Design and Analysis of Algorithms
Part III: Greedy Algorithms

Lecture 9: Fraction Knapsack, Huffman Coding and Activity Selection Problems

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Outline

- Introduction to Part III
- The Fractional Knapsack Problem
  - Problem Definition
  - A Greedy Algorithm and correctness
- The Huffman Coding Problem
  - Problem Definition
  - A Greedy Algorithm
- The Activity Selection Problem
  - Problem Definition
  - A Greedy Algorithm and correctness
  - Extended: Weighted Activity Selection

Introduction to Greedy Algorithm

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- Greedy algorithms don't always yield optimal solutions but, when they do, they're usually the simplest and most efficient algorithms available.

Introduction to Part III

- In Part III, we will illustrate greedy strategies using several examples:
  - Fractional Knapsack Problem (部分背包问题)
  - Huffman Coding Problem (赫夫曼编码问题)
  - Activity Selection Problem (活动选择问题)

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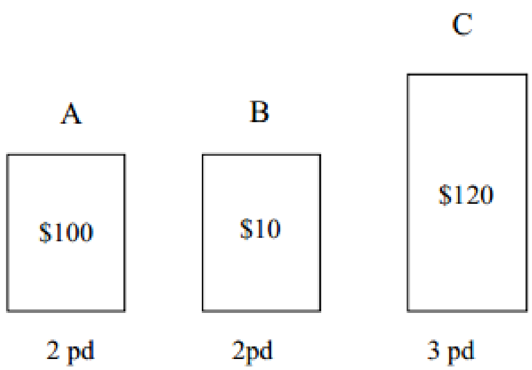
The Knapsack Problem...



Capacity of knapsack: $K = 4$

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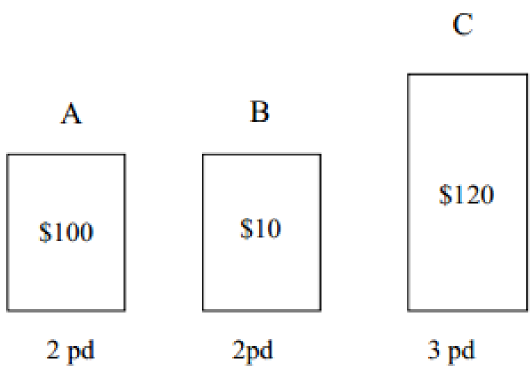

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Fractional Knapsack Problem:
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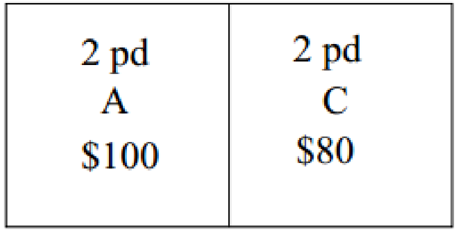
Solution:



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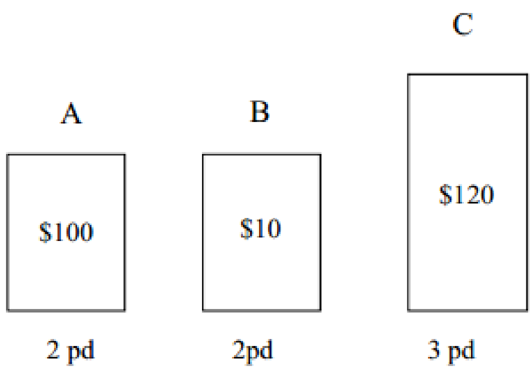

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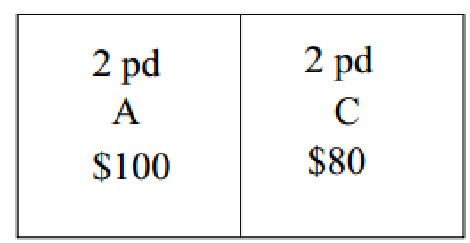
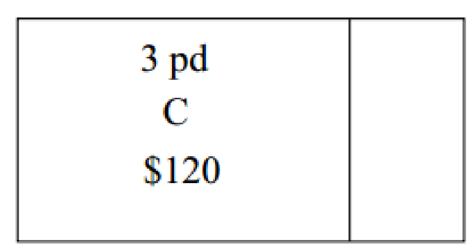


0-1 Knapsack Problem:
Can only take or leave item. You can't take a fraction.

Solution:



The Fractional Knapsack Problem: Formal Definition

- Given K and a set of n items:

weight	w_1	w_2	$\ldots$	w_n
value	v_1	v_2	$\ldots$	v_n

- Find: $0 \le x_i \le 1$, $i = 1,2,\ldots, n$ such that

$$\sum_{i=1}^{n} x_i w_i \le K$$

and the following is maximized:

$$\sum_{i=1}^{n} x_i v_i$$

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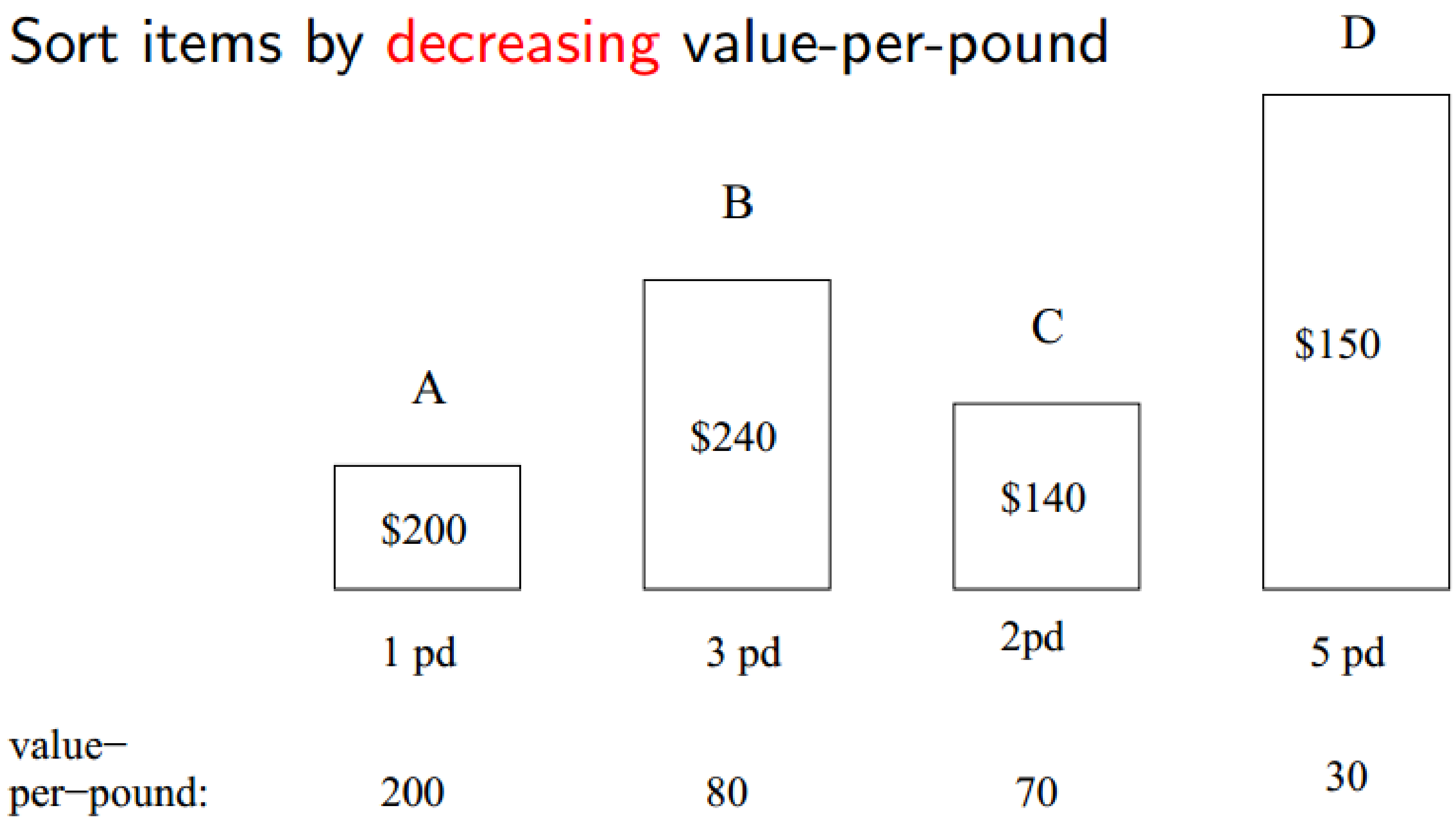

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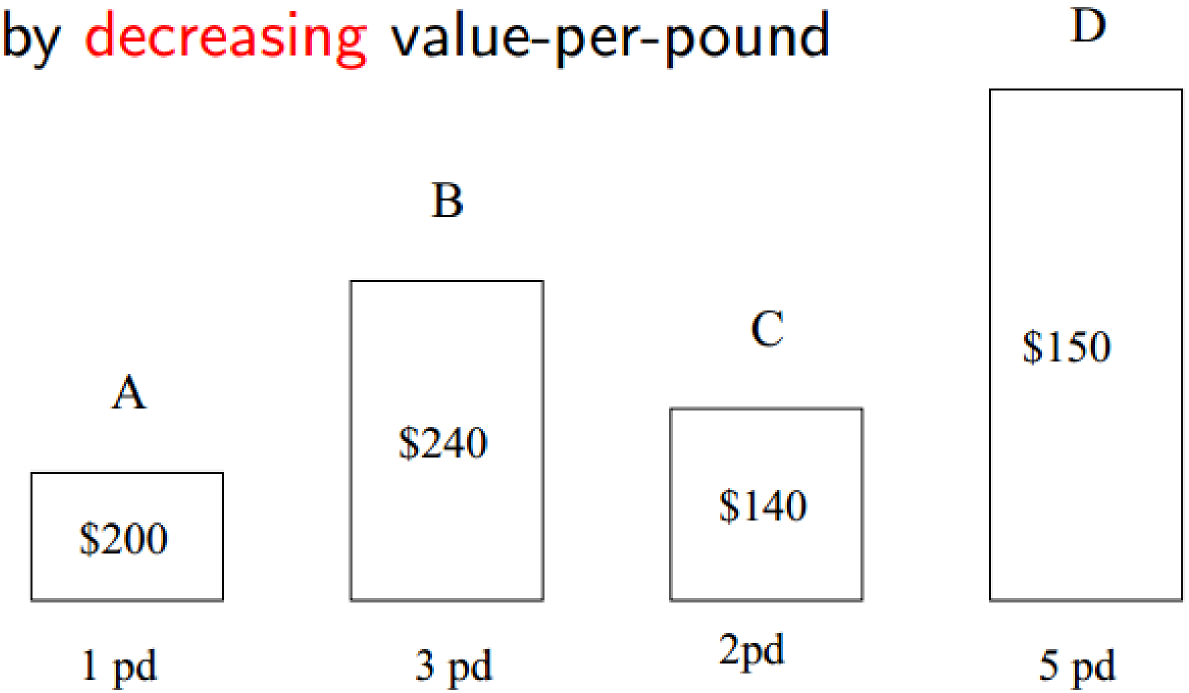

value–per–pound:	200	80	70	30

If knapsack holds $K = 5$ pd, solution is:

1	pd	A
3	pd	B
1	pd	C

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Running time: O(n log n).

Pseudocode

Fraction-Knapsack(n,v,w,K)

```
Input: Value array v and weight array w of n items, capacity of
       knapsack K.
Output: Solution of maximum value.
Let r[1..n],x[1..n] be two new arrays;
for i ← 1 to n do
|   r[i] ← v[i]/w[i];
|   x[i] ← 0;
end
Sort the items in decreasing order of their ratios r, rename the items if
 necessary so that the sorted order of items is ⟨1, 2, ..., n⟩;
j ← 0;
while K > 0 and j ≤ n do
|   j ← j + 1;
|   if K > w[j] then
|   |   x[j] ← 1;
|   |   K ← K − w[j];
|   end
|   else
|   |   x[j] ← k/w[j];
|   |   break;
|   end
end
return x;
```

Example of Optimal Solution Construction

	1	**2**	**3**	**4**
v_i	60	75	100	120
w_i	10	25	20	30

K = 50

Example of Optimal Solution Construction

	1	**2**	**3**	**4**
v_i	60	75	100	120
w_i	10	25	20	30
r_i	6	3	5	4

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K = 50

$w_i < K$

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v_i	60	100	120	75
w_i	10	20	30	25
r_i	6	5	4	3
x_i	1	0	0	0

K = 40

Example of Optimal Solution Construction

	1	**3**	**4**	**2**
v_i	60	100	120	75
w_i	10	20	30	25
r_i	6	5	4	3
x_i	1	1	0	0

K = 20

Example of Optimal Solution Construction

	1	**3**	**4**	**2**
v_i	60	100	120	75
w_i	10	20	30	25
r_i	6	5	4	
x_i	1	1	0	0

$w_i \geq K$

K = 20

Example of Optimal Solution Construction

	1	**3**	**4**	**2**
v_i	60	100	120	75
w_i	10	20	30	25
r_i	6	5	4	3
x_i	1	1	2/3	0

K = 20

Example of Optimal Solution Construction

	1	**3**	**4**	**2**
v_i	60	100	120	75
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Result

Correctness

- We are given a set of n items {1, 2, …, n}.

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  - Note that $0 \leq y_i \leq 1$, $1 \leq i \leq n$, and the knapsack must be full in both G and O: $\sum_{i=1}^{n} x_i w_i = \sum_{i=1}^{n} y_i w_i = K$.

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- Consider the first item i where the two selections differ.
  - By definition, solution G takes a greater amount of item i than solution O (because the greedy solution always takes as much as it can).

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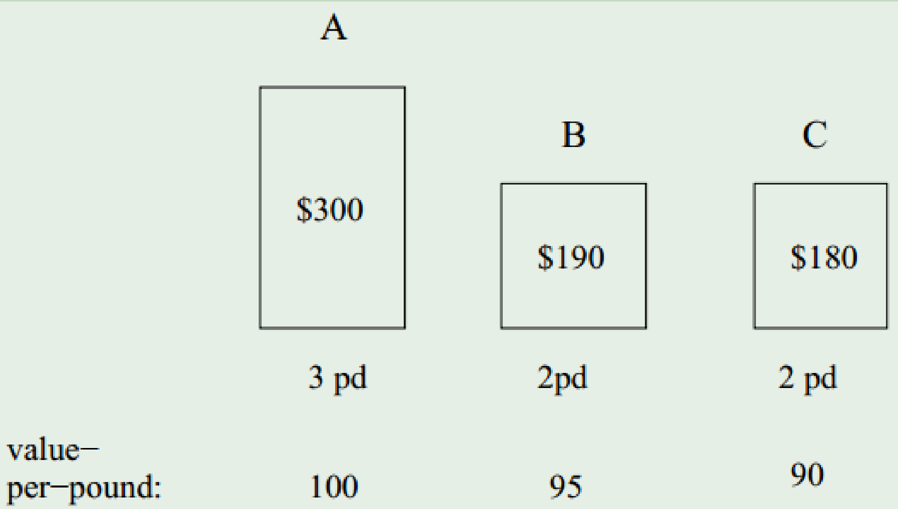
  - Thus solution O' is also optimal.
- By repeating this process, we will eventually convert O into G, without changing the total value of the selection. Therefore G is also optimal.

Greedy solution for 0-1 Knapsack Problem?

The 0-1 Knapsack Problem does not have a greedy solution!

Example



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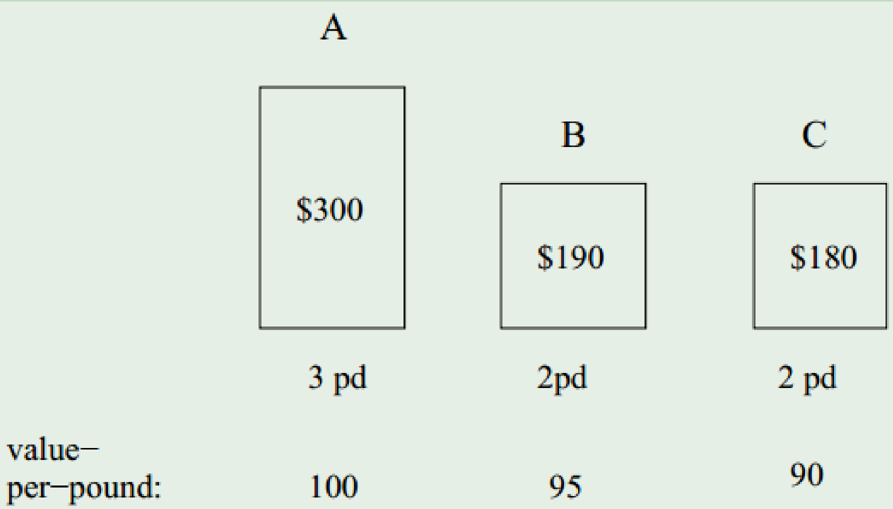

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Suppose we try to prove the greedy algorithm for 0-1 knapsack problem is correct. We follow exactly the same lines of arguments as fractional knapsack problem.

