Design and Analysis of Algorithms
Part IV: Graph Algorithms
Lecture 12: Single-Source Shortest Paths Problem

Yongxin Tong (童咏昕)

School of CSE, Beihang University

yxtong@buaa.edu.cn

Outline

- Review to Part IV
- Single-Source Shortest Paths Problem
- Dijkstra’s Algorithm
  - The idea
  - The algorithm
  - Analysis of Dijkstra’s algorithm
- The Bellman-Ford Algorithm
  - The algorithm
  - Analysis of Bellman-Ford algorithm

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Introduction to Part IV

- In Part IV, we will illustrate several graph algorithm problems using several examples:
  - Basic Concepts of Graphs (图的基本概念)
  - Breadth-First Search [BFS] (广度优先搜索)
  - Depth-First Search [DFS] (深度优先搜索)
  - Topological Sort (拓扑排序)
  - Strongly Connected Components (强联通分量)
  - Minimum Spanning Trees (最小生成树)
  - Single-Source Shortest Paths (单源最短路径)
  - All-Pairs Shortest Paths (所有结点对的最短路径)
  - Maximum/Network Flows (最大流/网络流)

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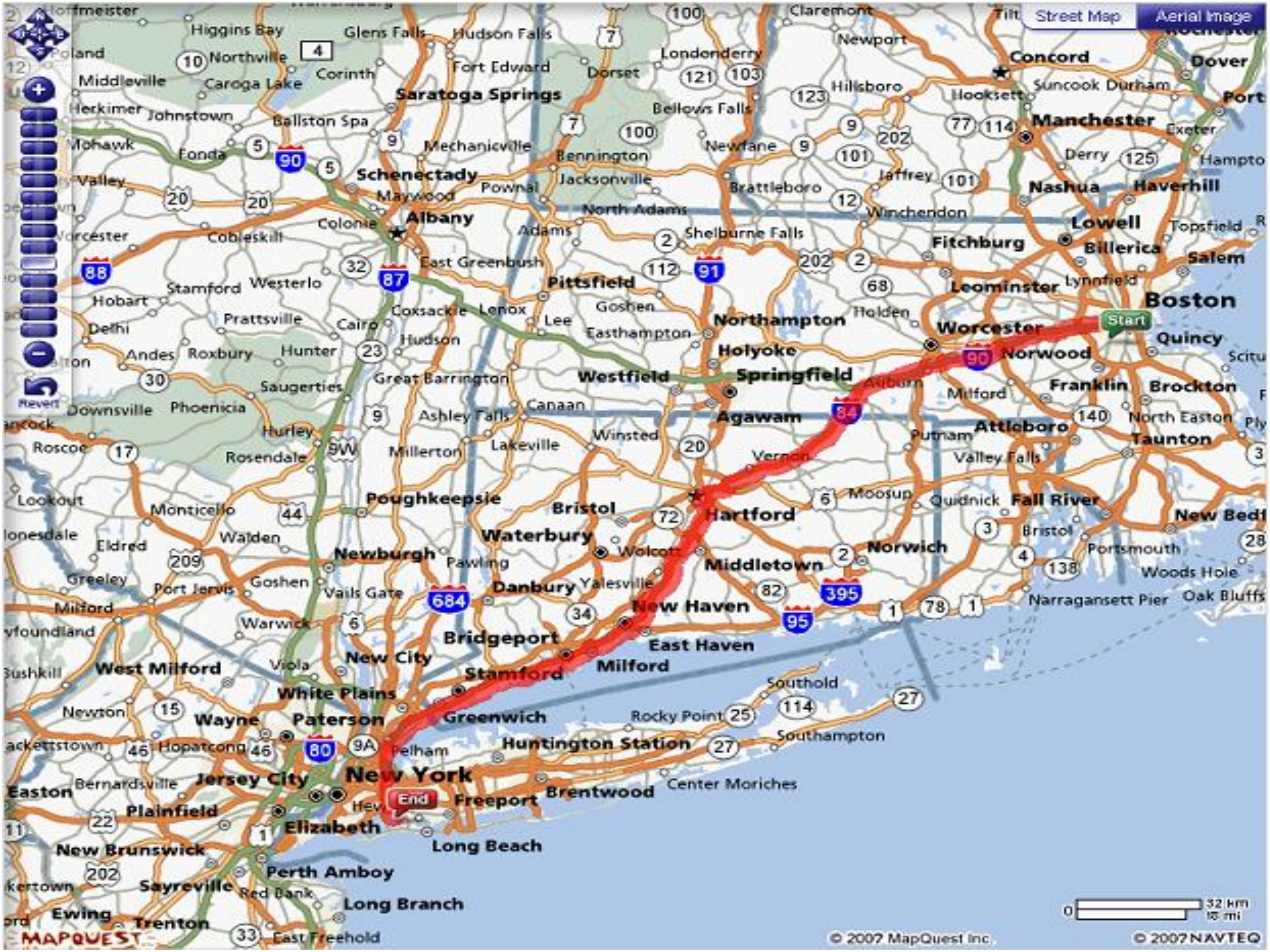
Single-Source Shortest Paths Problem



Shortest Path Problem for Weighted Graphs

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Definition

The length of a path $p = \langle v_0, v_1, \ldots, v_k \rangle$ is the sum of the weights of its constituent edges:

$$\text{length}(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i).$$

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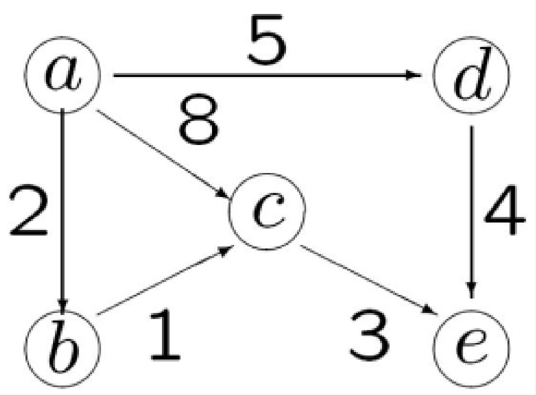

- $\text{length}(\langle a, b, c, e \rangle) =$

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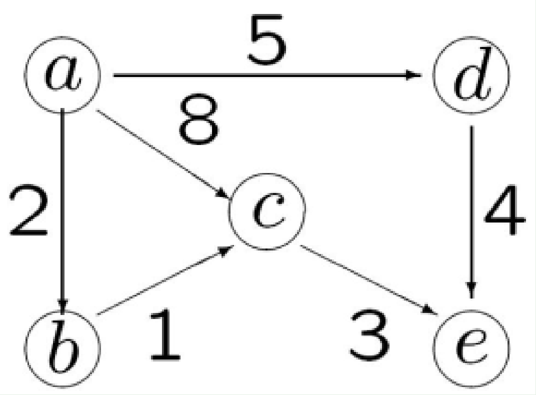

- length($\langle a, b, c, e\rangle$) $= 6$; distance from a to e is 6

Single-Source Shortest-Paths Problem

- Definition: Single-source shortest-paths problem
  - Given a digraph $G = (V, E)$ with no-negative edge weights w and a designated source vertex $s \in V$, determine the distance and a shortest path from the source vertex to every vertex in the digraph.

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Question

How do you design an efficient algorithm for this problem?

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The Rough Idea of Dijkstra’s Algorithm

- Maintain d[v] and S:
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- Initially
  - $S = \emptyset$
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- One by one we select vertices from $V \setminus S$ to add to S.
- Questions to answer at each step:
  - Which vertex do we select?
  - How do we update the distance upper bounds after a vertex is added to S?

Selection of Vertex

Question

How does the algorithm select which vertex among the vertices of $V \setminus S$ to process next?

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- For each vertex in u ∈ V \ S, we have computed a distance upper bound d[u].
- The next vertex processed is always a vertex u ∈ V \ S for which d[u] is minimum
  - that is, we take the unprocessed vertex that is closest (by our estimate) to s.

Updating Distance Estimates

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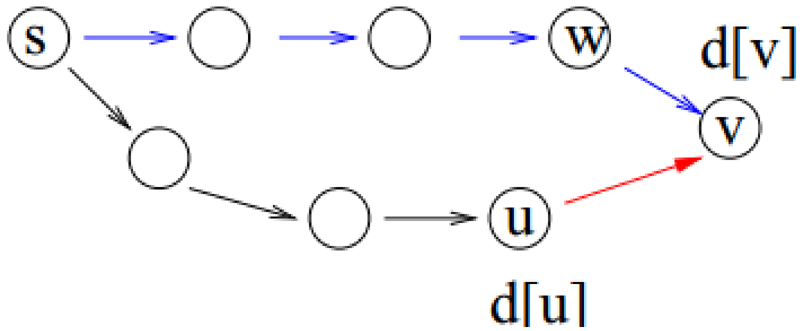
- How do we update d[v]?



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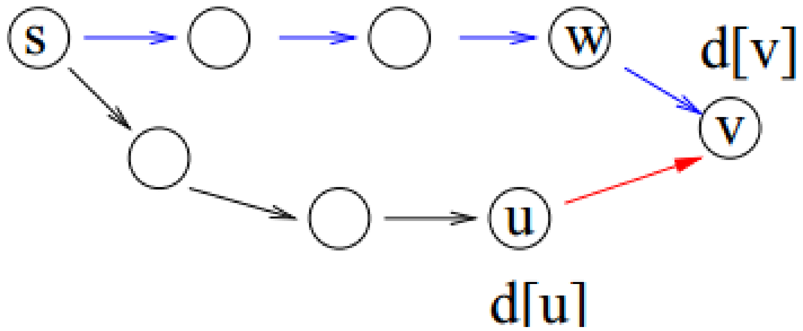

- Current shortest path to v: <s, …, w, v>, length d[v].
- New path to v: <s, …, u, v>, length d[u] + w(u,v).

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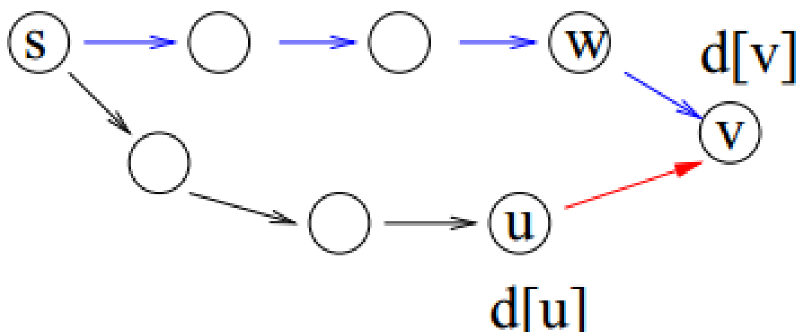

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$$d[v] = d[u] + w(u, v)$$

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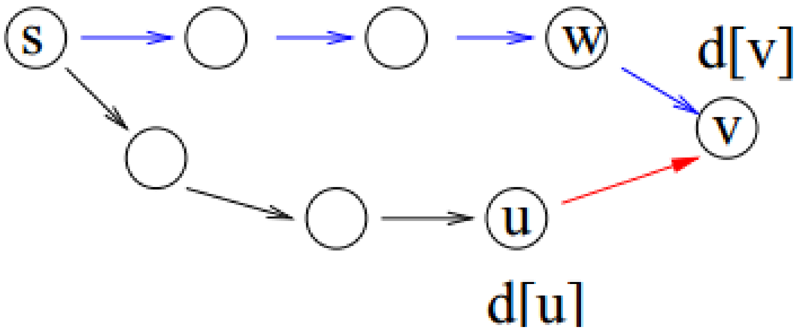

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$$d[v] = d[u] + w(u, v)$$

Now we have a better (tighter) upper bound for d[v]. This is called relaxing the edge (u, v).

The Algorithm for Relaxing an Edge

Relax(u,v)

Input: Update estimation of $\boldsymbol{u}$ according to distance of $\boldsymbol{v}$
Output: None
if $d[u] + w(u, v) < d[v]$ **then**

end

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The Algorithm for Relaxing an Edge: An Example

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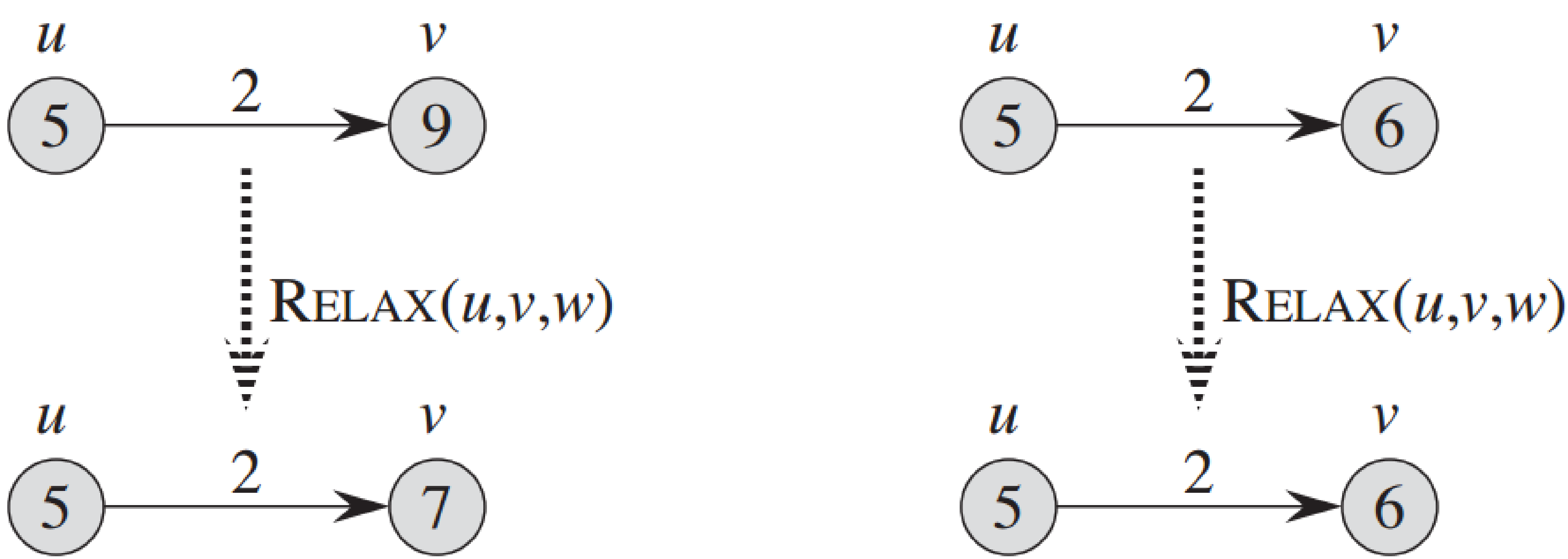

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- Remark 1: The predecessor pointer pred[ ] is for determining the shortest paths.
- Remark 2: After edge (u, v) is relaxed, we have

$$d[v] \leq d[u] + w(u, v)$$

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- Note: if we implement the priority queue using a heap, we can perform the operations Insert(), Extract-Min(), and Decrease-Key(), each in O(   ) time.

