Design and Analysis of Algorithms
Part IV: Graph Algorithms
Lecture 11: Minimum Spanning Trees

Yongxin Tong (童咏昕)

School of CSE, Beihang University

yxtong@buaa.edu.cn

Outline

- Review to Part IV
- Minimum Spanning Trees
  - Spanning trees
  - Minimum spanning trees
- Prim’s algorithm
  - The idea
  - The algorithm
  - Analysis for Prim’s algorithm
- Kruskal’s algorithm
  - The idea
  - The algorithm
  - The Disjoint Set Union-Find data structure
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Introduction to Part IV

- In Part IV, we will illustrate several graph algorithm problems using several examples:
  - Basic Concepts of Graphs (图的基本概念)
  - Breadth-First Search [BFS] (广度优先搜索)
  - Depth-First Search [DFS] (深度优先搜索)
  - Topological Sort (拓扑排序)
  - Strongly Connected Components (强联通分量)
  - Minimum Spanning Trees (最小生成树)
  - Shortest Path (最短路径)
  - All-Pairs Shortest Paths (所有结点对的最短路径)
  - Maximum/Network Flows (最大流/网络流)

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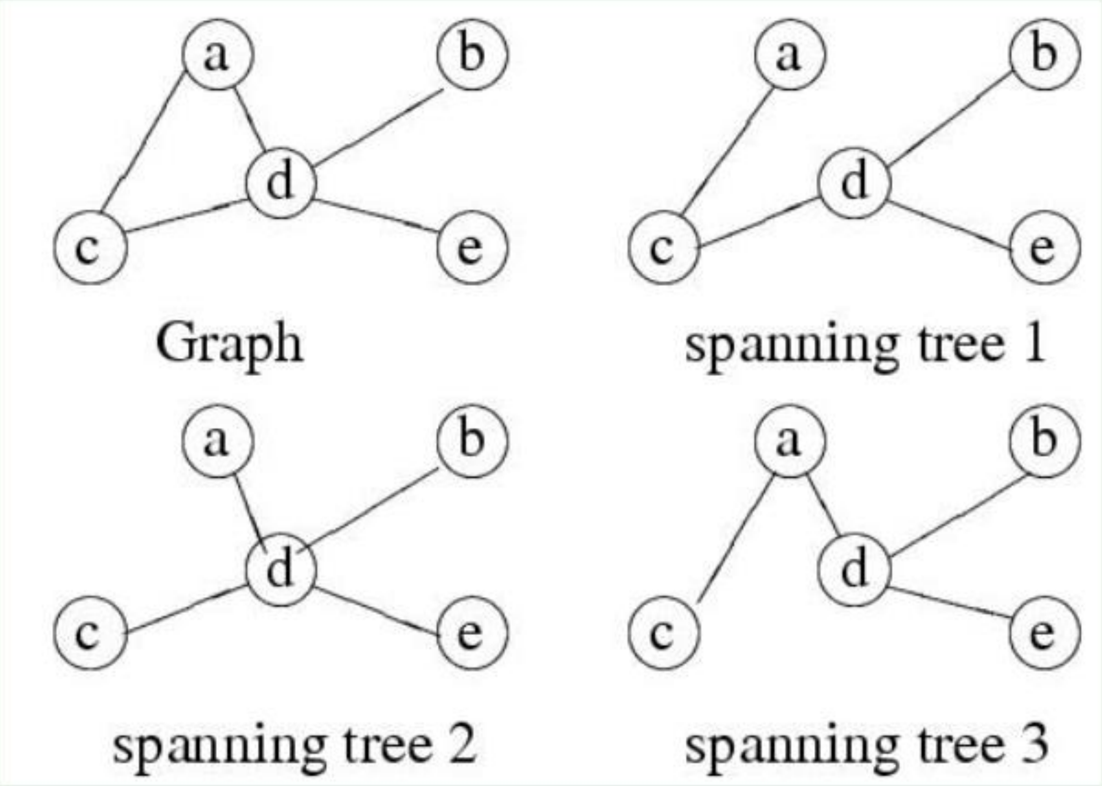

Spanning Trees

Theorem

Every connected graph has a spanning tree.

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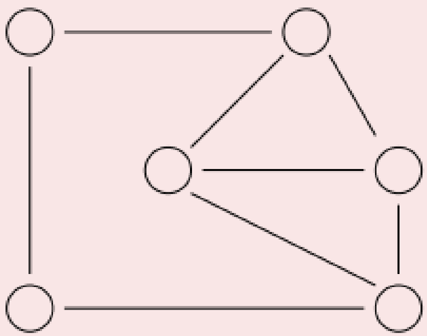
Given a connected graph G, how can you find a spanning tree of G?



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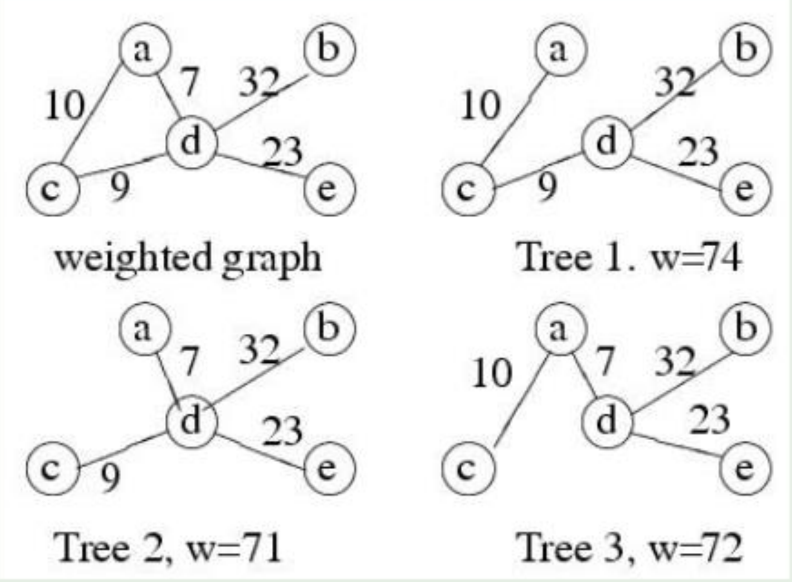

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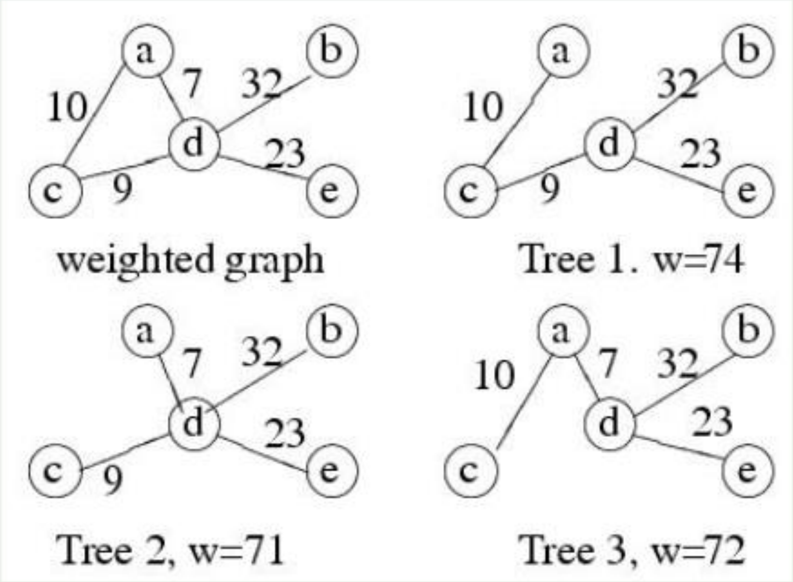

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Weight of a graph: The sum of the weights of all edges

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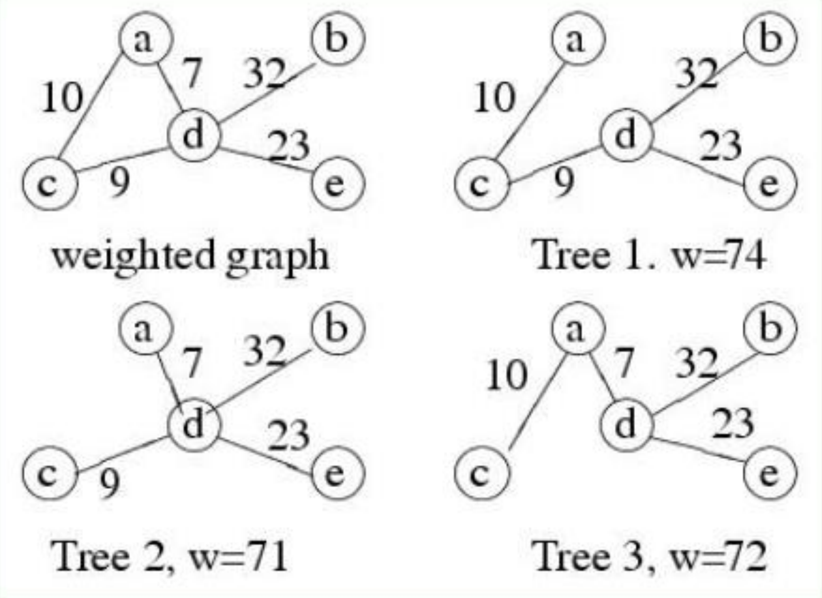

Remark

The minimum spanning tree may not be unique

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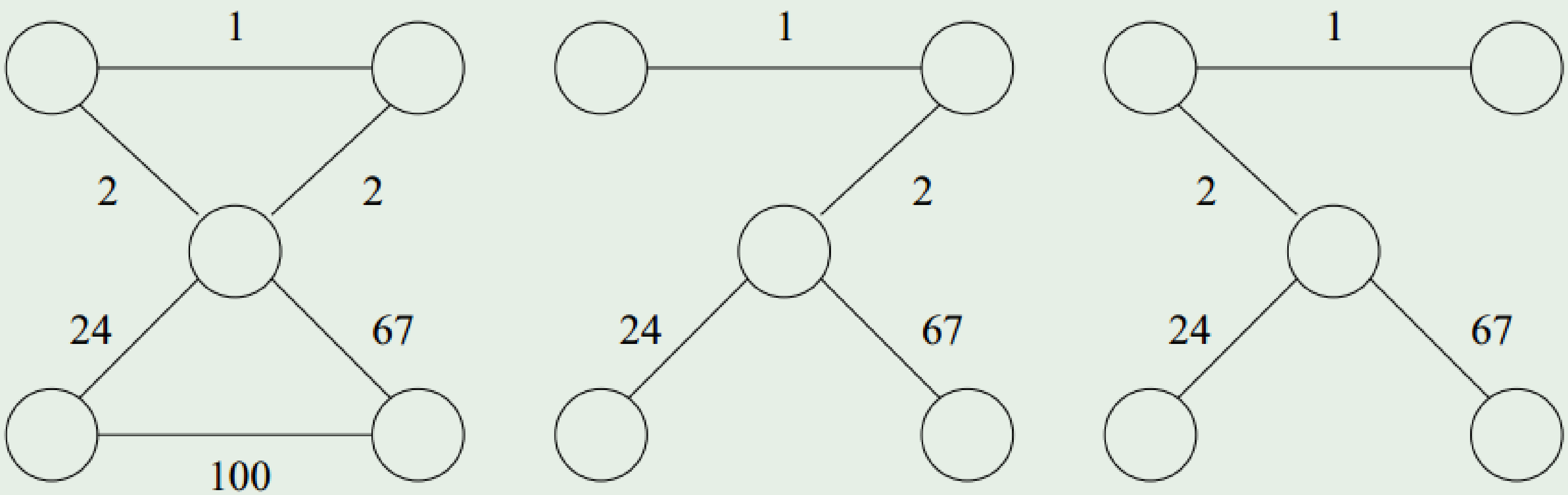

weighted graph MST1 MST2

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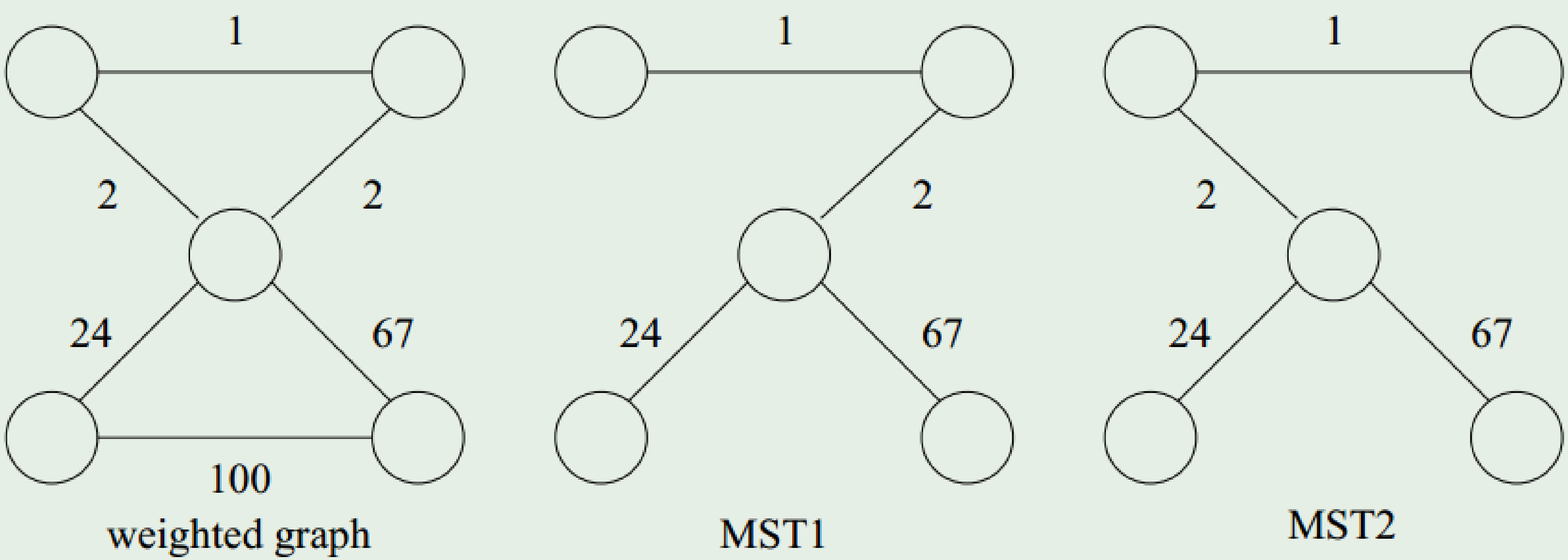

However, if the weights of all the edges are distinct, it is indeed unique (we won't prove this now)

Minimum Spanning Tree Problem

Definition (MST Problem)

Given a connected weighted undirected graph G, design an algorithm that outputs a minimum spanning tree (MST) of G.

General strategy for solving the MST Problem

A tree is an acyclic graph

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- if after adding each edge we are sure that the resulting graph is a subset of some minimum spanning trees, we are done.

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Input: A graph $\boldsymbol{G}$
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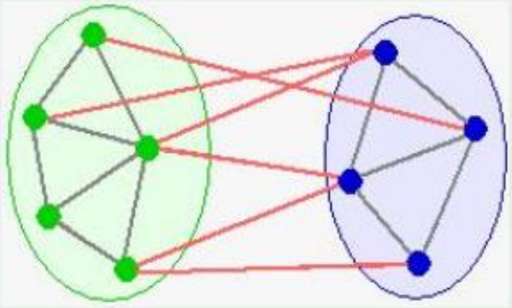

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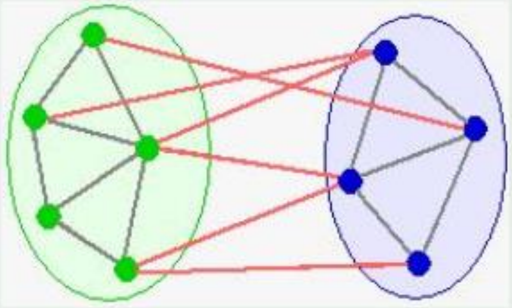

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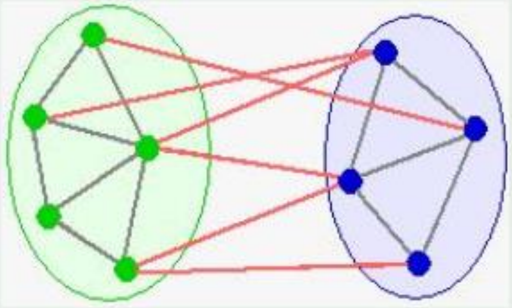

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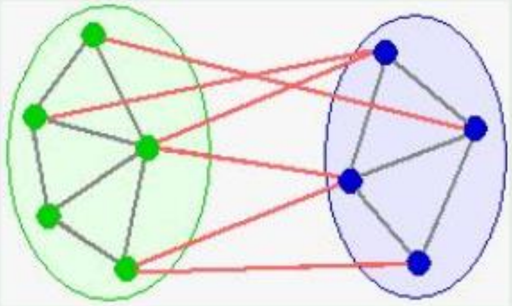

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An edge is a light edge crossing a cut if its weight is the minimum of any edge crossing the cut.

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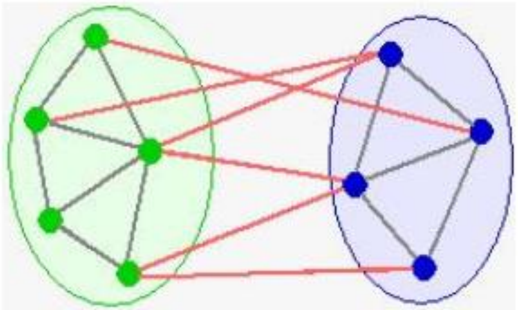

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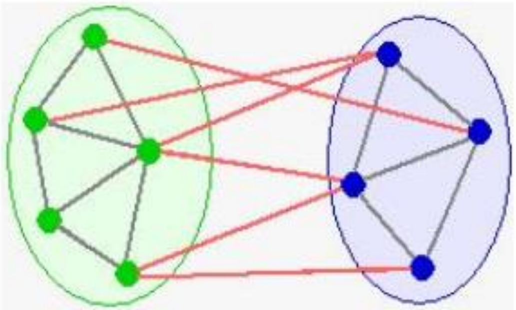

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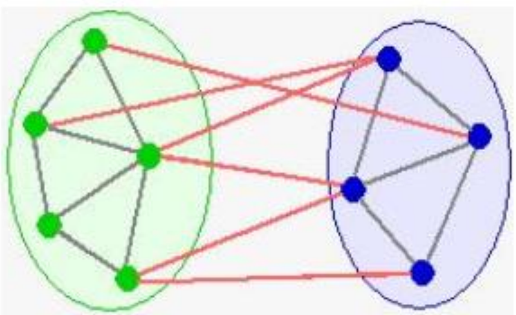

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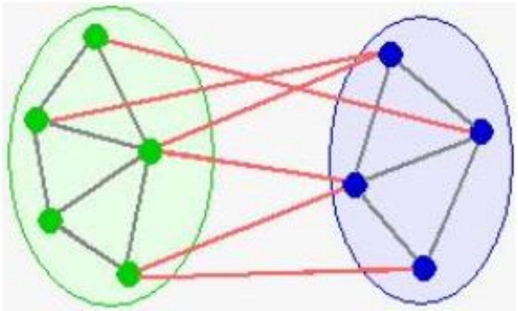

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That light edge is a safe edge.

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  - $A \cup \{(u, v)\} \subseteq T$.
  - Hence (u, v) is safe for A.

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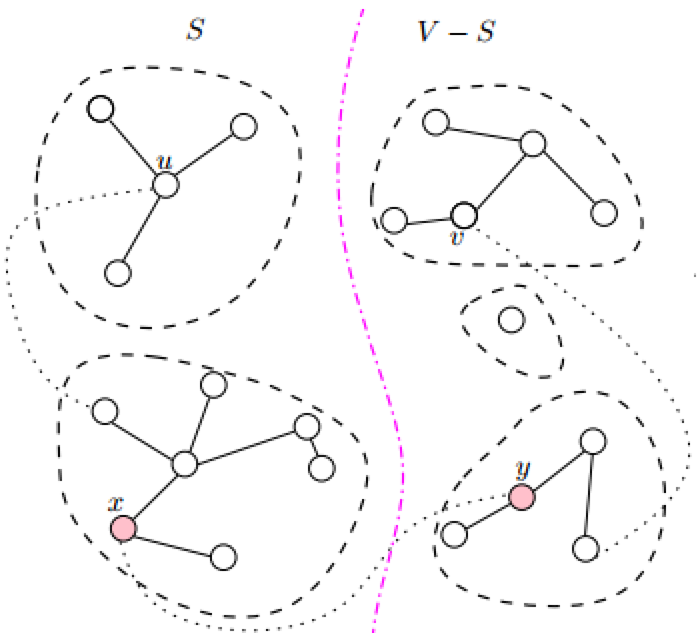

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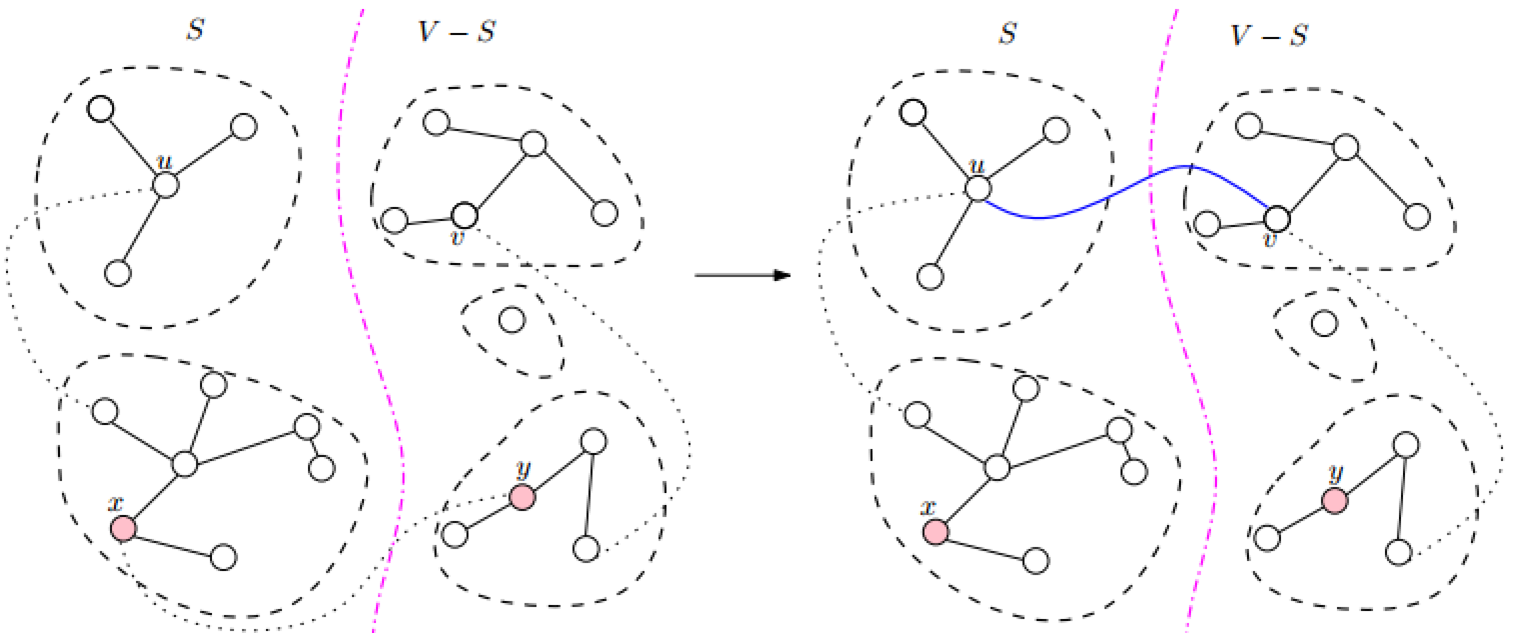

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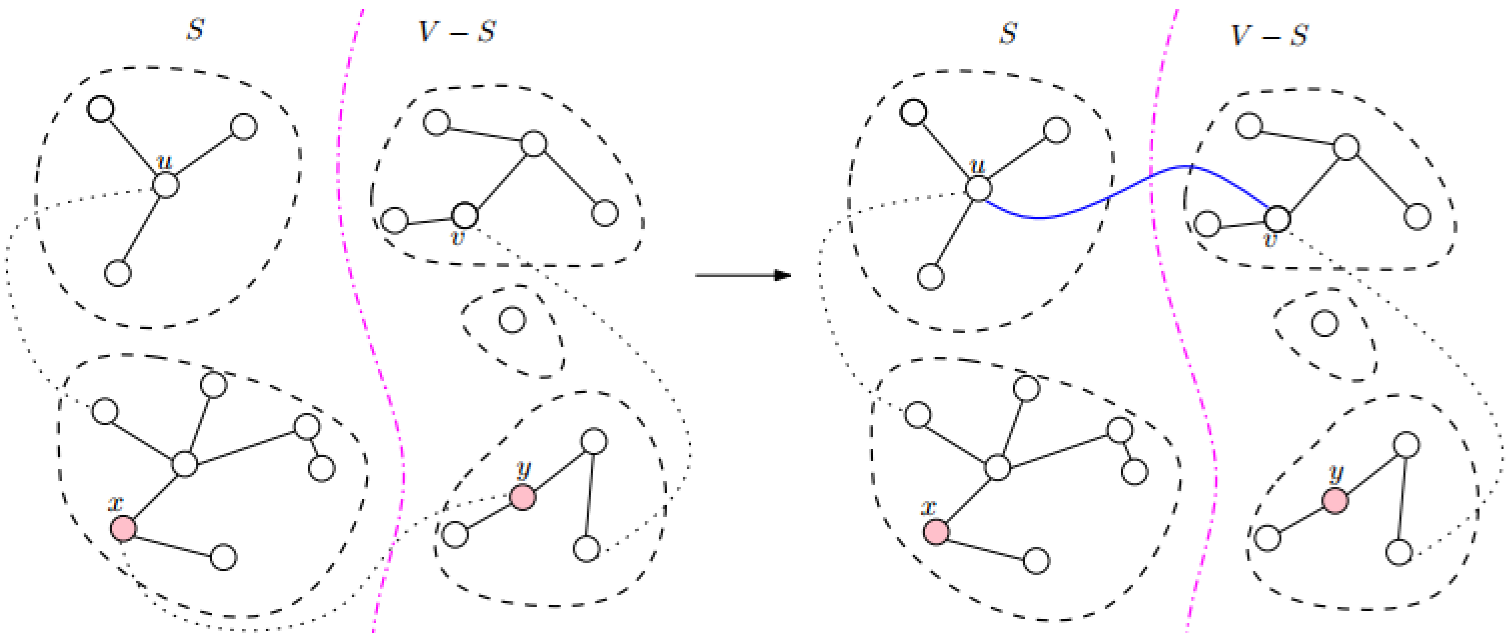

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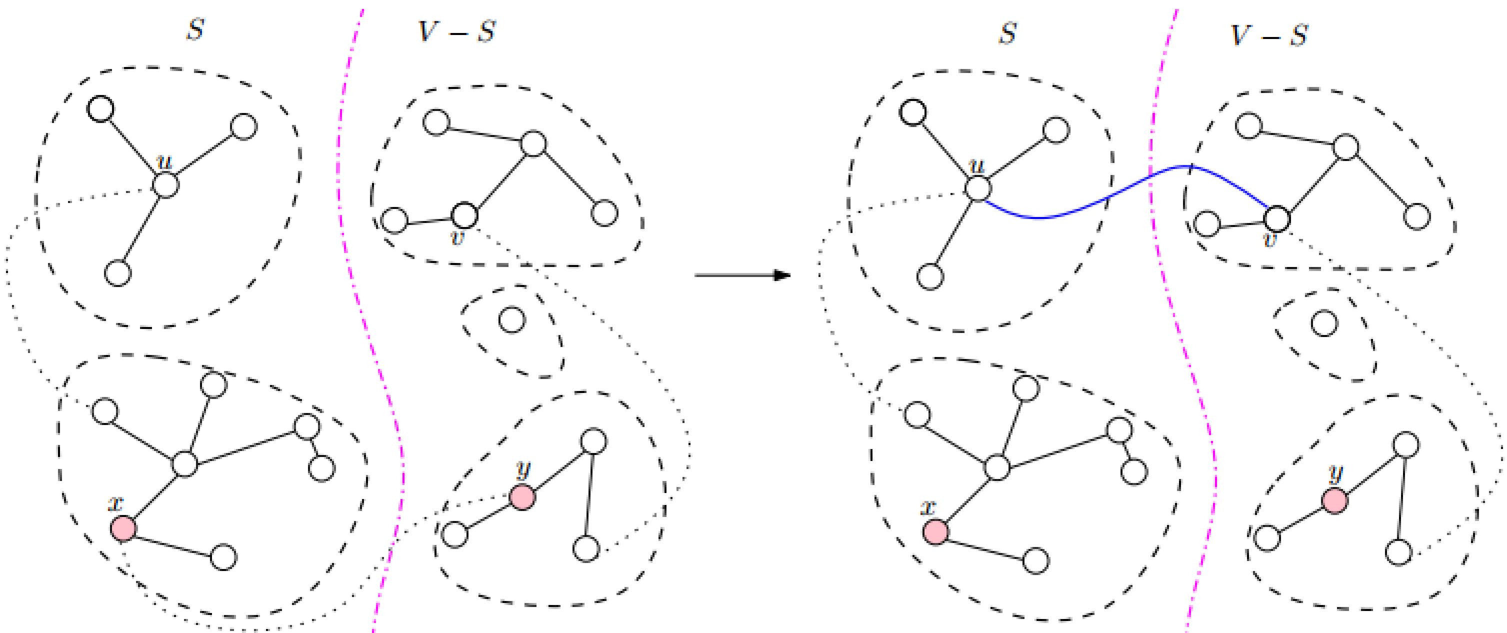

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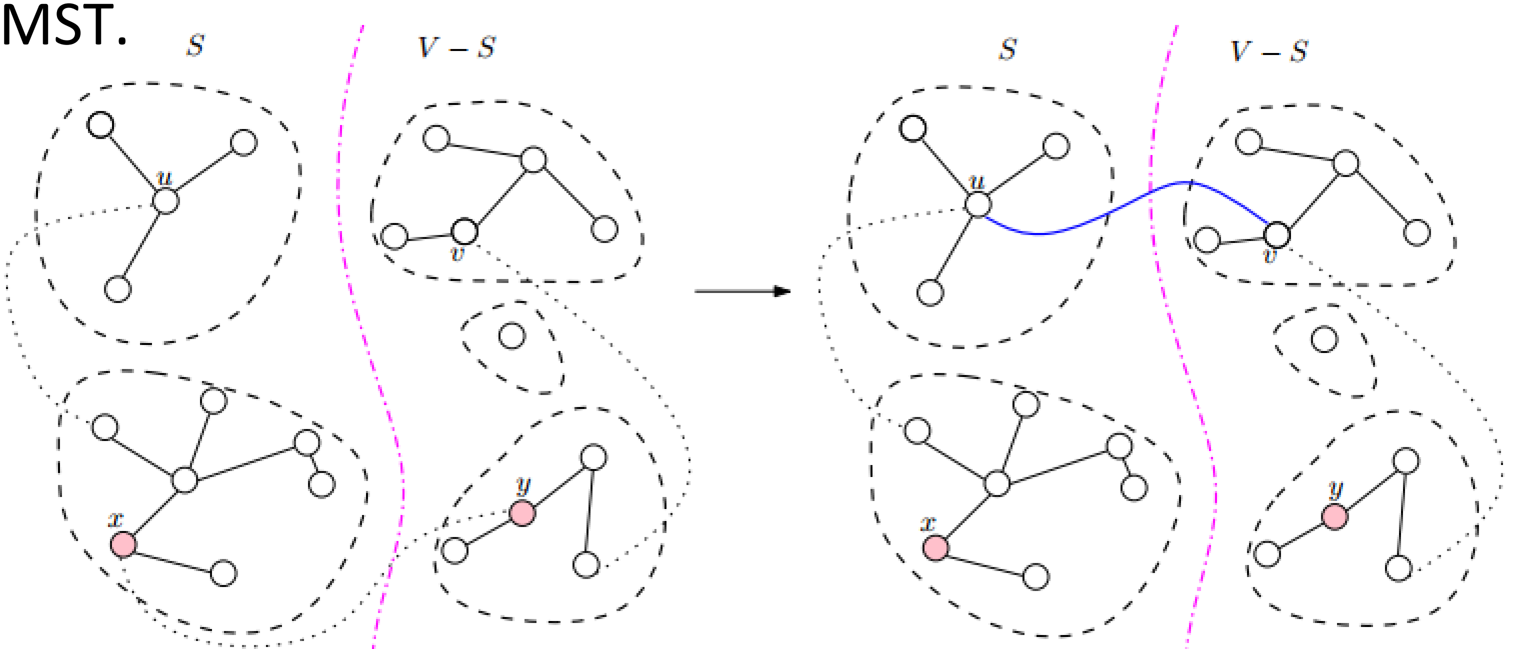

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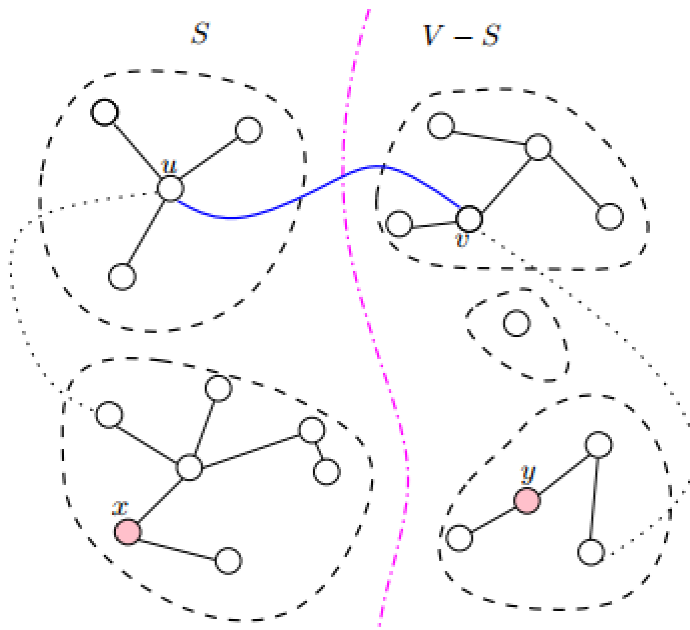
- The Lemma is proved.



