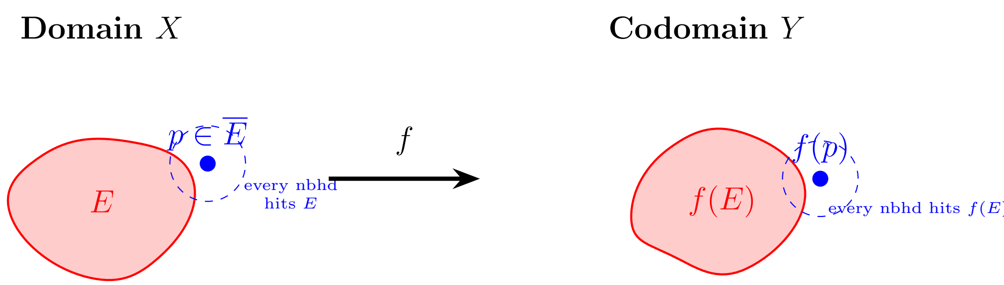
Continuity Preserves "Closeness"

A visual guide to Rudin 4.2

1 What Does $f(\overline{E}) \subset \overline{f(E)}$ Mean?

In plain English: **if a point is near E, its image is near $f(E)$.**



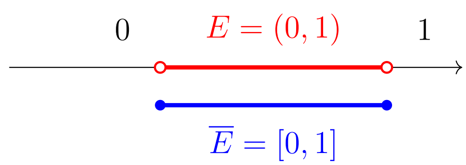
Key idea: Continuous functions can't "jump away" from a set. If p is close to E, then $f(p)$ must be close to $f(E)$.

2 Review: What is the Closure?

The **closure** $\overline{E}$ of a set E consists of:

- All points *in* E, plus
- All *limit points* of E (points that E "approaches")

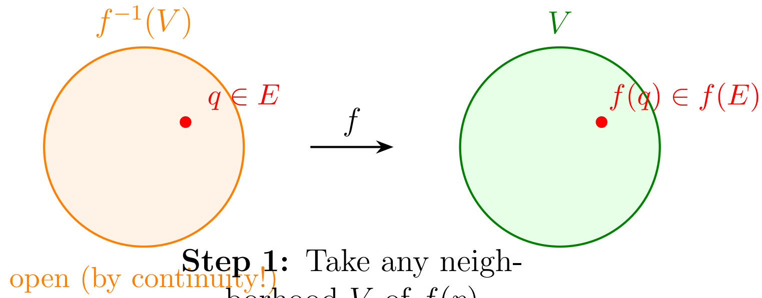
Equivalently: $p \in \overline{E}$ if and only if **every neighborhood of** p **intersects** E.



3 The Proof Strategy

To show $f(\overline{E}) \subset \overline{f(E)}$:

1. Pick any $p \in \overline{E}$ (so every neighborhood of p hits E)
2. Goal: show $f(p) \in \overline{f(E)}$ (every neighborhood of $f(p)$ hits $f(E)$)
3. Use continuity as the bridge!



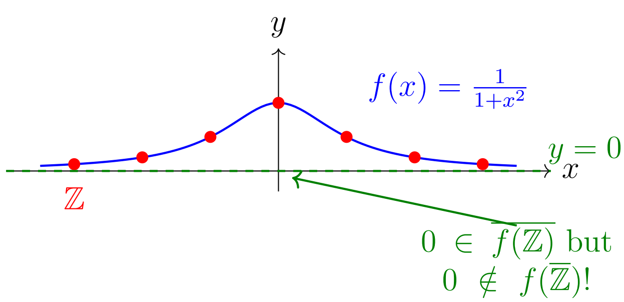
Step 1: Take any neighborhood V of $f(p)$
Step 2: $f^{-1}(V)$ is open (continuity!) and contains p
Step 3: Since $p \in \overline{E}$, grab $q \in E \cap f^{-1}(V)$
Step 4: Then $f(q) \in f(E) \cap V$ ✓

4 Why the Converse Fails

The converse would say: $\overline{f(E)} \subset f(\overline{E})$.

Translation: "Every point near $f(E)$ is the image of some point near E."

This can fail when a continuous function "sends points to infinity" — points of $f(E)$ accumulate at a value that f never reaches.



Why it fails:

- $\overline{\mathbb{Z}} = \mathbb{Z}$ (integers are closed — no limit points)
- $f(\mathbb{Z}) = \{1, 1/2, 1/5, 1/10, \ldots\}$ and these values approach 0
- So $0 \in \overline{f(\mathbb{Z})}$
- But $f(x) = \frac{1}{1+x^2} > 0$ for all x, so 0 is never in the image
- Therefore $0 \notin f(\overline{\mathbb{Z}}) = f(\mathbb{Z})$

The converse fails because f "uses up" its domain approaching 0 asymptotically but never getting there.

5 Summary

Continuity + closeness in domain

$\Downarrow$

closeness in codomain

But NOT the reverse!