_y)$ is open (it is a basis element), and an arbitrary union of open sets is open by the second topology axiom. Thus U is open. $\square$

4.3 Closed Sets

Section Overview: We define closed sets as complements of open sets and establish their dual properties (arbitrary intersections, finite unions). We then introduce closures, limit points, and isolated points, proving that $\overline{E} = E \cup E'$. Examples include the Cantor set and sets with isolated points. We also define boundedness (a metric-dependent property), convexity, and convergence of sequences in topological spaces.

Definition 4.11. A set $C \subseteq X$ is **closed** if its complement $X \setminus C$ is open.

Properties of closed sets (dual to the open set axioms):

1. $\emptyset$ and X are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

Definition 4.12. The **closure** of a set $E \subseteq X$ is

$$\overline{E} = \bigcap \{C \subseteq X \mid C \text{ is closed and } E \subseteq C\}.$$

Theorem 4.13. *$x \in \overline{E}$ if and only if for every open set U containing x, $U \cap E \neq \emptyset$. (We say U is a **neighborhood** of x.)*

Proof. We prove the contrapositive: $x \notin \overline{E}$ if and only if there exists an open set U containing x with $U \cap E = \emptyset$.

($\Rightarrow$) If $x \notin \overline{E}$, then there exists a closed set $C \supseteq E$ with $x \notin C$. Let $U = X \setminus C$. Then U is open, $x \in U$, and $U \cap E \subseteq U \cap C = \emptyset$.

($\Leftarrow$) If there exists an open set U with $x \in U$ and $U \cap E = \emptyset$, then $C = X \setminus U$ is closed, $E \subseteq C$, and $x \notin C$. Thus $x \notin \overline{E}$. $\square$

Proof by contrapositive. To prove $P \Rightarrow Q$, it is equivalent to prove $\neg Q \Rightarrow \neg P$. This is often easier when the negation is more concrete to work with. In the proof above, showing "$x \notin \overline{E}$ implies there exists an open set missing E" is more direct than working with the original statement, because $x \notin \overline{E}$ hands us a specific closed set to work with. As a rule of thumb: if the statement you want to prove starts with "for all," its contrapositive starts with "there exists" — and existential statements are often easier to prove since you only need to produce one witness.

P	Q	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

The columns for $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are identical — the two statements are logically equivalent.

Definition 4.14. A point x is a **limit point** of E if every neighborhood U of x contains some $y \neq x$ with $y \in U \cap E$.

Theorem 4.15. *$\overline{E} = E \cup E'$, where E' denotes the set of all* ***limit points*** *of E.*

Proof. ($\supseteq$) If $x \in E$, then for every open U containing x, we have $x \in U \cap E \neq \emptyset$, so $x \in \overline{E}$. If $x \in E'$, then every neighborhood U of x contains some $y \neq x$ with $y \in E$, so $U \cap E \neq \emptyset$, and thus $x \in \overline{E}$.

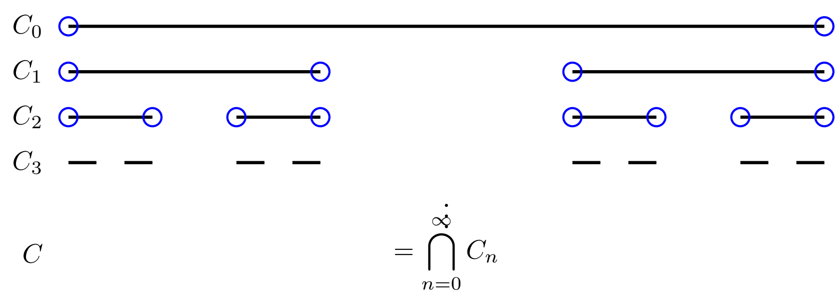
($\subseteq$) Suppose $x \in \overline{E}$ and $x \notin E$. Then for every neighborhood U of x, $U \cap E \neq \emptyset$. Since $x \notin E$, the point witnessing $U \cap E \neq \emptyset$ must be some $y \neq x$ with $y \in E$. Thus $x \in E'$. $\square$

Keeping E, $\overline{E}$, and E' straight.

- E — the original set.
- E' — the *limit points* (or *derived set*) of E: points x such that every neighborhood of x contains some *other* point of E. Note x need not belong to E itself.
- $\overline{E} = E \cup E'$ — the *closure* of E: the smallest closed set containing E. It includes the points of E together with any "boundary" points that E accumulates toward.

For example, if $E = (0,1) \subset \mathbb{R}$, then $E' = [0,1]$ (every neighborhood of 0 or 1 hits $(0,1)$), and $\overline{E} = E \cup E' = [0,1]$.

Example. The Cantor set. Let C be the Cantor set, constructed by repeatedly removing the open middle third of each interval. Despite the intervals shrinking at each stage, the endpoints of every removed interval remain in C and are limit points of C.