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Description of Dijkstra's Algorithm

Dijkstra(G,w,s)

Input: A graph $\boldsymbol{G}$, a matrix $\boldsymbol{w}$ representing the weights between vertices in G, source vertex $\boldsymbol{s}$

Output: None

for $u \in V$ **do**

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        end
    end
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    u ←Extract-Min(Q);// Find new vertex
    for v ∈ Adj[u] do
        if d[u] + w(u, v) < d[v] then
            // If estimate improves
            d[v] ← d[u] + w(u, v);// relax
            Decrease-Key(Q, v, d[v]);
            pred[v] ← u;
        end
    end
    color[u] ←BLACK;
end
```

An Example of Dijkstra's Algorithm

color

W	W	W	W	W

pred

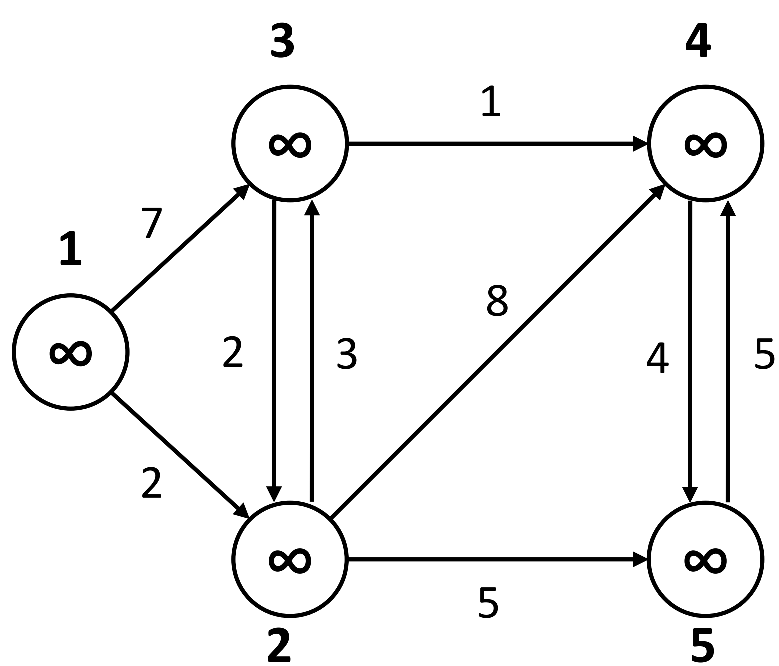
N	N	N	N	N

d

∞	∞	∞	∞	∞

Q(Priority Queue)

1	2	3	4	5



An Example of Dijkstra's Algorithm

color

W	**W**	**W**	**W**	**W**

pred

N	**N**	**N**	**N**	**N**

d

∞	**∞**	**∞**	**∞**	**∞**

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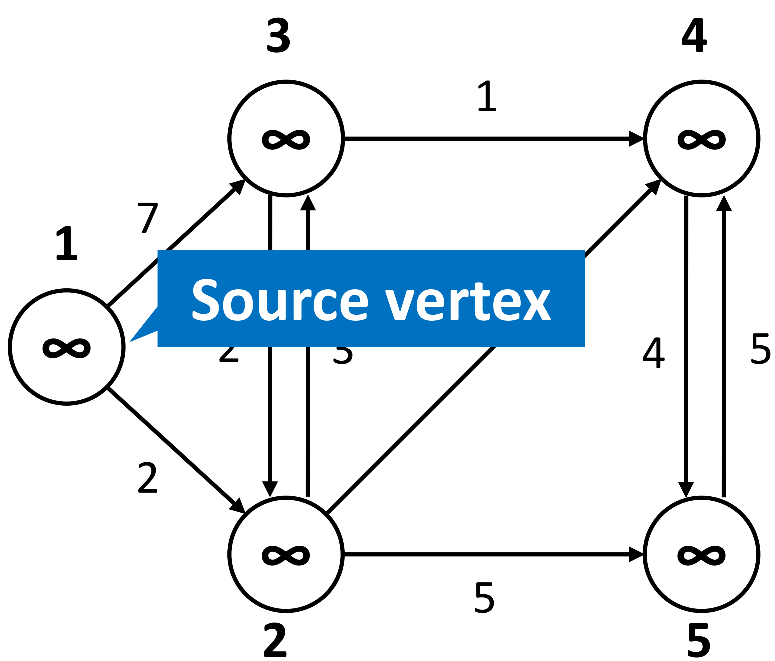


An Example of Dijkstra's Algorithm

color

W	W	W	W	W

d

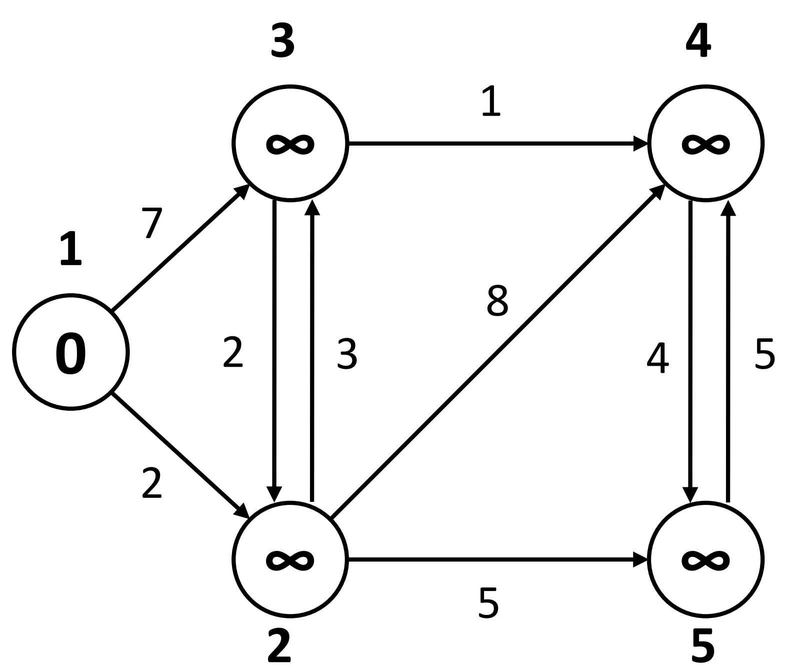
0	∞	∞	∞	∞

pred

N	N	N	N	N

Q(Priority Queue)

1	2	3	4	5



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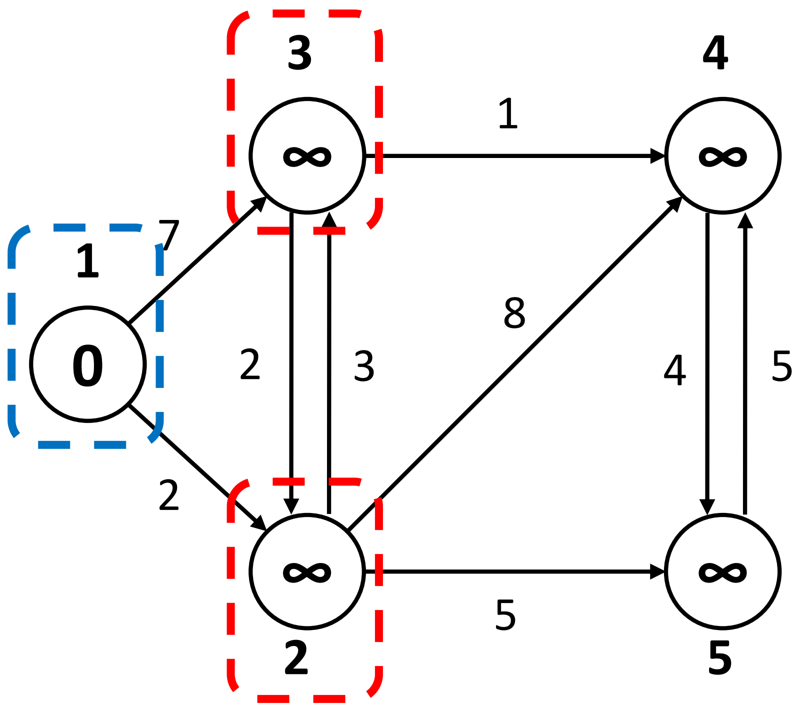


An Example of Dijkstra's Algorithm

color

W	W	W	W	W

pred

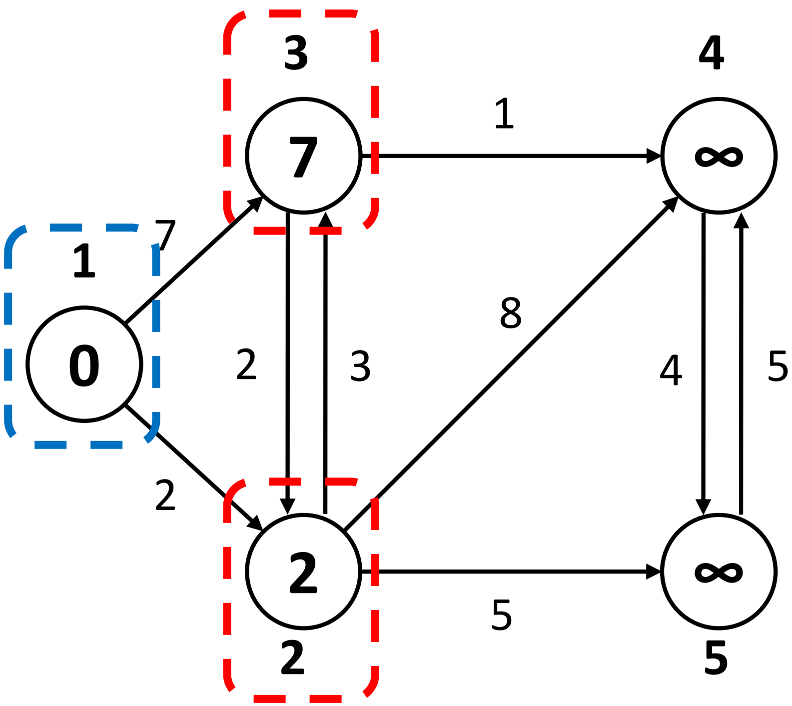
N	1	1	N	N

d

0	2	7	∞	∞

Q(Priority Queue)

1	2	3	4	5



An Example of Dijkstra's Algorithm

color

B	W	W	W	W

pred

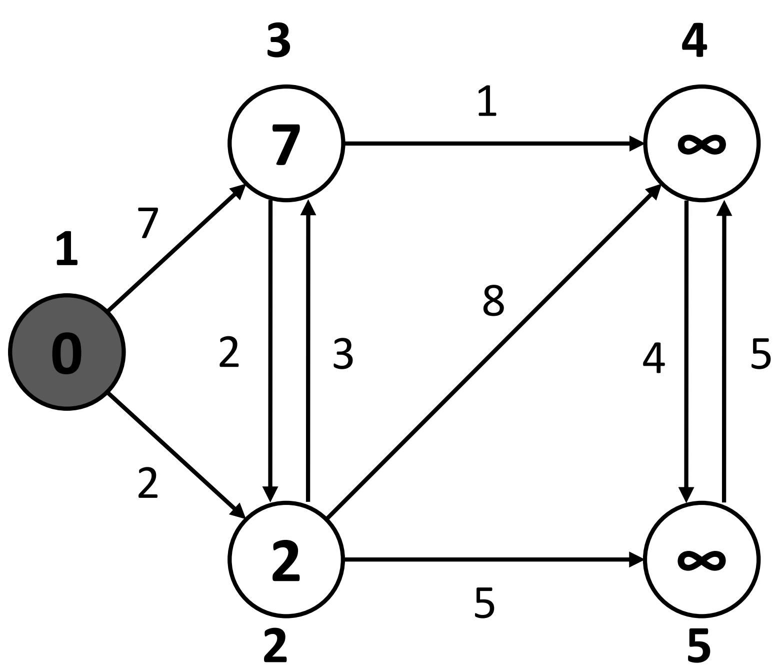
N	1	1	N	N

d

0	2	7	∞	∞

Q(Priority Queue)

1	2	3	4	5



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B	W	W	W	W

pred

N	1	1	N	N

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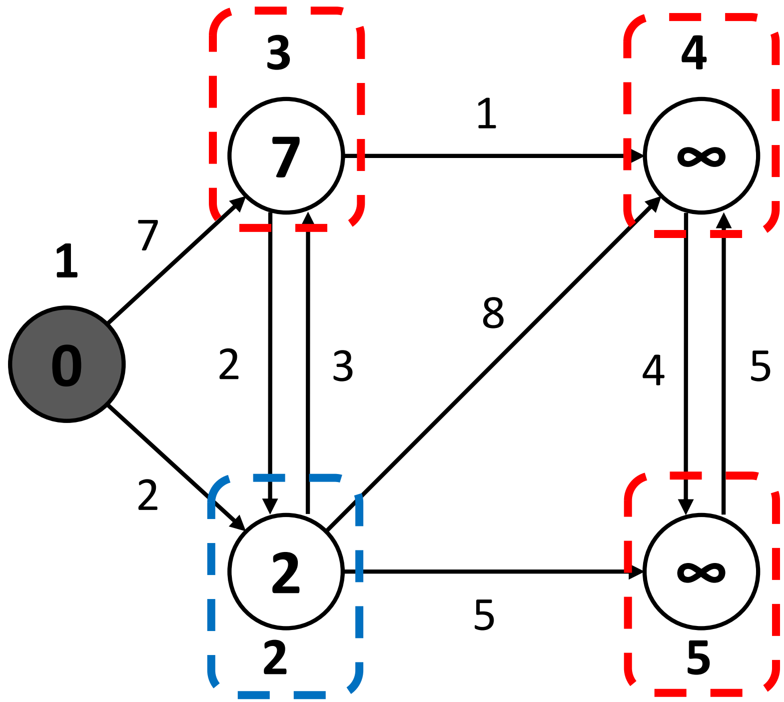

An Example of Dijkstra's Algorithm

color

B	W	W	W	W

pred

N	1	2	2	2

d

0	2	5	10	7

Q(Priority Queue)

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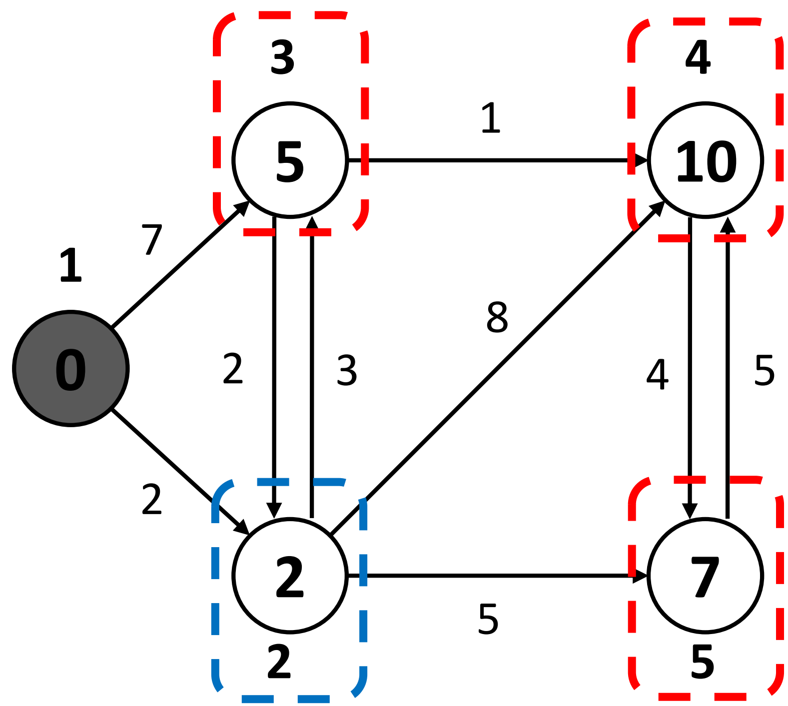

An Example of Dijkstra's Algorithm

color

B	B	W	W	W

d

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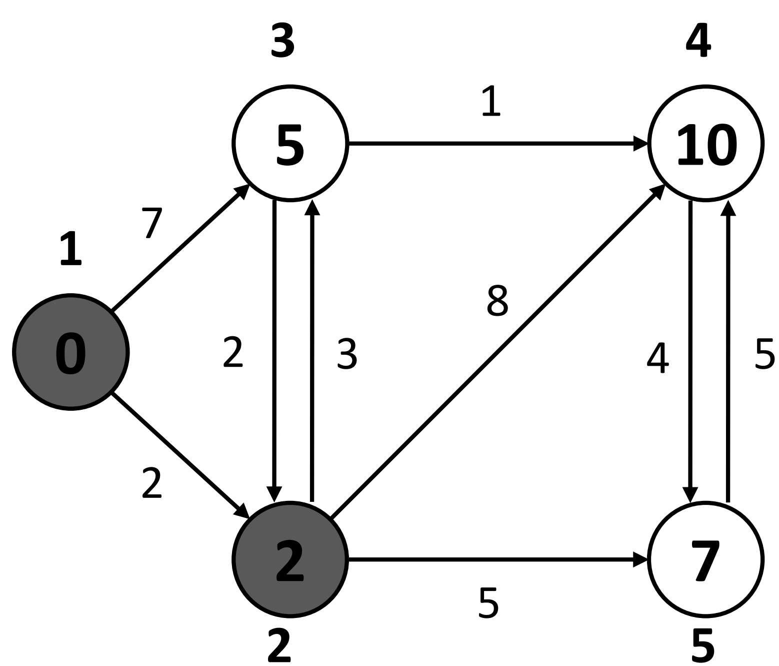


An Example of Dijkstra's Algorithm

color

B	B	W	W	W

pred

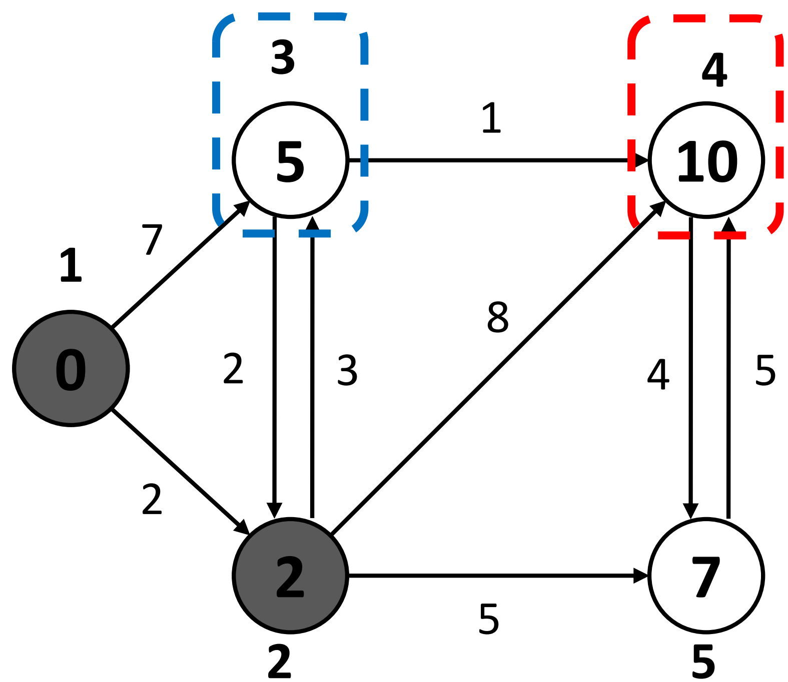
N	1	2	2	2

d

0	2	5	10	7

Q(Priority Queue)

1	2	3	4	5



An Example of Dijkstra's Algorithm

color

B	B	W	W	W

pred

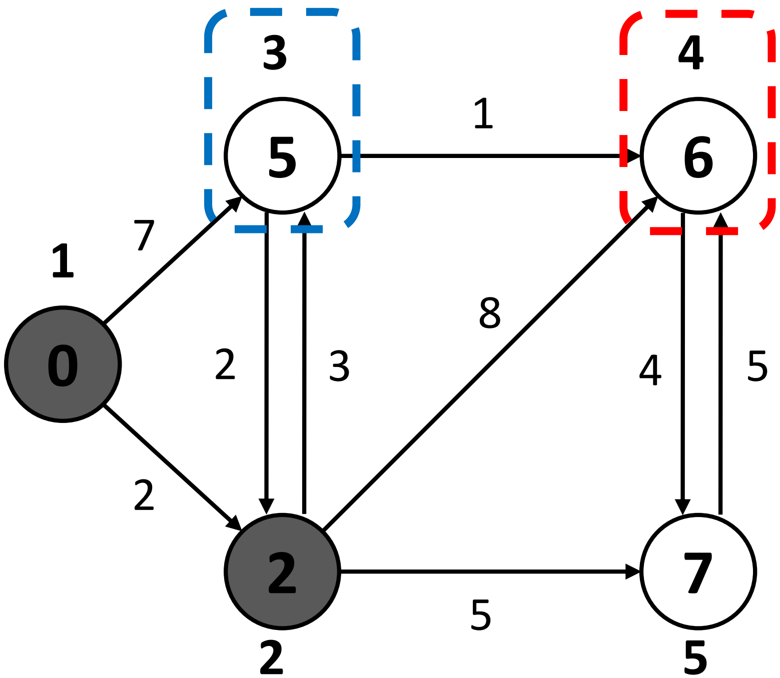
N	1	2	3	2

d

0	2	5	6	7

Q(Priority Queue)

1	2	3	4	5



An Example of Dijkstra's Algorithm

color

B	B	B	W	W

d

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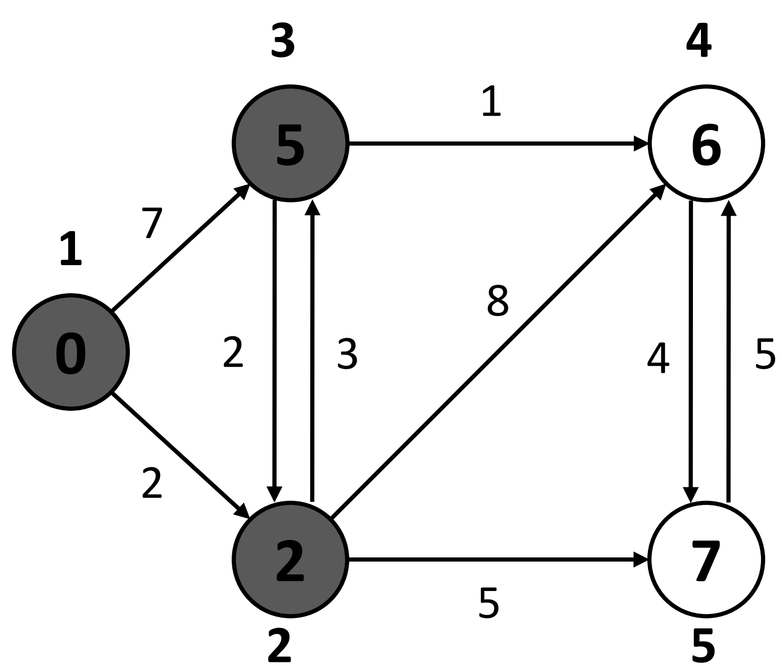

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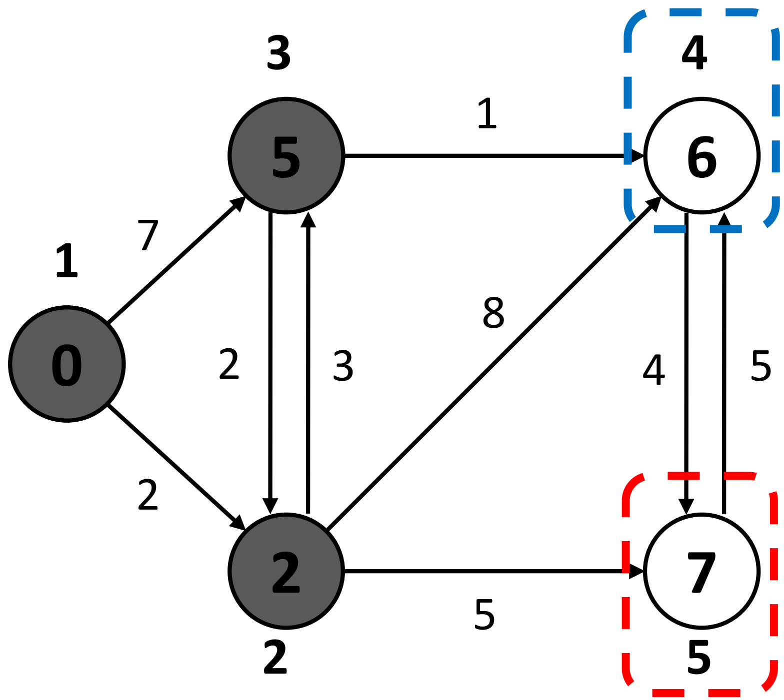

An Example of Dijkstra's Algorithm

color

B	**B**	**B**	**W**	**W**

pred

N	**1**	**2**	**3**	**2**

d

0	**2**	**5**	**6**	**7**

Q(Priority Queue)

