Design and Analysis of Algorithms
Part I: Divide and Conquer
Lecture 3: Maximum Contiguous Subarray Problem and Counting Inversion Problem


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Outline

- Introduction to Part I
- Maximum Contiguous Subarray Problem
  - Problem definition
  - A brute force algorithm
  - A data-reuse algorithm
  - A divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm
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  - **Conquer**
    Solving each subproblem (directly if small enough or recursively)
  - **Combine**
    Combining the solutions of the subproblems into a global solution

Introduction to Part I

- In Part I, we will illustrate Divide-and-Conquer using several examples:
  - Maximum Contiguous Subarray (最大子数组)
  - Counting Inversions (逆序计数)
  - Integer Multiplication (整数乘法)
  - Polynomial Multiplication (多项式乘法)
  - QuickSort and Partition (快速排序与划分)
  - Deterministic and Randomized Selection (确定性与随机化选择)

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Maximum Contiguous Subarray (MCS) Problem

ACME Corp[1] – PROFIT HISTORY

Year	1	2	3	4	5	6	7	8	9
Profit M$	-3	2	1	-4	5	2	-1	3	-1

[1] A Company that Makes Everything

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如果所有数组元素都是非负数，整个数组和肯定是最大

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Answer: Year 5-8, 9 M$

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Definition (Maximum Contiguous Subarray Problem)

Find $i \leq j$ such that $V(i,j)$ is maximized.

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VMAX ← A[1];
for i ← 1 to n do
    for j ← i to n do
        // calculate V(i,j)
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$O(n^3)$ arithmetic additions

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    end
end
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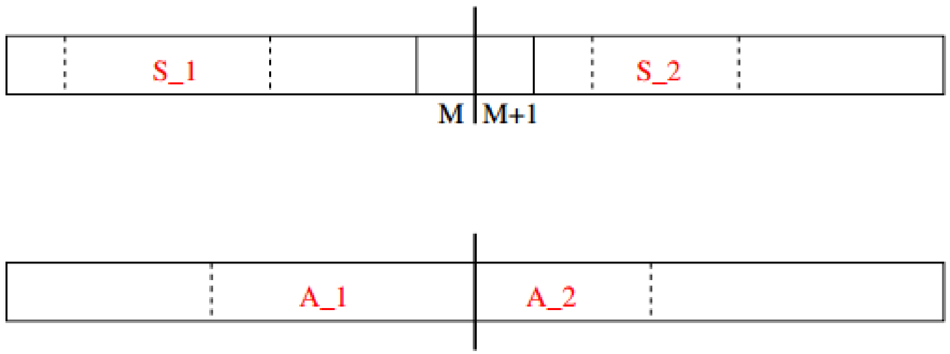
A Divide-and-Conquer Algorithm

Set $m = \lfloor (n+1)/2 \rfloor$



A_1 = MCS on left containing A[M]   A_2 = MCS on right containing A[M+1]

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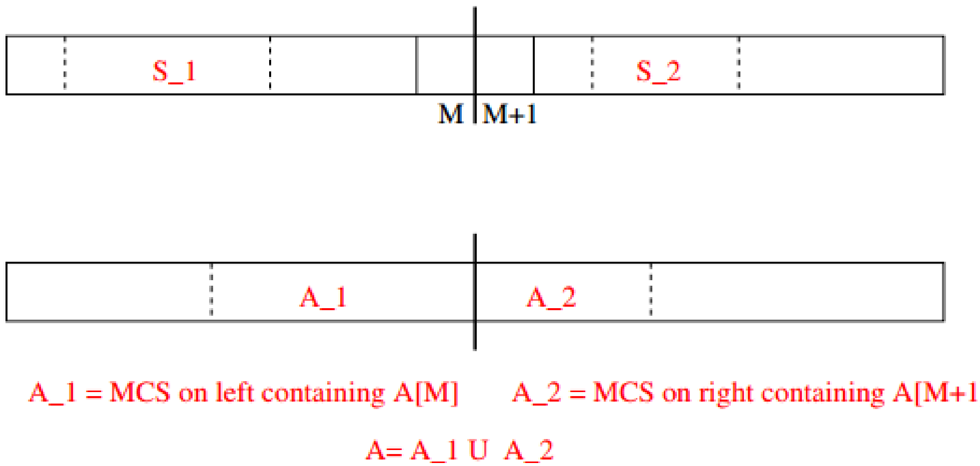

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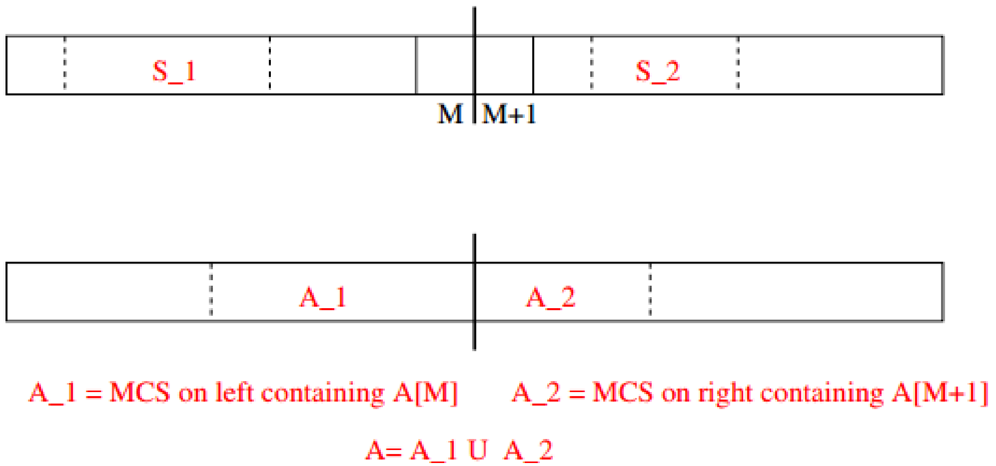

The MCS S must be one of

1. S_1: the MCS in $A[1 \ldots m]$

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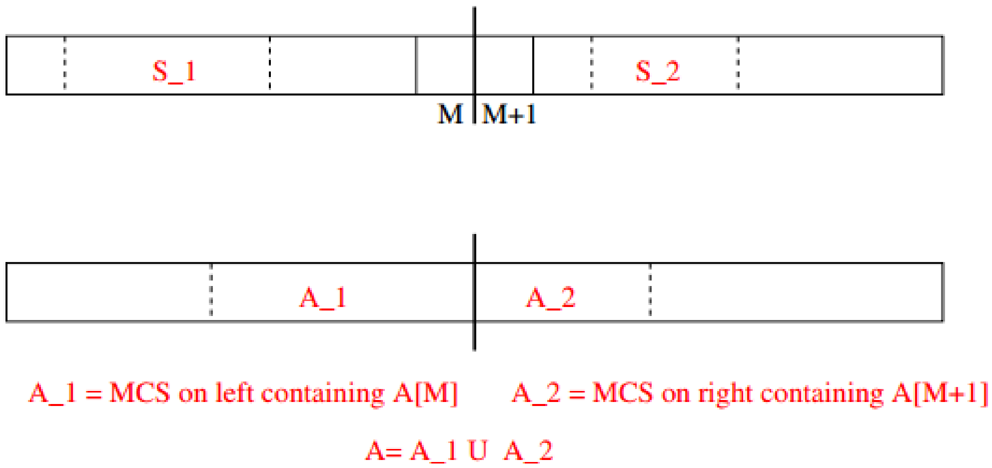

The MCS S must be one of

1. S_1: the MCS in $A[1 \ldots m]$
2. S_2: the MCS in $A[m+1 \ldots n]$

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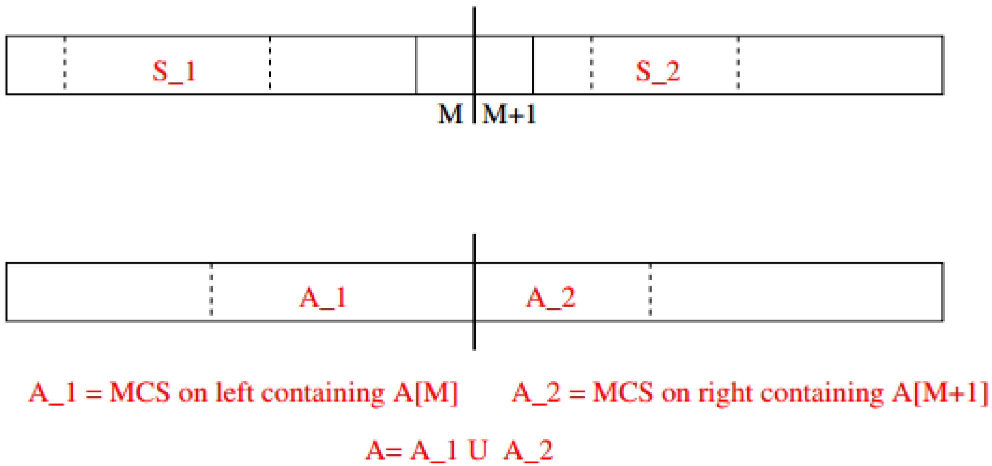

The MCS S must be one of

1. S_1: the MCS in $A[1 \ldots m]$
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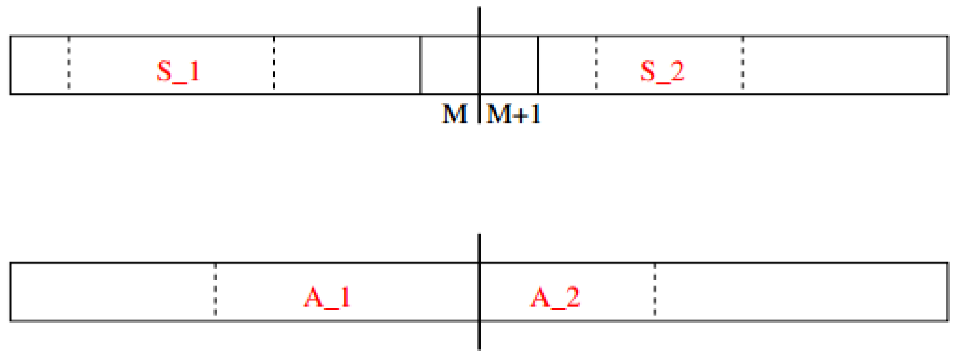

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So,

最终，在S_1, S_2和A(跨越中点的最大子数组)这三种情况中选取和最大者

$$S = \text{the best among } \{S_1, S_2, A\}$$

An Example of Divide-and-Conquer Algorithm

1	-5	4	2	-7	3	6	-1	2	-4	7	-10	2	6	1	-3

- $S_1 =$

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- $A_1 = [3, 6, -1]$ and $A_2 =$

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- $A_1 = [3, 6, -1]$ and $A_2 = [2, -4, 7]$
- $A = A_1 \cup A_2 =$

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- $Value(S_1) = 9;\ Value(S_2) = 9;\ Value(A) = 13$
- solution:

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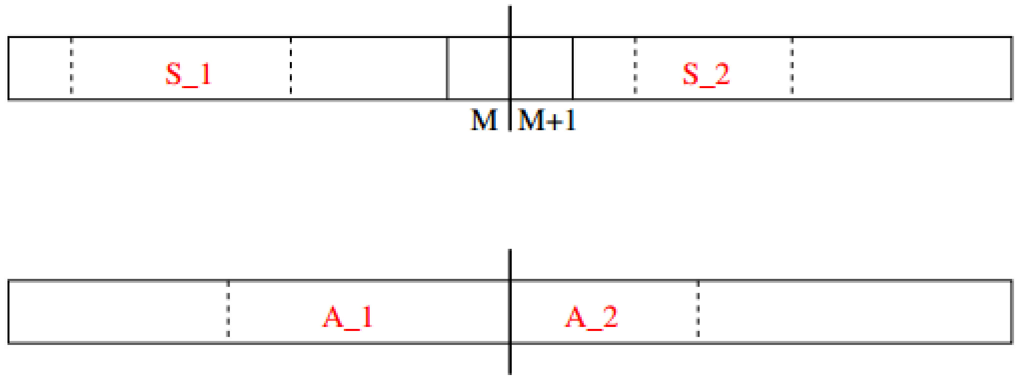
Divide: MCS across The Cut

Set $m = \lfloor (n+1)/2 \rfloor$



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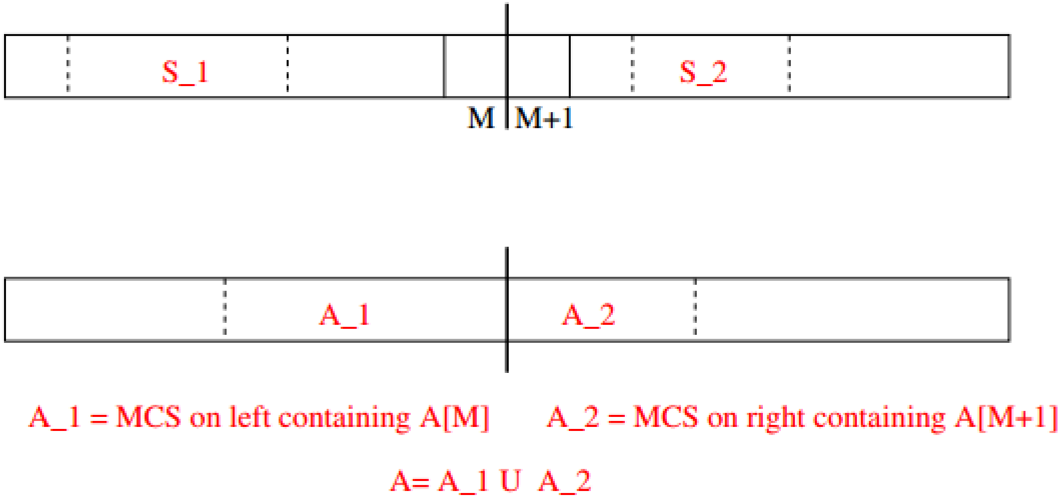

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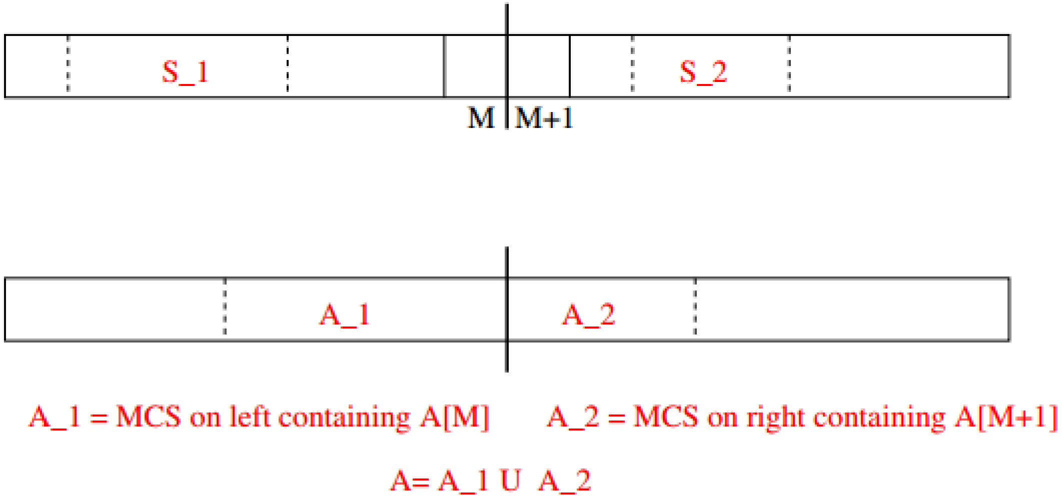

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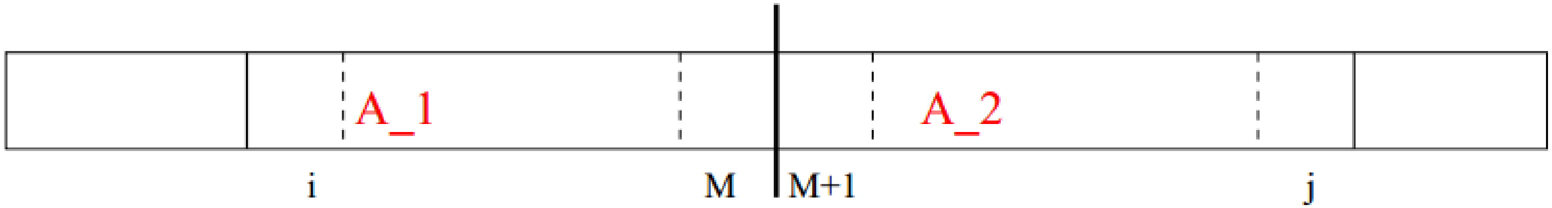
Conquer: Finding the "A_1" Subarrays



A_1 is in the form $A[i \ldots m]$, $V(i, m) = V(i+1, m) + A[i]$

MAX $\leftarrow A[m]$;
SUM $\leftarrow A[m]$;

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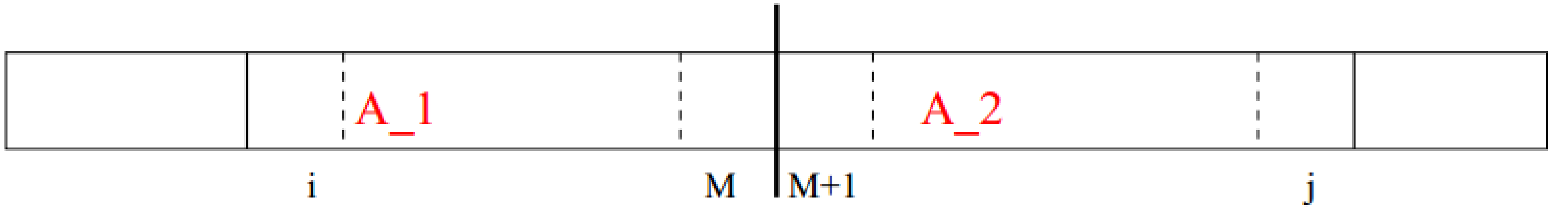

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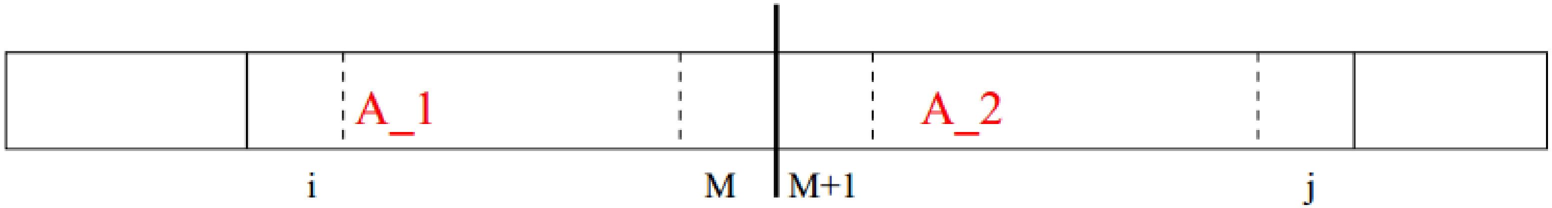
Conquer: Finding the "A_1" Subarrays



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  if $SUM > MAX$ **then**
    MAX $\leftarrow$ SUM;
  end
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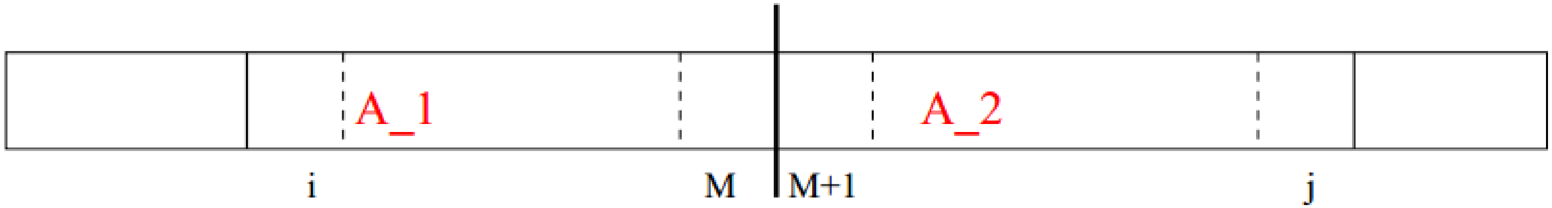

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for $i \leftarrow m-1$ **downto** *1* **do**
  SUM $\leftarrow$ SUM $+A[i]$;
  if *SUM* $>$ *MAX* **then**
    MAX $\leftarrow$ SUM;
  end
end
A_1=MAX;

Conquer: Finding "*A*" with A Linear Time

- There are only *m* sequences of the form

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  - linear to the input size

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$MCS(A, s, t)$

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    Find $MCS(A, s, m)$;

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Output: MCS of $A[s \dots t]$
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    Find $MCS(A, m+1, t)$;

The Complete Divide-and-Conquer Algorithm

$MCS(A, s, t)$

Input: $A[s \ldots t]$ with $s \leq t$
Output: MCS of $A[s \ldots t]$
begin
    if $s = t$ **then return** $A[s]$;
    else
        $m \leftarrow \lfloor \frac{s+t}{2} \rfloor$;
        Find $MCS(A, s, m)$;
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        Find MCS that contains both $A[m]$ and $A[m+1]$;
        return maximum of the three sequences found
    end
end

First Call: $MCS(A, 1, n)$

A Full Illustration of the D&C Algorithm

6 -4 7 -4 0 1

A Full Illustration of the D&C Algorithm

6 -4 7 ¦ -4 0 1

A Full Illustration of the D&C Algorithm

6 -4 7 | -4 0 1

6 -4 7

-4 0 1

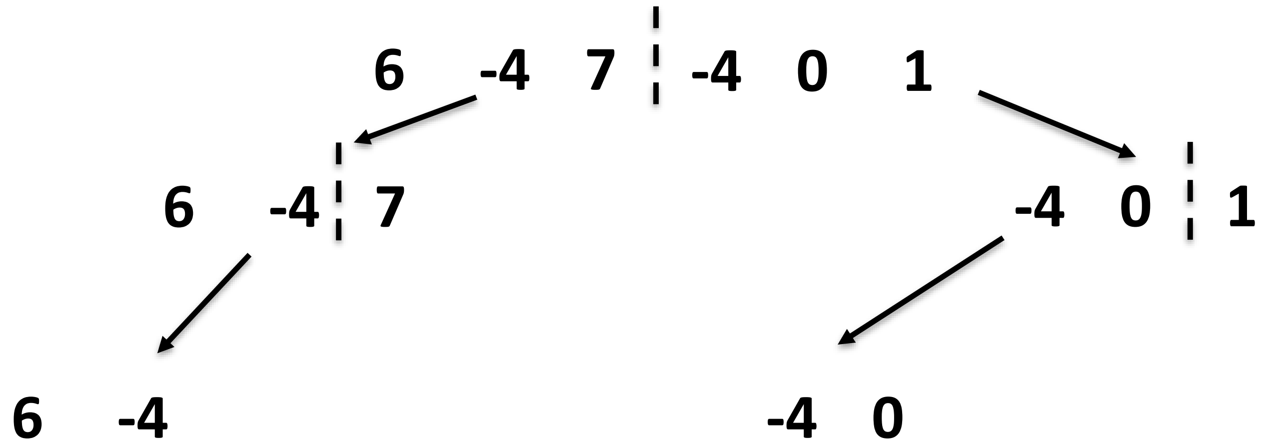
A Full Illustration of the D&C Algorithm

6 -4 7 ¦ -4 0 1

6 -4 ¦ 7

-4 0 ¦ 1

A Full Illustration of the D&C Algorithm



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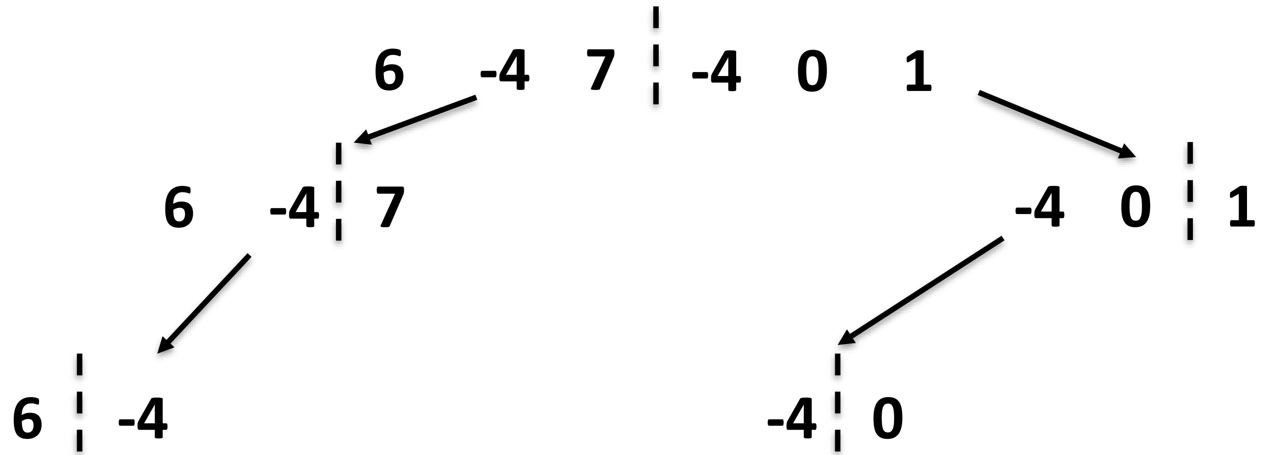