The circled endpoints at each stage are never removed — they persist through every C_n and thus belong to $C = \bigcap_{n=0}^{\infty} C_n$. Moreover, every neighborhood of such an endpoint contains points from the remaining intervals, making them limit points of C. In fact, C is closed ($C = \overline{C}$) and every point of C is a limit point: $C' = C$.

Corollary 4.16. *E is closed if and only if $E' \subseteq E$.*

Points of $E \setminus E'$ are called **isolated points**.

Example. Let $E = (0, \frac{1}{2}) \cup \{1\}$. The set of limit points is $E' = [0, \frac{1}{2}]$. Note that $1 \in E$ but $1 \notin E'$: for instance, the neighborhood $(\frac{3}{4}, \frac{5}{4})$ contains no point of E other than 1 itself. Thus 1 is an isolated point of E.

Conversely, $0 \notin E$ and $\frac{1}{2} \notin E$, yet both are limit points: for any $\varepsilon > 0$, the neighborhoods $(0 - \varepsilon, 0 + \varepsilon)$ and $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ each contain points of $(0, \frac{1}{2}) \subseteq E$. The key takeaway: limit points can lie *outside* the set, and points *inside* the set need not be limit points.

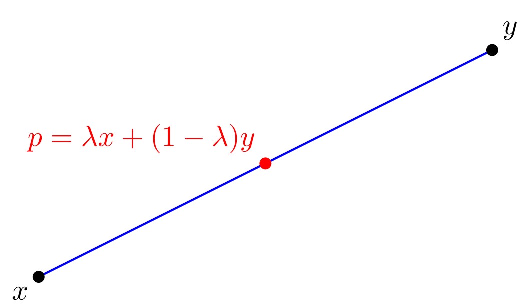
Example. Consider the set $E = (0,1) \setminus \{\frac{1}{2}\}$: the open interval $(0,1)$ with a hole at $\frac{1}{2}$. Although $\frac{1}{2} \notin E$, it is a limit point of E: for any $\varepsilon > 0$, the neighborhood $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ contains points of E other than $\frac{1}{2}$. Being in the set and being a limit point are independent properties.

Definition 4.17. A subset E of a metric space (X, d) is **bounded** if there exist $x \in X$ and $M > 0$ such that $E \subseteq B(x, M)$, i.e., $d(x, y) < M$ for all $y \in E$.

Remark 4.18. Boundedness is a metric-dependent property, not a topological one. The same set can be bounded under one metric and unbounded under another. For example, $\mathbb{R}$ with the usual metric $d(x,y) = |x - y|$ is unbounded. However, define the *bounded metric* $\bar{d}(x,y) = \min(|x-y|, 1)$. This induces the same topology on $\mathbb{R}$ (the same sets are open), but now $\mathbb{R}$ is bounded: $\bar{d}(x,y) \leq 1$ for all x, y, so $\mathbb{R} \subseteq B(0,2)$. Since two metrics can generate the same topology yet disagree on boundedness, boundedness is not a topological invariant — it depends on the choice of metric.

Definition 4.19. A set $E \subseteq \mathbb{R}$ is **convex** if for all $x, y \in E$ and for all $\lambda \in (0,1)$,

$$\lambda x + (1 - \lambda) y \in E.$$



Definition 4.20. A sequence (x_n) in a topological space X **converges** to $y \in X$ if for every neighborhood U of y, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$.

Theorem 4.21. *The product topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ equals the metric topology induced by the Euclidean metric.*

Proof. We show both topologies have the same open sets by showing each basis element of one is open in the other.

Product basis elements are open in the metric topology. A basic open set in the product topology is $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Let $x = (x_1, \ldots, x_n)$ be a point in this set. For each i, choose $\varepsilon_i = \min(x_i - a_i,\, b_i - x_i) > 0$. Set $\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_n) > 0$. Then $B(x, \varepsilon) \subseteq (a_1, b_1) \times \cdots \times (a_n, b_n)$, since if $d(x, y) < \varepsilon$ then $|x_i - y_i| \leq d(x, y) < \varepsilon \leq \varepsilon_i$ for each i, so $y_i \in (a_i, b_i)$.

Open balls are open in the product topology. Let $x \in B(y, \varepsilon)$ and set $\delta = \varepsilon - d(x, y) > 0$. Consider the product of intervals

$$U = \left(x_1 - \tfrac{\delta}{\sqrt{n}},\, x_1 + \tfrac{\delta}{\sqrt{n}}\right) \times \cdots \times \left(x_n - \tfrac{\delta}{\sqrt{n}},\, x_n + \tfrac{\delta}{\sqrt{n}}\right).$$

This is open in the product topology, $x \in U$, and for any $z \in U$ we have

$$d(x, z) = \sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} < \sqrt{n \cdot \tfrac{\delta^2}{n}} = \delta,$$

so $z \in B(x, \delta) \subseteq B(y, \varepsilon)$. Thus $U \subseteq B(y, \varepsilon)$, and $B(y, \varepsilon)$ is open in the product topology. $\square$

Theorem 4.22. *If $p \in E'$, then every neighborhood U of p contains infinitely many points of E.*