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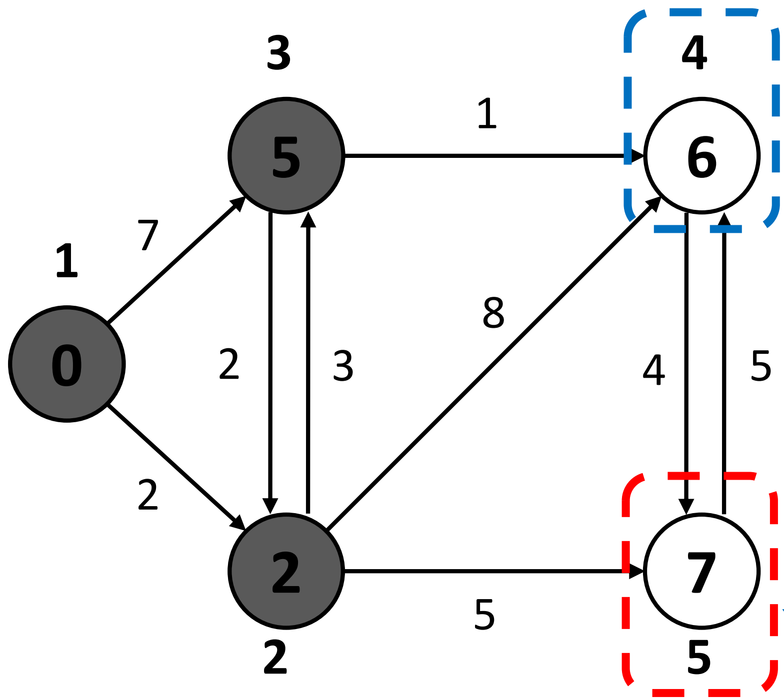

d[5] is not needed to be updated because d[4]+4 > d[5]

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color

B	B	B	B	W

pred

N	1	2	3	2

d

0	2	5	6	7

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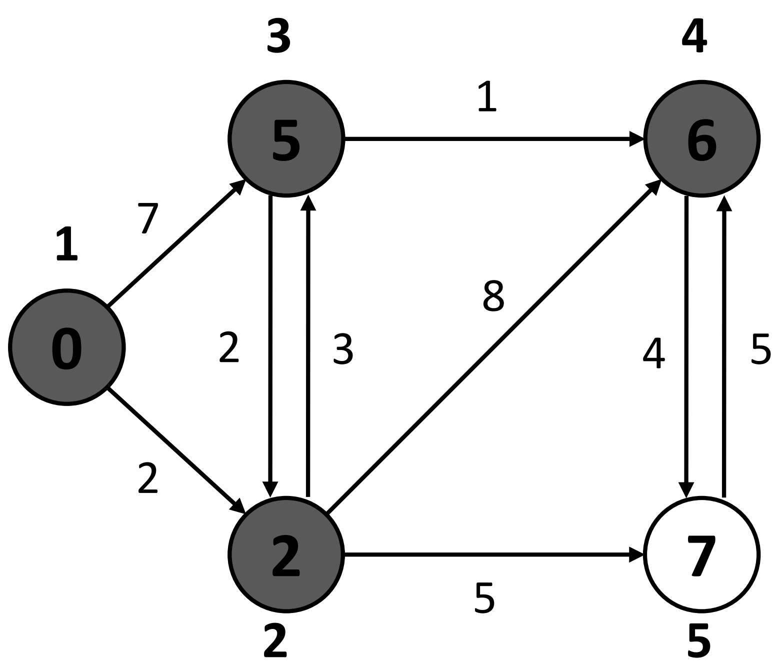

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B	B	B	B	W

d

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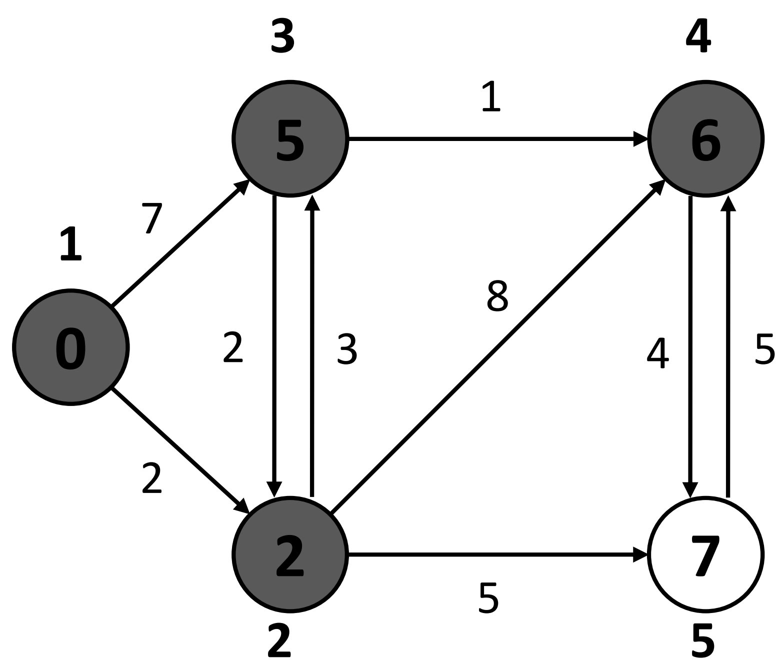

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pred

N	1	2	3	2

d

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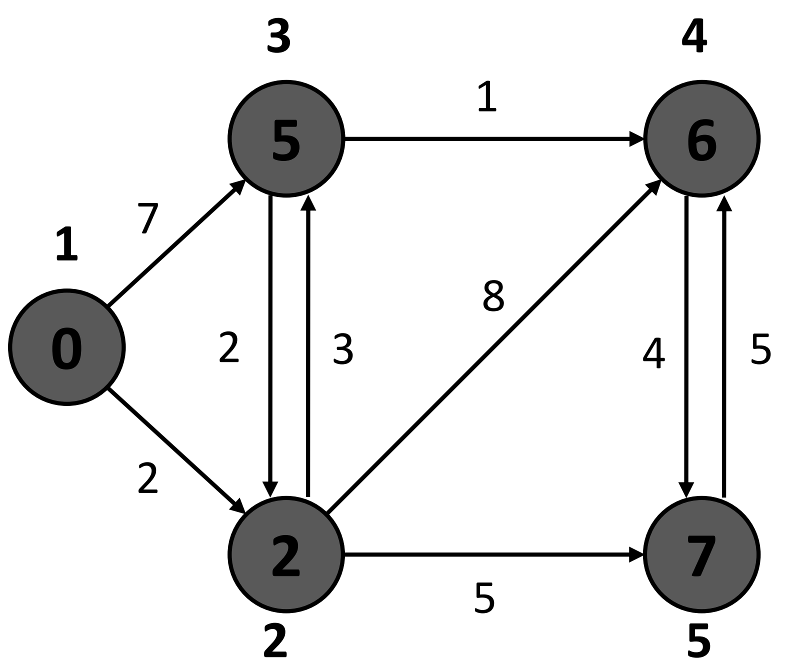

An Example of Dijkstra's Algorithm

color

B	B	B	B	B

pred

N	1	2	3	2

d

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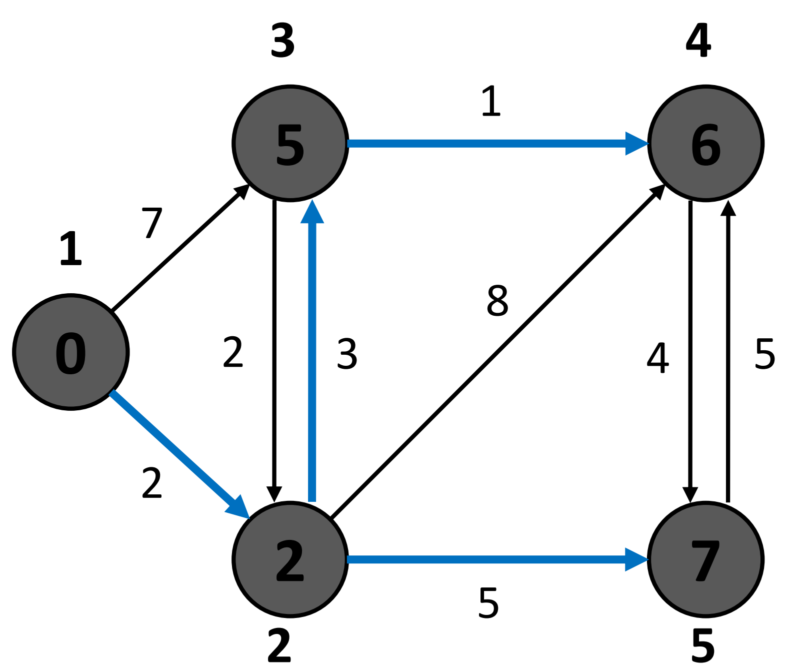

Shortest Path Tree for Dijkstra's Algorithm

Shortest Path Tree: T = (V ; A), where

$$A = \{(pred[v], v) | v \in V \backslash \{s\}\}$$

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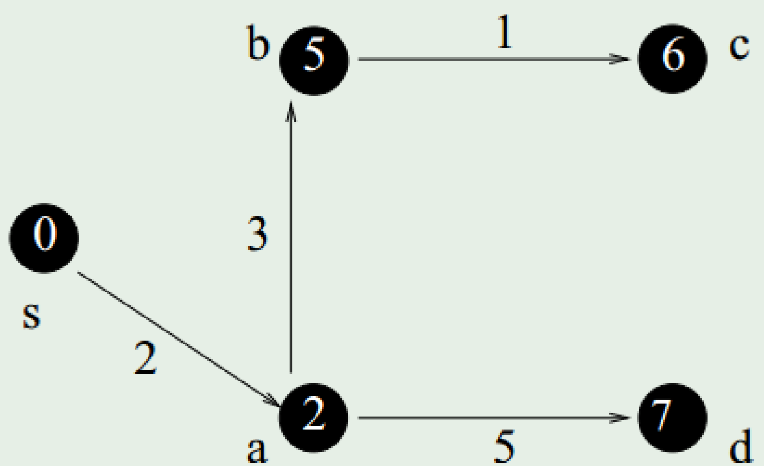
Example



v	s	a	b	c	d
$d[v]$	0	2	5	6	7
$pred[v]$	NIL	s	a	b	a

Outline

- Review to Part IV
- Single-Source Shortest Paths Problem
- Dijkstra’s Algorithm
  - The idea
  - The algorithm
  - Analysis of Dijkstra’s algorithm
- The Bellman-Ford Algorithm
  - The algorithm
  - Analysis of Bellman-Ford algorithm

Observation

Lemma

Any sub-path of a shortest path must also be a shortest path

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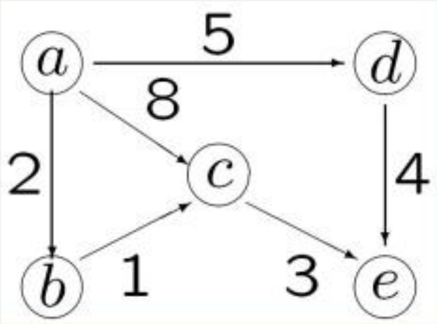
Example



$\langle a, b, c, e \rangle$ is a shortest path;

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Any sub-path of a shortest path must also be a shortest path

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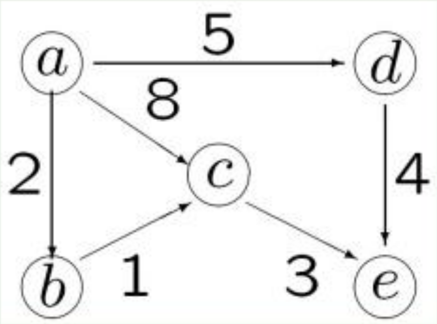

$\langle a, b, c, e \rangle$ is a shortest path; sub-path $\langle a, b, c \rangle$ is also a shortest path.

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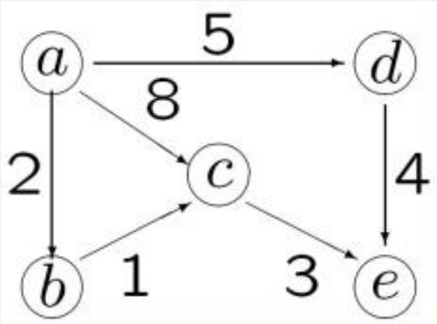

$\langle a, b, c, e \rangle$ is a shortest path; sub-path $\langle a, b, c \rangle$ is also a shortest path.

Question

Why?

Correctness of Dijkstra's Algorithm

Theorem: When a vertex u is added to S (i.e., dequeued from the queue), $d[u] = \delta(s, u)$.

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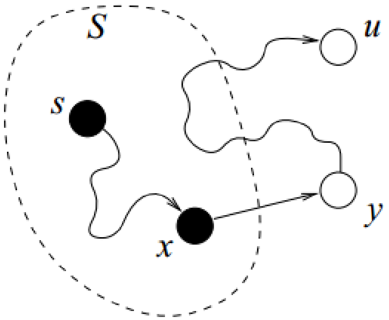

Proof:

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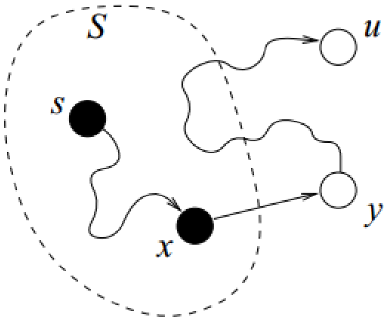

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- Suppose to the contrary that at some point Dijkstra's algorithm first attempts to add a vertex u to S for which $d[u] \neq \delta(s, u)$.
- Since $d[u] = \infty$ initially, and remains an upper bound of the true shortest distance, we have $d[u] > \delta(s, u)$.

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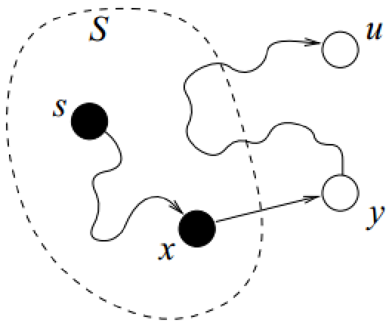

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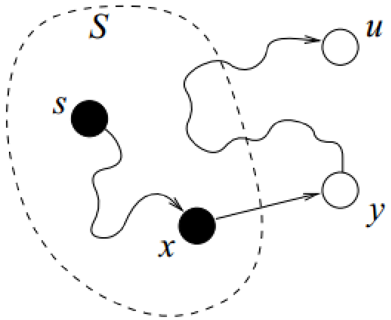