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  - If you read the theorem and the proof carefully, you will notice that the choice of a cut (and hence the corresponding light edge) in each iteration is arbitrary.
  - We can select any cut (that respects the selected edges) and find the light edge crossing that cut to proceed.
- Prim’s algorithm makes a nature choice of the cut in each Iteration
  - grows a single tree and adds a light edge in each iteration.
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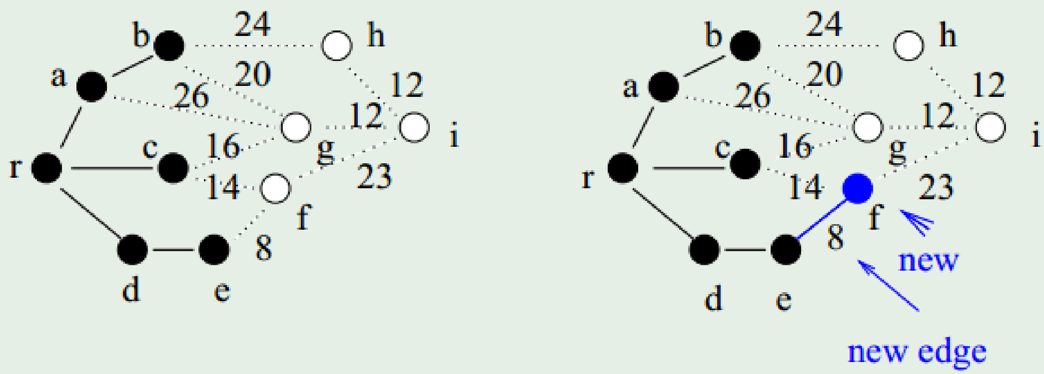
    - While the tree does not contain all vertices in the graph: find shortest edge leaving the tree and add it to the tree.

More Details

Example



Step 0:

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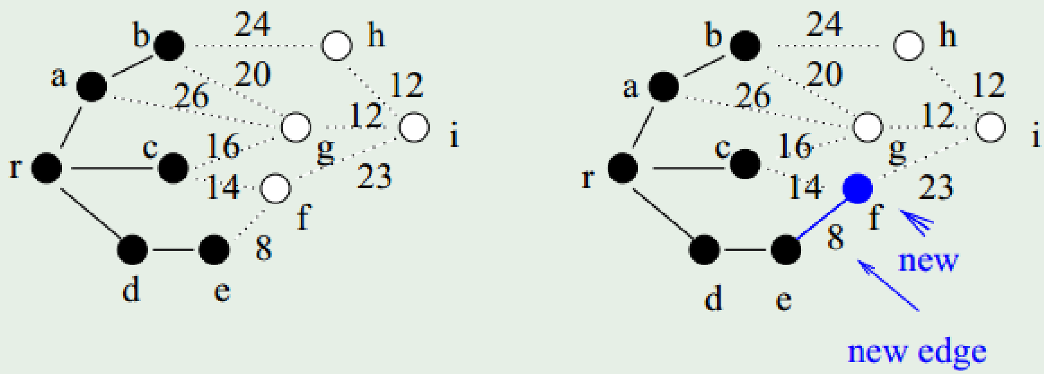
More Details

Example



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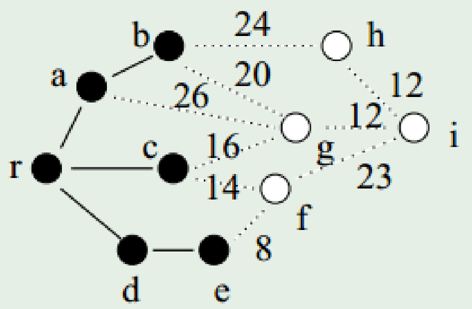

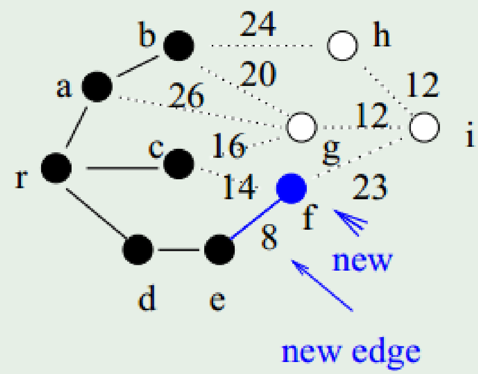

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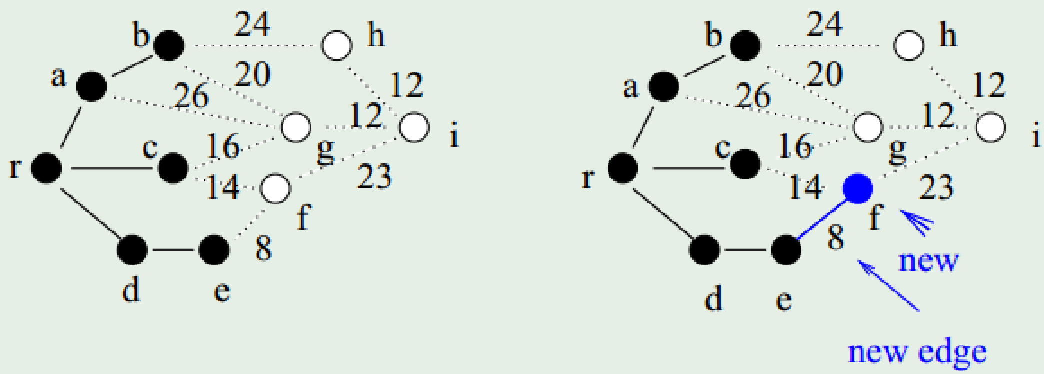

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Step 2:

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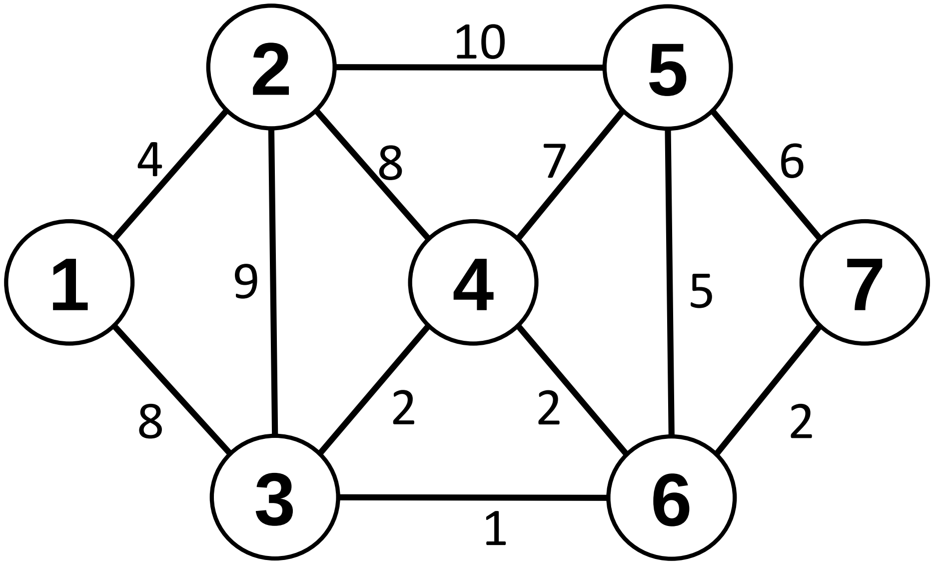
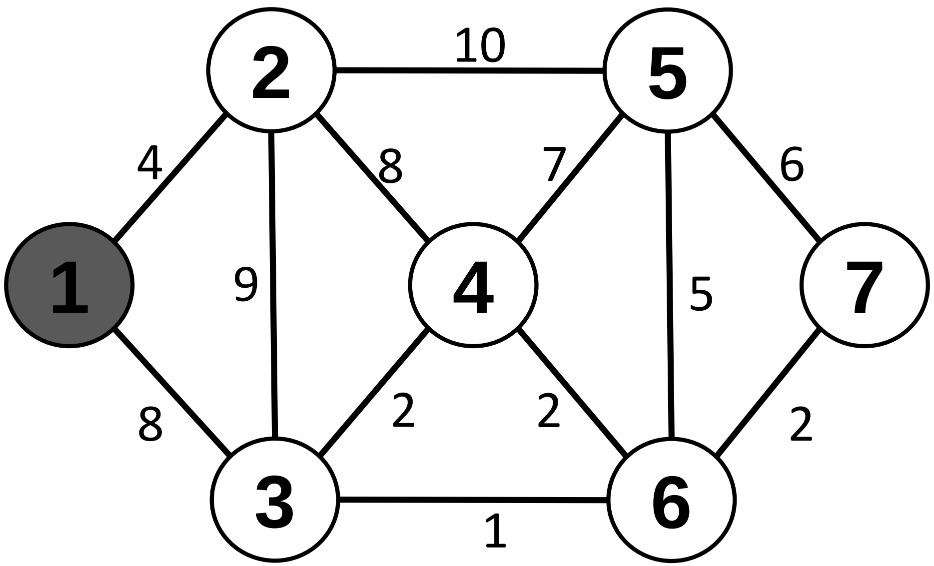
Prim's Example



Connected graph



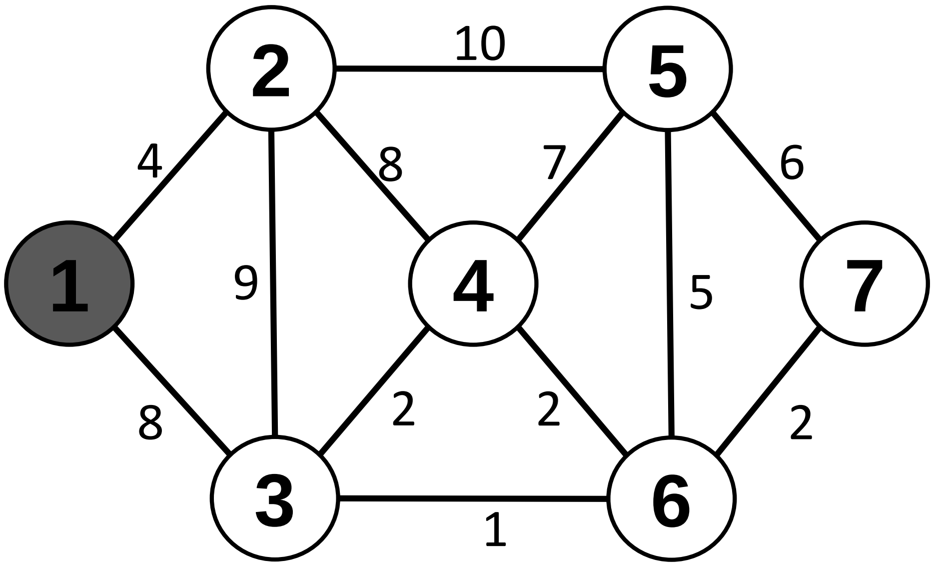
Step 0

S = {1}

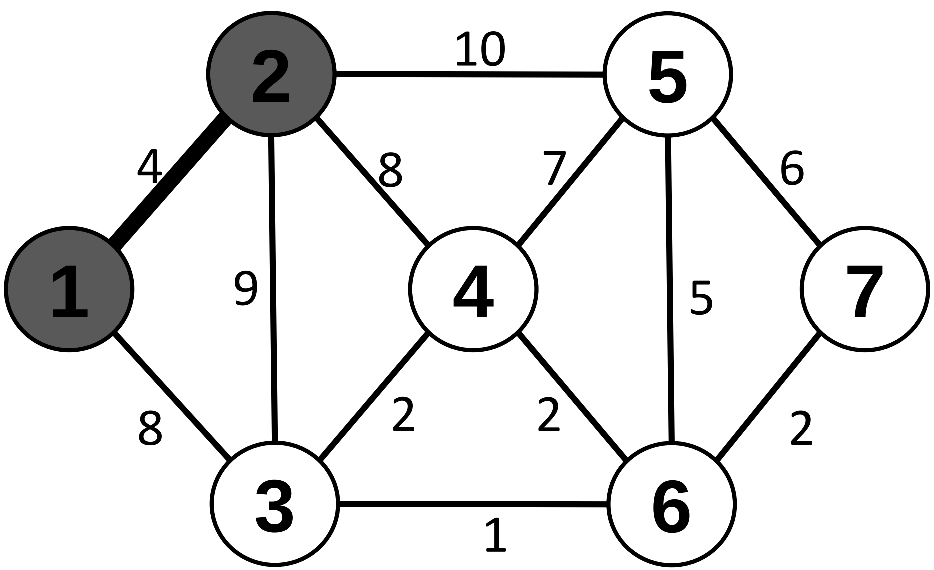
V \ S = {2,3,4,5,6,7}

lightest edge = {1,2}

Prim's Example

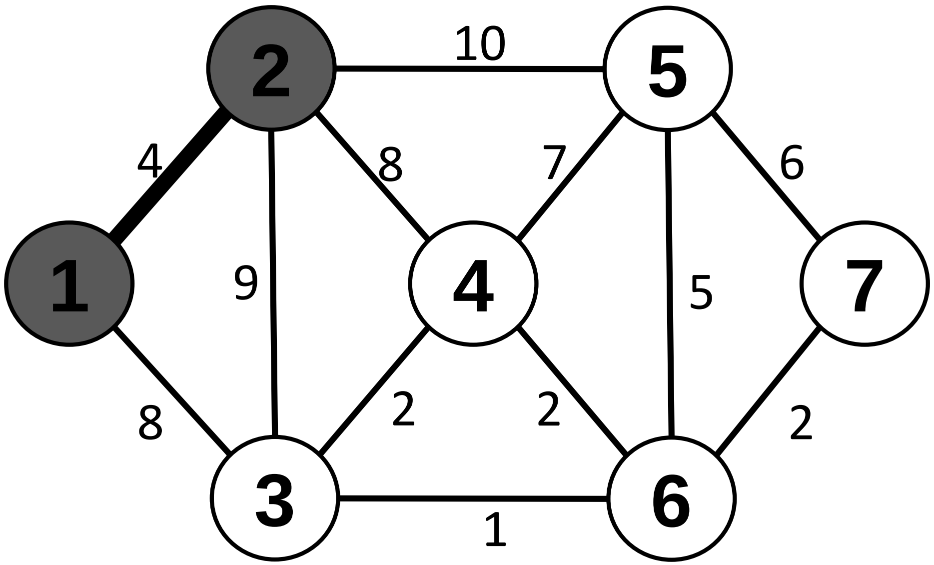


Step 1.1 before
S = {1}
V \ S = {2,3,4,5,6,7}
A={}
lightest edge = {1,2}

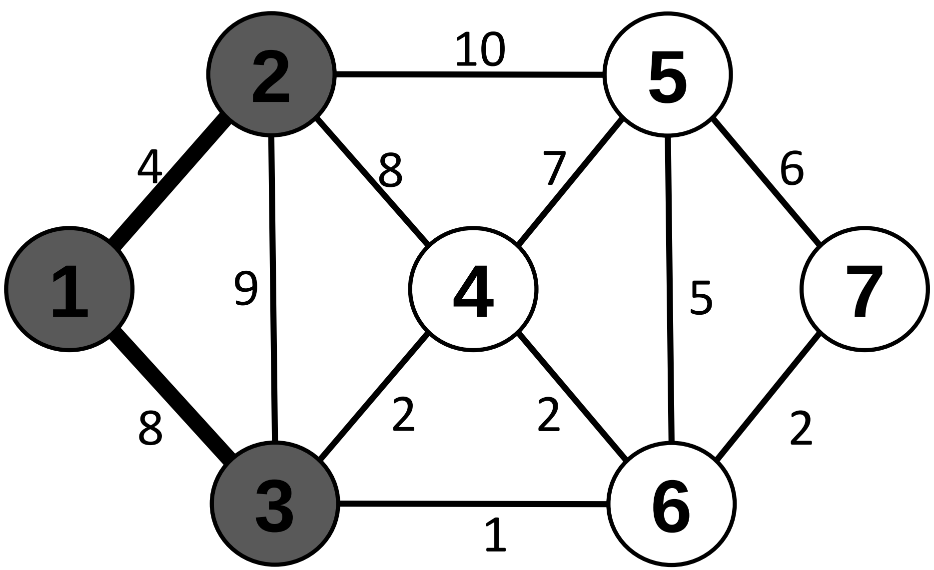


Step 1.1 after
S = {1,2}
V \ S = {3,4,5,6,7}
A = { {1,2} }
lightest edge = {1,3},{2,4}

Prim's Example

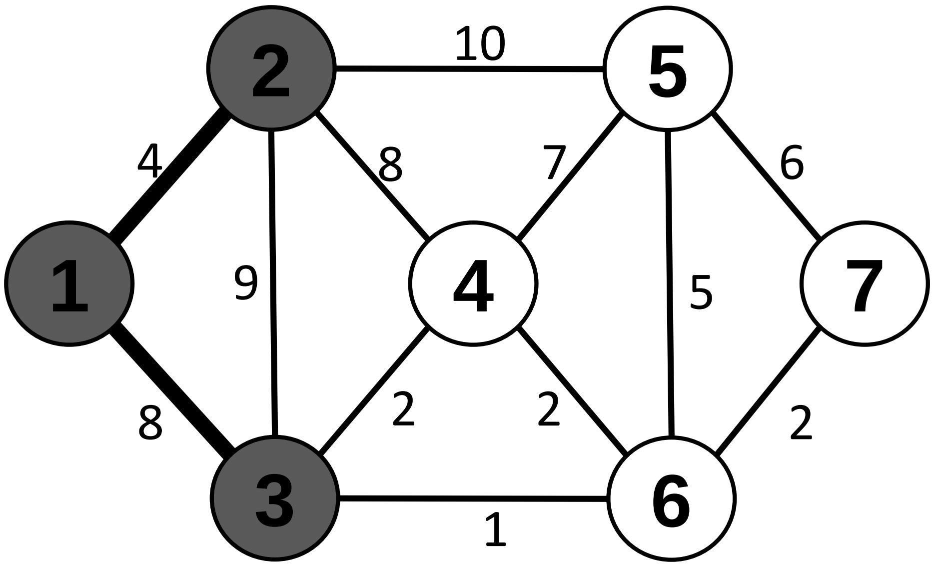


Step 1.2 before
S = {1,2}
V \ S = {3,4,5,6,7}
A = { {1,2} }
lightest edge = {1,3},{2,4}

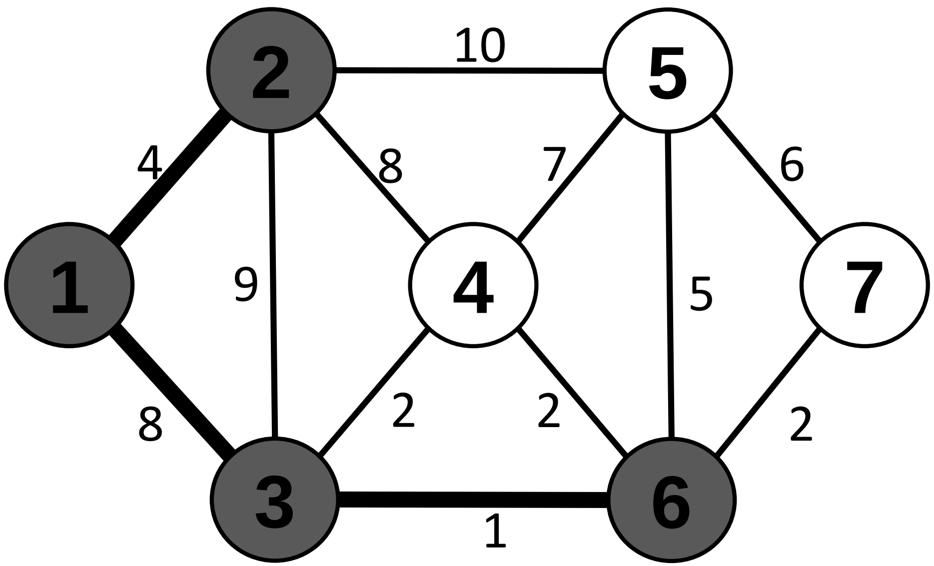


Step 1.2 after
S = {1,2,3}
V \ S = {4,5,6,7}
A = { {1,2},{1,3} }
lightest edge = {3,6}

Prim's Example

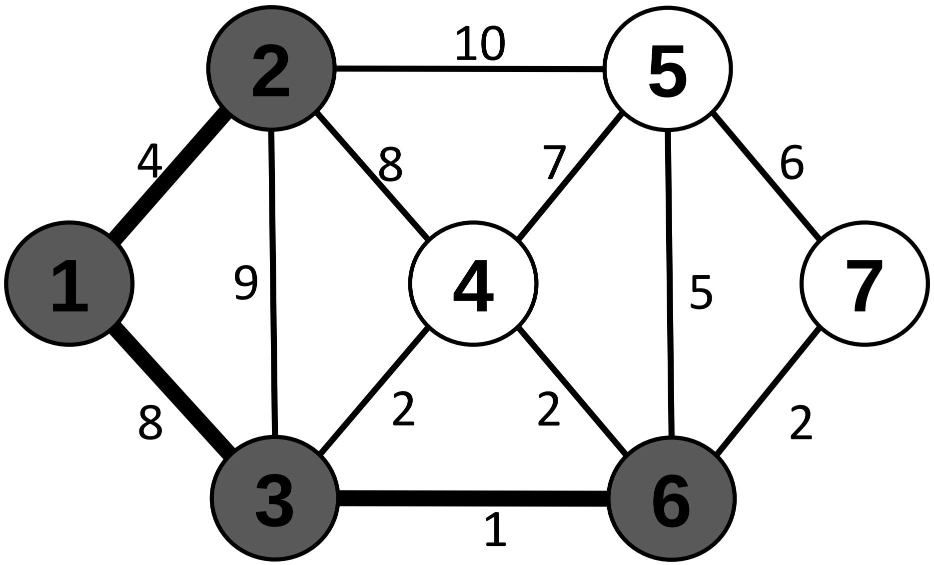


Step 1.3 before
S = {1,2,3}
V \ S = {4,5,6,7}
A = { {1,2},{1,3} }
lightest edge = {3,6}

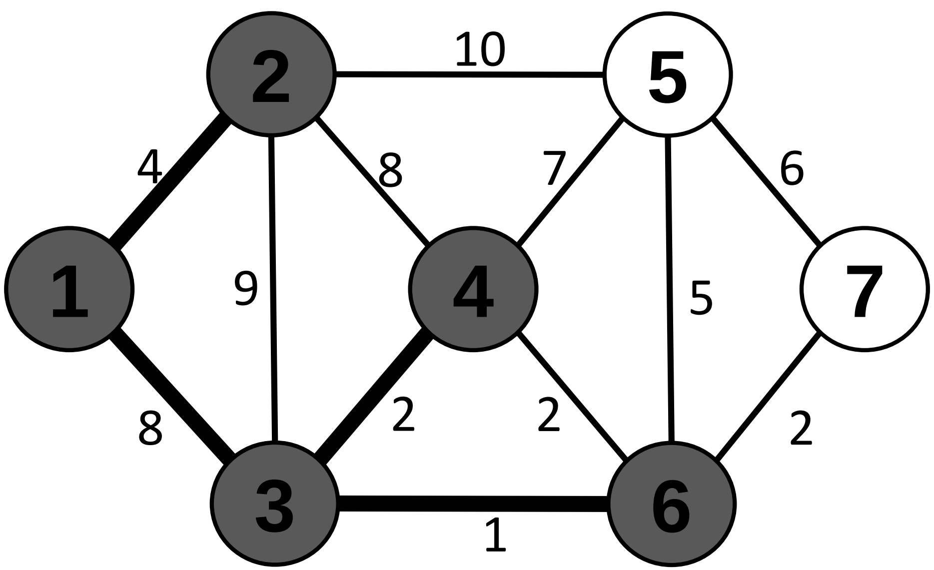


Step 1.3 after
S = {1,2,3,6}
V \ S = {4,5,7}
A = { {1,2},{1,3},{3,6} }
lightest edge = {3,4},{6,4},{6,7}

Prim's Example

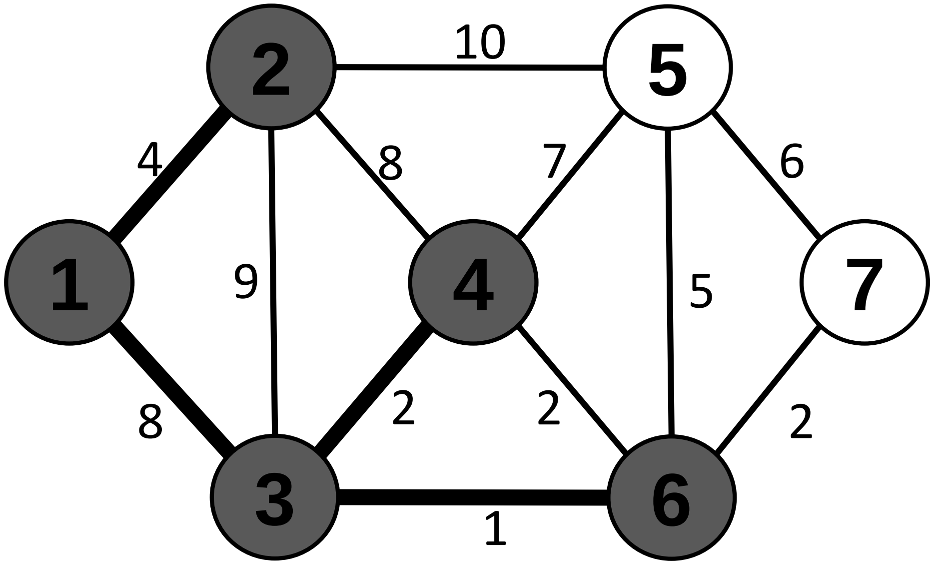


Step 1.4 before
S = {1,2,3,6}
V \ S = {4,5,7}
A = { {1,2},{1,3},{3,6} }
lightest edge = {3,4},{6,4},{6,7}

