1 Lecture 9: February 19, 2026

Lecture Overview: This lecture introduces series as a special type of sequence (partial sums) and develops the main convergence tests. We review the Cauchy criterion and monotonic sequences, then establish the comparison test and Cauchy condensation test. We study key examples: the geometric series, p-series, and the harmonic series. We define the number $e = \sum 1/n!$, prove it converges, and show e is irrational. Finally, we develop the root and ratio tests for convergence.

1.1 Series

Section Overview: A series is defined as the sequence of partial sums of a given sequence. Convergence of a series means convergence of its partial sums.

Given a sequence $\{a_n\}$, we can consider the **partial sums** $\{s_n\}$ where

$$s_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k.$$

We say this series **converges** to s if

$$s = \sum_{n=1}^{\infty} a_n,$$

that is, if $s_n \to s$, i.e. the partial sums converge. This means that many of the same properties of sequences apply to series.

1.2 Review: Cauchy Sequences

Section Overview: The Cauchy criterion lets us test convergence without knowing the limit. For series, the Cauchy condition on partial sums reduces to bounding tail sums $\left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon$.

Last time we discussed Cauchy sequences: a sequence $\{s_n\}$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$|s_m - s_n| < \varepsilon.$$

Reader's Note: The Cauchy criterion gives us a way to test for convergence *without knowing the limit*. Instead of checking that terms get close to some value s, we check that terms get close to *each other*. This is especially useful for series, where computing the exact sum is often difficult or impossible.

Without loss of generality we can assume $m > n$. Then

$$|s_m - s_n| = \left|\sum_{k=n+1}^{m} a_k\right|.$$

1.3 Monotonic Sequences

Section Overview: Monotonic sequences converge if and only if they are bounded. For series with non-negative terms, the partial sums are monotonically increasing, so convergence reduces to showing boundedness. A key corollary: if a series converges, its terms must tend to zero.

Definition 1.1. A monotonic sequence converges if and only if it is bounded.

$\{s_n\}$ is **monotonically increasing** if and only if $s_n \leq s_{n+1}$, i.e. all the terms a_n are non-negative.

Reader's Note: Why does $s_n \leq s_{n+1}$ mean all terms are non-negative? Since $s_{n+1} - s_n = a_{n+1}$, requiring $s_n \leq s_{n+1}$ is the same as requiring $a_{n+1} \geq 0$. If all terms were negative, the partial sums would be monotonically *decreasing*, not increasing.

Corollary 1.2. *If $m = n + 1$ and $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Example. The **harmonic series**

$$\sum \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

1.4 Comparison Tests for Convergence

Section Overview: The comparison test establishes convergence or divergence by comparing a series term-by-term against a known series. The Cauchy condensation test is a powerful refinement: for decreasing non-negative sequences, $\sum a_n$ converges iff $\sum 2^k a_{2^k}$ converges, reducing many problems to geometric series.

Theorem 1.3. *Given two sequences $\{a_n\}$ and $\{c_n\}$:*

(a) If $|a_n| \leq c_n$ for all $n \geq N$ and $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for all $n \geq N$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof. (a) Since $\sum c_n$ converges, by the Cauchy criterion, for every $\varepsilon > 0$ there exists N_0 such that for all $m > n \geq N_0$,

$$\sum_{k=n+1}^{m} c_k < \varepsilon.$$

For $n \geq \max(N, N_0)$, we have

$$\left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k| \leq \sum_{k=n+1}^{m} c_k < \varepsilon.$$

So $\sum a_n$ converges by the Cauchy criterion.

(b) Since $a_n \geq d_n \geq 0$ for $n \geq N$, we have $s_m - s_n = \sum_{k=n+1}^{m} a_k \geq \sum_{k=n+1}^{m} d_k$. Since $\sum d_n$ diverges, the partial sums of d_n are not Cauchy, so the partial sums of a_n are not Cauchy either. Thus $\sum a_n$ diverges.