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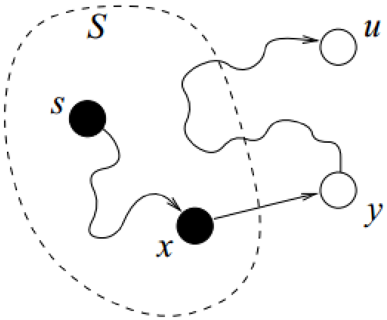

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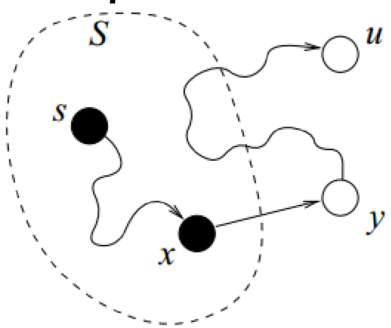

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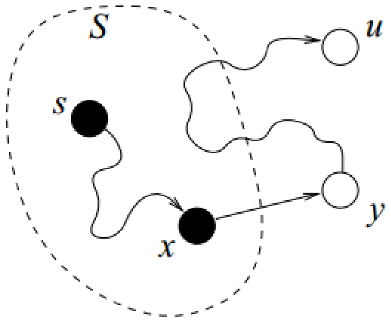

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- Consider the situation just prior to the insertion of u.
- Consider the true shortest path from s to u: <s,..., x, y,..., u>.
- Because $s \in S$ and u ∈ $V \backslash S$, at some point this path must first take a jump out of S. Let (x, y) be the edge taken by the path, where $x \in S$ and y ∈ $V \backslash S$ (it may be that x=s or/and y=u).

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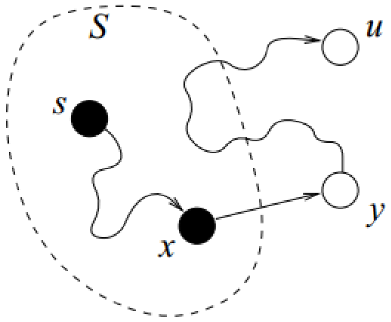

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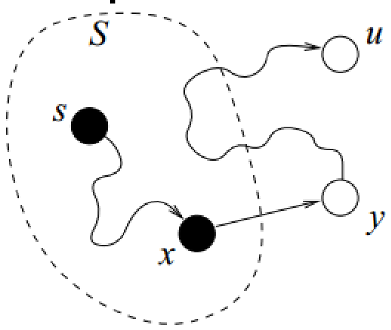

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$$d[y] \leq d[x] + w(x, y) \quad (1)$$

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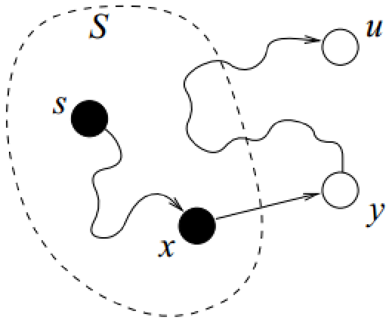

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- Since x is added to S earlier, by hypothesis,
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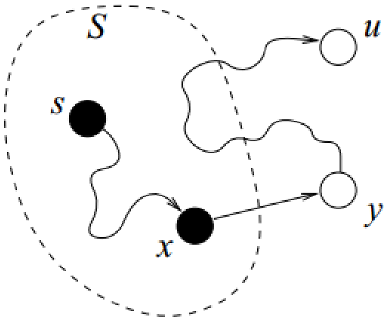

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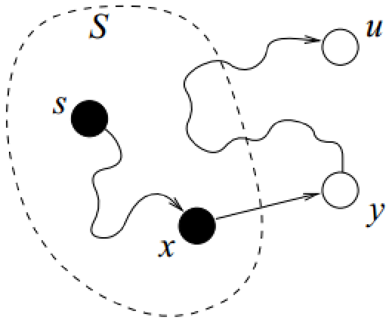

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- By (1) and (3), $d[y] \leq \delta(s, y)$. Hence $d[y] = \delta(s, y)$.

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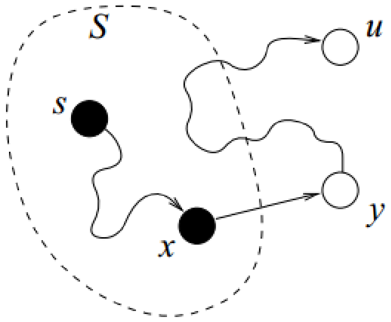

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- Since x is added to S earlier, by hypothesis,
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- Since <s, …, x, y> is sub-path of a shortest path,
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- By (1) and (3), $d[y] \leq \delta(s,y)$. Hence $d[y] = \delta(s,y)$. So $y \neq u$ (because we suppose $d[u] > \delta(s,u)$).

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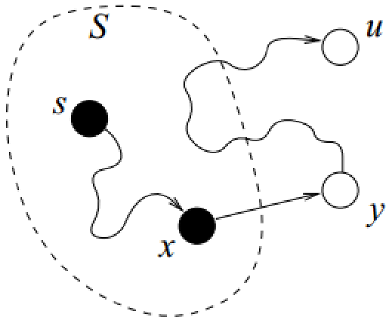

Proof:

Since y appears midway on the path from s to u, and all subsequent edges have non-negative weights, we have $\delta(s, y) \leq \delta(s, u)$, and thus

$$d[y] =$$

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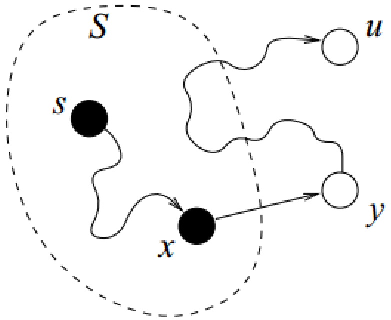

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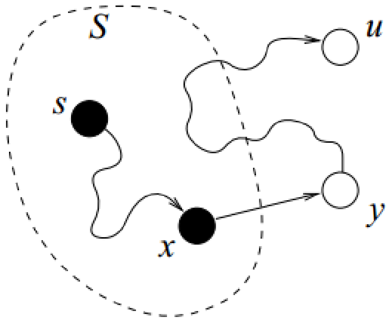

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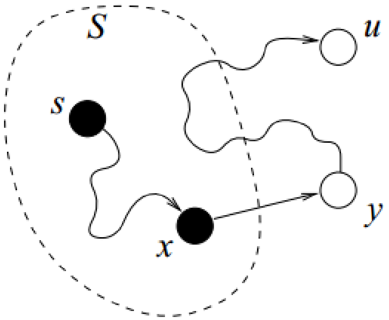

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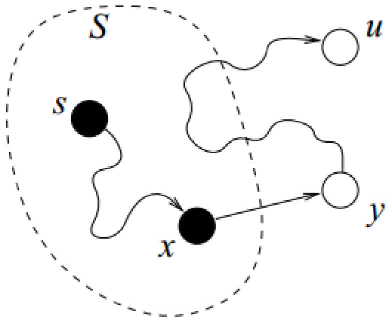

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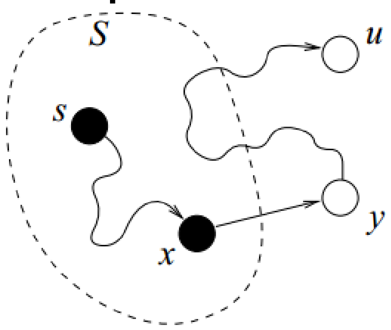

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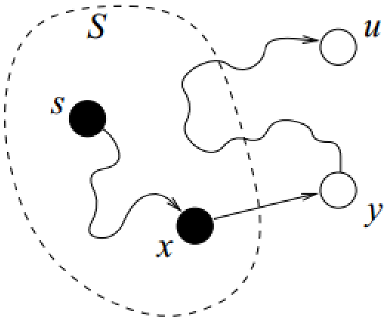

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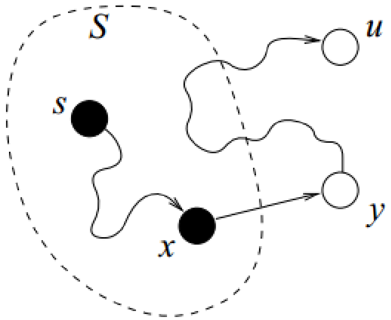