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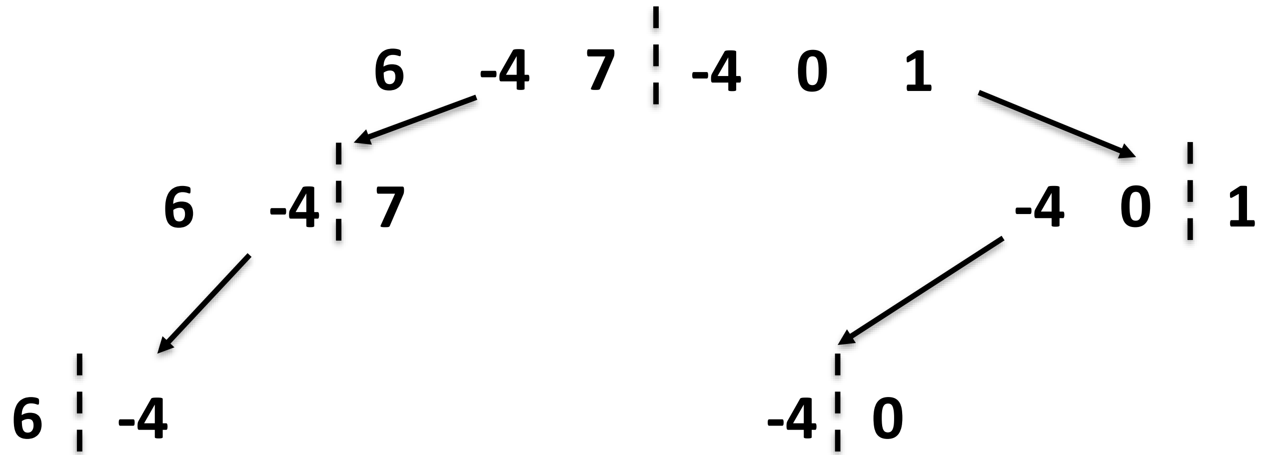
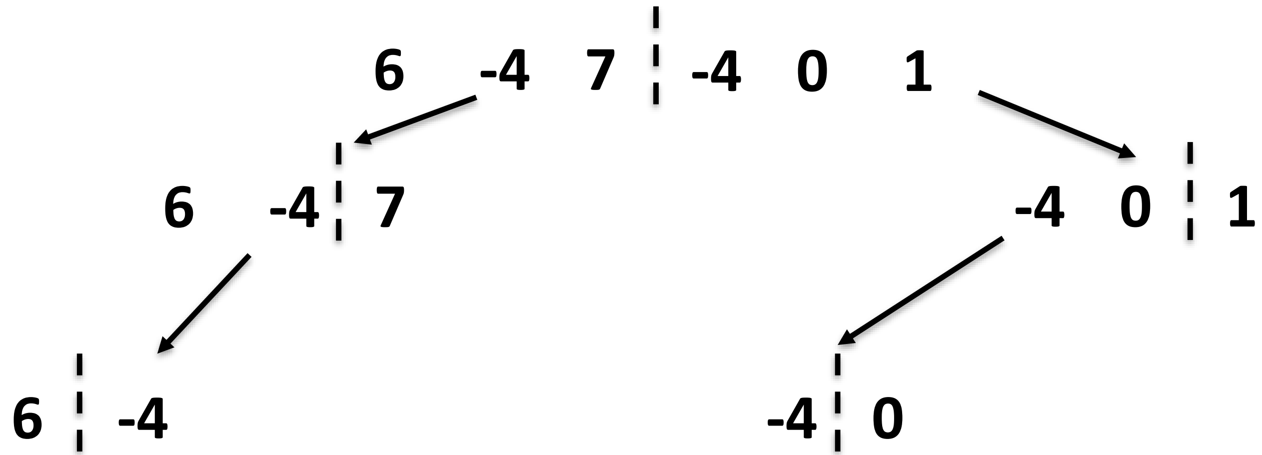


Divide

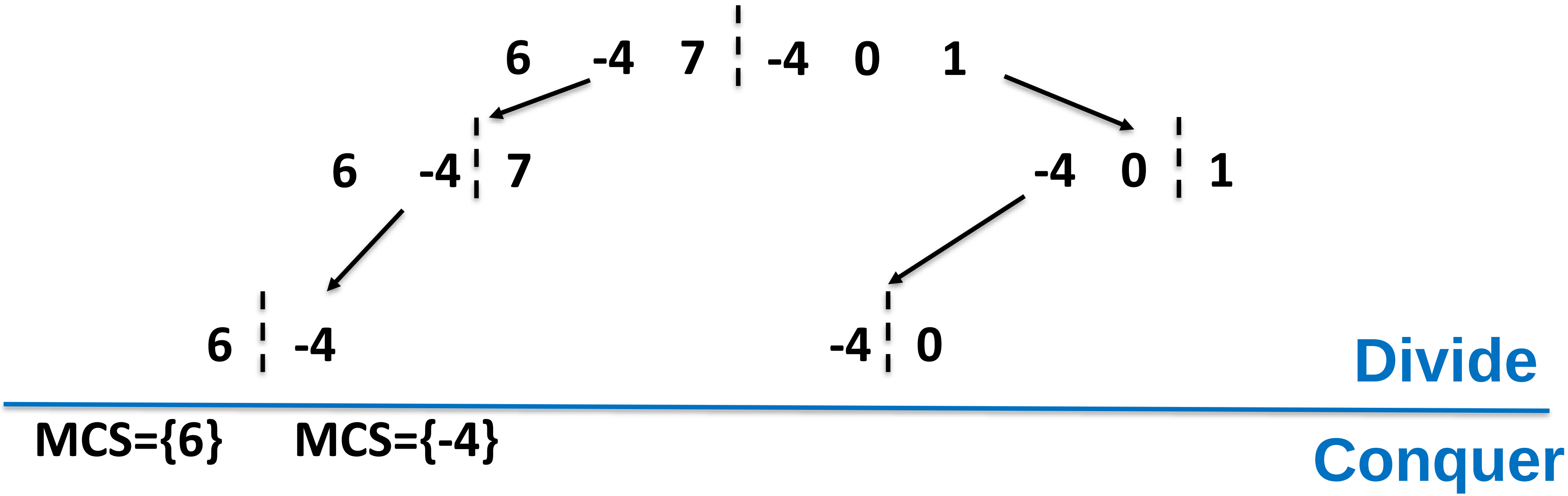
A Full Illustration of the D&C Algorithm



Divide

Conquer

A Full Illustration of the D&C Algorithm



A Full Illustration of the D&C Algorithm

6 -4 7 ¦ -4 0 1

6 -4 ¦ 7

-4 0 ¦ 1

6 ¦ -4

-4 ¦ 0

Divide

Conquer

MCS={6} **MCS={-4}**

A={6,-4}

Value(A)=2

A Full Illustration of the D&C Algorithm

6 -4 7 ¦ -4 0 1

6 -4 ¦ 7

-4 0 ¦ 1

6 ¦ -4

-4 ¦ 0

Divide

Conquer

MCS={6}

MCS={-4}

A={6,-4}

Value(A)=2

MCS={6}

A Full Illustration of the D&C Algorithm

6 -4 7 ¦ -4 0 1

6 -4 ¦ 7

-4 0 ¦ 1

6 ¦ -4

-4 ¦ 0

Divide

Conquer

MCS={6} MCS={-4}

A={6,-4}

Value(A)=2

MCS={6} MCS={7}

A Full Illustration of the D&C Algorithm

6 -4 7 ¦ -4 0 1

6 -4 ¦ 7 **-4 0 ¦ 1**

6 ¦ -4 **-4 ¦ 0**

Divide

Conquer

MCS={6} **MCS={-4}**

A={6,-4}

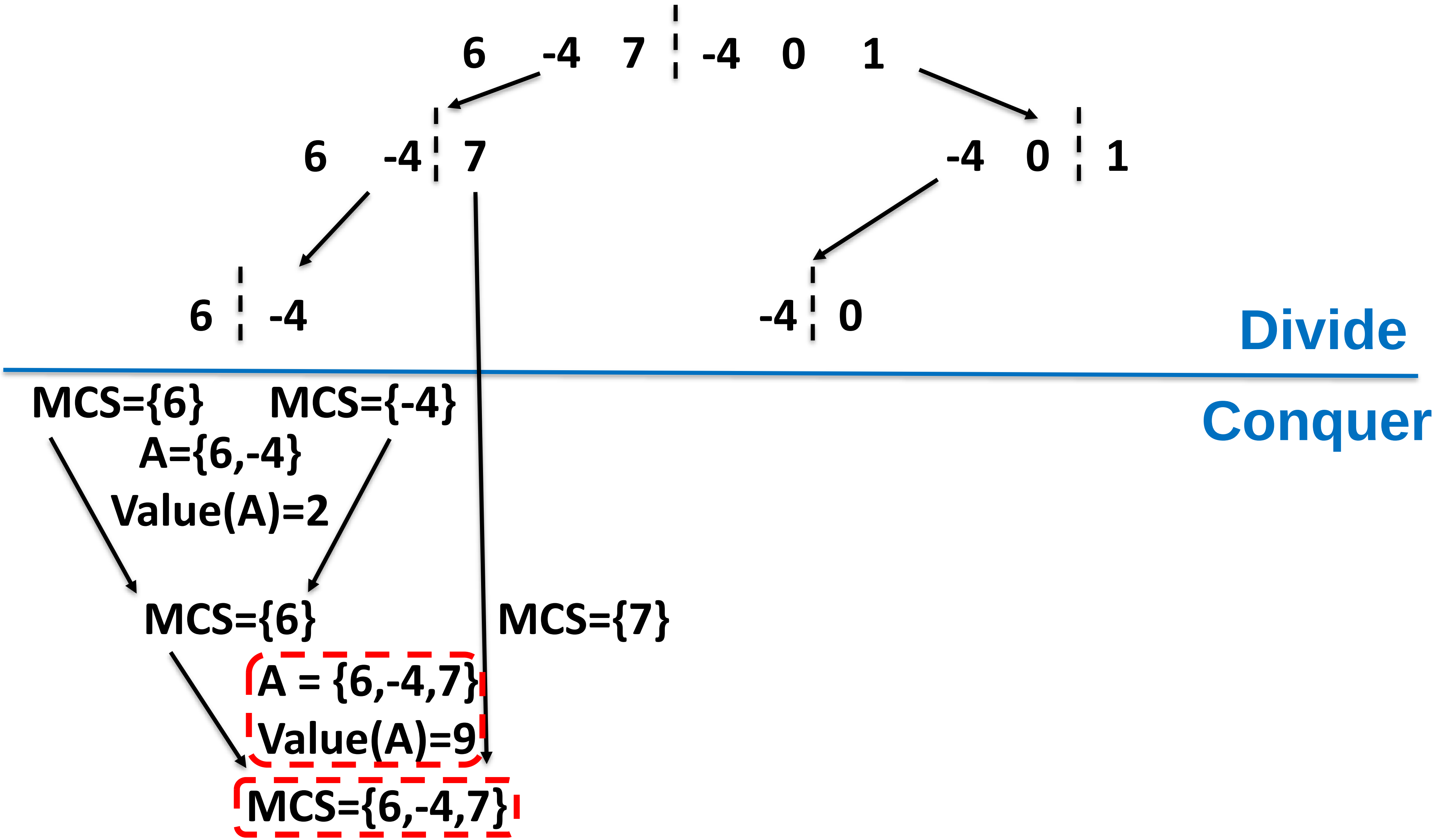
Value(A)=2

MCS={6} **MCS={7}**

A = {6,-4,7}

Value(A)=9

A Full Illustration of the D&C Algorithm



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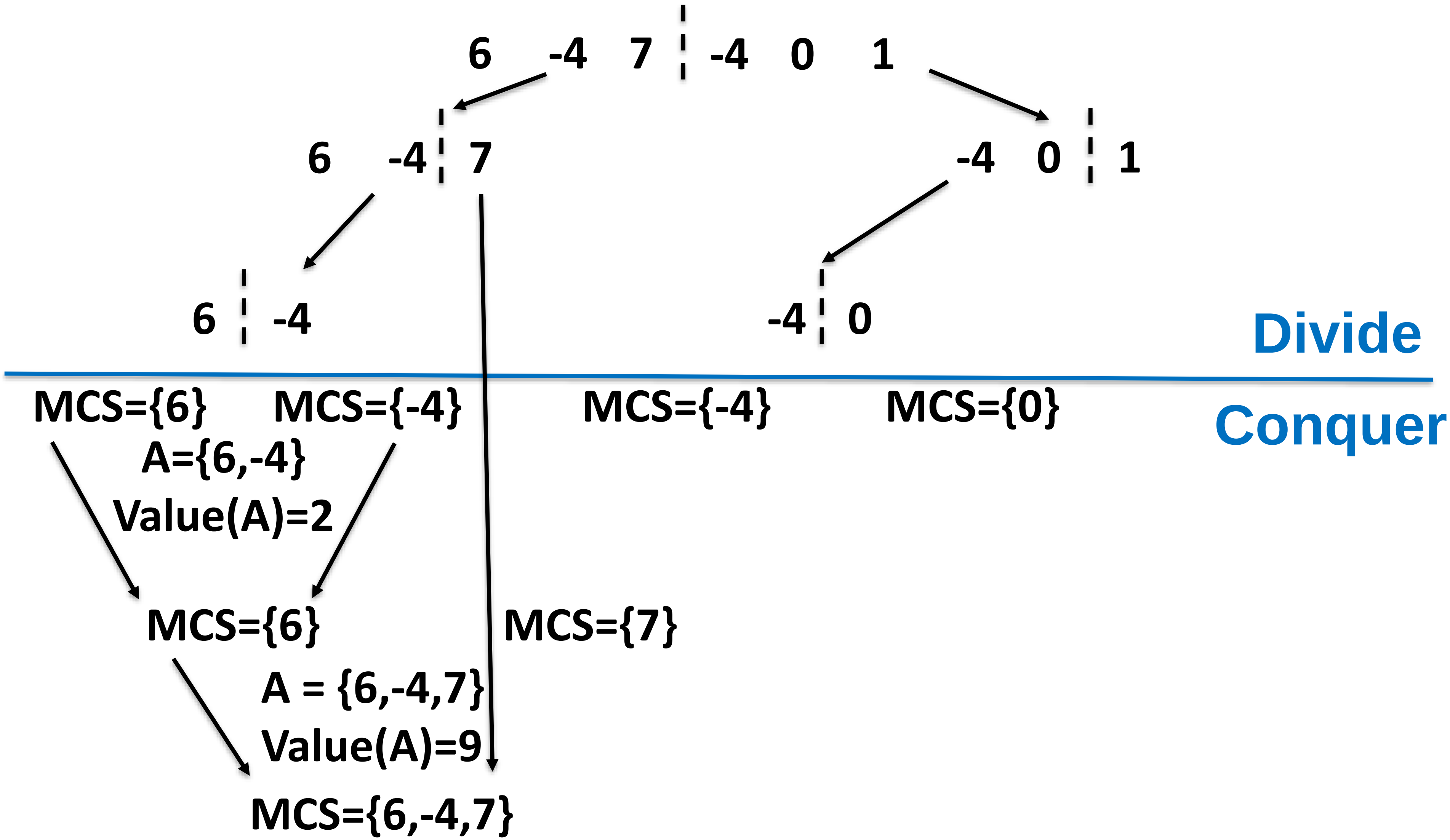

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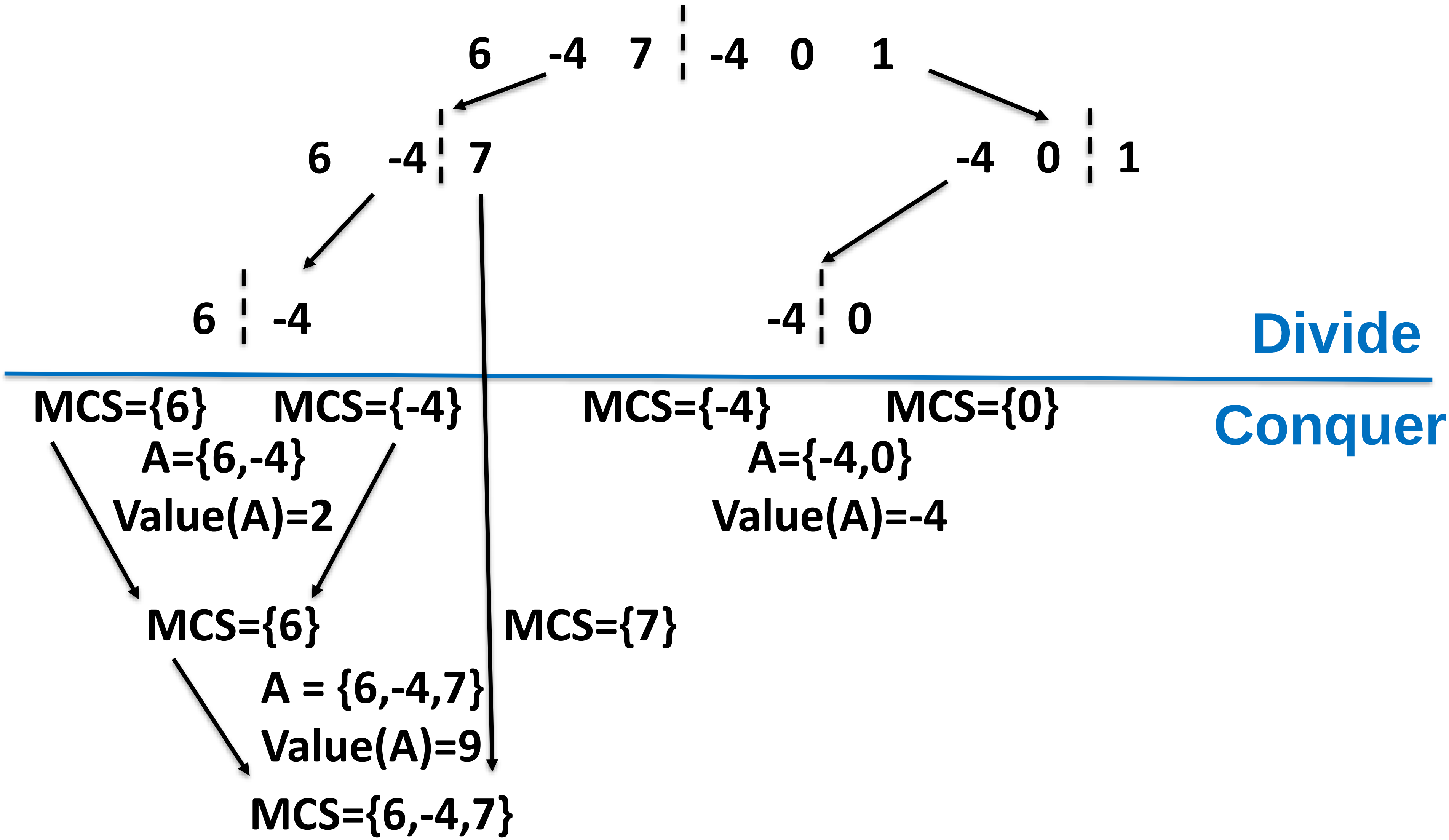

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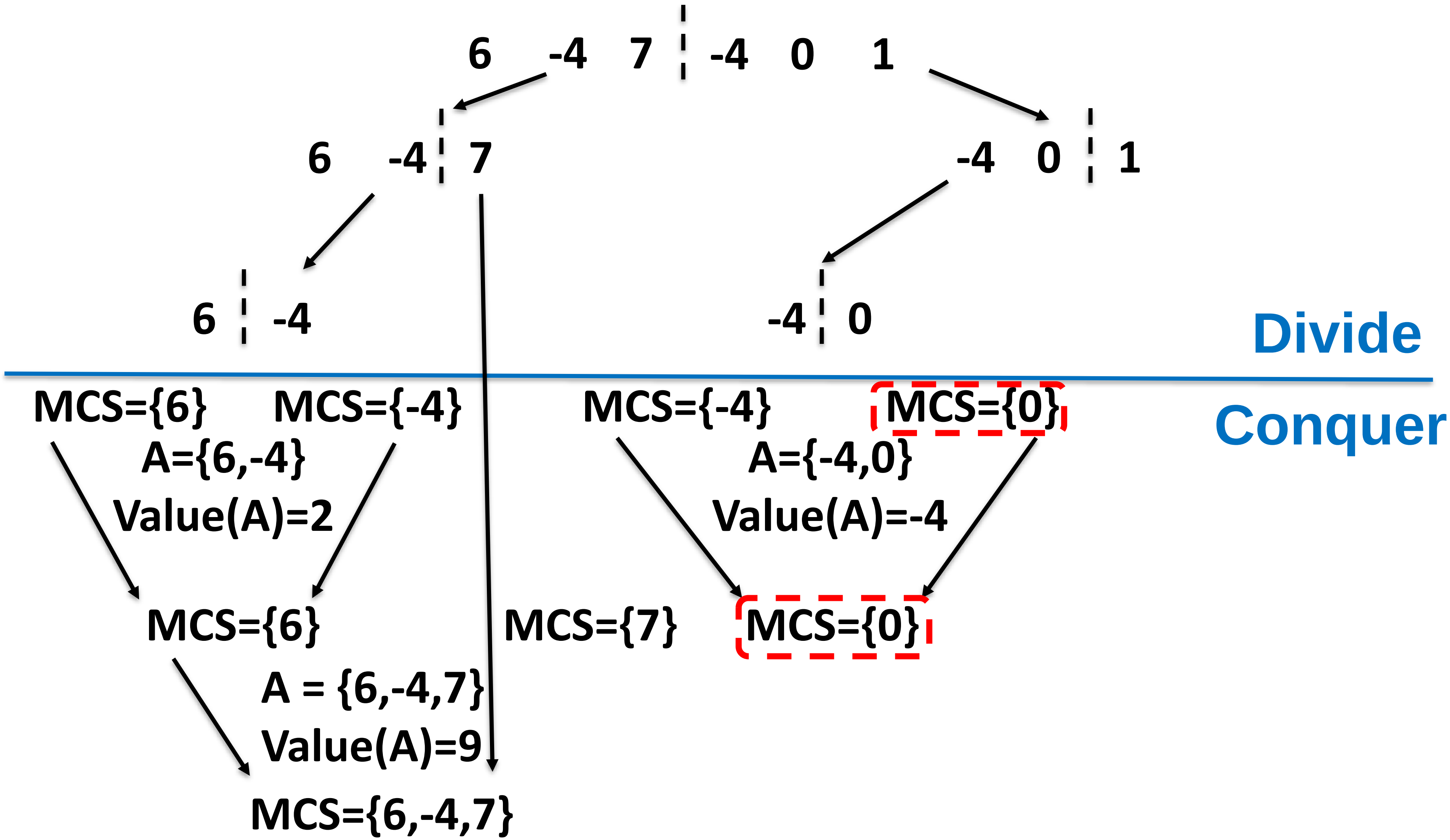