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Theorem 1.4 (Cauchy Condensation Test)**.** *If $a_0 \geq a_1 \geq \cdots \geq 0$, then $\sum a_n$ converges if and only if $\sum 2^k a_{2^k}$ converges.*

Proof. Since a_n is decreasing and non-negative, we group terms in blocks of powers of 2. For the partial sums $s_n = \sum_{k=0}^{n} a_k$, consider s_{2^n}:

$$\begin{aligned} s_{2^n} &= a_0 + a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^{n-1}} + \cdots + a_{2^n - 1}) \\ &\leq a_0 + a_1 + 2a_2 + 4a_4 + \cdots + 2^{n-1} a_{2^{n-1}} \\ &= a_0 + \sum_{k=0}^{n-1} 2^k a_{2^k}. \end{aligned}$$

So if $\sum 2^k a_{2^k}$ converges, then the partial sums s_{2^n} are bounded. Since $\{s_n\}$ is monotonically increasing (all terms are non-negative), $\{s_n\}$ is bounded, so $\sum a_n$ converges.

Conversely,

$$\begin{aligned} s_{2^n} &= a_0 + a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^{n-1}} + \cdots + a_{2^n - 1}) \\ &\geq a_0 + a_1 + 2a_3 + 4a_7 + \cdots + 2^{n-1} a_{2^n - 1} \\ &\geq a_0 + \frac{1}{2}(2a_2 + 4a_4 + \cdots + 2^n a_{2^n}) \\ &= a_0 + \frac{1}{2} \sum_{k=1}^{n} 2^k a_{2^k}. \end{aligned}$$

So if $\sum a_n$ converges (i.e. the partial sums are bounded), then $\sum 2^k a_{2^k}$ is bounded and monotonically increasing, hence converges. □

1.5 Special Series

Section Overview: Three important families of series: the geometric series $\sum x^n$ (converges iff $|x| < 1$ to $1/(1-x)$), the p-series $\sum 1/n^p$ (converges iff $p > 1$), and the series $\sum 1/(n(\log n)^p)$ (converges iff $p > 1$). The latter two are proved elegantly via Cauchy condensation.

Definition 1.5. The **geometric series** is

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n.$$

Let $s_n = 1 + x + x^2 + \cdots + x^n$. Then

$$x \cdot s_n = x + x^2 + x^3 + \cdots + x^{n+1}.$$

Subtracting,

$$x \cdot s_n - s_n = x^{n+1} - 1,$$

so $s_n(x-1) = x^{n+1} - 1$, and thus for $x \neq 1$,

$$s_n = \frac{x^{n+1} - 1}{x - 1}.$$

The geometric series converges if and only if $|x| < 1$, in which case $s_n \to \dfrac{1}{1-x}$.

Proof. ($\Rightarrow$) If $|x| \geq 1$, then $|x^n| \geq 1$ for all n, so $a_n = x^n \not\to 0$. Since the terms do not tend to zero, the series diverges.

($\Leftarrow$) If $|x| < 1$, we have

$$\left| s_n - \frac{1}{1-x} \right| = \left| \frac{1 - x^{n+1}}{1-x} - \frac{1}{1-x} \right| = \frac{|x|^{n+1}}{|1-x|}.$$

Since $|x| < 1$, we have $|x|^{n+1} \to 0$ as $n \to \infty$, so $s_n \to \dfrac{1}{1-x}$. □

Definition 1.6. The ***p*-series** is

$$\sum \frac{1}{n^p}.$$

If $p > 1$ it converges; if $p \leq 1$ it diverges.

Proof. By the Cauchy condensation test, $\sum \frac{1}{n^p}$ converges if and only if

$$\sum 2^k \cdot \frac{1}{(2^k)^p} = \sum 2^k \cdot 2^{-kp} = \sum 2^{k(1-p)}$$

converges. This is a geometric series with ratio $r = 2^{1-p}$. It converges if and only if $|r| < 1$, i.e. $2^{1-p} < 1$, which holds if and only if $1 - p < 0$, i.e. $p > 1$. □

Theorem 1.7. *If $p > 1$, then $\sum \dfrac{1}{n(\log n)^p}$ converges.*

Proof. By the Cauchy condensation test, $\sum \frac{1}{n(\log n)^p}$ converges if and only if

$$\sum 2^k \cdot \frac{1}{2^k (\log 2^k)^p} = \sum \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum \frac{1}{k^p}$$

converges. Since $p > 1$, the p-series $\sum \frac{1}{k^p}$ converges, so the original series converges. □

1.6 The Number e

Section Overview: We define $e = \sum 1/n!$, prove the series converges by comparison with the geometric series, and show e is irrational via a clever argument: assuming $e = p/q$ leads to a positive integer strictly less than 1, a contradiction.