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Thus y should have been added to S before u, in contradiction to our assumption that u is the next vertex to be added to S.

Analysis of Dijkstra's Algorithm

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$$O(|V|) + |E| \cdot O(1 + \log|V|) = O(|E| \log |V|)$$

time.

Description of Dijkstra's Algorithm

Dijkstra(G,w,s)

Input: A graph $\boldsymbol{G}$, a matrix $\boldsymbol{w}$ representing the weights between vertices in G, source vertex $\boldsymbol{s}$

Output: None

for $u \in V$ **do**
    $d[u] \leftarrow \infty$, $color[u] \leftarrow$ WHITE;// Initialize
end
$d[s] \leftarrow 0$;
$pred[s] \leftarrow$ NULL;
$Q \leftarrow$queue with all vertices;
while *Non-Empty(Q)* **do**
    // Process all vertices
    $u \leftarrow$Extract-Min(Q);// Find new vertex
    for $v \in Adj[u]$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            // If estimate improves
            $d[v] \leftarrow d[u] + w(u, v)$;// relax
            Decrease-Key($Q, v, d[v]$);
            $pred[v] \leftarrow u$;
        end
    end
    $color[u] \leftarrow$BLACK;
end

$\boldsymbol{O(|V|)}$

$\boldsymbol{O(\log |V|) \cdot |E|}$

Prim’s Algorithm vs. Dijkstra’s Algorithm

- Dijkstra’s algorithm looks similar to Prim’s algorithm.

- To understand the differences clearly, try them both on some example.

Outline

- Review to Part IV
- Single-Source Shortest Paths Problem
- Dijkstra’s Algorithm
  - The idea
  - The algorithm
  - Analysis of Dijkstra’s algorithm
- The Bellman-Ford Algorithm
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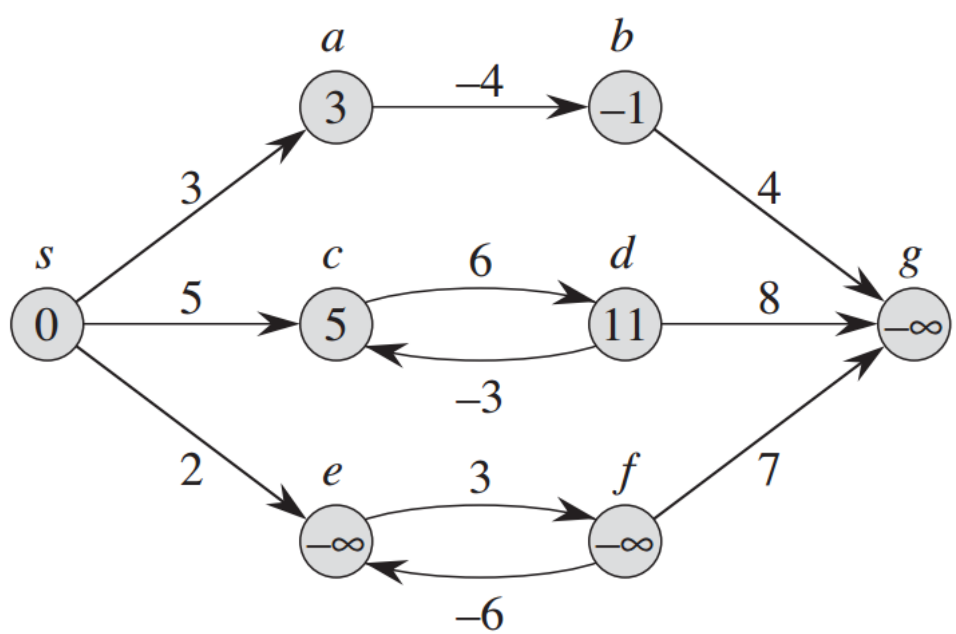

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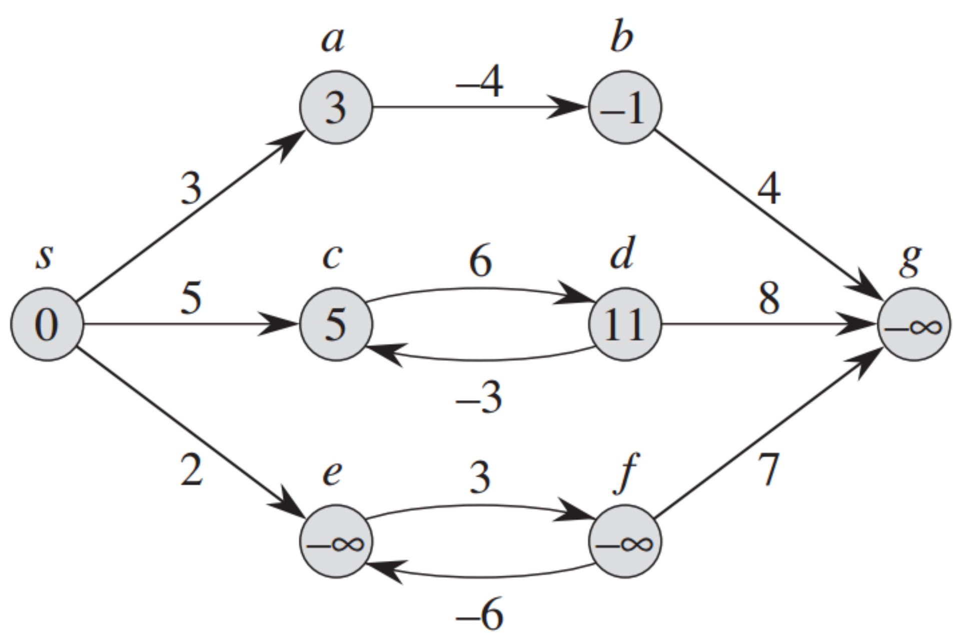
- By following the proposed "shortest" path and then traversing the negative-weight cycle, we can always find a path with weight.



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- If the graph contains a negative-weight cycle reachable from the source vertex s, shortest-path weights are not well defined.
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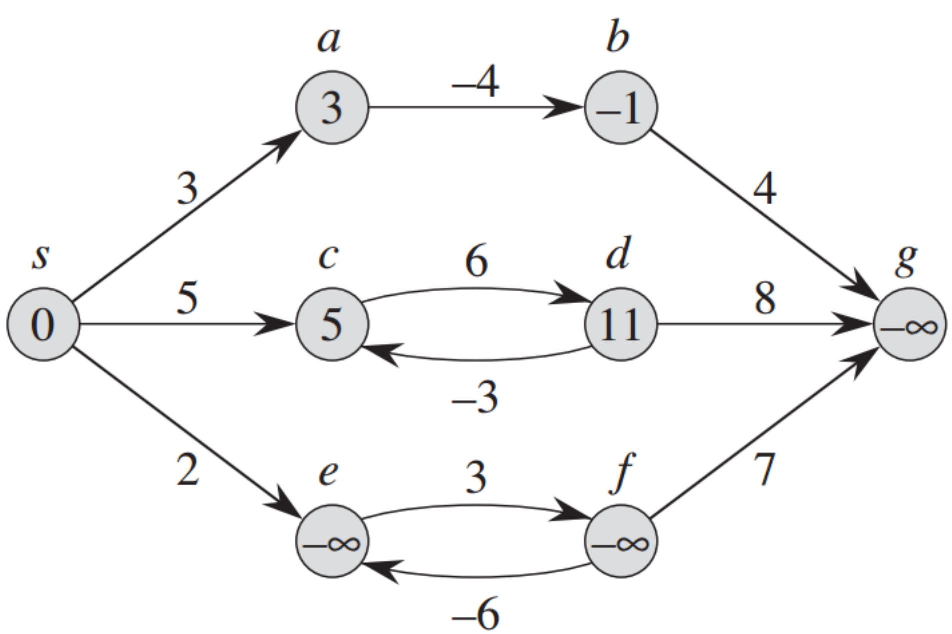
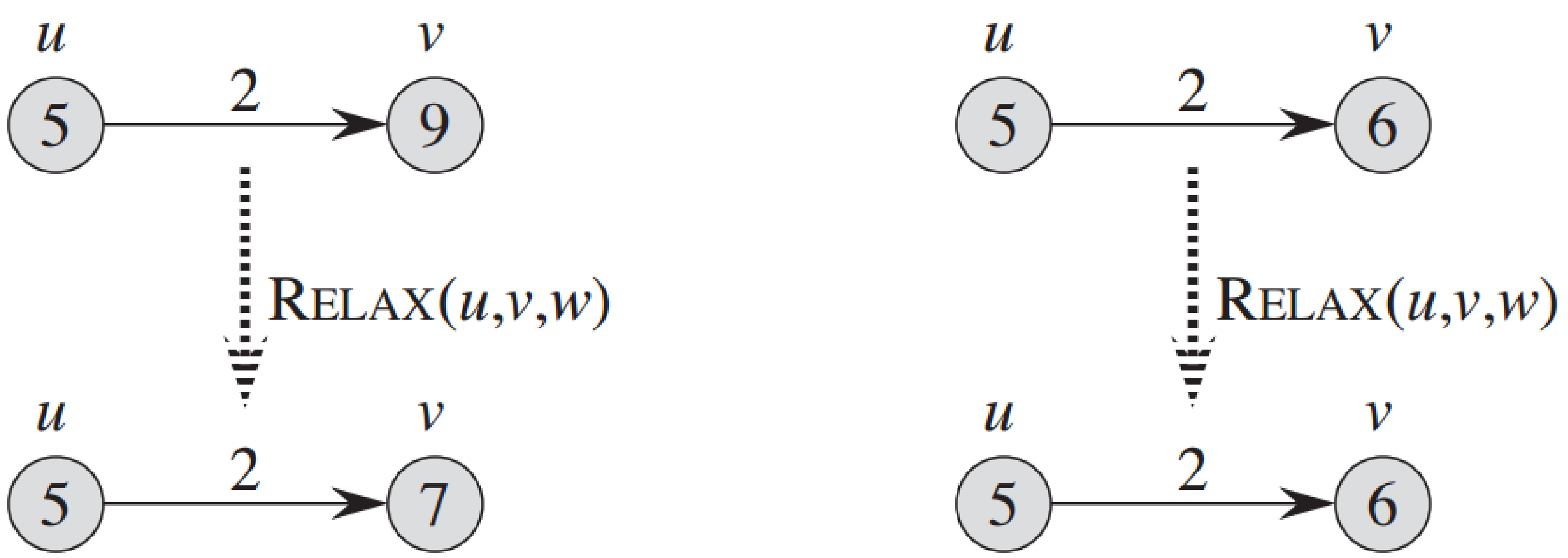


Review of The Algorithm for Relaxing an Edge

Relax(u,v)

Input: Update estimation of $\boldsymbol{u}$ according to distance of $\boldsymbol{v}$
Output: None
if $d[u] + w(u, v) < d[v]$ **then**
  $d[v] \leftarrow d[u] + w(u, v)$;
  $pred[v] \leftarrow u$;
end



Description of Bellman-Ford Algorithm

- The algorithm relaxes edges, progressively decreasing an estimate $v.d$ on the weight of a shortest path from the source s to each vertex $v \in V$ until it achieves the actual shortest-path weight $\delta(s, v)$.

Bellman-Ford(G,w,s)

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 | end
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 | end
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  | end
end
return TRUE;
```