Proof. By contradiction. Suppose U is a neighborhood of p containing only finitely many points $q_1, \ldots, q_n \in U \cap E$ with $q_i \neq p$. Let $r = \min_{1 \leq i \leq n} d(p, q_i) > 0$. Then $B(p, r)$ contains no point of E other than p itself. But $p \in E'$, so every neighborhood of p must contain some $y \neq p$ with $y \in E$ — a contradiction. $\square$

5 Lecture 5: February 3, 2026

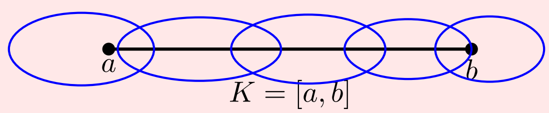
5.1 Compactness

Definition 5.1. Let $E \subseteq X$ be a topological space. If $\{G_\alpha\}$ is a collection of open sets of X such that $E \subseteq \bigcup G_\alpha$, then $\{G_\alpha\}$ is an **open cover** of E.

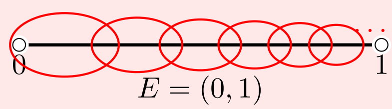
Definition 5.2. A subset $K \subseteq X$ is **compact** if every open cover contains a finite subcover. That is, given $\{G_\alpha\}$, there exist $G_{\alpha_1}, \ldots, G_{\alpha_n} \subseteq \{G_\alpha\}$ such that $K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$.

Understanding covers and compactness. An open cover is a collection of open sets that together contain every point of E. Think of it as "blanketing" the set with open sets. The cover may have infinitely many sets — in fact, that's the interesting case.
Compactness says: no matter how you cover K with open sets, you can always throw away all but finitely many and still cover K. This is a strong condition — it fails for many sets.

Open cover: finitely many suffice



Not compact: needs infinitely many



The closed interval $[a, b]$ is compact: any open cover has a finite subcover. The open interval $(0, 1)$ is not compact: the cover $\{(\frac{1}{n}, 1) : n \geq 2\}$ has no finite subcover, since points near 0 escape any finite subcollection.

5.2 Subspace Topology

$Y \subseteq X$. U is open in Y if and only if there exists V open in X such that $U = V \cap Y$.

Theorem 5.3. *If $K \subseteq Y \subseteq X$, then K compact relative to $X \Leftrightarrow K$ compact relative to Y.*

Proof. ($\Rightarrow$) Suppose K is compact relative to X. Let $\{U_\alpha\}$ be an open cover of K in Y. By the subspace topology, for each U_α there exists V_α open in X such that $U_\alpha = V_\alpha \cap Y$. Then $\{V_\alpha\}$ is an open cover of K in X. Since K is compact relative to X, there exist $V_{\alpha_1}, \ldots, V_{\alpha_n}$ such that $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. Then

$$K \subseteq (V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}) \cap Y = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

So K is compact relative to Y.

($\Leftarrow$) Suppose K is compact relative to Y. Let $\{V_\alpha\}$ be an open cover of K in X. Then $\{V_\alpha \cap Y\}$ is an open cover of K in Y (each $V_\alpha \cap Y$ is open in Y by the subspace topology). Since K is compact relative to Y, there exist $V_{\alpha_1} \cap Y, \ldots, V_{\alpha_n} \cap Y$ such that $K \subseteq (V_{\alpha_1} \cap Y) \cup \cdots \cup (V_{\alpha_n} \cap Y)$. Since $K \subseteq Y$, we have $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. So K is compact relative to X. $\square$

Theorem 5.4. *Compact subsets of metric spaces are closed.*

Proof. It suffices to show that K^c is open. Since we are in a metric space, we can use the open ball definition of open sets. We will show for every point $p \in K^c$, there exists $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq K^c$.

Fix $p \in K^c$. For each $q \in K$, let $r_q = \frac{1}{2}d(p,q) > 0$. The balls $B(p, r_q)$ and $B(q, r_q)$ are disjoint. The collection $\{B(q, r_q) : q \in K\}$ is an open cover of K. By compactness, there exist $q_1, \ldots, q_n \in K$ such that

$$K \subseteq B(q_1, r_{q_1}) \cup \cdots \cup B(q_n, r_{q_n}).$$

Let $\varepsilon = \min(r_{q_1}, \ldots, r_{q_n}) > 0$. Then $B(p, \varepsilon) \subseteq K^c$, since $B(p, \varepsilon) \subseteq B(p, r_{q_i})$ is disjoint from $B(q_i, r_{q_i})$ for each i, and K is covered by these balls. □

Theorem 5.5. *Closed subsets of compact sets are compact.*

Proof. Let F be a closed subset of a compact set K in X. Let $\{G_\alpha\}$ be an open cover of F. Then $\{G_\alpha\} \cup \{F^c\}$ is an open cover of K (F is closed so F^c is open, and $\{G_\alpha\}$ covers F). Since K is compact, there exists a finite subcover: $G_{\alpha_1}, \ldots, G_{\alpha_n}$, and possibly F^c. Removing F^c (if present), we have $F \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. Thus F is compact. □