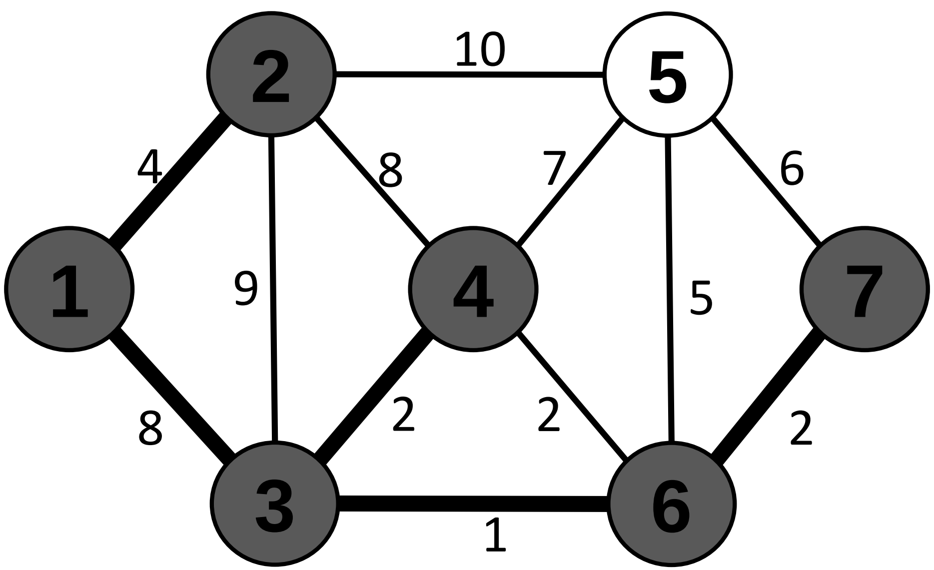


Step 1.4 after
S = {1,2,3,4,6}
V \ S = {5,7}
A = { {1,2},{1,3},{3,6},{3,4} }
lightest edge = {6,7}

Prim's Example

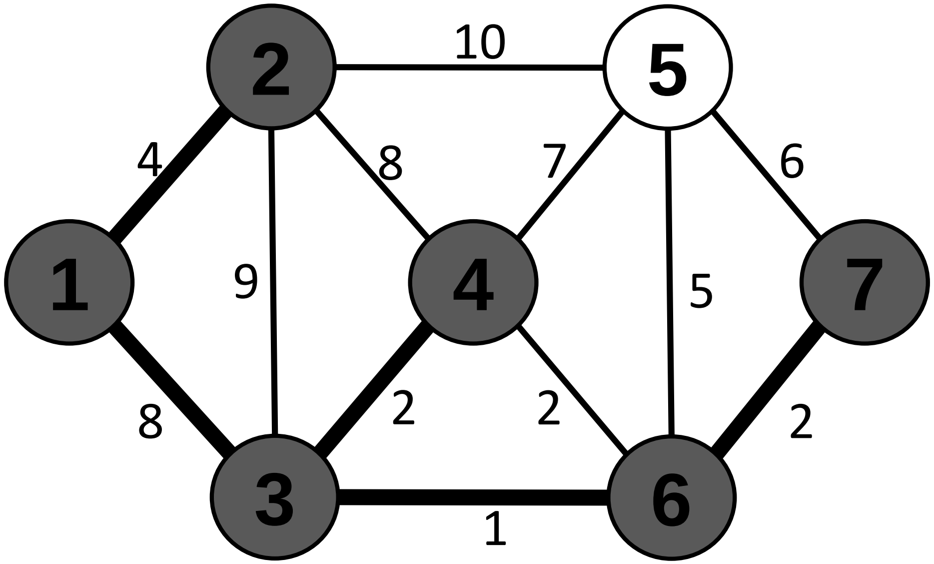


Step 1.5 before
S = {1,2,3,4,6}
V \ S = {5,7}
A = { {1,2},{1,3},{3,6},{3,4} }
lightest edge = {6,7}

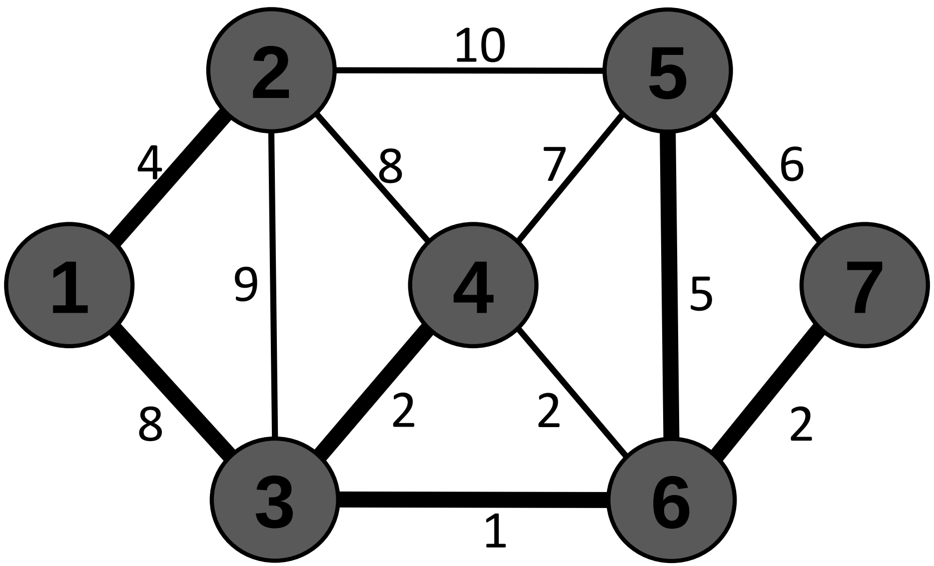


Step 1.5 after
S = {1,2,3,4,6,7}
V \ S = {5}
A = { {1,2},{1,3},{3,6},{3,4},{6,7} }
lightest edge = {6,5}

Prim's Example



Step 1.6 before
S = {1,2,3,4,6,7}
V \ S = {5}
A = { {1,2},{1,3},{3,6},{3,4},{6,7} }
lightest edge = {6,5}



Step 1.6 after
S = {1,2,3,4,5,6,7}
V \ S = {}
A = { {1,2},{1,3},{3,6},{3,4},{6,7}, {6,5} }
MST completed

Outline

- Review to Part IV
- Minimum Spanning Trees
  - Spanning trees
  - Minimum spanning trees
- Prim’s algorithm
  - The idea
  - The algorithm
  - Analysis for Prim’s algorithm
- Kruskal’s algorithm
  - The idea
  - The algorithm
  - The Disjoint Set Union-Find data structure
  - Analysis for Kruskal’s algorithm

Recall Idea of Prim's Algorithm

Step 0: Choose any element r and set S = {r} and A = ∅. (Take r as the root of our spanning tree.)

Step 1: Find a lightest edge such that one endpoint is in S and the other is in V \ S. Add this edge to A and its (other) endpoint to S.

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Questions:

- How does the algorithm update S efficiently?
- How does the algorithm find the lightest edge and update A efficiently?

Prim’s Algorithm

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Answer: Color the vertices.

- Initially all are white.

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Answer:

1. Use a priority queue to find the lightest edge.

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How does the algorithm find the lightest edge and update A efficiently?

Answer:

1. Use a priority queue to find the lightest edge.
2. Use pred$[v]$ to update A.

Reviewing Priority Queues

Priority Queue is a data structure

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- can be implemented as a heap

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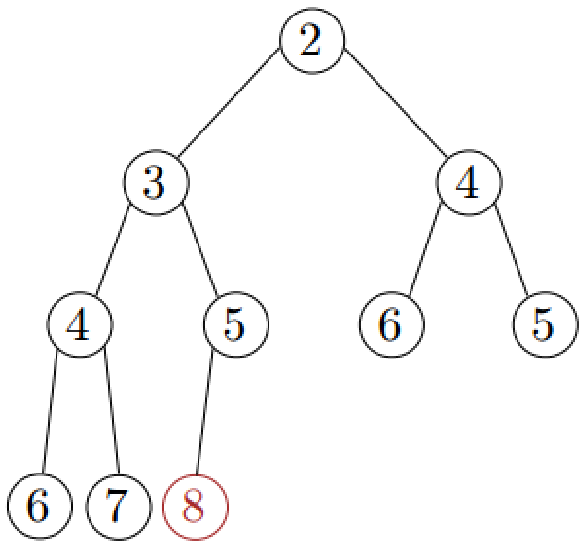
Remark: Priority Queues can be implemented so that each operation takes time O(log |Q|). See Lecture 5 & Chapter 6.5 CLRS.

Reviewing Extract-Min

- Copy the last element to the root (i.e., overwrite the minimum element stored there)
- Restore the min-heap property by percolate down (or bubble down): if the element is larger than either of its children, then interchange it with the smaller of its children.



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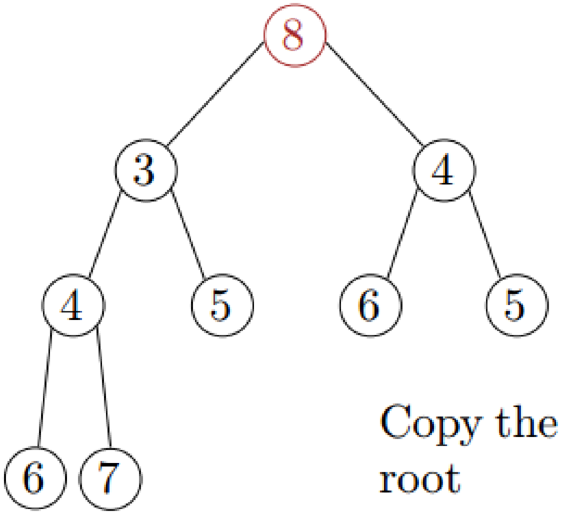

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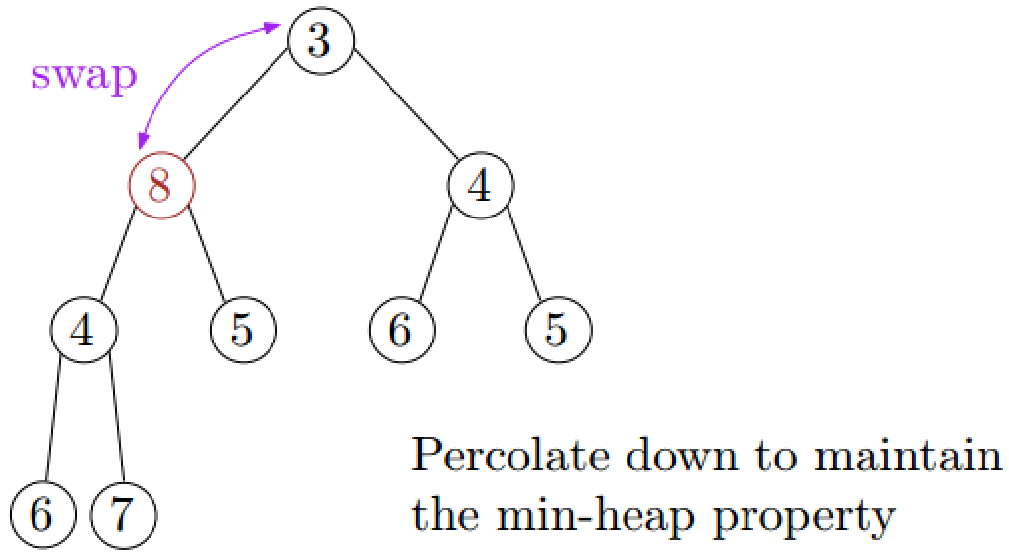

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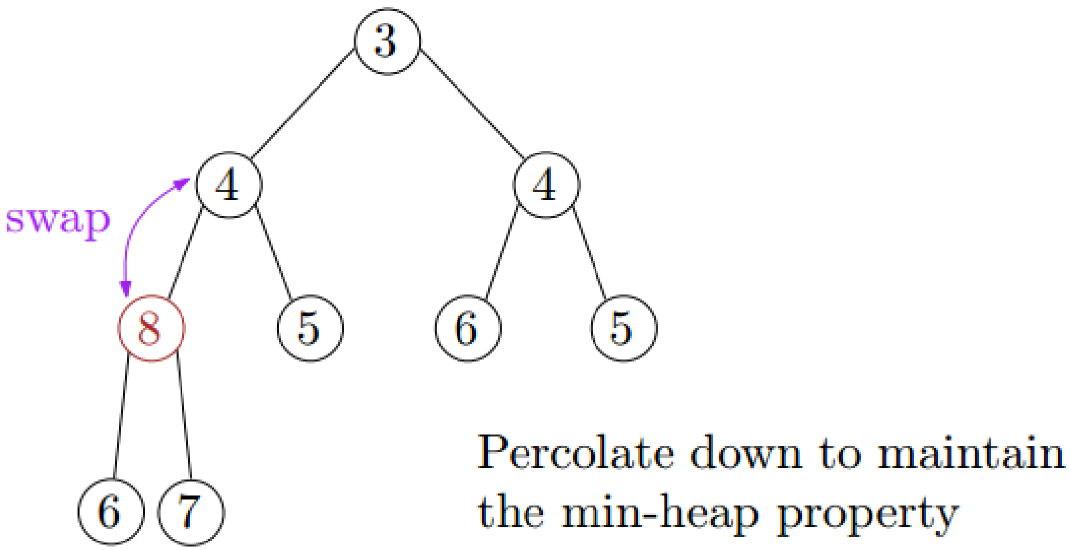

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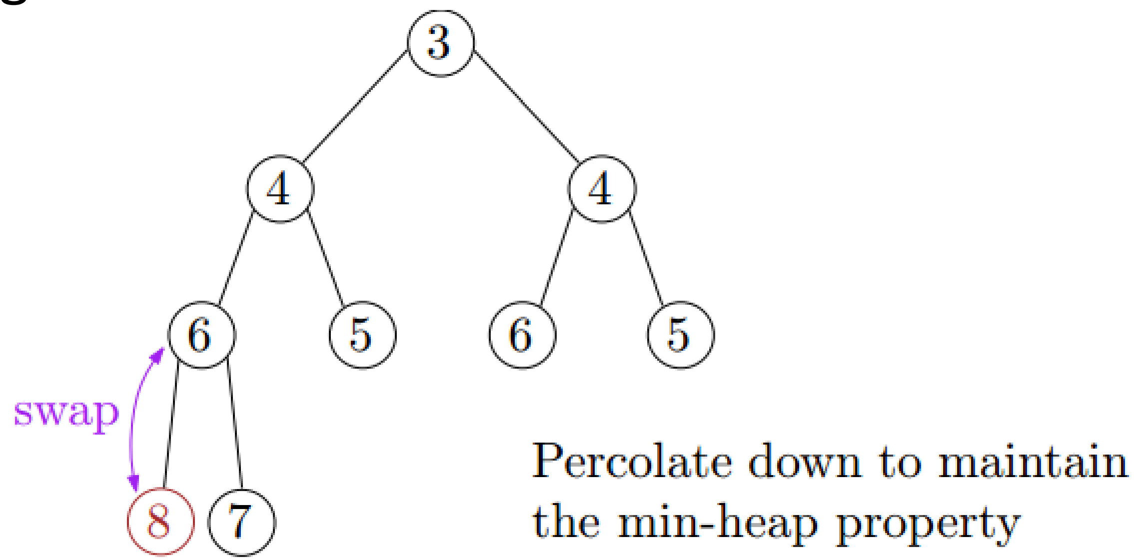

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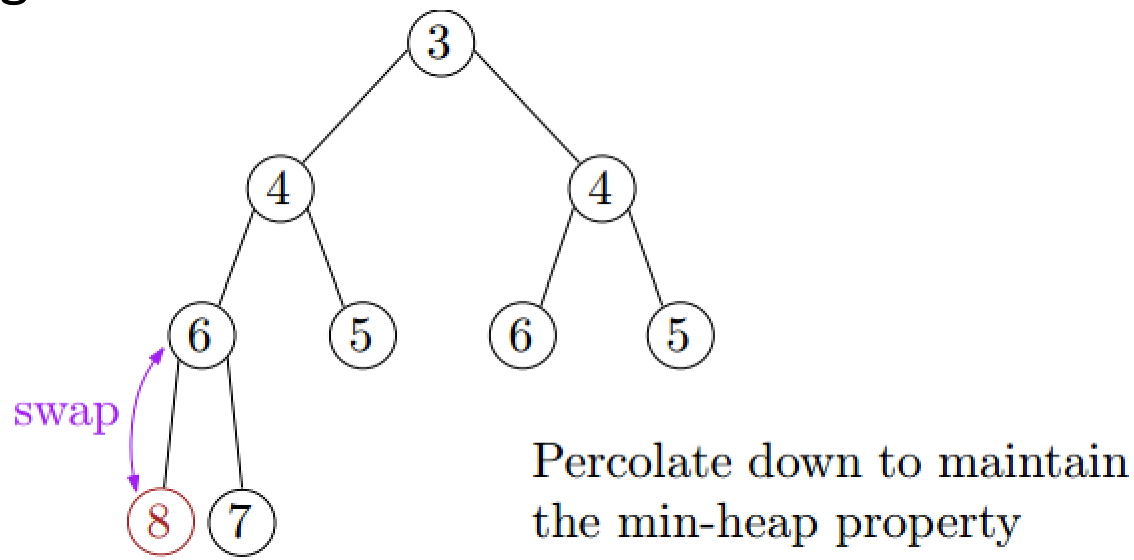

- Correctness: after each swap, the min-heap property is satisfied for all nodes except the node containing the element (with respect to its children)
- Time complexity = O(height) = O(log n)

Using a Priority Queue to Find the Lightest Edge

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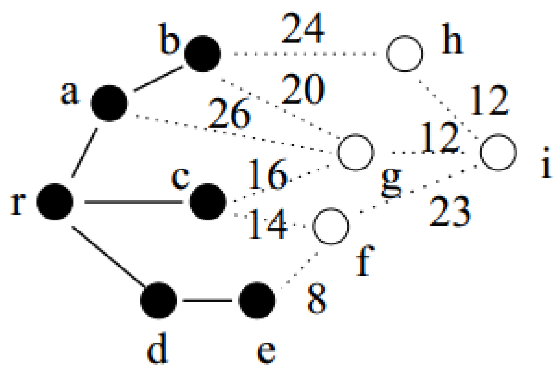
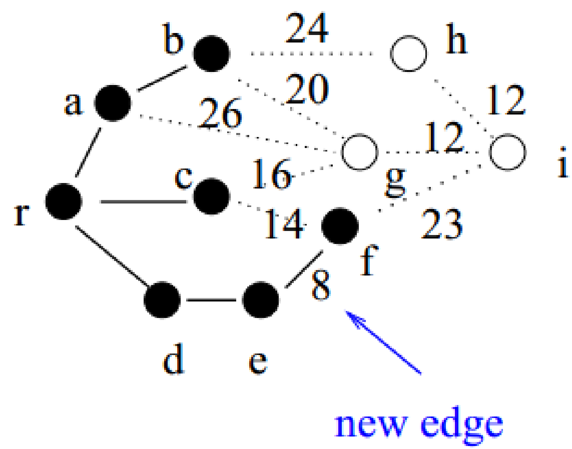
Using a Priority Queue to Find the Lightest Edge

Each item of the queue is a pair (u, key[u]), where

- u is a vertex in V\S,
- key[u] is the weight of the lightest edge from u to any vertex in S. (The endpoint of this edge in S is stored in pred[u], which is used to build the MST tree.)





key[f] = 8, pred[f] = e

key[i] = infinity, pred[i] = nil

key[g] = 16, pred[g] = c

key[h] = 24, pred[h] = b

→ f has the minimum key

key[i] = 23, pred[i] = f

After adding the new edge and vertex f, update the key[v] and pred[v] for each vertex v adjacent to f

Description of Prim's Algorithm

Prim(G,w,r)

Input: A graph $\boldsymbol{G}$, a matrix $\boldsymbol{w}$ representing the weights between vertices in G, the algorithm will start at root vertex $\boldsymbol{r}$

Output: None

Let $color[1...|V|], key[1...|V|], pred[1...|V|]$ be new arrays;

for $u \in V$ **do**

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while Q *is nonempty* **do**

  $u \leftarrow Q$.Extract-Min();// lightest edge

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Q ← new PriQueue(V); // put vertices in Q
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    for v ∈ adj[u] do
        if (color[v] ← WHITE)&&(w[u,v] < key[v]) then
            key[v] ← w[u,v]; // new lightest edge
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|  |  if (color[v] ← WHITE)&&(w[u, v] < key[v]) then
|  |  |  key[v] ← w[u, v]; // new lightest edge
|  |  |  Q.Decrease-Key(v, key[v]);
```

Description of Prim's Algorithm

Prim(G,w,r)

Input: A graph ***G***, a matrix ***w*** representing the weights between vertices in G, the algorithm will start at root vertex ***r***

Output: None

Let $color[1...|V|], key[1...|V|], pred[1...|V|]$ be new arrays;

for $u \in V$ **do**

$\quad color[u] \leftarrow$ WHITE,$key[u] \leftarrow +\infty$;// Initialize

end

$key[r] \leftarrow 0$, $pred[r] \leftarrow$ NULL;// Start at root vertex

$Q \leftarrow$ new PriQueue(V);// put vertices in Q

while Q *is nonempty* **do**

$\quad u \leftarrow Q$.Extract-Min();// lightest edge

$\quad$ **for** $v \in adj[u]$ **do**

$\quad\quad$ **if** $(color[v] \leftarrow WHITE)\&\&(w[u,v] < key[v])$ **then**

$\quad\quad\quad key[v] \leftarrow w[u,v]$;// new lightest edge

$\quad\quad\quad Q$.Decrease-Key$(v, key[v])$;

$\quad\quad\quad pred[v] \leftarrow u$;

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Input: A graph G, a matrix w representing the weights between vertices
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Let color[1...|V|], key[1...|V|], pred[1...|V|] be new arrays;
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|   |   |   pred[v] ← u;
|   |   end
|   end
|   color[u] ← BLACK;
end
```

