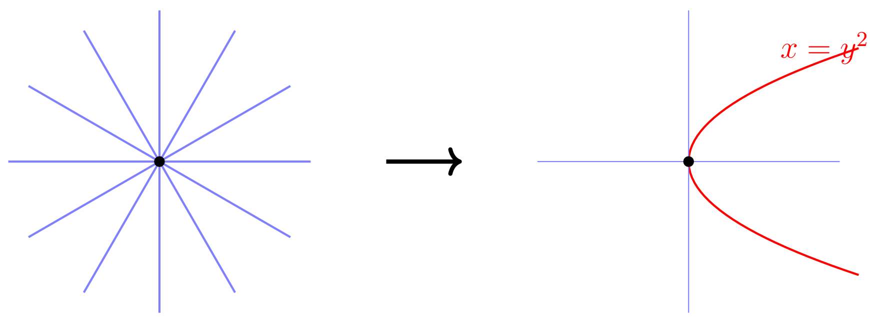
Continuous on Every Line $\neq$ Continuous at a Point

A visual guide to Rudin 4.7

1 The Big Picture

This problem shows a **surprising** fact: a function on $\mathbb{R}^2$ can be continuous along *every* straight line through the origin, and yet *not* be continuous at the origin. Lines don't "see" everything!



Every line says: "$f \to 0$ along me!" But the parabola says: "$f = 1/2$ along me!"

2 The Functions

$$f(x, y) = \frac{xy^2}{x^2 + y^4}, \qquad g(x, y) = \frac{xy^2}{x^2 + y^6}$$

with $f(0, 0) = g(0, 0) = 0$.

Notice the structure: Both have the same numerator xy^2, but different denominators. The denominator controls how fast it grows, which determines boundedness and continuity.

3 Part 1: Why is f Bounded?

3.1 The AM-GM Trick

The **AM-GM inequality** says: for non-negative numbers a, b,

$$a + b \geq 2\sqrt{ab}$$

Apply this to the denominator with $a = x^2$ and $b = y^4$:

$$x^2 + y^4 \geq 2\sqrt{x^2 \cdot y^4} = 2|x|y^2$$

$$\boxed{|f(x,y)| = \frac{|x|y^2}{x^2+y^4} \leq \frac{|x|y^2}{2|x|y^2} = \frac{1}{2}}$$

The numerator *perfectly cancels* with the AM-GM bound!

3.2 Why Doesn't This Work for g?

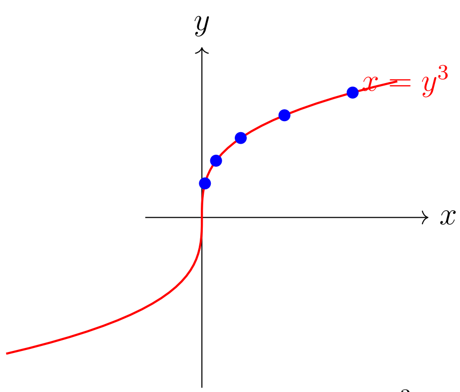
For g, the denominator is $x^2 + y^6$. AM-GM gives $x^2 + y^6 \geq 2|x|y^3$, so:

$$|g(x,y)| \leq \frac{|x|y^2}{2|x|y^3} = \frac{1}{2|y|}$$

This blows up as $y \to 0$! The AM-GM bound is not enough to control g.

4 Part 2: Why is g Unbounded Near the Origin?

The key idea: approach $(0,0)$ along a **curve**, not a line.



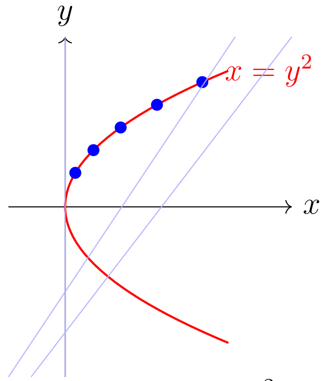
Along $x = y^3$:

$$g(y^3, y) = \frac{y^5}{2y^6} = \frac{1}{2y} \to \infty$$

Why does this work? On the curve $x = y^3$, the two terms in the denominator become *equal*: $x^2 = y^6$. This is the "worst case" — the denominator is as small as possible relative to the numerator.

5 Part 3: Why is f Not Continuous at $(0,0)$?

Same idea, but now use the parabola $x = y^2$ for f:



Along $x = y^2$:
$$f(y^2, y) = \frac{y^4}{2y^4} = \frac{1}{2} \text{ always!}$$

On the parabola $x = y^2$, the denominator terms become equal: $x^2 = y^4$. So $f = \frac{y^4}{2y^4} = \frac{1}{2}$ everywhere on this curve. Since $f(0,0) = 0 \neq 1/2$, the function is not continuous at the origin.

6 Part 4: Why Are Line Restrictions Continuous?

6.1 Lines Not Through the Origin

Away from $(0,0)$, both f and g are ratios of polynomials with nonzero denominators. Rational functions are continuous wherever their denominators don't vanish.

6.2 Lines Through the Origin

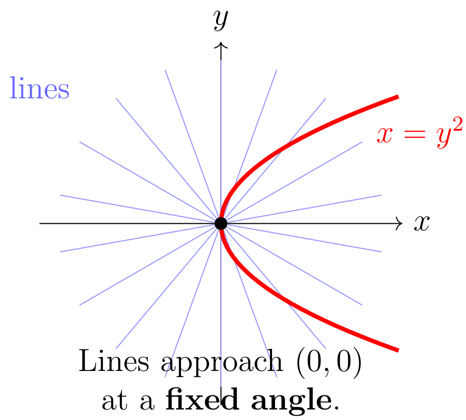
On a line $y = mx$ (with $m \neq 0$), substitute:

$$f(x, mx) = \frac{x \cdot m^2x^2}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

$$\text{As } x \to 0: \quad \frac{m^2 \cdot 0}{1 + 0} = 0 = f(0,0) \checkmark$$

The crucial point: On any line $y = mx$, the substitution reduces the function to a nice rational function of one variable that equals 0 at the origin. The "dangerous" behavior only appears along *curves* (like $x = y^2$), never along *lines*.

6.3 Why Lines Miss the Problem



Lines approach $(0, 0)$ at a **fixed angle**. The parabola $x = y^2$ approaches at a **changing angle** — it "curves in" to the origin faster than any line can follow.

On any fixed line $y = mx$: the variable x controls both coordinates, and f simplifies to something well-behaved.

On the parabola $x = y^2$: the relationship between x and y is nonlinear, which "matches the degree" of the denominator and creates a constant value.

7 Degree Matching: The Deep Reason

Look at the degrees in $f = \frac{xy^2}{x^2+y^4}$:

- Numerator: xy^2 has "weighted degree" $1 + 2 = 3$ (if x counts as 2, y as 1)
- Actually, think of it as: if $x \sim y^2$, then $xy^2 \sim y^4$ and $x^2 + y^4 \sim 2y^4$
- So $f \sim y^4/(2y^4) = 1/2$ — a nonzero constant!

On a line $y = mx$: $x \sim x$ (not $\sim y^2$), so the degrees *don't* match, and $f \to 0$.

The parabola $x = y^2$ is the exact curve where the degrees balance, creating a nonzero limit.