Corollary 5.6. *The intersection of a closed set and a compact set is compact (in a metric space).*

Theorem 5.7. *Suppose $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of any finite subcollection is nonempty. Then $\bigcap_\alpha K_\alpha \neq \emptyset$.*

Proof. Suppose $\bigcap_\alpha K_\alpha = \emptyset$. Each K_α^c is open. By De Morgan's law,

$$\left(\bigcap_\alpha K_\alpha\right)^c = \bigcup_\alpha K_\alpha^c = X.$$

Fix $K_1 \in \{K_\alpha\}$. Then $\{K_\alpha^c\}$ is an open cover of K_1. By compactness, there exist $K_{\alpha_1}, \ldots, K_{\alpha_n}$ such that $K_1 \subseteq K_{\alpha_1}^c \cup \cdots \cup K_{\alpha_n}^c$. By De Morgan's law,

$$K_1 \subseteq (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n})^c.$$

Thus $K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \emptyset$, contradicting the finite intersection property. □

Corollary 5.8. *If $\{K_n\}$ is a sequence of compact subsets of a metric space X such that $K_n \supseteq K_{n+1}$, then $\bigcap_n K_n \neq \emptyset$.*

Theorem 5.9. *If E is an infinite subset of a compact set K, then E has a limit point in K.*

Proof. (By contradiction) Assume that E has no limit points. Then for all $p \in K$, there exists a neighborhood U_p of p where either $U_p \cap E = \emptyset$ or $U_p \cap E = \{p\}$. Therefore, $\{U_p\}$ forms an open cover of K. By compactness of K, there is a finite subcover $U_{p_1} \cup \cdots \cup U_{p_n} \supseteq K$. Each U_{p_i} contains at most one point of E, so E has at most n points. This contradicts the fact that E is infinite. □

The Cantor set is nonempty. Recall the Cantor set is defined as $C = \bigcap_{n=0}^\infty C_n$, where each C_n is a finite union of closed intervals. Each C_n is compact. The sets are nested: $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$, so any finite intersection equals the smallest set in the subcollection, which is nonempty. By the theorem above, $C = \bigcap_{n=0}^\infty C_n \neq \emptyset$.

Theorem 5.10. *Suppose $\{I_n\}$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_n \supseteq I_{n+1}$. Then $\bigcap_k I_k \neq \emptyset$.*

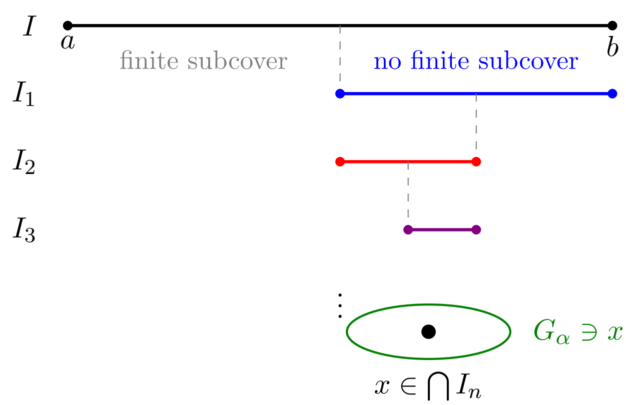
Proof. Let $I_n = [a_n, b_n]$. Let $E = \{a_n\}$. Since the intervals are nested, $a_n \leq b_m$ for all n, m, so E is bounded above. There exists $x = \sup E$.

For all n, we have $a_n \leq x$ (since x is an upper bound of E). Also $x \leq b_n$ for all n (since each b_n is an upper bound for E, and x is the least upper bound). Thus $a_n \leq x \leq b_n$, so $x \in I_n$ for all n. Therefore $x \in \bigcap_k I_k$. □

Theorem 5.11. *Closed intervals (and therefore closed boxes) are compact.*

Proof. Let $I = [a, b]$ and let $\{G_\alpha\}$ be an open cover of I. Suppose this open cover does not reduce to a finite subcover. Cut the interval in half: at least one half cannot be covered by finitely many G_α (if both halves could, we could combine them to cover I). Call this half I_1. Repeat: bisect I_1 and choose a half I_2 with no finite subcover. Continuing, we obtain nested closed intervals $I \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with $|I_n| = (b - a)/2^n$, each having no finite subcover.

By the nested intervals theorem, there exists $x \in \bigcap_n I_n$. Since $\{G_\alpha\}$ covers I, we have $x \in G_\alpha$ for some α. Since G_α is open, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq G_\alpha$. For large n, $|I_n| < \varepsilon$ and $x \in I_n$, so $I_n \subseteq (x - \varepsilon, x + \varepsilon) \subseteq G_\alpha$. But then I_n is covered by a single open set, contradicting that I_n has no finite subcover.