A Full Illustration of the D&C Algorithm

6 -4 7 ¦ -4 0 1

6 -4 ¦ 7

-4 0 ¦ 1

6 ¦ -4

-4 ¦ 0

Divide

Conquer

MCS={6} **MCS={-4}**

A={6,-4}

Value(A)=2

MCS={-4} **MCS={0}**

A={-4,0}

Value(A)=-4

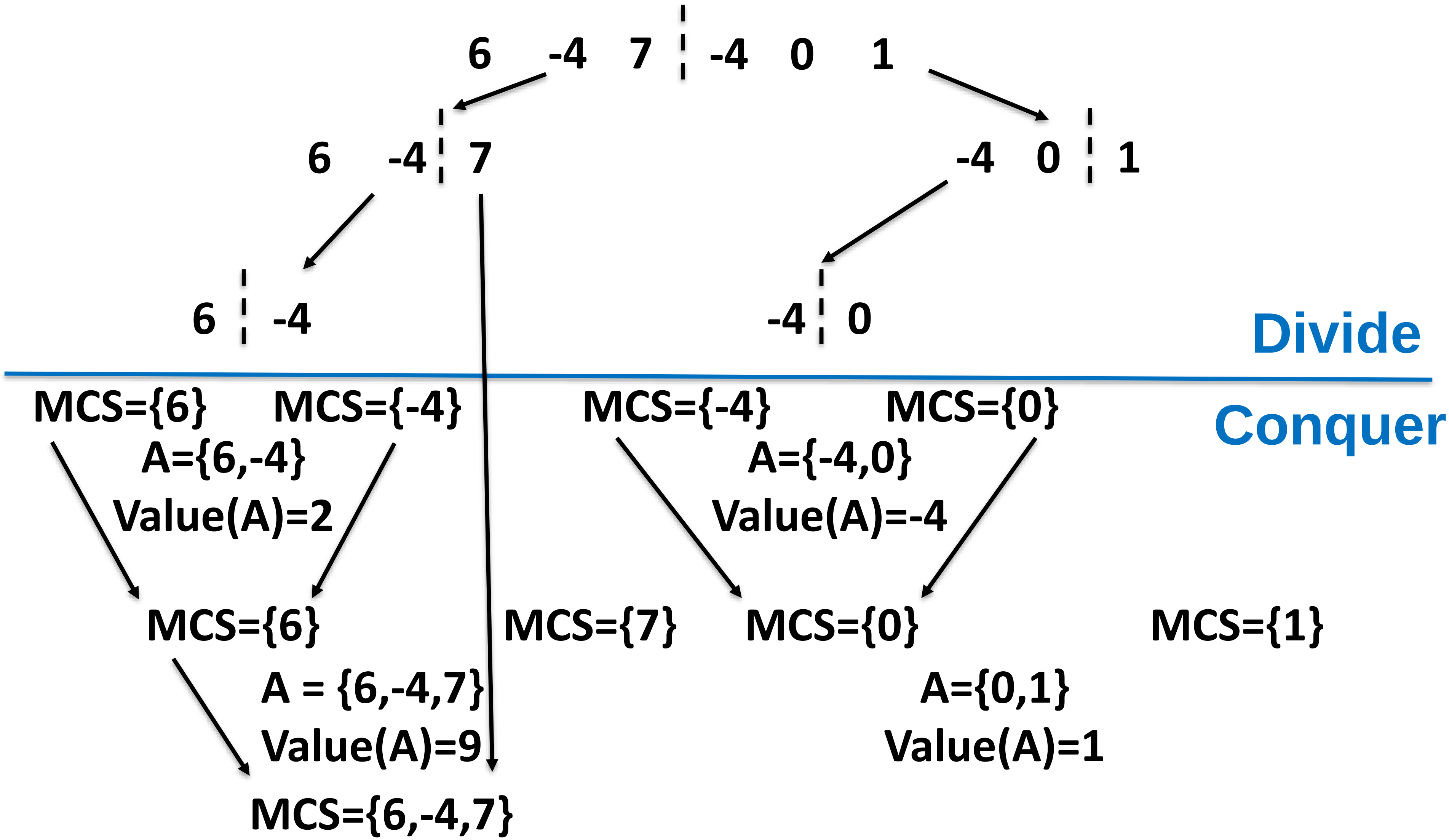
MCS={6} **MCS={7}** **MCS={0}** **MCS={1}**

A = {6,-4,7}

Value(A)=9

MCS={6,-4,7}

A Full Illustration of the D&C Algorithm



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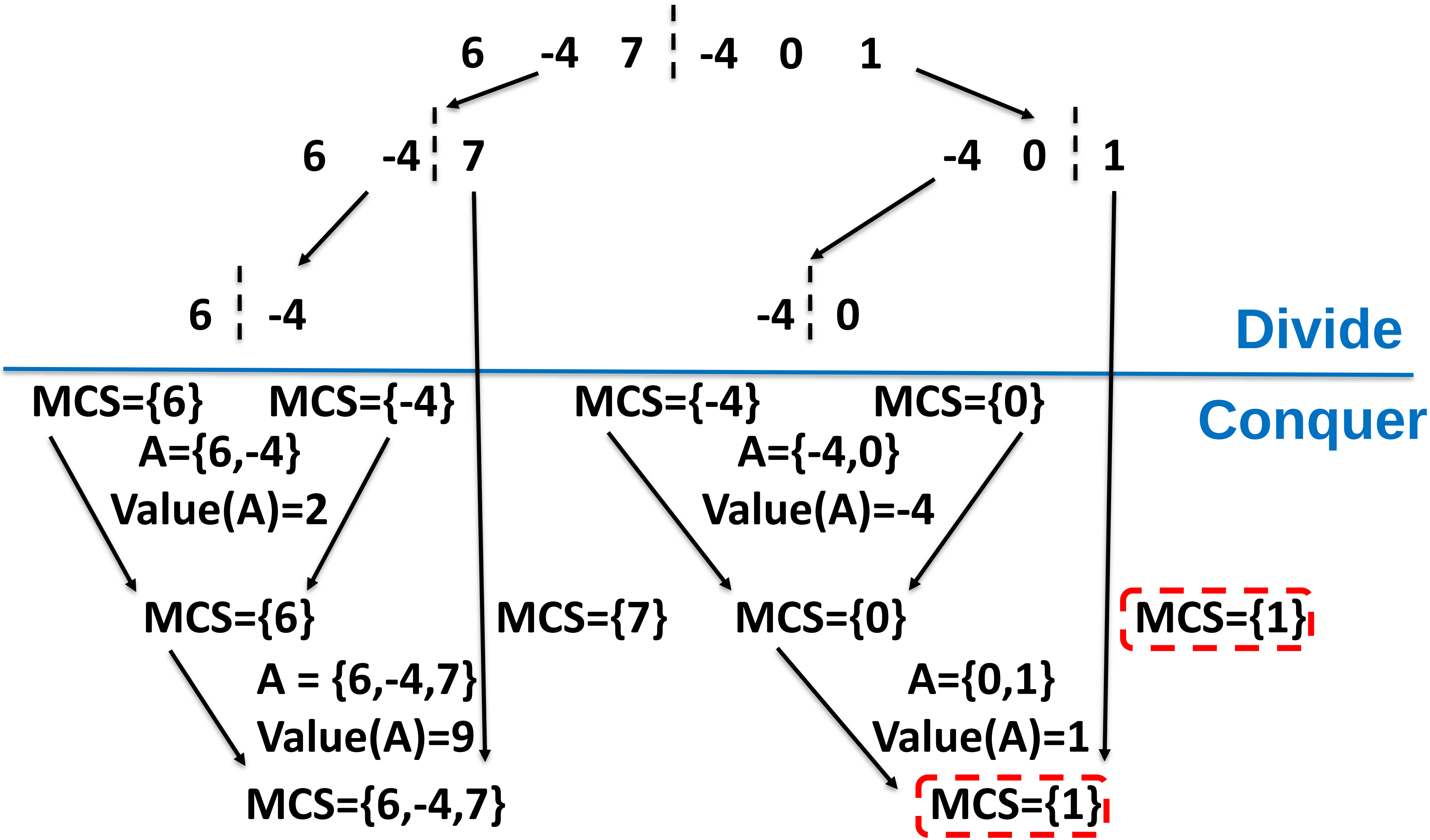

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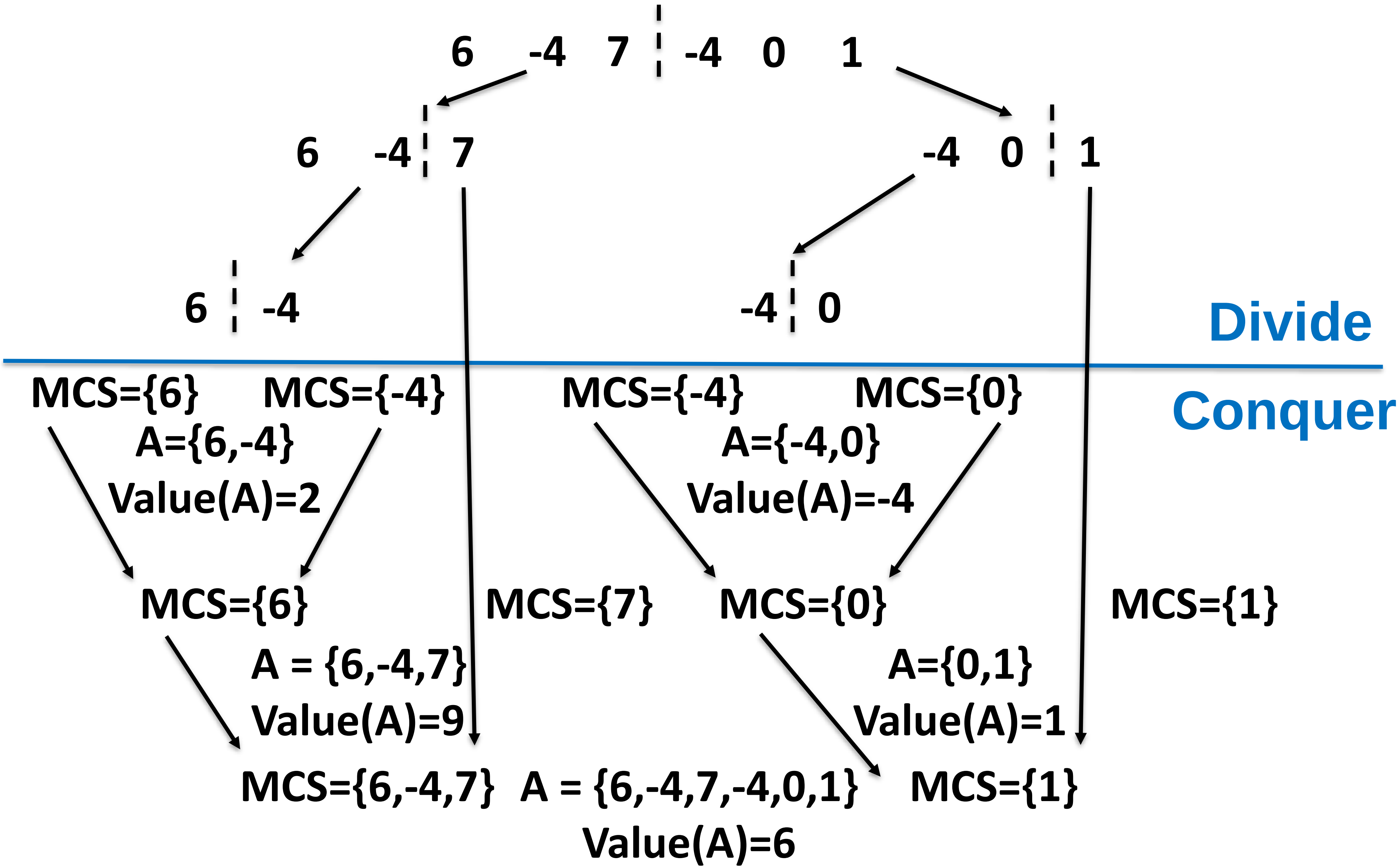

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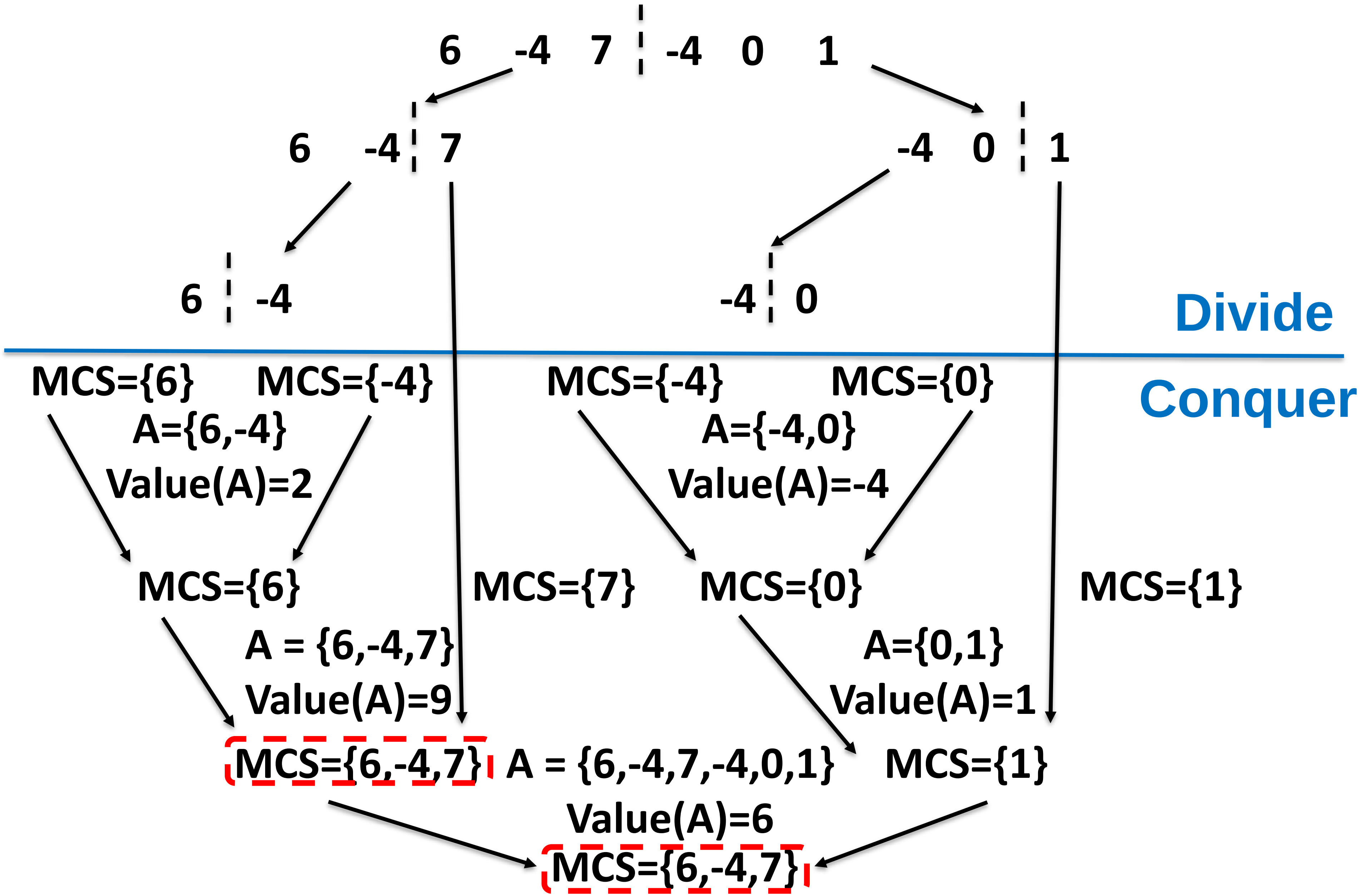

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Analysis of the D&C Algorithm

- n: problem size ($n = t - s + 1$)
- $T(n)$: time needed to run $MCS(A, s, t)$

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  else

    $m \leftarrow \lfloor \frac{s+t}{2} \rfloor$;

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```
begin
    if s = t then return A[s] // O(1)
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        m ← ⌊(s+t)/2⌋;
        Find MCS(A, s, m); // T(⌈n/2⌉)
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```

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end

$$T(1) = O(1)$$

$$T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n) \quad \text{for } n > 1$$

Analysis of the D&C Algorithm

To simplify the analysis, we assume that n is a power of 2

$$T(n) = \begin{cases} 2T(n/2) + n, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$

Analysis of the D&C Algorithm

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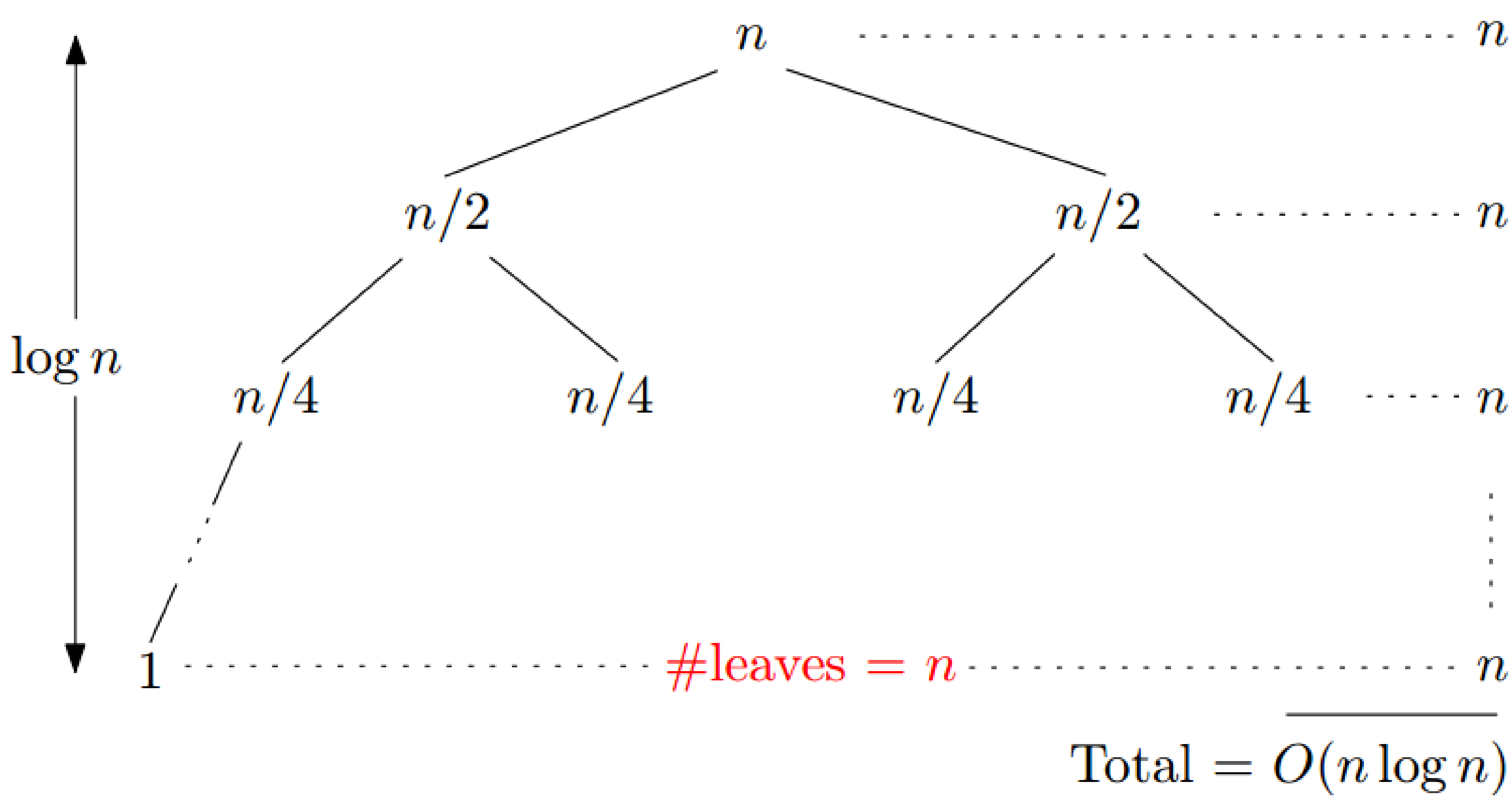

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- A $O(n^3)$ brute force algorithm
- A $O(n^2)$ algorithm that reuses data
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Can you solve the problem in $O(n)$ time?

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Counting inversions

Music site tries to match your song preferences with others.

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Similarity metric: number of inversions between two rankings.

	A	B	C	D	E
Me	1	2	3	4	5
You	1	3	4	2	5

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Similarity metric: number of inversions between two rankings.

- My rank: 1, 2, …, n.

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Similarity metric: number of inversions between two rankings.

- My rank: 1, 2, …, n.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j are inverted if i < j, but $a_i > a_j$

	A	B	C	D	E
Me	1	2	3	4	5
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A Brute Force Algorithm

List each pair i < j and count the inversions.

Input: L
Output: r
$r \leftarrow 0;$

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Output: r
$r \leftarrow 0;$
for $i \leftarrow 1$ *to* $L.length$ **do**

A Brute Force Algorithm

List each pair i < j and count the inversions.

Input: L
Output: r
$r \leftarrow 0$;
for $i \leftarrow 1$ *to* $L.length$ **do**
  for $j \leftarrow i+1$ *to* $L.length$ **do**
    if $L[i] > L[j]$ **then**

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List each pair i < j and count the inversions.

Input: L
Output: r
$r \leftarrow 0$;
for $i \leftarrow 1$ *to* $L.length$ **do**
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        if $L[i] > L[j]$ **then**
            $r \leftarrow r + 1$;

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List each pair i < j and count the inversions.

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$r \leftarrow 0;$
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        if $L[i] > L[j]$ **then**
            $r \leftarrow r + 1;$
        end
    end
end
return

A Brute Force Algorithm

List each pair i < j and count the inversions.

```
Input: L
Output: r
r ← 0;
for i ← 1 to L.length do
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        end
    end
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return r;
```

$O(n^2)$ comparisons and additions.

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Review to Merge Sort

Mergesort(A, left, right)

if *left* $<$ *right* **then**
    center $\leftarrow \lfloor(\text{left} + \text{right})/2\rfloor$;
    Mergesort(A, left, center);
    Mergesort(A, center+1, right);
    "Merge" the two sorted arrays;
end

- To sort the entire array $A[1 \ldots n]$, we make the initial call Mergesort(A, 1, n).
- Key subroutine: "Merge"

Counting inversions: divide-and-conquer

Input

14	**7**	**18**	**3**	**10**	**19**	**11**	**23**	**2**	**25**	**16**	**17**

Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.

Input

14	7	18	3	10	19	11	23	2	25	16	17

Counting inversions: divide-and-conquer

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Input

14	7	18	3	10	19	11	23	2	25	16	17

14	7	18	3	10	19

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Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.

Input

14	**7**	**18**	**3**	**10**	**19**	**11**	**23**	**2**	**25**	**16**	**17**

14	**7**	**18**	**3**	**10**	**19**

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Count inversions in left half A

14-7,14-3,14-10,7-3,18-3,18-10

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Input

14	7	18	3	10	19	11	23	2	25	16	17

14	7	18	3	10	19

Count inversions in left half A

14-7,14-3,14-10,7-3,18-3,18-10

11	23	2	25	16	17

Count inversions in right half B

11-2,23-2,23-16,23-17,25-16,25-17

Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with a∈A and b∈B.

Input

14	7	18	3	10	19	11	23	2	25	16	17

14	7	18	3	10	19

Count inversions in left half A

14-7,14-3,14-10,7-3,18-3,18-10

11	23	2	25	16	17

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14	7	18	3	10	19	11	23	2	25	16	17

14	7	18	3	10	19

Count inversions in left half A

14-7,14-3,14-10,7-3,18-3,18-10

11	23	2	25	16	17

Count inversions in right half B

11-2,23-2,23-16,23-17,25-16,25-17

Count inversions (a,b) with a∈A and b∈B

14-11,14-2,7-2,18-11,18-2,18-16,18-17,3-2,10-2,19-11,19-2,19-16,19-17

Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with a∈A and b∈B.
- Return sum of three counts.

Input

14	7	18	3	10	19	11	23	2	25	16	17

14	7	18	3	10	19

11	23	2	25	16	17

Count inversions in left half A

14-7,14-3,14-10,7-3,18-3,18-10

Count inversions in right half B

11-2,23-2,23-16,23-17,25-16,25-17

Count inversions (a,b) with a∈A and b∈B

14-11,14-2,7-2,18-11,18-2,18-16,18-17,3-2,10-2,19-11,19-2,19-16,19-17

Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with a∈A and b∈B.
- Return sum of three counts.

Input

14	7	18	3	10	19	11	23	2	25	16	17

14	7	18	3	10	19

11	23	2	25	16	17

Count inversions in left half A

14-7,14-3,14-10,7-3,18-3,18-10

Count inversions in right half B

11-2,23-2,23-16,23-17,25-16,25-17

Count inversions (a,b) with a∈A and b∈B

14-11,14-2,7-2,18-11,18-2,18-16,18-17,3-2,10-2,19-11,19-2,19-16,19-17

Output

6+6+13 =25

How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

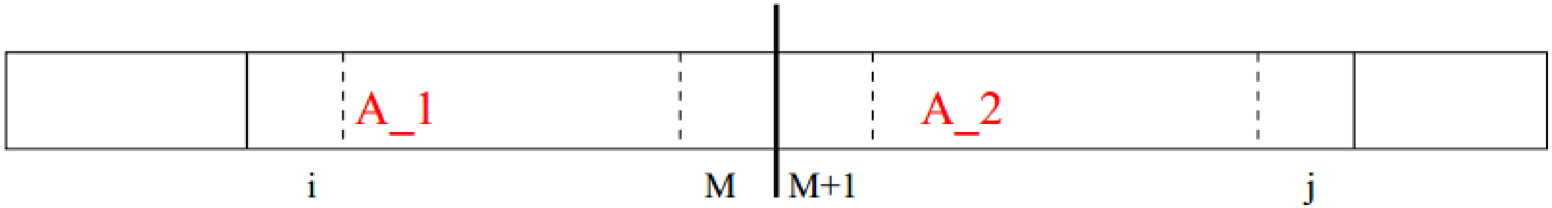
List A

14	7	18	3	10	19

List B

11	23	2	25	16	17

Review to the Conquer Step of MCS Problem



A_1 is in the form $A[i \ldots m]$, $V(i, m) = V(i+1, m) + A[i]$

MAX $\leftarrow A[m]$;
SUM $\leftarrow A[m]$;
for $i \leftarrow m - 1$ **downto** *1* **do**
  SUM $\leftarrow$ SUM $+A[i]$;
  if $SUM > MAX$ **then**
    MAX $\leftarrow$ SUM;
  end
end
A_1=MAX;

How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

A. Easy if A and B are sorted!

Warmup algorithm.