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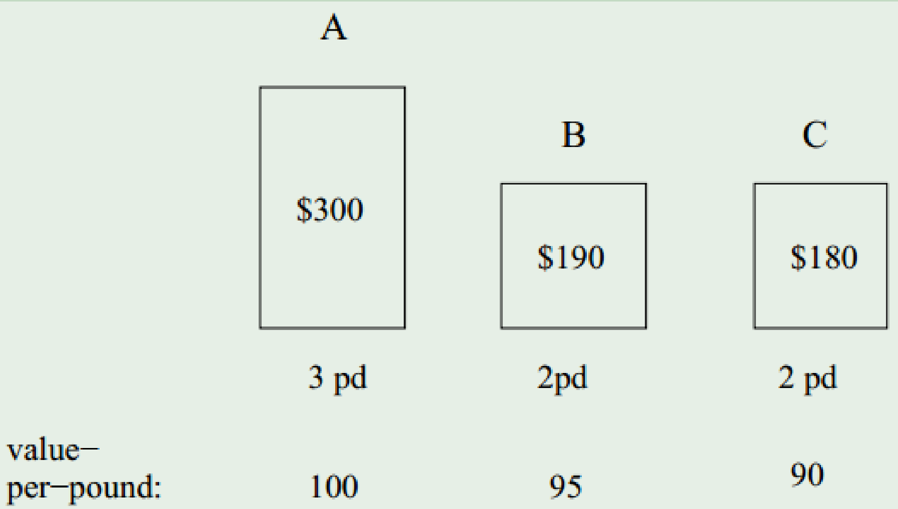

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Suppose we try to prove the greedy algorithm for 0-1 knapsack problem is correct. We follow exactly the same lines of arguments as fractional knapsack problem. Of course, it must fail. Where is the problem in the proof?

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Encoding

Example

Suppose that we have a $100,000$ character data file that we wish to store. The file contains only 6 characters, appearing with the following frequencies:

	a	b	c	d	e	f
Frequency in '000s	45	13	12	16	9	5

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  - a code is a set of codewords
  - e.g., {000, 001, 010, 011, 100, 101} and {0, 101, 100, 111, 1101, 1100}

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Given a code (corresponding to some alphabet Σ) and a message it is easy to encode the message. Just replace the characters by the codewords.

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Example

$\Sigma = \{a, b, c, d\}$
If the code is

$$C_1 = \{a = 00,\ b = 01,\ c = 10,\ d = 11\}$$

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then bad is encoded into

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If the code is

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then bad is encoded into 01

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Given a code (corresponding to some alphabet Σ) and a message it is easy to encode the message. Just replace the characters by the codewords.

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If the code is

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then bad is encoded into 01 00 11
If the code is

$$C_2 = \{a = 0,\ b = 110,\ c = 10,\ d = 111\}$$

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$\Sigma = \{a, b, c, d\}$
If the code is

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then bad is encoded into 01 00 11
If the code is

$$C_2 = \{a = 0,\ b = 110,\ c = 10,\ d = 111\}$$

then bad is encoded into

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$\Sigma = \{a, b, c, d\}$
If the code is

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If the code is

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then bad is encoded into 110

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If the code is

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If the code is

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then bad is encoded into 110 0 111

Decoding

$$C_1 = \{a = 00, b = 01, c = 10, d = 11\}.$$
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In fact, one can show that every message encoded using C_1 or C_2 is uniquely decipherable. The unique decipherability property is needed in order for a code to be useful.

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We are therefore interested in finding good (best compression) prefix-free codes.

The Optimal Source Coding Problem

Problem

Given an alphabet $A = \{a_1, \ldots, a_n\}$ with frequency distribution $f(a_i)$, find a binary prefix code C for A that minimizes the number of bits

$$B(C) = \sum_{i=1}^{n} f(a_i) L(c(a_i))$$

needed to encode a message of $\sum_{i=1}^{n} f(a_i)$ characters, where

- $c(a_i)$ is the codeword for encoding a_i, and
- $L(c(a_i))$ is the length of the codeword $c(a_i)$.

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Remark: We will see later that this is the optimum (lowest cost) prefix code.

Correspondence between Binary Trees and Prefix Codes

1-1 correspondence between leaves and characters.

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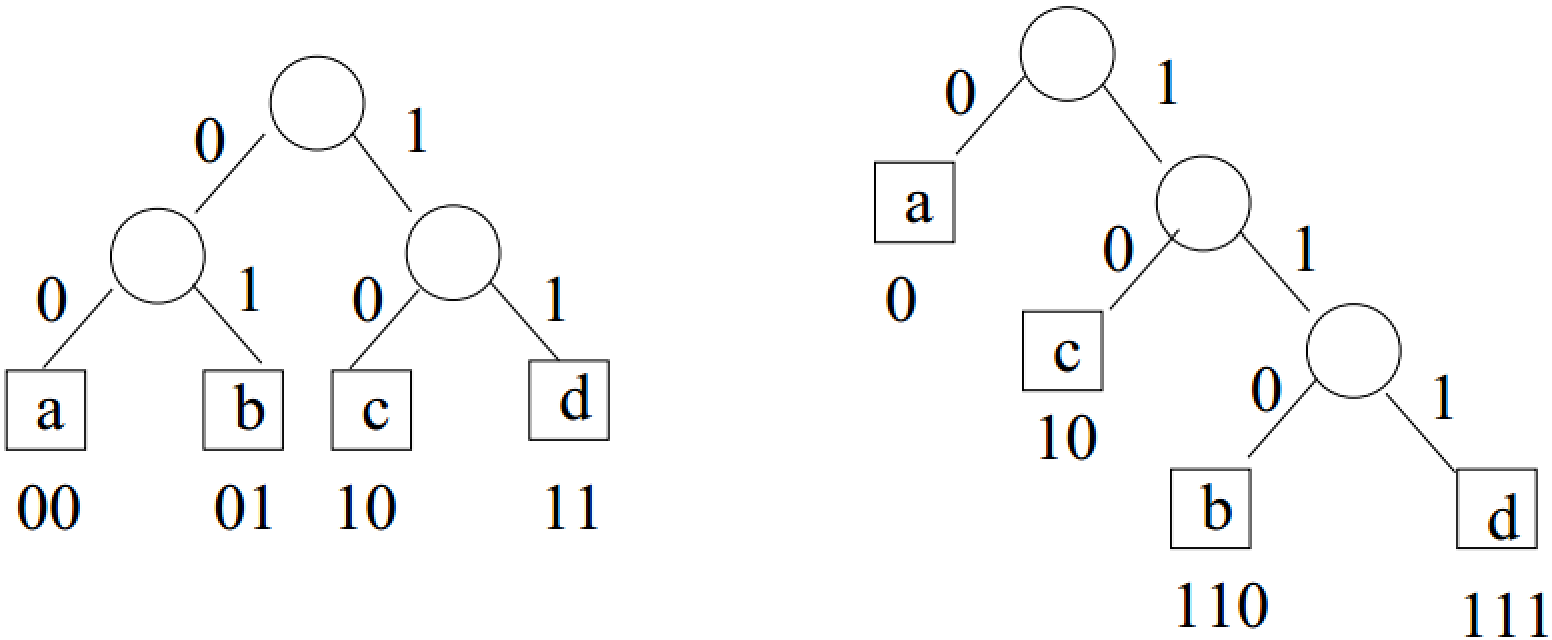

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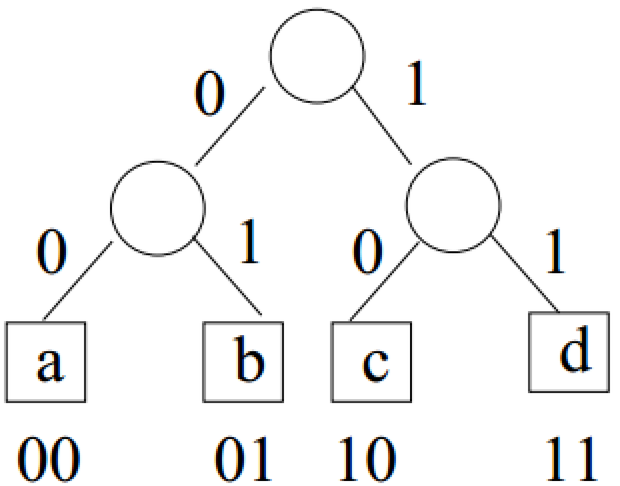

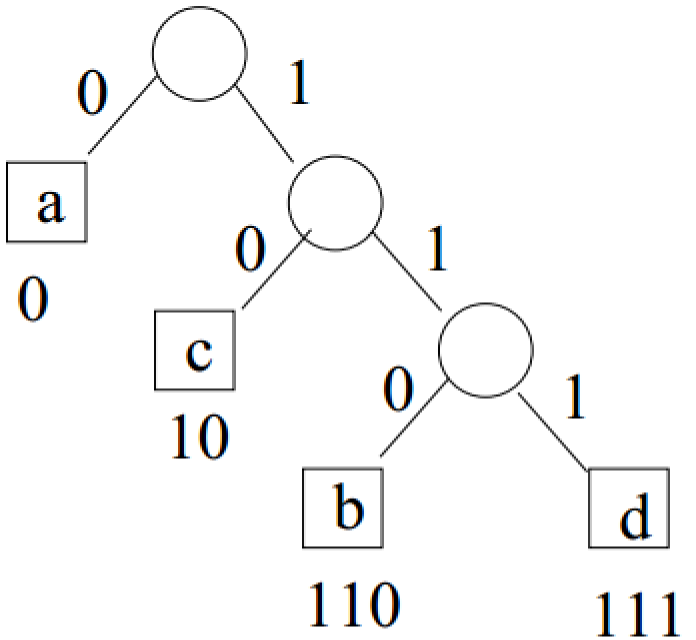

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- The binary string on a path from the root to a leaf is the codeword associated with the character at the leaf.

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The Huffman encoding problem is equivalent to the minimum weighted external path length problem.

Outline

- Introduction to Part III
- Fractional Knapsack Problem
  - Problem Definition
  - A Greedy Algorithm and correctness
- Huffman Coding Problem
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- Activity Selection Problem
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  - Extended: Weighted Activity Selection

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- Repeat this procedure, called merge, with the new alphabet A' until an alphabet with only one symbol is left. The resulting tree is the Huffman code.

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    - $A' = A \cup \{z\} - \{x,y\}$.
    - Note that $|A'| = |A| - 1$.
- Repeat this procedure, called merge, with the new alphabet A' until an alphabet with only one symbol is left. The resulting tree is the Huffman code.
- The optimum (minimum-cost) prefix code for the given frequency distribution.

Huffman Coding Algorithm

Given an alphabet A with frequency distribution $\{f(a_i):a_i \in A\}$. The binary Huffman tree is constructed using a priority queue, Q, of nodes, with frequencies as keys.

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    Insert(Q, z);
end
return *Extract-Min(Q)*;

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return *Extract-Min(Q)*;

Running time is O(n log n), as each priority queue operation takes time O(log n).

Example of Optimal Solution Construction

	a	b	c	d	e
$\boldsymbol{freq}$	18	16	5	15	45

Example of Optimal Solution Construction

	a	**b**	**c**	**d**	**e**
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a/18 **b/16** **c/5** **d/15** **e/45**

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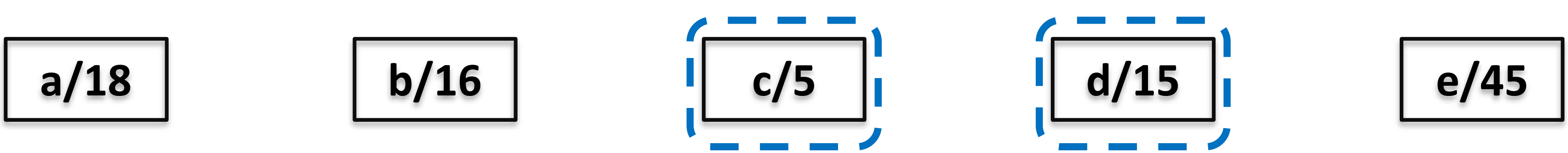

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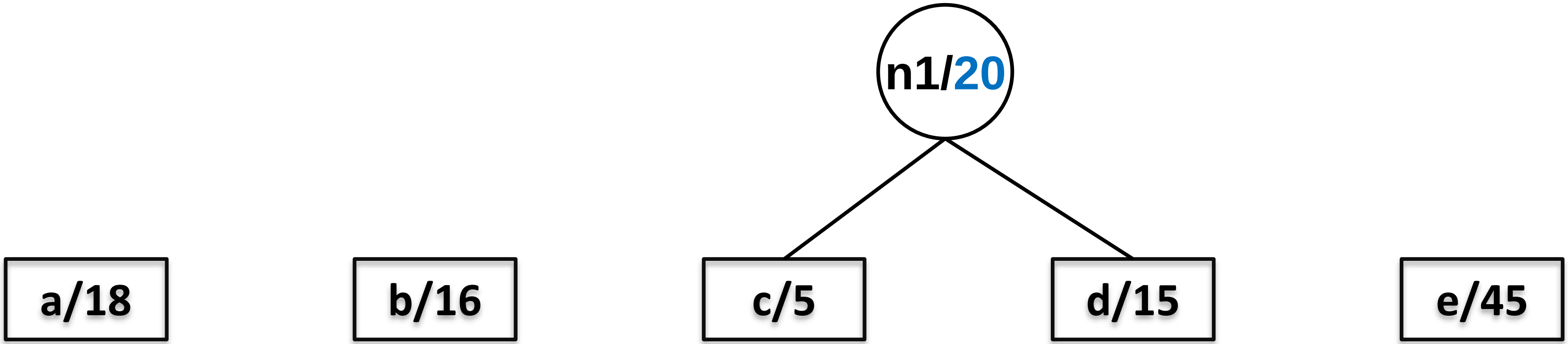

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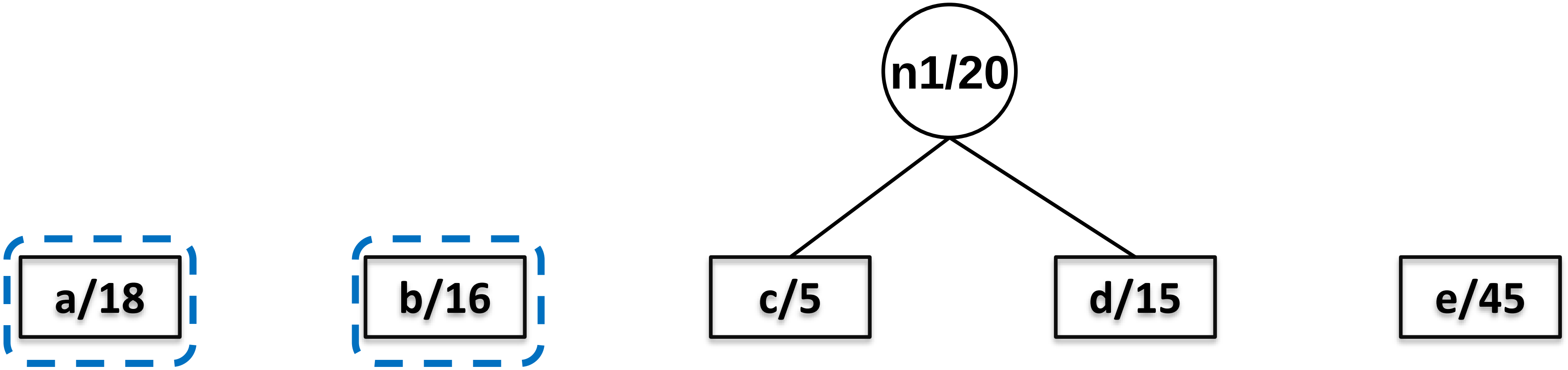

