Math 104 - Homework 3

Problem 1 (Rudin 4.2). *If f is a continuous mapping of a metric space X into a metric space Y, prove that*

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that the converse is not necessarily true.

Proof. Let $p \in \overline{E}$. We show $f(p) \in \overline{f(E)}$ by showing every neighborhood of $f(p)$ intersects $f(E)$.

Let V be any neighborhood of $f(p)$ in Y. Since f is continuous, $f^{-1}(V)$ is open in X and contains p. Since $p \in \overline{E}$, every open set containing p intersects E. Thus there exists $q \in E \cap f^{-1}(V)$. Then $f(q) \in f(E) \cap V$, so V intersects $f(E)$.

Since every neighborhood of $f(p)$ intersects $f(E)$, we have $f(p) \in \overline{f(E)}$. Since $p \in \overline{E}$ was arbitrary, $f(\overline{E}) \subset \overline{f(E)}$.

Counterexample for the converse: Let $X = Y = \mathbb{R}$ with the standard metric, and define $f(x) = \frac{1}{1+x^2}$. Let $E = \mathbb{Z}$. Then $\overline{E} = \mathbb{Z}$ (since $\mathbb{Z}$ is closed), so

$$f(\overline{E}) = f(\mathbb{Z}) = \left\{ \frac{1}{1+n^2} : n \in \mathbb{Z} \right\}.$$

Since $\frac{1}{1+n^2} \to 0$ as $|n| \to \infty$, we have $0 \in \overline{f(E)}$. But $f(x) = \frac{1}{1+x^2} > 0$ for all $x \in \mathbb{R}$, so $0 \notin f(\mathbb{R}) \supset f(\overline{E})$. Therefore $\overline{f(E)} \not\subset f(\overline{E})$. □

Problem 2 (Rudin 4.3). *Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.*

Proof. We show $Z(f)^c$ is open. Let $p \notin Z(f)$, so $f(p) \neq 0$. Let $\varepsilon = |f(p)| > 0$. Since f is continuous, there exists $\delta > 0$ such that $d_X(x, p) < \delta$ implies $|f(x) - f(p)| < \varepsilon$.

For any $x \in B(p, \delta)$, the triangle inequality gives

$$|f(x)| \geq |f(p)| - |f(x) - f(p)| > \varepsilon - \varepsilon = 0,$$

so $f(x) \neq 0$, meaning $x \notin Z(f)$. Thus $B(p, \delta) \subset Z(f)^c$.

Since every point of $Z(f)^c$ has a neighborhood contained in $Z(f)^c$, the set $Z(f)^c$ is open, and therefore $Z(f)$ is closed. □

Problem 3 (Rudin 4.7). *Define f and g on $\mathbb{R}^2$ by: $f(0,0) = g(0,0) = 0$, and*

$$f(x,y) = \frac{xy^2}{x^2 + y^4}, \quad g(x,y) = \frac{xy^2}{x^2 + y^6} \quad \text{if } (x,y) \neq (0,0).$$

Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous.

Proof. f **is bounded.** At $(0,0)$, $f=0$. For $(x,y)\neq(0,0)$: if $xy=0$, then $f(x,y)=0$. If $xy\neq 0$, by AM-GM,

$$x^2+y^4\geq 2\sqrt{x^2\cdot y^4}=2|x|y^2,$$

so

$$|f(x,y)|=\frac{|x|y^2}{x^2+y^4}\leq\frac{|x|y^2}{2|x|y^2}=\frac{1}{2}.$$

Thus $|f(x,y)|\leq\frac{1}{2}$ for all $(x,y)\in\mathbb{R}^2$.

g **is unbounded in every neighborhood of** $(0,0)$**.** Along the curve $x=y^3$ with $y>0$:

$$g(y^3,y)=\frac{y^3\cdot y^2}{y^6+y^6}=\frac{y^5}{2y^6}=\frac{1}{2y}.$$

For any $\delta>0$, choose $y>0$ small enough that $|(y^3,y)|=\sqrt{y^6+y^2}<\delta$. Then $g(y^3,y)=\frac{1}{2y}$, which can be made arbitrarily large by taking y small. Thus g is unbounded in every neighborhood of $(0,0)$.