List A

14	7	18	3	10	19

List B

11	23	2	25	16	17

How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

A. Easy if A and B are sorted!

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- Sort A and B.

List A

14	7	18	3	10	19

List B

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How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.

Sort A

3	7	10	14	18	19

List B

11	23	2	25	16	17

How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.

Sort A

3	7	10	14	18	19

Sort B

2	11	16	17	23	25

How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element b ∈ B,

Sort A

3	7	10	14	18	19

Sort B

2	11	16	17	23	25

How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element b ∈ B,
  - binary search in A to find how many elements in A are greater than b.

Sort A

3	**7**	**10**	**14**	**18**	**19**

Sort B

2	**11**	**16**	**17**	**23**	**25**

How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

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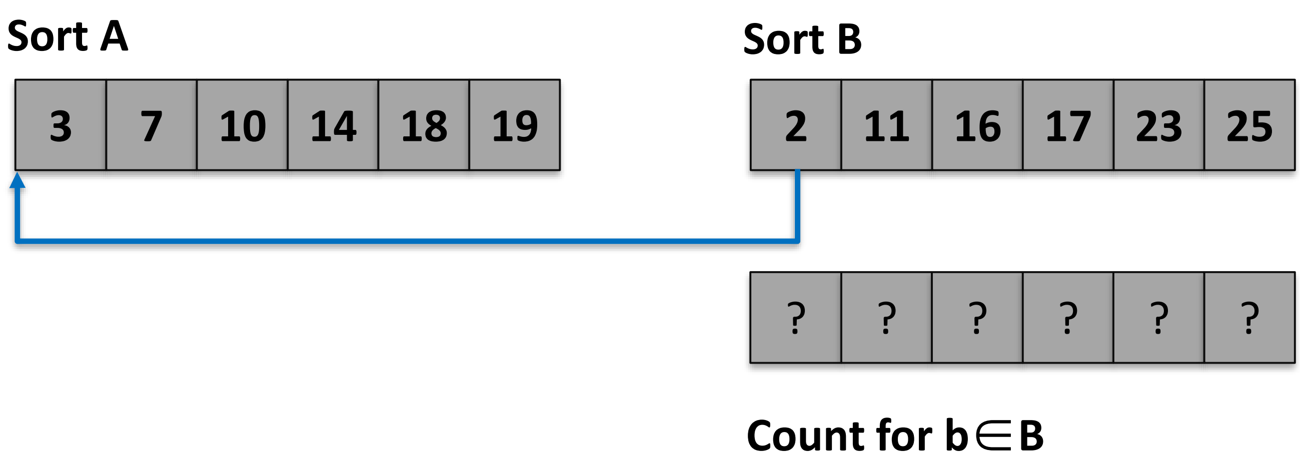

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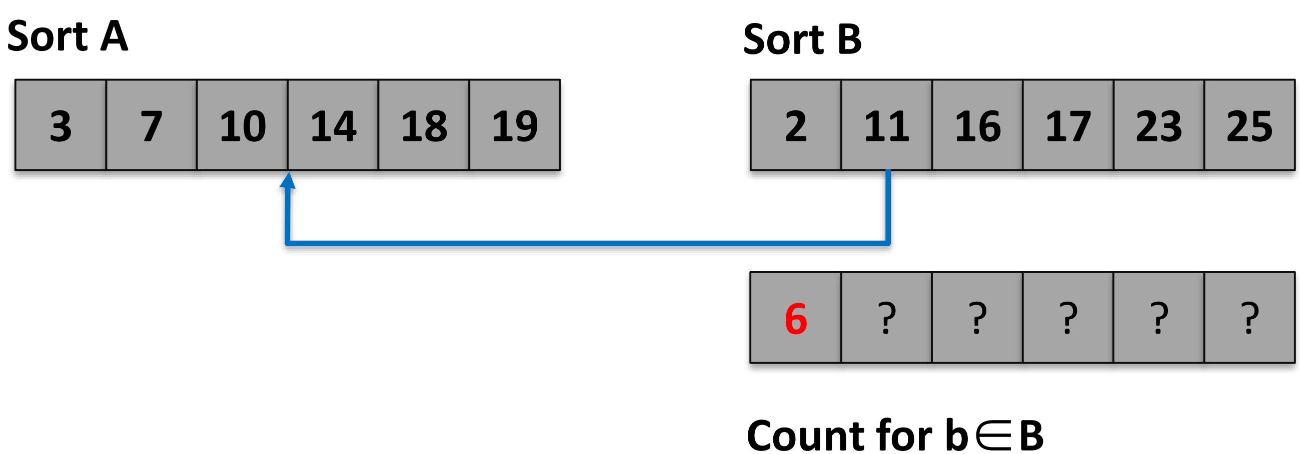

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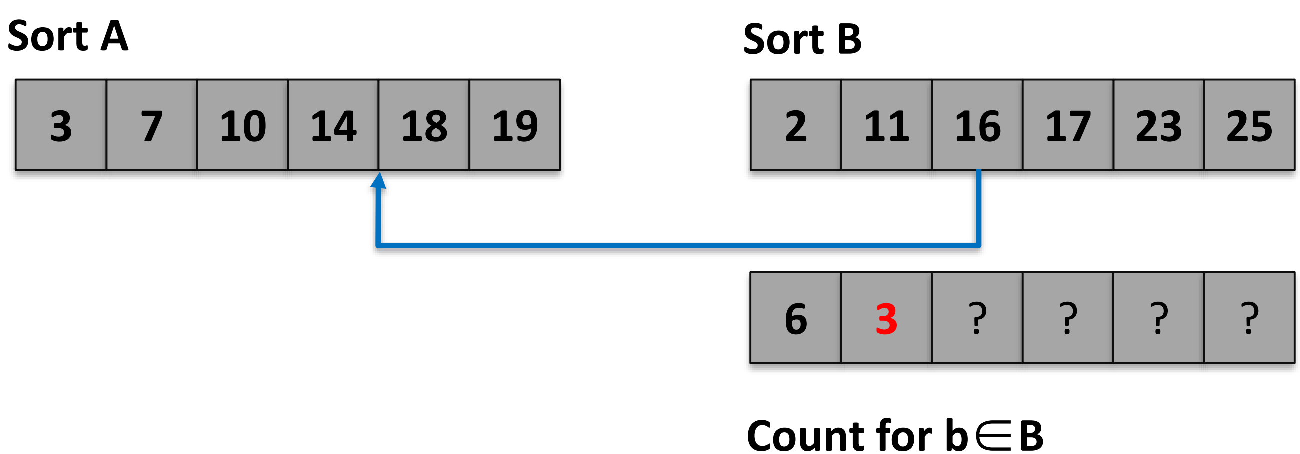

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- For each element b ∈ B,
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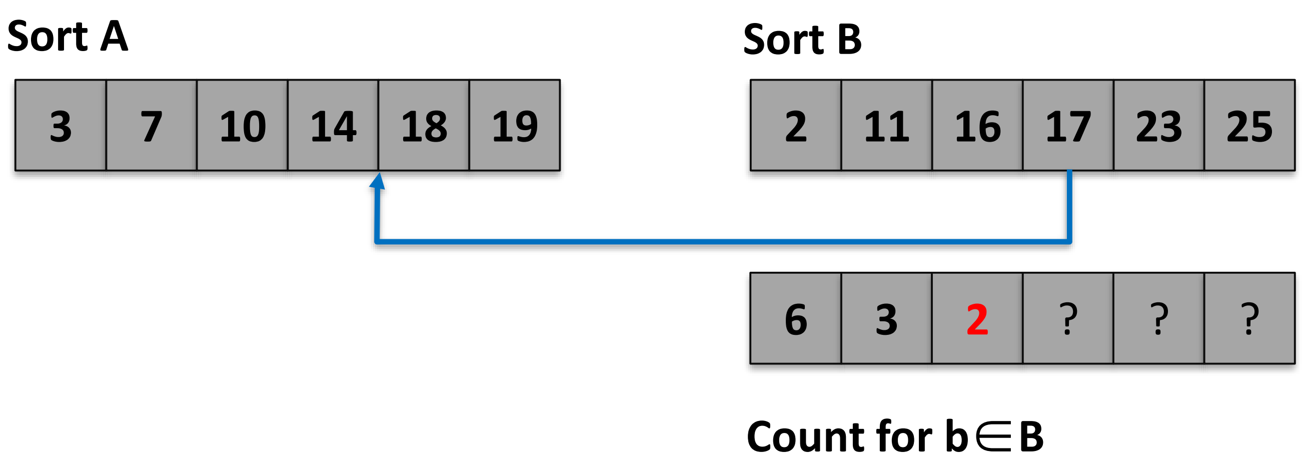

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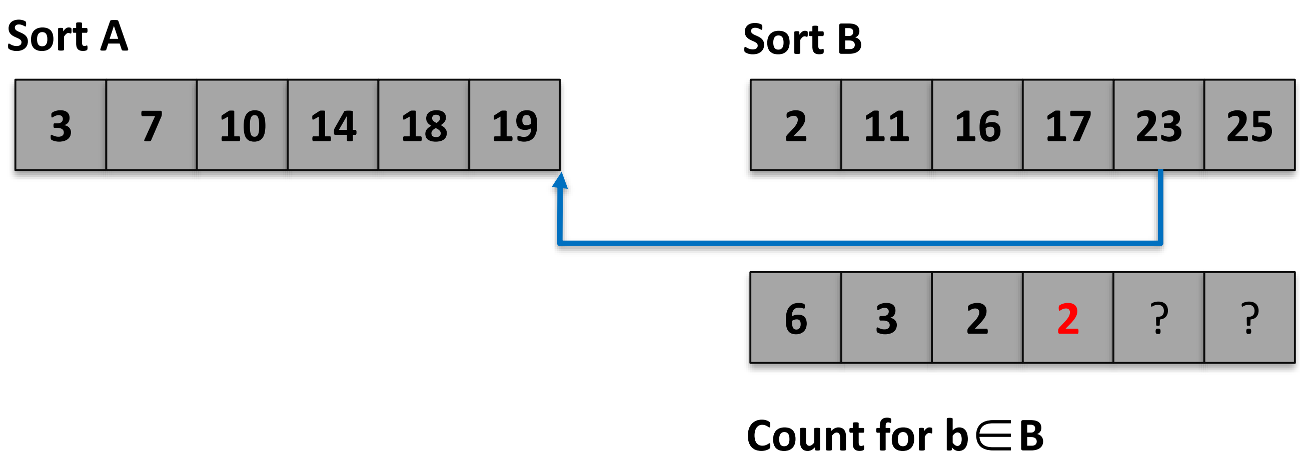
- For each element b ∈ B,
  - binary search in A to find how many elements in A are greater than b.



How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element b ∈ B,
  - binary search in A to find how many elements in A are greater than b.

Sort A

3	7	10	14	18	19

Sort B

2	11	16	17	23	25

6	3	2	2	0	?

Count for b∈B

How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element b ∈ B,
  - binary search in A to find how many elements in A are greater than b.

Sort A

3	7	10	14	18	19

Sort B

2	11	16	17	23	25

6	3	2	2	0	0

Count for b∈B

How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element b ∈ B,
  - binary search in A to find how many elements in A are greater than b.

Sort A

3	7	10	14	18	19

Sort B

2	11	16	17	23	25

Inversions between A and B: 6+3+2+2=13

6	3	2	2	0	0

Count for b∈B

Combine two subproblems: Improvement

Count inversions (a, b) with a $\in$ A and b $\in$ B, assuming A and B are sorted.

Combine two subproblems: Improvement

Count inversions (a, b) with a $\in$ A and b $\in$ B, assuming A and B are sorted.

- Scan A and B from left to right.

Combine two subproblems: Improvement

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then

Combine two subproblems: Improvement

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then

Combine two subproblems: Improvement

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.

Combine two subproblems: Improvement

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

3	7	10	14	18	19	2	11	16	17	23	25	**Two sorted halves**
												Auxiliary array

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
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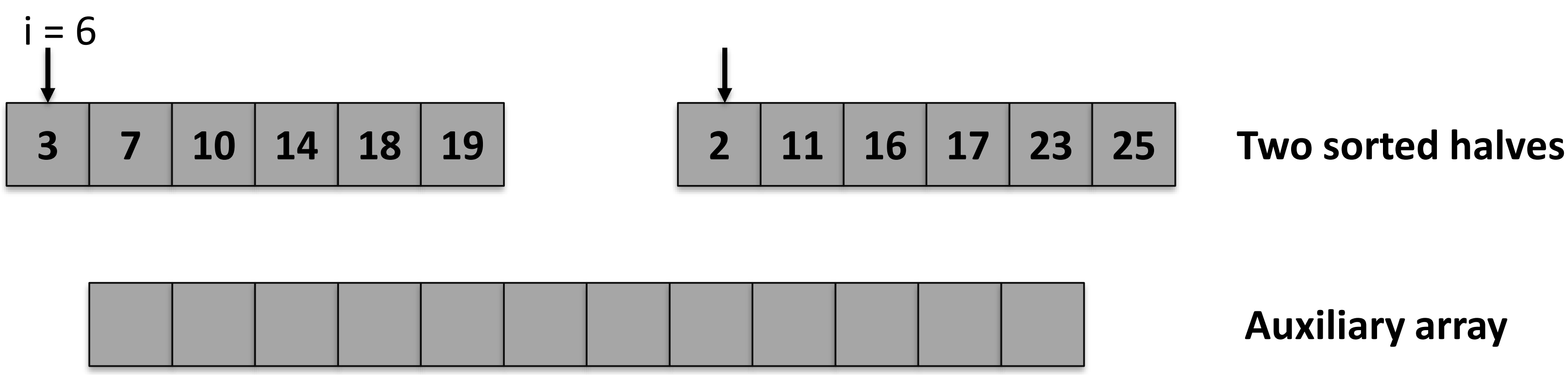
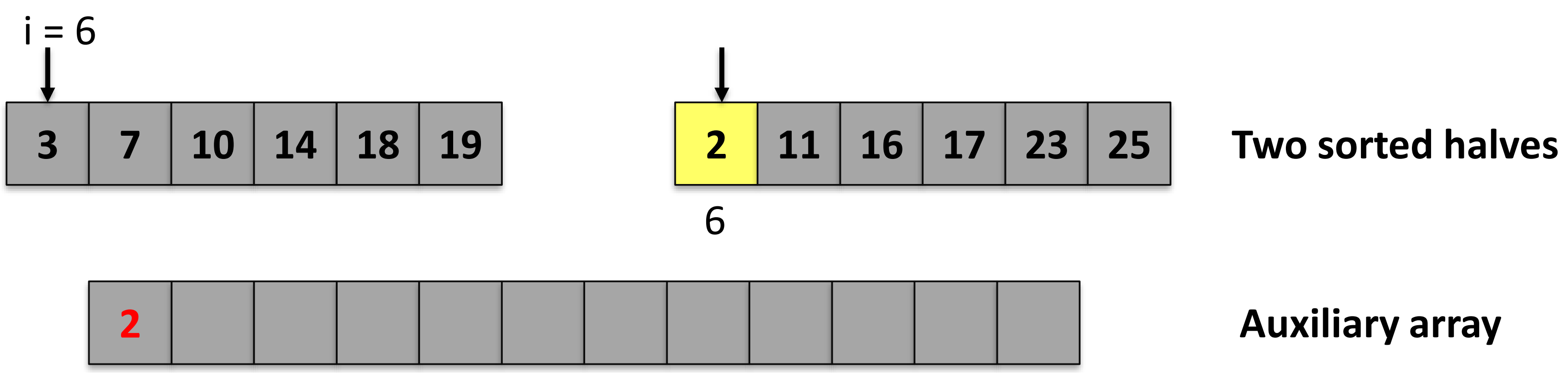


Total:

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

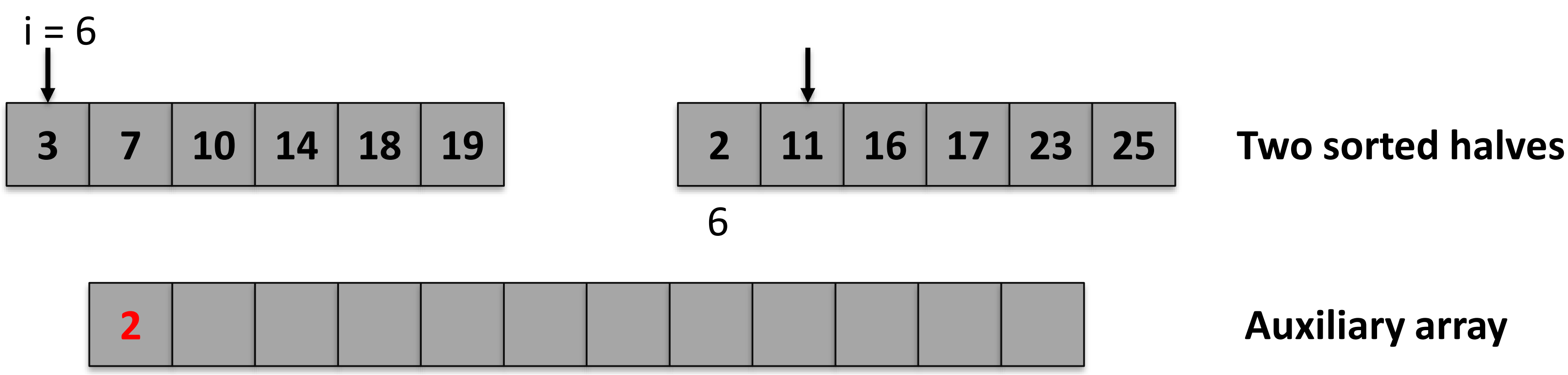


Total: 6

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

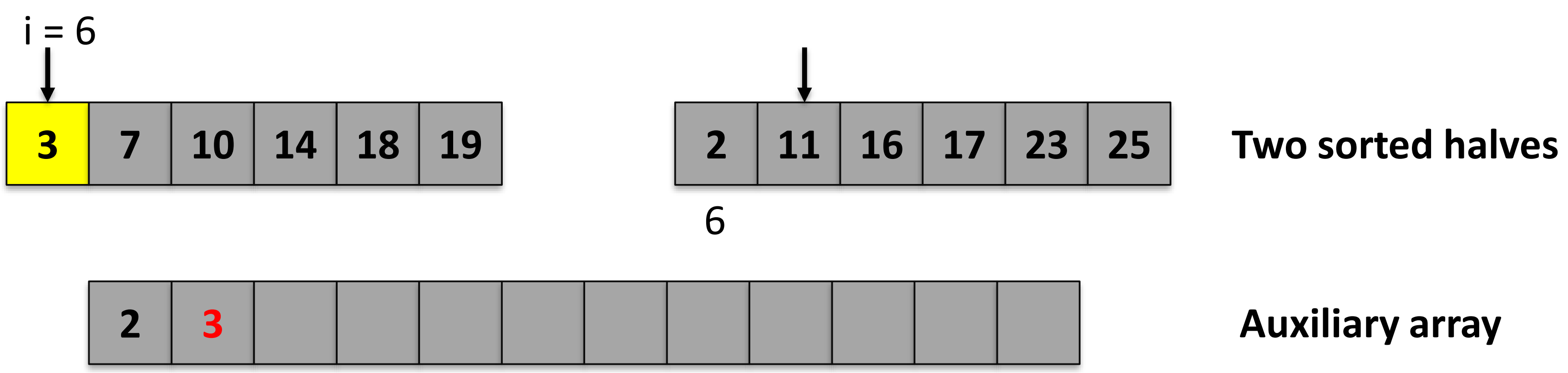


Total: 6

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

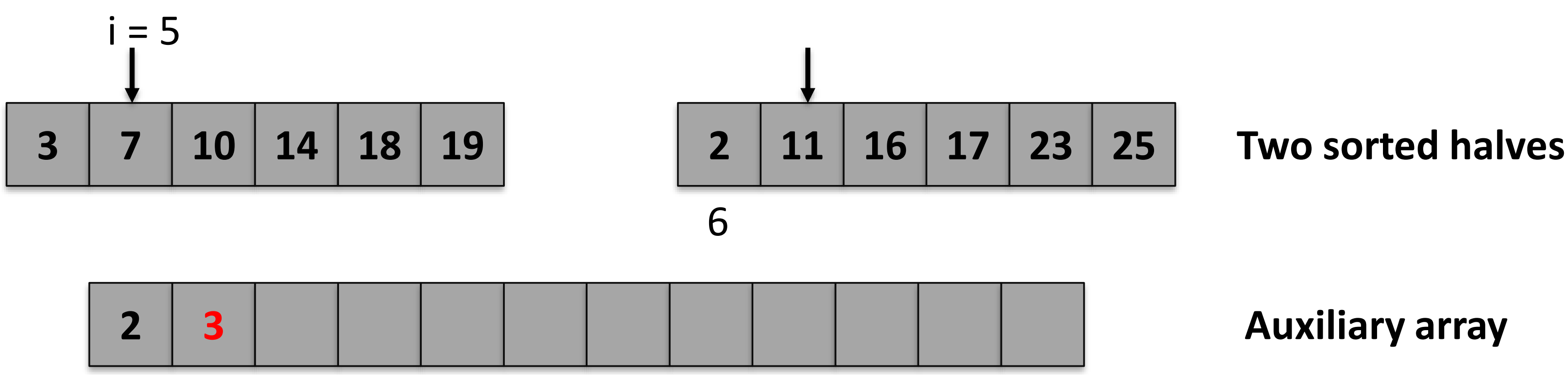


Total: 6

Combine two subproblems: Improvement

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.