□

Key ideas in this proof:

1. **Proof by contradiction:** Assume no finite subcover exists and derive a contradiction.
2. **Bisection argument:** If a set has no finite subcover, at least one half doesn't either. This lets us build nested intervals.
3. **Nested intervals theorem:** The intersection $\bigcap I_n \neq \emptyset$, giving us a point x.
4. **Open set definition:** Since $x \in G_\alpha$ and G_α is open, there exists $\varepsilon > 0$ with $(x - \varepsilon, x + \varepsilon) \subseteq G_\alpha$.
5. **Intervals shrink to zero:** $|I_n| = (b - a)/2^n \to 0$, so eventually I_n fits inside the ε-neighborhood, giving a single-set cover — contradiction.

Theorem 5.12 (**Heine-Borel**). *Let $E \subseteq \mathbb{R}^n$. The following are equivalent:*

1. *E is closed and bounded.*
2. *E is compact.*
3. *Every infinite subset of E has a limit point in E.*

Proof. ($1 \Rightarrow 2$): Since E is bounded, $E \subseteq [-M, M]^n$ for some $M > 0$. The closed box $[-M, M]^n$ is compact. Since E is a closed subset of a compact set, E is compact.

($2 \Rightarrow 3$): This follows from the theorem: if E is an infinite subset of a compact set K, then E has a limit point in K. Taking $K = E$, every infinite subset of E has a limit point in E.

($3 \Rightarrow 1$): *Closed:* Let p be a limit point of E. Every neighborhood of p contains a point of E distinct from p. We can construct a sequence (x_n) in E with $x_n \to p$. The set $\{x_n\}$ is infinite, so by (3) it has a limit point in E. This limit point must be p, so $p \in E$. Thus E contains all its limit points, so E is closed.

Bounded: Suppose E is unbounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in E$ with $|x_n| > n$. The set $\{x_1, x_2, \dots\}$ is infinite. By (3), it has a limit point $p \in E$. But for any $\varepsilon > 0$, only finitely many x_n lie in $B(p, \varepsilon)$ (since $|x_n| \to \infty$), contradicting that p is a limit point. Thus E is bounded. ☐

What Heine-Borel means and how to use it.
In $\mathbb{R}^n$, compactness has a simple characterization: *closed and bounded.* This is easy to check! You don't need to verify that every open cover has a finite subcover — just check two conditions.
Common uses:

- **Proving a set is compact:** Show it's closed (contains its limit points) and bounded (fits in some ball). Examples: $[0, 1]$, closed balls $\overline{B}(x, r)$, the Cantor set.
- **Proving a set is NOT compact:** Show it's either not closed or not bounded. Examples: $(0, 1)$ is not closed; $\mathbb{R}$ is not bounded.
- **Extracting convergent subsequences:** Condition (3) says infinite subsets have limit points. This is the key to proving the Bolzano-Weierstrass theorem: every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Warning: Heine-Borel is specific to $\mathbb{R}^n$. In general metric spaces, compact implies closed and bounded, but the converse can fail.

Theorem 5.13 (**Weierstrass**). *Every bounded infinite subset has a limit point in $\mathbb{R}^n$.*

5.3 Perfect Sets

Recall: E is **perfect** if E is closed and has no isolated points. If E is perfect, then $E = \overline{E} = E'$.

Theorem 5.14. *Every nonempty perfect subset of $\mathbb{R}^n$ is uncountable.*

Proof. Let $P \subseteq \mathbb{R}^n$ be nonempty and perfect. Suppose for contradiction that P is countable, say $P = \{x_1, x_2, x_3, \dots\}$.

We construct nested closed sets $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ such that:

1. $V_n \cap P \neq \emptyset$ for all n,
2. $x_n \notin V_n$ for all n.

Base case: Since P has no isolated points, x_1 is a limit point of P. Choose $y_1 \in P$ with $y_1 \neq x_1$. Let $V_1 = \overline{B}(y_1, r_1)$ where $r_1 = \frac{1}{2}d(x_1, y_1)$. Then $y_1 \in V_1 \cap P$ and $x_1 \notin V_1$.

Inductive step: Suppose V_n is constructed with $V_n \cap P \neq \emptyset$ and $x_n \notin V_n$. Pick any $y \in V_n \cap P$. Since P is perfect, y is a limit point of P, so there exists $y_{n+1} \in P \cap V_n$ with $y_{n+1} \neq x_{n+1}$ (if $x_{n+1} \notin V_n$, any point works; if $x_{n+1} \in V_n$, choose a different point). Let $V_{n+1} = \overline{B}(y_{n+1}, r_{n+1}) \cap V_n$ where r_{n+1} is small enough that $x_{n+1} \notin V_{n+1}$ and $V_{n+1} \subseteq V_n$.