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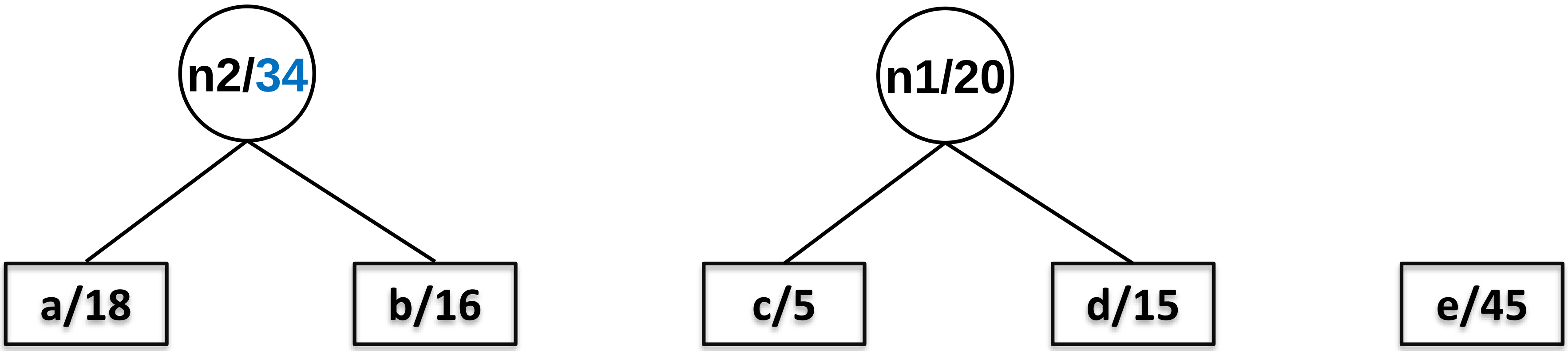

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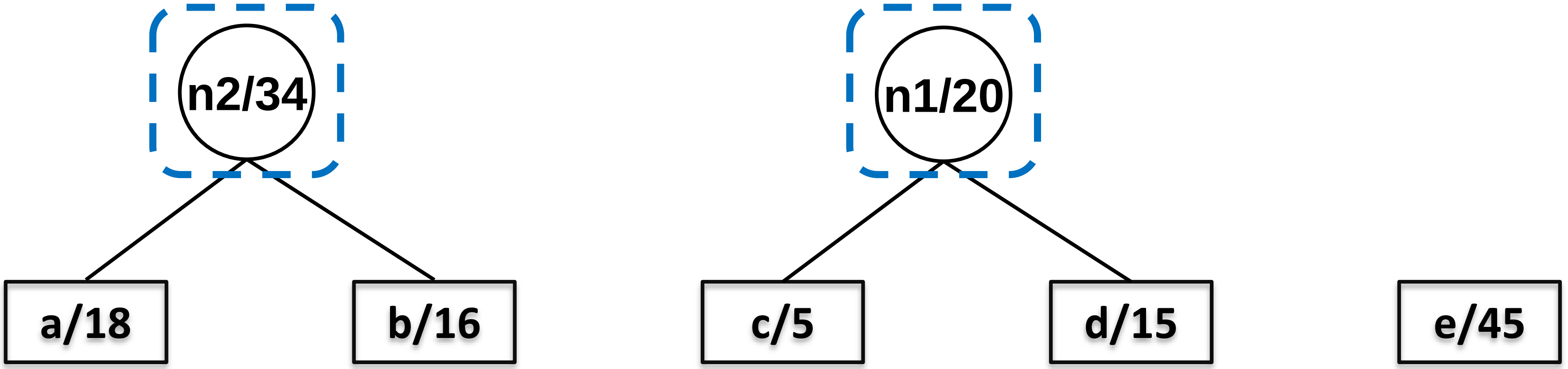

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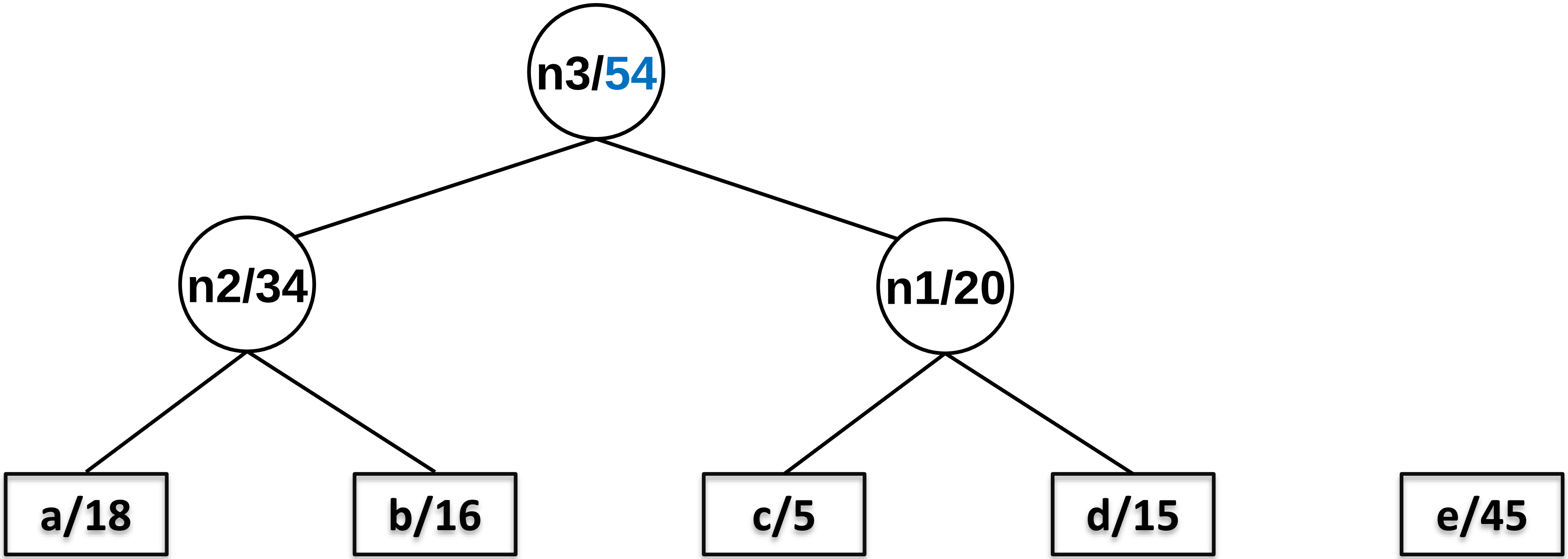

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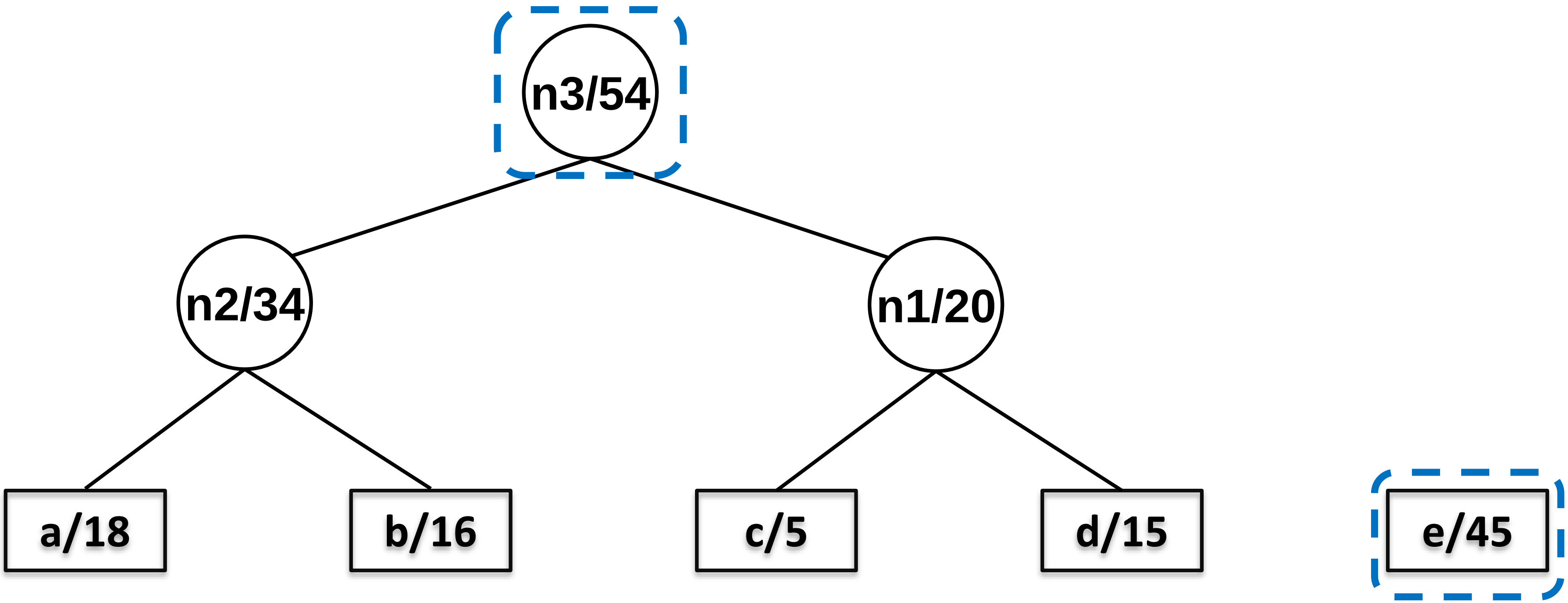

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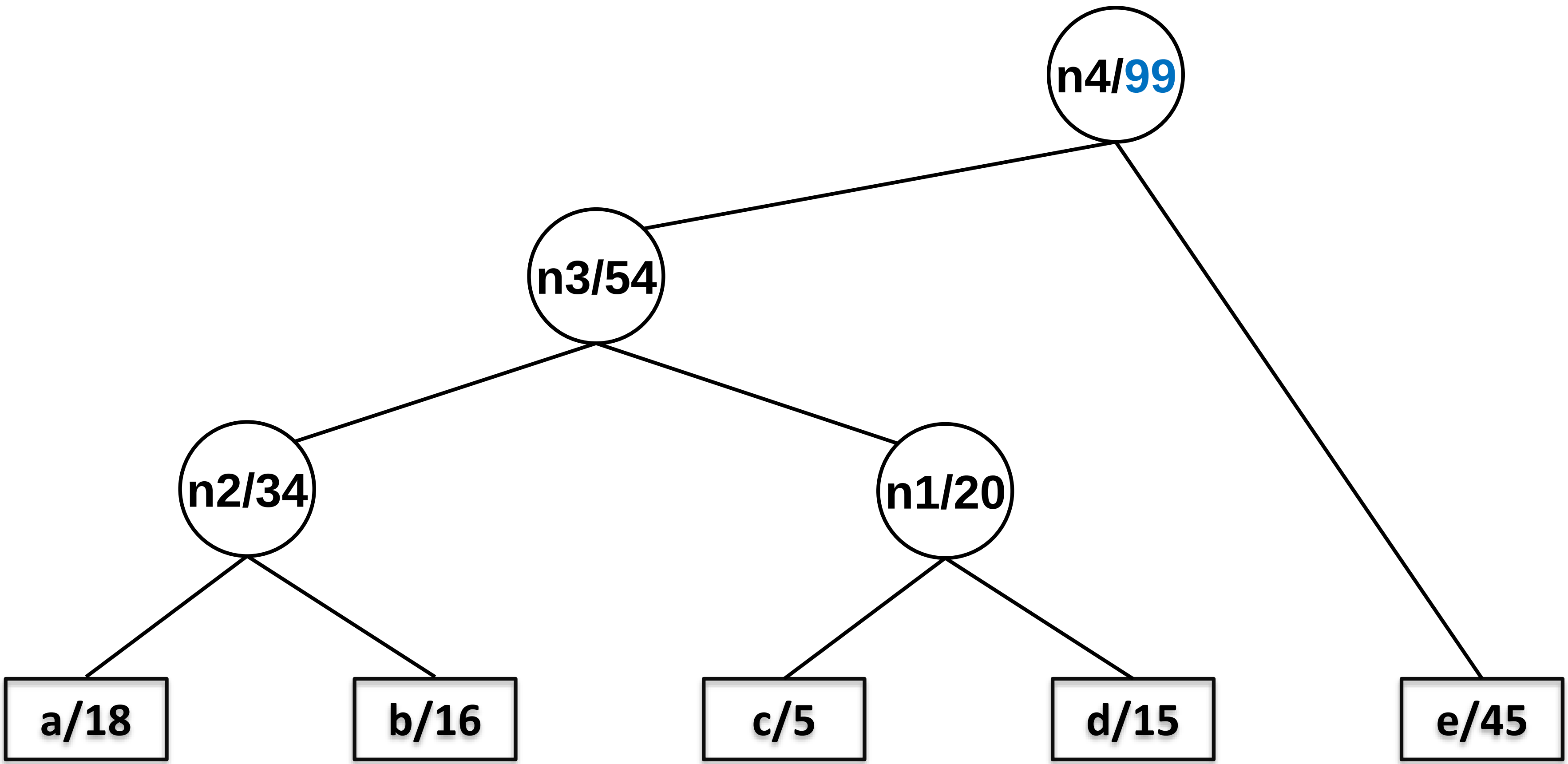

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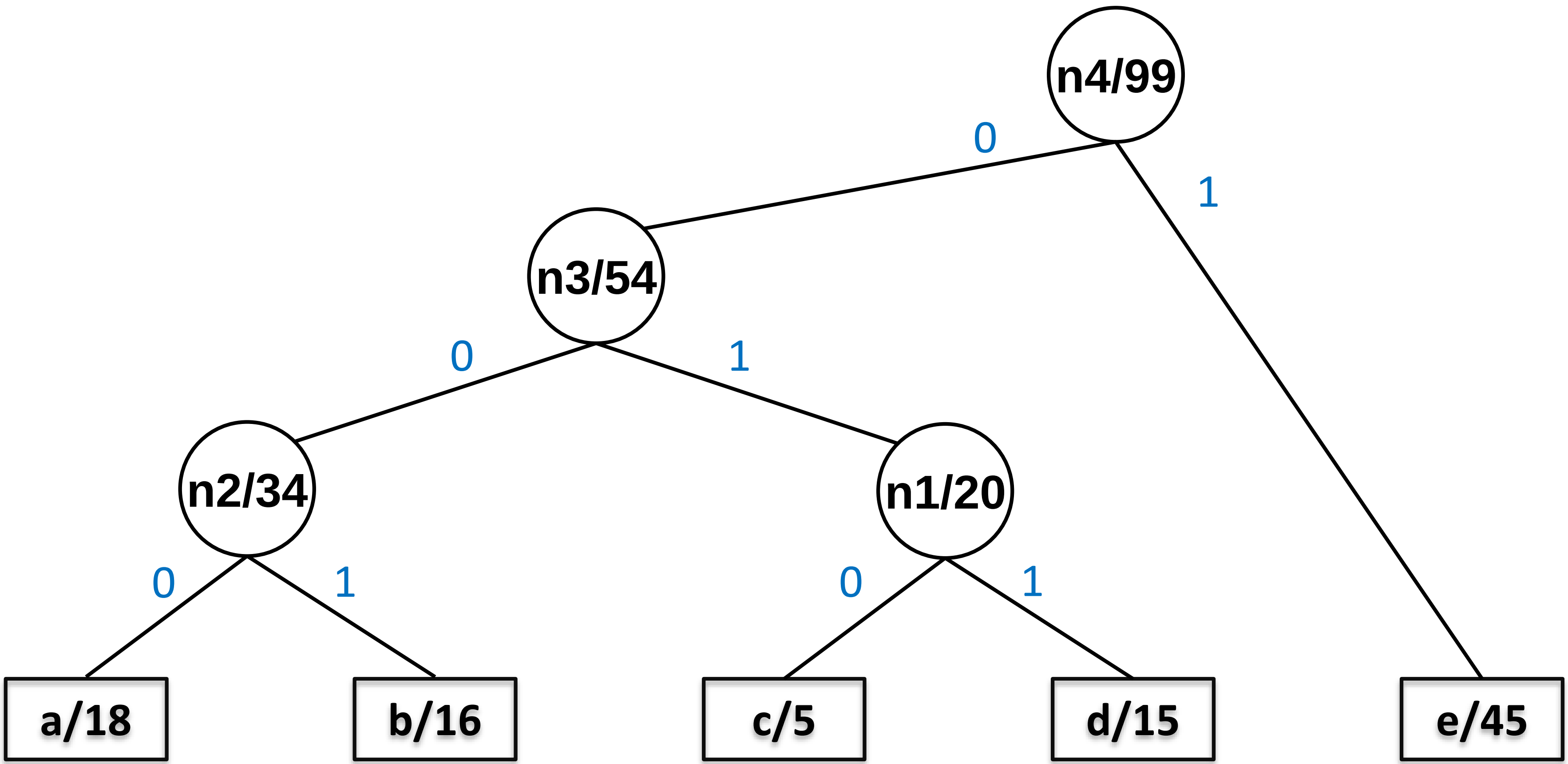
	a	b	c	d	e
$freq$	18	16	5	15	45
$code$	000	001	010	011	1