Total: 6

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

i = 5

3	7	10	14	18	19

2	11	16	17	23	25

Two sorted halves

6

2	3	7									

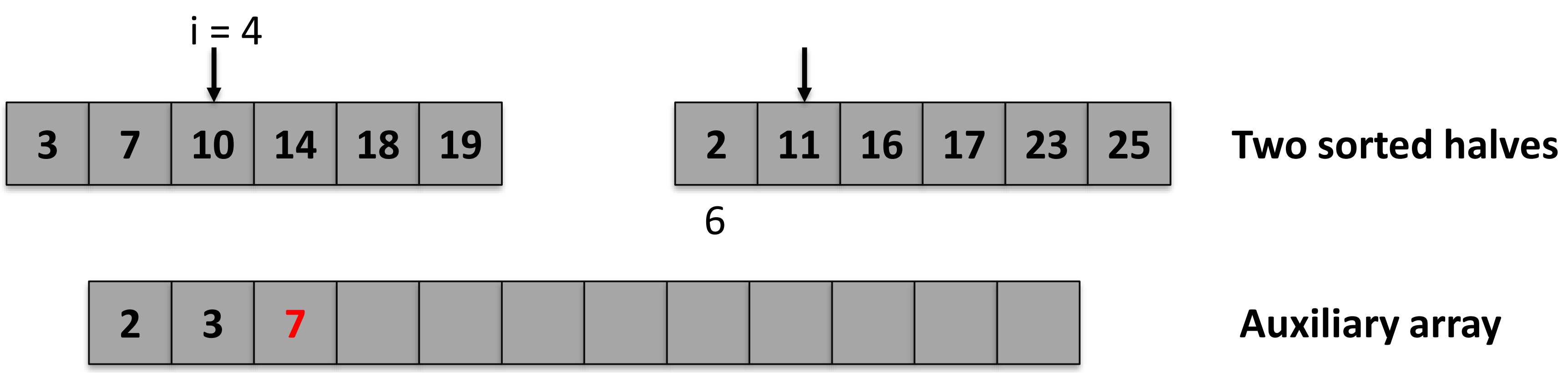
Auxiliary array

Total: 6

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

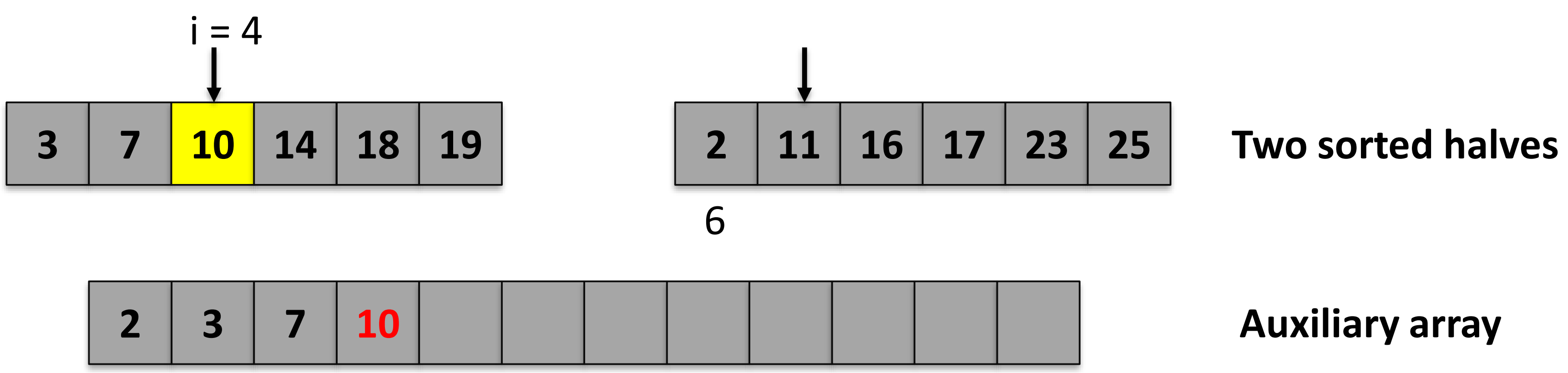


Total: 6

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

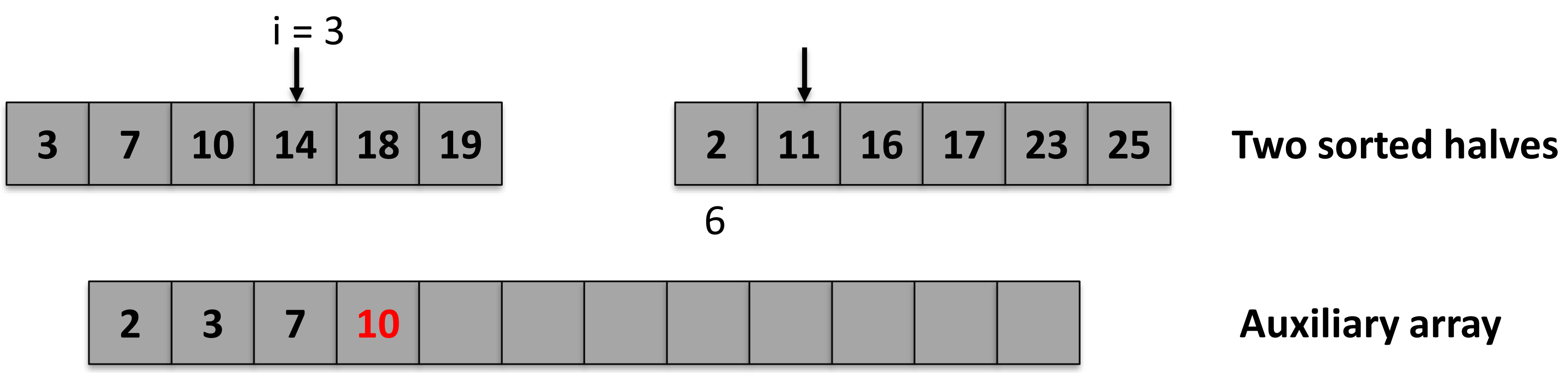


Total: 6

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.



Total: 6

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
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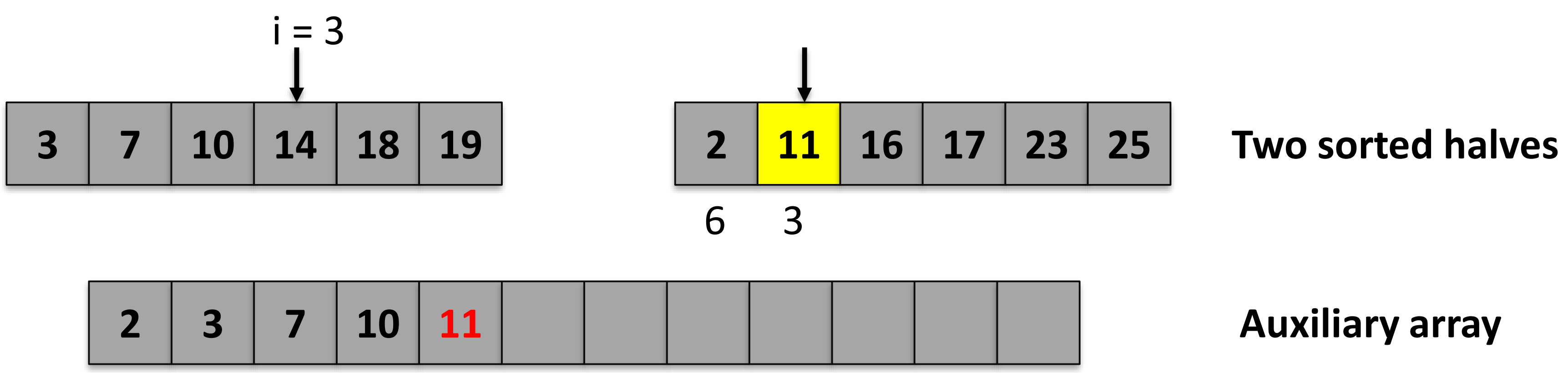

Total: 6+3

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
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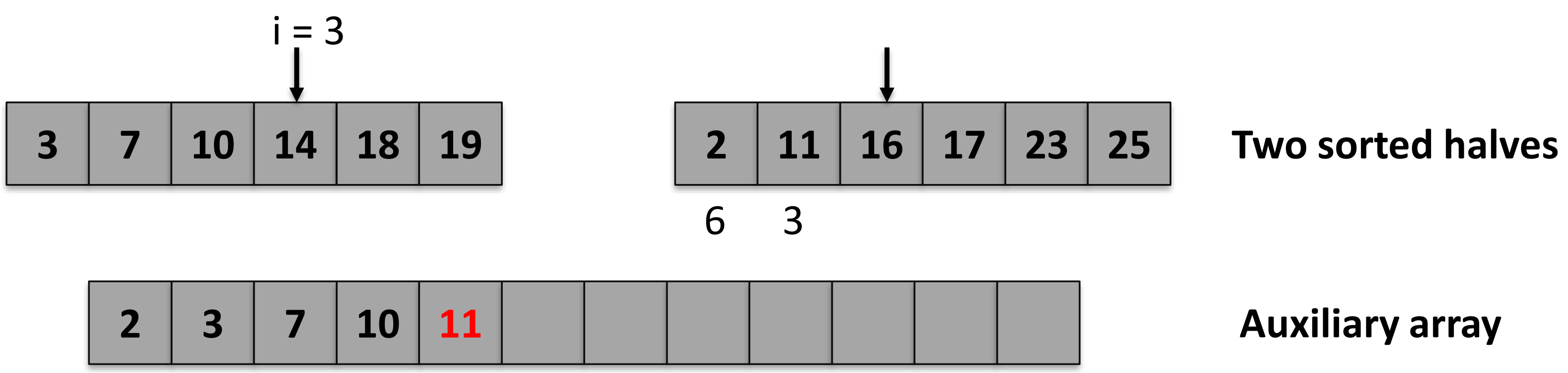

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Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

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  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

i = 3

3	7	10	**14**	18	19	2	11	16	17	23	25	**Two sorted halves**

6 3

2	3	7	10	11	14							**Auxiliary array**

Total: 6+3

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
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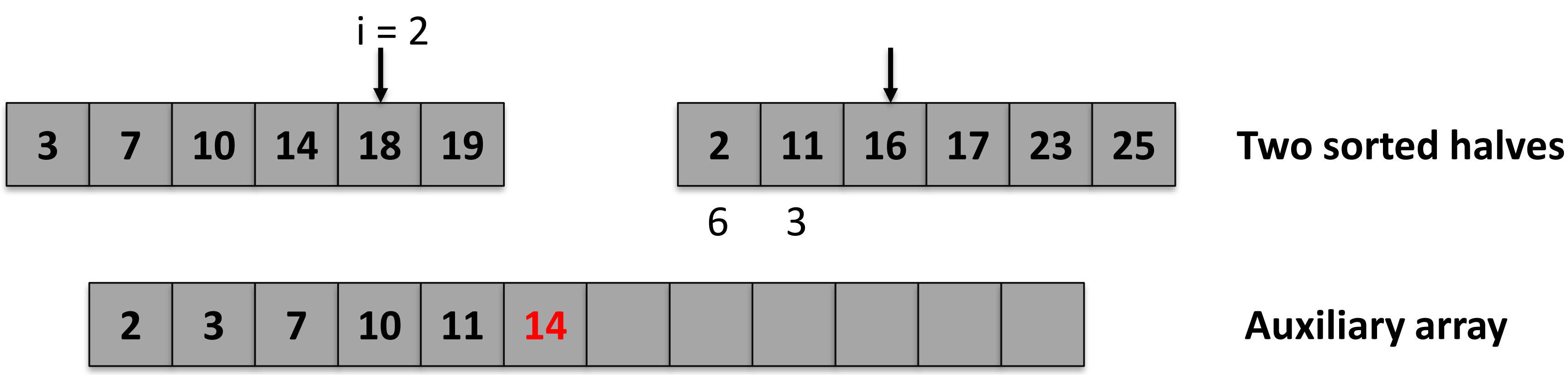

Total: 6+3

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Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

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- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

i = 2

Two sorted halves

3	7	10	14	18	19

2	11	16	17	23	25
6	3	2			

Auxiliary array

2	3	7	10	11	14	16					

Total: 6+3+2

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
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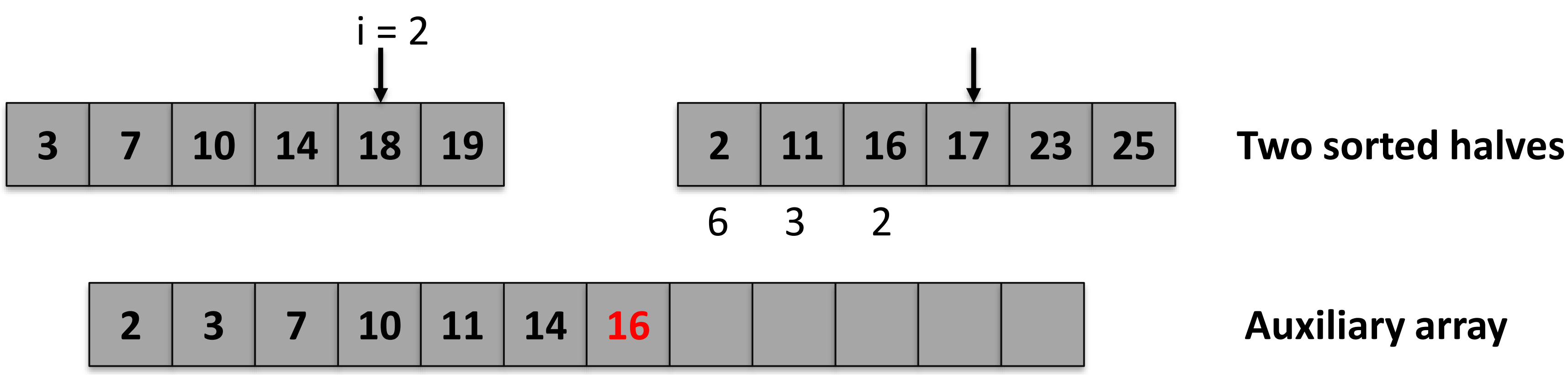
- Append smaller element to sorted list C.



Total: 6+3+2

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

i = 2

Two sorted halves

3	**7**	**10**	**14**	**18**	**19**

2	**11**	**16**	**17**	**23**	**25**
6	3	2	2		

Auxiliary array

2	**3**	**7**	**10**	**11**	**14**	**16**	**17**				

Total: 6+3+2+2

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

i = 2

3	7	10	14	18	19

2	11	16	17	23	25
6	3	2	2		

Two sorted halves

2	3	7	10	11	14	16	17				

Auxiliary array

Total: 6+3+2+2

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
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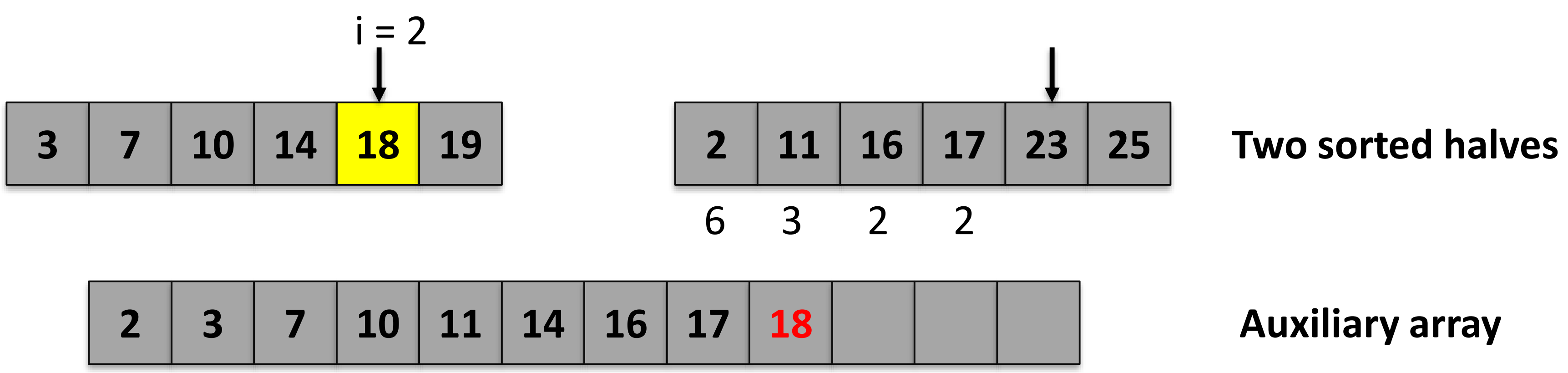
- Append smaller element to sorted list C.



Total: 6+3+2+2

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
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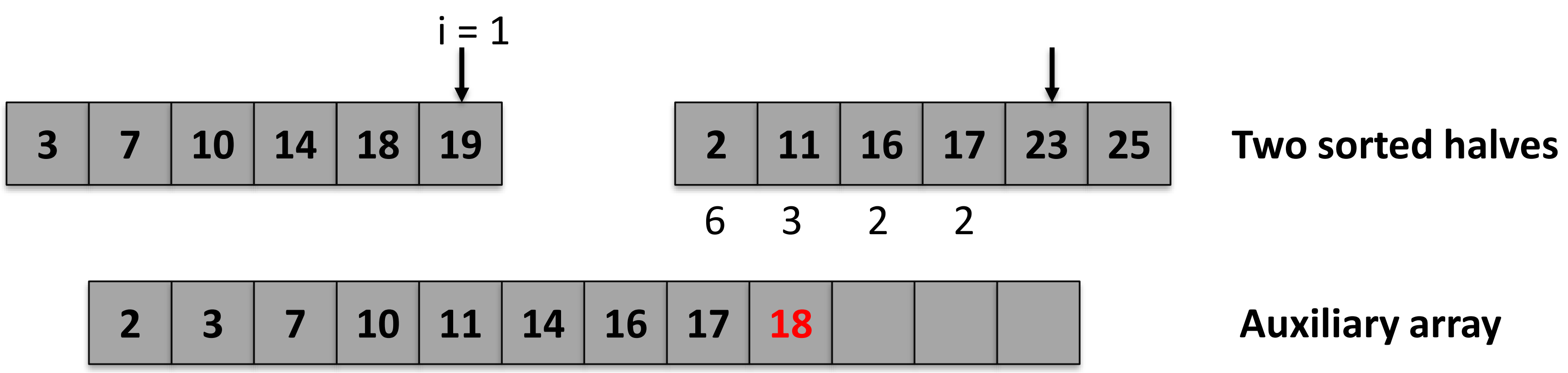

Total: 6+3+2+2

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
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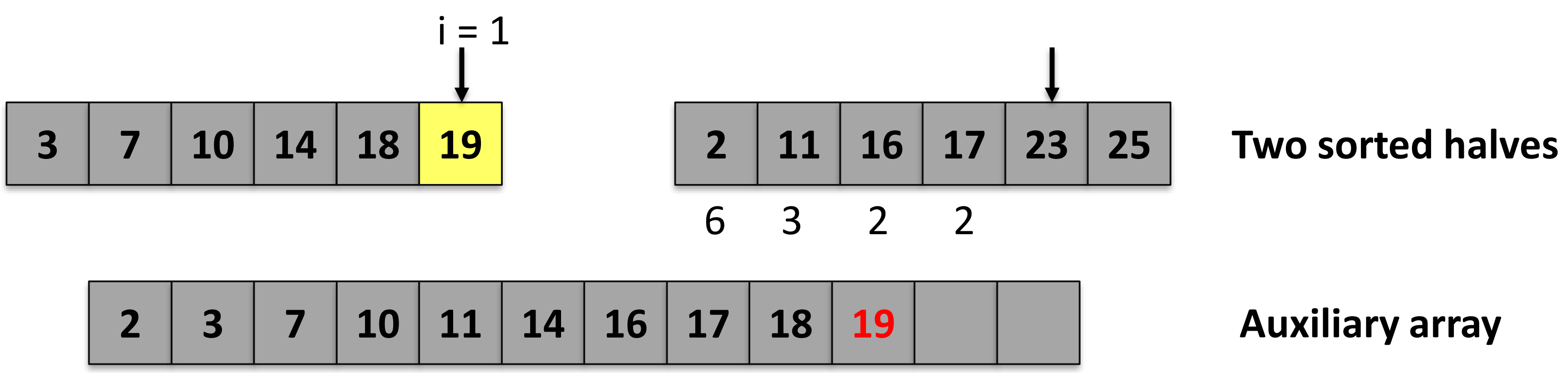
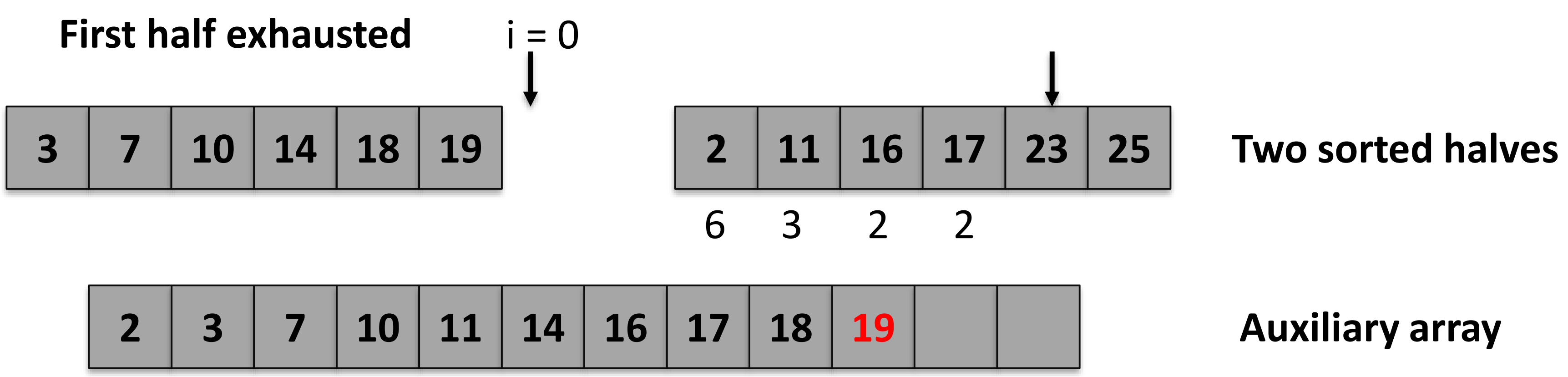


Total: 6+3+2+2

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.



Total: 6+3+2+2

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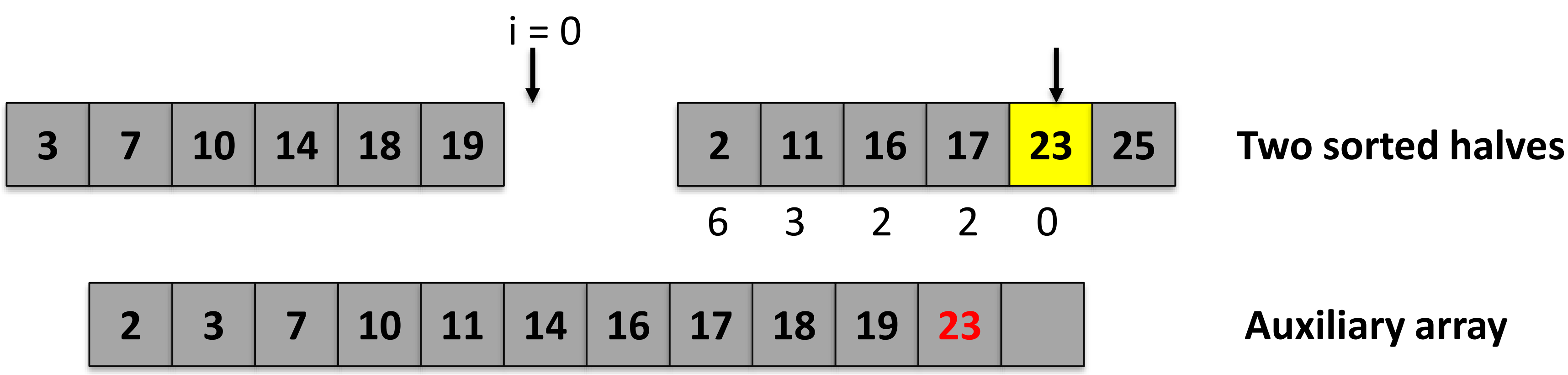

Total: 6+3+2+2+0

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

i = 0

Two sorted halves

3	7	10	14	18	19

2	11	16	17	23	25
6	3	2	2	0	

Auxiliary array

2	3	7	10	11	14	16	17	18	19	23	25

Total: 6+3+2+2+0+0

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
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i = 0

3	**7**	**10**	**14**	**18**	**19**

2	**11**	**16**	**17**	**23**	**25**
6	3	2	2	0	

Two sorted halves

2	**3**	**7**	**10**	**11**	**14**	**16**	**17**	**18**	**19**	**23**	**25**

Auxiliary array

Total: 6+3+2+2+0+0 = 13

Combine two subproblems: Improvement

Merge-and-Count(A, B)

Input: A, B
Output: r, L
$r \leftarrow 0, L \leftarrow \emptyset;$

Combine two subproblems: Improvement

Merge-and-Count(A, B)

Input: A, B
Output: r, L
$r \leftarrow 0, L \leftarrow \emptyset$;
while *both* A *and* B *are not empty* **do**
  // Let a and b represent the first element of A and B, repectively

Combine two subproblems: Improvement

Merge-and-Count(A, B)

Input: A, B
Output: r, L
$r \leftarrow 0, L \leftarrow \emptyset$;
while *both A and B are not empty* **do**
    // Let a and b represent the first element of A and B, repectively
    if $a < b$ **then**

Combine two subproblems: Improvement

Merge-and-Count(A, B)

Input: A, B
Output: r, L
$r \leftarrow 0, L \leftarrow \emptyset$;
while *both A and B are not empty* **do**
  // Let a and b represent the first element of A and B, repectively
  if $a < b$ **then**
    Move a to the back of L;//$A.length$ is decreased by 1;
  end
  else

Combine two subproblems: Improvement

Merge-and-Count(A, B)

```
Input: A, B
Output: r, L
r ← 0, L ← ∅;
while both A and B are not empty do
    // Let a and b represent the first element of A and B, repectively
    if a < b then
        Move a to the back of L; //A.length is decreased by 1;
    end
    else
        Increase r by A.length;
        Move b to the back of L;
    end
end
```

Combine two subproblems: Improvement

Merge-and-Count(A, B)

Input: A, B
Output: r, L
$r \leftarrow 0, L \leftarrow \emptyset$;
while *both A and B are not empty* **do**
  // Let a and b represent the first element of A and B, repectively
  if $a < b$ **then**
    Move a to the back of L;//$A.length$ is decreased by 1;
  end
  else
    Increase r by $A.length$;
    Move b to the back of L;
  end
end
if A *is not empty* **then**

Combine two subproblems: Improvement

Merge-and-Count(A, B)

```
Input: A, B
Output: r, L
r ← 0, L ← ∅;
while both A and B are not empty do
    // Let a and b represent the first element of A and B, repectively
    if a < b then
        Move a to the back of L;//A.length is decreased by 1;
    end
    else
        Increase r by A.length;
        Move b to the back of L;
    end
end
if A is not empty then
    Move A to the back of L;
end
else
```

Combine two subproblems: Improvement

Merge-and-Count(A, B)

