Zero Sets of Continuous Functions are Closed

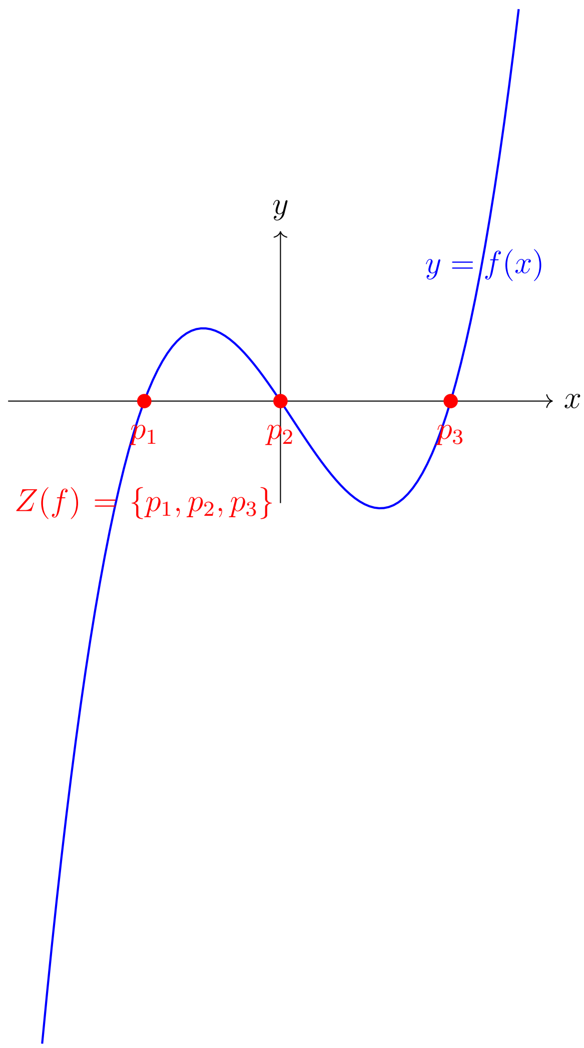
A visual guide to Rudin 4.3

1 What is a Zero Set?

Given a continuous function $f : X \to \mathbb{R}$, the **zero set** is:

$$Z(f) = \{p \in X : f(p) = 0\} = f^{-1}(\{0\})$$

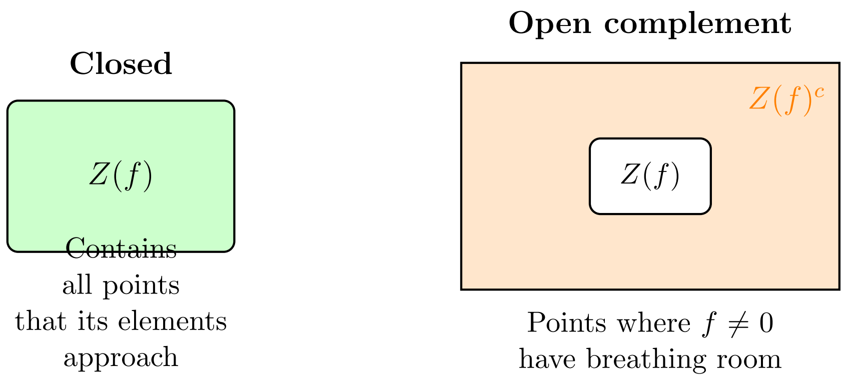
It's simply the set of points where f equals zero.



2 What Does "Closed" Mean Again?

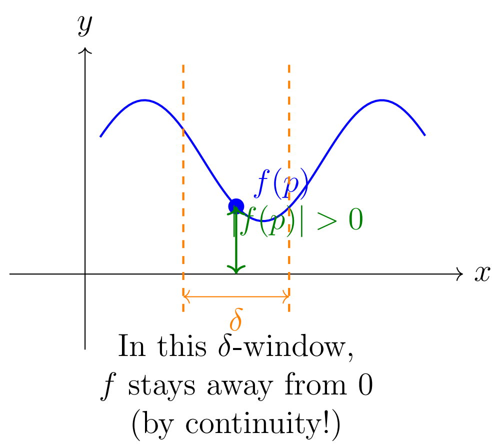
A set is **closed** if its complement is open.

Equivalently: a set is closed if it contains all its limit points.



3 The Intuition: Why is $Z(f)$ Closed?

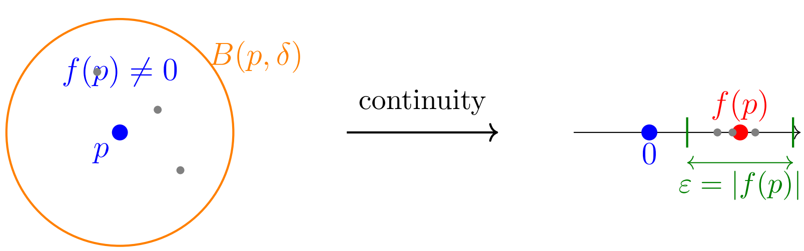
If f is continuous and $f(p) \neq 0$, then f is "stuck away from zero" near p.



Key insight: Continuity means f can't "jump." If $f(p)$ is some distance from 0, then nearby values of f are also bounded away from 0.

4 The Proof Strategy

We show $Z(f)^c = \{p : f(p) \neq 0\}$ is open.



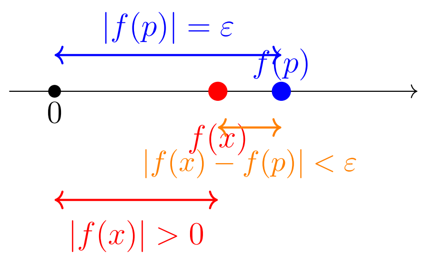
The argument:

1. Let $p \notin Z(f)$, so $|f(p)| > 0$

2. Set $\varepsilon = |f(p)|$
3. By continuity: there exists $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ when $d(x,p) < \delta$
4. Triangle inequality: $|f(x)| \geq |f(p)| - |f(x) - f(p)| > \varepsilon - \varepsilon = 0$
5. So $f(x) \neq 0$ for all x near p — meaning $B(p, \delta) \subset Z(f)^c$

5 The Triangle Inequality Trick

This is a standard technique worth remembering:



$f(x)$ can't reach 0 because it's within ε of $f(p)$, which is ε away from 0!

6 Alternative Viewpoint: Preimages

There's an even slicker way to see this:

- $Z(f) = f^{-1}(\{0\})$
- $\{0\}$ is a closed subset of $\mathbb{R}$
- Continuous functions pull back closed sets to closed sets
- Therefore $Z(f)$ is closed!

Both approaches work. The ε-δ proof is more hands-on; the preimage proof is more conceptual.