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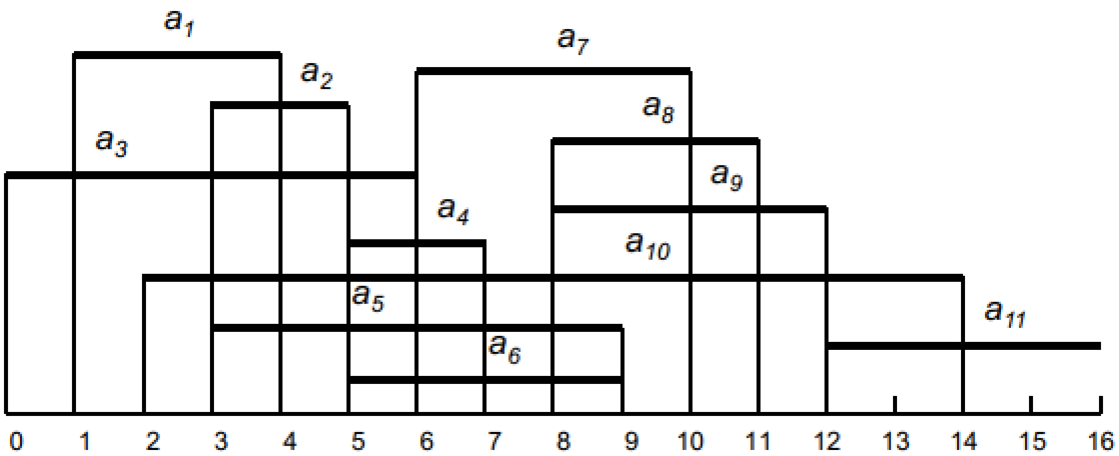
Activity Selection Problem



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  - Buy a wristband that lets you take part in any activity

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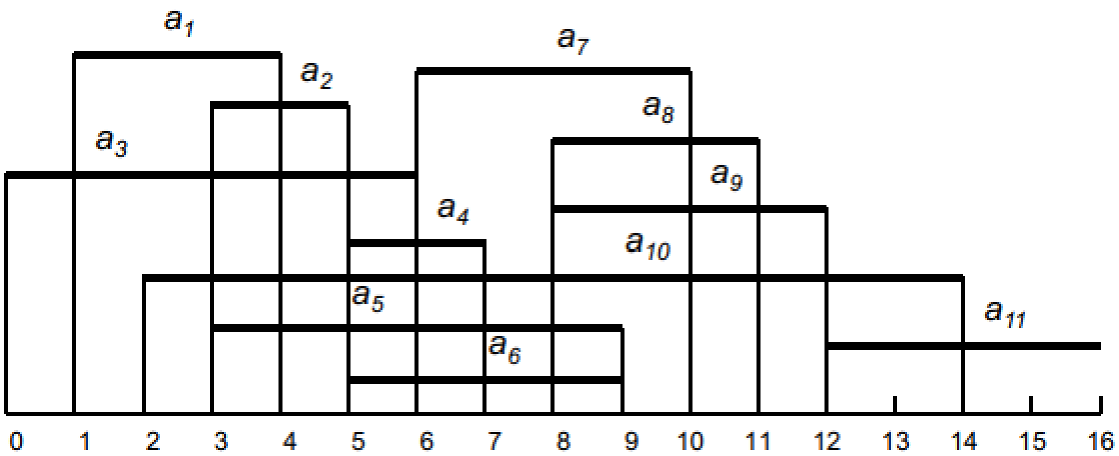

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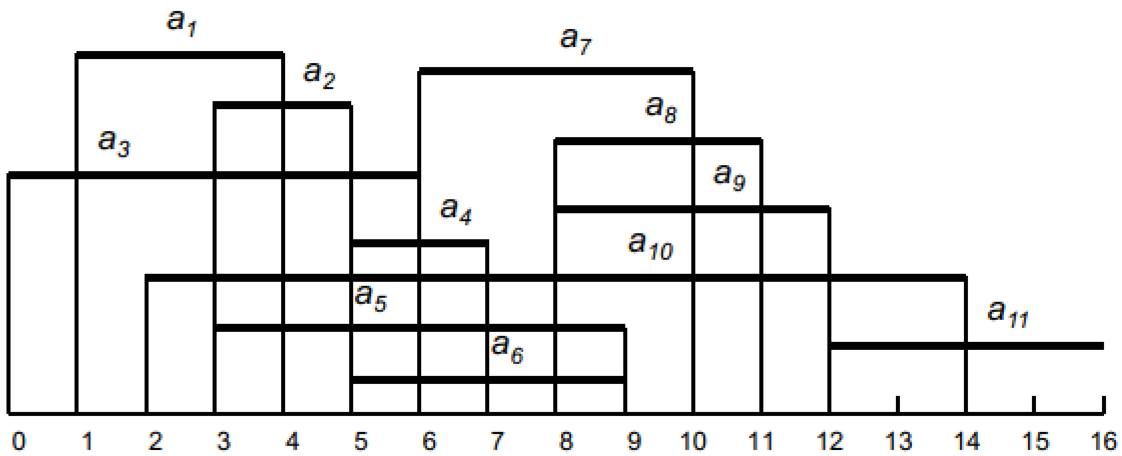

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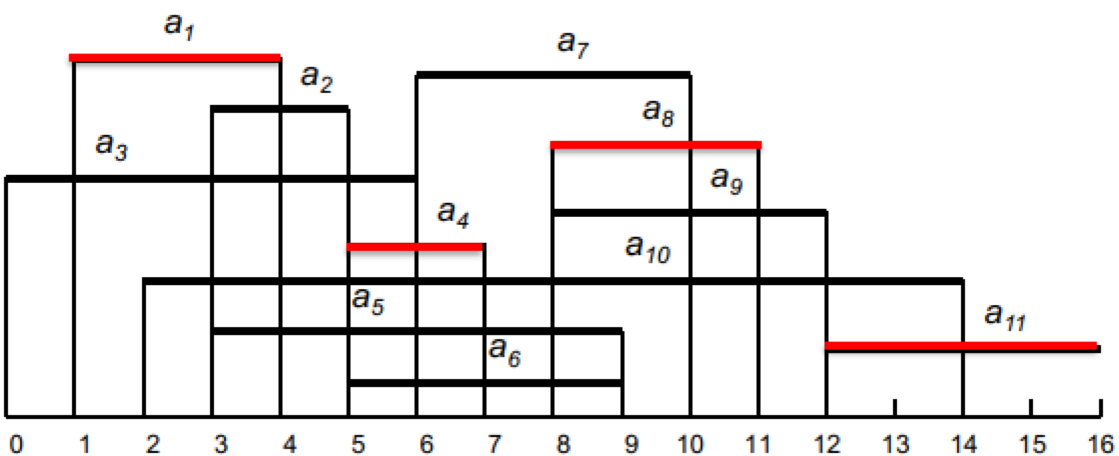

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a_1, a_4, a_8, a_{11}

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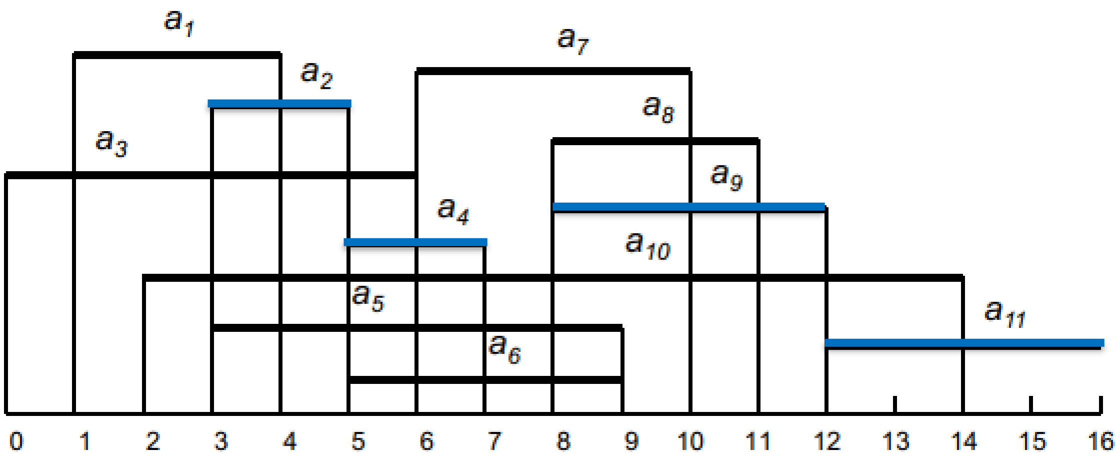

Some optimal solutions:
a_1, a_4, a_8, a_{11} or
a_2, a_4, a_9, a_{11}

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Activity Selection Problem : Formal Definition

- Given a set A of activities, such that each activity a_i has a start time s_i and a finishing time f_i:

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and the cardinality of S, $|S|$, is maximized.

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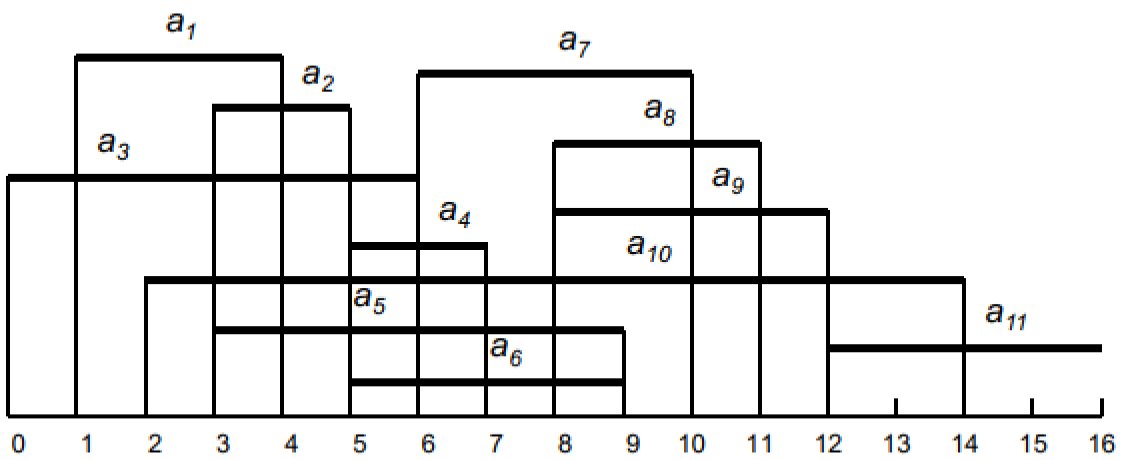

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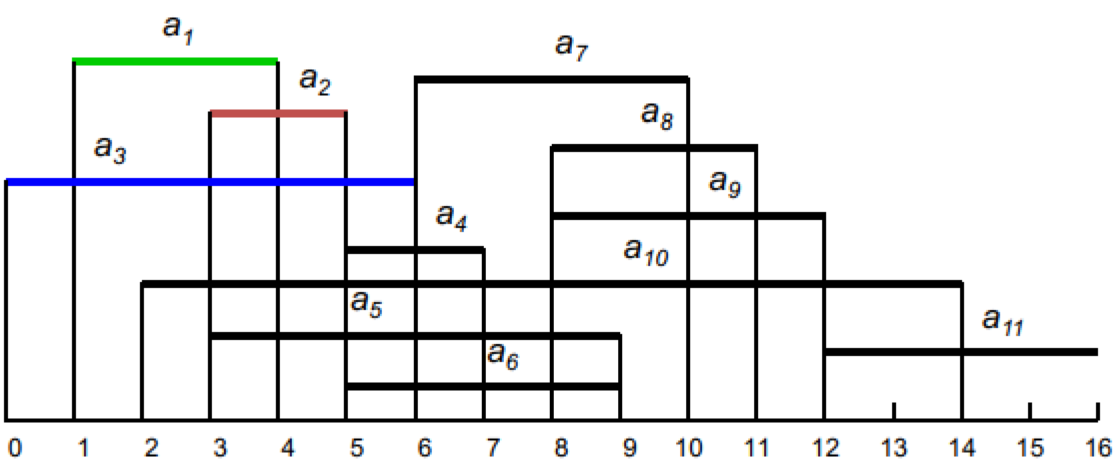
  - The shortest activity?
  - The one that starts at the earliest time?
  - The one that finishes at the earliest time?



Comparison of Different Greedy Strategies

- The shortest activity first?
  - Counterexample for shortest activity

This is optimal →

We will choose this →

- The activity with earliest start time first?
  - Counterexample for earliest start time

This is optimal →

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  - Once making the choice, delete all non-compatible activities
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  - Repeat the algorithm for the remaining activities
    - Either using iterations or recursion
    - We created one sub-problem to solve(Find the optimal schedule after the selected activity)

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for $i \leftarrow 2$ ***to*** n **do**

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for $i \leftarrow 2$ ***to*** n **do**
    if $s[i] \geq f[k]$ **then**
        `//` i `starts after` k `finishes - no overlap`
        $P \leftarrow P \cup a_i$;
        $k \leftarrow i$;
    end
end

Pseudocode

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Input: a set of activities A = a_1, a_2 ..., a_n
Output: the largest subset of A that do not overlap
Sort activities in increasing order of finishing time;
P = a_1; // insert the activity with earliest finishing time
k = 1; // index to the last activity in A
for i ← 2 to n do
    if s[i] ≥ f[k] then
        // i starts after k finishes - no overlap
        P ← P ∪ a_i;
        k ← i;
    end
end
return P;
```

Pseudocode

Greedy Activity Selection(A)

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Input: a set of activities A = a_1, a_2 ..., a_n
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Sort activities in increasing order of finishing time;  // cost: O(n log n)
P = a_1; // insert the activity with earliest finishing time
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for i ← 2 to n do                                        // cost: O(n)
    if s[i] ≥ f[k] then
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        P ← P ∪ a_i;
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    end
end
return P;
```

- Total Time Complexity: $O(n \log n)$

Example of Greedy Activity Selection Algorithm

- Sort activities in increasing order of finishing time(f_i)

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
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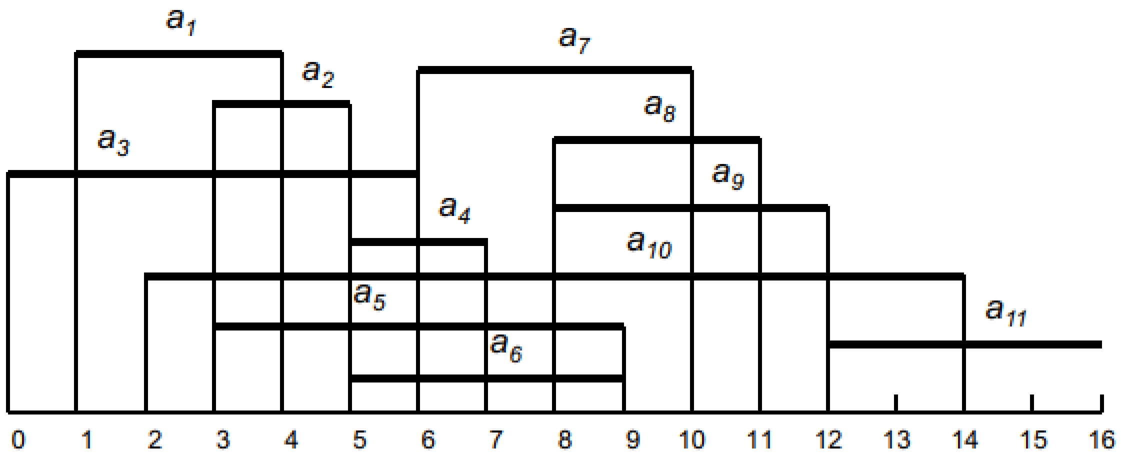