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Input: A, B
Output: r, L
r ← 0, L ← ∅;
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if A is not empty then
    Move A to the back of L;
end
else
    Move B to the back of L;
end
return
```

Combine two subproblems: Improvement

Merge-and-Count(A, B)

```
Input: A, B
Output: r, L
r ← 0, L ← ∅;
while both A and B are not empty do
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    if a < b then
        Move a to the back of L;//A.length is decreased by 1;
    end
    else
        Increase r by A.length;
        Move b to the back of L;
    end
end
if A is not empty then
    Move A to the back of L;
end
else
    Move B to the back of L;
end
return L, r;
```

Combine two subproblems: Improvement

- For every element in A and B,

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- For every element in A and B,
  - Only O( ) times operations are executed.

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- For every element in A and B,
  - Only O(1) times operations are executed.

- Function *Sort-and-Count(A,B)* can be executed in O(n) time where n is the number of elements in A and B.

The Complete Divide-and-Conquer Algorithm

Review of The Complete MCS Algorithm

$MCS(A, s, t)$

Input: $A[s \dots t]$ with $s \leq t$
Output: MCS of $A[s \dots t]$
begin
    if $s = t$ **then return** $A[s]$;
    else
        $m \leftarrow \lfloor \frac{s+t}{2} \rfloor$;
        Find $MCS(A, s, m)$;
        Find $MCS(A, m+1, t)$;
        Find MCS that contains both $A[m]$ and $A[m+1]$;
        return maximum of the three sequences found
    end
end

The Complete Divide-and-Conquer Algorithm

Sort-and-Count(L)

Input: L
Output: r_L, L

The Complete Divide-and-Conquer Algorithm

Sort-and-Count(L)

Input: L
Output: r_L, L
if L *is empty* **then**
  return $0, L$;
end
Divide L into two halves A and B;

The Complete Divide-and-Conquer Algorithm

Sort-and-Count(L)

Input: L
Output: r_L, L
if L *is empty* **then**
  return $0, L$;
end
Divide L into two halves A and B;
$(r_A, A) \leftarrow$Sort-and-Count(A); $//T(\lceil \frac{n}{2} \rceil)$

The Complete Divide-and-Conquer Algorithm

Sort-and-Count(L)

Input: L
Output: r_L, L
if L *is empty* **then**
  return $0, L$;
end
Divide L into two halves A and B;