Definition 1.8. We define the number e by

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

Theorem 1.9. *The series $\sum \frac{1}{n!}$ converges.*

Proof. For $n \geq 1$, we have $n! \geq 2^{n-1}$, so

$$\frac{1}{n!} \leq \frac{1}{2^{n-1}} = \frac{2}{2^n}.$$

Since $\sum \frac{1}{2^n}$ is a convergent geometric series (with $|x| = \frac{1}{2} < 1$), the series $\sum \frac{1}{n!}$ converges by the comparison test. □

Theorem 1.10. *e is irrational.*

Proof. Suppose for contradiction that $e = p/q$ for some $p, q \in \mathbb{N}$ with $q \geq 1$. Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

Then $q!\, s_q$ is an integer (since $q!/k!$ is an integer for $0 \leq k \leq q$), and $q!\, e = q!\, p/q$ is also an integer. Therefore

$$q!\,(e - s_q) = q! \sum_{k=q+1}^{\infty} \frac{1}{k!}$$

is a positive integer. But we can bound this:

$$\begin{aligned} q! \sum_{k=q+1}^{\infty} \frac{1}{k!} &= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots \\ &\leq \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots \\ &= \frac{1}{q+1} \cdot \frac{1}{1 - \frac{1}{q+1}} = \frac{1}{q} \leq 1. \end{aligned}$$

So $q!(e - s_q)$ is a positive integer that is at most $\frac{1}{q} \leq 1$. Since it is strictly less than 1 when $q \geq 2$ (and one checks e is not an integer directly), this is a contradiction. $\square$

1.7 Root and Ratio Tests

Section Overview: The root test uses $\limsup \sqrt[n]{|a_n|}$ and the ratio test uses $\limsup |a_{n+1}/a_n|$ to determine convergence by comparison with geometric series. Both are inconclusive when the limit is 1. The root test is at least as powerful as the ratio test.

Theorem 1.11 (Root Test). *Given $\sum a_n$, let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then:*

(a) If $\alpha < 1$, the series converges.

(b) If $\alpha > 1$, the series diverges.

(c) If $\alpha = 1$, the test is inconclusive.

Proof. **(a)** If $\alpha < 1$, choose β with $\alpha < \beta < 1$. Since $\alpha = \limsup \sqrt[n]{|a_n|}$, there exists N such that $\sqrt[n]{|a_n|} < \beta$ for all $n \geq N$, i.e. $|a_n| < \beta^n$. Since $\sum \beta^n$ is a convergent geometric series ($\beta < 1$), $\sum a_n$ converges by the comparison test.

(b) If $\alpha > 1$, then $\sqrt[n]{|a_n|} > 1$ for infinitely many n, so $|a_n| > 1$ for infinitely many n. Thus $a_n \not\to 0$, so the series diverges.

(c) Both $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ have $\alpha = 1$, but one diverges and the other converges. $\square$

Theorem 1.12 (Ratio Test). *Given $\sum a_n$ with $a_n \neq 0$:*

(a) If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series converges.

(b) If $\left|\dfrac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq N$*, the series diverges.*

(c) If $\limsup_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$*, the test is inconclusive.*

Proof. **(a)** If $\limsup \left|\frac{a_{n+1}}{a_n}\right| = r < 1$, choose β with $r < \beta < 1$. There exists N such that $\left|\frac{a_{n+1}}{a_n}\right| < \beta$ for all $n \geq N$. Then by induction,

$$|a_{N+k}| < \beta^k |a_N|$$

for all $k \geq 0$. Since $\sum \beta^k |a_N|$ is a convergent geometric series, $\sum a_n$ converges by comparison.

(b) If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq N$, then $|a_n| \geq |a_N| > 0$ for all $n \geq N$, so $a_n \not\to 0$ and the series diverges.

(c) Same examples as the root test: $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. □

Remark 1.13. The root test is at least as powerful as the ratio test: whenever the ratio test gives a conclusion, so does the root test. This is because

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| \geq \limsup \sqrt[n]{|a_n|}.$$

However, there are series where the root test succeeds but the ratio test is inconclusive.