Prim's Example

color

W	**W**	**W**	**W**	**W**	**W**	**W**

pred

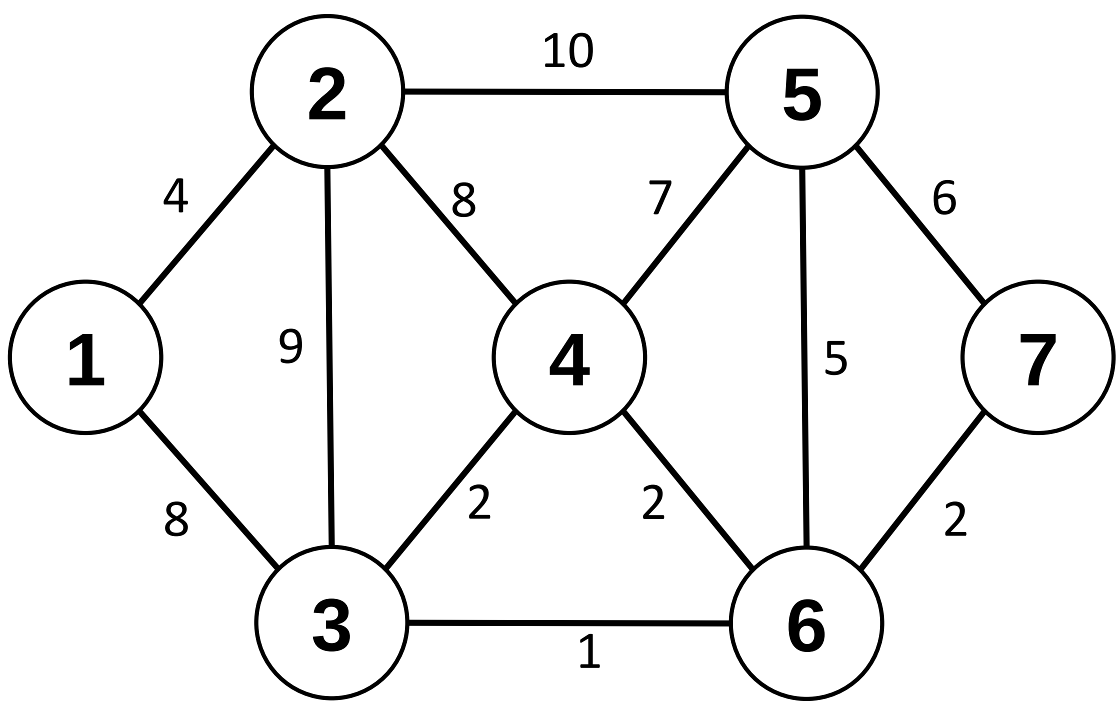
N	**N**	**N**	**N**	**N**	**N**	**N**

key

∞	**∞**	**∞**	**∞**	**∞**	**∞**	**∞**

Q

1,∞	**2,∞**	**3,∞**	**4,∞**	**5,∞**	**6,∞**	**7,∞**



Prim's Example

color

W	**W**	**W**	**W**	**W**	**W**	**W**

pred

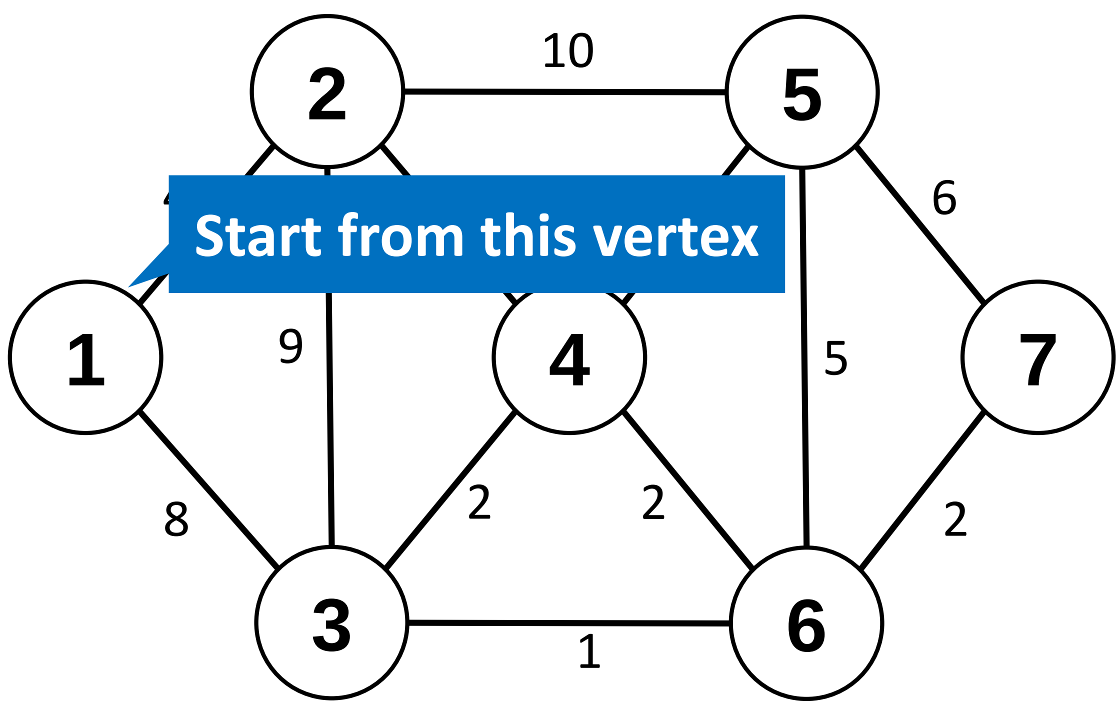
N	**N**	**N**	**N**	**N**	**N**	**N**

key

∞	∞	∞	∞	∞	∞	∞

Q

1,∞	2,∞	3,∞	4,∞	5,∞	6,∞	7,∞



Prim's Example

color

B	W	W	W	W	W	W

pred

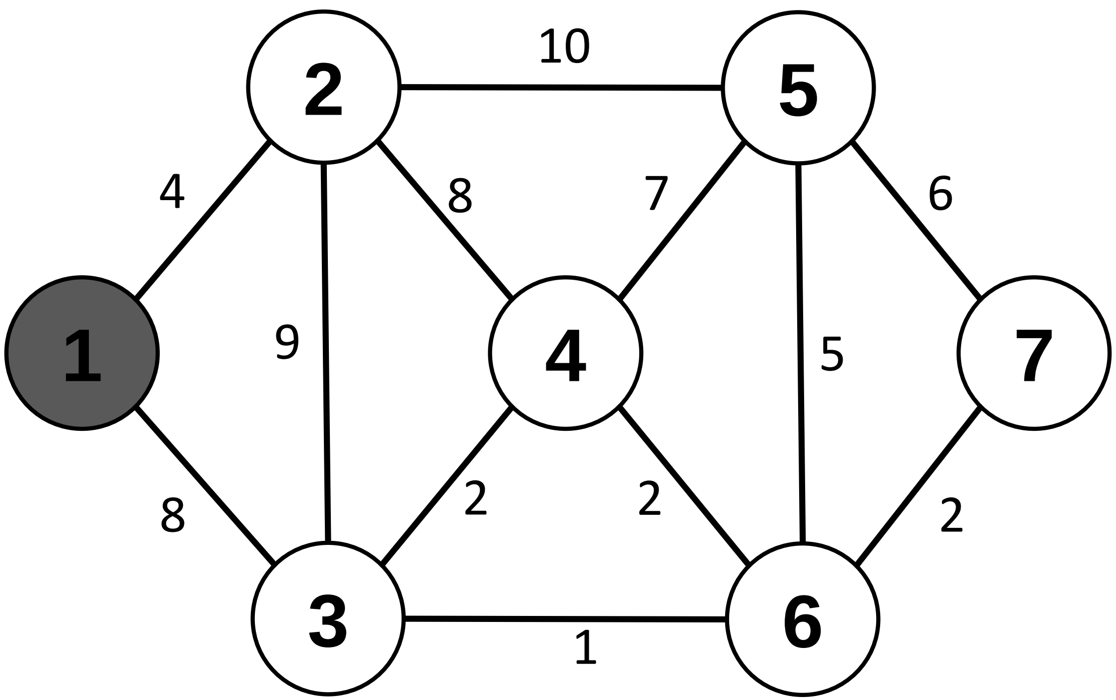
N	N	N	N	N	N	N

key

0	∞	∞	∞	∞	∞	∞

Q

1,0	2,∞	3,∞	4,∞	5,∞	6,∞	7,∞



Prim's Example

color

B	W	W	W	W	W	W

pred

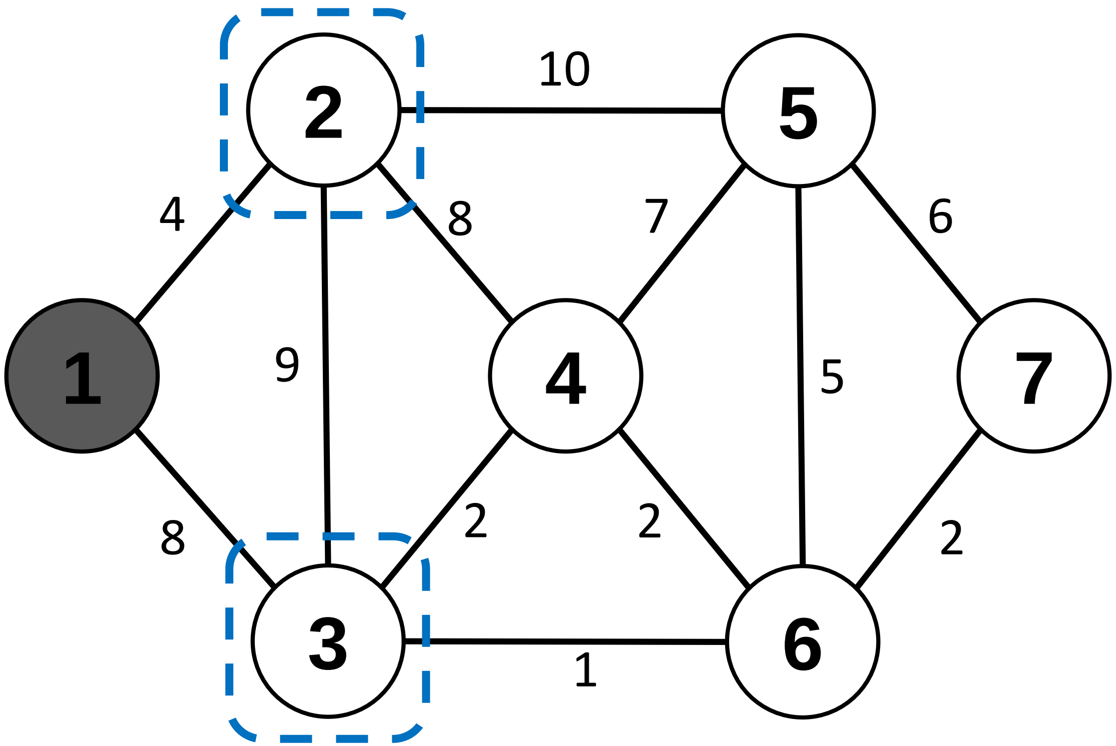
N	N	N	N	N	N	N

key

0	∞	∞	∞	∞	∞	∞

Q

7,∞	2,∞	3,∞	4,∞	5,∞	6,∞



Prim's Example

color

B	W	W	W	W	W	W

pred

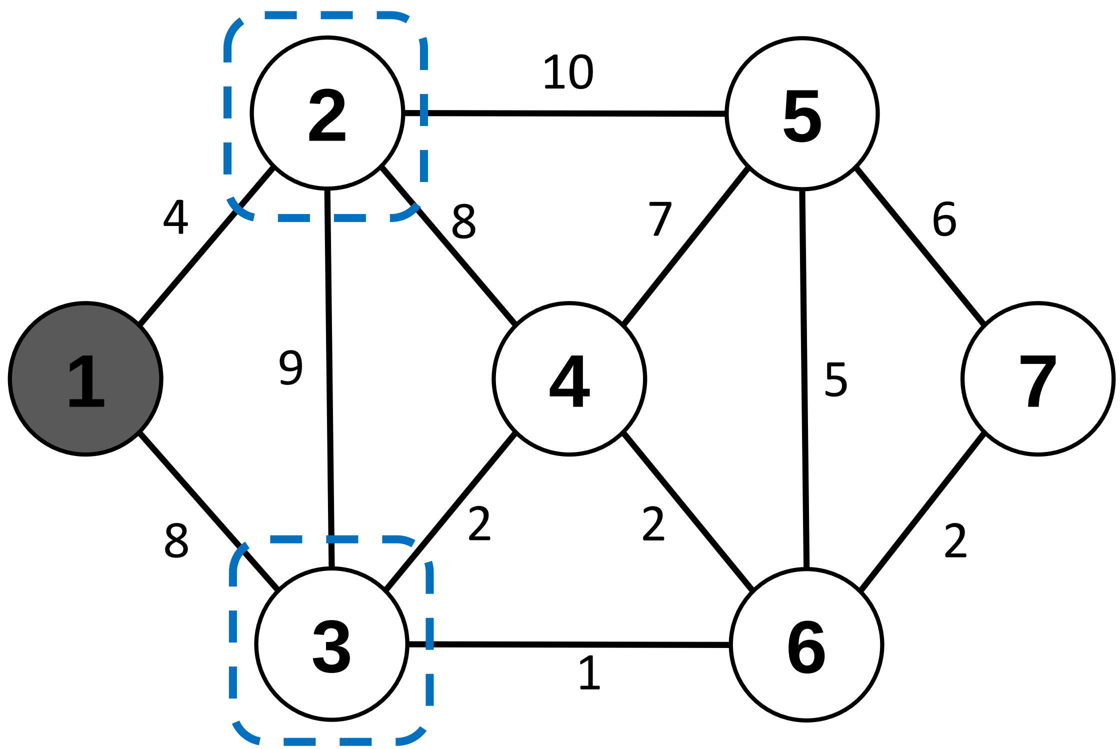
N	1	1	N	N	N	N

key

0	4	8	∞	∞	∞	∞

Q

7,∞	2,4	3,8	4,∞	5,∞	6,∞



Prim's Example

color

B	W	W	W	W	W	W

pred

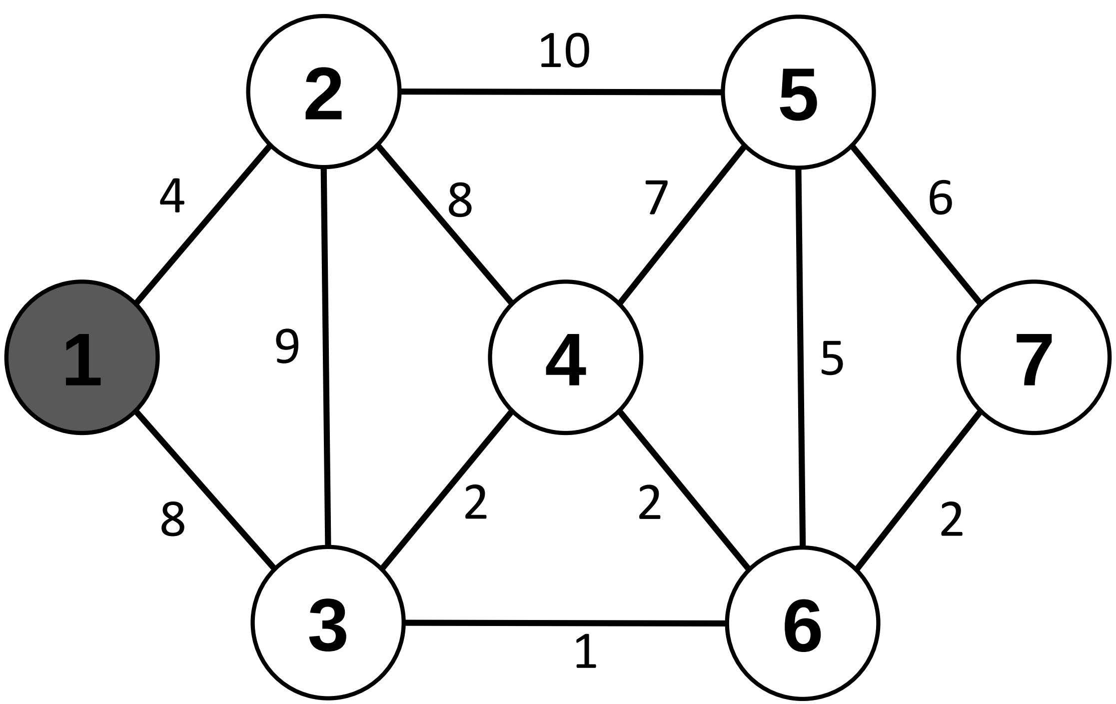
N	1	1	N	N	N	N

key

0	4	8	∞	∞	∞	∞

Q

2,4	7,∞	3,8	4,∞	5,∞	6,∞



Prim's Example

color

B	B	W	W	W	W	W

pred

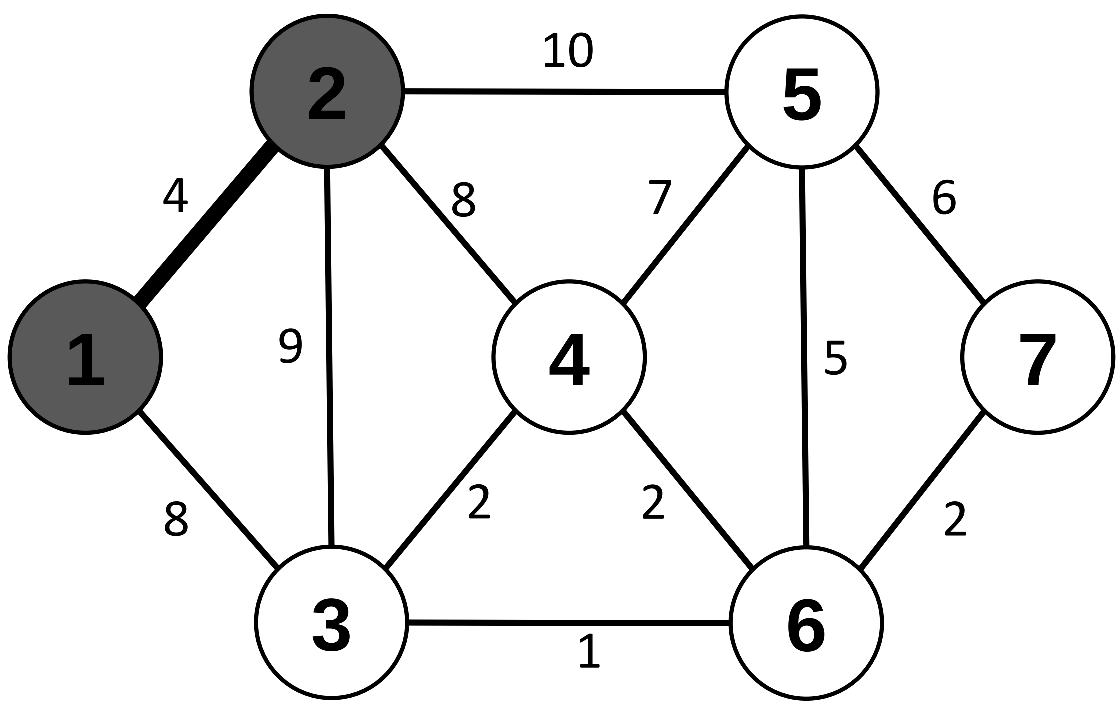
N	1	1	N	N	N	N

key

0	4	8	∞	∞	∞	∞

Q

3,8	7,∞	6,∞	4,∞	5,∞



Prim's Example

color

B	B	W	W	W	W	W

pred

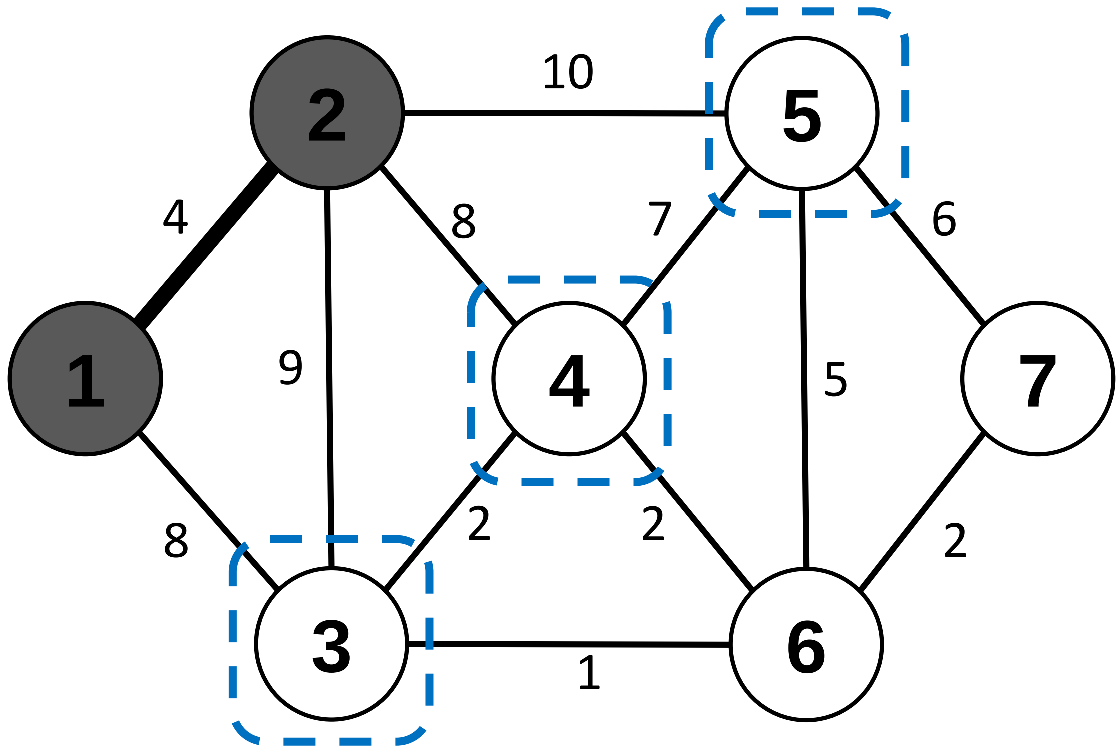
N	1	1	N	N	N	N

key

0	4	8	∞	∞	∞	∞

Q

3,8	7,∞	6,∞	4,∞	5,∞



Prim's Example

color

B	B	W	W	W	W	W

pred

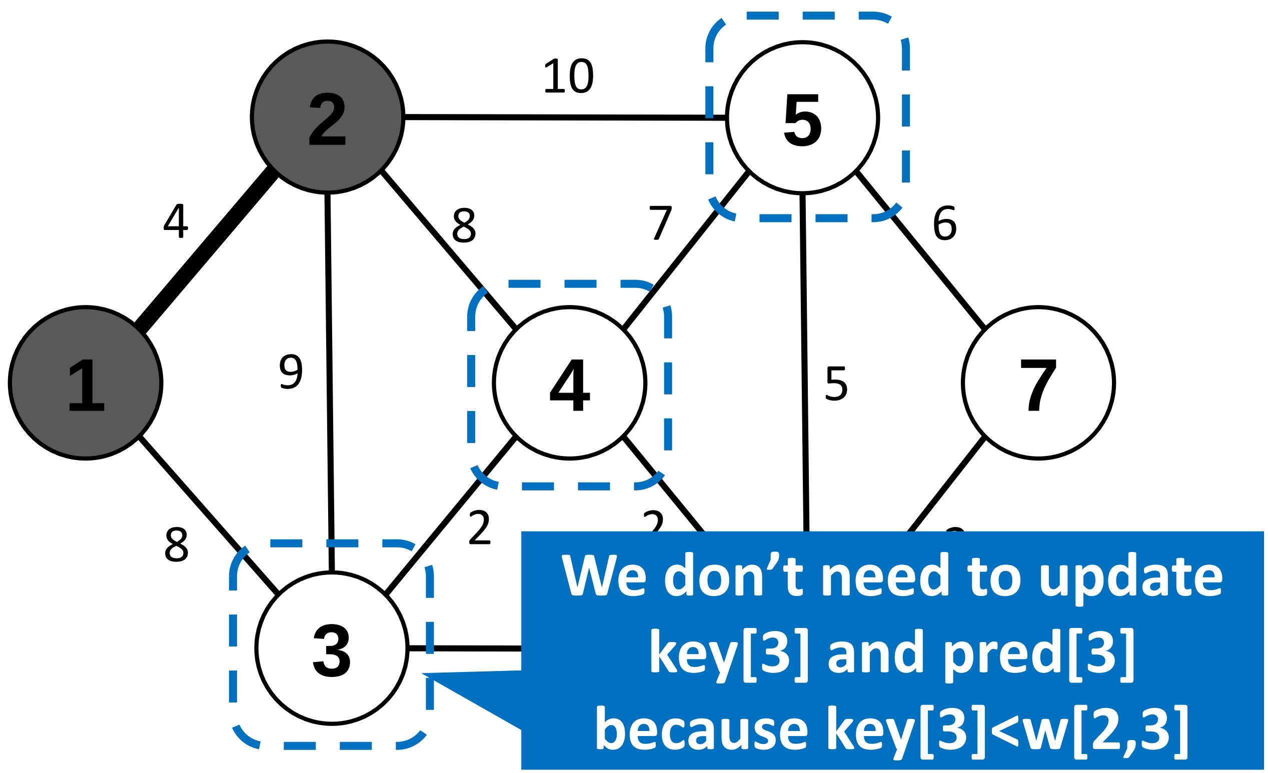
N	1	1	2	2	N	N

key

0	4	8	8	10	∞	∞

Q

3,8	7,∞	6,∞	4,8	5,10



Prim's Example

color

B	B	W	W	W	W	W

pred

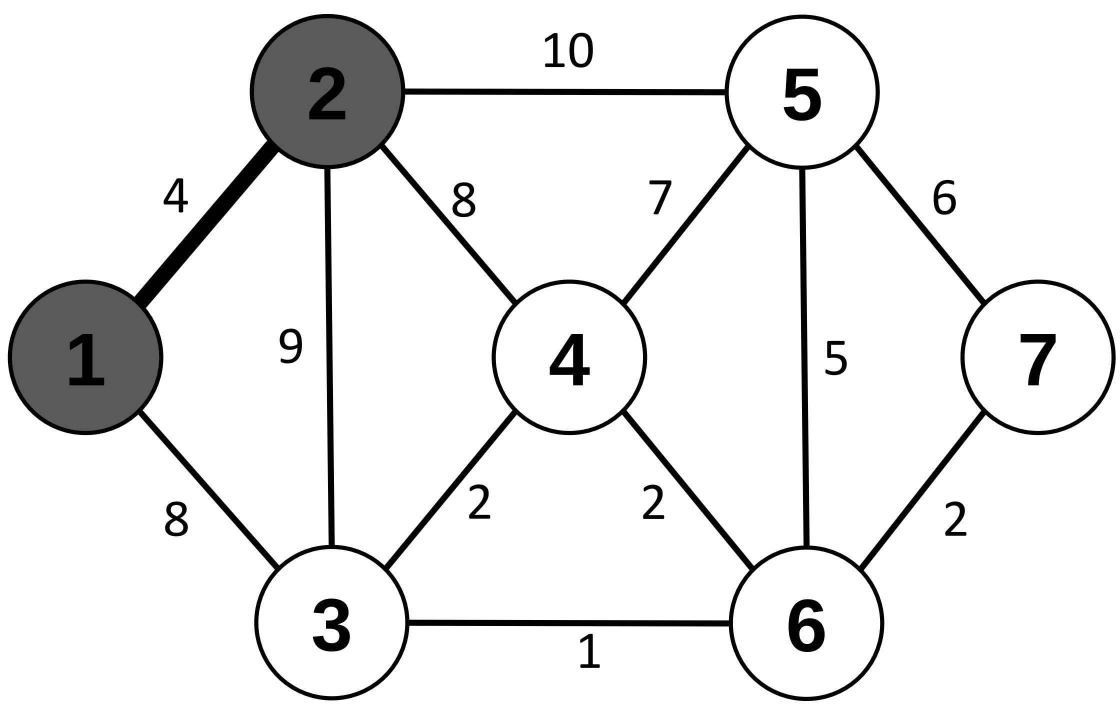
N	1	1	2	2	N	N

key

0	4	8	8	10	∞	∞

Q

3,8	4,8	6,∞	7,∞	5,10



Prim's Example

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pred

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key

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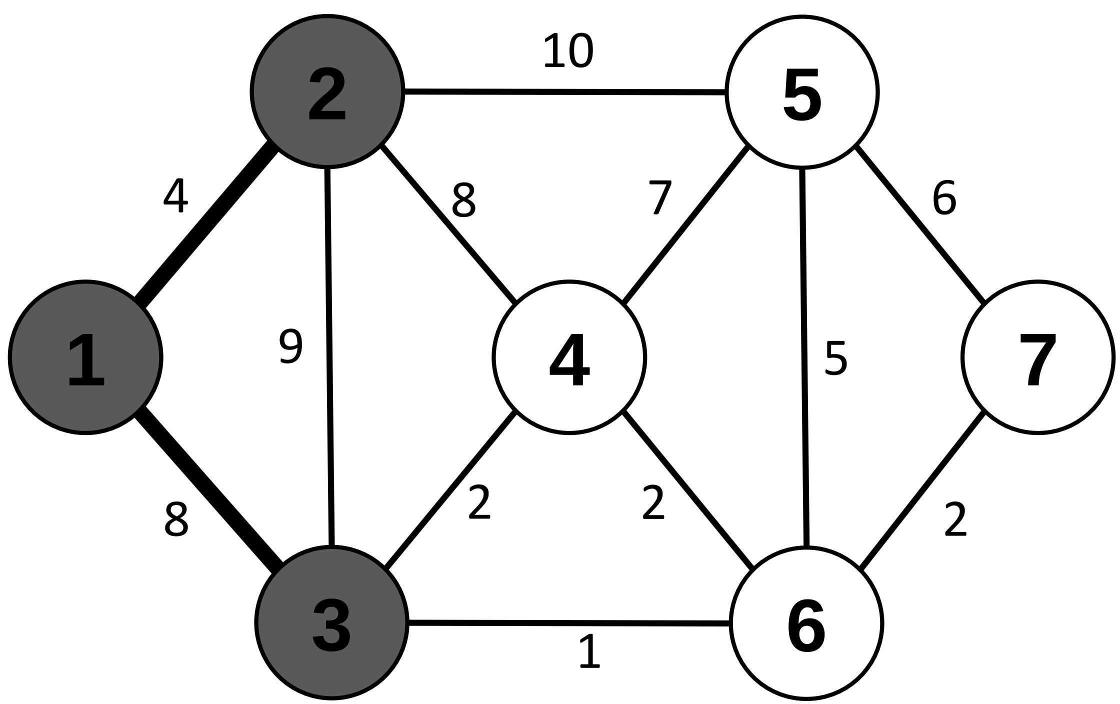
Q