An Example of Bellman-Ford Algorithm

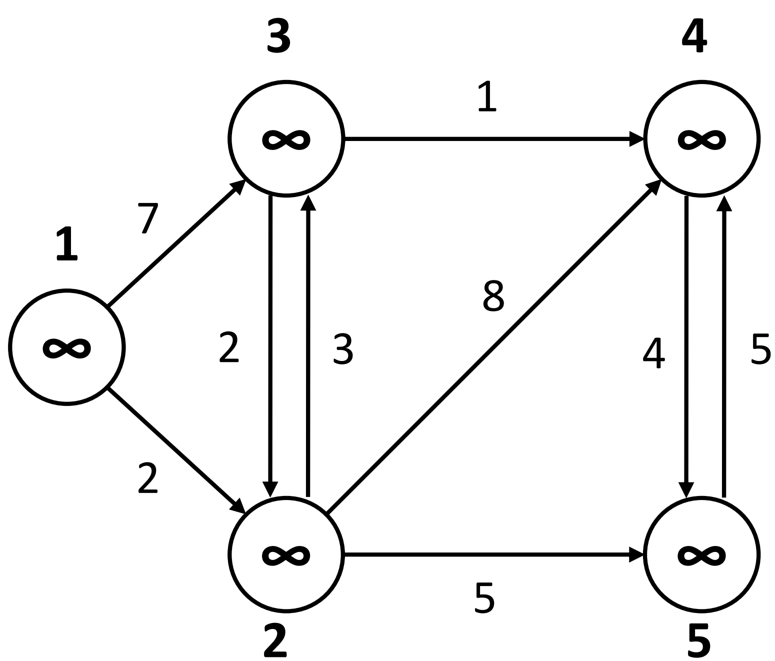
Initialization

d

∞	∞	∞	∞	∞

pred

N	N	N	N	N



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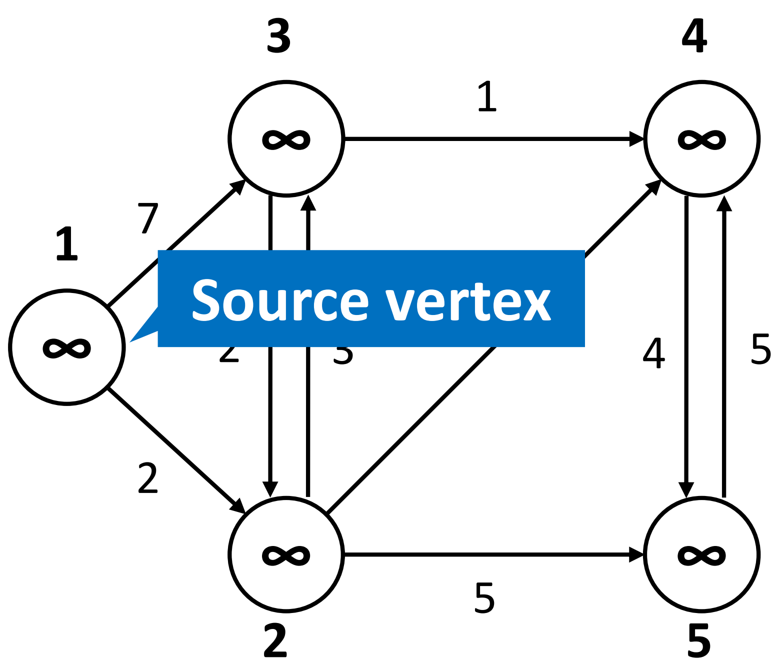

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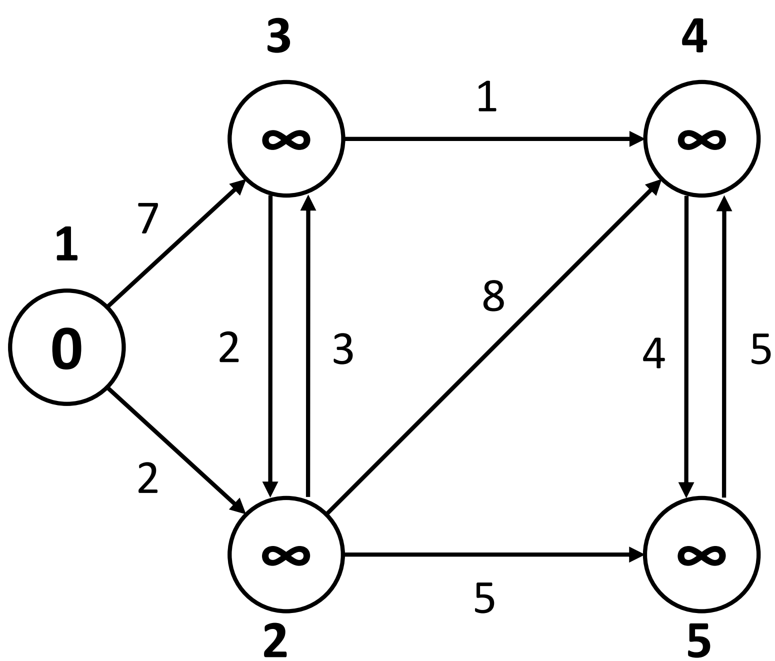
0	∞	∞	∞	∞

pred

N	**N**	**N**	**N**	**N**



An Example of Bellman-Ford Algorithm

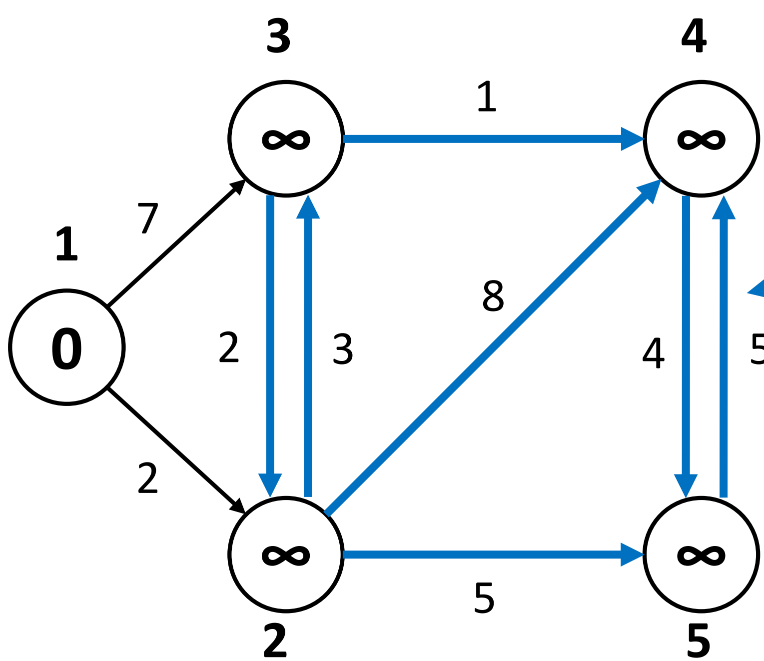
1st round

d

0	**∞**	**∞**	**∞**	**∞**

pred

N	**N**	**N**	**N**	**N**



These edges do not cause Relaxation.