- $P = \{\}$

Example of Greedy Activity Selection Algorithm

- Insert the first activity(a_1) in the optimal solution

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
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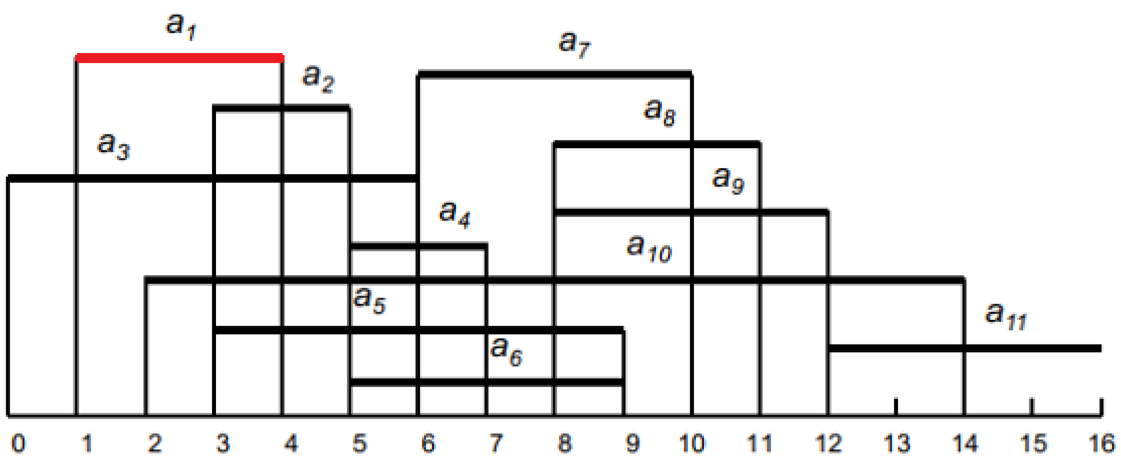

- $P = \{a_1\}$

Example of Greedy Activity Selection Algorithm

- $i = 2$, a_2 is non-compatible because $s_2 < f_1$, skip

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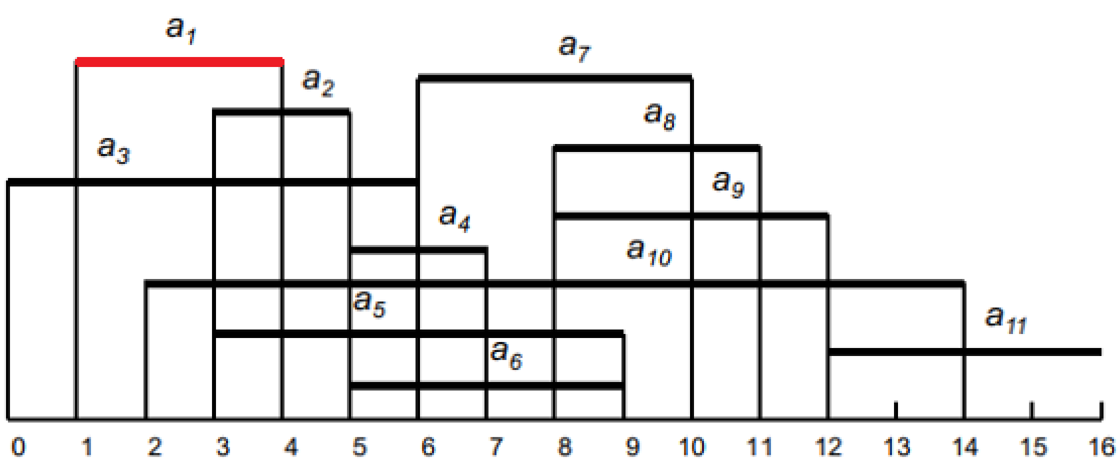
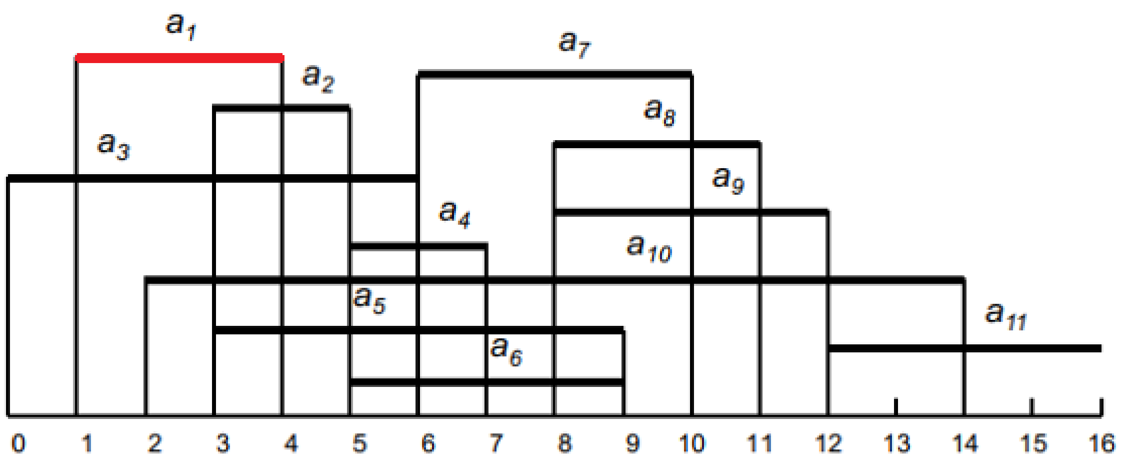


- $P = \{\mathrm{a}_1\}$

Example of Greedy Activity Selection Algorithm

- $i = 3, \mathrm{a}_3$ is also non-compatible because $s_3 < f_1$, skip

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

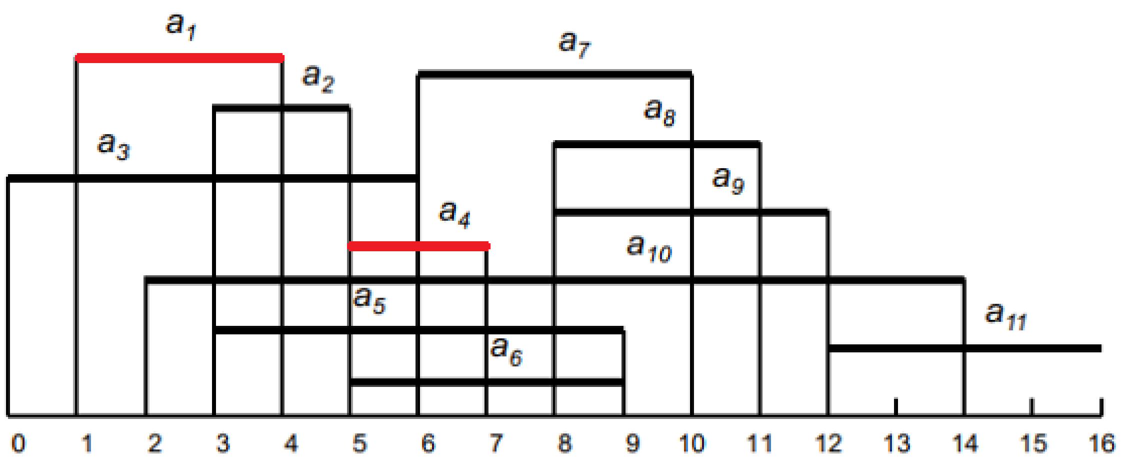


- $P = \{\mathrm{a}_1\}$

Example of Greedy Activity Selection Algorithm

- $i = 4, \mathrm{a}_4$ is a compatible choice because it satisfies $s_4 \geq f_1$, insert it in optimal solution

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

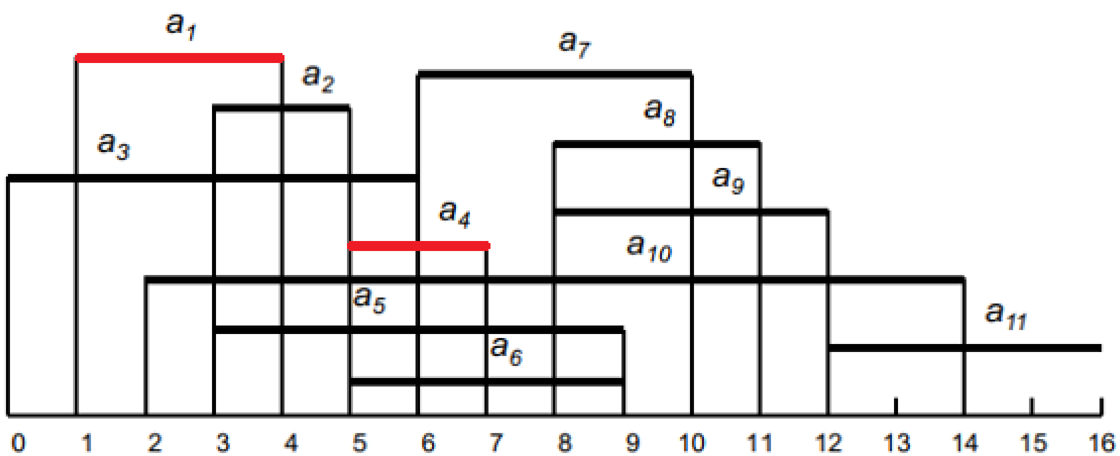


- $P = \{\mathrm{a}_1, \mathrm{a}_4\}$

Example of Greedy Activity Selection Algorithm

- $i = 5$, a_5 is non-compatible because $s_5 < f_4$, skip

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

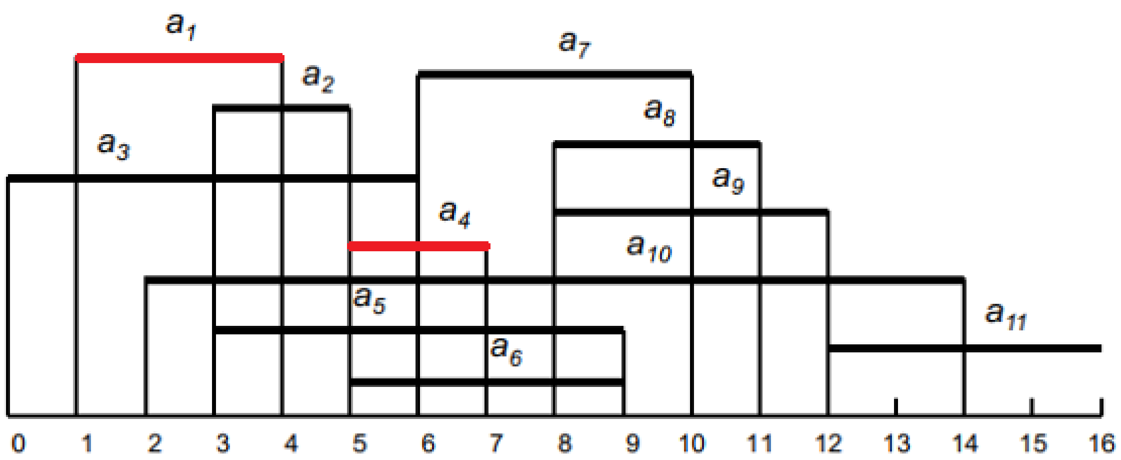


- $P = \{\mathrm{a}_1, \mathrm{a}_4\}$

Example of Greedy Activity Selection Algorithm

- $i = 6, \mathrm{a}_6$ is non-compatible because $s_6 < f_4$, skip

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

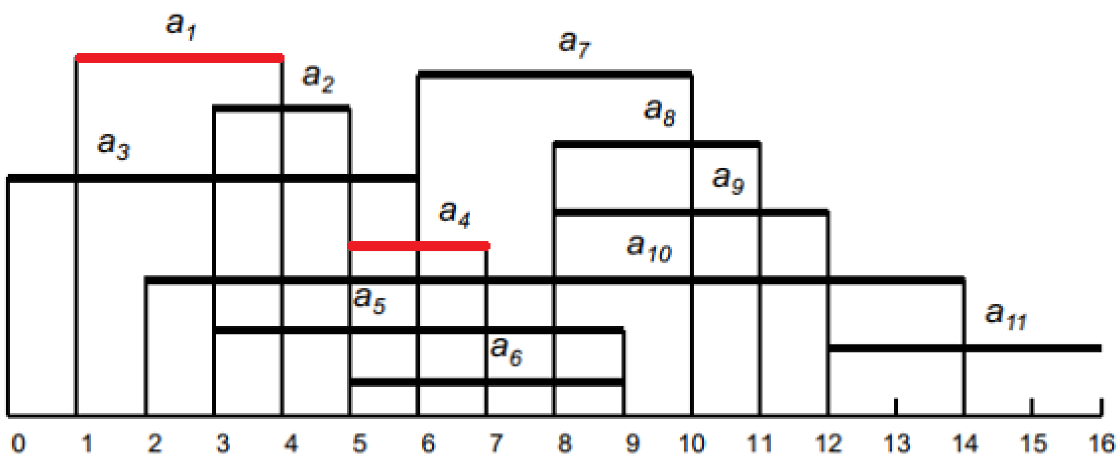


- $P = \{\mathrm{a}_1, \mathrm{a}_4\}$

Example of Greedy Activity Selection Algorithm

- $i = 7, \mathrm{a}_7$ is non-compatible because $s_7 < f_4$, skip

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

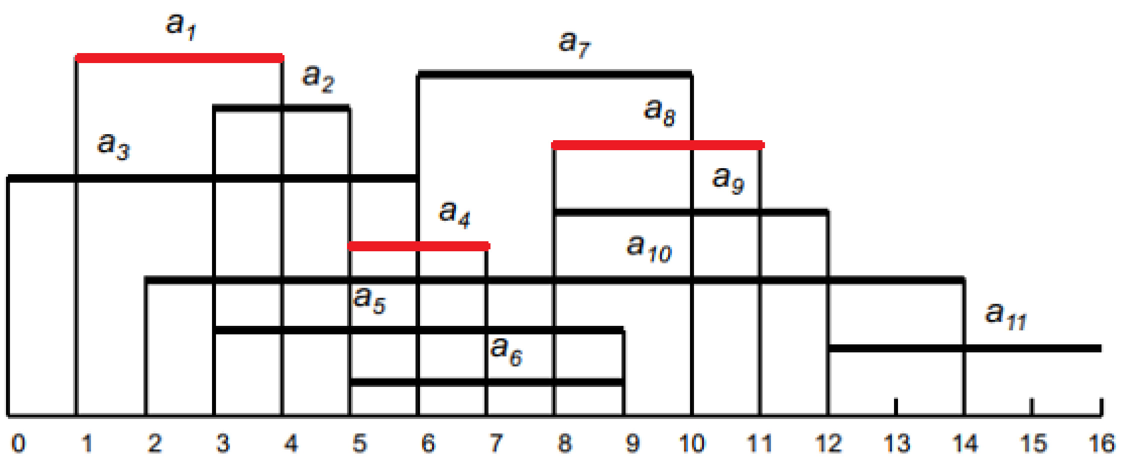


- $P = \{\mathrm{a}_1, \mathrm{a}_4\}$

Example of Greedy Activity Selection Algorithm

- $i = 8, \mathrm{a}_8$ is a compatible choice because it satisfies $s_8 \geq f_4$, insert it in optimal solution

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16



- $P = \{\mathrm{a}_1, \mathrm{a}_4, \mathrm{a}_8\}$

Example of Greedy Activity Selection Algorithm

- $i = 9$, a_9 is non-compatible because $s_9 < f_8$, skip

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

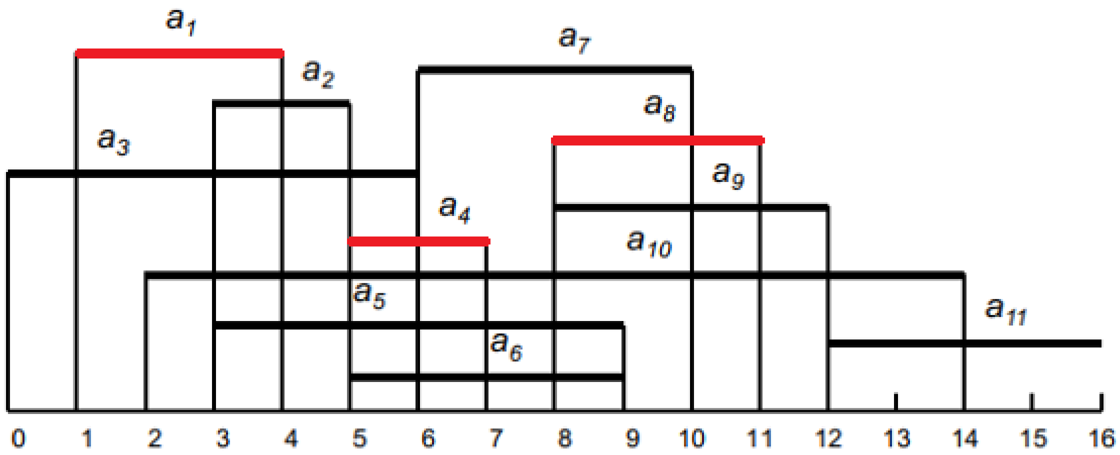
- $i = 9$, a_9 is non-compatible because $s_9 < f_8$, skip