$(r_A, A) \leftarrow$Sort-and-Count(A);$//T(\lceil \frac{n}{2} \rceil)$
$(r_B, B) \leftarrow$Sort-and-Count(B);$//T(\lfloor \frac{n}{2} \rfloor)$
$(r_L, L) \leftarrow$

The Complete Divide-and-Conquer Algorithm

Sort-and-Count(L)

Input: L
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end
Divide L into two halves A and B;
$(r_A, A) \leftarrow$Sort-and-Count(A);$//T(\lceil \frac{n}{2} \rceil)$
$(r_B, B) \leftarrow$Sort-and-Count(B);$//T(\lfloor \frac{n}{2} \rfloor)$
$(r_L, L) \leftarrow$Merge-and-Count(A, B);$//O(n)$
return

The Complete Divide-and-Conquer Algorithm

Sort-and-Count(L)

Input: L
Output: r_L, L
if L *is empty* **then**
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end
Divide L into two halves A and B;
$(r_A, A) \leftarrow$Sort-and-Count(A);$//T(\lceil \frac{n}{2} \rceil)$
$(r_B, B) \leftarrow$Sort-and-Count(B);$//T(\lfloor \frac{n}{2} \rfloor)$
$(r_L, L) \leftarrow$Merge-and-Count(A, B);$//O(n)$
return $r_A + r_B + r_L, L$;

The Complete Divide-and-Conquer Algorithm

Sort-and-Count(L)

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$(r_L, L) \leftarrow$Merge-and-Count(A, B);$//O(n)$
return $r_A + r_B + r_L, L$;

$$T(n) = \begin{cases} O(1), & if\ n = 1 \\ T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + O(n) & otherwise \end{cases}$$

Example

Divide

14	**7**	**18**	**3**	**10**	**19**	**11**	**23**	**2**	**25**	**16**	**17**

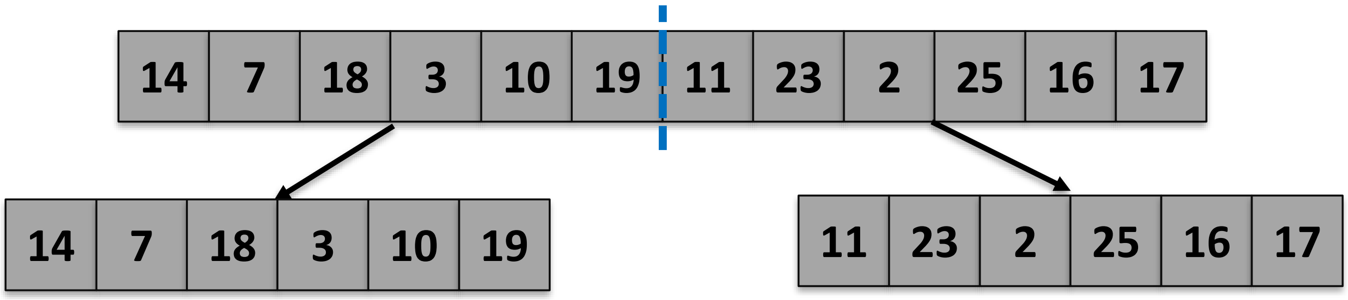
Example

Divide

14	7	18	3	10	19	11	23	2	25	16	17

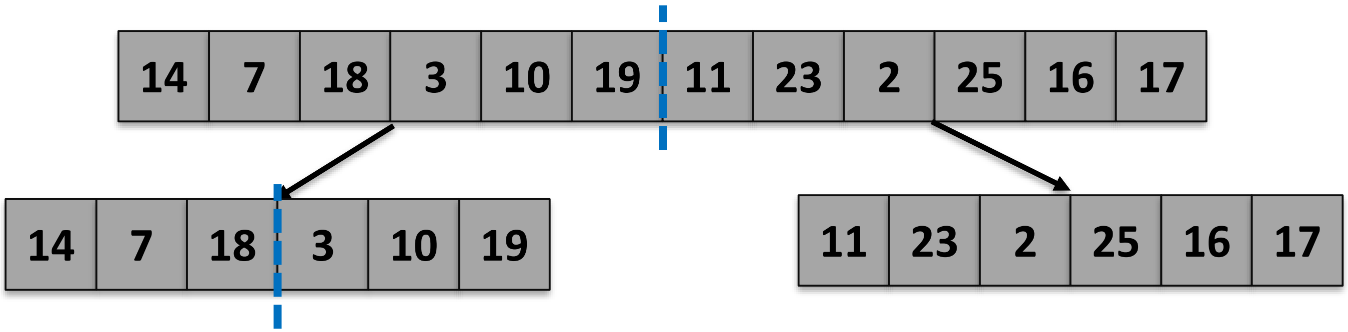
Example

Divide



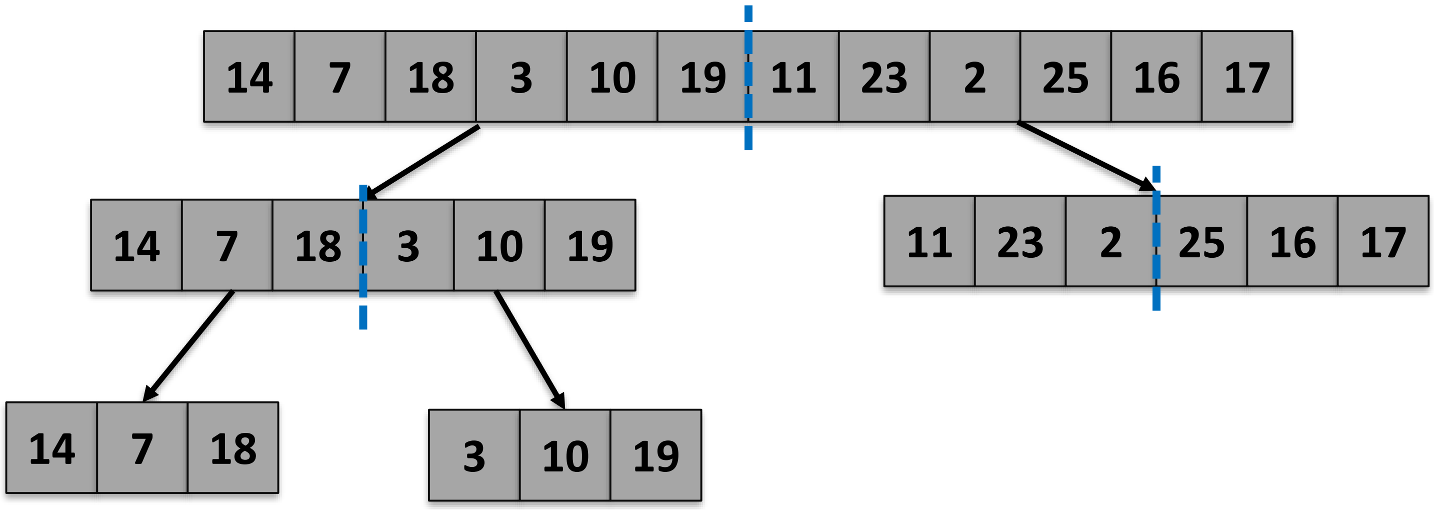
Example

Divide



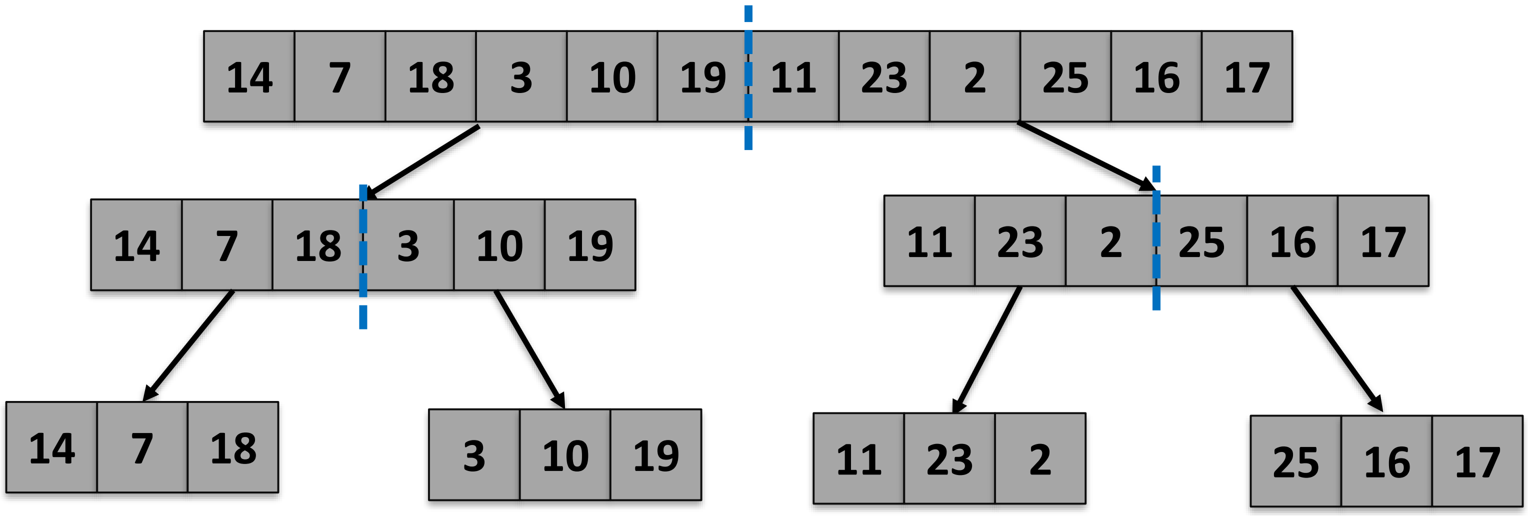
Example

Divide



Example

Divide



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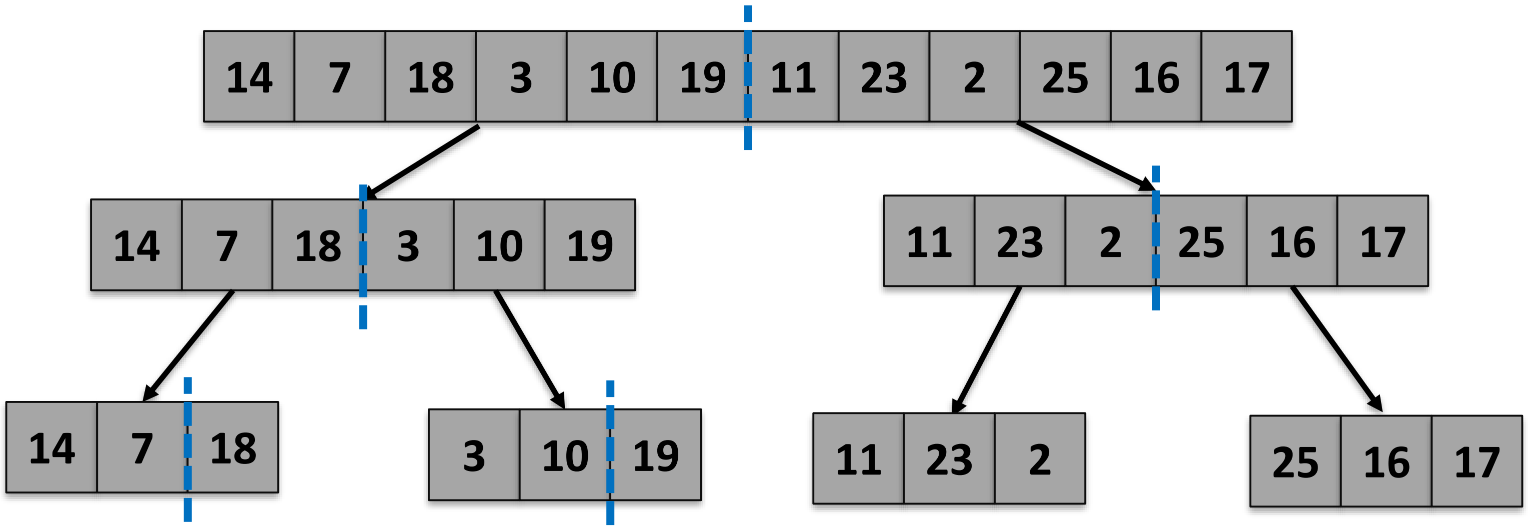
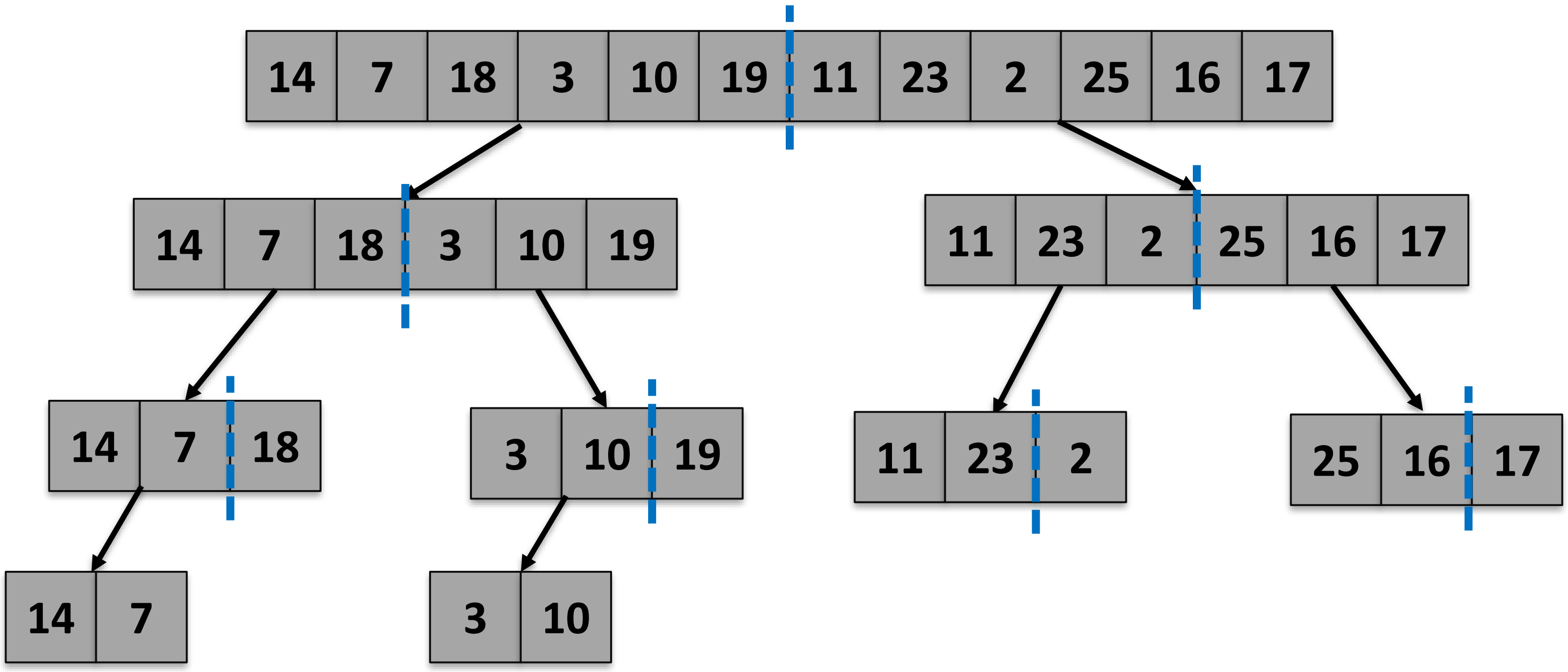
Divide



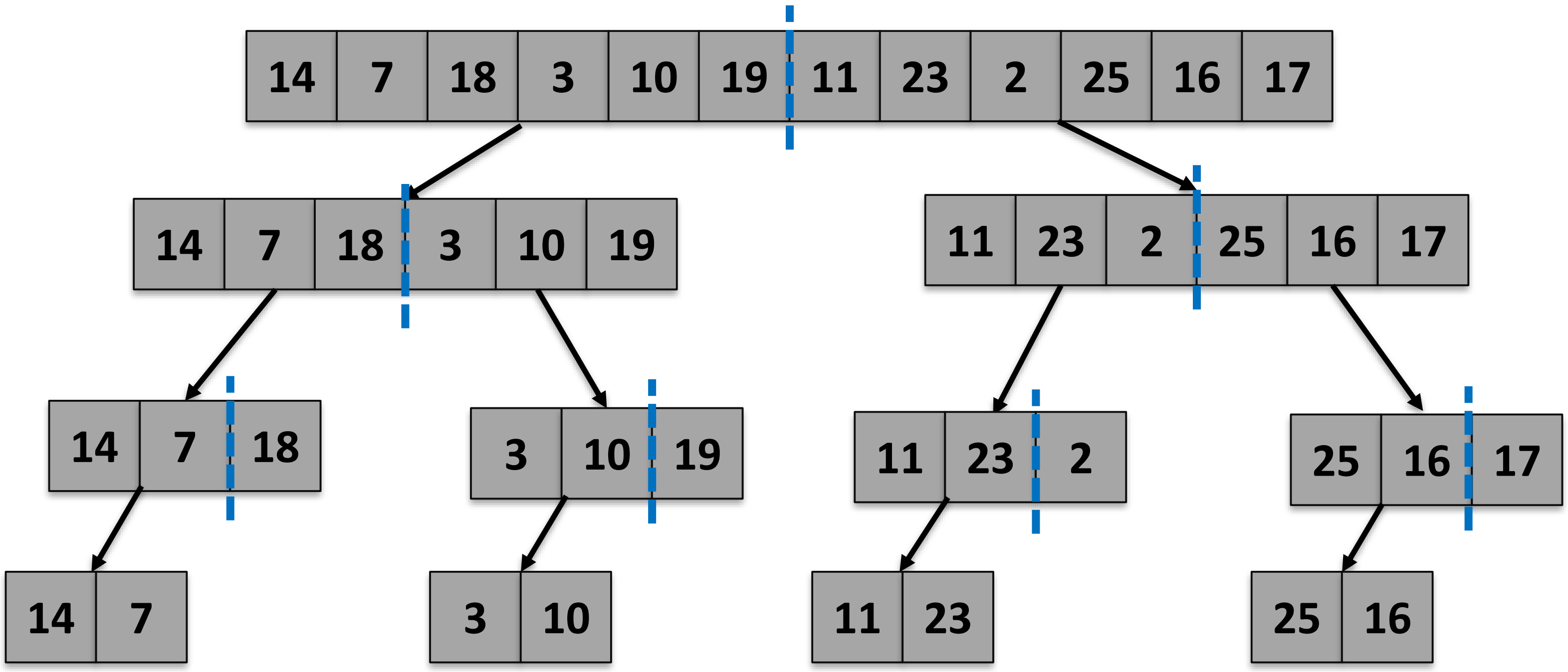
Example

Divide



Example

Divide



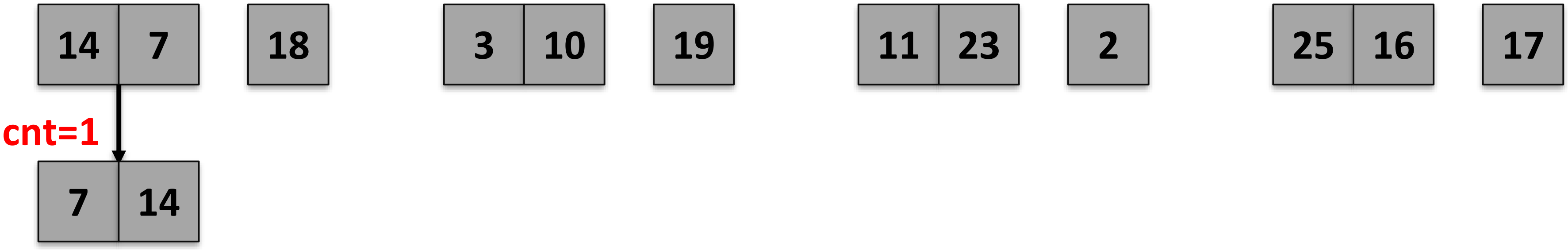
Example

Conquer

14	7	18	3	10	19	11	23	2	25	16	17

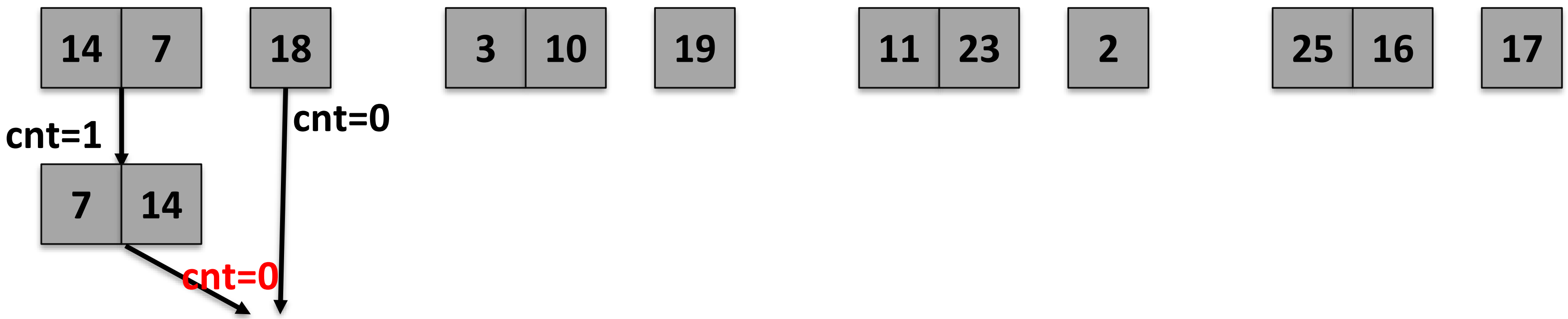
Example

Conquer



Example

Conquer



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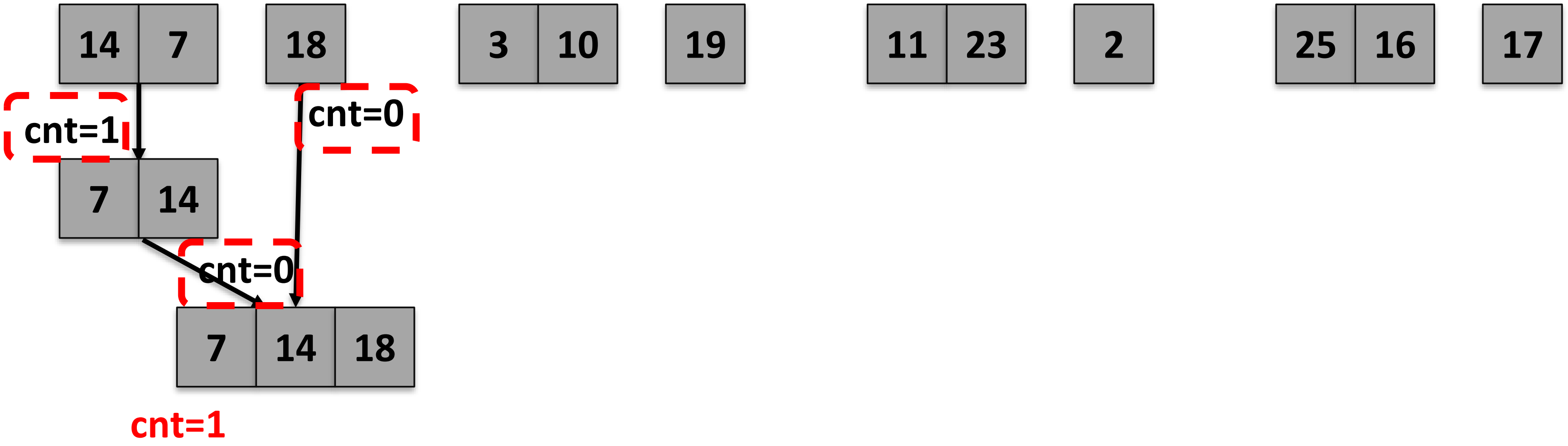

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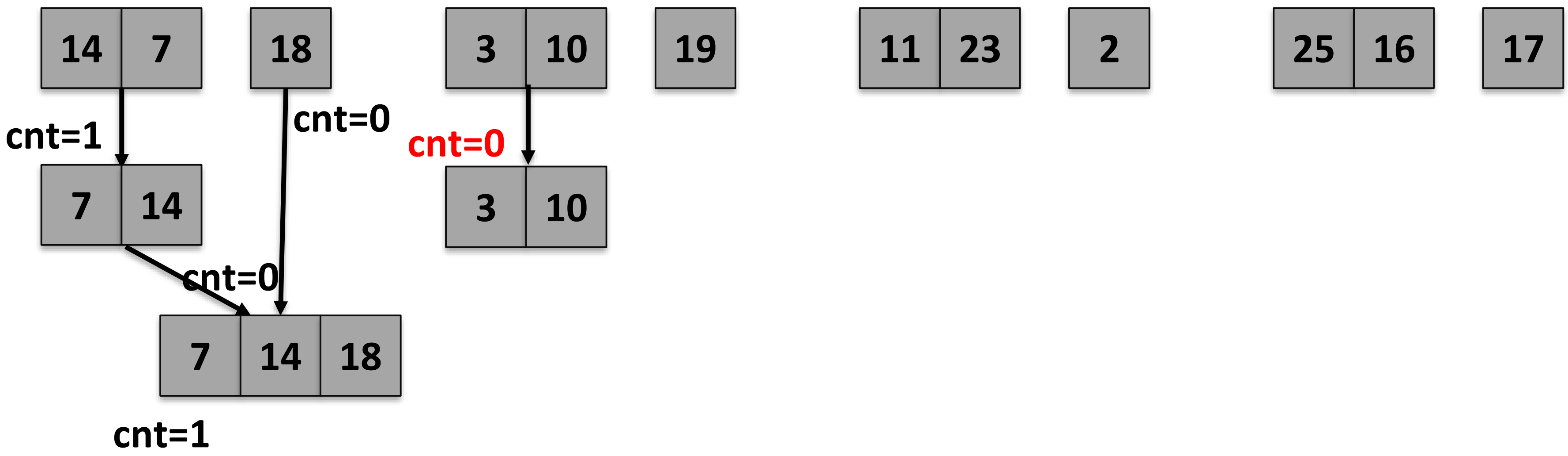

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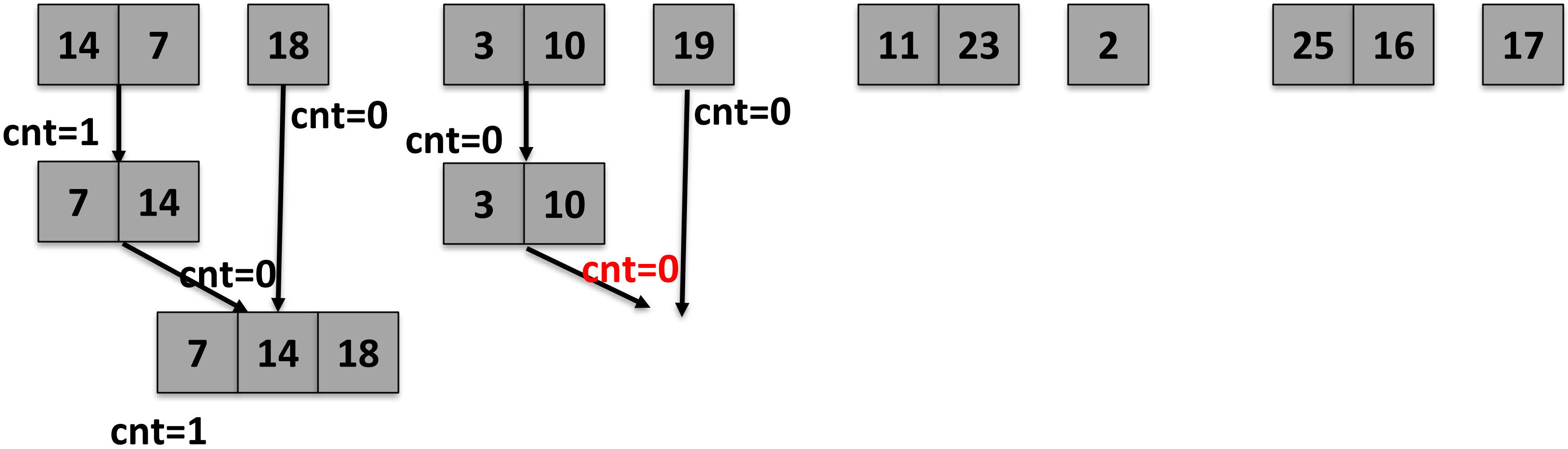