Each V_n is closed and bounded, hence compact. The V_n are nested and nonempty, so by the finite intersection property, $\bigcap_n V_n \neq \emptyset$. Let $x \in \bigcap_n V_n$. Since each $V_n \cap P$ is closed (intersection of closed sets) and the V_n are nested, we have $x \in P$. But $x \neq x_n$ for all n (since $x_n \notin V_n$). This contradicts $P = \{x_1, x_2, \dots\}$. □

Theorem 5.15. *The Cantor set is perfect.*

Proof. The Cantor set C is closed. Additionally, using the ternary expansion: for any $x \in C$, we can truncate x at the n-th digit and define a sequence (x_n) in C with $|x - x_n| \leq 3^{-n}$. Thus $x_n \to x$, so x is a limit point of C. Hence C has no isolated points, and C is perfect.

$$
\begin{aligned}
x &= 0.\underbrace{0202\overset{\text{truncate here}}{\downarrow}0}_{n\text{ digits}}2002\cdots \\
x_n &= 0.02020\underbrace{0000\cdots}_{\text{zeros}} \\
|x - x_n| &= 0.\quad\underbrace{00000}_{\text{first } n \text{ digits}}\quad 2002\cdots \\
&\leq 0.00000\overline{22} = \tfrac{2}{3^{n+1}} \cdot \tfrac{1}{1-1/3} = \tfrac{1}{3^n}
\end{aligned}
$$

□

6 Lecture 6: February 10, 2026

Lecture Overview: We begin with the definitions of separated and connected sets, and characterize connected subsets of $\mathbb{R}$ as those with no gaps. We then define continuity (via preimages of open sets, bases, locality, and ε-δ), homeomorphisms, and prove that continuous functions preserve compactness and connectedness. From these we derive the extreme value theorem, the intermediate value theorem, and that continuous functions on compact sets are uniformly continuous.

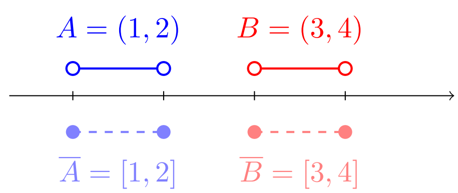
6.1 Separated and Connected Sets

Section Overview: We define separated sets and connected sets. A subset $E \subset \mathbb{R}$ is connected if and only if it has no "gaps" — whenever $x, y \in E$ and $x < z < y$, then $z \in E$. We prove this characterization in both directions.

Definition 6.1. Two subsets $A, B \subset X$ are **separated** if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

Definition 6.2. A set $E \subset X$ is **connected** if E is *not* a union $A \cup B$ where A and B are nonempty and separated.

Example. Consider the sets $A = (1, 2)$ and $B = (3, 4)$ in $\mathbb{R}$.



$$\overline{A} \cap B = [1, 2] \cap (3, 4) = \emptyset$$

$$A \cap \overline{B} = (1, 2) \cap [3, 4] = \emptyset$$

$$\textbf{A and B are separated}$$

Theorem 6.3. *$E \subset \mathbb{R}$ is connected if and only if for all $x, y \in E$ and for all z with $x < z < y$, we have $z \in E$.*

Proof. ($\Rightarrow$) Suppose E is connected. We proceed by contradiction. Assume there exist $x, y \in E$ and z with $x < z < y$ but $z \notin E$. Let

$$A = E \cap (-\infty, z), \qquad B = E \cap (z, \infty).$$

Then A is nonempty (since $x \in A$) and B is nonempty (since $y \in B$). Since $z \notin E$, we have $A \cup B = E$. Moreover, $\overline{A} \subset (-\infty, z]$ and $\overline{B} \subset [z, \infty)$, so

$$\overline{A} \cap B = \emptyset \quad \text{and} \quad A \cap \overline{B} = \emptyset.$$

Thus A and B are nonempty, separated, and $E = A \cup B$, contradicting the assumption that E is connected.

($\Leftarrow$) We prove the contrapositive. Suppose E is not connected, so $E = A \cup B$ where A, B are nonempty and separated. Pick $x \in A$, $y \in B$; without loss of generality assume $x < y$. Let

$$z = \sup(A \cap [x, y]).$$

This exists since $A \cap [x, y]$ is nonempty (contains x) and bounded above by y.

Since $z = \sup(A \cap [x, y])$, we have $z \in \overline{A}$. Since $\overline{A} \cap B = \emptyset$, we get $z \notin B$. Since $y \in B$ and $z \notin B$, we have $z \neq y$, so $z < y$.

Case 1: $z \notin A$. Then $z \notin A \cup B = E$, but $x, y \in E$ and $x \leq z < y$. If $z = x$ then $z = x \in A$, a contradiction, so $x < z < y$ and $z \notin E$.

Case 2: $z \in A$. Then since $A \cap \overline{B} = \emptyset$, we have $z \notin \overline{B}$, so there exists $\varepsilon > 0$ such that $(z - \varepsilon, z + \varepsilon) \cap B = \emptyset$. Choose ε small enough so that $z + \varepsilon < y$. Since $z = \sup(A \cap [x, y])$, no element of A lies in $(z, z + \varepsilon)$. So any point $z' \in (z, z + \varepsilon)$ satisfies $z' \notin A$ and $z' \notin B$, hence $z' \notin E$. But $x < z' < y$ with $x, y \in E$.

