Exercises in Metric Spaces and Continuity

Analysis Problem Set

2. If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that the converse is not necessarily true.

3. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

7. If $E \subset X$ and if f is a function defined on X, the *restriction* of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$.

   Define f and g on $\mathbb{R}^2$ by: $f(0,0) = g(0,0) = 0$, and

$$f(x,y) = \frac{xy^2}{x^2 + y^4}, \quad g(x,y) = \frac{xy^2}{x^2 + y^6} \quad \text{if } (x,y) \neq (0,0).$$

   Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!