- $P = \{\mathrm{a}_1, \mathrm{a}_4, \mathrm{a}_8\}$

Example of Greedy Activity Selection Algorithm

- $i = 10, \mathrm{a}_{10}$ is non-compatible because $s_{10} < f_8$, skip

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

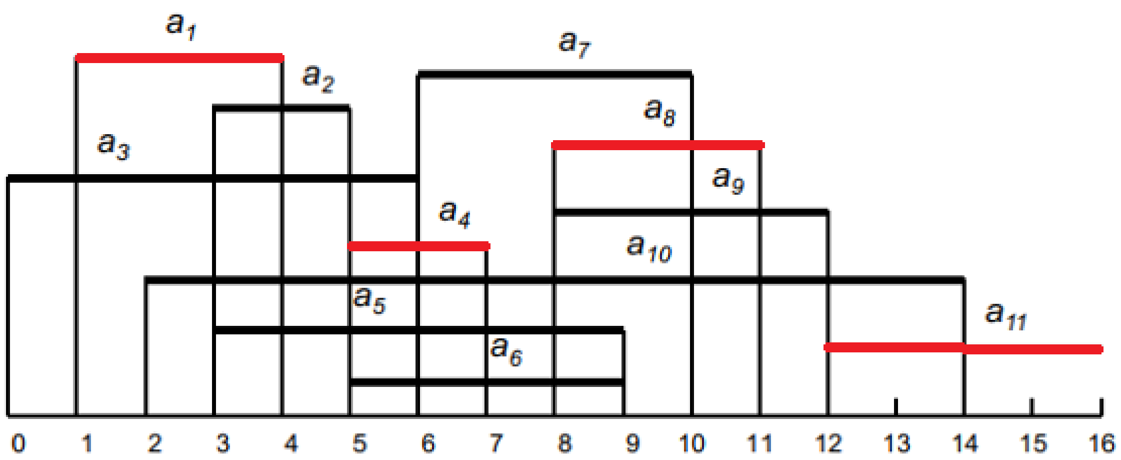


- $P = \{\mathrm{a}_1, \mathrm{a}_4, \mathrm{a}_8\}$

Example of Greedy Activity Selection Algorithm

- $i = 11, \mathrm{a}_{11}$ is a compatible choice because it satisfies $s_{11} \geq f_8$, insert it in optimal solution

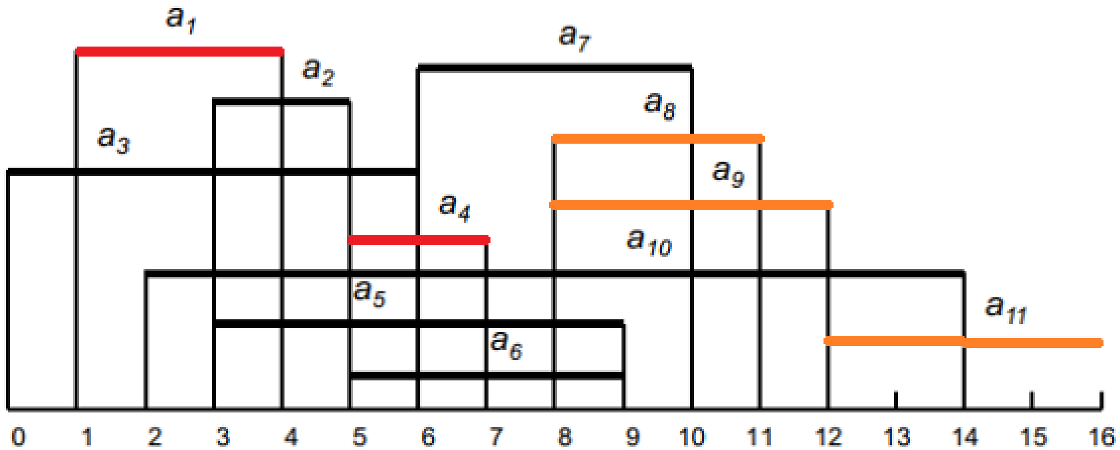
a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16



- $P = \{\mathrm{a}_1, \mathrm{a}_4, \mathrm{a}_8, \mathrm{a}_{11}\}$

Correctness

- **Theorem**:
  - If a_k is a selected activity by Greedy Activity Selection algorithm, S_k (activities that start after a_k finishes) is nonempty and a_m has the earliest finishing time in S_k, then a_m is included in some optimal solution.



When we insert a_4 in optimal solution,

$S_4 = \{\mathrm{a}_8, \mathrm{a}_9, \mathrm{a}_{11}\}$

$\mathrm{a}_m = a_8$

Correctness

- How can we prove it?

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  - If they match in the selected activity $\rightarrow$ skip

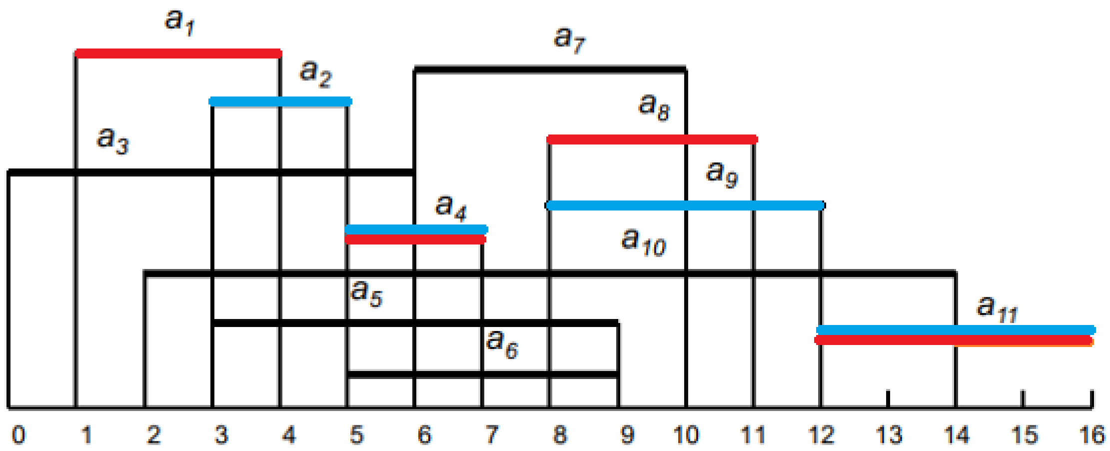
Correctness

- How can we prove it?
  - We can convert any other optimal solution(P') to the greedy algorithm solution(P)

- Idea:
  - Compare the activities in P' and P from left to right
  - If they match in the selected activity $\rightarrow$ skip
  - If they do not match, we can replace the activity in P' by that in P because the one in P finish first

Correctness

- $P = \{\mathrm{a}_1, \mathrm{a}_4, \mathrm{a}_8, \mathrm{a}_{11}\}$
- $P' = \{\mathrm{a}_2, \mathrm{a}_4, \mathrm{a}_9, \mathrm{a}_{11}\}$



Correctness

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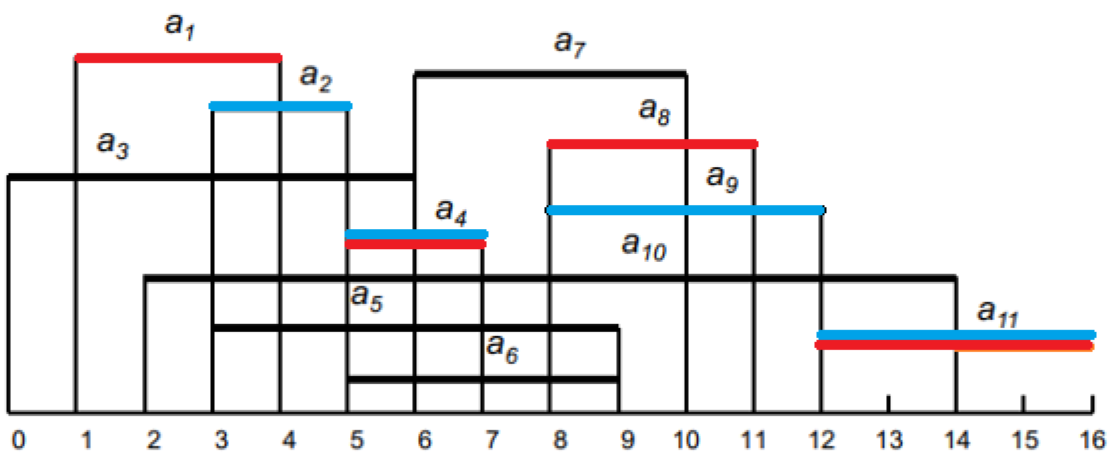
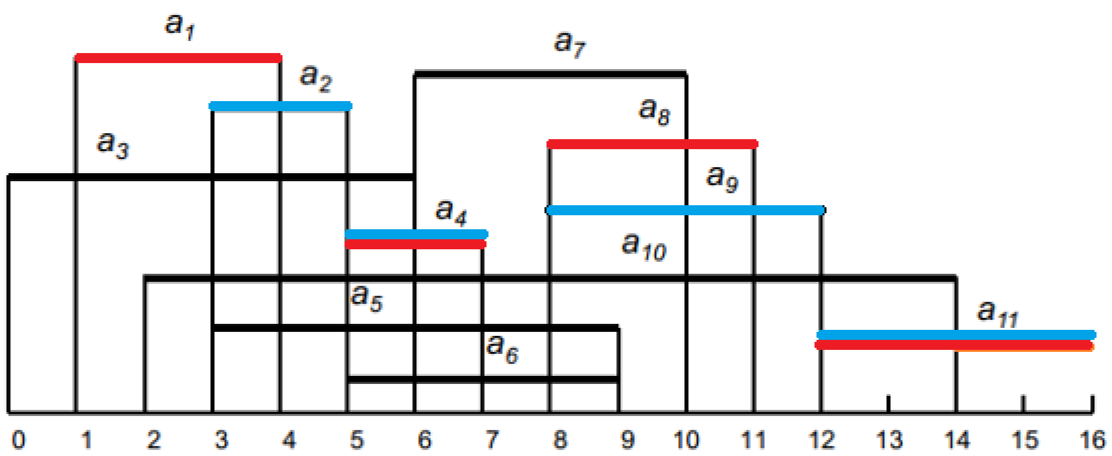


- We can map P' to P

Correctness

- $P = \{a_1, a_4, a_8, a_{11}\}$
- $P' = \{a_2, a_4, a_9, a_{11}\}$



- We can map P' to P
- a_2, a_9 in P' can be replaced by a_1, a_8 from P (finishes earlier)

Correctness

- Proving a greedy solution is optimal:

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  - Remember: Not all problems have optimal greedy solution.

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Correctness

- Proving a greedy solution is optimal:
  - Remember: Not all problems have optimal greedy solution.
  - If it does, you need to prove it.
  - Usually the proof includes mapping or converting any other optimal solution to the greedy solution.

Outline

- Introduction to Part III
- Fractional Knapsack Problem
    - Problem Definition
    - A Greedy Algorithm and correctness
- Huffman Coding Problem
    - Problem Definition
    - A Greedy Algorithm
- Activity Selection Problem
    - Problem Definition
    - A Greedy Algorithm and correctness
    - Extended: Weighted Activity Selection

Weighted Activity Selection Problem

- Given a set A of activities, such that each activity a_i has a start time s_i, a finishing time f_i and a weight $w_i > 0$.

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- Goal: find max-weighted subset of mutually compatible activities.

Weighted Activity Selection Problem

- Given a set A of activities, such that each activity a_i has a start time s_i, a finishing time f_i and a weight $w_i > 0$.

- Two activities are compatible if they do not overlap.

- Goal: find max-weighted subset of mutually compatible activities.

- That is, find a subset S of A, such that:

$$s_i \geq f_j \; or \; f_i \leq s_j \qquad \forall \mathrm{a}_i, \mathrm{a}_j \in S, i \neq j$$

  and the following is maximized:

$$\sum_{a_i \in S} w_i$$

Greedy Activity Selection Algorithm

- Can we use the Greedy Activity Selection algorithm to solve the Weighted Activity Selection Problem?

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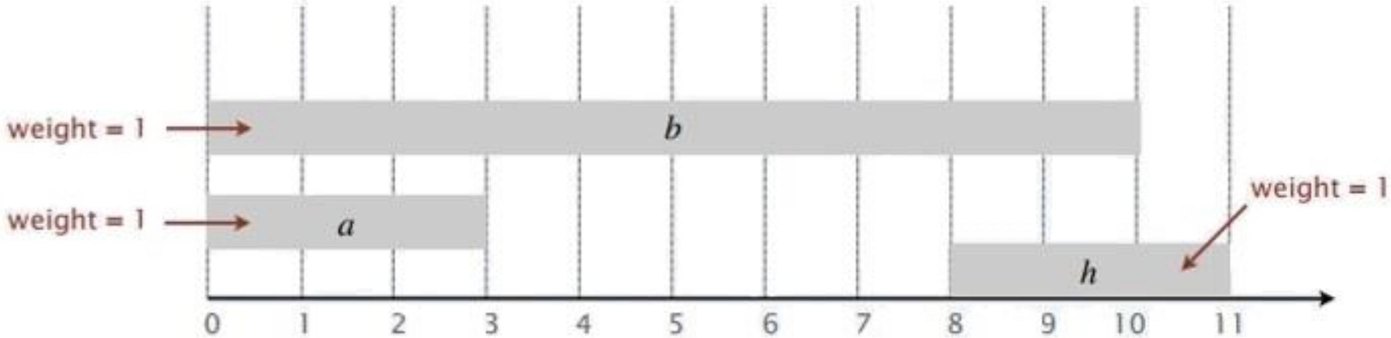

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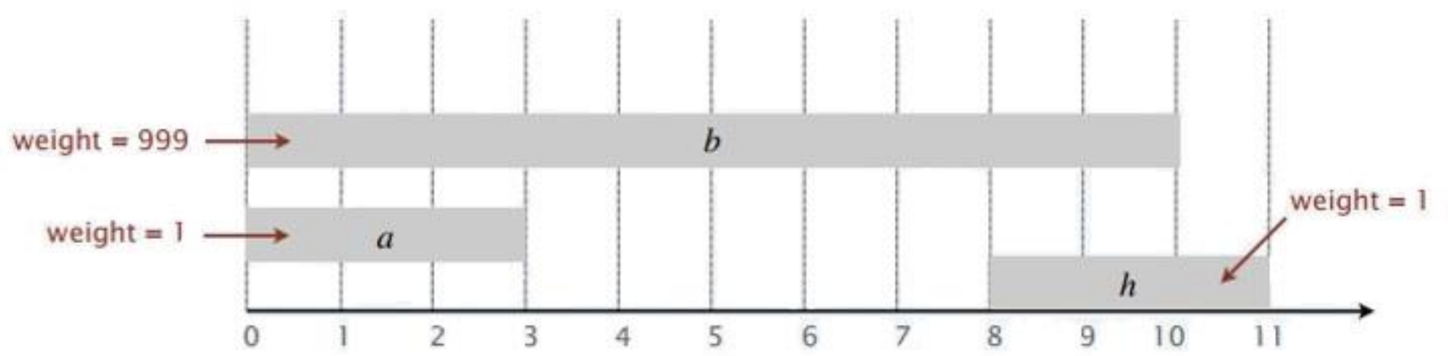

Greedy Activity Selection Algorithm

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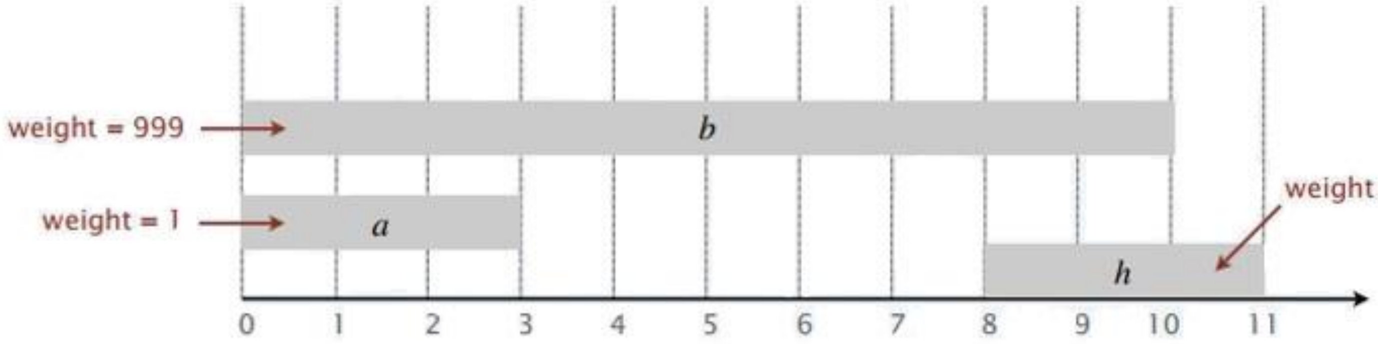
- Main Idea.
  - Consider activities in ascending order of finishing time.
  - Add an activity to subset if it is compatible with previously chosen activities.
- Recall. Greedy algorithm is correct if all weights are 1.
- Observation. Greedy algorithm fails spectacularly for weighted version.