In both cases, we find a point strictly between two elements of E that is not in E. $\square$

6.2 Continuous Functions

Section Overview: We define continuity via preimages of open sets and give three equivalent formulations: via a basis, locally, and via ε-δ. We show that sums, compositions, and quotients of continuous functions are continuous, and define homeomorphisms.

Remark 6.4. Connectedness and compactness are described purely in set-theoretic terms. From them, we obtain topological properties.

Definition 6.5. A function $f : X \to Y$ is **continuous** if for all V open in Y, $f^{-1}(V)$ is open in X.

Properties of continuous functions:

1. **Basis:** Given a basis $\mathcal{B}$ of Y, to check that f is continuous it suffices to check that $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$.

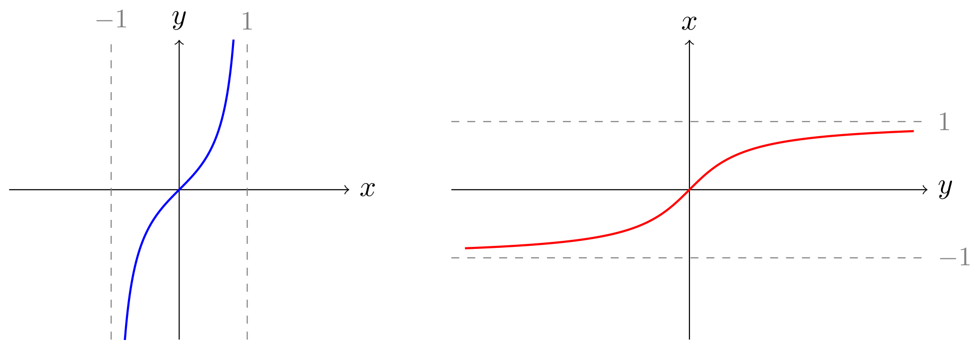
2. **Locality:** f is **continuous at α** if there exists an open neighborhood U_α of α such that $f|_{U_\alpha}$ is continuous.

3. **ε-δ (open balls):** In $\mathbb{R}^n$, f is continuous at x if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(y) - f(x)| < \varepsilon.$$

Example. $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = \dfrac{x}{1 - x^2}$.

This mapping is one-to-one. Its inverse is $f^{-1}(y) = \dfrac{2y}{1 + \sqrt{1 + 4y^2}}$.



$f(x) = \dfrac{x}{1-x^2}$ $\qquad$ $f^{-1}(y) = \dfrac{2y}{1+\sqrt{1+4y^2}}$

Consider the projection mappings $\pi_i : \mathbb{R}^n \to \mathbb{R}$ defined by $(x_1, \dots, x_n) \mapsto x_i$.

Theorem 6.6. *If $f, g : X \to \mathbb{R}$ are continuous, then:*

1. *$f + g$ is continuous,*
2. *$f \circ g$ is continuous,*
3. *f/g is continuous (wherever $g \neq 0$).*

Definition 6.7. If $f : X \to Y$ is a bijection and both f and f^{-1} are continuous, then f is a **homeomorphism**. We write $X \sim Y$.

6.3 Topology of Continuous Functions

Section Overview: Continuous functions preserve key topological properties: compact sets map to compact sets, and connected sets map to connected sets. As a consequence, continuous functions on compact sets attain their maximum and minimum.

Theorem 6.8. *Continuous functions preserve compactness and connectedness. If $E \subset X$ is compact (resp. connected) and $f : X \to Y$ is continuous, then $f(E)$ is compact (resp. connected).*

Take the interval $E = [0, 1] \subset \mathbb{R}$. Since E is compact, $f(E)$ is compact, hence closed and bounded (by Heine–Borel). In particular, f attains its maximum and minimum on E.

Proof (compactness). Let $\{V_\alpha\}$ be an open cover of $f(K)$. Since f is continuous, each $f^{-1}(V_\alpha)$ is open in X. The collection $\{f^{-1}(V_\alpha)\}$ is an open cover of K. Since K is compact, there exist finitely many indices $\alpha_1, \dots, \alpha_n$ such that

$$K \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}).$$

Applying f to both sides,

$$f(K) \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

Thus $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ is a finite subcover of $f(K)$. $\square$

Proof (connectedness). Suppose for contradiction that $f(E)$ is not connected. Then $f(E) = A \cup B$ where A, B are nonempty and separated. Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. Then G and H are nonempty (since A and B are nonempty subsets of $f(E)$) and $G \cup H = E$ (since every point of E maps into A or B).

We claim G and H are separated. Since A and B are separated, $\overline{A} \cap B = \emptyset$. If $p \in \overline{G} \cap H$, then $f(p) \in B$. Since f is continuous and p is a limit point of G (or in G), we have $f(p) \in \overline{f(G)} \subset \overline{A}$. But then $f(p) \in \overline{A} \cap B = \emptyset$, a contradiction. Thus $\overline{G} \cap H = \emptyset$. By symmetry, $G \cap \overline{H} = \emptyset$.