4,8	5,10	6,∞	7,∞



Prim's Example

color

B	B	B	W	W	W	W

pred

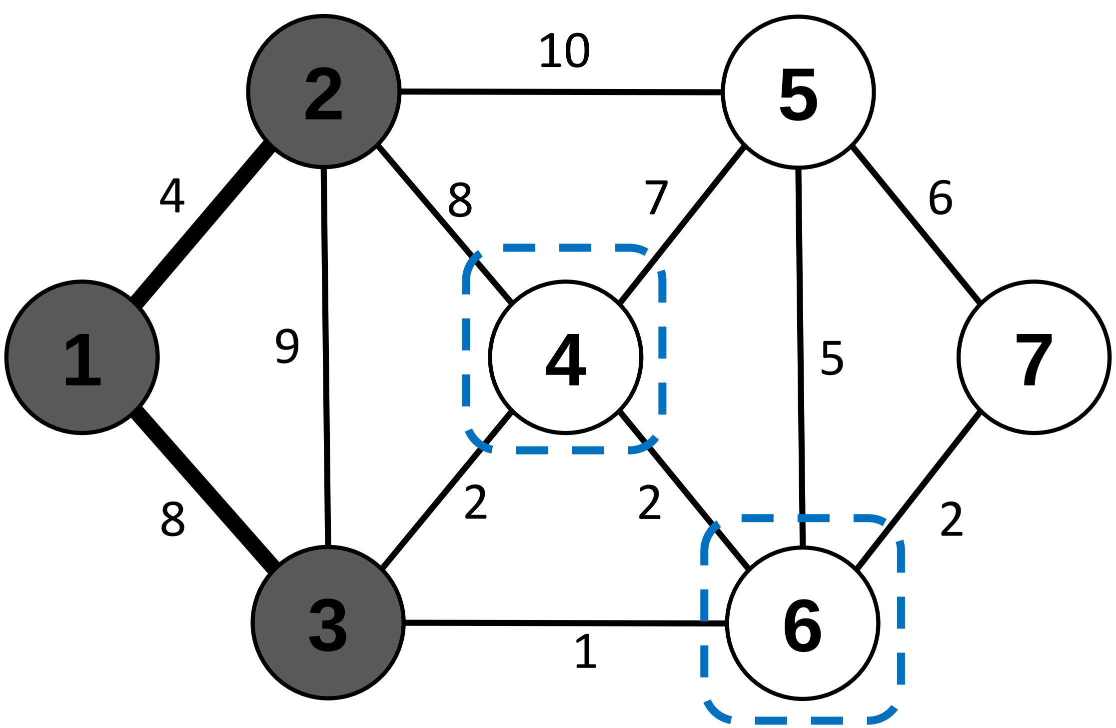
N	1	1	2	2	N	N

key

0	4	8	8	10	∞	∞

Q

4,8	5,10	6,∞	7,∞



Prim's Example

color

B	B	B	W	W	W	W

pred

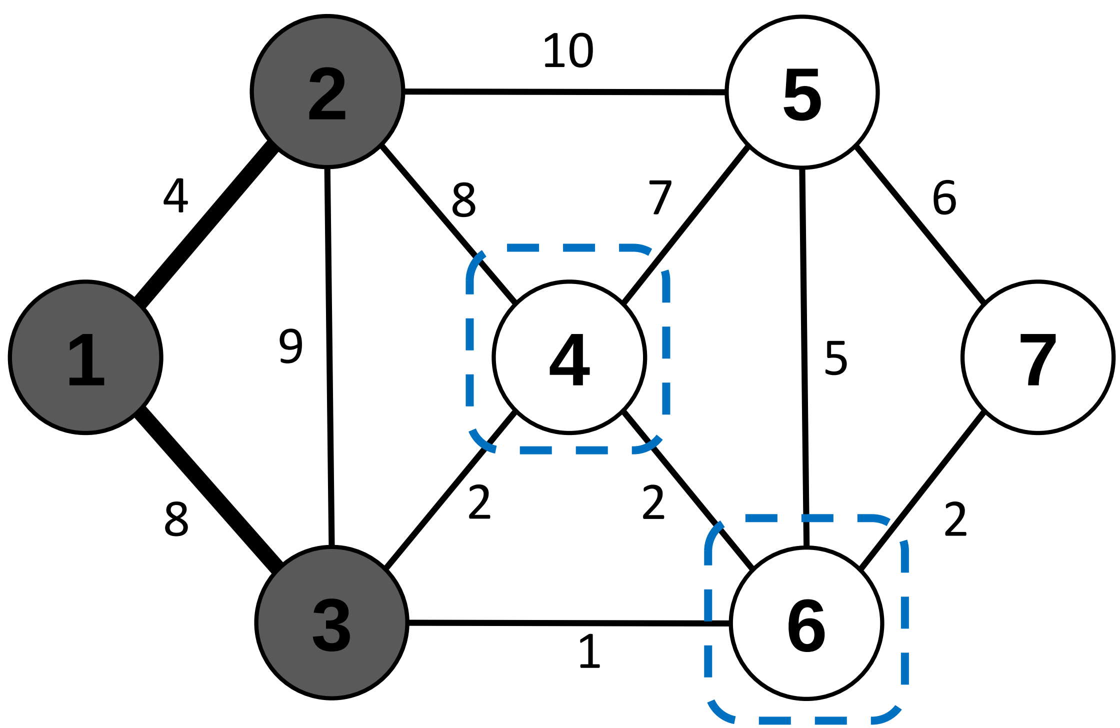
N	1	1	3	2	3	N

key

0	4	8	2	10	1	∞

Q

4,2	5,10	6,1	7,∞



Prim's Example

color

B	B	B	W	W	W	W

pred

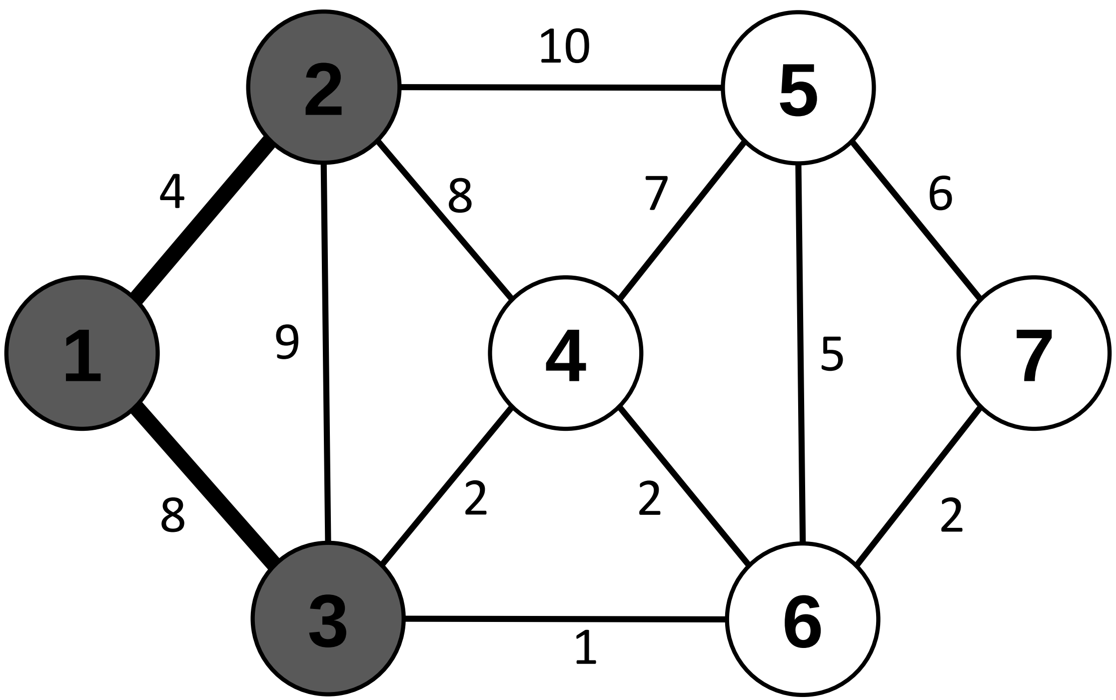
N	1	1	3	2	3	N

key

0	4	8	2	10	1	∞

Q

6,1	5,10	4,2	7,∞



Prim's Example

color

B	B	B	W	W	B	W

pred

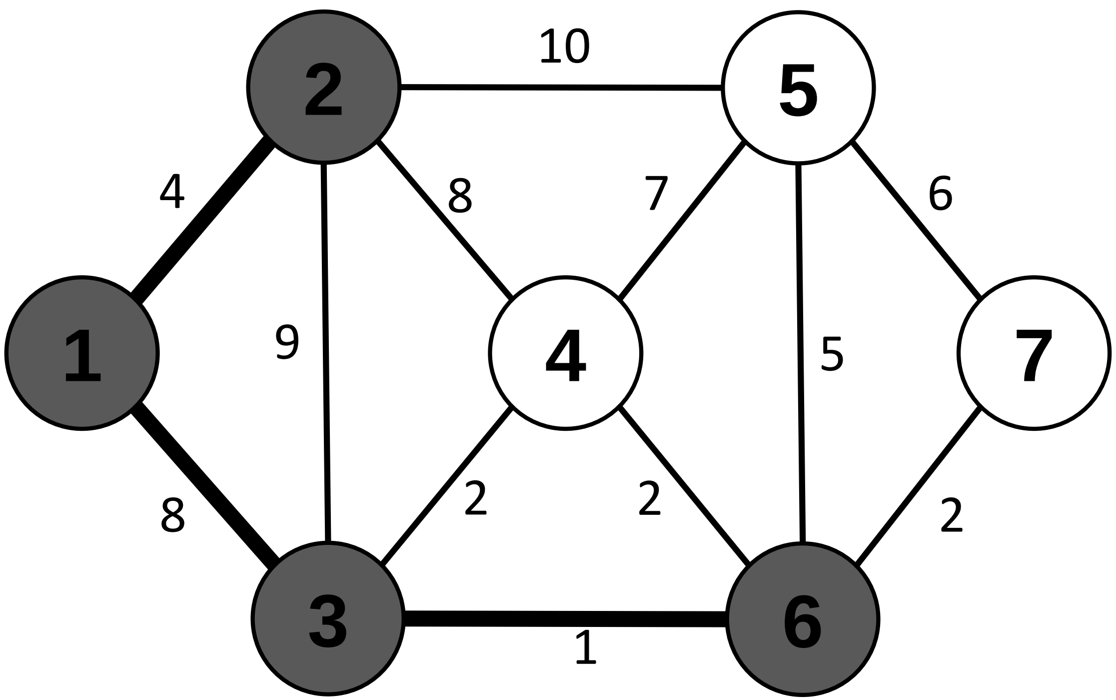
N	1	1	3	2	3	N

key

0	4	8	2	10	1	∞

Q

4,2	5,10	7,∞



Prim's Example

color

B	B	B	W	W	B	W

pred

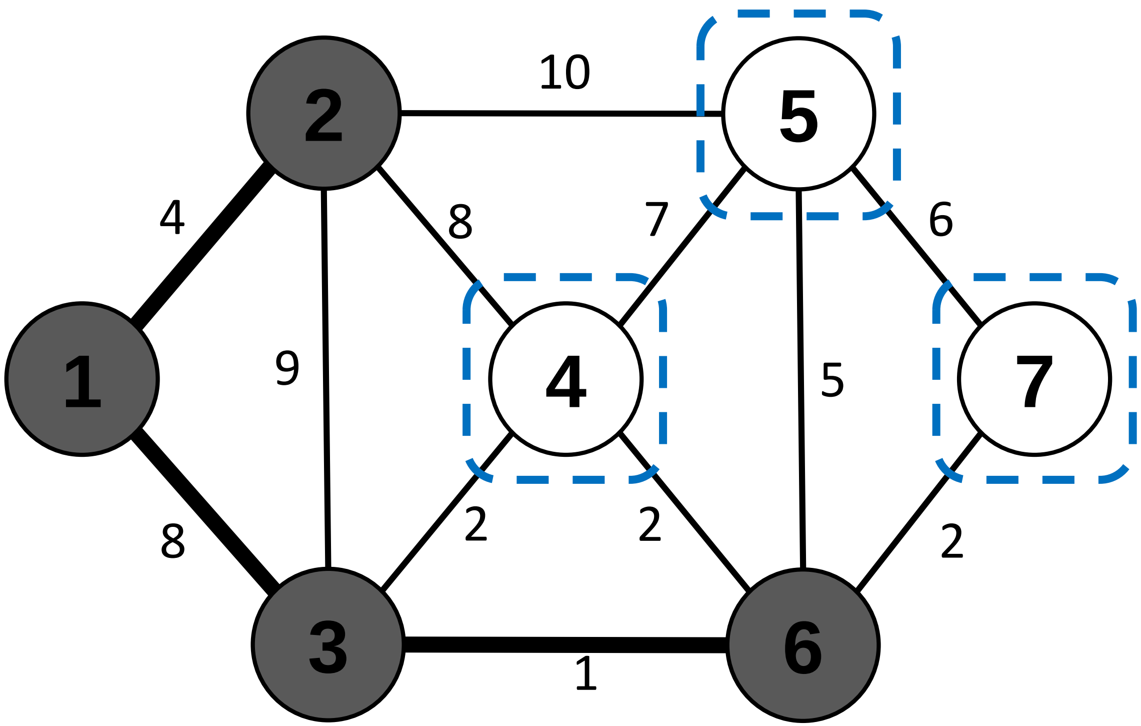
N	1	1	3	2	3	N

key

0	4	8	2	10	1	∞

Q

4,2	5,10	7,∞



Prim's Example

color

B	B	B	W	W	B	W

pred

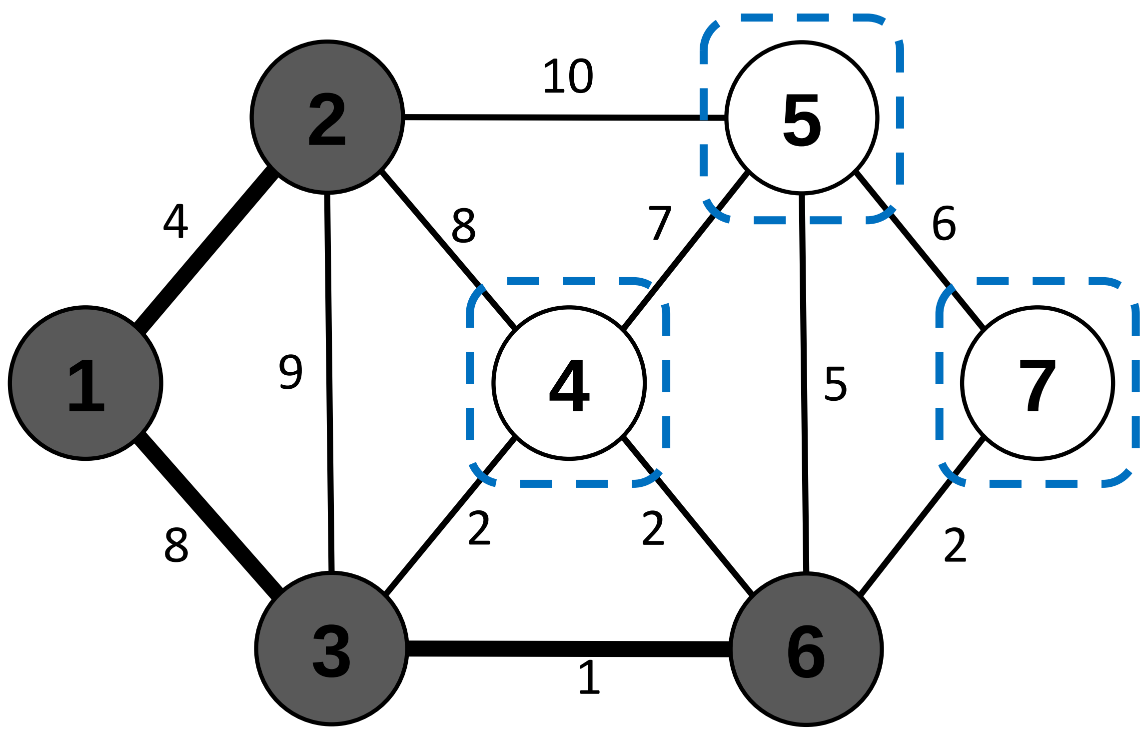
N	1	1	3	6	3	6

key

0	4	8	2	5	1	2

Q

4,2	5,5	7,2



Prim's Example

color

B	B	B	W	W	B	W

pred

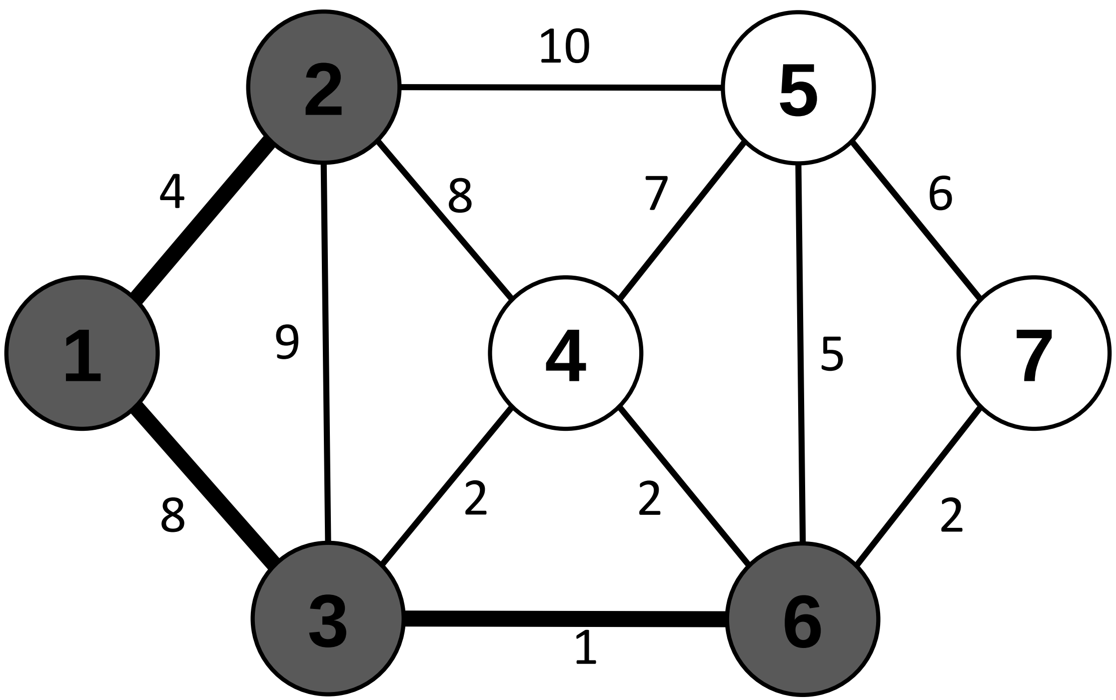
N	1	1	3	6	3	6

key

0	4	8	2	5	1	2

Q

4,2	5,5	7,2



Prim's Example

color

B	B	B	B	W	B	W

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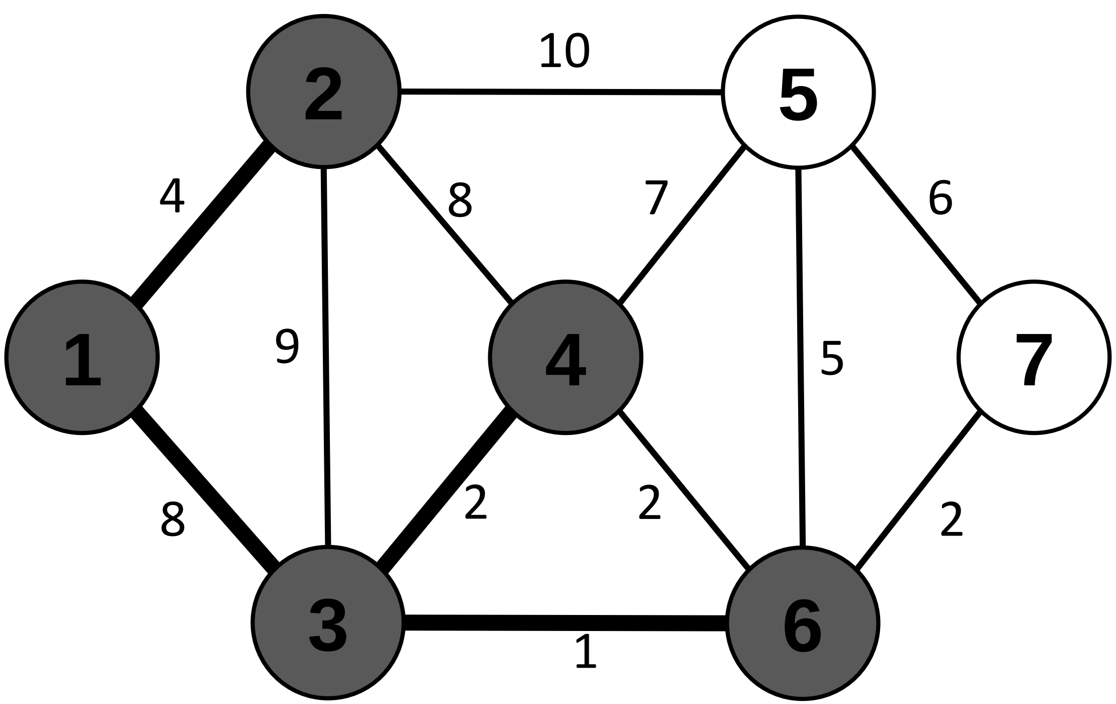
N	1	1	3	6	3	6

key

0	4	8	2	5	1	2

Q

7,2	5,5



Prim's Example

color

B	B	B	B	W	B	W

pred

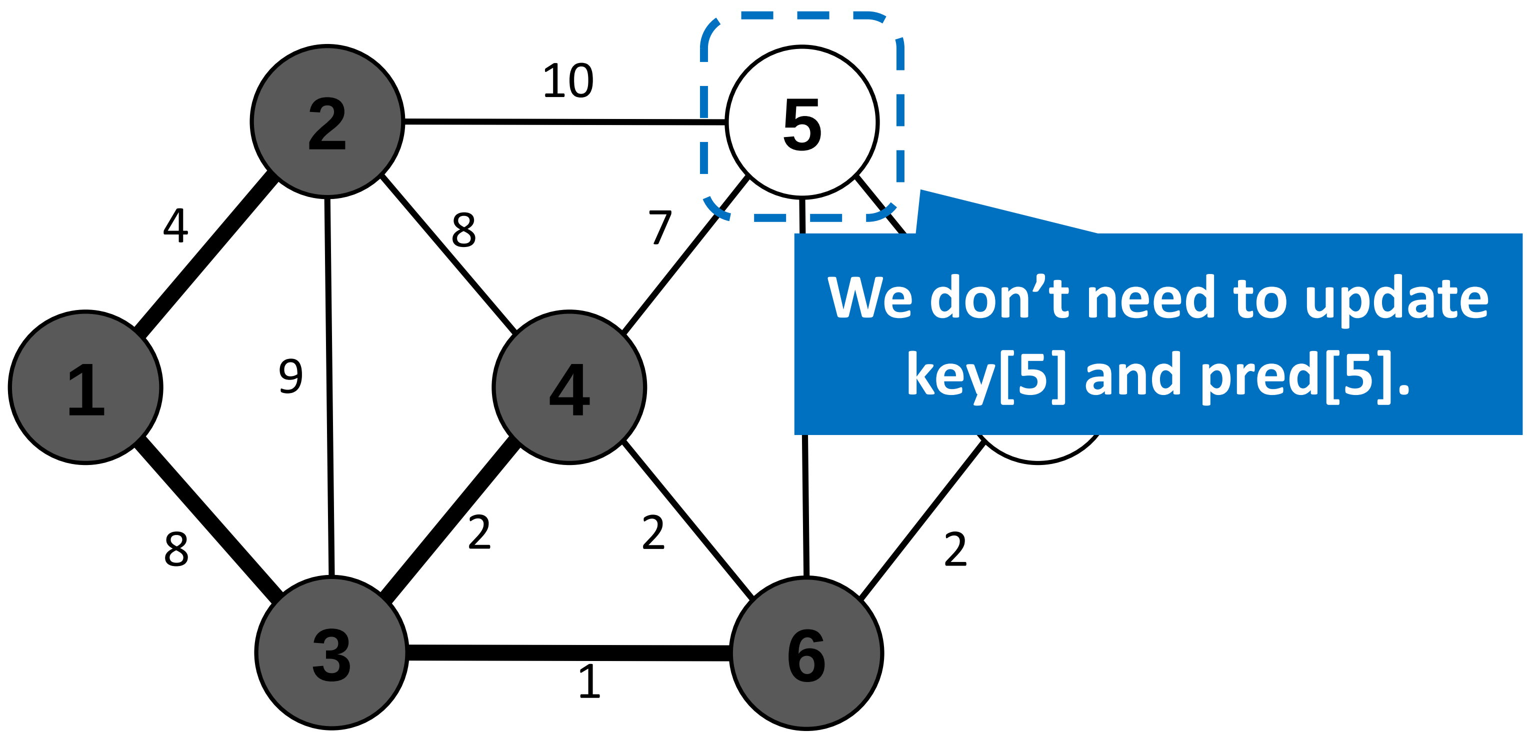
N	1	1	3	6	3	6

key

0	4	8	2	5	1	2

Q

7,2	5,5



Prim's Example

color

B	B	B	B	W	B	W

pred

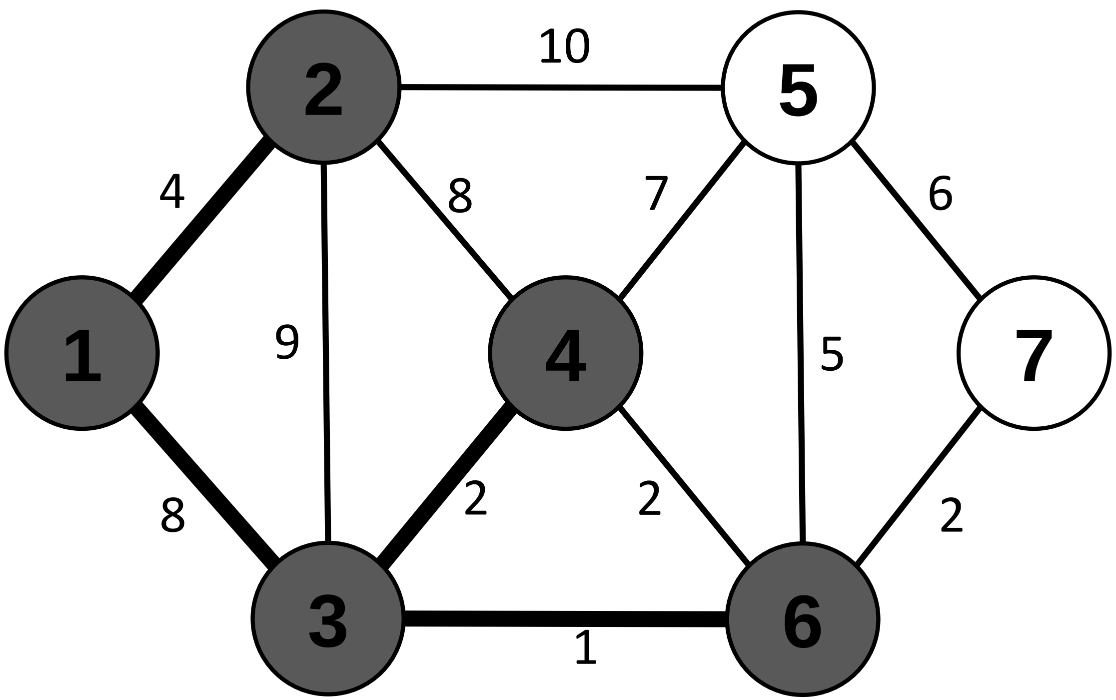
N	1	1	3	6	3	6

key

0	4	8	2	5	1	2

Q

7,2	5,5



Prim's Example

color

B	B	B	B	W	B	B

pred

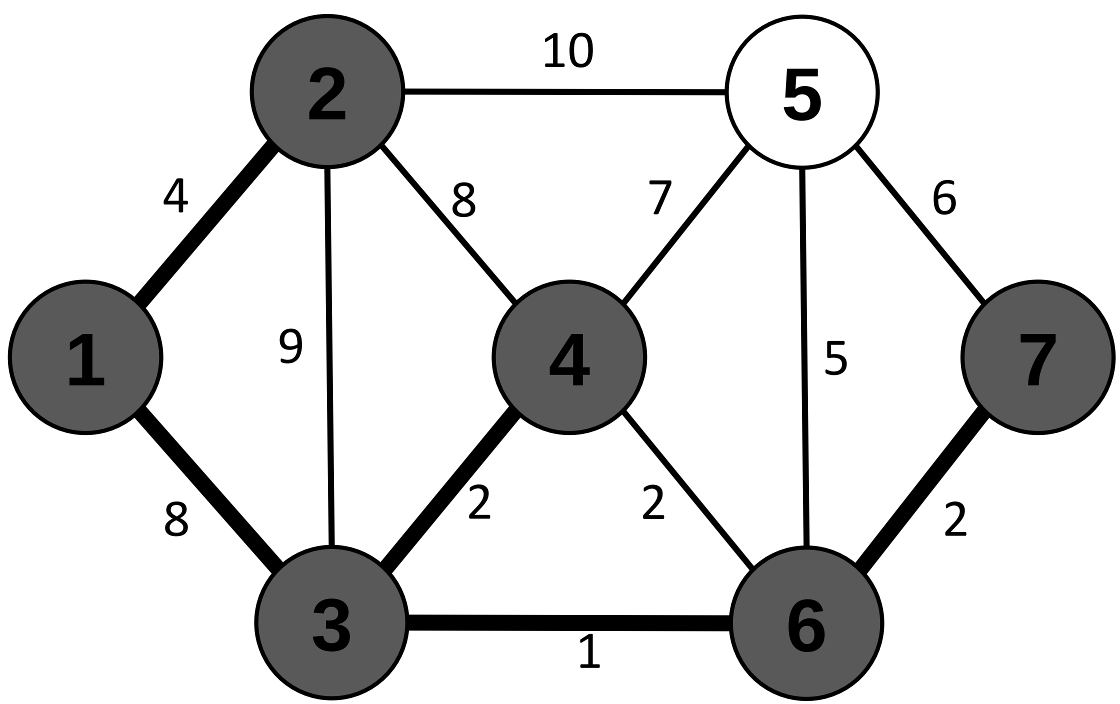
N	1	1	3	6	3	6

key

0	4	8	2	5	1	2

Q

5,5



Prim's Example

color

B	B	B	B	W	B	B

pred

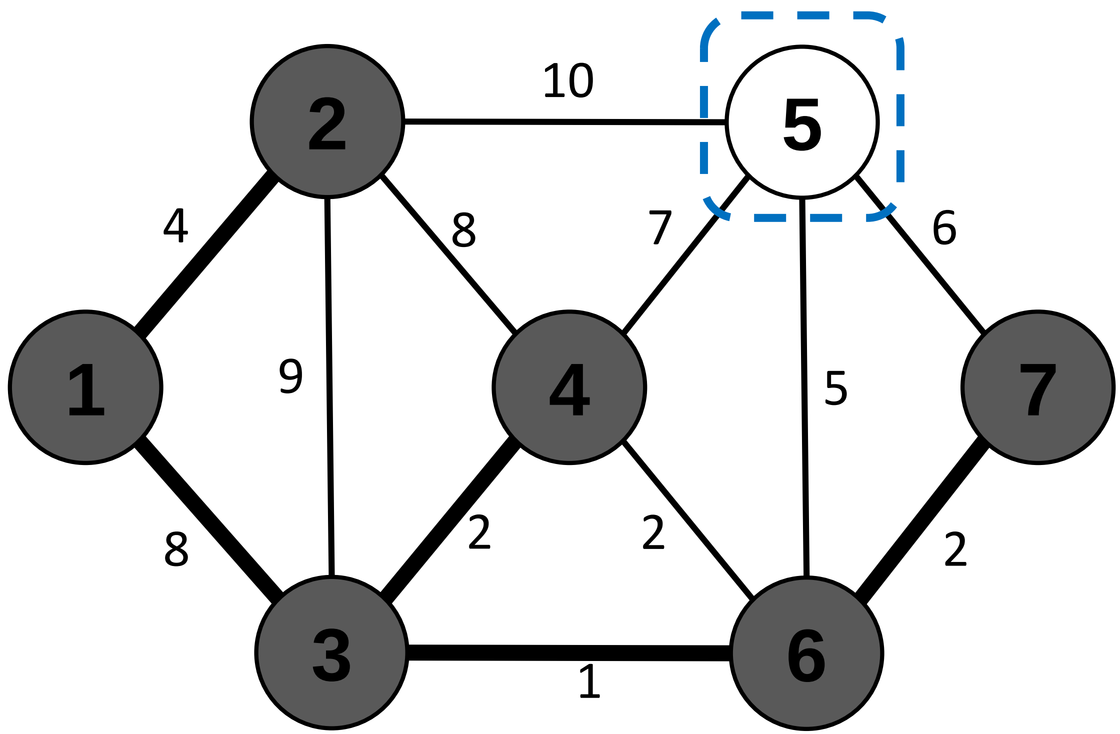
N	1	1	3	6	3	6

key

0	4	8	2	5	1	2

Q

5,5



Prim's Example

color

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pred

N	1	1	3	6	3	6

key

0	4	8	2	5	1	2

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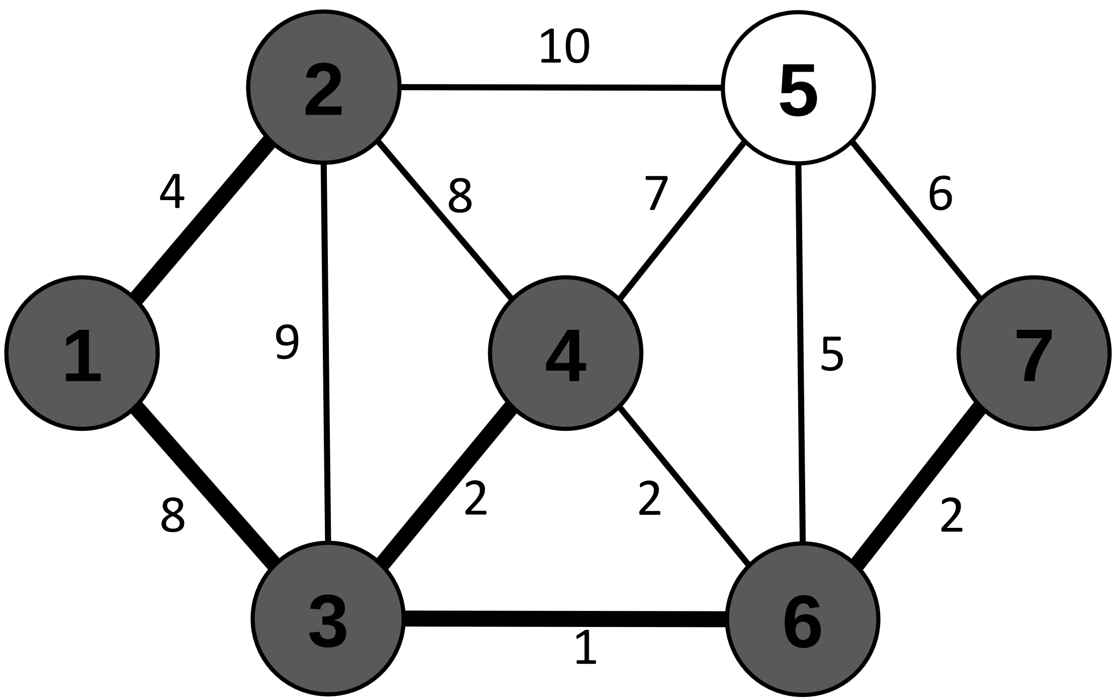