A Dynamic Programming Algorithm

Step 1: Space of Subproblems (State)

- Convention. Activities are in ascending order of finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$

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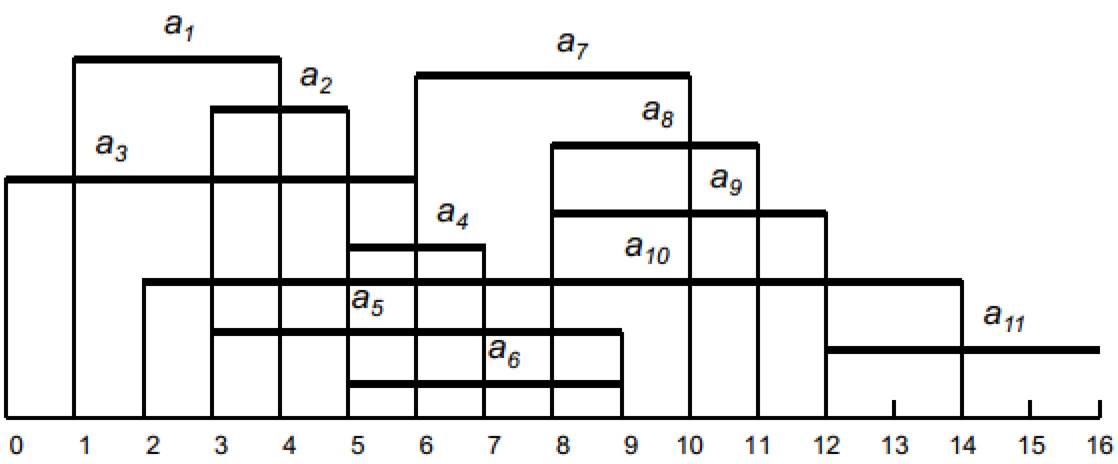
- Boundary case. $OPT(0) = 0$

A Dynamic Programming Algorithm

Step 2: Relating a problem to its subproblems

- Definition. $p(j)$ is the largest index $i < j$ such that activity a_i is compatible with j.
  - Ex. $p(7) = 3, p(8) = 4, p(11) = 9$
  - $p(j)$ can be easily computed via binary search



A Dynamic Programming Algorithm

Step 2: Relating a problem to its subproblems

- Case 1. $OPT(j)$ does not select activity a_j.
  - Must be an optimal solution to problem consisting of remaining activities $\mathrm{a}_1, \mathrm{a}_2, \cdots, \mathrm{a}_{j-1}$

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- Case 1. $OPT(j)$ does not select activity a_j.
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- Case 2. $OPT(j)$ selects activity a_j.
  - Collect profit w_j
  - Can't use incompatible activities $\{\mathrm{a}_{p(j)+1}, \mathrm{a}_{p(j)+2}, \cdots, \mathrm{a}_{j-1}\}$
  - Must include optimal solution to problem consisting of remaining compatible activities $\mathrm{a}_1, \mathrm{a}_2, \cdots, \mathrm{a}_{p(j)}$

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  - Must include optimal solution to problem consisting of remaining compatible activities $\mathrm{a}_1, \mathrm{a}_2, \cdots, \mathrm{a}_{p(j)}$
- Dynamic Programming Equation:

$$OPT(j) = \begin{cases} 0 & if\ j = 0 \\ \max\{OPT(j-1), w_j + OPT(p(j))\} & if\ j > 0 \end{cases}$$

A Dynamic Programming Algorithm

Step 3: Bottom-up computation

- Dynamic Programming Equation:

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Step 3: Bottom-up computation

- Dynamic Programming Equation:

$$OPT(j) = \begin{cases} 0 & if\ j = 0 \\ \max\{OPT(j-1), w_j + OPT(p(j))\} & if\ j > 0 \end{cases}$$

- We compute and save $OPT(j)$ in such an order that: When it is time to compute $OPT(j)$, the values of $OPT(j-1)$ and $OPT(p(j))$ are available

A Dynamic Programming Algorithm: Pseudocode

DP Activity Selection(A)

Input: a set of activities $A = a_1, a_2, \cdots, a_n$
Output: the max weight of any subset of mutually compatible activities

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Compute $p[1], p[2], \cdots, p[n]$

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Compute $p[1], p[2], \cdots, p[n]$
$OPT[0] \leftarrow 0$;
for $j = 1$ ***to*** n **do**
  $OPT[j] \leftarrow \max\{OPT[j-1], w_j + OPT[p[j]]\}$;
end
return $OPT[n]$;

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end
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Previously computed values

A Dynamic Programming Algorithm: Pseudocode

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Input: a set of activities $A = a_1, a_2, \cdots, a_n$
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end
return $OPT[n]$;

cost: $O(n \log n)$

A Dynamic Programming Algorithm: Pseudocode

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Sort activities by finishing time and renumber so that $f_1 \le f_2 \le \cdots \le f_n$;
Compute $p[1], p[2], \cdots, p[n]$
$OPT[0] \leftarrow 0$;
for $j = 1$ ***to*** n **do** $\longrightarrow$ cost: $O(n)$
   $OPT[j] \leftarrow \max\{OPT[j-1], w_j + OPT[p[j]]\}$;
end
return $OPT[n]$;

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DP Activity Selection(A)

Input: a set of activities $A = a_1, a_2, \cdots, a_n$
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end
return $OPT[n]$;

- Total Time Complexity: $O(n \log n)$

A Dynamic Programming Algorithm: Pseudocode

DP Activity Selection(A)

Input: a set of activities $A = a_1, a_2, \cdots, a_n$
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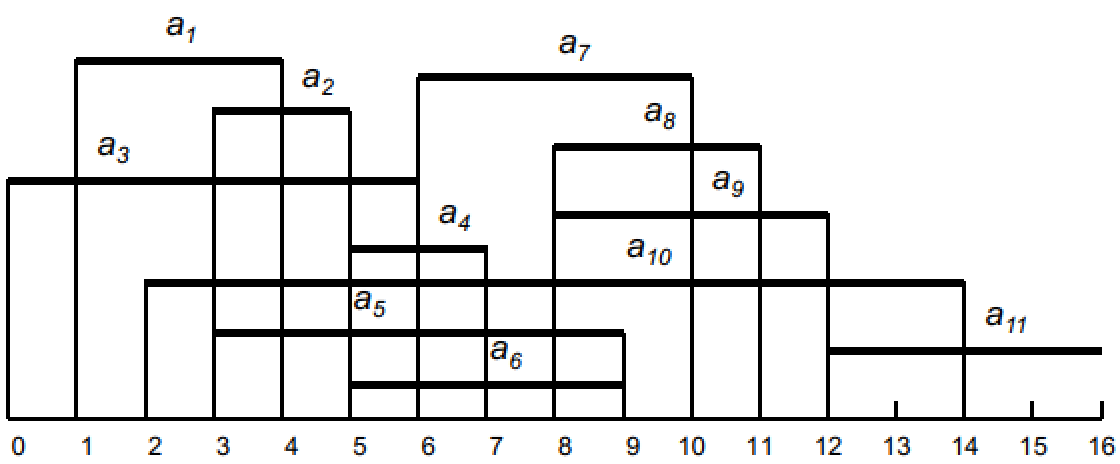
Compute $p[1], p[2], \cdots, p[n]$
$OPT[0] \leftarrow 0$;
for $j = 1$ ***to*** n **do**
    $OPT[j] \leftarrow \max\{OPT[j-1], w_j + OPT[p[j]]\}$;
end
return $OPT[n]$;

- Total Time Complexity: $O(n \log n)$
- How to construct the optimal solution (Step 4)?
    - Try by yourself

Example of Optimal Solution Construction

- Sort activities in increasing order of finishing time(f_i)

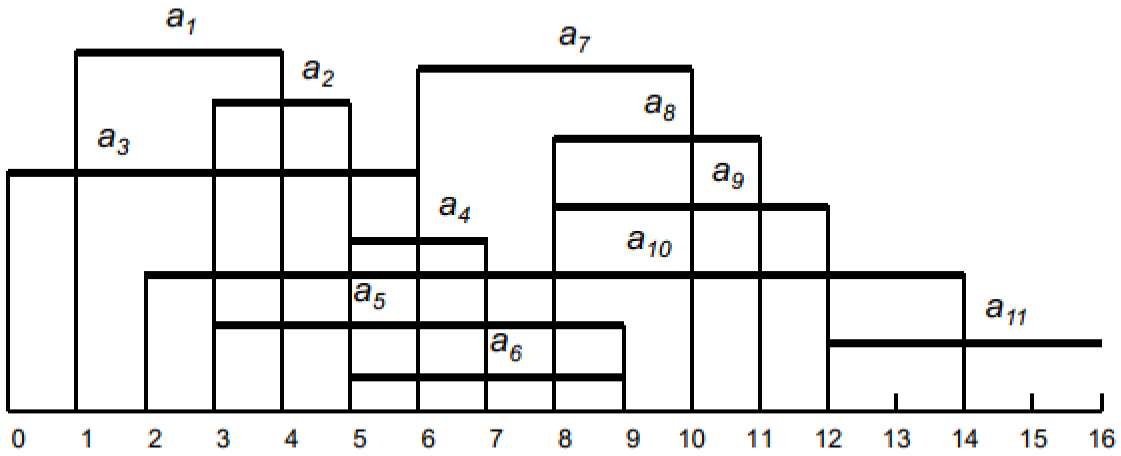
a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1



Example of Optimal Solution Construction

- Compute $p(1), p(2), \cdots, p(n)$ via binary search

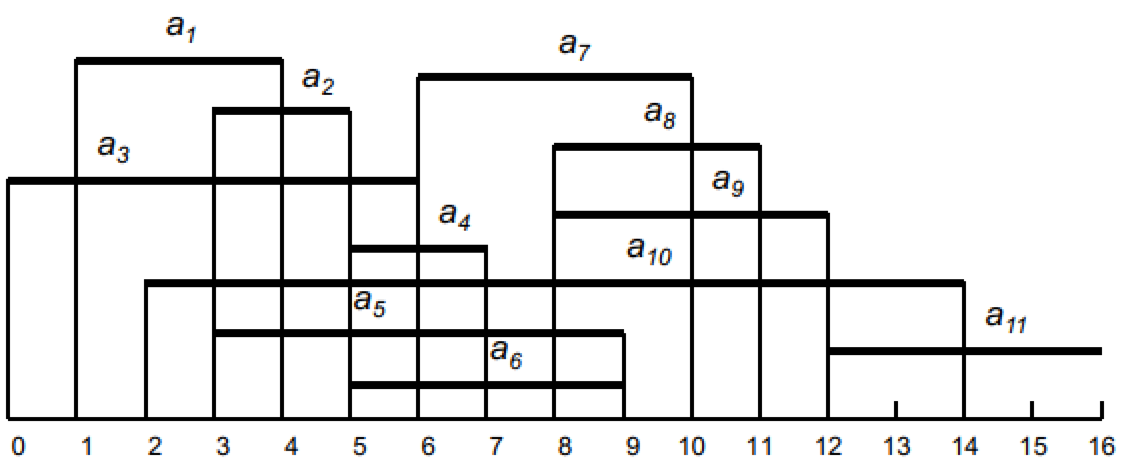
a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9



Example of Optimal Solution Construction

- Boundary case. $OPT(0) = 0$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

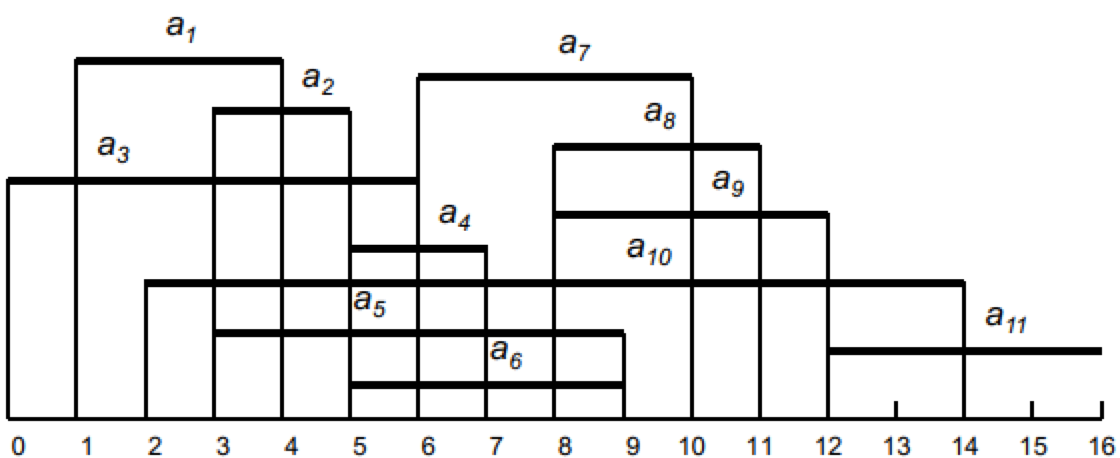


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0											

Example of Optimal Solution Construction

- $OPT(1) = \max\big(OPT(0), w_1 + OPT(p(1))\big) = 5$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

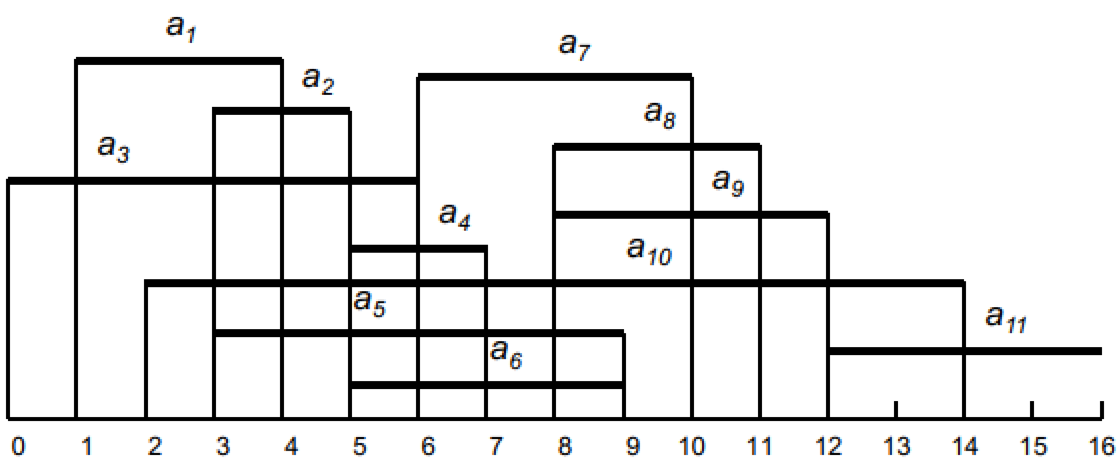


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5										

Example of Optimal Solution Construction

- $OPT(2) = \max\big(OPT(1), w_2 + OPT(p(2))\big) = 8$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

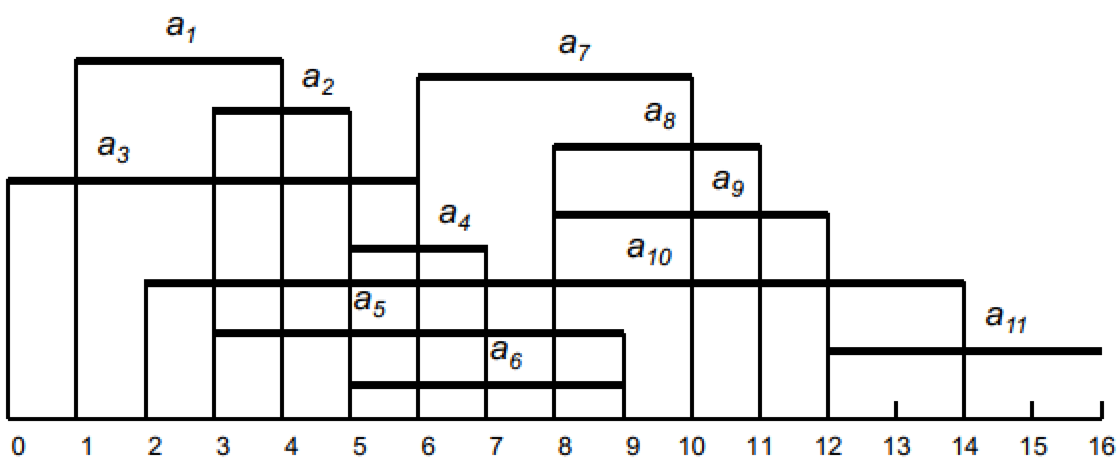


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5	8									

Example of Optimal Solution Construction

- $OPT(3) = \max\big(OPT(2), w_3 + OPT(p(3))\big) = 8$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