So $E = G \cup H$ with G, H nonempty and separated, contradicting the connectedness of E. □

Theorem 6.9. *Let f be a continuous function on a compact set K. Define $M = \sup_{x \in K} f(x)$ and $m = \inf_{x \in K} f(x)$. Then there exist $p, q \in K$ such that $f(p) = M$ and $f(q) = m$.*

Proof. Since K is compact and f is continuous, $f(K)$ is compact (by the previous theorem), hence closed and bounded by Heine–Borel. Since $f(K)$ is bounded, $M = \sup f(K)$ and $m = \inf f(K)$ are finite. Since $f(K)$ is closed, it contains all its limit points. But $M = \sup f(K)$ is a limit point of $f(K)$ (for every $\varepsilon > 0$, there exists $y \in f(K)$ with $M - \varepsilon < y \leq M$), so $M \in f(K)$. Thus there exists $p \in K$ with $f(p) = M$. Similarly, $m \in f(K)$, so there exists $q \in K$ with $f(q) = m$. □

Theorem 6.10 (Intermediate Value Theorem)**.** *If f is continuous on $[a, b]$ and $f(a) < c < f(b)$, then there exists $p \in (a, b)$ such that $f(p) = c$.*

Proof. Since f is continuous and $[a, b]$ is connected, $f([a, b])$ is connected (by the connectedness theorem above). Since $f([a, b]) \subset \mathbb{R}$ is connected, it has no gaps: for any two values in $f([a, b])$ and anything between them, that value is also in $f([a, b])$. We have $f(a), f(b) \in f([a, b])$ and $f(a) < c < f(b)$, so $c \in f([a, b])$. Thus there exists $p \in [a, b]$ with $f(p) = c$. Since $f(a) < c < f(b)$, we have $f(p) \neq f(a)$ and $f(p) \neq f(b)$, so $p \neq a$ and $p \neq b$, hence $p \in (a, b)$. □

Definition 6.11. A function $f : X \to Y$ is **uniformly continuous** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $p, q \in X$,

$$d_X(p, q) < \delta \implies d_Y(f(p), f(q)) < \varepsilon.$$

Theorem 6.12. *If K is compact and $f : K \to Y$ is continuous, then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$. Since f is continuous, for each $p \in K$ there exists $\delta_p > 0$ such that

$$d_X(x, p) < \delta_p \implies d_Y(f(x), f(p)) < \frac{\varepsilon}{2}.$$

The collection $\{B(p, \delta_p/2) : p \in K\}$ is an open cover of K. Since K is compact, there exist finitely many $p_1, \ldots, p_n \in K$ such that

$$K \subset B(p_1, \delta_{p_1}/2) \cup \cdots \cup B(p_n, \delta_{p_n}/2).$$

Let $\delta = \frac{1}{2} \min(\delta_{p_1}, \ldots, \delta_{p_n}) > 0$. Now suppose $p, q \in K$ with $d_X(p, q) < \delta$. Since the balls cover K, there exists p_i with $p \in B(p_i, \delta_{p_i}/2)$, i.e. $d_X(p, p_i) < \delta_{p_i}/2$. Then

$$d_X(q, p_i) \leq d_X(q, p) + d_X(p, p_i) < \delta + \frac{\delta_{p_i}}{2} \leq \frac{\delta_{p_i}}{2} + \frac{\delta_{p_i}}{2} = \delta_{p_i}.$$

So $d_Y(f(p), f(p_i)) < \varepsilon/2$ and $d_Y(f(q), f(p_i)) < \varepsilon/2$, and by the triangle inequality,

$$d_Y(f(p), f(q)) \leq d_Y(f(p), f(p_i)) + d_Y(f(p_i), f(q)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ □

Theorem 6.13. *Let $E \subset \mathbb{R}$ be a non-compact set. Then there exists a continuous function on E that is:*

1. *not bounded,*
2. *bounded but does not attain its maximum,*
3. *not uniformly continuous.*

Proof. Since E is not compact, by Heine–Borel E is either not closed or not bounded.

Case 1: E is not bounded. Then $f(x) = x$ is continuous on E and not bounded. The function $g(x) = \dfrac{x^2}{1+x^2}$ is continuous and bounded with $0 \leq g < 1$, but $\sup g = 1$ is never attained. The function $h(x) = x^2$ is continuous but not uniformly continuous (for any $\delta > 0$, choose x large enough so that $|x^2 - (x + \delta/2)^2| > 1$).

Case 2: E is bounded but not closed. Then there exists a limit point x_0 of E with $x_0 \notin E$. Define $f(x) = \dfrac{1}{x - x_0}$, which is continuous on E but not bounded. The function $g(x) = \dfrac{1}{1 + (x - x_0)^{-2}}$ is continuous and bounded with $0 < g < 1$, but $\sup g = 1$ is never attained (it would require $|x - x_0| \to \infty$, but E is bounded). The function $h(x) = \dfrac{1}{x - x_0}$ is not uniformly continuous (as $x \to x_0$, arbitrarily close points have arbitrarily far images). $\square$

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