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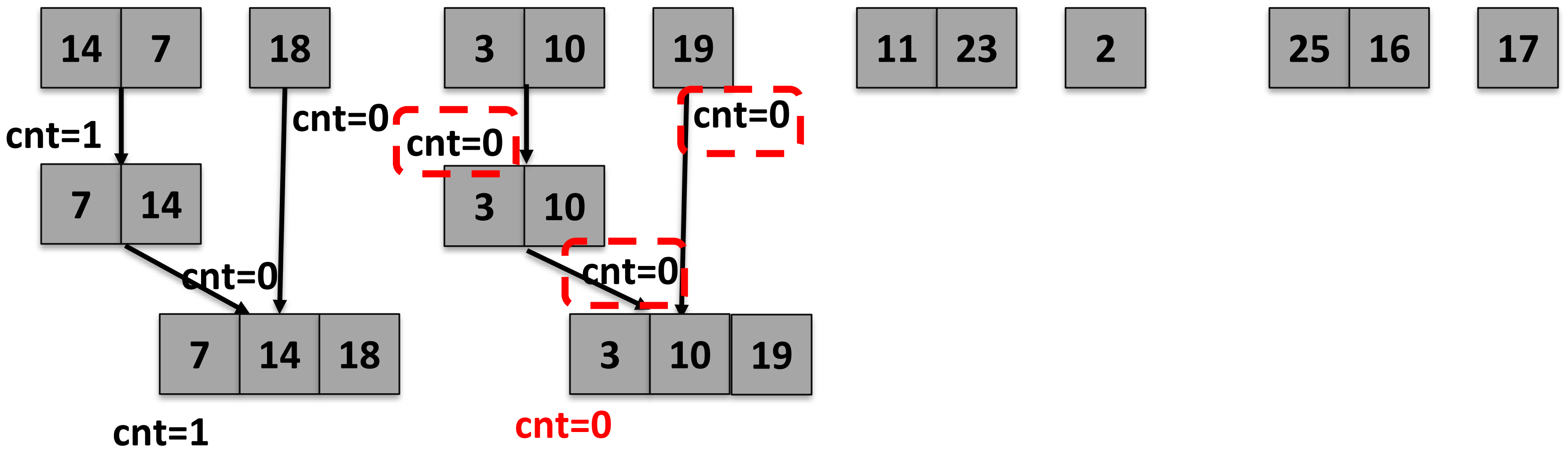

Example

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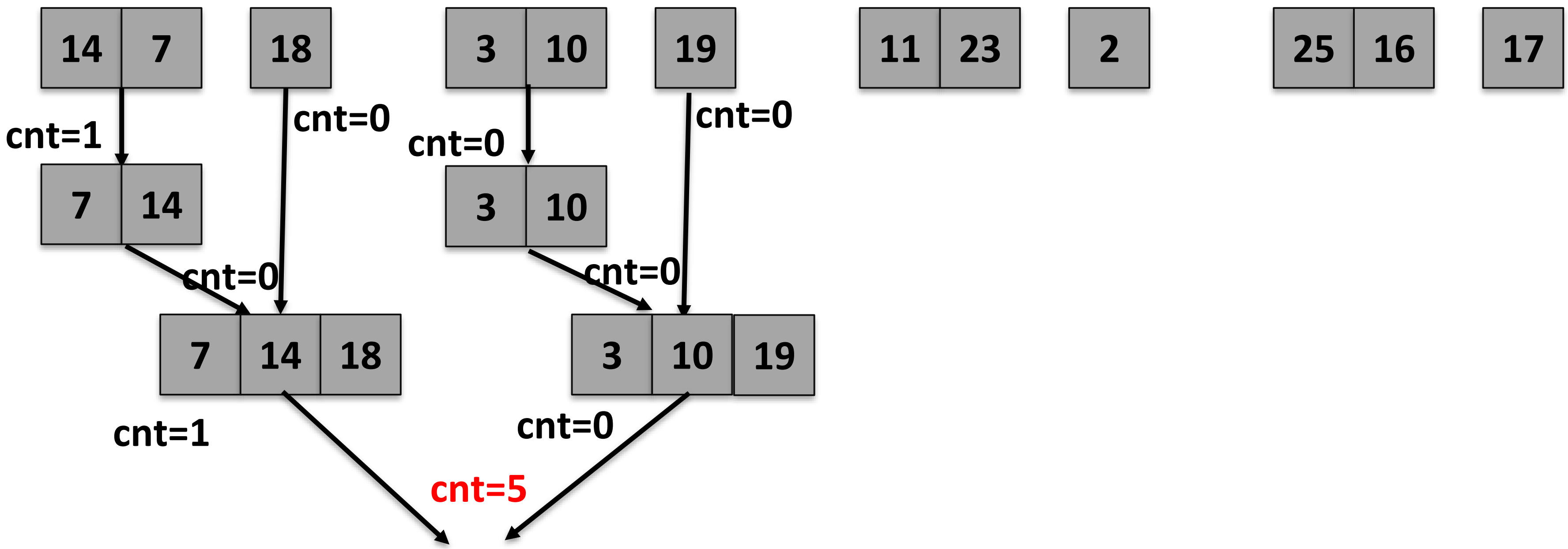

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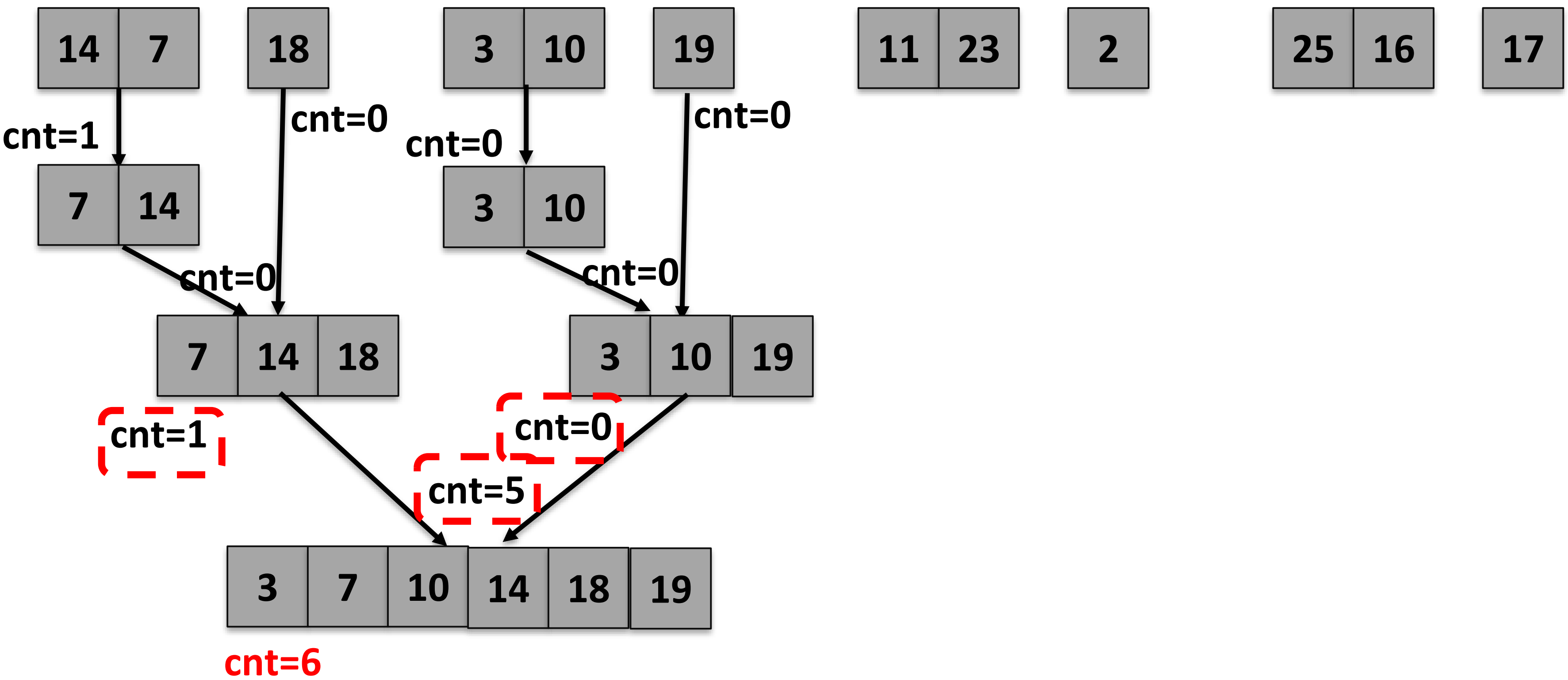

Example

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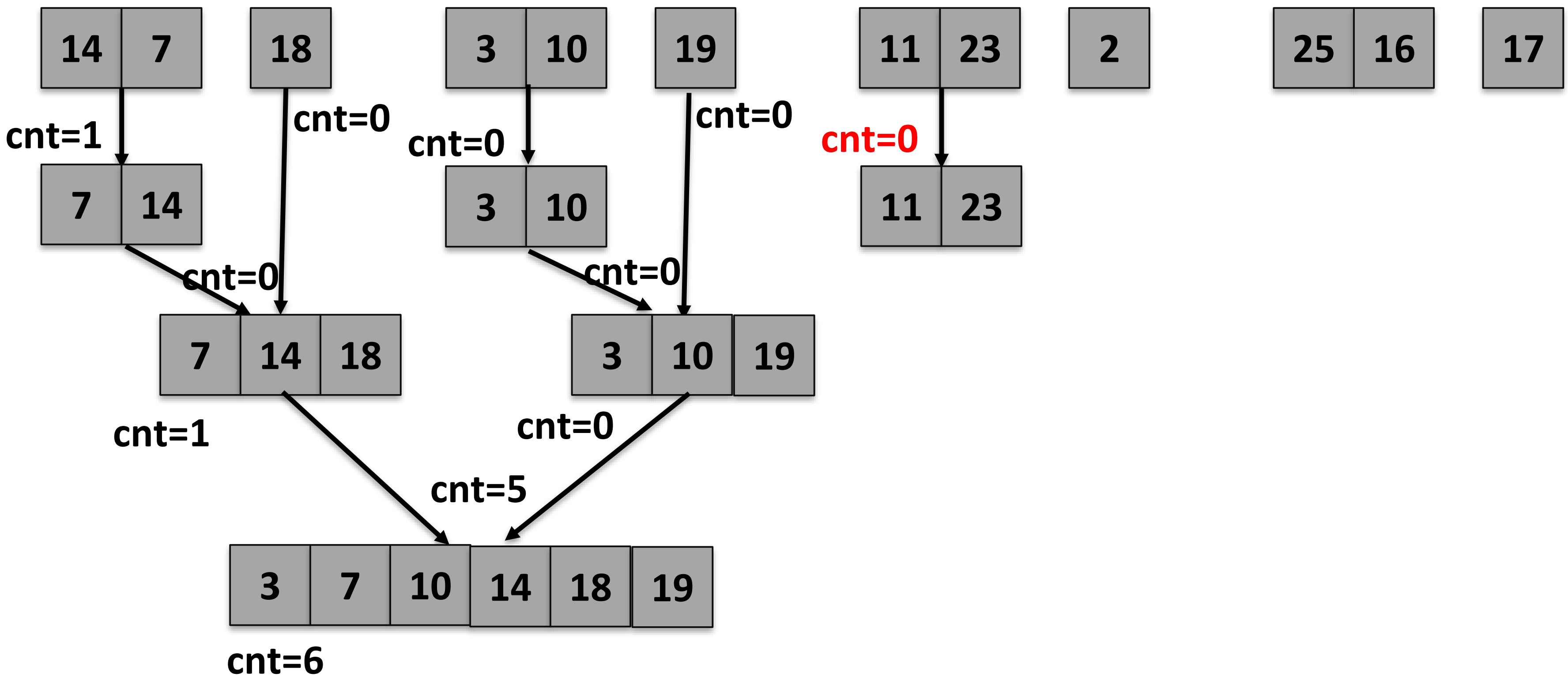

Example

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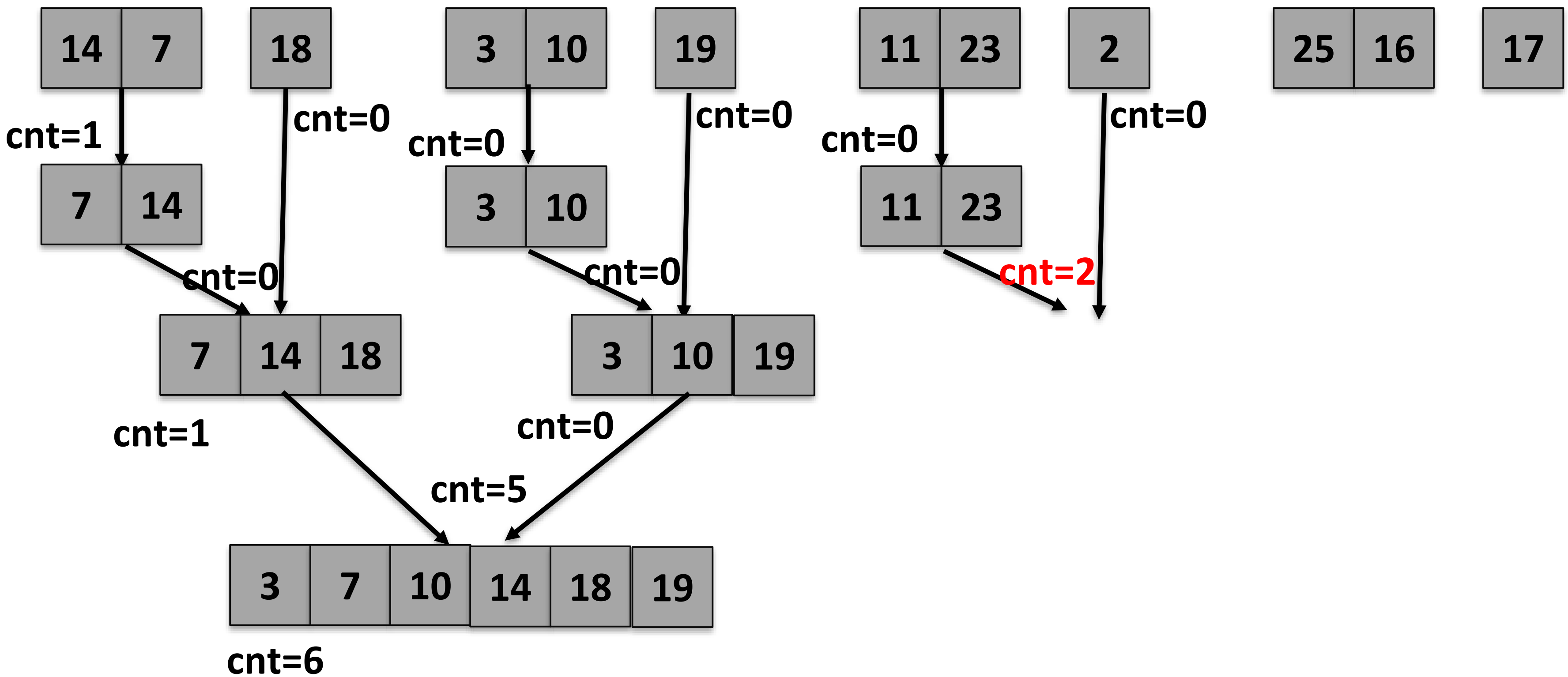

Example

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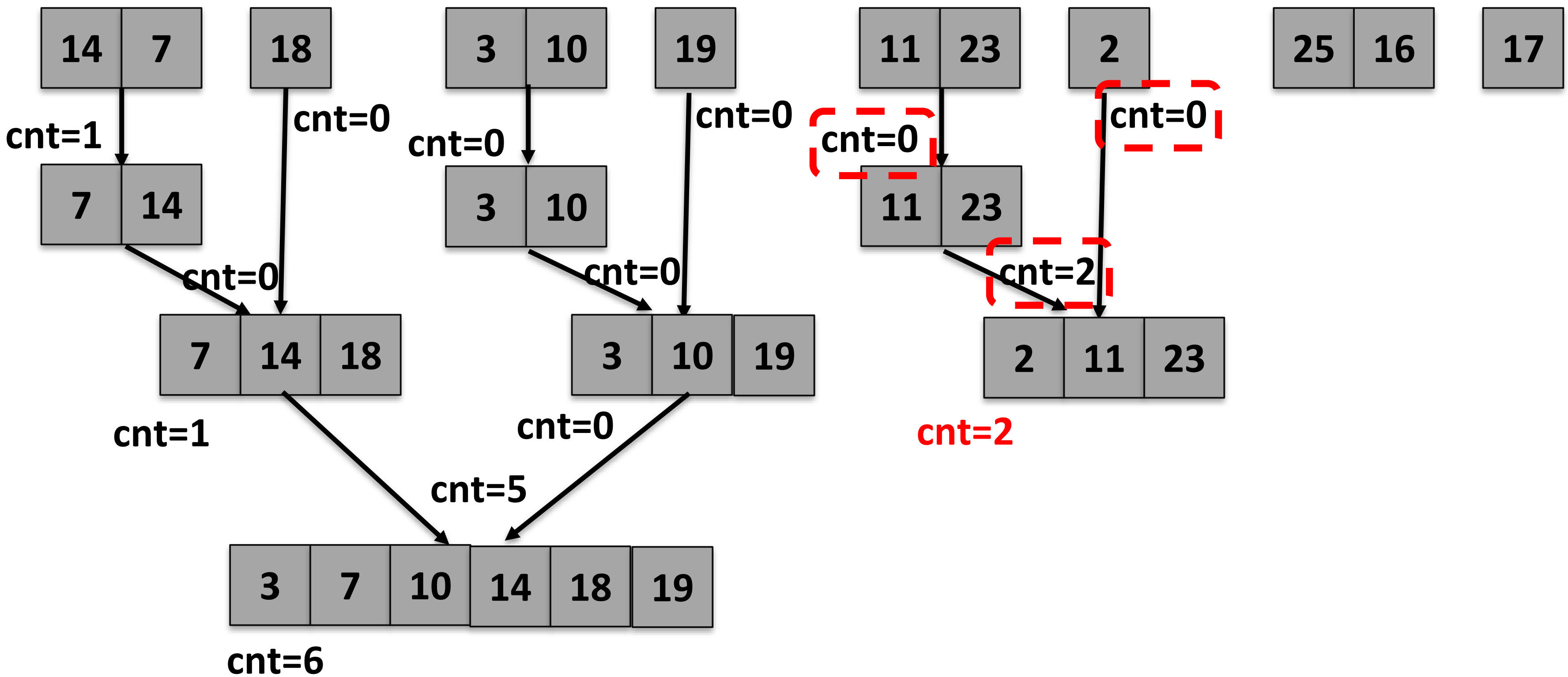

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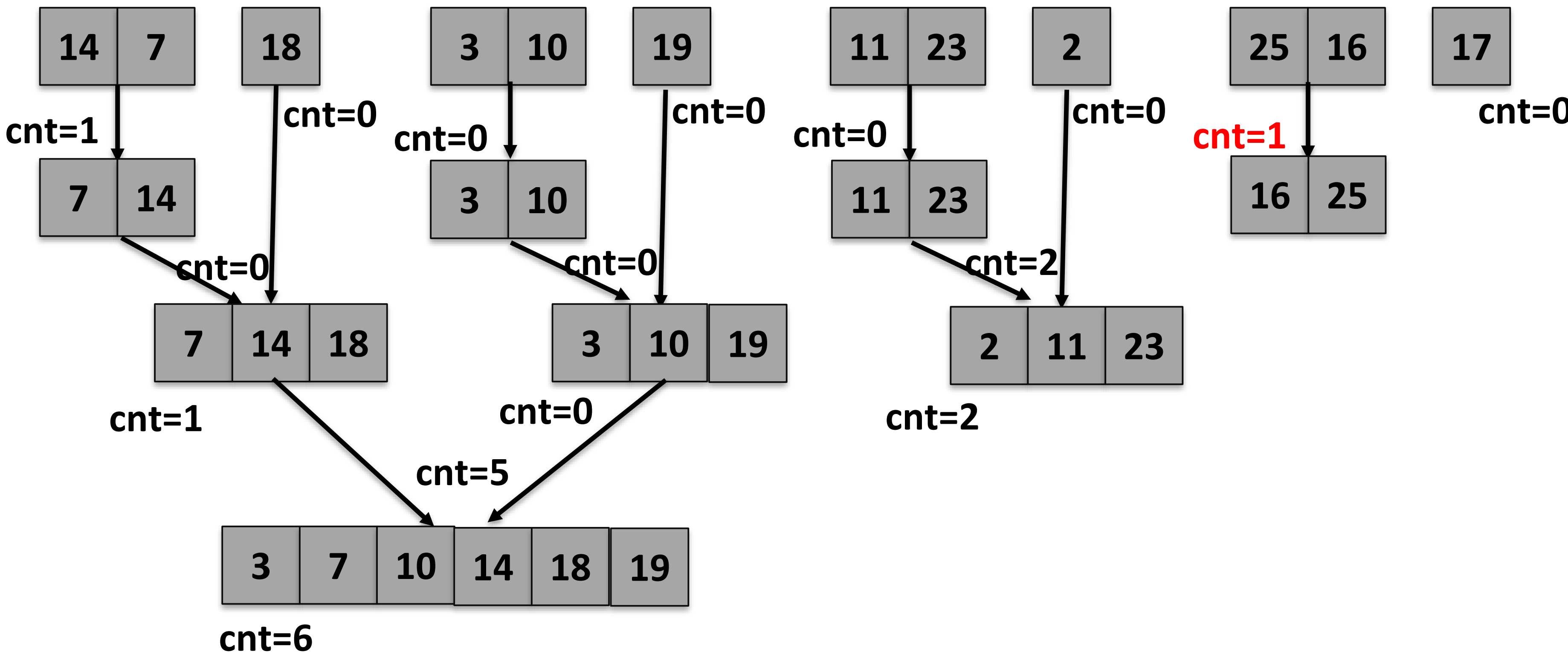

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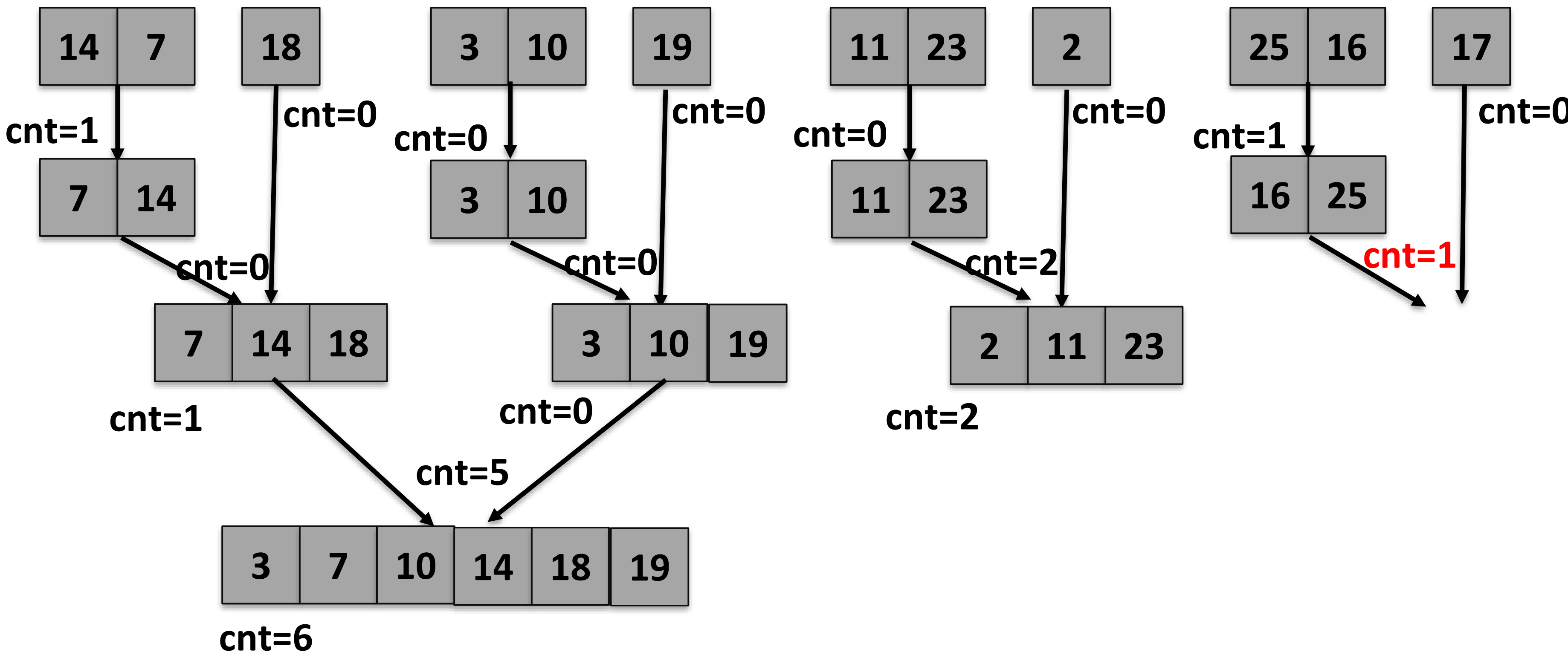

Example

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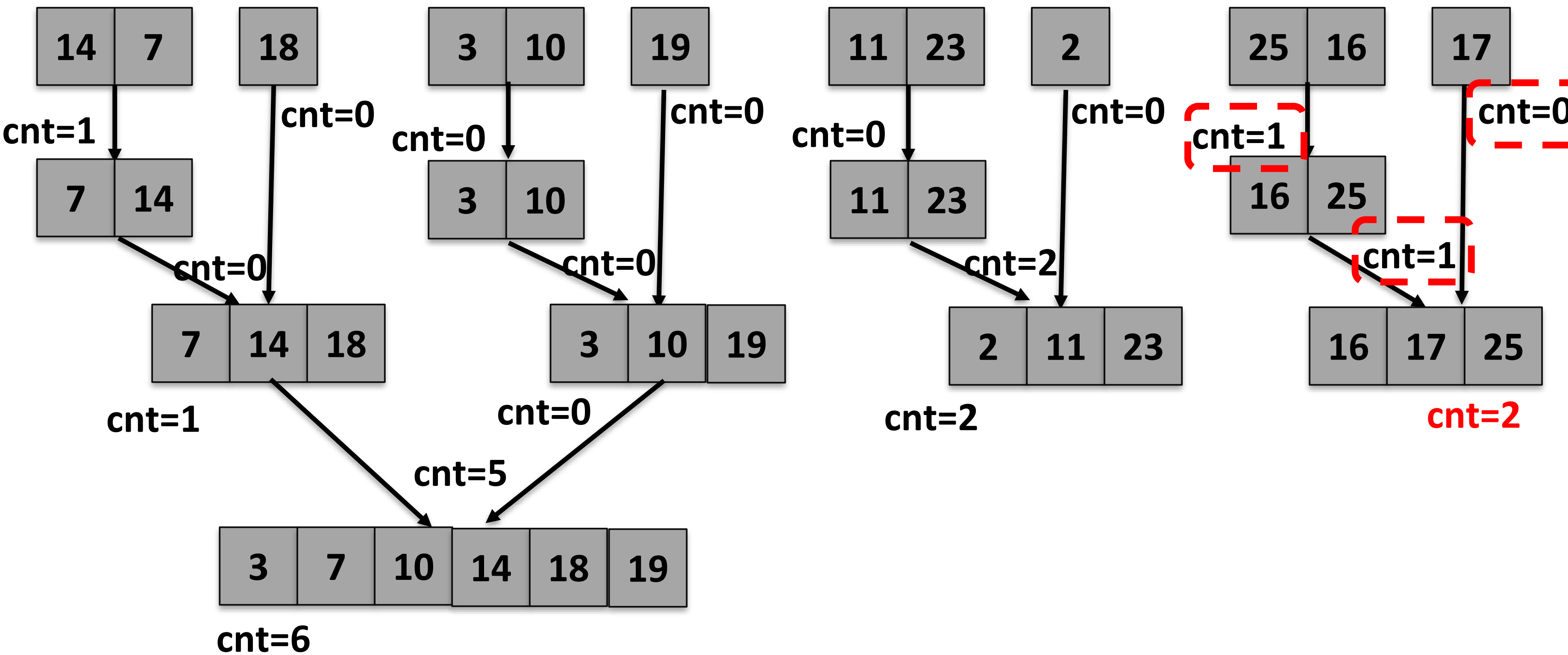

Example

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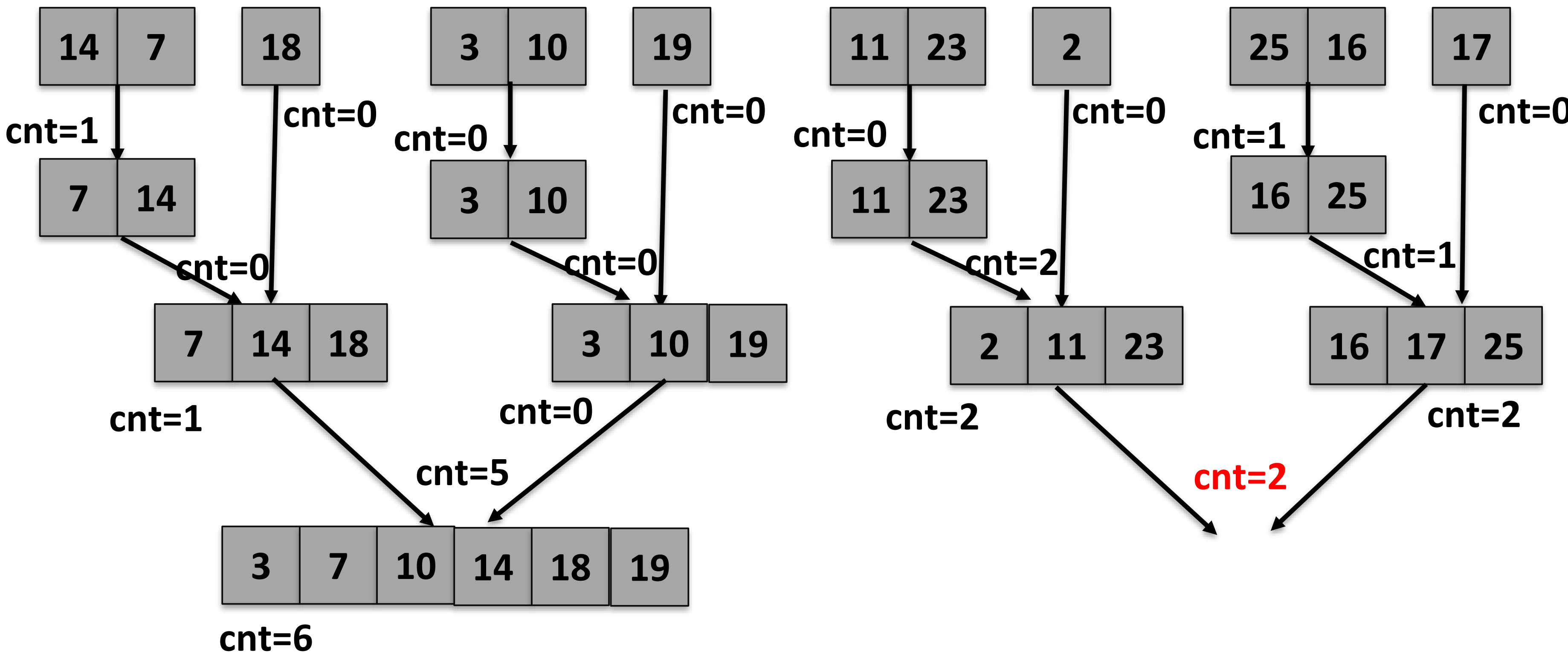
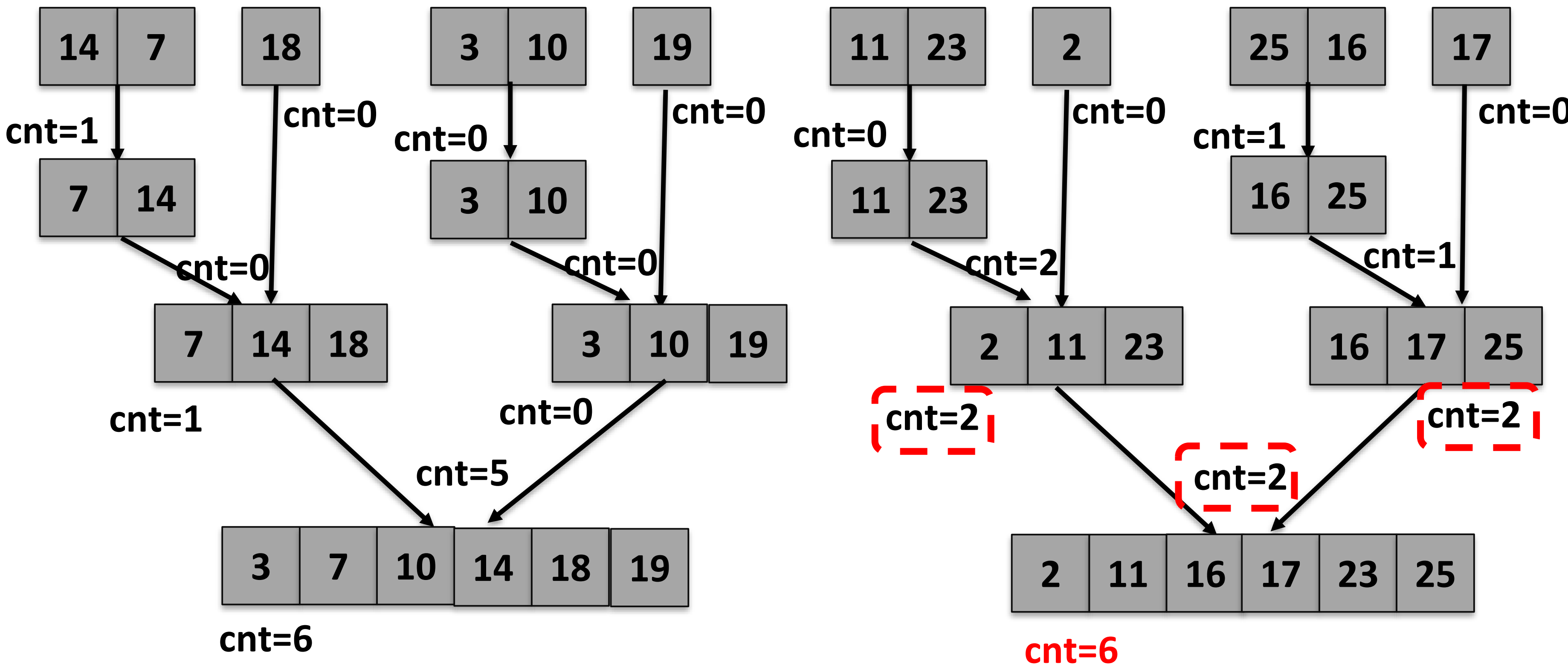


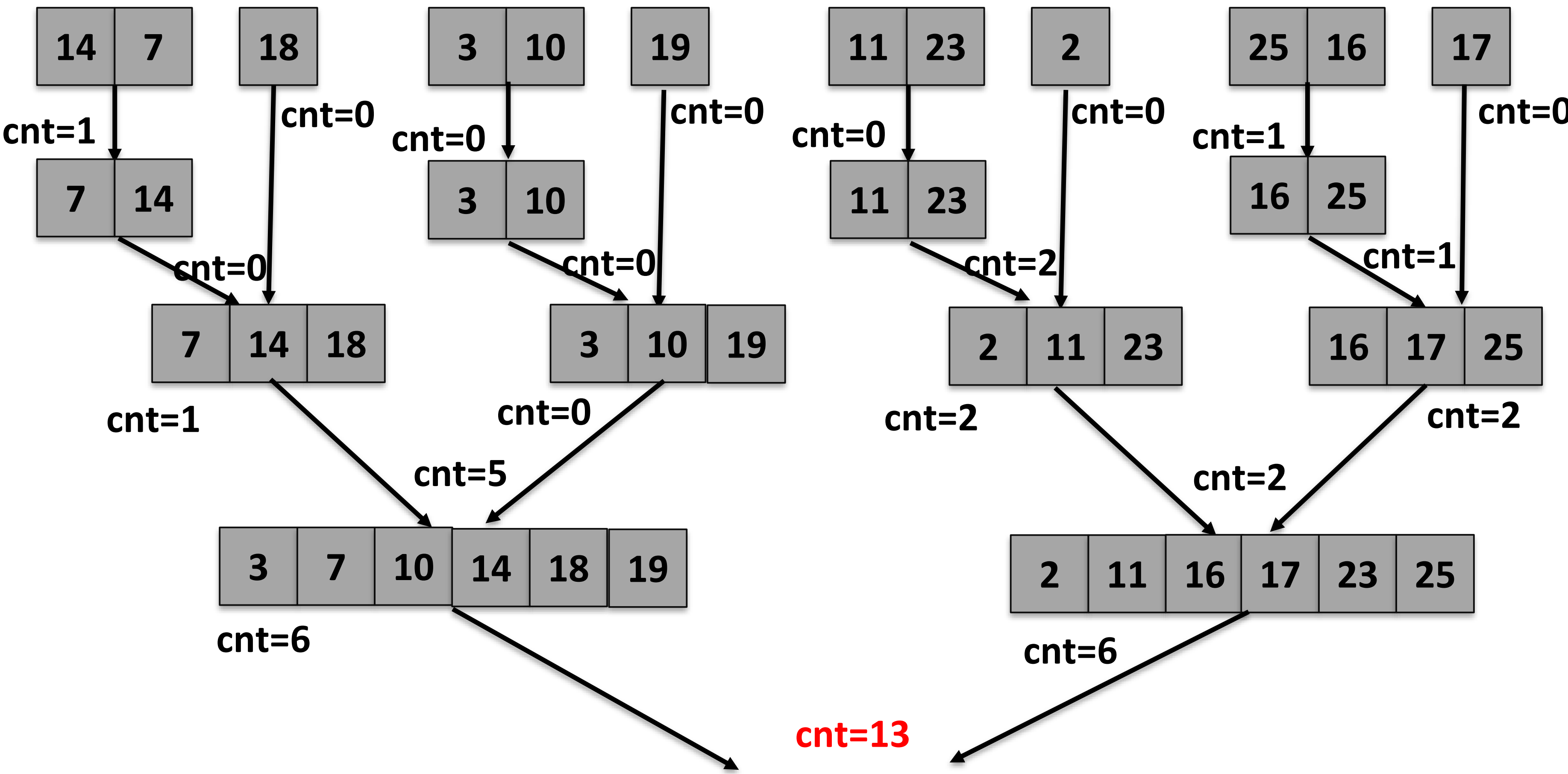
Example

Conquer



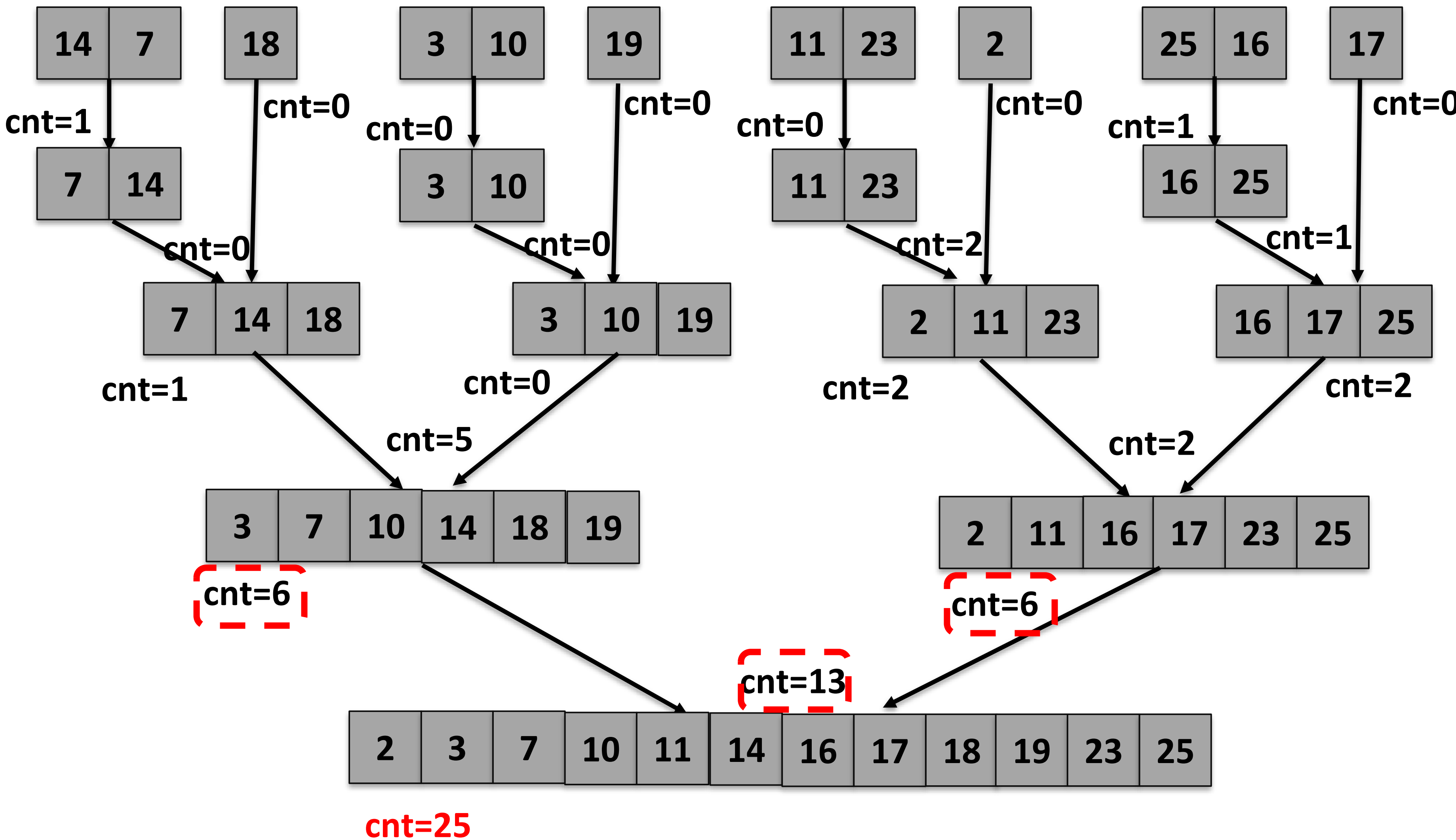
Example

Conquer



Example

Conquer



Outline

- Review to Divide-and-Conquer Paradigm
- Polynomial Multiplication Problem
  - Problem definition
  - A brute force algorithm
  - A first divide-and-conquer algorithm
  - An improved divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm
- Counting Inversion Problem
  - Problem definition
  - A brute force algorithm
  - A divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm

Analysis of the D&C Algorithm

Proposition. The sort-and-count algorithm counts the number of inversions in a permutation of size n in $O(n \log n)$ time.

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Proof. The worst-case running time T(n) satisfies the recurrence:

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Proof. The worst-case running time T(n) satisfies the recurrence:

$$T(n) = \begin{cases} O(1), & if\ n = 1 \\ T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + O(n) & otherwise \end{cases}$$


dank u
ju faleminderit
Tack
Asante
谢谢
Tak
mulţumesc
kiitos
Salamat!
Gracias
Terima kasih
Aliquam
Merci
Dankie
Obrigado
ありがとう
köszönöm
grazie
Aliquam
Go raibh maith agat
děkuji
Thank you
gam