Two Faces of Connectedness

Why the separated-sets and relative-open definitions agree

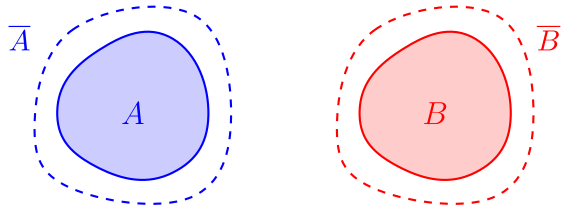
1 The Two Definitions

Both definitions capture the idea that a connected set is "one piece," but they phrase it differently.

1.1 Definition 1: Separated Sets (Rudin's Definition)

E is **disconnected** if $E = A \cup B$ where A, B are nonempty and **separated**:

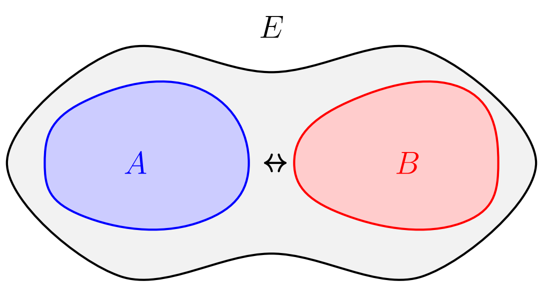
$$\overline{A} \cap B = \varnothing \quad \text{and} \quad A \cap \overline{B} = \varnothing$$



$\overline{A}$ doesn't touch B, and $\overline{B}$ doesn't touch A

1.2 Definition 2: Relative Open Sets

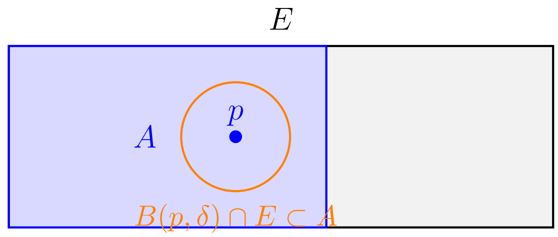
E is **disconnected** if $E = A \cup B$ where A, B are nonempty, disjoint, and both **open relative to** E.



Both A and B are "open within E" and they split E

2 What Does "Open Relative to E" Mean?

A is **open relative to** E if every point of A has a neighborhood (within E) contained in A.

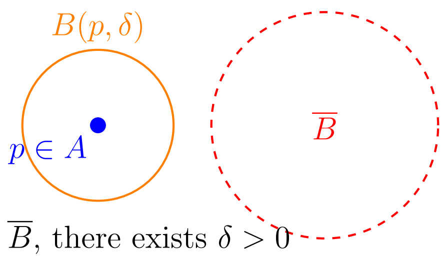


Equivalently, A is open relative to E if $A = E \cap U$ for some open set U in the ambient space.

3 Why Are These Definitions Equivalent?

3.1 Direction 1: Separated $\Rightarrow$ Relatively Open

If $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$:

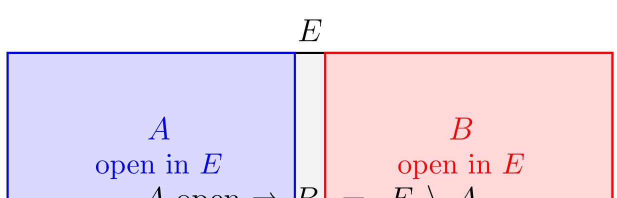


Since $p \notin \overline{B}$, there exists $\delta > 0$
with $B(p,\delta) \cap B = \varnothing$.
So $B(p,\delta) \cap E \subset A$. ✓

Key idea: "$A \cap \overline{B} = \varnothing$" means every point of A is at positive distance from B, so A is open relative to E. By symmetry, B is also open relative to E.

3.2 Direction 2: Relatively Open $\Rightarrow$ Separated

If A, B are both open relative to E, disjoint, and $A \cup B = E$:



A open $\Rightarrow B = E \setminus A$
is **closed** relative to E
Closed relative to E
means: $\overline{B} \cap E \subset B$
So: $A \cap \overline{B} \subset E \cap \overline{B} \subset B$, and $A \cap B = \varnothing$
Therefore $A \cap \overline{B} = \varnothing$ ✓

Key idea: If A is open in E, then $B = E \setminus A$ is closed in E, meaning $\overline{B} \cap E \subset B$. So A can't touch $\overline{B}$ (within E, and $A \subset E$). By symmetry, B can't touch $\overline{A}$.

4 The Open/Closed Duality

The magic ingredient: when $E = A \cup B$ with $A \cap B = \varnothing$:

$$A \text{ open in } E \iff B = E \setminus A \text{ closed in } E$$

So if *both* A and B are open in E, then both are *also* closed in E! This "clopen" property is what makes them separated.

Both open AND closed in E
=
Separated from each other

5 Summary

Def 1: Separated
$\overline{A} \cap B = \varnothing$
$A \cap \overline{B} = \varnothing$

$\Longleftrightarrow$

Def 2: Relative open
A, B disjoint and open in E

The bridge between them: **relative closure**. In a disjoint partition $E = A \cup B$:

- Separated $\Rightarrow$ each piece avoids the other's closure $\Rightarrow$ each piece is open in E
- Relatively open $\Rightarrow$ each piece is also closed in E $\Rightarrow$ separated