Prim's Example

color

B	**B**	**B**	**B**	**B**	**B**	**B**

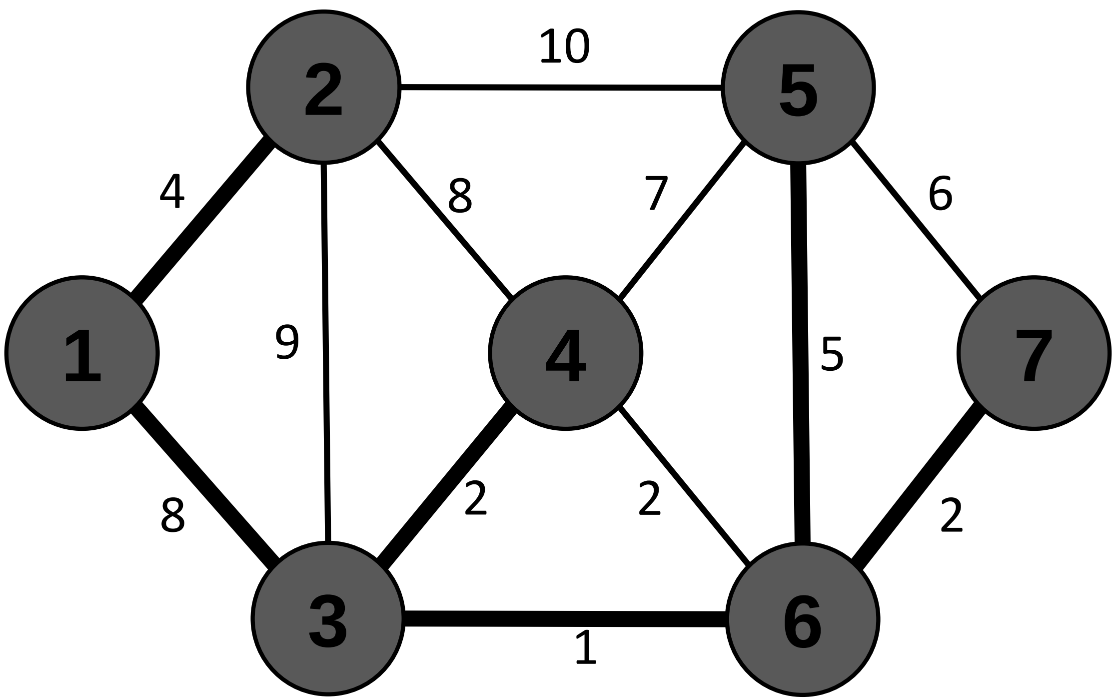
pred

N	**1**	**1**	**3**	**6**	**3**	**6**

key

0	4	8	2	5	1	2

Q



Prim's Example

color

B	**B**	**B**	**B**	**B**	**B**	**B**

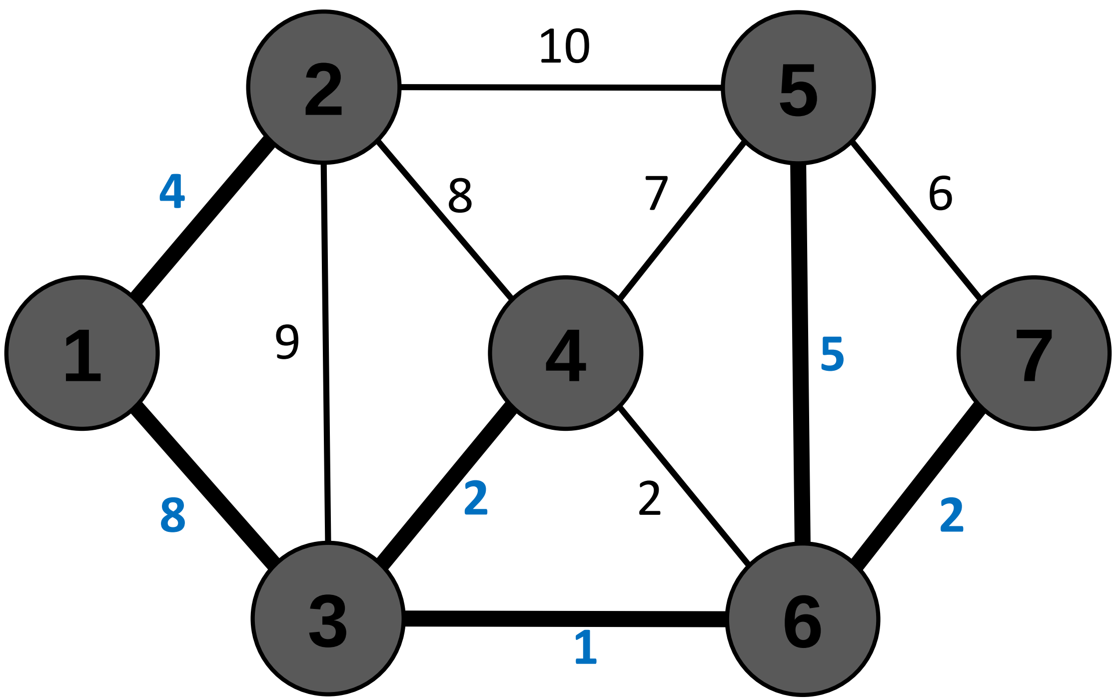
pred

N	**1**	**1**	**3**	**6**	**3**	**6**

key

0	**4**	**8**	**2**	**5**	**1**	**2**

Weight of MST = **22**



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  - The algorithm
  - Analysis for Prim's algorithm
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 |   |  if (color[v] ← WHITE)&&(w[u, v] < key[v]) then
 |   |   |  key[v] ← w[u, v];
 |
```

Analysis of Prim's Algorithm...

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end
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while Q *is nonempty* **do**
  $u \leftarrow Q$.Extract-Min();// $O(logV)$
  for $v \in adj[u]$ **do**
    if $(color[v] \leftarrow WHITE)\&\&(w[u,v] < key[v])$ **then**
      $key[v] \leftarrow w[u,v]$;
      Q.Decrease-Key($v, key[v]$);// $O(logV)$

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```
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|   |   |   key[v] ← w[u, v];
|   |   |   Q.Decrease-Key(v, key[v]); // O(logV)
|   |   |   pred[v] ← u;
|   |   end
|   end
|   color[u] ← BLACK;
end
```

Analysis of Prim's Algorithm

The data structure PriQueue (heap) supports the following two operations: (See CLRS)

- O(log V) for Extract-Min on a PriQueue of size at most V .

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- O(log V) time for Decrease-Key on a PriQueue of size at most E.
  Total cost: O(E log V).

Analysis of Prim's Algorithm

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  Total cost: O(V log V)
- O(log V) time for Decrease-Key on a PriQueue of size at most E.
  Total cost: O(E log V).

Total cost is then O((V + E) log V) = O(E log V)

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Recalling the Generic Algorithm

- Start with an empty graph.
- Try to add edges one at a time, always making sure that what is built remains acyclic.
- If we are sure at each step that the resulting graph is a subset of some minimum spanning tree, we are done.

Lemma

- *Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E*
- *Let A be a subset of E that is included in some minimum spanning tree for G.*

Let

- *$(S, V - S)$ be any cut of G that respects A*
- *(u, v) be a light edge crossing the cut $(S, V - S)$*

Then, edge (u, v) is safe for A.

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- Continue until the forest 'merges into' a single tree.

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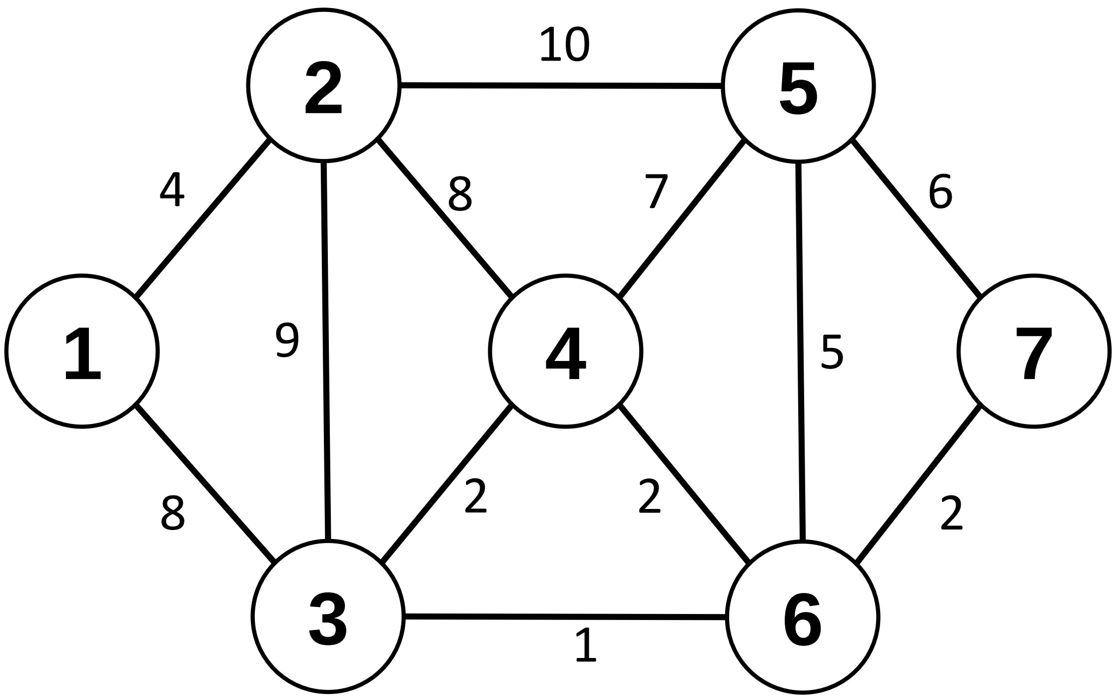

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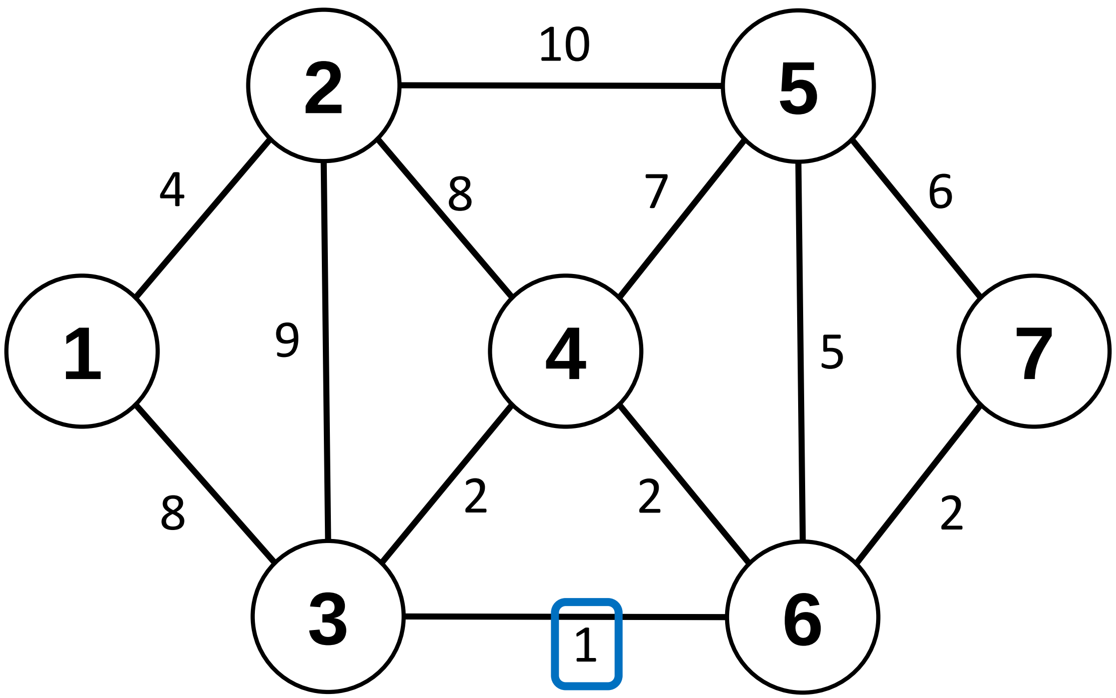

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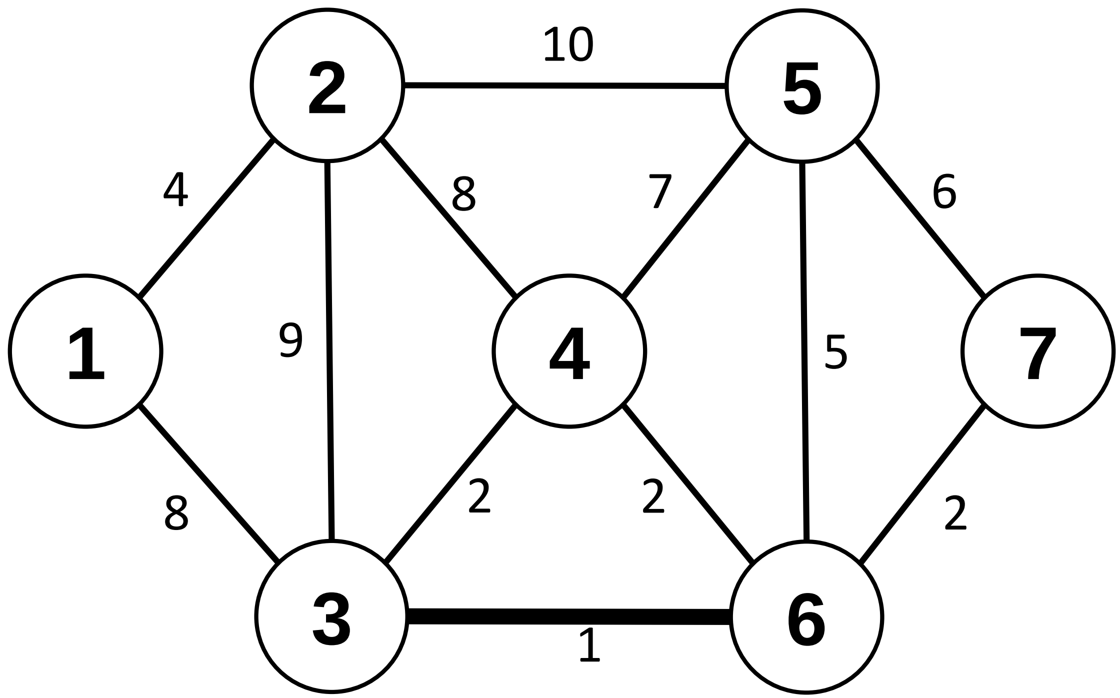

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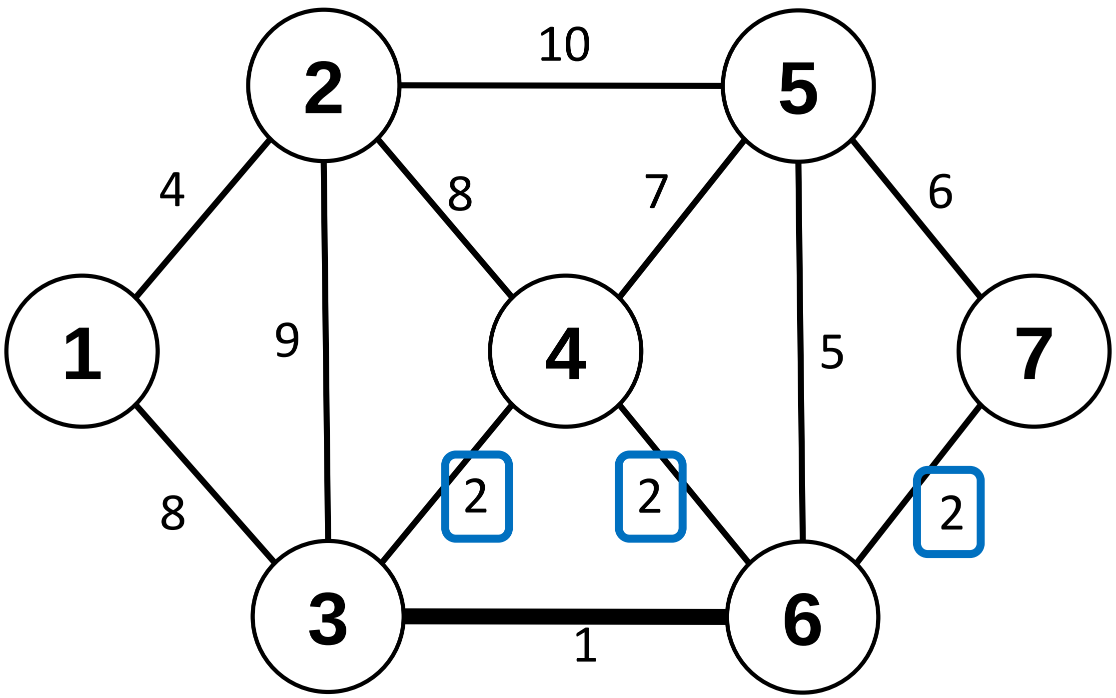

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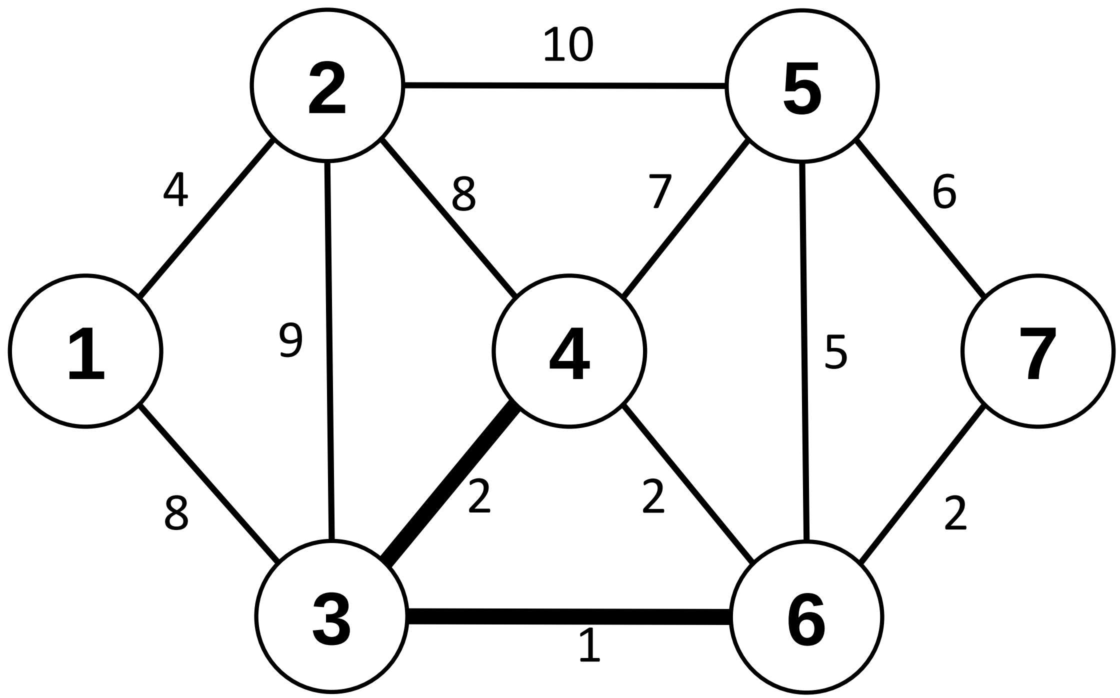

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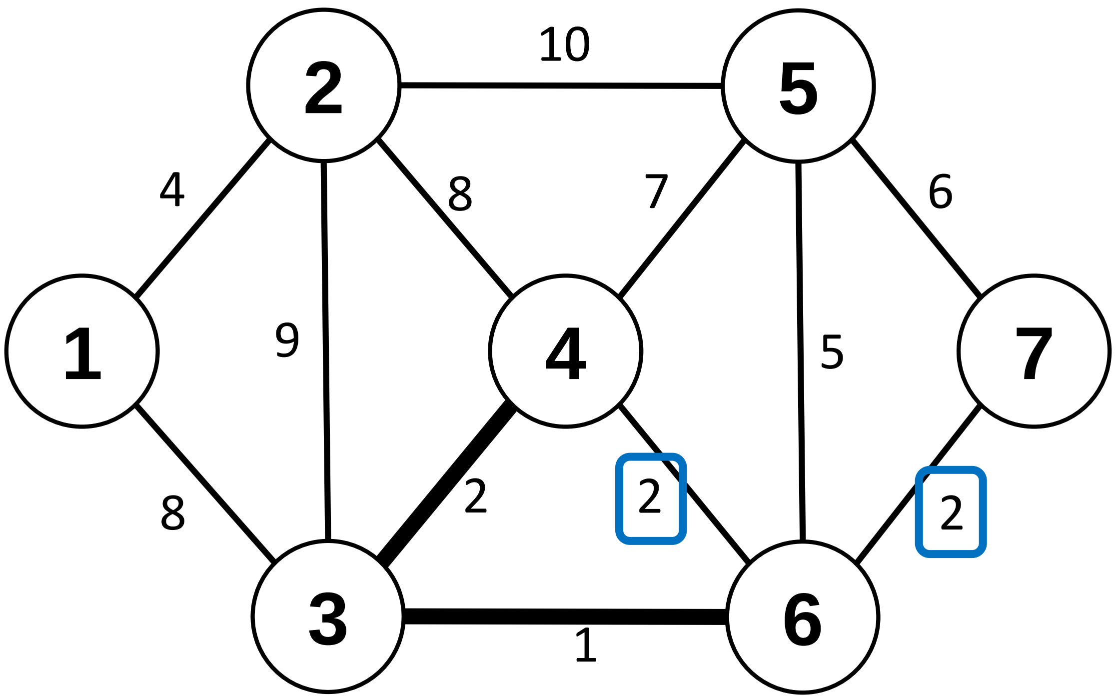

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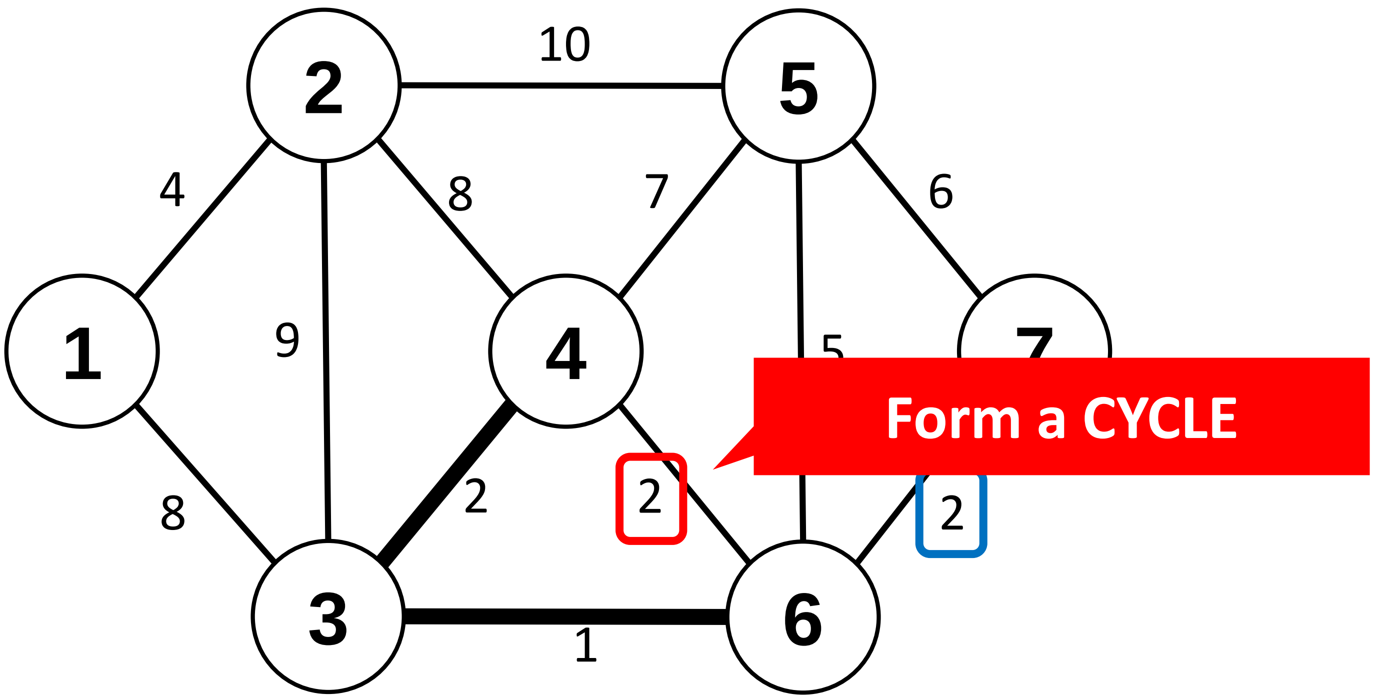

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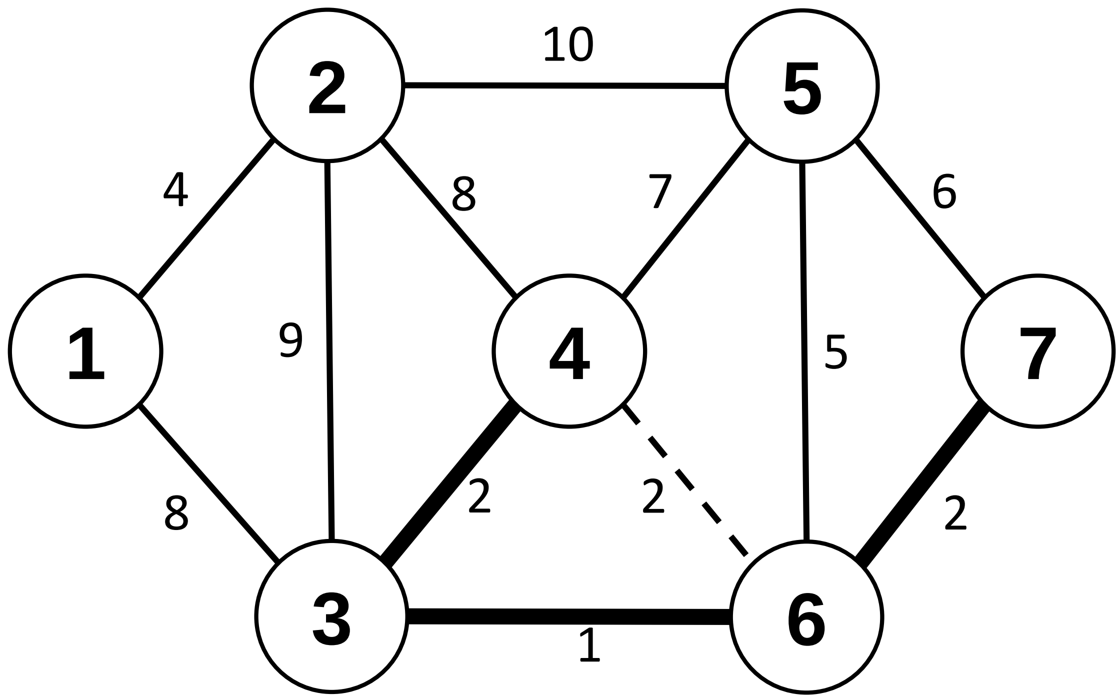

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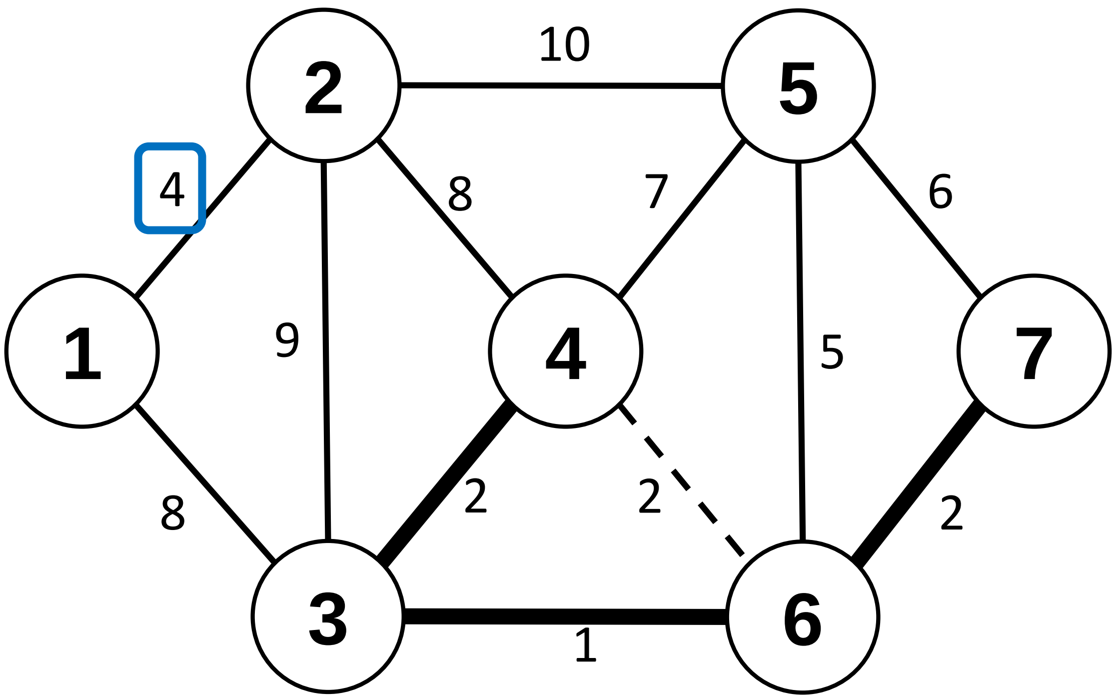

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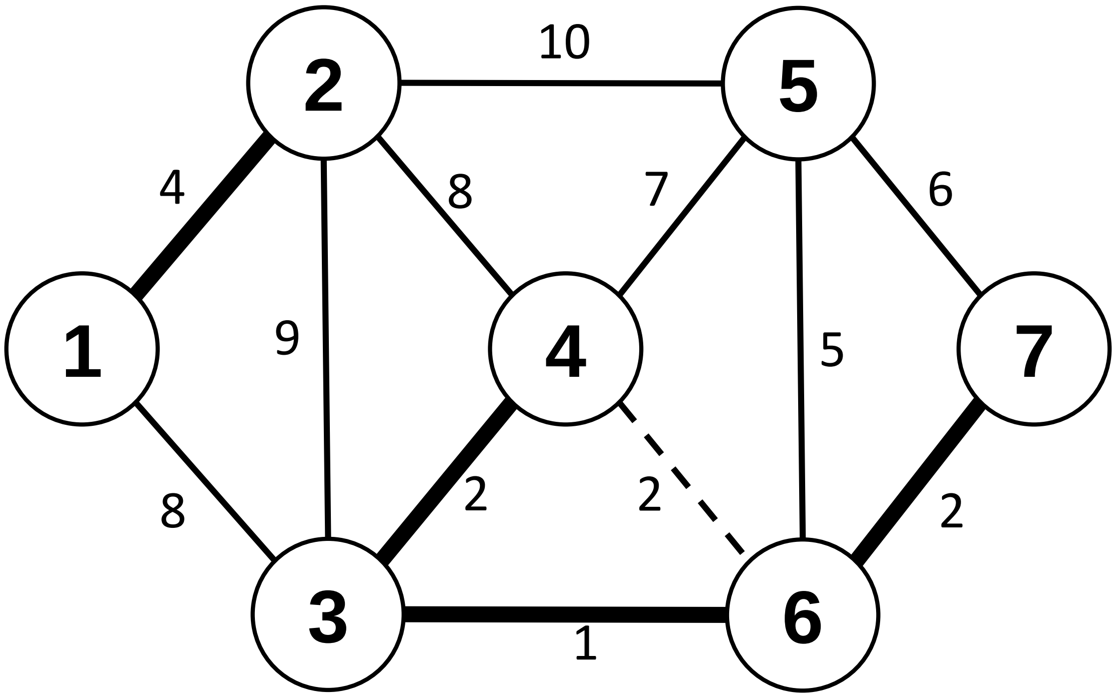