An Example of Bellman-Ford Algorithm

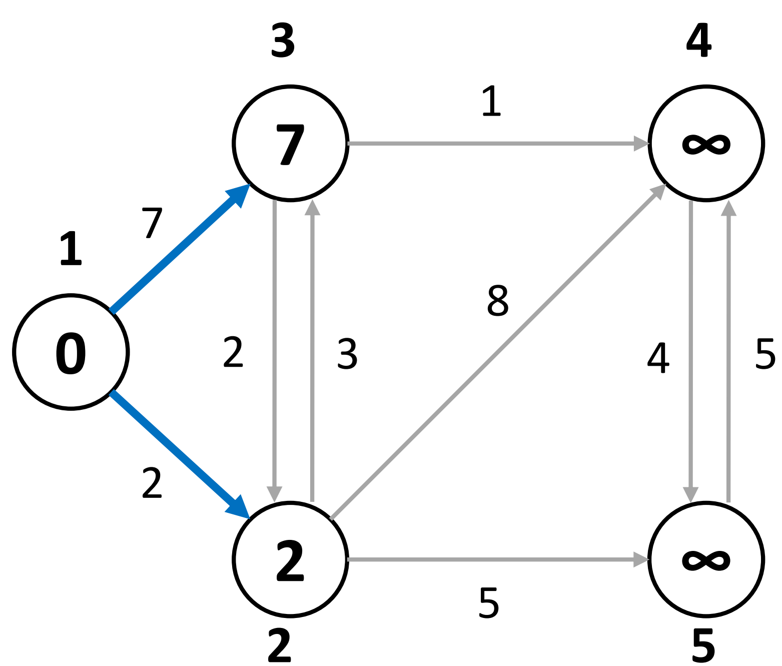
1st round

d

0	**2**	**7**	**∞**	**∞**

pred

N	**1**	**1**	**N**	**N**



An Example of Bellman-Ford Algorithm

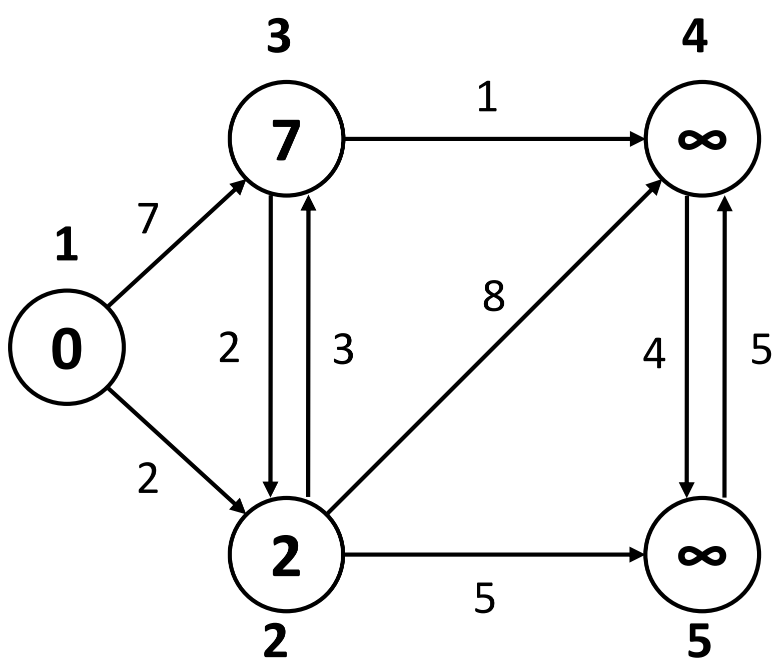
2nd round

d

0	2	7	∞	∞

pred

N	1	1	N	N



An Example of Bellman-Ford Algorithm

2nd round

d

0	**2**	**7**	**∞**	**∞**

pred

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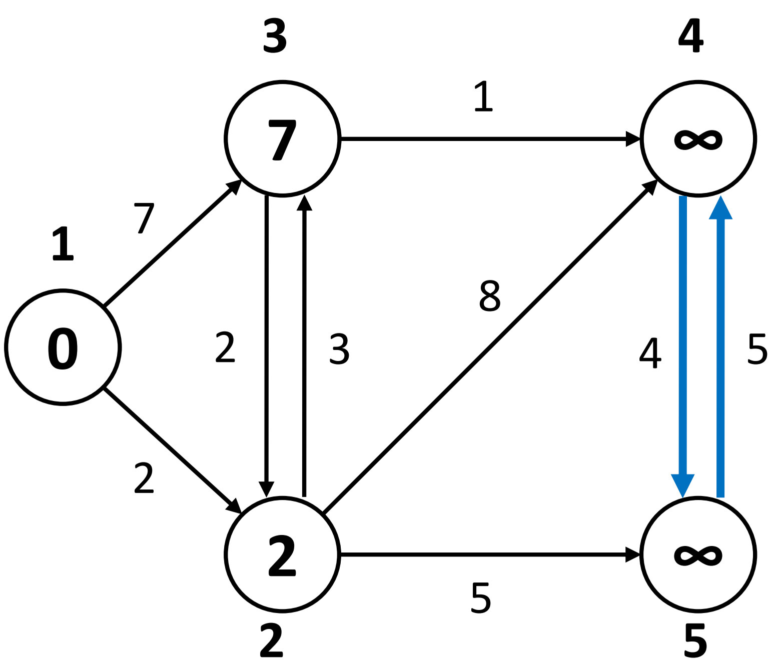

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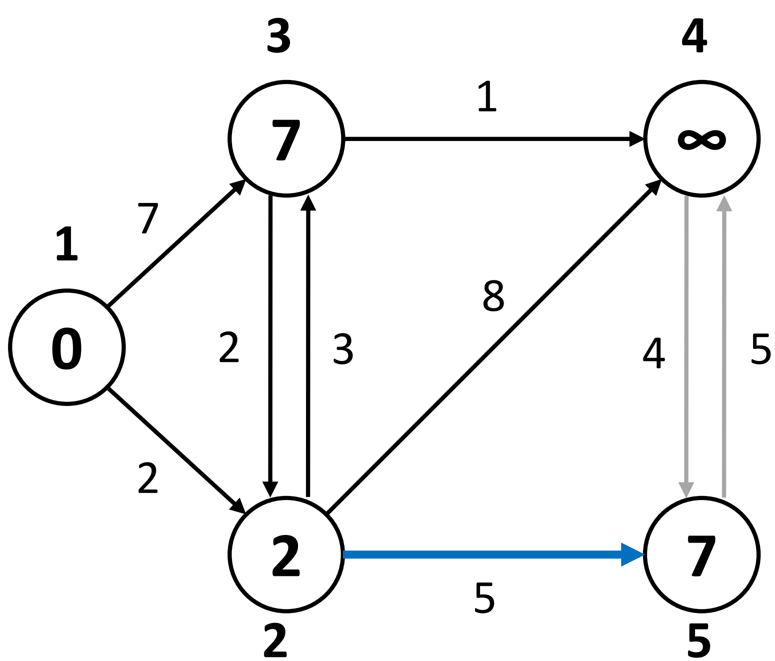
d

0	**2**	**7**	**∞**	**7**

pred

N	**1**	**1**	**N**	**2**



An Example of Bellman-Ford Algorithm

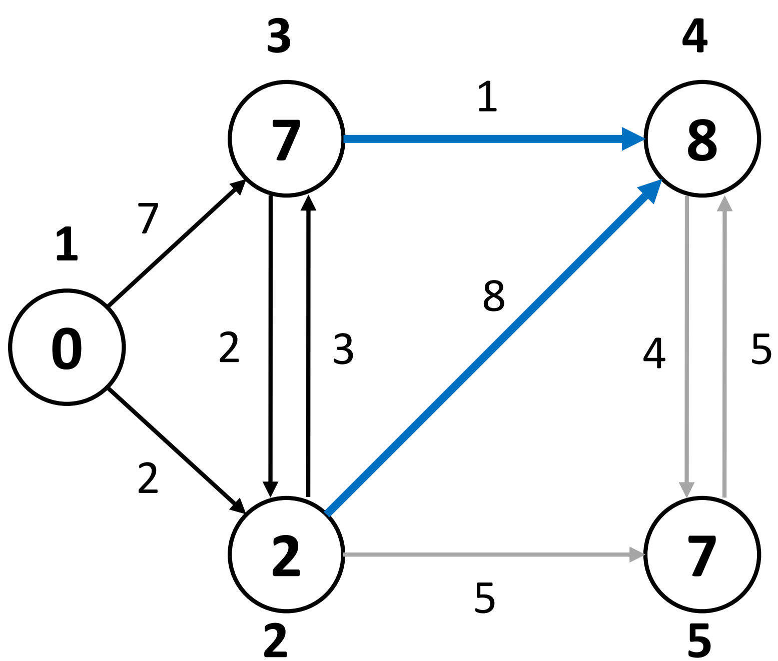
2nd round

d

0	**2**	**7**	**8**	**7**

pred

N	**1**	**1**	**3**	**2**



An Example of Bellman-Ford Algorithm

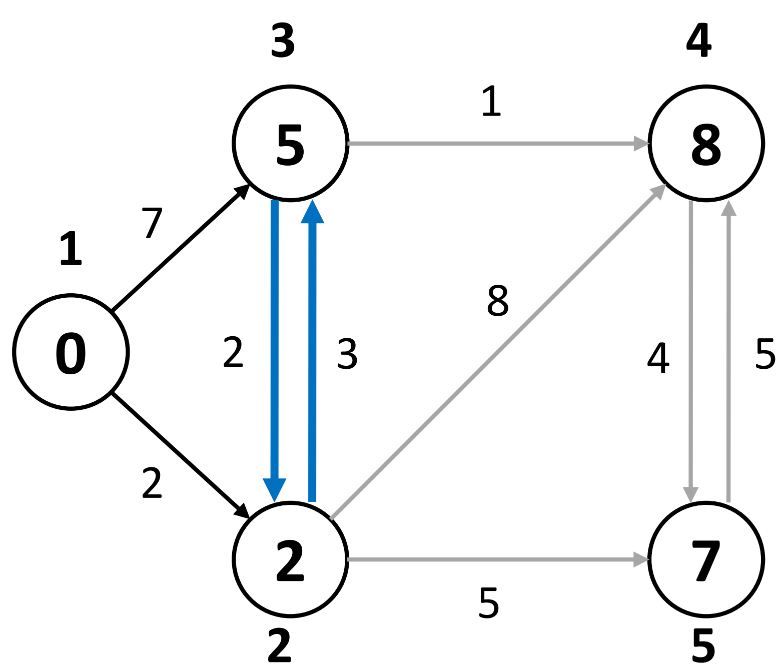
2nd round

d

0	**2**	**5**	**8**	**7**

pred

N	**1**	**2**	**3**	**2**



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d

0	**2**	**5**	**8**	**7**

pred

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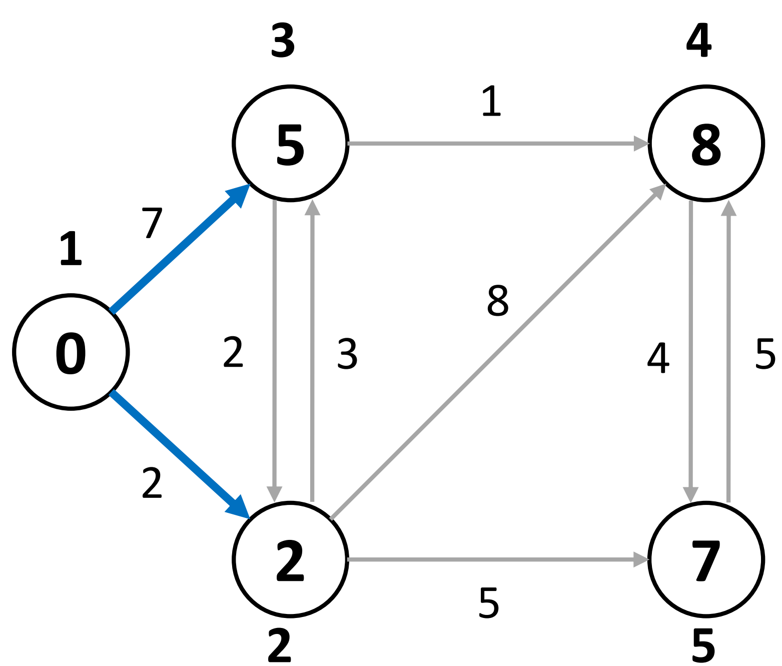