f **is not continuous at** $(0,0)$**.** Along the parabola $x=y^2$:

$$f(y^2,y)=\frac{y^2\cdot y^2}{y^4+y^4}=\frac{y^4}{2y^4}=\frac{1}{2}\quad\text{for all } y\neq 0.$$

So along this path, $f\to\frac{1}{2}$ as $(x,y)\to(0,0)$, but $f(0,0)=0\neq\frac{1}{2}$. Hence f is not continuous at $(0,0)$.

Restrictions to straight lines are continuous. Any line that does not pass through the origin avoids $(0,0)$, and both f and g are rational functions with nonzero denominators away from the origin, hence continuous there.

For lines through the origin, there are three cases:

Case 1: The x-axis ($y=0$). $f(x,0)=0$ and $g(x,0)=0$ for all x, so both restrictions are identically zero, hence continuous.

Case 2: The y-axis ($x=0$). $f(0,y)=0$ and $g(0,y)=0$ for all y. Continuous.

Case 3: $y=mx$ for $m\neq 0$. For $x\neq 0$:

$$f(x,mx)=\frac{x\cdot m^2x^2}{x^2+m^4x^4}=\frac{m^2x}{1+m^4x^2}.$$

This is a continuous function of x for all x, and as $x\to 0$ it tends to $0=f(0,0)$. So the restriction of f to $y=mx$ is continuous everywhere.

Similarly:

$$g(x,mx)=\frac{x\cdot m^2x^2}{x^2+m^6x^6}=\frac{m^2x}{1+m^6x^4}.$$

This also tends to $0=g(0,0)$ as $x\to 0$ and is continuous for all x. So the restriction of g to $y=mx$ is continuous everywhere. $\square$

Bonus Problems

Problem 4 (Bonus: Equivalence of Connectedness Definitions)**.** *Prove that the following two definitions of connectedness are equivalent for a subset E of a metric space X:*

(i) E is not *the union of two nonempty separated sets (i.e., sets A, B with $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$).*

(ii) E is not *the union of two nonempty disjoint sets that are both open relative to E.*

Proof. We prove the contrapositive in both directions: E is disconnected in sense (i) if and only if E is disconnected in sense (ii).

(i $\Rightarrow$ ii): Suppose $E = A \cup B$ where A, B are nonempty and separated, i.e., $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$. Since $A \cap B \subset A \cap \overline{B} = \varnothing$, the sets A and B are disjoint.

We show A is open relative to E. Let $p \in A$. Since $A \cap \overline{B} = \varnothing$, we have $p \notin \overline{B}$, so there exists $\delta > 0$ such that $B(p, \delta) \cap B = \varnothing$. Then for any $x \in B(p, \delta) \cap E$, since $x \in E = A \cup B$ and $x \notin B$, we have $x \in A$. Thus $B(p, \delta) \cap E \subset A$, so A is open relative to E.

By symmetry (using $\overline{A} \cap B = \varnothing$), B is open relative to E.

(ii $\Rightarrow$ i): Suppose $E = A \cup B$ where A, B are nonempty, disjoint, and both open relative to E.

Since A is open relative to E, the set $B = E \setminus A$ is closed relative to E, meaning $\overline{B} \cap E \subset B$. Then

$$A \cap \overline{B} \subset E \cap \overline{B} = (\overline{B} \cap E) \subset B,$$

so $A \cap \overline{B} \subset A \cap B = \varnothing$.

By symmetry, since B is open relative to E, the set $A = E \setminus B$ is closed relative to E, so $\overline{A} \cap E \subset A$. Then

$$\overline{A} \cap B \subset \overline{A} \cap E \subset A,$$

so $\overline{A} \cap B \subset A \cap B = \varnothing$.

Thus A and B are separated. □

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