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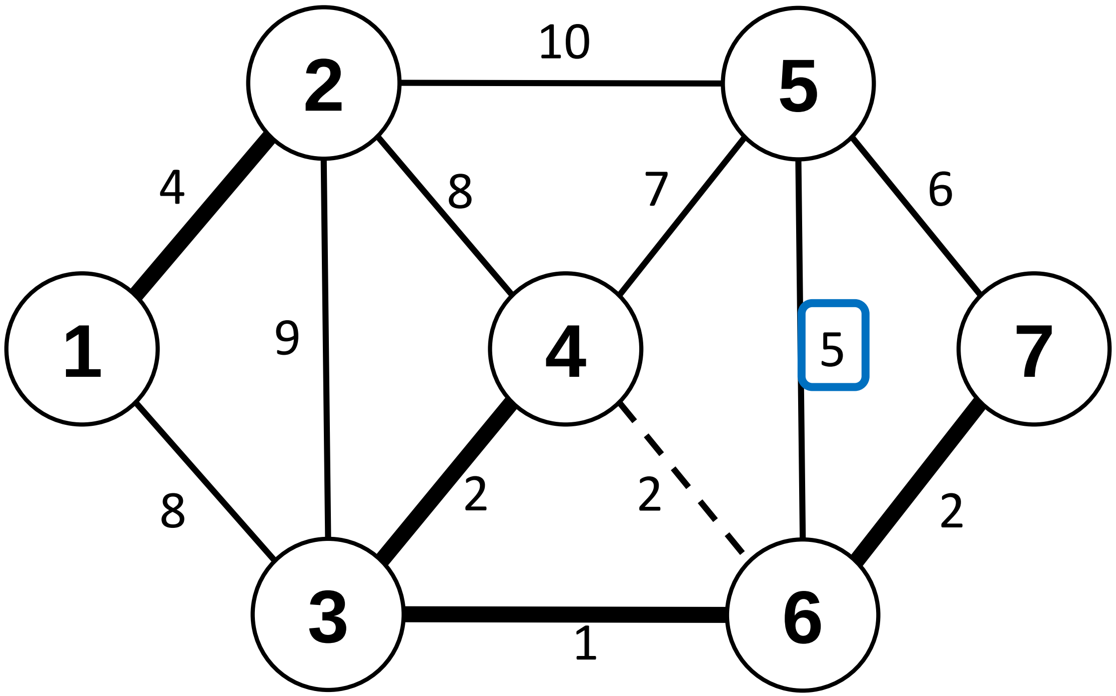

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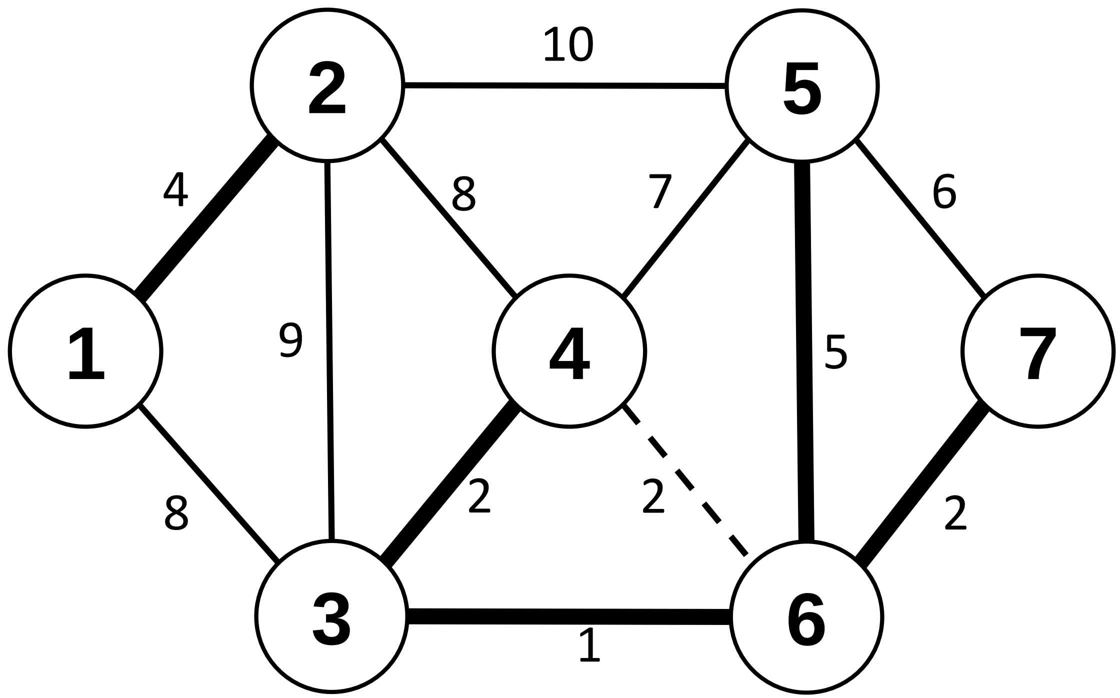

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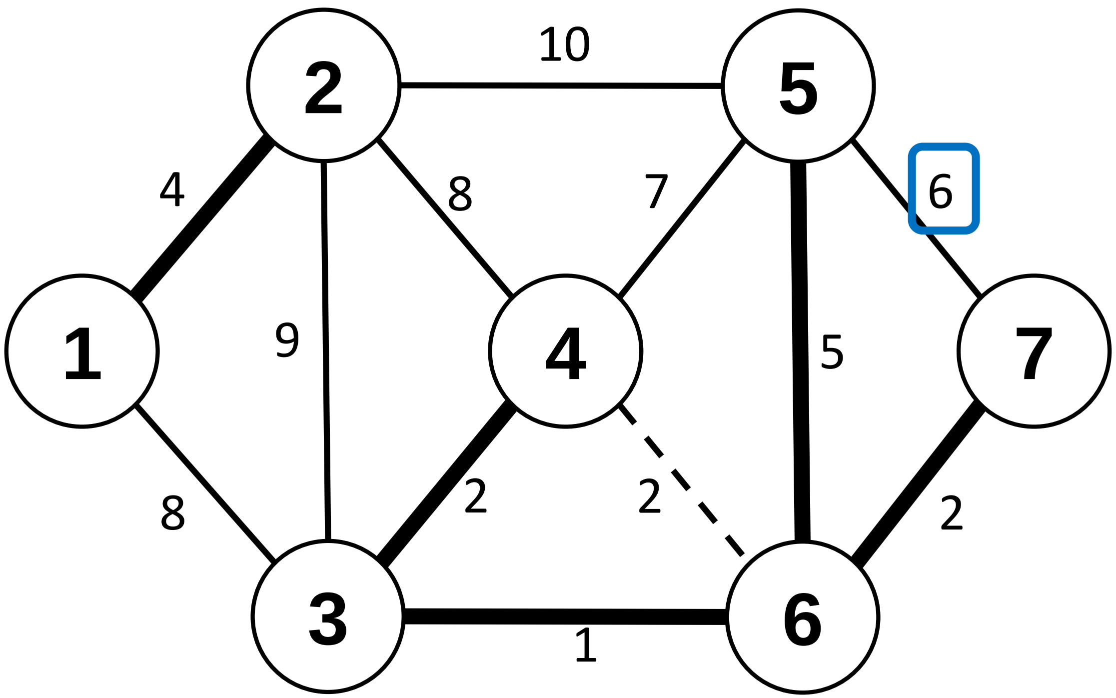

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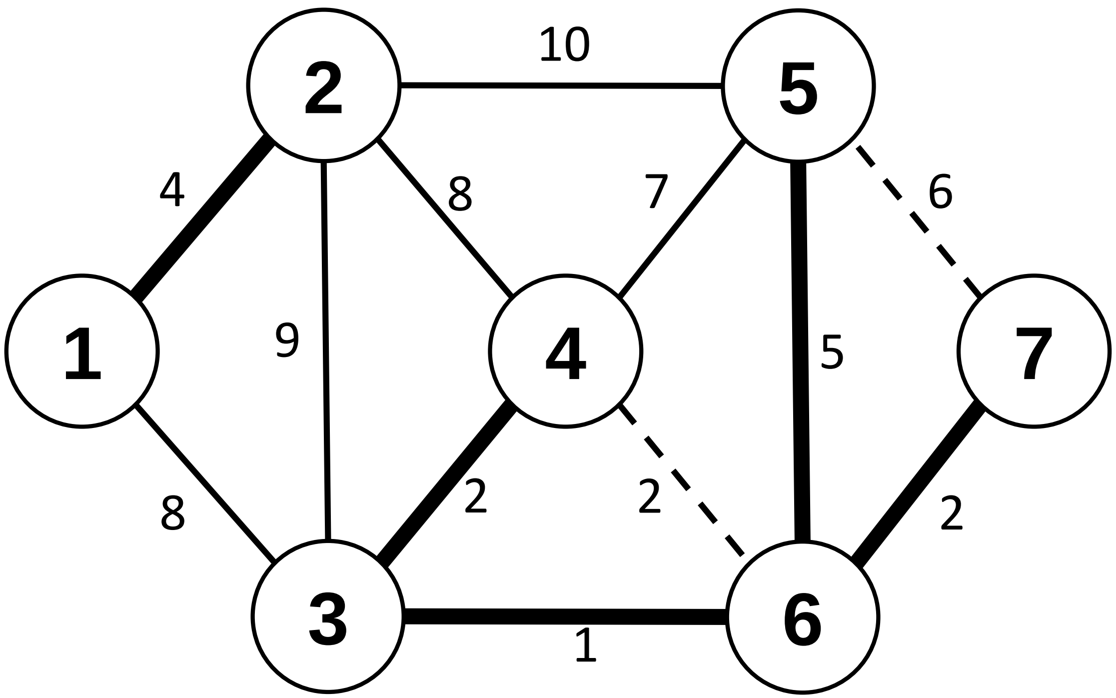

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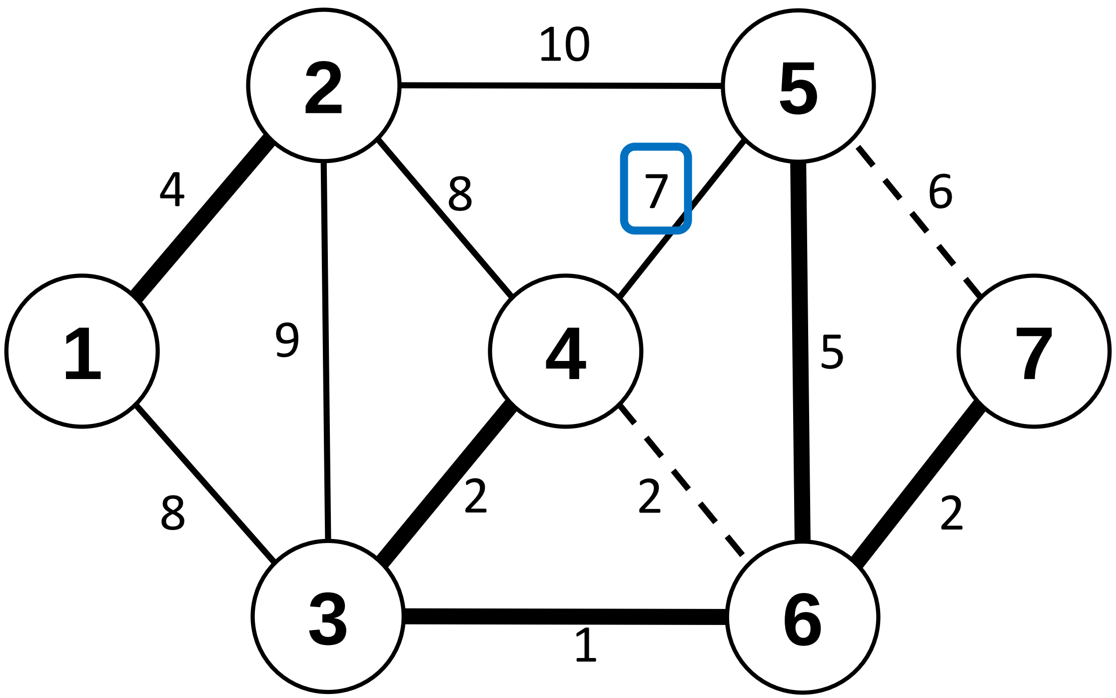

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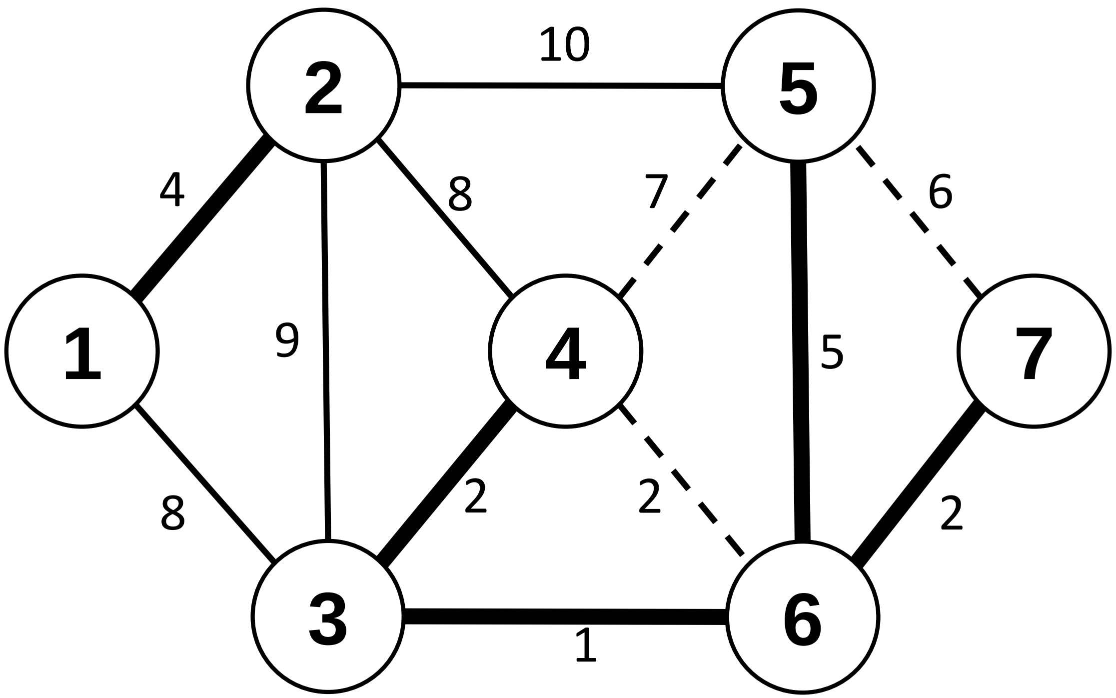

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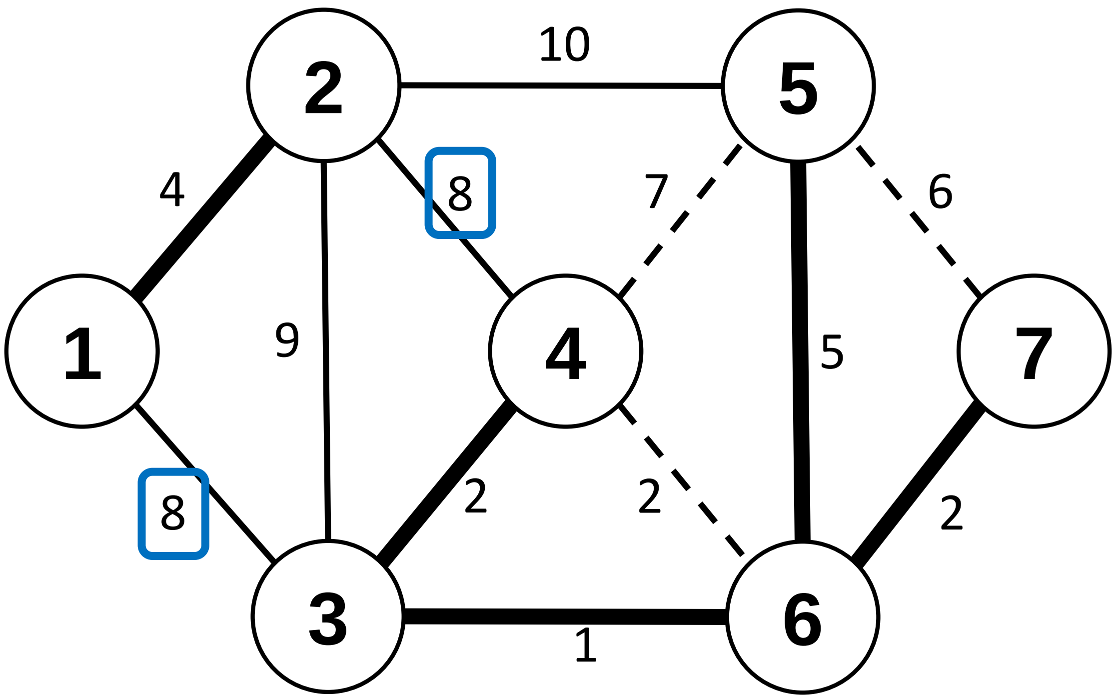

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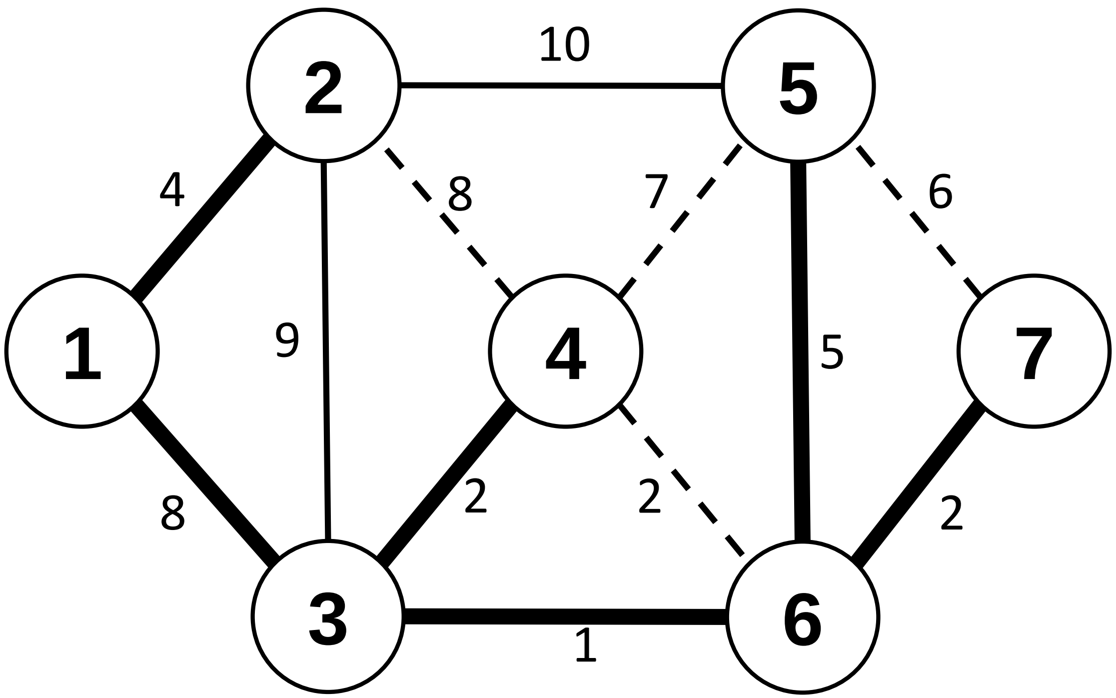

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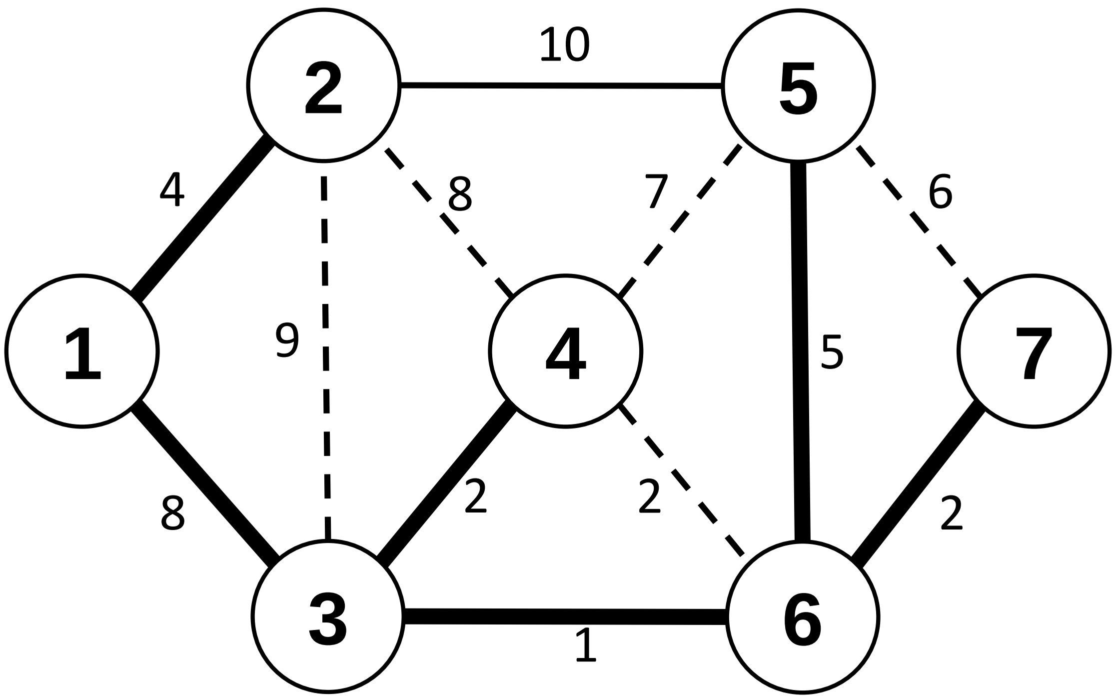

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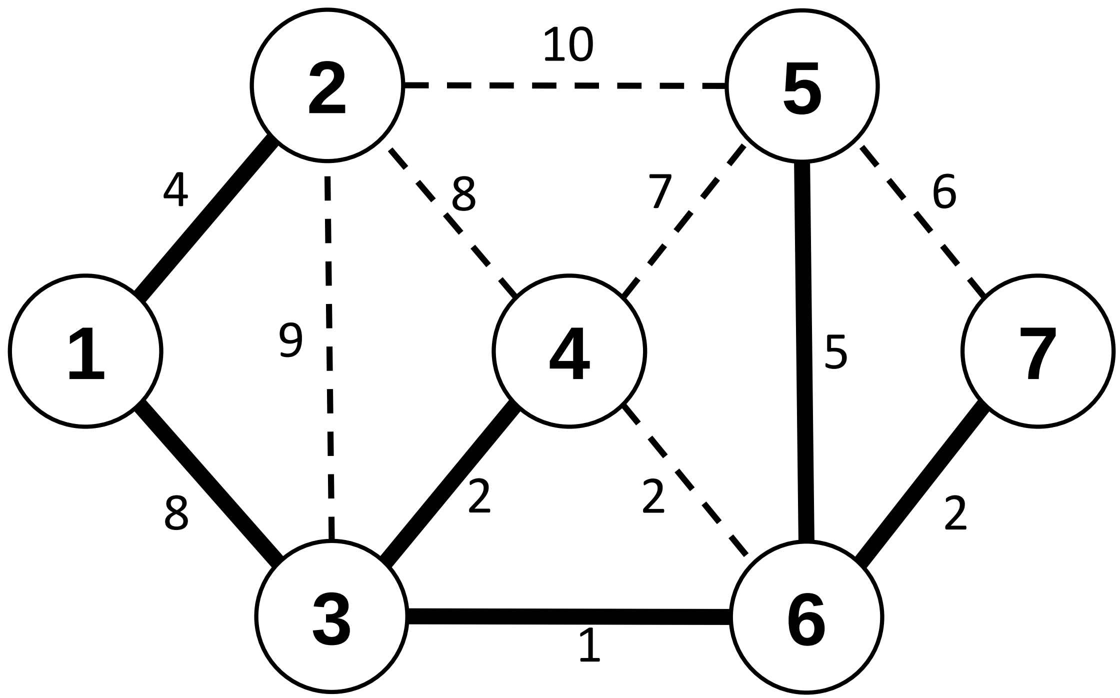

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- How does algorithm check whether adding e to A creates a cycle?

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- (Once edge e causes a cycle it will always cause a cycle, so it can be thrown away.)

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Low-Level Answer:

- The Union-Find data structure implements this

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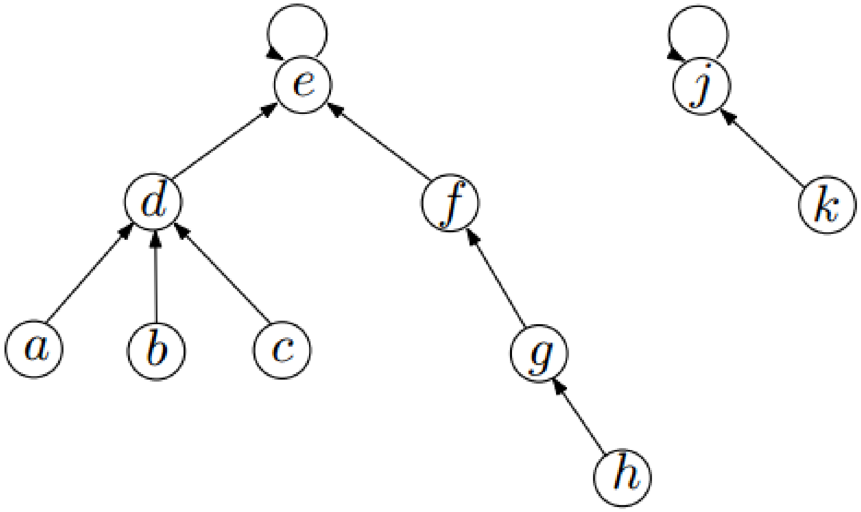
For now we treat Union-Find as a black box. We will present its implementation.