An Example of Bellman-Ford Algorithm

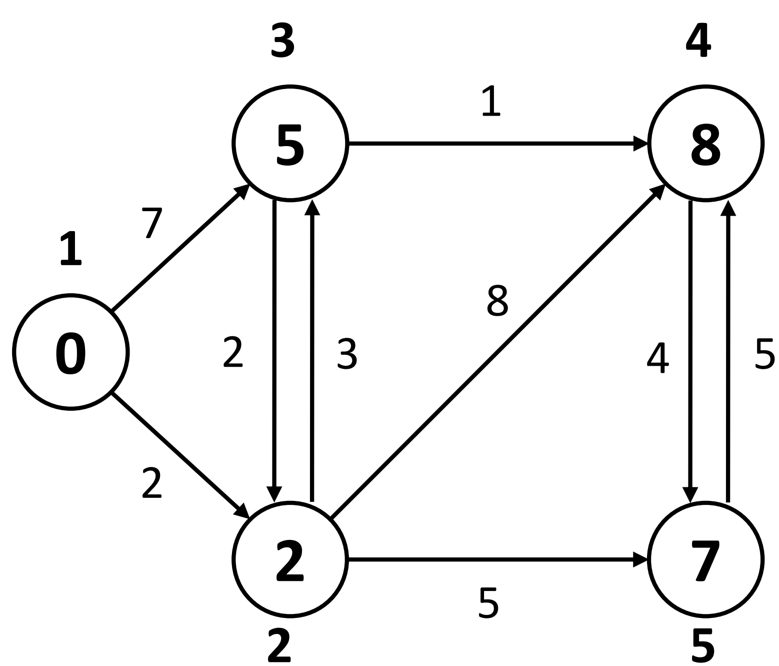
3rd round

d

0	2	5	8	7

pred

N	1	2	3	2



An Example of Bellman-Ford Algorithm

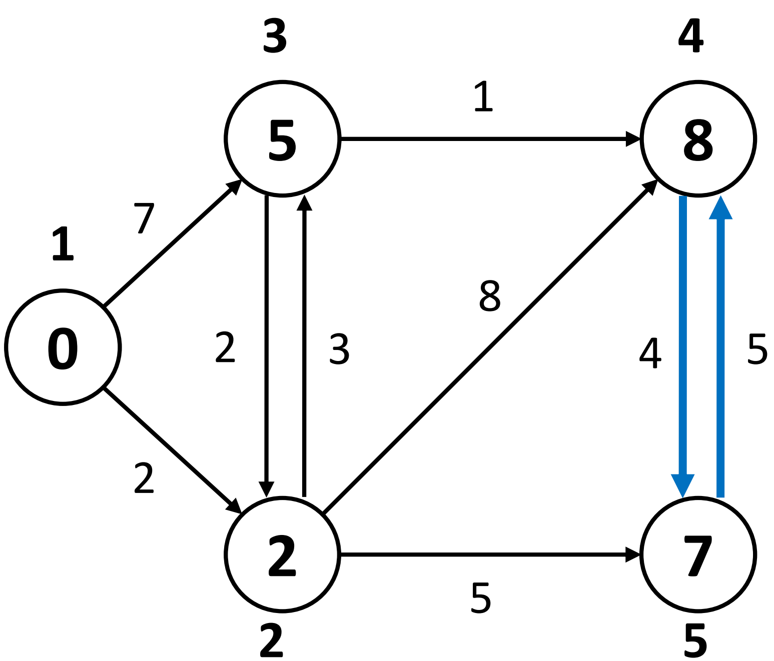
3rd round

d

0	2	5	8	7

pred

N	1	2	3	2



An Example of Bellman-Ford Algorithm

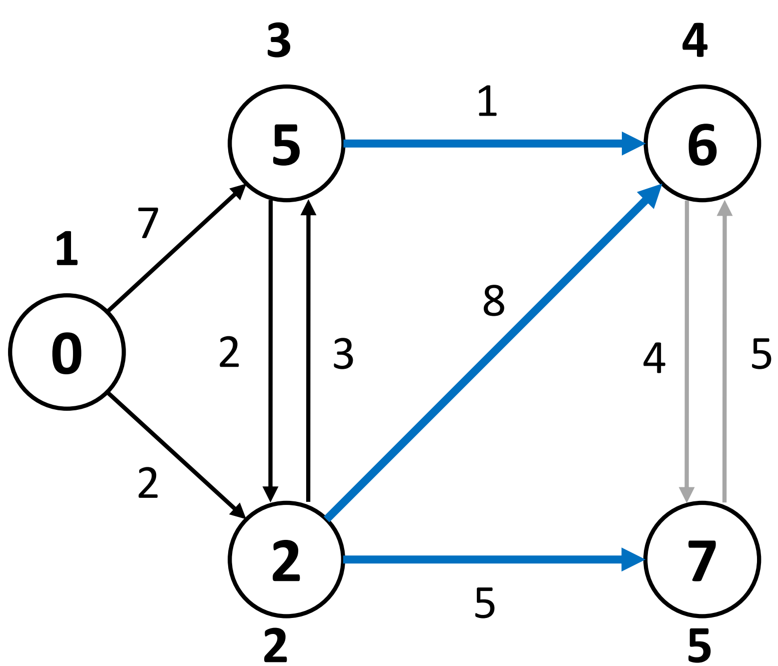
3rd round

d

0	**2**	**5**	**6**	**7**

pred

N	**1**	**2**	**3**	**2**



An Example of Bellman-Ford Algorithm

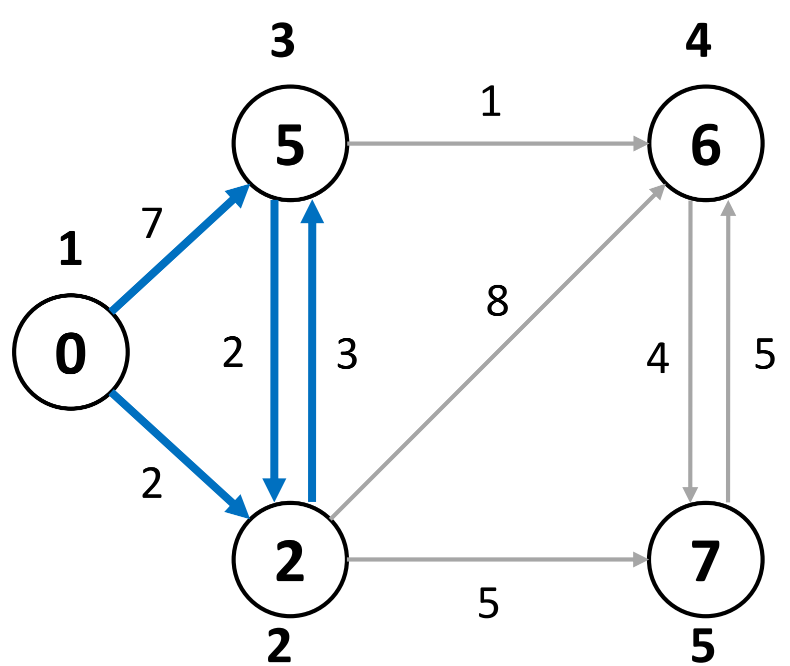
3rd round

d

0	2	5	6	7

pred

N	1	2	3	2



An Example of Bellman-Ford Algorithm

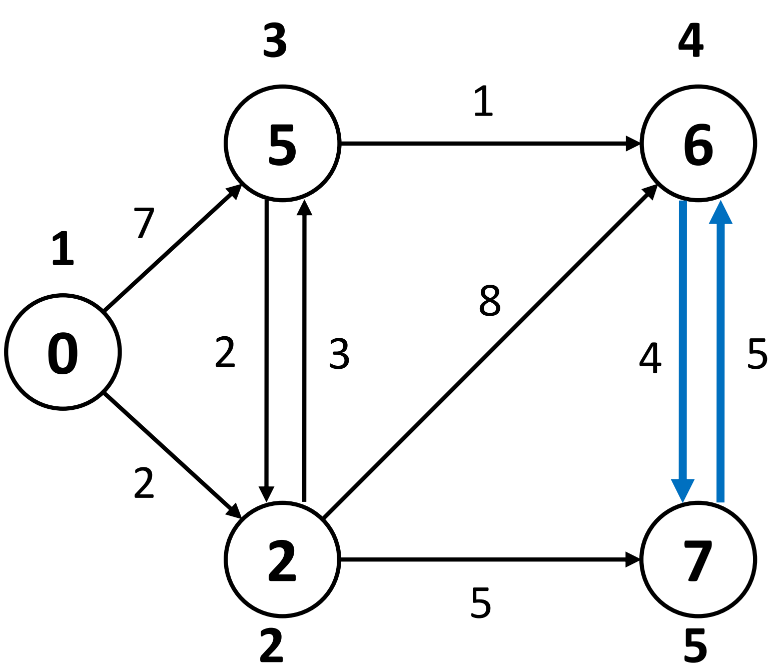
4th round

d

0	2	5	6	7

pred

N	1	2	3	2



An Example of Bellman-Ford Algorithm

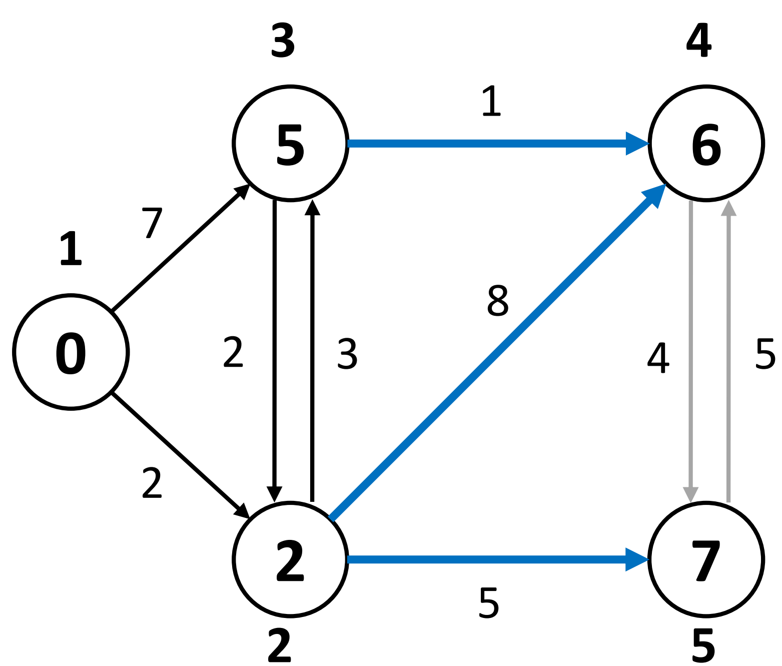
4th round

d

0	**2**	**5**	**6**	**7**

pred

N	**1**	**2**	**3**	**2**



An Example of Bellman-Ford Algorithm

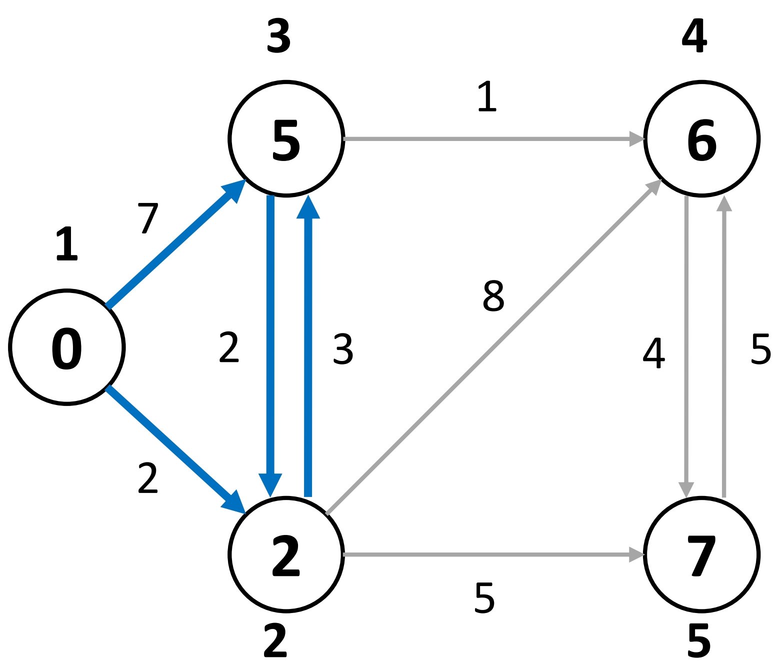
4th round

d

0	**2**	**5**	**6**	**7**

pred

N	**1**	**2**	**3**	**2**



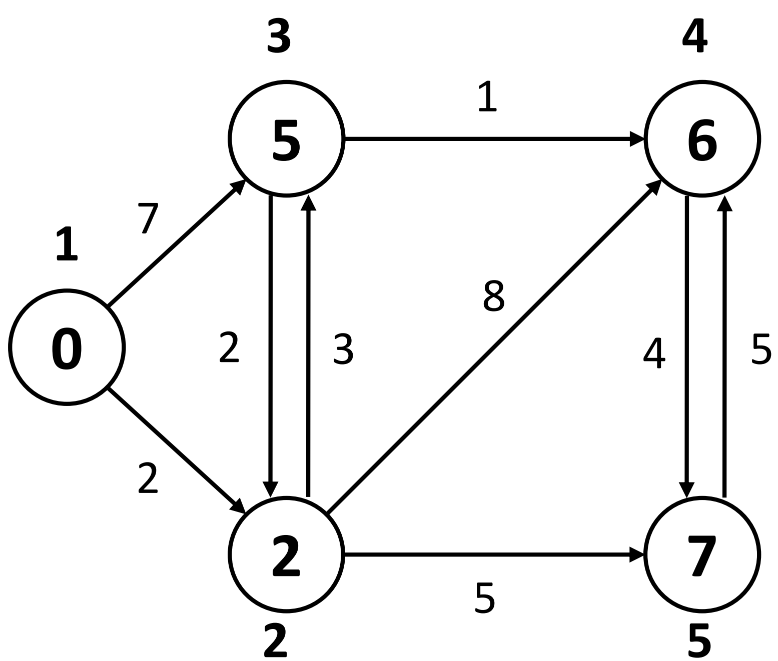
An Example of Bellman-Ford Algorithm

d

0	**2**	**5**	**6**	**7**

pred

N	**1**	**2**	**3**	**2**



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Analysis of Bellman-Ford Algorithm

- The Bellman-Ford algorithm runs in time $O(|V| \cdot |E|)$ since the initialization takes $O(|V|)$ time, each of the $|V| - 1$ passes over the edges takes $O(|E|)$ time, and the **for** loop takes $O(|E|)$ time.

Bellman-Ford(G,w,s)

```
Input: A directed graph G,weights w, and the source vertex s
Output: Return FALSE if G contains negative cycle, return TRUE if
        shortest paths from s to any other vertices obtained.
for u ∈ V do
  | d[u] ← ∞, pred[u] ←NIL;// Initialize
end
for i ← 1 to |V| − 1 do
  | for e ∈ E do
  |   | RELAX(u, v, w);
  | end
end
for e ∈ E do
  | if d[v] > d[u] + w(u, v) then
  |   | return FALSE;
  | end
end
return TRUE;
```

Analysis of Bellman-Ford Algorithm

- The Bellman-Ford algorithm runs in time $O(|V| \cdot |E|)$ since the initialization takes $O(|V|)$ time, each of the $|V| - 1$ passes over the edges takes $O(|E|)$ time, and the **for** loop takes $O(|E|)$ time.

Bellman-Ford(G,w,s)

Input: A directed graph $\boldsymbol{G}$,weights $\boldsymbol{w}$, and the source vertex $\boldsymbol{s}$
Output: Return FALSE if G contains negative cycle, return TRUE if shortest paths from s to any other vertices obtained.

```
for u ∈ V do
 | d[u] ← ∞, pred[u] ←NIL;// Initialize        O(|V|)
end
for i ← 1 to |V| − 1 do                        O(|V|·|E|)
 | for e ∈ E do
 |  | RELAX(u, v, w);
 | end
end
for e ∈ E do                                   O(|E|)
 | if d[v] > d[u] + w(u, v) then
 |  | return FALSE;
 | end
end
return TRUE;
```


dank u
ju faleminderit
Tack
谢谢
Tak
mulţumesc
Asante
kiitos
Gracias
Salamat!
Terima kasih
Aliquam
Merci
Dankie
Obrigado
ありがとう
köszönöm
grazie
Aliquam
Go raibh maith agat
děkuji
Thank you
gam