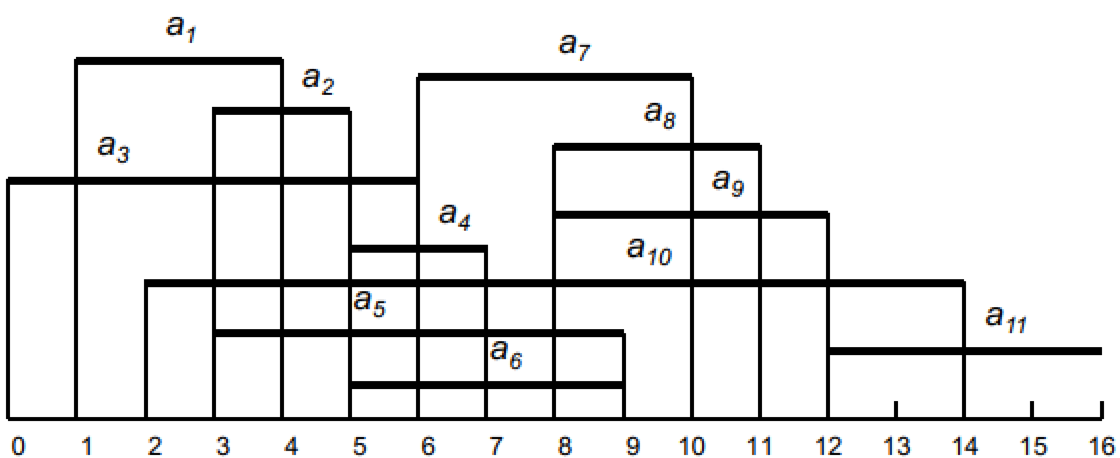


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5	8	8								

Example of Optimal Solution Construction

- $OPT(4) = \max\left(OPT(3), w_4 + OPT(p(4))\right) = 12$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

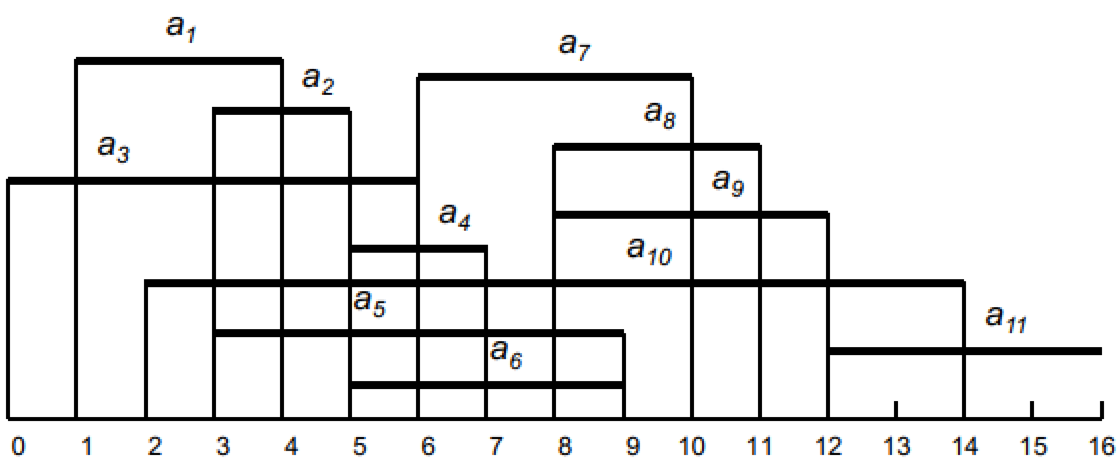


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5	8	8	12							

Example of Optimal Solution Construction

- $OPT(5) = \max\big(OPT(4), w_5 + OPT(p(5))\big) = 12$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

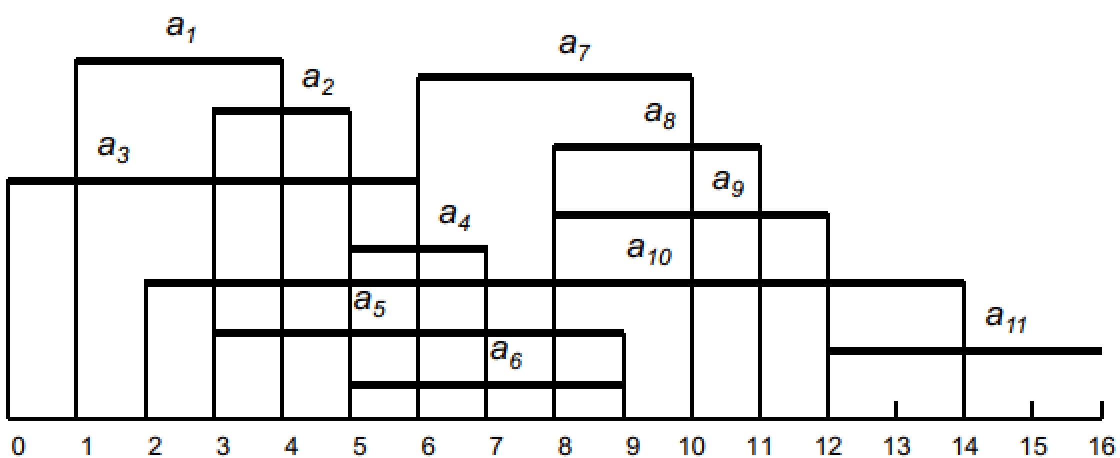


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5	8	8	12	12						

Example of Optimal Solution Construction

- $OPT(6) = \max\big(OPT(5), w_6 + OPT(p(6))\big) = 12$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

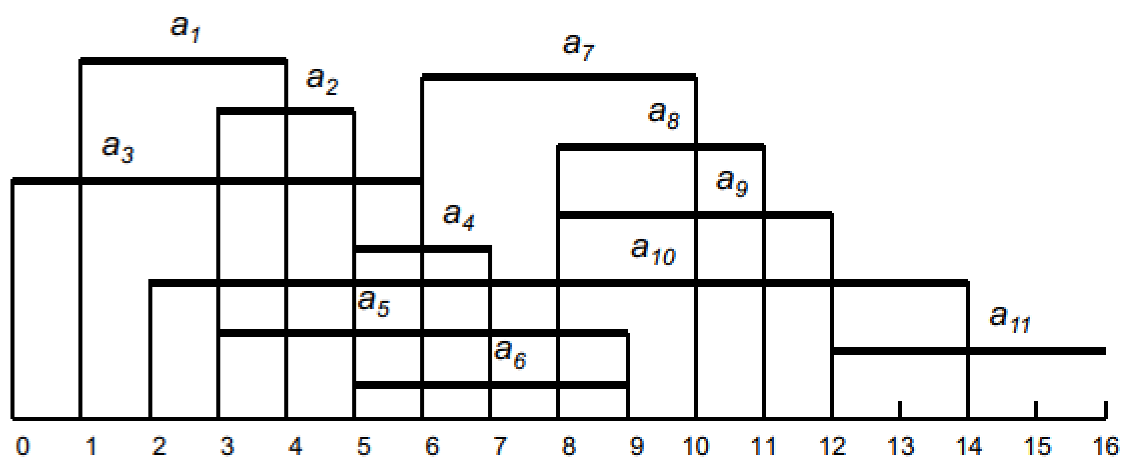


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5	8	8	12	12	12					

Example of Optimal Solution Construction

- $OPT(7) = \max\big(OPT(6), w_7 + OPT(p(7))\big) = 13$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

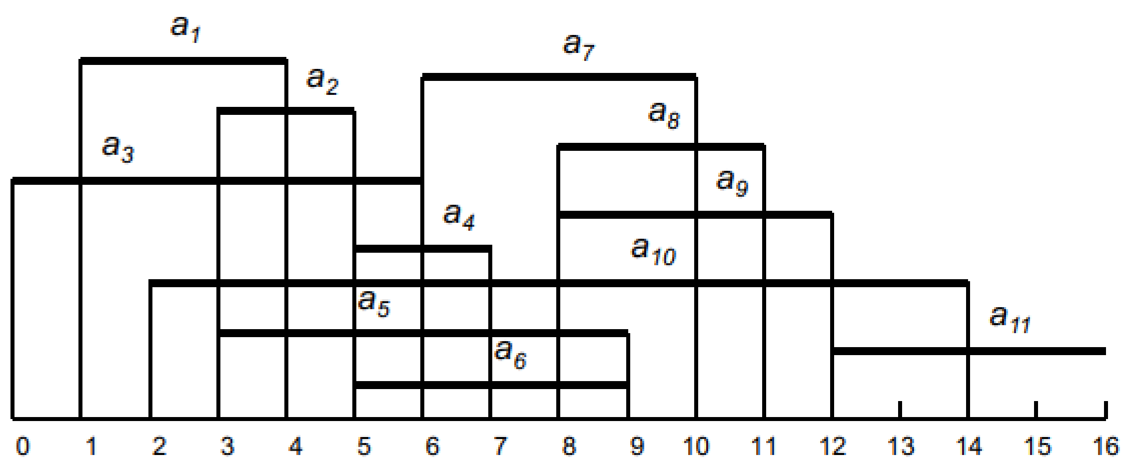


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5	8	8	12	12	12	13				

Example of Optimal Solution Construction

- $OPT(8) = \max\big(OPT(7), w_8 + OPT(p(8))\big) = 20$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

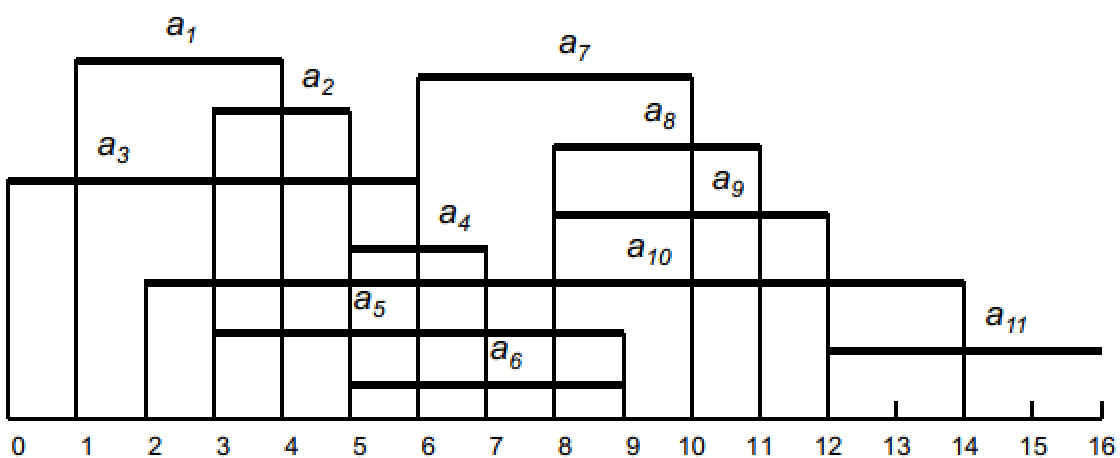


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5	8	8	12	12	12	13	20			

Example of Optimal Solution Construction

- $OPT(9) = \max\big(OPT(8), w_9 + OPT(p(9))\big) = 20$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

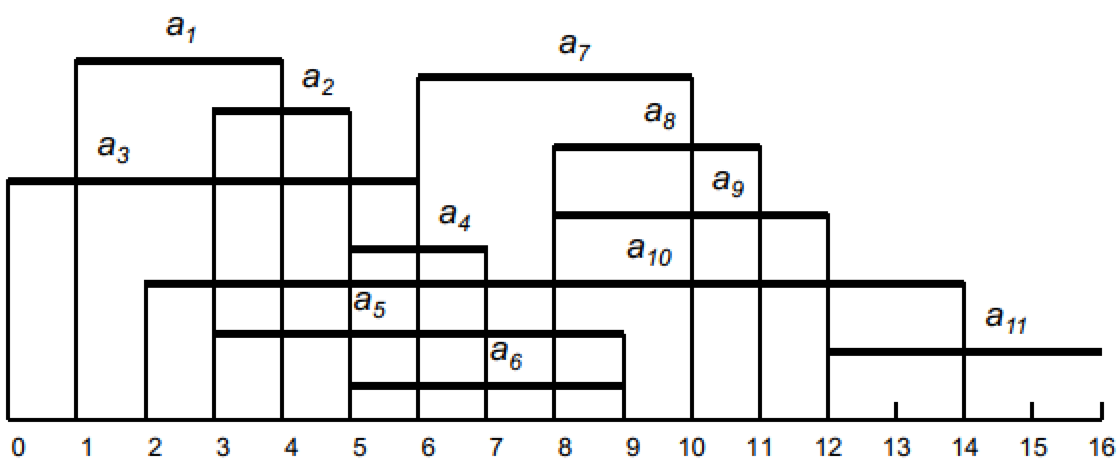


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5	8	8	12	12	12	13	20	20		

Example of Optimal Solution Construction

- $OPT(10) = \max\big(OPT(9), w_{10} + OPT(p(10))\big) = 20$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9

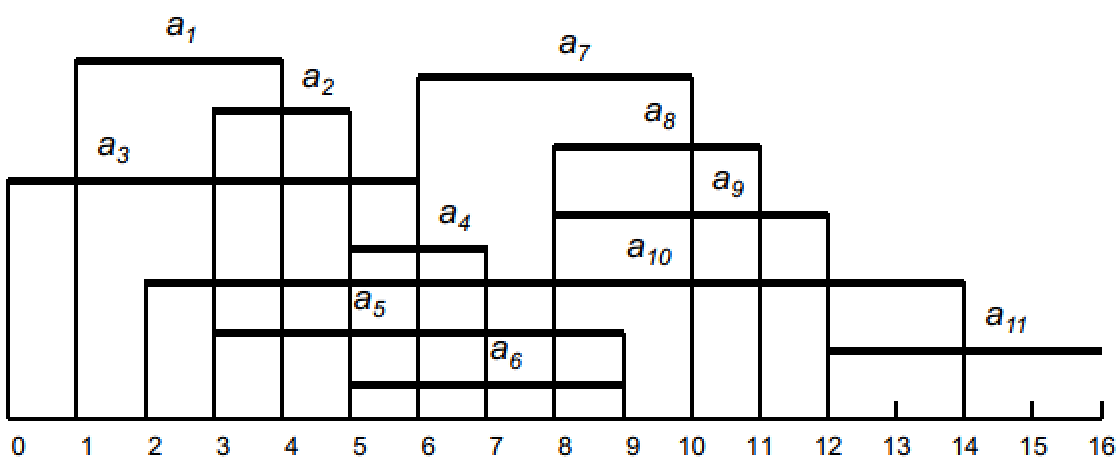


i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5	8	8	12	12	12	13	20	20	20	

Example of Optimal Solution Construction

- $OPT(11) = \max\big(OPT(10), w_{11} + OPT(p(11))\big) = 21$

a_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16
w_i	5	8	2	4	9	2	5	8	3	6	1
$p(i)$	0	0	0	2	0	2	3	4	4	0	9



i	0	1	2	3	4	5	6	7	8	9	10	11
OPT	0	5	8	8	12	12	12	13	20	20	20	21

Greedy vs. Dynamic Programming

- Greedy
  - Build up a solution incrementally, myopically optimizing some local criterion

- Dynamic Programming
  - Break up a problem into a series of overlapping subproblems; combine solutions to smaller subproblems to form solution to large subproblem

Greedy	never reconsiders its choices	focus on the present
Dynamic Programming	based on previous decisions	record the history


thank you
danke
рахмат
謝謝
ngiyabonga
teşekkür ederim
Баярлалаа
спасибо
faafetai lava
mersi
kia ora
barka
welalin
tack
spas
dank je
misaotra
matondo
paldies
grazzi
mahalo
tapadh leat
gracias
vinaka
спасибі
blagodaram
kiitos
dankie
nanni
nandri
dhanyavad
mauruuru
bedankt
hvala
köszönöm
bayarlalaa
gràçie
enkosi
dziękuję
dankon
ačiū
akun
хвала
asante
manana
obrigada
tenki
murakoze
djiere dieuf
tau
mochchakkeram
дякую
mamnun
chokrane
chnorakaloutioun
gratias ago
gràcies
sulpáy
go raibh maith agat
sobodi
děkuji
obrigado
sagolun
sukriya
kop khun krap
taiku
arigatô
takk
dakujem
trugarez
mèsi
didi madloba
najis tuke
ありがとう
tanemirt
grazie
kam sah hamnida
rahmat
terima kasih
rahmet
diolch
dhanyavadagalu
shukriya
mercé
мерси
তোমাকে ধন্যবাদ
감사합니다
xièxie
ευχαριστώ
merci