Up-Tree Implementation



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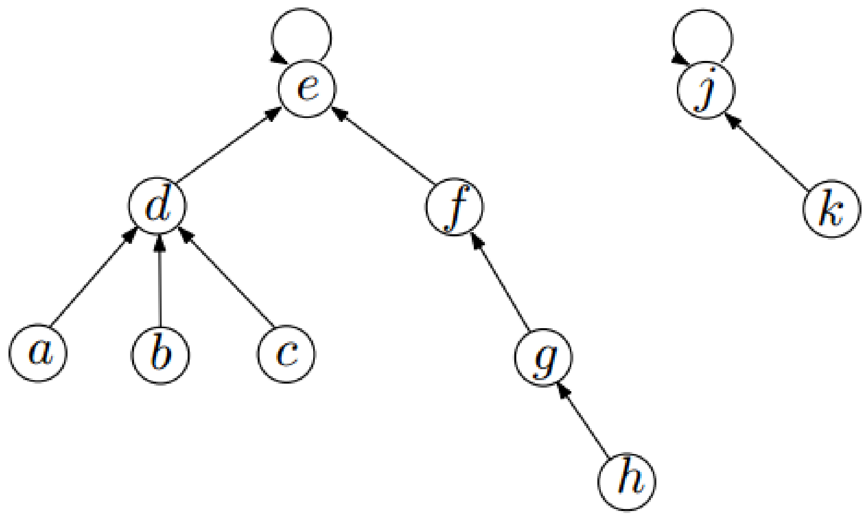

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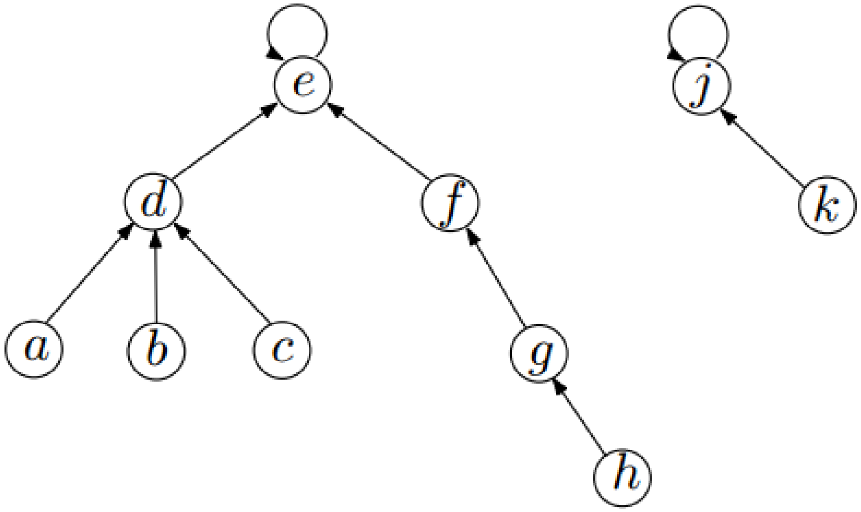

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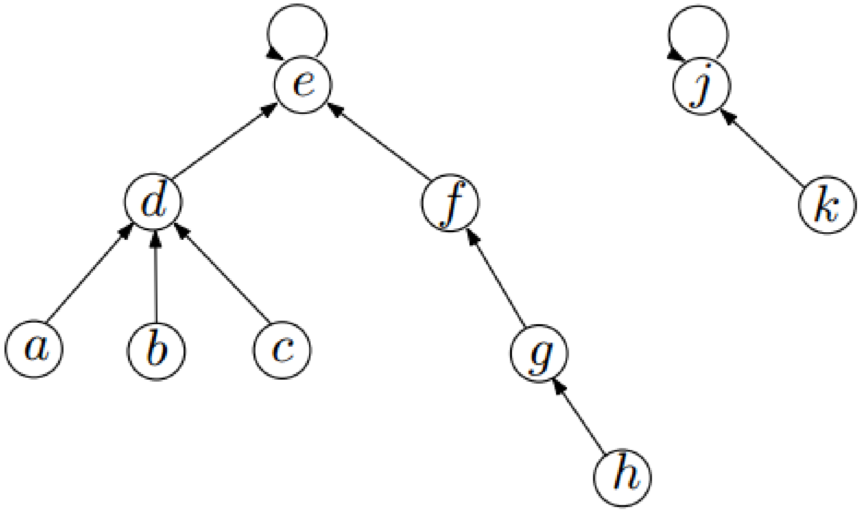

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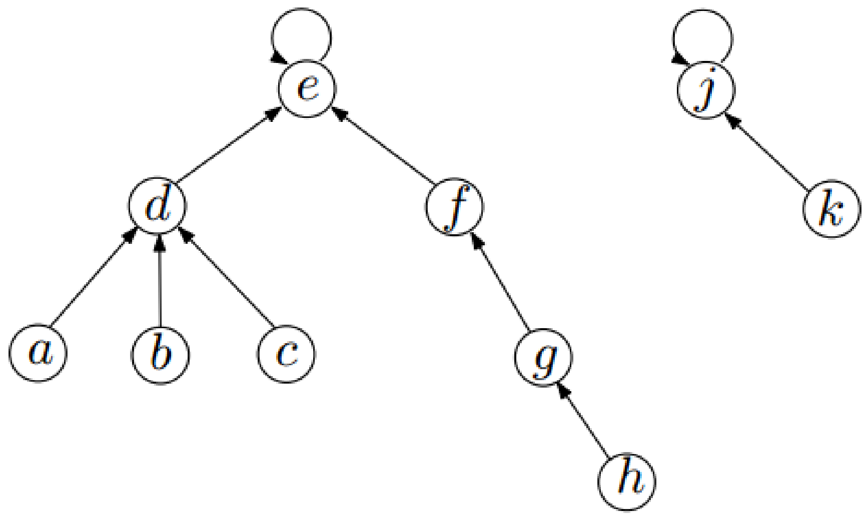

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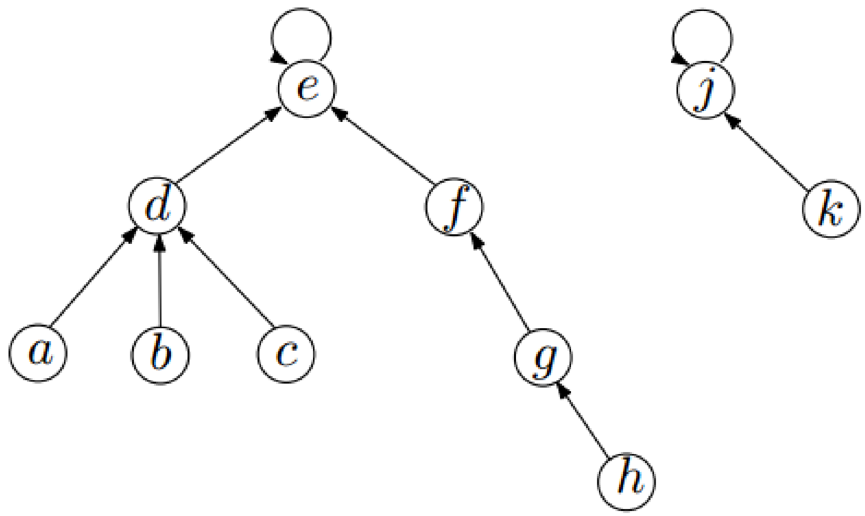

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while $x \neq x.parent$ **do**
$\quad x \leftarrow x$.parent;
end

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Naive solution:

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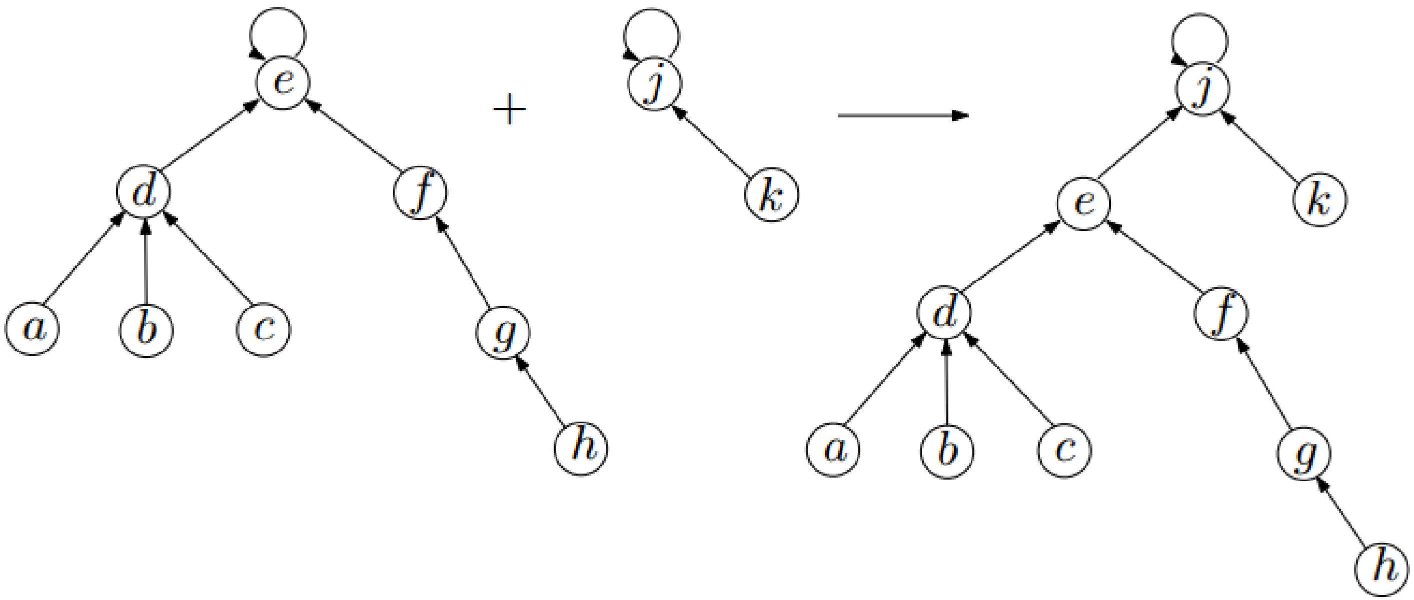

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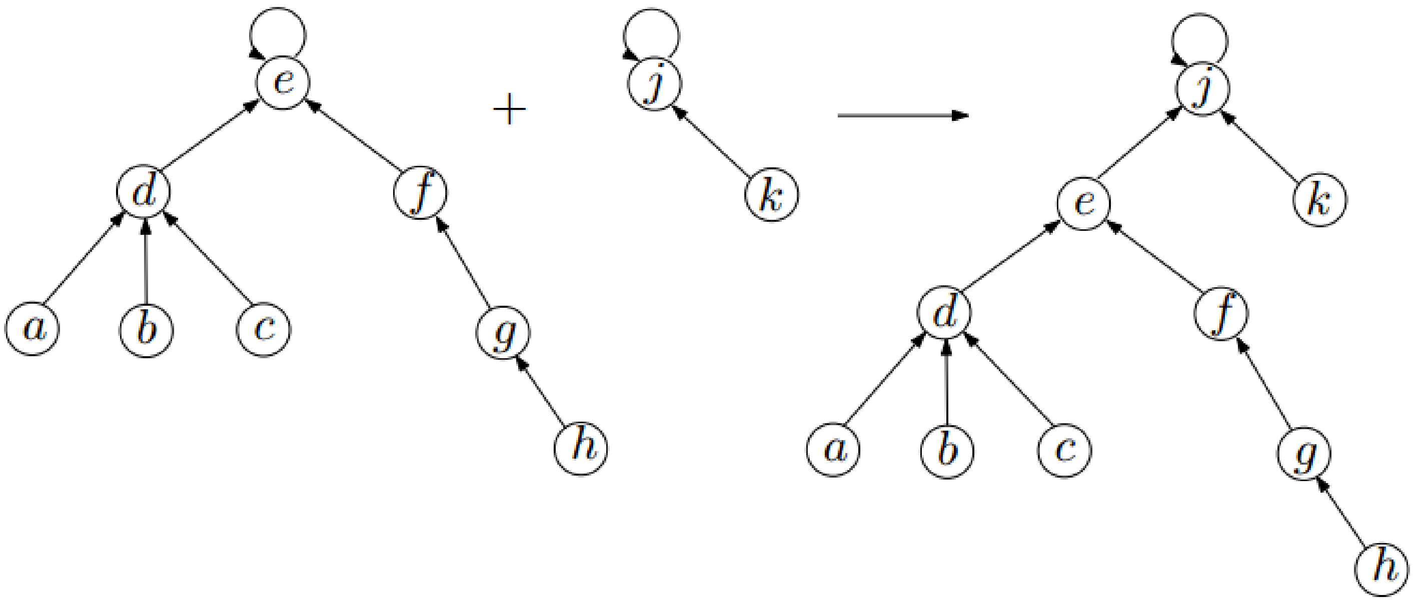

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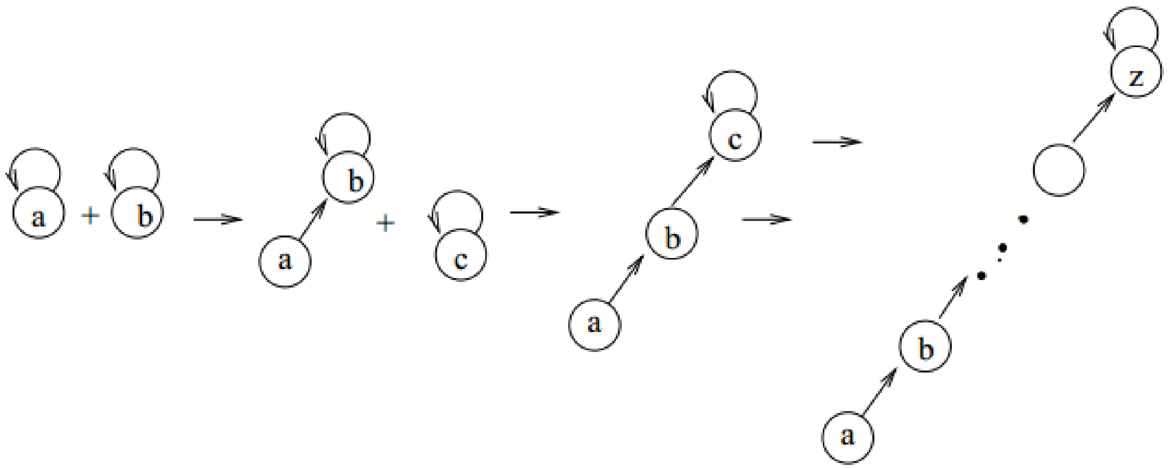

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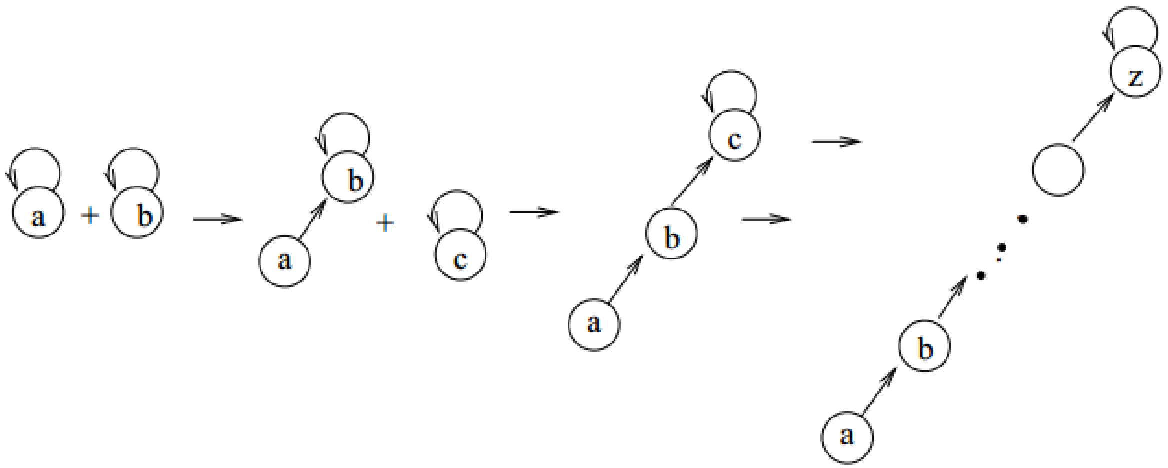

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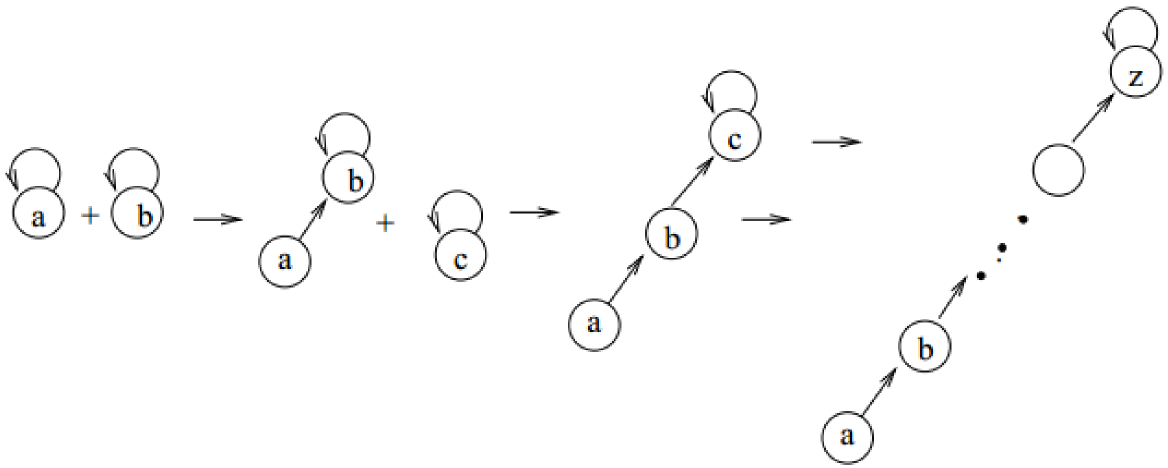

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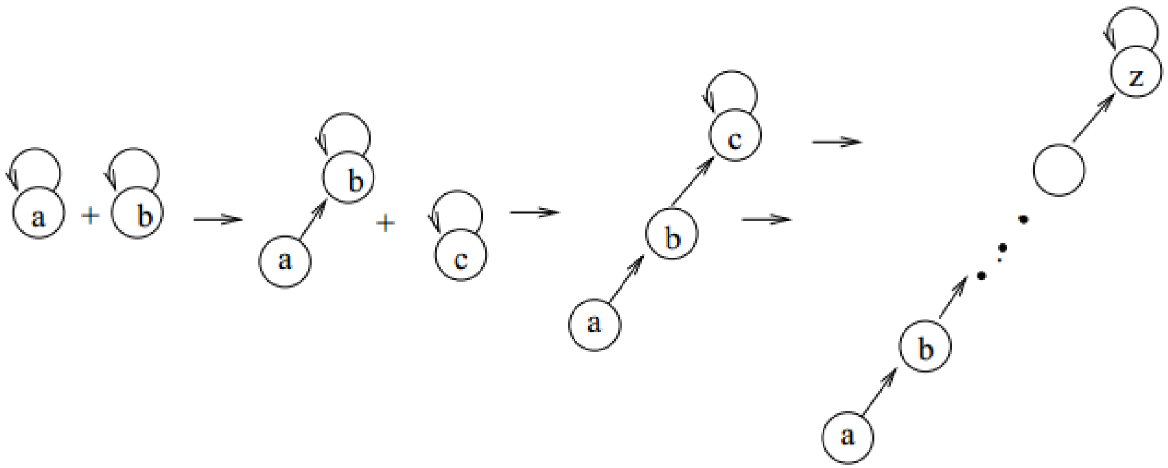

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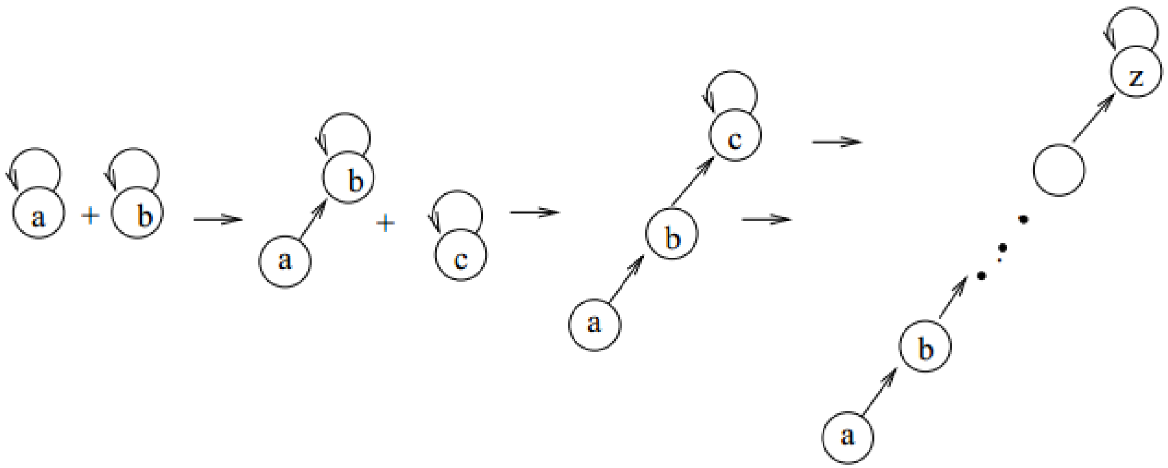

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- when we union two trees together, we always make the root of the taller tree the parent of shorter tree.

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- In case two trees have the same height, we choose the root of the first tree point to the root of the second. And the tree height is increased by 1.

Union(x,y)

Input: Two elements $\boldsymbol{x}, \boldsymbol{y} \in U$
Output: None
$a \leftarrow$Find-Set(x);
$b \leftarrow$Find-Set(y);
if *a.height* $\leq$ *b.height* **then**
    if *a.height is equal to b.height* **then**
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Hence we have Find-Set(x) $= O(\log n)$. $\square$

Outline

- Review to Part IV
- Minimum Spanning Trees
  - Spanning trees
  - Minimum spanning trees
- Prim’s algorithm
  - The idea
  - The algorithm
  - Analysis for Prim’s algorithm
- Kruskal’s algorithm
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  - The Disjoint Set Union-Find data structure
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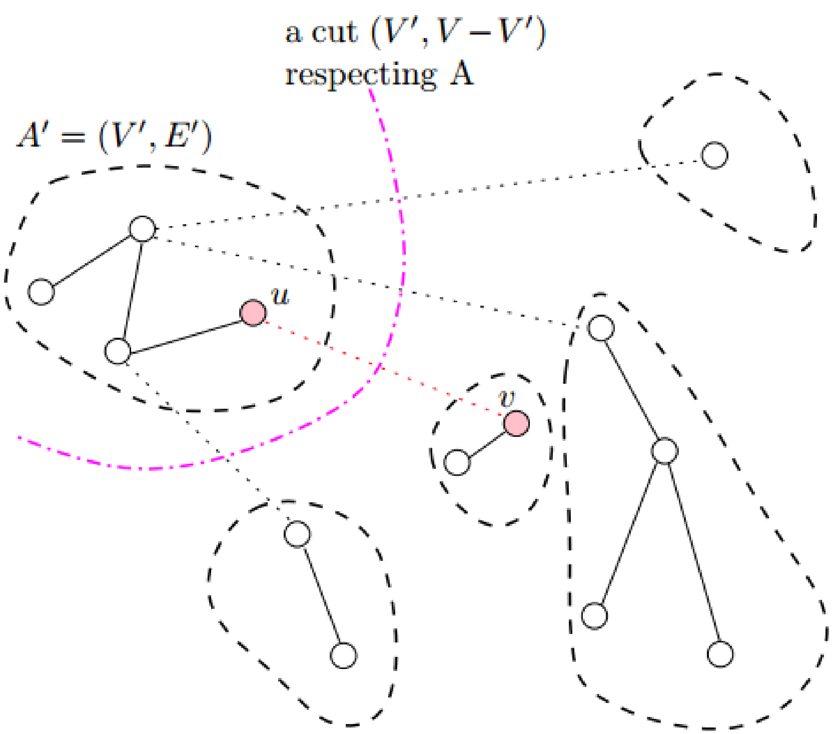

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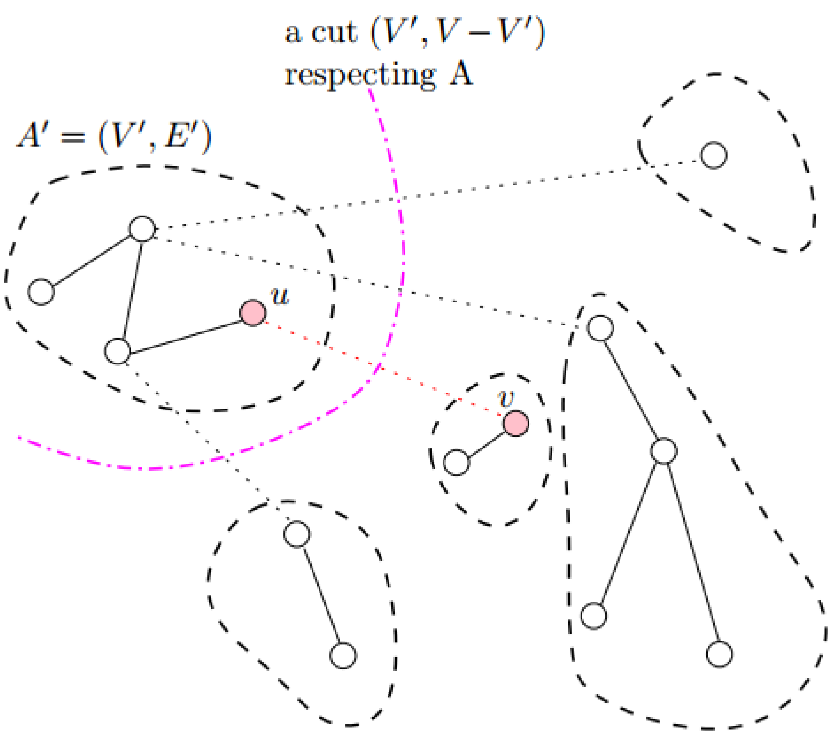

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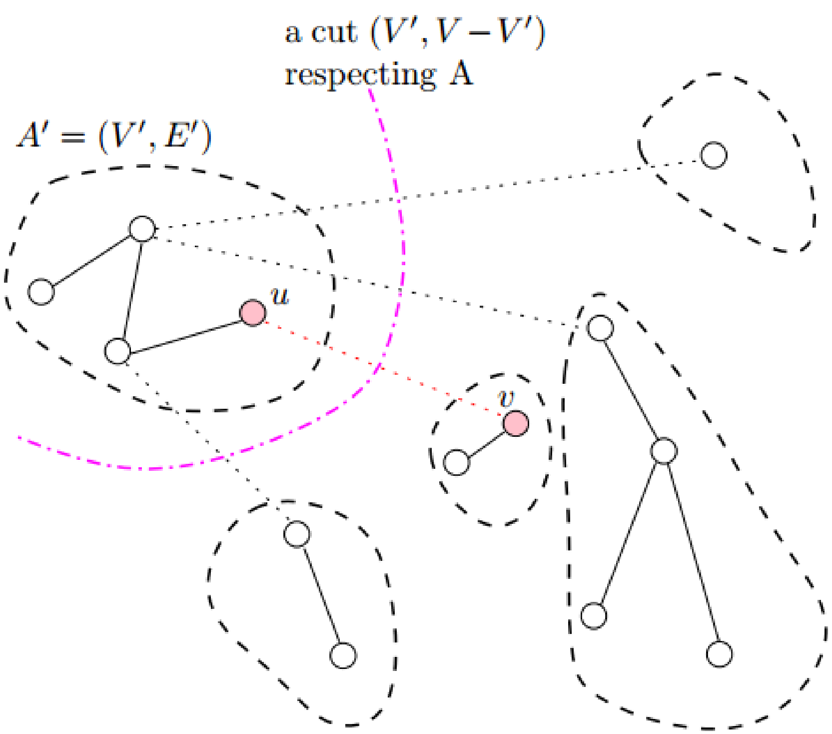

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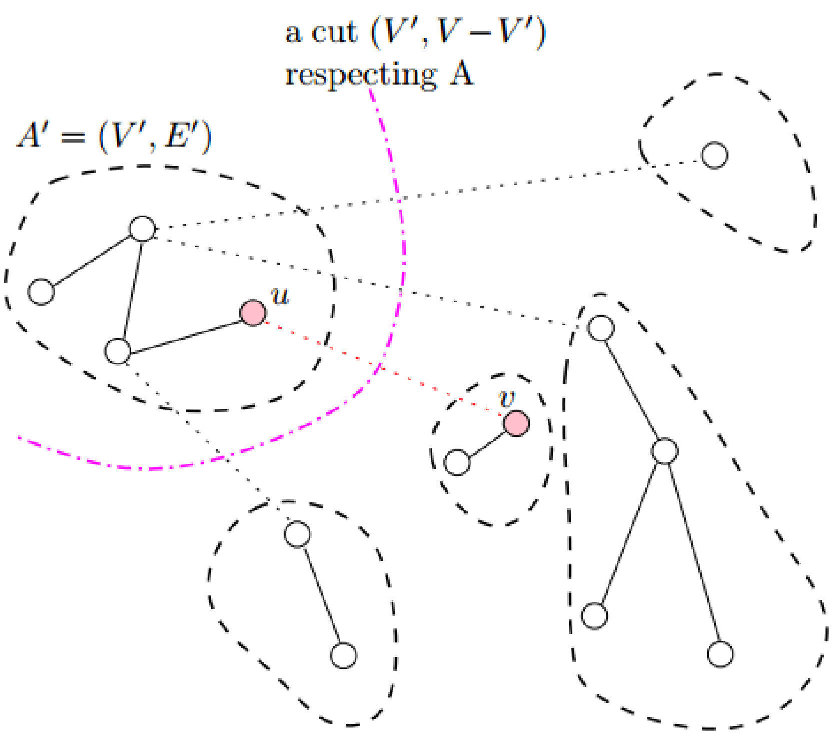

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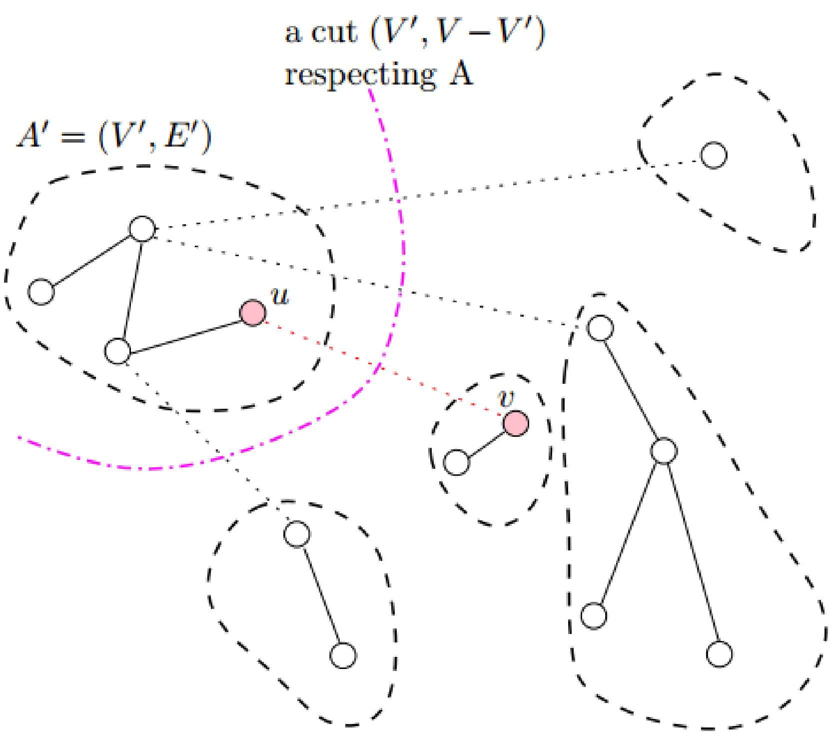

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- Moreover, since (u, v) is currently the smallest edge, (u, v) is the light edge crossing the cut.

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A ← {};
for u ∈ V do
 | Create-Set(u);// O(|V|)
end
for e_i ∈ E do
 | // O(|E| log |V|)
 | if Find-Set(u_i)≠Find-Set(v_i) then
 | | add {u_i, v_i} to A;
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 | end
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Remark: With a proper implementation of Union-Find, Kruskal's algorithm has running time $O(|E|\log|E|) = O(|E|\log|V|)$.

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  if *Find-Set*$(u_i)\neq$*Find-Set*(v_i) **then**

    add $\{u_i, v_i\}$ to A;

    Union(u_i, v_i);

  end

end

return A;

$\log|E| \leq \log|V|^2 = 2\log|V| = O(\log|V|)$

Remark: With a proper implementation of Union-Find, Kruskal's algorithm has running time $O(|E|\log|E|) = O(|E|\log|V|)$.

Summary

- Prim's algorithm always grows one tree.
- Kruskal's algorithm grows a collection of trees, namely a forest.
- Both Prim's algorithm and Kruskal's algorithm take O($|E|\log|V|$) time, but they adopt different data structures.


thank you
danke
gracias
merci
obrigado
謝謝
teşekkür ederim
ngiyabonga
go raibh maith agat
mochchakkeram
dziękuje
bedankt
sukriya
kop khun krap
terima kasih
tapadh leat
dank je
arigatô
dakujem
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